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HARNACK TYPE INEQUALITIES FOR SOME DOUBLY NONLINEAR SINGULAR PARABOLIC EQUATIONS

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Abstract. We prove Harnack type inequalities for a wide class of parabolic doubly nonlinear equations including $u_t = \mathrm{div}(|u|^{m-1}|Du|^{p-2}Du)$. We will distinguish between the *supercritical* range $3 - \frac{p}{N} < p + m < 3$ and the *subcritical* $2 < p+m \leq 3 - \frac{p}{N}$ range. Our results extend similar estimates holding for general equations having the same structure as the parabolic p-Laplace or the porous medium equation and recently collected in [6].

1. **Introduction.** Consider an open set $E \subset \mathbf{R}^N$, $T > 0$, and quasi-linear parabolic differential equations of the form

$$u_t - \mathrm{div} A\big(x, t, u, D(|u|^{\frac{m-1}{p-1}} u)\big) = 0 \tag{1}$$

in $E_T = E \times (0, T]$, with $p+m > 2$. The function $A : E_T \times \mathbf{R}^{N+1} \to \mathbf{R}^N$ is assumed to be measurable and subject to the structure conditions

$$\begin{cases} A(x,t,z,\eta) \cdot \eta \geq C_0 |\eta|^p \\ |A(x,t,z,\eta)| \leq C_1 |\eta|^{p-1} \end{cases} \tag{2}$$

for almost all $(x,t) \in E_T$, for all $z \in \mathbf{R}$ and $\eta \in \mathbf{R}^N$, with C_0, C_1 positive constants. Assume also that the function A is monotone in the variable η

$$(A(x,t,z,\eta_1) - A(x,t,z,\eta_2)) \cdot (\eta_1 - \eta_2) \geq 0 \tag{3}$$

and Lipschitz continuous in the variable $|z|^{\frac{m-1}{p-1}} z$ in the following sense

$$|A(x,t,z_1,\eta) - A(x,t,z_2,\eta)| \leq \Lambda \Big| |z_1|^{\frac{m-1}{p-1}} z_1 - |z_2|^{\frac{m-1}{p-1}} z_2 \Big| (1 + |\eta|^{p-1}) \tag{4}$$

for some given $\Lambda > 0$, and for the variables in the indicated domains. A prototype example is

$$u_t - \Big(m\, a_{ij}(x,t) \big| D |u|^{\frac{m-1}{p-1}} u \big|^{p-2} (|u|^{\frac{m-1}{p-1}} u)_{x_j} \Big)_{x_i} = 0$$

which can be written also as

$$u_t - \big(m^{p-1} a_{ij}(x,t) |u|^{m-1} |Du|^{p-2} u_{x_j} \big)_{x_i} = 0 \tag{5}$$

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weakly in E_T, where the matrix (a_{ij}) is only measurable and positive definite in E_T. In the literature these equations are classified as doubly non–linear, due to the fact that the diffusion term depends non-linearly on both the unknown function u and its gradient Du. A particularly interesting equation belonging to this class is

$$u_t - \mathrm{div}(|u|^{m-1}|Du|^{p-2}Du) = 0. \tag{6}$$

Such gradient-dependent diffusion equations appear in many different physical contexts such as, for instance, the description of turbulent filtration in porous media, or the flow of a gas through a porous medium in a turbulent regime ([5]) or in glaciology ([4]). In [12], a one dimensional version of (6) has been proposed to describe the filtration of water in porous building materials, such as bricks.

Besides its physical interpretation, equation (6) has been widely studied also from a theoretical point of view, since it unifies the well–known p–Laplace equation ($m = 1$) with the classical porous media equation ($p = 2$). Therefore, it is interesting to see how much of the regularity properties of the solutions of the p–Laplace and the porous media equations is preserved in this more general case. The literature related to doubly non-linear equations is quite rich and this fact shows that these equations have attracted a lot of attention. Existence, uniqueness and regularity of solutions expecially for the model (6) have been studied by several authors, see e.g. [1], [2], [3], [10], [11], [14], [15], [16], [17], [18] and [19].

We will deal with the singular range for the parameters m, p, namely $m + p < 3$ and we will establish some Harnack type inequalities for non-negative local weak solutions to the wide class of equations (1)-(2) in a specific geometry which is intrinsic to the solution and reflects the singularity of the equation. We will show that there is a critical threshold for such Harnack estimates, which occurs when $m + p + \frac{p}{N} = 3$. In the singular *supercritical* range

$$3 - \frac{p}{N} < p + m < 3,$$

a Harnack inequality holds in the same intrinsic form of the degenerate case $m+p > 3$ (see [7]); in addition, another family of Harnack inequalities will be proved. These will be simultaneously *forward* in time, *backward* in time, and *elliptic*. In the *critical* and *subcritical* range

$$2 < p + m \leq 3 - \frac{p}{N}$$

no Harnack estimate in any of the forms mentioned above holds. However, we will prove a different form of Harnack estimate for p, m in such a range, with constants depending on the ratio of some integral norms of the solution u (see Section 4). This inequality can be seen as a weak form of the Harnack inequality given in Section 3 for the supercritical range. Although the results are formally identical to those for equations of p-Laplacian and porous medium type and thus well expected, their proofs are not a straightforward extension, due to the double singularity exhibited by the equation. The literature does not always contain complete formal proofs of the results regarding doubly non-linear equations. For this reason, we have started a project aiming at providing all the technical details of the proofs of the Harnack inequalities for weak solutions of the singular equations (1)-(2). Such project started with the previous papers [8] and [9], where several preliminary results have been proved (even for more general equations), as DeGiorgi type lemmas, expansion of positivity, L^∞ local bounds, and ends with the present one, devoted to Harnack estimates. We have used the same ideas as in [6] where the cases $p = 2$ or $m = 1$ have been considered.

Concerning assumptions (3) and (4), we remark that they are needed to apply the comparison principle (see Corollary 1). We require such result in our approach for the supercritical range. However, to simplify the exposition, we have assumed such hypotheses since the beginning for the whole singular range $2 < m + p < 3$.

2. **Preliminaries.** A function $u : E_T \to \mathbf{R}$ is a local weak solution of (1)-(2) if

$$u \in C(0, T; L^2_{\rm loc}(E)) \cap W^{1,1}_{\rm loc}(0, T; L^1_{\rm loc}(E)), \qquad |u|^{\frac{m-1}{p-1}} u \in L^p_{\rm loc}(0, T; W^{1,p}_{\rm loc}(E)),$$

and

$$\int_K u\psi dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K \Big(- u\psi_t + A(x, t, u, D(|u|^{\frac{m-1}{p-1}} u)) \cdot D\psi \Big) dxdt = 0 \qquad (7)$$

for every compact set $K \subset E$, for every sub-interval $[t_1, t_2] \subset (0, T]$ and for every test function

$$\psi \in W^{1,2}_{\rm loc}(0, T; L^2(K)) \cap L^p_{\rm loc}(0, T; W^{1,p}_0(K)).$$

The integrability hypothesis on u ensures that the integrals in (7) are well defined. Sharp integrability hypothesis can be assumed by distinguishing the cases $m > 1$ and $m < 1$. For simplicity, we prefer to maintain a univalent definition.

We notice that weak solutions are also required to be in the class $W^{1,1}_{\rm loc}(0, T; L^1_{\rm loc}(E))$ as we need, in the supercritical range, the comparison principle (see Corollary 1), which is related to the uniqueness of solutions. In the subcritical range we can remove this hypothesis.

We denote by $K_\rho(y)$ the cube of $\mathbf{R}^N$ centered at y with edge 2ρ. If $y = 0$, we simply write K_ρ instead of $K_\rho(0)$. If Ω is a bounded open set contained in E with smooth boundary $\partial\Omega$, consider the boundary value problem

$$\begin{cases} u \in C(0, T; L^2(\Omega)) \cap W^{1,1}(0, T; L^1(\Omega)), \quad |u|^{\frac{m-1}{p-1}} u \in L^p(0, T; W^{1,p}(\Omega)) \\ u_t - {\rm div} A(x, t, u, D(|u|^{\frac{m-1}{p-1}} u)) = 0 \quad \text{weakly in } \Omega \times (0, T) \\ (|u|^{\frac{m-1}{p-1}} u)(\cdot, t) \big|_{\partial\Omega} = g(\cdot, t) \in L^p(0, T; W^{1-\frac{1}{p}}(\partial\Omega)) \\ u(\cdot, 0) = u_0 \in L^2(\Omega). \end{cases} \qquad (8)$$

Proposition 1. *Let A satisfy (2)-(3)-(4). Then the boundary value problem (8) has at most one solution.*

Proof. For $\varepsilon > 0$ let $H_\varepsilon(\cdot)$ be the approximation to the Heaviside function defined by

$$H_\varepsilon(s) = \begin{cases} 0 & \text{if } s \le 0 \\ \dfrac{s}{\varepsilon} & \text{if } 0 < s < \varepsilon \\ 1 & \text{if } s \ge \varepsilon. \end{cases}$$

If u and v are two weak solutions to (8), in their respective weak formulation take the test function $H_\varepsilon(\xi)$, with $\xi = |u|^{\frac{m-1}{p-1}} u - |v|^{\frac{m-1}{p-1}} v$, and subtract the expressions so obtained to get

$$\int_0^t \int_\Omega \partial_\tau (u - v) H_\varepsilon(\xi) dxd\tau$$

$$+\int_0^t\int_\Omega H'_\varepsilon(\xi)\big(A(x,\tau,u,D(|u|^{\frac{m-1}{p-1}}u))-A(x,\tau,u,D(|v|^{\frac{m-1}{p-1}}v))\big)\cdot D\xi dxd\tau$$
$$=\int_0^t\int_\Omega H'_\varepsilon(\xi)\big(A(x,\tau,v,D(|v|^{\frac{m-1}{p-1}}v))-A(x,\tau,u,D(|v|^{\frac{m-1}{p-1}}v))\big)\cdot D\xi dxd\tau$$

for all $t\in(0,T)$. The second term on the left-hand side is discarded by the monotonicity assumption (3). As $\varepsilon\to 0$, the first term tends to

$$\int_0^t\int_\Omega \partial_\tau(u-v)_+dxd\tau=\int_\Omega(u-v)_+(x,t)dx,$$

for all $t\in(0,T)$. The term on the right-hand side is estimated by making use of the Lipschitz continuity (4), and is majorized by

$$\frac{\Lambda}{\varepsilon}\iint_{\Omega\times(0,T)\cap[0<\xi<\varepsilon]}\xi\big(1+\big|D|v|^{\frac{m-1}{p-1}}v\big|^{p-1}\big)|D\xi|dxd\tau$$
$$\le\Lambda\iint_{\Omega\times(0,T)\cap[0<\xi<\varepsilon]}\big(1+\big|D|v|^{\frac{m-1}{p-1}}v\big|^{p-1}\big)|D\xi|dxd\tau\ \to 0\text{ as }\varepsilon\to 0.$$

This proves that $u\le v$ in $\Omega\times(0,T)$. Analogously $v\le u$ in $\Omega\times(0,T)$ and thus the thesis follows. □

As an immediate consequence we have the comparison principle for weak solutions.

Corollary 1 (Weak Comparison Principle)**.** *Let A satisfy (2)-(3)-(4). Let u_i for $i=1,2$ be weak solutions to (8) corresponding to initial and boundary data $u_{0,i}$ and g_i in the indicated functional classes. If $u_{0,1}\le u_{0,2}$ a.e. in Ω and $g_1\le g_2$ a.e. in $\partial\Omega\times(0,T)$, then $u_1\le u_2$ a.e. in $\Omega\times(0,T)$.*

In the proofs of our results we will make use of some local properties of non-negative, local, weak solution to the singular equations (1)-(2). For the sake of completeness, we recall the statements below.

In [8, Theorem 5.1] the following $L^1_{\rm loc}$ form of the Harnack inequality has been proved in the range $2<m+p<3$.

Theorem 2.1. *Let u be a non-negative, local, weak solution to the singular equations (1)-(2) in E_T. There exists a positive constant $\tilde\gamma$ depending only upon the data $\{p,m,N,C_0,C_1\}$, such that for all cylinders $K_{2\rho}(y)\times[s,t]\subset E_T$*

$$\sup_{s<\tau<t}\int_{K_\rho(y)}u(x,\tau)dx\le\tilde\gamma\inf_{s<\tau<t}\int_{K_{2\rho}(y)}u(x,\tau)dx+\tilde\gamma\left(\frac{t-s}{\rho^\lambda}\right)^{\frac{1}{3-m-p}},\tag{9}$$

where

$$\lambda=N(p+m-3)+p.$$

The constant $\tilde\gamma=\tilde\gamma(p,m)\to\infty$ as either $m+p\to 3,2$.

For $\lambda>0$, the parameters m,p are in the singular, super-critical range $m+p>3-\frac{p}{N}$, and if $\lambda\le 0$, m,p are in the critical and sub-critical range $m+p\le 3-\frac{p}{N}$. However the Harnack type estimate in the topology of $L^1_{\rm loc}$ (9) holds true for all $2<m+p<3$ and accordingly, λ could be of either sign.

In the sequel we also need the $L^r_{\rm loc}-L^\infty_{\rm loc}$ estimates below, which have been proved in [9, Theorem 3.1].

Theorem 2.2. *Let u be a non-negative, locally bounded, local, weak solution to the singular equations (1)-(2) in E_T, and let $r \geq 1$ be such that*

$$\lambda_r = N(p+m-3) + rp > 0.$$

Then there exists a positive constant γ depending only on the data $\{p, m, N, C_0, C_1\}$ such that for all cylinders $K_\rho(y) \times [2s-t, t] \subset E_T$

$$\sup_{K_{\frac{1}{2}\rho}(y)\times[s,t]} u \leq \gamma \left(\frac{\rho^p}{t-s}\right)^{\frac{N}{\lambda_r}} \left(\frac{1}{\rho^N(t-s)}\int_{2s-t}^{t}\int_{K_\rho(y)} u^r dx d\tau\right)^{\frac{p}{\lambda_r}} + \left(\frac{t-s}{\rho^p}\right)^{\frac{1}{3-m-p}}.$$

Combining Theorem 2.1 with Theorem 2.2 with $r = 1$ we obtain the following $L^1_{\rm loc} - L^\infty_{\rm loc}$ Harnack type estimate valid for $\lambda > 0$.

Theorem 2.3. *Let u be a non-negative, locally bounded, local, weak solution to the singular equations (1)-(2) in E_T and assume that $\lambda = N(p+m-3)+p > 0$. Then there exists a positive constant γ depending only upon the data $\{p, m, N, C_0, C_1\}$, such that for all cylinders $K_{2\rho}(y) \times [2s-t, t] \subset E_T$*

$$\sup_{K_\rho(y)\times[s,t]} u \leq \frac{\gamma}{(t-s)^{\frac{N}{\lambda}}}\left(\inf_{2s-t<\tau<t}\int_{K_{2\rho}(y)} u(x,\tau)dx\right)^{\frac{p}{\lambda}} + \gamma\left(\frac{t-s}{\rho^p}\right)^{\frac{1}{3-m-p}}.$$

The expansion of positivity is the content of the next proposition. It is at the heart of any form of Harnack inequality. We refer to [9] for the proof.

For $(y, s) \in E_T$ and some given positive constant M, consider the cylinder

$$K_{16\rho}(y) \times (s, s + \delta M^{3-m-p}\rho^p]$$

where $\delta > 0$ is given by Proposition 2 below and $\rho > 0$ is so small that the cylinder is included in E_T.

Proposition 2. *Assume that u is a non-negative local weak solution to (1)-(2) and that there holds*

$$|[u(\cdot, s) \geq M] \cap K_\rho(y)| \geq \alpha|K_\rho(y)|$$

for some $M > 0$ and some $\alpha \in (0, 1)$. Then there exist constants $\varepsilon, \delta, \eta \in (0, 1)$ and $\gamma > 1$ depending only upon the data $\{p, m, N, C_0, C_1\}$ and α, and independent of $(y, s), \rho, M$, such that

$$u(\cdot, t) \geq \eta M \qquad \text{in } K_{2\rho}(y)$$

for all times

$$s + (1-\varepsilon)\delta M^{3-m-p}\rho^p \leq t \leq s + \delta M^{3-m-p}\rho^p.$$

Finally, we need the following DeGiorgi type lemma (see [8] for the proof).

Lemma 2.4. *Let u be a non-negative, locally bounded, local, weak solution to the equation (1)-(2) in E_T. Let $(y, s) \in E_T$ and assume that the cylinder $K_{2\rho}(y) \times (s, s + \theta(2\rho)^p] \subset E_T$ and*

$$u(x, s) \geq \xi M \qquad \text{for a.e. } x \in K_{2\rho}(y),$$

for some $M > 0$ and $\xi \in (0, 1]$. Let $a \in (0, 1)$. Then there exists $\nu_0 \in (0, 1)$, depending only upon a and the data $\{N, m, p, C_0, C_1\}$, such that, if

$$|[u \leq \xi M] \cap [K_{2\rho}(y) \times (s, s+\theta(2\rho)^p]]| \leq \frac{\nu_0}{\theta(\xi M)^{p+m-3}}|K_{2\rho}(y) \times (s, s+\theta(2\rho)^p]|, \quad (10)$$

then

$$u \geq a\xi M \qquad \text{in } K_\rho(y) \times (s, s + \theta(2\rho)^p].$$

3. **Intrinsic Harnack inequality for super-critical, singular equations.** Let u be a continuous, non-negative, local, weak solution to the singular equations (1)-(2) in E_T, for p, m in the super-critical range

$$3 - \frac{p}{N} < m + p < 3. \tag{11}$$

Fix $(x_0, t_0) \in E_T$ such that $u(x_0, t_0) > 0$ and construct the cylinders

$$(x_0, t_0) + Q^{\pm}_{\rho}(\theta) = K_\rho(x_0) \times (t_0 - \theta\rho^p, t_0 + \theta\rho^p), \qquad \theta = \left(\frac{u(x_0, t_0)}{c}\right)^{3-m-p} \tag{12}$$

where c is a constant that will be fixed later. These cylinders are "intrinsic" to the solution since their length is determined by the value of u at (x_0, t_0). The Harnack inequality holds in such an intrinsic geometry, as made precise in Theorems 3.1–3.2 below. The first is an intrinsic, Harnack inequality similar to that for degenerate equations (see [7]). This Harnack estimate is stable as $p + m \to 3$. The second is a "time insensitive" Harnack inequality, valid for all times t ranging in a whole interval according to the intrinsic geometry of (12) and including t_0. This inequality is unstable as $p+m \to 3$. Indeed, it is well known that the elliptic Harnack inequality is false for the heat equation.

Theorem 3.1 (The Intrinsic Harnack Inequality)**.** *There exist constants $\epsilon, c \in (0, 1)$ and $\gamma > 1$ depending only upon the data $\{p, m, N, C_0, C_1\}$, such that for all $(x_0, t_0) \in E_T$ such that $u(x_0, t_0) > 0$, and all the intrinsic cylinders $(x_0, t_0) + Q^{\pm}_{8\rho}(\theta)$ as in (12), contained in E_T,*

$$\gamma^{-1} \sup_{K_\rho(x_0)} u\big(\cdot, t_0 - \epsilon u(x_0, t_0)^{3-m-p}\rho^p\big) \le u(x_0, t_0) \le \gamma \inf_{K_\rho(x_0)} u\big(\cdot, t_0 + \epsilon u(x_0, t_0)^{3-m-p}\rho^p\big). \tag{13}$$

The constants $\gamma, c \to \infty$ as $m + p + \frac{p}{N} \to 3$, but they are stable as $m + p \to 3$.

Theorem 3.2 (Time insensitive, Intrinsic Harnack Inequalities)**.** *There exists constants $\bar\epsilon \in (0, 1)$ and $\bar\gamma > 1$, depending only upon the data $\{p, m, N, C_0, C_1\}$, such that for all $(x_0, t_0) \in E_T$ such that $u(x_0, t_0) > 0$, and all the intrinsic cylinders $(x_0, t_0) + Q^{\pm}_{8\rho}(\theta)$ as in (12), where c is the constant of Theorem 3.1, contained in E_T,*

$$\bar\gamma^{-1} \sup_{K_\rho(x_0)} u(\cdot, \sigma) \le u(x_0, t_0) \le \bar\gamma \inf_{K_\rho(x_0)} u(\cdot, \tau) \tag{14}$$

for any pair of time levels σ, τ in the range

$$t_0 - \bar\epsilon u(x_0, t_0)^{3-m-p}\rho^p \le \sigma, \tau \le t_0 + \bar\epsilon u(x_0, t_0)^{3-m-p}\rho^p.$$

The constants $\bar\epsilon$ and $\bar\gamma^{-1}$ tend to zero as either $p + m + \frac{p}{N} \to 3$ or as $p + m \to 3$.

Choosing $\sigma = \tau = t_0$ in the last theorem, we are led to the *elliptic* Harnack inequality.

Corollary 2 (Elliptic Harnack Inequality)**.** *For all $(x_0, t_0) \in E_T$ with $u(x_0, t_0) > 0$, and all the intrinsic cylinders $(x_0, t_0) + Q^{\pm}_{8\rho}(\theta)$ as in (12), where c is the constant of Theorem 3.1, contained in E_T,*

$$\bar\gamma^{-1} \sup_{K_\rho(x_0)} u(\cdot, t_0) \le u(x_0, t_0) \le \bar\gamma \inf_{K_\rho(x_0)} u(\cdot, t_0),$$

for the same constant $\bar\gamma$ of Theorem 3.2.

Remark 1. The Theorems have been stated for continuous solutions, to give meaning to $u(x_0, t_0)$. Actually locally bounded, local, weak solutions to (1)-(2), for all $m + p > 2$ are locally Hölder continuous. The proof of such result can be found in [20], up to some adjustments needed to correct a mistake contained therein. To this aim, one can give a look to the recent monograph [6], where the p-Laplacian and porous medium equations are treated in full details, and try to imitate the tecnique. The reader may also consider the reference [13], where the same issue is studied for a specific type of doubly non linear equation.

The intrinsic Harnack inequality, in turn, can be used to prove local Hölder continuity of weak solutions. This can be seen by combining the arguments used to prove Hölder regularity of weak solutions to p-Laplacian and porous medium type.

The proofs of Theorem 3.1 and Theorem 3.2 are interwined. In either case the key inequalities to establish are the right-hand side estimates in (13) and (14). The left estimates will follow from these by geometrical arguments.

3.1. **Proof of the right-hand side Harnack inequality of Theorem 3.2.** We start by stating and proving independently the right-hand side of (14).

Proposition 3. *Let u be a continuous, locally bounded, non-negative, local, weak solution to the singular equation (1)-(2) in the super-critical range (11). Then there exist positive constants $\bar{\epsilon}$ and $\bar{\gamma}$, that can be determined quantitatively, a priori only in terms of the data $\{p, m, N, C_0, C_1\}$, such that for all $(x_0, t_0) \in E_T$ such that $u(x_0, t_0) > 0$, and all the intrinsic cylinders $K_{8\rho}(x_0) \times \big(t_0 - u(x_0, t_0)^{3-m-p}(8\rho)^p, t_0 + u(x_0, t_0)^{3-m-p}(8\rho)^p\big)$ contained in E_T,*

$$u(x_0, t_0) \leq \bar{\gamma} \inf_{K_\rho(x_0)} u(\cdot, t)$$

for all times

$$t_0 - \bar{\epsilon} u(x_0, t_0)^{3-m-p} \rho^p \leq t \leq t_0 + \bar{\epsilon} u(x_0, t_0)^{3-m-p} \rho^p.$$

The constant $\bar{\epsilon}$ and $\bar{\gamma}$ tend to zero as either $m + p \to 3$ or $\lambda \to 0$.

3.1.1. *Changing variables.* Introduce the change of variables and unknown function

$$z = \frac{x - x_0}{\rho}, \qquad \tau = \frac{t - t_0}{u(x_0, t_0)^{3-m-p} \rho^p}, \qquad v(z, \tau) = \frac{u(x, t)}{u(x_0, t_0)}.$$

This maps the original cylinder into

$$Q = K_8 \times (-8^p, 8^p].$$

The function v is a weak solution to

$$v_\tau - \mathrm{div} \bar{A}(z, \tau, v, D(|v|^{\frac{m-1}{p-1}} v)) = 0 \qquad \text{in } Q,$$

where the transformed function $\bar{A}$ satisfies (2) with the same constants C_0, C_1. Establishing Proposition 3 consists in finding positive constants $\bar{\epsilon}$ and $\bar{\gamma}$, depending only upon the data, such that

$$v(\cdot, \tau) \geq \bar{\gamma}^{-1} \qquad \text{in } K_1 \text{ for all } \tau \in [-\bar{\epsilon}, \bar{\epsilon}].$$

Hereafter we relabel by x, t the new coordinates z, τ.

Remark 2. We notice that the function $\bar{A}$ satisfies also (3) and (4) with a new constant $\bar{\Lambda}$ depending on Λ and $u(x_0, t_0)$.

3.1.2. *Locating the supremum of v in K_1.* For $s \in (0,1)$ introduce the family of nested expanding cubes $\{K_s\}$ in $\mathbf{R}^N$ centered at the origin, and the increasing family of positive numbers

$$M_s = \sup_{K_s} v(\cdot,0), \qquad N_s = (1-s)^{-\frac{p}{3-m-p}}.$$

By definition, $M_0 = N_0$ and $N_s \to +\infty$, as $s \to 1$, whereas M_s remains finite. Therefore the equation $M_s = N_s$ has roots. Denoting by τ_* the largest root we have

$$M_{\tau_*} = (1-\tau_*)^{-\frac{p}{3-m-p}} \qquad \text{and} \qquad M_s \le N_s \text{ for all } s \ge \tau_*.$$

Since v is continuous, the supremum M_{τ_*} is achieved at some $\bar{x} \in K_{\tau_*}$. Choose $\bar{\tau} \in (0,1)$ from

$$(1-\bar{\tau})^{-\frac{p}{3-m-p}} = 4(1-\tau_*)^{-\frac{p}{3-m-p}} \qquad \text{i.e.} \qquad \bar{\tau} = 1 - 4^{-\frac{3-m-p}{p}}(1-\tau_*).$$

Set also

$$2r := \bar{\tau} - \tau_* = (1 - 4^{-\frac{3-m-p}{p}})(1-\tau_*).$$

For those choices, $K_{2r}(\bar{x}) \subset K_{\bar{\tau}}, M_{\bar{\tau}} \le N_{\bar{\tau}}$, and

$$\sup_{K_{\tau_*}} v(\cdot,0) = (1-\tau_*)^{-\frac{p}{3-m-p}} = v(\bar{x},0) \le \sup_{K_{2r}(\bar{x})} v(\cdot,0) \le \sup_{K_{\bar{\tau}}} v(\cdot,0) \le 4(1-\tau_*)^{-\frac{p}{3-m-p}}.$$

3.1.3. *Estimating the supremum of v in some intrinsic neighbourhood about $(\bar{x},0)$.* Consider the cylinder centered at $(\bar{x},0)$

$$Q_{2r} = K_{2r}(\bar{x}) \times (-\theta_*(2r)^p, \theta_*(2r)^p], \qquad \theta_* = (1-\tau_*)^{-p}.$$

Such a cylinder is included in the box Q since

$$\theta_*(2r)^p = (1-\tau_*)^{-p}(1-4^{-\frac{3-m-p}{p}})^p(1-\tau_*)^p \le 1 < 8.$$

Lemma 3.3. *There exists a positive constant γ_1, depending only on the data $\{p, m, N, C_0, C_1\}$, and independent of ρ, such that*

$$\sup_{Q_r} v \le \gamma_1(1-\tau_*)^{-\frac{p}{3-m-p}}.$$

The constant $\gamma_1 \to +\infty$ as either $p+m \to 3$ or $\lambda \to 0$.

Proof. Apply Theorem 2.3 to the function v over the pair of cylinders $Q_r \subset Q_{2r}$. Apply it first for the choice

$$\rho = r, \quad y = \bar{x}, \quad s = 0, \quad t = \theta_*(2r)^p,$$

and we obtain

$$\begin{aligned}
&\sup_{K_r(\bar{x})\times[0,\theta_*(2r)^p]} v \\
&\le \frac{\gamma}{(\theta_*(2r)^p)^{\frac{N}{\lambda}}}\Big(\inf_{-\theta_*(2r)^p<\tau<\theta_*(2r)^p}\int_{K_{2r}(\bar{x})} v(x,\tau)dx\Big)^{\frac{p}{\lambda}} + \gamma(2^p\theta_*)^{\frac{1}{3-m-p}} \\
&\le \gamma(1-\tau_*)^{p\frac{N}{\lambda}}\Big(⨍_{K_{2r}(\bar{x})} v(x,0)dx\Big)^{\frac{p}{\lambda}} + \gamma 2^{\frac{p}{3-m-p}}(1-\tau_*)^{-\frac{p}{3-m-p}} \\
&\le \gamma(1-\tau_*)^{-\frac{p}{3-m-p}}[4^{\frac{p}{\lambda}} + 2^{\frac{p}{3-m-p}}] = \gamma_1(1-\tau_*)^{-\frac{p}{3-m-p}}.
\end{aligned}$$

Then apply it again, for the choice

$$\rho = r, \quad y = \bar{x}, \quad s = -\theta_* r^p, \quad t = 0.$$

We obtain

$$\sup_{K_r(\bar{x})\times[-\theta_* r^p,0]} v \le \frac{\gamma}{(\theta_* r^p)^{\frac{N}{\lambda}}}\Big(\inf_{-2\theta_* r^p<\tau<0}\int_{K_{2r}(\bar{x})} v(x,\tau)dx\Big)^{\frac{p}{\lambda}} + \gamma(\theta_*)^{\frac{1}{3-m-p}}$$
$$\le \gamma_1(1-\tau_*)^{-\frac{p}{3-m-p}},$$

by possibly changing the constant γ_1. Hence the statement is proved. □

Introduce next the cylinder

$$Q_r(\bar{\delta}) = K_r(\bar{x})\times(-\bar{\delta}\theta_* r^p, \bar{\delta}\theta_* r^p] \subset Q_{2r},$$

where $\bar{\delta}\in(0,1)$ is to be chosen.

Lemma 3.4. *There exist numbers $\bar{\delta}, \bar{c}$, and $\alpha\in(0,1)$, depending only upon the data $\{p,m,N,C_0,C_1\}$, and independent of ρ, such that*

$$|[v(\cdot,t)\ge \bar{c}(1-\tau_*)^{-\frac{p}{3-m-p}}\cap K_r(\bar{x})| > \alpha|K_r| \qquad \textit{for all } t\in[-\bar{\delta}\theta_* r^p, \bar{\delta}\theta_* r^p].$$

The constants $\bar{c}$ and α tend to zero as either $p+m\to 3$ or $\lambda\to 0$. The constant $\bar{\delta}$ tends to zero as $p+m\to 3$.

Proof. Apply Theorem 2.3 to the function v over the pair of cylinders $Q_{\frac{r}{2}}(\bar{\delta})\subset Q_r(\bar{\delta})$, for the choices $\rho=\frac{r}{2}$, $y=\bar{x}$, $s=0$, $t=\bar{\delta}\theta_* r^p$. For all $t\in[-\bar{\delta}\theta_* r^p, \bar{\delta}\theta_* r^p]$

$$(1-\tau_*)^{-\frac{p}{3-m-p}} = v(\bar{x},0) \le \sup_{K_{\frac{r}{2}}(\bar{x})} v(\cdot,0)$$
$$\le \frac{\gamma}{(\bar{\delta}\theta_* r^p)^{\frac{N}{\lambda}}}\Big(\int_{K_r(\bar{x})} v(x,t)dx\Big)^{\frac{p}{\lambda}} + \gamma\Big(\frac{\bar{\delta}\theta_* r^p}{r^p}\Big)^{\frac{1}{3-m-p}}$$
$$\le \gamma\frac{(1-\tau_*)^{p\frac{N}{\lambda}}}{\bar{\delta}^{\frac{N}{\lambda}}}\Big(\fint_{K_r(\bar{x})} v(x,t)dx\Big)^{\frac{p}{\lambda}} + \gamma(\bar{\delta})^{\frac{1}{3-m-p}}(1-\tau_*)^{-\frac{p}{3-m-p}}.$$

Choose $\bar{\delta}$ from

$$\gamma(\bar{\delta})^{\frac{1}{3-m-p}} \le \frac{1}{2},$$

and set

$$\gamma_2 = 2\gamma, \qquad \gamma_3 = \frac{2^{\frac{N}{\lambda}(3-m-p)}\gamma_2}{\bar{\delta}^{\frac{N}{\lambda}}}.$$

For such choices, the constants $\bar{\delta}$, γ_2, γ_3 depend only upon the data p,m,N,C_0,C_1. Then, for all $t\in[-\bar{\delta}\theta_* r^p, \bar{\delta}\theta_* r^p]$

$$\frac{1}{\gamma_2}(1-\tau_*)^{-\frac{p}{3-m-p}} \le \frac{(1-\tau_*)^{p\frac{N}{\lambda}}}{\bar{\delta}^{\frac{N}{\lambda}}}\Big(\fint_{K_r(\bar{x})} v(x,t)dx\Big)^{\frac{p}{\lambda}}.$$

From this, for $\bar{c}\in(0,1)$,

$$\frac{1}{\gamma_3}(1-\tau_*)^{-\frac{p}{3-m-p}} \le \frac{(1-\tau_*)^{p\frac{N}{\lambda}}}{2^{\frac{N}{\lambda}(3-m-p)}}\Big(\fint_{K_r(\bar{x})} v(x,t)dx\Big)^{\frac{p}{\lambda}}$$

$$
\begin{aligned}
&\leq \frac{(1-\tau_*)^{p\frac{N}{\lambda}}}{2^{\frac{N}{\lambda}(3-m-p)}}\Bigg(\fint_{K_r(\bar{x})} v(x,t)\chi_{[v(\cdot,t)<\bar{c}(1-\tau_*)^{-\frac{p}{3-m-p}}]}dx \\
&\qquad\qquad\qquad + \fint_{K_r(\bar{x})} v(x,t)\chi_{[v(\cdot,t)\geq\bar{c}(1-\tau_*)^{-\frac{p}{3-m-p}}]}dx\Bigg)^{\frac{p}{\lambda}} \\
&\leq (1-\tau_*)^{p\frac{N}{\lambda}}\Bigg(\fint_{K_r(\bar{x})} v(x,t)\chi_{[v(\cdot,t)<\bar{c}(1-\tau_*)^{-\frac{p}{3-m-p}}]}dx\Bigg)^{\frac{p}{\lambda}} \\
&\quad + (1-\tau_*)^{p\frac{N}{\lambda}}\Bigg(\fint_{K_r(\bar{x})} v(x,t)\chi_{[v(\cdot,t)\geq\bar{c}(1-\tau_*)^{-\frac{p}{3-m-p}}]}dx\Bigg)^{\frac{p}{\lambda}} \\
&\leq \bar{c}^{\frac{p}{\lambda}}(1-\tau_*)^{-\frac{p}{3-m-p}} \\
&\quad + \gamma_1^{\frac{p}{\lambda}}(1-\tau_*)^{-\frac{p}{3-m-p}}\frac{\left|K_r(\bar{x})\cap\left[v(\cdot,t)\geq\bar{c}(1-\tau_*)^{-\frac{p}{3-m-p}}\right]\right|^{\frac{p}{\lambda}}}{\left|K_r\right|^{\frac{p}{\lambda}}}.
\end{aligned}
$$

To prove the thesis choose

$$
\bar{c}^{\frac{p}{\lambda}} = \frac{1}{2\gamma_3} \qquad \text{and set} \qquad \alpha = \frac{1}{\gamma_1}\Big(\frac{1}{2\gamma_3}\Big)^{\frac{\lambda}{p}}. \tag{15}
$$

□

3.1.4. *Expanding the positivity of v.* The information provided by Lemma 3.4 is the assumption required by the expansion of positivity for all

$$
-\bar{\delta}\theta_* r^p \leq s \leq \bar{\delta}\theta_* r^p.
$$

Apply then the expansion of positivity (Proposition 2) to v with $\rho = r$, $M = \bar{c}(1-\tau_*)^{-\frac{p}{3-m-p}}$ and for s ranging in the indicated interval. It gives

$$
v(\cdot,t) > \eta\bar{c}(1-\tau_*)^{-\frac{p}{3-m-p}} \qquad \text{in } K_{2r}(\bar{x}) \tag{16}
$$

and for all times

$$
-\bar{\delta}\theta_* r^p + (1-\varepsilon)\delta M^{3-m-p} r^p < t < \bar{\delta}\theta_* r^p,
$$

for constants $\delta, \varepsilon \in (0,1)$ depending only upon the data $\{p, m, N, C_0, C_1\}$ and the constant α in (15), which itself is determined only in terms of the data. In order to expand the positivity of v to the full cube K_1 we apply the comparison principle. Consider the boundary value problem

$$
\begin{cases}
w \in C(-t_0, 1; L^2(K_4(\bar{x}))) \cap W^{1,1}(-t_0, 1; L^1(K_4(\bar{x}))), & \\
w^{\frac{m+p-2}{p-1}} \in L^p(-t_0, 1; W_0^{1,p}(K_4(\bar{x}))), & \\
w_t - \operatorname{div}\bar{A}(x,t,w,D(w^{\frac{m+p-2}{p-1}})) = 0, & \text{in } K_4(\bar{x}) \times [-t_0, 1], \\
w = 0, & \text{on } \partial K_4(\bar{x}) \times [-t_0, 1], \\
w(\cdot, -t_0) = w_0 & \text{in } K_4(\bar{x})
\end{cases}
$$

where

$$
w_0(x) = \begin{cases} \eta\bar{c}(1-\tau_*)^{-N}, & x \in K_{2r}(\bar{x}), \\ 0, & x \in K_4(\bar{x}) \setminus K_{2r}(\bar{x}), \end{cases}
$$

and $-t_0$ is any time in the interval $(-\bar{\delta}\theta_* r^p + (1-\varepsilon)\delta M^{3-m-p} r^p, 0)$. The problem has a unique solution w. Moreover

$$
w \leq v \quad \text{on } \partial K_4(\bar{x}) \times [-t_0, 1]
$$

and if $x \in K_{2r}(\bar{x})$ by (16)

$$v(x,-t_0) - w(x,-t_0) \geq \eta\bar{c}(1-\tau_*)^{-\frac{p}{3-m-p}} - \eta\bar{c}(1-\tau_*)^{-N} = \eta\bar{c}(1-\tau_*)^{-N}[(1-\tau_*)^{-\frac{\lambda}{3-m-p}} - 1] > 0,$$

since $\lambda > 0$. Therefore, by the comparison principle Corollary 1 (see also Remark 2)

$$v \geq w \text{ in } K_4(\bar{x}) \times [-t_0, 1].$$

To prove the Proposition 3, it suffices to show that we can determine two constants $\bar{\gamma}$ and $\bar{\varepsilon}$, depending only upon the data, such that

$$w(x,t) \geq \bar{\gamma}^{-1} \qquad \text{in } K_1 \text{ for all } t \in [-\bar{\varepsilon}, \bar{\varepsilon}].$$

By the definition

$$\int_{K_4(\bar{x})} w(x,-t_0)dx = \eta\bar{c}\nu_0,$$

where $\nu_0 = (1 - 4^{-\frac{3-m-p}{p}})^N$. From Theorem 2.3, we deduce

$$\sup_{K_2(\bar{x})\times[-\frac{t_0}{2},\frac{t_0}{2}]} w(\cdot,t) \leq \gamma\, t_0^{-\frac{N}{\lambda}} \left(\int_{K_4(\bar{x})} w(x,-t_0)dx\right)^{\frac{p}{\lambda}} + \gamma\, t_0^{\frac{1}{3-m-p}} = \gamma\, t_0^{-\frac{N}{\lambda}} (\eta\bar{c}\nu_0)^{\frac{p}{\lambda}} + \gamma\, t_0^{\frac{1}{3-m-p}}. \tag{17}$$

On the other hand, by Theorem 2.1, for all $t \in [-t_0, t_0]$

$$\eta\bar{c}\nu_0 = \int_{K_1(\bar{x})} w(x,-t_0)dx \leq \tilde{\gamma}\int_{K_2(\bar{x})} w(x,t)dx + \tilde{\gamma}\, t_0^{\frac{1}{3-m-p}},$$

We choose t_0 such that in the last line

$$\tilde{\gamma}\, t_0^{\frac{1}{3-m-p}} = \frac{1}{2}\eta\bar{c}\nu_0,$$

namely

$$t_0 = \left(\frac{1}{2}\tilde{\gamma}^{-1}\eta\bar{c}\nu_0\right)^{3-m-p}$$

This is clearly possible if $(\frac{1}{2}\gamma^{-1}\eta\bar{c}\nu_0)^{3-m-p} \leq \bar{\delta}\theta_* r^p - (1-\varepsilon)\delta M^{3-m-p} r^p$. If not, it suffices to choose a smaller η which does not change the validity of (16). At this point

$$\int_{K_2(\bar{x})} w(x,t)dx \geq \frac{1}{2}\eta\bar{c}\nu_0\tilde{\gamma}^{-1},$$

for all $t \in [-t_0, t_0]$. Moreover, with the choice of t_0 above, (17) yields

$$\sup_{K_2(\bar{x})\times[-\frac{t_0}{2},\frac{t_0}{2}]} w(\cdot,t) \leq \gamma^*\eta\bar{c}\nu_0 \overset{\text{def}}{=} \gamma_*,$$

for a suitable γ^* depending on the data. For any $t \in [-\frac{t_0}{2}, \frac{t_0}{2}]$ we have

$$\frac{1}{2}\eta\bar{c}\nu_0\tilde{\gamma}^{-1} \leq \int_{K_2(\bar{x})} w(x,t)dx = \int_{K_2(\bar{x})\cap[w(\cdot,t)<c_0]} w(x,t)dx + \int_{K_2(\bar{x})\cap[w(\cdot,t)\geq c_0]} w(x,t)dx \leq c_0|K_2| + \gamma_*|[w(\cdot,t)\geq c_0]\cap K_2(\bar{x})|,$$

where c_0 is any positive number. Choosing

$$c_0 = \frac{1}{4|K_2|}\eta\bar{c}\nu_0\tilde{\gamma}^{-1},$$

the previous inequality gives

$$|[w(\cdot,t) \geq c_0] \cap K_2(\bar{x})| \geq \bar{\alpha}|K_2|, \qquad \bar{\alpha} = \frac{1}{4\gamma^*\tilde{\gamma}|K_2|},$$

for all $t \in [-\frac{t_0}{2}, \frac{t_0}{2}]$. By the expansion of positivity (Proposition 2)

$$w(x,t) \geq \eta c_0 \qquad \text{in } K_4(\bar{x}) \qquad \text{for all } t \in [-\bar{\varepsilon}, \bar{\varepsilon}],$$

for a sufficiently small $\bar{\varepsilon}$ depending only the data. This ends the proof. □

3.2. **Proof of the right-hand side Harnack inequality of Theorem 3.1.** The estimate in the proof of Theorem 3.2 deteriorate as $p+m \to 3$ and as $m+p+\frac{p}{N} \to 3$. Stable estimates for $p+m \to 3$ required in the proof of the right-hand side inequality of Theorem 3.1 can be derived as in the case of equations of p-Laplacian type (see [6]). As remarked in that contest, the proof is based on the stability of the expansion of positivity and some geometrical arguments. □

3.3. **Proof of the left-hand side Harnack inequality of Theorem 3.1.** Let ϵ, γ be the constants appearing on the right-hand side Harnack inequality of Theorem 3.1. Set

$$\bar{t} = t_0 - \epsilon u(x_0,t_0)^{3-m-p}\rho^p$$
$$\alpha = (2\gamma)^{\frac{m+p-3}{p}}.$$

Consider the cube $K_{\alpha\rho}(x_0)$, and introduce the set

$$\mathcal{U}_\alpha = K_{\alpha\rho}(x_0) \cap [u(\cdot,\bar{t}) \leq \gamma u(x_0,t_0)].$$

Since u is continuous, $\mathcal{U}_\alpha$ is closed. We will prove that the choice of the parameter α yields that $\mathcal{U}_\alpha$ is also open. Then, if $\mathcal{U}_\alpha$ is not empty, it coincides with $K_{\alpha\rho}$, thereby establishing the left-hand side, intrinsic Harnack inequality in (13), modulo a suitable re-definition of ρ and ϵ.

Assume momentarily that $\mathcal{U}_\alpha$ is not empty, and fix $z \in \mathcal{U}_\alpha$. Since u is continuous, there exists a cube $K_\varepsilon(z) \subset K_{\alpha\rho}(x_0)$ such that

$$u(y,\bar{t}) \leq 2\gamma u(x_0,t_0) \qquad \text{for all } y \in K_\varepsilon(z). \tag{18}$$

For each $y \in K_\varepsilon(z)$ construct the intrinsic p-paraboloid

$$\mathcal{P}(y,\bar{t}) = \{(x,t)|\, |t-\bar{t}| \geq \epsilon u(y,\bar{t})^{3-m-p}|x-y|^p\}.$$

Due to the choice of α and (18) we have that $(x_0,t_0) \in \mathcal{P}(y,\bar{t})$, so that by the right-hand side Harnack inequality in (13)

$$u(y,\bar{t}) \leq \gamma\, u(x_0,t_0)$$

and hence $y \in \mathcal{U}_\alpha$, proving $\mathcal{U}_\alpha$ to be open. Notice that the right-hand side Harnack inequality can be applied since, in view of (18), the cylinder

$$(y,\bar{t}) + Q^{\pm}_{8\rho}(\bar{\theta}) \qquad \text{with} \qquad \bar{\theta} = \left(\frac{u(y,\bar{t})}{c}\right)^{3-m-p}$$

can be assumed to be contained in E_T, by possibly redefining the constant c appearing in (12).

It remains to show that $\mathcal{U}_\alpha \neq \emptyset$. Having determined α, consider the cylinder

$$K_{\alpha\rho}(x_0) \times (\bar{t}, \bar{t} + \nu(\gamma u(x_0,t_0))^{3-m-p}(\alpha\rho)^p], \tag{19}$$

where $\nu \in (0,1)$ is to be chosen, depending only on the data $\{p, m, N, C_0, C_1\}$. Such a cylinder crosses the time level t_0 if

$$t_0 - \epsilon u(x_0,t_0)^{3-m-p}\rho^p + \nu(\gamma u(x_0,t_0))^{3-m-p}(\alpha\rho)^p > t_0.$$

Recalling the value of α, this occurs if

$$\nu\gamma^{3-m-p}\alpha^p > \epsilon \qquad \Rightarrow \qquad \epsilon < \nu 2^{p+m-3},$$

which, by reducing ϵ if necessary, we assume. If $\mathcal{U}_\alpha = \emptyset$, then

$$u(\cdot,\bar{t}) > \gamma u(x_0,t_0) \qquad \text{in } K_{\alpha\rho}(x_0).$$

Apply Lemma 2.4 in the cylinder (19) with

$$a = \frac{1}{2}, \qquad \xi = 1, \qquad M = \gamma u(x_0,t_0), \qquad \nu = \nu_0, \qquad \theta = \nu_0(\gamma u(x_0,t_0))^{3-m-p},$$

where ν_0 is the number in the hypothesis (10) of Lemma 2.4. For such a choice of θ, (10) is satisfied and the lemma yields

$$u(x,t_0) > \frac{1}{2}\gamma u(x_0,t_0) \qquad \text{for all } x \in K_{\frac{1}{2}\alpha\rho}(x_0).$$

Computing this for $x = x_0$ gives a contradiction if $\gamma > 2$, which without loss of generality we may assume. □

3.4. **Proof of the left-hand side Harnack inequality of Theorem 3.2.** Let the assumptions of Theorem 3.2 be in force and consider first the left-hand side inequality (14) for the specific value of σ

$$\bar{\sigma} = t_0 - \bar{\epsilon} u(x_0,t_0)^{3-m-p}\rho^p.$$

For such fixed value of σ, the left-hand side inequality in (14) can be derived exactly as in the case of the left-hand side inequality (13) of Theorem 3.1 as established in the previous section. Thus, by possibly redefining $\bar{\gamma}$ and $\bar{\epsilon}$,

$$\sup_{K_\rho(x_0)} u(\cdot,\bar{\sigma}) \le \bar{\gamma} u(x_0,t_0).$$

Apply Theorem 2.3 over the cubes $K_{\frac{1}{2}\rho}(x_0) \subset K_\rho(x_0)$ for the time levels

$$\begin{aligned} t_0 - \bar{\epsilon} u(x_0,t_0)^{3-m-p}\rho^p < s < t_0 - \frac{1}{2}\bar{\epsilon} u(x_0,t_0)^{3-m-p}\rho^p \\ < t_0 < t < t_0 + \bar{\epsilon} u(x_0,t_0)^{3-m-p}\rho^p \end{aligned} \tag{20}$$

so that

$$\frac{1}{2}\bar{\epsilon} u(x_0,t_0)^{3-m-p}\rho^p \le t - s \le 2\bar{\epsilon} u(x_0,t_0)^{3-m-p}\rho^p.$$

With these choices,

$$\begin{aligned} \sup_{K_{\frac{1}{2}\rho}(x_0)} u(\cdot,t) &\le \frac{\gamma}{\bar{\epsilon}^{\frac{N}{\lambda}} u(x_0,t_0)^{\frac{N(3-m-p)}{\lambda}}} \left(\fint_{K_\rho(x_0)} u(x,\bar{\sigma})dx \right)^{\frac{p}{\lambda}} + \gamma(2\bar{\epsilon})^{\frac{1}{(3-m-p)}} u(x_0,t_0) \\ &\le (\gamma\bar{\gamma}^{\frac{p}{\lambda}}\bar{\epsilon}^{\frac{-N}{\lambda}} + \gamma(2\bar{\epsilon})^{\frac{1}{3-m-p}}) u(x_0,t_0) \\ &= \bar{\bar{\gamma}} u(x_0,t_0). \end{aligned}$$

This establishes the left-hand side inequality (14) for all $\sigma = t$ in the range (20), by possibly redefining $\bar{\gamma}$ and $\bar{\epsilon}$. □

4. **Harnack estimates for sub-critical singular equations.** For the sake of simplicity, we continue to deal with the homogeneous equation (1), but the result can be extended, by minor changes, to slightly more general non homogeneous equations (see [8] and [9]). In this section we will not need assumptions (3) and (4) since we will not make use of the comparison principle. Therefore weak solutions do not need to belong to $W^{1,1}_{\mathrm{loc}}(0,T;L^1_{\mathrm{loc}}(E))$.

Let u be a non-negative, local, weak solutions to the singular equation (1) in E_T, for p,m in the critical and sub-critical range

$$2 < p+m \leq 3-\frac{p}{N}. \tag{21}$$

Now weak solutions are not required to belong to $W^{1,1}_{\mathrm{loc}}(0,T;L^1_{\mathrm{loc}}(E))$. An analysis of the model equation (5) suggests that neither of the previous Harnack inequalities holds in the sub-critical range ($2<m+p<3-\frac{p}{N}$); as discussed by Vespri in [21], the solutions to the Cauchy problem

$$\begin{cases} u_t = \mathrm{div}(|u|^{m-1}|Du|^{p-2}Du) & \text{in } \mathbf{R}^N\times(0,\infty) \\ u(x,0)=u_0(x)\in L^1(\mathbf{R}^N)\cap L^{(3-m-p)(N/p)}(\mathbf{R}^N), & u_0(x)\geq 0 \end{cases} \tag{22}$$

become extinct after a finite time, and this contradicts the Harnack estimate in any of the forms (13)-(14). Nevertheless a different form of Harnack estimate holds for p,m in the range (21), with constants depending on the ratio of some integral norms of the solution u. Fix $(x_0,t_0)\in E_T$ and ρ such that $K_{4\rho}(x_0)\subset E$, and introduce the quantity

$$\theta = \left[\varepsilon\left(\fint_{K_\rho(x_0)} u^q(\cdot,t_0)dx\right)^{\frac{1}{q}}\right]^{3-m-p}, \tag{23}$$

where $\varepsilon\in(0,1)$ is to be chosen, and $q\geq 1$ is arbitrary. If $\theta>0$ assume that

$$(x_0,t_0)+Q^-_{8\rho}(\theta) = K_{8\rho}(\theta)\times(t_0-\theta(8\rho)^p,t_0]\subset E_T,$$

and set

$$\sigma = \left[\frac{\left(\fint_{K_\rho(x_0)} u^q(\cdot,t_0)dx\right)^{\frac{1}{q}}}{\left(\fint_{K_{4\rho}(x_0)} u^r(\cdot,t_0-\theta\rho^p)dx\right)^{\frac{1}{r}}}\right]^{\frac{rp}{\lambda_r}} \tag{24}$$

where $r\geq 1$ is any number such that

$$\lambda_r = N(p+m-3)+rp>0. \tag{25}$$

Theorem 4.1. *Let u be a non-negative, locally bounded, local, weak solution to the singular equation (1)-(2) in E_T, for $2<m+p<3$. Introduce θ as in (23) and assume that $\theta>0$. There exist constants $\varepsilon\in(0,1)$, and $\gamma,\beta>1$, depending only on the data $\{p,m,N,C_0,C_1\}$ and the parameters q,r, such that*

$$\inf_{(x_0,t_0)+Q^-_\rho(\frac{1}{2}\theta)} u \geq \gamma\sigma^\beta \sup_{(x_0,t_0)+Q^-_\rho(\theta)} u, \tag{26}$$

where σ is defined in (24), $q\geq 1$ and $r\geq 1$ satisfies (25). The constant $\varepsilon\to 0$, and $\gamma,\beta\to\infty$ as either $\lambda_r\to 0$ or $\lambda_r\to\infty$.

Remark 3. Inequality (26) is not a true Harnack inequality, since σ depends upon the solution itself. It would reduce to a Harnack inequality if $\sigma \geq \sigma_0$ for some absolute constant σ_0 depending only upon the data. This however cannot occur since a Harnack inequality for solutions to (22) does not hold. Inequality (26) can be regarded as a "weak" form of a Harnack estimate valid for all $2 < m + p < 3$.

4.1. **Components of the proof of Theorem 4.1.** We will need the expansion of positivity (Proposition 2) and some L^r_{loc} estimates backward in time proved in [9]. For the sake of completeness we recall them.

Proposition 4. *Let u be a locally bounded, local, weak solution to (1)–(2) in E_T, and let $\kappa > 1$. Assume that the cylinder $K_{2\rho}(y) \times [s,t]$ is included in E_T. Then there exists a positive constant γ, depending only upon the data $\{p, m, N, C_0, C_1\}$ and κ, such that*

$$\sup_{s\leq\tau\leq t}\int_{K_\rho(y)} u^\kappa_\pm(x,\tau)dx \leq \gamma\left(\int_{K_{2\rho}(y)} u^\kappa_\pm(x,s)dx + \Big(\frac{(t-s)^\kappa}{\rho^{\lambda_\kappa}}\Big)^{\frac{1}{3-m-p}}\right).$$

Theorem 4.2. *Let u be a non-negative, locally bounded, local, weak solutions to the singular equation (1) in E_T, for $2 < p + m < 3$, and let $r \geq 1$ satisfy (25). There exists a positive constant γ_r, depending only upon the data $\{p, m, N, C_0, C_1\}$, and r, such that*

$$\sup_{K_\rho(y)\times[s,t]} u \leq \frac{\gamma_r}{(t-s)^{\frac{N}{\lambda_r}}}\Big(\int_{K_{2\rho}(y)} u^r(x, 2s-t)dx\Big)^{\frac{p}{\lambda_r}} + \gamma_r\Big(\frac{t-s}{\rho^p}\Big)^{\frac{1}{3-m-p}}$$

for all cylinders

$$K_{2\rho}(y) \times [s-(t-s), s+(t-s)] \subset E_T.$$

The constant $\gamma_r \to \infty$ if either $\lambda_r \to 0$ or $\lambda_r \to \infty$.

4.2. **Estimating the positivity set of the solution.** Having fixed $(x_0,t_0) \in E_T$, assume it coincides with the origin, write $K_\rho(0) = K_\rho$ and introduce the quantity θ as in (23), which is assumed to be positive. Apply Proposition 4 (Theorem 2.1 if $q = 1$) for $\kappa = q, y = 0$, and $s \in (-\theta\rho^p, 0]$. Using the definition (23) of θ gives

$$\begin{aligned}\int_{K_\rho} u^q(x,0)dx &\leq \gamma_q\int_{K_{2\rho}} u^q(x,s)dx + \gamma_q\Big(\frac{(\theta\rho^p)^q}{\rho^{\lambda_q}}\Big)^{\frac{1}{3-m-p}}\\ &\leq \gamma_q\int_{K_{2\rho}} u^q(x,s)dx + \gamma_q\varepsilon^q\int_{K_\rho} u^q(x,0)dx,\end{aligned}$$

for a constant γ_q depending only on the data $\{p, m, N, C_0, C_1\}$ and q. Choosing ε from

$$\gamma_q\varepsilon^q \leq \frac{1}{2},$$

yields

$$\int_{K_{2\rho}} u^q(x,s)dx \geq \frac{1}{2\gamma_q}\int_{K_\rho} u^q(x,0)dx \tag{27}$$

for all $s \in (-\theta\rho^p, 0]$. Next apply Theorem 4.2 over the cylinder

$$K_{2\rho} \times (-\frac{1}{2}\theta\rho^p, 0]$$

with $r \geq 1$ such that $\lambda_r > 0$, to get

$$\begin{aligned}\sup_{K_{2\rho}\times(-\frac{1}{2}\theta\rho^p,0]} u &\leq \gamma_r \frac{(4\rho)^{\frac{Np}{\lambda_r}}}{(\theta\rho^p)^{\frac{N}{\lambda_r}}}\Big(\fint_{K_{4\rho}} u^r(x,-\theta\rho^p)dx\Big)^{\frac{1}{r}\frac{rp}{\lambda_r}} + \gamma_r\theta^{\frac{1}{3-m-p}} \\ &\leq \frac{\gamma_r'}{\varepsilon^{\frac{N(3-m-p)}{\lambda_r}}}\frac{1}{\sigma}\Big(\fint_{K_\rho} u^q(x,0)dx\Big)^{\frac{1}{q}} + \gamma_r'\varepsilon\Big(\fint_{K_\rho} u^q(x,0)dx\Big)^{\frac{1}{q}} \\ &= \gamma_r'\varepsilon\Big(1+\frac{1}{\sigma\varepsilon^{\frac{rp}{\lambda_r}}}\Big)\Big(\fint_{K_\rho} u^q(x,0)dx\Big)^{\frac{1}{q}}\end{aligned}$$

for a constant γ_r' depending only upon the data $\{p, m, N, C_0, C_1\}$ and r. One verifies that $\gamma_r' \to \infty$, as either $\lambda_r \to 0$ or $\lambda_r \to \infty$.

Assume momentarily that $0 < \sigma < 1$ so that in the round brackets containing σ, the second term dominates the first one. In such a case

$$\sup_{K_{2\rho}\times(-\frac{1}{2}\theta\rho^p,0]} u \leq \frac{1}{\varepsilon'\sigma}\Big(\fint_{K_\rho} u^q(x,0)dx\Big)^{\frac{1}{q}} =: M,$$

where

$$\varepsilon' = \frac{\varepsilon^{\frac{N(3-m-p)}{\lambda_r}}}{2\gamma_r'}.$$

From this

$$\varepsilon'\sigma M = \Big(\fint_{K_\rho} u^q(x,0)dx\Big)^{\frac{1}{q}}. \tag{28}$$

Let $\nu \in (0,1)$ to be chosen. Using (27) and (28) estimate

$$\begin{aligned}(\varepsilon'\sigma M)^q &\leq 2^{N+1}\gamma_q\fint_{K_{2\rho}} u^q(x,s)dx \\ &\leq 2^{N+1}\gamma_q\left(\fint_{K_{2\rho}\cap[u<\nu\sigma M]} u^q(x,s)dx + \fint_{K_{2\rho}\cap[u\geq\nu\sigma M]} u^q(x,s)dx\right) \\ &\leq 2^{N+1}\gamma_q\nu^q(\sigma M)^q + 2^{N+1}\gamma_q M^q\frac{|[u(\cdot,s)>\nu\sigma M]\cap K_{2\rho}|}{|K_{2\rho}|}\end{aligned}$$

for all $s \in (-\frac{1}{2}\theta\rho^p, 0]$. From this

$$|[u(\cdot,s) > \nu\sigma M] \cap K_{2\rho}| \geq \alpha\sigma^q|K_{2\rho}|,$$

where

$$\alpha = \frac{\varepsilon'^q - \nu^q 2^{N+1}\gamma_q}{2^{N+1}\gamma_q},$$

for all $s \in (-\frac{1}{2}\theta\rho^p, 0]$. By choosing $\nu \in (0,1)$ sufficiently small, only dependent on the data $\{p, m, N, C_0, C_1\}$ and γ_q, we can ensure that $\alpha \in (0,1)$ depends only upon the data $\{p, m, N, C_0, C_1\}$ and q, and is independent of σ. We summarize

Proposition 5. *Let u be a non-negative, locally bounded, local, weak solution to the singular equations (1)-(2), for $2 < p+m < 3$. Fix $(x_0,t_0) \in E_T$, let $K_{4\rho}(x_0) \subset E$ and let θ and σ be defined by (23), (24) respectively, for some $\varepsilon \in (0,1)$. Suppose $0 < \sigma < 1$. For every $r \geq 1$ satisfying (25) and every $q \geq 1$, there exist constants $\varepsilon, \nu, \alpha \in (0,1)$, depending only upon the data $\{p, m, N, C_0, C_1\}$, q and r, such that*

$$|[u(\cdot,t) > \nu\sigma M] \cap K_{2\rho}(x_0)| \geq \alpha\sigma^q|K_{2\rho}(x_0)|$$

for all $t \in (t_0 - \frac{1}{2}\theta\rho^p, t_0]$.

4.3. **A first form of the Harnack inequality.** The definitions (23) of θ and the parameters ε' and α imply that

$$\frac{1}{2}\theta = \epsilon(\nu\sigma M)^{3-m-p}, \qquad \text{where } \epsilon = \frac{1}{2}\Big(\frac{\varepsilon\varepsilon'}{\nu}\Big)^{3-m-p}.$$

By Proposition 2 with M replaced by $\nu\sigma M$ and α replaced by $\alpha\sigma^q$, there exist constants η and δ in $(0,1)$, depending upon the data $\{p,m,N,C_0,C_1\}$ and α,σ and ϵ such that

$$u(\cdot,t) > \eta\,\nu\sigma M \qquad \text{in } K_{4\rho}(x_0),$$

for all times

$$t \in (t_0 - \frac{1}{2}\theta\rho^p + \delta(\nu\sigma M)^{3-m-p}(2\rho)^p, t_0]$$

where δ includes the quantity $1-\varepsilon$ of Proposition 2. Without loss of generality we can assume that this time interval contains $(t_0 - \frac{1}{4}\theta\rho^p, t_0]$.

Proposition 6 (A first form of the Harnack inequality)**.** *Let u be a non-negative, locally bounded, local, weak solution to the singular equations (1)-(2), for $2 < p + m < 3$. Fix $(x_0,t_0) \in E_T$, let $K_{4\rho}(x_0) \subset E$ and let θ and σ be defined by (23)-(24) for some $\varepsilon \in (0,1)$. Suppose $0 < \sigma < 1$. For every $r \geq 1$ satisfying (25) and every $q \geq 1$, there exist $\varepsilon \in (0,1)$, and a continuous, increasing function $\sigma \to f(\sigma)$ defined in $\mathbf{R}^+$ and vanishing at $\sigma = 0$, that can be quantitatively determined a priori only in terms of the data $\{p,m,N,C_0,C_1\}$, q, and r, such that*

$$\inf_{K_{4\rho}(x_0)} u(\cdot,t) \geq f(\sigma) \sup_{(x_0,t_0)+Q^-_{2\rho(\frac{1}{4}\theta)}} u, \tag{29}$$

for all $t \in (t_0 - \frac{1}{4}\theta\rho^p, t_0]$, provided $(x_0,t_0) + Q^-_{8\rho(\theta)} \subset E_T$.

Remark 4. The proof of Proposition 2 shows that the function $f(\cdot)$ can be taken of the form

$$f(\sigma) \approx \sigma B^{-\frac{1}{\sigma^d}},$$

for constants $B, d > 1$ depending only upon the data, q and r. The function $f(\cdot)$ depends on θ only through the parameter ε in the definition (23) of θ.

Remark 5. The inequality (29) has been derived by assuming that $0 < \sigma < 1$. If $\sigma \geq 1$ the same proof gives (29) where $f(\sigma) \geq f(1)$, thereby establishing a strong form of the Harnack estimate for these solutions. Such a strong form is false for p, m in the critical, and sub-critical range $2 < p + m \leq 3 - \frac{p}{N}$.

4.4. **Proof of Theorem 4.1 concluded.** The final step in the proof of Theorem 4.1 consists in improving the dependence on σ so that $f(\sigma)$ can be replaced by σ^β, for some β depending on the data. This can be done by using the local Hölder continuity of u (see Remark 1) and some technical arguments which are independent of the partial differential equations and can be found in [6] for the singular p–Laplacian equation.

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