



Invariant Measures for Stochastic Damped 2D Euler Equations

Hakima Bessaih¹, Benedetta Ferrario² 

¹ Department of Mathematics, University of Wyoming, Dept. 3036, 1000 East University Avenue, Laramie, WY 82071, USA. E-mail: bessaih@uwyo.edu

² Dipartimento di Matematica, Università di Pavia, Via Ferrata 5, 27100 Pavia, Italy.
E-mail: benedetta.ferrario@unipv.it

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Abstract: We study the two-dimensional Euler equations, damped by a linear term and driven by an additive noise. The existence of weak solutions has already been studied; pathwise uniqueness is known for solutions that have vorticity in L^∞ . In this paper, we prove the Markov property and then the existence of an invariant measure in the space L^∞ by means of a Krylov–Bogoliubov’s type method, working with the weak* and the bounded weak* topologies in L^∞ .

1. Introduction

Two dimensional hydrodynamics is largely studied from the theoretical as well as from the applied point of view. Both the analysis of individual solutions or statistical solutions have been developed. In particular turbulence theory, which analyzes the equations of motion of a fluid by introducing statistical means, asks for existence/uniqueness of statistically stationary solutions. They describe the motion of fluids at equilibrium, for large time.

For the bidimensional equations of viscous fluids, that is the Navier-Stokes equations forced by a random forcing term, existence and uniqueness of invariant measures have been proved under many different assumptions on the noise term (see, among the others, [20] and references therein).

Moreover, these equations with a weaker dissipation have been considered more recently by Constantin, Glatt-Holtz and Vicol [14] proving existence and uniqueness of invariant measures; they are called the fractionally dissipated Euler equations.

On the other hand, for the stochastically forced 2D Euler equations with a linear damping, which is a wave-number independent dissipation, the only so far known results on the longtime behavior are through their weak random attractors and stationary solutions (see [2, 3, 5, 7]). These equations are given by

$$\begin{cases} du + (u \cdot \nabla)u \, dt + \gamma u \, dt + \nabla p \, dt = dW \\ \nabla \cdot u = 0 \end{cases} \quad (1)$$

The unknowns are the velocity vector $u = u(t, x)$ and the pressure $p = p(t, x)$; here t is the time variable and $x \in D \subset \mathbb{R}^2$ the space variable. $W = W(t, x)$ is a given Wiener process. We assume $\gamma > 0$. With respect to the vorticity $\xi = \nabla^\perp \cdot u \equiv \partial_1 u_2 - \partial_2 u_1$ they are

$$\begin{cases} d\xi + u \cdot \nabla \xi \, dt + \gamma \xi \, dt = dW^{curl} \\ \xi = \nabla^\perp \cdot u, \quad \nabla \cdot u = 0 \end{cases} \quad (2)$$

with $W^{curl} = \nabla^\perp \cdot W$.

When $\gamma = 0$ the above are the Euler equations governing the motion of an incompressible inviscid fluid that have been extensively studied. When $\gamma > 0$, the linear damping, although not regularizing, introduces some dissipative feature, discussed in [8, 19]. Kupianinen in [23] points out how these randomly forced damped Euler equations are related to 2D turbulence theory and to the viscous case (see also [5]); moreover, interesting scaling limits on the vanishing viscosity and/or the damping are discussed, giving some open conjectures on the limits problems.

We recall that in [3] the existence of stationary solutions to these stochastic damped Euler equations has been proved in the space $L^2(D)$ for the vorticity. In particular this is a space where the uniqueness does not hold. Let us recall that there is no need to define the associated transition semigroup in order to define stationary solutions. Hence, having stationary solutions is a weaker result than having an invariant measure where a proper dynamics is needed. Here we improve that result by defining a transition semigroup in the space $L^\infty(D)$, which is the space where uniqueness is proved for Eq. (2). The drawback of working in the space $L^\infty(D)$ is that it is not separable, and weak* measurability and strong measurability do not coincide. In this paper we prove that the transition semigroup is sequentially weak* Feller and Markov in $L^\infty(D)$ equipped with the bounded weak* topology. Then, we construct an invariant measure by means of Krylov–Bogoliubov’s technique but dealing with weak* topologies, in a similar way as done by Maslowski and Seidler in [24] (however they worked in a separable Hilbert space).

As far as we know, this is the first result for the damped Euler equation (2) and the first result for any fluid dynamic equation in a non separable space setting, like $L^\infty(D)$. We hope that our method could be used to tackle other models with similar problems.

The paper is organized as follows. In Sect. 2, we introduce the functional spaces and assumptions. The space $L^\infty(D)$ with its various topologies is described in some detail in Sect. 2.2. A particular attention will be devoted to the bounded weak* topology; this is a crucial point that will be used in the Krylov–Bogoliubov’s technique for the passage to the limit in order to get the invariant measure. We also recall some well posedness results, that are not new but contain some improvements for the measurability of the solutions in $L^\infty(D)$. In Sect. 3, we prove the continuous dependence of the vorticity solution with respect to the initial data and a spatial regularity result in the Sobolev space $W^{1,4}(D)$. This leads to the “weak” Feller property for the transition Markov semigroup that is defined afterwards. In Sect. 4, we prove the Markov property in the space $L^\infty(D)$ for system (2). In particular, we first prove the Markov property in $W^{1,4}(D)$ and then conclude by a density argument. Finally in Sect. 5 we prove existence of an invariant measure; this is the only part in which the assumption $\gamma > 0$ is required, otherwise all the previous results hold for any $\gamma \geq 0$.

2. Preliminaries and Assumptions

2.1. Mathematical setting. Let D be the torus $\mathbb{R}^2 \setminus \mathbb{Z}^2$. This means that the spatial domain is a square and periodic boundary conditions are assumed. The results remain true in a bounded domain, see [1, 4, 7].

We define the space H of periodic vector fields which are square integrable, divergence free and have zero mean value on D . This is a separable Hilbert space, with the $[L^2(D)]^2$ -scalar product. We denote by $|\cdot|$ the H -norm and by (\cdot, \cdot) the H -scalar product; $|\cdot|_p$ is the $[L^p(D)]^2$ -norm.

We define $V = [H^1(D)]^2 \cap H$ and denote by $\|\cdot\|$ its norm.

For $k \geq 1$ and $p > 2$ we define $V^{k,p} = [W^{k,p}(D)]^2 \cap V$, being $W^{k,p}(D)$ the Sobolev space. We denote by $\|\cdot\|_{k,p}$ the $V^{k,p}$ -norm. $V^{k_1,p}$ is a dense subspace of $V^{k_2,p}$ for $k_1 > k_2$ and the embedding is compact. For simplicity we write V^k for $V^{k,2}$.

Let V' be the dual space of V with respect to the H scalar product. Identifying H with its dual space H' , and H' with the corresponding natural subspace of the dual space V' , we have the Gelfand triple $V \subset H \subset V'$ with continuous dense injections. We denote the dual pairing between $u \in V$ and $v \in V'$ by $\langle u, v \rangle$. When $v \in H$, we have $\langle u, v \rangle = (u, v)$. For other duality pairings the spaces will be specified when necessary.

Let $b(\cdot, \cdot, \cdot) : V \times V \times V \rightarrow \mathbb{R}$ be the continuous trilinear form defined as

$$b(u, v, z) = \int_D ([u(x) \cdot \nabla]v(x)) \cdot z(x) dx.$$

It is well known that there exists a continuous bilinear operator $B(\cdot, \cdot) : V \times V \rightarrow V'$ such that $\langle B(u, v), z \rangle = b(u, v, z)$, for all $z \in V$. By the incompressibility condition, for $u, v, z \in V$ we have (see, e.g., [28])

$$\langle B(u, v), z \rangle = -\langle B(u, z), v \rangle \quad \text{and} \quad \langle B(u, v), v \rangle = 0. \tag{3}$$

Working on the torus we can develop the velocity and the vorticity in Fourier series, so to easily express the relationship between u and ξ (see, e.g., details in [5]), proving that for any $p \in [2, \infty)$ the norms $|\nabla u|_p$ and $|\xi|_p$ are equivalent and that the norm $|u|_p$ is bounded by the norm $|\xi|_p$.

As far as the stochastic part is concerned, we are given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence $\{\tilde{\beta}_j(t); t \geq 0\}_{j \in \mathbb{N}}$ of independent standard 1-dimensional Wiener processes defined on it. Then we consider a new sequence of i.i.d. Wiener processes defined for any time $t \in \mathbb{R}$:

$$\beta_i(t) = \begin{cases} \tilde{\beta}_{2i-1}(t) & \text{for } t \geq 0 \\ \tilde{\beta}_{2i}(-t) & \text{for } t \leq 0 \end{cases}$$

The noise forcing term in Eq. (1) is taken of the form

$$W(t, x) = \sum_{i \in \mathbb{N}} c_i \beta_i(t) e_i(x) \tag{4}$$

for some $c_i \in \mathbb{R}$ (see, e.g., [15]), where $\{e_i\}_i$ is a complete orthonormal system of H . We define the filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ by $\mathcal{F}_t = \sigma\{W(t_2) - W(t_1), -\infty < t_1 < t_2 \leq t\}$.

In the sequel we shall require W to take values in the space $C(\mathbb{R}; V^{k,\infty})$ for $k = 2$ or $k = 3$; by Sobolev embedding we know that it is sufficient that for some $h > k + 1$ the paths $W \in C(\mathbb{R}; V^h)$ a.s.; a sufficient condition for this is that

$$\sum_i c_i^2 \|e_i\|_{V^h}^2 < \infty. \tag{5}$$

2.2. *The space $L^\infty(D)$.* To shorten notation we write L^p for the space $L^p(D)$. The space L^∞ is the dual of the space L^1 ; moreover the space L^∞ is not separable whereas the space L^1 is separable. This is a crucial property which makes the analysis of the dynamics (1) a delicate matter with respect to some issues. Indeed, main results available in the literature about stochastic PDE's are based on the assumption that the state space is separable (see e.g. [15, 16]).

We recall the meaning of convergence in L^∞ with respect to the weak* topology: $\xi_n \xrightarrow{*} \xi$ in L^∞ means

$$L^\infty \langle \xi_n, \phi \rangle_{L^1} \rightarrow L^\infty \langle \xi, \phi \rangle_{L^1} \quad \forall \phi \in L^1.$$

Here we collect basic results on topologies and related Borelian subsets of L^∞ (see, e.g., [26]).

We denote by $\mathcal{T}_n, \mathcal{T}_{bw*}, \mathcal{T}_{w*}$ the strong (or norm) topology, the bounded weak* topology and the weak* topology of L^∞ , respectively. We have that

$$\mathcal{T}_{w*} \subsetneq \mathcal{T}_{bw*} \subsetneq \mathcal{T}_n. \tag{6}$$

We recall that the bounded weak* topology is the finest topology on L^∞ that coincides with the weak* topology on every norm bounded subset of L^∞ .

Let us note that $f : L^\infty \rightarrow \mathbb{R}$ is \mathcal{T}_{bw*} -continuous if and only if it is sequentially \mathcal{T}_{w*} -continuous.¹ Indeed, set $K_n = \{\xi \in L^\infty : \|\xi\|_{L^\infty} \leq n\}, n \in \mathbb{N}$, and note that K_n are metrizable \mathcal{T}_{bw*} -compact spaces. If f is \mathcal{T}_{bw*} -continuous and $\xi_j \rightarrow \xi$ weakly*, then for some n we have $\xi_j, \xi \in K_n$; the weak* continuity of $f|_{K_n}$ implies $f(\xi_j) \rightarrow f(\xi)$. In the opposite direction, let f be sequentially weakly* continuous. Then $f|_{K_n}$ is weakly*-continuous on any K_n by metrizability of the weak* topology on bounded subsets. If $U \subset \mathbb{R}$ is an arbitrary open set, then $f^{-1}(U) \cap K_n = (f|_{K_n})^{-1}(U)$ is \mathcal{T}_{w*} -open in K_n , so $f^{-1}(U)$ is \mathcal{T}_{bw*} -open and \mathcal{T}_{bw*} -continuity of f follows.

Denoting by $C(L^\infty, \mathcal{T})$ the space of all functions $f : L^\infty \rightarrow \mathbb{R}$ which are \mathcal{T} -continuous, we thus have that

$$C(L^\infty, \mathcal{T}_{w*}) \subsetneq C(L^\infty, \mathcal{T}_{bw*}) = SC(L^\infty, \mathcal{T}_{w*}) \subsetneq C(L^\infty, \mathcal{T}_n).$$

We recall that by Alaoglu-Banach theorem, the set $\{\xi \in L^\infty : \|\xi\|_{L^\infty} \leq R\}$ is \mathcal{T}_{w*} -compact. Hence it is also \mathcal{T}_{bw*} -compact, since the \mathcal{T}_{w*} -compact subsets coincide with the \mathcal{T}_{bw*} -compact subsets.

As far as measurability with respect to these topologies is concerned, let us denote by $\mathcal{B}(\mathcal{T})$ the σ -algebra of Borelian subsets of L^∞ w.r.t. the a given topology \mathcal{T} . According to (6) we have that $\mathcal{B}(\mathcal{T}_{w*}) \subseteq \mathcal{B}(\mathcal{T}_{bw*}) \subseteq \mathcal{B}(\mathcal{T}_n)$. Moreover

Lemma 1. *For the space L^∞ we have*

$$\mathcal{B}(\mathcal{T}_{w*}) = \mathcal{B}(\mathcal{T}_{bw*}).$$

Proof. From (6) it follows that $\mathcal{B}(\mathcal{T}_{w*}) \subseteq \mathcal{B}(\mathcal{T}_{bw*})$. Let us show the reverse inclusion.

Recall that a basis for the weak* topology \mathcal{T}_{w*} of L^∞ is given by the collection of all subsets

$$B(\eta; g_1, \dots, g_m) = \{\xi \in L^\infty : |\langle \xi - \eta, g_i \rangle| < 1 \text{ for } i = 1, \dots, m\} \tag{7}$$

¹ The space $SC(L^\infty, \mathcal{T}_{w*})$ of sequentially weakly* continuous functions is the space of all functions $f : L^\infty \rightarrow \mathbb{R}$ such that $f(\xi_n) \rightarrow f(\xi)$ if $\xi_n \rightarrow \xi$ weakly* in L^∞ , i.e. $\langle \xi_n, g \rangle \rightarrow \langle \xi, g \rangle$ for any $g \in L^1$.

for any $\eta \in L^\infty$, for any $m \in \mathbb{N}$ and $g_i \in L^1$ (see page 224 in [26]), and a basis for the bounded weak* topology \mathcal{T}_{bw^*} of L^∞ is given by the collection of all subsets

$$B(\eta; \{g_i\}_{i \in \mathbb{N}}) = \{\xi \in L^\infty : |\langle \xi - \eta, g_i \rangle| < 1 \text{ for each } i\} \tag{8}$$

for any $\eta \in L^\infty$, for any sequence $\{g_i\}_{i \in \mathbb{N}}$ in L^1 that converges to 0 (see page 235 in [26]).

The mapping $\theta_m : L^\infty \ni \xi \mapsto \sup_{i=1, \dots, m} |\langle \xi - \eta, g_i \rangle| \in \mathbb{R}$ is \mathcal{T}_{w^*} -continuous, hence $\mathcal{B}(\mathcal{T}_{w^*})$ -measurable.² Therefore, letting $m \rightarrow \infty$ we get that the limit mapping $\theta : L^\infty \ni \xi \mapsto \sup_{i \in \mathbb{N}} |\langle \xi - \eta, g_i \rangle| \in \mathbb{R}$ is $\mathcal{B}(\mathcal{T}_{w^*})$ -measurable. This shows that any element (8) of the basis of open subsets with respect to the topology \mathcal{T}_{bw^*} belongs to $\mathcal{B}(\mathcal{T}_{w^*})$. This implies that $\mathcal{B}(\mathcal{T}_{bw^*}) \subseteq \mathcal{B}(\mathcal{T}_{w^*})$. \square

Since in L^∞ the Borelian subsets w.r.t. the weak* and the norm topology do not coincide (see [27]), we conclude that

$$\mathcal{B}(\mathcal{T}_{w^*}) = \mathcal{B}(\mathcal{T}_{bw^*}) \subsetneq \mathcal{B}(\mathcal{T}_n).$$

Let us remind that in a separable Banach space X the Borelian subsets w.r.t. the weak and the norm topology coincide; hence we speak of measurability meaning that one w.r.t. the (weak=strong) Borelian subsets of X .

Finally we deal with the measurability property. Given the mapping $\omega \in (\Omega, \mathcal{F}) \rightarrow \xi(\omega) \in L^\infty$ we say that it is weakly* measurable if for any $g \in L^1$ the mapping

$$\omega \in \Omega \rightarrow \langle \xi(\omega), g \rangle \in \mathbb{R}$$

is $\mathcal{F} \setminus \mathcal{B}^1$ -measurable. This is equivalent to say that the mapping $\omega \mapsto \xi(\omega)$ is $\mathcal{F} \setminus \mathcal{B}(\mathcal{T}_{w^*})$ -measurable.

2.3. Existence and uniqueness results. In this section we collect the basic known results on existence and uniqueness for the Euler equation. For $\gamma = 0$, these results are stated in a Hilbert setting in [4, 6] and in a more general Banach setting in [13]. The extension to the case $\gamma > 0$ is trivial. We work on any finite time interval $[t_0, T]$; then the results hold on \mathbb{R} .

Theorem 2. *Let $\gamma \geq 0$ and assume (5) with $h > 3$.*

*i) If $u_0 \in V$, then on each interval $[t_0, T]$ there exists at least a weak global solution for (1) with the initial condition $u(t_0) = u_0$ satisfying **P**-a.s.*

$$u \in C([t_0, T]; H) \cap L^2(t_0, T; V)$$

and, for every $\varphi \in V$ and every $t \in [t_0, T]$

$$\langle u(t), \varphi \rangle - \int_{t_0}^t \langle [u(s) \cdot \nabla] \varphi, u(s) \rangle ds + \gamma \int_{t_0}^t \langle u(s), \varphi \rangle ds = \langle u_0, \varphi \rangle + \langle W(t) - W(t_0), \varphi \rangle$$

P-a.s.

*Moreover, u is measurable in these topologies and satisfies $u \in L^\infty(t_0, T; V)$ **P**-a.s.*

ii) Let $p \in]2, \infty[$. If $u_0 \in V^{1,p}$, then the weak global solution u obtained in i) satisfies

$$u \in L^\infty(t_0, T; V^{1,p}) \quad \mathbf{P} - a.s.$$

² We point out that on the space \mathbb{R} we always consider the Borel σ -algebra \mathcal{B}^1 . This is not stated at each instance but tacitely assumed.

iii) If $u_0 \in V$ and $\xi_0 = \nabla^\perp \cdot u_0 \in L^\infty$, then $\xi = \nabla^\perp \cdot u$ (with u the weak global solution obtained in i)) satisfies

$$\xi \in L^\infty([t_0, T] \times D) \quad \mathbf{P} - a.s.$$

and pathwise uniqueness holds. Moreover \mathbf{P} -a.s.

$$u \in C_w([0, T]; V), \quad \xi \in C([t_0, T]; (L^\infty, \mathcal{T}_{w^\star})),$$

and the mapping

$$(\omega, t) \mapsto \xi(t, \omega)$$

is jointly measurable, that is $\mathcal{F} \otimes \mathcal{B}([t_0, T]) \setminus \mathcal{B}(\mathcal{T}_{w^\star})$ measurable.

The important results are about existence; indeed, when the noise is additive pathwise uniqueness is easily obtained as in the deterministic setting (see [29,30]).

Remark 3. a) Here $C_w([0, T]; V)$ denotes the space of vectors u which are weakly continuous from $[0, T]$ into V , i.e. for any $\phi \in V'$ the real mapping $t \mapsto \langle u(t), \phi \rangle$ is continuous.

b) We say that the mapping

$$t \in [t_0, T] \mapsto \xi(t) \in L^\infty$$

is weakly \star continuous if it is continuous when on L^∞ we consider the weak \star topology \mathcal{T}_{w^\star} . This means that for any $g \in L^1$ the mapping

$$t \in [t_0, T] \mapsto \langle \xi(t), g \rangle \in \mathbb{R}$$

is continuous.

c) The measurability of the process ξ , defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbf{P})$, is obtained by proving that the mapping

$$(\Omega, \mathcal{F}_t) \ni \omega \rightarrow W^{curl}(\cdot)(\omega) \in C((-\infty, t]; H^{h-1})$$

is measurable, and for every $g \in L^1$ the mapping

$$C((-\infty, t]; H^{h-1}) \ni W^{curl} \mapsto \langle \xi(t), g \rangle \in \mathbb{R}$$

is continuous. Hence, composing these two mappings we find that the mapping $(\Omega, \mathcal{F}_t) \ni \omega \mapsto \langle \xi(t)(\omega), g \rangle \in \mathbb{R}$ is measurable, which means that $\omega \mapsto \xi(t)(\omega)$ is $\mathcal{F}_t \setminus \mathcal{B}(\mathcal{T}_{w^\star})$ measurable.

Since for \mathbf{P} -a.e. $\omega \in \Omega$ the mapping $t \mapsto \langle \xi(t)(\omega), g \rangle$ is continuous, then the mapping $(\omega, t) \mapsto \langle \xi(t)(\omega), g \rangle$ is jointly measurable, that is the mapping

$$(\Omega, (\infty, T]) \ni (\omega, t) \mapsto \xi(t)(\omega) \in L^\infty$$

is $\mathcal{F}_T \otimes \mathcal{B}((-\infty, T]) \setminus \mathcal{B}(\mathcal{T}_{w^\star})$ measurable.

3. Continuous Dependence with Respect to the Initial Data and Regularity

The vorticity equation (2) can also be rewritten using the Biot-Savart kernel K as follows:

$$d\xi + (K * \xi) \cdot \nabla \xi \, dt + \gamma \xi \, dt = dW^{curl} \tag{9}$$

For every $\chi \in L^\infty$, let $\xi(t; \chi)$ be the unique solution of Eq. (9) evaluated at time $t > t_0$ given the initial value χ at time t_0 . By Theorem 2 we have $\mathbf{P}(\xi(t; \chi) \in L^\infty) = 1$.

Moreover, we can prove a weak form of continuous dependence on the initial data.

Theorem 4. *Let $\gamma \geq 0$ and assume (5) with $h > 3$.*

Given a sequence $\{\chi^n\}_n \subset L^\infty$ which converges weakly in L^∞ to $\chi \in L^\infty$, we have that, \mathbf{P} -a.s., for every $t > t_0$ the sequence $\{\xi(t; \chi^n)\}_n$ converges weakly* in L^∞ to $\xi(t; \chi)$.*

Proof. In the sequel we work pathwise, that is ω is fixed in Ω on a set of \mathbf{P} -measure 1. We also fix $t_0 < T$, and will prove the result for $t \in [t_0, T]$. So all the constants appearing later depend on ω, t_0 and T .

By assumption, we have $\chi^n \in L^\infty$ hence $\chi^n \in L^p$ for any $p \geq 1$; moreover $L^\infty \langle \chi^n, g \rangle_{L^1} \rightarrow L^\infty \langle \chi, g \rangle_{L^1}$ for all $g \in L^1$.

Set $v^n = u^n - W$, and $\eta^n = \xi^n - W^{curl}$. Then

$$\frac{\partial v^n}{\partial t} + \gamma v^n + B(v^n + W, v^n + W) = -\gamma W \tag{10}$$

and

$$\frac{\partial \eta^n}{\partial t} + \gamma \eta^n + (v^n + W) \cdot \nabla \eta^n = -\gamma W - (v^n + W) \cdot \nabla W^{curl}. \tag{11}$$

Since the initial vorticities are bounded in L^∞ , then the initial velocities are bounded in $V^{1,p}$ for any finite p . As in Theorem 2, we get that \mathbf{P} -a.s.

$$\sup_n \sup_{t_0 \leq t \leq T} |v^n(t)|^2 < \infty, \quad \sup_n \sup_{t_0 \leq t \leq T} \|v^n(t)\|^2 < \infty, \quad \sup_n \left\| \frac{\partial v^n}{\partial t} \right\|_{L^2(t_0, T; V')} < \infty,$$

and

$$\sup_n \sup_{t_0 \leq t \leq T} |\eta^n(t)|_\infty^2 < \infty. \tag{12}$$

From these estimates, following [28], we have that v^n is bounded in $L^\infty(t_0, T; V) \cap H^1(t_0, T; V')$. So, we can extract a subsequence, still denoted by $\{v^n\}_n$, such that v^n converges to some function v strongly in $L^2(t_0, T; H)$ and weakly* in $L^\infty(t_0, T; V)$, $v^n(t)$ converges strongly in H for a.e. t , and v has the same regularity as v^n . Moreover $v \in C([t_0, T]; H)$.

We also deduce that η^n converges to some function η weakly* in $L^\infty((t_0, T) \times D)$. In particular, for any $g \in L^1$ we have $\int_{t_0}^T L^\infty \langle \eta^n(t), g \rangle_{L^1} dt \rightarrow \int_{t_0}^T L^\infty \langle \eta(t), g \rangle_{L^1} dt$ and for a.e. $t \in [t_0, T]$ $L^\infty \langle \eta^n(t), g \rangle_{L^1} \rightarrow L^\infty \langle \eta(t), g \rangle_{L^1}$. The same holds for the sequence $\{\xi^n\}_n$, that is for any $g \in L^1$

$$\int_{t_0}^T L^\infty \langle \xi^n(t), g \rangle_{L^1} dt \rightarrow \int_{t_0}^T L^\infty \langle \xi(t), g \rangle_{L^1} dt \tag{13}$$

and for a.e. $t \in [t_0, T]$

$$L^\infty \langle \xi^n(t), g \rangle_{L^1} \rightarrow L^\infty \langle \xi(t), g \rangle_{L^1}.$$

Now we show that the limit function ξ is the solution of system (2) with initial vorticity χ and that the convergence holds for any time t .

Let $g \in C^1(D)$; then for a.e. $t \in [t_0, T]$

$$\langle \xi^n(t), g \rangle + \gamma \int_{t_0}^t \langle \xi^n(s), g \rangle ds + \int_{t_0}^t \langle u^n(s) \cdot \nabla \xi^n(s), g \rangle ds = \langle \chi^n, g \rangle + (W^{curl}(t) - W^{curl}(t_0), g). \tag{14}$$

Writing

$$\begin{aligned} & \int_{t_0}^t \langle u^n(s) \cdot \nabla \xi^n(s), g \rangle ds - \int_{t_0}^t \langle u(s) \cdot \nabla \xi(s), g \rangle ds \\ &= - \int_{t_0}^t \langle u^n(s) \cdot \nabla g, \xi^n(s) \rangle ds + \int_{t_0}^t \langle u(s) \cdot \nabla g, \xi(s) \rangle ds \\ &= - \int_{t_0}^t \langle [u^n(s) - u(s)] \cdot \nabla g, \xi^n(s) \rangle ds - \int_{t_0}^t \langle u(s) \cdot \nabla g, \xi^n(s) - \xi(s) \rangle ds \end{aligned}$$

and using the strong convergence of u^n and the weak convergence of ξ^n , in the limit as $n \rightarrow \infty$ we get for any $g \in C^1(D)$ for a.e. $t \in [t_0, T]$

$$\langle \xi(t), g \rangle + \gamma \int_{t_0}^t \langle \xi(s), g \rangle ds - \int_{t_0}^t \langle u(s) \cdot \nabla g, \xi(s) \rangle ds = \langle \chi, g \rangle + (W^{curl}(t) - W^{curl}(t_0), g). \tag{15}$$

Moreover $t \mapsto \langle \xi(t), g \rangle$ is continuous; hence the result holds for any $t \in [t_0, T]$.

Now, by (12) the sequence $\{\xi^n(t)\}$ and $\xi(t)$ are bounded in $L^\infty(D)$; since $C^1(D)$ is dense in $L^1(D)$, the Hahn-Banach theorem provides that for any t

$$L^\infty \langle \xi^n(t) - \xi(t), g \rangle_{L^1} \longrightarrow 0 \quad \forall g \in L^1(D).$$

□

Now, we state a regularity result on any finite time interval $[t_0, T]$; the state space is now $W^{1,4}(D)$ which is smaller than $L^\infty(D)$. Hence uniqueness holds true. The upside of working in $W^{1,4}(D)$ is that this is a separable space, whereas $L^\infty(D)$ is not. This will be used in the next section. The downside is that in $W^{1,4}(D)$ we are not able to prove a uniform bound needed for the proof of existence of invariant measures, whereas we prove it in $L^\infty(D)$ (see Proposition 13).

Theorem 5. *Let $\gamma \geq 0$ and assume (5) with $h > 4$.*

*If $\xi_0 \in W^{1,4}(D)$, then $\xi \in L^\infty(t_0, T; W^{1,4}(D)) \cap C_w([t_0, T]; W^{1,4}(D))$ **P**-a.s.*

Moreover for every $t \in [t_0, T]$, the map $(\Omega, \mathcal{F}_t) \ni \omega \rightarrow \xi(t)(\omega) \in W^{1,4}(D)$ is measurable.

Proof. We have $W^{1,4}(D) \subset L^\infty(D)$. Hence, by the results of Theorem 2, we only need to prove the estimate for $\nabla \xi$. Let us take the gradient of Eq. (2):

$$d\nabla \xi + \gamma \nabla \xi + \nabla(u \cdot \nabla \xi) dt = d\nabla W^{curl}. \tag{16}$$

that can be rewritten for each component of the gradient as

$$d\partial_i \xi + \gamma \partial_i \xi + \partial_i(u \cdot \nabla \xi) dt = d\partial_i W^{curl}, \quad i = 1, 2. \tag{17}$$

We look for $|\nabla\xi| \in L^\infty([t_0, T]; L^4(D))$. Defining $\eta = \xi - W^{curl}$, we get

$$\frac{\partial}{\partial t} \partial_i \eta + \gamma \partial_i \eta + \partial_i [u \cdot \nabla \eta] = -\partial_i [u \cdot \nabla W^{curl}] - \gamma \partial_i W^{curl}, \quad i = 1, 2. \tag{18}$$

Let us multiply this equation by $\partial_i \eta |\nabla \eta|^2$, sum over i and then integrate over D ; we get

$$\begin{aligned} \frac{1}{4} \frac{d}{dt} |\nabla \eta(t)|_4^4 + \gamma |\nabla \eta(t)|_4^4 &= - \sum_{i=1}^2 \langle \partial_i [u \cdot \nabla \eta] + \partial_i [u \cdot \nabla W^{curl}], \partial_i \eta |\nabla \eta|^2 \rangle \\ &\quad - \gamma \sum_{i=1}^2 \langle \partial_i W^{curl}, \partial_i \eta |\nabla \eta|^2 \rangle. \end{aligned} \tag{19}$$

We have

$$\begin{aligned} \sum_i \langle \partial_i [u \cdot \nabla \eta], \partial_i \eta |\nabla \eta|^2 \rangle &= \sum_{i,j} \langle \partial_i u_j \partial_j \eta, \partial_i \eta |\nabla \eta|^2 \rangle + \sum_{i,j} \langle u_j \partial_{i,j}^2 \eta, \partial_i \eta |\nabla \eta|^2 \rangle \\ &=: I + II \end{aligned}$$

We use the following result

Lemma 6. $II = 0$.

Proof. By integration by parts

$$\begin{aligned} II &= \sum_{i,j} \int_D u_j [\partial_j \partial_i \eta] \partial_i \eta |\nabla \eta|^2 \\ &= - \sum_i \int_D [\sum_j \partial_j u_j] [\partial_i \eta] |\nabla \eta|^2 - \sum_{i,j} \int_D u_j [\partial_i \eta] \partial_j [\partial_i \eta |\nabla \eta|^2] \\ &= 0 - \sum_{i,j} \int_D u_j [\partial_i \eta] [\partial_j \partial_i \eta] |\nabla \eta|^2 - \sum_{i,j} \int_D u_j [\partial_i \eta] [\partial_i \eta] \partial_j [|\nabla \eta|^2] \\ &= -II - \sum_j \int_D u_j |\nabla \eta|^2 \partial_j [|\nabla \eta|^2] = -3II \end{aligned}$$

Hence, $II = 0$. \square

Now we go back to Eq. (19) and estimate each term in the r.h.s.:

$$|I| = \left| \sum_{i,j} \langle \partial_i u_j \partial_j \eta, \partial_i \eta |\nabla \eta|^2 \rangle \right| \leq C \int_D |\nabla u| |\nabla \eta|^4 \leq C \|\nabla u\|_\infty |\nabla \eta|_4^4.$$

Using the Hölder inequality and then the Young inequality, we estimate another term in (19)

$$\begin{aligned} \left| \sum_i \langle \partial_i [u \cdot \nabla W^{curl}], \partial_i \eta |\nabla \eta|^2 \rangle \right| &\leq C \|W^{curl}\|_{V^{2,\infty}} |\nabla u|_4 |\nabla \eta|_4^3 \\ &= C \|W^{curl}\|_{V^{2,\infty}} |\nabla u|_4 |\nabla \eta|_4^3 \\ &\leq |\nabla \eta|_4^4 + C \|\nabla u\|_4^4 \|W^{curl}\|_{V^{2,\infty}}^4 \end{aligned}$$

and we already know that the $|\nabla u|_4^4$ -norm is bounded by Theorem 2.ii.

Similarly for the other term in (19) we get

$$\gamma \sum_{i=1}^2 \langle \partial_i W^{curl}, \partial_i \eta |\nabla \eta|^2 \rangle \leq \gamma |W^{curl}|_4 |\nabla \eta|_4^3 \leq \frac{\gamma}{2} |\nabla \eta|_4^4 + C(\gamma) |W^{curl}|_4^4.$$

Now, we need an estimate for $|\nabla u|_\infty$. We can find it in Kato [22], which deals with the Euler equations in the whole plane, or in Ferrari [17], which deals with the Euler equations in a smooth bounded domain of the space; looking at the proofs of these papers we get for a smooth bounded domain of the plane that

$$|\nabla u|_\infty \leq C |\xi|_\infty \left[1 + \log \left(1 + \frac{|\nabla \xi|_4}{|\xi|_\infty} \right) \right]. \tag{20}$$

Thus, from (19) with the above estimates we get that for any $t \in [0, T]$

$$\begin{aligned} \frac{1}{4} \frac{d}{dt} |\nabla \eta(t)|_4^4 &\leq |\nabla \eta(t)|_4^4 + C |\xi(t)|_\infty \left[1 + \log \left(1 + \frac{|\nabla \xi(t)|_4}{|\xi(t)|_\infty} \right) \right] |\nabla \eta(t)|_4^4 \\ &+ C(\gamma, t_0, T, |\xi_0|_\infty, \|W\|_{C([0, \infty); V^{3, \infty})}). \end{aligned} \tag{21}$$

Gronwall lemma yields

$$|\nabla \eta(t)|_4^4 \leq (|\nabla \eta(t_0)|_4^4 + C(t - t_0)) e^{4 \int_{t_0}^t \{1 + C|\xi(s)|_\infty [1 + \log(1 + \frac{|\nabla \xi(s)|_4}{|\xi(s)|_\infty})\]} ds. \tag{22}$$

Taking the log of both sides we get

$$\begin{aligned} 4 \log(|\nabla \eta(t)|_4) &\leq \log(|\nabla \eta(t_0)|_4^4 + C(t - t_0)) \\ &+ 4 \int_{t_0}^t \{1 + C|\xi(s)|_\infty [1 + \log \left(1 + \frac{|\nabla \xi(s)|_4}{|\xi(s)|_\infty} \right)]\} ds. \end{aligned} \tag{23}$$

Now we use that $\log(x + y) \leq \log_+(x + y) \leq \log 2 + \log_+ x + \log_+ y$ and $-x \log x \leq \frac{1}{e}$ (for any $x, y > 0$). Therefore, since $\xi = \eta + W^{curl}$

$$\begin{aligned} \log(|\nabla \eta(t)|_4) &\leq \frac{1}{4} \log(|\nabla \eta(t_0)|_4^4 + C(t - t_0)) \\ &+ C \int_{t_0}^t \{1 + |\xi(s)|_\infty [C + \log_+(|\nabla \eta(s)|_4) + \log_+(|\nabla W^{curl}(s)|_4)]\} ds. \end{aligned} \tag{24}$$

Using again Gronwall lemma we get

$$\sup_{t_0 \leq t \leq T} |\nabla \eta(t)|_4 \leq C(\gamma, t_0, T, |\xi_0|_\infty, \|W\|_{C([t_0, T]; V^{3, \infty})}).$$

Going back to Eq. (18) and using the regularity of η obtained so far we get that **P**-a.s. $\partial_i \eta \in H^1(t_0, T; W^{-1,2}(D))$; combining with the fact that **P**-a.s. $\partial_i \eta \in L^\infty(t_0, T; L^4(D))$, then we conclude that $\partial_i \eta \in C_w([t_0, T]; L^4(D))$, **P**-a.s. (use Lemma 1.4 of Chapter 3 in [28]).

Finally, since $\xi = \eta + W^{curl}$ and using the regularity of the process W concludes the proof.

As far the measurability is concerned, this is obtained in a classical way when working in separable Banach space see [6]. For a more general theory see e.g. [15]. \square

4. Markov Property

We denote by $B_b(L^\infty, \mathcal{T}_{w^\star})$ the set of functions $\phi : L^\infty \rightarrow \mathbb{R}$ which are bounded and $\mathcal{B}(\mathcal{T}_{w^\star}) \setminus \mathcal{B}^1$ -measurable. Let $\xi(\cdot; \chi)$ be the solution of the vorticity equation (9) with initial vorticity χ at time 0. We define the family of operators (for each $t \geq 0$) as

$$(P_t \phi)(\chi) = \mathbb{E}[\phi(\xi(t; \chi))]$$

for any $\phi \in B_b(L^\infty, \mathcal{T}_{w^\star}) = B_b(L^\infty, \mathcal{T}_{bw^\star})$.

As a consequence of Theorem 4 and the Lebesgue dominated convergence theorem, we infer

Proposition 7. *The operator P_t is sequentially weakly \star Feller in L^∞ (see [24]), that is*

$$P_t : SC_b(L^\infty, \mathcal{T}_{w^\star}) \rightarrow SC_b(L^\infty, \mathcal{T}_{w^\star}). \tag{25}$$

This is equivalent to say that

$$P_t : C_b(L^\infty, \mathcal{T}_{bw^\star}) \rightarrow C_b(L^\infty, \mathcal{T}_{bw^\star}). \tag{26}$$

This property will be used in the proof of the main Theorem 11. Notice that the bounded weak \star topology is not metrizable; hence, continuity and sequential continuity are different. So one proves \mathcal{T}_{bw^\star} -continuity by means of sequential \mathcal{T}_{w^\star} -continuity, which is more feasible.

Now we want to show that P_t defines a Markov semigroup. As far as we know, we have not seen the Markov property stated or proved before for the stochastic Euler equations. This requires some care since the classical theory for Markov processes is usually set in Polish spaces.

We proceed in this way. First we state an auxiliary result working in the separable Banach space $W^{1,4}(D)$.

Lemma 8. *Let $\gamma \geq 0$ and assume (5) with $h > 4$.*

For every $\phi \in SC_b(L^\infty, \mathcal{T}_{w^\star})$, $\chi \in W^{1,4}(D)$ and $t, s > 0$ we have

$$\mathbf{E}[\phi(\xi(t+s; \chi)) | \mathcal{F}_t] = (P_s \phi)(\xi(t; \chi)) \quad \mathbf{P}\text{-a.s.} \tag{27}$$

Proof. We divide the proof in four parts. For short let $\xi(t; \chi)$ be denoted by ξ_t^χ ; moreover we use the notation $\xi_{t,t+s}^\eta$ to denote the solution of (9) (on the time interval $[t, t+s]$) evaluated at time $t+s$ and started from η at time t .

Step 1. Given $\phi \in SC_b(L^\infty, \mathcal{T}_{w^\star})$, $\chi \in W^{1,4}(D)$ and $t, s > 0$, (27) is equivalent to

$$\mathbf{E}[\phi(\xi_{t+s}^\chi) Z] = \mathbf{E}[(P_s \phi)(\xi_t^\chi) Z] \tag{28}$$

for every bounded \mathcal{F}_t -measurable random variable Z .

Given a $W^{1,4}(D)$ -valued \mathcal{F}_t -measurable random variable η , denote by $\xi_{t,t+s}^\eta$ the unique solution of (2) on the time interval $[t, t+s]$ with initial vorticity $\xi(t) = \eta$. Since by uniqueness

$$\xi_{t+s}^\chi = \xi_{t,t+s}^{\xi_t^\chi} \quad (\mathbf{P}\text{-a.s.})$$

and $\mathbf{P}(\xi_t^\chi \in W^{1,4}(D)) = 1$ by Theorem 5, in order to get (28) it is sufficient to prove that

$$\mathbf{E}[\phi(\xi_{t,t+s}^\eta) Z] = \mathbf{E}[(P_s \phi)(\eta) Z] \tag{29}$$

for every $W^{1,4}(D)$ -valued \mathcal{F}_t -measurable random variable η .

Step 2. Given such a random variable η and since $W^{1,4}(D)$ is a separable metric space (in contrast to $L^\infty(D)$), there exists a sequence $\{\eta_n\}_n$ of \mathcal{F}_t -measurable $W^{1,4}(D)$ -valued random variables of the form

$$\eta_n = \sum_{i=1}^{k_n} \eta_n^{(i)} 1_{A_n^{(i)}}$$

with $\eta_n^{(i)} \in W^{1,4}(D)$ and $A_n^{(i)} \in \mathcal{F}_t$ with $\{A_n^{(1)}, A_n^{(2)}, \dots, A_n^{(k_n)}\}$ a partition of Ω , such that $\{\eta_n\}$ converges **P**-a.s. strongly in $W^{1,4}(D)$ to η . If we assume that

$$\mathbf{E} [\phi (\xi_{t,t+s}^{\eta_n}) Z] = \mathbf{E} [(P_s \phi) (\eta_n) Z] \quad \forall n$$

then, since the strong convergence of η_n in $W^{1,4}(D)$ implies the weak \star convergence in L^∞ , using Proposition 7 we have that $(P_s \phi) (\eta_n)$ converges **P**-a.s. to $(P_s \phi) (\eta)$. On the other side, using Theorem 4 $\xi_{t,t+s}^{\eta_n}$ converges weakly \star in L^∞ to $\xi_{t,t+s}^\eta$, so $\phi (\xi_{t,t+s}^{\eta_n})$ also converges to $\phi (\xi_{t,t+s}^\eta)$ **P**-a.s. The proof of (29) is completed by using the Lebesgue dominated convergence theorem.

Step 3. Therefore, it is sufficient to prove (29) for every random variable η of the form

$$\eta = \sum_{i=1}^k \eta^{(i)} 1_{A^{(i)}}$$

with $\eta^{(i)} \in W^{1,4}(D)$, $A^{(i)} \in \mathcal{F}_t$ and $\{A^{(1)}, A^{(2)}, \dots, A^{(k)}\}$ a partition of Ω . Notice that

$$(P_s \phi) (\eta) = \sum_{i=1}^k (P_s \phi) (\eta^{(i)}) 1_{A^{(i)}} \quad \mathbf{P} - a.s.$$

Moreover $\xi_{t,t+s}^\eta = \sum_{i=1}^k \xi_{t,t+s}^{\eta^{(i)}} 1_{A^{(i)}}$, since we have solved the equation pathwise. Hence

$$\phi (\xi_{t,t+s}^\eta) = \sum_{i=1}^k \phi (\xi_{t,t+s}^{\eta^{(i)}}) 1_{A^{(i)}}.$$

Thus it is sufficient to prove

$$\mathbf{E} \left[\phi (\xi_{t,t+s}^{\eta^{(i)}}) 1_{A^{(i)}} Z \right] = \mathbf{E} \left[(P_s \phi) (\eta^{(i)}) 1_{A^{(i)}} Z \right] \tag{30}$$

for every i .

Step 4. Since $1_{A^{(i)}}$ is a bounded \mathcal{F}_t -measurable random variable, in order to prove (30) it is sufficient that

$$\mathbf{E} [\phi (\xi_{t,t+s}^\eta) Z] = \mathbf{E} [(P_s \phi) (\eta) Z]$$

for every bounded \mathcal{F}_t -measurable random variable Z and every deterministic element $\eta \in W^{1,4}(D)$. The random variable $\xi_{t,t+s}^\eta$ depends only on the increments of the Wiener process between t and $t + s$, hence it is independent of \mathcal{F}_t . Therefore

$$\begin{aligned} \mathbf{E} [\phi (\xi_{t,t+s}^\eta) Z] &= \mathbf{E} [\mathbf{E}[\phi (\xi_{t,t+s}^\eta) Z | \mathcal{F}_t]] = \mathbf{E} [Z \mathbf{E}[\phi (\xi_{t,t+s}^\eta) | \mathcal{F}_t]] \\ &= \mathbf{E} [Z \mathbf{E}[\phi (\xi_{t,t+s}^\eta)]] = \mathbf{E} [\phi (\xi_{t,t+s}^\eta)] \mathbf{E} [Z]. \end{aligned}$$

Since $\xi_{t,t+s}^\eta$ and ξ_s^η have the same law, we have $\mathbf{E} [\phi (\xi_{t,t+s}^\eta)] = \mathbf{E} [\phi (\xi_s^\eta)]$ and thus we have proved that

$$\mathbf{E} [\phi (\xi_{t,t+s}^\eta) Z] = \mathbf{E} [\phi (\xi_s^\eta)] \mathbf{E} [Z] = (P_s \phi) (\eta) \mathbf{E} [Z] = \mathbf{E} [(P_s \phi) (\eta) Z].$$

The proof is complete. \square

Now, we are ready to state the main result related to the Markov property. The following proposition is one possible Markov property for the family of solutions to Eq. (9).

Proposition 9. *Let $\gamma \geq 0$ and assume (5) with $h > 4$. For every $\phi \in SC_b(L^\infty, \mathcal{T}_{w^\star})$, $\chi \in L^\infty$ and $t, s > 0$, we have*

$$\mathbf{E} [\phi (\xi(t + s; \chi)) | \mathcal{F}_t] = (P_s \phi) (\xi(t; \chi)) \quad \mathbf{P} - a.s. \tag{31}$$

Proof. The space $W^{1,4}(D)$ is densely embedded in the space $(L^\infty, \mathcal{T}_{w^\star})$, see [9]. By the way, this shows that $(L^\infty, \mathcal{T}_{w^\star})$ is a separable space.

Thus, given $\chi \in L^\infty$ there is a sequence $\{\chi^n\} \subset W^{1,4}(D)$ which converges weakly \star in L^∞ to χ . Lemma 8 infers that, given $\phi \in SC_b(L^\infty, \mathcal{T}_{w^\star})$ and $t, s > 0$,

$$\mathbf{E} [\phi (\xi(t + s; \chi^n)) | \mathcal{F}_t] = (P_s \phi) (\xi(t; \chi^n)) \quad \mathbf{P} - a.s.$$

This means that

$$\mathbf{E} [\phi (\xi(t + s; \chi^n)) Z] = \mathbf{E} [(P_s \phi) (\xi(t; \chi^n)) Z]$$

for every bounded \mathcal{F}_t -measurable random variable Z .

From Theorem 4 we know that for any $r > 0$ $\{\xi(r; \chi^n)\}$ converges weakly \star in L^∞ to $\xi(r; \chi)$, \mathbf{P} -a.s. Hence $(P_s \phi) (\xi(t; \chi^n))$ converges to $(P_s \phi) (\xi(t; \chi))$, \mathbf{P} -a.s., and $\phi (\xi(t + s; \chi^n))$ converges to $\phi (\xi(t + s; \chi))$, \mathbf{P} -a.s., and thus by Lebesgue dominated convergence theorem we can pass to the limit in the previous equation and get

$$\mathbf{E} [\phi (\xi(t + s; \chi)) Z] = \mathbf{E} [(P_s \phi) (\xi(t; \chi)) Z] \quad \mathbf{P} - a.s.$$

This is equivalent to (31). \square

Corollary 10. *For any $s, t \geq 0$ we have $P_{t+s} = P_t P_s$ on $SC_b(L^\infty, \mathcal{T}_{w^\star})$.*

Proof. Taking the expectation in (31), we have

$$\mathbf{E} [\phi (\xi(t + s; \chi))] = \mathbf{E} [(P_s \phi) (\xi(t; \chi))]$$

which can be rewritten as $(P_{t+s} \phi)(\chi) = (P_t(P_s \phi))(\chi)$. \square

5. Invariant Measures

Let us consider the Markov semigroup $\{P_t\}_{t \geq 0}$ acting in $SC_b(L^\infty, \mathcal{T}_{w\star}) = C_b(L^\infty, \mathcal{T}_{bw\star})$, associated to the Eq. (9). We say that a probability measure μ on $\mathcal{B}(\mathcal{T}_{bw\star})$ is an invariant measure for it if

$$\int P_t \phi \, d\mu = \int \phi \, d\mu \quad \forall t \geq 0, \forall \phi \in C_b(L^\infty, \mathcal{T}_{bw\star}) \tag{32}$$

We want to prove existence of an invariant measure by means of Krylov–Bogoliubov’s method. We recall that already Maslowski and Seidler in [24] used this method with weak topologies, but assuming that the state space is a separable Hilbert space. Anyway also when dealing with the space L^∞ we can proceed along the lines of Krylov–Bogoliubov’s method in order to prove existence of invariant measures defined on $\mathcal{B}(\mathcal{T}_{bw\star})$.

This is our result

Theorem 11. *Let $\gamma > 0$ and assume (5) with $h > 4$. Then there exists at least one invariant measure for the stochastic equation (9).*

Proof. The idea is to construct a sequence of measures $\{\mu_n\}_{n \in \mathbb{N}}$, which is $\mathcal{T}_{bw\star}$ -tight; from it we can extract a subsequence converging to a measure μ ; then we show that the limit measure μ is an invariant measure, thanks to (26).

We denote by m_t the law of the random variable $\xi(t; 0)$ on $\mathcal{B}(\mathcal{T}_{bw\star})$; since the mapping $(\omega, t) \mapsto \xi(t; x)(\omega)$ is jointly measurable, we can integrate with respect to both variables and define the probability measure on $\mathcal{B}(\mathcal{T}_{bw\star})$

$$\mu_n = \frac{1}{n} \int_0^n m_t \, dt$$

for any $n > 0$.

We recall that the set $\{\|x\|_{L^\infty} \leq R\}$ is $\mathcal{T}_{bw\star}$ -compact. From Corollary 14, which will be proved in the next subsection, we have that the sequence $\{\mu_n\}_{n \in \mathbb{N}}$ is $\mathcal{T}_{bw\star}$ -tight, that is

$$\forall \epsilon > 0 \exists K_\epsilon \text{ } \mathcal{T}_{bw\star}\text{-compact subset of } L^\infty : \inf_n \mu_n(K_\epsilon) > 1 - \epsilon.$$

Now we apply Prokhorov’s theorem in the version given by Jakubowski (see Theorem 3 in [21]), which allows to work in non metric spaces. This requires that the space L^∞ with the bounded weak \star topology $\mathcal{T}_{bw\star}$ is countably separated, that is there exists a countable family $\{g_i : L^\infty \rightarrow [-1, 1]\}_{i \in \mathbb{N}}$ of $\mathcal{T}_{bw\star}$ -continuous functions which separate points of L^∞ . This is our case, since L^1 is separable, so there exists a countable sequence $\{h_i\}_i \subset L^1$ separating the points of L^∞ , that is for any two elements $x \neq y$ in L^∞ there exists h_i such that $\langle x, h_i \rangle \neq \langle y, h_i \rangle$. Since the mapping $x \mapsto \langle x, h_i \rangle$ is $\mathcal{T}_{w\star}$ -continuous, then it is also $\mathcal{T}_{bw\star}$ -continuous.

Therefore there exists a subsequence $\{\mu_{n_k}\}_k$ and a probability measure μ on $\mathcal{B}(\mathcal{T}_{bw\star})$ such that μ_{n_k} converges narrowly to μ as $k \rightarrow \infty$ ($n_k \rightarrow \infty$), that is

$$\int \phi \, d\mu_{n_k} \rightarrow \int \phi \, d\mu \quad \forall \phi \in C_b(L^\infty, \mathcal{T}_{bw\star}).$$

On the other hand we have that

$$\langle P_t \phi, \mu_{n_k} \rangle = \langle \phi, \mu_{n_k} \rangle + \frac{1}{n_k} \int_{n_k}^{t+n_k} \langle \phi, m_u \rangle du - \frac{1}{n_k} \int_0^t \langle \phi, m_u \rangle du.$$

Letting $k \rightarrow \infty$, the two latter terms vanish. From (26) we know that $P_t\phi \in C_b(L^\infty, \mathcal{T}_{bw\star})$ if $\phi \in C_b(L^\infty, \mathcal{T}_{bw\star})$. Hence in the limit we obtain

$$\langle P_t\phi, \mu \rangle = \langle \phi, \mu \rangle$$

for each $\phi \in C_b(L^\infty, \mathcal{T}_{bw\star})$ and each $t \geq 0$. \square

Remark 12. Maslowski and Seidler in [24] proved existence of an invariant measure dealing with weak topologies; applications can be found in that work and also in some papers by Brzeźniak and collaborators, see [10–12]; in all these works the state space is a separable Hilbert space. Working with the weak topologies is an improvement in applications, since it is easier to prove the tightness with respect to weak topologies than with respect to the strong ones. For instance we prove the weak tightness for the damped Euler equation (9) whereas the tightness with respect to the strong topology requires a dissipative term of the form $-\Delta\xi$ (or a fractional power of the Laplacian operator), that is it holds for the Navier-Stokes equations or fractional Navier-Stokes equations but not for the Euler equations.

The classical Krylov–Bogoliubov’s method is based on the tightness and the Feller property (see, e.g., [15, 16]). Therefore Maslowski and Seidler realized that dealing with weak topologies for the tightness called for a “weak” Feller property too. Actually, working in a separable Hilbert space H they considered the weak topology \mathcal{T}_w and the strong topology \mathcal{T}_n , and proved the existence of an invariant measure by assuming

1. $P_t : SC_b(H, \mathcal{T}_w) \rightarrow SC_b(H, \mathcal{T}_w)$
2. the family $\{\mu_n\}_{n \in \mathbb{N}}$ is \mathcal{T}_w -tight

Let us point out that taking into account the bounded weak topology \mathcal{T}_{bw} (which they considered in a subsequent paper [25]), one can write the two assumptions in an equivalent way as

1. $P_t : C_b(H, \mathcal{T}_{bw}) \rightarrow C_b(H, \mathcal{T}_{bw})$
2. the family $\{\mu_n\}_{n \in \mathbb{N}}$ is \mathcal{T}_{bw} -tight

since $SC(H, \mathcal{T}_w) = C(H, \mathcal{T}_{bw})$ and \mathcal{T}_w -compact subsets coincide with the \mathcal{T}_{bw} -compact subsets of H . This simplifies a bit the proof of Theorem 3.1 in [24], looking more similar to that of the classical Krylov–Bogoliubov’s theorem. So in principle the weak topology \mathcal{T}_w does not appear in the assumptions. However, one proves \mathcal{T}_{bw} -continuity by means of sequential \mathcal{T}_w -continuity, which is easier to prove.

5.1. Boundedness in probability. Here we prove the uniform bound in probability needed in the last proof.

Proposition 13. *Let $\gamma > 0$ and assume (5) with $h > 3$. Then, there exists a real random variable r (\mathbf{P} -a.s. finite) such that*

$$\sup_{t_0 \leq 0} |\xi(0; \xi(t_0) = 0)|_\infty \leq r \quad \mathbf{P} - a.s. \tag{33}$$

Proof. Our proof will follow a similar result introduced by Flandoli in [18] which uses dissipative features of the Navier-Stokes equations and the ergodic properties of an auxiliary Ornstein-Uhlenbeck process.

We introduce the linear equation

$$d\zeta_\lambda(t) + \lambda\zeta_\lambda(t)dt = dW^{curl}(t) \tag{34}$$

for $\lambda > 0$; its stationary solution is

$$\zeta_\lambda(t) = \int_{-\infty}^t e^{-\lambda(t-s)} dW^{curl}(s). \tag{35}$$

Set $\eta_\lambda = \xi - \zeta_\lambda$. Then η_λ fulfils the following equation

$$\frac{\partial \eta_\lambda}{\partial t} + \gamma \eta_\lambda + [K \star (\eta_\lambda + \zeta_\lambda)] \cdot \nabla \eta_\lambda = -[K \star (\eta_\lambda + \zeta_\lambda)] \cdot \nabla \zeta_\lambda + (\lambda - \gamma) \zeta_\lambda. \tag{36}$$

We multiply Eq. (36) by $|\eta_\lambda|^{p-2} \eta_\lambda$, $p \geq 2$, and integrate over the spatial domain D ; using that $\langle u \cdot \nabla \eta_\lambda, |\eta_\lambda|^{p-2} \eta_\lambda \rangle = 0$ (here the estimates are first performed on more regular solutions, the Navier-Stokes approximations, and then pass to the limit for vanishing viscosity), we infer that

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} |\eta_\lambda(t)|_p^p + \gamma |\eta_\lambda(t)|_p^p &= - \langle [K \star (\eta_\lambda(t) + \zeta_\lambda(t))] \cdot \nabla \eta_\lambda(t), |\eta_\lambda(t)|^{p-2} \eta_\lambda(t) \rangle \\ &\quad - \langle [K \star (\eta_\lambda(t) + \zeta_\lambda(t))] \cdot \nabla \zeta_\lambda(t), |\eta_\lambda(t)|^{p-2} \eta_\lambda(t) \rangle + (\lambda - \gamma) \langle \zeta_\lambda(t), |\eta_\lambda(t)|^{p-2} \eta_\lambda(t) \rangle \\ &\leq |K \star (\eta_\lambda(t) + \zeta_\lambda(t))|_p |\nabla \zeta_\lambda(t)|_\infty |\eta_\lambda(t)|_p^{p-1} + |\lambda - \gamma| |\zeta_\lambda(t)|_p |\eta_\lambda(t)|_p^{p-1} \\ &\leq \left[C(|\eta_\lambda(t)|_p + |\zeta_\lambda(t)|_p) |\nabla \zeta_\lambda(t)|_\infty + |\lambda - \gamma| |\zeta_\lambda(t)|_p \right] |\eta_\lambda(t)|_p^{p-1}. \end{aligned}$$

On the other side, we have that $\frac{d}{dt} |\eta_\lambda(t)|_p^p = p |\eta_\lambda(t)|_p^{p-1} \frac{d}{dt} |\eta_\lambda(t)|_p$; we deduce that for any arbitrary $p \geq 1$

$$\frac{d}{dt} |\eta_\lambda(t)|_p + \gamma |\eta_\lambda(t)|_p \leq C(|\eta_\lambda(t)|_p + |\zeta_\lambda(t)|_p) |\nabla \zeta_\lambda(t)|_\infty + |\lambda - \gamma| |\zeta_\lambda(t)|_p.$$

Hence

$$\frac{d}{dt} |\eta_\lambda(t)|_p + (\gamma - C|\nabla \zeta_\lambda(t)|_\infty) |\eta_\lambda(t)|_p \leq (C|\nabla \zeta_\lambda(t)|_\infty + |\lambda - \gamma|) |\zeta_\lambda(t)|_p.$$

Now Gronwall’s inequality yields on the interval $[t_0, 0]$

$$\begin{aligned} |\eta_\lambda(0)|_p &\leq |\eta_\lambda(t_0)|_p e^{-\int_{t_0}^0 (\gamma - C|\nabla \zeta_\lambda(s)|_\infty) ds} \\ &\quad + \int_{t_0}^0 (C|\nabla \zeta_\lambda(s)|_\infty + |\lambda - \gamma|) |\zeta_\lambda(s)|_p e^{-\int_s^0 (\gamma - C|\nabla \zeta_\lambda(r)|_\infty) dr} ds. \end{aligned} \tag{37}$$

Using that $H^{a-1} \subset L^\infty$ for any $a > 2$ and taking $p \rightarrow \infty$, we get that

$$\begin{aligned} |\eta_\lambda(0)|_\infty &\leq |\eta_\lambda(t_0)|_\infty e^{-\int_{t_0}^0 (\gamma - \tilde{C} \|\zeta_\lambda(s)\|_{H^a}) ds} \\ &\quad + \int_{t_0}^0 C (\|\zeta_\lambda(s)\|_{H^a} + |\lambda - \gamma|) \|\zeta_\lambda(s)\|_{H^a} e^{-\int_s^0 (\gamma - \tilde{C} \|\zeta_\lambda(r)\|_{H^a}) dr} ds \end{aligned}$$

for some positive constants C and \tilde{C} . Since $\xi(t_0) = 0$, we have

$$\begin{aligned} |\eta_\lambda(0)|_\infty &\leq C \|\zeta_\lambda(t_0)\|_{H^a} e^{-\int_{t_0}^0 (\gamma - \tilde{C} \|\zeta_\lambda(s)\|_{H^a}) ds} \\ &\quad + \int_{t_0}^0 C (\|\zeta_\lambda(s)\|_{H^a} + |\lambda - \gamma|) \|\zeta_\lambda(s)\|_{H^a} e^{-\int_s^0 (\gamma - \tilde{C} \|\zeta_\lambda(r)\|_{H^a}) dr} ds. \end{aligned} \tag{38}$$

Now we choose λ large enough in order to have a uniform bound. First of all we require that $\int_{t_0}^0 (\gamma - \tilde{C} \|\zeta_\lambda(s)\|_{H^a}) ds > 0$. To this end, we notice that the process ζ_λ has the same regularity as W^{curl} and using that $\mathbf{E} \left[\int_{-\infty}^t e^{-\lambda(t-s)} d\beta_i(s) \int_{-\infty}^t e^{-\lambda(t-r)} d\beta_j(r) \right] = \delta_{ij} \frac{1}{2\lambda}$, we compute

$$\mathbf{E} \left[\|\zeta_\lambda(t)\|_{H^a}^2 \right] = \frac{1}{2\lambda} \mathbf{E} \left[\|W^{curl}(1)\|_{H^a}^2 \right].$$

Since ζ_λ is an ergodic process (see, e.g., [16]) we have

$$\lim_{t_0 \rightarrow -\infty} \frac{1}{-t_0} \int_{t_0}^0 \|\zeta_\lambda(s)\|_{H^a} ds = \mathbf{E} \|\zeta_\lambda(0)\|_{H^a} \quad \mathbf{P} - a.s.$$

We choose λ large enough such that

$$\tilde{C} \mathbf{E} \|\zeta_\lambda(0)\|_{H^a} \leq \frac{\tilde{C}}{\sqrt{2\lambda}} \sqrt{\mathbf{E} \|W^{curl}(1)\|_{H^a}^2} < \frac{\gamma}{2} \tag{39}$$

where \tilde{C} is the constant appearing in (38); thus

$$\lim_{t_0 \rightarrow -\infty} \frac{1}{-t_0} \int_{t_0}^0 \tilde{C} \|\zeta_\lambda(s)\|_{H^a} ds < \frac{\gamma}{2} \quad \mathbf{P} - a.s.$$

Then, given $\omega \in \Omega$ there exists $\tau(\omega) < 0$ such that

$$\int_{t_0}^0 \tilde{C} \|\zeta_\lambda(s)\|_{H^a} ds \leq \frac{\gamma}{2} (-t_0), \quad \forall t_0 < \tau(\omega). \tag{40}$$

Moreover, by the continuity of the trajectories of ζ_λ , there exists a (random) constant r_1 , \mathbf{P} -a.s. finite, such that

$$\sup_{\tau(\omega) < t_0 \leq 0} \int_{t_0}^0 \tilde{C} \|\zeta_\lambda(s)\|_{H^a} ds \leq r_1$$

\mathbf{P} -a.s. Hence

$$e^{-\int_{t_0}^0 (\gamma - \tilde{C} \|\zeta_\lambda(s)\|_{H^a}) ds}$$

is (pathwise) uniformly bounded for $t_0 < 0$ and vanishes exponentially fast as $t_0 \rightarrow -\infty$.

Now, arguing as before we get that there exists a random variable r_2 (\mathbf{P} -a.s. finite) such that \mathbf{P} -a.s. we have

$$\|\zeta_\lambda(t)\|_{H^a} \leq r_2(|t| + 1) \quad t < 0. \tag{41}$$

Thus we have proved a uniform bound for each term in the r.h.s. of estimate (38), that is we have obtained that there exists a random variable r_3 (\mathbf{P} -a.s. finite) such that

$$\sup_{t_0 \leq 0} |\eta_\lambda(0; \eta(t_0) = -\zeta_\lambda(t_0))|_\infty \leq r_3 \quad \mathbf{P} - a.s.$$

Since $\xi = \eta_\lambda + \zeta_\lambda$, we obtain (33). \square

From this we get

Corollary 14. *Let $\gamma > 0$ and assume (5) with $h > 3$. Then, for any $\epsilon > 0$ there exists $R_\epsilon > 0$ such that*

$$\inf_{t \geq 0} \mathbf{P}\{|\xi(t; \xi(0) = 0)|_\infty \leq R_\epsilon\} \geq 1 - \epsilon.$$

Proof. First, let us note that for any $t_0 < 0$ the random variables $\xi(0; \xi(t_0) = 0)$ and $\xi(-t_0; \xi(0) = 0)$ have the same law (homogeneity). Moreover, given a random variable r which is non negative and finite, we have that for any $\epsilon > 0$ there exists $R_\epsilon > 0$ such that

$$\mathbf{P}\{r \leq R_\epsilon\} \geq 1 - \epsilon.$$

Therefore, keeping in mind the result of Proposition 13 we get

$$\mathbf{P}\{|\xi(t; \xi(0) = 0)|_\infty \leq R_\epsilon\} = \mathbf{P}\{|\xi(0; \xi(-t) = 0)|_\infty \leq R_\epsilon\} \geq \mathbf{P}\{r \leq R_\epsilon\} \geq 1 - \epsilon$$

and this estimate is uniform in time. \square

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