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On 2D Eulerian limits à la Kuksin

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Abstract

We prove the existence of stochastic processes solving the deterministic Euler equations for an inviscid fluid on the 2D torus. In [20] Kuksin obtained this result by approximating the Euler equations by the stochastic Navier-Stokes equations with viscous term $-\nu\Delta v$ and intensity of the noise vanishing as $\sqrt{\nu}$; then in the limit as $\nu \to 0$ non trivial stationary processes solving the deterministic Euler equations were obtained. In this paper we modify the approximating viscous equations by considering a dissipative term $\nu(-\Delta)^p v$ for $p > 0$ and $p \neq 1$. We prove that the Eulerian limit process depends on the noise and on the parameter p; hence the Eulerian limits obtained for $p \neq 1$ are different from those obtained by Kuksin when $p = 1$.

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Keywords: Euler equations; Inviscid limit; Stationary solutions

1. Introduction

We are interested in stationary solutions of the Euler equations from the point of view of turbulence. We recall that the Euler equations describe the motion of inviscid fluids, whereas the Navier-Stokes equations model the motion of viscous fluids. In a two dimensional domain, these equations are well studied as far as existence, uniqueness and regularity are concerned. It is clear that they are different from the physical as well as from the mathematical point of view. However they are related to each other in the sense that the Euler equations are obtained from

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the Navier-Stokes equations by setting the viscosity equal to 0, at least formally and when the spatial domain is $\mathbb{R}^d$ or a box with periodic boundary conditions; otherwise the two equations have different boundary conditions. Rigorously one can recover the Euler equations as the limit of the Navier-Stokes equations for vanishing viscosity.

The link with the turbulence theory of fluids lies in the fact that the K41 theory of Kolmogorov is a statistical description of fluids; it involves averaged quantities and the average is understood in the statistical sense with respect to some stationary measure (see, e.g., [15,16]). These stationary measures can be studied by means of stochastic analysis. We recall that many results on invariant measures have been proved for the Navier-Stokes equations with a stochastic forcing term (see, among the others, [14,24,7,19,10]). However few results are known for the deterministic Euler equations. Working on the 2D torus, for the Euler equations explicit expression for stationary measures has been given by Albeverio and collaborators [2,3,1] (actually they work with the Gibbs measure of the enstrophy, see also [13]), by Cipriano [8] and by Biryuk [6]. Moreover in a series of papers [20–23] Kuksin proves existence of stationary solutions of the deterministic unforced Euler equations

$$\begin{cases} \partial_t v + (v \cdot \nabla) v + \nabla q = 0 \\ \nabla \cdot v = 0 \end{cases} \tag{1}$$

as the limit of stationary solutions of the Navier-Stokes equations with stochastic forcing term

$$\begin{cases} \partial_t v + [\nu(-\Delta) v + (v \cdot \nabla) v + \nabla q]\, dt = \sqrt{\nu} dW(t) \\ \nabla \cdot v = 0 \end{cases} \tag{2}$$

as $\nu \to 0$ (through a vanishing sequence). Here v is the velocity, q the pressure and ν the kinematic viscosity.

This means that the stationary space-periodic 2D turbulence is described by the limit for vanishing viscosity in the 2D Navier-Stokes equations perturbed by a stationary random force. Kuksin calls these limits the Eulerian limits and they describe the transition to turbulence for space-periodic 2D flows that are stationary in time.

Kuksin proves that any Eulerian limit is a stationary process which is not the trivial zero solution of the Euler equations (1); moreover it keeps track of the structure of the noise term. So the Eulerian limits obtained in this way depend on the vanishing sequence $\{v^{\nu_j}\}_j$ and on the noise term. This provides a lot of non trivial stationary processes solving the deterministic Euler (1). Actually they correspond to invariant measures for the Euler equations, as proved in [20] (Theorem 3.6).

For all these results the two invariants of the energy and the enstrophy play a prominent role.

Here we want to prove the same result as Kuksin, by considering the stochastic Navier-Stokes equations with fractional Laplacian

$$\begin{cases} \partial_t v + [\nu(-\Delta)^p v + (v \cdot \nabla) v + \nabla q]\, dt = \sqrt{\nu} dW(t) \\ \nabla \cdot v = 0 \end{cases} \tag{3}$$

for $p \neq 1$. If $0 < p < 1$, these are the so called stochastic hypoviscous equations, studied in [9,17], where existence and uniqueness of the invariant measure is proven under suitable assumptions on the noise term. If $p > 1$, these are the so called stochastic hyperviscous equations, whose analysis is simpler and similar to the case $p = 1$.

The aim of the present paper is to show that any Eulerian limit obtained from the viscous equations (3) depends on the vanishing sequence $\{\nu^{\nu_j}\}_j$, on the noise term *and* on p. Thus we enrich the set of Eulerian limits. We will focus mainly on the case $p \in (0, 1)$, since the case $p > 1$ presents no novelty with respect to the case $p = 1$. Only in the last section we will consider any $p \neq 1$.

As far as the structure of the paper is concerned, in Section 2 we introduce the mathematical setting. In Section 3 we present the basic results on the stochastic hypoviscous equation (3). The passage to the limit for vanishing viscosity is considered in Section 4; we will comment on the hyperviscous approximation for $p > 1$ in Remark 4.2. Finally Section 5 presents the dependence of the Eulerian limit process on the parameter p.

2. Mathematical framework

The spatial domain in which we consider the fluid is the square $\mathbb{T} = [-\pi, +\pi]^2$ and we assume periodic boundary conditions. In the book by Temam [27] there are the main results for the deterministic Navier-Stokes equations in this setting. We refer the reader to this book for the basic notations and results that we now list.

We consider the space $\mathbb{Z}_0^2$ of nonzero vectors $h = (h_1, h_2)$ with integer components, and define $|h| = \sqrt{h_1^2 + h_2^2}$ and $h^\perp = (-h_2, h_1)$. We set $\mathbb{Z}_+^2 = \{h \in \mathbb{Z}_0^2 : h_1 > 0\} \cup \{h \in \mathbb{Z}_0^2 : h_1 = 0, h_2 > 0\}$ and $\mathbb{Z}_-^2 = \mathbb{Z}_0^2 \setminus \mathbb{Z}_+^2$. We introduce the sequence

$$e_h(x) = \begin{cases} \frac{h^\perp}{\sqrt{2}\pi|h|} \sin(h \cdot x), & h \in \mathbb{Z}_+^2 \\ \frac{h^\perp}{\sqrt{2}\pi|h|} \cos(h \cdot x), & h \in \mathbb{Z}_-^2 \end{cases}$$

The basic space we work with is the Hilbert space $H^0 = \text{span}\{e_h : h \in \mathbb{Z}_0^2\}$. This is the $[L^2]^2$-closure of the space of smooth divergence free vector fields, periodic and with zero spatial mean. For $u, v \in H^0$ of the form $u = \sum_h u_h e_h$ and $v = \sum_h v_h e_h$, the scalar product is given by $\langle u, v\rangle = \sum_h u_h v_h$.

Next we introduce the Laplace operator $A = -\Delta$ which is a linear unbounded self-adjoint operator in H^0. We have $Ae_h = |h|^2 e_h$ for any h and $D(A) = \{u = \sum_h u_h e_h \in H^0 : \sum_h |h|^4 u_h^2 < \infty\}$. We denote by A^p its p power: $A^p e_h = |h|^{2p} e_h$ and $D(A^p) = \{u = \sum_h u_h e_h : \sum_h |h|^{4p} u_h^2 < \infty\}$. We set $H^s = D(A^{s/2})$; this is a Hilbert space with norm $\|u\|_{H^s} = \left(\sum_h |h|^{2s} u_h^2\right)^{1/2}$. With some abuse of notation we denote by $\langle \cdot, \cdot\rangle$ also the $H^s - H^{-s}$ duality pairing. We recall the continuous and compact embedding $H^{s_1} \subset H^{s_2}$ for $s_1 > s_2$, and the Sobolev embedding theorem

$$\begin{aligned} &H^s \subset L^q && \text{for } s \in (0,1), q \in (1,\infty), \tfrac{1}{q} = \tfrac{1-s}{2} > 0 \\ &H^1 \subset L^q && \text{for any } q \in (1,\infty) \\ &H^s \subset L^\infty && \text{for any } s > 1 \end{aligned}$$

where all the embeddings are continuous.

We recall a compactness result. Let X_0, X, X_1 be reflexive Banach spaces such that the embeddings $X_0 \subset X \subset X_1$ are continuous and the embedding $X_0 \subset X$ is compact. Let $T > 0$ be a fixed finite number. The following result is due to Dubinsky (see Theorem IV.4.1 in [28])

Theorem 2.1. *A bounded set in $L^2(0, T; X_0)$ consisting of functions equicontinuous in $C([0, T]; X_1)$ is relatively compact in $L^2(0, T; X)$ and $C([0, T]; X_1)$.*

We define the projection operator Π from $[L^2]^2$ onto H^0, and the bilinear term $B(u, v) = \Pi[(u \cdot \nabla)v]$ from $H^1 \times H^1$ to H^{-1}. Basic results are (see [27])

$$\langle B(u, v), v\rangle = 0 \qquad \forall u, v \in H^1 \tag{4}$$

$$\langle B(v, v), Av\rangle = 0 \qquad \forall v \in H^2 \tag{5}$$

and, for any $\epsilon > 0$

$$\begin{aligned} &\|B(u, v)\|_{H^0} \le C\|u\|_{H^1}\|v\|_{H^{1+\epsilon}} \qquad \forall u \in H^1, v \in H^{1+\epsilon} \\ &\|B(u, v)\|_{H^0} \le C\|u\|_{H^{1+\epsilon}}\|v\|_{H^1} \qquad \forall u \in H^{1+\epsilon}, v \in H^1 \end{aligned} \tag{6}$$

Here is another estimate for the bilinear term.

Lemma 2.2. *Let $0 < p < 1$. Then, for any $q > 1$ there exists a finite constant C (depending on p and q) such that*

$$\|B(u, v)\|_{H^p} \le C\|u\|_{H^q}\|v\|_{H^{1+p}} \tag{7}$$

for any $u \in H^q, v \in H^{1+p}$.

Proof. First we have, by Hölder and Sobolev inequalities

$$\|B(u, v)\|_{H^0} = \|(u \cdot \nabla)v\|_{L^2} \le \|u\|_{L^\infty}\|\nabla v\|_{L^2} \le C\|u\|_{H^q}\|v\|_{H^1}$$

and

$$\begin{aligned} \|B(u, v)\|_{H^1} = \|\nabla[(u \cdot \nabla)v]\|_{L^2} &\le \|\nabla u\|_{L^{\frac{2}{2-q}}}\|\nabla v\|_{L^{\frac{2}{1+q}}} + \|u\|_{L^\infty}\|v\|_{H^2} \\ &\le C\|\nabla u\|_{H^{q-1}}\|\nabla v\|_{L^2} + C\|u\|_{H^q}\|v\|_{H^2} \le C\|u\|_{H^q}\|v\|_{H^2} \end{aligned}$$

Therefore the estimate (7) follows by Marcinkievicz interpolation theorem. □

As far as the noise term is concerned, we define an H^0-cylindrical Wiener process as

$$w(t) = \sum_{h \in \mathbb{Z}_0^2} \beta_h(t)e_h$$

where $\{\beta_h\}_{h \in \mathbb{Z}_0^2}$ is a sequence of i.i.d. standard Wiener processes defined on a probability basis $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t \ge 0}, \mathbb{P})$ (see, e.g., [11]). Then given a non-negative linear operator Q in H^0 we set

$$W(t) = \sqrt{Q}w(t).$$

This means that the noise is white in time and colored in space (for $Q \neq I$). We assume that $Qe_h = q_h e_h$ for all $h \in \mathbb{Z}_0^2$; hence Q and A commute.

For each $m \geq 0$ we set

$$K_m = Tr(A^m Q) = \sum_{h \in \mathbb{Z}_0^2} q_h |h|^{2m}. \tag{8}$$

If the noise is finite dimensional, all the K_m are finite, whereas if infinitely many q_h's do not vanish the condition $K_m < \infty$ requires a suitable behavior of the sequence $\{q_h\}_h$ for large $|h|$.

We have $\sqrt{Q}w \in C(\mathbb{R}; H^a)$ if $Tr(A^a Q) < \infty$, i.e. if $K_a < \infty$; and for any a and t

$$\mathbb{E} \sup_{0 \leq t \leq T} \|\sqrt{Q}w(t)\|_{H^a} \leq \sqrt{K_a}\sqrt{T}. \tag{9}$$

Finally, we project equation (3) onto the space H^0 so to get rid of the pressure term and we obtain the stochastic viscous equation (3) in the abstract form

$$dv(t) + [\nu A^p v(t) + B(v(t), v(t))]\, dt = \sqrt{\nu Q} dw(t). \tag{10}$$

In the same way, the abstract form for the inviscid equation (1) is

$$\frac{dv}{dt}(t) + B(v(t), v(t)) = 0. \tag{11}$$

In Sections 3 and 4 we analyze in details the case $0 < p < 1$, since the case $p > 1$ presents no big differences with respect to the case $p = 1$ (see also Remark 4.2).

3. Basic results on the hypoviscous equation

We recall from [9] the known results on the stochastic equation (10). From now on we assume that only finitely many of the q_h's are nonzero, as in [9]. Otherwise an appropriate decay property for the q_h's has to be assumed.

We fix $\nu > 0$ and $p \in (0, 1)$. In the sequel r is any value larger than 3.

1. For any $v_0 \in H^r$ there exists a unique strong solution with initial velocity v_0; its paths are in $L^\infty(0, \infty; H^r) \cap L^2_{loc}(0, \infty; H^{r+p})$, a.s. This is Proposition 1.1 of [9], written there with respect to the vorticity and here with respect to the velocity.
2. The Markov semigroup $\{P_t^{\nu,p}\}_{t \geq 0}$ is Feller in H^r. We recall its definition

$$(P_t^{\nu,p} f)(x) = \mathbb{E}[f(v_p^\nu(t; x))]$$

where $v_p^\nu(t; x)$ the solution of equation (10) at time t which started from x at time 0 and $f \in C_b(H^r)$.

3. For any fixed $\nu > 0$, there exists at least one invariant measure μ_p^ν, that is a measure on H^r fulfilling

$$\int P_t^{\nu,p} f d\mu_p^\nu = \int f d\mu_p^\nu \qquad \forall t \geq 0, f \in C_b(H^r).$$

Every invariant measure is supported on C^∞ and

$$\int_{H^r} \|v\|_{H^s}^n d\mu_p^\nu(v) < \infty$$

for every $s \geq r$ and $n \geq 2$.

Remark 3.1. A finite dimensional noise, acting on all determining modes, appears in the papers [9,17], where existence and uniqueness of the invariant measure is proven. More precisely, there exists $N = N(\nu, p)$ such that if

$$q_h \neq 0 \qquad \forall |h| \leq N,$$

then (10) is ergodic. In both papers the number N diverges as the viscosity ν vanishes. Hence for fixed $p \in (0, 1)$, using the results of [9,17] we cannot prove the uniqueness of the invariant measure for any $\nu > 0$ and a finite dimensional noise.

As in [20] (which considers equation (10) for $p = 1$) we can get some a-priori estimates for any stationary solution V_p^ν, associated to an invariant measure μ_p^ν, which are uniform with respect to $\nu > 0$:

$$\mathbb{E}\|V_p^\nu(t)\|_{H^p}^2 = \tfrac{1}{2}K_0 \qquad \forall t \tag{12}$$

$$\mathbb{E}\|V_p^\nu(t)\|_{H^{p+1}}^2 = \tfrac{1}{2}K_1 \qquad \forall t \tag{13}$$

$$\tfrac{1}{2}K_0\Big(\frac{K_0}{K_1}\Big)^p \leq \mathbb{E}\|V_p^\nu(t)\|_{H^0}^2 \leq \tfrac{1}{2}K_0 \qquad \forall t \tag{14}$$

and for any $n \in \mathbb{N}$

$$\mathbb{E}\|V_p^\nu(t)\|_{H^1}^{2n} \leq C_n \qquad \forall t \tag{15}$$

for suitable constants $C_n > 0$ independent of ν.

Let us comment on these crucial estimates. For (12): Itô formula for $\|V_p^\nu(t)\|_{H^0}^2$, using the relationship (4) gives

$$\tfrac{1}{2}d\|V_p^\nu(t)\|_{H^0}^2 = -\nu\|V_p^\nu(t)\|_{H^p}^2 dt + \sqrt{\nu}\langle V_p^\nu(t), \sqrt{Q}dw(t)\rangle + \tfrac{1}{2}\nu K_0 dt$$

Taking the time integral and then the mathematical expectation, we have

$$\frac{1}{2}\mathbb{E}\|V_p^\nu(t_2)\|_{H^0}^2 - \frac{1}{2}\mathbb{E}\|V_p^\nu(t_1)\|_{H^0}^2 = -\nu\int_{t_1}^{t_2}\mathbb{E}\|V_p^\nu(s)\|_{H^p}^2 ds + \frac{1}{2}\nu K_0(t_2-t_1)$$

for any $0 \le t_1 < t_2$. Using the stationarity we get that the l.h.s. vanishes and therefore we obtain (12).

Similarly, for (13): use Itô formula for $\|V_p^\nu(t)\|_{H^1}^2$ and (5), or equivalently use the Itô formula for the L^2-norm of the vorticity.

Moreover, using the interpolation inequality

$$\|v\|_{H^p} \le \|v\|_{H^0}^{\frac{1}{p+1}}\|v\|_{H^{p+1}}^{\frac{p}{p+1}}$$

and the Hölder inequality, we get

$$\mathbb{E}\|V_p^\nu(t)\|_{H^p}^2 \le \mathbb{E}[\|V_p^\nu(t)\|_{H^0}^{\frac{2}{p+1}}\|V_p^\nu(t)\|_{H^{p+1}}^{\frac{2p}{p+1}}] \le (\mathbb{E}[\|V_p^\nu(t)\|_{H^0}^2])^{\frac{1}{p+1}}(\mathbb{E}[\|V_p^\nu(t)\|_{H^{p+1}}^2])^{\frac{p}{p+1}}.$$

This provides

$$\mathbb{E}\|V_p^\nu(t)\|_{H^0}^2 \ge \frac{(\mathbb{E}[\|V_p^\nu(t)\|_{H^p}^2])^{p+1}}{(\mathbb{E}[\|V_p^\nu(t)\|_{H^{p+1}}^2])^p}$$

which gives the lower bound in (14), keeping in mind (12) and (13). The upper bound comes from (12) and the continuous embedding $H^p \subset H^0$.

The estimate (15) is proved in the next subsection.

3.1. Exponential estimates

In [20] the author uses that any stationary solution of the Navier-Stokes equations satisfies a uniform exponential moment estimate. We get the same result here for the hypoviscous equation (10), with some minimal differences; we provide the proofs in order to highlight the role of p.

Theorem 3.2. *Let* $\tilde{q} := \max_{h\in\mathbb{Z}_0^2}(q_h|h|^2)$. *If*

$$0 < \gamma \le \frac{1}{2\tilde{q}} \tag{16}$$

then, for any stationary solution V_p^ν *of equations* (10) *we have that*

$$\mathbb{E}e^{\gamma\|V_p^\nu(t)\|_{H^1}^2} \le K_1 e^{\gamma K_1}$$

for any $t > 0$, $\nu > 0$ *and* $p > 0$.

This is based on the two following results:

Proposition 3.3. *Let $\tilde{q} := \max_{h\in\mathbb{Z}_0^2}(q_h|h|^2)$. If* (16) *holds true, then the solution of equation* (10) *(for initial velocity in H^r, $r > 3$) satisfies*

$$\mathbb{E}e^{\gamma\|v_p^\nu(t)\|^2_{H^1}} \le e^{-\gamma\nu t}\mathbb{E}e^{\gamma\|v_p^\nu(0)\|^2_{H^1}} + K_1e^{\gamma K_1} \tag{17}$$

for any $t \ge 0$, $\nu > 0$ and $p > 0$.

and

Proposition 3.4. *Assume that there exist constants $\gamma, K > 0$ such that*

$$\mathbb{E}e^{\gamma\|v_p^\nu(t)\|^2_{H^1}} \le e^{-\gamma\nu t}\mathbb{E}e^{\gamma\|v_p^\nu(0)\|^2_{H^1}} + K \tag{18}$$

for all $t \ge 0$, $\nu > 0$ and $p > 0$. Then, considering any stationary solution V_p^ν of equations (10) *with support of the invariant measure contained in H^1 we have that*

$$\mathbb{E}e^{\gamma\|V_p^\nu(t)\|^2_{H^1}} \le K$$

for any $t > 0$, $\nu > 0$ and $p > 0$.

Let us prove them.

Proposition 3.3. If $\mathbb{E}e^{\gamma\|v_p^\nu(0)\|^2_{H^1}} = +\infty$, we have nothing to prove. Hence, we assume that $\mathbb{E}e^{\gamma\|v_p^\nu(0)\|^2_{H^1}} < +\infty$.

We follow Lemma 3.1 in [25]. We write Itô formula for $e^{\gamma\|v_p^\nu(t)\|^2_{H^1}}$:

$$\begin{aligned} de^{\gamma\|v_p^\nu(t)\|^2_{H^1}} = e^{\gamma\|v_p^\nu(t)\|^2_{H^1}} 2\gamma\langle Av_p^\nu(t), -\nu A^p v_p^\nu(t) - B(v_p^\nu(t), v_p^\nu(t))\rangle dt \\ + e^{\gamma\|v_p^\nu(t)\|^2_{H^1}} 2\gamma\langle Av_p^\nu(t), \sqrt{\nu Q}dw(t)\rangle \\ + e^{\gamma\|v_p^\nu(t)\|^2_{H^1}}\left[\nu\gamma K_1 + 2\gamma^2\nu\|\sqrt{Q}Av_p^\nu(t)\|^2_{H^0}\right]dt \end{aligned} \tag{19}$$

Taking the time integral and then the mathematical expectation, we get thanks to (5)

$$\begin{aligned} \mathbb{E}e^{\gamma\|v_p^\nu(t)\|^2_{H^1}} &= \mathbb{E}e^{\gamma\|v_p^\nu(0)\|^2_{H^1}} \\ &\quad - \gamma\nu\mathbb{E}\int_0^t e^{\gamma\|v_p^\nu(s)\|^2_{H^1}}[2\|v_p^\nu(s)\|^2_{H^{p+1}} - 2\gamma\|\sqrt{Q}Av_p^\nu(s)\|^2_{H^0} - K_1]ds. \end{aligned}$$

We use that for any $p > 0$ we have that $\|v_p^\nu\|_{H^1} \le \|v_p^\nu\|_{H^{p+1}}$ and $\|\sqrt{Q}Av\|^2_{H^0} = \sum_h q_h|h|^4v_h^2 \le \tilde{q}\sum_h |h|^2v_h^2 = \tilde{q}\|v_p^\nu\|^2_{H^1}$; then

$$\mathbb{E}e^{\gamma\|v_p^\nu(t)\|_{H^1}^2} \le \mathbb{E}e^{\gamma\|v_p^\nu(0)\|_{H^1}^2} - \gamma\nu\mathbb{E}\int_0^t e^{\gamma\|v_p^\nu(s)\|_{H^1}^2}[2(1-\gamma\tilde{q})\|v_p^\nu(s)\|_{H^1}^2 - K_1]ds.$$

Assume that $2(1-\gamma\tilde{q}) \ge 1$, i.e. (16). Then

$$\mathbb{E}e^{\gamma\|v_p^\nu(t)\|_{H^1}^2} \le \mathbb{E}e^{\gamma\|v_p^\nu(0)\|_{H^1}^2} - \gamma\nu\mathbb{E}\int_0^t e^{\gamma\|v_p^\nu(s)\|_{H^1}^2}[\|v_p^\nu(s)\|_{H^1}^2 - K_1]ds \tag{20}$$

Now use the estimate

$$-e^{\gamma y^2}[y^2-K_1] \le \begin{cases} -e^{\gamma y^2} & \text{if } \ y^2-K_1 \ge 1 \\ 0 & \text{if } \ 0 < y^2-K_1 \le 1 \\ K_1 e^{\gamma K_1} & \text{if } \ y^2-K_1 \le 0 \end{cases}$$

to get

$$-e^{\gamma y^2}[y^2-K_1] \le -e^{\gamma y^2} + K_1 e^{\gamma K_1}.$$

With these estimates, from (20) we obtain

$$\mathbb{E}e^{\gamma\|v_p^\nu(t)\|_{H^1}^2} \le \mathbb{E}e^{\gamma\|v_p^\nu(0)\|_{H^1}^2} - \gamma\nu\int_0^t \mathbb{E}e^{\gamma\|v_p^\nu(s)\|_{H^1}^2}ds + \gamma\nu K_1 e^{\gamma K_1}t$$

and Gronwall Lemma allows to conclude that

$$\mathbb{E}e^{\gamma\|v_p^\nu(t)\|_{H^1}^2} \le e^{-\gamma\nu t}\mathbb{E}e^{\gamma\|v_p^\nu(0)\|_{H^1}^2} + K_1 e^{\gamma K_1} \quad \square$$

Proposition 3.4. This is very similar to the proof of Theorem 2.2 in [25]. For any $R > 0$ we define

$$f_R(v) = \begin{cases} e^{\gamma\|v\|_{H^1}^2} & \text{if } \|v\|_{H^1} \le R \\ e^{\gamma R^2} & \text{if } \|v\|_{H^1} > R \end{cases}$$

Notice that $f_R \in C_b(H^1)$, its restriction to H^r (for any $r > 1$) is $C_b(H^r)$; $f_R(v) \le e^{\gamma\|v\|_{H^1}^2}$ and $f_R(v) \le e^{\gamma R^2}$ for any $v \in H^1$.

Applying the Markov semigroup we get

$$(P_t^{\nu,p} f_R)(x) \le \mathbb{E}e^{\gamma\|v_p^\nu(t;x)\|_{H^1}^2} \underbrace{\le}_{by (18)} e^{-\gamma\nu t}\mathbb{E}e^{\gamma\|x\|_{H^1}^2} + K$$

We consider a stationary solution V_p^ν, and denote by μ_p^ν the law of its marginal at fixed time; this is an invariant measure. This invariance implies that

$$\int f_R(x)\mu_p^\nu(dx) = \int (P_t^{\nu,p} f_R)(x)\mu_p^\nu(dx) \tag{21}$$

for all $t \geq 0$.

We now estimate the r.h.s.

$$\begin{aligned}\int (P_t^{\nu,p} f_R)(x)\mu_p^\nu(dx) &= \int\limits_{\{\|x\|_{H^1}\leq r\}} (P_t^{\nu,p} f_R)(x)\mu_p^\nu(dx) + \int\limits_{\{\|x\|_{H^1}> r\}} (P_t^{\nu,p} f_R)(x)\mu_p^\nu(dx)\\ &\leq \int\limits_{\{\|x\|_{H^1}\leq r\}} [e^{-\gamma\nu t+\gamma r^2} + K]\mu_p^\nu(dx) + \int\limits_{\{\|x\|_{H^1}> r\}} e^{\gamma R^2}\mu_p^\nu(dx)\\ &\leq e^{-\gamma\nu t+\gamma r^2} + K + e^{\gamma R^2}\mu_p^\nu\{\|x\|_{H^1} > r\}\end{aligned}$$

Keeping in mind (21) and letting $t \to \infty$, we get

$$\int f_R(x)\mu_p^\nu(dx) \leq K + e^{\gamma R^2}\mu_p^\nu\{\|x\|_{H^1} > r\}.$$

Letting $r \to \infty$, since the measure μ_p^ν has support contained in H^1 we get $\mu_p^\nu\{\|x\|_{H^1} > r\} \to 0$; hence

$$\int f_R(x)\mu_p^\nu(dx) \leq K.$$

By monotone convergence as $R \to \infty$, we obtain

$$\int e^{\gamma\|x\|_{H^1}^2}\mu_p^\nu(dx) \leq K.$$

Since $V_p^\nu(t)$ has law μ_p^ν, this finishes the proof. □

From the exponential estimate, we obtain the polynomial estimate as usual.

Corollary 3.5. *Let $p > 0$ and $n \in \mathbb{N}$. Then there exists a suitable constant $\gamma > 0$ such that for any stationary solution of* (10) *we have*

$$\mathbb{E}\|V_p^\nu(t)\|_{H^1}^{2n} \leq \frac{n!}{\gamma^n} K_1 e^{\gamma K_1}$$

for any $t \geq 0$ and $\nu > 0$.

In particular, for $0 < p < 1$ we have

$$\sup_{\nu>0}\mathbb{E}\|V_p^\nu(t)\|_{H^p}^4 \leq \sup_{\nu>0}\mathbb{E}\|V_p^\nu(t)\|_{H^1}^4 < \infty. \tag{22}$$

This is a sufficient condition for the uniform integrability of the family $\{\|V_p^\nu(t)\|_{H^p}^2\}_{\nu>0}$.

4. Inviscid limit

The uniform a-priori estimates obtained so far allow to pass to the limit in the stochastic viscous equation (10) as the viscosity vanishes. In this section we fix $0 < p < 1$. Moreover we fix a parameter a very close to 1, let us say $a \in (\frac{1}{2}, 1)$, and define the space $Y_p = L^2_{loc}(\mathbb{R}_+; H^{1+ap}) \cap C(\mathbb{R}_+; H^0)$. This is a Polish space and the distance in Y_p is defined as $d(u, v) = \sum_n 2^{-n} \dfrac{\|u - v\|_{Y_{p,n}}}{1 + \|u - v\|_{Y_{p,n}}}$, where $Y_{p,n}$ is the norm for the Banach space $Y_{p,n} := L^2(0, n; H^{1+ap}) \cap C([0, n]; H^0)$.

This is our main result

Theorem 4.1. *Fix* $0 < p < 1$.
For any $\nu > 0$, *let* V_p^ν *be a stationary solution of the viscous equation* (10). *We have:*

i) *The family of the laws of* $\{V_p^\nu\}_{0<\nu\le 1}$ *is tight in the space* Y_p.
Any sequence $\nu_n \to 0$ *contains a subsequence such that the law of* $V_p^{\nu_{n_j}}$ *converges to a measure* m_p; *this limit defines a stationary stochastic process* V_p *on a probability basis* $(\tilde{\Omega}, \{\tilde{F}_t\}_t, \tilde{\mathbb{P}})$; *a.e. path of* V_p *is in* Y_p *and solves the deterministic Euler equation*

$$\frac{dV}{dt}(t) + B(V(t), V(t)) = 0. \tag{23}$$

Moreover, we have (a.s.) that $V_p \in L^2_{loc}(\mathbb{R}_+; H^{1+p})$ *and* $\frac{dV_p}{dt} \in L^1_{loc}(\mathbb{R}_+; H^p)$.

ii) *For any* t *we have*

$$\tilde{\mathbb{E}}\|V_p(t)\|^2_{H^{p+1}} \le \frac{1}{2} K_1 \tag{24}$$

$$\tfrac{1}{2} K_0 \left(\frac{K_0}{K_1}\right)^p \le \tilde{\mathbb{E}}\|V_p(t)\|^2_{H^0} \le \tfrac{1}{2} K_0 \tag{25}$$

$$\tilde{\mathbb{E}}\|V_p(t)\|^2_{H^p} = \tfrac{1}{2} K_0 \tag{26}$$

$$\tilde{\mathbb{E}} e^{\gamma \|V_p(t)\|^2_{H^1}} \le K_1 e^{\gamma K_1} \tag{27}$$

for γ *as in* (16). *In particular* V_p *is not the zero-process.*

iii) *For a.e.* t, *the energy* $\frac{1}{2}\|V_p(t)\|^2_{H^0}$ *and the enstrophy* $\frac{1}{2}\|V_p(t)\|^2_{H^1}$ *are random variables independent of time.*

iv) *If* $q_{-k} = q_k$ *for all* k, *then the process* V_p *is spatially homogeneous.*

Proof. We divide the proof into three steps.

Step 1 [Tightness]

We define the process $z_p^\nu = V_p^\nu - \sqrt{\nu Q} w$. Bearing in mind (13) and (9) to estimate the two terms in the right hand side, we get that for any $n > 0$ there exists a finite constant $C(p, n)$ independent of ν such that

$$\mathbb{E}\|z_p^\nu\|^2_{L^2(0,n; H^{p+1})} \le C(p, n) \tag{28}$$

for any $\nu \in (0, 1]$.

Moreover, recalling the equation fulfilled by V_p^ν we have that

$$\frac{dz_p^\nu}{dt}(t) = -\nu A^p V_p^\nu(t) - B(V_p^\nu(t), V_p^\nu(t)). \tag{29}$$

Let us analyze the regularity of the right hand side. Thanks to (13) we have that

$$\mathbb{E}\|A^p V_p^\nu\|^2_{L^2(0,n;H^{1-p})} = \mathbb{E}\|V_p^\nu\|^2_{L^2(0,n;H^{1+p})} = nK_1.$$

Moreover, by Lemma 2.2 and the inequalities coming from the interpolation $H^{1+p/4} = [H^1, H^{1+p}]_{1/4}$ and $H^{1+p/2} = [H^1, H^{1+p}]_{1/2}$, we get

$$\begin{aligned}\mathbb{E}[\|B(V_p^\nu, V_p^\nu)\|^2_{H^{p/2}}] &\le C_p \mathbb{E}[\|V_p^\nu\|^2_{H^{1+p/4}}\|V_p^\nu\|^2_{H^{1+p/2}}]\\ &\le C_p \mathbb{E}[\|V_p^\nu\|^{5/2}_{H^1}\|V_p^\nu\|^{3/2}_{H^{1+p}}]\\ &\le C_p \left(\mathbb{E}[\|V_p^\nu\|^{10}_{H^1}]\right)^{1/4}\left(\mathbb{E}[\|V_p^\nu\|^2_{H^{1+p}}]\right)^{3/4}\end{aligned}$$

Using (13) and (15) we obtain that there exists a constant $C(n, p, K_1)$ independent of ν such that

$$\mathbb{E}[\|B(V_p^\nu, V_p^\nu)\|^2_{L^2(0,n;H^{p/2})}] \le C(n, p, K_1). \tag{30}$$

Therefore, the r.h.s. of (29) belongs to $L^2(\Omega; L^2(0, n; H^q))$ where $q = \min(1 - p, \frac{p}{2})$, and the estimate is uniform in ν, i.e.

$$\mathbb{E}\left\|\frac{dz_p^\nu}{dt}\right\|^2_{L^2(0,n;H^q)} \le C(n, p, K_1) \tag{31}$$

for any $\nu \in (0, 1]$. For simplicity we choose $q = 0$; this is enough to get the basic result.

Now we recall that the space $\{z \in L^2(0, n; H^{1+p}) : \frac{dz}{dt} \in L^2(0, n; H^0)\}$ is continuously embedded into the space $C([0, n]; H^{\frac{1+p}{2}})$ (see [26] Lemma III.1.2) and

$$\frac{d}{dt}\|z_p^\nu(t)\|^2_{H^{\frac{1+p}{2}}} = 2\langle\frac{dz_p^\nu}{dt}(t), A^{\frac{1+p}{2}} z_p^\nu(t)\rangle \le 2\|\frac{dz_p^\nu}{dt}(t)\|_{H^0}\|z_p^\nu(t)\|_{H^{1+p}}.$$

Therefore integrating in time, taking the supremum and the expected value, it follows

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le n}\|z_p^\nu(t)\|^2_{H^{\frac{1+p}{2}}} &\le \Big(\int_0^n \mathbb{E}\|z_p^\nu(t)\|^2_{H^{1+p}}dt\Big)^{1/2}\Big(\int_0^n \mathbb{E}\left\|\frac{dz_p^\nu}{dt}(t)\right\|^2_{H^0}dt\Big)^{1/2}\\ &\qquad + \mathbb{E}\|v_p^\nu(0)\|^2_{H^{\frac{1+p}{2}}}\\ &\le C(n, p, K_1)\end{aligned}$$

Moreover, if $\frac{dz_p^\nu}{dt} \in L^2(0,n;H^0)$, then z_p^ν is equicontinuous in time with values in H^0. Therefore, applying Theorem 2.1 we have that the space $\{z \in L^2(0,n;H^{1+p}) : \frac{dz}{dt} \in L^2(0,n;H^0)\}$ is compactly embedded into the space $L^2(0,n;H^{1+\epsilon}) \cap C([0,n];H^0)$ for any $\epsilon < p$. In particular for $\epsilon = ap$ we have obtained that the space $\{z \in L^2(0,n;H^{1+p}) : \frac{dz}{dt} \in L^2(0,n;H^0)\}$ is compactly embedded into the space $Y_{p,n}$.

Hence, (28) and (31) prove that the family of the laws of the processes $\{z_p^\nu\}_{0<\nu\le 1}$, when considered on the time interval $[0,n]$, is tight in $Y_{p,n}$.

Step 2 [Limit process]

Denote by $m_{p,n}^\nu$ the law of the process z_p^ν on the time interval $[0,n]$. Then, by Prokhorov's theorem there exists a subsequence weakly convergent to a measure $m_{p,n}$, i.e.

$$\lim_{\nu_k\to 0} \int \phi \, dm_{p,n}^{\nu_k} = \int \phi \, dm_{p,n}$$

for any $\phi \in C_b(Y_{p,n})$.

Now we show that also the sequence of laws of $V_p^{\nu_k}$ weakly converges to $m_{p,n}$. Indeed, for any bounded Lipschitz function $\phi : Y_{p,n} \to \mathbb{R}$

$$\begin{aligned}
\left|\mathbb{E}\phi(V_p^{\nu_k}) - \int \phi \, dm_{p,n}\right| &\le \left|\mathbb{E}\phi(V_p^{\nu_k}) - \mathbb{E}\phi(z_p^{\nu_k})\right| + \left|\mathbb{E}\phi(z_p^{\nu_k}) - \int \phi \, dm_{p,n}\right| \\
&\le \mathbb{E}L\|\sqrt{\nu_k Q}w\|_{Y_{p,n}} + \left|\int \phi \, dm_{p,n}^{\nu_k} - \int \phi \, dm_{p,n}\right| \\
&\le \sqrt{\nu_k} L\mathbb{E}\|\sqrt{Q}w\|_{C([0,n];H^{1+p})} + \left|\int \phi \, dm_{p,n}^{\nu_k} - \int \phi \, dm_{p,n}\right|
\end{aligned}$$

and both terms in the latter line vanish as $\nu_k \to 0$.

Hence $\{m_{p,n}\}_{n\in\mathbb{N}}$ is a compatible family of measures and by a diagonalization procedure we define a measure m_p on Y_p such that the law of $V_p^{\nu_k}$ weakly converges to m_p.

By Shorokhod's theorem there exist a probability basis $(\tilde{\Omega}, \{\tilde{\mathcal{F}}_t\}_t, \tilde{P})$, random variables $\tilde{V}_p^k, V_p : \tilde{\Omega} \to Y_p$ such that $\tilde{V}_p^k$ and $V_p^{\nu_k}$ have the same law and

$$\lim_{k\to\infty} \tilde{V}_p^k = V_p \qquad \tilde{\mathbb{P}} - a.s. \text{ in } Y_p.$$

Moreover the limit process V_p is stationary. The bounds (12), (13) and (17) for the sequence $\{\tilde{V}_p^k\}_\nu$ still hold for the limit

$$\tilde{\mathbb{E}}\|V_p(t)\|_{H^p}^2 \le \tfrac{1}{2}K_0, \qquad \tilde{\mathbb{E}}\|V_p(t)\|_{H^{p+1}}^2 \le \tfrac{1}{2}K_1 \tag{32}$$

$$\tilde{\mathbb{E}}\|V_p(t)\|_{H^1}^{2n} \le C_n \qquad \forall n \in \mathbb{N} \tag{33}$$

for all t.

In addition there is a better result on the first relationship: this is indeed an equality

$$\tilde{\mathbb{E}}\|V_p(t)\|_{H^p}^2 = \tfrac{1}{2}K_0.$$

To show it, we bear in mind the a.s. convergence in $L^2(0,T;H^{1+ap})$, hence in $L^2(0,T;H^p)$; moreover

$$\sup_k \mathbb{E}\left(\int_0^T \|V_p^k(t)\|^2_{H^p} dt\right)^2 \le T \sup_k \mathbb{E}\left(\int_0^T \|V_p^k(t)\|^4_{H^p} dt\right) = T^2 \sup_k \mathbb{E}\|V_p^k(t)\|^4_{H^p}$$

which is finite, thanks to (22). This gives uniform integrability; hence from the convergence a.s. we get the convergence in the mean (see, e.g., [29]), namely

$$\tfrac{1}{2} K_0 T = \tilde{\mathbb{E}} \int_0^T \|V_p^k(t)\|^2_{H^p} dt \to \tilde{\mathbb{E}} \int_0^T \|V_p(t)\|^2_{H^p} dt = \int_0^T \tilde{\mathbb{E}}\|V_p(t)\|^2_{H^p} dt$$

for any T. Using stationarity we get (26). This finishes the proof of part ii).

Step 3 [The limit equation]

If $\tilde{V}_p^k$ converges to V_p in Y_p, $\tilde{\mathbb{P}}$-a.s., then what is the equation fulfilled by V_p? We have

$$\tilde{V}_p^k(t_2) - \tilde{V}_p^k(t_1) + \int_{t_1}^{t_2} B(\tilde{V}_p^k(s), \tilde{V}_p^k(s)) ds = -\nu_k \int_{t_1}^{t_2} A^p \tilde{V}_p^k(s) ds + \sqrt{\nu_k}\sqrt{Q}[\tilde{w}(t_2) - \tilde{w}(t_1)] \quad (34)$$

Both sides are understood as elements of H^0 (at least).

We have the uniform bound for the linear term

$$\tilde{\mathbb{E}}\| \int_{t_1}^{t_2} A^p \tilde{V}_p^k(s) ds \|_{H^{1-p}} ds \le \int_{t_1}^{t_2} \left(\tilde{\mathbb{E}}\|\tilde{V}_p^k(s)\|^2_{H^{1+p}}\right)^{\frac{1}{2}} ds = \left(\frac{K_1}{2}\right)^{\frac{1}{2}} (t_2 - t_1),$$

and by (9) a similar one for the stochastic part. Hence the r.h.s. of (34) vanishes as $k \to \infty$, considering the limit in the space $L^1(\tilde{\Omega}; H^{1-p})$.

For the l.h.s. of (34): for $\tilde{\mathbb{P}}$-a.e. path we have that $\tilde{V}_p^k(t) \to V_p(t)$ in H^0 for any t; moreover, using (6)

$$
\begin{aligned}
&\|B(\tilde V_p^k,\tilde V_p^k)-B(V_p,V_p)\|_{L^1(t_1,t_2:H^0)}\\
&\qquad\le \|B(\tilde V_p^k,\tilde V_p^k-V_p)\|_{L^1(t_1,t_2:H^0)}+\|B(\tilde V_p^k-V_p,V_p)\|_{L^1(t_1,t_2:H^0)}\\
&\qquad\le C\int_{t_1}^{t_2}\|\tilde V_p^k(s)\|_{H^1}\|\tilde V_p^k(s)-V_p(s)\|_{H^{1+\frac p2}}ds\\
&\qquad\quad+C\int_{t_1}^{t_2}\|V_p(s)\|_{H^1}\|\tilde V_p^k(s)-V_p(s)\|_{H^{1+\frac p2}}ds\\
&\qquad\le C\left(\|\tilde V_p^k\|_{L^2(t_1,t_2:H^1)}+\|V_p\|_{L^2(t_1,t_2:H^1)}\right)\|\tilde V_p^k-V_p\|_{L^2(t_1,t_2:H^{1+\frac p2})}
\end{aligned}
$$

This proves that for any $0<t_1<t_2$ the l.h.s. of equation (34) converges to the Euler equation in H^0, $\tilde{\mathbb P}$-a.s. Using the time continuity in the H^0-norm we get that $\tilde{\mathbb P}$-a.s. the limit process satisfies

$$
V_p(t_2)-V_p(t_1)+\int_{t_1}^{t_2}B(V_p(s),V_p(s))ds=0 \tag{35}
$$

for any $0<t_1<t_2$. Now we investigate the properties of the limit process V_p. We know that $V_p\in Y_p$, $\tilde{\mathbb P}$-a.s. Moreover, we have already proved that

$$
\tilde{\mathbb E}\|V_p(t)\|^2_{H^{1+p}}\le \tfrac12 K_1 \qquad \forall t.
$$

Hence we get that $V_p\in L^1(\tilde\Omega;L^2_{loc}(\mathbb R_+;H^{1+p}))$. Moreover, since a.a. paths of V_p are in $L^2_{loc}(\mathbb R_+;H^{1+p})$, from Lemma 2.2 we get that $\frac{dV_p}{dt}=-B(V_p,V_p)\in L^1_{loc}(\mathbb R_+;H^p)$. This proves i).

As far as iii) is concerned, the time independence of energy and enstrophy is obtained as in Lemma 3.2 of [20], starting from the fact that the oscillation of the H^0 and H^1 norms vanishes in the mean as the viscosity vanishes, thanks to (4) and (5). Hence there exist two random variables $X_{p,0}$ and $X_{p,1}$ such that a.s.

$$
\|V_p(t)\|^2_{H^0}=X_{p,0},\qquad \|V_p(t)\|^2_{H^1}=X_{p,1}
$$

for a.a. t. Actually

$$
X_{p,0}=\int_0^1\|V_p(s)\|^2_{H^0}ds,\qquad X_{p,1}=\int_0^1\|V_p(s)\|^2_{H^1}ds.
$$

Finally, for iv) we notice that the homogeneity in space is the same as in Section 3.5 of [20]. □

To conclude this section, let us comment on the results obtained and compare them with those of the case $p=1$.

Remark 4.2. For $p > 1$ the analysis of the problem is easier and very similar to that done by Kuksin for $p = 1$. The main difference is in getting the uniform integrability; indeed the exponential moment estimate (18) involves the H^1-norm, which does not allow to deal with the H^p-norm. Here are the details. From the interpolation inequality $\|x\|_{H^p} \le \|x\|_{H^{p+1}}^{\frac{p-1}{p}} \|x\|_{H^1}^{\frac{1}{p}}$, we obtain

$$\|x\|_{H^p}^{2+\epsilon} \le \|x\|_{H^{p+1}}^{(2+\epsilon)\frac{p-1}{p}} \|x\|_{H^1}^{\frac{2+\epsilon}{p}} \le C_1 \|x\|_{H^{p+1}}^2 + C_2 \|x\|_{H^1}^{\frac{2(2+\epsilon)}{2-\epsilon(p-1)}}$$

thanks to Young inequality when $(2+\epsilon)\frac{p-1}{p} < 2$, i.e. $\epsilon < \frac{2}{p-1}$. Hence we estimate the mean for the H^p-norm with $p > 1$ as follows:

$$\begin{aligned}\sup_k \mathbb{E}\left(\int_0^T \|V_p^k(t)\|_{H^p}^2 dt\right)^{1+\frac{\epsilon}{2}} &\le T^{\frac{\epsilon}{2}} \sup_k \mathbb{E}\left(\int_0^T \|V_p^k(t)\|_{H^p}^{2+\epsilon} dt\right)\\ &\le T^{\frac{\epsilon}{2}} \sup_k \mathbb{E}\left(\int_0^T C_1 \|V_p^k(t)\|_{H^{p+1}}^2 dt + \int_0^T C_2 \|V_p^k(t)\|_{H^1}^{\frac{2(2+\epsilon)}{2-\epsilon(p-1)}} dt\right)\end{aligned}$$

By stationarity the latter line equals

$$T^{\frac{\epsilon}{2}+1} \sup_k \left[C_1 \mathbb{E}\|V_p^k(t)\|_{H^{p+1}}^2 + C_2 \mathbb{E}\|V_p^k(t)\|_{H^1}^{\frac{2(2+\epsilon)}{2-\epsilon(p-1)}}\right]$$

which is finite, thanks to (13) and (17).

We will postpone to a future work the analysis of the case $p = 0$ for which existence of stationary solutions and invariant measures μ_ν is known (see [4,5]).

Remark 4.3. Let μ_p denote the law of the vorticity $\nabla^\perp \cdot V_p(t)$ of the stationary Eulerian limit at any fixed time. Theorem 4.2 in [18] proves that for $p = 1$ the support of μ_p is in L^∞, which is a space where existence and uniqueness of solutions for the Euler equations is known. Their Remark 4.5 extends this result to any $0 < p < 1$. The result is based on De Giorgi-Moser regularization for parabolic equations. Let us notice that this trivially holds for $p > 1$ since the Eulerian limit V_p is more regular, namely (32) implies that $\tilde{\mathbb{E}}\|\nabla^\perp \cdot V_p(t)\|_{H^p}^2 < \infty$. Hence $\mu_p(H^p) = 1$ but $H^p \subset L^\infty$ for any $p > 1$.

Therefore for any $p > 0$ the marginal law of $V_p(t)$ is an invariant measure for the Euler equations (23).

Remark 4.4. In order to compare our result with that of Kuksin, it is worth to point out that when $0 < p < 1$, for the random variables $X_{p,0}$ and $X_{p,1}$ appearing iii) in the previous Theorem we have less information than in [20]. Indeed we cannot identify what is the mathematical expectation but we have the following bounds

$$\tfrac{1}{2}K_0\left(\frac{K_0}{K_1}\right)^p \le \mathbb{E}X_{p,0} \le \tfrac{1}{2}K_0, \qquad \mathbb{E}X_{p,1} \le \tfrac{1}{2}K_1$$

thanks to ii).

Remark 4.5. Choosing other scalings. i.e. considering ν^a in front of the noise term in equation (10) for $a \neq \frac{1}{2}$, one can obtain the same results as in [21][Theorem 10.8].

5. Dependence of the Eulerian limit on p

When p is fixed, any Eulerian limit keeps track of the noise term since $\mathbb{E}\|V_p(t)\|^2_{H^p} = \frac{K_0}{2}$. So noises with different values of K_0 provide different Eulerian limits. In principle the limit may depend also on the vanishing sequence $\{\nu_j\}_{j\in\mathbb{N}}$.

Now we want to show that for a suitable noise the limit depends also on the power p. To this end we ask the noise to act at least on some suitable Fourier components.

Before stating the result let us analyze our problem when the noise acts on four particular Fourier components (or a couple of them, chosen symmetrically), i.e. the noise in equation (10) is defined by setting $q_h = 0$ for any $|h| > 1$ and $q_h \neq 0$ for $|h| = 1$.

We set

$$M_4 = \text{span}\{e_h : |h| = 1\}.$$

The role of the subspace M_4 has been pointed out by Kuksin, see [20][§3.6]. The stationary Ornstein-Uhlenbeck process

$$U_p^\nu(t) = \sqrt{\nu} \sum_{h\in M_4} \sqrt{q_h} \int_{-\infty}^{t} e^{-\nu(t-s)} d\beta_h(s) e_h \tag{36}$$

solves the stochastic linear problem and leaves in M_4. However, it solves also the nonlinear equation (10) for any p, since the nonlinear term $B(v, v)$ vanishes when $v \in M_4$. The law of (36) is centered and normally distributed. In particular at fixed time the variance is given by $\mathbb{E}\|V_p^\nu(t)\|^2_{H^p} = \frac{1}{2}\sum_{h\in M_4} q_h$, namely it is independent of ν and p. This is the marginal distribution of the Eulerian limit too, for any p. This is an example of a trivial dynamics of (11) for the vanishing limit, which is indeed independent of p.

However, for a different choice of activated Fourier modes in the noise, we get that the Eulerian limit depends on p. To prove this, now we set

$$D_1 = \{(1, 0), (1, 1), (-1, 0), (-1, -1)\}$$
$$D_2 = \{(0, 1), (1, 1), (0, -1), (-1, -1)\}$$

We have the following result.

Theorem 5.1. *Fix $0 < p_1 < p_2$. If*

$$q_h \neq 0 \text{ for any } h \in D_i \tag{37}$$

for $i = 1$ or $i = 2$, then the Eulerian limits V_{p_1} and V_{p_2} of the previous section do not coincide.

Proof. Here we consider the Eulerian limit processes V_{p_1} and V_{p_2} for a fixed vanishing sequence and a fixed noise term.

Given $p_1 \neq p_2$ we have $\|v\|_{H^{p_1}} = \|v\|_{H^{p_2}}$ if and only if $v \in M_4$. As soon as there exists a non vanishing component of v out of M_4, we get that the two norms are different: $\|v\|_{H^{p_1}} < \|v\|_{H^{p_2}}$ for $p_1 < p_2$. Therefore to prove our result it is enough to show that the Eulerian limit is not concentrated on M_4.

On the one hand the linear dynamics

$$dU_p^\nu(t) + \nu A^p U_p^\nu(t)\, dt = \sqrt{\nu Q} dw(t)$$

lives on the subspace of the activated modes for the noise, which is not M_4; its law is normally distributed, is independent of the viscosity but depends on p. There is dependence on p for its vanishing viscosity limit too. On the other hand we are going to show that if the noise is chosen as in (37) then this is true for the nonlinear problem as well, since the nonlinearity mixes the dynamics in such a way that the Eulerian limit too is not concentrated on M_4.

We bear in mind a result by E and Mattingly. In [12] they consider the finite-dimensional approximation of the Navier-Stokes, i.e. the state space is isomorphic to $\mathbb{R}^{d_N}$, where $d_N = 4N(N+1)$ is the dimension of the space spanned by $\mathbb{Z}^2_N = \{h \in \mathbb{Z}^2_0 : |h|_1 \leq N, |h|_2 \leq N\}$. So the Fourier representation $u = \sum_h u_h e_h$ involves only the indices $h \in \mathbb{Z}^2_N$. Under the assumption (37) they prove that for any N the Galerkin approximation of the stochastic Navier-Stokes equations has a C^∞-transition density $p_N(t, x, y)$ ($N \in \mathbb{N}$, x and y in $\mathbb{R}^{d_N}$, $t > 0$); hence it cannot be supported in a proper linear subspace of $\mathbb{R}^{d_N}$. Therefore considering any finite N the dynamics is not concentrated on M_4. This result relies heavily on the structure of the nonlinearity, which "mixes" different Fourier modes in a suitable way (see (1.4)-(1.5) in [12]): even if the noise acts on few but suitable modes, the nonlinearity activates all the Fourier modes for the velocity in $\mathbb{R}^{d_N}$. E and Mattingly remark that this is true also for the Galerkin approximation of the stochastic damped Euler equations, which are the equations formally obtained from (10) by setting $p = 0$. Indeed the linear dissipative part in the Galerkin approximation of equation (2) is given by $-\nu\Delta v$, corresponding to $\nu|h|^2 v_h$ in each Fourier component; but as far as the finite dimensional approximation is considered there is no qualitative difference with respect to the case $p = 0$, where the linear dissipative part is given by νv, corresponding to νv_h in each Fourier component. The "weights" $|h|^2$ play no role in the finite dimensional approximation equation.

Therefore we can extend the same remark to any $p \neq 1$, since in the equation for the Galerkin approximation the Fourier components of the linear dissipative term are now $\nu|h|^{2p} v_h$.

Finally, given the structure of the nonlinearity, we deduce that under the assumption (37) the (full) dynamics of (10) is not concentrated on M_4 or any subspace of M_4. This holds for its Eulerian limit too, since this limit process has a dynamics governed by the nonlinearity.

Suppose now that two Eulerian limits V_{p_1} and V_{p_2} coincide: $V_{p_1} = V_{p_2} =: V$. Since V is not concentrated on M_4 we have

$$\tilde{\mathbb{E}}\|V(t)\|^2_{H^{p_1}} < \tilde{\mathbb{E}}\|V(t)\|^2_{H^{p_2}}. \tag{38}$$

On the other hand, we have the estimate (26) for p_1 and p_2 providing

$$\tilde{\mathbb{E}}\|V(t)\|^2_{H^{p_1}} = \tfrac{1}{2}K_0 = \tilde{\mathbb{E}}\|V(t)\|^2_{H^{p_2}}.$$

This contradicts (38). Hence we conclude that the two Eulerian limits V_{p_1} and V_{p_2} must be different. □

Data availability

No data was used for the research described in the article.

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