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EXPONENTIAL CONVERGENCE OF *hp*-FEM FOR THE INTEGRAL FRACTIONAL LAPLACIAN IN POLYGONS[*]

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Abstract. We prove exponential convergence in the energy norm of *hp*-finite element discretizations for the integral fractional Laplacian of order $2s \in (0,2)$ subject to homogeneous Dirichlet boundary conditions in bounded polygonal domains $\Omega \subset \mathbb{R}^2$. Key ingredients in the analysis are the weighted analytic regularity from [M. Faustmann, C. Marcati, J. M. Melenk, and C. Schwab, *SIAM J. Math. Anal.,* 54 (2022), pp. 6323–6357] and meshes that feature anisotropic geometric refinement towards $\partial\Omega$.

Key words. fractional Laplacian, corner domains, *hp*-FEM, exponential convergence

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1. Introduction.

In recent years, mathematical and computational modelling in engineering and natural sciences has witnessed the emergence of nonlocal boundary value problems and their mathematical and numerical analysis. For applications of fractional models, we refer the reader to the surveys [14, 9, 32, 28] and the references therein.

A typical nonlocal, elliptic operator is the so-called fractional Laplacian. In a bounded domain $\Omega \subset \mathbb{R}^d$, and for $s \in (0,1)$, the Dirichlet problem of the fractional Laplacian reads, informally, for given $f : \Omega \to \mathbb{R}$, as follows: find $u : \mathbb{R}^d \to \mathbb{R}$ such that

$$(-\Delta)^s u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{in } \Omega^c := \mathbb{R}^d \backslash \overline{\Omega}. \tag{1.1}$$

Nonlocality manifests here in that the operator $(-\Delta)^s$ acts on u globally (see (1.2)), and that the Dirichlet "boundary" condition is, in fact, a condition on the unknown on the whole exterior of Ω.

1.1. Integral fractional diffusion.

We consider a bounded, open polygon $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$ consisting of a finite number of straight sides (the edges of the polygon) and vertices. For $s \in (0,1)$, there are various different possible definitions of the fractional Laplacian $(-\Delta)^s$ (cf. [27]), which are equivalent on the full-space, but may differ on bounded domains. Here, we study the integral (Dirichlet) fractional Laplacian $(-\Delta)^s$ that, acting on a sufficiently regular function u in Ω, reads

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$$(-\Delta)^s u(x) := C(s)\,\text{P.V.}\int_{\mathbb{R}^2} \frac{u(x)-u(z)}{|x-z|^{2+2s}}dz\,, \quad C(s) := -2^{2s}\frac{\Gamma(s+1)}{\pi\Gamma(-s)}. \tag{1.2}$$

Here, P.V. denotes the Cauchy principal value integral.

In order to state a variational formulation of (1.1), fractional order Sobolev spaces are required. For an integer order $t \in \mathbb{N}_0$ and a domain $\omega \subset \mathbb{R}^2$, we denote by $H^t(\omega)$ the Hilbertian Sobolev spaces. Fractional order Sobolev spaces for $t \in (0,1)$ are defined through the Slobodeckij seminorm $|\cdot|_{H^t(\omega)}$ and the corresponding norm $\|\cdot\|_{H^t(\omega)}$ given by

$$|v|^2_{H^t(\omega)} = \int_\omega\int_\omega \frac{|v(x)-v(z)|^2}{|x-z|^{2+2t}}\,dz\,dx, \;\; \|v\|^2_{H^t(\omega)} = \|v\|^2_{L^2(\omega)} + |v|^2_{H^t(\omega)}. \tag{1.3}$$

For $t \in (0,1)$, we employ the spaces

$$\widetilde{H}^t(\Omega) := \left\{u \in H^t(\mathbb{R}^2) : u \equiv 0 \text{ on } \mathbb{R}^2\backslash\overline{\Omega}\right\}, \; \|v\|^2_{\widetilde{H}^t(\Omega)} := \|v\|^2_{H^t(\Omega)} + \left\|v/r^t\right\|^2_{L^2(\Omega)}. \tag{1.4}$$

Here and throughout, $r(x) := \operatorname{dist}(x,\partial\Omega)$ denotes the Euclidean distance of a point $x \in \Omega$ from the boundary $\partial\Omega$. For $t > 0$, the space $H^{-t}(\Omega)$ denotes the dual space of $\widetilde{H}^t(\Omega)$, and $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$ denotes the duality pairing that extends the $L^2(\Omega)$-inner product.

The variational form of (1.1) reads as follows: find $u \in \widetilde{H}^s(\Omega)$ such that, for all $v \in \widetilde{H}^s(\Omega)$,

$$a(u,v) := \frac{C(s)}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{(u(x)-u(z))(v(x)-v(z))}{|x-z|^{2+2s}}\,dz\,dx = \langle f, v\rangle_{L^2(\Omega)}\,. \tag{1.5}$$

Existence and uniqueness of $u \in \widetilde{H}^s(\Omega)$ follow from the Lax–Milgram lemma for any $f \in H^{-s}(\Omega)$, upon the observation that the bilinear form $a(\cdot,\cdot) : \widetilde{H}^s(\Omega) \times \widetilde{H}^s(\Omega) \to \mathbb{R}$ is continuous and coercive; see, e.g., [2, sect. 2.1].

This observation implies that, for any subspace $V_N \subset \widetilde{H}^s(\Omega)$ of finite dimension N, the Galerkin discretization

$$u_N \in V_N : \quad a(u_N, v) = \langle f, v\rangle_{L^2(\Omega)} \quad \forall v \in V_N \tag{1.6}$$

admits a unique solution $u_N \in V_N$. Whence

$$\forall v_N \in V_N : \quad \|u - u_N\|_{\widetilde{H}^s(\Omega)} \leq C\|u - v_N\|_{\widetilde{H}^s(\Omega)}\,. \tag{1.7}$$

Convergence rates depend on the regularity of u and on the structure of $\{V_N\}_{N\in\mathbb{N}}$. We establish *exponential convergence rate bounds* for the right-hand side of (1.7) under *weighted, analytic regularity* of u in vertex- and edge-weighted spaces in Ω. This requires $\{V_N\}_{N\in\mathbb{N}}$ to be a family of finite-dimensional subspaces of *hp*-type. In particular, we construct a family $\{\Pi_N\}_{N\in\mathbb{N}}$ of spectral element approximation operators such that exponential convergence rate bounds are attained in (1.7) with $v_N = \Pi_N u_N$.

1.2. Previous results.

In the recent work [11], the regularity of the solution u of (1.1) in a certain (isotropic) Besov space on Lipschitz domains Ω was shown. This was subsequently used in [12] to infer algebraic convergence rates of Galerkin finite element methods (FEMs) in (1.7), where, in [12], the spaces $\{V_N\}_{N\in\mathbb{N}}$ are a family of continuous, piecewise affine Lagrangian first order finite elements (FEs), on a sequence of shape-regular triangulations in Ω with judicious, *isotropic boundary refinement.*

The necessity of such refinement can be expected by the boundary asymptotics of the solution shown, e.g., in [33], where $u(x) \sim \operatorname{dist}(x, \partial\Omega)^s$ was established.

The anisotropic nature of the edge-singularities of the solution u in Ω precludes high convergence rates (in terms of error versus number of degrees of freedom) for FE-discretizations based on shape-regular mesh families: *anisotropic boundary refinement* is necessary to this end.

The regularity of solutions to (1.1) has been studied intensively in recent years. The works [33, 1] established Hölder regularity of solutions in $\overline{\Omega}$, when $\partial\Omega$ is C^1, with asymptotic behavior as $\operatorname{dist}(x, \partial\Omega)^s$ for $x \in \Omega$ (corner domains $\Omega \subset \mathbb{R}^2$ as considered here are not covered by these results). In [19, 36] vertex- and edge-singularities of solutions to (1.1) have been investigated formally, and the dominant singular terms of weak solutions $u \in \widetilde{H}^s(\Omega)$ of (1.5) have been calculated, under provision of sufficiently high (finite) regularity of f in (1.1).

In [17], we studied elliptic regularity for (1.1) in the case that (a) $\Omega \subset \mathbb{R}^2$ is a polygon, with (a finite number of) straight sides, and (b) the data f in (1.1) is analytic in $\overline{\Omega}$. We detail the results of [17] in section 2; they constitute the basis of the proof of our main result, the exponential convergence rate bound (1.8).

We recall that there are are several constructions of fractional powers of the (Dirichlet) Laplacian. In addition to the integral fractional Laplacian considered here and, e.g., in [33, 11, 19], we mention the related, so-called *spectral fractional Laplacian,* for which regularity and FE-analysis was considered, e.g., in [31, 4, 6, 8]. In [6, 8], exponential convergence of *hp*-FEMs for spectral fractional diffusion problems in so-called curvilinear polygonal domains, subject to analytic data, was proved. The mathematical analysis and the numerical method in these references leveraged the reformulation of the nonlocal boundary value problem in terms of a degenerate, elliptic *local* boundary value problem, which can be approximated by a collection of (still local) elliptic singular perturbation problems, for which *hp*-FEMs have been shown to deliver exponential convergence rates in [30, 7].

Numerical analysis for the integral fractional Laplacian was developed also in the recent contributions [10, 2, 16, 26]. We refer the reader to the surveys [9, 28] and the references therein for a comprehensive presentation and references. None of these references establishes, in space dimension $d > 1$, exponential rates of convergence. In space dimension $d = 1$, higher order spectral methods have been proposed and analyzed, in particular for settings that imply an explicit, separable singularity structure of solutions in [3, 29] and the references therein. In the presently considered polygonal domains Ω with analytic right-hand side, such approaches [3, 29] are not applicable, due to the more involved singularity structure in Ω.

1.3. Contributions.

We prove exponential rate of convergence of *hp*-FEMs to solutions of the homogeneous Dirichlet problem for the integral fractional Laplacian of order $2s \in (0, 2)$ in polygonal domains $\Omega \subset \mathbb{R}^2$, subject to a source term f that is analytic in $\overline{\Omega}$.

We resolve the vertex- and edge-singularities, which are well known to occur due to the singular support of the solution u being all of $\partial\Omega$ (see, e.g., [22, 1, 19]) by *anisotropic, geometric mesh refinement towards* $\partial\Omega$. The class of admissible geometric meshes in Ω will consist of a finite union of patchwise structured geometric partitions that are images of partitions from a finite catalog $\mathfrak{P}$, as depicted in Figure 2, similar to the construction in [7, 8]. The structured, anisotropic geometric partitions in the patches are assumed to be obtained by a finite number L of bisections. On the corresponding global geometric partition in Ω, the *hp*-approximation space V_N in (1.6), (1.7) consists of continuous, piecewise polynomials of degree $q \sim L \geq 1$.

The principal result of the present paper can be stated as follows.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a polygon. There is a sequence $\{V_N\}_{N\geq 1}$ of hp-FE spaces, with dimension not exceeding N, such that for f that is analytic in $\overline{\Omega}$ and the solution u of* (1.5)*, the Galerkin approximations $u_N \in V_N$ of* (1.6) *converge exponentially to u, i.e., there are constants b, $C > 0$ (depending on s, Ω, and f) such that*

$$\|u - u_N\|_{\widetilde{H}^s(\Omega)} \leq C \exp(-b\sqrt[4]{N}). \tag{1.8}$$

The spaces V_N can be taken as the spaces W^L_q (see (5.1) *for the precise definition), which are spaces of globally continuous, piecewise mapped polynomials of degree q on boundary-refined meshes $\mathcal{T}^L_{\mathrm{geo},\sigma}$ (see Definition* 4.2*) with L layers of geometric refinement, for $L \sim q \sim N^{1/4}$.*

We remark that the exponential rates of convergence (1.8) have recently been reported in computations in [20, sect. 8]. See also [5] and [36].

1.4. Layout. In section 2, we recapitulate the weighted, analytic regularity results of [17], which form the basis of the proofs of the exponential convergence. In section 3, we state an embedding result of weighted, integer order spaces $H^1_\beta(\Omega)$ into fractional ones, which will be instrumental in the ensuing analysis as global error bounds are obtained from adding scaled local error bounds in these spaces. Section 4 contains the definition of the *hp*-FE spaces, in particular of the structured geometric meshes on the reference mesh patches. They are simplifications of the constructions used in [7, 8]. Section 5 has the key exponential approximation error bounds in the weighted, *local* $H^1_\beta(\Omega)$-norm for the *hp*-FE spaces on the geometric, boundary-refined meshes in the patches. This is followed by the proof of Theorem 1.1.

Appendix A recapitulates the Gauss–Lobatto interpolants in the reference elements together with their basic approximation and stability properties from [30, 7]. In Appendix B, we show some technical lemmas used in the proof of the main result.

1.5. Notation. Constants C may be different in each occurrence but are independent of critical parameters of the discretization such as N, p, L. We denote by $\widehat{S} := (0,1)^2$ the reference square and by $\widehat{T} := \{(x,y) \in (0,1)^2 : y < x\}$ the reference triangle. Sets of the form $\{x = y\}$, $\{x = 0\}$, $\{x = y\}$, etc. refer to edges and diagonals of $\widehat{S}$ or $\widehat{T}$ and analogously $\{y \leq x\} = \{(x,y) \in \widehat{S} : y \leq x\}$.

For $q \in \mathbb{N}$, $\mathbb{P}_q = \operatorname{span}\{x^i y^j \,|\, i,j \geq 0, i+j \leq q\}$ denotes the space of polynomials of total degree q and $\mathbb{Q}_q = \operatorname{span}\{x^i y^j \,|\, 0 \leq i,j \leq q\}$ denotes the tensor product space of polynomial of maximum degree q in each variable separately.

For $x \in \Omega$, we recall $r(x) = \operatorname{dist}(x, \partial\Omega)$. Finally, for $t > 0$, we denote a t-neighborhood of $\partial\Omega$ by

$$S_t = \{x \in \Omega : r(x) < t\}.$$

2. Analytic regularity in polygons with straight sides.

We start by recapitulating the weighted spaces from [17] used to describe the analytic regularity.

Recall that $\Omega \subset \mathbb{R}^2$ is a bounded polygon with a finite number of straight sides, whose boundary $\partial\Omega$ is Lipschitz. By $\mathcal{V}$, we denote the set of vertices of the polygon $\Omega \subset \mathbb{R}^2$ and by $\mathcal{E}$ the set of its (open) edges. For $\mathbf{v} \in \mathcal{V}$ and $\mathbf{e} \in \mathcal{E}$, we define the distance functions

$$r_{\mathbf{v}}(x) := |x - \mathbf{v}|, \qquad r_{\mathbf{e}}(x) := \inf_{y\in\mathbf{e}} |x - y|, \qquad \rho_{\mathbf{ve}}(x) := r_{\mathbf{e}}(x)/r_{\mathbf{v}}(x).$$

For each vertex $\mathbf{v}\in\mathcal{V}$, we denote by $\mathcal{E}_{\mathbf{v}} := \{\mathbf{e}\in\mathcal{E} : \mathbf{v}\in\overline{\mathbf{e}}\}$ the set of all edges that meet at $\mathbf{v}$. For any $\mathbf{e}\in\mathcal{E}$, we define $\mathcal{V}_{\mathbf{e}} := \{\mathbf{v}\in\mathcal{V} : \mathbf{v}\in\overline{\mathbf{e}}\}$ as the set of endpoints of $\mathbf{e}$. For fixed, sufficiently small $\xi>0$ and for $\mathbf{v}\in\mathcal{V}$, $\mathbf{e}\in\mathcal{E}$, we define vertex, vertex-edge, and edge neighborhoods by

$$\omega^{\xi}_{\mathbf{v}} := \{x\in\Omega : r_{\mathbf{v}}(x)<\xi \quad\wedge\quad \rho_{\mathbf{ve}}(x)\geq\xi \quad \forall\mathbf{e}\in\mathcal{E}_{\mathbf{v}}\}, \tag{2.1}$$

$$\omega^{\xi}_{\mathbf{ve}} := \{x\in\Omega : r_{\mathbf{v}}(x)<\xi \quad\wedge\quad \rho_{\mathbf{ve}}(x)<\xi\}, \tag{2.2}$$

$$\omega^{\xi}_{\mathbf{e}} := \{x\in\Omega : r_{\mathbf{v}}(x)\geq\xi \quad\wedge\quad r_{\mathbf{e}}(x)<\xi^2 \quad \forall\mathbf{v}\in\mathcal{V}_{\mathbf{e}}\}. \tag{2.3}$$

Figure 1, taken from [17], illustrates this notation near a vertex $\mathbf{v}\in\mathcal{V}$ of the polygon. Throughout the paper, we will assume that ξ is small enough so that $\omega^{\xi}_{\mathbf{v}}\cap\omega^{\xi}_{\mathbf{v}'}=\emptyset$ for all $\mathbf{v}\neq\mathbf{v}'$, that $\omega^{\xi}_{\mathbf{e}}\cap\omega^{\xi}_{\mathbf{e}'}=\emptyset$ for all $\mathbf{e}\neq\mathbf{e}'$, and $\omega^{\xi}_{\mathbf{ve}}\cap\omega^{\xi}_{\mathbf{v}'\mathbf{e}'}=\emptyset$ for all $(\mathbf{v},\mathbf{e})\neq(\mathbf{v}',\mathbf{e}')$. We will also drop the superscripts ξ unless strictly necessary.

The polygon Ω may be decomposed into sectoral neighborhoods of vertices $\mathbf{v}$, which are unions of vertex neighborhoods $\omega_{\mathbf{v}}$ and vertex-edge neighborhoods $\omega_{\mathbf{ve}}$ (as depicted in Figure 1), edge neighborhoods $\omega_{\mathbf{e}}$ (that are properly separated from vertices $\mathbf{v}$), and an interior part Ω_{int}, i.e., we may write

$$\Omega = \bigcup_{\mathbf{v}\in\mathcal{V}}\left(\omega_{\mathbf{v}}\cup\bigcup_{\mathbf{e}\in\mathcal{E}_{\mathbf{v}}}\omega_{\mathbf{ve}}\right)\cup\bigcup_{\mathbf{e}\in\mathcal{E}}\omega_{\mathbf{e}}\cup\Omega_{\mathrm{int}}.$$

Each sectoral and edge neighborhood may have a different value ξ, but we shall work with one common (positive) value for all neighborhoods. The set $\Omega_{\mathrm{int}}\subset\Omega$ has a positive distance from the boundary $\partial\Omega$.

In a neighborhood $\omega_{\mathbf{e}}$ or $\omega_{\mathbf{ve}}$, we denote by $\mathbf{e}_{\parallel}$ and $\mathbf{e}_{\perp}$ unit vectors such that $\mathbf{e}_{\parallel}$ is tangential to $\mathbf{e}$ and $\mathbf{e}_{\perp}$ is normal to $\mathbf{e}$. We introduce the differential operators

$$D_{x_{\parallel}}v := \mathbf{e}_{\parallel}\cdot\nabla_x v, \qquad\qquad D_{x_{\perp}}v := \mathbf{e}_{\perp}\cdot\nabla_x v$$

corresponding to differentiation in the tangential and the normal direction. Higher order tangential and normal derivatives in $\omega_{\mathbf{e}}$ or $\omega_{\mathbf{ve}}$ are defined by $D^j_{x_{\parallel}}v := D_{x_{\parallel}}(D^{j-1}_{x_{\parallel}}v)$ and $D^j_{x_{\perp}}v := D_{x_{\perp}}(D^{j-1}_{x_{\perp}}v)$ for $j>1$.

The analytic regularity result in weighted local norms is [17, Thm. 2.1].

THEOREM 2.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded polygonal Lipschitz domain. Let the data $f\in C^{\infty}(\overline{\Omega})$ satisfy with a constant $\gamma_f>0$*

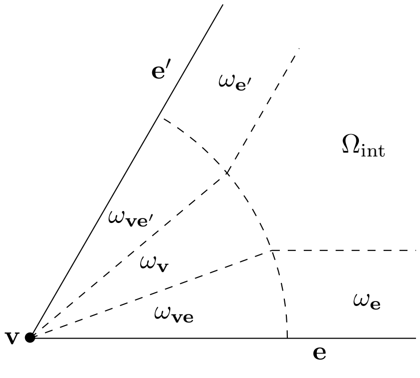

FIG. 1. *Notation near vertex $\mathbf{v}\in\mathcal{V}$.*

$$\forall j\in\mathbb{N}_0:\quad \sum_{|\alpha|=j}\|\partial_x^\alpha f\|_{L^2(\Omega)}\le\gamma_f^{j+1}j^j. \tag{2.4}$$

Let u be the solution of (1.5)*. Let $\mathbf{v}\in\mathcal{V}$, $\mathbf{e}\in\mathcal{E}$, and $\omega_{\mathbf{v}}$, $\omega_{\mathbf{ve}}$, $\omega_{\mathbf{e}}$ be fixed vertex, vertex-edge, and edge neighborhoods. Then, there is $\gamma>0$ depending only on γ_f, s, and Ω such that for every $\varepsilon>0$ there exists $C_\varepsilon>0$ (depending only on ε and Ω) such that the following holds:*

(i) *For all $\alpha\in\mathbb{N}_0^2$,*

$$\left\|r_{\mathbf{v}}^{|\alpha|-1/2-s+\varepsilon}\partial_x^\alpha u\right\|_{L^2(\omega_{\mathbf{v}})}\le C_\varepsilon\gamma^{|\alpha|+1}|\alpha|^{|\alpha|}. \tag{2.5}$$

(ii) *For all $(p_\perp,p_\parallel)\in\mathbb{N}_0^2$, it holds, with $p=p_\perp+p_\parallel$, that*

$$\left\|r_{\mathbf{e}}^{p_\perp-1/2-s+\varepsilon}D_{x_\perp}^{p_\perp}D_{x_\parallel}^{p_\parallel}u\right\|_{L^2(\omega_{\mathbf{e}})}\le C_\varepsilon\gamma^{p+1}p^p, \tag{2.6}$$

$$\left\|r_{\mathbf{e}}^{p_\perp-1/2-s+\varepsilon}r_{\mathbf{v}}^{p_\parallel+\varepsilon}D_{x_\perp}^{p_\perp}D_{x_\parallel}^{p_\parallel}u\right\|_{L^2(\omega_{\mathbf{ve}})}\le C_\varepsilon\gamma^{p+1}p^p. \tag{2.7}$$

(iii) *In the interior Ω_{int}, for all $\alpha\in\mathbb{N}_0^2$,*

$$\|\partial_x^\alpha u\|_{L^2(\Omega_{\mathrm{int}})}\le\gamma^{|\alpha|+1}|\alpha|^{|\alpha|}. \tag{2.8}$$

3. Embedding into weighted integer order space.

The nonlocal nature of the $\widetilde{H}^s(\Omega)$-norm (1.3) is well known to obstruct the common FE-approximation strategy to obtain global error bounds by adding scaled, local error estimates on subdomains. Accordingly, as proposed in [5, sect. 3.4], we localize this norm via an embedding into a weighted integer order space. While such embeddings are known (e.g., [37, sect. 3.4]), we provide a short proof to render the exposition self-contained.

Recall $r(x):=\operatorname{dist}(x,\partial\Omega)$ for $x\in\Omega$. For $\beta\in[0,1)$ and an open set $\omega\subseteq\Omega$, denote by $H^1_\beta(\omega)$ the local Sobolev space defined via the weighted norm $\|\cdot\|_{H^1_\beta(\omega)}$ given by

$$\|v\|^2_{H^1_\beta(\omega)}:=\|r^\beta\nabla v\|^2_{L^2(\omega)}+\|r^{\beta-1}v\|^2_{L^2(\omega)}. \tag{3.1}$$

PROPOSITION 3.1 ([5, Lem. 8]). *Let $\Omega\subset\mathbb{R}^d$ denote a bounded domain with Lipschitz boundary $\partial\Omega$, and assume $\sigma\in(0,1]$. Denote by $H^1_\beta(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^1_\beta(\Omega)}$ in* (3.1)*.*

Then, for all $\beta\in[0,1-\sigma)$, $H^1_\beta(\Omega)$ is continuously embedded into $\widetilde{H}^\sigma(\Omega)$, and there exists a constant $C_{\beta,\sigma}(\Omega)>0$ such that

$$\forall v\in H^1_\beta(\Omega):\quad \|v\|_{\widetilde{H}^\sigma(\Omega)}\le C_{\beta,\sigma}\|v\|_{H^1_\beta(\Omega)}\,. \tag{3.2}$$

The estimate (3.2) *remains valid in the limit case $(\sigma,\beta)=(1,0)$.*

Proof. We present the argument from the univariate case [5, Lem. 8], with the minor adaptations to the present setting. For Banach spaces $X_1\subset X_0$ with continuous injection, and for $v\in X_0$, $t>0$, the K-functional is given by $K(v,t;X_0,X_1):=\inf_{w\in X_1}\|v-w\|_{X_0}+t\|w\|_{X_1}$. For $\theta\in(0,1)$ and $q\in[1,\infty)$, the interpolation spaces (e.g., [37, Chap. 1.3]) $X_{\theta,q}:=(X_0,X_1)_{\theta,q}$ are given by the norm

$$\|v\|^q_{X_{\theta,q}}:=\int_{t=0}^\infty\left(t^{-\theta}K(v,t;X_0,X_1)\right)^q\frac{dt}{t}. \tag{3.3}$$

We now choose $X_0=L^2(\Omega)$ and $X_1=H^1_0(\Omega)$ and fix $\beta\in(0,1-\sigma)$. We note that the function r is Lipschitz. For each $t>0$ sufficiently small, we may choose

$\chi_t \in C^\infty(\mathbb{R})$ such that $\chi_t \circ r \equiv 0$ on the strip $S_{t/2}$ and $\chi_t \circ r \equiv 1$ on $\Omega \setminus S_t$ as well as $\|\nabla^j(\chi_t \circ r)\|_{L^\infty(\mathbb{R}^2)} \le Ct^{-j}$, $j \in \{0,1\}$. Decomposing $v = (\chi_t \circ r)v + (1-(\chi_t \circ r))v$, we have $(\chi_t \circ r)v \in H_0^1(\Omega)$ and $(1-(\chi_t \circ r))v \in L^2(\Omega)$ for $v \in H_\beta^1(\Omega)$.

A calculation shows that there exists a constant $C > 0$ such that

$$\begin{aligned} \forall v \in C_0^\infty(\Omega): \quad & \|\nabla((\chi_t \circ r)v)\|_{L^2(\Omega)} \le Ct^{-\beta}\|v\|_{H_\beta^1(\Omega)}, \\ & \|(1-(\chi_t \circ r))v\|_{L^2(\Omega)} \le Ct^{1-\beta}\|r^{\beta-1}v\|_{L^2(\Omega)}. \end{aligned}$$

This implies that $K(v,t;X_0,X_1) \le Ct^{1-\beta}\|v\|_{H_\beta^1(\Omega)}$ for small $t > 0$. Since $X_1 \subset X_0$, replacing the integration limit ∞ in (3.3) by a finite number T leads to an equivalent norm [15, Chap. 6, sect. 7]. Hence,

$$\|v\|_{X_{\sigma,2}}^2 \simeq \int_0^T \left[t^{-\sigma}K(v,t;X_0,X_1)\right]^2 \frac{dt}{t} \le C\|v\|_{H_\beta^1(\Omega)}^2 \int_0^T t^{1-2\beta-2\sigma}dt,$$

and the latter integral is bounded for all $\beta < 1-\sigma$. We conclude by remarking that $\widetilde{H}^\sigma(\Omega) = X_{\sigma,2}$ with equivalent norms [13, Prop. 4.1 and Thm. 4.10].

The validity of the assertion in the limiting case $(\sigma,\beta) = (1,0)$ follows from [21, Thm. 1.4.4.3]. ◻

4. Geometrically refined meshes.

We review here briefly the patchwise construction of geometrically refined meshes from [7, 8]. We admit both triangular and quadrilateral elements $K \in \mathcal{T}$, but *do not* assume shape regularity: *anisotropic, geometric mesh refinement towards* $\partial\Omega$ is essential to resolving edge-singularities (generically present in solutions of fractional PDEs) at exponential rate.

4.1. Macro triangulation. Mesh patches.

We recapitulate the *hp*-FE approximation theory on geometrically refined meshes generated as push-forwards of a small number of so-called *mesh patches,* similar to those introduced (for the *hp*-approximation of singularly perturbed, linear elliptic boundary value problems) in [30, sect. 3.3.3] and [18]. These mesh families are based on a *fixed macro-triangulation* $\mathcal{T}^{\mathcal{M}}$ of the domain Ω. The macro-triangulation $\mathcal{T}^{\mathcal{M}}$ consists of mapped triangles and quadrilaterals $K^{\mathcal{M}}$ which are endowed with *patch maps* (to be distinguished from the actual element maps) $F_{K^{\mathcal{M}}}: \widehat{S} \to K^{\mathcal{M}}$, for quadrilateral patches, and $F_{K^{\mathcal{M}}}: \widehat{T} \to K^{\mathcal{M}}$, for triangular patches, that satisfy the usual compatibility conditions.[1] Each element of the fixed macro-triangulation $\mathcal{T}^{\mathcal{M}}$ is further subdivided according to one of the refinement patterns in Definition 4.1 (see also [30, sect. 3.3.3] or [18]). The actual triangulation is then obtained by transplanting refinement patterns on the reference patch into the physical domain Ω by means of the patch maps $F_{K^{\mathcal{M}}}$ of the macro-triangulation. That is, for any element $K \in \mathcal{T}$, at refinement level $L \in \mathbb{N}$, the element map F_K^L is the concatenation of an affine map—which realizes the mapping from the reference square or triangle to the elements in the patch-refinement pattern and will be denoted by A_K^L— and the patch map (denoted by $F_{K^{\mathcal{M}}}$), i.e., $F_K^L = F_{K^{\mathcal{M}}} \circ A_K^L : \hat{K} \to K$. We introduce the refinement patterns; see also [25] and [7, Def. 2.1].

DEFINITION 4.1 (catalog $\mathfrak{P}$ of refinement patterns). *Given* $\sigma \in (0,1)$, $L \in \mathbb{N}_0$, *the catalog* $\mathfrak{P}$ *consists of the following patterns:*

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[1] $\mathcal{T}^{\mathcal{M}}$ does not have hanging nodes and, for any two distinct elements $K_1^{\mathcal{M}}, K_2^{\mathcal{M}} \in \mathcal{T}^{\mathcal{M}}$ that share an edge e, their respective element maps induce compatible parametrizations of e (cf., e.g., [30, Def. 2.4.1] for the precise conditions).

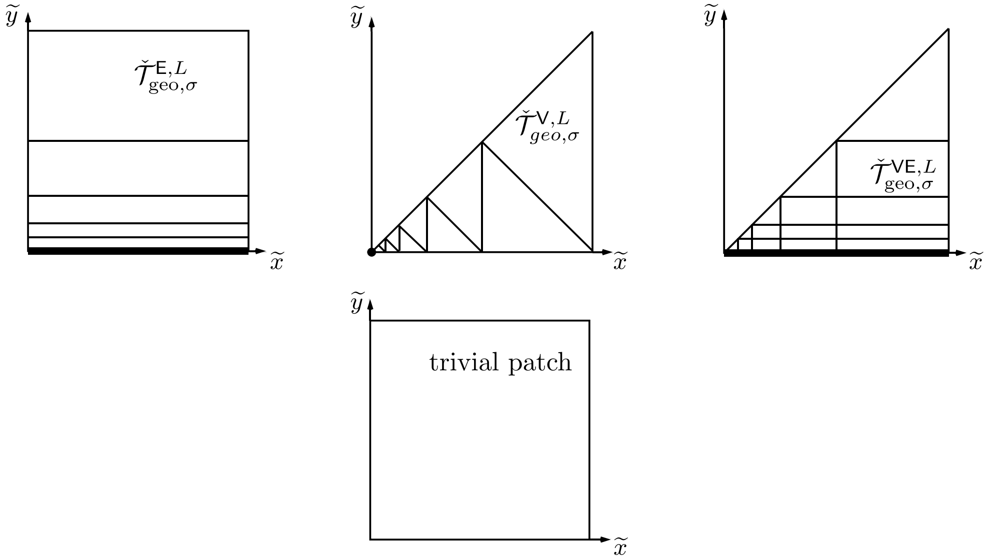


FIG. 2. *Catalog $\mathfrak{P}$ of reference refinement patterns. Top row: reference edge patch $\check{\mathcal{T}}^{\mathsf{E},L}_{\mathrm{geo},\sigma}$ with L layers of geometric refinement towards $\{\widetilde{y}=0\}$; reference vertex patch $\check{\mathcal{T}}^{\mathsf{V},L}_{\mathrm{geo},\sigma}$ with L layers of geometric refinement towards $(0,0)$; vertex-edge patch $\check{\mathcal{T}}^{\mathsf{VE},L}_{\mathrm{geo},\sigma}$ with L layers of refinement towards $(0,0)$ and L layers of refinement towards $\{\widetilde{y}=0\}$. Bottom row: trivial patch. Geometric entities shown in boldface indicate parts of $\partial\widehat{S}$ that are mapped to $\partial\Omega$. These patch meshes are transported into the polygon Ω via patch maps $F_{K^{\mathcal{M}}}$.*

1. *The trivial patch: The reference square $\widehat{S}=(0,1)^2$ is not further refined. The corresponding triangulation of $\widehat{S}$ consists of the single element: $\check{\mathcal{T}}^{\mathrm{trivial}}=\{\widehat{S}\}$.*
2. *The geometric edge patch $\check{\mathcal{T}}^{\mathsf{E},L}_{\mathrm{geo},\sigma}$: $\widehat{S}$ is refined anisotropically towards $\{\widehat{y}=0\}$ into L elements as depicted in Figure 2 (top left). The mesh $\check{\mathcal{T}}^{\mathsf{E},L}_{\mathrm{geo},\sigma}$ is characterized by the nodes $(0,0)$, $(0,\sigma^i)$, $(1,0)$, $(1,\sigma^i)$, $i=0,\dots,L$, and the corresponding rectangular elements generated by these nodes.*
3. *The geometric vertex patch $\check{\mathcal{T}}^{\mathsf{V},L}_{\mathrm{geo},\sigma}$: $\widehat{T}$ is refined isotropically toward $(0,0)$ as depicted in Figure 2 (top middle). The reference geometric vertex patch mesh $\check{\mathcal{T}}^{\mathsf{V},L}_{\mathrm{geo},\sigma}$ in $\widehat{T}$ with geometric refinement toward $(0,0)$ and L layers is given by triangles determined by the nodes $(0,0)$, $(\sigma^i,0)$, and (σ^i,σ^i), $i=0,\dots,L$.*
4. *The geometric vertex-edge patch $\check{\mathcal{T}}^{\mathsf{VE},L}_{\mathrm{geo},\sigma}$: the triangulation, depicted in Figure 2 (top right), consists of both anisotropic elements and isotropic elements. It is given by the nodes $(0,0)$, $(\sigma^i,0)$, (σ^i,σ^j), $0\le i\le L$, $i\le j\le L$, and consists of anisotropic rectangles and uniformly shape-regular triangles.*

4.2. Geometric boundary-refined mesh $\mathcal{T}^{L}_{\mathrm{geo},\sigma}$.

We now define the global, boundary-refined meshes $\mathcal{T}^{L}_{\mathrm{geo},\sigma}$, which will be used in the definition of the FE-space (5.1). These meshes are built by assembling possibly anisotropic, geometric patch partitions from the catalog $\mathfrak{P}$ in Definition 4.1. To ensure inter-patch compatibility, all partitions from $\mathfrak{P}$ are taken with the same values of σ and L. The resulting partitions of Ω are regular and feature anisotropic, geometric refinement toward the edges $\mathbf{e}\subset\partial\Omega$ and isotropic geometric refinement toward the vertices $\mathbf{v}\subset\partial\Omega$.

DEFINITION 4.2 (geometric boundary-refined mesh [7, Def. 2.3]). *Let $\mathcal{T}^{\mathcal{M}}$ be a fixed macro-triangulation consisting of quadrilateral or triangular patches with bilinear*

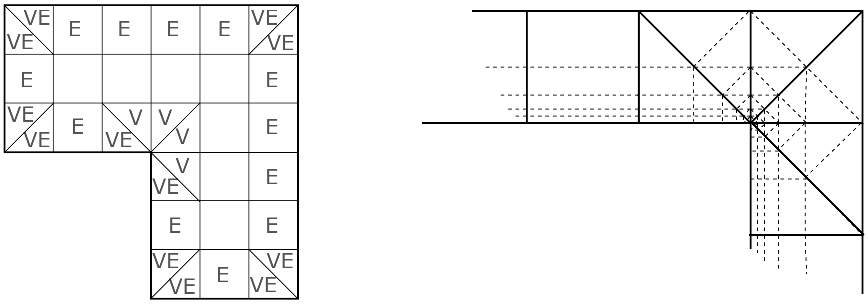


FIG. 3. *Patch arrangement in Ω. Left panel: example of an L-shaped domain decomposed into patches* (V, E, VE *indicate vertex, edge, vertex-edge patches, empty squares signify trivial patches). Right panel: zoom-in near the reentrant corner* $\mathbf{v}$. *Solid lines indicate patch boundaries, dashed lines indicate mesh lines.*

or affine patch maps, depending on the patch (see the end of this definition). Patch-refinement patterns are specified in terms of parameters σ *and* L.

Given $\sigma \in (0,1)$, $L \in \mathbb{N}_0$, $\mathcal{T}^L_{\text{geo},\sigma}$ *is called a geometric boundary-refined mesh if the following conditions hold:*

1. $\mathcal{T}^L_{\text{geo},\sigma}$ *is obtained by refining each element* $K^{\mathcal{M}} \in \mathcal{T}^{\mathcal{M}}$ *according to the finite catalog* $\mathfrak{P}$ *of patch-refinement patterns as specified in Definition* 4.1.
2. $\mathcal{T}^L_{\text{geo},\sigma}$ *is a regular partition of* Ω, *i.e., it does not have hanging nodes. Since the element maps for the refinement patterns are assumed to be affine or bilinear, this requirement ensures that the resulting triangulation satisfies* [30, Def. 2.4.1].

For each macro-patch $K^{\mathcal{M}} \in \mathcal{T}^{\mathcal{M}}$, *exactly one of the following cases is possible:*

3. $\overline{K^{\mathcal{M}}} \cap \partial\Omega = \emptyset$. *Then, the trivial patch is selected as the reference patch. We denote* $\mathcal{T}^{\mathcal{M}}_{\text{int}}$ *the set of such macro-elements.*
4. $\overline{K^{\mathcal{M}}} \cap \partial\Omega = \{\mathbf{P}\}$ *is a single point, where* $\mathbf{P}$ *can be a vertex of* Ω *or a point on the boundary. The refinement pattern is the vertex patch* $\widetilde{\mathcal{T}}^{\mathsf{V},L}_{\text{geo},\sigma}$ *with* L *layers of geometric mesh refinement toward the origin* $\mathbf{O}$; *it is assumed that* $F_{K^{\mathcal{M}}}(\mathbf{O}) = \mathbf{P} \in \partial\Omega$. *We denote* $\mathcal{T}^{\mathcal{M}}_{\mathsf{V}}$ *the set of such macro-elements.*
5. $\overline{K^{\mathcal{M}}} \cap \partial\Omega = \overline{\mathbf{e}}$ *for an edge* $\mathbf{e}$ *of* $K^{\mathcal{M}}$, *and neither endpoint of* $\mathbf{e}$ *is a vertex of* Ω. *Then, the refinement pattern is the edge patch* $\widetilde{\mathcal{T}}^{\mathsf{E},L}_{\text{geo},\sigma}$ *and additionally* $F_{K^{\mathcal{M}}}(\{\widetilde{y} = 0\}) \subset \partial\Omega$. *We denote* $\mathcal{T}^{\mathcal{M}}_{\mathsf{E}}$ *the set of such macro-elements.*
6. $\overline{K^{\mathcal{M}}} \cap \partial\Omega = \overline{\mathbf{e}}$ *for an edge* $\mathbf{e}$ *of* $K^{\mathcal{M}}$, *and exactly one endpoint of* $\mathbf{e}$ *is a vertex* $\mathbf{v}$ *of* Ω. *The refinement pattern is the vertex-edge patch* $\widetilde{\mathcal{T}}^{\mathsf{VE},L}_{\text{geo},\sigma}$ *and additionally* $F_{K^{\mathcal{M}}}(\{\widetilde{y} = 0\}) \subset \partial\Omega$ *as well as* $F_{K^{\mathcal{M}}}(\mathbf{O}) = \mathbf{v}$. *We denote* $\mathcal{T}^{\mathcal{M}}_{\mathsf{VE}}$ *the set of such macro-elements.*

We assume that $F_{K^{\mathcal{M}}}$ *is bilinear for all* $K^{\mathcal{M}} \in \mathcal{T}^{\mathcal{M}}_{\text{int}}$ *and that it is affine for all* $K^{\mathcal{M}} \in \mathcal{T}^{\mathcal{M}}_{\mathsf{E}} \cup \mathcal{T}^{\mathcal{M}}_{\mathsf{V}} \cup \mathcal{T}^{\mathcal{M}}_{\mathsf{VE}}$.

Example 4.3. Figure 3 shows a so-called "L-shaped domain" with macro-triangulation and patch-refinement patterns in the vicinity of a reentrant corner $\mathbf{v}$.

5. *hp*-Approximation on geometric boundary-refined meshes.

The exponential convergence of *hp*-approximations for functions $u \in \widetilde{H}^s(\Omega)$ that satisfy the weighted analytic regularity (2.5)–(2.8) will be developed in several steps. As is customary in proofs of FE-error bounds, we shall obtain exponential convergence from the quasioptimality (1.7) by constructing $v_N = \Pi_N u$ in a subspace $V_N \subset \widetilde{H}^s(\Omega)$ which is designed to exploit (2.5)–(2.8). Specifically, we shall use an *hp*-patch framework similar to the one developed in [7, 8] for exponentially convergent approximations of solutions to singular perturbation problems and of spectral fractional diffusion in Ω. We recapitulate in section 5.1 this *hp*-approximation framework.

5.1. *hp*-FE spaces in Ω.

On the geometric partitions $\mathcal{T}^L_{\mathrm{geo},\sigma}$ introduced in section 4, we consider Lagrangian FEs of uniform polynomial degree $q \geq 1$, i.e., we choose the global FE-space W^L_q in (1.7) as

(5.1)
$$W^L_q := \mathcal{S}^q_0(\Omega, \mathcal{T}^L_{\mathrm{geo},\sigma}) := \left\{ v \in C(\overline{\Omega}) : v|_K \circ F^L_K \in \mathbb{V}_q(\widehat{K}) \quad \forall K \in \mathcal{T}^L_{\mathrm{geo},\sigma},\ v|_{\partial\Omega} = 0 \right\}.$$

Here, for $q \geq 1$, the local polynomial space is

$$\mathbb{V}_q(\widehat{K}) = \begin{cases} \mathbb{P}_q & \text{if } \widehat{K} = \widehat{T}, \\ \mathbb{Q}_q & \text{if } \widehat{K} = \widehat{S}. \end{cases}$$

5.2. Definition of the *hp*-interpolation operator Π^L_q.

The global *hp*-interpolator $\Pi^L_q : H^1_\beta(\Omega) \to W^L_q$ will be obtained by assembling local Gauss–Lobatto–Legendre (GLL) interpolants in the reference patches. The global error estimate will follow from adding patchwise error bounds in H^1_β in the reference patches. Addition of elementwise and patchwise error bounds is possible due to the locality of the H^1_β-norm. In possibly anisotropic quadrilateral elements, the GLL interpolants are generated by tensorization of univariate GLL interpolants. We review their definition and properties briefly in Appendix A. Recall that all triangular elements are shape-regular. Only quadrilateral elements may be anisotropic.

5.2.1. Definition of the *hp*-interpolator $\widetilde{\Pi}^L_q$ on reference patches.

The *hp*-approximation operators on reference patches are obtained by assembling elementwise GLL interpolants (cf. [7, eq. (3.6)]). Recalling $A^L_{\widetilde{K}} : \widehat{K} \to \widetilde{K} = A^L_{\widetilde{K}}(\widehat{K}) \in \check{\mathcal{T}}^{\bullet,L}_{\mathrm{geo},\sigma} \cup \check{\mathcal{T}}^{\mathrm{trivial}}$ with $\bullet \in \{\mathsf{V}, \mathsf{E}, \mathsf{VE}\}$ the *affine bijection between the reference element* $\widehat{K}$ *and the corresponding element* $\widetilde{K}$ *on the reference patch,* we set

$$(\widetilde{\Pi}^L_q)|_{\widetilde{K}} v := \begin{cases} \widehat{\Pi}^{\triangle}_q (v \circ A^L_{\widetilde{K}}) & \text{if } \widetilde{K} \text{ is a triangle}, \\ \widehat{\Pi}^{\square}_q (v \circ A^L_{\widetilde{K}}) & \text{if } \widetilde{K} \text{ is a rectangle}. \end{cases} \tag{5.2}$$

The elemental GLL interpolators $\widehat{\Pi}^{\triangle}_q$ and $\widehat{\Pi}^{\square}_q$ defined in Lemmas A.1 and A.2 coincide with univariate GLL interpolants of traces on the edges of $\widehat{K}$. This ensures global H^1_β-conformity of the reference patch interpolator $\widetilde{\Pi}^L_q$.

5.2.2. Definition of the global *hp*-interpolator Π^L_q.

With the *hp*-patch-interpolants in (5.2) in place, the global *hp*-interpolator Π^L_q is assembled from elementwise projectors on an element K via

$$(\Pi^L_q u)|_K \circ F^L_K := \begin{cases} \widehat{\Pi}^{\triangle}_q (u \circ F^L_K) & \text{if } K \text{ is a triangle}, \\ \widehat{\Pi}^{\square}_q (u \circ F^L_K) & \text{if } K \text{ is a rectangle}, \end{cases}$$

where $\widehat{\Pi}^{\triangle}_q$ is defined in Lemma A.1 and $\widehat{\Pi}^{\square}_q$ in Lemma A.2. Since $\widehat{\Pi}^{\triangle}_q$ and $\widehat{\Pi}^{\square}_q$ reduce to the Gauss–Lobatto interpolation operator on the edges of the reference element, the operator Π^L_q indeed maps into $S^q_0(\Omega, \mathcal{T}^L_{\mathrm{geo},\sigma})$. We recall that the element maps F^L_K have the form

$$F^L_K = F_{K^{\mathcal{M}}} \circ A^L_K,$$

where $A_K^L : \widehat{K} \to \widetilde{K} := A_K^L(\widehat{K}) = F_{K^{\mathcal{M}}}^{-1}(K) \in \check{\mathcal{T}}_{\mathrm{geo},\sigma}^{\bullet,L} \cup \check{\mathcal{T}}^{\mathrm{trivial}}$ is an affine bijection, and $F_{K^{\mathcal{M}}}$ is the patch map.

Furthermore, $\widehat{u}$ denotes the pull-back of u to the reference element, i.e.,

$$\widehat{u} := u|_K \circ F_K^L, \tag{5.3}$$

whereas

$$\widetilde{u} := (u \circ F_{K^{\mathcal{M}}})|_{\widetilde{K}} = \widehat{u} \circ (A_K^L)^{-1} \tag{5.4}$$

is the corresponding function on $\widetilde{K}$. With the patch-interpolant $\widetilde{\Pi}_q^L$ from (5.2), we obtain on a macro-element $K^{\mathcal{M}} \in \mathcal{T}^{\mathcal{M}}$

$$(\Pi_q^L u) \circ F_{K^{\mathcal{M}}} = \widetilde{\Pi}_q^L \widetilde{u}. \tag{5.5}$$

For $k \in \mathbb{N}_0$, we have the following for all elements $K \subset K^{\mathcal{M}}$ with $\widetilde{K} = F_{K^{\mathcal{M}}}^{-1}(K)$:

$$\forall v \in H^k(K) : \ \ \|v \circ F_{K^{\mathcal{M}}}\|_{H^k(\widetilde{K})} \sim \|v\|_{H^k(K)}, \tag{5.6a}$$

$$\forall v \in W^{k,\infty}(K) : \ \ \|v \circ F_{K^{\mathcal{M}}}\|_{W^{k,\infty}(\widetilde{K})} \sim \|v\|_{W^{k,\infty}(K)}, \tag{5.6b}$$

where in both cases the constants implied in $\sim$ depend solely on k, the patch maps $F_{K^{\mathcal{M}}}$, and the macro-element $K^{\mathcal{M}}$.

The equivalences (5.6) show that the approximation error $v - \Pi_q^L v$ on K is equivalent to the corresponding error $\widetilde{v} - \widetilde{\Pi}_q^L \widetilde{v}$ on $\widetilde{K}$.

5.3. Mesh layers and cutoff function. For $L \in \mathbb{N}$, we subdivide the mesh $\mathcal{T}_{\mathrm{geo},\sigma}^L$ into boundary layer $\mathcal{L}_0^L$, transition layer $\mathcal{L}_1^L$, and internal mesh elements $\mathcal{L}_{\mathrm{int}}^L$. Specifically, we let

$$\begin{aligned}
\mathcal{L}_0^L &:= \left\{K \in \mathcal{T}_{\mathrm{geo},\sigma}^L : \overline{K} \cap \partial\Omega \neq \emptyset\right\},\\
\mathcal{L}_1^L &:= \left\{K \in \mathcal{T}_{\mathrm{geo},\sigma}^L \setminus \mathcal{L}_0^L : \exists\, J \in \mathcal{L}_0^L \text{ such that } \overline{K} \cap \overline{J} \neq \emptyset\right\},\\
\mathcal{L}_{\mathrm{int}}^L &:= \mathcal{T}_{\mathrm{geo},\sigma}^L \setminus \left(\mathcal{L}_0^L \cup \mathcal{L}_1^L\right).
\end{aligned}$$

Furthermore, we introduce the continuous, piecewise linear, cutoff function $g^L : \Omega \to [0,1]$ satisfying

$$g^L \in \mathcal{S}_0^1(\Omega, \mathcal{T}_{\mathrm{geo},\sigma}^L), \qquad g^L \equiv 0 \text{ on all } K \in \mathcal{L}_0^L, \qquad g^L \equiv 1 \text{ on all } K \in \mathcal{L}_{\mathrm{int}}^L. \tag{5.7}$$

Finally, the subdomain comprising the union of all mesh elements touching the boundary is

$$\Omega_0^L = \bigcup_{K \in \mathcal{L}_0^L} \overline{K}\,. \tag{5.8}$$

5.4. Exponential convergence of the *hp*-approximation. We aim to construct an approximation $v \in W_q^L$, with W_q^L as defined in (5.1), to the weak solution u of the fractional PDE (1.1) that converges exponentially in the $\widetilde{H}^s(\Omega)$-norm. By Proposition 3.1, we fix $\beta \in [0, 1-s)$ and apply the triangle inequality to obtain

$$\begin{aligned}
\inf_{v \in W_q^L} \|u - v\|_{\widetilde{H}^s(\Omega)} &\leq \inf_{v \in W_q^L} \|g^L u - v\|_{\widetilde{H}^s(\Omega)} + \|(1-g^L)u\|_{\widetilde{H}^s(\Omega)}\\
&\leq C_{\beta,s} \|g^L(u - \Pi_{q-1}^L)u\|_{H^1_\beta(\Omega)} + \|(1-g^L)u\|_{\widetilde{H}^s(\Omega)},
\end{aligned} \tag{5.9}$$

where we have used $g^L \in \mathcal{S}_0^1(\Omega, \mathcal{T}^L_{\mathrm{geo},\sigma})$ so that $g^L \Pi^L_{q-1} u \in W^L_q$ for $q \geq 2$. In the next section, we estimate the second term in the right-hand side of the above inequality. Then, in the following sections, we proceed with an estimate of the first term in the right-hand side of (5.9). We will consider separately the reference vertex (section 5.4.2), edge (section 5.4.3), and vertex-edge (section 5.4.4) patches. Finally, in section 5.4.5 we bring all estimates together in Ω.

5.4.1. Estimate of the term $(1-g^L)u$. The following statement is an estimate of the $\widetilde{H}^s(\Omega)$-norm of the term $u - g^L u$.

LEMMA 5.1. *Let u be the solution to* (1.5) *for $s \in (0,1)$. Let $L \in \mathbb{N}$ and g^L be defined as in* (5.7). *Then, there exist $C, b > 0$ independent of L such that*

$$\|u - g^L u\|_{\widetilde{H}^s(\Omega)} \leq C \exp(-bL). \tag{5.10}$$

Proof. We fix $\beta \in [0,1)$ additionally satisfying $\beta \in (1/2 - s, 1 - s)$ and estimate the $H^1_\beta(\Omega)$-norm of $u - g^L u$. From Lemma B.2 it follows that there exist constants $c, C > 0$ independent of L such that

$$\|(1-g^L)u\|_{H^1_\beta(\Omega)} \leq C \|u\|_{H^1_\beta(S_{c\sigma^L})}.$$

We now decompose $S_{c\sigma^L}$ into its components belonging to vertex, edge, vertex-edge, and internal neighborhoods:

$$S_{c\sigma^L} = \bigcup_{\mathbf{v}\in\mathcal{V}} \left((\omega_{\mathbf{v}} \cap S_{c\sigma^L}) \cup \bigcup_{\mathbf{e}\in\mathcal{E}_{\mathbf{v}}} (\omega_{\mathbf{ve}} \cap S_{c\sigma^L}) \right) \cup \bigcup_{\mathbf{e}\in\mathcal{E}} (\omega_{\mathbf{e}} \cap S_{c\sigma^L}) \cup (\Omega_{\mathrm{int}} \cap S_{c\sigma^L}).$$

We start with vertex neighborhoods $\omega_{\mathbf{v}}$: Since $\beta > 1/2 - s$, we may choose ε sufficiently small such that $\beta - 1/2 + s - \varepsilon > 0$. For any $\mathbf{v} \in \mathcal{V}$, we obtain using the weighted regularity estimate (2.5) for $p = 0, 1$

$$\begin{aligned}
\|u\|_{H^1_\beta(\omega_{\mathbf{v}} \cap S_{c\sigma^L})} &\lesssim \|r_{\mathbf{v}}^{-1/2-s+\varepsilon+\beta-1+1/2+s-\varepsilon} u\|_{L^2(\omega_{\mathbf{v}} \cap S_{c\sigma^L})} \\
&\quad + \|r_{\mathbf{v}}^{1/2-s+\varepsilon+\beta-1/2+s-\varepsilon} \nabla u\|_{L^2(\omega_{\mathbf{v}} \cap S_{c\sigma^L})} \\
&\leq (c\sigma^L)^{\beta-1/2+s-\varepsilon} \|u\|_{H^1_{1/2-s+\varepsilon}(\omega_{\mathbf{v}} \cap S_{c\sigma^L})} \\
&\overset{(2.5)}{\lesssim} (c\sigma^L)^{\beta-1/2+s-\varepsilon}.
\end{aligned}$$

We next estimate the H^1_β-norm of the interpolation error on edge neighborhoods $\omega_{\mathbf{e}}$: for any $\mathbf{e} \in \mathcal{E}$, we use the weighted regularity (2.6) with $p_\parallel = 0$ and $p_\perp = 0, 1$ to bound

$$\begin{aligned}
&\|u\|_{H^1_\beta(\omega_{\mathbf{e}} \cap S_{c\sigma^L})} \\
&\quad \simeq \|r_{\mathbf{e}}^{\beta-1} u\|_{L^2(\omega_{\mathbf{e}} \cap S_{c\sigma^L})} + \|r_{\mathbf{e}}^{\beta} D_{x_\parallel} u\|_{L^2(\omega_{\mathbf{e}} \cap S_{c\sigma^L})} + \|r_{\mathbf{e}}^{\beta} D_{x_\perp} u\|_{L^2(\omega_{\mathbf{e}} \cap S_{c\sigma^L})} \\
&\quad \leq \left\| r_{\mathbf{e}}^{-1/2-s+\varepsilon+\beta-1+1/2+s-\varepsilon} u \right\|_{L^2(\omega_{\mathbf{e}} \cap S_{c\sigma^L})} \\
&\qquad + \left\| r_{\mathbf{e}}^{-1/2-s+\varepsilon+\beta+1/2+s-\varepsilon} D_{x_\parallel} u \right\|_{L^2(\omega_{\mathbf{e}} \cap S_{c\sigma^L})} \\
&\qquad + \left\| r_{\mathbf{e}}^{1/2-s+\varepsilon+\beta-1/2+s-\varepsilon} D_{x_\perp} u \right\|_{L^2(\omega_{\mathbf{e}} \cap S_{c\sigma^L})} \\
&\quad \leq (c\sigma^L)^{\beta+1/2+s-\varepsilon} \left\| r_{\mathbf{e}}^{-1/2-s+\varepsilon} D_{x_\parallel} u \right\|_{L^2(\omega_{\mathbf{e}} \cap S_{c\sigma^L})} \\
&\qquad + (c\sigma^L)^{\beta-1/2+s-\varepsilon} \left(\left\| r_{\mathbf{e}}^{-1/2-s+\varepsilon} u \right\|_{L^2(\omega_{\mathbf{e}} \cap S_{c\sigma^L})} + \left\| r_{\mathbf{e}}^{1/2-s+\varepsilon} D_{x_\perp} u \right\|_{L^2(\omega_{\mathbf{e}} \cap S_{c\sigma^L})} \right) \\
&\quad \overset{(2.6)}{\lesssim} (c\sigma^L)^{\beta-1/2+s-\varepsilon}.
\end{aligned}$$

The error on the vertex-edge neighborhood $\omega_{\mathbf{ve}}$ can be bounded for any $\mathbf{v} \in \mathcal{V}$ and any $\mathbf{e} \in \mathcal{E}_{\mathbf{v}}$ using $r_{\mathbf{v}}(x) \gtrsim r_{\mathbf{e}}(x)$ for all $x \in \omega_{\mathbf{ve}}$ as well as the weighted regularity (2.7) with $p_\parallel, p_\perp$ satisfying $p_\parallel + p_\perp \leq 1$. We have

$$\begin{aligned}
&\|u\|_{H^1_\beta(\omega_{\mathbf{ve}} \cap S_{c\sigma^L})} \\
&\quad \simeq \left\|r_{\mathbf{e}}^{\beta-1} u\right\|_{L^2(\omega_{\mathbf{ve}} \cap S_{c\sigma^L})} + \left\|r_{\mathbf{e}}^{\beta} D_{x_\parallel} u\right\|_{L^2(\omega_{\mathbf{ve}} \cap S_{c\sigma^L})} + \left\|r_{\mathbf{e}}^{\beta} D_{x_\perp} u\right\|_{L^2(\omega_{\mathbf{ve}} \cap S_{c\sigma^L})} \\
&\quad \leq \left\|r_{\mathbf{e}}^{-1/2-s+\varepsilon+\beta-1+s+1/2-\varepsilon} r_{\mathbf{v}}^{\varepsilon-\varepsilon} u\right\|_{L^2(\omega_{\mathbf{ve}} \cap S_{c\sigma^L})} \\
&\qquad + \left\|r_{\mathbf{e}}^{-1/2-s+\varepsilon+\beta+s+1/2-\varepsilon} r_{\mathbf{v}}^{1+\varepsilon-1-\varepsilon} D_{x_\parallel} u\right\|_{L^2(\omega_{\mathbf{ve}} \cap S_{c\sigma^L})} \\
&\qquad + \left\|r_{\mathbf{e}}^{1/2-s+\varepsilon+\beta-1/2+s-\varepsilon} r_{\mathbf{v}}^{\varepsilon-\varepsilon} D_{x_\perp} u\right\|_{L^2(\omega_{\mathbf{ve}} \cap S_{c\sigma^L})} \\
&\quad \overset{(2.7)}{\lesssim} (c\sigma^L)^{\beta-1/2+s-2\varepsilon},
\end{aligned}$$

where we assumed ε to be chosen small enough such that $\beta - 1/2 + s - 2\varepsilon > 0$. Finally, as c, σ are fixed, we may assume that $\Omega_{\mathrm{int}} \cap S_{c\sigma^L} = \emptyset$ by replacing L by $L + L_0$ with a fixed $L_0 \in \mathbb{N}$ large enough and independent of L, which only changes the constant b in the exponential estimate. We have thus obtained that

$$\|(1-g^L)u\|_{H^1_\beta(\Omega)} \leq C\|u\|_{H^1_\beta(S_{c\sigma^L})} \leq C' \exp(-bL).$$

Applying Proposition 3.1 concludes the proof. ∎

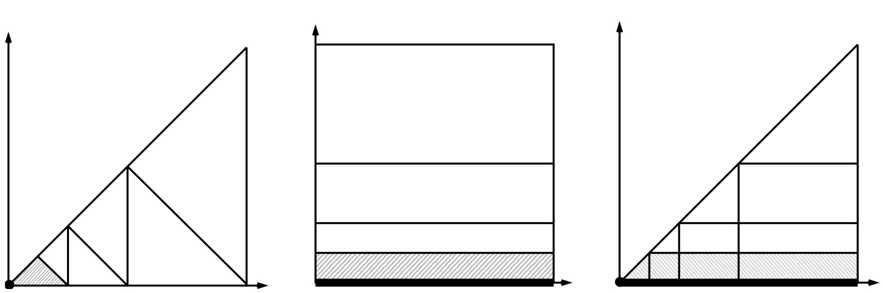
5.4.2. *hp*-FE approximation in reference vertex patch $\boldsymbol{\check{\mathcal{T}}^{\mathsf{V},L}_{\mathrm{geo},\sigma}}$. We denote $\mathbf{v} = (0,0)$ and $\widetilde{r}_{\mathbf{v}} = \operatorname{dist}(\mathbf{v}, \cdot)$. Furthermore, let

$$\widetilde{\mathcal{L}}_0^{\mathsf{V},L} = \left\{K \in \check{\mathcal{T}}^{\mathsf{V},L}_{\mathrm{geo},\sigma} : \overline{K} \cap \mathbf{v} \neq \emptyset\right\}, \qquad \widetilde{T}^{\mathsf{V},L}_{\mathrm{int}} = \widehat{T} \setminus \bigcup_{K \in \widetilde{\mathcal{L}}_0^{\mathsf{V},L}} \overline{K}$$

be, respectively, the elements abutting the singular vertex and the interior part of the vertex reference patch; see Figure 4a.

LEMMA 5.2 (*hp*-FE approximation in reference vertex patch $\check{\mathcal{T}}^{\mathsf{V},L}_{\mathrm{geo},\sigma}$). *For fixed* $s \in (0,1)$ *and* $\gamma > 0$, *let* $\widetilde{u}$ *satisfy the following: for all* $\varepsilon > 0$ *there exists a constant* $C_\varepsilon > 0$ *such that for all* $\alpha \in \mathbb{N}_0^2$ *it holds that, with* $|\alpha| = p$,

$$\left\|\widetilde{r}_{\mathbf{v}}^{\,p-1/2-s+\varepsilon} \partial^\alpha \widetilde{u}\right\|_{L^2(\widehat{T})} \leq C_\varepsilon \gamma^{p+1} p^p. \tag{5.11}$$



(a) (b) (c)

FIG. 4. *Boundary elements (displayed shaded)* $\widetilde{\mathcal{L}}_0^{\mathsf{V},L}$, $\widetilde{\mathcal{L}}_0^{\mathsf{E},L}$, *and* $\widetilde{\mathcal{L}}_0^{\mathsf{VE},L}$ *in the* (a) *vertex,* (b) *edge, and* (c) *vertex-edge reference patches.*

Then, for all $\beta > 1/2 - s$ *and all* $0 < \varepsilon < \beta + s - 1/2$*, there exist constants* $b_{\mathsf V} > 0$ *(depending only on* γ*,* β*,* s*,* σ*) and* $C_{\mathsf V} > 0$ *(depending additionally on* ε*) such that for every* L*,* $q \in \mathbb N$

$$\left\|\widetilde r_{\mathbf v}^{\beta-1}\left(\widetilde u - \widetilde\Pi_q^L \widetilde u\right)\right\|_{L^2(\widetilde T^{\mathsf V,L}_{\mathrm{int}})} + \left\|\widetilde r_{\mathbf v}^{\beta}\nabla\left(\widetilde u - \widetilde\Pi_q^L \widetilde u\right)\right\|_{L^2(\widetilde T^{\mathsf V,L}_{\mathrm{int}})} \le C_{\mathsf V} C_\varepsilon \exp(-b_{\mathsf V} q). \tag{5.12}$$

Proof. All elements $\widetilde K \in \check{\mathcal T}^{\mathsf V,L}_{\mathrm{geo},\sigma}$ are shape-regular: we denote by $h_{\widetilde K}$ their diameter. For all $\widetilde K \in \check{\mathcal T}^{\mathsf V,L}_{\mathrm{geo},\sigma} \setminus \widetilde{\mathcal L}_0^{\mathsf V,L}$, we have $\widetilde r_{\mathbf v}|_{\widetilde K} \simeq h_{\widetilde K}$ with equivalence constant uniform over $\check{\mathcal T}^{\mathsf V,L}_{\mathrm{geo},\sigma} \setminus \widetilde{\mathcal L}_0^{\mathsf V,L}$. From this equivalence and (5.11) it follows that, for all $\widetilde K \in \check{\mathcal T}^{\mathsf V,L}_{\mathrm{geo},\sigma} \setminus \widetilde{\mathcal L}_0^{\mathsf V,L}$ and all $\alpha \in \mathbb N_0^2$, there exists a constant $C_1 > 0$ such that

$$\left\|\widetilde r_{\mathbf v}^{|\alpha|}\partial^\alpha \widetilde u\right\|_{L^2(\widetilde K)} \le C_1 C_\varepsilon h_{\widetilde K}^{1/2+s-\varepsilon}\gamma^{|\alpha|+1}|\alpha|^{|\alpha|}.$$

By a scaling argument, then, there exists a constant $\gamma_1 > 0$ such that for all $\alpha \in \mathbb N_0^2$ and all $\widetilde K \in \check{\mathcal T}^{\mathsf V,L}_{\mathrm{geo},\sigma} \setminus \widetilde{\mathcal L}_0^{\mathsf V,L}$,

$$\left\|\widehat\partial^\alpha\left(\widetilde u \circ A^L_{\widetilde K}\right)\right\|_{L^2(\widehat K)} \le C_1 C_\varepsilon h_{\widetilde K}^{-1/2+s-\varepsilon}\gamma_1^{|\alpha|+1}\gamma^{|\alpha|+1}|\alpha|^{|\alpha|}$$

with $\widehat K = \widehat T = (A^L_{\widetilde K})^{-1}(\widetilde K)$. Recalling $\widehat u = \widetilde u \circ A^L_{\widetilde K}$, we can now exploit the embedding of $H^2(\widehat K)$ into $L^\infty(\widehat K)$ to obtain the existence of constants $C_2, \gamma_2 > 0$ such that

$$\forall \alpha \in \mathbb N_0^2\colon \quad \|\widehat\partial^\alpha \widehat u\|_{L^\infty(\widehat K)} \le C_2 h_{\widetilde K}^{s-1/2-\varepsilon}\gamma_2^{|\alpha|+3}(|\alpha|+2)^{|\alpha|+2}.$$

It follows that there exist C_3, $\gamma_3 > 0$ such that

$$\forall \widetilde K \in \check{\mathcal T}^{\mathsf V,L}_{\mathrm{geo},\sigma} \setminus \widetilde{\mathcal L}_0^{\mathsf V,L}, \quad \forall \alpha \in \mathbb N_0^2\colon \quad \|\widehat\partial^\alpha \widehat u\|_{L^\infty(\widehat K)} \le C_3 h_{\widetilde K}^{s-1/2-\varepsilon}\gamma_3^{|\alpha|+1}|\alpha|^{|\alpha|}. \tag{5.13}$$

From Lemma A.1 and a scaling argument, it then follows that, for all $L, q \in \mathbb N$,

$$\left\|\widetilde r_{\mathbf v}^{\beta-1}\left(\widetilde u - \widetilde\Pi_q^L \widetilde u\right)\right\|^2_{L^2(\widetilde K)} + \left\|\widetilde r_{\mathbf v}^{\beta}\nabla\left(\widetilde u - \widetilde\Pi_q^L \widetilde u\right)\right\|^2_{L^2(\widetilde K)} \lesssim h_{\widetilde K}^{2\beta+2s-1-2\varepsilon}e^{-2bq}.$$

Since $\beta > 1/2 - s$, the power of $h_{\widetilde K}$ is nonnegative for every $\varepsilon < \beta + s - 1/2$ and summing the bound over all elements $\widetilde K \in \check{\mathcal T}^{\mathsf V,L}_{\mathrm{geo},\sigma} \setminus \widetilde{\mathcal L}_0^{\mathsf V,L}$ concludes the proof by a geometric series argument. ☐

5.4.3. *hp*-FE approximation in the reference edge patch $\check{\mathcal T}^{\mathsf E,L}_{\mathrm{geo},\sigma}$.

In this section, we denote $\mathbf e = \{y = 0\}$ and $\widetilde r_{\mathbf e} = \mathrm{dist}(\mathbf e, \cdot)$. Let $\widetilde D_{x_\parallel} = \partial_x$ and $\widetilde D_{x_\perp} = \partial_y$. Furthermore, let

$$\widetilde{\mathcal L}_0^{\mathsf E,L} = \left\{K \in \check{\mathcal T}^{\mathsf E,L}_{\mathrm{geo},\sigma} : \overline K \cap \mathbf e \neq \emptyset\right\}, \qquad \widetilde S^{\mathsf E,L}_{\mathrm{int}} = \widehat S \setminus \bigcup_{K \in \widetilde{\mathcal L}_0^{\mathsf E,L}} \overline K$$

be, respectively, the elements abutting the singular boundary and the interior part of the edge reference patch; see Figure 4b.

LEMMA 5.3 (*hp*-FE approximation in reference edge patch $\check{\mathcal T}^{\mathsf E,L}_{\mathrm{geo},\sigma}$). *Let* $s \in (0,1)$ *and* $\gamma > 0$ *be fixed, and let* $\widetilde u$ *be such that for all* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that*

$$\forall (p_\perp, p_\parallel) \in \mathbb N_0^2\colon \quad \left\|\widetilde r_{\mathbf e}^{p_\perp - 1/2 - s + \varepsilon}\widetilde D_{x_\perp}^{p_\perp}\widetilde D_{x_\parallel}^{p_\parallel}\widetilde u\right\|_{L^2(\widehat S)} \le C_\varepsilon \gamma^{p+1}p^p, \tag{5.14}$$

with $p = p_\perp + p_\parallel$*. Then, for all* $\beta > 1/2 - s$ *and all* $0 < \varepsilon < \beta + s - 1/2$*, there exist constants* $b_{\mathsf E} > 0$ *(depending only on* γ*,* β*,* s*,* σ*) and* $C_{\mathsf E} > 0$ *(depending additionally on* ε*) such that for every* L*,* $q \in \mathbb N$

$$\left\|\widetilde r_{\mathbf e}^{\beta-1}\left(\widetilde u - \widetilde\Pi_q^L \widetilde u\right)\right\|_{L^2(\widetilde S^{\mathsf E,L}_{\mathrm{int}})} + \left\|\widetilde r_{\mathbf e}^{\beta}\nabla\left(\widetilde u - \widetilde\Pi_q^L \widetilde u\right)\right\|_{L^2(\widetilde S^{\mathsf E,L}_{\mathrm{int}})} \le C_{\mathsf E} C_\varepsilon \exp(-b_{\mathsf E} q). \tag{5.15}$$

Proof. We denote by $h_{\parallel,\widetilde{K}}$ and $h_{\perp,\widetilde{K}}$ the edge-lengths of the rectangle $\widetilde{K}\in\check{\mathcal{T}}^{\mathsf{E},L}_{\mathrm{geo},\sigma}$ in, respectively, parallel and perpendicular directions to $\mathbf{e}$. For all $\widetilde{K}\in\check{\mathcal{T}}^{\mathsf{E},L}_{\mathrm{geo},\sigma}\setminus\widetilde{\mathcal{L}}^{\mathsf{E},L}_0$, we have $\widetilde{r}_{\mathbf{e}}|_{\widetilde{K}}\simeq h_{\perp,\widetilde{K}}$ with equivalence constant uniform over $\check{\mathcal{T}}^{\mathsf{E},L}_{\mathrm{geo},\sigma}\setminus\widetilde{\mathcal{L}}^{\mathsf{E},L}_0$. From (5.14), an anisotropic scaling argument, and a Sobolev embedding it follows that there exist $\widetilde{C},\widetilde{\gamma}>0$ such that

$$\forall\widetilde{K}\in\check{\mathcal{T}}^{\mathsf{E},L}_{\mathrm{geo},\sigma}\setminus\widetilde{\mathcal{L}}^{\mathsf{E},L}_0,\quad\forall(p_\perp,p_\parallel)\in\mathbb{N}^2_0:\qquad\|\widehat{\partial}^{(p_\parallel,p_\perp)}\widehat{u}\|_{L^\infty(\widehat{K})}\le\widetilde{C}h^{s-\varepsilon}_{\perp,\widetilde{K}}\widetilde{\gamma}^{p+1}p^p \tag{5.16}$$

with $\widehat{K}=\widehat{S}=(A^L_{\widetilde{K}})^{-1}(\widetilde{K})$ and $p=p_\parallel+p_\perp$ (see the derivation of (5.13) for the detailed steps). From Lemma A.2 and a scaling argument, it then follows that

$$\left\|\widetilde{r}^{\beta-1}_{\mathbf{e}}\left(\widetilde{u}-\widetilde{\Pi}^L_q\widetilde{u}\right)\right\|^2_{L^2(\widetilde{K})}+\left\|\widetilde{r}^{\beta}_{\mathbf{e}}\nabla\left(\widetilde{u}-\widetilde{\Pi}^L_q\widetilde{u}\right)\right\|^2_{L^2(\widetilde{K})}\lesssim h^{2\beta+2s-1-2\varepsilon}_{\perp,\widetilde{K}}e^{-2bq}.$$

Summing this bound over all $\widetilde{K}\in\check{\mathcal{T}}^{\mathsf{E},L}_{\mathrm{geo},\sigma}\setminus\widetilde{\mathcal{L}}^{\mathsf{E},L}_0$ using a geometric series argument concludes the proof since $\beta+s-1/2-\varepsilon>0$. ∎

5.4.4. *hp*-FE approximation in the reference vertex-edge patch $\check{\mathcal{T}}^{\mathsf{VE},L}_{\mathrm{geo},\sigma}$.

In this section, we denote $\mathbf{v}=(0,0)$, $\mathbf{e}=\{y=0\}$, $\widetilde{r}_{\mathbf{v}}=\mathrm{dist}(\mathbf{v},\cdot)$, and $\widetilde{r}_{\mathbf{e}}=\mathrm{dist}(\mathbf{e},\cdot)$. Let $\widetilde{D}_{x_\parallel}=\partial_x$ and $\widetilde{D}_{x_\perp}=\partial_y$. Furthermore, let

$$\widetilde{\mathcal{L}}^{\mathsf{VE},L}_0=\left\{K\in\check{\mathcal{T}}^{\mathsf{VE},L}_{\mathrm{geo},\sigma}:\overline{K}\cap(\mathbf{e}\cup\mathbf{v})\neq\emptyset\right\},\qquad\widetilde{T}^{\mathsf{VE},L}_{\mathrm{int}}=\widehat{T}\setminus\bigcup_{K\in\widetilde{\mathcal{L}}^{\mathsf{VE},L}_0}\overline{K}$$

be, respectively, the elements abutting the singular boundary and the interior part of the vertex-edge reference patch; see Figure 4c.

Lemma 5.4 (*hp*-FE approximation in reference vertex-edge patch $\check{\mathcal{T}}^{\mathsf{VE},L}_{\mathrm{geo},\sigma}$). *Let $s\in(0,1)$ and $\gamma>0$ be fixed, and let $\widetilde{u}$ be such that for all $\varepsilon>0$ there exists $C_\varepsilon>0$ such that for all $(p_\parallel,p_\perp)\in\mathbb{N}^2_0$ with $p_\parallel+p_\perp=p$,*

$$\left\|\widetilde{r}^{p_\perp-1/2-s+\varepsilon}_{\mathbf{e}}\widetilde{r}^{p_\parallel+\varepsilon}_{\mathbf{v}}\widetilde{D}^{p_\perp}_{x_\perp}\widetilde{D}^{p_\parallel}_{x_\parallel}\widetilde{u}\right\|_{L^2(\widehat{T})}\le C_\varepsilon\gamma^{p+1}p^p. \tag{5.17}$$

Then, for all $\beta>1/2-s$ and all $0<\varepsilon<\beta/2+s/2-1/4$, there exist constants $b_{\mathsf{VE}}>0$ (depending only on γ, β, s, σ) and $C_{\mathsf{VE}}>0$ (depending additionally on ε) such that for every L, $q\in\mathbb{N}$

$$\|\widetilde{r}^{\beta-1}_{\mathbf{e}}(\tilde{u}-\widetilde{\Pi}^L_q\tilde{u})\|_{L^2(\widetilde{T}^{\mathsf{VE},L}_{\mathrm{int}})}+\|\widetilde{r}^{\beta}_{\mathbf{e}}\nabla(\tilde{u}-\widetilde{\Pi}^L_q\tilde{u})\|_{L^2(\widetilde{T}^{\mathsf{VE},L}_{\mathrm{int}})}\le C_{\mathsf{VE}}C_\varepsilon\exp(-b_{\mathsf{VE}}q). \tag{5.18}$$

Proof. Let $\widetilde{K}$ be an element not belonging to $\widetilde{\mathcal{L}}^{\mathsf{VE},L}_0$. We denote by $h_{\parallel,\widetilde{K}}$ and $h_{\perp,\widetilde{K}}$ the size of $\widetilde{K}$ in, respectively, parallel and perpendicular directions to $\mathbf{e}$. We have $\widetilde{r}_{\mathbf{e}}|_{\widetilde{K}}\simeq h_{\perp,\widetilde{K}}$ and $\widetilde{r}_{\mathbf{v}}|_{\widetilde{K}}\simeq h_{\parallel,\widetilde{K}}$ with uniform equivalence constants. From (5.17), a scaling argument, and a Sobolev imbedding, it follows that there exist $\widetilde{C},\widetilde{\gamma}>0$ such that for all $(p_\perp,p_\parallel)\in\mathbb{N}^2_0$ with $p=p_\parallel+p_\perp$

$$\forall\widetilde{K}\in\check{\mathcal{T}}^{\mathsf{VE},L}_{\mathrm{geo},\sigma}\setminus\widetilde{\mathcal{L}}^{\mathsf{VE},L}_0:\quad\|\widehat{D}^{p_\parallel}_{x_\parallel}\widehat{D}^{p_\perp}_{x_\perp}\widehat{u}\|_{L^\infty(\widehat{K})}\le\widetilde{C}h^{s-\varepsilon}_{\perp,\widetilde{K}}h^{-1/2-\varepsilon}_{\parallel,\widetilde{K}}\widetilde{\gamma}^{p+1}p^p. \tag{5.19}$$

By a scaling argument (dropping temporarily the subscript $\cdot_{,\widetilde K}$)

$$\begin{aligned}
&\|\widetilde r_{\mathbf e}^{\beta-1}(\tilde u - \widetilde\Pi_q^L \tilde u)\|^2_{L^2(\widetilde K)} + \|\widetilde r_{\mathbf e}^{\beta}\nabla(\tilde u - \widetilde\Pi_q^L\tilde u)\|^2_{L^2(\widetilde K)}\\
&\quad\lesssim h_\perp^{2\beta}\left(\frac{h_\|}{h_\perp}\left(\|\widehat u - \widehat\Pi_q\widehat u\|^2_{L^2(\widehat K)} + \|\widehat D_\perp(\widehat u-\widehat\Pi_q\widehat u)\|^2_{L^2(\widehat K)}\right)\right.\\
&\qquad\left. + \frac{h_\perp}{h_\|}\|\widehat D_\|(\widehat u - \widehat \Pi_q\widehat u)\|^2_{L^2(\widehat K)}\right)\\
&\quad\le h_\perp^{2\beta-1}h_\|\|\widehat u-\widehat\Pi_q\widehat u\|^2_{W^{1,\infty}(\widehat K)},
\end{aligned}$$

where the penultimate estimate follows from $h_\perp \simeq \widetilde r_{\mathbf e}\lesssim \widetilde r_{\mathbf v}\simeq h_\|$ in $\check{\mathcal T}^{\mathsf{VE},L}_{\mathrm{geo},\sigma}\backslash\widetilde{\mathcal L}_0^{\mathsf{VE},L}$. From Lemmas A.1 and A.2, using (5.19) then gives

$$\left\|\widetilde r_{\mathbf e}^{\beta-1}\left(\tilde u - \widetilde\Pi_q^L\tilde u\right)\right\|^2_{L^2(\widetilde K)} + \left\|\widetilde r_{\mathbf e}^{\beta}\nabla\left(\tilde u - \widetilde\Pi_q^L\tilde u\right)\right\|^2_{L^2(\widetilde K)} \lesssim h_{\perp,\widetilde K}^{2\beta+2s-1-2\varepsilon}h_{\|,\widetilde K}^{-2\varepsilon}e^{-2bq}. \tag{5.20}$$

From $\beta > 1/2 - s$ it follows that $\varepsilon > 0$ can be chosen so that $2\beta+2s-1 > 4\varepsilon$. In addition, $h_{\perp,\widetilde K}\le h_{\|,\widetilde K}$. Hence, there exists $\delta > 0$ such that, for all ε as specified above, $h_{\perp,\widetilde K}^{2\beta+2s-1-2\varepsilon}h_{\|,\widetilde K}^{-2\varepsilon}\le h_{\perp,\widetilde K}^{\delta}$. Then,

$$\sum_{\widetilde K\in\check{\mathcal T}^{\mathsf{VE},L}_{\mathrm{geo},\sigma}\backslash\widetilde{\mathcal L}_0^{\mathsf{VE},L}} h^\delta_{\perp,\widetilde K} = (1-\sigma)^\delta\sum_{i=0}^{L-1}\sum_{j=i}^{L-1}\sigma^{\delta j} = \frac{(1-\sigma)^\delta}{1-\sigma^\delta}\sum_{i=0}^{L-1}\left(\sigma^{\delta i}-\sigma^{\delta L}\right) \le \frac{(1-\sigma)^\delta}{(1-\sigma^\delta)^2}.$$

Summing (5.20) over all elements in $\check{\mathcal T}^{\mathsf{VE},L}_{\mathrm{geo},\sigma}\setminus\widetilde{\mathcal L}_0^{\mathsf{VE},L}$ concludes the proof. ☐

Remark 5.5. The dependence on ε of the constants $C_{\mathsf V}$, $C_{\mathsf E}$, C_{VE} of Lemmas 5.2, 5.3, and 5.4 can be dropped if, for $\bullet\in\{\mathsf V,\mathsf E,\mathsf{VE}\}$, the constant $C_\bullet$ is replaced by $\widetilde C_{\mathsf V}L$ in (5.12), by $\widetilde C_{\mathsf E}L$ in (5.15), and by $\widetilde C_{\mathsf{VE}}L^2$ in (5.18). The newly introduced constants $\widetilde C_\bullet$ are independent of the choice of ε.

This has no effect on the final result. Considering the dependence of $C_\bullet$ on ε, a fixed value of ε is chosen in the proof of Theorem 1.1, independently of β and s. If one were to use instead the results with the constants $\widetilde C_\bullet$, the terms in L and L^2 can be absorbed in the exponential $e^{-b_\bullet q}$ after having set $q\sim L$.

5.4.5. Global error bound (proof of Theorem 1.1).

Recall that $W_q^L = \mathcal S_0^q(\Omega,\mathcal T^L_{\mathrm{geo},\sigma})$ is the space of continuous, piecewise polynomials of maximum degree q on a mesh with L levels of refinement. From (5.9), Lemma 5.1, and Lemma B.1 it follows that for all $\beta\in[0,1-s)$

$$\inf_{v\in W_q^L}\|u-v\|_{\widetilde H^s(\Omega)}\lesssim \|u-\Pi^L_{q-1}u\|_{H^1_\beta(\Omega\backslash\Omega_0^L)} + \exp(-b_1L). \tag{5.21}$$

Remark that we can choose (potentially overlapping) $\omega_{\mathbf v}$, $\omega_{\mathbf e}$, and $\omega_{\mathbf{ve}}$ so that for all $K^{\mathcal M}\in\mathcal T^{\mathcal M}_{\mathsf E}$, $K^{\mathcal M}\subset\omega_{\mathbf e}$; for all $K^{\mathcal M}\in\mathcal T^{\mathcal M}_{\mathsf{VE}}$, $K^{\mathcal M}\subset\omega_{\mathbf{ve}}$; and for all $K^{\mathcal M}\in\mathcal T^{\mathcal M}_{\mathsf V}$, either $K^{\mathcal M}\subset\omega_{\mathbf v}$ or $K^{\mathcal M}\subset\omega_{\mathbf e}$. In other words, the edge and vertex-edge patches in the domain Ω are, respectively, contained in $\omega_{\mathbf e}$ and $\omega_{\mathbf{ve}}$; the vertex patch is either contained in $\omega_{\mathbf v}$ or $\omega_{\mathbf e}$, with origin mapped to a point on a vertex or along an edge.

Suppose now that u satisfies (2.5)–(2.8). Consider a patch $K^{\mathcal M}\in\mathcal T^{\mathcal M}$ and denote $\widetilde u = u\circ F_{K^{\mathcal M}}$. Let $\partial_{\widetilde x}$ be differentiation with respect to the variable $\widetilde x = F^{-1}_{K^{\mathcal M}}(x)$, and let $D_{\widetilde x_\perp}$ and $D_{\widetilde x_\|}$ be differentiation in directions respectively perpendicular and parallel to an edge pulled back to the reference patch. Let also $\widetilde{\mathbf v} = (0,0)$, $\widetilde{\mathbf e} = (0,1)\times\{0\}$ and denote $\widetilde r_{\mathbf v}(\widetilde x) = |\widetilde x - \widetilde{\mathbf v}|$ and $\widetilde r_{\mathbf e}(\widetilde x) = \mathrm{dist}(\widetilde x,\widetilde{\mathbf e})$ for all $\widetilde x\in F^{-1}_{K^{\mathcal M}}(K^{\mathcal M})$.

Case $K^{\mathcal{M}} \in \mathcal{T}^{\mathcal{M}}_{\mathsf{E}}$. Since $F_{K^{\mathcal{M}}}$ is affine and since it maps the closure of $\{(x_1,x_2)\in \widehat{S}: x_2=0\}$ to $\partial\Omega\cap\partial K^{\mathcal{M}}$, its Jacobian $J_{F_{K^{\mathcal{M}}}}$ can be written as the composition of an upper triangular matrix $U_{K^{\mathcal{M}}}$ and a rotation $R_{K^{\mathcal{M}}}$:

$$J_{F_{K^{\mathcal{M}}}} = R_{K^{\mathcal{M}}} U_{K^{\mathcal{M}}}.$$

Without loss of generality, we may assume the coordinate systems are oriented such that the vector $(1,0)^\top$, parallel to the singular edge in $\widehat{S}$, is mapped to $\mathbf{e}_\parallel = R_{K^{\mathcal{M}}}(1,0)^\top$. We remark that, since $U_{K^{\mathcal{M}}}$ is upper triangular, there exists $\eta_{K^{\mathcal{M}}}\in\mathbb{R}$ such that

$$U_{K^{\mathcal{M}}}(1,0)^\top = \eta_{K^{\mathcal{M}}}(1,0)^\top.$$

Hence,

$$D_{\widetilde{x}_\parallel} = \begin{pmatrix}1\\0\end{pmatrix}\cdot\nabla_{\widetilde{x}} = \begin{pmatrix}1\\0\end{pmatrix}\cdot(J^\top_{F_{K^{\mathcal{M}}}}\nabla_x) = \eta_{K^{\mathcal{M}}}\begin{pmatrix}1\\0\end{pmatrix}\cdot(R^\top_{K^{\mathcal{M}}}\nabla_x) = \eta_{K^{\mathcal{M}}}\mathbf{e}_\parallel\cdot\nabla_x = \eta_{K^{\mathcal{M}}} D_{x_\parallel}.$$

By a similar argument, there exist $\beta_1,\beta_2\in\mathbb{R}$ such that

$$D_{\widetilde{x}_\perp} = \beta_1 D_{x_\parallel} + \beta_2 D_{x_\perp}.$$

Finally, there exists $c_{K_M}>0$ such that for all $x\in\widehat{S}$,

$$\frac{1}{c_{K^{\mathcal{M}}}}\widetilde{r}_{\mathbf{e}}(x) \le r_{\mathbf{e}}(F_{K^{\mathcal{M}}}(x)) \le c_{K^{\mathcal{M}}}\widetilde{r}_{\mathbf{e}}(x).$$

Hence,

$$\begin{aligned}
&\left\|\tilde{r}_{\mathbf{e}}^{p_\perp-1/2-s+\varepsilon} D^{p_\perp}_{\widetilde{x}_\perp} D^{p_\parallel}_{\widetilde{x}_\parallel}\tilde{u}\right\|_{L^2(F^{-1}_{K^{\mathcal{M}}}(K^{\mathcal{M}}))}\\
&\quad\le C c_{K^{\mathcal{M}}}^{p_\perp-1/2-s+\varepsilon}\eta_{K^{\mathcal{M}}}^{p_\parallel}\|r_{\mathbf{e}}^{p_\perp-1/2-s+\varepsilon}(\beta_1 D_{x_\parallel}+\beta_2 D_{x_\perp})^{p_\perp} D^{p_\parallel}_{x_\parallel} u\|_{L^2(K^{\mathcal{M}})}\\
&\quad\le C c_{K^{\mathcal{M}}}^{p_\perp-1/2-s+\varepsilon}\eta_{K^{\mathcal{M}}}^{p_\parallel}(\beta_1+\beta_2)^{p_\perp}\max_{j=0,\dots,p_\perp}\|r_{\mathbf{e}}^{p_\perp-1/2-s+\varepsilon} D^j_{x_\perp} D^{p_\parallel+p_\perp-j}_{x_\parallel} u\|_{L^2(K^{\mathcal{M}})}.
\end{aligned}$$

It follows then from (2.6) that there exist $\widetilde{C}_\varepsilon,\widetilde{\gamma}>0$ such that, for all $(p_\perp,p_\parallel)\in\mathbb{N}_0^2$ with $p=p_\perp+p_\parallel$ and for all $K^{\mathcal{M}}\in\mathcal{T}^{\mathcal{M}}_{\mathsf{E}}$, we obtain

$$\left\|\tilde{r}_{\mathbf{e}}^{p_\perp-1/2-s+\varepsilon} D^{p_\perp}_{\widetilde{x}_\perp} D^{p_\parallel}_{\widetilde{x}_\parallel}\tilde{u}\right\|_{L^2(F^{-1}_{K^{\mathcal{M}}}(K^{\mathcal{M}}))} \le \widetilde{C}_\varepsilon\widetilde{\gamma}^{p+1}p^p.$$

Case $K^{\mathcal{M}} \in \mathcal{T}^{\mathcal{M}}_{\mathsf{VE}}$. This case is treated as the previous one, noting that in addition

$$\frac{1}{c_{K^{\mathcal{M}}}}\widetilde{r}_{\mathbf{v}}(x) \le r_{\mathbf{v}}(F_{K^{\mathcal{M}}}(x)) \le c_{K^{\mathcal{M}}}\widetilde{r}_{\mathbf{v}}(x). \tag{5.22}$$

We obtain from (2.7) that there exist $\widetilde{C}_\varepsilon,\widetilde{\gamma}>0$ such that, for all $(p_\perp,p_\parallel)\in\mathbb{N}_0^2$ with $p=p_\perp+p_\parallel$ and for all $K^{\mathcal{M}}\in\mathcal{T}^{\mathcal{M}}_{\mathsf{VE}}$,

$$\left\|\tilde{r}_{\mathbf{e}}^{p_\perp-1/2-s+\varepsilon}\tilde{r}_{\mathbf{v}}^{p_\parallel+\varepsilon} D^{p_\perp}_{\widetilde{x}_\perp} D^{p_\parallel}_{\widetilde{x}_\parallel}\tilde{u}\right\|_{L^2(F^{-1}_{K^{\mathcal{M}}}(K^{\mathcal{M}}))} \le \widetilde{C}_\varepsilon\widetilde{\gamma}^{p+1}p^p.$$

Case $K^{\mathcal{M}} \in \mathcal{T}^{\mathcal{M}}_{\mathsf{V}}$. If $K^{\mathcal{M}}\subset\omega_{\mathbf{v}}$, then (5.22) holds. In addition, there exists a constant $\tilde{c}$ such that, for all $\alpha\in\mathbb{N}_0^2$,

(5.23)
$$\left\| \tilde{r}_{\mathbf{v}}^{|\alpha|-1/2-s+\varepsilon} \partial_{\tilde{x}}^{\alpha} \widetilde{u} \right\|_{L^2(F^{-1}_{K^{\mathcal{M}}}(K^{\mathcal{M}}))} \leq c_{K^{\mathcal{M}}}^{|\alpha|-1/2-s+\varepsilon} \tilde{c}^{|\alpha|} \max_{\beta \leq \alpha} \left\| r_{\mathbf{v}}^{|\alpha|-1/2-s+\varepsilon} \partial_x^{\beta} u \right\|_{L^2(K^{\mathcal{M}})}.$$

Therefore, (2.5) implies

$$\left\| \tilde{r}_{\mathbf{v}}^{|\alpha|-1/2-s+\varepsilon} \partial_{\tilde{x}}^{\alpha} \tilde{u} \right\|_{L^2(F^{-1}_{K^{\mathcal{M}}}(K^{\mathcal{M}}))} \leq \widetilde{C}_{\varepsilon} \widetilde{\gamma}^{|\alpha|+1} |\alpha|^{|\alpha|}.$$

If instead $K^{\mathcal{M}} \subset \omega_{\mathbf{e}}$, there exists $c_{K^{\mathcal{M}}}$ such that

$$\frac{1}{c_{K^{\mathcal{M}}}} \widetilde{r}_{\mathbf{v}}(x) \leq r_{\mathbf{e}}(F_{K^{\mathcal{M}}}(x)) \leq c_{K^{\mathcal{M}}} \widetilde{r}_{\mathbf{v}}(x)$$

for all $x \in K^{\mathcal{M}}$, with $\mathbf{e}$ being the edge such that $\mathbf{e} \cap \partial K^{\mathcal{M}} \neq \emptyset$. It follows from (5.23) and (2.6) that, for all $\alpha \in \mathbb{N}_0^2$,

$$\left\| \tilde{r}_{\mathbf{v}}^{|\alpha|-1/2-s+\varepsilon} \partial_{\tilde{x}}^{\alpha} \tilde{u} \right\|_{L^2(F^{-1}_{K^{\mathcal{M}}}(K^{\mathcal{M}}))} \leq \widetilde{C}_{\varepsilon} \widetilde{\gamma}^{|\alpha|+1} |\alpha|^{|\alpha|}.$$

Case $K^{\mathcal{M}} \in \mathcal{T}^{\mathcal{M}}_{\mathrm{int}}$. If $K^{\mathcal{M}} \in \mathcal{T}^{\mathcal{M}}_{\mathrm{int}}$, then $\widetilde{u}|_{F^{-1}_{K^{\mathcal{M}}}(K^{\mathcal{M}})}$ is analytic.

Since the macro-triangulation $\mathcal{T}^{\mathcal{M}}$ is fixed, all constants depending on $K^{\mathcal{M}}$ can be taken uniformly over the macro-triangulation. We have obtained that for all $K^{\mathcal{M}} \in \mathcal{T}^{\mathcal{M}}$, the restriction of $\widetilde{u}$ to $F^{-1}_{K^{\mathcal{M}}}(K^{\mathcal{M}})$ satisfies the hypotheses of Lemmas 5.2–5.4.

Restricting $\beta \in (1/2 - s, 1 - s)$ in (5.21) and using Lemmas 5.2–5.4 therefore gives

$$\inf_{v \in W_q^L} \|u - v\|_{\widetilde{H}^s(\Omega)} \leq C \left(\exp(-b_2 q) + \exp(-b_1 L) \right).$$

Choosing $q \simeq L$, $V_N := W_q^L$ and remarking that $\dim(W_q^L) \simeq q^2 L^2$ concludes the proof.

6. Conclusions.

We proved *exponential rates of convergence for a class of hp-finite element approximations* of the Dirichlet problem for the integral fractional Laplacian in a bounded, polygonal domain $\Omega \subset \mathbb{R}^2$, with analytic source term f, based on anisotropic, geometric boundary-refined meshes. The *realization of corresponding hp-FE algorithms* will incur significant issues of *numerical quadrature for stable numerical evaluation of the bilinear form* $a(\cdot,\cdot)$ in (1.5) on pairs of large aspect ratio rectangles in the geometric boundary mesh patches shown in Figure 2. While being in principle known (see, e.g., [34] for a related discussion in *hp*-Galerkin boundary element methods on polyhedral domains), the corresponding consistency analysis for the form $a(\cdot,\cdot)$ in (1.5) in the space $\widetilde{H}^s(\Omega)$ in (1.4) will be the topic of a forthcoming work. We remark also that, even for the *h*- or *p*-version of the FEM, the matrices associated with the bilinear form are dense. This makes the importance of the reduction in the number of degrees of freedom obtained with *hp*-methods even greater. Even if the dense linear system resulting from the presently proposed Galerkin discretization were to be solved directly with a Cholesky algorithm, the overall complexity of the *hp*-FEM would grow like a polynomial of the logarithm of the $\widetilde{H}^s(\Omega)$-norm of the error.

Here, we analyzed only the convergence rate of the *hp*-Galerkin discretization (1.6) based on the subspaces W_q^L in (5.1) with geometric, boundary-refined triangulations $\mathcal{T}^L_{\mathrm{geo},\sigma}$ and with uniform polynomial degree q in all elements. As it is well known (see, e.g., [23, 24, 35, 20]), exponential rates of convergence of the *hp*-FEM for problems with weighted analytic solutions are also achievable with *variable, so-called linear polynomial degree distributions* in the geometric partition $\mathcal{T}^L_{\mathrm{geo},\sigma}$. Similar techniques, based again on the anisotropic, weighted high order Sobolev regularity of the solution u of (1.1) proved in [17], allow us to infer optimal algebraic rates of convergence $O(h^{q+1-s}) = O(N^{-(q+1-s)/2})$ in the $\widetilde{H}^s(\Omega)$-norm for continuous, piecewise polynomial Lagrangian FEs of order $q \geq 1$. To this end, however, the geometric boundary-refined partitions $\mathcal{T}^L_{\mathrm{geo},\sigma}$ in the design of the spaces W^L_q in (5.1) must be replaced by *boundary-refined graded partitions.* Details shall be reported elsewhere.

Appendix A. Polynomial approximation operators on the reference element. The following two lemmas are consequences of [7, Lemma 3.1, 3.2].

LEMMA A.1 (approximation on triangles). *Let $\widehat{T}$ be the reference triangle. Then, for every $q \in \mathbb{N}$, there exists a linear operator $\widehat{\Pi}^{\triangle}_q : C^0(\overline{\widehat{T}}) \to \mathbb{P}_q$ with the following properties:*

1. *For each edge e of $\widehat{T}$, $(\widehat{\Pi}^{\triangle}_q u)|_e$ coincides with the Gauss–Lobatto interpolant $i_q(u|_e)$ of degree q on the edge e.*
2. *(projection property) $\widehat{\Pi}^{\triangle}_q v = v$ for all $v \in \mathbb{P}_q$.*
3. *Let $u \in C^\infty(\widehat{T})$ satisfy, for some C_u, $\gamma > 0$*

$$\forall n \in \mathbb{N}_0 : \ \ \|\nabla^n u\|_{L^\infty(\widehat{T})} \leq C_u \gamma^n (n+1)^n.$$

Then, there exist $C, b > 0$ such that for all $q \geq 1$

$$\|u - \widehat{\Pi}^{\triangle}_q u\|_{W^{1,\infty}(\widehat{T})} \leq C C_u e^{-bq}.$$

LEMMA A.2 (approximation on quadrilaterals). *Let $\widehat{S}$ be the reference square. For each $q \in \mathbb{N}$, the tensor-product Gauss–Lobatto interpolation operator $\widehat{\Pi}^{\square}_q : C^0(\overline{\widehat{S}}) \to \mathbb{Q}_q$ satisfies the following:*

1. *For each edge $e \subset \partial\widehat{S}$, $(\widehat{\Pi}^{\square}_q u)|_e$ coincides with the univariate Gauss–Lobatto interpolant $i_q(u|_e)$ on e.*
2. *(projection property) $\widehat{\Pi}^{\square}_q v = v$ for all $v \in \mathbb{Q}_q$.*
3. *Let $u \in C^\infty(\widehat{S})$ satisfy for some C_u, $\gamma > 0$ and all $(n, m) \in \mathbb{N}^2_0$*

$$\|\partial^m_{\hat{x}} \partial^n_{\hat{y}} u\|_{L^\infty(\widehat{S})} \leq C_u \gamma^{n+m} (n+1)^n (m+1)^m. \tag{A.1}$$

Then, there exist $C, b > 0$ such that for all $q \geq 1$

$$\|u - \widehat{\Pi}^{\square}_q u\|_{W^{1,\infty}(\widehat{S})} \leq C C_u e^{-bq}.$$

Appendix B. Estimates of norms with cutoff function. We introduce two technical lemmas.

LEMMA B.1. *Let g^L be defined as in (5.7) and let $\beta \in [0,1)$. Then, there exists $C > 0$ such that with Ω^L_0 defined in (5.8), it holds that, for all $w \in H^1_\beta(\Omega)$ and all $L \in \mathbb{N}$,*

$$\|g^L w\|_{H^1_\beta(\Omega)} \leq C \|w\|_{H^1_\beta(\Omega \setminus \Omega^L_0)}.$$

Proof. By definition of g^L we have for all $L \geq 1$

$$\|g^L\|_{L^\infty(\Omega)} = 1, \qquad \|\nabla g^L\|_{L^\infty(\Omega)} \simeq \sigma^{-L}.$$

In addition, there exists $c > 0$ such that for all $L \geq 1$

$$\operatorname{supp}(\nabla g^L) \subset S_{c\sigma^L} \setminus \Omega^L_0, \qquad \operatorname{supp}(1 - g^L) \subset S_{c\sigma^L}, \qquad \operatorname{supp}(g^L) \subset \Omega \setminus \Omega^L_0.$$

Hence, for all $L \geq 1$,

$$\begin{aligned}
&\left\|r^{\beta-1}g^L w\right\|^2_{L^2(\Omega)} + \left\|r^{\beta}\nabla(g^L w)\right\|^2_{L^2(\Omega)}\\
&\quad\leq \left\|r^{\beta-1}w\right\|^2_{L^2(\Omega\setminus\Omega_0^L)} + \left\|\nabla g^L\right\|^2_{L^\infty(\Omega)}\left\|r^{\beta}w\right\|^2_{L^2(S_{c\sigma^L}\setminus\Omega_0^L)} + \left\|r^{\beta}\nabla w\right\|^2_{L^2(\Omega\setminus\Omega_0^L)}\\
&\quad\lesssim \|w\|^2_{H^1_\beta(\Omega\setminus\Omega_0^L)} + \sigma^{-2L}\left\|r^{\beta}w\right\|^2_{L^2(S_{c\sigma^L}\setminus\Omega_0^L)}\\
&\quad\lesssim \|w\|^2_{H^1_\beta(\Omega\setminus\Omega_0^L)} + c^2\left\|r^{\beta-1}w\right\|^2_{L^2(S_{c\sigma^L}\setminus\Omega_0^L)},
\end{aligned}$$

with constants hidden in $\lesssim$ independent of L. ∎

LEMMA B.2. *Let g^L be defined as in (5.7) and let $\beta \in [0,1)$. Then, there exist $C, c > 0$ independent of L such that, for all $w \in H^1_\beta(\Omega)$ and all $L \in \mathbb{N}$,*

$$\|(1-g^L)w\|_{H^1_\beta(\Omega)} \leq C\|w\|_{H^1_\beta(S_{c\sigma^L})}.$$

Proof. The proof proceeds along the same lines as the proof of the previous lemma. We have

$$\|r^{\beta-1}(1-g^L)w\|_{L^2(\Omega)} \leq \|r^{\beta-1}w\|_{L^2(S_{c\sigma^L})},$$

where c is defined as in the preceding proof such that $r(x) \leq c\sigma^L$ holds for all $x \in \operatorname{supp}(1-g^L)$. Similarly, we obtain

$$\|r^{\beta}\nabla((1-g^L)w)\|^2_{L^2(\Omega)} \lesssim \|r^{\beta}\nabla w\|^2_{L^2(S_{c\sigma^L})} + c^2\|r^{\beta-1}w\|^2_{L^2(S_{c\sigma^L})},$$

which finishes the proof. ∎

REFERENCES

[1] H. Abels and G. Grubb, *Fractional-Order Operators on Nonsmooth Domains*, preprint, https://arxiv.org/abs/2004.10134, 2020.

[2] G. Acosta and J. P. Borthagaray, *A fractional Laplace equation: Regularity of solutions and finite element approximations*, SIAM J. Numer. Anal., 55 (2017), pp. 472–495, https://doi.org/10.1137/15M1033952.

[3] G. Acosta, J. P. Borthagaray, O. Bruno, and M. Maas, *Regularity theory and high order numerical methods for the (1D)-fractional Laplacian*, Math. Comp., 87 (2018), pp. 1821–1857, https://doi.org/10.1090/mcom/3276.

[4] M. Ainsworth and C. Glusa, *Aspects of an adaptive finite element method for the fractional Laplacian: A priori and a posteriori error estimates, efficient implementation and multigrid solver*, Comput. Methods Appl. Mech. Engrg., 327 (2017), pp. 4–35.

[5] B. Bahr, M. Faustmann, C. Marcati, J. M. Melenk, and C. Schwab, *Exponential convergence of hp-FEM for the integral fractional Laplacian in 1D*, in Spectral and High Order Methods for Partial Differential Equations ICOSAHOM 2020+1, J. M. Melenk, I. Perugia, J. Schöberl, and C. Schwab, eds., Springer, Cham, 2023, pp. 291–306.

[6] L. Banjai, J. Melenk, R. Nochetto, E. Otárola, A. Salgado, and C. Schwab, *Tensor FEM for spectral fractional diffusion*, Found. Comput. Math., 19 (2019), pp. 901–962, https://doi.org/10.1007/s10208-018-9402-3.

[7] L. Banjai, J. M. Melenk, and C. Schwab, *hp-FEM for reaction-diffusion equations.* II*: Robust exponential convergence for multiple length scales in corner domains*, IMA J. Numer. Anal., (2022), https://doi.org/10.1093/imanum/drac070.

[8] L. Banjai, J. M. Melenk, and C. Schwab, *Exponential convergence of hp FEM for spectral fractional diffusion in polygons*, Numer. Math., 153 (2023), pp. 1–47, https://doi.org/10.1007/s00211-022-01329-5.

[9] A. Bonito, J. Borthagaray, R. Nochetto, E. Otárola, and A. Salgado, *Numerical methods for fractional diffusion*, Comput. Vis. Sci., 19 (2018), pp. 19–46.

[10] A. Bonito, W. Lei, and J. Pasciak, *Numerical approximation of the integral fractional Laplacian*, Numer. Math., 142 (2019), pp. 235–278.

[11] J. Borthagaray and R. Nochetto, *Besov regularity for the Dirichlet integral fractional Laplacian in Lipschitz domains*, J. Funct. Anal., 284 (2023), 109829, https://doi.org/10.1016/j.jfa.2022.109829.

[12] J. Borthagaray and R. Nochetto, *Constructive approximation on graded meshes for the integral fractional Laplacian*, Constr. Approx., 57 (2023), pp. 463–487.

[13] L. Brasco and A. Salort, *A note on homogeneous Sobolev spaces of fractional order*, Ann. Mat. Pura Appl. (4), 198 (2019), pp. 1295–1330, https://doi.org/10.1007/s10231-018-0817-x.

[14] M. D'Elia, Q. Du, C. Glusa, M. Gunzburger, X. Tian, and Z. Zhou, *Numerical methods for nonlocal and fractional models*, Acta Numer., 29 (2020), pp. 1–124, https://doi.org/10.1017/s096249292000001x.

[15] R. DeVore and G. Lorentz, *Constructive Approximation*, Springer, Verlag, 1993.

[16] M. Faustmann, M. Karkulik, and M. J. Melenk, *Local convergence of the FEM for the integral fractional Laplacian*, SIAM J. Numer. Anal., 60 (2022), pp. 1055–1082, https://doi.org/10.1137/20M1343853.

[17] M. Faustmann, C. Marcati, J. M. Melenk, and C. Schwab, *Weighted analytic regularity for the integral fractional Laplacian in polygons*, SIAM J. Math. Anal., 54 (2022), pp. 6323–6357, https://doi.org/10.1137/21M146569X.

[18] M. Faustmann and J. M. Melenk, *Robust exponential convergence of hp-FEM in balanced norms for singularly perturbed reaction-diffusion problems: Corner domains*, Comput. Math. Appl., 74 (2017), pp. 1576–1589, https://doi.org/10.1016/j.camwa.2017.03.015.

[19] H. Gimperlein, E. Stephan, and J. Štoček, *Corner Singularities for the Fractional Laplacian and Finite Element Approximation*, preprint, 2021, https://mat1.uibk.ac.at/heiko/corners.pdf.

[20] H. Gimperlein, E. Stephan, and J. Stocek, *hp finite element approximation for the fractional Laplacian on uniform and geometrically graded meshes*, preprint, 2022, https://mat1.uibk.ac.at/heiko/HpAndExpoNumerics_public.pdf.

[21] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Classics Appl. Math. 69, SIAM, Philadelphia, 2011, https://doi.org/10.1137/1.9781611972030.ch1; reprint of the 1985 original.

[22] G. Grubb, *Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators*, Adv. Math., 268 (2015), pp. 478–528.

[23] B. Guo and I. Babuška, *The h-p version of the finite element method. Part 1: The basic approximation results*, Comput. Mech., 1 (1986), pp. 21–41.

[24] B. Guo and I. Babuška, *The h-p version of the finite element method. Part 2: General results and applications*, Comput. Mech., 1 (1986), pp. 203–220.

[25] H. Holm, M. Maischak, and E. P. Stephan, *Exponential convergence of the h-p version BEM for mixed boundary value problems on polyhedrons*, Math. Methods Appl. Sci., 31 (2008), pp. 2069–2093, https://doi.org/10.1002/mma.1009.

[26] M. Karkulik and J. Melenk, *$\mathcal{H}$-matrix approximability of inverses of discretizations of the fractional Laplacian*, Adv. Comput. Math., 45 (2019), pp. 2893–2919.

[27] M. Kwaśnicki, *Ten equivalent definitions of the fractional Laplace operator*, Fract. Calc. Appl. Anal., 20 (2017), pp. 7–51.

[28] A. Lischke, G. Pang, M. Gulian, F. Song, C. Glusa, X. Zheng, Z. Mao, W. Cai, M. M. Meerschaert, M. Ainsworth, and G. E. Karniadakis, *What is the fractional Laplacian? A comparative review with new results*, J. Comput. Phys., 404 (2020), 109009, https://doi.org/10.1016/j.jcp.2019.109009.

[29] Z. Mao and G. E. Karniadakis, *A spectral method (of exponential convergence) for singular solutions of the diffusion equation with general two-sided fractional derivative*, SIAM J. Numer. Anal., 56 (2018), pp. 24–49, https://doi.org/10.1137/16M1103622.

[30] J. M. Melenk, *hp-Finite Element Methods for Singular Perturbations*, Lecture Notes in Math. 1796, Springer-Verlag, Berlin, 2002, https://doi.org/10.1007/b84212.

[31] R. Nochetto, E. Otárola, and A. Salgado, *A PDE approach to fractional diffusion in general domains: A priori error analysis*, Found. Comput. Math., 15 (2015), pp. 733–791, https://doi.org/10.1007/s10208-014-9208-x.

[32] X. Ros-Oton, *Nonlocal elliptic equations in bounded domains: A survey*, Publ. Mat., 60 (2016), pp. 3–26, http://projecteuclid.org/euclid.pm/1450818481.

[33] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: Regularity up to the boundary*, J. Math. Pures Appl. (9), 101 (2014), pp. 275–302, https://doi.org/10.1016/j.matpur.2013.06.003.

[34] S. Sauter and C. Schwab, *Quadrature for hp-Galerkin BEM in* R^3, Numer. Math., 78 (1997), pp. 211–258, https://doi.org/10.1007/s002110050311.

[35] D. Schötzau and C. Schwab, *Exponential convergence of hp-FEM for elliptic problems in polyhedra: Mixed boundary conditions and anisotropic polynomial degrees*, Found. Comput. Math., 18 (2018), pp. 595–660, https://doi.org/10.1007/s10208-017-9349-9.

[36] J. Štoček, *Efficient Finite Element Methods for the Integral Fractional Laplacian and Applications*, Ph.D. thesis, Heriot-Watt University, Edinburgh, 2020.

[37] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, 2nd ed., Johann Ambrosius Barth, Heidelberg, 1995.