Commun. Math. Phys. (2024) 405:288
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-024-05165-6

Communications in Mathematical Physics

On the Stochastic Sine-Gordon Model: An Interacting Field Theory Approach

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Received: 12 November 2023 / Accepted: 15 October 2024

Abstract: We investigate the massive sine-Gordon model in the finite ultraviolet regime on the two-dimensional Minkowski spacetime $(\mathbb{R}^2, \eta)$ with an additive Gaussian white noise. In particular we construct the expectation value and the correlation functions of a solution of the underlying stochastic partial differential equation (SPDE) as a power series in the coupling constant, proving ultimately uniform convergence. This result is obtained combining an approach first devised in Dappiaggi et al. (Commun Contemp Math 24(07):2150075, 2022. arXiv:2009.07640 [math-ph]) to study SPDEs at a perturbative level with the one discussed in Bahns and Rejzner (Commun Math Phys 357(1):421, 2018. arXiv:1609.08530 [math-ph]) to construct the quantum sine-Gordon model using techniques proper of the perturbative, algebraic approach to quantum field theory (pAQFT). At a formal level the relevant expectation values are realized as the evaluation of suitably constructed functionals over $C^\infty(\mathbb{R}^2)$. In turn, these are elements of a distinguished algebra whose product is a deformation of the pointwise one, by means of a kernel which is a linear combination of two components. The first encompasses the information of the Feynmann propagator built out of an underlying Hadamard, quantum state, while the second encodes the correlation codified by the Gaussian white noise. In our analysis, first of all we extend the results obtained in Bahns et al. (J Math Anal Appl 526:127249, 2023. arXiv:2103.09328 [math-ph]) and Bahns and Rejzner (Commun Math Phys 357(1):421, 2018. arXiv:1609.08530 [math-ph]) proving the existence of a convergent modified version of the S-matrix and of an interacting field as elements of the underlying algebra of functionals. Subsequently we show that it is possible to remove the contribution due to the Feynmann propagator by taking a suitable $\hbar \to 0^+$-limit, hence obtaining the sought expectation value of the solution and of the correlation functions of the SPDE associated to the stochastic sine-Gordon model.

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Published online: 13 November 2024

1. Introduction

The investigation of nonlinear stochastic partial differential equations (SPDEs) represents one of the most thriving branches of research in mathematics, mostly thanks to the formulation of different successful frameworks aimed at studying their underlying solution space. Regularity structures [21] or paracontrolled calculus [18] have proven to be two complementary, albeit rather different, approaches which have allowed to prove existence and uniqueness of the solutions of a large class of nonlinear elliptic or parabolic SPDEs. More recently, a novel approach using flow equations has been introduced to handle this class of equations and related problems [12,13]. At the same time these equations are remarkably important in many physical models, stemming from interface dynamics to stochastic quantization. Yet, in order to build a solid and longstanding bridge between the probabilistic and analytic approach to SPDEs and the physical models, it is mandatory to be able to provide as much explicit information as possible on the underlying solutions and on their correlation functions. With this goal in mind, in [11] it has been proposed a new method for the construction of the solutions and of the correlation functions of nonlinear SPDEs, largely inspired by the algebraic approach to quantum field theory. On the one hand this framework has the advantage of allowing to encompass the renormalization procedure and freedoms which are often a key ingredient in the solution theory of a nonlinear SPDE, without resorting to any specific ϵ-regularization scheme. On the other hand, it allows to establish an algorithmic procedure to construct both the solutions and the n-point correlation functions, though as a formal power series in the coupling constant which rules the nonlinear term in the equation of motion.

It is important to highlight that the algebraic approach, devised in [11] has the additional net advantage to be applicable also to nonlinear SPDEs which do not lie in the subcritical regime, a necessary prerequisite instead when applying the theory of regularity structures or of paracontrolled calculus, see, e.g., [5] and [6]. One might be tempted to perceive subcriticality essentially as yielding a constraint on the nonlinear potential ruling the underlying dynamics, but this viewpoint is correct provided that the fundamental solutions associated to the operator ruling the linear contribution to the underlying SPDE is regularizing. This is indeed the case when considering second order elliptic or parabolic differential operators with smooth coefficients, as it occurs in the vast majority of the models in the literature. On the contrary this feature is no longer present when one considers hyperbolic SPDEs, for example the stochastic, nonlinear wave equation. In this case one has often to resort to a case by case analysis, see for example [19,20] in order to prove existence and uniqueness of the underlying solutions.

For this reason, in comparison to the elliptic and parabolic scenarios, hyperbolic SPDEs have been analyzed less in depth. In this paper we shall focus our attention on a specific instance of this class of equations, which is known as the stochastic massive sine-Gordon model in the so called finite ultraviolet regime. Its parabolic counterpart has been studied in [8,21], while the hyperbolic scenario in two space dimensions has been investigated recently in [27]. If one focuses instead the attention on one space dimension, the sine-Gordon equation without an additive Gaussian, white noise as a source, is a very important and thoroughly studied model in quantum field theory, especially due to the underlying integrability properties and to its connections with the Thirring model, which is at the heart of a phenomenon known as Bosonization [9,30]. In particular, within the framework of the perturbative, algebraic approach to quantum field theory, it has gained a lot of attention in the past decade since it represents one of the few notable examples where it is possible to prove convergence of the perturbative series defining the S-matrix of the model, see [3,4]. This result prompts naturally the question whether the techniques used in these papers can be adapted to be applicable also to the analysis of the stochastic sine-Gordon equation on the two-dimensional Minkowski spacetime. In this work we shall prove that this is indeed the case and this allows us to obtain two notable, connected results. On the one hand we are able to establish for the first time the convergence of the perturbative series which lies at the heart of the algebraic approach to the construction of the solution and of the correlation functions of the stochastic sine-Gordon equation. On the other hand, this opens the path to the possibility of combining the recent analysis in [5] on the stochastic Thirring model to establish a stochastic counterpart of the phenomenon of Bosonization using techniques inspired by the algebraic approach to quantum field theory.

More in detail, our approach to the stochastic sine-Gordon equation can be divided in two main steps. In the first one, we follow the rationale of [3,4], namely we consider a suitable algebra of functionals defined on $C^\infty(\mathbb{R}^2)$, where $\mathbb{R}^2$ plays here the rôle of the underlying Minkowski spacetime. At the beginning we consider a commutative pointwise product and, subsequently, in the spirit of the perturbative, algebraic approach to quantum field theory, we deform it by means of a suitable kernel which encompasses the information both of an underlying Feynmann propagator, the building block of the time-ordered product in an interacting quantum theory, and of the correlation function of the Gaussian process codified by the underlying white noise. It is important to stress that this entails a deviation from the approach in [3] since, in this paper, no stochastic effect has been considered. Yet we are able to show that all convergence results for the S-matrix and for the interacting quantum field can be generalized to this scenario, although they hold true in an arbitrary, but fixed compact region of the two-dimensional Minkowski spacetime. The outcome is a model which mixes both a quantum and a stochastic behaviour. Yet the effect of the former can be sharply disentangled from the latter since the action of the Feynman propagator is always tagged by the presence of a multiplicative constant, namely $\hbar$. This feature leads to the second step of our approach in which we investigate the limit as $\hbar$ tends to 0 of all relevant functionals and we prove that such limit is always well-defined. In this way we are also able to show that we are actually constructing explicitly the functionals encoding the information on the expectation value of the interacting solution of the stochastic sine-Gordon equation and on the associated n-point correlation functions. Observe that taking the expectation value corresponds in the algebraic approach to considering the evaluation of the corresponding functional on

the zero configuration. The detour over the quantum world, in order to prove a classical result, might appear as surprising. At an intuitive level, the advantage stems combining the Bogoliubov formula for the interacting field with the complex exponential form of the interaction. Indeed, since thee S-matrix and the quantum products are built out of exponential structures, the arising series is convergent in a suitable topology. To our knowledge, this exponential structure does not translate slavishly at the classical level. It is also worth recalling that the low dimension of the underlying spacetime plays a pivotal rôle in the whole construction.
The paper is organized as follows: in Sect. 2.1 we review the building blocks of the algebraic approach to an interacting quantum field theory, in particular the S-matrix and the Bogoliubov map in Sect. 2.1.1. The goal of Sect. 2.2 is instead to present succinctly the content of [11] and the interplay with microlocal analysis. The specialization of the structures outlined in the first sections to the specific case of the sine-Gordon model is the content of Sect. 2.3, while in Sect. 2.4 we present the strategy that we plan to follow to construct the solutions and the n-point correlation functions of the stochastic sine-Gordon equation. Section 3 contains the first of our main results since we introduce the notion of the $Q-S$-matrix first as a formal power series in the underlying coupling constant. Subsequently in Sect. 3.1 we prove uniform convergence in Theorem 3.4 and in Corollary 3.5. This allows us in turn to establish also convergence both of the interacting field in Sect. 3.2 and of the n-point correlation functions in Sect. 3.2.1. The main result of our work is discussed instead in Sect. 4, namely in Theorem 4.7, we prove that the limit as $\hbar \to 0^+$ of all relevant functionals exist and this allows us to establish the existence of the expectation value of the solutions of the stochastic sine-Gordon equation as well as of the associated n-point correlation functions, as suitably convergent power series in the underlying coupling constant.

1.1. Notation and conventions. In this short section we introduce the stochastic sine-Gordon equation and we take the chance to fix the notation and the conventions that we use in this work. Throughout the paper we denote by $\mathbb{M}$, a generic d-dimensional globally hyperbolic spacetime. With $\mathcal{E}(\mathbb{M}) := C^\infty(\mathbb{M})$, $\mathcal{D}(\mathbb{M}) := C_0^\infty(\mathbb{M})$ we indicate respectively the space of smooth field configurations and of test functions. In addition $\mathcal{D}'(\mathbb{M})$ is the space of distributions over $\mathbb{M}$, dual to $\mathcal{D}(M)$, while $\mathcal{E}'(\mathbb{M})$ is the space of compactly supported distributions dual to $\mathcal{E}(\mathbb{M})$. We will be particularly interested in the case where the rôle of $\mathbb{M}$ is played by the two-dimensional Minkowski spacetime $\mathbb{R}^2$ which is endowed with the standard Minkowski metric η of signature $(+,-)$. On top of it we consider the *stochastic sine-Gordon equation*

$$(\Box + m^2)\hat{\psi} + \lambda g a \sin(a\hat{\psi}) = \hat{\xi}, \tag{1.1}$$

where $\Box = \partial_t^2 - \partial_x^2$ is the d'Alembert wave operator, $\hat{\psi}$ is a random valued distribution, whereas $a \in \mathbb{R}$ is chosen according to the finite ultraviolet regime, namely $a^2 < 4\pi/\hbar$, while $g \in \mathcal{D}(\mathbb{R}^2)$. In addition $\hat{\xi}$ denotes a space-time white noise, namely a Gaussian centered random distribution whose two-point correlation function is, at the level of integral kernel,

$$\mathbb{E}[\hat{\xi}(z)\hat{\xi}(z')] = \delta(z - z'), \tag{1.2}$$

where we adopt the notation $z = (t, x)$, t being the time coordinate.

2. Setting

The goal of this section is both to fix the notation and the conventions adopted in this work, and to give a succinct overview of the key definitions and results concerning the algebraic approach to interacting quantum fields (AQFT). For more information, see [3,7,10,29]. Subsequently we sketch the key ideas at the heart of an approach to stochastic PDEs inspired by AQFT and analyzed in [5,6,11]. These frameworks will serve as the foundation for the analysis of the stochastic sine-Gordon model.

2.1. Interacting algebraic quantum field theory. Algebraic Quantum Field Theory (AQFT) is a two step approach to quantization, which is tailored to be applicable to as many models as possible, regardless whether they are defined on a Lorentzian or an Euclidean manifold. At first, given a physical system, one needs to construct a suitable $*$-algebra of observables, say $\mathcal{A}$ which encompasses all structural properties, ranging from dynamics to causality or to the canonical commutation or anti-commutation relations. Subsequently, on top of $\mathcal{A}$, one must identify an *algebraic state*, that is a linear, normalized and positive functional $\omega : \mathcal{A} \to \mathbb{C}$. As a consequence of the renown GNS theorem one can recover from the pair $(\mathcal{A}, \omega)$ the standard probabilistic interpretation, proper of quantum theories. In the past decade, especially in connection to interacting quantum field theories, it has become clear that an advantageous way to construct a concrete $*$-algebra $\mathcal{A}$ consists of identifying it as a collection of suitable functionals on the underlying space of smooth field configurations. This procedure allows, on the one hand, to encompass the above mentioned structural properties in terms of a deformation of the pointwise product among functionals, while, on the other hand, it facilitates the possibility of including specific constraints on the existing singular structures, a feature which is of paramount relevance to deal with renormalization in the algebraic setting – see, e.g., [7,29].

We denote by $\mathcal{F}(\mathbb{M})$ the space of complex-valued, continuous, linear functionals over $\mathcal{E}(\mathbb{M})$. It is worth recalling that $\mathcal{F}(\mathbb{M})$ comes with a natural notion of functional derivative, which allows to identify a class of distinguished functionals, namely the polynomial ones.

Definition 2.1. Given $F \in \mathcal{F}(\mathbb{M})$, we call $F^{(k)}$, $k \geq 1$, its k-th order functional derivative, namely $F^{(k)} \in \mathcal{E}'(\underbrace{\mathbb{M} \times \cdots \times \mathbb{M}}_{k}; \mathcal{F}(\mathbb{M}))$ such that

$$F^{(k)}(\eta_1 \otimes \cdots \otimes \eta_k; \eta) := \frac{\partial^k}{\partial s_1 \ldots \partial s_k} F(\eta + s_1\eta_1 + \cdots + s_k\eta_k)\Big|_{s_1=\cdots=s_k=0}, \tag{2.1}$$

for all $\eta, \eta_1, \ldots, \eta_k \in \mathcal{E}(\mathbb{M})$. Accordingly, we define the **directional derivative** along $\varphi \in \mathcal{E}(\mathbb{M})$ as

$$\delta_\varphi : \mathcal{F}(\mathbb{M}) \to \mathcal{F}(\mathbb{M}), \qquad [\delta_\varphi F](\eta) := F^{(1)}(\varphi; \eta).$$

A functional $F \in \mathcal{F}(\mathbb{M})$ is said to be **polynomial**, $F \in \mathcal{F}_{\mathrm{Pol}}(\mathbb{M})$, if there exists $n \in \mathbb{N}_0$ such that $F^{(k)} = 0$ for all $\{k \geq n\}$. In addition we define the (spacetime) **support** of a functional F as

$$\begin{aligned} \mathrm{supp}(F) := \{x \in \mathbb{M} \mid \forall U \in \mathcal{N}_x\ \exists \varphi, \psi \in \mathcal{E}(\mathbb{M}) \text{ with } \mathrm{supp}(\varphi) \subset U, \\ \text{such that } F(\varphi + \psi) \neq F(\varphi)\}, \end{aligned} \tag{2.2}$$

where $\mathcal{N}_x$ denotes the family of all open subsets of $\mathbb{M}$ such that $x \in \mathcal{N}_x$.

In the applications both to interacting quantum field theories and to non-linear stochastic partial differential equations, we will be forced to consider either products among the derivatives of suitable functionals or their composition with specific propagators. Some of these operations are *a priori* ill-defined and one needs to resort to techniques proper of microlocal analysis to overcome these hurdles. In order to keep the length of this work at bay we shall assume that the reader is familiar with the basic concepts of this framework and we refer to [25] for all information or to [11, Appendix B] for a succinct summary of the key ingredients. In view of these considerations, we look for a restricted class of functionals, characterized by a constraint on the singular structure of the functional derivatives.

Definition 2.2. Denoting with $\mathcal{F}(\mathbb{M})$ the space of continuous, complex-valued functionals on $\mathcal{E}(\mathbb{M})$, we define

- the **microcausal functionals** as

$$\mathcal{F}_{\mu c}(\mathbb{M}) := \left\{ F \in \mathcal{F}(\mathbb{M}) \mid F^{(n)} \in \mathcal{E}'(\mathbb{M}^n), \quad \mathrm{WF}(F^{(n)}) \cap \Big[\bigcup_{p\in\mathbb{M}} (\bar{V}_p^+ \cup \bar{V}_p^-) \Big] = \emptyset, \quad n \in \mathbb{N} \right\},$$

  where $F^{(n)}$ is the n-th functional derivative of F as per Definition 2.1, while $\bar{V}_p^+$ and $\bar{V}_p^-$ are, respectively, the sets of future-pointing and past-pointing covectors in $T_p^*\mathbb{M}$;
- the **regular functionals** as

$$\mathcal{F}_{\mathrm{reg}}(\mathbb{M}) := \left\{ F \in \mathcal{F}_{\mu c}(\mathbb{M}) \mid F^{(n)} \in \mathcal{D}(\mathbb{M}^n) \hookrightarrow \mathcal{E}'(\mathbb{M}^n), \quad n \in \mathbb{N} \right\};$$

- the **local functionals** as

$$\mathcal{F}_{\mathrm{loc}}(\mathbb{M}) := \left\{ F \in \mathcal{F}_{\mu c}(\mathbb{M}) \mid F^{(1)} \in \mathcal{E}(\mathbb{M}) \hookrightarrow \mathcal{D}'(\mathbb{M}), \quad \mathrm{supp}(F^{(n)}) \subset \mathrm{Diag}_n \subset \mathbb{M}^n, \quad n \in \mathbb{N} \right\},$$

  where Diag_n denotes the total diagonal of $\mathbb{M}^n$, namely

$$\mathrm{Diag}_n := \{(x, \dots, x) \in \mathbb{M}^n \mid x \in \mathbb{M}\}.$$

Remark 2.3. With reference to Definition 2.1, when we need to consider in addition only polynomial functionals, we shall employ the symbol $\mathcal{F}^p_{\mu c/reg/loc}(\mathbb{M}) \doteq \mathcal{F}_{\mu c/reg/loc}(\mathbb{M}) \cap \mathcal{F}_{\mathrm{Pol}}(\mathbb{M})$.

Example 2.4. In order to introduce functionals which will play a prominent rôle in our construction, let us delve in two simple, yet informative examples. The first one is the so-called **smeared linear field**: Given $f \in \mathcal{D}(\mathbb{M})$, we set

$$\Phi_f : \varphi \mapsto \Phi_f(\varphi) := \int_{\mathbb{M}} d\mu_x\, f(x)\varphi(x),$$

with $d\mu_x$ is the metric induced measure. A direct computation of the first functional derivative, see Eq. (2.1), entails that its integral kernel reads

$$\Phi_f^{(1)}(x, y) = f(x)\delta_{Diag_2}(x, y),$$

where $\delta_{Diag_2} \in \mathcal{D}'(\mathbb{M} \times \mathbb{M})$ acts as

$$\delta_{Diag_2}(h) := \int_{\mathbb{M}} d\mu_x\, h(x, x), \qquad \forall h \in \mathcal{D}(\mathbb{M} \times \mathbb{M}), \tag{2.3}$$

while all higher order derivatives vanish. Hence the smeared linear field is a polynomial local functional, $\Phi_f \in \mathcal{F}^p_{loc}(\mathbb{M})$ for all $f \in \mathcal{D}(\mathbb{M})$. As a second example of local functional we consider the **smeared vertex operator**

$$V_{a,f} : \varphi \mapsto V_{a,f}(\varphi) := \int_{\mathbb{M}} \mathrm{d}\mu_x f(x) e^{ia\varphi(x)}, \quad a \in \mathbb{R}, \text{ and } f \in \mathcal{D}(\mathbb{M}), \tag{2.4}$$

where locality is once more a by-product of Definition 2.1. We observe that Eq. (2.4) does not identify a polynomial functional, but it is of primary relevance since it encodes the information of the interaction term in the Lagrangian of the sine-Gordon model, *cf.* Eq. (1.1),

$$V_g := \frac{V_{a,g} + V_{-a,g}}{2},$$

with $g \in \mathcal{D}(\mathbb{M})$ and $a \in \mathbb{R}_+$.

Starting from Definition 2.2, we can endow $\mathcal{F}_{\mu c}(\mathbb{M})$ with the structure of a commutative $*$-algebra denoted by $\mathcal{A}_{\mu c}(\mathbb{M}) \doteq (\mathcal{F}_{\mu c}(\mathbb{M}), \cdot, *)$ and constituted by the following data:

- a $*$-operation $* : \mathcal{F}_{\mu c}(\mathbb{M}) \to \mathcal{F}_{\mu c}(\mathbb{M})$ such that, for all $F \in \mathcal{F}_{\mu c}(\mathbb{M})$

$$F^*(\varphi) := \overline{F(\varphi)}.$$

- a product $\cdot : \mathcal{F}_{\mu c}(\mathbb{M}) \times \mathcal{F}_{\mu c}(\mathbb{M}) \to \mathcal{F}_{\mu c}(\mathbb{M})$ such that, for all $F, G \in \mathcal{F}_{\mu c}(\mathbb{M})$,

$$F \cdot G = \mathcal{M}(F \otimes G), \tag{2.5}$$

  where $\mathcal{M}$ denotes the pullback on $\mathcal{F}_{\mu c} \otimes \mathcal{F}_{\mu c}$ via the diagonal map

$$\begin{aligned} \iota : \mathcal{E}(\mathbb{M}) &\to \mathcal{E}(\mathbb{M}) \times \mathcal{E}(\mathbb{M}) \\ \iota(\varphi) &:= (\varphi, \varphi). \end{aligned}$$

Observe that we call $\cdot$, as per Eq. (2.5), the pointwise product between functionals since, for all $\varphi \in \mathcal{E}(\mathbb{M})$,

$$(F \cdot G)(\varphi) = F(\varphi)G(\varphi). \tag{2.6}$$

Definition 2.5. Let $\mathcal{A}_{\mathrm{cl}}(\mathbb{M})$ be the algebra of microcausal functionals. We say that a family $\{F_n\}_{n\in\mathbb{N}}$, $F_n \in \mathcal{A}_{\mathrm{cl}}(\mathbb{M})$ converges to a functional $F \in \mathcal{A}_{\mathrm{cl}}(\mathbb{M})$ for $n \to \infty$ if, for all $\ell \in \mathbb{N}$ and for any $\varphi \in \mathcal{E}(\mathbb{M})$ it holds that $F_n^{(\ell)}(\varphi)$ converges to $F^{(\ell)}(\varphi)$ as $n \to \infty$ in the weak $*$-topology of $\mathcal{E}'(\mathbb{M}^\ell)$.

We observe that in the above definition the subscript cl stands for classic since the product is the classical pointwise one.

Up to this point, all algebras that we have considered do not carry any specific information either on an underlying dynamics or on a quantization scheme. In order to encode eventually these data, the algebraic approach calls for considering a deformation of the product $\cdot$ introduced above. This is codified by means of a formal deformation parameter which is denoted by $\hbar$ and by a bidistribution $K \in \mathcal{D}'(\mathbb{M} \times \mathbb{M})$ whose explicit form depends on the case in hand. More precisely one switches from $\mathcal{A}_{\mu c}(\mathbb{M})$ to $(\mathcal{F}_{\mu c}(\mathbb{M}), \star_{\hbar K}, *)$ such that, for all $F, G \in \mathcal{F}_{\mu c}(\mathbb{M})$,

$$F \star_{\hbar K} G = \mathcal{M} \circ e^{D_{\hbar K}}[F \otimes G], \quad D_{\hbar K} := \left\langle \hbar K, \frac{\delta}{\delta \varphi} \otimes \frac{\delta}{\delta \varphi} \right\rangle$$
$$:= \int_{\mathbb{M}^2} \mathrm{d}\mu_x \mathrm{d}\mu_y \hbar K(x, y) \frac{\delta}{\delta \varphi(x)} \otimes \frac{\delta}{\delta \varphi(y)}, \tag{2.7}$$

where $K(x, y)$ is the formal integral kernel of K. Henceforth we shall also refer to this class of products as *exponential products*.

Remark 2.6. Observe that the exponential product in Eq. (2.7) can be conveniently rewritten as

$$F \star_{\hbar K} G = \Gamma_{\hbar K}[\Gamma^{-1}_{\hbar K}(F)\Gamma^{-1}_{\hbar K}(G)], \tag{2.8}$$

where we introduced the deformation map $\Gamma_{\hbar K} : \mathcal{F}_{\mu c}(\mathbb{M}) \to \mathcal{F}_{\mu c}(\mathbb{M})$ defined as

$$\Gamma_{\hbar K} = e^{\frac{1}{2}\mathcal{D}_{\hbar K}}, \qquad \mathcal{D}_{\hbar K} = \left\langle \hbar K, \frac{\delta^2}{\delta \varphi^2} \right\rangle = \int_{\mathbb{M}^2} \mathrm{d}\mu_x \mathrm{d}\mu_y \hbar K(x, y) \frac{\delta^2}{\delta \varphi(x) \delta \varphi(y)}. \tag{2.9}$$

Observe that Eq. (2.7) is a priori only a formal expression on account of two potential, distinct issues:

1. since $K \in \mathcal{D}'(\mathbb{M} \times \mathbb{M})$, the action of $D_{\hbar K}$ and of its powers might be ill-defined. This is subordinated to the singular structure both of K and of the functional derivatives of the underlying functionals. As we shall see, in our investigation, being the background of low dimension, this will not be an issue. More in general, this is handled by resorting, if necessary, to a renormalization procedure.
2. the action $\mathcal{M} \circ e^{D_{\hbar K}}$ yields a priori only a formal power series in $\hbar$ unless one proves convergence with respect to a suitable topology or considers polynomial functionals which entail that only a finite number of non vanishing contributions exist.

Remark 2.7. In the preceding discussion, particularly in defining $\mathcal{F}(\mathbb{M})$ and its distinguished subspaces as per Definition 2.2, we have only considered kinematic configurations $\varphi \in \mathcal{E}(\mathbb{M})$ and no information on an underlying dynamics has been assumed. Yet, in many concrete models, among which the sine-Gordon, one assumes that the field abides by suitable equations of motions of the form

$$P\varphi + V^{(1)}[\varphi] = 0, \tag{2.10}$$

where $V : \mathbb{R} \to \mathbb{R}$ is a non linear potential while P is a normally hyperbolic operator, see [1]. In the following and in view of the application to the Lorentzian sine-Grordon model, we shall always assume that an underlying dynamics of the form of Eq. (2.10) has been chosen. For definiteness a reader can think of P as being the Klein-Gordon operator, namely $P = \Box - m^2$ where $\Box$ is the d'Alembert wave operator built out of the underlying metric, although such assumption is not strictly necessary as far as the content of this section is concerned.

This entails that, being $\mathbb{M}$ globally hyperbolic, there exists unique advanced and retarded fundamental solutions $\Delta^{A/R} : \mathcal{D}(\mathbb{M}) \to \mathcal{E}(\mathbb{M})$ such that $\mathrm{supp}(\Delta^{A/R}(f)) \subseteq J^{\mp}(\mathrm{supp}(f))$ for all $f \in \mathcal{D}(\mathbb{M})$.

As discussed, e.g. in [2], these propagators are the building block for implementing in a covariant way the canonical commutation relations (CCRs). More precisely, with reference to Eq. (2.7), we can make the identification $K = \frac{i}{2}\Delta$, where $\Delta := \Delta^R - \Delta^A$ again is the *causal propagator*. As a matter of fact, one can observe that, considering the smeared linear fields as per Example 2.4, it holds that $\forall f_1, f_2 \in \mathcal{D}(\mathbb{M})$

$$\begin{aligned}[\Phi_{f_1}, \Phi_{f_2}]_{\star_{i2^{-1}\hbar\Delta}} &= \Phi_{f_1} \star_{i2^{-1}\hbar\Delta} \Phi_{f_2} - \Phi_{f_2} \star_{i2^{-1}\hbar\Delta} \Phi_{f_1} \\ &= i\hbar\{\Phi_{f_1}, \Phi_{f_2}\} = i\hbar\langle f_1, \Delta f_2\rangle, . \end{aligned} \tag{2.11}$$

where we have also introduced the symbol $\{,\}$ denoting the classical **Poisson brackets**.

Observe that, if we stick with the choice of $K = \frac{i}{2}\Delta$ as deformation kernel, the product $\star_{i2^{-1}\hbar\Delta}$ as per Eq. (2.7) is well-defined provided that we consider functionals lying in $\mathcal{F}^{p}_{\mathrm{reg}}(\mathbb{M})$ as per Remark 2.3. Yet, such class is too small to encompass all relevant observables proper of a quantum field theory and, therefore, one needs to account for more singular functionals. Observe that [28]

$$\mathrm{WF}(\Delta) = \{(x, k_x, y, -k_y) \in T^*(\mathbb{M} \times \mathbb{M}) \setminus \{0\} \mid (x, k_x) \sim (y, k_y)\}, \tag{2.12}$$

where $\sim$ entails that the point x and y are connected by a lightlike geodesic γ such that k_x is coparallel to γ at x while k_y is obtained by means of a parallel transport of k_x along γ. This implies that, in general, given $F, G \in \mathcal{F}_{\mu c}(\mathbb{M})$, $F \star_{i2^{-1}\hbar\Delta} G$ is ill-defined at a microlocal level. This hurdle can be circumvented by means of a *normal ordering procedure* [10,23]. Restricting to regular functionals, it amounts to considering, at the level of formal power series in $\hbar$,

$$: F :_\omega := \Gamma^{-1}_{\hbar\omega} F \in \mathcal{F}_{\mathrm{reg}}[\![\hbar]\!], \qquad F \in \mathcal{F}_{\mathrm{reg}}, \tag{2.13}$$

where $\Gamma_{\hbar\omega}$ is defined as in Eq. (2.9). Here $\omega \in \mathcal{D}'(\mathbb{M} \times \mathbb{M})$ denotes the so-called *Hadamard parametrix*, characterized by $(P \otimes \mathbb{I})\omega = (\mathbb{I} \otimes P)\omega \in C^\infty(\mathbb{M} \times \mathbb{M})$ and whose anti-symmetric part is $i2^{-1}\Delta$. Moreover, it satisfies the *microlocal spectrum condition*, namely

$$\mathrm{WF}(\omega) = \{(x, y; k_x, k_y) \in T^*\mathbb{M}^2 \setminus \{0\} \mid (x, k_x) \sim (y, k_y), k_x \triangleright 0\}, \tag{2.14}$$

where $\sim$ has the same meaning as in Eq. (2.12), while $k_x \triangleright 0$ entails that the covector k_x is future pointing. This condition on the wavefront set of the Hadamard parametric entails that Eq. (2.13) is well-defined also for microcausal functionals $\mathcal{F}_{\mu c}(\mathbb{M})$ as in Definition 2.2. This feature is thoroughly examined in [16] and therefore we omit entering into the technical details. To conclude this succinct overview, we highlight the following two comments:

1. In view of the preceding comment we shall denoted by $\mathcal{F}_{\mu c}[\![\hbar]\!]$ the normal ordered space of microcausal functionals and the ensuing algebra as the triple $(\mathcal{F}_{\mu c}[\![\hbar]\!], \star_{\hbar\omega}, *)$
2. most notably the product $\star_{\hbar\omega}$ does not affect the canonical commutation relations, since, in view of Example 2.4, for any $f_1, f_2 \in \mathcal{D}(\mathbb{M})$, it holds that

$$[\Phi_{f_1}, \Phi_{f_2}]_{\star_{\hbar\omega}} = i\hbar\langle f_1, \Delta f_2\rangle.$$

2.1.1. S-matrix and Bogoliubov map Keeping in mind Eq. (2.10), the algebras of functionals defined in the previous section do not allow to account for the non linear contribution encoded by $V^{(1)}[\varphi]$. To encompass this information a key structure in the perturbative approach to quantum field theory is the S-matrix and the associated Bogoliubov map. In order to keep the length of this paper at bay, we shall not discuss the whole framework in detail, leaving an interested reader to [7,14,29] for a thorough analysis. We shall limit ourselves to introducing all the relevant structures necessary for our analysis to be self-consistent.

The starting point is the *time ordered product* which is here defined in terms of a *time ordering map* $\mathcal{T}$ acting on the space of multi-local functionals, which are tensor products of elements lying in $\mathcal{F}_{\mathrm{loc}}(\mathbb{M})$ as in Definition 2.2. More precisely $\mathcal{T}$ is constructed out of a family of multi-linear maps

$$\mathcal{T}_n : \mathcal{F}_{\mathrm{loc}}^{\otimes n}(\mathbb{M}) \to \mathcal{F}_{\mu c}(\mathbb{M}),$$

which satisfy the constraints $\mathcal{T}_0 = 1$ and $\mathcal{T}_1 = \mathrm{id}$. The link between $\mathcal{T}$ and $\mathcal{T}_n$ is codified by the identity

$$\mathcal{T}\left(\prod_{j=1}^{n} F_j\right) = \mathcal{T}_n\left(\bigotimes_{j=1}^{n} F_j\right). \tag{2.15}$$

In addition, one requires the maps $\mathcal{T}_n$ to be such that $\mathcal{T}$ is symmetric and to satisfy a *causal factorization property*. This can be stated as follows: consider $\{F_i\}_{i=1,\ldots,n}, \{G_j\}_{j=1,\ldots,m} \subset \mathcal{F}_{\mathrm{loc}}(\mathbb{M})$ two arbitrary families of local functionals such that $F_i \gtrsim G_j$ for any i, j, where the symbol $\gtrsim$ entails that $\mathrm{supp}(F_i) \cap J^-(\mathrm{supp}(G_j)) = \emptyset$, where the support of a functional is as per Eq. (2.2). It descends that

$$\mathcal{T}\left(\bigotimes_i F_i \bigotimes_j G_j\right) = \mathcal{T}\left(\bigotimes_i F_i\right) \star \mathcal{T}\left(\bigotimes_j G_j\right), \tag{2.16}$$

where $\star$ is defined in Eq. (2.7). We stress that, whenever the hypotheses at the heart of Eq. (2.16) are not met, one needs to devise a suitable extension criterion for $\mathcal{T}$, which requires in turn a renormalization procedure. In this work we abide by the Epstein-Glaser inductive procedure [10,15,24], although, as highlighted in [3,4], when one considers two-dimensional self-interacting, scalar field theories, renormalization is unnecessary if one restricts the attention to local functionals not containing derivatives of the field. As a consequence, for these classes of local functionals the causal factorization property suffices to fully determine the map $\mathcal{T}$.

As discussed in [3,29], if we work with the algebra $(\mathcal{F}_{\mu c}[\![\hbar]\!], \star_{\hbar\omega}, *)$, an explicit realization of the time-ordering map is completely characterized by the identity

$$\mathcal{T}^{\hbar\omega_F}(F_1 \otimes \cdots \otimes F_n) := F_1 \star_{\hbar\Delta_F} \ldots \star_{\hbar\Delta_F} F_n := \mathcal{M} \circ e^{\sum_{\ell<j}^{n} D^{\ell j}_{\hbar\Delta_F}}(F_1 \otimes \cdots \otimes F_n), \tag{2.17}$$

where $F_i \in \mathcal{F}_{\mu c}[\![\hbar]\!]$, for all $i = 1, \ldots, n$ while $\Delta_F \in \mathcal{D}'(\mathbb{M} \times \mathbb{M})$ denotes the *Feynman parametrix*, linked to $\omega \in \mathcal{D}'(\mathbb{M} \times \mathbb{M})$ and to the advanced propagator $\Delta^A \in \mathcal{D}'(\mathbb{M} \times \mathbb{M})$ associated to P as per Remark 2.7 via the defining relation

$$\Delta_F = \omega + i\Delta^A. \tag{2.18}$$

In addition in Eq. (2.17) $\mathcal{M}$ is defined as in Eq. (2.5) while

$$D^{\ell j}_{\hbar\omega_F} := \left\langle \hbar\omega_F, \frac{\delta^2}{\delta\varphi_\ell \delta\varphi_j} \right\rangle,$$

which is manifestly symmetric under exchange of i and ℓ. We have all the data necessary to define two key ingredients in the perturbative investigation of a self-interacting, scalar field theory whose dynamics is ruled by Eq. (2.10):

1. the S-matrix as

$$S(\lambda V) := \exp_{\star_{\hbar\Delta_F}}\left(\frac{i}{\hbar}\lambda V\right) := \sum_{n\geqslant 0} \frac{1}{n!}\left(\frac{i\lambda}{\hbar}\right)^n \underbrace{V \star_{\hbar\Delta_F} \dots \star_{\hbar\Delta_F} V}_{n}$$
$$= \sum_{n\geqslant 0} \frac{1}{n!}\left(\frac{i\lambda}{\hbar}\right)^n \mathcal{T}^{\hbar\Delta_F}(V^{\otimes n}), \tag{2.19}$$

   where here with $V \equiv V[\varphi]$ we denote the interaction potential lying in $\mathcal{F}_{\text{loc}}(\mathbb{M})$. *A priori*, the S-matrix is a formal power series in the coupling constant λ and a Laurent series in $\hbar$.
2. The interacting classical field, which is a perturbative solution of Eq. (2.10) with vanishing initial conditions, written as a formal power series in λ with coefficients lying in $\mathcal{F}_{\text{loc}}[\mathbb{M}]$:

$$r_{\lambda V_g}(\varphi)(x)$$
$$= \sum_{n\geqslant 0} \lambda^n \int_{t_1\leqslant\dots\leqslant t_n\leqslant t} \mathrm{d}\mu_{x_1}\dots\mathrm{d}\mu_{x_n} g(x_1)\dots g(x_n)\{V(x_1), \{V(x_2), \dots \{V(x_n), \varphi(x)\}\dots\}\}, \tag{2.20}$$

   where $d\mu_x$ is the metric induced measure, the curly brackets are the classical Poisson brackets as per Remark 2.7, while $g \in \mathcal{D}(\mathbb{M})$ is a cut-off function which is introduced to avoid infrared divergences. In addition, the dependence of the interaction vertices from the field configuration has been omitted to simplify the notation. Observe that $r_{\lambda V_g}$ is also referred to as classical Möller map [22,29].

Example 2.8. The formal expression in Eq. (2.20) is analogous to the standard notion of perturbative solution of a nonlinear partial differential equation adopted, *e.g.*, in [11]. As an example, consider

$$V_g(\varphi) = \frac{1}{4}\int_{\mathbb{M}} \mathrm{d}\mu_x\, g(x)\varphi^4(x).$$

The perturbative solution is defined in terms of the following power series in the coupling constant λ

$$\Phi[\![\lambda]\!](\varphi)(x) = \sum_{n\geqslant 0} \lambda^n F_n(\varphi)(x),$$

where

$$F_0(\varphi)(x) = \Phi(\varphi)(x) = \varphi(x), \quad F_n(\varphi)(x) := -\sum_{j_1+j_2+j_3=n-1} \Delta^R * [F_{j_1}(\varphi)F_{j_2}(\varphi)F_{j_3}(\varphi)](x),$$

where $*$ here denotes the convolution while Δ^R denotes the retarded fundamental solution associated to P as in Remark 2.7. By direct inspection, it can be seen that, order by order in λ, it holds that

$$\Phi[\![\lambda]\!](\varphi)(x) = r_{\lambda V_g}(\varphi)(x).$$

The information carried by the S-matrix and by the classical Møller map can be brought together extending Eq. (2.20) to the whole algebra of observables of the theory. More precisely, given $F \in \mathcal{F}_{\mathrm{loc}}^{\otimes m}(\mathbb{M})$ and, adopting the notation $Y = (y_1, \ldots, y_m) \in \mathbb{M}^m$, the associated interacting classical observable can be written as

$$r_{\lambda V_g}(F)(Y) = \sum_{n \geqslant 0} \lambda^n \int_{t_1 \leqslant \cdots \leqslant t_n \leqslant t} \mathrm{d}\mu_{x_1} \ldots \mathrm{d}\mu_{x_n} g(x_1) \ldots g(x_n) \{V(x_1), \{V(x_2), \ldots \{V(x_n), F(Y)\} \ldots\}\}. \tag{2.21}$$

Having established the algebraic formulation of a classical interacting field theory, an analogous procedure carries over to the quantum scenario, with the notable difference that one needs to work with $(\mathcal{F}_{\mu c}[\![\hbar]\!], \star_{\hbar H}, *)$. To this end we introduce the *Bogoliubov map* $R_{\lambda V}$ associated with the interaction $V \in \mathcal{F}_{loc}(\mathbb{M})$, which maps any observable of the free field theory whose dynamics is ruled by P into its interacting quantum counterpart [29]. The action on any $F \in \mathcal{F}_{\mathrm{loc}}^{\otimes m}(\mathbb{M})$ reads as

$$\begin{aligned} R_{\lambda V}(F) &:= -i\hbar \frac{d}{d\alpha} S(\lambda V)^{\star_{\hbar\omega}-1} \star_{\hbar\omega} S(\lambda V + \alpha F)|_{\alpha=0} \\ &= S(\lambda V)^{\star_{\hbar\omega}-1} \star_{\hbar\omega} (S(\lambda V) \star_{\omega} F), \end{aligned} \tag{2.22}$$

where $S(\lambda V)^{\star_{\hbar\omega}-1}$ denotes the inverse (in the sense of formal power series) of $S(\lambda V)$ with respect to the product $\star_{\hbar\omega}$.

Remark 2.9. For later convenience we need to give an explicit expression of $S(\lambda V)^{\star_{\hbar\omega}-1}$. This requires the anti-Feynman parametrix

$$\Delta_{\mathrm{AF}} = \omega - i\Delta^R, \tag{2.23}$$

where Δ^R is the retarded fundamental solution associated to P as per Remark 2.7. Consequently it holds that

$$S(\lambda V)^{\star_{\hbar\omega}-1} := \sum_{n \geqslant 0} \frac{1}{n!} \left(-\frac{i\lambda}{\hbar}\right)^n \underbrace{V \star_{\hbar\Delta_{\mathrm{AF}}} \cdots \star_{\hbar\Delta_{\mathrm{AF}}} V}_{n} = \sum_{n \geqslant 0} \frac{1}{n!} \left(-\frac{i\lambda}{\hbar}\right)^n \mathcal{T}^{\hbar\Delta_{\mathrm{AF}}}(V^{\otimes n}).$$

As a consequence, Eq. (2.22) can be written as a formal power series in the coupling constant λ:

$$R_{\lambda V}(F) = \sum_{n \geqslant 0} \frac{\lambda^n}{n!} R_{n,m}(V^{\otimes n}, F),$$

where $R_{n,m}(V^{\otimes n}, F)$ are the so-called *retarded products*

$$R_{n,m}(V^{\otimes n}, F) = \left(\frac{i}{\hbar}\right)^n \sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^\ell \mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}}(V \otimes \cdots \otimes V) \star_{\hbar\omega} \mathcal{T}_{n-\ell,m}^{\hbar\Delta_F}(V \otimes \ldots \otimes V \otimes F).$$

The notation $\mathcal{T}^{\hbar\Delta_F}_{n-\ell,m}(V\otimes\cdots\otimes V\otimes F)$ keeps track of the fact that its argument contain $n-\ell$ copies of the interaction V and that F is a multi-local functional of order $m\in\mathbb{N}$. Hence, on account of Eq. (2.17)

$$\mathcal{T}^{\hbar\Delta_F}_{n-\ell,m}(V\otimes\cdots\otimes V\otimes F)=V\star_{\hbar\Delta_F}\cdots\star_{\hbar\Delta_F}V\star_{\hbar\Delta_F}F.$$

Remark 2.10. An important feature of the retarded products, which underpins their name, concerns their support properties. More precisely, $R_{n,m}(V^{\otimes n},F)$ vanishes if at least one of the first n arguments is not supported in the past light cone of one of the last m ones [14,15].

2.2. Microlocal approach to SPDEs. In this section we give a succinct overview of a recent approach to the construction at a perturbative level of both the expectation value of solutions and the correlation functions of a class of nonlinear stochastic partial differential equations (SPDEs), first introduced in [11] for scalar theories and later extended in [6] to the analysis of the nonlinear stochastic Schrödinger equation and in [5] to that of spinors. This framework can be seen as a transliteration to the analysis of SPDEs of the algebraic approach to quantum field theory outlined in Sect. 2.1 and, as such, it has the main advantage of allowing to encode all renormalization ambiguities intrinsically without resorting to any specific ϵ-regularization scheme. The reason lies in the possibility of adapting to the case in hand the microlocal approach to Epstein-Glaser renormalization, see e.g., [10,24,26].

It is worth emphasizing that, in [5,6,11], the focus has always been on the analysis of elliptic or parabolic SPDEs and therefore we feel worth revisiting the content of these works when the linear part of the dynamics is ruled by an hyperbolic partial differential operator. For this reason, in this section, we start by considering a simple toy model, namely a linear, scalar, SPDE on $\mathbb{R}^d$ endowed with the flat Minkowski metric of signature $(-,\underbrace{+,\ldots,+}_{d-1})$:

$$(\Box-m^2)\hat{\psi}=\chi\hat{\xi}, \tag{2.24}$$

with vanishing initial condition. Here $\Box$ is the d'Alembert wave operator, while $m^2>0$ is a fixed parameter, which can be interpreted in concrete models as playing the rôle of a mass term. Furthermore $\hat{\xi}$ denotes a space-time white noise as per Eq. (1.2). The last ingredient $\chi:\mathbb{M}\to\mathbb{R}$ is the smooth cutoff function

$$\chi(t,x)=\tilde{\chi}(t)1(x),$$

where $\tilde{\chi}:\mathbb{R}\to\mathbb{R}$ is a positive smooth function such that there exists $T\in\mathbb{R}$ for which

$$\tilde{\chi}(t)=\begin{cases}0 & \text{if} \quad t<T,\\ 1 & \text{if} \quad t\geqslant T+1.\end{cases} \tag{2.25}$$

Observe that $\Box-m^2$ plays the rôle of the operator P in Remark 2.7 and, in particular, we can associate to it unique advanced and retarded fundamental solutions, still denoted by Δ^A and Δ^R respectively. The latter allows to solve Eq. (2.24) forward in time as, with vanishing initial condition for simplicity,

$$\hat{\phi}=\Delta^R(\chi\hat{\xi}).$$

We can infer that the solution $\hat{\phi}$ is a Gaussian random distribution with vanishing mean, while, given $f, f' \in \mathcal{D}(\mathbb{R}^d)$, the covariance reads

$$\begin{aligned} Q(f,f') &:= \int_{\mathbb{R}^d\times\mathbb{R}^d} d\mu_z d\mu_{z'} \mathbb{E}[\hat{\phi}(z)\hat{\phi}(z')] f(z) f(z') \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} d\mu_z d\mu_{z'} \int_{\mathbb{R}^d} d\mu_{z_1} \chi^2(t_1) \Delta^R(z-z_1)\Delta^A(z_1-z') f(z) f(z') \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} d\mu_z d\mu_{z'} \int_{\mathbb{R}^d} d\mu_{z_1} \chi^2(t_1) \Delta^R(z-z_1)\Delta^R(z'-z_1) f(z) f(z'), \end{aligned} \tag{2.26}$$

where, in the last equality, we used the following structural properties of the advanced and retarded fundamental solutions associated to a second order, hyperbolic, partial differential operator on a globally hyperbolic manifold $\mathbb{M}$, $\langle f, \Delta^R f'\rangle = \langle \Delta^A f, f'\rangle$ for all $f, f' \in \mathcal{D}(\mathbb{M})$, see [2].

Observe that Eq. (2.26) entails that the covariance $Q \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ can be written as

$$Q := \Delta^R \circ_\chi \Delta^A, \tag{2.27}$$

where the symbol $\circ$ denotes the composition of distributions, while the subscript χ keeps track of the cut-off function. A direct application of [25, Thm. 8.2.14], combined with a propagation of singularity argument entails that

$$\mathrm{WF}(Q) = \{(z, k_z, z', k_{z'}) \in T^*\left(\mathbb{R}^d \times \mathbb{R}^d\right) \setminus \{0\} \mid (z, k_z) \sim (z', -k_{z'})\} \cup \mathrm{WF}(\delta_2), \tag{2.28}$$

where $\sim$ means that z and z' are connected by a lightlike geodesic γ, $-k_{z'}$ is the parallel transport of k_z along γ, while k_z is cotangent to γ. In addition δ_2 is the Dirac delta along the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$ and, for the sake of completeness, we recall that

$$\mathrm{WF}(\delta_2) = \{(z, k_z, z, -k_z) \in T^*(\mathbb{R}^d \times \mathbb{R}^d) \setminus \{0\}\}.$$

In addition the covariance Q enjoys the property of being positive, a feature which will be used extensively in the following sections.

Lemma 2.11. *The bi-distribution $Q \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ as in Eq. (2.27) is positive, namely, for all $f \in \mathcal{D}(\mathbb{R}^d)$,*

$$Q(f, f) \geqslant 0.$$

Proof. Working at the level of integral kernels, it holds that

$$\begin{aligned} Q(f,f) &= \int_{\mathbb{R}^{3d}} d\mu_{z_1} d\mu_{z_2} d\mu_{z_3} \chi^2(z_2)\Delta^R(z_1-z_2)\Delta^A(z_2-z_3) f(z_1) f(z_3) \\ &= \int_{\mathbb{R}^d} d\mu_{z_2} \chi^2(z_2) \left(\int_{\mathbb{R}^d} d\mu_{z_1} \Delta^R(z_1-z_2) f(z_1)\right)\left(\int_{\mathbb{R}^d} d\mu_{z_3} \Delta^A(z_2-z_3) f(z_3)\right) \\ &= \int_{\mathbb{R}^d} d\mu_{z_2} \chi^2(z_2) \left(\int_{\mathbb{R}^d} d\mu_{z_1} \Delta^A(z_2-z_1) f(z_1)\right)\left(\int_{\mathbb{R}^d} d\mu_{z_3} \Delta^A(z_2-z_3) f(z_3)\right) \\ &= \int_{\mathbb{R}^d} d\mu_{z_2} \chi^2(z_2)(\Delta^A f)(z_2)(\Delta^A f)(z_2) \\ &= (\chi(\Delta^A f), \chi(\Delta^A f))_{L^2(\mathbb{R}^d)} \geq 0, \end{aligned}$$

where, in the third line, we used the identity $\Delta^R(z_1 - z_2) = \Delta^A(z_2 - z_1)$. □

Equation (2.26) gives a complete control on the solution theory of Eq. (2.24) and, in the following, we reformulate its content using the rationale at the heart of the approach to SPDEs inspired by algebraic quantum field theory, adapting to the case in hand the procedure of [11]. More precisely we define the *functional-valued distribution* $F : \mathcal{E}(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d) \to \mathbb{C}$ on the space of smooth configurations

$$\begin{aligned}\mathcal{E}(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d) \ni (\varphi, f, f') \mapsto F[\varphi](f, f')\\ = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{d}\mu_z \mathrm{d}\mu_{z'} \varphi(z)\varphi(z') f(z) f(z') \in \mathbb{C}.\end{aligned} \tag{2.29}$$

Observe that, on the one hand the functional F, despite being regular, can also be interpreted as an element lying in the space of microcausal functional as per Definition 2.2 which is endowed with an algebra structure in terms of the pointwise product as per Eq. (2.6). On the other hand, this rationale allows us to follow Eq. (2.7) as well as Remark 2.6 to introduce a deformed product encompassing the information carried by the underlying white noise $\hat{\xi}$ by means of the map

$$\Gamma_Q = e^{\frac{1}{2}\mathcal{D}_Q}, \qquad \mathcal{D}_Q = \left\langle Q, \frac{\delta^2}{\delta\varphi^2} \right\rangle = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{d}\mu_z \mathrm{d}\mu_{z'} Q(z, z') \frac{\delta^2}{\delta\varphi(z)\delta\varphi(z')}. \tag{2.30}$$

Applying Eqs. (2.30)–(2.29) yields

$$\begin{aligned}(\Gamma_Q F)[\varphi](f, f') = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{d}\mu_z \mathrm{d}\mu_{z'} \varphi(z)\varphi(z') f(z) f(z')\\ + \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{d}\mu_z \mathrm{d}\mu_{z'} Q(z, z') f(z) f(z'),\end{aligned} \tag{2.31}$$

where $Q(z, z')$ denotes the integral kernel of $Q \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ introduced in Eq. (2.26). As one can infer by direct inspection, the action of Γ_Q is that of encoding the information of the correlation function on the underlying stochastic process. Yet, a last bit of information is missing, namely the expectation value $\mathbb{E}$ corresponds to evaluating Eq. (2.31) at the configuration $\varphi = 0$. In other words

$$\begin{aligned}\mathrm{ev}_0(\Gamma_Q F)(f, f') := (\Gamma_Q F)[0](f, f') &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{d}\mu_z \mathrm{d}\mu_{z'} Q(z, z') f(z) f(z')\\ &= Q(f, f').\end{aligned}$$

Observe that the procedure, just highlighted, strongly enjoys from F being regular and, if one tries to apply it to a local, non-linear polynomial functional, such as

$$\Phi^2[\varphi](f) = \int_{\mathbb{R}^d} \mathrm{d}\mu_z \varphi^2(z) f(z),$$

it does not carry over slavishly. A formal application of Eq. (2.30) yields

$$(\Gamma_Q \Phi^2)[\varphi](f) = \Phi^2[\varphi](f) + \int_{\mathbb{R}^d} \mathrm{d}\mu_z Q(z, z) f(z),$$

which is a priori ill-defined since, on account of Eq. (2.26), this would be equivalent to testing Q against a distribution of the form $f(z)\delta(z - z')$. This yields

$$\int_{\mathbb{R}^d} \mathrm{d}\mu_z Q(z, z) f(z) = \int_{\mathbb{R}^d} \mathrm{d}\mu_z \int_{\mathbb{R}^d} \mathrm{d}\mu_{z'} \chi^2(t')[\Delta^R(z - z')]^2 f(z) = (\chi\Delta^R)^2(f \otimes 1). \tag{2.32}$$

Yet the last expression in Eq. (2.32) encompasses the square of $\chi\Delta^R$ which is not a well-defined distribution due to the singular structure of Δ^R. Bypassing this hurdle requires a renormalization procedure and this feature cannot be ascribed to the hyperbolic nature of the problem in hand, since it occurs also when analyzing elliptic or parabolic models, see e.g. [5,6,11].

Yet it is important to stress that, in lower dimensional scenarios, typically when $d \leq 2$, the singularities encoded in the fundamental solutions of the Klein-Gordon operator are rather mild, being of logarithmic type, and, in this case, a renormalization procedure is not necessary. This is exactly the scenario that we shall consider when working with the sine-Gordon model.

2.2.1. On the covariance Q in two-dimensional Minkowski spacetime In the following we shall investigate some notable structural properties of the bi-distribution Q in the case when the spacetime dimension is $d = 2$. It is also convenient to consider in this scenario the massless d'Alembert wave operator $\Box$ and, repeating in this case the analysis of Sect. 2.2, we denote the counterpart of Q as $Q_0 = \Delta_0^R \circ_\chi \Delta_0^A$, $\Delta_0^{R/A}$ being the advanced and retarded fundamental solutions associated to $\Box$. Denoting with $z = (t, x)$ an arbitrary point of $\mathbb{R}^2$, the integral kernel of Δ_0^R reads [4]

$$\Delta_0^R(z) = -\frac{1}{2}\theta(t - |x|) = -\frac{1}{2}\theta(t - x)\theta(t + x), \tag{2.33}$$

where θ denotes the Heaviside step function. The counterpart in the massive case reads instead [3]

$$\Delta^R(z) = \Delta_0^R(z) + \left(1 - I_0\left(m\sqrt{z^2}\right)\right)\Delta_0^R(z) = \left(2 - I_0\left(m\sqrt{z^2}\right)\right)\Delta_0^R(z),$$

where I_0 is the modified Bessel function of first kind, while $z^2 = -t^2 + x^2$ is the Minkowskian square. Since $\mathrm{supp}(\Delta_0^R(z)) \subseteq J^+(0)$, we consider only the region $-t^2 + x^2 \leqslant 0$ and, thereon, $I_0\left(m\sqrt{z^2}\right) = I_0\left(im\sqrt{|z^2|}\right)$. Since $I_0\left(m\sqrt{z^2}\right)$ is a real, smooth, damped oscillating function in z with $\sup_z \left|I_0\left(m\sqrt{z^2}\right)\right| = 1$, it holds that

$$\begin{aligned} Q(z, z') &= \int_{\mathbb{R}^2} d\mu_{\tilde{z}} \chi^2(\tilde{z}) \Delta^R(z - \tilde{z}) \Delta^R(z' - \tilde{z}) \\ &= \int_{\mathbb{R}^2} d\mu_{\tilde{z}} \chi^2(\tilde{z}) \Delta_0^R(z - \tilde{z}) \Delta_0^R(z' - \tilde{z}) \left(2 - I_0\left(im\sqrt{|(z - \tilde{z})^2|}\right)\right) \\ &\quad \times \left(2 - I_0\left(im\sqrt{|(z' - \tilde{z})^2|}\right)\right). \end{aligned} \tag{2.34}$$

Since $\chi^2(\tilde{z})$ has past-compact support according to Eq. (2.25) and since Eq. (2.33) entails that

$$\Delta_0^R(z - \tilde{z}) \Delta_0^R(z' - \tilde{z}) = 1_{\{J^-(z) \cap J^-(z')\}}(\tilde{z}),$$

where $1_{\{J^-(z) \cap J^-(z')\}}(\tilde{z})$ denotes the characteristic function on the subset $J^-(z) \cap J^-(z') \subset \mathbb{R}^2$, we can conclude that the integral in Eq. (2.34) is convergent. In addition, $Q(z, z')$ is a continuous and positive function, which is translation invariant along the space direction. These data can be summarized in the following lemma.

Lemma 2.12. *On the two-dimensional Minkowski spacetime the covariance $Q \in \mathcal{D}'(\mathbb{R}^2 \times \mathbb{R}^2)$ introduced in Eq. (2.27) is positive and in turn it is generated by a continuous and positive function $Q(z, z')$. The coincidence limit $Q(z, z)$ is a smooth positive function, namely $Q(z, z) \in \mathcal{E}(\mathbb{R}^2)$ and $Q(z, z) \geqslant 0$ for all $z \in \mathbb{R}^2$.*

Proof. We only discuss the smoothness with respect to z of the coincidence limit $Q(z,z)$. We observe that, on account of Eq. (2.34), we can write at the level of integral kernel

$$Q(z,z) = \int_{\mathbb{R}^2} \mathrm{d}\mu_{\tilde{z}} \chi^2(\tilde{z})(\Delta^R(z-\tilde{z}))^2.$$

On account of the above discussion and of the explicit form of the retarded propagator in two spacetime dimensions, the distribution generated by $(\Delta^R)^2$ is well defined even though not using a microlocal argument. The smoothness of $Q(z,z)$ follows from the form of the wave-front set of Δ^R together with [25, Thm. 8.2.12] since the integration with respect to the variable $\tilde{z}$ plays the rôle of a composition with the function identically equal to 1. □

2.3. The algebra of functionals for the sine-Gordon model. In Sect. 2.1 we have emphasized how the analysis of an interacting field theory requires in the language of algebraic quantum field theory the identification of a distinguished algebra of functionals and in Sect. 2.2 we have emphasized how the same structures come into play when studying a nonlinear stochastic partial differential equation along the lines of [11]. Hence, in the following we introduce the set of functionals which is necessary to investigate the sine-Gordon model as per Sect. 1.1.

The rationale that we follow is strongly inspired by to the one discussed in [3, Sec. 1.4] and, for this very reason, we shall refer to this reference for the proof of many statements, limiting ourselves to highlighting the main differences with our setting.

Definition 2.13. We denote with $\mathcal{F}^V(\mathbb{R}^2)$ the vector space of functionals generated by elements of the form

$$F_{a,n,m}(\zeta) = \int_{\mathbb{M}^{n+m}} \mathrm{d}\mu_{X,Y} e^{i\sum_{j=1}^n a_j\varphi(x_j)} e^{-i\sum_{1\leqslant \ell<j\leqslant n} a_\ell a_j (Q(x_\ell,x_j)+\hbar H_0(x_\ell,x_j))} \zeta(X,Y) \prod_{k=1}^m \varphi(y_k),$$

where $\mathrm{d}\mu_{X,Y} = \mathrm{d}\mu_{x_1}\ldots\mathrm{d}\mu_{y_m}$, $X=(x_1,\ldots,x_n)$, $Y=(y_1,\ldots,y_m)$, $a\subset(-\eta,\eta)^n$ is a multi-index with $\eta < 4\pi\hbar^{-1}$. Furthermore ζ is a distribution generated by bounded and compactly supported function such that

$$\mathrm{WF}(\zeta) \subset W_{m+n} := \left\{ (x_1,\ldots,x_{n+m};k_1,\ldots,k_{n+m}) \in T^*\mathbb{R}^{d(n+m)} \setminus \{0\} \mid \sum_{\ell=1}^{n+m} k_\ell = 0 \right\}. \quad (2.35)$$

Observe that, in Eq. (2.35), the condition

$$\sum_{\ell=1}^{n+m} k_\ell = 0,$$

surmises that all covectors have been parallel transported at a common base point.

Remark 2.14. For future convenience, it is worth highlighting that a comparison between Eqs. (2.35) and (2.28) entails that

$$\mathrm{WF}(Q) \subset W_2. \quad (2.36)$$

Proposition 2.15. *Denoting with $\mathcal{F}_{\mu c}(\mathbb{R}^2)$ the set of microcausal functionals on $\mathbb{R}^2$ as per Definition 2.2 and with $\mathcal{F}^V(\mathbb{R}^2)$ that introduced in Definition 2.13, it holds that*

1. $\mathcal{F}^V \subset \mathcal{F}_{\mu c}$*;*
2. $\mathcal{F}^V(\mathbb{R}^2)$ *identifies a* $*$*-subalgebra of* $(\mathcal{F}_{\mu c}(\mathbb{R}^2), \star_{\hbar K}, *)$ *if* K *is one among the following bi-distributions* $Q+\hbar\omega$, $\mathrm{Re}(Q+\hbar\omega)$, $Q+\hbar\Delta_F$, $\mathrm{Re}(Q+\hbar\Delta_F)$ *where* ω *is the two-point function of the Poincaré invariant ground state on the two-dimensional Minkowski spacetime and* Δ_F *the associated Feynmann propagator.*

In particular we set

$$\mathcal{A}^V_{\hbar\omega+Q}(\mathbb{R}^2) := (\mathcal{F}^V(\mathbb{R}^2), \star_{\hbar\omega+Q}, *), \tag{2.37}$$

Proof. The proof follows the same lines of [3, Prop. 1.2, 1.3], the only difference being that the generators in Definition 2.13 encompass also $Q \in \mathcal{D}'(\mathbb{R}^2\times\mathbb{R}^2)$ defined in Eq. (2.26). Nonetheless, the line of reasoning extends slavishly to our scenario on account of Eq. (2.36). □

2.4. The model and the strategy. We reckon that it is desirable to outline succinctly the goal of our analysis as well as the strategy employed to reach it, before delving in all the technical details. Henceforth we shall fix the background $\mathbb{M}$ to be the two-dimensional Minkowski spacetime $(\mathbb{R}^2, \eta)$ and, as already mentioned in Sect. 1.1, on top of it we consider the *stochastic sine-Gordon equation* with coupling constant λ

$$(\Box + m^2)\hat{\psi} + \lambda g a \sin(a\hat{\psi}) = \chi\hat{\xi}, \tag{2.38}$$

where $\Box = \partial_t^2 - \partial_x^2$ is the d'Alembert wave operator and where we assume vanishing initial condition. In addition, $a \in \mathbb{R}$, while $g \in \mathcal{D}(\mathbb{R}^2)$. Finally the spacetime with noise $\hat{\xi}$ and $\chi \in \mathcal{E}(\mathbb{R}^2)$ enjoy the properties discussed in Sect. 2.2.

Following and actually combining the rationale of Sect. 2.2 and in particular of Sect. 2.1, we start from Eq. (2.38) and we construct the classical interacting field

$$\begin{aligned} &r_{\lambda V_g}(\varphi)(x) \\ &= \sum_{n\geqslant 0} \lambda^n \int_{t_1\leqslant\cdots\leqslant t_n\leqslant t} \mathrm{d}\mu_{x_1}\ldots\mathrm{d}\mu_{x_n} g(x_1)\ldots g(x_n)\{V(x_1), \{V(x_2), \ldots\{V(x_n), \varphi(x)\}\ldots\}\}. \end{aligned} \tag{2.39}$$

This is an application of Eq. (2.20) in which the rôle of $V[\varphi](x)$ is played by $\cos(a\varphi(x))$ and, heuristically, this amounts to (perturbatively) constructing a solution of the classical PDE $(\Box + m^2)\varphi + \lambda g a \sin(a\varphi) = 0$. Subsequently, as seen in Sect. 2.2, we can start from Eq. (2.39) encoding the stochastic properties of the noise by acting with the map Γ_Q defined in Eq. (2.30), namely

$$\begin{aligned} &\Gamma_Q[r_{\lambda V_g}(\varphi)](x) \\ &= \sum_{n\geqslant 0} \lambda^n \Gamma_Q \int_{t_1\leqslant\ldots\leqslant t_n\leqslant t} \mathrm{d}\mu_X g(x_1)\ldots g(x_n)\{V(x_1), \{V(x_2), \ldots\{V(x_n), \varphi(x)\}\ldots\}\}. \end{aligned} \tag{2.40}$$

In agreement with the formulation of [11], evaluating $\Gamma_Q[r_{\lambda V_g}(\varphi)](x)$ at $\varphi = 0$ yields the expectation value of the solution of Eq. (2.38) as a formal power series in λ. Yet, a natural question pertains the convergence of the formal power series in Eq. (2.40). To answer this question, we adopt an unconventional strategy which consists of building $\Gamma_Q[r_{\lambda V_g}(\varphi)]$ as the classical limit of $\Gamma_Q[R_{\lambda V_g}(\varphi)]$, where $R_{\lambda V_g}(\varphi)$ is the interacting quantum field defined in Eq. (2.22) via the Bogoliubov map for the specific choice of the observable $F \equiv \Phi$. In other words, we shall prove that

$$\Gamma_Q[r_{\lambda V_g}(\varphi)] = \lim_{\hbar \to 0^+} \Gamma_Q[R_{\lambda V_g}(\varphi)]. \tag{2.41}$$

The net advantage of this unusual strategy is that, adapting to the case in hand techniques introduced in [3,17], we are able to prove absolute convergence of the formal power series defining $\Gamma_Q[R_{\lambda V_g}(\varphi)]$ for any field configuration $\varphi \in \mathcal{E}(\mathbb{M})$. The subsequent investigation of the classical limit procedure will be discussed in Sect. 4. As a consequence, the non-perturbative expectation value of the solution can be written as

$$\Gamma_Q[r_{\lambda V_g}(\varphi)]_{\varphi=0} = \lim_{\hbar \to 0^+} \Gamma_Q[R_{\lambda V_g}(\varphi)]_{\varphi=0}.$$

The same line of reasoning shall be shown to be applicable also to the analysis of the n-point correlation functions of the solution of Eq. (2.38), see Sect. 3.2.1.

3. Interplay Between Quantum and Stochastic

As outlined in Sect. 2.4, we shall investigate the construction of the expectation value and of the correlations of the solutions of Eq. (2.38) following a two-steps procedure. In the following we address the analysis of the first part. This consists of an unconventional combination of the frameworks outlined in Sects. 2.1.1 and 2.2. More precisely we start by considering $F \in \mathcal{F}_{\text{loc}}(\mathbb{R}^2)$ as per Definition 2.13 and we associate to it its quantum counterpart via the Bogoliubov map as per Eq. (2.22):

$$R_{\lambda V}(F) = S(\lambda V)^{\star_{\hbar\omega} -1} \star_{\hbar\omega} (S(\lambda V) \star_{\hbar\Delta_F} F),$$

where here we are implicitly thinking about $V = \cos(a\varphi)$ and where ω and Δ_F are in the previous sections. On account of Remark 2.6, the last expression can be written as

$$R_{\lambda V}(F) = \Gamma_{\hbar\omega}\Big[\Gamma_{\hbar\omega}^{-1}((S(\lambda V))^{\star_{\hbar\omega}-1})\Gamma_{\hbar\omega}^{-1}(\Gamma_{\hbar\Delta_F}[\Gamma_{\hbar\Delta_F}^{-1}(S(\lambda V))\Gamma_{\hbar\Delta_F}^{-1}(F)])\Big].$$

Subsequently, by applying the rationale outlined in Sect. 2.2, we can codify the information on the correlations of the Gaussian white noise in Eq. (2.38) by applying the map Γ_Q as per Eq. (2.30):

$$\begin{aligned}
\Gamma_Q[R_{\lambda V}(F)] &= \Gamma_Q\Gamma_{\hbar\omega}\Big[\Gamma_{\hbar\omega}^{-1}((S(\lambda V))^{\star_{\hbar\omega}-1})\Gamma_{\hbar\omega}^{-1}(\Gamma_{\hbar\Delta_F}[\Gamma_{\hbar\Delta_F}^{-1}(S(\lambda V))\Gamma_{\hbar\Delta_F}^{-1}(F)])\Big] \\
&= \Gamma_{Q+\hbar\omega}\Big[\Gamma_{\hbar\omega}^{-1}(\Gamma_Q^{-1}\Gamma_Q(S(\lambda V))^{\star_{\hbar\omega}-1})\Gamma_{\hbar\omega}^{-1}\Gamma_Q^{-1}\Gamma_Q(\Gamma_{\hbar\Delta_F}[\Gamma_{\hbar\Delta_F}^{-1}(S(\lambda V))\Gamma_{\hbar\Delta_F}^{-1}(F)])\Big] \\
&= \Gamma_{Q+\hbar\omega}\Big[\Gamma_{Q+\hbar\omega}^{-1}(\Gamma_Q(S(\lambda V))^{\star_{\hbar\omega}-1})\Gamma_{Q+\hbar\omega}^{-1}(\Gamma_{Q+\hbar\Delta_F}[\Gamma_{\hbar\Delta_F}^{-1}(S(\lambda V))\Gamma_{\hbar\Delta_F}^{-1}(F)])\Big] \\
&= \Gamma_{Q+\hbar\omega}\Big[\Gamma_{Q+\hbar\omega}^{-1}(\Gamma_Q(S(\lambda V))^{\star_{\hbar\omega}-1})\Gamma_{Q+\hbar\omega}^{-1}(\Gamma_{Q+\hbar\Delta_F}[\Gamma_{\hbar\Delta_F}^{-1}\Gamma_Q^{-1}\Gamma_Q(S(\lambda V))\Gamma_{\hbar\Delta_F}^{-1}\Gamma_Q^{-1}\Gamma_Q(F)])\Big] \\
&= \Gamma_{Q+\hbar\omega}\Big[\Gamma_{Q+\hbar\omega}^{-1}(\Gamma_Q(S(\lambda V))^{\star_{\hbar\omega}-1})\Gamma_{Q+\hbar\omega}^{-1}(\Gamma_{Q+\hbar\Delta_F}[\Gamma_{Q+\hbar\Delta_F}^{-1}\Gamma_Q(S(\lambda V))\Gamma_{Q+\hbar\Delta_F}^{-1}\Gamma_Q(F)])\Big].
\end{aligned}$$

Exploiting once more Eq. 2.8, $\Gamma_Q[R_{\lambda V}(F)]$ can be rewritten as

$$\Gamma_Q[R_{\lambda V}(F)] = \Gamma_Q((S(\lambda V))^{\star_{\hbar\omega}-1}) \star_{Q+\hbar\omega} [(\Gamma_Q(S(\lambda V)) \star_{Q+\hbar\Delta_F} \Gamma_Q(F))]. \tag{3.1}$$

Giving a closer look to this expression and focusing on $\Gamma_Q(S(\lambda V))$, on account of Eq. (2.19) we can decompose this term as

$$\Gamma_Q(S(\lambda V)) = \sum_{n\geqslant 0} \frac{1}{n!}\left(\frac{i\lambda}{\hbar}\right)^n \Gamma_Q \mathcal{T}_n^{\hbar\Delta_F}(V^{\otimes n}), \tag{3.2}$$

where, in view of Eq. (2.17),

$$\mathcal{T}_n^{\hbar\Delta_F}(V\otimes\cdots\otimes V) = \Gamma_{\hbar\Delta_F}[\Gamma^{-1}_{\hbar\Delta_F}(V)\ldots\Gamma^{-1}_{\hbar\Delta_F}(V)].$$

The action of Γ_Q yields

$$\begin{aligned}
\Gamma_Q\mathcal{T}_n^{\hbar\Delta_F}(V\otimes\cdots\otimes V) &= \Gamma_Q\Gamma_{\hbar\Delta_F}[\Gamma^{-1}_{\hbar\Delta_F}(V)\ldots\Gamma^{-1}_{\hbar\Delta_F}(V)]\\
&= \Gamma_Q\Gamma_{\hbar\Delta_F}[\Gamma^{-1}_{\hbar\Delta_F}\Gamma_Q^{-1}\Gamma_Q(V)\ldots\Gamma^{-1}_{\hbar\Delta_F}\Gamma_Q^{-1}\Gamma_Q(V)]\\
&= \Gamma_{Q+\hbar\Delta_F}[\Gamma^{-1}_{Q+\hbar\Delta_F}\Gamma_Q(V)\ldots\Gamma^{-1}_{Q+\hbar\Delta_F}\Gamma_Q(V)]\\
&= \Gamma_Q(V)\star_{Q+\hbar\Delta_F}\ldots\star_{Q+\hbar\Delta_F}\Gamma_Q(V)\\
&= \mathcal{T}_n^{\hbar\Delta_F+Q}(\Gamma_Q(V)\otimes\ldots\otimes\Gamma_Q(V)).
\end{aligned} \tag{3.3}$$

To summarize we have constructed the $Q-S$*-matrix* as

$$\Gamma_Q(S(\lambda V)) = \sum_{n\geqslant 0} \frac{1}{n!}\left(\frac{i\lambda}{\hbar}\right)^n \mathcal{T}_n^{\hbar\Delta_F+Q}(\Gamma_Q(V)\otimes\cdots\otimes\Gamma_Q(V)). \tag{3.4}$$

Bearing in mind that our primary goal is to prove the convergence of the series defining an interacting field associated to the sine-Gordon model, it is convenient to give an analogous characterization of the $\star_{\hbar\omega}$-inverse of S in terms of anti-time ordered products, namely

$$\begin{aligned}
S(\lambda V)^{\star_{\hbar\omega}-1} &:= \sum_{n\geqslant 0} \frac{1}{n!}\left(\frac{-i\lambda}{\hbar}\right)^n V\star_{\hbar\Delta_{\mathrm{AF}}}\ldots\star_{\hbar\Delta_{\mathrm{AF}}} V\\
&= \sum_{n\geqslant 0} \frac{1}{n!}\left(\frac{-i\lambda}{\hbar}\right)^n \mathcal{T}_n^{\hbar\Delta_{\mathrm{AF}}}(V\otimes\cdots\otimes V),
\end{aligned}$$

where Δ_{AF} denotes the anti-Feynman propagator, see Eq. (2.23). As a matter of fact, mirroring the computations in Eq. (3.3), it descends that we can rewrite the $\star_{\hbar\omega}$-inverse of the $Q-S$-matrix

$$\Gamma_Q(S(\lambda V)^{\star_{\hbar\omega}-1}) = \sum_{n\geqslant 0} \frac{1}{n!}\left(\frac{-i\lambda}{\hbar}\right)^n \mathcal{T}_n^{\hbar\Delta_{\mathrm{AF}}+Q}(\Gamma_Q(V)\otimes\ldots\otimes\Gamma_Q(V)), \tag{3.5}$$

where

$$\mathcal{T}_n^{\hbar\Delta_{\mathrm{AF}}+Q}(\Gamma_Q(V)\otimes\ldots\otimes\Gamma_Q(V)) = \Gamma_Q(V)\star_{Q+\hbar\Delta_{\mathrm{AF}}}\ldots\star_{Q+\hbar\Delta_{\mathrm{AF}}}\Gamma_Q(V).$$

3.1. Convergence of the $Q - S$-matrix. In the preceding section the main result has been the definition in Eq. (3.4) of the $Q - S$-matrix $\Gamma_Q(S(\lambda V))$ as a formal power series expansion in the coupling constant λ. In this section we prove absolute convergence of the series for the two-dimensional sine-Gordon model. This amount to choosing

$$\begin{aligned} V_g(\varphi) &:= \int_{\mathbb{R}^2} d\mu_z \cos(a\varphi(z))g(z) = \frac{1}{2}(V_{a,g} + V_{-a,g}), \quad V_{a,g}(\varphi) \\ &:= \int_{\mathbb{R}^2} \mathrm{d}\mu_z e^{ia\varphi(z)} g(z). \end{aligned} \tag{3.6}$$

Henceforth, for simplicity and without loss of generality, we shall assume that $g \geq 0$. Our strategy consists of extending to the case in hand the methods devised in [3], so to account also for the contribution of the correlations codified by Q, see Eq. (2.27). To this end a few preliminary considerations are necessary. First of all, in view of Eqs. (3.6), (3.4) takes the form

$$\Gamma_Q(S(\lambda V_g)) = \sum_{n \geqslant 0} \frac{1}{n!} \left(\frac{i\lambda}{2\hbar}\right)^n \sum_{k=0}^{n} \binom{n}{k} \mathcal{T}_n^{\hbar\Delta_F + Q}((\Gamma_Q(V_{a,g}))^{\otimes k} \otimes (\Gamma_Q(V_{-a,g}))^{\otimes(n-k)}). \tag{3.7}$$

Focusing on the single elements appearing in the tensor products and recalling Eq. (2.30), it descends

$$\begin{aligned} \Gamma_Q(V_{\pm a,g})(\varphi) &= \exp\left\{\frac{1}{2}\left\langle Q(z,y), \frac{\delta^2}{\delta\varphi(z)\delta\varphi(y)}\right\rangle\right\} \int_{\mathbb{R}^2} \mathrm{d}\mu_x e^{\pm ia\varphi(x)} g(x) \\ &= \sum_{k \geqslant 0} \frac{(\pm ia)^{2k}}{k! 2^k} \int_{\mathbb{R}^{2(2k+1)}} \mathrm{d}\mu_x \prod_{i=1}^{\ell} \mathrm{d}\mu_{z_i} \mathrm{d}\mu_{y_i} \prod_{\ell=1} \delta(x - z_\ell)\delta(x - y_\ell) Q(z_\ell, y_\ell) e^{\pm ia\varphi(x)} g(x) \\ &= \sum_{k \geqslant 0} \frac{1}{k! 2^k} \int_{\mathbb{R}^2} \mathrm{d}\mu_x e^{\pm ia\varphi(x)} (-a^2 Q(x,x))^k g(x) \\ &= \int_{\mathbb{R}^2} \mathrm{d}\mu_x g(x) e^{\pm ia\varphi(x)} \sum_{k \geqslant 0} \frac{(-\frac{a^2}{2} Q(x,x))^k}{k!} \\ &= \int_{\mathbb{R}^2} \mathrm{d}\mu_x g(x) e^{\pm ia\varphi(x)} e^{-\frac{a^2}{2} Q(x,x)}, \end{aligned}$$

where the last identity is a byproduct of the regularity of $Q(x, x)$, see Lemma 2.12, and of g lying in $\mathcal{D}(\mathbb{R}^2)$. These entail that the series $\sum_{k \geqslant 0} g(x) \frac{(-\frac{a^2}{2} Q(x,x))^k}{k!}$ is uniformly convergent. In other words, introducing

$$g_Q(x) := g(x) e^{-\frac{a^2}{2} Q(x,x)} \in \mathcal{D}(\mathbb{M}), \tag{3.8}$$

we have thus proved that

$$\Gamma_Q(V_{\pm a,g}) = V_{\pm a, g_Q}, \tag{3.9}$$

which entails in turn that

$$\Gamma_Q(V_g) = V_{g_Q}. \tag{3.10}$$

Furthermore, the n-th order coefficient of the formal power series in Eq. (3.7) can be written as

$$\begin{aligned}&\mathcal{T}_n^{\hbar\Delta_F+Q}(\Gamma_Q(V_{a_1,g})\otimes\cdots\otimes\Gamma_Q(V_{a_n,g}))\\&\quad=\int_{\mathbb{R}^{2n}}\mathrm{d}\mu_{x_1}\dots\mathrm{d}\mu_{x_n}e^{i\sum_k a_k\varphi(x_k)}e^{-\sum_{1\leq i<j\leq n}a_ia_j(\hbar\Delta^F(x_i,x_j)+Q(x_i,x_j))}g_Q(x_1)\dots g_Q(x_n),\end{aligned}$$

where a_i can acquire the values $\pm a$ depending on the case under investigation. Observing that the positivity of $Q(x,x)$ as per Lemma 2.12 entails that $0\leqslant g_Q(x)\leqslant g(x)$ and denoting by $H:=\mathrm{Re}(\Delta_F)$, we can estimate the absolute value of the time-ordered product as follows

$$\begin{aligned}&|\mathcal{T}_n^{\hbar\Delta_F+Q}(\Gamma_Q(V_{a_1,g})\otimes\cdots\otimes\Gamma_Q(V_{a_n,g}))|\\&\quad\leqslant\int_{\mathbb{R}^{2n}}\mathrm{d}\mu_X e^{-\sum_{1\leq i<j\leq n}a_ia_j(\hbar H(x_i,x_j)+Q(x_i,x_j))}g_Q(x_1)\dots g_Q(x_n)\\&\quad\leqslant\int_{\mathbb{R}^{2n}}\mathrm{d}\mu_X e^{-\sum_{1\leq i<j\leq n}a_ia_j(\hbar H(x_i,x_j)+Q(x_i,x_j))}g(x_1)\dots g(x_n)\\&\quad=\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{a_1,g}\otimes\ldots\otimes V_{a_n,g})),\end{aligned}$$

where $\mathrm{d}\mu_X=\mathrm{d}\mu_{x_1}\dots\mathrm{d}\mu_{x_n}$ and where $\mathrm{ev}_0(F)=F(0)$ is the evaluation map at the field configuration $\varphi=0$. As a consequence, denoting with $[\Gamma_Q(S(\lambda V_g))]_n$ the n-th order coefficient in λ of the expansion in Eq. (3.4), for any configuration $\varphi\in\mathcal{E}(\mathbb{R}^2)$, it holds that

$$|[\Gamma_Q(S(\lambda V_g))]_n|\leqslant\frac{1}{n!}\left(\frac{\lambda}{\hbar}\right)^n\mathrm{ev}_0[\mathcal{T}_n^{\hbar H+Q}(V_g\otimes\cdots\otimes V_g)].\tag{3.11}$$

Remark 3.1. Both $H:=\mathrm{Re}(\Delta_F)$ and Q are symmetric bi-distribution. This entails that convergence of the $Q-S$-matrix, which is built out of a non-commutative product, is tantamount to that of the exponential series of V_g computed with respect to the commutative product $\star_{\hbar H+Q}$, eventually evaluated at the zero configuration.

The analysis in [3] is mainly based on techniques of conditioning and inverse conditioning borrowed from Euclidean quantum field theory [17]. They allow one to control the convergence of the S-matrix of a massive theory in terms of the one associated to the massless counterpart. Before delving into the application of such methods to the case in hand, we report for the sake of completeness a notable result.

Theorem 3.2. (Thm. 2.2, [3]) *Let* $w_0,\ w_1\in\mathcal{D}'(\mathbb{R}^2\times\mathbb{R}^2)$ *be positive, real and symmetric. Let*

$$w_0-w_1=P-N,$$

where $P,N\in\mathcal{D}'(\mathbb{R}^2\times\mathbb{R}^2)$ *are once more positive and symmetric. Moreover, assume both that* $\exists\ N|_{\mathrm{Diag}_2}\in L^\infty(\mathbb{R}^2)$ *where* Diag_2 *is the diagonal of* $\mathbb{R}^2\times\mathbb{R}^2$*. Hence, for* V_g *as in Eq.* (3.6)*, it holds that, setting* $v_0=w_0+N$*,*

$$\mathrm{ev}_0[\exp_{w_1}(\lambda V_g)]\leqslant\mathrm{ev}_0[\exp_{v_0}(\lambda V_g)]\leqslant 2\,\mathrm{ev}_0[\exp_{w_0}(2\lambda e^{\frac{a^2}{2}K}V_g)].$$

We review succinctly the strategy adopted in [3] and, subsequently, we show how to adapt it to the case of our interest. In [3] the authors consider the distributions in Theorem 3.2 as being

$$w_0 = H_0 := \mathrm{Re}(\Delta_{F,0}), \qquad\qquad w_1 = H := \mathrm{Re}(\Delta_F),$$

where $\Delta_{F,0}$ denotes the Feynman propagator associated with the massless theory while Δ_F is the one associated with the massive counterpart. To avoid possible sources of confusion, we shall adopt the following convention to distinguish a bi-distribution from its integral kernel: $\forall f, f' \in \mathcal{D}(\mathbb{R}^2)$

$$\begin{aligned} H_0(f\otimes f') &= \int_{\mathbb{R}^4} \mathrm{d}\mu_z \mathrm{d}\mu_{z_1} \mathcal{H}_0(z-z_1) f(z) f'(z_1), \\ H(f\otimes f') &= \int_{\mathbb{R}^4} \mathrm{d}\mu_z \mathrm{d}\mu_{z_1} \mathcal{H}(z-z_1) f(z) f'(z_1). \end{aligned} \tag{3.12}$$

It can be seen that $H \in \mathcal{D}'(\mathbb{R}^2\times\mathbb{R}^2)$ is positive, translation invariant and symmetric being generated by $\mathcal{H}(x) = \frac{1}{2\pi}\mathrm{Re}\left(K_0(m\sqrt{x^2})\right)$, where K_0 is a modified Bessel function of second kind and where x^2 is the Lorentzian distance between the point $x \in \mathbb{R}^2$ and the origin, *cf.* [3, Prop. 2.4]. Considering instead the massless case, the integral kernel of H_0 reads

$$\mathcal{H}_0(x) = -\frac{1}{4\pi}\log\left|\frac{x^2}{4\mu^2}\right|, \tag{3.13}$$

where μ is a positive, reference length scale. By direct inspection, $H_0 \in \mathcal{D}'(\mathbb{R}^2\times\mathbb{R}^2)$ is symmetric in its arguments, but it fails to be positive. Nonetheless, as proven in [3, Prop. 2.3], if we consider the space-time diamond of size μ

$$D_\mu := \left\{(t,s)\in\mathbb{R}^2 \,|\, -\mu < t-s < \mu, \text{ and } -\mu < t+s < \mu\right\}, \tag{3.14}$$

then the restriction thereon of H_0 is positive. In other words, H_0 is both symmetric and positive if we consider test-functions lying in $\mathcal{D}(D_\mu^2)$. This lack of positivity of H_0 on the whole space-time requires to restrict the support of the interaction to the space-time diamond defined in Eq. (3.14). This is tantamount to choosing the cutoff function g in Eq. (3.6) to be positive and such that $\mathrm{supp}(g) \subseteq D_\mu$.

In order to apply Theorem 3.2, we must decompose the restriction of $H_0 - H$ to $\mathcal{D}(D_\mu^2)$ in the positive and negative components. To this end, let us consider a smooth, symmetric, positive and compactly supported function $\Omega \in \mathcal{D}(\mathbb{R}^2)$ such that, denoting by $\iota : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^2$ the map defined as $\iota(x,y) = x-y$, the function $\Omega\circ\iota|_{D_{2\mu}\times D_{2\mu}} = 1$. Focusing on $(H_0 - H)\Omega \in \mathcal{D}'(\mathbb{R}^2\times\mathbb{R}^2)$, we set

$$W(f\otimes f') := \int_{\mathbb{R}^4} \mathrm{d}\mu_z \mathrm{d}\mu_{z_1} (\mathcal{H}_0(z-z_1) - \mathcal{H}(z-z_1))\Omega(z-z_1) f(z) f'(z_1).$$

By direct inspection and on account of Eq. (3.12), $W(f\otimes f') = (H_0 - H)(f\otimes f')$ for $f, f' \in \mathcal{D}(D_\mu)$. Being $\mathcal{H}_0 - \mathcal{H}$ locally integrable, $(\mathcal{H}_0 - \mathcal{H})\Omega$ is a measurable function and thus $\exists C > 0$ such that

$$|W(f\otimes f')| \leqslant C\|f\|_{L^2(\mathbb{R}^2)}\|f'\|_{L^2(\mathbb{R}^2)}.$$

Therefore one can extend per continuity W to a bounded quadratic form over $L^2(D_\mu)$. In turn, on account of the Riesz representation theorem W can be written in terms of a bounded linear operator over $L^2(D_\mu)$ which acts as a multiplication in Fourier space, i.e., there exists $\hat{\mathcal{W}} \in L^\infty(\mathbb{R}^2)$ such that, for all $f, f' \in L^2(\mathbb{R}^2)$,

$$W(f \otimes f') = \int_{\mathbb{R}^2} \mathrm{d}\mu_k \hat{\mathcal{W}}(k) \hat{f}(k) \hat{f}'(k).$$

Observe that, combining the Riemann-Lesbegue theorem with $(\mathcal{H}_0 - \mathcal{H})\Omega$ being symmetric and real-valued, one can show that $\hat{\mathcal{W}}$ is in addition continuous, it vanishes at infinity and it is real-valued.

With these premises, the decomposition in positive and negative parts of $(\mathcal{H}_0 - \mathcal{H})\Omega$ follows suit. As a matter of facts, it suffices to decompose the real and continuous function $\hat{\mathcal{W}}$ in its positive and negative parts, *i.e.*,

$$\hat{\mathcal{W}} = \hat{\mathcal{P}} - \hat{\mathcal{N}},$$

where $\hat{\mathcal{P}} \geqslant 0$ and $\hat{\mathcal{N}} \geqslant 0$. In turn, it descends that for all $f, f' \in \mathcal{D}(\mathbb{R}^2)$

$$N(f, f') := \int_{\mathbb{R}^4} \mathrm{d}\mu_x \mathrm{d}\mu_y \mathcal{N}(x-y) f(x) f'(y),$$
$$P(f, f') := \int_{\mathbb{R}^4} \mathrm{d}\mu_x \mathrm{d}\mu_y \mathcal{P}(x-y) f(x) f'(y).$$

We also observe that

$$N(x, x) := \mathcal{N}(0) = \int_{\mathbb{R}} \mathrm{d}\mu_k \hat{\mathcal{N}}(k) = \|\hat{\mathcal{N}}\|_{L^1}.$$

In addition, as discussed in [3, Prop. 2.5], $\hat{\mathcal{W}} \in L^1(\mathbb{R}^2)$, implying in turn that $\hat{\mathcal{N}}$ is integrable. Thus $\exists K > 0$ such that

$$N(x, x) = \|\hat{\mathcal{N}}\|_{L^1} \leqslant K.$$

It is noteworthy that this constant can be chosen to be $K = \|\hat{\mathcal{W}}\|_{L^1}$.

The analysis sketched above is tailored to the investigation of an interacting quantum field theory, but, since we are interested in proving the convergence of the $Q-S$-matrix, we need to improve it. To start with, we underline that, in view of Lemma 2.11 and restricting ourselves to the compact region D_μ as per Eq. 3.14, both $Q + \hbar H_0$ and $Q + \hbar H$ are positive, symmetric and real bi-distributions as required by Theorem 3.2. In addition we observe that the difference between $Q + \hbar H_0$ and $Q + \hbar H$ is nothing but $H_0 - H$. This entails that the operators N and P appearing in Theorem 3.2 are left unmodified by the kernel Q.

Before delving into the detailed proof of convergence of the $Q - S$ matrix, we need a technical result which will play a pivotal rôle in the forthcoming discussion.

Lemma 3.3 (Cauchy determinant). *Consider $\mu > 0$ and $g \in \mathcal{D}(D_\mu)$, where D_μ is chosen as per Eq.* (3.14)*. Moreover, denoting by V_a^g the vertex functional introduced in Eq.* (2.4)*, let us define*

$$\mathcal{O} := \mathrm{ev}_0(V_a^g \star_{\hbar H_0 + Q_m} \dots \star_{\hbar H_0 + Q_m} V_a^g \star_{\hbar H_0 + Q_m} V_{-a}^g \star_{\hbar H_0 + Q_m} \dots \star_{\hbar H_0 + Q_m} V_{-a}^g)$$

where $a_i = a$ *if* $i \in \{1, \ldots n\}$, $a_i = -a$ *if* $i \in \{n+1, \ldots, 2n\}$ *and where* a *is chosen such that* $\alpha := \frac{a^2\hbar}{4\pi} < 1$. *Then, for any* $p \in [1, \alpha^{-1})$, *there exist two constants* $C_Q(\mu)$, *depending on* μ *in Eq.* (3.14) *and* $\tilde{C}$ *such that*

$$|\mathcal{O}| \leqslant (C_Q(\mu))^{2n^2}(4\mu^2)^{n\alpha}\|g\|^{2n}_{L^q}(\tilde{C}^{2n}(n!)^2)^{1/p}, \tag{3.15}$$

where q *satisfies* $1/q + 1/p = 1$.

Proof We follow [3, Thm. 2.7], though we need to account also for Q. First of all we observe that, on account of the specific form of $\mathcal{H}_0$, *cf.* Eq. (3.13),

$$\begin{aligned}
\mathcal{O} &= \mathrm{ev}_0\Big(\underbrace{V^g_a \star_{\hbar H_0+Q} \ldots \star_{\hbar_0+Q} V^g_a}_{n} \star_{\hbar H_0+Q} \underbrace{V^g_{-a} \star_{\hbar H_0+Q} \ldots \star_{\hbar H_0+Q} V^g_{-a}}_{n}\Big) \\
&= \int_{\mathbb{R}^{4n}} \mathrm{d}\mu_{z_1} \ldots \mathrm{d}\mu_{z_{2n}} e^{-\sum_{1\le i<j\le 2n} a_i a_j (\hbar\mathcal{H}_0(z_i,z_j)+Q(z_i,z_j))} g(z_1)\ldots g(z_{2n}) \\
&= (4\mu^2)^{n\alpha}\int_{\mathbb{R}^{4n}} \mathrm{d}\mu_{X,Y} e^{\sum_{1\le i,j\le n} a^2 Q(x_i,y_j)} e^{-\sum_{0\le i<j\le n} a^2[Q(x_i,x_j)+Q(y_i,y_j)]} \\
&\quad\times\left(\frac{\prod_{1\le i<j\le n}|(x_i-x_j)^2||(y_i-y_j)^2|}{\prod_{i=1}^n\prod_{j=1}^n|(x_i-y_j)^2|}\right)^{\alpha} G(X,Y)
\end{aligned} \tag{3.16}$$

where we introduced the notation $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n)$, $\mathrm{d}\mu_{X,Y} = \mathrm{d}\mu_{x_1} \ldots \mathrm{d}\mu_{y_n}$, and $G(X, Y) = g(x_1) \ldots g(y_n)$. Furthermore, we exploited the fact that for $i \leqslant n$ and $j > n$ it holds that $a_i a_j = -a^2$ while, in all the other allowed regimes, $a_i a_j = a^2$. Considering a point $z = (t, s) \in \mathbb{R}^2$ in Euclidean coordinates, we can represent it with respect to null counterparts by setting $z^u = t - s$ and $z^v = t + s$. Hence $z = (z^u, z^v)$ and the squared Lorentzian distance factorizes as $|z^2| = |z^u||z^v|$. By defining the $n \times n$ matrices

$$\mathcal{D}^v_{ij} = \frac{1}{(x^v_i - y^v_j)}, \qquad \mathcal{D}^u_{ij} = \frac{1}{(x^u_i - y^u_j)},$$

one can write [3,4]

$$\frac{\prod_{1\le i<j\le n}|(x_i-x_j)^2||(y_i-y_j)^2|}{\prod_{i=1}^n\prod_{j=1}^n|(x_i-y_j)^2|} = |\det\mathcal{D}^v||\det\mathcal{D}^u|,$$

where

$$\det\mathcal{D}^b = \sum_\pi\prod_{i=1}^n\frac{(-1)^{|\pi|}}{(x^b_i - y^b_{\pi(i)})}, \quad b \in \{u, v\},$$

and where π runs over all partitions of $\{1, \ldots, n\}$. As a consequence Eq. (3.16) can be rewritten in terms of Cauchy determinants as

$$\mathcal{O} = (4\mu^2)^{n\alpha}\int_{\mathbb{R}^{4n}}\mathrm{d}\mu_{X,Y} e^{\sum_{1\le i<j\le n} a^2 Q(x_i,y_j)} e^{-\sum_{1\le i<j\le n} a^2[Q(x_i,x_j)+Q(y_i,y_j)]}|\det\mathcal{D}^v|^{\alpha}|\det\mathcal{D}^u|^{\alpha} G(X,Y). \tag{3.17}$$

In order to obtain a bound on the absolute value of $\mathcal{O}$, first of all we introduce

$$C_Q(\mu) := \left(\sup_{x,y \in D_\mu} e^{a^2 Q(x,y)} \right)^{\frac{1}{2}},$$

observing that, on account of the local boundedness of Q inherited from its continuity, $C_Q(\mu) < \infty$ for every finite μ. This entails that

$$e^{\sum_{1\le i,j\le n} a^2 Q(x_i,y_j)} = \prod_{1\le i,j\le n} e^{a^2 Q(x_i,y_j)} \leqslant (C_Q^2(\mu))^{n^2}$$

Combining this information with the estimate

$$e^{-a^2 Q(x,y)} \leqslant 1.$$

we can rewrite Eq. (3.17) to obtain

$$\mathcal{O} \leqslant (C_Q(\mu))^{2n^2} (4\mu^2)^{n\alpha} \int_{\mathbb{R}^{4n}} d\mu_{X,Y} |\det \mathcal{D}^v|^\alpha |\det \mathcal{D}^u|^\alpha G(X,Y).$$

In view of [17, Thm. 3.4] it holds that, for any p and q such that $1/q + 1/p = 1$,

$$\mathcal{O} \leqslant (C_Q(\mu))^{2n^2} (4\mu^2)^{n\alpha} \|G\|_{L^q(D_\mu^{2n})} \| |\det \mathcal{D}^v|^\alpha |\det \mathcal{D}^u|^\alpha \|_{L^p(D_\mu^{2n})}.$$

Having assumed that $\alpha = \frac{a^2\hbar}{4\pi} < 1$, we choose $p \geqslant 1$ so that $\alpha p < 1$. This entails that the product

$$\prod_{i=1}^n \frac{1}{|x_i^b - y_{\pi(i)}^b|^{\alpha p}},$$

is locally integrable and, thus, $\| |\det \mathcal{D}^v|^\alpha |\det \mathcal{D}^u|^\alpha \|_{L^p(D_\mu^{2n})} \lesssim 1$. More precisely,

$$\begin{aligned}\mathcal{O} \leqslant& (C_Q(\mu))^{2n^2} (4\mu^2)^{n\alpha} \|G\|_{L^q(D_\mu^{2n})} \left(\sum_\pi \sum_{\pi'} \int_{\mathbb{R}^{4n}} d\mu_{X,Y} \prod_{i=1}^n \frac{1}{|x_i^u - y_{\pi(i)}^u|^{\alpha p}} \prod_{\ell=1}^n \frac{1}{|x_\ell^v - y_{\pi'(\ell)}^v|^{\alpha p}} \right)^{\frac{1}{p}} \\ \leqslant& (C_Q(\mu))^{2n^2} (4\mu^2)^{n\alpha} \|G\|_{L^q(D_\mu^{2n})} (\tilde{C}^{2n} (n!)^2)^{1/p},\end{aligned}$$

where the factor $(n!)^2$ comes from counting the number of admissible permutations π and π', while K is a constant such that

$$\int_{\mathbb{R}^{4n}} d\mu_{X,Y} \prod_{i=1}^n \frac{1}{|x_i^u - y_{\pi(i)}^u|^{\alpha p}} \prod_{\ell=1}^n \frac{1}{|x_\ell^v - y_{\pi'(\ell)}^v|^{\alpha p}} \leqslant K^{2n}.$$

Finally, observing that

$$\|G\|_{L^q(D_\mu^{2n})} = \|g\|_{L^q(D_\mu)}^{2n},$$

we conclude that

$$\mathcal{O} \leqslant (C_Q)^{2n^2} (4\mu^2)^{n\alpha} \|g\|_{L^q(D_\mu)}^{2n} (\tilde{C}^{2n} (n!)^2)^{1/p}.$$

□

We are in a position to prove a key result on the convergence of the $Q - S$-matrix introduced in Eq. (3.4).

Theorem 3.4 *Let $g \in \mathcal{D}(D_\mu)$ be a positive test-function supported on the compact domain D_μ as per Eq. (3.14) and let $[\Gamma_Q(S(\lambda V))]_n$ be the n-th perturbative coefficient of the $Q - S$-matrix introduced in Eq. (3.4) with the interaction functional*

$$V_g(\varphi) := \int_{\mathbb{R}^2} \mathrm{d}\mu_z \cos(a\varphi(z)),$$

with $0 < a < 4\pi\hbar^{-1}$. Then, setting $\alpha := \frac{a^2\hbar}{4\pi}$, there exist positive constants $\tilde{C}$, $C_Q(\mu)$ and K such that

$$|[\Gamma_Q(S(\lambda V))(\varphi)]_n| \leqslant \frac{2(2\mu)^{n\alpha}(C_Q)^{n^2}}{(n!)^{1-1/p}} \left(\frac{2\lambda e^{2^{-1}a^2K}}{\hbar}\right)^n \|g\|^n_{L^q} C^{n/p}, \qquad \forall\varphi \in \mathcal{E}(\mathbb{R}^2), \tag{3.18}$$

for any $p \in [1, \alpha^{-1})$ such that $1/q + 1/p = 1$.

Proof As highlighted by Eq. (3.11), it suffices to exhibit a suitable bound for the n-th term in the expansion of

$$\mathrm{ev}_0\left[\exp_{\hbar H+Q}\left(\frac{\lambda}{\hbar}V_g\right)\right],$$

namely

$$\frac{1}{n!}\left(\frac{\lambda}{\hbar}\right)^n \mathrm{ev}_0[\mathcal{T}_n^{\hbar H+Q}(V_g \otimes \cdots \otimes V_g)].$$

On account of the discussion about the positive and negative parts of the integral kernels under scrutiny as per Theorem 3.2 and as per the following discussion, we can conclude that

$$\mathrm{ev}_0\left[\exp_{\hbar H+Q}\left(\frac{\lambda}{\hbar}V_g\right)\right] \leqslant 2\,\mathrm{ev}_0\left[\exp_{\hbar H_0+Q}\left(\frac{2\lambda e^{2^{-1}a^2K}}{\hbar}V_g\right)\right],$$

with K a suitable constant, whose dependence on the underlying data has no relevance in the current analysis. Hence, thanks to conditioning, it suffices to control the series associated to the massless theory in order to bound the massive one. Following the strategy outlined in [3], we can exploit the Cauchy-Schwartz inequality, whose validity is guaranteed by the positivity of $\hbar H_0 + Q$ on the support of $g^{\otimes 2}$. More precisely, for any functionals A and B in the algebra $\mathcal{A}^V_{\hbar H_0+Q}$, see Eq. (2.37), it holds that

$$|\,\mathrm{ev}_0(A \star_{\hbar H_0+Q} B)| \leqslant \sqrt{|\,\mathrm{ev}_0(A^* \star_{\hbar H_0+Q} A)|}\sqrt{|\,\mathrm{ev}_0(B^* \star_{\hbar H_0+Q} B)|},$$

which yields, setting $B = 1$,

$$|\,\mathrm{ev}_0(A)|^2 \leqslant |\,\mathrm{ev}_0(A^* \star_{\hbar H_0+Q} A)|. \tag{3.19}$$

As a consequence, observing that $V_{a,g}^* = V_{-a,g}$, a direct application of Eq. (3.19) implies

$$\begin{aligned}
&\mathrm{ev}_0(V_g \star_{\hbar H_0+Q} \dots \star_{\hbar H_0+Q} V_g) = \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\mathrm{ev}_0\Bigg(\underbrace{V_{a,g}\star_{\hbar H_0+Q}\dots\star_{\hbar H_0+Q}V_{a,g}}_{k}\star_{\hbar H_0+Q}\underbrace{V_{-a,g}\star_{\hbar H_0+Q}\dots\star_{\hbar H_0+Q}V_{-a,g}}_{n-k}\Bigg) \\
&\leqslant \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left[\mathrm{ev}_0\Bigg(\underbrace{V_{a,g}\star_{\hbar H_0+Q}\dots\star_{\hbar H_0+Q}V_{a,g}}_{n}\star_{\hbar H_0+Q}\underbrace{V_{-a,g}\star_{\hbar H_0+Q}\dots\star_{\hbar H_0+Q}V_{-a,g}}_{n}\Bigg)\right]^{\frac{1}{2}} \\
&= \left[\mathrm{ev}_0\Bigg(\underbrace{V_{a,g}\star_{\hbar H_0+Q}\dots\star_{\hbar H_0+Q}V_{a,g}}_{n}\star_{\hbar H_0+Q}\underbrace{V_{-a,g}\star_{\hbar H_0+Q}\dots\star_{\hbar H_0+Q}V_{-a,g}}_{n}\Bigg)\right]^{\frac{1}{2}}.
\end{aligned}$$

Since all hypotheses of Lemma 3.3 are met, we apply the estimate in Eq. (3.15) to obtain

$$\mathrm{ev}_0(V_g \star_{\hbar H_0+Q} \dots \star_{\hbar H_0+Q} V_g) \leqslant (C_Q)^{n^2}(2\mu)^{n\alpha}\|g\|_{L^q}^n (C^n n!)^{1/p}.$$

As a consequence, it descends that

$$\begin{aligned}
|[\Gamma_Q(S(\lambda V))(\varphi)]_n| &\leqslant \frac{2}{n!}\left(\frac{2\lambda e^{2^{-1}a^2K}}{\hbar}\right)^n (C_Q)^{n^2}(2\mu)^{n\alpha}\|g\|_{L^q}^n (C^n n!)^{1/p} \\
&= \frac{2(2\mu)^{n\alpha}(C_Q)^{n^2}}{(n!)^{1-1/p}}\left(\frac{2\lambda e^{2^{-1}a^2K}}{\hbar}\right)^n \|g\|_{L^q}^n C^{n/p},
\end{aligned}$$

which concludes the proof. □

As a consequence Theorem 3.4, we state the following crucial corollary.

Corollary 3.5 *For any, but fixed $p \in [1, \alpha^{-1})$ with α defined as in Theorem 3.4, the series*

$$\Gamma_Q(S(\lambda V))(\varphi) = \sum_{n\geqslant 0}[\Gamma_Q(S(\lambda V))(\varphi)]_n,$$

is uniformly convergent for any $\varphi \in \mathcal{E}(\mathbb{M})$.

Proof Absolute convergence directly follows from the observation that, as inferred from Stirling formula, the factorial $(n!)^{1-1/p}$ at the denominator of Eq. (3.18) for $p > 1$, tames the exponential growth due to the other factors. □

Remark 3.6 We observe that the argument yielding absolute convergence of the $Q-S$-matrix holds true also for the case of $\Gamma_Q((S(\lambda V))^{\star_{\hbar\omega}-1})$. As we have shown in Eq. (3.5),

$$\Gamma_Q(S(\lambda V)^{\star_{\hbar\omega}-1}) = \sum_{n\geqslant 0}\frac{1}{n!}\left(\frac{-i\lambda}{\hbar}\right)^n \mathcal{T}_n^{\hbar\Delta_{\mathrm{AF}}+Q}(\Gamma_Q(V)\otimes\dots\otimes\Gamma_Q(V)).$$

The only difference with respect to $\Gamma_Q(S(\lambda V))$ lies in the fact that we have to work with the anti-Feynman rather than the Feynman propagator. Nonetheless, since $\mathrm{Re}(\Delta_{\mathrm{AF}}) = \mathrm{Re}(\Delta_F) = H$, the convergence result follows suit.

Remark 3.7 Observe that the convergence of the $Q-S$-matrix proven in Theorem 3.4 is guaranteed only if we restrict the attention to the diamond D_μ as per Eq. (3.14), where μ is arbitrary but fixed. In [3] this limitation has been removed, though the relevant proof strongly relies on the underlying kernels being translation invariant both in time and in space. Alas, we cannot extend this result to the case under scrutiny since the covariance Q as per Eq. (2.27) is defined in terms of cut-off along the time directions which inhibits translation invariance. This limitation is not a major hurdle as far as the construction of solutions for the stochastic sine-Gordon model is concerned as we discuss in Sect. 4.

3.2. Convergence of the interacting field. Goal of this section is to elaborate on Theorem 3.4 deriving a convergence result for the formal power series in Eq. (3.1). We recall that the Q-deformed version of the Bogoliubov map, for $F \in \mathcal{F}_{\text{loc}}^{\otimes m}(\mathbb{R}^2)$:

$$\Gamma_Q[R_{\lambda V}(F)] = \Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \star_{Q+\hbar\omega} [(\Gamma_Q(S(\lambda V_g)) \star_{Q+\hbar\Delta_F} \Gamma_Q(F))]. \tag{3.20}$$

In this section we focus on the functional $F_f = \Phi_f$, $f \in \mathcal{D}(\mathbb{R}^2)$, see Example 2.4, whose interacting version reads

$$\Gamma_Q[R_{\lambda V}(\Phi_f)] = \Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \star_{Q+\hbar\omega} [(\Gamma_Q(S(\lambda V_g)) \star_{Q+\hbar\Delta_F} \Gamma_Q(\Phi_f))].$$

In order to simplify the notation, we write

$$\Phi_{I,f} := \Gamma_Q[R_{\lambda V}(\Phi_f)].$$

The starting point consists of exhibiting a series expansion for the interacting field which is suitable for studying its convergence. On account of the linearity of the field functional, it descends that, since only the first order functional derivative of Φ_f is non vanishing,

$$\begin{aligned}\Gamma_Q(S(\lambda V_g)) \star_{Q+\hbar\Delta_F} \Gamma_Q(\Phi_f) &= \Gamma_Q(S(\lambda V_g)) \star_{Q+\hbar\Delta_F} \Phi_f \\ &= \Gamma_Q(S(\lambda V_g))\Phi_f + \langle \Gamma_Q(S(\lambda V_g))^{(1)}, (Q+\hbar\Delta_F)\Phi_f^{(1)}\rangle,\end{aligned}$$

where, in the first step, we used that $\Gamma_Q(\Phi_f) = \Phi_f$. As a consequence

$$\begin{aligned}\Phi_{I,f} = {} & \Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \star_{Q+\hbar\omega} (\Gamma_Q(S(\lambda V_g))\Phi_f) \\ & + \Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \star_{Q+\hbar\omega} (\langle \Gamma_Q(S(\lambda V_g))^{(1)}, (Q+\hbar\Delta_F)\Phi_f^{(1)}\rangle) \\ =: {} & J(V_g, \Phi_f) + M(V_g, \Phi_f).\end{aligned} \tag{3.21}$$

We study separately $J(V_g, \Phi_f)$ and $M(V_g, \Phi_f)$. Starting with the first term, we shall exploit the following lemma, which is an immediate consequence of the Leibniz rule. In the next section we shall prove Lemma 3.14 as a generalization.

Lemma 3.8 (Lemma 9, [4]). *Let K be a generic integral kernel and let A, B, C be smooth functionals as per Definition 2.2, with C linear. Then*

$$\mathcal{M} \circ e^{D_K}[A \otimes (BC)] = [\mathcal{M} \circ e^{D_K}(A \otimes B)]C + \mathcal{M} \circ e^{D_K}[\langle A^{(1)}, KC^{(1)}\rangle \otimes B], \tag{3.22}$$

where $A^{(1)}$, $C^{(1)}$ are first-order functional derivatives.

Thanks to the defining properties of the deformation map Γ_Q as per Eq. (2.8) and applying Lemma 3.8, it descends that

$$\begin{aligned} J(V_g, \Phi_f) &= \Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \star_{Q+\hbar\omega} (\Gamma_Q(S(\lambda V_g))\Phi_f) \\ &= \mathcal{M} \circ e^{D_{Q+\hbar\omega}}[\Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \otimes (\Gamma_Q(S(\lambda V_g))\Phi_f)] \\ &= \mathcal{M} \circ e^{D_{Q+\hbar\omega}}[\Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \otimes (\Gamma_Q(S(\lambda V_g))]\Phi_f + \\ &\quad + \mathcal{M} \circ e^{D_{Q+\hbar\omega}}[\langle \Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1})^{(1)}, (Q+\hbar\omega)\Phi_f^{(1)}\rangle \otimes \Gamma_Q(S(\lambda V_g))]. \end{aligned} \tag{3.23}$$

Exploiting once again that the deformation map acts as an homomorphism on the deformed algebra, see Remark 2.6, the first term in Eq. (3.23) can be further expanded as

$$\begin{aligned} &\mathcal{M} \circ e^{D_{Q+\hbar\omega}}[\Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \otimes (\Gamma_Q(S(\lambda V_g))]\Phi_f \\ &= \Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \star_{Q+\hbar\omega} \Gamma_Q(S(\lambda V_g))\Phi_f \\ &= \Gamma_{Q+\hbar\omega}[\Gamma^{-1}_{Q+\hbar\omega}(\Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}))\Gamma^{-1}_{Q+\hbar\omega}(\Gamma_Q(S(\lambda V_g)))]\Phi_f \\ &= \Gamma_Q\Gamma_{\hbar\omega}[\Gamma^{-1}_{\hbar\omega}((S(\lambda V_g))^{\star_{\hbar\omega}-1})\Gamma^{-1}_{\hbar\omega}(S(\lambda V_g))]\Phi_f \\ &= \Gamma_Q[(S(\lambda V_g))^{\star_{\hbar\omega}-1} \star_{\hbar\omega} S(\lambda V_g)]\Phi_f = \Gamma_Q(1)\Phi_f = \Phi_f. \end{aligned}$$

Therefore the first term on the right hand side of Eq. (3.23) coincides with Φ_f itself. Focusing on the second contribution, we observe that

$$\begin{aligned} &\mathcal{M} \circ e^{D_{Q+\hbar\omega}}[\langle \Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1})^{(1)}, (Q+\hbar\omega)\Phi_f^{(1)}\rangle \otimes \Gamma_Q(S(\lambda V_g))] \\ &= \langle \Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1})^{(1)}, (Q+\hbar\omega)\Phi_f^{(1)}\rangle \star_{Q+\hbar\omega} \Gamma_Q(S(\lambda V_g)). \end{aligned}$$

Replacing the $Q-S$ matrix and its $\star_{\hbar\omega}$ inverse with their formal power series in λ as per Eqs. (3.4) and (3.5) we obtain

$$\begin{aligned} J(V_g, \Phi_f) &= \Phi_f + \sum_{n\geqslant 0}\left(\frac{i\lambda}{\hbar}\right)^n \sum_{\ell=0}^{n}\binom{n}{\ell}(-1)^\ell \ell \\ &\quad \times \langle \mathcal{T}^{\hbar\Delta_{\mathrm{AF}}+Q}_\ell(\Gamma_Q(V_g)\otimes\ldots\otimes\Gamma_Q(V_g)^{(1)}\otimes\ldots\otimes\Gamma_Q(V_g)), (Q+\hbar\omega)\Phi_f^{(1)}\rangle \\ &\quad \star_{Q+\hbar\omega} \mathcal{T}^{\hbar\Delta_F+Q}_{n-\ell}, \end{aligned} \tag{3.24}$$

where the symmetry property of the map $\mathcal{T}^{\hbar\Delta_{\mathrm{AF}}+Q}_\ell$ in its arguments has been used. Observe that, on account of Example 2.4, it holds by direct inspection that

$$((Q+\hbar\omega)\Phi_f^{(1)})(y) = \int_{\mathbb{R}^2} \mathrm{d}\mu_z (Q+\hbar\omega)(y-z)f(z).$$

Focusing on the interacting vertex functional

$$V_g = \int_{\mathbb{R}^2} \mathrm{d}\mu_x \frac{e^{ia\varphi(x)} + e^{-ia\varphi(x)}}{2} g(x),$$

a formal computation entails that

$$\begin{aligned} V_g^{(1)}(y) &= \int_{\mathbb{R}^2} \mathrm{d}\mu_x \frac{e^{ia\varphi(x)} - e^{-ia\varphi(x)}}{2} ia\delta(x-y)g(x) \\ &= \frac{e^{ia\varphi(x)} - e^{-ia\varphi(x)}}{2} iag(y) = V'(y)g(y), \end{aligned} \tag{3.25}$$

where we introduced

$$V'(y) := -a\sin(a\varphi(y)) = -a\frac{e^{ia\varphi(y)} - e^{-ia\varphi(y)}}{2i}.$$

Since any deformation map acts as the identity on local functionals, Eq. (3.10) allows us to write the n-th order contribution to Eq. (3.24) as

$$\begin{aligned}&\langle \mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(\Gamma_Q(V_g)\otimes\cdots\otimes\Gamma_Q(V_g)^{(1)}\otimes\ldots\otimes\Gamma_Q(V_g)),(Q+\hbar\omega)\Phi_f^{(1)}\rangle\\ &\quad=\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{g_Q}\otimes\ldots\otimes\langle V'g_Q,(Q+\hbar\omega)f\rangle\otimes\ldots\otimes V_{g_Q}),\end{aligned}$$

yielding

$$\begin{aligned}J(V_g,\Phi_f)=\Phi_f+\sum_{n\geqslant 0}\left(\frac{i\lambda}{\hbar}\right)^n\sum_{\ell=0}^n\binom{n}{\ell}(-1)^\ell\ell\\ \times\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{g_Q}\otimes\ldots\otimes\langle V'g_Q,(Q+\hbar\omega)f\rangle\\ \otimes\cdots\otimes V_{g_Q})\star_{Q+\hbar\omega}\mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{g_Q}\otimes\ldots\otimes V_{g_Q}).\end{aligned}\tag{3.26}$$

Focusing on $M(V_g,\Phi_f)$ in Eq. (3.21), its analysis is similar to that of $J(V_g,\Phi_f)$. It can be expressed as a formal power series, since

$$\langle\Gamma_Q(V_g)^{(1)},(Q+\hbar\Delta_F)\Phi_f^{(1)}\rangle=\langle V'g_Q,(Q+\hbar\Delta_F)f\rangle,$$

and thus, *mutatis mutandis*,

$$\begin{aligned}M(V_g,\Phi_f)=\sum_{n\geqslant 0}\left(\frac{i\lambda}{\hbar}\right)^n\sum_{\ell=0}^n\binom{n}{\ell}(-1)^\ell(n-\ell)\\ \times\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{g_Q}\otimes\cdots\otimes V_{g_Q})\star_{Q+\hbar\omega}[\mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{g_Q}\otimes\ldots\otimes\langle V'g_Q,(Q+\hbar\Delta_F)f\rangle\otimes\ldots\otimes V_{g_Q})].\end{aligned}$$

Overall, we proved the following result.

Theorem 3.9 *Let*

$$\Phi_{I,f}:=\Gamma_Q[R_{\lambda V}(\Phi_f)]=\Gamma_Q((S(\lambda V))^{\star_{\hbar\omega}-1})\star_{Q+\hbar\omega}[(\Gamma_Q(S(\lambda V))\star_{Q+\hbar\Delta_F}\Gamma_Q(\Phi_f))].$$

It holds that

$$\begin{aligned}\Phi_{I,f}&=J(V_g,\Phi_f)+M(V_g,\Phi_f)\\ &=\Phi_f+\textstyle\sum_{n\geqslant 0}\left(\frac{i\lambda}{\hbar}\right)^n J_n(V_g^{\otimes n},\Phi_f)+\sum_{n\geqslant 0}\left(\frac{i\lambda}{\hbar}\right)^n M_n(V_g^{\otimes n},\Phi_f),\end{aligned}\tag{3.27}$$

where

$$\begin{aligned}J_n(V_g^{\otimes n},\Phi_f)=\sum_{\ell=0}^n\binom{n}{\ell}(-1)^\ell\ell\\ \times\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{g_Q}\otimes\cdots\otimes\langle V'g_Q,(Q+\hbar\omega)f\rangle\otimes\cdots\otimes V_{g_Q})\star_{Q+\hbar\omega}\mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{g_Q}\otimes\ldots\otimes V_{g_Q}),\end{aligned}\tag{3.28}$$

and

$$\begin{aligned}M_n(V_g^{\otimes n},\Phi_f)=\sum_{\ell=0}^n\binom{n}{\ell}(-1)^\ell(n-\ell)\\ \times\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{g_Q}\otimes\cdots\otimes V_{g_Q})\star_{Q+\hbar\omega}\mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{g_Q}\otimes\ldots\otimes\langle V'g_Q,(Q+\hbar\Delta_F)f\rangle\otimes\ldots\otimes V_{g_Q}).\end{aligned}\tag{3.29}$$

The rest of this section is devoted to proving that the series expansion of the interacting field introduced in Theorem 3.9 is absolutely convergent. Eventually, we shall show that the series in Eq. (3.27) shares the same form as the one appearing in the construction of the $Q-S$-matrix. Once this correspondence has been settled, the proof is analogous to the one of Theorem 3.4 and therefore we shall omit it.

Since the perturbative expansion of the interacting field is split in two separate contributions, see Eq. (3.27), we shall start by further manipulating the n-th perturbative order of $J(V_g, \Phi_f)$ as per Eq. (3.28). Discarding the combinatorial coefficients, we consider a generic element of the sum, namely

$$\begin{aligned} &\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{g_Q} \\ &\quad \otimes \ldots \otimes \langle V' g_Q, (Q+\hbar\omega) f\rangle \otimes \ldots \otimes V_{g_Q}) \star_{Q+\hbar\omega} \mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{g_Q} \otimes \cdots \otimes V_{g_Q}). \end{aligned} \tag{3.30}$$

For later convenience, let us introduce the modified test function

$$\tilde{g}(y) := g_Q(y)[(Q+\hbar\omega)f](y), \tag{3.31}$$

which inherits from g_Q the property of being smooth and compactly supported. Hence Eq. (3.30) can be rewritten in a concise form as

$$\begin{aligned} &\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{g_Q} \otimes \ldots \otimes V'_{\tilde{g}} \otimes \cdots \otimes V_{g_Q}) \star_{Q+\hbar\omega} \mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{g_Q} \otimes \ldots \otimes V_{g_Q}) \\ &\quad = \mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V'_{\tilde{g}} \otimes \cdots \otimes V_{g_Q} \otimes \cdots \otimes V_{g_Q}) \star_{Q+\hbar\omega} \mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{g_Q} \otimes \cdots \otimes V_{g_Q}), \end{aligned} \tag{3.32}$$

where we used that the time-ordered product is commutative. As in the analysis of the S-matrix in Sect. 3.1 and recalling the explicit form of the vertex functional $V_{g_Q} = 2^{-1}(V_{a,g_Q} + V_{-a,g_Q})$, a direct computation shows that

$$\mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{g_Q} \otimes \ldots \otimes V_{g_Q}) = \frac{1}{2^{n-\ell}} \sum_{p=0}^{n-\ell} \binom{n-\ell}{p} \mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{a,g_Q}^{\otimes p} \otimes V_{-a,g_Q}^{\otimes(n-\ell-p)}),$$

while

$$\begin{aligned} &\mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{a,g_Q}^{\otimes p} \otimes V_{-a,g_Q}^{\otimes(n-\ell-p)}) \\ &\quad = \mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}\left[\int_{\mathbb{R}^{2(n-\ell)}} e^{i\sum_{j=1}^{n-\ell} a_j^{(p)}\varphi(y_j)} g_Q(y_1)\ldots g_Q(y_{n-\ell}) \mathrm{d}\mu_{Y_{n-\ell}}\right] \\ &\quad = \int_{\mathbb{R}^{2(n-\ell)}} e^{i\sum_{j=1}^{n-\ell} a_j^{(p)}\varphi(y_j)} e^{-\sum_{1\leqslant m<j\leqslant n-\ell} a_m^{(p)} a_j^{(p)} (Q+\hbar\Delta_F)(y_m,y_j)} g_Q(Y_{n-\ell}) \mathrm{d}\mu_{Y_{n-\ell}}. \end{aligned}$$

where we adopted the notation

$$a^{(p)} = \{a_1^{(p)}, \ldots, a_\ell^{(n-\ell-p)}\} = \{\underbrace{a, \ldots, a}_{p}, \underbrace{-a, \ldots, -a}_{n-\ell-p}\}.$$

Starting from the first factor $\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V'_{\tilde{g}}\otimes\cdots\otimes V_{g_Q}\otimes\cdots\otimes V_{g_Q})$ it holds that

$$\begin{aligned}
&\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V'_{\tilde{g}}\otimes\cdots\otimes V_{g_Q}\otimes\cdots\otimes V_{g_Q})\\
&=\frac{1}{2^\ell i}\sum_{k=0}^{\ell-1}\binom{\ell-1}{k}\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}((V_{a,\tilde{g}}-V_{-a,\tilde{g}})\otimes V_{a,g_Q}^{\otimes k}\otimes V_{-a,g_Q}^{\otimes(\ell-1-k)})\\
&=\frac{1}{2^\ell i}\sum_{k=0}^{\ell-1}\binom{\ell-1}{k}[\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{a,\tilde{g}}\otimes V_{a,g_Q}^{\otimes k}\otimes V_{-a,g_Q}^{\otimes(\ell-1-k)})+\\
&\quad-\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{-a,\tilde{g}}\otimes V_{a,g_Q}^{\otimes k}\otimes V_{-a,g_Q}^{\otimes(\ell-1-k)})].
\end{aligned}\tag{3.33}$$

Considering separately the two terms on the right hand side of Eq. (3.33), the first one reads

$$\begin{aligned}
&\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{a,\tilde{g}}\otimes V_{a,g_Q}^{\otimes k}\otimes V_{-a,g_Q}^{\otimes(\ell-1-k)})\\
&=\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}\left[\int_{\mathbb{R}^{2\ell}}e^{ia\varphi(x_1)+i\sum_{j=2}^{\ell}a_j^{(k)}\varphi(x_j)}\tilde{g}(x_1)\dots g_Q(x_\ell)\mathrm{d}\mu_{X_\ell}\right]\\
&=\int_{\mathbb{R}^{2\ell}}\mathrm{d}\mu_{X_\ell}\tilde{G}_Q(X_\ell)e^{i\left[a\varphi(x_1)+\sum_{j=2}^{\ell}a_j^{(k)}\varphi(x_j)\right]}\\
&\quad\times e^{-\sum_{2\leqslant j\leqslant\ell}aa_j^{(k)}(Q+\hbar\Delta_{\mathrm{AF}})(x_1,x_j)}e^{-\sum_{2\leqslant m<j\leqslant\ell}a_m^{(k)}a_j^{(k)}(Q+\hbar\Delta_{\mathrm{AF}})(x_m,x_j)},
\end{aligned}$$

where $\mathrm{d}\mu_{X_\ell}:=\mathrm{d}\mu_{x_1}\dots\mathrm{d}\mu_{x_\ell}$, $\tilde{G}_1(X_\ell):=\tilde{g}(x_1)\dots g_Q(x_\ell)$ and where this time

$$a^{(k)}=\{a_2^{(k)},\dots,a_\ell^{(k)}\}=\{\underbrace{a,\dots,a}_{k},\underbrace{-a,\dots,-a}_{\ell-1-k}\}.$$

Focusing on the second term in Eq. (3.33), we obtain

$$\begin{aligned}
&\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{-a,\tilde{g}}\otimes V_{a,g_Q}^{\otimes k}\otimes V_{-a,g_Q}^{\otimes(\ell-1-k)})\\
&=\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}\left[\int_{\mathbb{R}^{2\ell}}e^{-ia\varphi(x_1)+i\sum_{j=2}^{\ell}a_j^{(k)}\varphi(x_j)}\tilde{g}(x_1)\dots g_Q(x_\ell)\mathrm{d}\mu_{X_\ell}\right]\\
&=\int_{\mathbb{R}^{2\ell}}\mathrm{d}\mu_{X_\ell}\tilde{G}_Q(X_\ell)e^{-ia\varphi(x_1)+i\sum_{j=2}^{\ell}a_j^{(k)}\varphi(x_j)}\\
&\quad\times e^{\sum_{2\leqslant j\leqslant\ell}aa_j^{(k)}(Q+\hbar\Delta_{\mathrm{AF}})(x_1,x_j)}e^{-\sum_{2\leqslant m<j\leqslant\ell}a_m^{(k)}a_j^{(k)}(Q+\hbar\Delta_{\mathrm{AF}})(x_m,x_j)}.
\end{aligned}$$

As a last step we consider the $\star_{Q+\hbar\omega}$-product of the last two identities computed above. This amounts to

$$\begin{aligned}
&\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{g_Q}\otimes\ldots\otimes V'_{\tilde{g}}\otimes\cdots\otimes V_{g_Q})\star_{Q+\hbar\omega}\mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_g\otimes\cdots\otimes V_g)\\
&=\frac{1}{2^n i}\sum_{k=0}^{\ell-1}\sum_{p=0}^{n-\ell}\binom{\ell-1}{k}\binom{n-\ell}{p}\\
&\quad\times[\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{a,\tilde{g}}\otimes V_{a,g_Q}^{\otimes k}\otimes V_{-a,g_Q}^{\otimes(\ell-1-k)})\star_{Q+\hbar\omega}\mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{a,g_Q}^{\otimes p}\otimes V_{-a,g_Q}^{\otimes(n-\ell-p)})+\\
&\quad-\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{-a,\tilde{g}}\otimes V_{a,g_Q}^{\otimes k}\otimes V_{-a,g_Q}^{\otimes(\ell-1-k)})\star_{Q+\hbar\omega}\mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{a,g_Q}^{\otimes p}\otimes V_{-a,g_Q}^{\otimes(n-\ell-p)})\Big].
\end{aligned}\tag{3.34}$$

From the point of view of convergence, the reader can easily convince him/herself that the two terms in Eq. (3.34) behave analogously. Thus we shall focus only on one of them, the convergence of the other following suit. A direct application of Eq. (2.7) entails that

$$\begin{aligned}&\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{a,\tilde{g}}\otimes V_{a,g_Q}^{\otimes k}\otimes V_{-a,g_Q}^{\otimes(\ell-1-k)})\star_{Q+\hbar\omega}\mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{a,g_Q}^{\otimes p}\otimes V_{-a,g_Q}^{\otimes(n-\ell-p)})\\&=\int_{\mathbb{R}^{2n}}\mathrm{d}\mu_{X_n}e^{ia\varphi(x_1)+i\sum_{j=2}^{\ell}a_j^{(k)}\varphi(x_j)}e^{i\sum_{j=\ell+1}^{n}a_j^{(p)}\varphi(x_j)}\tilde{G}_Q(X_n)e^{-\sum_{2\leqslant j\leqslant\ell}aa_j^{(k)}(Q+\hbar\Delta_{\mathrm{AF}})(x_1,x_j)}\\&\quad\times e^{-\sum_{2\leqslant m<j\leqslant\ell}a_m^{(k)}a_j^{(k)}(Q+\hbar\Delta_{\mathrm{AF}})(x_m,x_j)}e^{-\sum_{\ell+1\leqslant m<j\leqslant n}a_m^{(p)}a_j^{(p)}(Q+\hbar\Delta_F)(x_m,x_j)}\\&\quad\times e^{-\sum_{2\leqslant m\leqslant\ell,1\leqslant j\leqslant n-\ell}a_m^{(k)}a_j^{(p)}(Q+\hbar\omega)(x_m,x_{\ell+j})}e^{-\sum_{1\leqslant j\leqslant n-\ell}aa_j^{(p)}(Q+\hbar\omega)(x_1,x_{\ell+j})}.\end{aligned}\tag{3.35}$$

We shall estimate the absolute value of the expression in Eq. (3.35) as follows

$$\begin{aligned}&|\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{a,\tilde{g}}\otimes V_{a,g_Q}^{\otimes k}\otimes V_{-a,g_Q}^{\otimes(\ell-1-k)})\star_{Q+\hbar\omega}\mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{a,g_Q}^{\otimes p}\otimes V_{-a,g_Q}^{\otimes(n-\ell-p)})|\\&\leqslant\int_{\mathbb{R}^{2n}}\mathrm{d}\mu_{X_n}|\tilde{G}(X_n)|e^{-\sum_{2\leqslant j\leqslant\ell}aa_j^{(k)}\mathrm{Re}(Q+\hbar\Delta_{\mathrm{AF}})(x_1,x_j)}e^{-\sum_{2\leqslant u<j\leqslant\ell}a_u^{(k)}a_j^{(k)}\mathrm{Re}(Q+\hbar\Delta_{\mathrm{AF}})(x_u,x_j)}\\&\quad\times e^{-\sum_{\ell+1\leqslant u<j\leqslant n}a_u^{(p)}a_j^{(p)}\mathrm{Re}(Q+\hbar\Delta_F)(x_u,x_j)}e^{-\sum_{2\leqslant u\leqslant\ell,1\leqslant j\leqslant n-\ell}a_u^{(k)}a_j^{(p)}\mathrm{Re}(Q+\hbar\omega)(x_u,x_{\ell+j})}\\&\quad\times e^{-\sum_{1\leqslant j\leqslant n-\ell}aa_j^{(p)}\mathrm{Re}(Q+\hbar\omega)(x_1,x_{\ell+j})}.\end{aligned}\tag{3.36}$$

Recalling that Q is real-valued and that

$$\begin{aligned}&\mathrm{Re}(\omega)=:H\qquad\mathrm{Re}(\Delta_F)\\&\quad=\mathrm{Re}\left(\frac{i}{2}(\Delta^R+\Delta^A)+H\right)=H,\qquad\mathrm{Re}(\Delta_{\mathrm{AF}})=\mathrm{Re}(\omega-i\Delta^R)=\mathrm{Re}(\omega)=H,\end{aligned}$$

Equation (3.36) can be further improved as follows

$$\begin{aligned}&|\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{a,\tilde{g}}\otimes V_{a,g_Q}^{\otimes k}\otimes V_{-a,g_Q}^{\otimes(\ell-1-k)})\star_{Q+\hbar\omega}\mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{a,g_Q}^{\otimes p}\otimes V_{-a,g_Q}^{\otimes(n-\ell-p)})|\\&\quad\leqslant\int_{\mathbb{M}^\ell}\mathrm{d}\mu_{X_n}|\tilde{G}(X_n)|e^{-\sum_{2\leqslant j\leqslant\ell}aa_j^{(k)}(Q+\hbar H)(x_1,x_j)}e^{-\sum_{2\leqslant u<j\leqslant\ell}a_u^{(k)}a_j^{(k)}(Q+\hbar H)(x_u,x_j)}\\&\qquad\times e^{-\sum_{\ell+1\leqslant u<j\leqslant n}a_u^{(p)}a_j^{(p)}(Q+\hbar H)(x_u,x_j)}e^{-\sum_{2\leqslant u\leqslant\ell,1\leqslant j\leqslant n-\ell}a_u^{(k)}a_j^{(p)}(Q+\hbar H)(x_u,x_{\ell+j})}\\&\qquad\times e^{-\sum_{1\leqslant j\leqslant n-\ell}aa_j^{(p)}(Q+\hbar H)(x_1,x_{\ell+j})}\\&\quad=\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{a,|\tilde{g}|}\otimes V_{a,g}^{\otimes k}\otimes V_{-a,g}^{\otimes(\ell-1-k)}\otimes V_{a,g}^{\otimes p}\otimes V_{-a,g}^{\otimes(n-\ell-p)})).\end{aligned}$$

Considering the second term in Eq. (3.34), an analogous procedure entails

$$\begin{aligned}&|\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{-a,\tilde{g}}\otimes V_{a,g_Q}^{\otimes k}\otimes V_{-a,g_Q}^{\otimes(\ell-1-k)})\star_{Q+\hbar\omega}\mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{a,g_Q}^{\otimes p}\otimes V_{-a,g_Q}^{\otimes(n-\ell-p)})|\\&\quad\leqslant\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{-a,|\tilde{g}|}\otimes V_{a,g}^{\otimes k}\otimes V_{-a,g}^{\otimes(\ell-1-k)}\otimes V_{a,g}^{\otimes p}\otimes V_{-a,g}^{\otimes(n-\ell-p)})).\end{aligned}$$

Using Eq. (3.34) as well as the estimates above, we obtain

$$
\begin{aligned}
&|\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}+Q}(V_{g_Q}\otimes\cdots\otimes V'_{\tilde{g}}\otimes\cdots\otimes V_{g_Q})\star_{Q+\hbar\omega}\mathcal{T}_{n-\ell}^{\hbar\Delta_F+Q}(V_{g_Q}\otimes\cdots\otimes V_{g_Q})|\\
&\quad\leqslant\frac{1}{2^n}\sum_{k=0}^{\ell-1}\sum_{p=0}^{n-\ell}\binom{\ell-1}{k}\binom{n-\ell}{p}\\
&\qquad\times\Big[\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{a,|\tilde{g}|}\otimes V_{a,g}^{\otimes k}\otimes V_{-a,g}^{\otimes(\ell-1-k)}\otimes V_{a,g}^{\otimes p}\otimes V_{-a,g}^{\otimes(n-\ell-p)}))\\
&\qquad+\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{-a,|\tilde{g}|}\otimes V_{a,g}^{\otimes k}\otimes V_{-a,g}^{\otimes(\ell-1-k)}\otimes V_{a,g}^{\otimes p}\otimes V_{-a,g}^{\otimes(n-\ell-p)}))\Big]\\
&\quad=\frac{1}{2}[\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{a,|\tilde{g}|}\otimes V_g^{\otimes n-1}))+\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{-a,|\tilde{g}|}\otimes V_g^{\otimes n-1}))].
\end{aligned}
$$

Eventually, using Eq. (3.28) we obtain

$$
\begin{aligned}
|J_n(V_g^{\otimes n},\Phi_f)|&\leqslant\frac{1}{2}\sum_{\ell=0}^{n}\binom{n}{\ell}\ell[\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{a,|\tilde{g}|}\otimes V_g^{\otimes n-1}))+\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{-a,|\tilde{g}|}\otimes V_g^{\otimes n-1}))]\\
&=n2^{n-2}[\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{a,|\tilde{g}|}\otimes V_g^{\otimes n-1}))+\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{-a,|\tilde{g}|}\otimes V_g^{\otimes n-1}))]. \quad (3.37)
\end{aligned}
$$

With this estimate we are in a position to prove a bound strictly related to the absolute convergence of the series defining $J(V_g,\Phi_f)$.

Theorem 3.10 *Under the assumptions of Theorem 3.4 and recalling $J_n(V_g^{\otimes n},\Phi_f)$ as per Eq. (3.28), the following bound holds true:*

$$
|J_n(V_g^{\otimes n},\Phi_f)|\leqslant\frac{n2^n(2\mu)^{n\alpha}(C_Q)^{n^2}}{2(n!)^{1-1/p}}\left(\frac{2\lambda e^{2^{-1}a^2K}}{\hbar}\right)^n\|g\|_{L^q}^{n-1}\|\tilde{g}\|_{L^q}C^{n/p}.
$$

Proof The proof of this result is closely related to the one of Theorem 3.4. Since we are going to adopt the same strategy, we discuss only the main steps of the proof. On account of the analysis performed in this section, in particular of Eq. (3.37), it suffices to exhibit a suitable bound for $\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{a,|\tilde{g}|}\otimes V_g^{\otimes n-1}))$ and for $\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{-a,|\tilde{g}|}\otimes V_g^{\otimes n-1}))$. Since the analysis of these two contributions is analogous, we limit ourselves to discussing the first one, as an identical argument applies to the other.

We observe that, as in the proof of Theorem 3.4, the conditioning and inverse conditioning results stated in Theorem 3.2 can be applied, guaranteeing the validity of the following inequality:

$$
\begin{aligned}
&\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{a,|\tilde{g}|}\otimes V_g^{\otimes n-1}))\\
&\quad\leqslant 2\,\mathrm{ev}_0\left\{\mathcal{T}_n^{\hbar H+Q}\left[\left(\frac{2\lambda e^{2^{-1}a^2K}}{\hbar}V_{a,|\tilde{g}|}\right)\otimes\left(\frac{2\lambda e^{2^{-1}a^2K}}{\hbar}V_g\right)^{\otimes n-1}\right]\right\}. \quad (3.38)
\end{aligned}
$$

As a matter of fact, Eq. (3.38) suggests that controlling the massless case entails an analogous control of the massive counterpart. Applying *mutatis mutandis* Lemma 3.3, the sought after estimate descends. This concludes the proof. □

Starting from Eq. (3.29) and iterating the analysis applied to $J_n(V_g^{\otimes n}, \Phi_f)$, an analogous expression can be obtained for M_n, the only difference being the explicit form of the effective test-function $\tilde{\tilde{g}}$, which is defined in this case in terms of the Feynman propagator as

$$\tilde{\tilde{g}}(y) := g_Q(y)[(Q + \hbar\Delta_F)f](y). \tag{3.39}$$

Without going through the detailed procedure once more, we directly state the resulting estimate on the absolute value of $M_n(V_g^{\otimes n}, \Phi_f)$:

$$\begin{aligned}|M_n(V_g^{\otimes n}, \Phi_f)| &\leqslant \frac{1}{2}\sum_{\ell=0}^{n}\binom{n}{\ell}(n-\ell)[\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{a,|\tilde{\tilde{g}}|} \otimes V_g^{\otimes n-1})) \\ &\quad + \mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{-a,|\tilde{\tilde{g}}|} \otimes V_g^{\otimes n-1}))] \\ &= n2^{n-2}[\mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{a,|\tilde{\tilde{g}}|} \otimes V_g^{\otimes n-1})) \\ &\quad + \mathrm{ev}_0(\mathcal{T}_n^{\hbar H+Q}(V_{-a,|\tilde{\tilde{g}}|} \otimes V_g^{\otimes n-1}))],\end{aligned}$$

where the second line descends from the identity

$$\sum_{\ell=0}^{n}\binom{n}{\ell}(n-\ell) = n2^{n-1}.$$

Theorem 3.11 *Under the assumptions of Theorem 3.4 and recalling $M_n(V_g^{\otimes n}, \Phi_f)$ as per Eq. (3.29), the following bound is satisfied:*

$$|M_n(V_g^{\otimes n}, \Phi_f)| \leqslant \frac{n2^n(2\mu)^{n\alpha}(C_Q)^{n^2}}{2(n!)^{1-1/p}}\left(\frac{2\lambda e^{2^{-1}a^2K}}{\hbar}\right)^n \|g\|_{L^q}^{n-1}\|\tilde{\tilde{g}}\|_{L^q}C^{n/p}.$$

From Theorems 3.10 and 3.11, the following corollary follows.

Corollary 3.12 *Under the assumptions of Theorem 3.9, the power series defining the interacting field $\Phi_{I,f}(\varphi)$ as per Eq. (3.27) is absolutely convergent, for all field configurations $\varphi \in \mathcal{E}(\mathbb{R}^2)$.*

Proof In complete analogy with the proof of Corollary 3.5, convergence is a direct consequence of the estimates in Theorems 3.10 and 3.11, combined with the decomposition in Eq. (3.27). □

3.2.1. Convergence of the n-point correlation functions In this section we shall extend the convergence result proven in Sect. 3.2 at the level of the n-point correlation functions of the interacting field. To wit, in this section we shall prove convergence of the power series defining the interacting counterpart of algebra elements of the form

$$\Phi_{f_1}\dots\Phi_{f_p}, \quad p \in \mathbb{N}, \quad f_i \in \mathcal{D}(\mathbb{R}^2), \quad \forall i \in \{1, \dots, p\}.$$

Throughout this section we adopt the notation $\Phi_1 \dots \Phi_p := \Phi_{f_1}\dots\Phi_{f_p}$. On account of Eq. (3.20), the Q-deformed interacting version of the product of fields $F = \Phi_1 \dots \Phi_p$ reads

$$\begin{aligned}&\Gamma_Q[R_{\lambda V}(\Phi_1\dots\Phi_p)] \\ &\quad = \Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \star_{Q+\hbar\omega} [(\Gamma_Q(S(\lambda V_g)) \star_{Q+\hbar\Delta_F} \Gamma_Q(\Phi_1\dots\Phi_p))].\end{aligned}$$

As a first step let us observe that, splitting the deformed product $\star_{Q+\hbar\Delta_F}$ into the sum between the pointwise counterpart and higher order contracted contributions, we obtain

$$\begin{aligned}&\Gamma_Q(S(\lambda V_g)) \star_{Q+\hbar\Delta_F} \Gamma_Q(\Phi_1\dots\Phi_p)\\ &= \Gamma_Q(S(\lambda V_g))\Gamma_Q(\Phi_1\dots\Phi_p)\\ &\quad+\sum_{n=1}^{p}\langle\Gamma_Q(S(\lambda V_g))^{(n)}, (Q+\hbar\Delta_F)^{\otimes n}\Gamma_Q(\Phi_1\dots\Phi_p)^{(n)}\rangle.\end{aligned}$$

As a consequence,

$$\begin{aligned}\Gamma_Q[R_{\lambda V}(\Phi_1\dots\Phi_p)] &= \Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \star_{Q+\hbar\omega} [\Gamma_Q(S(\lambda V_g))\Gamma_Q(\Phi_1\dots\Phi_p)]\\ &\quad+\sum_{n=1}^{p}\Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \star_{Q+\hbar\omega} \langle\Gamma_Q(S(\lambda V_g))^{(n)}, (Q+\hbar\Delta_F)^{\otimes n}\Gamma_Q(\Phi_1\dots\Phi_p)^{(n)}\rangle\\ &=: J(V_g,\Phi_1\dots\Phi_p)+\sum_{n=1}^{p}M^n(V_g,\Phi_1\dots\Phi_p),\end{aligned} \tag{3.40}$$

where the arguments of J and M^n remind us of the specific functional under scrutiny. Mirroring the analysis of the interacting field, let us start working with $J(V_g,\Phi_1\dots\Phi_p)$, namely

$$J(V_g,\Phi_1\dots\Phi_p) = \Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \star_{Q+\hbar\omega} [\Gamma_Q(S(\lambda V_g))\Gamma_Q(\Phi_1\dots\Phi_p)]. \tag{3.41}$$

Note that, setting $I=\{1,\dots,p\}$, $I_1 = I\setminus\{i_1, j_1\}$ and, iteratively, $I_\ell = I_{\ell-1}\setminus\{i_\ell, j_\ell\}$, the deformed product $\Gamma_Q(\Phi_1\dots\Phi_p)$ can be written as

$$\begin{aligned}\Gamma_Q(\Phi_1\dots\Phi_p) &= \Phi_1\dots\Phi_p\\ &\quad+\sum_{i_1<j_1;i_1,j_1\in I} Q(f_{i_1},f_{j_1})\Bigg[\prod_{k_1\in I_1}\Phi_{k_1}+\sum_{i_2<j_2;i_2,j_2\in I_1} Q(f_{i_2},f_{j_2})\Bigg[\prod_{k_2\in I_2}\Phi_{k_2}+\dots\\ &\quad+\sum_{i_{\lfloor p/2\rfloor}<j_{\lfloor p/2\rfloor};i_{\lfloor p/2\rfloor},j_{\lfloor p/2\rfloor}\in I_{\lfloor p/2\rfloor}} Q(f_{i_{\lfloor p/2\rfloor}},f_{j_{\lfloor p/2\rfloor}})\Phi_{\lfloor p/2\rfloor}\Bigg]\Bigg],\end{aligned} \tag{3.42}$$

which is nothing but Wick's theorem adapted to the map Γ_Q, see Eq. (2.30). In Eq. (3.42), with the notation $\Phi_{\lfloor p/2\rfloor}=1$ we mean that, whenever p is even, then $\Phi_{\lfloor p/2\rfloor}=1$. At this level, Eqs. (3.41) and (3.42) suggest that, in order to prove the convergence of the series defining $J(V_g,\Phi_1\dots\Phi_p)$, it suffices to check convergence of the products

$$\Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \star_{Q+\hbar\omega} [\Gamma_Q(S(\lambda V_g))\Phi_1\dots\Phi_m], \tag{3.43}$$

for $m\leqslant p$. Indeed, this statement stems from replacing Eqs. (3.42) into (3.41), observing that the factors $Q(f_{i_\ell}, f_{j_\ell})$ are constant functionals. Hence they are not affected by the contractions defining the product $\star_{Q+\hbar\omega}$.

To better grasp the underlying rationale, an example is in due order.

Example 3.13 The simplest contribution to Eq. (3.42) is the one where a single contraction occurs, namely

$$\sum_{i_1<j_1;i_1,j_1\in I}\Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1})\star_{Q+\hbar\omega}\left[Q(f_{i_1},f_{j_1})\prod_{k_1\in I_1}\Phi_{k_1}\right]$$
$$=\sum_{i_1<j_1;i_1,j_1\in I}Q(f_{i_1},f_{j_1})\left[\Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1})\star_{Q+\hbar\omega}\prod_{k_1\in I_1}\Phi_{k_1}\right].$$

The only term to be tamed is the one under square brackets which falls in the class of those considered in Eq. (3.43).

As a consequence, the analysis of $J(V_g,\Phi_1\dots\Phi_p)$ boils down to that of terms of the form

$$\Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1})\star_{Q+\hbar\omega}[\Gamma_Q(S(\lambda V_g))\Phi_1\dots\Phi_m],\tag{3.44}$$

for $m\leqslant p$. Observe that $J(V_g,\Phi_1\dots\Phi_p)$ is a finite sum of contributions of this kind, see Eq. (3.41) and (3.42). Hence, we need a generalization of Lemma 3.8 to deal with this scenario.

Lemma 3.14 *Let K be an integral kernel and let $A,B\in\mathcal{F}_{loc}$ be regular functionals, $m\in\mathbb{N}$ and $I=\{1,\dots,m\}$. Then*

$$A\star_K(B\Phi_1\dots\Phi_m)=(A\star_K B)\Phi_1\dots\Phi_m + \sum_{\ell=1}^{m}\frac{1}{\ell!}\sum_{i_1,\dots,i_\ell\in I,i_1<\dots<i_\ell}\left[\left\langle A^{(\ell)},K^{\otimes\ell}\Phi^{(1)}_{i_1}\dots\Phi^{(1)}_{i_\ell}\right\rangle\star_K B\right]\prod_{j\in I\setminus\{i_1,\dots,i_\ell\}}\Phi_j,\tag{3.45}$$

where

$$\prod_{j\in\emptyset}\Phi_j=1.$$

Proof The statement descends directly from an iterative application of the Leibniz rule. It suffices to recall the identity

$$(B\Phi_1\dots\Phi_m)^{(n)}=\sum_{k=0}^{n}\binom{n}{k}B^{(n-k)}(\Phi_1\dots\Phi_m)^{(k)},$$

observing that it is non-vanishing only for $k\leqslant m$. On account of the explicit expression of $\star_K$ as per Eq. (2.7), it holds that

$$A\star_K(B\Phi_1\dots\Phi_m)=\sum_{n\geqslant0}\frac{1}{n!}\langle A^{(n)},K^{\otimes n}(B\Phi_1\dots\Phi_m)^{(n)}\rangle\tag{3.46}$$

$$=\sum_{n\geqslant0}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\langle A^{(n)},K^{\otimes n}B^{(n-k)}(\Phi_1\dots\Phi_m)^{(k)}\rangle\tag{3.47}$$

We observe that the case $k = 0$ in Eq. (3.46) coincides with the first contribution on the right hand side of Eq. (3.45), obtained when all the functional derivatives act on B. Considering instead a generic k, such tat $1 \leqslant k \leqslant m$, and exchanging the sums in Eq. (3.46) we write the inner series as

$$\sum_{n\geqslant k} \frac{1}{n!}\binom{n}{k}\langle A^{(n)}, K^{\otimes n} B^{(n-k)}(\Phi_1 \dots \Phi_m)^{(k)}\rangle$$
$$= \sum_{n\geqslant k} \frac{1}{k!(n-k)!} \sum_{\substack{i_1,\dots,i_k\in I\\ i_1<\dots<i_k}} \left\langle A^{(n)}, K^{\otimes n}\left(\Phi^{(1)}_{i_1}\dots\Phi^{(1)}_{i_k} B^{(n-k)}\right)\right\rangle \prod_{j\in I\setminus\{i_1,\dots,i_k\}} \Phi_j$$
$$= \sum_{h\geqslant 0} \frac{1}{k!h!} \sum_{\substack{i_1,\dots,i_k\in I\\ i_1<\dots<i_k}} \left\langle A^{(h+k)}, K^{\otimes h+k}\left(\Phi^{(1)}_{i_1}\dots\Phi^{(1)}_{i_k} B^{(h)}\right)\right\rangle \prod_{j\in I\setminus\{i_1,\dots,i_k\}} \Phi_j$$
$$= \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k\in I\\ i_1<\dots<i_k}} \left[\left\langle A^{(k)}, K^{\otimes k}\Phi^{(1)}_{i_1}\dots\Phi^{(1)}_{i_k}\right\rangle \star_K B\right] \prod_{j\in I\setminus\{i_1,\dots,i_k\}} \Phi_j .$$

Summing over all values of k, the proof is concluded. □

We can apply Lemma 3.14 directly to Eq. (3.44) by choosing $A = \Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1})$, $B = \Gamma_Q(S(\lambda V_g))$ and $K = Q+\hbar\omega$. As a result, the deformed products of interest assume the form

$$\Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \star_{Q+\hbar\omega} [\Gamma_Q(S(\lambda V_g))\Phi_1 \dots \Phi_m] = \Phi_1 \dots \Phi_m$$
$$+ \sum_{\ell=1}^{m} \frac{1}{\ell!} \sum_{\substack{i_1,\dots,i_\ell\in I\\ i_1<\dots<i_\ell}} \prod_{j\in I\setminus\{i_1,\dots,i_\ell\}} \Phi_j \underbrace{\left\langle \Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1})^{(\ell)}, (Q+\hbar\omega)^{\otimes\ell}\Phi^{(1)}_{i_1}\dots\Phi^{(1)}_{i_\ell}\right\rangle \star_{Q+\hbar\omega} \Gamma_Q S(\lambda V_g)}_{\ell-\text{th term}} . \tag{3.48}$$

Considering the ℓ-th term in Eq. (3.48), we can rewrite it as

$$\left\langle \Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1})^{(\ell)}, (Q+\hbar\omega)^{\otimes\ell}\Phi^{(1)}_{i_1}\dots\Phi^{(1)}_{i_\ell}\right\rangle \star_{Q+\hbar\omega} \Gamma_Q S(\lambda V_g)$$
$$= \sum_{n\geqslant 0}\left(\frac{i\lambda}{\hbar}\right)^n \sum_{j=0}^{n}\binom{n}{j}(-1)^j$$
$$\times \left\langle \mathcal{T}^{\hbar\Delta_{\mathrm{AF}}+Q}_j (V^{\otimes j}_{gQ})^{(\ell)}, (Q+\hbar\omega)^{\otimes\ell}\Phi^{(1)}_{i_1}\dots\Phi^{(1)}_{i_\ell}\right\rangle \star_{Q+\hbar\omega} \mathcal{T}^{\hbar\Delta_F+Q}_{n-j}(V^{\otimes(n-j)}_{gQ}), \tag{3.49}$$

where we have exploited Eq. (3.10). Observing that the j-th functional derivatives acting on $V^{\otimes j}_{gQ}$ are distributed on the factors according to the Leibniz rule, the ℓ-th derivative of the modified time ordered product of the vertex functionals reads

$$\mathcal{T}^{\hbar\Delta_{\mathrm{AF}}+Q}_j(V^{\otimes j}_{gQ})^{(\ell)} = \mathcal{T}^{\hbar\Delta_{\mathrm{AF}}+Q}_j(V_{gQ}\otimes\cdots\otimes V_{gQ})^{(\ell)}$$
$$= \sum_{i_1=0}^{\ell}\cdots\sum_{i_{j-1}=0}^{\ell-i_1-\dots-i_{j-2}}\binom{\ell}{i_1}\binom{\ell-i_1}{i_2}\cdots\binom{\ell-i_1-\dots-i_{j-2}}{i_{j-1}}\mathcal{T}^{\hbar\Delta_{\mathrm{AF}}+Q}_j(V^{(i_1)}_{gQ}$$
$$\otimes\cdots\otimes V^{(\ell-i_1-\dots-i_{j-1})}_{gQ}),$$

giving rise to a finite number of contributions. These ought to be contracted with $(Q + \hbar\omega)^{\otimes\ell}\Phi^{(1)}_{i_1}\dots\Phi^{(1)}_{i_\ell}$, yielding

$$
\begin{aligned}
(3.49) = \sum_{n\geqslant 0}\left(\frac{i\lambda}{\hbar}\right)^n \sum_{\ell=0}^{n}\binom{n}{j}(-1)^j \sum_{i_1=0}^{\ell}\sum_{i_2=0}^{\ell-i_1} \\
\cdots \sum_{i_{j-1}=0}^{\ell-i_1-\ldots-i_{j-2}} \binom{\ell}{i_1}\binom{\ell-i_1}{i_2}\cdots\binom{\ell-i_1-\ldots-i_{j-2}}{i_{j-1}}\times \\
\times\left\langle \mathcal{T}_j^{\hbar\Delta_{\mathrm{AF}}+Q}\left(V^{(i_1)}_{\tilde{g}_{i_1}}\otimes\cdots\otimes V^{(\ell-i_1-\ldots-i_{j-1})}_{\tilde{g}_{i_\ell}}\right)\right\rangle \star_{Q+\hbar\omega} \mathcal{T}_{n-j}^{\hbar\Delta_F+Q}(V_{g_Q}^{\otimes n-j}),
\end{aligned}
\tag{3.50}
$$

where, for simplicity of the notation, we introduced

$$
\tilde{g}_{i_k}(y) := g_Q(y)[(Q+\hbar\omega)f_{i_k}]^{i_k}(y) \in \mathcal{D}(\mathbb{R}^2).
$$

Remark 3.15 At this point, convergence of the series in Eq. (3.50) descends by mirroring the analysis of the interacting field in Sect. 3.2. This can be seen by noticing that, apart from an irrelevant coefficient, functional derivatives of V_{g_Q} are either cosine or sine functionals. Hence we have to cope with a finite number of terms whose structure closely resembles the one of the interacting field.

This concludes our investigation on the absolute convergence of the series defining $J(V_g, \Phi_1\dots\Phi_p)$. As in the case of the interacting field, an identical procedure applies to the analysis of M^n introduced in Eq. (3.40). Indeed, by means of the same argument, discussed after Eq. (3.42), one can restrict the attention to studying

$$
\Gamma_Q((S(\lambda V_g))^{\star_{\hbar\omega}-1}) \star_{Q+\hbar\omega} \langle \Gamma_Q(S(\lambda V_g))^{(n)}, (Q+\hbar\Delta_F)^{\otimes n}(\Phi_1\dots\Phi_p)^{(n)}\rangle,
$$

for $n \leqslant p$. As highlighted by Eq. (3.39), the only difference with respect to the analysis of J lies in the fact that the modified test-functions are built out of the Feynman propagator, namely

$$
\tilde{g}^F_{i_k}(y) := g_Q(y)[(Q+\hbar\Delta_F)f_{i_k}]^{i_k}(y) \in \mathcal{D}(\mathbb{R}^2).
$$

The content of this section culminates in the following result.

Corollary 3.16 *The power series defining the interacting observable* $\Gamma_Q[R_{\lambda V}(\Phi_1\dots\Phi_p)]$ *is absolutely convergent for any field configuration* $\varphi \in \mathcal{E}(\mathbb{R}^2)$ *and for any* $p \in \mathbb{N}$.

4. The Classical Limit $\hbar \to 0^+$

The previous section has been entirely devoted to showing absolute convergence of the formal power series defining the interacting field and the associated correlation functions within the perturbative quantum approach to the stochastic sine-Gordon model. Our next objective is to retrieve the information concerning the classical stochastic Sine-Gordon model via a limit procedure. In this endeavor we shall adapt to our setting the approach discussed in [14].

As anticipated in Sect. 2.4, our ultimate goal is to compute the expectation value of the solution of the stochastic sine-Gordon equation on the two-dimensional Minkowski space-time. As extensively discussed in [5,6,10] this is achieved by applying the deformation map Γ_Q, whose effect is to codify the probabilistic information of the free stochastic equation at the level of the algebra of classical interacting observables. In turn, these can be seen as the classical limit of their quantum interacting counterparts. Having proven absolute convergence of the stochastic, quantum interacting field, existence of the classical limit would imply the absolute convergence of the formal power series representing the expectation value of the perturbative solution of the stochastic sine-Gordon model.

Before investigating classical limit of Eq. (3.20), one must check that such limit is well-defined. In the following we shall prove that the only non-vanishing contributions to the series defining Eq. (3.20) involve non-negative powers of $\hbar$ ruling out possible blow-ups as $\hbar \to 0^+$. To keep the discussion as general as possible, in this section we shall consider arbitrary multi-local observables $F \in \mathcal{F}_{\text{loc}}^{\otimes m}(\mathbb{R}^2)$, which encompasses the relevant examples of the interacting field and of its correlation functions.

As discussed in Sect. 2.1, the starting point is the power series expansion in $\lambda > 0$ of the quantum interacting observable $F \in \mathcal{F}_{\text{loc}}^{\otimes m}(\mathbb{R}^2)$, $m \in \mathbb{N}$,

$$R_{\lambda V_g}(F) = \sum_{n \geqslant 0} \frac{\lambda^n}{n!} R_{n,m}(V_g^{\otimes n}, F), \tag{4.1}$$

where $R_n(V_g^{\otimes n}, F)$ are the so-called *retarded products*, namely

$$\begin{aligned} & R_{n,m}(V_g^{\otimes n}, F_f) \\ & = \left(\frac{i}{\hbar}\right)^n \sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^\ell \mathcal{T}_\ell^{\hbar \Delta_{\text{AF}}}(V_g \otimes \ldots \otimes V_g) \star_{\hbar\omega} \mathcal{T}_{n-\ell,m}^{\hbar \Delta_F}(V_g \otimes \cdots \otimes V_g \otimes F_f), \end{aligned} \tag{4.2}$$

where with $\mathcal{T}_{n-\ell,m}^{\hbar\Delta_F}(V_g \otimes \ldots \otimes V_g \otimes F)$ we mean that the argument of $\mathcal{T}_{n-\ell,m}^{\hbar\Delta_F}$ is given by $n-\ell$ copies of V_g, so to keep track of the fact that F depends on m points. Accordingly, per consistency f must lie in $\mathcal{D}(\mathbb{R}^{2m})$. In the following we prove that all contributions to Eq. (4.2) are of order $O(\hbar^0)$, which is tantamount to the existence and finiteness for any $n \geqslant 0$ of the limit

$$\lim_{\hbar \to 0^+} R_{n,m}(V_g^{\otimes n}, F_f).$$

Remark 4.1 If we consider in the previous discussion the functionals codifying the interacting field and the n-point correlation functions, we can combine the existence of the limit as $\hbar \to 0$ to the proof that the underlying power series in the coupling constant λ is absolutely convergent is a suitable regime. This two ingredients are the cornerstone of our construction of the solutions and of the correlation functions of the stochastic sine-Gordon equation on two-dimensional Minkowski spacetime.

A central notion in this section is that of connected products. Hereafter, a connection is represented by a contraction of fields by means of a suitable integral kernel.

Definition 4.2 Let $F_1, \ldots, F_n \in \mathcal{F}_{\mathrm{loc}}(\mathbb{R}^2)$ and let $\star_{\hbar\omega}$ be the deformed product introduced in Eq. (2.7). In addition, for any $n > 1$, we denote by σ_P the collection of all partitions $P \equiv \bigcup_{j=1}^k P_j$ of the set $\{1, \ldots, n\}$ into k disjoint, non-empty subsets P_j with $1 < k < n$. We call connected product between the functionals $F_1, \ldots, F_n$

$$(F_1 \star_{\hbar\omega} \ldots \star_{\hbar\omega} F_n)^c = F_1 \star_{\hbar\omega} \ldots \star_{\hbar\omega} F_n - \sum_{|P|\geqslant 2} \prod_{p\in P} (F_{p_1} \star_{\hbar\omega} \ldots \star_{\hbar\omega} F_{p_{|p|}})^c, \tag{4.3}$$

where $p = (p_1, \ldots, p_{|p|})$, $p_1 < \ldots < p_{|p|}$, the sum runs over the set of all partitions P of the set $\{1, \ldots, n\}$ containing at least two sets and where $\prod$ denotes the classical pointwise product. Similarly, given the time-ordering map $\mathcal{T}_n$ as per Eq. (2.17), we define the connected time-ordered and anti-time-ordered products as

$$\mathcal{T}_n^K(F_1 \otimes \cdots \otimes F_n)^c = \mathcal{T}_n^K(F_1 \otimes \cdots \otimes F_n) - \sum_{|P|\geqslant 2} \prod_{p\in P} \mathcal{T}_{|p|}^K(F_{p_1} \otimes \cdots \otimes F_{p_{|p|}})^c,$$

for $K = \hbar\Delta_F$ and $K = \hbar\Delta_{\mathrm{AF}}$.

As proven in [14, Prop. 1] and in the following comments, whenever one considers local functionals $F_1, \ldots, F_n$ of order $\hbar^0$, then

$$(F_1 \star_{\hbar\omega} \ldots \star_{\hbar\omega} F_n)^c = \mathcal{O}(\hbar^{n-1}), \tag{4.4}$$

$$\mathcal{T}_n^K(F_1 \otimes \cdots \otimes F_n)^c = \mathcal{O}(\hbar^{n-1}), \tag{4.5}$$

regardless whether $K = \hbar\Delta_F$ or $K = \hbar\Delta_{\mathrm{AF}}$. Heuristically, this statement relies on the observation that a fully connected product of functionals accounts for at least $n-1$ contractions, each of which carries a factor $\hbar$.

Proposition 4.3 *For any $F \in \mathcal{F}_{loc}^{\otimes m}(\mathbb{R}^2)$ it holds that*

1. *any non-vanishing contribution to $R_{n,m}(V_g^{\otimes n}, F)$ in Eq. (4.2) is such that each one among the n functionals V_g is contracted at least once with one of the entries of F;*
2. *for any $n \geqslant 0$,*

$$R_{n,m}(V_g^{\otimes n}, F) = \mathcal{O}(\hbar^0).$$

Proof Starting from 1., observe that it suffices to prove that, if one of the n factors V_g in $R_{n,m}(V_g^{\otimes n}, F)$ is not contracted with all the other factors V_g and with F, then its contribution to $R_{n,m}(V_g^{\otimes n}, F)$ vanishes.

Denoting the non-contracted vertex functional by $\tilde{V}_g$ to distinguish it from the others, Eq. (4.2), takes the form

$$R_{n,m}(V_g^{\otimes n}, F_f) = \left(\frac{i}{\hbar}\right)^n \sum_{\ell=0}^{n} \sum_{(p_1,p_2)\in\mathcal{P}_{\ell,n-\ell}} (-1)^\ell \mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}}(V_g^{\otimes p_1}) \star_{\hbar\omega} \mathcal{T}_{n-\ell,m}^{\hbar\Delta_F}(V_g^{\otimes p_2} \otimes F_f), \tag{4.6}$$

where $\mathcal{P}_{\ell,n-\ell}$ is the collection of partitions of the set $\{1,\dots,n\}$ into subsets of size ℓ and $n-\ell$, respectively. We stress that, if all the factors V_g are identical, then Eq. (4.6) turns into Eq. (4.2) due to the relation

$$\sum_{(p_1,p_2)\in\mathcal{P}_{\ell,n-\ell}} = \binom{n}{\ell}.$$

Nonetheless, the expression in Eq. (4.6) is more convenient since it allows to keep track of the fact that one of the factor V_g is different from the other ones as it is the only one not connected with the other ones. For a fixed $n \geq 1$ and $0 < \ell < n$, the case $n = 0$ being trivial, we consider

$$\sum_{(p_1,p_2)\in\mathcal{P}_{\ell,n-\ell}} (-1)^\ell \mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}}(V_g^{\otimes p_1}) \star_{\hbar\omega} \mathcal{T}_{n-\ell,m}^{\hbar\Delta_F}(V_g^{\otimes p_2}\otimes F_f). \tag{4.7}$$

Notice that there are precisely $\binom{n-1}{\ell-1}$ cases where the factor $\tilde{V}_g$ lies in the argument of $\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}}$ and

$$\binom{n}{\ell} - \binom{n-1}{\ell-1} = \binom{n-1}{\ell},$$

cases where $\tilde{V}_g$ is acted upon by $\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{F}}}$. This combinatorial argument is direct consequence of the property of the $n-1$ factors V_g of being indistinguishable. As a result it holds that

$$\begin{aligned}
(4.7) = (-1)^\ell &\Bigg[\binom{n-1}{\ell-1}\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}}(\tilde{V}_g\otimes V_g^{\otimes(\ell-1)}) \star_{\hbar\omega} \mathcal{T}_{n-\ell,m}^{\hbar\Delta_F}(V_g^{\otimes(n-\ell)}\otimes F_f)\\
&+\binom{n-1}{\ell}\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}}(V_g^{\otimes\ell}) \star_{\hbar\omega} \mathcal{T}_{n-\ell,m}^{\hbar\Delta_F}(\tilde{V}_g\otimes V_g^{\otimes(n-\ell-1)}\otimes F_f)\Bigg]\\
= (-1)^\ell \tilde{V}_g &\Bigg[\binom{n-1}{\ell-1}\mathcal{T}_{\ell-1}^{\hbar\Delta_{\mathrm{AF}}}(V_g^{\otimes(\ell-1)}) \star_{\hbar\omega} \mathcal{T}_{n-\ell,m}^{\hbar\Delta_F}(V_g^{\otimes(n-\ell)}\otimes F_f)\\
&+\binom{n-1}{\ell}\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}}(V_g^{\otimes\ell}) \star_{\hbar\omega} \mathcal{T}_{n-\ell-1,m}^{\hbar\Delta_F}(V_g^{\otimes(n-\ell-1)}\otimes F_f)\Bigg],
\end{aligned}$$

where we exploited that the (anti)time-ordered products are symmetric and that $\tilde{V}_g$ is not contracted with the other vertices, i.e., it multiplies them via the pointwise product. Focusing once more on the retarded products, Eq. (4.6) reads

$$\begin{aligned}
R_{n,m}(V_g^{\otimes n}, F_f) = \left(\frac{i}{\hbar}\right)^n \tilde{V}_g\Bigg[&\mathcal{T}_{n-1,m}^{\hbar\Delta_F}(V_g^{\otimes(n-1)}\otimes F_f) + (-1)^n\mathcal{T}_{n-1}^{\hbar\Delta_{\mathrm{AF}}}(V_g^{\otimes(n-1)}) \star_{\hbar\omega} \mathcal{T}_{0,m}^{\hbar\Delta_F}(F_f)\\
&+\sum_{\ell=1}^{n-1}\Bigg((-1)^\ell\binom{n-1}{\ell-1}\mathcal{T}_{\ell-1}^{\hbar\Delta_{\mathrm{AF}}}(V_g^{\otimes(\ell-1)}) \star_{\hbar\omega} \mathcal{T}_{n-\ell,m}^{\hbar\Delta_F}(V_g^{\otimes(n-\ell)}\otimes F_f)\\
&+(-1)^\ell\binom{n-1}{\ell}\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}}(V_g^{\otimes\ell}) \star_{\hbar\omega} \mathcal{T}_{n-\ell-1,m}^{\hbar\Delta_F}(V_g^{\otimes(n-\ell-1)}\otimes F_f)\Bigg)\Bigg],
\end{aligned} \tag{4.8}$$

where, for later convenience, in the first line of Eq. (4.8) we have isolated the terms due to $\ell = 0$ and $\ell = n$. Observe that the sum in Eq. (4.8) is telescopic: the contribution from $\ell = j$ in the first term under the sum is

$$(-1)^j \binom{n-1}{j-1} \mathcal{T}_{j-1}^{\hbar\Delta_{\mathrm{AF}}}(V_g^{\otimes(j-1)}) \star_{\hbar\omega} \mathcal{T}_{n-j,m}^{\hbar\Delta_F}(V_g^{\otimes(n-j)} \otimes F_f),$$

while the case $\ell = j-1$ in the second term under the sum reads

$$(-1)^{j-1} \binom{n-1}{j-1} \mathcal{T}_{j-1}^{\hbar\Delta_{\mathrm{AF}}}(V_g^{\otimes(j-1)}) \star_{\hbar\omega} \mathcal{T}_{n-j,m}^{\hbar\Delta_F}(V_g^{\otimes(n-j)} \otimes F_f).$$

A direct inspection entails that these two contributions cancel each other. As a consequence, only the case $\ell = 1$ in the first term and the case $\ell = n-1$ in the second term survive yielding

$$\begin{aligned} &R_{n,m}(V_g^{\otimes n}, F_f) \\ &= \left(\frac{i}{\hbar}\right)^n \tilde{V}_g \Big[\mathcal{T}_{n-1,m}^{\hbar\Delta_F}(V_g^{\otimes(n-1)} \otimes F_f) + (-1)^n \mathcal{T}_{n-1}^{\hbar\Delta_{\mathrm{AF}}}(V_g^{\otimes(n-1)}) \star_{\hbar\omega} \mathcal{T}_{0,m}^{\hbar\Delta_F}(F_f) \\ &\quad - \mathcal{T}_{n-1,m}^{\hbar\Delta_F}(V_g^{\otimes(n-1)} \otimes F_f) + (-1)^{n-1} \mathcal{T}_{n-1}^{\hbar\Delta_{\mathrm{AF}}}(V_g^{\otimes(n-1)}) \star_{\hbar\omega} \mathcal{T}_{0,m}^{\hbar\Delta_F}(F_f) \Big] = 0\,. \end{aligned}$$

This concludes the proof of item 1.
Focusing instead on item 2., the rationale at the heart of the proof consists of writing

$$\mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}}(V_g \otimes \ldots \otimes V_g) \star_{\hbar\omega} \mathcal{T}_{n-\ell,m}^{\hbar\Delta_F}(V_g \otimes \cdots \otimes V_g \otimes F),$$

in terms of connected time-ordered products, see Definition 4.2, and to use the ensuing expression in Eq. (4.2) obtaining

$$\begin{aligned} R_{n,m}(V_g^{\otimes n}, F) = \left(\frac{i}{\hbar}\right)^n \sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|} \sum_{P \subset \mathrm{Part}\{I\}} \sum_{Q \subset \mathrm{Part}\{I^c \bigsqcup \{1,\ldots,m\}\}} \\ \left(\prod_{p \in P} \mathcal{T}_{|p|}^{\hbar\Delta_{\mathrm{AF}}}(p)^c \right) \star_{\hbar\omega} \left(\prod_{q \in Q} \mathcal{T}_{|q|}^{\hbar\Delta_F}(q)^c \right). \end{aligned}$$

On account of the dependence of (anti)time-ordered products on $\hbar$ as per Eq. (4.5), it follows that

$$\prod_{p \in P} \mathcal{T}_{|p|}^{\hbar\Delta_{\mathrm{AF}}}(p)^c = \mathcal{O}(\hbar^{|I|-|P|}), \qquad \prod_{q \in Q} \mathcal{T}_{|q|}^{\hbar\Delta_F}(q)^c = \mathcal{O}(\hbar^{|I^c|+m-|Q|}). \quad (4.9)$$

Yet, resorting to point 1. of this proposition, the only non-vanishing contributions are those where all the vertices associated with V_g are contracted with at least one of F. Since F has m vertices, among these contributions the one with lowest order in $\hbar$ are those with m disconnected components, each of which encompasses precisely one vertex of F. Hence we can infer that, overall, there exist at least $n \geqslant |P| + |Q| - m$ contractions. This is due to the fact that, denoting by $(X_j, Y_j, y_j)_{j=1}^m$ the vertices associated with such connected components, there are at least $|X_j| + |Y_j| - 1$ contractions for any

$j = 1, \ldots, m$. As a consequence, since $\sum_{j=1}^{m} |X_j| = |P|$ and $\sum_{j=1}^{m} |Y_j| = |Q|$, we conclude that the minimum number of contractions is $n \geqslant |P| + |Q| - m$.

To conclude, on account of Eq. (4.9), the lowest order in $\hbar$ contributing to the sum is

$$(|I| - |P|) + (|I^c| + m - |Q|) + (|P| + |Q| - m) = |I| + |I^c| = n,$$

compensating exactly the overall factor $\left(\frac{i}{\hbar}\right)^n$. □

Remark 4.4 It is important to observe that Proposition 4.3, and in particular point 2. entails that, at any perturbative order in $\lambda > 0$, the classical limit exists.

On top of Remark 4.4, being the classical limit of $R_{\lambda V_g}(F)$ well defined, the remaining task is to prove that it coincides with $r_{\lambda V_g}(F)$ obtained via the classical Möller map as per Eq. (2.20). To this end, we recall that the algebra of classical observables is endowed with the Poisson brackets

$$\{\varphi(x), \varphi(y)\} = \Delta(x - y).$$

Classical interacting observables are obtained as formal power series in the coupling parameter λ having coefficients in $\mathcal{F}^V(\mathbb{R}^2)$, see Eq. (2.20). For the sake of readability we recall the formula here, at the level of integral kernel and adopting the notation $Y = (y_1, \ldots, y_m)$,

$$r_{\lambda V_g}(F)(Y) = \sum_{n\geqslant 0} \lambda^n \int_{t_1 \leqslant \ldots \leqslant t_n \leqslant t} \mathrm{d}\mu_{x_1} \ldots \mathrm{d}\mu_{x_n} g(x_1) \ldots g(x_n) \{V(x_1), \{V(x_2), \ldots \{V(x_n), F(Y)\} \ldots\}\}. \tag{4.10}$$

where the dependence of the interacting vertices from the field configuration has been omitted to simplify the notation. As already stated above, we focus on proving that the quantum interacting observables built in terms of retarded products via the Bogoliubov map, see Eq. (4.2), converge to their classical counterpart represented by Eq. (4.10), see [14, Sec. 5.3] for further details. The main ingredient that we employ is the following lemma.

Lemma 4.5 *Let $h, f, g \in \mathcal{D}'(\mathbb{R}^2)$ be such that* $\mathrm{supp}(h)$ *is contained in the past of a fixed Cauchy surface Σ, while* $\mathrm{supp}(f)$ *and* $\mathrm{supp}(g)$ *are contained in its future. Then, for any $n \in \mathbb{N}$, if holds that*

$$R_{n+1,m}(V_h \otimes V_g^{\otimes n}, F_f) = -\frac{i}{\hbar}[V_h, R_{n,m}(V_g^{\otimes n}, F_f)]_{\hbar\omega}.$$

Proof A key rôle in this proof is played by the combinatorial argument in the proof of item 1. of Proposition 4.3, the factor $\tilde{V}_g$ being replaced by V_h. *Mutatis mutandis*, it holds that

$$\begin{aligned} &R_{n+1,m}(V_h \otimes V_g^{\otimes n}, F_f) \\ &= \left(\frac{i}{\hbar}\right)^{n+1} \Bigg[\mathcal{T}_{n+1,m}^{\hbar\Delta_F}(V_h \otimes V_g^{\otimes n} \otimes F_f) + (-1)^{n+1} \mathcal{T}_{n+1}^{\hbar\Delta_{\mathrm{AF}}}(V_h \otimes V_g^{\otimes n}) \star_{\hbar\omega} \mathcal{T}_{0,m}^{\hbar\Delta_F}(F_f) \\ &\quad + \sum_{\ell=1}^{n} (-1)^\ell \binom{n}{\ell - 1} \mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}}(V_h \otimes V_g^{\otimes(\ell-1)}) \star_{\hbar\omega} \mathcal{T}_{n+1-\ell,m}^{\hbar\Delta_F}(V_g^{\otimes(n+1-\ell)} \otimes F_f) \\ &\quad + (-1)^\ell \binom{n}{\ell} \mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}}(V_g^{\otimes\ell}) \star_{\hbar\omega} \mathcal{T}_{n+1-\ell,m}^{\hbar\Delta_F}(V_h \otimes V_g^{\otimes(n-\ell-1)} \otimes F_f) \Bigg]. \end{aligned} \tag{4.11}$$

Resorting now to Eq. (2.16) and to its anti-time ordered counterpart, we can reformulate Eq. (4.11) as

$$\begin{aligned}
&R_{n+1,m}(V_h \otimes V_g^{\otimes n}, F_f) \\
&= \left(\frac{i}{\hbar}\right)^{n+1} \Big[\mathcal{T}^{\hbar\Delta_F}_{n,m}(V_g^{\otimes n} \otimes F_f) \star_{\hbar\omega} V_h + (-1)^{n+1} V_h \star_{\hbar\omega} \mathcal{T}^{\hbar\Delta_{\mathrm{AF}}}_n(V_g^{\otimes n}) \star_{\hbar\omega} \mathcal{T}^{\hbar\Delta_F}_{0,m}(F_f) \\
&+ \sum_{\ell=1}^{n} (-1)^{\ell} \binom{n}{\ell-1} V_h \star_{\hbar\omega} \mathcal{T}^{\hbar\Delta_{\mathrm{AF}}}_{\ell}(V_g^{\otimes(\ell-1)}) \star_{\hbar\omega} \mathcal{T}^{\hbar\Delta_F}_{n+1-\ell,m}(V_g^{\otimes(n+1-\ell)} \otimes F_f) \\
&+ (-1)^{\ell}\binom{n}{\ell} \mathcal{T}^{\hbar\Delta_{\mathrm{AF}}}_{\ell}(V_g^{\otimes \ell}) \star_{\hbar\omega} \mathcal{T}^{\hbar\Delta_F}_{n+1-\ell,m}(V_g^{\otimes(n-\ell-1)} \otimes F_f) \star_{\hbar\omega} V_h \Big] \\
&= \frac{i}{\hbar}\Big[R_{n,m}(V_g^{\otimes n}, F_f) \Big] \star_{\hbar\omega} V_h - \frac{i}{\hbar} V_h \star_{\hbar\omega} \Big[R_{n,m}(V_g^{\otimes n}, F_f) \Big] \\
&= -\frac{i}{\hbar} [V_h, R_{n,m}(V_g^{\otimes n}, F_f)]_{\hbar\omega},
\end{aligned}$$

which entails the sought statement. □

Let us consider a family of points $(x_1, \ldots, x_n) \in \mathbb{R}^{2n}$ such that $x_i \neq x_j$ if $i \neq j$. At the level of integral kernels and, out of an iterative application of Lemma 4.5, *cf.* [14, Sec. 5.3], denoting by $X' = (x'_1, \ldots, x'_m)$ and by $t' := \min\{t'_1, \ldots, t'_m\}$ we obtain

$$\begin{aligned}
&R_{n,m}(V(x_1)\ldots V(x_n), F(X')) \\
&= \left(-\frac{i}{\hbar}\right)^n \sum_{\pi \in \mathcal{S}_n} \vartheta(t' - t_{\pi(n)})\vartheta(t_{\pi(n)} - t_{\pi(n-1)}) \ldots \vartheta(t_{\pi(2)} - t_{\pi(1)}) \times \\
&\times [V(x_{\pi(1)}), [V(x_{\pi(2)}), \ldots [V(x_{\pi(n)}), F(X')]_{\hbar\omega} \ldots]_{\hbar\omega}]_{\hbar\omega}, \qquad (4.12)
\end{aligned}$$

where $\mathcal{S}_n$ is the group of permutations of n-indices. Eq. (4.12), together with the convergence result for the series defining the quantum interacting observables, see Remark 4.4, entails that [14]

$$\lim_{\hbar \to 0^+} R_{\lambda V_g}(F) = r_{\lambda V_g}(F), \tag{4.13}$$

since the quantisation procedure is designed in such a way that

$$-\frac{i}{\hbar}[\cdot, \cdot]_{\hbar\omega} \to \{\cdot, \cdot\}, \qquad \text{as} \quad \hbar \to 0^+.$$

Remark 4.6 This argument confirms the result of Proposition 4.3 since it shows that the above is a Taylor series expansion in $\hbar$ and not a Laurent one as for the S-matrix. We stress that, for a finite value of $\hbar$, Eq. (4.12) involves a number of contributions which is strictly larger than that of those involved in its classical counterpart, Eq. (4.10). This is due to the fact that the quantum commutators $[\cdot, \cdot]_{\hbar\omega}$ are constructed with respect to an exponential product, while this is not the case for the classical Poisson bracket $\{\cdot,\cdot\}$. Using a language typical of quantum field theory, we could rephrase the statement as a consequence of the fact that, in an interacting quantum theory, at a perturbative level, one has to take into account loop diagrams, which do not have a counterpart in the classical scenario.

As a last step the existence of the classical limit must be translated to the study of the underlying stochastic sine-Gordon model.

Theorem 4.7 *Let $F = \Phi^{\otimes m} \in \mathcal{F}_{\mu c}(\mathbb{R}^2)$, $m \in \mathbb{N}$. Denoting by Γ_Q the deformation map as per Eq. (2.30), by $R_{\lambda V_g}(F)$ the interacting version of the observable $F \in \mathcal{F}_{\mu c}(\mathbb{R}^2)$, see Eq. (2.22) and by $r_{\lambda V_g}(F)$ the corresponding classical interacting observable obtained via the classical Möller map introduced in Eq. (2.20), it holds that*

$$\lim_{\hbar\to 0^+} \Gamma_Q[R_{\lambda V_g}(F)] = \Gamma_Q[r_{\lambda V_g}(F)].$$

Proof This result is a direct consequence of Proposition 4.3. As a matter of fact we observe that

$$\lim_{\hbar\to 0^+} \Gamma_Q[R_{\lambda V_g}(F)] = \lim_{\hbar\to 0^+} \lim_{k\to\infty} \sum_{n=0}^{k} \Gamma_Q[R_{\lambda V_g}(F)]_n = \lim_{\hbar\to 0^+} \lim_{k\to\infty} \sum_{n=0}^{k} \Gamma_Q[R_{n,m}(F)],$$

and we recall that $R_{n,m}(F)$ can be decomposed in the sum of a term of order zero in $\hbar$, which coincides with $r_{n,m}(F)$, and of additional terms all of order $\mathcal{O}(\hbar)$, denoted by $\mathfrak{O}_n$:

$$R_{n,m}(F) = r_{n,m}(F) + \mathfrak{O}_{n,m}(F).$$

Thus,

$$\lim_{\hbar\to 0^+} \lim_{k\to\infty} \sum_{n=0}^{k} \Gamma_Q[r_{n,m}(F)] = \lim_{k\to\infty} \sum_{n=0}^{k} \Gamma_Q[r_{n,m}(F)] = \Gamma_Q[r_{\lambda V_g}(F)],$$

where, in the first step, we exploited that $\lim_{k\to\infty} \sum_{n=0}^{k} \Gamma_Q[r_{n,m}(F)]$ does not depend on $\hbar$ while the second one is a direct consequence of Eq. 4.13) together with the convergence results proven in Sect. 3. Then, the following chain of identities is satisfied

$$\begin{aligned}\lim_{\hbar\to 0^+} \Gamma_Q[R_{\lambda V_g}(F)] &= \Gamma_Q[r_{\lambda V_g}(F)] + \lim_{\hbar\to 0^+} \lim_{k\to\infty} \sum_{n=0}^{k} \Gamma_Q[\mathfrak{O}_n] \\ &= \Gamma_Q[r_{\lambda V_g}(F)] + \lim_{\hbar\to 0^+} \lim_{k\to\infty} \sum_{n=0}^{k} \hbar\Gamma_Q[\hbar^{-1}\mathfrak{O}_n] \\ &= \Gamma_Q[r_{\lambda V_g}(F)] + \lim_{\hbar\to 0^+} \hbar \lim_{k\to\infty} \sum_{n=0}^{k} \Gamma_Q[\hbar^{-1}\mathfrak{O}_n].\end{aligned}$$

We observe that, on account of Proposition 4.3, $\lim_{k\to\infty} \sum_{n=0}^{k} \Gamma_Q[\hbar^{-1}\mathfrak{O}_n]$ is absolutely convergent and bounded for $\hbar \to 0^+$ and therefore $\lim_{\hbar\to 0^+} \hbar \lim_{k\to\infty} \sum_{n=0}^{k} \Gamma_Q [\hbar^{-1}\mathfrak{O}_n] = 0$. As a consequence

$$\lim_{\hbar\to 0^+} \Gamma_Q[R_{\lambda V_g}(F)] = \Gamma_Q[r_{\lambda V_g}(F)],$$

which is the sought after identity. □

Acknowledgements The authors are grateful to Nicola Pinamonti for the enlightening discussions about the topic. We are grateful to Beatrice Costeri for the comments on this manuscript. The work of A.B is supported by a PhD fellowship of the University of Pavia and partly by the GNFM-Indam Progetto Giovani *Feynman propagator for Dirac fields: a microlocal analytic approach*, CUP E53C22001930001, whose support is gratefully acknowledged. A.B. is also grateful to the Sorbonne Université for the kind hospitality during the realization of part of this project. The research of P.R. is supported by a postdoc fellowship of the University of Bonn, which is gratefully acknowledged.

Data Availability Statement We declare that the manuscript has no associated data and hence no data set has been used in the realization of this work.

Declarations

Conflict of interest There are no financial and non-financial Conflict of interest.

Ethical Approval & Informed Consent Not applicable.

A. Comparison with a Perturbative Approach

In this short appendix, we compare the non-perturbative results concerning the stochastic sine-Gordon model constructed in this work with the one that would be obtained by means of a purely perturbative approach to the solution theory of the stochastic Sine-Gordon equation (2.38). We observe that Proposition 4.3 provides a rationale for computing at a perturbative level the coefficients $\Gamma_Q[r_{n,m}(F)]$ of the expectation value of any observable F. Yet, for the sake of simplicity, we shall discuss here only the case of an interacting field $\Gamma_Q[r_{\lambda V_g}(\Phi_f)]$, i.e., a solution to Eq. (2.38). Our goal is to confirm that the results in this paper are in agreement with a perturbative approach at any order in the coupling parameter. Hence we first compute the lower order contributions to the solution of the functional version of Eq. (2.38), i.e.,

$$\begin{aligned}\Psi &= \lambda a \Delta^R \circledast g \sin(a\Psi) + \Phi_f \\ &= -\lambda \Delta^R \circledast V_g^{(1)}(\Psi) + \Phi_f, \quad \Psi[\![\lambda]\!] = \sum_{n=0}^{\infty} \lambda^n \Psi_n, \quad \Psi_n \in \mathcal{F}_{loc},\end{aligned} \tag{A.1}$$

where $\circledast$ denotes the stochastic convolution, see e.g. [11], Φ_f is the functional introduced in Example 2.4 with $f \in \mathcal{D}(\mathbb{R}^2)$, while $V_g^{(1)}$ is the first order functional derivative of the vertex functional introduced in Eq. (2.4). The coefficients of the perturbative expansion $\Psi[\![\lambda]\!]$ can be computed via the formula

$$\Psi_n := \frac{1}{n!} \frac{d^n}{d\lambda^n} \Psi[\![\lambda]\!] \Big|_{\lambda=0}. \tag{A.2}$$

Focusing the attention on to the lower order contributions, Eq. (A.2) yields

$$\begin{aligned}\Psi_0 &= \Phi, \\ \Psi_1 &= \Delta^R \circledast V_g^{(1)} \\ \Psi_2 &= \Delta^R \circledast (V_g^{(2)} \Delta^R \circledast V_g^{(1)}),\end{aligned} \tag{A.3}$$

where $V_g^{(1)}$ is as per Eq. (3.25), while the second order functional derivative $V_g^{(2)}$ can be defined accordingly. We claim that these coefficients coincide with those obtained via the classical Möller map as per Eq. (2.39). To wit, we calculate the quantum counterpart of the interacting field at a given order and, subsequently, we take the classical limit which has been proven to coincide with $r_{\lambda V_g}(\psi)$, see [14]. Specializing Eqs. (4.1) and (4.2) to $F = \Phi_f$, it holds that

$$R_{\lambda V_g}(\Phi_f) = \sum_{n \geqslant 0} \frac{\lambda^n}{n!} R_{n,m}(V_g^{\otimes n}, \Phi_f),$$

where

$$\begin{aligned} &R_{n,m}(V_g^{\otimes n}, \Phi_f) \\ &= \left(\frac{i}{\hbar}\right)^n \sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^\ell \mathcal{T}_\ell^{\hbar\Delta_{\mathrm{AF}}}(V_g \otimes \ldots \otimes V_g) \star_{\hbar\omega} \mathcal{T}_{n-\ell,m}^{\hbar\Delta_F}(V_g \otimes \cdots \otimes V_g \otimes \Phi_f), \end{aligned}$$

is the n-th perturbative contribution to the interacting field, up to a factor $n!$ at the denominator. Proposition 4.3 highlights how the only non-vanishing contributions are those for which all the n terms V_g are contracted at least once with the linear field Φ_f. This implies that $R_{n,m}(V_g^{\otimes n}, F) = \mathcal{O}(\hbar^0)$. The only contributions surviving the limiting procedure are those with *exactly* n-contractions, the terms with more being multiplied by a positive power of $\hbar$, in agreement with the absence of loops in the classical framework. The trivial case $n = 0$ yields $R_{0,1}(\Phi_f) = \Phi_f$. Considering the case $n = 1$, it holds that

$$\begin{aligned} R_{1,1}(V_g, \Phi_f) &= \frac{i}{\hbar}[\mathcal{T}_{1,1}^{\hbar\Delta_F}(V_g \otimes \Phi_f) - \mathcal{T}_1^{\hbar\Delta_{\mathrm{AF}}}(V_g) \star_{\hbar\omega} \mathcal{T}_{0,1}^{\hbar\Delta_F}(\Phi_f)] \\ &= \frac{i}{\hbar}[V_g \star_{\hbar\Delta_F} \Phi_f - V_g \star_{\hbar\omega} \Phi_f]. \end{aligned}$$

Since only the contributions with exactly one contraction contribute to the classical case, it descends that

$$r_{1,1}(V_g, \Phi_f) = \frac{i}{\hbar}\langle V_g^{(1)}, \hbar(\Delta_F - \omega) f\rangle = -\langle V_g^{(1)}, \Delta^A f\rangle,$$

where, in the last identity, we used that $\Delta_F - \omega = i\Delta^A$, see Eq. (2.18). We conclude that

$$r_{1,1}(V_g, \Phi_f) = -\langle \Delta^R V_g^{(1)}, f\rangle. \tag{A.4}$$

To better grasp the structure of a perturbative solution of Eq. (2.38), the analysis of the second order is way more enlightening. At the quantum level it reads

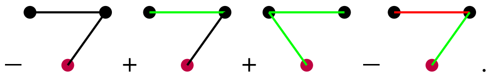
$$\begin{aligned} R_{2,1}(V_g \otimes V_g, \Phi_f) = &-\frac{1}{\hbar^2}[\mathcal{T}_{2,1}^{\hbar\Delta_F}(V_g \otimes V_g \otimes \Phi_f) - 2\mathcal{T}_1^{\hbar\Delta_{\mathrm{AF}}}(V_g) \star_{\hbar\omega} \mathcal{T}_{1,1}^{\hbar\Delta_F}(V_g \otimes \Phi_f) \\ &+ \mathcal{T}_2^{\hbar\Delta_{\mathrm{AF}}}(V_g \otimes V_g) \star_{\hbar\omega} \mathcal{T}_{0,1}^{\hbar\Delta_F}(\Phi_f)] \\ = &-\frac{1}{\hbar^2}[V_g \star_{\hbar\Delta_F} V_g \star_{\hbar\Delta_F} \Phi_f - 2V_g \star_{\hbar\omega} (V_G \star_{\hbar\Delta_F} \Phi_f) \\ &+ V_g \star_{\hbar\Delta_{AF}} (V_g \star_{\hbar\omega} \Phi_f)]. \end{aligned}$$

As proven, the only non-vanishing contributions to the classical limit are those where there are exactly two contractions and where all the interacting vertices are contracted with Φ_f. To analyze this scenario, we can resort to a graphical representation. The rationale is to encode the information carried by the underlying kernels within graphs subordinated to the following rules:

- black dots represent occurrences of the vertex functionals V_g, while purple dots indicate the linear field Φ_f,
- the occurrence of a kernel is denoted by a segment joining the contracted functionals. To distinguish all possible bi-distributions involved in the computation, we associate black edges to $\hbar\Delta_F$, green edges to $\hbar\omega$ and red ones to $\hbar\Delta_{AF}$.

In view of these rules, taking into account the relevant multiplicities and the Leibniz rule, we obtain

$$\frac{1}{2!}r_{2,1}(V_g \otimes V_g, \Phi_f) = \qquad \text{(A.5)}$$



At the level of integral kernels the expression for the classical solution at second order in perturbation theory reads

$$\begin{aligned}
&\frac{1}{2!}r_{2,1}(V_g \otimes V_g, \Phi_f)\\
&= -\langle V_g^{(1)}, \Delta_F \langle V_g^{(2)}, \Delta_F f\rangle\rangle + \langle V_g^{(1)}, \omega\langle V_g^{(2)}, \Delta_F f\rangle\rangle + \langle\langle V_g^{(2)}\rangle, \omega f, \omega V_g^{(1)}\rangle\\
&\quad -\langle V_g^{(1)}, \Delta_{AF}\langle V_g^{(2)}, \omega f\rangle\rangle\\
&= -\langle V_g^{(1)}, (\Delta_F - \omega)\langle V_g^{(2)}, \Delta_F f\rangle\rangle + \langle\langle V_g^{(2)}\rangle, \omega f, \omega V_g^{(1)}\rangle - \langle V_g^{(1)}, \Delta_{AF}\langle V_g^{(2)}, \omega f\rangle\rangle\\
&= -\langle V_g^{(1)}, i\Delta^A\langle V_g^{(2)}, \Delta_F f\rangle\rangle + \langle V_g^{(1)}, (\Delta_{AF} - i\Delta^A)\langle V_g^{(2)}, \omega f\rangle\rangle - \langle V_g^{(1)}, \Delta_{AF}\langle V_g^{(2)}, \omega f\rangle\rangle\\
&= -\langle V_g^{(1)}, i\Delta^A\langle V_g^{(2)}, \Delta_F f\rangle\rangle + \langle V_g^{(1)}, i\Delta^A\langle V_g^{(2)}, \omega f\rangle\rangle\\
&= \langle V_g^{(1)}, \Delta^A\langle V_g^{(2)}, \Delta^A f\rangle\rangle,
\end{aligned} \qquad \text{(A.6)}$$

where, in the third equality, we used once more the identity $\Delta_F - \omega = i\Delta^A$ as well as

$$\begin{aligned}
\langle\langle V_g^{(2)}, \omega f\rangle, \omega V_g^{(1)}\rangle &= \langle \omega V_g^{(1)}, \langle V_g^{(2)}, \omega f\rangle\rangle = \langle(\Delta_{AF} + i\Delta^R)V_g^{(1)}, \langle V_g^{(2)}, \omega f\rangle\rangle\\
&= \langle V_g^{(1)}, (\Delta_{AF} + i\Delta^A)\langle V_g^{(2)}, \omega f\rangle\rangle.
\end{aligned}$$

A direct inspection shows that, up to tje second order in λ, the perturbative coefficients calculated both via the perturbative expansion as per Eq. (A.3) and via a classical limiting procedure, as per Eqs. (A.4) and (A.6), coincide.

Eventually, an analogous comparison can be carried over at the level of expectation values, the only difference being the action of the map Γ_Q implementing additional contractions. Even if from the standpoint of a perturbative analysis of the SPDE this boils down to an algorithmic procedure, its field theoretical counterpart relies heavily on Theorem 4.7.

Remark A.1 This graphical representation of the contributions to the classical version of the interacting field has the net advantage of allowing for an immediate extension to the computation of expectation values. As explained in Sect. 2.2, such information is encoded at the algebraic level by contracting pairs of fields into a kernel $Q \in \mathcal{D}'(\mathbb{R}^2 \times \mathbb{R}^2)$ which encodes the stochastic properties of the free random field. Hence the graphical counterpart of this operation amounts to adding a colored edge representing Q. At a practical level the problem of computing expectation values of products of fields boils down to finding all admissible maximally connected graphs having a fixed number of vertices and edges, including the two-point function of the free random field. This viewpoint, which sheds light on how the stochastic information is handled at the same level of the quantum features of the problem, simplifies a two-steps procedure within the perturbative study of the solution, namely building the desired perturbative coefficients and applying the deformation map $\Gamma_{\star_Q}$ evaluated at the zero configuration.

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Communicated by C. Schweigert