Brief paper

Active fault diagnosis: A multi-parametric approach☆

G. Roberto Marseglia, Davide M. Raimondo

Department of Electrical, Computer and Biomedical Engineering, University of Pavia, Pavia, Italy

ARTICLE INFO

ABSTRACT

This paper considers the design of an input signal for minimizing the time and energy required to detect and isolate faults in the outputs of a system. Faults are represented by discrete switches between affine models with bounded disturbances and bounded measurement errors. Within this framework, previous work has demonstrated that a minimally harmful input guaranteeing fault diagnosis can be obtained by solving a Mixed Integer Quadratic Program (MIQP). A closed-loop approach allows to reduce the length and/or norm of this input by solving an MIQP at each time instant with the newly available measurements. However, solving such programs online can be computationally demanding. In this paper, we employ multi-parametric (mp) programming to move most of the computation offline, thus allowing the application of the closed-loop approach to fast processes. Still, the mp-MIQP complexity becomes quickly prohibitive as the number of faulty models increases. In order to overcome this problem, we propose a strategy based on mp-optimization and graph theory that takes into account only two models at a time. While this approach is suboptimal compared to the case in which all the models are considered simultaneously, simulations show that, in practice, the performance is comparable.

© 2017 Elsevier Ltd. All rights reserved.

1. Introduction

The increasing complexity of industrial processes has made Fault Detection and Isolation (FDI) difficult for human operators. For this reason, several automated procedures have been developed over the years, e.g. Chen and Patton (2012), Frank and Seliger (2012), Patton, Frank, and Clark (2013), Russell, Chiang, and Braatz (2012). *Passive* FDI schemes, for example, determine the presence of faults by exploiting input–output data and prior knowledge of the process. Although often effective, these schemes may be slow in detecting anomalies since based on observations only. A significant reduction in the time required for diagnosis may be obtained by suitably modifying the inputs of the process. This type of approaches, named "active FDI", has received increasing attention in recent years (Ashari, Nikoukhah, & Campbell, 2012a, 2011; Cheong & Manchester, 2015; Nikoukhah, 1998; Poulsen & Niemann, 2008; Punčochář & Šimandl, 2014; Punčochář, Širokỳ, & Šimandl, 2015; Scott, Findeisen, Braatz, & Raimondo, 2014; Streif, Petzke, Mesbah, Findeisen, & Braatz, 2014; Tabatabaeipour, 2015; Xu, Olaru, Puig, Ocampo-Martínez, & Niculescu, 2014; Yang, Hamelin, & Sauter, 2014). The focus of this paper is on deterministic active FDI. In particular, we build on the method proposed in Scott

et al. (2014) where faults are described by a set of affine models (with bounded uncertainties/disturbances) and a minimally harmful (in terms of length and norm) input sequence guaranteeing fault diagnosis is obtained by solving a set of MIQPs. According to Ashari, Nikoukhah, and Campbell (2012b), Raimondo, Braatz, and Scott (2013), rather than injecting the entire sequence, better performance can be obtained by re-solving the optimization problem at each time step using the newly available measurements (closed-loop approach). Still, solving such optimization at each time step can be computationally prohibitive for many applications. In this paper we use multi-parametric programming (Dua, Bozinis, & Pistikopoulos, 2002) to alleviate the online complexity of closed-loop active FDI. Mp-programming allows to express the solution of an optimization problem as an explicit function of the parameters, thus reducing the online computation to a simple function evaluation. To the best of our knowledge, mp-programming has never been used in the context of active FDI. However, an attempt to give an explicit solution to the problem was made in Raimondo et al. (2013), where the proposed method was based on the gridding of the parameter space. Even though simulations show that mp-optimization outperforms gridding, the offline complexity of the mp-approach scales badly with the number of faults taken into account. To improve scalability, we also propose an approach that considers mp-programs involving only two models at a time. This strategy makes use of a graph to select in which order to execute the different programs to still provide a guaranteed FDI and maximize performance. Simulations show that the proposed approach

outperforms online strategies involving only two models at a time and provides comparable performance to the online approach that considers all models simultaneously.

2. Problem statement

Consider a discrete-time affine system whose dynamics obeys one of n_m possible models (known and observable)

$$\mathbf{x}_{k+1}^{[i]} = \mathbf{A}^{[i]}\mathbf{x}_k^{[i]} + \mathbf{B}^{[i]}\mathbf{u}_k + \mathbf{B}_w^{[i]}\mathbf{w}_k + \mathbf{r}^{[i]}, \tag{1}$$

$$\mathbf{y}_k^{[i]} = \mathbf{C}^{[i]}\mathbf{x}_k^{[i]} + \mathbf{D}_v^{[i]}\mathbf{v}_k + \mathbf{s}^{[i]} \tag{2}$$

where $\mathbf{x}_k^{[i]} \in \mathbb{R}^{n_x}, \mathbf{y}_k^{[i]} \in \mathbb{R}^{n_y}$ and $\mathbf{u}_k \in U \subset \mathbb{R}^{n_u}$ denote the states, the measured outputs and the inputs respectively, with $\mathbf{x}_0^{[i]} \in X_0^{[i]} \subset \mathbb{R}^{n_x}$ the state initial condition ($k \geq 0$). Vectors $\mathbf{w}_k \in W \subset \mathbb{R}^{n_w}$, $\mathbf{v}_k \in V \subset \mathbb{R}^{n_v}$ represent the disturbance and the measurement noise. Constant vectors $\mathbf{r}^{[i]}$ and $\mathbf{s}^{[i]}$ are used to model additive faults, e.g. actuator offset and sensor bias and are considered to be known. $X_0^{[i]}, U, W$ and V are zonotopes (see Section 3) known a priori. In the following, we assume $\mathbf{C}^{[i]}$ is invertible, for all $i \in \mathcal{M}$.

The objective of active FDI is to determine which dynamics the process obeys to. To achieve this goal, we look for the shortest sequence $(\mathbf{u}_0, \ldots, \mathbf{u}_{N-1})$ such that any possible output at time N is consistent with only one $i \in \mathcal{M}$. Since multiple input sequences of minimal length N may satisfy this requirement, we select among them the one which minimizes a given cost. Moreover, rather than applying the entire sequence, we do inject only the first element and re-compute a new input sequence at the next time step, taking advantage of the newly available information (closed-loop approach). In the following, it is assumed that only one model is active during $[0, \ldots, N]$, i.e. the diagnosis is fast enough to avoid the switching between models in this time window.

3. Notation and basic definitions

Below, a tilde is used to indicate a sequence associated with (1)–(2). When referring to $\tilde{\mathbf{u}}_{l:k}$ or $\tilde{\mathbf{w}}_{l:k}$, the notation stands for $\tilde{\sigma}_{l:k} = (\sigma_l, \ldots, \sigma_{k-1})$ while, for $\tilde{\mathbf{x}}_{l:k}, \tilde{\mathbf{y}}_{l:k}, \tilde{\mathbf{v}}_{l:k}, \tilde{\sigma}_{l:k} = (\sigma_l, \ldots, \sigma_k)$. Similarly, one has $\tilde{\sigma}_k = (\sigma_0, \ldots, \sigma_{k-1})$ or $\tilde{\sigma}_k = (\sigma_0, \ldots, \sigma_k)$. The notation $\tilde{\sigma}_{l:k|l}$ indicates that the sequence is computed at time l. With $\tilde{\Sigma}_k$ we denote the kth cartesian product of a set Σ ($\Sigma \times \cdots \times \Sigma$). Given $(\tilde{\mathbf{u}}_k, \mathbf{x}_0^{[i]}, \tilde{\mathbf{w}}_k, \mathbf{v}_k) \in \mathbb{R}^{n_u k} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_w k} \times \mathbb{R}^{n_v}$, the state and output of model i, k-steps ahead, are given by the functions $\boldsymbol{\phi}_{(k)}^{[i]}(\tilde{\mathbf{u}}_k, \mathbf{x}_0^{[i]}, \tilde{\mathbf{w}}_k)$ and $\boldsymbol{\psi}_{(k)}^{[i]}(\tilde{\mathbf{u}}_k, \mathbf{x}_0^{[i]}, \tilde{\mathbf{w}}_k, \mathbf{v}_k)$ respectively, with $\boldsymbol{\phi}_{(k)}^{[i]}(\cdot, \cdot, \cdot)$ and $\boldsymbol{\psi}_{(k)}^{[i]}(\cdot, \cdot, \cdot, \cdot)$ the solution maps. *Zonotopes* are centrally symmetric convex polytopes (Guibas, Nguyen, & Zhang, 2003). Denoting with $\mathbf{G} = [\mathbf{g}_1 \ldots \mathbf{g}_{n_g}] \in \mathbb{R}^{n \times n_g}$ the generator matrix and with $\mathbf{c} \in \mathbb{R}^n$ the zonotope center, a set can be expressed as $Z = \{\mathbf{G}\boldsymbol{\xi} + \mathbf{c} : \|\boldsymbol{\xi}\|_\infty \leq 1\}$, and compactly indicated as $Z = \{\mathbf{G}, \mathbf{c}\}$. The order of a zonotope is defined as n_g/n. Define $\Omega_{k|k_0}^{[i]} = \bar{X}_{k_0}^{[i]} \times \tilde{W}_{k-k_0} \times V$ with $\bar{X}_{k_0}^{[i]} = \{\mathbf{G}_{\bar{X}_{k_0}}^{[i]}, \mathbf{c}_{\bar{X}_{k_0}}^{[i]}\}$, $\tilde{W}_{k-k_0} = \{\mathbf{G}_{\tilde{W}_{k-k_0}}, \mathbf{0}\}$, $V = \{\mathbf{G}_V, \mathbf{0}\}$. For each $i \in \mathcal{M}$, given $\Omega_{k|k_0}^{[i]}$ and $\tilde{\mathbf{u}}_{k_0:k|k_0}$, the *reachable sets* are defined as

$X_{k|k_0}^{[i]}(\tilde{\mathbf{u}}_{k_0:k|k_0}, \Omega_{k|k_0}^{[i]})$ (*state reachable set*)

$= \{\boldsymbol{\phi}_{(k-k_0)}^{[i]}(\tilde{\mathbf{u}}_{k_0:k|k_0}, \bar{\mathbf{x}}_{k_0}^{[i]}, \tilde{\mathbf{w}}_{0:k}) : (\bar{\mathbf{x}}_{k_0}^{[i]}, \tilde{\mathbf{w}}_{0:k}, \mathbf{v}_k) \in \Omega_{k|k_0}^{[i]}\}$

$Y_{k|k_0}^{[i]}(\tilde{\mathbf{u}}_{k_0:k|k_0}, \Omega_{k|k_0}^{[i]})$ (*output reachable set*)

$= \{\boldsymbol{\psi}_{(k-k_0)}^{[i]}(\tilde{\mathbf{u}}_{k_0:k|k_0}, \bar{\mathbf{x}}_{k_0}^{[i]}, \tilde{\mathbf{w}}_{0:k}, \mathbf{v}_k) : (\bar{\mathbf{x}}_{k_0}^{[i]}, \tilde{\mathbf{w}}_{0:k}, \mathbf{v}_k) \in \Omega_{k|k_0}^{[i]}\}$.

The dependence of $X_{k|k_0}^{[i]}$ on $\mathbf{v}_k$ is specified only to simplify notation. When clear from the context, the arguments of sets and maps will be omitted. The set $\bar{X}_{k_0}^{[i]}$ is the result of a *set-valued observer*

initialized at $k_0 = 0$, as $\bar{X}_{k_0}^{[i]} = X_0^{[i]} \cap \{(\mathbf{C}^{[i]})^{-1}\mathbf{D}_v^{[i]}\mathbf{G}_V, (\mathbf{C}^{[i]})^{-1}(\mathbf{y}_{k_0} - \mathbf{s}^{[i]})\}$ and, for $k_0 > 0$, obtained recursively

$$\bar{X}_{k_0+1}^{[i]} \supseteq \{\boldsymbol{\phi}_{(1)}^{[i]}(\mathbf{u}_{k_0|k_0}, \bar{\mathbf{x}}_{k_0}^{[i]}, \mathbf{w}_{k_0}) : (\bar{\mathbf{x}}_{k_0}^{[i]}, \mathbf{w}_{k_0}, \mathbf{v}_{k_0+1}) \in \Omega_{k_0+1|k_0}^{[i]}\}$$
$$\cap \{(\mathbf{C}^{[i]})^{-1}\mathbf{D}_v^{[i]}\mathbf{G}_V, (\mathbf{C}^{[i]})^{-1}(\mathbf{y}_{k_0+1} - \mathbf{s}^{[i]})\}. \tag{3}$$

Since the set on the right hand side of the equation above is difficult to compute exactly and may not be a zonotope (zonotopes are not closed under intersection (Scott, Raimondo, Marseglia, & Braatz, 2016)), it is conservatively outer approximated using, e.g., zonotopes with low complexity (Combastel, 2005), parallelotopes (Bravo, Alamo, & Camacho, 2006), or constrained zonotopes (Scott et al., 2016). Thanks to zonotope properties (5)–(7) in Scott, Marseglia, Magni, Braatz, and Raimondo (2013), by iterating (1)–(2), one obtains suitable matrices $\tilde{\mathbf{A}}_{k-k_0}^{[i]}, \tilde{\mathbf{B}}_{k-k_0}^{[i]}, \tilde{\mathbf{B}}_{w_{k-k_0}}^{[i]}$, etc. such that

$$X_{k|k_0}^{[i]} = \left\{ \overbrace{\left[\tilde{\mathbf{A}}_{k-k_0}^{[i]}\mathbf{G}_{\bar{X}_{k_0}}^{[i]} \quad \tilde{\mathbf{B}}_{w_{k-k_0}}^{[i]}\mathbf{G}_{\tilde{W}_{k-k_0}}\right]}^{\mathbf{G}_{X_{k|k_0}}^{[i]} \in \mathbb{R}^{n_x \times n_{gX}^{[i]}}}, \overbrace{\boldsymbol{\phi}_{(k-k_0)}^{[i]}(\tilde{\mathbf{u}}_{k_0:k|k_0}, \mathbf{c}_{\bar{X}_{k_0}}^{[i]}, \mathbf{0})}^{\mathbf{c}_{X_{k|k_0}}^{[i]}(\tilde{\mathbf{u}}_{k_0:k|k_0}) \in \mathbb{R}^{n_x}} \right\}$$

$$Y_{k|k_0}^{[i]} = \left\{ \underbrace{\left[\mathbf{C}^{[i]}\mathbf{G}_{X_{k|k_0}}^{[i]} \quad \mathbf{D}_v^{[i]}\mathbf{G}_V\right]}_{\mathbf{G}_{Y_{k|k_0}}^{[i]} \in \mathbb{R}^{n_y \times n_{gY}^{[i]}}}, \underbrace{\boldsymbol{\psi}_{(k-k_0)}^{[i]}(\tilde{\mathbf{u}}_{k_0:k|k_0}, \mathbf{c}_{\bar{X}_{k_0}}^{[i]}, \mathbf{0}, \mathbf{0})}_{\mathbf{c}_{Y_{k|k_0}}^{[i]}(\tilde{\mathbf{u}}_{k_0:k|k_0}) \in \mathbb{R}^{n_y}} \right\}. \tag{4}$$

Note that $\tilde{\mathbf{u}}_{k_0:k|k_0}$ affects only the center of these sets. For ease of reading, only the dependence of $\mathbf{c}_{X_{k|k_0}}^{[i]}$ and $\mathbf{c}_{Y_{k|k_0}}^{[i]}$ on $\tilde{\mathbf{u}}_{k_0:k|k_0}$ is made explicit (unless differently needed). In the following, we denote as *separating input sequence* of length N, any input sequence $\tilde{\mathbf{u}}_{k_0:k_0+N|k_0}$ able to guarantee the separation of all the output reachable sets after N steps, i.e. $Y_{k_0+N|k_0}^{[i]} \bigcap Y_{k_0+N|k_0}^{[j]} = \emptyset, \forall(i, j) \in \mathcal{M}, i \neq j$. By separating these sets, measurements will become consistent with one model only, thus leading to a guaranteed diagnosis.

4. The multi-parametric approach

The search for the shortest *separating input sequence* can be performed, for the case of zonotopic uncertain sets, as described in Scott et al. (2014). This approach, here named *open-loop approach* (OL), requires the solution of a set of MIQPs (for increasing horizon $N = 1, 2, \ldots$, until the separating condition is satisfied or a threshold N_{max} is attained) and provides, for the shortest feasible horizon, the input sequence of minimum norm. As suggested in Raimondo et al. (2013), rather than applying the entire sequence in open-loop, it is possible to enhance performance with a closed-loop approach (here named *closed-loop online approach*, CL-O). At each time step, the set-valued-observer in (3) is updated with the newly available measurements and a new input sequence is computed online (see Raimondo et al. (2013) for further detail). In order to reduce the computational complexity of CL-O, in Raimondo et al. (2013) the authors suggest a way to move most of the computation offline. The approach, (here named *closed-loop approach based on gridding*, CL-G), requires measurements to lie in a known hyperrectangle $\bar{Y}, \forall k_0 \geq 0$. By defining a partition $\{P_\sigma\}_{\sigma \in S}$ of $\bar{Y}$ into sets $P_\sigma = \hat{\mathbf{y}}_\sigma + Y_{\text{grid}}$ (with Y_{grid} a zero-centered hyper-rectangle whose size impacts the grid refinement), offline, for each P_σ, one has to solve the separation problem with $\bar{X}_{k_0}^{[i]} = X_\sigma^{[i]} = \{\mathbf{x} : \hat{\mathbf{y}}_\sigma - \mathbf{C}^{[i]}\mathbf{x} - \mathbf{s}^{[i]} \in \mathbf{D}_v^{[i]}V + Y_{\text{grid}}\}, \forall i \in \mathcal{M}$. By doing so, one obtains a solution for any $\mathbf{y} \in \{P_\sigma\}_{\sigma \in S}$. Online, at each time step, one has just to figure out which P_σ the obtained measurement belongs to. While CL-G reduces the online complexity to a simple function evaluation, it has some drawbacks: (i) since based on partitioning the measurement space, it could result in poor performance,

i.e. high input norm and/or long input sequence (ii) the use of a very simple observer, based on measurements only (no prior knowledge of $\bar{X}_{k_0}^{[i]}$ is used when computing $\bar{X}_{k_0+1}^{[i]}$), limits the benefit of a closed-loop strategy. A method that partially overcomes these problems is presented in the following.

4.1. Multi-parametric programming

Multi-parametric programming aims to provide the optimal solution of an optimization problem as an explicit function of its parameters $\gamma \in \Gamma \subset \mathbb{R}^{n_\gamma}$ (with Γ here assumed polyhedral) (Dua et al., 2002). The multi-parametric programs we will consider in the remainder of the paper have the following form

$$\min_{\mathbf{x}} \frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x} + \mathbf{f}^\mathsf{T}\mathbf{x} \tag{5}$$

subj. to $\mathbf{A}\mathbf{x} \le \mathbf{S}\gamma + \mathbf{b}$

with $\mathbf{x} \in \mathbb{R}^{n_x}$ the optimization variables. Matrix $\mathbf{H} \in \mathbb{R}^{n_x \times n_x}$ is assumed to be positive semidefinite. If $\mathbf{H} = \mathbf{0}$, (5) is a multi-parametric LP (mp-LP), otherwise it is an mp-QP. When $\mathbf{x} = [\mathbf{x}_c^\mathsf{T}; \mathbf{x}_b^\mathsf{T}]^\mathsf{T}$, with $\mathbf{x}_c \in \mathbb{R}^{n_c}$ and $\mathbf{x}_b \in \{0, 1\}^{n_b}$, (5) becomes an mp-MIQP. The mp-solution of these programs can be computed using, e.g., MPT (Herceg, Kvasnica, Jones, & Morari, 2013) and evaluated online through efficient data structures (see e.g. Fuchs, Axehill, and Morari (2015)).

4.2. Closed-loop approach based on mp-programming

Mp-programming can be used to obtain an active FDI scheme, here named *closed-loop approach based on mp-programming* (CL-mp) able to guarantee fast online computation and better performance compared to CL-G. Rather than partitioning the measurement space, we parametrize the output reachable sets in $\bar{X}_{k_0}^{[i]}$, $\forall i \in \mathcal{M}$. By doing so, $\mathbf{G}_{Y_{k|k_0}}^{[i]}$ and $\mathbf{c}_{Y_{k|k_0}}^{[i]}$ (defined in (4)) can be rewritten as

$$\mathbf{G}_{Y_{k|k_0}}^{[i]} = \left[\overbrace{\mathbf{C}^{[i]}\tilde{\mathbf{A}}_{k-k_0}^{[i]}\mathbf{G}_{\bar{X}_{k_0}}^{[i]}}^{(\bar{x})\mathbf{G}_{Y_{k|k_0}}^{[i]}} \quad \overbrace{\mathbf{C}^{[i]}\tilde{\mathbf{B}}_{w_{k-k_0}}^{[i]}\mathbf{G}_{\tilde{W}_{k-k_0}} \quad \mathbf{D}_v^{[i]}\mathbf{G}_V}^{(w,v)\mathbf{G}_{Y_{k|k_0}}^{[i]}} \right]$$

$$\mathbf{c}_{Y_{k|k_0}}^{[i]} = \underbrace{\psi_{(k-k_0)}^{[i]}(\mathbf{0}, \mathbf{c}_{\bar{X}_{k_0}}^{[i]}, \mathbf{0}, \mathbf{0})}_{(\bar{x})\mathbf{c}_{Y_{k|k_0}}^{[i]}} + \underbrace{\psi_{(k-k_0)}^{[i]}(\tilde{\mathbf{u}}_{k_0:k|k_0}, \mathbf{0}, \mathbf{0}, \mathbf{0})}_{(\bar{u})\mathbf{c}_{Y_{k|k_0}}^{[i]}}$$

isolating the effect of the initial condition. To keep the number of parameters small, $\bar{X}_{k_0}^{[i]}$ is chosen hyper-cubic, $\bar{X}_{k_0}^{[i]} = \{\mathbf{I}(1 + \sigma_{\bar{X}_{k_0}}^{[i]}), \mathbf{c}_{\bar{X}_{k_0}}^{[i]}\}$, with $\mathbf{c}_{\bar{X}_{k_0}}^{[i]} \in \mathbb{R}^{n_x}$, $\sigma_{\bar{X}_{k_0}}^{[i]} \in \mathbb{R}$, $\sigma_{\bar{X}_{k_0}}^{[i]} \ge -1$, the center and the size of the parametrized set (hyper-cubic outer approximation of the exact set can be computed using, e.g., Bravo et al. (2006)). Note that the parametrization is more flexible, and thus less conservative, than a fixed grid (used in CL-G) since it allows the center of $\bar{X}_{k_0}^{[i]}$ to freely move within the parameter bounds and its size to shrink/grow to get the tightest set consistent with measurements and prior information. In order to see if $\tilde{\mathbf{u}}_{k_0:k_0+N|k_0}$ is a *separating input sequence* for the different parameters, one has to solve a set of mp-LPs

$$\hat{\delta}_N^{[i,j]}(\tilde{\mathbf{u}}_{k_0:k_0+N|k_0}, \sigma_{\bar{X}_{k_0}}^{[i]}, \mathbf{c}_{\bar{X}_{k_0}}^{[i]}, \sigma_{\bar{X}_{k_0}}^{[j]}, \mathbf{c}_{\bar{X}_{k_0}}^{[j]}) = \tag{6}$$

$$\min_{\xi, \chi, \gamma, \lambda, \delta} \delta$$

subj. to $\mathbf{C}^{[i]}\tilde{\mathbf{A}}_N^{[i]}\xi + {}^{(w,v)}\mathbf{G}_{Y_{k_0+N|k_0}}^{[i]}\chi + {}^{(\bar{u})}\mathbf{c}_{Y_{k_0+N|k_0}}^{[i]} + {}^{(\bar{x})}\mathbf{c}_{Y_{k_0+N|k_0}}^{[i]} = $

$\mathbf{C}^{[j]}\tilde{\mathbf{A}}_N^{[j]}\gamma + {}^{(w,v)}\mathbf{G}_{Y_{k_0+N|k_0}}^{[j]}\lambda + {}^{(\bar{u})}\mathbf{c}_{Y_{k_0+N|k_0}}^{[j]} + {}^{(\bar{x})}\mathbf{c}_{Y_{k_0+N|k_0}}^{[j]}$

$$\|\xi\|_\infty \le \delta + \sigma_{\bar{X}_{k_0}}^{[i]}$$
$$\|\gamma\|_\infty \le \delta + \sigma_{\bar{X}_{k_0}}^{[j]}$$
$$\|\chi\|_\infty \le \delta$$
$$\|\lambda\|_\infty \le \delta$$
$$\sigma_{\bar{X}_{k_0}}^{[i]}, \sigma_{\bar{X}_{k_0}}^{[j]} \in \Sigma, \quad \mathbf{c}_{\bar{X}_{k_0}}^{[i]}, \mathbf{c}_{\bar{X}_{k_0}}^{[j]} \in \bar{P}$$

with $\hat{\delta}_N^{[i,j]}$ representing a measure of distance between zonotopes (see Scott et al. (2014)) and check for which parameter values

$$\hat{\delta}_N^{[i,j]}(\tilde{\mathbf{u}}_{k_0:k_0+N|k_0}, \sigma_{\bar{X}_{k_0}}^{[i]}, \mathbf{c}_{\bar{X}_{k_0}}^{[i]}, \sigma_{\bar{X}_{k_0}}^{[j]}, \mathbf{c}_{\bar{X}_{k_0}}^{[j]}) > 1 \tag{7}$$

hold for all $i, j \in \mathcal{M}$, $i \ne j$. In order the parameters to appear linearly in (6), the initial condition has been rewritten as $\bar{X}_{k_0}^{[i]} = \{\xi + \mathbf{c}_{\bar{X}_{k_0}}^{[i]} : \|\xi\|_\infty \le 1 + \sigma_{\bar{X}_{k_0}}^{[i]}\}$ (the two formulations are equivalent). By doing so, when writing $Y_{k_0+N|k_0}^{[i]}$, one uses of ${}^{(\bar{x})}\mathbf{G}_{Y_{k_0+N|k_0}}^{[i]}$ only the columns $\mathbf{C}^{[i]}\tilde{\mathbf{A}}_N^{[i]}$ and replaces the generators constraint, $\|\xi\|_\infty \le 1$, with $\|\xi\|_\infty \le 1 + \sigma_{\bar{X}_{k_0}}^{[i]}$. The same applies to index j. Note that Σ and $\bar{P}$ bound the size and center of the initial conditions space.

According to Borrelli (2003), it is always possible to find a solution of a mp-LP which can be expressed in terms of linear constraints involving both continuous and binary variables. Using this property, similarly to Scott et al. (2014), for a given N, it is possible to obtain a parametrized separating input sequence of minimum quadratic norm by solving offline a mp-MIQP. In order to provide the shortest feasible separating sequence, a set of mp-MIQPs is solved offline from $N = 1, \ldots, N_{max}$ (or until separation is guaranteed for all parameters within bounds). All the related maps get stored. Note that the minimum horizon guaranteeing diagnosis may be different for different parameter values.

In order to implement the CL-mp approach, online, at each $k_0 \ge 0$, given the parameters consistent with $\bar{X}_{k_0}^{[i]}$ ($i \in \mathcal{M}$), one has to query the maps starting from horizon 1 until a feasible solution is obtained. Then, according to the closed-loop paradigm, only the first input element is applied and a new input sequence is obtained at time $k_0 + 1$. In the following, we assume $\bar{X}_{k_0}^{[i]}$ to lie within the parametric bounds for all $k_0 \ge 0$.

Remark 4.1. For both CL-G and CL-mp approaches, the requirement to lie within given bounds for all $k_0 \ge 0$ may be relaxed to $k_0 = 0$ only. By doing so, if at any $k_0 > 0$ measurements/initial conditions get out of bounds, it is still possible to provide a guaranteed diagnosis by applying the last feasible input sequence in open-loop.

Remark 4.2. The complexity of the parametrized output reachable sets can be alleviated by computing zonotopic outer approximations with less generators, see e.g. Althoff, Stursberg, and Buss (2010). However, in order to keep the dependence on the parameters, only the columns ${}^{(w,v)}\mathbf{G}_{Y_{k|k_0}}^{[i]}$ of $\mathbf{G}_{Y_{k|k_0}}^{[i]}$ can be reduced.

Even though mp-programming shifts most of the computation offline (online one has to query a set of lookup tables only), the complexity of computing the maps grows exponentially with the number of binary variables of the mp-MIQP. This number is proportional to the number of mp-LPs involved in the separation problem, which is equal to $\binom{n_m}{2}$, with n_m the number of models. In order to reduce complexity, in Section 4.3, we propose a sub-optimal approach, here named *closed-loop couple-based mp-approach* (CL-CBmp), based on mp-programs involving only two models at a time. The use of a graph allows to select in which order to execute the different programs to still provide a guaranteed FDI and maximize performance.

4.3. Closed-loop couple-based mp-approach

Rather than designing an input sequence able to separate all the reachable sets, the approach proposed in the following looks at the separation of all the models considering, however, just a couple of models at a time. We therefore look for an input sequence guaranteeing the separation of a couple (i, j). We apply the sequence until at least one of the two models is discarded. Then, we move to the separation of a new couple. This procedure continues until only one model (the true model) remains consistent with the measurements. By doing so, diagnosis is still attained, but at the price of a more expensive (in length/norm) input sequence. On the other side, the advantage is a significant complexity reduction. Indeed, we replace the offline computation of an mp-MIQP involving several binary variables with the computation of $\binom{n_m}{2}$ mp-MIQP involving a negligible number of binaries. Still, this approach raises some issues. Remark 4.1, for example, does not hold for CL-CBmp. When only two models are considered at a time, the requirement that $\bar{X}_{k_0}^{[i]}$ lies within bounds should at least hold for all the time instants at which we switch to the separation of a new couple (the previously computed sequence does not in general guarantee the separation of the remaining models). In order to satisfy such requirement, the parameter space has to be chosen suitably. Since $\{(\mathbf{C}^{[i]})^{-1}\mathbf{D}_v^{[i]}\mathbf{G}_V, (\mathbf{C}^{[i]})^{-1}(\mathbf{y}_{k_0} - \mathbf{s}^{[i]})\}$ is the set of states consistent with measurement $\mathbf{y}_{k_0}$ at time k_0, an upper bound on $\sigma_{\bar{X}_{k_0}}^{[i]}$ is given by the minimum $\sigma^u \in \mathbb{R}$ such that $\{\mathbf{I}(1 + \sigma^u), \mathbf{0}\} \supseteq \{(\mathbf{C}^{[i]})^{-1}\mathbf{D}_v^{[i]}\mathbf{G}_V, \mathbf{0}\}$, for all $i \in \mathcal{M}$. The use of a hyper-cubic set-valued observer guarantees that the size of any new initial conditions will never exceed σ^u. On the other side, $\sigma^l = -1$ represents the tightest lower bound for $\sigma_{\bar{X}_{k_0}}^{[i]}$ since, for that value, $\bar{X}_{k_0}^{[i]}$ becomes a point. Summarizing, the initial condition size can be bounded by $\Sigma = [\sigma^l, \sigma^u]$. The bounds on the parameter space $\bar{P}$ are provided in Section 4.3.2. In order to construct such a set, we first compute, without solving any mp-program, an upper bound N^* on the time necessary to separate all the reachable sets $i \in \mathcal{M}$ (see Section 4.3.1).

4.3.1. Existence of N^*

In this section we demonstrate the existence of an upper bound N^* on the time necessary for guaranteeing FDI with CL-CBmp. In this sense, two preliminary theorems and two lemmas are needed.

Lemma 4.3. Given $N > 0$, a separating input sequence $\tilde{\mathbf{u}}_N \in \tilde{U}_N$ for models (i, j) exists if and only if $\mathbf{L}_N^{[i,j]}\tilde{U}_N \not\subseteq \mathbf{S}_N^{[i,j]}$, where $\mathbf{S}_N^{[i,j]} = \{[\mathbf{G}_{Y_N}^{[i]} - \mathbf{G}_{Y_N}^{[j]}], {}^{(\bar{x})}\mathbf{c}_{Y_N}^{[i]} - {}^{(\bar{x})}\mathbf{c}_{Y_N}^{[j]}\}$, and $\mathbf{L}_N^{[i,j]} = \mathbf{C}^{[j]}\tilde{\mathbf{B}}_N^{[j]} - \mathbf{C}^{[i]}\tilde{\mathbf{B}}_N^{[i]}$.

Theorem 4.4. Given two zonotopes in $\mathbb{R}^{n_y}$, $A = \{\mathbf{G}_A, \mathbf{c}_A\}$, $B = \{\mathbf{G}_B, \mathbf{c}_B\}$ ($\mathbf{G}_A \in \mathbb{R}^{n_y \times n_{gA}}$, $\mathbf{G}_B \in \mathbb{R}^{n_y \times n_{gB}}$, $\mathbf{c}_A, \mathbf{c}_B \in \mathbb{R}^{n_y}$), if $\exists \mathbf{c}_A \neq \mathbf{c}_B$ for which $B \subseteq A$ then $\mathbf{c}_A = \mathbf{c}_B \rightarrow B \subseteq A$.

Lemma 4.5. Given zonotopes A, B as defined in Theorem 4.4. $\{\mathbf{G}_B, \mathbf{0}\} \not\subseteq \{\mathbf{G}_A, \mathbf{0}\} \rightarrow \forall \mathbf{c}_B \in \mathbb{R}_y^n, \{\mathbf{G}_B, \mathbf{c}_B\} \not\subseteq \{\mathbf{G}_A, \mathbf{0}\}$.

The proofs can be found in the Appendix.

Theorem 4.6. Assume $N^*_{(i,j)} > 0$ is the shortest horizon for which $\exists \tilde{\mathbf{u}}_{N^*_{(i,j)}} \in \tilde{U}_{N^*_{(i,j)}}$ ($U = \{\mathbf{G}_U, \mathbf{0}\}$) able to separate $Y_{N^*_{(i,j)}}^{[i]}$ from $Y_{N^*_{(i,j)}}^{[j]}$ when

- ${}^{(\bar{x})}\mathbf{c}_{Y_{N^*_{(i,j)}}}^{[i]} = {}^{(\bar{x})}\mathbf{c}_{Y_{N^*_{(i,j)}}}^{[j]}$ • $\sigma_{\bar{X}_{k_0}}^{[i]} = \sigma_{\bar{X}_{k_0}}^{[j]} = \sigma^u$.

Then $N^*_{(i,j)}$ is an upper bound on the number of steps required for separating (i, j) also when

- ${}^{(\bar{x})}\mathbf{c}_{Y_{N^*_{(i,j)}}}^{[i]} \neq {}^{(\bar{x})}\mathbf{c}_{Y_{N^*_{(i,j)}}}^{[j]}$ with ${}^{(\bar{x})}\mathbf{c}_{Y_{N^*_{(i,j)}}}^{[i]}, {}^{(\bar{x})}\mathbf{c}_{Y_{N^*_{(i,j)}}}^{[j]} \in \mathbb{R}^{n_y}$

- $\sigma_{\bar{X}_{k_0}}^{[i]}, \sigma_{\bar{X}_{k_0}}^{[j]} \neq \sigma^u$, with $\sigma_{\bar{X}_{k_0}}^{[i]}, \sigma_{\bar{X}_{k_0}}^{[j]} \in \Sigma$

Proof. First of all, if the output reachable sets are separated when the initial conditions have maximum size, i.e. σ^u, then they will be separated for any other value in Σ. Thus, we can focus on finding the smallest k for which $\exists \tilde{\mathbf{u}}_k \in \tilde{U}_k$ guaranteeing separation for any ${}^{(\bar{x})}\mathbf{c}_{Y_k}^{[i]}, {}^{(\bar{x})}\mathbf{c}_{Y_k}^{[j]} \in \mathbb{R}^{n_y}$ when $\sigma_{\bar{X}_{k_0}}^{[i]} = \sigma_{\bar{X}_{k_0}}^{[j]} = \sigma^u$. Defining $\mathbf{L}_k^{[i,j]} = \mathbf{C}^{[j]}\tilde{\mathbf{B}}_k^{[j]} - \mathbf{C}^{[i]}\tilde{\mathbf{B}}_k^{[i]}$, this is equivalent to finding the first instant k satisfying

$$\exists \tilde{\mathbf{u}}_k \in \tilde{U}_k : \forall (\gamma_1, \gamma_2) \in \mathbf{1}[-(1 - \sigma^u), (1 + \sigma^u)], \forall ({}^{(\bar{x})}\mathbf{c}_{Y_k}^{[i]}, {}^{(\bar{x})}\mathbf{c}_{Y_k}^{[j]})$$

$$[\mathbf{G}_{Y_k}^{[i]} - \mathbf{G}_{Y_k}^{[j]}][\gamma_1\ \gamma_2]^T + {}^{(\bar{x})}\mathbf{c}_{Y_k}^{[i]} - {}^{(\bar{x})}\mathbf{c}_{Y_k}^{[j]} \neq \mathbf{L}_k^{[i,j]}\tilde{\mathbf{u}}_k \quad (8)$$

Let $\mathbf{S}_k^{[i,j]} = \{[\mathbf{G}_{Y_k}^{[i]} - \mathbf{G}_{Y_k}^{[j]}], {}^{(\bar{x})}\mathbf{c}_{Y_k}^{[i]} - {}^{(\bar{x})}\mathbf{c}_{Y_k}^{[j]}\}$. According to Lemma 4.3, the condition above holds iff $\mathbf{S}_k^{[i,j]} \not\supseteq \mathbf{L}_k^{[i,j]}\tilde{U}_k$. Finally, thanks to Lemma 4.5, if $\exists k = N^*_{(i,j)}$ such that this latter condition is verified when ${}^{(\bar{x})}\mathbf{c}_{Y_k}^{[i]} = {}^{(\bar{x})}\mathbf{c}_{Y_k}^{[j]}$, then it also hold when ${}^{(\bar{x})}\mathbf{c}_{Y_k}^{[i]} \neq {}^{(\bar{x})}\mathbf{c}_{Y_k}^{[j]}$, thus proving the theorem. ∎

The following lemma extends the result of the previous theorem to the case $U = \{\mathbf{G}_U, \mathbf{c}_U\}$, with $\mathbf{c}_U \neq \mathbf{0}$.

Lemma 4.7. Assume the input admissible set is not zero centered, i.e. $U = \{\mathbf{G}_U, \mathbf{c}_U\}$. An upper bound on the number of steps required to separate the couple (i, j) is given by the smallest k for which $\exists \tilde{\mathbf{u}}_k$ able to separate the sets when ${}^{(\bar{x})}\mathbf{c}_{Y_k}^{[i]} - {}^{(\bar{x})}\mathbf{c}_{Y_k}^{[j]} = \left[\mathbf{C}^{[j]}\tilde{\mathbf{B}}_k^{[j]} - \mathbf{C}^{[i]}\tilde{\mathbf{B}}_k^{[i]}\right][\mathbf{c}_U^T\ \mathbf{c}_U^T \dots \mathbf{c}_U^T]^T$.

The lemma can be easily demonstrated replacing the value $\mathbf{0}$ with $\mathbf{L}_k^{[i,j]}[\mathbf{c}_U^T\ \mathbf{c}_U^T \dots \mathbf{c}_U^T]^T$ in the proof of Theorem 4.6. Theorem 4.6 and Lemma 4.7 allow to prove the following theorem.

Theorem 4.8. Consider a CL-CBmp approach. Assume that $\bar{P}$ and Σ are chosen in a way that the parameters $\mathbf{c}_{\bar{X}_{k_0}}^{[i]}$ and $\sigma_{\bar{X}_{k_0}}^{[i]}$ stay within bounds $\forall i \in \mathcal{M}$, at least at the time instants when the switch to the separation of a new couple is made. Then, if $N^*_{(i,j)}$ is an upper bound on the time required to separate each couple (i, j), $N^* = \sum_{i}^{n_m-1}\sum_{j=i+1}^{n_m} N^*_{(i,j)}$ is an upper bound on the time required to separate all models.

Proof. Consider, at a given time k, the generic couple (i, j). Since $\mathbf{c}_{\bar{X}_{k_0}}^{[i]}$ and $\sigma_{\bar{X}_{k_0}}^{[i]}$ are assumed to be within bounds, the two output reachable sets can be separated in at most $N^*_{(i,j)}$ steps. Thus, at time $N^*_{(i,j)}$, model i and/or model j will be discarded. Consequently, after these steps, the set of possibly active models becomes $\{1, \dots, n_m\} \setminus \{i\}$, $\{1, \dots, n_m\} \setminus \{j\}$ or $\{1, \dots, n_m\} \setminus \{i, j\}$. By considering iteratively the separation of the remaining couples, all models will be discarded except for one (the active model), thus providing a diagnosis. An overestimation of the time required to separate all models is then $N^* = \sum^{n_c} N^*_{(i,j)}$, $n_c = \binom{n_m}{2}$. ∎

Note that Theorem 4.8 requires $\mathbf{c}_{\bar{X}_{k_0}}^{[i]}$ to stay within bounds, but a suitable dimension of $\bar{P}$ has not been computed yet (see Section 4.3.2). However, the bound N^* does not depend on the size of $\bar{P}$ but requires only its existence.

In Corollary 4.13 we show how to compute a tighter bound on N^* making use of a graph. In order to do so, we first introduce the following definitions and theorem.

Definition 4.9. We call system graph $\mathcal{G}(E, V)$ a non-oriented graph that has a node $v_i \in V$ for each possible model that can be active on the system and an edge $e_{i,j} \in E$ of weight $s(e_{i,j}) = N^*_{(i,j)}$ linking each couple of nodes.

Definition 4.10. Given a graph $\mathcal{G}$ (E, V), a spanning tree is a simple connected graph with no cycles containing every vertex of $\mathcal{G}$. A minimum spanning tree is a spanning tree of a weighted graph having minimum weight.

Note that the minimum spanning tree of a system graph can be computed by negating the weights of each edge and applying the Kruskal's algorithm (Kruskal, 1956).

Definition 4.11. Given a system graph $\mathcal{G}$ (E, V), we define *system graph exploration* as any of its sub-graphs $\mathcal{G}(\tilde{E}, V)$, where $\tilde{E} \subseteq E$ is any subset of edges $e_{i,j}$ that, if taken iteratively into account, is sufficient to provide a diagnosis.

Theorem 4.12. *Given a system graph $\mathcal{G}$ (E, V), all the possible system graph explorations $\mathcal{G}(\tilde{E}, V)$ are acyclic not necessarily connected sub-graphs of the system graph.*

The proof can be found in the Appendix.

Corollary 4.13. *Consider a system graph $\mathcal{G}(E, V)$ with nodes $V = \{1, \ldots, n_m\}$ and edges corresponding to all the links connecting each couple of nodes. A better estimate of N^* is given by the sum of the edges of the maximum spanning tree of the system graph.*

Proof. Let start considering three models, i, j, k. The estimate of N^* given in Theorem 4.8 would be very conservative, being the sum of $N^*_{(i,j)}, N^*_{(i,k)}, N^*_{(j,k)}$. Indeed, as stated in the proof of Theorem 4.12, if the couple (i, j) is considered first, either i or j would be discarded and therefore either the couple (i, k) or (j, k) will not have to be considered any further. A better estimate of N^* can be obtained as follows. Recall that, according to Theorem 4.12, system graph explorations are sufficient to provide a diagnosis. In order to estimate N^*, we consider only connected graph explorations. In fact, the case when a diagnosis is obtained by discarding models that were not currently under investigation (or the case when both models of a couple get discarded) is a fortunate case that clearly would not provide an upper bound on the overall time required for diagnosis. Among all possible graph explorations, we therefore consider the one whose edges sum is the highest. Such result can be obtained exactly by computing the maximum spanning tree of the system graph.

Remark 4.14. Note that N^* upper bounds the time required to separate all models also when an *online closed-loop couple-based approach* (CL-CBO) is used instead of a multi-parametric one (i.e. CL-CBmp). In this case, one couple of models is separated at a time by solving optimization problems online. The only requirement for N^* to hold is that a hyper-cubic set-valued observer is employed and its size does not exceed σ^u.

Remark 4.15. In CL-CBmp, the use of a couple of models at a time reduces significantly the computational complexity of the separation problem. For this reason, one might be tempted to avoid the parametrization and use CL-CBO instead, which requires the solution of MIQPs online but allows the use of a more complex observer. In Section 4.3.3, we will show that CL-CBmp can dramatically benefit of information computed offline using a graph. This allows CL-CBmp to outperform CL-CBO as shown in Section 5.

4.3.2. Bounding the parameter space $\bar{p}$

The computation of N^*, described in the previous section, provides an upper bound on the number of steps required to guarantee diagnosis when using CL-CBmp. This upper bound allows to estimate the output reachable sets which could be generated in N^* steps by the different models using any of the possible inputs in U. By choosing $\bar{P}$ as the smallest sets including all these reachable sets, we do guarantee that the center of $\bar{X}^{[i]}_{k_0}$ will belong to $\bar{P}$, for all $k_0 > 0$, and all $i \in \mathcal{M}$.

Theorem 4.16. *Assume measurements at time $k = 0$ lie in known set $\bar{Y}_0 = \{\mathbf{G}_{\bar{Y}_0}, \mathbf{0}\}$. Then, given N^*, it is possible to compute $\bar{P}$ so that, when applying CL-CBmp for $k = 0, \ldots, N^*$, the center of any new initial condition never leaves such a set.*

Proof. Define $\bar{P}_0 = \{\mathbf{G}_P, \mathbf{c}_p\}$ as the smallest set containing $(\mathbf{C}^{[i]})^{-1}(\bar{Y}_0 - \mathbf{s}^{[i]})$, $\forall i \in \mathcal{M}$. Given the assumptions, the center of any $\bar{X}^{[i]}_{k_0}$ consistent with $\bar{Y}_0$ will lie in $\bar{P}_0$. Since σ^u upper bounds the initial condition size, the initial condition sets will for sure lie in $\bar{X}^{[i]}_{k_0} = \{\mathbf{G}_{\bar{X}_{k_0}}, \mathbf{c}_{\bar{X}_{k_0}}\} = \{\mathbf{I}(1 + \sigma^u), \mathbf{0}\} \oplus \{\mathbf{G}_P, \mathbf{c}_p\}, \forall i \in \mathcal{M}$, where $\oplus$ stands for Minkowski sum. Given $\bar{X}^{[i]}_{k_0}, U, V, W$, it is possible to compute the output reachable sets for all $i \in \mathcal{M}$, all inputs in U and all $k = 0, \ldots, N^*$ as $\bar{Y}^{[i]}_k = \left\{ \mathbf{G}_{\bar{Y}^{[i]}_k}, \boldsymbol{\psi}^{[i]}_k(\tilde{\mathbf{c}}_{U_k}, \mathbf{c}^{[i]}_{\bar{X}_{k_0}}, \mathbf{0}, \mathbf{0}) \right\}$ with $\mathbf{G}_{\bar{Y}^{[i]}_k} = \left[\mathbf{C}^{[i]}_k \left[\tilde{\mathbf{A}}^{[i]}_k \mathbf{G}_{\bar{X}_{k_0}}^{[i]} \quad \tilde{\mathbf{B}}^{[i]}_k \mathbf{G}_{\tilde{U}_k} \quad \tilde{\mathbf{B}}^{[i]}_{w_k} \mathbf{G}_{\bar{W}_k} \right] \quad \mathbf{D}^{[i]}_v \mathbf{G}_V \right]$. According to this, the center of any new initial condition at $k \in [0, N^*]$, will lie in $(\mathbf{C}^{[i]})^{-1}(\bar{Y}^{[i]}_k - \mathbf{s}^{[i]})$, $i \in \mathcal{M}$. Now, we compute the smallest zonotope $\bar{P}$ containing all $(\mathbf{C}^{[i]})^{-1}(\bar{Y}^{[i]}_k - \mathbf{s}^{[i]})$, for all $k \in [0, N^*]$. Being an over bound of all possible initial conditions center, the parametrization of CL-CBmp over the obtained $\bar{P}$ and $\Sigma = [-1, \sigma^u]$, allows to positively query the maps at any $k \in [0, N^*]$, also when the switching to the separation of a new couple occurs.

In conclusion, assuming that measurements at time $k = 0$ belong to $\bar{Y}_0$, CL-CBmp parametrized over $\bar{P}$ and Σ guarantees a diagnosis within N^* steps. Finally, we show that the use of a graph allows to select the order of execution of the couple-based mp-programs to minimize the overall time/energy required for diagnosis.

4.3.3. Selecting the order of execution of the mp-programs

Compared to CL-CBO, the main advantage of CL-CBmp is the knowledge of an upper bound on the time required to separate each couple of models. This allows CL-CBmp to select which couples to separate first as described in Alg. 1.

Algorithm 1 Graph-based strategy

1: Set $k = 0$.
2: For each model $i \in \mathcal{M}$, use a hyper-cubic set-valued observer to obtain parameters $\mathbf{c}^{[i]}, \sigma^{[i]}$ describing the new state initial condition.
3: Initialize system graph $\mathcal{G}(E, V)$ with all models $i \in \mathcal{M}$ and with edges of length $N^*_{(i,j)}, (i, j \in \mathcal{M}, i \neq j)$.
4: For each $(i, j) \in \mathcal{M}, i \neq j$, compute $\tilde{\mathbf{u}}^{(i,j)}$ as the best (in terms of length/norm) between the remaining of the sequence obtained at time $k-1$ (if able to separate couple (i, j)) and the one obtained by querying mp-map (i, j) with parameters $\mathbf{c}^{[i]}, \mathbf{c}^{[j]}, \sigma^{[i]}, \sigma^{[j]}$.
5: For each couple, compute a modified copy $\mathcal{G}^m_{(i,j)}$ of system graph $\mathcal{G}(E, V)$ where $s(e_{i,j})$ is replaced by the length of $\tilde{\mathbf{u}}^{(i,j)}$. For each $\mathcal{G}^m_{(i,j)}$, compute the maximum spanning tree containing edge (i, j).
6: Choose the couple (i^*, j^*) providing the maximum spanning tree of minimum length. If more than one couple has minimum length, choose the one with the best $\tilde{\mathbf{u}}^{(i^*, j^*)}$. Inject the first input of $\tilde{\mathbf{u}}^{(i^*, j^*)}$. Set $k = k+1$.
7: Discard all models not consistent with the measurement and eliminate them from $\mathcal{M}$. If only one model is left, then the procedure ends. Otherwise, return to Step 2.

In practice, the approach allows to decide in which order to separate couples by selecting the maximum spanning tree of minimum length according to edges values $N^*_{(i,j)}$. Note that, $N^*_{(i,j)}$ are conservative since hold for any parameter values within bounds. Thanks to step 5 of Alg. 1, such bounds can be replaced for each new time instant with the length of the minimum input sequence able to separate couple (i, j) for the parameter values of the new initial condition. This way, a better estimate of the worst-case time
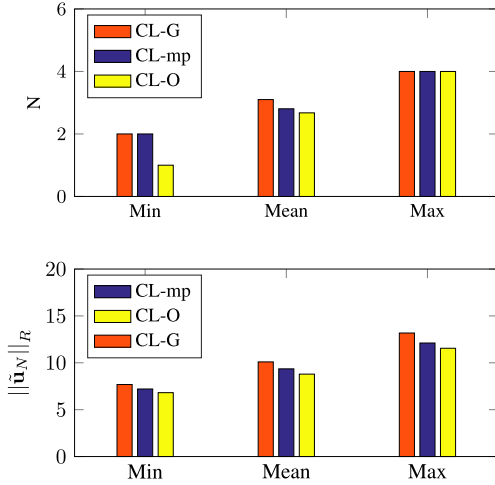
Fig. 1. Comparison in terms of input length and norm between CL-O, CL-G and CL-mp.

Fig. 2. Mean steps needed for separation for model 4.

Fig. 3. Mean steps needed for separation for model 4.

required to diagnose a fault can be obtained and a less conservative strategy applied. Note that, the inclusion of edge (i, j) when computing the maximum spanning tree can be obtained by replacing $s(e_{i,j})$ with a value bigger than the one of any other edge before applying Kruskal's algorithm.

The advantages of using Alg. 1 when applying CL-CBmp are shown on numerical examples in Section 5.

5. Numerical examples

Consider four models defined by matrices

$$A^{[1,2]} = \begin{bmatrix} 0.8 & 0.2 \\ -0.2 & 0.8 \end{bmatrix} A^{[3]} = \begin{bmatrix} 0.8 & 0 \\ -0.2 & 0.8 \end{bmatrix} A^{[4]} = \begin{bmatrix} 0.8 & 0.2 \\ 0 & 0.8 \end{bmatrix}$$

$$B^{[1]} = \begin{bmatrix} -0.3861 & 0 \\ -0.1994 & 0 \end{bmatrix} B^{[2]} = \begin{bmatrix} 0 & 0.1994 \\ 0 & 0.3861 \end{bmatrix}$$

$$B^{[3,4]} = \begin{bmatrix} -0.3861 & 0.1994 \\ -0.1994 & 0.3861 \end{bmatrix}$$

$$C^{[1,2,3,4]} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} D_v^{[1,2,3,4]} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B_w^{[1,2,3,4]} = \begin{bmatrix} 0.1215 & 0.0598 \\ 0.0598 & 0.1215 \end{bmatrix}$$

with $r^{[1,2,3,4]} = s^{[1,2,3,4]} = 0$. Uncertainties are bounded in zonotopes $X_0^{[i]} = \{3\mathbf{I}, 0\}, \forall i \in \mathcal{M}, W = \{\mathbf{I}, 0\}, V = \{0.7\mathbf{I}, 0\}$. We aim to compute for the shortest horizon N guaranteeing separation the input sequence of minimum quadratic norm $\|\tilde{\mathbf{u}}_{k_0+N|k_0}\|_R$ with $R = \mathbf{I}$. The inputs are constrained in $U = \{\mathbf{u} : \|\mathbf{u}\|_\infty \leq 5\}$. To alleviate complexity, the zonotopes order is limited to 2 (zonotope order reduction techniques are used to guarantee such constraint).

5.1. Example 1

In this example we assume that only models 3 and 4 can be active. CL-mp, CL-G and CL-O are compared in Fig. 1. For CL-G, $Y_{grid} = \{0.1\mathbf{I}, 0\}$. The explicit map has been defined over $\bar{Y} = \{6\mathbf{I}, 0\}$ while the one of CL-mp over $\bar{P} = \{6\mathbf{I}, 0\}$ and with $\Sigma = [-1, -0.3]$. The results have been obtained performing 400 random simulations and checking the effective time required to diagnose the fault (half of the cases had model 3 as real dynamics, the other half
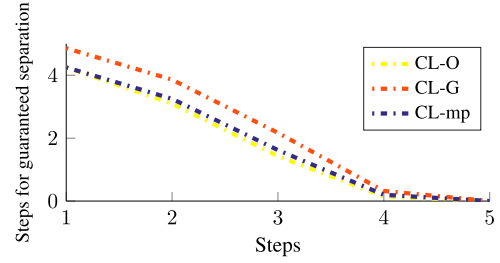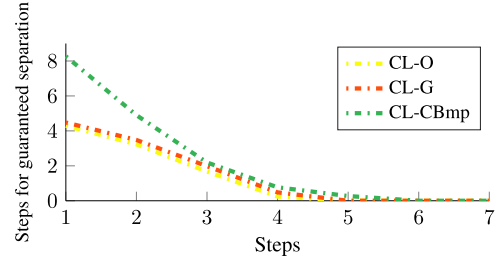
model 4). As expected, CL-O provides the best performance but requires the solution at each time step of an optimization. CL-mp provides reasonable performance and outperforms CL-G since not based on gridding and supported by a better set-valued observer. Fig. 2 shows the average (over 200 random simulations) evolution over time of the guarantees (in terms of number of steps) for the different approaches when considering model 4 active. At each step k, the guarantees are provided by the length of the open-loop sequence computed at that time. At the beginning, being the most conservative, CL-G has a significant gap compared with the other approaches. However, the conservativeness is partially beneficial in a closed loop framework cause, being more aggressive, could lead to an *earlier termination*. This is demonstrated by the reduction of gap in the guarantees along the time. A similar result holds when comparing CL-O to CL-mp.

5.2. Example 2

In this example, we consider all models and compare the closed-loop performance CL-O, CL-G and CL-CBmp (this latter in combination with Algorithm 1). The bounds on the mp-maps are the same of Example 1 and fulfill the requirements of Sections 4.3.1 and 4.3.2. Fig. 3 summarizes the results obtained over 800 random simulations. At time $k = 0$, CL-CBmp shows the most conservative guarantees, being based on the separation of one couple at a time only. However, thanks to Algorithm 1, the approach promptly improves over time thus resulting in closed-loop performance comparable to the two methods involving all models. Finally, Fig. 4 compares the performance of CL-CBO and CL-CBmp. As expected, the use of a graph allows to schedule the mp-programs to maximize performance and provide better results than CL-CBO.
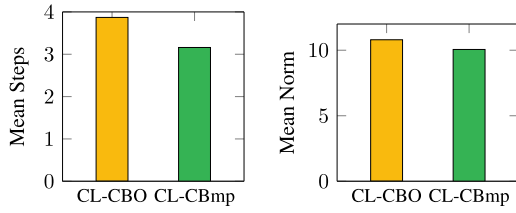
Fig. 4. Performance of couple-based (CB) approaches.

Appendix

Proof of Lemma 4.3. $\mathbf{L}_N^{[i,j]}\tilde{U}_N \subseteq \mathbf{S}_N^{[i,j]}$ is equivalent to

$\forall \tilde{\mathbf{u}}_N \in \tilde{U}_N,\ \exists(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2),\ \|\boldsymbol{\gamma}_1\|_\infty \le 1,\ \|\boldsymbol{\gamma}_2\|_\infty \le 1$ such that
$\mathbf{G}_{Y_N}^{[i]}\boldsymbol{\gamma}_1 + {}^{(\bar{x})}\mathbf{c}_{Y_N}^{[i]} + \mathbf{C}^{[i]}\tilde{\mathbf{B}}^{[i]}\tilde{\mathbf{u}}_N = \mathbf{G}_{Y_N}^{[j]}\boldsymbol{\gamma}_2 + {}^{(\bar{x})}\mathbf{c}_{Y_N}^{[j]} + \mathbf{C}^{[j]}\tilde{\mathbf{B}}^{[j]}\tilde{\mathbf{u}}_N.$

This expression is actually the negation of the separation condition, where this latter can be rewritten as

$\exists \tilde{\mathbf{u}}_N \in \tilde{U}_N :\ \forall(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2),\ \|\boldsymbol{\gamma}_1\|_\infty \le 1,\ \|\boldsymbol{\gamma}_2\|_\infty \le 1$
$\mathbf{G}_{Y_N}^{[i]}\boldsymbol{\gamma}_1 + {}^{(\bar{x})}\mathbf{c}_{Y_N}^{[i]} + \mathbf{C}^{[i]}\tilde{\mathbf{B}}^{[i]}\tilde{\mathbf{u}}_N \neq \mathbf{G}_{Y_N}^{[j]}\boldsymbol{\gamma}_2 + {}^{(\bar{x})}\mathbf{c}_{Y_N}^{[j]} + \mathbf{C}^{[j]}\tilde{\mathbf{B}}^{[j]}\tilde{\mathbf{u}}_N.$

For this reason, the separation condition and $\mathbf{L}_N^{[i,j]}\tilde{U}_N \nsubseteq \mathbf{S}_N^{[i,j]}$ are equivalent.

Proof of Theorem 4.4. Without loosing generality, we set the origin of the axes at the center of set A. In this reference system $\mathbf{c}_A = \mathbf{0}$. Suppose $\exists \mathbf{c}_B \neq \mathbf{c}_A$ such that $B \subseteq A$. Since A is symmetric by definition and $\mathbf{c}_A = \mathbf{0}$, also $B^* = \{-\mathbf{G}_B, -\mathbf{c}_B\}$ is subset of A. Additionally, because A is convex, all the segments connecting an element of B to an element of B^* are inside A. To prove the theorem is then sufficient to demonstrate that $B^0 = \{\mathbf{G}_B, \mathbf{0}\}$ (same center as A) is a linear combination of B and B^*. To do so, let us define X as the zero-centered unitary hypercube in $\mathbb{R}^{n_{gB}}$ ($X = \{\mathbf{I}, \mathbf{0}\}$, with $\mathbf{I}$ the identity matrix). Since B^0, B, B^* are zonotopes (i.e. their elements can be written as $\mathbf{G}_B\mathbf{x}_1$, $\mathbf{G}_B\mathbf{x}_2 + \mathbf{c}_B$ and $-\mathbf{G}_B\mathbf{x}_3 - \mathbf{c}_B$, with $\|\mathbf{x}_1\|_\infty \le 1$, $\|\mathbf{x}_2\|_\infty \le 1$, $\|\mathbf{x}_3\|_\infty \le 1$ respectively), to prove that elements of B^0 can be obtained as linear combinations of elements of B and B^*, it is enough to show that $\forall \mathbf{x}_1 \in X$, $\exists(\mathbf{x}_2, \mathbf{x}_3) \in X \times X$ and a $k \in [0, 1]$ such that:

$\mathbf{G}_B\mathbf{x}_1 = k \cdot \mathbf{G}_B\mathbf{x}_2 + k\mathbf{c}_B - (1-k)\cdot\mathbf{G}_B\mathbf{x}_3 - (1-k)\mathbf{c}_B$

holds. Let us fix $k = 0.5$. Then one has to show that $\forall \mathbf{x}_1 \in X$, $\exists(\mathbf{x}_2, \mathbf{x}_3) \in X \times X$ such that $\mathbf{G}_B\mathbf{x}_1 = \mathbf{G}_B\left(\frac{\mathbf{x}_2 - \mathbf{x}_3}{2}\right)$. Since the generator matrix is the same, one has to show that $\forall \mathbf{x}_1 \in X$, $\exists(\mathbf{x}_2, \mathbf{x}_3) \in X \times X$ such that $\mathbf{x}_1 = \frac{1}{2}(\mathbf{x}_2 - \mathbf{x}_3)$. Such condition can be rewritten as $\mathbf{I}\mathbf{x}_1 = \frac{1}{2}[\mathbf{I}\ -\mathbf{I}][\mathbf{x}_2\ \mathbf{x}_3]^T$ with $\mathbf{I}$ the identity matrix of suitable dimension. Recalling the operations on zonotopes and the definition of Minkowski sum, let us denote $\mathcal{U}_1 = \{\mathbf{I}, \mathbf{0}\}$ and $\mathcal{U}_2 = \{\frac{1}{2}[\mathbf{I}\ -\mathbf{I}], \mathbf{0}\}$. To prove that B^0 can be obtained as linear combination of B and B^*, one has to prove that $\mathcal{U}_1 \subseteq \mathcal{U}_2$. In this case, $\mathcal{U}_2 = \{\frac{1}{2}\cdot 2\mathbf{I}, \mathbf{0}\} = \mathcal{U}_1$. Therefore the theorem is proven.

Proof of Lemma 4.5. From Theorem 4.4 we have that:
$\exists \mathbf{c}_B \in \mathbb{R}^{n_y}, \{\mathbf{G}_B, \mathbf{c}_B\} \subseteq \{\mathbf{G}_A, \mathbf{0}\} \to \{\mathbf{G}_B, \mathbf{0}\} \subseteq \{\mathbf{G}_A, \mathbf{0}\}$. Given, thus, that the implication is true, if the second predicate is false, then the first has also to be false. Namely, $\{\mathbf{G}_B, \mathbf{0}\} \nsubseteq \{\mathbf{G}_A, \mathbf{0}\} \to \forall \mathbf{c}_B \in \mathbb{R}^{n_y}, \{\mathbf{G}_B, \mathbf{c}_B\} \nsubseteq \{\mathbf{G}_A, \mathbf{0}\}$.

Proof of Theorem 4.12. Assume by contradiction there exists a system graph exploration containing a cycle (i, j, k) (to simplify the proof let us consider three models only but the following applies also to a larger number of models). The presence of the cycle implies that all the couples (i, j), (j, k) and (i, k) have been taken into account to provide a diagnosis. In order this to be possible, it is necessary, when considering the first couple (e.g. (i, j)), that the input sequence $\tilde{\mathbf{u}}$ is not able to discard any model. If, in fact, model i would have been discarded, the couple (i, k) would not have been considered. If, instead, model j would have been discarded, the separation of couple (j, k) would not have been useful. By definition, the input sequence $\tilde{\mathbf{u}}$ is able to separate the couple (i, j) and therefore able to discard at least one of the two models. This proves that a system graph exploration cannot contain a cycle.

Let us prove now that the system graph exploration is not necessarily connected. This may happen since it is not necessary to actively separate all couple of models to obtain a diagnosis. Indeed, if, while trying to separate (i, j), the measurement acquired is not consistent with a third model k, model k gets discarded and its node v_k disconnected from all to other nodes. Similarly, if both model i and j get discarded while trying the separation, both nodes v_i and v_j get disconnected and all the edges involving i and/or j discarded.

References

Althoff, M., Stursberg, O., & Buss, M. (2010). Computing reachable sets of hybrid systems using a combination of zonotopes and polytopes. *Nonlinear Analysis-Hybrid Systems*, 4(2), 233–249.

Ashari, A. E., Nikoukhah, R., & Campbell, S. L. (2011). Auxiliary signal design for robust active fault detection of linear discrete-time systems. *Automatica*, 47(9), 1887–1895.

Ashari, A. E., Nikoukhah, R., & Campbell, S. L. (2012a). Active robust fault detection in closed-loop systems: Quadratic optimization approach. *IEEE Transactions on Automatic Control*, 57(10), 2532–2544.

Ashari, A. E., Nikoukhah, R., & Campbell, S. L. (2012b). Effects of feedback on active fault detection. *Automatica*, 48, 866–872.

Borrelli, F. (2003). Constrained optimal control of linear and hybrid systems. Springer.

Bravo, J. M., Alamo, T., & Camacho, E. F. (2006). Bounded error identification of systems with time-varying parameters. *IEEE Transactions on Automatic Control*, (ISSN: 0018-9286) 51, 1144–1150. http://dx.doi.org/10.1109/TAC.2006.878750.

Chen, J., & Patton, R. J. (2012). Robust model-based fault diagnosis for dynamic systems. Springer Publishing Company, Incorporated.

Cheong, S., & Manchester, I. R. (2015). Input design for discrimination between classes of LTI models. *Automatica*, 53, 103–110.

Combastel, C. (2005). A state bounding observer for uncertain non-linear continuous-time systems based on zonotopes. In *Proc. 44th IEEE conference on decision and control* (pp. 7228–7234).

Dua, V., Bozinis, N. A., & Pistikopoulos, E. N. (2002). A multiparametric programming approach for mixed-integer quadratic engineering problems. *Computers & Chemical Engineering*, 26(4), 715–733.

Frank, P. M., & Seliger, R. (2012). Fault detection and isolation in automatic processes. In Leondes, C. (Ed.), *Control and Dynamic Systems*, 49, 241–287.

Fuchs, A., Axehill, D., & Morari, M. (2015). Lifted evaluation of mp-MIQP solutions. *IEEE Transactions on Automatic Control*, 60(12), 3328–3331.

Guibas, L. J., Nguyen, A., & Zhang, L. (2003). Zonotopes as bounding volumes. In *14th ACM-SIAM symp. on discrete algorithms*.

Herceg, M., Kvasnica, M., Jones, C. N., & Morari, M. (2013). Multi-parametric toolbox 3.0. In *Proc. of the European control conference* (pp. 502–510).

Kruskal, J. B. (1956). On the shortest spanning subtree of a graph and the traveling salesman problem. *Proceedings of the American mathematical society*, 7(1) 48–50.

Nikoukhah, R. (1998). Guaranteed active failure detection and isolation for linear dynamical systems. *Automatica*, 34(11), 1345–1358.

Patton, R. J., Frank, P. M., & Clark, R. N. (2013). Issues of fault diagnosis for dynamic systems. Springer Science & Business Media.

Poulsen, N., & Niemann, H. (2008). Active fault diagnosis based on stochastic tests. *International Journal of Applied Mathematics and Computer Science*, 18(4) 487–496.

Punčochář, I., & Šimandl, M. (2014). On infinite horizon active fault diagnosis for a class of non-linear non-Gaussian systems. *International Journal of Applied Mathematics and Computer Science*, 24(4), 795–807.

Punčochář, I., Širokỳ, J., & Šimandl, M. (2015). Constrained active fault detection and control. *IEEE Transactions on Automatic Control*, 60(1), 253–258.

Raimondo, D. M., Braatz, R. D., & Scott, J. K. (2013). Active fault diagnosis using moving horizon input design. In *Proc. of the European control conference* (pp. 3131–3136).

Russell, E. L., Chiang, L. H., & Braatz, R. D. (2012). Data-driven methods for fault detection and diagnosis in chemical processes. Springer Science & Business Media.

Scott, J. K., Findeisen, R., Braatz, R. D., & Raimondo, D. M. (2014). Input design for guaranteed fault diagnosis using zonotopes. *Automatica*, 50(6), 1580–1589.

Scott, J. K., Marseglia, G. R., Magni, L., Braatz, R. D., & Raimondo, D. M. (2013). A hybrid stochastic-deterministic input design method for active fault diagnosis. In *Decision and control (CDC), 2013 IEEE 52nd annual conference on* (pp. 5656–5661). IEEE.

Scott, J. K., Raimondo, D. M., Marseglia, G. R., & Braatz, R. D. (2016). Constrained zonotopes: A new tool for set-based estimation and fault detection. *Automatica*, 69, 126–136.

Streif, S., Petzke, F., Mesbah, A., Findeisen, R., & Braatz, R. D. (2014). Optimal experimental design for probabilistic model discrimination using polynomial chaos. In *Proc. of the IFAC world congress*.

Tabatabaeipour, S. M. (2015). Active fault detection and isolation of discrete-time linear time-varying systems: a set-membership approach. *International Journal of Systems Science*, 46(11), 1917–1933.

Xu, F., Olaru, S., Puig, V., Ocampo-Martínez, C., & Niculescu, S. (2014). Sensor-fault tolerance using robust MPC with set-based state estimation and active fault isolation. In *Decision and control (CDC), 53rd annual conference on* (pp. 4953–4958). IEEE.

Yang, J., Hamelin, F., & Sauter, D. (2014). Active fault diagnosis based on a framework of optimization for closed loop system. In *Control, decision and information technologies (CoDIT), 2014 international conference on* (pp. 387–392). IEEE.

G. Roberto Marseglia was born in Milano (Italy) in 1988. He received the B.Sc. and M.Sc. in Computer Engineering and the Ph.D. in Electronic, Computer Science and Electric Engineering from the University of Pavia (Italy) respectively in 2010, 2012, 2016. As Ph.D. student he held a visiting position at the Department of Chemical Engineering at the Massachusetts Institute of Technology (MIT), USA. Currently he is visiting Mesbah's group at the Department of Chemical and Biomolecular Engineering, University of California Berkeley, USA. He is author of several papers published in refereed journals and conference proceedings. His current research interests include fault-detection and diagnosis and fault tolerant control.

Davide M. Raimondo is an associate professor in the Department of Electrical, Computer and Biomedical Engineering at University of Pavia, Italy. He received the Ph.D. in Electronics, Computer Science and Electrical Engineering from the University of Pavia, Italy, in 2009. From January 2009 to December 2010 he was a post-doctoral fellow in the Automatic Control Laboratory, ETH Zürich, Switzerland. In 2012 (March to June), 2013 (August to September) and 2014 (September to November) he was visiting scholar in Prof. Braatz Group, Department of Chemical Engineering, MIT, USA. He is the author or co-author of more than 70 papers published in refereed journals, edited books, and refereed conference proceedings. He is subject editor of the Journal of Optimal Control Applications and Methods and CEB Associate Editor of the IEEE CSS. His current research interests include model predictive control, fault diagnosis and fault-tolerant control, advanced battery management systems, renewable energy and control of glycemia in diabetics.