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Hadamard states and tunnelling effects for a BTZ black hole

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Cover: Artist concept of a black hole.

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Abstract

This thesis analyses the behaviour of a real massive scalar field in BTZ spacetime, a 2+1-dimensional black hole solution of the Einstein's field equations with a negative cosmological constant.

The analysis is performed for a large class of Robin boundary conditions that can be imposed at infinity and we show whether, for a given boundary condition, there exists a ground state by constructing explicitly its two-point function. We demonstrate that, for a subclass of such boundary conditions, it is possible to construct a ground state that locally satisfies the Hadamard property. In all other cases we show that bound state mode solutions exist, a novel feature in literature. Moreover we show that the presence of bound state mode solutions prevents the construction of a physically acceptable ground state.

Subsequently we focus our attention in a neighborhood of a Killing horizon in a 2+1-dimensional spacetime, so to analyse the local behaviour of the two-point correlation function of a quantum state for a scalar field. In particular we show that, if the state is of Hadamard form in such neighbourhood, the two-point correlation function exhibits a thermal behaviour at the Hawking temperature, under a suitable scaling limit towards the horizon. This results are then specified to the case of a massive, real scalar field subject to Robin boundary conditions in the the non-extremal, rotating BTZ spacetime.

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I also thank Igor Khavkine, Hugo R. C. Ferreira and Nicolò Drago for mentoring me and for their friendship: their insights and suggestions have been extremely valuable during these years. In particular I owe Hugo R. C. Ferreira a great deal for his guidance, while significant results were obtained thanks to the contribution of Igor Khavkine, who is almost entirely responsible for the spectral analysis in Appendix C.
Many others took care for me in this three years. I'm grateful and there is no need to write much: each one of you knows how much I care in return.

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Preface

The subject presented in this work has been also published in

1 - Francesco Bussola, Claudio Dappiaggi, Hugo R. C. Ferreira, and Igor Khavkine. Ground state for a massive scalar field in the BTZ spacetime with Robin boundary conditions. *Phys. Rev.*, D96(10):105016, 2017

2 - Francesco Bussola and Claudio Dappiaggi. Tunnelling processes for hadamard states through a 2+1 dimensional black hole and hawking radiation. *Classical and Quantum Gravity*, 36(1):015020, dec 2018

Notation

$\bar{a}$	Complex conjugate of a
Ω_H	Angular velocity of the black hole horizon
TM	The tangent bundle of the manifold M
T_pM	The tangent space at $p \in M$
κ	The surface gravity of a black hole
κ_B	Boltzmann constant
$g_{\mu\nu}$, $g^{\mu\nu}$	Metric and inverse metric matrix elements
$a^\mu a_\mu$	Repeated indices mean summation $\sum g^{\mu\nu} a^\mu a_\mu$
$J^\pm(p)$	*Causal future* $(+)$, resp. *Causal past* $(-)$ of $p \in \mathcal{M}$
$J^\pm(K)$	$\bigcup_j \{J^\pm(q_j) \mid q_j \in K\}$
$D^\pm_{\mathcal{M}}(A)$	*Future* $(+)$, resp. *Past* $(-)$, *domain of dependence*. For any $A \in \mathcal{M}$, the collection of all points $q \in \mathcal{M}$ such that every past $(+)$, resp. future $(-)$ inextensible causal curve passing through q intersects A
$D_{\mathcal{M}}(A)$	*Domain of dependence* $D^+_{\mathcal{M}}(A) \cup D^-_{\mathcal{M}}(A)$

The $2+1$ dimensional BTZ metric adopts the signature convention $(-++)$. The AdS_3 singature convention is $(--++)$. We also adopt the subsequent simplified notation for the following notable constants by setting $c = \hbar = 1$.

Chapter 1

Introduction

In the framework of General relativity, the geodesics describing the motion of free falling particles are deformed by gravity or, to be more accurate, gravity is the result of the geometrical curvature of the spacetime. In this picture, a *black hole* can be naively described as a region of *no escape*, in which the gravitational acceleration produces an escape velocity greater than the speed of light. When this condition applies, neither matter nor light can break out of such region, which appears completely black for any external observer. The seed of this notion turns out to be surprisingly old. In 1784 Rev. John Michell submitted a paper to Philosophical Transactions [3] in which he attempted to couple a rudimental corpuscular description of light with Newton gravitational laws. At the time the speed of light had been estimated by analysing the aberration of light coming from fixed stars, and was assessed around 0.3×10^6 kilometers per second [4]. Far from understanding the invariance of the speed of light or the real nature of photons, Mitchell pondered the existence of a cosmological object with enough mass ($1.25 \times 10^8 M_\odot$) to attract light particles to its surface.

After being ignored for almost 300 years, the notion of black hole reappeared as a consequence of gravitational collapse of matter in General relativity. A first encounter occurs when studying the interior solutions of Einstein field equations for a static (that is, non-rotating), spherically symmetric, perfect fluid body. Here, sufficiently massive objects are unable to support themselves against their own gravitational attraction and they undergo a complete collapse. In 1916 Karl Schwarzschild published a paper showing what would then become the first static vacuum solution for Einstein field equations: a static spherically symmetric metric describing the behaviour of a gravitational field around a spherical body of mass M. This metric, when imposing an asymptotical Newtonian (flat) behaviour, reads the well-known expression

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\ . \quad (1.1)$$

The Schwarzschild solution is valid only in the exterior region ($r > 2M$) of the spherical body. However, if one presumes that the whole mass is concentrated in the central point, called *singularity*, one can interpret (1.1) as a black hole solution where the surface at $r = 2M$ plays the role of a *event horizon*. This solution can then be continuously extended to the interior region of the black hole ($r < 2M$) by means of a coordinate transformation. This extension, named after Kruskal and Szekeres, reflects the fact that the domain of validity of (1.1) is bounded by a singular point, $r = 2M$, dominated by coordinate artifacts and makes the singular nature of the black hole horizon disappear (see Fig. 1.1).

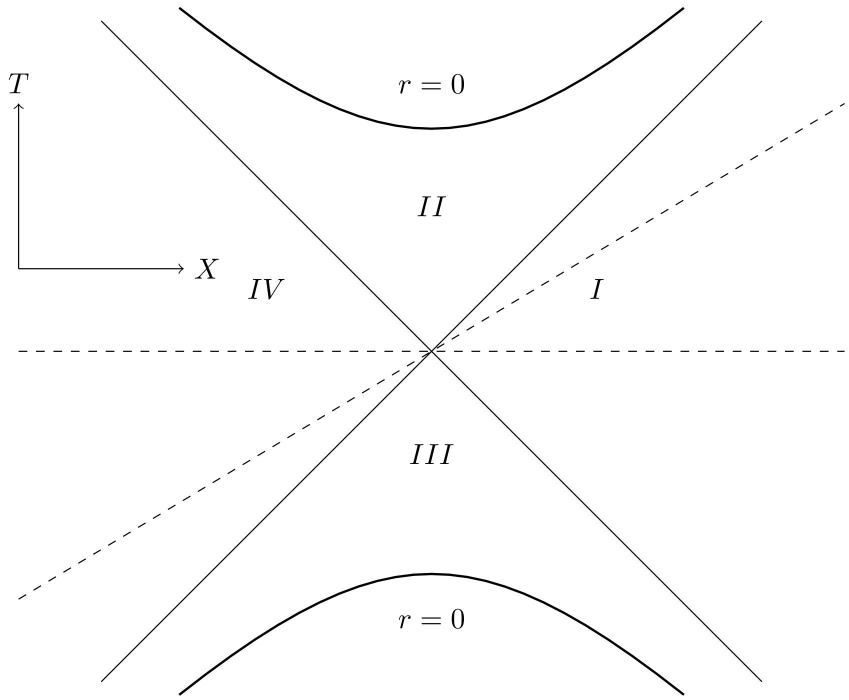

Figure 1.1: Kruskal-Szekeres extension of Schwarzschild spacetime. Any point of the diagram represents a two-dimensional sphere. Dashed lines represent surfaces at constant t. Thick lines describe the singularity $r = 0$. Diagonal continuous lines identify the horizon and anti-horizon surfaces $T - X = 0$ and $T + X = 0$ respectively. The coordinates X and T span all the original spacetime region $r > 0$, that is $X^2 - T^2 > -1$, $T > -X$, $X \in \mathbb{R}$. Moreover a brand new region appears which duplicates the universe structure for $T < -X$. Region *I*, $-X < T < +X$, comprises the exterior region of the spacetime, Region *II*, $|X| < T < \sqrt{1 + X^2}$, the no escape region of the spherical body and is referred as a *black hole*. Regions *IV* and *III* duplicate a time reversed version of the spacetime

From a physical point of view it is very unlikely that such extended solution, which assumes the existence of two asymptotically flat regions connected by a unique singularity acting as a causal "wormhole", can be observed in our universe. A reasonable scenario to achieve the formation of a black hole is in fact the collapse of a spherical body. Most likely, then, Regions *III* and *IV*

in Fig. 1.1 are unphysical, while Regions I and II are partially overlaid by collapsing matter, as in Fig. 1.2.
On the other hand, in 1967, W. Israel proved that, given an asymptotically flat spacetime, any static[1] vacuum solution of Einstein field equations exhibiting a bifurcate Killing horizon[2] reduces to the Schwarzschild solution [5, 6]. In this sense, the maximally extended Kruskal-Szekeres solution remains the only spherically symmetric, static vacuum solution of Einstein field equation. Therefore, as long as we consider static configurations, there is no reason to believe that any gravitational collapse would lead to a different equilibrium state.

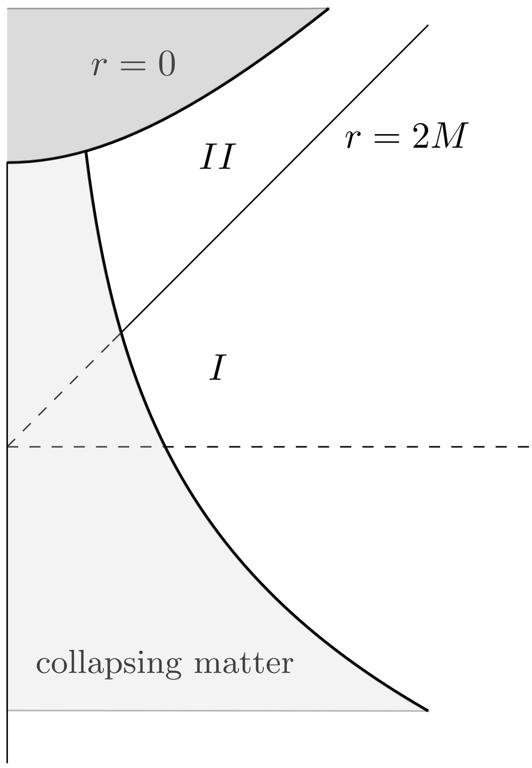

Figure 1.2: The complete gravitational collapse of a spherical body.

Of course these comments are no more valid whenever we are considering non-static, but stationary solutions. Non-static solutions are of particular interest, since one expects any black hole to rotate, especially in the unverified hypothesis that this configuration is the result of a collapse of rotating spherical body. As found by Kerr, there exists a two-parameter family of black hole solutions, characterised by an angular momentum J and a mass M, and it was possible to prove that this is the only vacuum stationary family of solutions of

[1]The term static refers to the existence of a non-vanishing, irrotational, timelike Killing vector field.

[2]From a mathematical point of view, other solutions, such as those leading to *naked singularities*, are possible. Those cases can be excluded by asking for the existence of the horizon.

Einstein field equations [7] [8] when the spacetime describes an asymptotically flat black hole. Kerr-like solutions [9] [10] might be regarded as the prototypical physical models for black holes, but they are quite convoluted. They are all axisymmetric, asymptotically flat (and of course stationary), with two Killing vector fields. Any non extremal Kerr black hole has an inner horizon, an outer horizon and an ergosphere. The inner horizon acts as a Cauchy horizon. The outer horizon is a Killing horizon and the Killing parameter associated to its Killing vector field can be interpreted, in analogy with the Schwarzschild solution, as a natural time coordinate. The ergosphere is the surface where the purely temporal component of the metric in Boyer–Lindquist coordinates [11] changes sign. Moreover its maximally extended solution comprises a globally hyperbolic black hole spacetime (see Fig. 1.3).

This last consideration is of notable importance if one wants to deal with quantum physics in the presence of a gravitational field, since the study of quantum field theory on a curved background often relies on the assumption that the underlying manifold is globally hyperbolic. In particular if one considers a dynamical field ruled by some normally hyperbolic equation (wave equation) and that the spacetime is globally hyperbolic, then the associated Cauchy problem is well-posed and there exists a unique solution, as well as unique advanced and retarded fundamental solutions [12].

Quantum field theory on curved backgrounds is nowadays a well-established branch of theoretical and mathematical physics [13] and it is quite suitable for studying fields in presence of black holes, permitting to unveil non classical phenomena, such as the Hawking radiation [14], which have no analogue in the flat case. The main assumption behind this approach is neglecting quantum gravitational effects as well as any backreaction effect in Einstein field equations.

Concerning the black hole scenarios, many results have been derived under the hypothesis of spherical symmetry. This assumption allowed to understand almost completely the structure of these matter systems and their physical behaviour [15, 16, 17, 18, 19, 20, 21].

One of the building block in the quantization of free field theories on curved spacetimes is the identification of a physical relevant state, which is used to construct the algebra of Wick polynomials and the multi-particle states under the action of the creation and annihilation operators. In the trivial case of a field theory in Minkowski spacetime, this is represented by the Poincaré invariant vacuum state, often dubbed by the quantum mechanical label $|0\rangle$.

The situation is totally different in curved spacetime, since we cannot rely on the Poincaré symmetry group and, as a consequence, there is no clear way to select a single state above another. A constructive way to select a physical relevant state for a free field theory is to identify an associated two-point function, that is a certain bisolution of the field equations, satisfying proper initial conditions and support properties while exhibiting the same short-range behaviour of the Poincaré vacuum.

Generally speaking, the two-point functions, such as the Feynman propagator, are expressed as a product of field configurations $\Phi(x)\Phi(x')$. As an example, if one considers the scalar field on flat spacetime, the Feynman propagator associated to the vacuum state $|0\rangle$ is defined as

$$G_F(x, x') = i \langle 0|\mathcal{T}(\Phi(x)\Phi(x'))|0\rangle \ ,$$

where $\mathcal{T}$ is the time ordering product and it plays a dominant role in the renormalization of the vacuum polarization $\langle\Phi^2\rangle$.

The selection of a physical relevant state and the construction of the associated two-point function is moreover one of the fundamental constituents to implement the dynamics while imposing canonical (anti-)commutation relations. This allows us to introduce a Fock quantization of the dynamics [22, 23, 24] and, consequently, to reconstruct the constitutive elements of quantum mechanics. This procedure does not only account for a complete description of quantum free field theories, but it is also useful to implement perturbative quantum models [25, 26, 27].

Unfortunately, the construction of a full-fledged physical state on a Kerr-like spacetime, is non trivial at all. The main obstacle is provided by the fact that the timelike Killing field normal to the horizon becomes spacelike at large distances. This undermines the possibility to unambiguously define the notion of positive frequencies and it prevents the explicit construction of the state by means of a distinguished two-point function. If a complete, everywhere timelike Killing field existed, it would allow for the identification of a unique full-fledged quantum state, dubbed the *ground state*, guaranteeing that all quantum observables have finite fluctuations and allowing the construction the algebra of all Wick polynomials [28, 29].

As we said, this is not the case for Kerr-like models. Even considering the extended Kerr spacetime (see Fig. 1.3), it can be proven [30, Section 6] that, whenever superradiant modes are present, there does not exist any proper ground state on the region which is invariant under the isometries generating the event horizon of the Kerr black hole (the grey area in Fig. 1.3). In this context, a proper ground state is asked to be a *Hadamard state*. Hadamard states are defined as those states whose two-point function exhibits the same short-range behaviour of the Poincaré vacuum. A thorough definition can be found in Chapter 3. As a consequence, the computation of physical observables, as well as the introduction of a legitimate renormalization scheme, is non straightforward [31, 32, 33, 34] and only in recent years a promising renormalization procedure has been introduced [35, 36].

Despite this dismal situation, an interesting case study for a non static black hole solutions is provided by solving the vacuum Einstein field equations in $2+1$ dimensions. General relativity in $2+1$ dimensions is indeed simpler. Generally speaking, the spacetime geometry is described by the Riemann curvature tensor $R_{\alpha\beta\mu\nu}$ and it is possible to extract three important quantities: the Weyl tensor $C_{\alpha\beta\mu\nu}$, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R. In a $2+1$

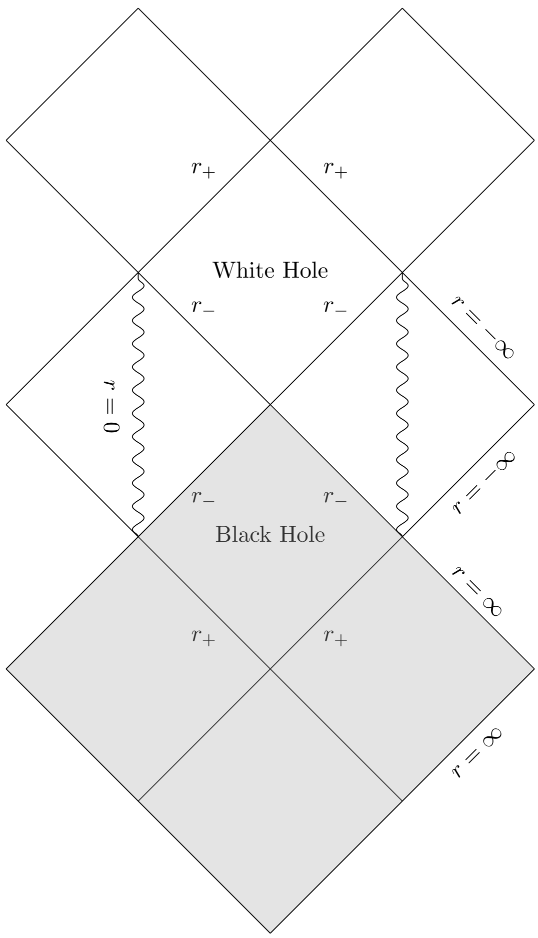

Figure 1.3: Portion of the Penrose diagram for the extended Kerr spacetime. Whenever the classical Hamiltonian exhibit a superradiant behavior, no proper physical state exists in the globally hyperbolic portion highlighted grey area.

dimensional scenario, the Weyl tensor vanishes identically. This entails that any solution of the vacuum Einstein field equations is flat, while any solution with non vanishing cosmological constant Λ has constant curvature. As a result, the Riemann curvature tensor in $2+1$ dimensions can be written as

$$R_{\alpha\beta\mu\nu} = g_{\alpha\mu}R_{\beta\nu} + g_{\beta\nu}R_{\alpha\mu} - g_{\beta\mu}R_{\alpha\nu} - g_{\alpha\nu}R_{\beta\mu} - \frac{1}{2}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})R \ .$$

In 1991, Bañados, Teitelboim and Zanelli [37] proved that there exists a $2+1$ dimensional black hole solution for negative values of the cosmological constant Λ. This solution, which goes under the name of BTZ *spacetime*, represents a $2+1$ dimensional rotating black hole in an asymptotically AdS_3 spacetime.
This setting presents many analogies with the $3+1$ dimensional Kerr spacetime: it is stationary and axisymmetric and it possesses an inner and an outer horizon, as well as two canonical Killing fields associated to the mentioned symmetries. Nonetheless it has some peculiar features that distinguish it from Kerr-like models. First of all it is locally isometric to AdS_3 and, as a result, it is locally of constant curvature. This also implies that there is no curvature singularity at the radial origin. Secondly, and this is the most notable difference, it is not extensible to a globally hyperbolic spacetime, since it possesses a conformal boundary at radial infinity. This major setback is somehow compensated by the fact that, even though none of the two Killing fields is everywhere timelike, there exists a suitable linear combination which is timelike in the whole exterior region of the black hole. This last property essentially allows to unambiguously define the notion of positive frequencies and, in the end, it might lead to explicitly construct a full-fledged ground state, if it exists. This is the main reason to try to consider the BTZ spacetime as an interesting arena to analyse the behaviour of quantum fields in a stationary black hole setting.

Given the multiple similarities with Kerr, the BTZ spacetime is still a debated topic in relation to various issues, as for example the AdS/CFT correspondence [38], the analysis of acoustic black hole simulators [39] or used as a test environment for probing the physics of quantum fields [40].
Regarding the construction of quantum states, it has been already proven that, when Dirichlet or Neumann boundary conditions are imposed at radial infinity, there exists a Hartle-Hawking state for the massless conformally coupled scalar field on the $2+1$ dimensional static hole background [41]. Further attention has been also devoted to the analysis of quasinormal modes [42], of superradiant modes [43] and of stationary clouds [44].

One of the main difficulties in studying quantum fields on BTZ background is related to the analysis of the spectrum associated to the equation of motion. Even when a mode-decomposition of the field equation is possible it happens that, whenever the black hole spacetime is rotating, the angular and temporal coordinates couple together and they produce a non-linear dependence on the spectral parameter which might not be eliminable.

As presented in this work, this is indeed the case when facing the problem of a real massive scalar field on a stationary BTZ spacetime. We will consider a real massive scalar field in the exterior region of a stationary $2+1$ dimensional black hole and we will show under which circumstances a full-fledged ground state exists. We will perform this analysis for a large class of boundary conditions of Robin type which, as we will see, guarantees that the spacetime can be regarded as an isolated system. The Robin boundary conditions comprise the particular cases of Dirichlet and Neumann boundary conditions. Moreover we will show that, when a ground state does not exist, this is due to the presence of bound states, that are solutions to the dynamical equation with non-real frequency which seem to appear in the spectrum as a result of the background rotation.
In order to obtain these results, the equation of motion, a Klein-Gordon equation, is solved by applying a mode-decomposition, which exploits the symmetry properties of the BTZ spacetime, so to obtain a one-dimensional radial differential equation. The spectrum analysis is then performed taking into account that the radial equation exhibits both a linear and a quadratic dependence on the spectral parameter. After that, the two-point functions associated to some physical states are constructed and we see that, when no bound states are present, these two-point function represent a full-fledged ground state. Lastly, we tackle the problem of analysing the high-energy behaviour of Hadamard states in the proximity of the black hole horizon and we show that they exhibit a thermal spectrum at the Hawking temperature T_H. As a result, we identify two Kubo–Martin–Schwinger (KMS) states [45] [46], global thermal equilibrium states generalising the notion of Gibbs ensemble, which seem to be intertwined with some tunnelling effect through the black hole horizon and which appear to be somehow related to the local formulation of the Hawking radiation proposed by Parikh and Wilczek [47].

Following the present introduction, in Chapter 2 one can find a general presentation of the $2+1$ black hole geometry. According to the original paper [37], the BTZ black hole is here defined as a quotient space of AdS_3 with respect to a suitable group of identifications. In particular, Section 2.3 presents some relevant properties of the BTZ spacetime, by analysing its causal structure and its symmetries. In Section 2.3.3 we also introduce the notion of bifurcate horizon, which will play a relevant role in Chapter 4. Moreover, Section 2.3.4 details the global structure of the spacetime, highlighting the presence of a timelike conformal boundary at radial infinity, which spoils the property of BTZ of being globally hyperbolic. The last part of the section is then devoted to the construction of a particular chart of Kruskal-like coordinates, needed in Chapter 4.

In Chapter 3 we tackle the problem of the construction of a two-point function for the ground state of a real, massive, scalar field in the external region of the BTZ black hole spacetime. In Section 3.1.2 the Klein-Gordon equation, which rules the dynamics of the field, is mode-expanded by exploiting the two

Killing vector fields of the underlying background. As a result we obtain a one-dimensional radial differential equation, which is reduced to a Sturm-Liouville problem. A brief overview of Sturm-Liouville problems and their relation to the two-point functions is presented in Appendix A.
In Section 3.2.1 the radial differential equation is solved and the radial mode solutions are expressed in terms of Gaussian hypergeometric functions. Gaussian hypergeometric functions are parametric one-dimensional solutions of a Gaussian hypergometric equation. An introduction to their properties can be found in Appendix B.
In order to understand, whether and which boundary conditions need to be applied at the conformal boundary so to obtain a unique and general solution, the asymptotic behaviour and the square integrability of radial modes is studied. Finally in Section 3.2.5 Robin boundary conditions are applied. Robin boundary conditions, which are a large class of non-dynamical boundary conditions imposing zero energy flux at the boundary, guarantee that the spacetime can be treated as an isolated system.
Section 3.3 is then completely devoted to the construction of the scalar field two-point function. After a general introduction to the role of the two-point functions in the quantization scheme for free fields, see Section 3.3.1, we focus the attention of the specific case of the scalar field on BTZ. In particular in Section 3.3.3 we identify the correct notion of "positive-frequency" and we show how to perform the spectral resolution of the identity operator in the case of a quadratic operator pencil, that is in the case of a differential equation with a linear and quadratic dependence on the spectral parameter. This procedure is presented in general in Appendix C, where we also prove that it is applicable in our case. Finally, see Sections 3.3.4 and 3.3.5, we explicitly display two ground state candidates, relative to two range values of the field parameters, while a third two-point function exhibits bound states mode solutions. The presence of bound states is discussed in Section 3.4, where we also debate whether the ground state candidates actually represent a full-fledged ground state, satisfying the so-called Hadamard condition.

In Chapter 4 the attention is devoted to the analysis of the high-energy behaviour of the two-point functions in proximity of the black hole horizon. In this chapter we first generalise to $2+1$ dimension the approach proposed by Moretti and Pinamonti [48] and we see that, under these circumstances, any physical state exhibits a short-range thermal distribution at the Hawking temperature T_H and that this thermal behaviour seems to be related to some tunnelling processes through the horizon. Therefore, in Section 4.2.1 we inquire whether this approach is applicable to the specific case of a scalar field on a BTZ background and we argue that this is indeed the case if one considers a KMS state. Finally in Section 4.2.1 we show the explicit form of such a thermal state in the exterior region of the spacetime.

In the Conclusion one can find some final remarks. The mentioned Appendices follow.

Chapter 2

Black holes in $2+1$ dimensions

In this chapter we are going to define the BTZ black hole, listing its fundamental geometrical properties and its physical features. The geometry of the $2+1$ dimensional BTZ black hole has been widely analysed by Bañados, Henneaux, Zanelli and Teitelboim [37] and a comprehensive review can be found in [49]. One of the most notable properties of BTZ is that it can be constructed as a quotient space starting from AdS_3 spacetime. This feature will play a distinguished role in Chapter 4 of this work, where we will present the results published in [2]. Another key geometrical feature is the presence of an inner r_- and an outer horizon r_+ when the black hole spacetime is rotating, which roughly mimics the behaviour of the most notable Kerr black hole. Moreover we will introduce a Killing vector field, timelike in the whole exterior region of the black hole, which will prove to be of primary importance when trying to build the two-point functions of the scalar field in Chapter 3.

In Section 2.1 the AdS_3 spacetime, as well as its universal covering CAdS_3, is defined. In Section 2.2, the BTZ spacetime is introduced as a quotient space of CAdS_3 with respect to a certain group of isometries and a natural set of coordinates is presented. In Section 2.3 we review some key features of BTZ, including some comments about the causal structure of the spacetime, the relation between the timelike Killing vector field and the event horizon and we will introduce various coordinate sets, which will become useful in the forthcoming chapters.

2.1 The AdS_3 spacetime

Let us consider $\mathbb{R}^4$ equipped with Cartesian coordinates (u, v, x, y) and with a symmetric covariant differentiable non degenerate tensor field $g \in T^*_p\mathbb{R}^4 \otimes T^*_p\mathbb{R}^4$ of signature $(--++)$, whose line element reads

$$\mathrm{d}s^2 = -\mathrm{d}u^2 - \mathrm{d}v^2 + \mathrm{d}x^2 + \mathrm{d}y^2 \ . \tag{2.1}$$

Given a positive constant ℓ^2, the hypersurface

$$-v^2 - u^2 + x^2 + y^2 = -\ell^2 \tag{2.2}$$

endowed with the induced metric is called three-dimensional *Anti-de Sitter space* and it is denoted as AdS_3. By construction, AdS_3 has a negative curvature with *curvature radius* ℓ and it is a maximally symmetric solution of vacuum Einstein field equations with negtive cosmological constant $\Lambda = -\frac{1}{\ell^2}$. The induced metric on AdS_3 is nondegenerate and it has Lorentzian signature $(-++)$. A global chart (μ, λ, θ) can be introduced by setting

$$u = \ell \cosh\mu \sin\lambda\,, \quad v = \ell\cosh\mu\cos\lambda\,, \quad x = \ell\sinh\mu\cos\theta\,, \quad y = \ell\sinh\mu\cos\theta\,,$$

with $0 \leq \mu < \infty$, $0 \leq \lambda < 2\pi$ and $0 \leq \theta < 2\pi$. Using this global chart, the metric of AdS_3 can be derived from (2.1) and (2.2) as

$$\mathrm{d}s^2 = \ell^2[-\cosh^2\mu\ \mathrm{d}\lambda^2 + \mathrm{d}\mu^2 + \sinh^2\mu\ \mathrm{d}\theta^2]\ , \tag{2.3}$$

where $d\theta^2$ is the standard line element of the unit 1-sphere. Here we notice that the time coordinate λ is periodic and hence AdS_3 admits closed timelike curves, for example all those identified by fixing $\mu = \mu_0$ and $\theta = \theta_0$. A common approach to overcome this situation is to unwrap the coordinate λ, avoiding to identify any $\lambda \in [0, 2\pi)$ with the value $\lambda + 2\pi$. The spacetime obtained in this way is called *universal covering of* AdS_3, denoted as $CAdS_3$ and it has locally the same metric (2.3), though with $\lambda \in \mathbb{R}$. Eventually, one can set $\lambda = t/\ell$ and $r = \ell\sinh\mu$ and the line element can be written in the more common form

$$\mathrm{d}s^2 = -\left(1 + \frac{r^2}{\ell^2}\right)\mathrm{d}t^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2\ . \tag{2.4}$$

Isometries

The AdS_3 spacetime is invariant[1] under the action of the isometry group $SO(2,2)$, which preserves any vector product $\tilde{\eta}^{\mu\nu}v_\mu v_\nu$, with $\tilde{\eta} = \mathrm{diag}(-1,-1,1,1)$. Given the set of coordinates (v,u,x,y), the algebra of $SO(2,2)$, $\mathfrak{so}(2,2)$, is generated by the Killing vector fields in the form

$$J_{\mu\nu} = x_\nu\frac{\partial}{\partial x_\mu} - x_\mu\frac{\partial}{\partial x_\nu}\ .$$

As we will see in the Section 2.2, the metric of a BTZ black hole will be locally invariant under the same algebra of isometries, though not all of them will be associated to a one-parameter symmetry group.

2.2 The BTZ black hole as a quotient of AdS_3

Starting from the realization of CAdS_3 spacetime as in the previous section, it is possible to derive the whole BTZ solution by patching three coordinate

[1] AdS_3 inherits its isometry group from the isometries of the original spacetime, which preserve the hyperboloid equation (2.2).

transformations covering separate regions. The black hole solution is indeed obtained via suitable identifications provided by the isometry subgroups of anti-de Sitter space.

Here we sketch the procedure. Let χ be a Killing vector for the AdS_3 space and let us introduce the one parameter subgroup $\varphi : \mathbb{R} \to SO(2,2)$ defined by $\varphi(s) = \exp[s\chi]$ and

$$\begin{aligned} \varphi(s) : \mathrm{CAdS}_3 &\to \mathrm{CAdS}_3 \\ P &\mapsto \varphi(s)P \ . \end{aligned}$$

Eventually, one would like to identify equivalent classes of orbits in CAdS_3. Given $p, q \in \mathrm{CAdS}_3$ and the equivalence relation $p \sim q$ if $q = e^{s\chi}p$, the relevant space will be the quotient space

$$\mathrm{CAdS}_3' := \frac{\mathrm{CAdS}_3}{\sim} \tag{2.5}$$

or, equivalently, $\mathrm{CAdS}_3/\{e^{s\chi}\}_{s\in\mathbb{R}}$.

Since this quotienting procedure closes all the curves joining equivalent points of AdS_3, it is imperative to check that those curves are not causal, that is neither timelike nor null.

The BTZ black hole can be in fact obtained by quotienting CAdS_3 by the equivalence relation built out of the isometry subgroup of $SO(2,2)$ generated by the Killing vector

$$\chi = \frac{r_+}{\ell} J_{12} - \frac{r_-}{\ell} J_{03} - J_{13} + J_{23} \tag{2.6}$$

where $J_{\mu\nu} = x_\nu \frac{\partial}{\partial x_\mu} - x_\mu \frac{\partial}{\partial x_\nu}$ and $x^\mu = (u, v, x, y)$. Here we focus our attention on the non extreme case[2] $r_+^2 - r_-^2 > 0$. The antisymmetric tensor $\omega^{\mu\nu}$ associated to the Killing vector χ via the relation $\chi = \frac{1}{2}\omega^{\mu\nu} J_{\mu\nu}$ has real eigenvalues $\pm r_\pm/\ell$. Therefore, it is possible to apply an $SO(2,2)$ isometry transformation reducing χ to

$$\chi' = \frac{r_+}{\ell} J_{12} - \frac{r_-}{\ell} J_{03} \ .$$

The Lorentzian norm of χ' is

$$\chi' \cdot \chi' = \frac{r_+}{\ell^2}(u^2 - x^2) + \frac{r_-}{\ell^2}(v^2 - y^2).$$

In order to avoid the creation of closed causal curves in the quotient space (2.5), we need to impose that

$$\chi' \cdot \chi' > 0 \ ,$$

[2]In the extremal case $r_+ = r_-$ a different argument applies, though it leads to the same result [37].

which reduces to the inequality

$$-u^2 + x^2 < \frac{r_-^2 \ell^2}{r_+^2 - r_-^2} \; .$$

Given this bound, it is possible to identify three types of regions, corresponding to these sectors:

- the inner region, for which $0 < \chi' \cdot \chi' < r_-^2$;
- the intermediate region, for which $r_-^2 < \chi' \cdot \chi' < r_+^2$;
- the outer region, for which $r_+^2 < \chi' \cdot \chi'$.

As a matter of fact there is an infinite number of these regions. They are bounded by the null surfaces $u^2 - x^2 = 0$, $\ell^2 - (u^2 - x^2) = 0$ and $v^2 - y^2 = 0$ and divided by the lightlike surfaces $r = r_+$ and $r = r_-$, which, as we will see, can be interpreted as the horizons of the BTZ black hole. The fact that we are dealing with an infinite number of regions separated by an infinite number of lightlike surfaces, is due to the fact that we started the realization of the BTZ black hole from the universal covering of the anti-de Sitter spacetime CAdS_3.

We select three neighbouring regions: an inner, an intermediate and an outer one. On these three regions we define a coordinate chart (t, r, ϕ) defined as follows. Set

$$A(r) = \ell^2 \left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right) , \quad B(r) = \ell^2 \left(\frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right) ,$$

$$\tilde{t} = \frac{1}{\ell} \left(r_+ \frac{t}{\ell} - r_- \phi \right) , \quad \tilde{\phi} = \frac{1}{\ell} \left(-r_- \frac{t}{\ell} + r_+ \phi \right)$$

with $t \in (-\infty, \infty)$, $\phi \in (-\infty, \infty)$. In the outer region, for $r > r_+$, selected by $u, y > 0$, we set

$$\begin{aligned}
u &= \sqrt{A(r)} \cosh \tilde{\phi}(t, \phi) \\
v &= \sqrt{B(r)} \sinh \tilde{t}(t, \phi) \\
x &= \sqrt{A(r)} \sinh \tilde{\phi}(t, \phi) \\
y &= \sqrt{B(r)} \cosh \tilde{t}(t, \phi) \; .
\end{aligned}$$

In the intermediate region, for $r \in (r_-, r_+)$, selected by $u >$ and $v < 0$, we fix

$$\begin{aligned}
u &= \sqrt{A(r)} \cosh \tilde{\phi}(t, \phi) \\
v &= -\sqrt{-B(r)} \cosh \tilde{t}(t, \phi) \\
x &= \sqrt{A(r)} \sinh \tilde{\phi}(t, \phi) \\
y &= -\sqrt{-B(r)} \sinh \tilde{t}(t, \phi) \; .
\end{aligned}$$

In the inner region, for $r \in (0, r_-)$, selected by $x > 0$ and $v < 0$, we set

$$\begin{aligned} u &= \sqrt{-A(r)} \sinh \tilde{\phi}(t, \phi) \\ v &= -\sqrt{-B(r)} \cosh \tilde{t}(t, \phi) \\ x &= \sqrt{-A(r)} \cosh \tilde{\phi}(t, \phi) \\ y &= -\sqrt{-B(r)} \sinh \tilde{t}(t, \phi) \ . \end{aligned}$$

Using these coordinate patches, one obtains the line element for the anti-de Sitter metric in CAdS_3' in the form

$$\mathrm{d}s^2 = -N(r)^2 \mathrm{d}t^2 + N^{-2}(r) \mathrm{d}r^2 + r^2 (N^\phi(r) \mathrm{d}t + \mathrm{d}\phi)^2 \ , \tag{2.7}$$

where $N(r)$ and $N^\phi(r)$ are smooth functions of $r \in (0, \infty)$, as it can be verified by inserting (2.7) in the 2+1 dimensional Einstein field equations with negative cosmological constant $\Lambda = -\frac{1}{\ell^2}$.

We also see that χ' and $\frac{\partial}{\partial t}$ are both Killing vectors for the quotient space. Moreover we can write χ' with respect to the new (t, r, ϕ) as

$$\chi' = \frac{\partial}{\partial \phi} \ .$$

In order to obtain the BTZ black hole we need to make a further identification to interpret the coordinate ϕ as an angle. Therefore, given the Killing vector $\frac{\partial}{\partial \phi}$, we further quotient the space via the of the symmetry group $\{e^{2k\pi \frac{\partial}{\partial \phi}}\}_{k \in \mathbb{Z}}$ so that $p \approx q$ if $\phi \mapsto \phi + 2k\pi$.

The BTZ spacetime is therefore the quotient space

$$\mathrm{BTZ} := \frac{\mathrm{CAdS}_3'}{\approx} \ .$$

2.3 Properties of BTZ

The BTZ black hole, as derived in Section 2.2 is written in Schwarzschild-like coordinates (t, r, ϕ). One can easily check by direct inspection that this is indeed a vacuum solution of Einstein field equations with a negative cosmological constant $\Lambda = -\frac{1}{\ell^2}$, whose curvature tensor is completely determined by the Ricci tensor $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$. The BTZ line element is

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + N^{-2} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\phi + N^\phi \mathrm{d}t\right)^2 \tag{2.8}$$

with the lapse and the shift functions being

$$N(r)^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} \ , \qquad N^\phi(r) = -\frac{r_+ r_-}{\ell r^2} \ ,$$

respectively. We can define two parameters

$$M = \frac{r_+^2 + r_-^2}{\ell^2} \ , \qquad J = \frac{2 r_+ r_-}{\ell}$$

to be interpreted as the total mass and the angular momentum of the black hole. By the constraints given in Section 2.2, they satisfy the bounds $M > 0$ and $|J| < M\ell$. The lapse and the shift functions become

$$N(r) = \left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)^{1/2} , \qquad N^\phi(r) = -\frac{J}{2r^2}$$

and the outer and inner horizons can be defined in terms of the zeros of the lapse function N. One gets

$$r_\pm^2 = \frac{\ell^2}{2}\left(M \pm \sqrt{M^2 - \frac{J^2}{\ell^2}}\right) .$$

2.3.1 Absence of closed timelike curves

The BTZ black hole spacetime has no closed causal curves, meaning that, when considering the Killing vector χ defined in Eq. (2.6) and its $SO(2,2)$ transformation χ' and when restricting our attention to the region of CAdS_3' where $\chi' \cdot \chi' > 0$, there is no timelike nor null future-directed curve, joining a point and its image generated by the isometry transformation $e^{2\pi\chi}$.

This is a direct result of the construction obtained by the quotienting procedure. When $J > 0$, the outer region $r > r_+$, region I in Fig. 2.2, has the intermediate region II, $r_- < r < r_+$, lying in its future. Conversely the inner region III, $0 < r < r_-$, has the intermediate region II lying in its past. As we said, the horizon surfaces $r = r_\pm$ are lightlike, therefore a causal curve leaving one of these regions through a horizon surface cannot re-enter it. This already guarantees that there is no closed timelike curve across the horizon surface. Anyway, one needs to check that no closed timelike curve is generated by $\sim$ and $\approx$. First of all, we notice that the quotienting procedure $\sim$ does not separate points belonging to the same regions, meaning that the image of any point $p \in$ BTZ built out of an isometry transformation $e^{s\chi}$, $s = 0, \pm 2\pi, \ldots$ belongs to the same region of p. Therefore, the causal structure of BTZ can be discussed for each region separately. Moreover, the identification $\sim$ does not create closed causal curve, since we restricted our focus to the regions of CAdS_3 where $\chi' \cdot \chi' > 0$. As for the second quotient, which leads to interpret the coordinate ϕ as an angle, it is sufficient to consider a causal curve $\Gamma_\lambda = (t_\lambda, r_\lambda, \phi_\lambda)$, smoothly parametrized in such a way that its tangent vector $(\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, \frac{d\phi}{d\lambda})$ does not vanish for any λ. Given the CAdS_3' metric (2.7), the causal property of this curve is

$$N^2\left(\frac{dt}{d\lambda}\right)^2 - N^{-2}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(N^\phi \frac{dt}{d\lambda} + \frac{d\phi}{d\lambda}\right)^2 \leq 0 .$$

Therefore, this causal curve joins the points (t_0, r_0, ϕ_0) and $(t_0, r_0, \phi_0 + 2k\pi)$ only if $\frac{dt}{d\lambda} = 0$ for a given lambda λ. A timelike curve would join these points only if

$$-N^{-2}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \ .$$

Since $N^2 > 0$, this is possible only when $\frac{dr}{d\lambda} = 0 = \frac{d\phi}{d\lambda}$, leading to a contradiction.

On the other hand, if $N^2 < 0$, the two points would be joined by a causal curve only if $\frac{dr}{d\lambda} = 0$ for some value of λ. If one considers timelike curves only, one gets that $\frac{dt}{d\lambda} = 0$ and $\frac{d\phi}{d\lambda} = 0$, which is again a contradiction.

2.3.2 Killing vector and symmetries

The AdS_3 spacetime, defined in Section 2.1 is by construction invariant under the transformations of the $SO(2,2)$ isometry group, which is generated by the Killing vectors in the form

$$J_{\mu\nu} = x_\nu \frac{\partial}{\partial x_\mu} - x_\mu \frac{\partial}{\partial x_\nu} \ .$$

Before the quotienting procedures, there are six linearly independent Killing vectors, which are

$$\begin{aligned} J_{01} &= v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v} & J_{02} &= x\frac{\partial}{\partial v} + v\frac{\partial}{\partial x} \\ J_{03} &= y\frac{\partial}{\partial v} + v\frac{\partial}{\partial y} & J_{12} &= x\frac{\partial}{\partial u} + u\frac{\partial}{\partial x} \\ J_{13} &= y\frac{\partial}{\partial u} + u\frac{\partial}{\partial y} & J_{23} &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \ . \end{aligned} \tag{2.9}$$

The aim is now to determine how many linearly independent Killing vectors BTZ has. Since BTZ is constructed directly from CAdS_3, we expect them to be induced directly by (2.9) and to be no more than six. In order to identify a well defined vector field in BTZ, we ask an AdS_3 Killing vector field ζ to be invariant under the isometry transformation generated by the identification subgroup introduced in Eq. (2.5), that is we ask that the pullback of $e^{2\pi\chi}$ leaves ζ unchanged

$$(e^{s\chi})^*\zeta = \zeta \ ,$$

where $(e^{s\chi})^*\eta := (e^{-s\chi})_*\eta$. Here, by looking at the Lie algebra $\mathfrak{so}(2,2)$, the Lie algebra $\mathfrak{h}$ of the quotient group $SO(2,2)/\{e^{s\chi}\}_s$, and considering the adjoint map $\mathrm{ad}_\chi : \mathfrak{so}(2,2) \to \mathfrak{h}$, we see that

$$e^{s\chi}\zeta e^{-s\chi} = \exp[s\,\mathrm{ad}_\chi(\zeta)] = (e^{s\chi})^*\zeta = \zeta \ .$$

In particular, this means that, given $e^{2\pi\chi}, \zeta \in \mathfrak{so}(2,2)$, then $[e^{2\pi\chi}, \zeta] = 0$.

For χ, one can apply the Jordan-Chevalley decomposition theorem [50] so to split it in a semisimple and a nilpotent part

$$\chi = S + N \ ,$$

where $[S, N] = 0$. Consequently, the exponential $e^{2\pi\chi}$ satisfies the decomposition

$$e^{2\pi\chi} = e^{2\pi S} + e^{2\pi S}(e^{2\pi N} - 1) \ .$$

Any element ζ commuting with $e^{2\pi\chi}$ must therefore commute with $e^{2\pi S}$ and $e^{2\pi N}$ separately. By inspecting the polynomial definition of the exponentials, one gets that this holds true when $[S, \zeta] = 0$ and $[N, \zeta] = 0$, therefore we derive that

$$[\chi, \zeta] = 0 \ .$$

Therefore, if one wants to to find which AdS_3 Killing vectors (2.9) induce a legitimate vector field in the quotient space, one needs to check those commuting with the vector χ. By observing that $\mathfrak{so}(2,2) = \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$, we can decompose both χ and ζ in their selfdual and anti-selfdual decomposition

$$\chi = \chi_+ + \chi_- \qquad \zeta = \zeta_+ + \zeta_- \ ,$$

and the equation becomes

$$[\chi_+, \zeta_+] = 0 \qquad [\chi_-, \zeta_-] = 0 \ .$$

Observe that χ_+ and χ_- are non zero for any value of the black hole parameters and everywhere in the manifold. The only admissible solutions in $\mathfrak{so}(2,1)$ are the real multiples

$$\zeta_+ = a\chi_+, \qquad \zeta_- = b\chi_-$$

.

Thus, we can conclude that the BTZ spacetime has at most two linearly independent Killing vectors associated with a one-parameter symmetry group, respectively.

In Section 2.2 we already identified $\frac{\partial}{\partial t}$ and of course $\frac{\partial}{\partial \phi}$ as two linearly independent Killing vectors. Therefore the most general Killing vector is a linear combination of those two vectors. As we will see, the linear combination

$$\frac{\partial}{\partial t} + N^{\phi}(r_+)\frac{\partial}{\partial \phi}$$

will play a distinguished role in the forthcoming chapters. In particular this is the Killing vector field generating the event horizon $r = r_+$. Moreover, this turns out to be a well-defined, global, timelike Killing vector field across the whole exterior region $r > r_+$ of BTZ spacetime. This is possibly the most notable difference in comparison to other models of rotating black hole spacetimes like, for example, the Kerr solution of the Einstein's field equations with vanishing cosmological constant.

2.3.3 Horizons

In order to analyse the relevant surfaces of the BTZ black hole we write the line element (2.8) in the more convenient form

$$\mathrm{d}s^2 = \left(M - \frac{r^2}{\ell^2}\right)\mathrm{d}t^2 - J\ \mathrm{d}t\mathrm{d}\phi + \frac{1}{-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2 \qquad (2.10)$$

The component g_{tt} vanishes for

$$r_{\text{erg}} = M^{1/2}\ell$$

and can be interpreted as an ergosphere surface. The Killing vector $\frac{\partial}{\partial t}$ is timelike for $r > r_{\text{erg}}$, null at $r = r_{\text{erg}}$, spacelike for $r \in (r_+, r_{\text{erg}})$ and no static observer on integral curves of $\frac{\partial}{\partial t}$ can exist for $r < r_{\text{erg}}$.

On the other hand, the component g^{rr} of (2.8) vanishes for two values, identifying two surfaces at $r = r_+$ and $r = r_-$.
In particular, $r = r_+$ is a Killing horizon for the Killing vector field

$$\xi = \frac{\partial}{\partial t} + N^{\phi}(r_+)\frac{\partial}{\partial \phi}\ . \qquad (2.11)$$

This surface acts as a boundary between region I and region II in Fig. 2.2, from which null curves do not escape to infinity. Therefore it can be considered as the event horizon for the BTZ black hole[3]. The constant $\Omega_H := N^{\phi}(r_+)$ is called the *angular velocity at the horizon*. Since $\xi^{\mu}\xi_{\mu} = 0$ constantly on the horizon, $\nabla^{\nu}(\xi^{\mu}\xi_{\mu})$ is normal to the horizon. Therefore there exists a function κ such that

$$\nabla^{\nu}(\xi^{\mu}\xi_{\mu}) = -2\kappa\xi^{\nu}\ ,$$

which is a non vanishing constant on all the orbits of ξ. This constant function, called the *surface gravity*, plays a dominant role in identifying the thermodynamic properties of the black hole. Its value for the BTZ black hole will be specified in Section 2.3.4.
The inner surface $r = r_-$ is a Cauchy horizon and it is unstable, meaning that it exhibits mass inflation similarly to the one of Kerr or Reissner-Nordström spacetimes [51, 52, 53]. In the static case $J = 0$, the inner horizon coincides with the singualirity $r_- = 0$.

Bifurcate horizon

The BTZ spacetime exhibits a *bifurcate Killing horizon*, that is a union of two intersecting Killing horizons [54]. As we will see later, the BTZ bifurcate horizon is generated by the Killing vector ξ.

Moreover, any bifurcate Killing horizon contains a C^1, spacelike, totally geodesic surface B called *bifurcation surface*. Around the surface B, the

[3]Notice that in the static case $J = 0$, the component of the metric g_{rr} and g^{tt} are equal, and the outer horizon $r = r_+$ coincides with the ergosphere surface $r = r_{\text{erg}}$.

branches of the horizon divide the BTZ spacetime into four disjoint parts such that ξ is spacelike in two of these regions and timelike in the others.

These notions will be particularly useful in Chapter 4. An explicit realization of the bifurcated horizon is proposed in Section 2.3.4.

2.3.4 Global structure

Kruskal-like coordinates

Let us start from the line element (2.8). We might consider a Kruskal-like coordinate patch around each root of the lapse function, that is the solutions of $N(r)^2 = 0$.

If we consider a neighborhood around $r = r_+$, the Kruskal-like coordinate patch can be defined as

$$
\begin{aligned}
r_- < r \leq r_+ \qquad & \begin{cases} U_+ = \left[\left(\frac{r_+ - r}{r + r_+}\right)\left(\frac{r + r_-}{r - r_-}\right)^{r_-/r_+}\right]^{1/2} \sinh A_+ t \\ V_+ = \left[\left(\frac{r_+ - r}{r + r_+}\right)\left(\frac{r + r_-}{r - r_-}\right)^{r_-/r_+}\right]^{1/2} \cosh A_+ t \end{cases} \\
r_+ \leq r < \infty \qquad & \begin{cases} U_+ = \left[\left(\frac{r - r_+}{r + r_+}\right)\left(\frac{r + r_-}{r - r_-}\right)^{r_-/r_+}\right]^{1/2} \cosh A_+ t \\ V_+ = \left[\left(\frac{r - r_+}{r + r_+}\right)\left(\frac{r + r_-}{r - r_-}\right)^{r_-/r_+}\right]^{1/2} \sinh A_+ t \end{cases}
\end{aligned}
\tag{2.12}
$$

where the frequency A_+ is defined as

$$
A_+ = \frac{r_+ - r_-^2}{\ell^2 r_+}
$$

and the angular coordinate of the metric, call it ϕ_+, is chosen on the patch such that the component $N^{\phi_+} dt$ is regular at r_+ and we impose

$$
N^{\phi_+}(r_+) = 0 \ .
$$

Using this coordinate patch around $r = r_+$, the line element becomes

$$
\mathrm{d}s^2 = \Omega^2(\mathrm{d}U_+^2 - \mathrm{d}V_+^2) + r^2(N^{\phi_+}\mathrm{d}t + \mathrm{d}\phi_+)^2 \tag{2.13}
$$

where $r = r(U, V)$ and $t = t(U, V)$ are now to be considered smooth functions of the coordinates, while

$$
\Omega^2(r) = \frac{(r^2 - r_-^2)(r + r_+)^2}{A_+^2 r^2 \ell^2}\left(\frac{r - r_-}{r + r_-}\right)^{r_-/r_+} \qquad \text{for } r_- < r < \infty
$$

We also notice that for $J = 0$, the Kruskal-like coordinates completely cover the space. On the other hand, for $J = M\ell$, the two roots $r_+ = r_-$ coincide and it is impossible to define Kruskal-like coordinates [55].

Conversely, around the other root r_-, one can use the coordinate patch

$$
\begin{aligned}
0 < r \leq r_- \qquad & \begin{cases} U_- = \left[\left(\frac{r_- - r}{r + r_-}\right)\left(\frac{r + r_+}{r_+ - r}\right)^{r_+/r_-}\right]^{1/2} \cosh A_- t \\ V_- = \left[\left(\frac{r_- - r}{r + r_-}\right)\left(\frac{r + r_+}{r_+ - r}\right)^{r_+/r_-}\right]^{1/2} \sinh A_- t \end{cases} \\
r_- \leq r < r_+ \qquad & \begin{cases} U_- = \left[\left(\frac{r - r_-}{r + r_-}\right)\left(\frac{r + r_+}{r_+ - r}\right)^{r_+/r_-}\right]^{1/2} \sinh A_- t \\ V_- = \left[\left(\frac{r - r_-}{r + r_-}\right)\left(\frac{r + r_+}{r_+ - r}\right)^{r_+/r_-}\right]^{1/2} \cosh A_- t \end{cases}
\end{aligned}
\tag{2.14}
$$

where the frequency A_- is defined as

$$A_- = \frac{r_-^2 - r_+^2}{\ell^2 r_-} \ .$$

Again we define the angular coordinate, call it ϕ_-, so that the term $N^{\phi_-} dt$ is regular at r_- and we impose

$$N^{\phi_-}(r_-) = 0 \ .$$

Using this coordinate patch around $r = r_-$, the line element becomes

$$\mathrm{d}s^2 = \Omega^2(\mathrm{d}U_-^2 - \mathrm{d}V_-^2) + r^2(N^{\phi_-}\mathrm{d}t + \mathrm{d}\phi_-)^2 \tag{2.15}$$

where $r = r(U, V)$ and $t = t(U, V)$ are again to be considered smooth functions of the coordinates, while this time

$$\Omega^2(r) = \frac{(r_+^2 - r^2)(r + r_-)^2}{A_-^2 r^2 \ell^2} \left(\frac{r_+ - r}{r_+ + r}\right)^{r_+/r_-} \qquad \text{for } 0 < r < r_+ \ .$$

As self-evident by (2.13) and (2.13), these patches are never lightlike in the domain of definition. A simple way to obtain a set of null coordinates is to define some linear combinations of them [56]. In the following we will introduce a different set of null coordinates, adapted to analyse the behaviour of fields in a neighbourhood of the horizon surface $r = r_+$.

Penrose diagrams and compactification

The most convenient way to visualise the causal structure of the spacetime is to adopt the point of view proposed by Roger Penrose [57], by drawing its conformal diagrams. Starting from the coordinate patch (U, V, ϕ) presented in the previous section, one can arrange the following change of coordinates

$$
\begin{aligned}
U + V &= \tan\left(\frac{p+q}{2}\right) \\
U - V &= \tan\left(\frac{p-q}{2}\right)
\end{aligned}
\tag{2.16}
$$

that is

$$\begin{aligned} p + q &= 2 \tan^{-1}(U + V) \\ p - q &= 2 \tan^{-1}(U - V) \end{aligned} \tag{2.17}$$

for $U + V, U - V \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

If one considers the static case $J = 0$, then $r_- = 0$ and the surface $r = \infty$ are mapped onto the lines $p = \pm\frac{1}{2}\pi$, while the singularity is mapped onto the lines $q = \pm\frac{1}{2}\pi$. The one-dimensional vertices i^+ and i^- are excluded from the diagram, so to avoid any artificial intersection between the conformal boundary $r = \infty$, the horizon $r = r_+$ and the singularity $r = 0$. This is represented in Fig. 2.1. Here we also see that the horizon is now represented by the diagonal lines $p = \pm q$.

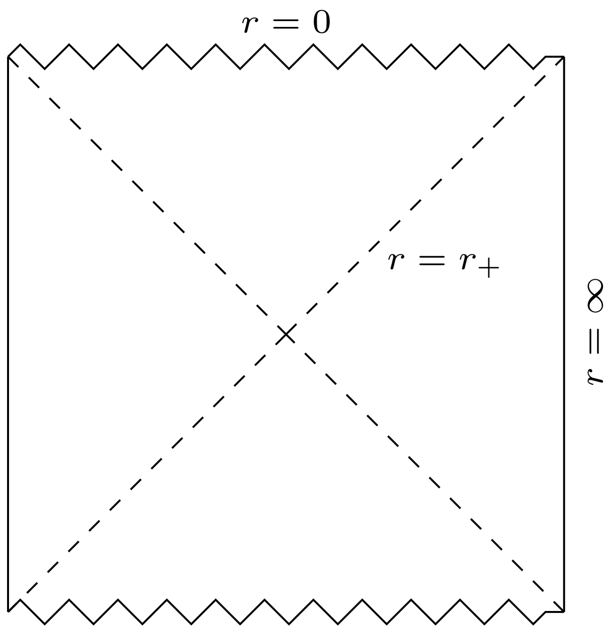

Figure 2.1: Penrose diagram showing the causal structure of the BTZ black hole, with an inner and an outer region only. Null curves move diagonally at 45° from the upward vertical. The surface $r = \infty$ will act as a conformal boundary for the spacetime.

In the non static case $J \neq 0$ $|J| < M\ell$, when applying (2.16) to the Kruskal-like patches (2.12) and (2.14), one obtains one Penrose diagram each. The maximal causal extension is obtained by gluing together an infinite sequence of regions. The result is shown in Fig. 2.2, where again the vertices i^+ and i_- are excluded from the diagram, so to avoid any artificial intersection between the conformal boundary $r = \infty$, the horizon $r = r_+$ and the singularity $r = 0$.

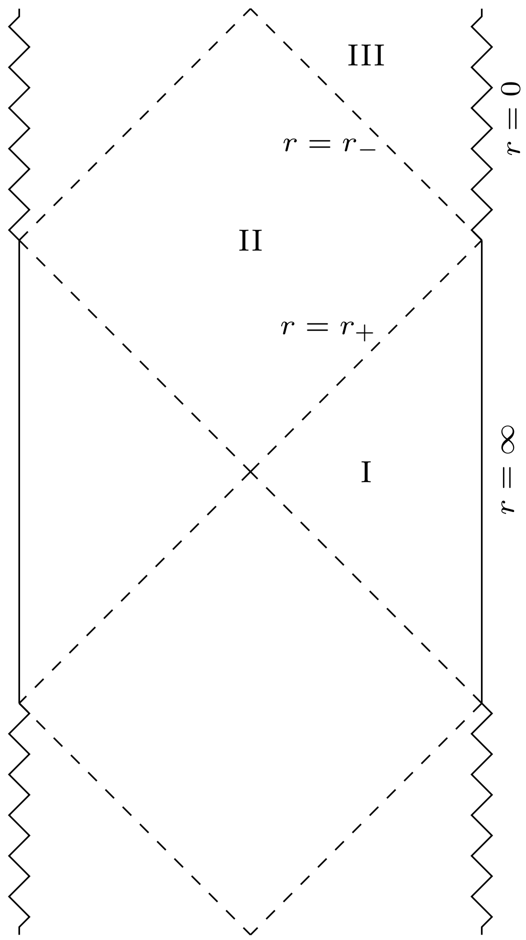

Figure 2.2: Penrose diagram showing the causal structure of the rotating BTZ black hole, with an inner, an intermediate and an outer region. Null curves move diagonally at 45° from the upward vertical. The surface $r = \infty$ again acts as a conformal boundary for the spacetime and has no intersection with $r = 0$ and $r = r_+$.

Ingoing and null coordinates

In this section we are going to introduce a particular set of null coordinates for the BTZ spacetime. These can be considered a second version of Kruskal-like coordinates. Nevertheless, the coordinate patches (2.12) and (2.14) are not null, therefore they are not adapted to analyse any physical phenomenon across the horizons. Eventually in Chapter 4 we are going to study the behaviour of fields across the surface $r = r_+$ and in particular the shape of the scalar field two-point function in a neighborhood sharpened around the event horizon.
The realisation of this set of null coordinates start by imposing the following general conditions[4]. Let $\mathcal{N}$ be the Killing horizon generated by a Killing vector

[4]Notice that all these conditions are satisfied for the BTZ spacetime, when setting $\xi^\mu = (\frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi})^\mu$.

field ξ^μ, such that all the orbits of ξ^μ on $\mathcal{N}$ are diffeomorphic to $\mathbb{R}$. Moreover we demand that $\mathcal{N}$ admits a smooth cross section, call it Σ, such that each orbit intersects Σ precisely one time. This last condition, in particular, implies that the topology of $\mathcal{N}$ is $\Sigma \times \mathbb{R}$. Note that, in the static case $J=0$, $\Sigma \simeq \mathbb{S}^1$. Let us assume that the surface gravity κ, defined by

$$\nabla^\nu(\xi^\mu \xi_\mu) = -2\kappa \xi^\nu \ ,$$

is non vanishing.
Under these hypotheses, any open neighborhood $\mathcal{U}$ of $\mathcal{N}$ can be extended to a spacetime (M^*, g^*) that contains a bifurcate Killing horizon $\mathcal{H}$, such that the image of $\mathcal{N}$ comprises a portion of $\mathcal{H}$.

The realisation works as follows. We cover Σ, with chart $\tilde{\Sigma}$ and we extend the coordinate x^3 on $\tilde{\Sigma}$ to $\tilde{\mathcal{N}}$ by keeping it constant on the null geodesics generators of $\tilde{\mathcal{N}}$.

On $\tilde{\mathcal{N}}$ we define a function τ by

$$\begin{cases} \xi^\mu \nabla_\mu \tau = 1 & \\ \tau = 0 & \text{on } \tilde{\Sigma} \end{cases} ,$$

so that $\xi^\mu = (\partial/\partial\tau)^\mu$, that is u is a Killing parameter for ξ. We also recall that $\xi^\mu \xi_\mu = 0$ on $\tilde{\mathcal{N}}$.

At each $p \in \tilde{\mathcal{N}}$, let η^μ be the unique null vector satisfying:

$$\begin{cases} \eta^\mu \xi_\mu = 1 & \text{on } \tilde{\mathcal{N}} \\ \eta^\mu X_\mu = 0 & \text{for all vectors } X^\mu \text{ which are tangent to } \tilde{\mathcal{N}} \text{ with } X^\mu \nabla_\mu \tau = 0 \end{cases} .$$

We aim at finding the affine parameter ρ associated to the vector η^μ, that is a parameter spanning the null geodesics starting at $p \in \tilde{\mathcal{N}}$ with tangent η^μ. Therefore we ask

$$\begin{cases} \rho(p) = 0 & \text{for } p \in \tilde{\mathcal{N}} \\ \eta^\mu = (\frac{\partial}{\partial \rho})^\mu & \\ \nabla_\eta \eta = 0 & \end{cases} .$$

Once ρ and τ are found we shall use the chart (τ, ρ, x^3) on an open neighborhood $\mathcal{O} \subset \tilde{\mathcal{N}}$. These coordinates will be of Eddington-Finkelstein type.

In the rotating BTZ spacetime, all hypotheses are fullfilled with $\xi^\mu = (\frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi})^\mu$ and $\kappa \neq 0$. Using the chart (t, r, ϕ) and the vector basis $(\partial_t, \partial_r, \partial_\phi)$, we know that $\eta^\mu = (A\partial_t + B\partial_r + C\partial_\phi)^\mu$ with A, B, C depending on the coordinates.

We want to check that

$$\eta^\mu \nabla_\mu \eta^\nu = 0$$

or at least $\eta^\mu \nabla_\mu \eta^\nu = \beta \eta^\mu$ where β is a non vanishing constant. In that case the parameter spanning the geodesics can be rescaled as $\rho' = \int \exp(\beta\lambda')\mathrm{d}\lambda'$.

The local properties $\eta^\mu \eta_\mu = 0$ and $\eta^\mu \xi_\mu = 1$ lead to the following solutions

$$\eta_\pm = \left(-\frac{1}{N^2}, \pm 1, \frac{N^\phi}{N^2}\right), \quad \text{i.e. } \eta^\mu_\pm = \left(-\frac{1}{N^2}\partial_t \pm \partial_r + \frac{N^\phi}{N^2}\partial_\phi\right)^\mu$$

Let us choose $\eta = \eta_-$. We can see that the parameter describing the geodesics generated by this vector is linear in r. In fact setting

$$\eta_- = \frac{\partial}{\partial\lambda} = \frac{\partial t}{\partial\lambda}\frac{\partial}{\partial t} - \frac{\partial r}{\partial\lambda}\frac{\partial}{\partial r} + \frac{\partial\phi}{\partial\lambda}\frac{\partial}{\partial\phi} \equiv \left(-\frac{1}{N^2}, -1, \frac{N^\phi}{N^2}\right)$$

we see that $\frac{\partial r}{\partial\lambda} = -1$ that is $r = -\lambda + \alpha$.

We can also check that η_- satisfies the null geodesic equation

$$\eta^\mu_- \nabla_\mu \eta^\nu_- = 0$$

This is tantamount to saying that the geodesic parameter λ is precisely the affine parameter ρ that we were searching for. In fact we expect it to be $\rho = \int_0^\lambda \exp(0)\mathrm{d}\lambda' = -\int_0^{-r} \mathrm{d}r' \equiv r + \mathit{const}$. Condition $\rho(p) = 0$ for $p \in \tilde{\mathcal{N}}$ implies that $\rho = r - r_+$

To complete the coordinate chart of Eddington-Finkelstein type, we need also the Killing parameter τ.

If we complete the parametrization of the geodesics in terms of r, we get

$$\frac{\partial t}{\partial r} = -\frac{1}{N^2} \quad \frac{\partial\phi}{\partial r} = \frac{N^\phi}{N^2}$$

We want these geodesics to be coordinate lines of our new system. Thus, one of our coordinates is ρ, while the others are quantities which are constant along a geodesic of the family. One of the remaining two coordinates is in fact τ, the Killing parameter of the Killing vector $\xi = \frac{\partial}{\partial\tau}$. By comparison

$$\xi = \frac{\partial}{\partial\tau} = \frac{\partial t}{\partial\tau}\frac{\partial}{\partial t} - \frac{\partial r}{\partial\tau}\frac{\partial}{\partial r} + \frac{\partial\phi}{\partial\tau}\frac{\partial}{\partial\phi} \equiv (1, 0, \Omega_H)$$

and we obtain $\frac{\partial t}{\partial\tau} = 1$, that is $t = \tau + \beta$. The vector ξ satisfies the geodesic equation, allowing us to identify the Killing parameter, which in this case is $\tau = \int_0^\tau \exp(0)\mathrm{d}\tau' = \int_0^t \mathrm{d}t' \equiv t + \mathit{const}$. From the comparison we also have

$$\frac{\partial r}{\partial\tau} = 0 \quad \frac{\partial\phi}{\partial\tau} = \Omega_H$$

The remaining two coordinates will be

$$u = t + T(r,\phi) \qquad \theta = \phi + \Phi(t,r)$$

The conditions to find T and Φ are given by the requirement that u and θ are constant along the geodesics parametrized by ρ and that ρ and θ are constant along the geodesics parametrized by u.

$$\frac{\partial u}{\partial r} = 0 \quad \frac{\partial\theta}{\partial r} = 0$$

$$\frac{\partial r}{\partial \tau} = 0 \quad \frac{\partial \theta}{\partial \tau} = 0$$

Setting

$$\frac{\partial T}{\partial r} = -\frac{\partial t}{\partial r} = \frac{1}{N^2} \quad \frac{\partial \Phi}{\partial r} = -\frac{\partial \phi}{\partial r} = -\frac{N^\phi}{N^2}$$

and

$$\frac{\partial T}{\partial \tau} = -\frac{\partial t}{\partial \tau} = 0 \quad \frac{\partial \Phi}{\partial \tau} = -\frac{\partial \phi}{\partial \tau} = -\Omega$$

we achieve the desired result.

Therefore, the new coordinate chart (τ, ρ, θ) is obtained by the following transformation.

$$\begin{cases} \mathrm{d}t = \mathrm{d}\tau + \frac{1}{N^2}\mathrm{d}\rho \\ \mathrm{d}r = \mathrm{d}\rho \\ \mathrm{d}\phi = \mathrm{d}\theta - \Omega_H \mathrm{d}\tau - \frac{N^\phi}{N^2}\mathrm{d}\rho \end{cases}$$

In this chart, the tangent vector of the ingoing null geodesics written above is

$$\eta^\mu = (0, -1, 0)$$

while the Killing vector is

$$\xi^\mu = (1, 0, 0).$$

We can now compute the line element of the metric *in Eddington-Finkelstein coordinates* (τ, ρ, θ):

$$\begin{aligned} \mathrm{d}s^2 =& -N^2 \mathrm{d}\tau^2 - 2\mathrm{d}\tau\mathrm{d}\rho + (\rho + r_+)^2((N^\phi - \Omega_H)\mathrm{d}\tau + \mathrm{d}\theta)^2 \\ =& (-N^2 + (\rho + r_+)^2(N^\phi - \Omega_H)^2)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}\rho \\ &+ 2(\rho + r_+)^2(N^\phi - \Omega_H)\mathrm{d}u\mathrm{d}\theta + (\rho + r_+)^2\mathrm{d}\theta^2 \ . \end{aligned} \tag{2.18}$$

Now we will show how, starting from (2.18), it is possible to build a chart of null coordinates of Kruskal type, highlighting the presence of a bifurcate Killing horizon.

By using the functions $N(\rho)$, $N^\phi(\rho)$ and $r(\rho)$ one can build the auxiliary function

$$F = -\xi^\mu \xi_\mu = -(-N^2 + (\rho + r_+)^2(N^\phi - \Omega_H)^2) = \frac{\rho(r_+^2 - r_-^2)(2r_+ + \rho)}{r_+^2 \ell^2}$$

As expected $\lim_{r \to r_+} F = 0$. We also see that the surface gravity of the BTZ black hole can be calculated directly as

$$\kappa = \frac{1}{2}\frac{\partial F}{\partial \rho}_{|\rho=0} = \frac{1}{2}\frac{\partial F}{\partial r}_{|r=r_+} = r(\ell^{-2} - \Omega_H^2)_{|horizon} = \frac{r_+^2 - r_-^2}{\ell^2 r_+} \ ,$$

which is a non vanishing constant. It also holds that

$$f := \frac{F}{\rho} = \frac{(r_+^2 - r_-^2)(2r_+ + \rho)}{r_+^2\ell^2} \qquad f(0) = 2\frac{(r_+^2 - r_-^2)}{r_+^2\ell^2} = 2\kappa \ .$$

We can now define the function $g(\rho)$ as

$$\rho g(\rho) = \frac{1}{f} - \frac{1}{2\kappa} = -\frac{r_+\ell^2\rho}{2(r_+^2 - r_-^2)(2r_+ + \rho)}$$

that is

$$g(\rho) = -\frac{r_+\ell^2}{2(r_+^2 - r_-^2)(2r_+ + \rho)}$$

Finally we define two null coordinates of Kruskal type U and V as

$$\begin{cases} U = e^{\kappa\tau} \\ V = -e^{-\kappa\tau}\rho\exp\left[2\kappa\int_0^\rho g(\rho')d\rho'\right] \end{cases} , \tag{2.19}$$

where

$$\int_0^\rho g(\rho')d\rho' = -\frac{r_+\ell^2\log(2r_+ + \rho)}{2(r_+^2 - r_-^2)} \ .$$

Hence

$$2\kappa\int_0^\rho g(\rho')d\rho' = -\log(2r_+ + \rho) \ ,$$

which yields

$$\begin{cases} U = e^{\kappa\tau} \\ V = -e^{-\kappa\tau}\rho e^{-\log(2r_+ + \rho)} = -e^{-\kappa\tau}\frac{\rho}{(2r_+ + \rho)} \end{cases} \tag{2.20}$$

or, equivalently,

$$\begin{cases} \mathrm{d}U = \kappa e^{\kappa\tau}\mathrm{d}\tau \\ \mathrm{d}V = \kappa e^{-\kappa\tau}\frac{\rho}{(2r_+ + \rho)}\mathrm{d}\tau - e^{-\kappa\tau}\frac{2r_+}{(2r_+ + \rho)^2}\mathrm{d}\rho \end{cases} . \tag{2.21}$$

In these coordinates, the surface $\{U = 0\} \cup \{V = 0\}$ is the bifurcate Killing horizon generated by ξ and it comprises the horizon $\mathcal{N} = \{V = 0\}$. The Killing vector ξ now reads

$$\xi = -\kappa\left(U\frac{\partial}{\partial U} - V\frac{\partial}{\partial V}\right)$$

and vanishes for the surface $\tilde{\Sigma} := \{U = V = 0\}$, which is the bifurcation surface B.

As a final part of this Chapter, we complete the null chart and we write the corresponding line element. Be

$$\varphi := UV = -\rho e^{-\log(2r_+ + \rho)} = -\frac{\rho}{2r_+ + \rho} \ .$$

There exists a function ψ such that

$$\psi(\varphi) = \frac{\rho(\varphi)}{\varphi} = -\frac{2r_+}{1 + \varphi}$$

with d $\rho = UV\psi(UV)$. Let

$$G = \frac{F}{\kappa UV} = \frac{\rho f}{\kappa^2 UV} = \frac{1}{\kappa^2} f\psi = -\frac{4\ell^2 r_+^2}{(r_+^2 - r_-^2)} \frac{1}{(1+\varphi)^2} \ ,$$

then

$$\mathrm{d}s^2 = G\mathrm{d}U\mathrm{d}V + VH\mathrm{d}U\mathrm{d}\theta + g_{\theta\theta}\mathrm{d}\theta^2$$

where

$$\begin{aligned} g_{\theta\theta} &= r^2 = (\rho + r_+)^2 = r_+^2 \left(\frac{1-\varphi}{1+\varphi}\right) \\ G &= -\frac{4r_+^2\ell^2}{(1+\varphi)^2(r_+^2 - r_-^2)} \\ \varphi &= UV \\ H &= 2\left[\frac{h}{\kappa} - \frac{\psi f}{\kappa}\int_0^\rho \frac{\partial g}{\partial \theta}\mathrm{d}\rho'\right] = 2\frac{\psi h}{\kappa} \end{aligned} \tag{2.22}$$

with $\frac{\partial g}{\partial \theta} = 0$ and $\psi = -\frac{2r_+}{1+\varphi}$, while h is defined by $g_{u\theta} = \rho h(\rho)$. Here $g_{u\theta} = 2(\rho + r_+)^2(N^\phi - \Omega)$, thus

$$\psi h = -\frac{r_+ r_-}{\ell\varphi}\left[1 - \left(\frac{1-\varphi}{1+\varphi}\right)^2\right] \ .$$

The *metric in Kruskal-like coordinates* (U, V, θ) is therefore

$$\begin{aligned} ds^2 = &- \frac{4r_+^2\ell^2}{(r_+^2 - r_-^2)}\frac{1}{(1+\varphi)^2}\mathrm{d}U\mathrm{d}V \\ &- \frac{4r_+\ell}{(r_+^2 - r_-)}\frac{r_+r_-}{U}\left(1 - \left(\frac{1-\varphi}{1+\varphi}\right)^2\right)\mathrm{d}U\mathrm{d}\theta + r_+^2\left(\frac{1-\varphi}{1+\varphi}\right)^2\mathrm{d}\theta^2 \ . \end{aligned}$$

Chapter 3

Scalar field around a BTZ black hole

The aim of this chapter is the construction of a two-point function for the ground state of a real, massive, scalar field in the external region of the BTZ black hole. As will become clear in Section 3.3, two-point functions play a distinguished role in the quantization of field, especially when dealing with quantum field theory in curved spacetime. The construction of a two-point function associated to a field theory often presuppose the knowledge of the basis of solutions for the equation of motion.

The equation of motion encoding the dynamics of the scalar field is the Klein-Gordon equation. The identification of a solution for the Klein-Gordon equation over a curved spacetime usually asks for the solution of a initial-valued Cauchy problem. As we will see in Section 3.1.1, the geometry of BTZ spacetime and in particular the presence of a timelike conformal boundary in the exterior region of the black hole poses the question of whether and which boundary conditions should be imposed so to have a unique solution. In Section 3.2.5 we will identify a specific class of boundary conditions, those of Robin type, as a distinguished class of non-dynamical boundary conditions ensuring the absence of energy flux through the boundary. This feature essentially guarantees that the spacetime can be interpreted as an isolated system.

In Section 3.1.2, the scalar field equation is solved by exploiting the symmetries of the background spacetime reconducing it to a one-dimensional radial differential equation, which can be expressed in Sturm-Liouville form and further reduced to a Gaussian hypergeometric differential equation. In Section 3.2.1, the set of radial solutions are then expressed in terms of Gaussian hypergeometric functions. The properties of the solutions are studied in Section 3.2.3 and in 3.2.4.

Finally in Section 3.3, a quantization scheme for the scalar field is presented. This quantization procedure essentially relies on the construction of a two-point function associated to a ground state. In Sections 3.3.4 and 3.3.5, two different ground states candidates, corresponding to two different ranges of value of the field parameters, are presented, while a third prototype fails to describe a ground state due to the presence of bound state mode solutions. Finally in Section 3.4 we argue whether the two two-point functions mentioned above actually describe a physically acceptable ground state, by verifying if they satisfy the Hadamard condition and we discuss the bound states and their physical meaning in this picture.

3.1 The Scalar field equation

3.1.1 The Cauchy problem

Let us consider the BTZ spacetime as defined in Chapter 2 with metric g provided by (2.8), restricted to regions I, II and III of Fig. 2.2, and a real, massive scalar field $\Phi : \mathrm{BTZ} \to \mathbb{R}$, whose action is given by

$$S = -\frac{1}{2}\int_{\mathrm{BTZ}} \mathrm{d}\mu_g \left(\nabla_\mu \Phi \nabla^\mu \Phi + (m^2 + \xi R)\Phi^2\right) \ , \tag{3.1}$$

where $m^2 \in \mathbb{R}$ is the mass parameter of the scalar field, $R = -\frac{6}{\ell^2}$ is the Ricci scalar curvature built out of g, $\xi \in \mathbb{R}$ is the curvature coupling constant and $\mathrm{d}\mu_g(x) = \sqrt{|\det g|}d^3x$. The dynamics is ruled by the Klein-Gordon equation

$$P\Phi := (\Box_g - m^2 - \xi R)\Phi = 0 \ , \tag{3.2}$$

where $\Box_g$ is the D'Alembert wave operator. This equation is often solved for globally hyperbolic spacetimes [58, 27].

A spacetime $(\mathcal{M}, g)$, $\dim \mathcal{M} \geq 2$, is called *globally hyperbolic* if and only if it admits a Cauchy surface[1], that is a subset $\Sigma \subset \mathcal{M}$ such that any inextensible (continuous, locally Lipschitz) timelike curve intersects Σ exactly once and such that its domain of dependence is $D_{\mathcal{M}}(\Sigma) = \mathcal{M}$.

For any globally hyperbolic spacetime, a solution for the differential equation (3.2) is in fact uniquely determined by assigning initial data on a Cauchy surface Σ

$$\begin{cases} P\Phi = 0 \\ \Phi_{|\Sigma} = f_0 \\ \nabla_n \Phi_\Sigma = f_1 \end{cases} \ , \tag{3.3}$$

where $f_0, f_1 \in C^\infty(\Sigma)$ and n is the future directed unit normal vector on Σ [59, §3.5.3].

However BTZ spacetime is not globally hyperbolic. This is manifest from the Penrose diagrams in Fig. 2.1 and 2.2, since in both cases the spatial infinity surface $r = \infty$ is a timelike conformal boundary through which data can propagate [60]. This property is essentially inherited by the universal covering CAdS_3, through the identifications of Section 2.2.

[1] An equivalent definition states that a time-oriented spacetime is globally hyperbolic if $J^+(p) \cap J^-(p) = \{p\}$ for any $p \in \mathcal{M}$ (causality) and, for any two $p, p' \in \mathcal{M}$, the causal diamond $J^+(p) \cap J^-(p')$ is either compact or empty.

In order to find a general and unique solution for (3.2), one must consequently impose boundary conditions at the conformal boundary. For a scalar field, this problem is usually analysed by considering three types of boundary conditions: *Dirichlet*, *Neumann*, and the so called *transparent* boundary conditions [41] [61] which simulate the absence of the boundary.
In the following, the analysis will be performed for all admissible boundary conditions of Robin type (see Section 3.2.5). For later convenience, we introduce the dimensionless parameter

$$\mu^2 := \frac{m^2}{\ell^2} - 6\xi \ ,$$

with $\mu^2 > -1$.

3.1.2 Scalar field expansion in (t, r, ϕ)

In order to solve the Cauchy problem, we work with coordinates (t, r, ϕ) as defined in Section 2.2. Starting from the line element (2.8), and knowing that

$$\det g = -r^2$$
$$(g^{\mu\nu})_{\mu\nu} = \begin{bmatrix} -N^{-2} & 0 & N^{-2}N^{\phi} \\ 0 & N^2 & 0 \\ N^{-2}N^{\phi} & 0 & r^{-2} - (N^{-1}N^{\phi})^2 \end{bmatrix} ,$$

the differential equation (3.2) reads

$$\frac{1}{r}\partial_r[rN^2]\partial_r\Phi + N^2\partial_r^2\Phi - N^{-2}\partial_t^2\Phi + \\ 2N^{-2}N^{\phi}\partial_t\partial_\phi\Phi + \frac{1}{r^2}\partial_\phi^2\Phi - (N^{-1}N^{\phi})^2\partial_\phi^2\Phi - \frac{\mu^2}{\ell^2}\Phi = 0 \ ,$$

with $\mu^2 := \frac{m^2}{\ell^2} - 6\xi$, $\mu^2 > -1$.

Since $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ are both Killing vectors fields for (2.8), we can Fourier expand the scalar field Φ as

$$\Phi(t, r, \phi) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \mathrm{d}\omega \ e^{-i\omega t + ik\phi} \ \Psi_{\omega k}(r) \ .$$

Therefore, one can reduce the differential equation to a one-dimensional ordinary differential equation (ODE)

$$\frac{1}{r}\partial_r[rN^2]\partial_r\Psi_{\omega k}(r) + N^2\partial_r^2\Psi_{\omega k}(r) + N^{-2}\omega^2\Psi_{\omega k}(r) + 2N^{-2}N^{\phi}\omega k\Psi_{\omega k}(r) \\ - \frac{1}{r^2}k^2\Psi_{\omega k}(r) + (N^{-1}N^{\phi})^2k^2\Psi_{\omega k}(r) - \frac{\mu^2}{\ell^2}\Psi_{\omega k}(r) = 0 \ .$$

Since we are mainly interested in finding solutions for the external region $r_+ < r < \infty$ of BTZ spacetime, we will need to clarify which are the admissible boundary conditions that can be assigned at the horizon or at the radial infinity. For ordinary differential equations this problem can be solved in full generality by using Sturm-Liouville theory (see e.g. [62, 63, 64] or [65] for an application to the study of a real, massive scalar field in the Poincaré patch of anti-de Sitter spacetime of arbitrary dimension). It is therefore convenient to introduce a new radial coordinate $z \in (0,1)$,

$$z = \frac{r^2 - r_+^2}{r^2 - r_-^2} \ . \tag{3.4}$$

This change of variable identifies the event horizon $r = r_+$ with $z = 0$ and radial infinity $r = \infty$ with $z = 1$.

The relevant functions now reduce to

$$r = \sqrt{\frac{zr_-^2 - r_+^2}{z-1}} \qquad N^2 = \frac{z(r_+^2 - r_-^2)^2}{(z-1)(zr_-^2 - r_+^2)} \qquad N^\phi = \frac{r_+ r_-(z-1)}{zr_-^2 - r_+^2}$$

and the differential equation for $\Psi_{\omega k}(z)$ can be written as

$$L\Psi_{\omega k}(z) := \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}\Psi_{\omega k}(z)}{\mathrm{d}z}\right) + q(z)\Psi_{\omega k}(z) = 0 \ , \tag{3.5}$$

with

$$q(z) = \frac{1}{4(1-z)}\left[\frac{\ell^2(\omega\ell r_+ - kr_-)^2}{(r_+^2 - r_-^2)^2 z} - \frac{\ell^2(\omega\ell r_- - kr_+)^2}{(r_+^2 - r_-^2)^2} - \frac{\mu^2}{1-z}\right] .$$

Eq. (3.5) is a second order ODE in Sturm-Liouville form with domain $z \in (0,1)$. This will be particularly useful in Section 3.3, when we will deal with the construction of the two-point function in the exterior region of the black hole spacetime. A brief recap of Sturm-Liouville problems is in Appendix A, while a more complete presentation can be found in [62]. We point out that the endpoints of the Sturm-Liuoville (3.5) problem are not included in the domain. This will be of particular importance in Section 3.2.5 when imposing some boundary conditions to identify the most general solution.

3.2 Solutions

3.2.1 Radial solutions

In order to obtain a basis of the vector space of solutions for Eq. 3.5 we apply Froebenius method [66] and we set

$$\Psi_{\omega k}(z) = z^\alpha (1-z)^\beta F_{\omega k}(z) \ . \tag{3.6}$$

Here the parameters α and β satisfy the second order algebraic equations

$$\alpha^2 = -\frac{\ell^4 r_+^2 \tilde{\omega}^2}{4(r_+^2 - r_-^2)^2} \, , \quad \beta^2 + \beta - \frac{\mu^2}{4} = 0 \, , \tag{3.7}$$

where we define $\tilde{\omega} := \omega - k\Omega_{\mathcal{H}}$ to be the square root $\tilde{\omega} = \sqrt{\tilde{\omega}^2}$, with $\mathrm{Im}\{\tilde{\omega}\} = \mathrm{Im}(\omega) \geq 0$. Four possible equivalent pairs of values (α, β) are admissible. Here we choose

$$\alpha = -i\,\frac{\ell^2 r_+ \tilde{\omega}}{2(r_+^2 - r_-^2)}\ , \quad \beta = \frac{1}{2}\left(1 + \sqrt{1+\mu^2}\right)\ . \tag{3.8}$$

It is important to notice that by picking a root for α, we are choosing one of the possible branch cuts of the square root in the complex plane. Up to strictly positive multiplicative constants, this choice is ruled only by $\tilde{\omega}$, which, as we will see in the following, is the Fourier parameter naturally associated to the Killing field ξ defined in (2.11).

When (3.6) is plugged into Eq. (3.5), one obtains the following Gaussian hypergeometric equation [67]

$$z(1-z)\partial_z^2 F_{\omega k} + [c - (a+b+1)z]\partial_z F_{\omega k} - abF_{\omega k} = 0\ , \tag{3.9}$$

where

$$\begin{cases} a = \dfrac{1}{2}\left(1 + \sqrt{1+\mu^2} - i\ell\,\dfrac{\omega\ell - k}{r_+ - r_-}\right)\ , \\ b = \dfrac{1}{2}\left(1 + \sqrt{1+\mu^2} - i\ell\,\dfrac{\omega\ell + k}{r_+ + r_-}\right)\ , \\ c = 1 - i\,\dfrac{\ell^2 r_+ \tilde{\omega}}{r_+^2 - r_-^2}\ . \end{cases} \tag{3.10}$$

The hypergeometric differential equation (3.9) has analytic solutions in the domain $(0,1)$, which can be expressed in closed form by means of Gaussian hypergeometric functions, depending on the three parameters a, b and c.

Gaussian hypergeometric functions are special functions represented by a Gauss series which include other special functions such as the rising factorial[2] $(x)_s$ or the Euler Gamma functions, i.e.

$$F(a,b,c;z) = \sum_{s=0}^{\infty} \frac{(a)_s (b)_s}{(c)_s s!} z^s = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{s=0}^{\infty} \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)s!} z^s \tag{3.11}$$

which is analytic in the region $|z| < 1$. Moreover Eq. (3.9) has regular singularities at $z = 0, 1$, therefore one can find pairs of linearly independent regular solutions around the poles at the endpoints. A brief review of hypergeometric function is presented in Appendix B, while a complete review can be found in [67].

When choosing a vector basis of solutions, the dependence of the solutions on the three parameters a, b and c forces us to distinguish two cases, discriminated by the values of μ^2.

[2]The rising factorial, also known also Pochhammer's symbol, is defined as $(x)_s := x(x+1)\ldots(x+s-1)$, with $x_0 := 1$.

General case

When none of c, $c-a-b$, $a-b$ is an integer, an admissible and convenient basis of solutions is given by

$$\begin{aligned}\Psi_1(z) &= z^\alpha(1-z)^\beta F(a,b,a+b-c+1;1-z)\ , &(3.12a)\\ \Psi_2(z) &= z^\alpha(1-z)^{1-\beta}\\ &\quad\times F(c-a,c-b,c-a-b+1;1-z)\ . &(3.12b)\end{aligned}$$

From (3.10), we get that the conditions on a, b and c identify exactly the special range of values for μ^2 which must be been excluded. The general case holds therefore for all the values $\mu^2 \neq (n-1)^2-1,\ n=1,2,3,\ldots$, with $\mu^2 > -1$.

The fundamental solutions $\Psi_1(z)$ and $\Psi_2(z)$ enjoy several useful properties. Using the conjugation identities (B.8), the symmetry $F(a,b,c;z) = F(b,a,c;z)$ and the second equality in (B.2) it is possible to check that $\Psi_1 \mapsto \overline{\Psi_1}$ and $\Psi_2 \mapsto \overline{\Psi_2}$, if $\tilde\omega \mapsto \overline{\tilde\omega}$.

They are also regular at the endpoint $z=1$, corresponding to radial infinity $r=\infty$, where, as we will show, they behave as

$$\Psi_1(z) \approx_1 (1-z)^\beta\ , \tag{3.13}$$

$$\Psi_2(z) \approx_1 (1-z)^{1-\beta}\ . \tag{3.14}$$

Notably, the behaviour at infinity is dominated by the sole parameter β and, therefore, by the value of μ^2 only. This is not surprising since one expects that the behaviour at infinity is not affected by the local properties of the black hole, such as the parameter Ω_H describing its angular velocity at the horizon.

Special case

In the special case $\mu^2 = (n-1)^2-1,\ n=2,3,\ldots$, solutions (3.12) cease to be analytic in $(0,1)$ and must be replaced. An admissible and convenient basis of solutions for (3.5) (see [67, §15.10.8]) is

$$\begin{aligned}\Psi_1(z) &= z^\alpha(1-z)^\beta F(a,b,n;1-z)\,, &(3.15a)\\ \Psi_2(z) &= z^\alpha(1-z)^\beta\\ &\quad\times \left[F(a,b,n;1-z)\log(1-z) + K_n(z)\right]\ , &(3.15b)\end{aligned}$$

where

$$\begin{aligned}K_n(z) := &-\sum_{p=1}^{n-1}\frac{(n-1)!(p-1)}{(n-p-1)!(1-a)_p(1-b)_p}(z-1)^{-p}\\ &+\sum_{p=0}^{\infty}\frac{(a)_p(b)_p}{(n)_p p!}f_{p,n}\,(1-z)^p\,,\end{aligned} \tag{3.16}$$

while

$$(a)_p = \Gamma(a+p)/\Gamma(a)\ ,$$

$$f_{p,n} = \psi(a+p) + \psi(b+p) - \psi(1+p) - \psi(n+p)\,,$$

and ψ is the digamma function

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}\ . \tag{3.17}$$

Note that $\Psi_1 \mapsto \overline{\Psi_1}$ under the replacement $\tilde{\omega} \mapsto \overline{\tilde{\omega}}$. Once more this is proven by using the conjugation identities (B.8), the symmetry $F(a,b,c;z) = F(b,a,c;z)$ and the second equality in (B.2). This property is not needed for Ψ_2 since, as it will be clear in next section, Ψ_1 is the only solution which will play a role for the admissible boundary conditions.

3.2.2 The asymptotic behaviour of the solutions and the principal solution

Inspired by the fact that the Gaussian hypergeometric equation (3.9) can be written in Sturm-Liouville form as in (3.5), we borrow the terminology used for Sturm-Liouville problems [62], introducing the notion of *principal solution*. We call a solution Ψ of (3.9) the *principal solution* at $z = 1$, if it is the unique[3] solution up to scalar multiples such that $\lim_{z\to 1}\Psi(z)/\Phi(z) = 0$ for every solution Φ that is not a scalar multiple of Ψ. As we will see in Section 3.2.5, the identification of the principal solution plays a primary role to determine which boundary conditions are admissible at the radial infinity $z = 1$.

As we will see, the Principal Solution at $z = 1$ will provide a straightforward generalization of the Dirichlet boundary conditions for a singular boundary value problem.

It is checked by direct inspection that the function Ψ_1 in (3.12a) and (3.15a) is the principal solution at $z = 1$, in the general and in the special case, respectively.

Proofs In the general case, the Principal solution is

$$\Psi_1(z) = z^{\alpha}(1-z)^{\beta} F(a,b,a+b+1;1-z)\ . \tag{3.18}$$

Notice that, by Definition 3.11, any hypergeometric function is defined in the disk $|z| < 1$ (and by analytic continuation, everywhere) and it holds that

$$\lim_{z\to 0} F(a,b,c,z) = 1$$

or, equivalently

$$\lim_{z\to 1} F(a,b,c,1-z) = 1$$

[3] See Appendix A for some insights.

for any a, b and c for which F is properly defined. Bearing in mind this property, it is manifest that the solutions in (3.12) are asymptotically equivalent to

$$\begin{aligned}\Psi_1(z) &\approx_1 (1-z)^{\beta}\\ \Psi_2(z) &\approx_1 (1-z)^{1-\beta}\end{aligned}$$

and we see that

$$\frac{\Psi_1(z)}{\Psi_2(z)} \approx_1 (1-z)^{2\beta-1} \to_1 0 \quad \text{for } \beta > \frac{1}{2} \text{ that is } \mu^2 > -1\ .$$

So, it follows that

$$\lim_{z\to 1} \frac{\Psi_1(z)}{A\Psi_1(z)+B\Psi_2(z)} = 0\ , \quad \text{for } B \neq 0\ .$$

Analogously, in the special case the principal solution is

$$\Psi_1(z) = z^{\alpha}(1-z)^{\beta} F(a,b,n;1-z)\ .$$

The solutions in (3.15) are in fact asymptotically equivalent to

$$\begin{aligned}\Psi_1(z) &\approx_1 (1-z)^{\beta}\\ \Psi_2(z) &\approx_{1^+} (1-z)^{\beta}\Big[\log(1-z) - \frac{(n-1)!(n-2)}{(1-a)_{n-1}(1-b)_{n-1}}(z-1)^{1-n}\\ &\quad + [\psi(a)+\psi(b)-1-\psi(n)]\Big]\\ &\approx_1 \Big[(1-z)^{\beta}\log(1-z) - \frac{(n-1)!(n-2)}{(1-a)_{n-1}(1-b)_{n-1}}(-1)^{1-n}(1-z)^{1+\beta-n} +\\ &\quad + [\psi(a)+\psi(b)-1-\psi(n)](1-z)^{\beta}\Big]\end{aligned}$$

And we see that that

$$\lim_{z\to 1} \frac{\Psi_1(z)}{A\Psi_1(z)+B\Psi_2(z)} = 0 \quad \text{for } B \neq 0$$

Stability of solutions

We call a solution *stable*, when its asymptotic behaviour at the boundary is not dominated by a complex power of z, such as $z^{i\varphi}$, $\varphi \in \mathbb{R}$.
By looking at the solutions (3.12) and (3.15) we notice that their asymptotic behaviours at $z=1$ are dominated by the factors $(1-z)^{\beta}$ and $(1-z)^{1-\beta}$, depending only on

$$\beta = \frac{1}{2}(1+\sqrt{1+\mu^2}).$$

In the considered case, a solution is therefore stable whenever β is real, that is when $\mu^2 \geq -1$. This constraint is in agreement with the Breitenlohner-Freedman bound [68] for having stable solutions. This bound was originally derived [68] in the AdS_{d+1} picture by demanding positivity of the conserved energy functional for scalar fluctuations vanishing sufficiently fast at radial infinity. For the AdS_3 case, the Bretenlohner-Freedman bound is $\mu^2 \geq \mu^2_{BF} = -1$.

3.2.3 Square-integrability at the endpoint $z = 1$

Having specified a basis of solutions of (3.5), we shall proceed to identify all the admissible boundary conditions at the end point[4] $z = 1$ for (3.2). These boundary conditions will depend on the square integrability of the solutions (3.12) and (3.15) in a neighborhood of the considered end point.

In particular we classify end points in the following way: We call an end point $z = 1$ *limit circle* if, for some $\tilde{\omega} \in \mathbb{C}$, all solutions of (3.5) lie in $L^2((z_1, 1); \mathcal{J}(z)\mathrm{d}z)$ for some $z_1 \in (0, 1)$, otherwise, we call it *limit point*. The measure $\mathcal{J}(z)\mathrm{d}z$, with

$$\mathcal{J}(z) = \frac{1}{1-z} + \frac{r_+^2}{z(r_+^2 - r_-^2)}\,, \tag{3.19}$$

is $\pi_I^* \mathrm{d}\nu(g)$, where, $\mathrm{d}\nu(g) = r/N^2\, \mathrm{d}r\mathrm{d}\varphi$ and $\pi_I : M \to I$ is the projection along the r-direction. Notice that the measure $\mathcal{J}(z)\mathrm{d}z$ has been fixed so that the operator L in (3.5) is Hermitian with respect to it.

A direct inspection of (3.12a) and (3.12b) as well as of (3.15a) and (3.15b), and the analysis of the asymptotic behaviour of the solutions, implies that a Robin boundary condition can only be applied when $\mu^2 \in (-1, 0)$.

In fact, as already anticipated in Section 3.2.2, since any hypergeometric function $F(1-z)$ equals 1 when evaluated at $z = 1$, the behavior of (3.12a) and (3.12b) can be inferred from that of $(1-z)^\beta$ and $(1-z)^{1-\beta}$ respectively. By accounting for the integration measure and using again (3.8), one finds that (3.12a) lies in $L^2((z_1, 1); \mathcal{J}(z)\mathrm{d}z)$ for all values of $\mu^2 > -1$ for any $z_1 \in (0, 1)$ and any $\tilde{\omega}$. On the contrary, (3.12b) lies in $L^2((z_1, 1); \mathcal{J}(z)\mathrm{d}z)$ if and only if $\mu^2 \in (-1, 0)$. According to the nomenclature introduced, $z = 1$ is therefore limit circle if $\mu^2 \in (-1, 0)$ while it is limit point if $\mu^2 \geqslant 0$. Hence, no boundary condition is required for $\mu^2 \geqslant 0$ and everything works as if the Dirichlet boundary condition had been chosen.

For the special case $\mu^2 = (n-1)^2 - 1$, $n = 2, 3, \ldots$, the first element of the basis Ψ_1, as in (3.15a), behaves exactly like (3.12a). On the contrary, Ψ_2, as in (3.15b), never lies in $L^2((z_1, 1); \mathcal{J}(z)\mathrm{d}z)$ on account of the singularities of $K(z)$. Hence, whenever $\mu^2 \geqslant 0$, $z = 1$ is always limit point and no boundary condition is required.

[4] A similar reasoning will be applied at $z = 0$ but, if one focuses only on square integrable solutions, only one exists, provided that $\mathrm{Im}[\tilde{\omega}] \neq 0$. Therefore, no boundary condition needs to be applied at $z = 0$.

Range of μ^2	Range of $\tilde{\omega}$	L^2 at $z=0$	L^2 at $z=1$
	$\mathrm{Im}[\tilde{\omega}] < 0$	Ψ_4	Ψ_1 and Ψ_2
$-1 < \mu^2 < 0$	$\mathrm{Im}[\tilde{\omega}] = 0$	none	Ψ_1 and Ψ_2
	$\mathrm{Im}[\tilde{\omega}] > 0$	Ψ_3	Ψ_1 and Ψ_2
	$\mathrm{Im}[\tilde{\omega}] < 0$	Ψ_4	Ψ_1
$\mu^2 \geqslant 0$	$\mathrm{Im}[\tilde{\omega}] = 0$	none	Ψ_1
	$\mathrm{Im}[\tilde{\omega}] > 0$	Ψ_3	Ψ_1

Table 3.1: Square integrability at the endpoints $z = 0$ and $z = 1$ of a basis of solutions for (3.5) depending on the parameters μ^2 and $\tilde{\omega}$ of the equation. The integration measure is $\mathrm{d}\nu(z) = \mathcal{J}(z)\mathrm{d}z$ as per (3.19).

3.2.4 Square-integrability at the endpoint $z = 0$

The behaviour of the solutions of (3.5) at $z = 0$ can be analysed by considering a more convenient basis of solutions

$$\begin{aligned}\Psi_3(z) &= z^{\alpha}(1-z)^{\beta} F(a,b,c;z)\,, &\text{(3.20a)}\\ \Psi_4(z) &= z^{-\alpha}(1-z)^{\beta}\\ &\qquad \times F(a-c+1, b-c+1, 2-c; z)\,, &\text{(3.20b)}\end{aligned}$$

where again a, b, c are the parameters defined in (3.10). The vectors Ψ_3 and Ψ_4 form a well-defined basis of solutions for all $\mu^2 > -1$, except when $c = 1$, corresponding to the value $\alpha = 0$.

Since the hypergeometric functions in the form $F(z)$ are equal to 1 when evaluated at $z = 0$, the leading behaviour of the two solutions at the origin is ruled by z^{α} and by $z^{-\alpha}$, respectively. It is easy to verify that $\Psi_3 \in L^2((0,z_0), \mathcal{J}(z)\mathrm{d}z)$ for $\mathrm{Im}[\tilde{\omega}] > 0$, independently on $z_0 \in (0,1)$, while $\Psi_4 \in L^2((0,z_0), \mathcal{J}(z)\mathrm{d}z)$ whenever $\mathrm{Im}[\tilde{\omega}] < 0$. On the other hand, none of the solutions is square integrable for $\mathrm{Im}[\tilde{\omega}] = 0$, since a logarithmic singularity dominates the leading term of the asymptotic behaviour. Since only one square integrable solution exists, provided that $\mathrm{Im}[\tilde{\omega}] \neq 0$, no boundary condition needs to be applied at $z = 0$. Therefore, we say that $z = 0$ is limit point.

The case $c = 1$ deserves a special mention. This value corresponds to $\omega = k\frac{r_-}{\ell r_+} = k\Omega_H$. This relation satisfies a synchronization condition with the black hole angular velocity and it has been extensively studied in [44]. In this case, the solutions Ψ_3 and Ψ_4 no longer form a basis of solutions of (3.5), hence, we consider the following new basis [67, §15.10.8]:

$$(1-z)^{\beta} F(a,b,1;z)\,,$$
$$(1-z)^{\beta}\left[F(a,b,1;z)\log(z) + K_1(1-z)\right]\,,$$

where K_1 is as in (3.16). The leading behaviour at $z = 0$ of these two solutions is dominated by a constant in the first case and by $\log(z)$ in the second one. Therefore, none of them lies in $L^2((0, z_0), \mathcal{J}(z)\mathrm{d}z)$ independently from $z_0 \in (0, 1)$.

Note that for $\tilde{\omega} \notin \mathbb{R}$, hence excluding the case $c = 1$, the above definitions obey $\Psi_3 \mapsto \overline{\Psi_4}$ and $\Psi_4 \mapsto \overline{\Psi_3}$ under the replacement $\tilde{\omega} \mapsto \overline{\tilde{\omega}}$. This can be checked using again the symmetry $F(a, b, c; z) = F(b, a, c; z)$ and the conjugation identities (B.8).

3.2.5 Robin boundary conditions

In the following we will focus on identifying which class of boundary conditions can be applied at the timelike conformal boundary $z = 1$ of the BTZ spacetime, so to obtain a general and unique solution for the Klein-Gordon equation (3.2). From a physical point of view, we would like to deal with a thermodynamical isolated system and the presence of the boundary might spoil this property. Therefore, the reasonable physical request is to impose that the boundary behaves like a perfect mirror, that is to impose that the energy flux through the boundary is zero [69, 70]). In the end, this approach is more general than the usual practice to impose Dirichlet boundary conditions.

Let us consider the the action (3.1)

$$S = -\frac{1}{2} \int_{\mathrm{BTZ}} \mathrm{d}\mu_g \left(\nabla_\mu \bar{\Phi} \nabla^\mu \Phi + \frac{\mu^2}{\ell^2} \Phi^2 \right) ,$$

with stress-energy tensor

$$T^{\mu\nu} = (g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} - g^{\mu\nu} g^{\alpha\beta}) \partial_\alpha \bar{\Phi} \partial_\beta \Phi - g^{\mu\nu} \bar{\Phi} \Phi .$$

The radial energy flux is

$$\mathfrak{F}_r = \int_{\mathbb{S}^1} T^r{}_t \sqrt{|\det g|} g^{rr} \mathrm{d}\phi . \tag{3.22}$$

Let us consider the asymptotic behaviour of (3.5) for $z \to 1$ $(r \to \infty)$, with an expansion of the form

$$\Phi(t, r, \phi) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \mathrm{d}\omega \; e^{-i\omega t + ik\phi} \; \Psi_{\omega k}(r) .$$

As we have seen in Section 3.2.3, if z is as in (3.4), one finds that boundary conditions are required only for $-1 < \mu^2 < 0$ and the principal and non-principal solutions (3.12) asymptotically behave as (3.13) and (3.14), respectively. In terms of the natural radial coordinate r one has that

$$\Psi_1(r) \approx_\infty r^{-1-\sqrt{1+\mu^2}} , \tag{3.23}$$

$$\Psi_2(r) \approx_\infty r^{-1+\sqrt{1+\mu^2}} . \tag{3.24}$$

A general solution can therefore be written as $\Psi = A\Psi_1 + B\Psi_2$ for some complex constants A and B. For convenience, here we introduce a new basis of solutions, given by

$$\Psi_+ := \Psi_1 - i\Psi_2 \ ,$$
$$\Psi_- := \Psi_1 + i\Psi_2 \ .$$

The general solution becomes $\Psi = C_+\Psi_+ + C_-\Psi_-$ and we substitute it into (3.22), so to obtain

$$\mathfrak{F}_r = \int_{\mathbb{S}^1} T^r{}_t \sqrt{|\det g|} g^{rr} \mathrm{d}\phi \ \propto \ (|C_+|^2 - |C_-|^2) \ .$$

We then make the hypothesis that the conformal boundary of BTZ may be regarded as a perfectly reflecting mirror, asking for the energy flux to vanish asymptotically when $r \to \infty$, that is

$$\lim_{r\to\infty} \mathfrak{F}_{r\to\infty} = 0 \ .$$

This requirement is satisfied for $|C_+| = |C_-|$. Writing the complex constants in polar coordinates as $C_\pm = \rho e^{i\theta_\pm}$, one obtains that

$$\frac{B}{A} = -i\frac{C_+ - C_-}{C_+ + C_-} = \tan\left(\frac{\theta_+ - \theta_-}{2}\right)$$

and we can define a variable $\zeta := \frac{\theta_+ - \theta_-}{2}$, $\zeta \in [0,\pi) \setminus \{\frac{\pi}{2}\}$, such that

$$\tan(\zeta) = \frac{B}{A} \ .$$

Therefore, imposing zero energy flux at the boundary is equivalent to ask for a general solution to be in the form

$$\cos(\zeta)\Psi_1(z) + \sin(\zeta)\Psi_2(\zeta), \quad \zeta \in [0,\pi) \setminus \{\frac{\pi}{2}\} \ ,$$

that is to satisfy *Robin boundary conditions.* Robin boundary conditions are usually introduced as a constraint over some linear combination between a solution of the differential equation and its derivative and, in this sense, they behave as a generalization of Dirichlet and Neumann boundary conditions.

The implementation of Robin boundary conditions at the conformal boundary $z = 1$ of BTZ spacetime is not straightforward since, as already pointed out in Section 3.2.1, the radial differential equation is singular at the endpoints. A brief comparison between the procedure to impose boundary conditions in a regular differential equation and in a singular one is presented in Appendix A. Here we just stress that it would be improper to specify the boundary conditions by assigning specific values at the boundary. These kind of problems are known as *singular* or *non regular problems.*

A convenient generalization is provided by the following approach.
We first choose as primary solution Ψ_1 of the basis the principal solution at the boundary identified in Section 3.2.2, while Ψ_2 can be any other linearly independent solution. Secondly, given the set Sol of solutions of (3.2), we introduce the notion of Wronskian between two solutions

$$\mathcal{W}_z[\varphi, \psi] := \varphi(\partial_z \psi) - (\partial_z \varphi)\psi \ .$$

Then we say that a solution Ψ_ζ of (3.9) satisfies the boundary conditions of Robin type at $z = 1$ if

$$\lim_{z\to 1}\{\cos(\zeta)\mathcal{W}_z[\Psi, \Psi_1](z) + \sin(\zeta)\mathcal{W}_z[\Psi, \Psi_2](z)\} = 0 \qquad \zeta \in [0, \pi) \ , \tag{3.25}$$

where Ψ is any solution of the differential equation, while the linear combination has been written in terms of a parameter ζ spanning all admissible values up to a normalization constant.

In Appendix A we show that this procedure also applies for a regular differential problem and reduces to the usual notion of Robin boundary conditions

$$\cos(\zeta)\Psi_\zeta(1) + \sin(\zeta)\partial_z\Psi_\zeta(1) = 0 \qquad \zeta \in [0, \pi) \ ,$$

up to a normalization constant.

Boundary conditions at $z = 1$ and general solution

At the singular point $z = 1$ we can apply boundary conditions (3.25) for a pair of solutions of the general case (3.12).

Given the boundary conditions (3.25), for any value of $\zeta \in [0, \pi)$ the most general solution of equation (3.9) with parameters (3.10) is then

$$\Psi_\zeta(z) = \mathcal{N}\{\sin(\zeta)\Psi_2(z) + \cos(\zeta)\Psi_1(z)\} \ . \tag{3.26}$$

The same holds for the special case, where the basis of solutions (3.12) is replaced by (3.15).

The value $\zeta = 0$ in (3.25) provides a generalization of the concept of Dirichlet boundary condition, since it guarantees that the general solution in that case is $\Psi(z) = \mathcal{N}\Psi_1(z)$. For any $\zeta \in (0, \pi)$ we have the Robin boundary conditions.
We stress that the notable case $\zeta = \frac{\pi}{2}$ cannot be unambiguously interpreted as a generalization of the Neumann boundary conditions, since it depends on the choice of Ψ_2.

3.3 The two-point functions

In field theory, a particular interest is given to the construction of two-point functions

$$\langle \Phi(x)\Phi(x') \rangle \ .$$

In our picture, we will read this object as an integral kernel built out of a real, massive, scalar field $\Phi : \mathcal{M} \to \mathbb{R}$ whose dynamics is ruled by a generic Klein-Gordon equation

$$P\Phi := (\Box_g - m^2 - \xi R)\Phi = 0 \ , \tag{3.27}$$

with $m^2, R, \xi \in \mathbb{R}$ over a $2+1$ dimensional, connected Lorentzian manifold $(\mathcal{M}, g)$.

The quantization of (3.27) has been studied in great details in the literature, both for flat and curved globally hyperbolic spacetimes. As a reference, one can refer for example to [71], while in [13] one can find a recent review, which adopts an algebraic point of view based on a set of axioms, first spelt out by Haag and Kastler [72]. This second approach gives an algebraic formulation to quantum field theory (AQFT) and it is particularly suitable for curved backgrounds, expecially when dealing with globally hyperbolic spacetimes [12].

This quantization scheme does not focus directly on fields. Instead it relies on the construction of an algebra of observables $\mathcal{A}(\mathcal{M})$ compatible with the dynamics of the system and satisfying the canonical commutations relations (CCRs). This abstract algebra of observables is then related to a Hilbert space and to a vector space of operators over it, by the identification of a positive and normalized linear functional $\omega : \mathcal{A}(\mathcal{M}) \to \mathbb{C}$ dubbed *algebraic state*. This realization is provided by the renown Gelfand-Naimark-Segal (GNS) theorem [29].

In this picture, expectation values of observables are computed in terms of *correlation functions*. These are distributions acting on the generators of the algebra of observables

$$\omega_n : C_0^\infty(\mathcal{M})^{\otimes n} \to \mathbb{C} \ .$$

A distinguished role is indeed played by the two-point correlation function ω_2. One the one hand, in non interacting field theory, it solely determines the n-point functions for Gaussian (quasifree) states by means of the Wick theorem [71]. On the other hand, the CCRs over the algebra are imposed in a covariant way [22, 58] by means of a bidistribution G dubbed *causal propagator*, whose integral kernel can be split in a symmetric and antisymmetric part

$$iG(x, x') = \omega_2(x, x') - \omega_2(x'x) \ ,$$

where $\omega_2(x, x') = \langle \Phi(x)\Phi(x') \rangle$.

In the following, after an introduction to the general quantization scheme, we specialise the discussion to the case of a BTZ spacetime, aiming to construct a class of two-point functions defining a ground state for a real, massive, scalar field.

3.3.1 Quantum field theory in curved spacetime

Eq. (3.27) can be solved in terms two fundamental solutions, called the *advanced* (A) and the *retarded* (R) solutions respectively,

$$G_{A/R} : C_0^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$$

such that

- $P \circ G_{A/R} = G_{A/R} \circ P = id|_{C_0^\infty(\mathcal{M})}$,
- $\text{supp}(G_{A/R}(f)) \subseteq J^{\mp}(\text{supp}(f))$ for all $f \in C_0^\infty(\mathcal{M})$.

These solutions exist for any normally hyperbolic differential operator acting over a globally hyperbolic Lorentzian manifold [13]. They are unique when the operator P and its formal adjoint P^*, defined by $\langle P^* f, g\rangle = \langle f, Pg\rangle$ for all $f, g \in C_0^\infty(\mathcal{M})$, $\langle f, g\rangle := \int fg$, both admit[5] advanced and retarded solutions. A partial second order differential operator over a 2+1 dimensional Lorentzian manifold is called *normally hyperbolic* if it can be written in the form

$$P\Phi = \left[-\sum_{i,j=1}^{3} g^{ij}\partial_i\partial_j + \sum_{i=1}^{3} A_i\partial_i + A\right]\Phi \ ,$$

for some smooth functions A, A_i, $i = 1, 2, 3$ and a partial differential equation

$$P\Phi = \mathrm{J} \ ,$$

where J is a compactly supported smooth function, is called *wave-like equation* if P is normally hyperbolic. The prototypical example of a wave-like equation is therefore the Klein-Gordon equation (3.27).

Given the advanced and retarded solutions, one can introduce the *causal propagator*

$$G := G_A - G_R$$

out of which one associates to the scalar field Φ an algebra of observables $\mathcal{A}(\mathcal{M})$, constructed as follows:

1. We define $\mathcal{T}(\mathcal{M}) := \bigoplus_{n=0}^{\infty}(C_0^\infty(\mathcal{M}))^{\otimes n}$, where $C_0^\infty(\mathcal{M})^{\otimes 0} \equiv \mathbb{C}$, as the universal tensor algebra of test-functions, endowed with the $*$-operation induced from complex conjugation,
2. We call $\mathcal{I}(\mathcal{M})$ the $*$-ideal of $\mathcal{T}(\mathcal{M})$ generated by elements of the form Pf, so to implement the dynamics of the equation of motion in the quantum system
3. For any $f, f' \in C_0^\infty(\mathcal{M})$ we impose the canonical commutation relations (CCRs) $f \otimes f' - f' \otimes f - iG(f, f')\mathbb{I}$, where $\mathbb{I}$ is the identity of $\mathcal{T}(\mathcal{M})$,
4. We define $\mathcal{A}(\mathcal{M}) = \frac{\mathcal{T}(\mathcal{M})}{\mathcal{I}(\mathcal{M})}$, to be the *algebra of observables.*

Notice that any $f \in C_0^\infty(\mathcal{M})$, the following properties hold true for G:

- $P \circ Gf = G \circ Pf = 0$,

[5] An operator admitting advanced and retarded solutions is sometimes referred as Green-hyperbolic [12].

- $\mathrm{supp}(G(f)) \subseteq J(\mathrm{supp}(f))$,

where $J(K) := J^+(K) \cup J^-(K)$ is the union of the causal future and the causal past of K. Furthermore, the causal propagator can be regarded as a bi-distribution, where

$$G \in C_0^\infty(\mathcal{M} \times \mathcal{M})' = \{T : C_0^\infty(\mathcal{M} \times \mathcal{M}) \to \mathbb{C} \mid T \text{ is continuous and linear}\}$$

and, for all $f, f' \in C_0^\infty(\mathcal{M})$

$$G(f, f') := \int_{\mathcal{M}} \mathrm{dvol}_{\mathcal{M}}(x) f(x)(Gf')(x) \ ,$$

with $(Gf)(x) := \int_{\mathcal{M}} \mathrm{dvol}_{\mathcal{M}}(x) f(x) G(x, x') f'(x)$ and $G(x, x')$ is interpreted in a distributional sense.
It is worth noticing that the algebra of observables $\mathcal{A}(\mathcal{M})$ is nothing but an abstract generalization of the algebra of second quantized fields $\hat{\Phi}(f)$, with $f \in C_0^\infty(\mathcal{M})_{\mathbb{C}} \simeq C_0^\infty(\mathcal{M}; \mathbb{C})$:

1. $\hat{\Phi}(af + bf') = a\hat{\Phi}(f) + b\hat{\Phi}(f')$ for all $a, b \in \mathbb{C}$, $f, f' \in C_0^\infty(\mathcal{M})$,

2. $\hat{\Phi}(Pf) = 0$, for all $f \in C_0^\infty(\mathcal{M})$,

3. $\hat{\Phi}(f)^* = \hat{\Phi}(\bar{f})$, for all $f \in C_0^\infty(\mathcal{M})$, where $\bar{f}$ denotes the complex conjugate of f,

4. $[\hat{\Phi}(f), \hat{\Phi}(f')] := \hat{\Phi}(f)\hat{\Phi}(f') - \hat{\Phi}(f')\hat{\Phi}(f) = iG(f, f')\mathbb{I}$.

At this point, it is possible to define an *algebraic state*, that is a linear functional $\omega : \mathcal{A}(\mathcal{M}) \to \mathbb{C}$ such that

$$\omega(\mathbb{I}) = 1 \qquad \omega(aa^*) \geq 0 \quad \forall a \in \mathcal{A}(\mathcal{M}) \ ,$$

which allows to realize a suitable Hilbert H_ω space and to interpret the algebra of observables in terms of linear operators acting on it. This realization is provided by the Gelfand-Naimark-Segal (GNS) theorem [29], which states that, given a state ω on a unital $*$-algebra $\mathcal{A}$, there exist a quadruple $(H_\omega, \mathcal{D}, \pi, \Psi_\omega)$

· $(H_\omega, \langle - | - \rangle)$ is a Hilbert space,

· $\mathcal{D} \subset H_\omega$ is a dense subspace,

· $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{D})$ is a $*$-representation of $\mathcal{A}$ on the linear operators over $\mathcal{D}$, that is a linear map such that $\pi(a)^\dagger_{|\mathcal{D}} = \pi(a^*)$.

· $\pi(\mathcal{A})\Psi_\omega = \mathcal{D}$

· $\omega(a) = \langle \Psi_\omega | \pi(a) \Psi_\omega \rangle$, for all $a \in \mathcal{A}$.

Expectation values of observables can then be computed in terms of *correlation functions*. These are distributions acting on the generators of the algebra of observables

$$\omega_n : C_0^\infty(\mathcal{M})^{\otimes n} \to \mathbb{C} \ .$$

Knowing a state ω is therefore equivalent to knowing all its correlation functions. As already anticipated, a distinguished role is indeed played by the two-point correlation function ω_2. If one considers the integral kernel of the causal propagator G one can split it in a symmetric and antisymmetric part

$$iG(x, x') = \omega_2(x, x') - \omega_2(x'x) \ ,$$

where the functions ω_2, usually called *Wightman functions*, are given by

$$\omega_2(x, x') = \langle \Phi(x)\Phi(x') \rangle \ .$$

In particular, G and ω_2 all satisfy the Klein-Gordon equation both for x and x', that is they are bisolutions for (3.27).

In the following, we specialise the discussion to the case of a BTZ black hole background with the aim to construct a class of two-point functions defining a state for the real massive scalar field of (3.2). In particular we would like to define a ground state, that is a state built out only of positive frequencies with respect to a preferred timelike Killing field. As we discussed in the Introduction, in curved spacetime we cannot rely on the Poincaré simmetry group and, as a consequence, there is no clear way to select a single vacuum state above another. Nevertheless, if a complete, everywhere timelike Killing field exists, it allows for the identification of a unique full-fledged quantum state, dubbed the *ground state*, guaranteeing that all quantum observables have finite fluctuations [28, 29].

3.3.2 Quantum field theory in BTZ spacetime

As already pointed out in Chapter 2, the BTZ spacetime is not globally hyperbolic due to the presence of the timelike surface $r = \infty$ acting as conformal boundary, while the quantization scheme presented in Section 3.3.1 is particularly suitable for globally hyperbolic spacetimes. Despite this discrepancy, our final interest will be to evaluate physical quantities defined in geodesically convex neighbourhoods of points on the Killing horizon. This will become clear in Chapter 4, where we will analyse the tunnelling processes through the black hole horizon and the local thermal behaviour of the two-point function. With regards to the quantization procedure, if one is interested in the local properties of the system in a submanifold $\widetilde{\mathcal{O}} \subset \mathcal{M}$, one might as well define a local algebra of observable as the restriction of $\mathcal{A}(\mathcal{M})$ on $\widetilde{\mathcal{O}}$. This is indeed the case of the present work, since we are only interested in studying local quantities. Nonetheless, the construction of a global algebra might be possible by adopting the approach used for the counterpart of a real, massive scalar field in the Poincaré patch of a $(d+1)-$dimensional AdS spacetime [73].

Given the Klein-Gordon field equation (3.2), we aim to build its *Wightman function*, that is a bidistribution $G^+ \in \mathcal{D}'(\mathrm{BTZ} \times \mathrm{BTZ})$ such that

$$(P \otimes \mathbb{I})G^+ = (\mathbb{I} \otimes P)G^+ = 0\,,$$

$$G^+(f,f) \geq 0\,, \quad \forall f \in C_0^\infty(\mathrm{BTZ})$$

and $\mathrm{supp}(G^+(f,-)) \subseteq J(\mathrm{supp}(f))$ for all $f \in C_0^\infty(\mathrm{BTZ})$ in a distributional sense. Let us then consider the coordinate system (t,z,ϕ) introduced in (2.8) with r replaced by the new radial coordinate z as in (3.4). When imposing the canonical commutation relations (CCRs), one requires that the antisymmetric part of the integral kernel of G^+,

$$iG(x,x') = G^+(x,x') - G^+(x',x) \qquad \text{with } x,x' \in \text{ BTZ}$$

satisfies (3.3.2) together with the initial conditions

$$G(x,x')|_{t=t'} = 0\ , \tag{3.28a}$$

$$-\partial_t G(x,x')|_{t=t'} = \partial_{t'} G(x,x')|_{t=t'} = \frac{\delta(z-z')\delta(\phi-\phi')}{\mathcal{J}(z)}\ , \tag{3.28b}$$

with $\mathcal{J}(z)$ as in (3.19). That is, the antisymmetric part of G^+ is constrained to coinciding with the commutator distribution if one wants to account for the CCRs of the underlying quantum field theory. In order to construct explicitly the two-point function we assume that G^+ can be mode expanded in the form

$$G^+(x,x') = \lim_{\epsilon\to 0^+} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{(2\pi)^2}\, e^{-i\omega(t-t'-i\epsilon)+ik(\phi-\phi')} \widehat{G}_{\omega k}(z,z')\,, \tag{3.29}$$

where $x,x' \in \mathrm{BTZ}$. Notice that a term $i\epsilon$ has been added as a regularization, while the limit must be interpreted in the weak sense.

3.3.3 The resolution of the identity

Initial conditions

As we already anticipated in Section 2.3, the Killing vector field (2.11)

$$\xi = \frac{\partial}{\partial t} + N^\phi(r_+)\frac{\partial}{\partial \phi}$$

plays a prominent physical role: it is timelike in the whole external region of the BTZ black hole and it generates the Killing horizon. Moreover, it is of paramount importance for the construction of a ground state, providing a preferred direction to define the notion of positive frequencies. In fact, the positive frequencies can be identified by its Killing parameter $\tilde{\omega} = \omega - k\Omega_{\mathcal{H}}$. This can be seen by changing the two-point function parameters from (ω, k)

to $(\tilde{\omega}, k)$ and by introducing a new coordinate system $(\tilde{t}, r, \tilde{\phi})$, which is related to (t, r, ϕ) so that $\partial_{\tilde{t}} = \xi$. The simplest choice consists in defining[6]

$$\begin{aligned} t =& \tilde{t} \\ \phi =& \tilde{\phi} + \Omega_{\mathcal{H}} \tilde{t} \,. \end{aligned}$$

Using this new coordinate chart, the line element becomes

$$\mathrm{d}s^2 = -N^2 \mathrm{d}\tilde{t}^2 + N^{-2}\mathrm{d}r^2 + \left(\mathrm{d}\tilde{\phi} + (N^\phi + \Omega_{\mathcal{H}})\mathrm{d}\tilde{t}\right)^2 \,. \tag{3.30}$$

Since only the positive $\tilde{\omega}$-frequencies contribute to the two-point function of the ground state, one can write $\widehat{G}_{\omega k}(z, z') := \widetilde{G}_{\tilde{\omega} k}(z, z')\Theta(\tilde{\omega})$, with $\widetilde{G}_{\tilde{\omega} k}(z, z')$ defined for all $\tilde{\omega} \in \mathbb{R}$.

In order to build the ground state, since the antisymmetric part must satisfy (3.28a), a natural requirement consists of looking for a two point function $\widetilde{G}_{\tilde{\omega} k}(z, z')$ which is symmetric under exchange of z and z' and such that $\widetilde{G}_{-\tilde{\omega},-k}(z, z') = -\widetilde{G}_{\tilde{\omega} k}(z, z')$. With this requirement, the commutator distribution reads

$$iG(x, x') = \lim_{\epsilon \to 0^+} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \frac{\mathrm{d}\tilde{\omega}}{(2\pi)^2} e^{-i\tilde{\omega}(t - t' - i|\tilde{\omega}|\epsilon) + ik(\tilde{\phi} - \tilde{\phi}')} \, \widetilde{G}_{\tilde{\omega} k}(z, z') \,. \tag{3.31}$$

Here, from Eq. (3.28b), one has that $\widetilde{G}_{\omega k}(z, z')$ is a mode bidistribution chosen in such a way that

$$\int_{\mathbb{R}} \frac{\mathrm{d}\tilde{\omega}}{2\pi} \, \tilde{\omega} \, \widetilde{G}_{\tilde{\omega} k}(z, z') = \frac{\delta(z - z')}{\mathcal{J}(z)} \,. \tag{3.32}$$

Quadratic operator pencils

Provided that positivity as in (3.3.2) is satisfied, the identity (3.32) and the Fourier series for the delta distribution along the angular coordinates

$$\frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{ik(\tilde{\phi} - \tilde{\phi}')} = \delta(\phi - \phi')$$

ensure that finding $\widetilde{G}_{\tilde{\omega} k}(z, z')$ is equivalent to constructing a full-fledged two-point function G^+.

Moreover (3.3.2) entails that the mode bidistribution is such that

$$(L_{\tilde{\omega}} \otimes \mathbb{I})\widetilde{G}_{\tilde{\omega} k}(z, z') = (\mathbb{I} \otimes L_{\tilde{\omega}})\widetilde{G}_{\tilde{\omega} k}(z, z') = 0 \,,$$

where $L_{\tilde{\omega}}$ is defined in (3.5).

[6]Notice that, while the range of $\tilde{t}$ is still $\mathbb{R}$, that of $\tilde{\phi}$ appears to be no longer the interval $(0, 2\pi)$, but $(-\Omega_H \tilde{t}, 2\pi - \Omega_H \tilde{t})$ instead. Eventually, this choice is purely a matter of convention: the interval $0 \leq \tilde{\phi} < 2\pi$ would be equivalently suited.

Our aim is now to obtain an integral representation for the delta distribution on the right hand side of (3.32), from which one can read off $\widetilde{G}_{\tilde{\omega}k}(z,z')$.

This result is quite straightforward when dealing with the static case $J=0$. The ODE (3.5) is indeed an eigenvalue problem with spectral parameter $\tilde{\omega}^2$ and if the operator $L_{\tilde{\omega}}$ is self-adjoint, for example by imposing suitable boundary conditions [74], one can express the delta distribution as an expansion in terms of the eigenfunctions of $L_{\tilde{\omega}}$. This procedure, known in literature as *resolution of the identity*, is not possible when dealing with the non static case $J \neq 0$ since, when the black hole is rotating, the ODE (3.5) presents both a quadratic and a linear term in $\tilde{\omega}$.

Therefore, the particular from of equation (3.5), forces us to treat $L_{\tilde{\omega}}$ as a *quadratic operator pencil*. Quadratic operator pencils are differential operators with quadratic dependence on the spectral parameter.

Quadratic operator pencils are a family of operators defined on a Hilbert space H of the form

$$S_{\tilde{\omega}} = P + \tilde{\omega}\mathcal{R}_1 + \tilde{\omega}^2\mathcal{R}_2\,, \tag{3.33}$$

where $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_2^{-1}$ are all bounded and self-adjoint operators, while P is unbounded, closed and hermitian on a dense domain $D(S_{\tilde{\omega}}) \subset H$. In our case, $S_{\tilde{\omega}} = \mathcal{J}^{-1}L_{\tilde{\omega}}$ on $H = L^2((0,1);\mathcal{J}(z)\,\mathrm{d}z)$, where $L_{\tilde{\omega}}$ is defined by (3.5) and $\mathcal{J}(z)$ is as in (3.19).

A detailed spectral analysis of this family of operators is presented in Appendix C.1. In the following we show what is the procedure to obtain the expansion of the delta distribution in terms of eigenfunctions of an operator of this type and, hence, the mode expansion of the two-point function (3.29) for the case in which the mass parameter is such that $-1<\mu^2<0$ and Robin boundary conditions apply at the conformal boundary $z=1$. The results for the range $\mu^2 \geqslant 0$ may be simply obtained by setting $\zeta = 0$.

Explicit calculation of the delta integral representation The strategy is to apply the discussion from Appendix C.1 to the differential operator $L_{\tilde{\omega}}$ introduced in (3.5)

$$\begin{aligned} L_{\tilde{\omega}}\Psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}\Psi(z)}{\mathrm{d}z}\right) - \Bigg[&\frac{\ell^2k^2(1-z)+r_+^2\mu^2}{4r_+^2(1-z)^2} \\ &- \frac{\tilde{\omega}\ell^3kr_-}{2r_+(r_+^2-r_-^2)(1-z)} - \frac{\tilde{\omega}^2\ell^4\mathcal{J}(z)}{4(r_+^2-r_-^2)}\Bigg]\Psi(z), \end{aligned} \tag{3.34}$$

with $\mathcal{J}(z)$ again as in (3.19). As already stated, the Hilbert space is $H = L^2((0,1);\mathcal{J}(z)\,\mathrm{d}z)$ and the quadratic operator pencil is

$$S_{\tilde{\omega}}\Psi(z) = \frac{1}{\mathcal{J}(z)}L_{\tilde{\omega}}\Psi(z). \tag{3.35}$$

This operator satisfies the hypotheses (S1), (S2) and (S3) from Appendix C.1. The verification of the hypotheses is relegated to Appendix C.2, C.3 and C.4, respectively.

Let us now focus the attention on the Green distribution $\mathcal{G}_{\tilde{\omega},\zeta}$ associated to the operator $L_{\tilde{\omega}}$. Such bidistribution can be constructed as a product of square integrable solutions of $L_{\tilde{\omega}}\Psi = 0$ at both the endpoints $z = 0$ and $z = 1$. For this reason we introduce the function

$$u_{\tilde{\omega}}(z) = \begin{cases} \Psi_3(z)\,, & \mathrm{Im}[\tilde{\omega}] > 0\,, \\ \Psi_4(z)\,, & \mathrm{Im}[\tilde{\omega}] < 0\,, \end{cases} \tag{3.36}$$

with Ψ_3 and Ψ_4 defined in (3.20). This function is chosen so as to be square integrable at $z = 0$. Analogously we introduce the function

$$\Psi_{\tilde{\omega},\zeta}(z) = \cos(\zeta)\Psi_1(z) + \sin(\zeta)\Psi_2(z)\,, \tag{3.37}$$

with Ψ_1 and Ψ_2 defined either by (3.12) or (3.15), accordingly to the values of μ^2. This function is chosen so as to be square integrable at $z = 1$ when $-1 < \mu^2 < 0$ and satisfying Robin boundary conditions parametrized by $\zeta \in [0, \pi)$. Note that, given the identity $\overline{L_{\tilde{\omega}}} = L_{\overline{\tilde{\omega}}}$, it follows that $u_{\overline{\tilde{\omega}}} = \overline{u_{\tilde{\omega}}}$ and $\Psi_{\overline{\tilde{\omega}},\zeta} = \overline{\Psi_{\tilde{\omega},\zeta}}$.

The Green distribution $\mathcal{G}_{\tilde{\omega},\zeta}$ associated to the operator $L_{\tilde{\omega}}$ is therefore

$$\mathcal{G}_{\tilde{\omega},\zeta}(z,z') = \begin{cases} \mathcal{N}_{\tilde{\omega},\zeta}^{-1}\, u_{\tilde{\omega}}(z)\Psi_{\tilde{\omega},\zeta}(z')\,, & z \leqslant z'\,, \\ \mathcal{N}_{\tilde{\omega},\zeta}^{-1}\, u_{\tilde{\omega}}(z')\Psi_{\tilde{\omega},\zeta}(z)\,, & z \geqslant z'\,, \end{cases} \tag{3.38}$$

where the normalization constant $\mathcal{N}_{\tilde{\omega},\zeta}$, depends on the parameter ζ spanning all admissible boundary conditions. Knowing that

$$\mathcal{W}_z[\Psi_1, \Psi_2] = \frac{a+b-c}{z} = \frac{\sqrt{1+\mu^2}}{z}\,,$$

and using formulae (B.3) of hypergeometric functions listed in Appendix B, a direct calculation provides the following values for $\mathcal{N}_{\tilde{\omega},\zeta}$

$$\begin{aligned} \mathcal{N}_{\tilde{\omega},\zeta} &= -z\mathcal{W}_z\left[u_{\tilde{\omega}}, \Psi_\zeta\right] \\ &= \begin{cases} \cos(\zeta)\,\dfrac{\Gamma(c)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)} \\ \qquad + \sin(\zeta)\,\dfrac{\Gamma(c)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}\,, & \mathrm{Im}[\tilde{\omega}] > 0\,, \\ \cos(\zeta)\,\dfrac{\Gamma(2-c)\Gamma(a+b-c+1)}{\Gamma(a-c+1)\Gamma(b-c+1)} \\ \qquad + \sin(\zeta)\,\dfrac{\Gamma(2-c)\Gamma(c-a-b+1)}{\Gamma(1-a)\Gamma(1-b)}\,, & \mathrm{Im}[\tilde{\omega}] < 0\,, \end{cases} \end{aligned} \tag{3.39}$$

where the parameters a, b, c are again as in (3.10).

By inspection of (3.38) and (3.39), it follows that $\overline{\mathcal{N}_{\tilde{\omega},\zeta}} = \mathcal{N}_{\overline{\tilde{\omega}},\zeta}$ and $\overline{\mathcal{G}_{\tilde{\omega},\zeta}}(z,z') = \mathcal{G}_{\overline{\tilde{\omega}},\zeta}(z',z)$. Furthermore, as made explicit in Appendix C.3, $\mathcal{N}_{\tilde{\omega}}$ is analytic on

$\mathrm{Im}[\tilde{\omega}] \neq 0$ and it has at most two isolated zeros. These, called the *bound state frequencies*, are symmetric with respect to the real axis and they form a set $\mathrm{BS}_\zeta \subset \mathbb{C}$, such that $\mathrm{BS}_\zeta = \mathrm{BS}^+_\zeta \cup \overline{\mathrm{BS}^+_\zeta}$ with $\mathrm{Im}[\mathrm{BS}^+_\zeta] > 0$.

The integral representation of the delta distribution can therefore be written by applying formula (C.2):

$$\frac{4(r_+^2 - r_-^2)}{\ell^4 \mathcal{J}(z)} \delta(z - z') = -\int_{\mathbf{R}} \frac{\mathrm{d}\tilde{\omega}}{2\pi i}\, \tilde{\omega}\, \Delta\mathcal{G}_{\tilde{\omega},\zeta}(z, z') + \oint_{\mathring{C}} \frac{\mathrm{d}\tilde{\omega}}{2\pi i}\, \tilde{\omega}\mathcal{G}_{\tilde{\omega},\zeta}(z, z')\,, \tag{3.40}$$

where the positively oriented contour $\mathring{C}$, illustrated in Figure C.1, encircles the bound state frequencies in BS_ζ, while

$$\Delta\mathcal{G}_{\tilde{\omega},\zeta}(z, z') := \lim_{\epsilon\to 0^+} [\mathcal{G}_{\tilde{\omega}+i\epsilon,\zeta}(z, z') - \mathcal{G}_{\tilde{\omega}-i\epsilon,\zeta}(z, z')] \tag{3.41}$$

has to be interpreted as a distribution in $\tilde{\omega}$.
An application of Cauchy residue theorem leads to the following integral representation

$$\frac{\delta(z - z')}{\mathcal{J}(z)} = -\frac{\ell^4}{4(r_+^2 - r_-^2)} \left[\int_{\mathbb{R}} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \tilde{\omega}\, \Delta\mathcal{G}_{\tilde{\omega},\zeta}(z, z') + \sum_{\tilde{\omega}' \in \mathrm{BS}_\zeta} \mathrm{Res}_{\tilde{\omega}=\tilde{\omega}'} [\tilde{\omega}\, \mathcal{G}_{\tilde{\omega},\zeta}(z, z')] \right]. \tag{3.42}$$

Both integrands in (3.42) can be computed rather explicitly, except for analytic expressions for the bound state frequencies (see Appendix C.3). Introducing the constants

$$A = \frac{\Gamma(c-1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad B = \frac{\Gamma(c-1)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)},$$

and using formulas (B.3a) and (B.3b), it is possible to write

$$u_{\tilde{\omega}}(z) = \begin{cases} (c-1)\left[A\Psi_1(z) + B\Psi_2(z)\right], & \mathrm{Im}[\tilde{\omega}] > 0\,, \\ (1-c)\left[\overline{A}\Psi_1(z) + \overline{B}\Psi_2(z)\right], & \mathrm{Im}[\tilde{\omega}] < 0\,, \end{cases}$$

and

$$\mathcal{N}_{\tilde{\omega},\zeta} = \begin{cases} (1-c)\sqrt{1+\mu^2}\left[\cos(\zeta)B - \sin(\zeta)A\right], & \mathrm{Im}[\tilde{\omega}] > 0, \\ (c-1)\sqrt{1+\mu^2}\left[\cos(\zeta)\overline{B} - \sin(\zeta)\overline{A}\right], & \mathrm{Im}[\tilde{\omega}] < 0. \end{cases}$$

Hence, for $z < z'$, the distribution (3.41) can be written as

$$\begin{aligned} \Delta\mathcal{G}_{\tilde{\omega}}(z, z') &= -\frac{1}{\sqrt{1+\mu^2}} \left[\frac{A\Psi_1(z) + B\Psi_2(z)}{\cos(\zeta)B - \sin(\zeta)A} \right. \\ &\quad \left. - \frac{\overline{A}\Psi_1(z) + \overline{B}\Psi_2(z)}{\cos(\zeta)\overline{B} - \sin(\zeta)\overline{A}} \right] \Psi_\zeta(z') \\ &= \frac{\overline{A}B - A\overline{B}}{|\cos(\zeta)B - \sin(\zeta)A|^2} \frac{\Psi_\zeta(z)\Psi_\zeta(z')}{\sqrt{1+\mu^2}}\,, \end{aligned} \tag{3.43}$$

and the result still holds for $z > z'$.

A separate analysis is needed when considering the possible presence of residues at a bound state frequency $\tilde{\omega}_\zeta \in \mathrm{BS}^+_\zeta$. In fact, when it exists, it is an isolated root of $\mathcal{N}_{\tilde{\omega},\zeta} = 0$ and

$$\mathrm{Res}_{\tilde{\omega}=\tilde{\omega}_\zeta}[\tilde{\omega}\mathcal{G}_{\tilde{\omega},\zeta}(z,z')] = \frac{\tilde{\omega}_\zeta}{2} D(\tilde{\omega}_\zeta)\Psi_{\tilde{\omega}_\zeta,\zeta}(z)\Psi_{\tilde{\omega}_\zeta,\zeta}(z'), \tag{3.44}$$

where $D(\tilde{\omega}_\zeta) = D_2(\tilde{\omega}_\zeta)/D_1(\tilde{\omega}_\zeta)$. Expanding $\mathcal{N}_{\tilde{\omega},\zeta}$ in the Laurent series, one gets that

$$\begin{aligned} D_1(\tilde{\omega_\zeta}) \doteq & \frac{\ell^2\sqrt{1+\mu^2}}{i(r_+^2 - r_-^2)} \big\{\sin(\zeta)\, A\big[(r_+ + r_-)\psi(c-a) \\ & + (r_+ - r_-)\psi(c-b) - 2r_+\psi(c)\big](1-c) \\ & - \cos(\zeta)\, B\big[(r_+ + r_-)\psi(b) + (r_+ - r_-)\psi(a) \\ & - 2r_+\psi(c)\big](1-c)\big\}|_{\tilde{\omega}=\tilde{\omega}_\zeta}\,, \end{aligned}$$

where ψ is again the digamma function introduced in (3.17). Since in this case $\mathcal{N}_{\tilde{\omega}_\zeta,\zeta} = 0$, the solutions $u_{\tilde{\omega}_\zeta}$ and $\Psi_{\tilde{\omega}_\zeta,\zeta}$ are no longer linearly independent. Recalling that $\mathrm{Im}[\tilde{\omega}_\zeta] > 0$, their ratio is

$$\begin{aligned} D_2(\tilde{\omega_\zeta}) & \doteq \frac{u_{\tilde{\omega}_\zeta}(z)}{\Psi_{\tilde{\omega}_\zeta,\zeta}(z)} \\ & = \begin{cases} \sec(\zeta)(c-1)A|_{\tilde{\omega}=\tilde{\omega}_\zeta}\,, & \cos(\zeta) \neq 0\,, \\ \csc(\zeta)(c-1)B|_{\tilde{\omega}=\tilde{\omega}_\zeta}\,, & \sin(\zeta) \neq 0\,. \end{cases} \end{aligned}$$

Finally, the spectral resolution of the delta distribution is

$$\begin{aligned} \frac{\delta(z-z')}{\mathcal{J}(z)} = & \frac{\ell^4}{4(r_+^2 - r_-^2)} \\ & \times \Bigg[\int_{\mathbf{R}} \frac{\mathrm{d}\tilde{\omega}}{2\pi i}\, \tilde{\omega}\, \frac{A\overline{B} - \overline{A}B}{|\cos(\zeta)B - \sin(\zeta)A|^2} \frac{\Psi_\zeta(z)\Psi_\zeta(z')}{\sqrt{1+\mu^2}} \\ & + \sum_{\tilde{\omega}_\zeta \in \mathrm{BS}^+_\zeta} \mathrm{Re}\left[\tilde{\omega}_\zeta D(\tilde{\omega}_\zeta)\Psi_{\tilde{\omega}_\zeta,\zeta}(z)\Psi_{\tilde{\omega}_\zeta,\zeta}(z')\right] \Bigg]\,, \end{aligned} \tag{3.45}$$

where we have taken advantage of two facts. Firstly the bound state frequencies appear in complex conjugate pairs, $\mathrm{BS}_\zeta = \mathrm{BS}^+_\zeta \cup \overline{\mathrm{BS}^+_\zeta}$. Secondly $\overline{D(\tilde{\omega}_\zeta)} = D(\overline{\tilde{\omega}_\zeta})$ and $\overline{\Psi_{\tilde{\omega}_\zeta,\zeta}(z)} = \Psi_{\overline{\tilde{\omega}_\zeta},\zeta}(z)$.

In the following sections, we present the results for the resolution of the identity and for the mode expansion of the two-point function for certain fixed Robin boundary conditions, identifying in which cases it is possible to construct a full-fledged ground state for the scalar field in the external region of BTZ. In the simplest scenario, corresponding to the range of values $\mu^2 \geqslant 0$, no boundary condition needs to be imposed at $z = 1$ and it is always possible to construct a ground state. In the second case, corresponding to the range of values $-1 < \mu^2 < 0$ a ground state will be admissible only for certain boundary conditions. The full details of the calculation can be consulted in Appendix 3.3.3.

3.3.4 The two point function for $\mu^2 \geqslant 0$

As shown in Sections 3.2.3 and 3.2.4, for the range of values $\mu^2 \geqslant 0$ both $z = 0$ and $z = 1$ are of limit point type. Using the results of Section 3.3.3 in the case $\zeta = 0$, it is possible to obtain an integral representation of $\delta(z - z')$ in terms of eigenfunctions of $L_{\tilde{\omega}}$,

$$\frac{\delta(z-z')}{\mathcal{J}(z)} = \int_{\mathbb{R}} \frac{\mathrm{d}\tilde{\omega}}{2\pi i}\, \tilde{\omega} \left(\frac{A}{B} - \frac{\overline{A}}{\overline{B}}\right) C\, \Psi_1(z)\Psi_1(z'),$$

where the constants A, B and C are defined as

$$A = \frac{\Gamma(c-1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{3.46a}$$

$$B = \frac{\Gamma(c-1)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \tag{3.46b}$$

$$C = \frac{\ell^4}{4(r_+^2 - r_-^2)\sqrt{1+\mu^2}}. \tag{3.46c}$$

Comparing this integral representation with (3.32), we can read off the formula for $\widetilde{G}_{\tilde{\omega}k}(z, z')$ and write the two-point function as

$$G^+(x, x') = \lim_{\epsilon\to 0^+} \sum_{k\in\mathbb{Z}} e^{ik(\tilde{\phi}-\tilde{\phi}')} \int_0^\infty \frac{\mathrm{d}\tilde{\omega}}{(2\pi)^2}\, e^{-i\tilde{\omega}(\tilde{t}-\tilde{t}'-i\epsilon)} \left(\frac{A}{B} - \frac{\overline{A}}{\overline{B}}\right) C\, \Psi_1(z)\Psi_1(z'). \tag{3.47}$$

3.3.5 The two-point function for $-1 < \mu^2 < 0$

For the range of values $-1 < \mu^2 < 0$, a Robin boundary condition needs to be imposed on solutions at $z = 1$ and we obtain a different two-point function for each possible Robin boundary condition. Two separate regimes must be considered. The boundary conditions identified by $\zeta \in [0, \zeta_c)$ and those identified by $\zeta \in [\zeta_c, \pi)$, where ζ_c is a critical value of ζ. As we will show shortly, for $\zeta \geq \zeta_c$ bound state frequencies $\tilde{\omega}_\zeta \in \mathrm{BS}_\zeta^+$ will be present, while no bound state occurs in the regime $\zeta \in [0, \zeta_c)$. The search for bound states is performed by looking at the isolated roots of $\mathcal{N}_{\tilde{\omega},\zeta} = 0$, by reversing the formula

$$\tan(\zeta) = \left.\frac{B}{A}\right|_{\tilde{\omega}=\tilde{\omega}_\zeta}. \tag{3.48}$$

In particular, the critical value ζ_c is

$$\zeta_c \doteq \arctan\left(\frac{\Gamma(2\beta-1)\left|\Gamma\left(1-\beta+i\ell\frac{k}{r_+}\right)\right|^2}{\Gamma(1-2\beta)\left|\Gamma\left(\beta+i\ell\frac{k}{r_+}\right)\right|^2}\right), \tag{3.49}$$

where $\beta = \frac{1}{2}+\frac{1}{2}\sqrt{1+\mu^2}$ is the Frobenius parameter defined in (3.8). Notice that $\mu^2 \in (-1,0)$, therefore $\beta \in (\frac{1}{2},1)$ and $\zeta_c \in (\frac{\pi}{2},\pi)$. The subsequent results follow.

Case $\zeta \in [0,\zeta_c)$

For Robin boundary conditions identified by $\zeta \in [0,\zeta_c)$, the spectrum of the operator $L_{\tilde\omega}$ in (3.5) is only $\tilde\omega \in \mathbb{R}$ and it does not include any isolated eigenvalue in $\mathbb{C}\backslash\mathbb{R}$. Therefore no pole is present in the Green distribution associated with $L_{\tilde\omega}$ calculated at the beginning of Section 3.3.3. Since the value ζ_c lies in the range $(\frac{\pi}{2},\pi)$, this regime includes both the Dirichlet and the Neumann-like boundary conditions. In this case, therefore, the result is structurally identical to the one investigated in the previous section for $\mu^2 \geqslant 0$ and we obtain the following resolution of the identity

$$\frac{\delta(z-z')}{\mathcal{J}(z)} = \int_{\mathbb{R}} \frac{\mathrm{d}\tilde\omega}{2\pi i}\,\tilde\omega\,\frac{\left(A\overline{B}-\overline{A}B\right)C}{|\cos(\zeta)B-\sin(\zeta)A|^2}\Psi_\zeta(z)\Psi_\zeta(z'), \tag{3.50}$$

where the constants A, B and C are the same as in (3.46).

Combining this result with (3.29) and (3.32) one obtains, for each $\zeta \in [0,\zeta_c)$,

$$G^+_\zeta(x,x') = \tag{3.51}$$
$$\lim_{\epsilon\to 0^+}\sum_{k\in\mathbb{Z}} e^{ik(\tilde\phi-\tilde\phi')}\int_0^\infty \frac{\mathrm{d}\tilde\omega}{(2\pi)^2}\,e^{-i\tilde\omega(\tilde t-\tilde t'-i\epsilon)}\frac{\left(A\overline{B}-\overline{A}B\right)C}{|\cos(\zeta)B-\sin(\zeta)A|^2}\Psi_\zeta(z)\Psi_\zeta(z').$$

Not surprisingly, if $\zeta = 0$ (that is, for Dirichlet boundary conditions), this two-point function structurally coincides with the one for scalar fields with $\mu^2 \geqslant 0$ obtained in (3.47).

Case $\zeta \in [\zeta_c,\pi)$

For Robin boundary condition in the regime $\zeta \in [\zeta_c,\pi)$, the spectrum of the operator $L_{\tilde\omega}$ in (3.5) contains all $\tilde\omega \in \mathbb{R}$ and also two isolated eigenvalues in $\mathbb{C}\setminus\mathbb{R}$. These eigenvalues are complex conjugate to each other and they correspond to poles in the Green distribution associated with $L_{\tilde\omega}$. We denote such eigenvalues by $\tilde\omega_\zeta$ and $\overline{\tilde\omega_\zeta}$ so that $\mathrm{Im}[\tilde\omega_\zeta] > 0$. We call them *bound state frequencies*. Consequently, their corresponding eigensolutions are called *bound state mode solutions*. An analytic expression for $\tilde\omega_\zeta$ cannot be found since, for $\mathrm{Im}[\tilde\omega_\zeta] > 0$ and fixed ζ, one needs to invert the equality (3.48) for $\tilde\omega_\zeta$.

This operation can only be completed numerically (except in very particular cases such as $\zeta = 0$ and $\zeta = \pi/2$. A representative example is shown in Fig. 3.1.

As a consequence of the presence of these bound state frequencies, the resolution of the identity acquires an extra term, which, accordingly to the procedure presented in Section 3.3.3, can be computed via Cauchy residue theorem. One obtains

$$\begin{aligned}\frac{\delta(z-z')}{\mathcal{J}(z)} = &\int_{\mathbb{R}} \frac{\mathrm{d}\tilde{\omega}}{2\pi i}\, \tilde{\omega}\, \frac{\left(A\overline{B} - \overline{A}B\right) C}{|\cos(\zeta)B - \sin(\zeta)A|^2} \Psi_\zeta(z)\Psi_\zeta(z') \\ &+ \mathrm{Re}\left[\tilde{\omega}\, CD(\tilde{\omega})\Psi_\zeta(z)\Psi_\zeta(z')\right]\big|_{\tilde{\omega}=\tilde{\omega}_\zeta}\,, \end{aligned} \tag{3.52}$$

where we used the identity $\Psi_\zeta(z)|_{\tilde{\omega}=\overline{\tilde{\omega}_\zeta}} = \overline{\Psi_\zeta(z)|_{\tilde{\omega}=\tilde{\omega}_\zeta}}$. The remaining term $D(\tilde{\omega}_\zeta)$ cannot be expressed analytically, but it can be defined implicitly by means of equation (3.44).

The bound state mode solutions will also contribute to the two-point function so that its antisymmetric part still obeys (3.28), so to guarantee that the CCRs are satisfied. The mode expanded two-point function, for each $\zeta \in [\zeta_c, \pi)$, is therefore

$$\begin{aligned} &G^+_\zeta(x,x') = \\ &\lim_{\epsilon\to 0^+} \sum_{k\in\mathbb{Z}} e^{ik\left(\tilde{\phi}-\tilde{\phi}'\right)} \int_0^\infty \frac{\mathrm{d}\tilde{\omega}}{(2\pi)^2}\, e^{-i\tilde{\omega}\left(\tilde{t}-\tilde{t}'-i\epsilon\right)} \frac{\left(A\overline{B} - \overline{A}B\right) C}{|\cos(\zeta)B - \sin(\zeta)A|^2} \Psi_\zeta(z)\Psi_\zeta(z') \\ &\quad + i\sum_{k\in\mathbb{Z}} e^{ik\left(\tilde{\phi}-\tilde{\phi}'\right)} \left(e^{-i\tilde{\omega}_\zeta(\tilde{t}-\tilde{t}')} + e^{-i\overline{\tilde{\omega}_\zeta}(\tilde{t}-\tilde{t}')}\right) \mathrm{Re}\left[CD(\tilde{\omega})\Psi_\zeta(z)\Psi_\zeta(z')\right]\big|_{\tilde{\omega}=\tilde{\omega}_\zeta}\,. \end{aligned} \tag{3.53}$$

In the special case of $\zeta = \zeta_c$, the frequency of both bound states becomes $\tilde{\omega} = 0$. One should interpret the integral over the positive $\tilde{\omega}$-frequencies as the Cauchy principal value for $\tilde{\omega} = 0$, and use the Sokhotsky-Plemelj formula for distributions in order to account for bound state mode solutions.

3.4 Ground states and bound states

3.4.1 The Hadamard condition

One of the most important questions about G^+ is if it represents a physically reasonable state. To this end, one might want to look at the situation in flat Minkowski spacetime. In this background, some examples of physically interesting states include the Fock vacuum state, the corresponding multiparticle states and states describing systems at thermal equilibrium. The main characteristics shared by all these states is the short-distance behaviour, that is they exhibit the same ultraviolet properties. As we will see, the Hadamard condition requires that a two-point function describing a physical state ω on some (curved) background must undergo the same high-energy behaviour of the Poincaré vacuum [71], ensuring the presence of finite quantum fluctuations for all the observables. Moreover, it is deeply connected to the presence of Hawking radiation [18].

In quantum field theory, the product $\Phi(x)\Phi(x)$ or any other pointwise product of fields like $\Phi(x)$ among themseleves, which is the building block of the Wick polynomial expansion for perturbative theory, might be ill-defined. The usual procedure to overcome this problem, known as *normal ordering*, consists in expanding the field into creation and annihilation operators and then to rearrange them in such a way to preserve the commutation relations between operators. This procedure usually leads to a well-defined object, denoted with the symbol

$$:\Phi(x)^2: \ .$$

If one considers the Minkowski vacuum state $|0\rangle$ for the scalar field, then the normal ordered product can be written as

$$:\Phi(x)^2 := \lim_{x'\to x} \left(\Phi(x)\Phi(x') - \langle 0|\Phi(x)\Phi(x')|0\rangle\, \mathbb{I}\right) \ .$$

In order for this product to make sense over a physical state $|\omega\rangle$, one needs $\langle\omega|\Phi(x)\Phi(x')|\omega\rangle$ and $\langle 0|\Phi(x)\Phi(x')|0\rangle$, to have the same singularities.

Let $(\mathcal{M}, g)$ be a smooth, D dimensional, connected Lorentzian manifold spacetime, $\widetilde{\mathcal{O}} \subset \mathcal{M}$ a geodesically convex open set and $\widetilde{U}_p \subset T_p\mathcal{M}$ an open set in which the exponential map $\exp : \widetilde{U}_p \to \widetilde{\mathcal{O}}$ is well defined. For any pair of points p and p' in $\widetilde{\mathcal{O}}$, with coordinates x and x' respectively, one can define the *half squared geodesic distance* $\sigma(p,q)$, also called *Synge world function* as

$$\sigma(x,x') := \frac{1}{2} g(\exp_x^{-1}(x'), \exp_x^{-1}(x')) \ , \tag{3.54}$$

which is both smooth and symmetric in $\widetilde{\mathcal{O}} \times \widetilde{\mathcal{O}}$. Let us now introduce the function

$$\sigma_\epsilon(x,x') := \sigma(x,x') + 2i\epsilon(T(x) - T(x')) + \epsilon^2 \ , \tag{3.55}$$

where $\epsilon > 0$ while T is any, but fixed time function on $\mathcal{M}$.

We say that a two-point function ω_2 is of *local Hadamard form*, namely it satisfies the *Hadamard condition* if, for every $x \in \mathcal{M}$ there exists a geodesically convex neighbourhood $\widetilde{\mathcal{O}}$ such that the restriction of its integral kernel to $\widetilde{\mathcal{O}}\times\widetilde{\mathcal{O}}$ reads [30]

$$\begin{aligned} \omega_{2,\epsilon}(x,x') =& \beta_D^{(1)} \frac{U(x,x')}{\sigma_\epsilon^{D/2-1}(x,x')} \\ &+ \beta_D^{(2)} V(x,x') \ln\frac{|\sigma_\epsilon(x,x')|}{\lambda^2} + w(x,x') \qquad \text{if } D \text{ is even} \ , \\ \omega_{2,\epsilon}(x,x') =& \beta_D^{(1)} \Theta(\sigma_\epsilon(x,x')) \frac{U(x,x')}{\sigma_\epsilon^{D/2-1}(x,x')} + w(x,x') \quad \text{if } D \text{ is odd} \ , \end{aligned} \tag{3.56}$$

where x, x' are two arbitrary points in $\widetilde{\mathcal{O}}$, Θ is the Heaviside function, the functions U and V are defined by recursive expansions [75] in powers of σ and are completely determined by the metric g and by the equations of motion, $\beta_D^{(i)}$ are numerical coefficients $\lambda > 0$ is an arbitrarily fixed length scale, while $w(x, x')$ is a smooth function on $\widetilde{\mathcal{O}} \times \widetilde{\mathcal{O}}$. The bidistributions

$$\begin{aligned} &\beta_D^{(1)} \frac{U(x,x')^{1/2}}{\sigma_\epsilon^{D/2-1}(x,x')} + \beta_D^{(2)} V(x,x') \ln \frac{|\sigma_\epsilon(x,x')|}{\lambda^2} \qquad &\text{if } D \text{ is even },\\ &\beta_D^{(1)} \Theta(\sigma_\epsilon(x,x')) \frac{\Delta(x,x')^{1/2}}{\sigma_\epsilon^{D/2-1}(x,x')} \qquad &\text{if } D \text{ is odd }, \end{aligned} \tag{3.57}$$

are called *Hadamard parametrices* and they are bisolutions of the D-dimensional Klein-Gordon equation up to smooth terms. States of local Hadamard form are a special case of the global Hadamard form, which requires that there exist no spacelike singularities other to the lighlike shown in the local form [30]. States of global Hadamard form are often referred as *Hadamard states* and they are usually characterized with advanced techniques of microlocal analysis [76]. Even though we are not going to introduce this topic in the present work, it is important to mention that these techniques allowed to build specific examples of Hadamard states, such as the Unruh state in Schwarzschild spacetime [77] or the asymptotic vacuum and KMS states in certain classes of Friedmann-Robertson-Walker spacetimes [78]. Eventually, all the ground states and the thermal equilibrium states on ultrastatic spacetimes are Hadamard states and consequently Hadamard states exist on any globally hyperbolic spacetime by means of a spacetime deformation argument [79]. The local and the global definitions are equivalent for globally hyperbolic spacetime. This is not the case of BTZ spacetime because of the presence of the timelike conformal boundary. Nonetheless an abstract characterisation by Sahlmann and Verch [28, Appendix A] proves that a ground state built out of positive frequencies is always of local Hadamard form and it identifies a Hadamard state in every globally hyperbolic subregion of BTZ [29]. Bearing in mind these considerations, it is possible to comment on the results obtained in Sections 3.3.4 and 3.3.5. In the following, will adopt the following definition of ground state, as stated by Sahlmann and Verch [28, Appendix A]. Given the function

$$\alpha_{\widetilde{t}} : C_0^\infty(\mathrm{BTZ}) \to C_0^\infty(\mathrm{BTZ}) \tag{3.58}$$

such that, for all $f \in C_0^\infty(\mathrm{BTZ})$ and for all $t \in \mathbb{R}$,

$$\alpha_t f(x) = f(\widetilde{\alpha}_{-t}(x)) , \tag{3.59}$$

where $\widetilde{\alpha}_{-t}(x)$ indicates the flow of a point $p \in \mathrm{BTZ}$ with coordinates x built out of the integral curves of a timelike Killing vector ξ of Eq. (2.11) and given $\hat{f}(t) := \frac{1}{\sqrt{2\pi}} \int e^{-ipt} f(p) dp$, $f \in C_0^\infty(\mathbb{R})$, a state $\omega : C_0^\infty(\mathrm{BTZ}) \to \mathbb{C}$ is called a *ground state* if $\mathbb{R} \ni t \to \omega(f' \alpha_t(f))$ is, for each $g, h \in C_0^\infty(\mathrm{BTZ})$, a bounded function and if

$$\int_{-\infty}^{\infty} \mathrm{d}t\, \omega(g, \alpha_t(h))\hat{f}(t) = 0$$

for all $f \in C_0^\infty((-\infty, 0))$.

As we have seen, the mode decomposition of G^+ in (3.47) contains only positive $\tilde{\omega}$-frequencies. Moreover its antisymmetric part satisfies (3.28). Hence, comparing it to the previous definition, it is legitimate to call the state associated with G^+ the ground state for a real, massive scalar field in the BTZ spacetime with $\mu^2 \geqslant 0$ and it is of Hadamard form in any globally hyperbolic subregion of the exterior region of the black hole spacetime.
Regarding the two-point functions obtained in (3.51) and (3.53), a distinction is needed. In the first case, we are dealing with a generalization of (3.47) to Robin boundary conditions. Hence, (3.51) is a genuine ground state built only out of positive $\tilde{\omega}$-frequencies satisfying the local Hadamard condition in the exterior region of the black hole spacetime. The same can not be stated for (3.53), where an additional contribution related to the presence of bound state frequencies $\tilde{\omega}_\zeta$ in the spectrum spoils the property of G_ζ^+ being a ground state.

In Chapter 4, we will focus our attention on the interplay between local Hadamard states just obtained and the thermal properties of fields in a neighbourhood of the BTZ Killing horizon.

3.4.2 Bound states

As we have seen, the resolution of the identity and the construction of a mode expanded two-point function take different form depending on the presence of bound state frequencies $\tilde{\omega}_\zeta \in \mathrm{BS}_\zeta^+$. To each bound state frequency is then associated a bound state mode solution, that is an exponentially decaying solution in $\tilde{t}$, such as

$$\Phi \propto e^{i\tilde{\omega}_\zeta \tilde{t}}$$

for $\mathrm{Im}\,\tilde{\omega}_\zeta < 0$. Bound state frequencies are complex frequencies of Eq. (3.5) and they occur in the range value $-1 < \mu^2 < 0$ for a specific regime of Robin boundary conditions. Essentially, they start to appear in fact at the critical value $\zeta_c \in (\frac{\pi}{2}, \pi)$ in (3.49), while on the contrary no bound states occur in the regime $\zeta \in [0, \zeta_c)$. A plot of the dependence of the bound state frequency ω_ζ as a function of ζ can be found in Fig. 3.1 for some test values of the system parameters.
In principle, bound state frequencies can be found by looking for isolated roots of $\mathcal{N}_{\tilde{\omega},\zeta} = 0$, i.e. by reversing the formula

$$\sin(\zeta)\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = \cos(\zeta)\frac{\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \ , \tag{3.60}$$

where a, b and c are as in (3.10) and $\zeta \in [0, \pi)$. Unfortunately this equation can be solved analytically only for the Dirichlet (+) and Neumann (−) case,

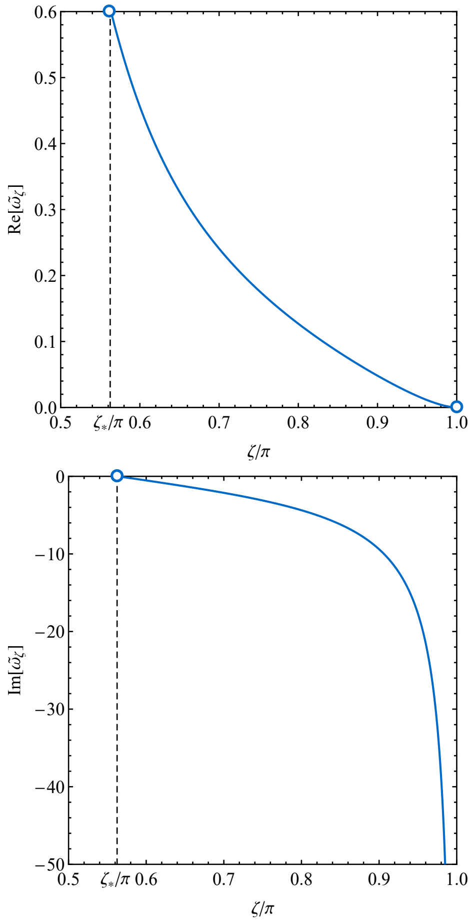

Figure 3.1: Real and imaginary part of the bound state frequency ω_ζ as a function of ζ for a sample BTZ black hole with $\ell = 1$, $r_+ = 5$ and $r_- = 3$ and a scalar field with $\mu^2 = -0.65$ and $k = 1$. The bound state mode solutions occurr for values of ζ between $\zeta_c \approx 0.5625\pi$ and π.

for which one gets

$$\omega_\zeta = \frac{\kappa}{\ell} - i\frac{(r_+ - r_-)}{\ell^2}(3 \pm \sqrt{1+\mu^2}) \ ,$$

while all the other cases must be studied numerically [43].

The presence of such complex frequencies for $-1 < \mu^2 < 0$ implies that the bare Fourier reconstruction of the two-point function G^+ does not represent the full solution to the equation when $\zeta \in [\zeta_c, \pi)$. A general solution for this class of boundary conditions must include bound state mode solutions, along the usual propagating modes, see (3.53).

As already stated, the presence of bound state mode solutions directly spoils the property of G^+ being a ground state for the system. Nonetheless the physical reason for their appearance is still unclear. In [43] it was observed that their presence is coupled to the presence of superradiant modes, extracting energy from the black hole, with a non vanishing energy flux through the horizon towards the exterior region. More interestingly, it was observed that only a subset of all modes growing up exponentially in time are superradiant, a feature possibly related to the bulk instability of the underlying AdS background itself [43].

Chapter 4

Thermal effects and tunnelling processes in $2+1$ dimensions

This chapter is devoted to analyzing the local thermal behaviour of the two-point correlation functions for the massive scalar field (3.2) in a neighborhood of the bifurcate Killing horizon of the BTZ spacetime introduced in Section 2.3.3, wondering if they can be related to some tunnelling processes through the horizon. The results are obtained by generalizing to $2+1$ dimensions the approach proposed in [48], which adopts the local point of view first proposed by Parikh and Wilczeck in [47]. The work of Parikh and Wilczeck has its roots in the founding paper of Stephen Hawking [14], which describes a thermal radiation detected at future null infinity of a collapsing black hole spacetime, but focuses instead on the local properties of the spacetime models. This local approach also relates the radiation to thermal effects, which seem to emerge as a result of a tunnelling process through the horizon. If one performs a WKB approximation, it is in fact possible to relate the tunnelling probability through the horizon to the characteristic Boltzmann thermal form e^{-ET_H}, where E describes the amount of energy coming out of the black hole horizon and T_H the so-called Hawking temperature. In recent years, this strategy, which involves a limit towards the horizon for an endpoint of the path of the classical particle of the field, has been applied to various scenarios, BTZ included [80], and some interesting results have been found also considering the backreaction on the event horizon [81, 82]. Rather than relying on a semi-classical framework, the approach proposed by Moretti and Pinamonti in [48] adopts the point of view of quantum field theory in curved spacetime and analyses the local behaviour of Hadamard states in a neighborhood smeared to the horizon of a $3+1$ dimensional spacetime. The limiting procedure, which shall be regarded as a limit towards the horizon, is particularly flexible, and can be adapted also to settings in which the horizon exists just locally and therefore momentarily. Moreover it seems particularly fit to be used in our scenario, since it focuses directly on the behaviour of the two-point correlation function ω_2 associated to a quantum state ω satisfying the local Hadamard condition.

In the following, see Section 4.1, we will extend the work of Pinamonti and

Moretti to a 2+1 dimensional spacetime. In Section 4.1.1 the basic setting and some notations are presented. Section 4.1.2 we analyse the limiting behaviour of the two-point correlation function of a Hadamard state in a neighborhood of a bifurcate horizon. In Section 4.1.3 we show that the two-point function actually exhibits a thermal behaviour exactly at the Hawking temperature and that this thermal behaviour might be associated to some tunnelling process through the horizon. In the second part of the chapter, see Section 4.2, the results are applied to the specific case of the BTZ spacetime, by taking advantage of some results outlined in Chapter 3. In particular we will prove the existence of a thermal Hadamard state in the considered neighborhood of the bifurcate horizon and we will show its two-point function in the exterior region of the spacetime.

4.1 Thermal effects near a bifurcate horizon

In the following, tunnelling processes are studied for any $2+1$ dimensional spacetime equipped with a (local) bifurcate Killing horizon.

4.1.1 Basic setting in $2+1$ dimensions

Let us consider a smooth, three dimensional, connected Lorentzian manifold $(\mathcal{M}, g)$, assuming that there exists an open subset $\mathcal{O} \subset \mathcal{M}$ such that

(i) there exists $\xi \in \Gamma(T\mathcal{O})$ for which $\mathcal{L}_\xi g = 0$,

(ii) the orbits of ξ in $\mathcal{O}$ are diffeomorphic to an open interval in $\mathbb{R}$,

(iii) there exists a two dimensional, connected submanifold $\mathcal{H} \subset \mathcal{O}$, called *local Killing horizon*, invariant under the action of the group of local isometries generated by ξ,

(iv) ξ is lightlike on $\mathcal{H}$ and the intersection between $\mathcal{H}$ and the integral curves of ξ identifies a smooth 2-dimensional submanifold of $\mathcal{H}$,

(v) κ, the surface gravity of $\mathcal{H}$ is a non vanishing, positive constant.

All these hypotheses comprise the case of a manifold $(\mathcal{M}, g)$ endowed with a Killing field ξ generating a (even local) bifurcate Killing horizon. In this case ξ is expected to vanish on a one-dimensional acausal submanifold $\mathcal{B}$ and to be lightlike on two ξ-invariant null submanifolds, $\mathcal{H}_+$ and $\mathcal{H}_-$. This is indeed the case of BTZ spacetime, see Chapter 2, where $\mathcal{B} = \mathcal{H}_+ \cap \mathcal{H}_-$ is the bifurcation surface defined in Section 2.3.3, while $\mathcal{H} = \mathcal{H}_+ \cup \mathcal{H}_-$ is the bifurcate Killing horizon.

As a matter of fact, any neighbourhood $\mathcal{O} \subset \mathcal{M}$ of a point $p \in \mathcal{H}_\pm$ with $\mathcal{O} \cap \mathcal{B} = \emptyset$, satisfies the geometric hypotheses and, whenever we consider $\mathcal{O}$ fulfilling the hypotheses (i-v), it is possible to deform smoothly $(\mathcal{M}, g)$ so for a bifurcate Killing horizon to exist. Therefore, it is not restrictive to focus the analysis to these types of neighbourhoods, expecially since our interest is to evaluate only quantities defined in $\mathcal{O}$.

Using the structural properties of a Killing horizon [30, 83] it is possible to identify per restriction in $\mathcal{O}$ a coordinate patch (V, U, x^3), where

- U denotes an affine parameter along the null integral lines of ξ with origin fixed at $\mathcal{B}$,
- V is the affine parameter, with origin at $\mathcal{B}$, of the integral curves of the future-pointing lightlike vector field[1] $n_{\mathcal{H}_+}$ of $\mathcal{H}_+$,
- x_3 denotes any, but fixed coordinate defined on an open neighbourhood of a point lying in $\mathcal{B}$.

The vector field ξ can be exprssed in terms of the local chart (U, V, x_3) as

$$\xi = \xi^1 \frac{\partial}{\partial V} + \xi^2 \frac{\partial}{\partial U} + \xi^3 \frac{\partial}{\partial x^3} \ . \tag{4.1}$$

If there exists a subset $\mathcal{O}' \subset \mathcal{O}$ with compact closure in $\mathcal{O}$, for any point $p \in \mathcal{O}'$ one has that[2] $\xi^1(p) = -\kappa V + V^2 R_1(p)$, $\xi^2(p) = \kappa U + V^2 R_2(p)$, $\xi^3(p) = V R_3(p)$, where R_1, R_2, R_3 are bounded smooth functions on $\mathcal{O}'$ and κ is the surface gravity. In this subset, the line element of the metric g, restricted to a neighbourhood of $\mathcal{H}^+$, reads

$$g_{\restriction \mathcal{H}^+} = -\frac{1}{2} dU \otimes dV - \frac{1}{2} dV \otimes dU + h(x_3) dx^3 \otimes dx^3 \ , \tag{4.2}$$

where h is a strictly positive function depending only on x_3.

The leading order in V of $g(\xi, \xi)$ in a neighbourhood of $\mathcal{H}^+$ can be evaluated by combining together Eq. (4.1) and Eq. (4.2). One obtains

$$g_{\restriction \mathcal{H}^+}(\xi, \xi) = \kappa^2 U V + O(V^2) \ .$$

For now on we assume that U is of positive sign in $\mathcal{O}$. Taking $\mathcal{O}$ small enough we can split it as the union of three disjoint regions $\mathcal{O} = \mathcal{O}_s \cup \mathcal{O}_0 \cup \mathcal{O}_t$ where

$$\begin{aligned} \mathcal{O}_0 &:= \mathcal{O} \cap \mathcal{H}_+ \\ \mathcal{O}_s &:= \{p \in \mathcal{O} \mid V(p) < 0\} \\ \mathcal{O}_t &:= \{p \in \mathcal{O} \mid V(p) > 0\} \end{aligned} \tag{4.3}$$

In the region $\mathcal{O}_s$, which can be referred as the interior region of $\mathcal{O}$ the vector field ξ is spacelike. Conversely, ξ is timelike in $\mathcal{O}_t$, which can be referred as the

[1] This vector field is built as the parallel transport of n, the unique, future pointing, lightlike vector at $\mathcal{B}$ such that $g(n, -\frac{\partial}{\partial U}) = -\frac{1}{2}$.

[2] This proposition is a straightforward generalization of [48, Prop. 2.1].

exterior region of $\mathcal{O}$. Equation (4.2) makes possible to study the properties of the geodesic distance in $\mathcal{O}'$. The following results are generalization of [48, Prop. 2.1] to the $2+1$ dimensional case. The proof follows exactly as in [48], simply removing the coordinate x^4 and the label $i=4$ from the text.
Let $p \in \mathcal{O}$ be a point of coordinates (U, V, x_3), such that $p \in \mathcal{H}^+$ if and only if $V=0$. Then the following statements hold true.

1. Let $\widetilde{\mathcal{O}} \subset \mathcal{O}'$ be any geodesically convex neighbourhood of $\mathcal{H}^+$ and let $p, q \in \mathcal{H}^+ \cap \widetilde{\mathcal{O}}$. The squared geodesic distance between these points is
$$\sigma(x(p), x(q)) \equiv \ell(x_3(p), x_3(q)) := \left(\int_{x_3(p)}^{x_3(q)} d\lambda\ f(\lambda) \right)^2 , \tag{4.4}$$
where $x(p)$ (*respectively* $x(q)$) indicates the representation of the point p (*respectively* of the point q) in terms of the coordinates (U, V, x_3). Moreover, $x_3(p)$ and $x_3(q)$ are respectively the evaluation of the points p and q along the coordinate x_3, while $f^2 = h$, where h is the function in Eq. (4.2).

2. Let now $p \in \widetilde{\mathcal{O}}$, where $\widetilde{\mathcal{O}} \subset \mathcal{O}'$ is like in 1. and, for any fixed, admissible value of the coordinates U', V', we define $S_{U',V'}$ as the collection of points q lying in the cross section of $\widetilde{\mathcal{O}}$ with V' and U' constant. For $\delta > 0$, we further define the set of points
$$G_\delta(p, V', U') = \{q \in S_{U',V'} \mid \ell(x_3(p), x_3(q)) < \delta^2\} , \tag{4.5}$$
where the distance ℓ is as in (4.4). Then δ can be chosen so that the smooth map
$$G_\delta(p, V', U') \ni q \mapsto \sigma(x(p), x(q))$$
has minimum in a unique point $q(p, V', U')$.
As a consequence of (4.4), $x_3(q) = x_3(p)$ if $p \in \mathcal{H}^+ \cap \widetilde{\mathcal{O}}$, although in general, there exist three bounded functions F_i, $i = 1, 2, 3$, depending smoothly on $x(p), U', V'$ such that
$$\sigma(x(p), x(q)) = \ell(x_3(p), x_3(q)) - (U - U')(V - V') + R(x(p), V, V', U') \tag{4.6}$$
where $R(x(p), U', V') = F_1 V^2 + F_2 V'^2 + F_3 V V'$.

Hadamard states

Bearing in mind the quantisation scheme presented in Chapter 3, we focus our attention on evaluating only quantities defined in geodesically convex neighbourhoods of points on the Killing horizon. For any globally hyperbolic subregion $\widetilde{\mathcal{O}}$ it is indeed possible to define a local algebra of observables. Nonetheless we recall that the assignment of an algebraic state is not sufficient to identify physically meaningful states. This statement is tantamount to saying that a two-point function associated to a state is physically meaningful if and only if some extra constraints on its singular structure are met. In Section 3.4.1, we already identified these constraints by introducing the Hadamard condition, which reflects a particular short distance behaviour of the two-point correlation function. In the following we specialise the local definition of Hadamard states presented in (3.56) for the $2+1$ dimensional case. Notice that this setting applies for any smooth, $2+1$ dimensional connected Lorentzian manifold, even though in the end we aim to analyse the case of a BTZ spacetime.

Let $(\mathcal{M}, g)$ be a $2+1$ dimensional spacetime as in Section 4.1.1, $\widetilde{\mathcal{O}} \subset \mathcal{M}$ a geodesically convex open set and $\widetilde{U}_p \subset T_p\mathcal{M}$ an open set in which the exponential map $\exp : \widetilde{U}_p \to \widetilde{\mathcal{O}}$ is well defined. For any pair of points p and p' in $\widetilde{\mathcal{O}}$, with coordinates x and x' respectively, let $\sigma(x, x')$ be the Synge world function defined in 3.54, and $\sigma_\epsilon(x, x')$ the function in 3.55.

A two-point function ω_2 is then of local Hadamard form (see Section 3.4.1, if, for every $x \in \mathcal{M}$ there exists a geodesically convex neighbourhood $\widetilde{\mathcal{O}}$ such that the restriction of its integral kernel to $\widetilde{\mathcal{O}} \times \widetilde{\mathcal{O}}$ reads

$$\omega_{2,\epsilon}(x, x') = \frac{\Delta(x, x')^{1/2}}{4\pi\sqrt{\sigma_\epsilon(x, x')}} + w(x, x') \ , \tag{4.7}$$

where x, x' are two arbitrary points in $\widetilde{\mathcal{O}}$, $\Delta \in C^\infty(\widetilde{\mathcal{O}} \times \widetilde{\mathcal{O}})$ is the Van Vleck-Morette determinant

$$\Delta(x, x') := \frac{\det(\nabla_\alpha \nabla_{\beta'} \sigma(x, x'))}{\sqrt{|\det g(x)||\det g(x')|}} \ ,$$

while $w(x, x')$ is a smooth function on $\widetilde{\mathcal{O}} \times \widetilde{\mathcal{O}}$. Therefore the Hadamard parametrix

$$h_\epsilon := \frac{\Delta(x, x')^{1/2}}{4\pi\sqrt{\sigma_\epsilon(x, x')}}$$

is a bisolution of the Klein-Gordon equation (3.2) up to smooth terms.

4.1.2 Limiting behaviour of the two-point correlation functions

In between the class of admissible correlation functions, we will be focusing our attention on those of local Hadamard form (4.7). Recall that we are considering a convex geodesic neighbourhood $\widetilde{\mathcal{O}}$ with non vanishing intersection with the Killing horizon $\mathcal{H}^+$.

Let us now extend the approach of [48] to the $2+1$ dimensional case[3]. For that, we consider two one-parameter families of test functions $f_\lambda, f'_\lambda \in C^\infty(\widetilde{\mathcal{O}})$,

[3] These results have been extensively presented in [2].

$\lambda \in \mathbb{R}$ such that both obey to the constraints

$$f_\lambda(V, U, x_3) = \frac{1}{\lambda} f\left(\frac{V}{\lambda}, U, x_3\right) \quad \text{and} \quad f = \frac{\partial F}{\partial V}, \text{ with } F \in C_0^\infty(\widetilde{\mathcal{O}}). \tag{4.8}$$

As anticipated, we are going to evaluate the limit behaviour of the two-point correlation function ω_2 in a sharp localized region near the horizon. The aim is to obtain the leading order of the expansion of ω_2 in terms of the spacetime coordinates, in order to study the tunnelling processes and the effects due to the presence of the Killing horizon. To this end, we take into account (4.7) and we evaluate the limit

$$\lim_{\lambda\to 0^+} \omega_2(f_\lambda, f'_\lambda) =$$

$$\lim_{\lambda\to 0^+} \lim_{\epsilon\to 0^+} \int_{\widetilde{\mathcal{O}}\times\widetilde{\mathcal{O}}} \left(\frac{\Delta(x,x')^{1/2}}{4\pi\sqrt{\sigma_\epsilon(x,x')}} + w(x,x') \right) f_\lambda(x) f'_\lambda(x') \mathrm{d}\mu_g(x)\mathrm{d}\mu_g(x') \ , \tag{4.9}$$

where $d\mu_g = \sqrt{h(x_3)}dx_3 dU dV$ is the volume form on $\widetilde{\mathcal{O}}$ induced by the metric (4.2).

In Eq. (4.9), only the contribution of the singular part of the two-point correlation function, the one corresponding to the Hadamard parametrix h_ϵ is relevant. In fact, due to the constraints (4.8), and being w smooth, the integral of the second term with respect to the coordinates V or V' vanishes. In order to evaluate the limit of the first term, we introduce an auxiliary cut-off as follows. Let $\delta > 0$ and let $G_\delta(p, V', U')$ be the set defined in (4.5). We define a smooth and compactly supported function

$$G_\delta(p, V', U') \ni p' \mapsto \chi_\delta(x, x') \geq 0 \ ,$$

where $x = (V, U, x_3)$ indicates the coordinates of p, while $x' = (V', U', x'_3)$ those of p', with constraint

$$\chi_\delta(x,x') = 1, \text{ for } 0 \leq \sqrt{\ell(x_3(p), x_3(p'))} \leq \frac{\delta}{2} + \frac{1}{2}\sqrt{\ell(x_3(p), x_3(q))}$$

Here q refers to the unique point $q(p, V', U') \in \widetilde{\mathcal{O}}$, minimizing the function $G_\delta(p, V', U') \ni p' \mapsto \sigma(x, x')$, that is the unique[4] point where the function has vanishing gradient with respect to the coordinates of p'. Hence, for any $p, p' \in \widetilde{\mathcal{O}}$ with coordinates x, x' respectively, we rewrite the contribution to the two-point correlation function coming from the singular part of (4.7) as

$$\int_{\widetilde{\mathcal{O}}\times\widetilde{\mathcal{O}}} \mathrm{d}\mu_g(x)\mathrm{d}\mu_g(x') \, \frac{\Delta(x,x')^{\frac{1}{2}} f_\lambda(x) f'_\lambda(x')}{4\pi\sqrt{\sigma_\epsilon(x,x')}} \chi_\delta(x,x') + \tag{4.10a}$$

$$\int_{\widetilde{\mathcal{O}}\times\widetilde{\mathcal{O}}} \mathrm{d}\mu_g(x)\mathrm{d}\mu_g(x') \, \frac{\Delta(x,x')^{\frac{1}{2}} f_\lambda(x) f'_\lambda(x')}{4\pi\sqrt{\sigma_\epsilon(x,x')}} (1 - \chi_\delta(x,x')) \ . \tag{4.10b}$$

We are going now to evaluate separately the limits of the above two terms as λ and ϵ tend to 0^+.

[4]Uniqueness can be proven by taking the Taylor expansion of the gradient of $\sigma(x, x')$ and by exploiting the Banach fix point theorem. More details can be found in [48].

Evaluation of (4.10b) By shrinking if necessary $\widetilde{\mathcal{O}}$, one finds out that the integrand is jointly smooth in all variables, including the case $\epsilon = 0$ and therefore is nowhere singular. This allows us to apply the Lesbegue dominated convergence theorem, so to exchange the order of both limits with the integrals. By taking the limits first, one obtains an integrand, which is the derivative with respect to V and V' of a compactly supported smooth function. Integration by parts makes the overall integral vanish.

Evaluation of (4.10a) The evaluation of the first integral is more complicated, since it involves the presence of a singularity due to $\sqrt{\sigma_\epsilon(p,p')}$ in the limit $\epsilon \to 0^+$. In order to deal with it, for every $p \in \widetilde{\mathcal{O}}$ we identify a smooth function

$$\rho : G_\delta(p, V', U') \to [0, \infty)$$

such that

$$\sqrt{\sigma(x,x')} = \sqrt{\rho(x')^2 + \sigma(x, x(q))},\, . \tag{4.11}$$

where $G_\delta(p, V', U')$ is the set defined in (4.5) while q is again the point where, for fixed p, the function σ attains its unique minimum. Moreover, in each $G_\delta(p, V', U')$ we change coordinates from (U', V', x_3') to (U', V', ρ). This change allows us to use (4.11) and to exploit the Taylor expansion (4.6), so that (4.10a) reduces to

$$\begin{aligned}\int_{\widetilde{\mathcal{O}}\times\widetilde{\mathcal{O}}} & \mathrm{d}\mu_g(x)\mathrm{d}\mu_g(x')\,\frac{\Delta(x,x')^{\frac{1}{2}}}{4\pi} \\ & \times \frac{\chi_\delta(x,x') f_\lambda(x) f'_\lambda(x')}{\sqrt{\rho^2 + \ell(x_3, x_3') - (U - U' - i\epsilon)(V - V' - i\epsilon) + R(x, V', U')}}\,,\end{aligned} \tag{4.12}$$

where $\ell(x_3, x_3')$ is the squared geodesic distance defined in Eq. (4.4). We observe now that the denominator of (4.12) is indeed the derivative with respect to ρ of

$$\begin{aligned}&\Xi(\rho, V, V', R) = \\ &\ln\Big(\big(\rho^2 + \ell(x_3(p), x_3(q)) - (U - U' - i\epsilon)(V - V' - i\epsilon) + R(x, V', U')\big)^{\frac{1}{2}} + \rho\Big)\,.\end{aligned}$$

In order to write the integral (4.12) in a simpler form, we rescale (V, V') to $(\lambda V, \lambda V')$. Labelling $\Delta_\lambda, R_\lambda, d\mu_{g_\lambda}$ the quantities transformed accordingly and considering the original hypothesis (4.8) according to which $f = \partial_V F$, $f' = \partial_{V'} F'$, we get

$$\begin{aligned}\int_{\widetilde{\mathcal{O}}_\lambda\times\widetilde{\mathcal{O}}_\lambda} & \mathrm{d}\mu_{g_\lambda}(x)\mathrm{d}\mu_{g_\lambda}(x')\frac{\Delta_\lambda(x,x')^{\frac{1}{2}}}{4\pi} \\ & \times \chi_\delta(x,x')\, \partial_V F(x)\, \partial_{V'} F'(x')\, \partial_\rho \Xi\left(\rho, \lambda V, \lambda V', R_\lambda\right).\end{aligned}$$

Since the domain of definition of the function ρ is $[0, \rho_0)$, we can integrate by parts in this variable. The result is that, of the ensuing boundary terms associated with the contribution $\partial_\rho \Xi$, the one due to ρ_0 vanishes because F' is compactly supported, while the one due to $\rho = 0$ yields, up to a rescaling of ϵ as $\lambda\epsilon$

$$-\int_{\widetilde{\mathcal{O}}_\lambda \times \widetilde{\mathcal{O}}_\lambda} \mathrm{d}\mu_{g_\lambda}(x) \mathrm{d}U'\mathrm{d}V' \sqrt{|\det g|}\Big|_{\rho=0} \frac{\Delta_\lambda(x,x')^{\frac{1}{2}}}{4\pi}$$
$$, \times \partial_V F(x)\, \partial_{V'} F'(x')|_{\rho=0} \left(\Xi\left(0, V, V', \frac{R_\lambda}{\lambda}\right) + \ln\lambda \right), \quad (4.13)$$

where we have now written the volume form as $\mathrm{d}\mu_g(x') = \sqrt{|\det g|}\mathrm{d}\rho\, \mathrm{d}U'\mathrm{d}V'$ and we have implicitly used that, when $\rho = 0$, $\ell(x_3, x_3') = 0$.

Our goal is now to take both the limits of (4.13) as λ and ϵ tend to 0^+. To this end is worth noticing that the diverging term $\ln\lambda$, gives no actual contribution to the integral. Again we stress that F is both smooth and compactly supported, therefore one can integrate by parts in V and the boundary terms vanish.

The remaining term in (4.13) can be evaluated as follows. Firstly, we notice that, because of the presence of the derivative of $\Delta_\lambda^{\frac{1}{2}}$, the remaining integral is proportional to $\lambda\ln\lambda$. Secondly, in view of Eq. (4.6), being R quadratic in V and V', there exists a constant $C \in \mathbb{R}$ such that $|\lambda^{-1}R_\lambda| < C\lambda$ in the limit $\lambda \to 0^+$. Thirdly, by direct inspection of (4.2) one can see that, when either x or x' tend to the horizon as $\lambda \to 0^+$, $\sqrt{|\det g_\lambda|} \to \frac{1}{2}$ while

$$\mathrm{d}\mu_g(x) = \frac{\mathrm{d}U\mathrm{d}V}{2}\mathrm{d}\mu(x_3) \quad \text{where} \quad \mathrm{d}\mu(x_3) = \sqrt{h(x_3)}\mathrm{d}x_3\ .$$

Finally we observe that from a distributional point of view, if $z = w + iy$, then

$$\lim_{y\to 0^+} \ln(z) = \ln|w| + i\pi(1 - \Theta(w))\ ,$$

where Θ is the Heaviside function.

As a result of the evaluations, the remaining contribution from (4.13) in the limit $\epsilon \to 0^+$, and also of (4.10) reads exactly as in the $3+1$ dimensional case [48]

$$-\frac{1}{32\pi}\lim_{\epsilon\to 0^+}\int_{\widetilde{\mathcal{O}}\times\widetilde{\mathcal{O}}} \partial_V F(x)\partial_{V'}F'(x')$$
$$\times \ln(-(U-U')(V-V'-i\epsilon))\mathrm{d}U'\mathrm{d}V'\mathrm{d}U\mathrm{d}V\mathrm{d}\mu(x_3)\,, \quad (4.14)$$

where we used that $\Delta_0 = 1$ when $x_3 = x_3'$ and $V = V' = 0$.

The new form 4.14 of Eq. (4.9) in the form 4.14 is particularly useful to investigate local behaviour of the two-point function in a sharpely localized neighbourhood of the horizon. More specifically, it is the proper tool to analyse the energy spectrum, as seen by an observer moving along the curves generated by ξ with respect to the associated Killing time[5] τ.

4.1.3 Thermal spectrum and tunnelling processes

In the previous section we have established the form (4.14) for the two-point correlation function of a real, massive scalar field in 2+1 dimensions. What we aim at now, is to compute the energy spectrum of (4.14) as seen by an observer moving along the integral curves of ξ. To this end, we are going to repeat the analysis of [48]. For the sake of clarity we summarize the most relevant point of the procedure. Let us consider two test functions which are squeezed on the Killing horizon as in (4.8). This is tantamount to focusing on the leading behaviour of (4.14) as V is close to 0. By direct inspection of (4.1), if τ is the affine parameter of the integral curves of ξ, then one can approximate the coordinate V as

$$V \approx -e^{-\kappa\tau} \text{ for V} < 0, \quad V \approx e^{-\kappa\tau} \text{ for V} > 0\ . \tag{4.15}$$

Therefore two cases are relevant when analysing $\omega_2(x, x')$: the first case consists in both points x, x' lying in the exterior region $\mathcal{O}_t$, as defined in (4.3), the second one consists of one point lying in $\mathcal{O}_t$ while the second lies in $\mathcal{O}_s$, the interior region.

Case 1) Let us consider two 1-parameter families of test functions f_λ and f'_λ obeying the constraints (4.8) and let us replace $\widetilde{\mathcal{O}}$ with $\mathcal{O}_t$. Eq. (4.14) can therefore be integrated by parts both in V and in V'. Since the support of both F and F' is compact, no boundary term gives contribution to the result. Eventually, replacing V with (4.15) in the limit of sharp localization near the horizon, we obtain

$$\lim_{\lambda\to 0} \omega_2(\Phi(f_\lambda)\Phi(f'_\lambda)) = \lim_{\epsilon\to 0^+} \frac{-\kappa^2}{128\pi} \int\limits_{\mathcal{R}^4\times\mathcal{B}} \mathrm{d}\tau \mathrm{d}U \mathrm{d}\tau' \mathrm{d}U' \mathrm{d}x_3\, \frac{F(\tau, U, x_3) F'(\tau', U', x_3)}{\left(\sinh\left(\frac{\kappa}{2}(\tau - \tau')\right) + i\epsilon\right)^2}\ ,$$

where we can extend[6] the domain of integration for the variables τ, τ', U, U' to the whole real axis, while $\mathcal{B}$ indicates a one-dimensional, connected, domain of integration, diffeomorphic to the bifurcation surface. The last expression can be rewritten in Fourier space with respect to τ and τ' as

$$\lim_{\lambda\to 0} \omega_2(\Phi(f_\lambda)\Phi(f'_\lambda)) = \frac{1}{64} \int\limits_{\mathbb{R}^2\times\mathcal{B}} \mathrm{d}U \mathrm{d}U' \mathrm{d}x_3 \left(\int\limits_{-\infty}^{\infty} \mathrm{d}E E\, \frac{\overline{\widehat{F}(E, U, x^3)} \widehat{F}'(E, U', x_3)}{1 - e^{-\beta_H E}} \right)\ ,$$

where $\beta_H = \frac{2\pi}{\kappa}$ while $\widehat{F}$ and $\widehat{F}'$ indicate the Fourier transform of F and F' respectively, where $f = \partial_V F$, $f' = \partial_{V'} F'$. A possible physical interpretation

[5] In the case of the BTZ spacetime, the Killing time associated to ξ has already been introduced in Chapter 2.

[6] This is again possible because the support of both F and F' is compact.

of this result is that, whenever a state for a real, massive Klein Gordon field is such that its two-point function is of Hadamard form in a geodesically convex neighbourhood of a point of a Killing horizon, then the mode expanded two-point correlation function, built with respect to the coordinate (4.15), obeys a thermal distribution at the *Hawking temperature* β_H^{-1} in the region external to the horizon.

Case 2) Once more, let us consider two 1-parameter families of test functions f_λ and f'_λ obeying (4.8), with $\widetilde{\mathcal{O}}$ replaced by $\mathcal{O}_t$ for f'_λ and by $\mathcal{O}_s$ for f_λ. Following the same calculations as in Case 1), the result is unchanged except for the hyperbolic sine being replaced by a hyperbolic cosine. Rewriting the result with respect to the variables τ and τ' in the Fourier space, returns

$$\lim_{\lambda\to 0}\omega_2(\Phi(f_\lambda)\Phi(f'_\lambda)) = \frac{1}{32}\int\limits_{\mathbb{R}^2\times\mathcal{B}} \mathrm{d}U\mathrm{d}U'\mathrm{d}x_3 \left(\int\limits_{-\infty}^{\infty} dEE\,\frac{\overline{\widehat{F}(E,U,x_3)}\widehat{F'}(E,U',x_3)}{\sinh(\beta_H E/2)}\right), \tag{4.16}$$

where $\beta_H = \frac{2\pi}{\kappa}$ and $\widehat{F}$ and $\widehat{F}'$ indicate the Fourier transform of F and F', respectively. This time, the support of the test functions and therefore the support of the observables is located once in the interior and once the exterior region with respect to the Killing horizon, it is possible to interpret the squared modulus

$$|\omega_2(f_\lambda, f'_\lambda)|^2$$

as a tunnelling probability through the horizon. If one considers wave packets peaked around $E_0 \gg 1$, it is worth noticing that Eq. (4.16) yields

$$\lim_{\lambda\to 0}|\omega_2(\Phi(f_\lambda), \Phi'(f'_\lambda))|^2 \approx E_0^2 e^{-\beta_H E_0} \quad ,$$

which is the original result provided by Parikh and Wilczek [47].

These results, therefore, suggest that the thermal behaviour of the two-point correlation function in the proximity of a Killing horizon is somehow connected to a non zero tunnelling probability through the horizon itself. One might wonder whether this phenomenon is related, in the specific case of a black hole spacetime equipped with a global Killing horizon, to the Hawking radiation, at least in the local formulation proposed by Parikh and Wilczek. As a matter of fact, in order for the underlying geometry to possess such property, one must consider only solutions to the Einstein field equation with negative cosmological constant, that is asymptotically AdS spacetimes. The prime example is therefore the BTZ spacetime, which, as we said, is not globally hyperbolic due to the presence of a timelike, conformal boundary (see Chapter 2).

4.2 Thermal effects near a BTZ black hole and Hawking radiation

Our goal now is to understand whether and under which circumstances the results of the previous section are applicable to the specific case of a BTZ spacetime. In particular we would like to find out if the leading behaviour of the ground state two-point function exhibits a thermal behaviour in a neighbourhood squeezed to the horizon and if it is possible to interpret it as a local version of the Hawking radiation.

The analysis performed in the previous section rests on two main hypotheses. The first one is the presence of a (local) Killing horizon $\mathcal{H}_+$, the second one being the existence of a state for a real, massive scalar field satisfying the Hadamard property in a geodesic neighbourhood of a point at $\mathcal{H}_+$.

While the first requirement seems to make black hole spacetimes particularly fit structures to be considered, the second one is directly connected to the underlying quantum theory and needs to be carefully addressed. The construction of Hadamard states, for example, is an established topic when the underlying manifold is globally hyperbolic [79]. Renown examples are the Unruh [77, 84] and the Hartle-Hawking states [85, 86] for the Klein-Gordon field in the four-dimensional Schwarzschild spacetime.

In $2 + 1$ dimensions, the situation is more convoluted since the arising BTZ black hole solution presented in Chapter 2 does possess a global Killing horizon but is not globally hyperbolic, as one can easily infer by the presence a timelike, conformal boundary.

Therefore it is not possible to invoke a general result and conclude that, for a real, massive scalar field, a state satisfying the Hadamard condition in a neighbourhood of the horizon exists. The existence of such a state is nonetheless granted by some specific conditions, as presented in the next section.

4.2.1 KMS state for a massive scalar field

We consider now a (rotating) BTZ black hole (2.8) and the real, massive scalar field $\Phi : \mathrm{BTZ} \to \mathbb{R}$ introduced in (3.2). Because the spacetime is not globally hyperbolic, solutions of the Klein-Gordon equation must be constructed not only by assigning initial data on a two-dimensional, smooth, spacelike hypersurface, but also imposing suitable boundary conditions as $r \to \infty$. We identified in Chapter 3 a large class of non-dynamical boundary conditions guaranteeing that the field energy flux vanishes at the conformal boundary is the one of Robin type introduced in Section 3.2.5. This result has been studied focusing on the stationary region $r > r_+$, that corresponds to the outer region coordinate patch introduced in Section 2.2.

Consequently, the space of classical solutions of the equation of motion has been used to construct, whenever possible and for each admissible Robin boundary condition, the two-point function of the associated ground state.

Notice that, even though BTZ is not globally hyperbolic, it is possible to construct the algebra of observables adopting the same approach used for the counterpart of a real, massive scalar field in the Poincaré patch of a $(d+1)-$dimensional AdS spacetime [73].

The ground state constructed in Chapter 3 corresponds to two separate ranges of the field parameter $\mu^2 := \frac{m^2}{\ell^2} - 6\xi$.

1. If $\mu^2 \geq 0$, the solutions of (3.2) possess only one admissible asymptotic behaviour at the conformal boundary and there is no need to impose any boundary condition at $z = 1$, corresponding to $r \to \infty$. In this case the two point function reads as in equation (3.47), which contains only positive $\tilde{\omega}$-frequencies. Therefore it is legitimate to call the state associated with it a ground state for the scalar field.

2. If $-1 < \mu^2 < 0$, there exists a one-parameter family of admissible boundary conditions which can be assigned at $z = 1$. These are conditions of Robin type as in (3.25). The class of Robin boundary conditions is ruled by a parameter $\zeta \in [0, \pi)$. By looking at the distribution of $\tilde{\omega}$-frequencies composing the two point function, it turns out that there exists a value $\zeta_* \in (0, \frac{\pi}{2})$ such that, whenever $\zeta \in [0, \zeta_*)$, the two-point function reads as in Eq. (3.51). This is the only case for which the two-point function is built out of positive $\tilde{\omega}$-frequencies and can be referred as a ground state for the scalar field. On the contrary, in the range $\zeta \in [\zeta_*, \pi)$, the spectrum exhibits non real bound state frequencies and this spoils the property of the two-point function (3.53) of being a ground state.

As previously pointed out (see Section 3.3), since (3.47) and (3.51) identify ground states, they are all of Hadamard form as proven in full generality in [28]. Our goal now is to build, for both (3.47) and (3.51), an associated thermal equilibrium state, satisfying the KMS condition. Firstly introduced by Kubo [45], Martin and Schwinger [87], as a class of boundary conditions for thermodinamic Green functions, the KMS condition has then been reproposed in the context of algebraic quantum field theory by Haag, Hugenholtz and Winnink [46] as a condition over test functions to identify quantum states at thermal equilibrium.

Following the standard procedure of [46] we reintroduce the function defined in (3.58) and (3.59)

$$\alpha_{\widetilde{t}} : C_0^\infty(\mathrm{BTZ}) \to C_0^\infty(\mathrm{BTZ})$$

such that, for all $f \in C_0^\infty(\mathrm{BTZ})$ and for all $\widetilde{t} \in \mathbb{R}$,

$$\alpha_{\widetilde{t}} f(x) = f(\widetilde{\alpha}_{-\widetilde{t}}(x)) \ ,$$

where $\widetilde{\alpha}_{-\widetilde{t}}(x)$ indicates the flow of a point $p \in \mathrm{BTZ}$ with coordinates x built out of the integral curves of the timelike Killing vector ξ of Eq. (2.11).

Therefore, we say that a two-point correlation function $\omega_{2,\beta} \in \mathcal{D}'(\mathrm{BTZ} \times \mathrm{BTZ})$ satisfies the KMS *condition* at inverse temperature $\beta > 0$ with respect to $\alpha_{\widetilde{t}}$ if, for every $f, f' \in C_0^\infty(\mathrm{BTZ})$, the following relation holds

$$\int_{\mathbb{R}} \mathrm{d}\widetilde{t}\, \omega_{2,\beta}(f, \alpha_{\widetilde{t}}(f'))e^{-i\widetilde{\omega}\widetilde{t}} = \int_{\mathbb{R}} \mathrm{d}\widetilde{t}\, \omega_{2,\beta}(\alpha_{\widetilde{t}}(f), f')e^{-i\widetilde{\omega}(\widetilde{t}+i\beta)}. \tag{4.17}$$

We already proved the existence of a ground state in the exterior region $\mathcal{O}_t^{BTZ}$ built out of the positive frequencies $\widetilde{\omega}$. This is given by (3.47) and (3.51), for $\mu^2 \geq 0$ and $-1 < \mu^2 < 0$, respectively. The construction of a two-point correlation function obeying (4.17) goes as follows.

If $\mu^2 \geq 0$, the integral kernel reads

$$\omega_{2,\beta}(x, x') = \lim_{\epsilon \to 0^+} \sum_{k \in \mathbb{Z}} e^{ik(\widetilde{\phi}-\widetilde{\phi}')} \int_0^\infty \frac{\mathrm{d}\widetilde{\omega}}{(2\pi)^2} \left(\frac{A}{B} - \frac{\overline{A}}{\overline{B}}\right) C$$
$$\left[e^{-i\widetilde{\omega}(\widetilde{t}-\widetilde{t}'-i\epsilon)} \frac{e^{\beta\widetilde{\omega}}}{e^{\beta\widetilde{\omega}}-1} + e^{i\widetilde{\omega}(\widetilde{t}-\widetilde{t}'+i\epsilon)} \frac{e^{-\beta\widetilde{\omega}}}{1-e^{-\beta\widetilde{\omega}}}\right] \Psi_1(z)\Psi_1(z'), \,. \tag{4.18}$$

while, if $-1 < \mu^2 < 0$ and if $\zeta \in [0, \zeta_*]$,

$$\omega_{2,\beta}^{\zeta}(x, x') = \lim_{\epsilon \to 0^+} \sum_{k \in \mathbb{Z}} e^{ik(\widetilde{\phi}-\widetilde{\phi}')} \int_0^\infty \frac{\mathrm{d}\widetilde{\omega}}{(2\pi)^2} \frac{\left(A\overline{B} - \overline{A}B\right) C}{|\cos(\zeta)B - \sin(\zeta)A|^2}$$
$$\left[e^{-i\widetilde{\omega}(\widetilde{t}-\widetilde{t}'-i\epsilon)} \frac{e^{\beta\widetilde{\omega}}}{e^{\beta\widetilde{\omega}}-1} + e^{i\widetilde{\omega}(\widetilde{t}-\widetilde{t}'+i\epsilon)} \frac{e^{-\beta\widetilde{\omega}}}{1-e^{-\beta\widetilde{\omega}}}\right] \Psi_\zeta(z)\Psi_\zeta(z'). \tag{4.19}$$

It is worth pointing out that, for all $\beta > 0$, both (4.18) and (4.19) identify two-point correlation functions satisfying the Hadamard condition. This can be inferred by a straightforward application of the results of [28]. In alternative, one could observe by direct inspection that the difference between $\omega_{2,\beta}$ and (3.47), as well as the difference between $\omega_{2,\beta}^{\zeta}$ and (3.51), are smooth functions.

Lastly, we wonder whether there exists a specific value of the inverse temperature β for which both (4.18) and (4.19) are the restriction to the external region of the BTZ black hole of the two-point correlation function of a state, which is Hadamard also in a neighbourhood of the outer horizon.

A solution to this problem has been already pointed out by the construction of the BTZ spacetime as described in [49, Ch. 12]. First of all let us remember that BTZ is realized form CAdS_3 (see Chapter 2), the universal cover of AdS_3 via the identifications presented in Section 2.2. As a consequence, starting from the integral kernel of any two-point function in CAdS_3, one can build a counterpart on BTZ by means of the method of images, which implements the periodic identification built in the relevant coordinate patches. Notice that both AdS_3 and its universal cover CAdS_3 are maximally symmetric solutions of the Einstein equations with a negative cosmological constant. In such context, it has been shown [88] that one can construct the two-point correlation function associated to the ground state of the Klein-Gordon field coupled to the scalar curvature, for a large class of Robin boundary condition and, most notably, all these two-point functions are locally of Hadamard form thanks to the analysis in [28]. Since the two-point function of the ground state is also maximally symmetric, independently from the specific Robin boundary condition, its integral kernel depends on the spacetime points x, x' only via the AdS_3 geodesic distance $\sigma_{AdS_3}(x, x')$ [89]. Moreover, one sees[7] that the resulting two-point function is periodic with respect to the time variable $\widetilde{t}$ under the shift $i\beta_H$, where the constant

$$\beta_H = \frac{2\pi r_+}{r_+^2 - r_-^2}$$

is proportional to the inverse Hawking temperature

$$T_H = \frac{\kappa}{2\pi} = \frac{r_+^2 - r_-^2}{r_+}$$

of the BTZ black hole[8]. The periodicity is obtained solely as a consequence of the explicit form of $\sigma_{AdS_3}(x, x')$ and thus it can be applied also to any two-point correlation function depending on the spacetime points only via the geodesic distance[9]. As a consequence, the restrictions of these two-point correlation functions to the external region $z \in (0, 1)$ (or equivalently $r > r_+$) enjoy the KMS property at the Hawking temperature and coincide with (4.18) and with (4.19).

[7]In [49] this procedure is displayed for the case of a ground state for a massless, conformally coupled real scalar field, with either Dirichlet or Neumann boundary conditions. The states constructed with this procedure are of (local) Hadamard form and so are the ensuing counterparts defined on the BTZ spacetime.

[8]Notice that this temperature is closely analogous to the more realistic case of a $3+1$ dimensional black hole.

[9]This is for example the case for the ground states shown in [88].

Chapter 5

Conclusions

In this thesis we have studied the behaviour of a real massive scalar field in the exterior region of a rotating 2+1 dimensional BTZ black hole. The analysis has been performed first by studying the classical solutions to the Klein-Gordon equation and by explicitly constructing the two-point functions associated to the ground states. In doing so, we have also tested some some tools that might be useful in other physical scenarios such as Robin boundary conditions for a singular Sturm-Liouville problem (Chapter 3, Section 3.2.1 and Appendix A) and the resolution of the identity for quadratic operator pencils (Chapter 3, Section 3.3 and Appendix C). The main results of this thesis have been presented in Chapters 3 and 4. Here we have shown under which conditions it is possible to construct a full-fledged ground state for the scalar field and we have also verified that any global Hadamard state on BTZ exhibits a high-energy thermal behaviour precisely at the Hawking temperature.

In detail, we presented the following results:

- we analysed a real massive scalar field in the exterior region of a rotating BTZ spacetime by applying for the first time in literature Robin boundary conditions, which guarantee that the spacetime can be regarded as an isolated system. These are boundary conditions generalising the most notorious Dirichlet and Neumann boundary conditions. In particular, Robin boundary conditions are a linear combination of Dirichlet and Neumann. The field equation has then been solved for any possible linear combination. We expressed all the possible linear combinations in terms of a parameter $\zeta \in [0, \pi)$;

- we constructed the two-point correlation function associated to the field equation, by means of a mode-expansion with respect to the spacetime coordinates. In particular, this result is obtained first by writing the spectral representation of the radial identity operator and then by facing the difficulty of dealing with a quadratic eigenvalue problem. Due to the rotation of the underlying spacetime, the radial mode equation has in fact both a linear and a quadratic dependence on the spectral parameter;

- the physical meaning of the constructed two-point correlation functions has been discussed. We found that, for values of the field parameter $\mu^2 \in [0, \infty)$, there is no need to imposing any boundary condition at radial infinity and the two-point function built out of positive $\tilde{\omega}$-frequencies represents indeed a legitimate ground-state satisfying the Hadamard condition in the external region of the black hole.
  When $\mu^2 \in (-1, 0)$, on the contrary, Robin boundary boundary conditions apply and it is possible to identify two different regimes: for $\zeta \in [0, \zeta_c)$, $\zeta_c \in (\frac{\pi}{2}, \pi)$, the two-point function again represent a full fledged ground state, while this is not true for the class of boundary conditions $\zeta \in [\zeta_c, \pi)$. These are indeed the first examples of a ground state for a quantum field theory in the exterior region of a rotating black hole;

- in the case $\zeta \in [\zeta_c, \pi)$ the two-point function fails to represent a ground state because of the presence of bound state mode solutions. These are mode solutions with non-real frequency $\tilde{\omega}$ and they represent decaying modes at radial infinity. The presence of bound states was an unexpected feature and they seem to appear as a result of the rotation of the black hole, since they are not present in the static scenario;

- in the last part of the thesis we analysed the high energy behaviour of a physical state in the proximity of the event horizon. The analysis has been performed by generalising to $2+1$ dimension the limiting procedure proposed by Moretti and Pinamonti [48] and we found that, as for the $3+1$ dimensional case, the state exhibits a thermal spectrum at the Hawking temperature and that this thermal behaviour seems to be related to some tunnelling processes through the horizon, in the sense of Parikh and Wilczeck [47];

- subsequently we showed that this result is indeed applicable to the case of a global KMS state for the scalar field in a rotating BTZ spacetime and, by retrieving the results of the previous chapters, we wrote its explicit form in the external region of the spacetime. This result was obtained by applying a general argument [28] to prove the existence of such a KMS state and again it is valid for a large class of admissible Robin boundary conditions.

Despite the encouraging results obtained in this work, many issues remain unsolved for the rotating case. One on the main open problems is the computation of physical observables and, first of all, the introduction of a legitimate renormalization procedure for $\langle \Phi^2 \rangle$, or the expectation value of the stress-energy tensor $\langle T_{\mu\nu} \rangle$. These results might be obtained by extending the work of [90] and it could be used in the analysis of the semiclassical Einstein field equations, leading to the investigation of other physical phenomena, such as the backreaction on the event horizon.

Appendix A

Appendix A

A.1 Sturm-Liouville theory

In Chapter 3, the Klein-Gordon equation 3.2 is reduced to a linear second order ODE via a Fourier mode expansion. As a result one obtains Eq. (3.5), defined in the interval $z \in (0,1)$. This type of equation can be dealt with the Sturm-Liouville theory [62, 64, 63] for formally self-adjoint second order differential equations. A second order ODE is called of Sturm-Liouville type if it reads

$$-\frac{d}{dz}\left(p(z)\frac{du}{dz}\right) + q(z)u - \lambda s(z)u = 0, \quad \text{for } a \leq z \leq b \ ,$$

where p, q and s are real valued, p, p' and q are continuous and p and s are positive functions. Such a problem is called *regular* Sturm Liouville problem. Here, the factor λ can be interpreted as an eigenvalue parameter, that is the equation

$$Lu - \lambda u = 0 \tag{A.1}$$

with

$$Lu := \frac{1}{s}[-(pu')' + qu] \tag{A.2}$$

can be treated as an eigenvalue problem[1] and can be solved in terms of eigenfunctions. The operator A.2 is formally self-adjoint with respect to the inner product

$$\langle u, v\rangle := \int_a^b s(z)\ \bar{u}(z)v(z)\ \mathrm{d}z \ ,$$

where the overline symbol denotes the complex conjugate of the eigenfunction u. The space of solutions can therefore be equipped with the norm

$$||u|| := \sqrt{\langle u, u\rangle} = \left(\int_a^b s(z)|u(z)|^2 \mathrm{d}z\right).$$

[1]Notice that Eq. 3.5 cannot be treated as a linear eigenvalue problem, since it has a quadratic dependence on the parameter ω. This difference is deeply discussed in Section 3.3.3 and in Appendix C. Despite this complication in the spectral resolution, any other result relative to the Sturm-Liouville theory is applicable to our case.

The inner product, therefore, provides the space of solutions with a Hilbert space structure. A more definite notation is needed when dealing with the endpoints of the interval. As already state in Chapter 3, endpoints are classified in the following way: we call the endpoint b (respectively a) *limit circle* if, for some $\lambda \in \mathbb{C}$, all solutions lie in $L^2((z_1, b); s(z)\mathrm{d}z)$ (respectively $L^2((a, z_1); s(z)\mathrm{d}z)$) for some $z_1 \in (a, b)$; otherwise, we call it *limit point*. Moreover, if the properties about the functions p, p', q and s hold only in the open interval (a, b), then the problem is called *singular* Sturm-Liouville problem. Singular and regular Sturm-Liouville problems must be addressed with different approaches, in particular when dealing with the choice of the boundary conditions. In the following we will show how to select the principal solution at the endpoint and how to impose boundary conditions of Robin type for a regular problem and for a singular problem in which one of the endpoints, say b, is limit circle.

A.2 Principal and non-principal solutions

A solution u of (A.1) is called the *principal solution* at one endpoint, say b, if

- $u(z) \neq 0$ for $z \in [d, b)$, with $d \in (a, b)$,
- $\lim_{z \to 1} u(z)/\Psi(z) = 0$ for every solution $\Psi \neq 0$ that is not a scalar multiple of u.

By definition, if u is a principal solution at b, then any non zero real multiple of u is also a principal solution. Therefore, if u_1 and u_2 are two linearly independent principal solutions, then $\frac{u_1}{u_2}\frac{u_2}{u_1} = 1$ in $z \in [d, b)$ for some $d \in (a, b)$. Taking the limit for $z \to b$ one gets that, if the principal solution exists, this is unique up to real multiplicative constant factors.

A solution v of (A.1) is called *non-principal solution* at one endpoint, say b, if

- $v(z) \neq 0$ for $z \in [d, b)$, with $d \in (a, b)$,
- v is not a principal solution at b.

Non-principal solutions are not unique since, if u is a principal solution and v is a non-principal solution, then $u + cv$ is a non principal solution for any real constant $c \neq 0$.

A.3 Robin boundary conditions for a regular problem

Let us consider a second-order regular Sturm-Liouville problem A.1 defined in a closed interval $[a, b]$ and two eigenfunctions u and v, such that one of them is square-integrable at both endpoints a and b, while the other is square-integrable only in a neighbourhood of b, but not of a.

Accordingly to the definition, the point a is limit point, while b is limit circle and boundary conditions are required at b. Let us assume that we want to identify a general solution by imposing homogeneous boundary conditions of Robin type. This is tantamount to asking for a general solution

$$f = \cos(\zeta)u(z) + \sin(\zeta)v(z) \quad \zeta \in [0, \pi)$$

to satisfy

$$B_b f = \cos(\zeta)f(b) + \sin(\zeta)\partial_z f(b) = 0 \quad \zeta \in [0, \pi) \ . \tag{A.3}$$

The parameter ζ spans all possible linear combinations and selects all possible ratios $\frac{\dot{f}(b)}{f(b)}$. Notice that the notable cases $\zeta = 0$ and $\zeta = \frac{\pi}{2}$ correspond to the Dirichlet and Neumann boundary conditions, respectively.

Given two solutions f, g, the following property holds true

$$gLf - fLg = \frac{1}{s(z)}[p(z)(f(z)\dot{g}(z) - \dot{f}(z)g(z))] = \frac{1}{s(z)}\partial_z[p(z)\mathcal{W}_z(f, g, z)]$$

where $\mathcal{W}_z(f, g, z) := f\partial_z g - \partial_z f g$ is the Wronskian of f and g.

A.4 Robin boundary conditions for a singular problem

If we are dealing with a singular problem, defined in the open interval (a, b), where one of the endpoints, say b is limit circle and therefore boundary conditions apply. Unfortunately the condition $B_b f = 0$ introduced in (A.3) is no longer valid, since one of the two solutions, say v, is expected to diverge because of the non regularity of the endpoint. A natural way to implement a Robin boundary condition for a singular limit circle endpoint b is to rewrite (A.3) as

$$B_b f = \lim_{z \to b} \cos(\zeta)\mathcal{W}_z[u, f](z) + \sin(\zeta)\mathcal{W}_z[v, f](z) = 0 \quad \zeta \in [0, \pi) \ , \tag{A.4}$$

which is also valid when the endpoint b is singular, since the Wronskians in (A.4) are non vanishing and independent from z. In particular, if one selects u to be the principal solution, as defined in Section 3.2.2, then v prescribes automatically the Dirichlet boundary condition, while u might be interpreted as a generalization of the Neumann boundary condition, even though it is not unique.

Notice that in the case of a regular eigenvalue problem with Robin boundary conditions, if $u(b) = \partial_z v(b) = 0$, $\partial_z u = c$ and $v(b) = -c$, Eq. (A.4) provides a natural generalization of Eq. (A.3):

$$
\begin{aligned}
0 = B_b f =& \lim_{z\to b} \sin(\zeta)\mathcal{W}_z[v,f](z) + \cos(\zeta)\mathcal{W}_z[u,f](z) = 0 \\
=& \lim_{z\to b} \sin(\zeta)[v(z)\dot{f}(z) - \dot{v}(z)f(z)] + \cos(\zeta)[u(z)\dot{f}(z) - \dot{u}(z)f(z)] = 0 \\
=& \lim_{z\to b} [\sin(\zeta)v(z) + \cos(\zeta)u(z)]\dot{f}(z) - [\sin(\zeta)\dot{v}(z) + \cos(\zeta)\dot{u}(z)]f(z) = 0 \\
=& - c\cos(\zeta)f(b) - c\sin(\zeta)\dot{f}(b) \\
=& - c[\cos(\zeta)f(b) + \sin(\zeta)\dot{f}(b)] \quad \zeta \in [0,\pi) \; .
\end{aligned}
$$

A.5 Green's functions and eigenfunctions

Let us consider a Green function $g_\lambda(z,z')$ satisfying the equation

$$-\partial_z(p(z)\partial_z g_\lambda(z,z')) + q(z)g_\lambda(z,z') - \lambda s(z)g_\lambda(z,z') = \delta(z-z') \quad B_b = 0$$

for $z, z' \in (a,b)$. The Green function can be Fourier expanded in eigenfunctions $f_n(z)$ of Eq. (A.1) as

$$g_\lambda(z,z') = \sum_n g_n(z',\lambda)f_n(z), \quad g_n = \langle g_\lambda, f_n\rangle = \int_a^b s(z)g_\lambda(z,z')f_n(z)\mathrm{d}z \; .$$

or as

$$g_\lambda(z,z') = \sum_n \frac{f_n(z)\bar{f}_n(z')}{\lambda - \lambda_n} \; .$$

Such an expansion can be constructed if the sets of eigenvalues $\{\lambda_n\}$ and the corresponding eigenfunctions $\{f_n\}$ are known. The previous series shows that $g_\lambda(z,z')$, as a function of the complex parameter λ has singularities at $\lambda = \lambda_n$. The identity operator can then be expressed as

$$\frac{\delta(z,z')}{s(z)} = \frac{1}{2\pi i}\int_{C_\infty} g_\lambda(z,z')\mathrm{d}\lambda = -\sum_n f_n(z)\bar{f}_n(z') \; ,$$

where the circle C_∞ positively oriented surrounds all the complex λ plane. Whenever the spectrum is only discrete, this formula takes into account all simple poles in the complex plane, and by the residues formula one gets

$$\frac{1}{2\pi i}\int_{C_\infty} g_\lambda(z,z')\mathrm{d}\lambda = -\sum_n f_n(z)\bar{f}_n(z') \; .$$

When the spectrum is at least partly continuous, a branch cut for $g_\lambda(z,z')$ must be taken into account and the integral around the large circle C_∞ can be divided in two parts: one relative to the contribution coming from the single poles the other given by a branch-cut integral over the continuous part of the spectrum. Hence one obtains

$$\frac{1}{2\pi i}\int_{C_\infty} g_\lambda(z,z')\mathrm{d}\lambda = -\sum_n f_n(z)\bar{f}_n(z') - \int_{b.cut} f_\nu(z)\bar{f}_\nu(z')\mathrm{d}\nu \; .$$

The construction of $g_\lambda(z,z')$ proceed by considering a product of square-integrable solutions of the Sturm-Liouville problem at both the endpoints $z = a$ and $z = b$, so to ensure the continuity at $z = z'$. Let $u(z)$ be square integrable at $z = a$, $v(z)$ square-integrable at $z = b$ and satisfying the boundary conditions $B_b v = 0$. Then

$$g_\lambda(z,z') = \mathcal{N} u_\lambda(z_<) v_\lambda(z_>) \ ,$$

where $z_< = \min(z,z')$ and $z_> = \max(z,z')$. The jump condition on $g_\lambda(z,z')$ is

$$\frac{\mathrm{d}g_\lambda}{\mathrm{d}z}_{|z=z'^+} - \frac{\mathrm{d}g_\lambda}{\mathrm{d}z}_{|z=z'^-} = -\frac{1}{p(z')} \ ,$$

or equivalently

$$\mathcal{N}\mathcal{W}_z(u,v) = -\frac{1}{p(z')} \ .$$

Finally one gets that

$$g_\lambda(z,z') = -\frac{u_\lambda(z_<)v_\lambda(z_>)}{p(z')\mathcal{W}_z(u,v)} \ .$$

Appendix B

Appendix B

B.1 Hypergeometric functions and hypergeometric equation

In Chapter 3, the radial Sturm-Liouville equation (3.5) deriving from the Klein-Gordon equation of motion is transformed, by means of the ansatz (3.6), into a Gaussian hypergeometric equation [67] of the form

$$z(1-z)\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}+[c-(a+b+1)z]\frac{\mathrm{d}w}{\mathrm{d}z}-abw(z)=0 \quad z\in(0,1),\ a,b,c\in\mathbb{R}\ . \quad \text{(B.1)}$$

The hypergeometric differential equation (B.1) has regular singualrities at $z = 0, 1$. Its closed form solutions are provided by Gaussian hypergeometric functions, depending on the three parameters a, b and c.

Gaussian hypergeometric functions are special functions defined via a Gauss series of other special functions via the general formula

$$F(a,b,c;z)=\sum_{s=0}^{\infty}\frac{(a)_s(b)_s}{(c)_s s!}z^s=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{s=0}^{\infty}\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)s!}z^s \quad |z|<1\ ,$$

where $(x)_s := x(x+1)\dots(x+s-1)$ indicates the rising factorial (also known as Pochhammer symbol) while $\Gamma(x)$ is the Gamma function. Generally speaking $F(a,b,c;z)$ does not exist when $c = 0, -1, -2, \dots$ and its definition can be extended elswhere by analytic continuation. A principal branch can be introduced by cutting the z plane from 1 to $+\infty$ on the real axis, which lies in the sector $|\operatorname{ph}(1-z)| \leq \pi$. Moreover, in the disk of radius $|z| < 1$, the Gauss series converges absolutely whenever $\mathrm{Re}\{c-a-b\} > 0$, converges conditionally for $-1 < \mathrm{Re}\{c-a-b\} \leq 0$ and $z \neq 1$, diverges for $\mathrm{Re}\{c-a-b\} \leq -1$. The case $z = 1$ takes the follwing form. If $\mathrm{Re}\{c-a-b\} > 0$, then

$$F(a,b,c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ .$$

If $c-a-b = 0$, then

$$\lim_{z\to 1^-}\frac{F(a,b,a+b;z)}{-\ln(1-z)}=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\ .$$

If $\mathrm{Re}\{c-a-b\}=0$ and $c-a-b\neq 0$, then

$$\lim_{z\to 1^-}(1-z)^{a+b-c}\left(F(a,b,c;z)-\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right)=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\ .$$

If $\mathrm{Re}(c-a-b)<0$, then

$$\lim_{z\to 1^-}\frac{F(a,b,c;z)}{(1-z)^{c-a-b)}}=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\ .$$

Finally we stress that $F(a,b,c;z)$ is symmetric under exchange of a and b and we denote

$$\mathbf{F}(a,b,c;z):=\frac{F(a,b,c;z)}{\Gamma(c)}\ .$$

B.2 Fundamental solutions

When none of c, $c-a-b$, $a-b$ is an integer, Eq. (B.1) has the following pairs of fundamental solutions. Each of these pairs is numerically satisfactory at the corresponding endpoint, meaning that one solution is recessive, while the other exhibits a dominant behaviour in the limit.

Singularity at $z=0$

$$\begin{aligned} f_1(z) &= F(a,b,c;z)\ ,\\ f_2(z) &= z^{1-c}F(a-c+1,b-c+1,2-c;z). \end{aligned}$$

Singularity at $z=1$

$$\begin{aligned} f_3(z) &= F(a,b,a+b-c+1;1-z)\ ,\\ f_4(z) &= (1-z)^{c-a-b}F(c-a,c-b,c-a-b+1;1-z)\ . \end{aligned}$$

For the the case of interest it is worth noticing that if $a+b+1-c$ is a positive integer n, then the fundamental solutions at $z=1$ are

Singularity at $z=1$

$$\begin{aligned} f_5(z) =& F(a,b,n;1-z)\ ,\\ f_6(z) =& F(a,b,n;1-z)\ln(1-z)-\sum_{p=1}^{n-1}\frac{(n-1)!(p-1)!}{(n-p-1)!(1-a)_p(1-b)_p}(z-1)^{-p}\\ &+\sum_{p=0}^{\infty}\frac{(a)_p(b)_p}{(n)_p p!}(1-z)^p(\psi(a+p)+\psi(b+p)-\psi(1-p)-\psi(n+p))\ . \end{aligned}$$

Other solutions

The pairs of fundamental solutions can be transformed into other solutions, dubbed Kummer solutions [67], via the linear transformation

$$\begin{aligned}\mathbf{F}\left(a,b,c;z\right) &= (1-z)^{-a}\mathbf{F}\left(a,c-b,c;\frac{z}{z-1}\right)\\ &= (1-z)^{-b}\mathbf{F}\left(c-a,b,c;\frac{z}{z-1}\right)\\ &= (1-z)^{c-a-b}\mathbf{F}\left(c-a,c-b,c;z\right), \quad \text{for } |\arg(1-z)|<\pi\ .\end{aligned}$$

In particular it is worth noticing that (see [67, Eq. 15.10.13])

$$F\left(a,b,a+b-c+1;1-z\right) = z^{1-c}F\left(a-c+1,b-c+1,a+b-c+1;1-z\right) \tag{B.2}$$

and, if Ψ_1, Ψ_2, Ψ_3 and Ψ_4 are the solutions listed in 3 for the radial Sturm-Liouville equation (3.5), then the following relations hold

$$\begin{aligned}\Psi_3(z) = &\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\Psi_1(z)\\ &+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\Psi_2(z)\,,\end{aligned} \tag{B.3a}$$

$$\begin{aligned}\Psi_4(z) = &\frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}\Psi_1(z)\\ &+\frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}\Psi_2(z)\,.\end{aligned} \tag{B.3b}$$

B.3 Useful relations

With respect to the Gaussian hypergeometric equation (3.9), the parameters (3.10), with α and β defined as (3.8), satisfy the following relations

$$a = \alpha+\beta+\gamma, \tag{B.4}$$

$$b = \alpha+\beta-\gamma, \tag{B.5}$$

$$c = 1+2\alpha, \tag{B.6}$$

$$\gamma = \frac{i}{2}\frac{r_-\omega - r_+\kappa}{r_-^2-r_+^2}\ . \tag{B.7}$$

We also note that under the substitution $\tilde{\omega} \mapsto \overline{\tilde{\omega}}$, these parameters behave as

$$\begin{aligned} a &\mapsto \overline{b-c+1}, \quad \alpha\mapsto -\overline{\alpha},\\ b &\mapsto \overline{a-c+1}, \quad \beta\mapsto\beta,\\ c &\mapsto \overline{2-c}.\end{aligned} \tag{B.8}$$

Appendix C

C.1 Resolution of the identity for a quadratic eigenvalue problem

In this appendix we build an expansion of the delta distribution for a quadratic operator pencil [91, 92] defined on a Hilbert space H as in Eq. (3.33). Namely, let us consider a family of operators defined on a Hilbert space H in the form

$$S_{\tilde{\omega}} = P + \tilde{\omega}\mathcal{R}_1 + \tilde{\omega}^2\mathcal{R}_2 \,, \tag{C.1}$$

where (S1) $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_2^{-1}$ are bounded and self-adjoint, while P is unbounded, closed and hermitian on a dense domain $D(S_{\tilde{\omega}}) \subset H$. These are differential operators with quadratic dependence on the spectral parameter. In particular we are concerned with obtaining the spectral resolution of the identity operator for the case of unbounded operators [93, 94] coming from Sturm-Liouville ODEs as in (3.5), where $S_{\tilde{\omega}} = \mathcal{J}^{-1}L_{\tilde{\omega}}$ on $H = L^2((0,1); \mathcal{J}(z)\ \mathrm{d}z)$ and $\mathcal{J}(z)$ is as in in (3.19). We start by defining the *resolvent set* of $S_{\tilde{\omega}}$, dubbed $\rho(S_{\tilde{\omega}}) \subset \mathbb{C}$ as the set of all values of $\tilde{\omega} \in \mathbb{C}$ such that $T_{\tilde{\omega}} = S_{\tilde{\omega}}^{-1}$ exists and is a bounded operator. Our first goal is to show that when (S2) the *spectrum* $\sigma(S_{\tilde{\omega}}) = \mathbb{C} \setminus \rho(S_{\tilde{\omega}})$ consists only of a subset of $\mathbb{R}$ together with a finite number of isolated points in $\mathbb{C} \setminus \mathbb{R}$ symmetric with respect to complex conjugation, the identity operator can be represented by the integral

$$\begin{aligned} \mathbb{I} = \lim_{\varsigma\to\infty} \int_{-\varsigma}^{\varsigma} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \lim_{\epsilon\to 0^+} \tilde{\omega}(T_{\tilde{\omega}-i\epsilon} - T_{\tilde{\omega}+i\epsilon})\mathcal{R}_2 \\ + \oint_{\mathring{C}} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \tilde{\omega} T_{\tilde{\omega}} \,. \end{aligned} \tag{C.2}$$

The contour $\mathring{C}$ is illustrated in Figure C.1. It is positively oriented and it surrounds the non-real part of the spectrum. The inner $\epsilon \to 0^+$ limit is taken in the sense of distributions in $\tilde{\omega}$ (boundary values of holomorphic functions define a special kind of distribution [95, Ch.IX]) and the outer $\varsigma \to \infty$ limit is taken in the sense of the strong operator topology.

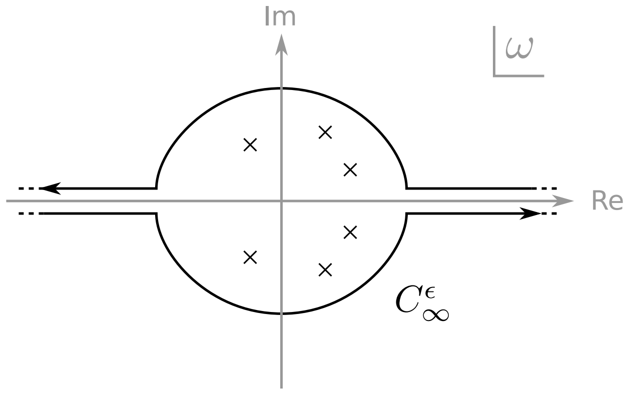

Figure C.1: Contour for the integral representation of the identity operator in (C.2).

Following a standard approach, we *linearize* the quadratic operator pencil to a linear operator pencil $\mathbf{S}_{\tilde{\omega}}$, and in the process we double the size of the Hilbert space, so that the spectral problems of $S_{\tilde{\omega}}$ and $\mathbf{S}_{\tilde{\omega}}$ remain equivalent.

Therefore, we consider the following linear operator pencil defined on $H^2 = H \oplus H$,

$$\mathbf{S}_{\tilde{\omega}} = \mathbf{P} + \tilde{\omega}\mathbf{R} = \begin{bmatrix} P & 0 \\ 0 & -\mathcal{R}_2 \end{bmatrix} + \tilde{\omega} \begin{bmatrix} \mathcal{R}_1 & \mathcal{R}_2 \\ \mathcal{R}_2 & 0 \end{bmatrix}, \tag{C.3}$$

which is related to $S_{\tilde{\omega}}$ by the identities

$$\mathbf{S}_{\tilde{\omega}} \begin{bmatrix} \mathbb{I} \\ \tilde{\omega} \end{bmatrix} \Psi = \begin{bmatrix} \mathbb{I} \\ 0 \end{bmatrix} S_{\tilde{\omega}} \Psi \,, \tag{C.4}$$

$$S_{\tilde{\omega}} \begin{bmatrix} \mathbb{I} & 0 \end{bmatrix} \begin{bmatrix} \Psi \\ \Phi \end{bmatrix} = \begin{bmatrix} \mathbb{I} & \tilde{\omega} \end{bmatrix} \mathbf{S}_{\tilde{\omega}} \begin{bmatrix} \Psi \\ \Phi \end{bmatrix}. \tag{C.5}$$

Among all possibilities, this linearization preserves the following self-adjointness properties. When $\mathcal{R}_1$ and $\mathcal{R}_2$ are bounded and self-adjoint, so is $\mathbf{R}$, and when in addition P is closed and hermitian on $D(S_{\tilde{\omega}})$, so is $\mathbf{P}$ on $D(\mathbf{S}_{\tilde{\omega}}) = D(S_{\tilde{\omega}}) \oplus H$. We define, when it exists, the resolvent $\mathbf{T}_{\tilde{\omega}} = \mathbf{S}_{\tilde{\omega}}^{-1}$ and the ensuing spectrum $\sigma(\mathbf{S}_{\tilde{\omega}})$ and resolvent set $\rho(\mathbf{S}_{\tilde{\omega}}) \subset \mathbb{C}$ are defined in the usual way, essentially exactly as above. The resolvents of $S_{\tilde{\omega}}$ and $\mathbf{S}_{\tilde{\omega}}$ are related to each other by

$$\begin{aligned} \mathbf{T}_{\tilde{\omega}} &= \begin{bmatrix} \mathbb{I} \\ \tilde{\omega} \end{bmatrix} T_{\tilde{\omega}} \begin{bmatrix} \mathbb{I} & \tilde{\omega} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\mathcal{R}_2^{-1} \end{bmatrix} \\ &= \begin{bmatrix} T_{\tilde{\omega}} & \tilde{\omega} T_{\tilde{\omega}} \\ \tilde{\omega} T_{\tilde{\omega}} & \tilde{\omega}^2 T_{\tilde{\omega}} - \mathcal{R}_2^{-1} \end{bmatrix}, \end{aligned} \tag{C.6}$$

$$T_{\tilde{\omega}} = \begin{bmatrix} \mathbb{I} & 0 \end{bmatrix} \mathbf{T}_{\tilde{\omega}} \begin{bmatrix} \mathbb{I} \\ 0 \end{bmatrix} = \frac{1}{\tilde{\omega}} \begin{bmatrix} \mathbb{I} & 0 \end{bmatrix} \mathbf{T}_{\tilde{\omega}} \begin{bmatrix} 0 \\ \mathbb{I} \end{bmatrix}. \tag{C.7}$$

These formulas show that $T_{\tilde{\omega}}$ exists and is bounded if and only if this is true for $\mathbf{T}_{\tilde{\omega}}$. Therefore, $\rho(S_{\tilde{\omega}}) = \rho(\mathbf{S}_{\tilde{\omega}})$ and, consequently, $\sigma(S_{\tilde{\omega}}) = \sigma(\mathbf{S}_{\tilde{\omega}})$, and the two spectral problems are equivalent. Once $\rho(S_{\tilde{\omega}})$ is found, we can use the boundedness[1] of $\mathbf{R}$ to show that $\mathbf{T}_{\tilde{\omega}}$ is analytic on $\rho(S_{\tilde{\omega}}) = \rho(\mathbf{S}_{\tilde{\omega}})$. This implies that $T_{\tilde{\omega}}$ is also analytic on $\rho(S_{\tilde{\omega}})$.

Let $\nu \in \rho(\mathbf{S}_{\tilde{\omega}}) = \rho(S_{\tilde{\omega}})$ and let $C_\nu \subset \rho(\mathbf{S}_{\tilde{\omega}}) = \rho(S_{\tilde{\omega}})$ be a contour encircling ν in the negative direction. Upon deformation, C_ν is positively oriented and it is a simple curve which surrounds $\sigma(S_{\tilde{\omega}})$. When $\sigma(S_{\tilde{\omega}})$ is unbounded, the deformation of the contour must go through a limiting procedure, by taking a connected component of C_ν in each connected component of $\rho(S_{\tilde{\omega}})$. If the resolvent $\mathbf{G}_{\tilde{\omega}}$ is analytic on $\rho(S_{\tilde{\omega}})$, the Cauchy residue formula yelds

$$\mathbf{T}_\nu \mathbf{R} = -\oint_{C_\nu} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \frac{1}{\tilde{\omega} - \nu} \mathbf{T}_{\tilde{\omega}} \mathbf{R} \,. \tag{C.8}$$

Multiplying both sides by $\mathbf{R}^{-1}\mathbf{S}_\nu$, one gets

$$\mathbf{I} = \oint_{C_\nu} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \left(\mathbf{T}_{\tilde{\omega}} \mathbf{R} - \frac{\mathbf{I}}{\tilde{\omega} - \nu} \right), \tag{C.9}$$

where the contour C_ν can be deformed at will, as long as it remains within $\rho(S_{\tilde{\omega}}) \setminus \{\nu\}$.

At this point, the contour C_ν can be deformed to the needed limiting form in (C.2). Let us consider the abstract spectral representation for the operator $\mathbf{R}^{-1}\mathbf{P}$, that is (S3) there exists a projection operator valued measure $\mathbf{E}(\nu)$ on $\sigma(\mathbf{S}_{\tilde{\omega}})$, which satisfies the usual commutation and monotonicity conditions and it gives the spectral representation $\mathbf{R}^{-1}\mathbf{P} = \int_{\sigma(\mathbf{S}_{\tilde{\omega}})} \nu \, \mathrm{d}\mathbf{E}(\nu)$. Consequently, we obtain also the spectral representation $\mathbf{T}_{\tilde{\omega}}\mathbf{R} = \int_{\sigma(\mathbf{S}_{\tilde{\omega}})} \frac{1}{\nu + \tilde{\omega}} \mathrm{d}\mathbf{E}(\nu)$. Let now $\mathbf{E}_\varsigma = \mathbf{E}(\{\nu \in \mathbb{C} \mid |\nu| < \varsigma\})$. Then $\mathbf{E}_\varsigma \to \mathbf{I}$ strongly as $\varsigma \to \infty$ and the set $\bigcup_{\varsigma > 0} \operatorname{ran} \mathbf{E}_\varsigma$ is dense in H^2.

As a consequence we also obtain that $\mathbf{T}_{\tilde{\omega}}\mathbf{R}\mathbf{E}_\varsigma$ is analytic for $|\tilde{\omega}| > \varsigma$ and it has the strong asymptotic expansion $\mathbf{T}_{\tilde{\omega}}\mathbf{R}\mathbf{E}_\varsigma = \frac{1}{\tilde{\omega}}\mathbf{E}_\varsigma + \mathcal{O}(\frac{1}{\tilde{\omega}^2})$. Multiplying both sides of (C.9) by $\mathbf{E}_\varsigma$ one gets

$$\mathbf{E}_\varsigma = \oint_{C_\nu} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \left(\mathbf{E}_\varsigma \mathbf{T}_{\tilde{\omega}} \mathbf{R} - \frac{\mathbf{E}_\varsigma}{\tilde{\omega} - \nu} \right) - \oint_{C_\nu} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \frac{\mathbf{I} - \mathbf{E}_\varsigma}{\tilde{\omega} - \nu} \,.$$

Notice now that the contour C_ν in the first integral can be deformed to the contour $C_\varsigma \cup C_\varsigma^\epsilon \cup \mathring{C}$, as illustrated in Figure C.1. Because the asymptotical evaluation of $\mathbf{E}_\varsigma$ mentioned above, the integral over anlarged circle C_ς contributes as $\mathcal{O}(\frac{1}{\varsigma})$. On the other hand, the term $\frac{\mathbf{E}_\varsigma}{\tilde{\omega} - \nu}$ is analytic over the contours $\mathring{C}$, C_ς^ϵ and also in their interior. Hence its contribution vanishes identically. We are left with the follwing expression

$$\mathbf{I} = \oint_{C_\varsigma^\epsilon} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}} \mathbf{R} \mathbf{E}_\varsigma + \oint_{\mathring{C}} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}} \mathbf{R} \mathbf{E}_\varsigma + \mathcal{O}(\varsigma^{-1}). \tag{C.10}$$

[1] See Theorem VI.5 of [93].

We multiply both sides of (C.10) by an arbitrary $\mathbf{v}_{\varsigma'} \in H^2$, such that $\mathbf{v}_{\varsigma'} = \mathbf{E}_\varsigma \mathbf{v}_{\varsigma'}$ for any $\varsigma > \varsigma'$. In this way we obtain

$$\begin{aligned}
\mathbf{v}_{\varsigma'} &= \oint_{C^\epsilon_\varsigma} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}} \mathbf{R} \mathbf{E}_\varsigma \mathbf{v}_{\varsigma'} + \oint_{\mathring{C}} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}} \mathbf{R} \mathbf{E}_\varsigma \mathbf{v}_{\varsigma'} + \mathcal{O}(\varsigma^{-1}) \\
&= \int_{-\varsigma}^{\varsigma} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \lim_{\epsilon\to 0^+} (\mathbf{T}_{\tilde{\omega}-i\epsilon} - \mathbf{T}_{\tilde{\omega}+i\epsilon}) \mathbf{R} \mathbf{E}_\varsigma \mathbf{v}_{\varsigma'} \\
&\quad + \oint_{\mathring{C}} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}} \mathbf{R} \mathbf{E}_\varsigma \mathbf{v}_{\varsigma'} + \mathcal{O}(\varsigma^{-1}) \\
&= \lim_{\varsigma\to\infty} \int_{-\varsigma}^{\varsigma} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \lim_{\epsilon\to 0^+} (\mathbf{T}_{\tilde{\omega}-i\epsilon} - \mathbf{T}_{\tilde{\omega}+i\epsilon}) \mathbf{R} \mathbf{v}_{\varsigma'} \\
&\quad + \oint_{\mathring{C}} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}} \mathbf{R} \mathbf{v}_{\varsigma'} \,.
\end{aligned}$$

We take the limits $\epsilon \to 0^+$ and $\varsigma \to \infty$, where the first limit is in the distributional sense with respect to $\tilde{\omega}$ while the second is in a strong sense. The following result is obtained[2] using a variant of the Banach-Steinhaus theorem (Theorem 2.11.4 of [96]).

$$\mathbf{R}^{-1} = \lim_{\varsigma\to\infty} \int_{-\varsigma}^{\varsigma} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \lim_{\epsilon\to 0^+} (\mathbf{T}_{\tilde{\omega}-i\epsilon} - \mathbf{T}_{\tilde{\omega}+i\epsilon}) + \oint_{\mathring{C}} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}} \,,$$

The desired result (C.2) is finally obtained by using the formula

$$\mathbf{R}^{-1} = \begin{bmatrix} \mathcal{R}_1 & \mathcal{R}_2 \\ \mathcal{R}_2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \mathcal{R}_2^{-1} \\ \mathcal{R}_2^{-1} & -\mathcal{R}_2^{-1}\mathcal{R}_1\mathcal{R}_2^{-1} \end{bmatrix} \tag{C.11}$$

and the second equality in (C.6). This argument for the linear operator pencil mimicks that of [63, Ch.9], the only difference being that the existence of the spectral measure $\mathbf{E}(\nu)$ from the spectral theorem for self-adjoint operators on a Krein space $\mathcal{K} = (H^2, [-,-])$, where the inner product $[\mathbf{v}, \mathbf{u}] = (\mathbf{v}, \mathbf{R}\mathbf{u})$ is indefinite [97, 98]. Nevertheless the spectral theorem is still applicable, provided that the operator $\mathbf{R}^{-1}\mathbf{P}$ is definitizable. This is indeed the case for the specific operators defined in Appendix 3.3.3. The hypotheses (S1,S2,S3) are verified in Appendices C.2, C.3 and C.4, respectively.

C.2 Check of hypothesis (S1)

In this Appendix we prove that hypothesis (S1) of Appendix C.1 is verified for the quadratic operator pencil $S_{\tilde{\omega}}$, coming from Sturm-Liouville ODEs as in (3.5), where $S_{\tilde{\omega}} = \mathcal{J}^{-1} L_{\tilde{\omega}}$ on $H = L^2((0,1); \mathcal{J}(z)\, \mathrm{d}z)$ and $\mathcal{J}(z)$ is as in

[2] Here we recall that finite linear combinations of vectors like $\mathbf{v}_{\varsigma'}$ are dense in H^2 and note that due to (C.10) the norms of the integrals $\int_{-\varsigma}^{\varsigma} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \lim_{\epsilon\to 0^+} (\mathbf{T}_{\tilde{\omega}-i\epsilon} - \mathbf{T}_{\tilde{\omega}+i\epsilon}) + \oint_{\mathring{C}} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}}$ are uniformly bounded for large σ.

in (3.19). We verify that $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_2^{-1}$ are bounded and self-adjoint, while P is unbounded, closed and hermitian on a dense domain $D(S_{\tilde{\omega}}) \subset H$

The domain of $S_{\tilde{\omega}}$, $D(S_{\tilde{\omega}}) \subset \mathcal{H} = L^2((0,1); \mathcal{J}(z)\,\mathrm{d}z)$, is strictly related to the choice of boundary conditions for $L_{\tilde{\omega}}$ in (3.35). In fact [63, Ch.3], each choice of boundary conditions defines a closed operator on a dense domain $D(S_{\tilde{\omega}})$. Let us assume that there exists at least one $\tilde{\omega} \in \mathbb{C}$ such that $\tilde{\omega}, \overline{\tilde{\omega}} \in \rho(S_{\tilde{\omega}})$ and that the corresponding bounded resolvents are self-adjoint $T^*_{\tilde{\omega}} = T_{\overline{\tilde{\omega}}}$. Then, the closed operator $S_{\tilde{\omega}}$ will be self-adjoint, that is $S^*_{\tilde{\omega}} = S_{\overline{\tilde{\omega}}}$ with $D(S^*_{\tilde{\omega}}) = D(S_{\tilde{\omega}})$.

Therefore, we need to check the following properties:

(a) the Green distribution associated to $L_{\tilde{\omega}}$, dubbed $\mathcal{G}_{\tilde{\omega}}$, exists for at least one $\tilde{\omega} \in \mathbb{C}$,

(b) $T_{\tilde{\omega}} = \mathcal{G}_{\tilde{\omega}}\mathcal{J}$ is bounded for at least one $\tilde{\omega} \in \mathbb{C}$,

(c) $\overline{\mathcal{G}_{\tilde{\omega}}}(z, z') = \mathcal{G}_{\overline{\tilde{\omega}}}(z', z)$ and consequently $T^*_{\tilde{\omega}} = T_{\overline{\tilde{\omega}}}$.

Properties (a) and (c) have been verified in Section 3.3.3 for each choice of Robin boundary conditions.

In order for property (b) to hold, the resolvent $T_{\tilde{\omega}} = \mathcal{G}_{\tilde{\omega}}\mathcal{J}$ must be bounded. Let us now consider the functions $u_{\tilde{\omega}}$ and $\Psi_{\tilde{\omega},\zeta}$ introduced in (3.36) and (3.37) of Section 3.3.3. For any given $\tilde{\omega}$ with $\mathrm{Im}[\tilde{\omega}] \neq 0$, these functions are linearly independent, that is $\mathcal{N}_{\tilde{\omega},\zeta}$ in Eq. (3.39) does not vanish. Moreover let us consider the asymptotic estimates

$$|u_{\tilde{\omega}}(z)| \lesssim z^{\lambda}(1-z)^{1-\beta-\epsilon} \,, \tag{C.12a}$$

$$|\Psi_{\tilde{\omega},\zeta}(z)| \lesssim \begin{cases} z^{-\lambda}(1-z)^{\beta} \,, & \zeta = 0 \,, \\ z^{-\lambda}(1-z)^{1-\beta-\epsilon} \,, & \zeta \neq 0 \,, \end{cases} \tag{C.12b}$$

$$|\mathcal{J}(z)| \lesssim z^{-1}(1-z)^{-1} \,. \tag{C.12c}$$

where $\lambda \doteq \ell^2 r_+ \left|\mathrm{Im}\{\tilde{\omega}\}\right| / 2(r_+^2 - r_-^2)$, the symbol $\lesssim$ denotes an inequality up to a multiplicative constant, uniform over $z \in (0,1)$, while the constant $\epsilon > 0$ takes care of the cases with logarithmic singularities.

In order to prove the boundedness of $T_{\tilde{\omega}}$ we apply the *weighted Schur test* (see Theorem 5.2 of [99]). Let us consider the following chain of inequalities,

$$
\begin{aligned}
&\|T_{\tilde\omega}\Psi\|^2\\
&=\int_0^1 \mathrm{d}z\,\mathcal{J}(z)\left|\int_0^1 \mathrm{d}z'\,\mathcal{G}_{\tilde\omega}(z,z')\mathcal{J}(z')\Psi(z')\right|^2\\
&\leqslant \int_0^1 \mathrm{d}z\,\mathcal{J}(z)\left(\int_0^1 \sqrt{|\mathcal{G}_{\tilde\omega}(z,z')|\,\mathcal{J}(z')\mathcal{J}_1(z')}\sqrt{|\mathcal{G}_{\tilde\omega}(z,z')|\,\frac{\mathcal{J}(z')}{\mathcal{J}_1(z')}\,|\Psi(z')|^2\,\mathrm{d}z'}\right)^2\\
&\leqslant \int_0^1 \mathrm{d}z\,\mathcal{J}(z)\left(\int_0^1 \mathrm{d}z'\,|\mathcal{G}_{\tilde\omega}(z,z')|\,\mathcal{J}(z')\mathcal{J}_1(z')\right)\left(\int_0^1 \mathrm{d}z'\,|\mathcal{G}_{\tilde\omega}(z,z')|\,\frac{\mathcal{J}(z')}{\mathcal{J}_1(z')}\,|\Psi(z)|^2\right)\\
&\leqslant \int_0^1 \mathrm{d}z'\left(\int_0^1 \mathrm{d}z\,\mathcal{J}(z)\mathcal{J}_2(z)\,|\mathcal{G}_{\tilde\omega}(z,z')|\right)\frac{\mathcal{J}(z')}{\mathcal{J}_1(z')}\,|\Psi(z')|^2\\
&\leqslant \int_0^1 \mathrm{d}z'\,\frac{\mathcal{J}_3(z')}{\mathcal{J}_1(z')}\mathcal{J}(z')\,|\Psi(z')|^2\,.
\end{aligned}
$$

Here $\|T_{\tilde\omega}\Psi\|^2 \lesssim \|\Psi\|^2$, only if there exist three functions $\mathcal{J}_1(z)$, $\mathcal{J}_2(z)$, $\mathcal{J}_3(z)$ satisfying the following estimates

$$
\begin{gathered}
\int_0^1 \mathrm{d}z'\,|\mathcal{G}_{\tilde\omega}(z,z')|\,\mathcal{J}(z')\mathcal{J}_1(z')\lesssim \mathcal{J}_2(z)\,,\\
\int_0^1 \mathrm{d}z\,\mathcal{J}(z)\mathcal{J}_2(z)\,|\mathcal{G}_{\tilde\omega}(z,z')|\lesssim \mathcal{J}_3(z')\,,\\
\frac{\mathcal{J}_3(z')}{\mathcal{J}_1(z')}\lesssim 1\,.
\end{gathered}
$$

A deeper analysis shows that the lower bounds of $\mathcal{J}_2$ and $\mathcal{J}_3$ are in fact determined by the properties of $\mathcal{G}_{\tilde\omega}(z,z')$ and that the only free choice is actually the function $\mathcal{J}_1$. Given the estimates (C.12) and formula (3.38), one sees that boundedness is obtained with the following choices:

$$
\zeta=0\colon\ \begin{cases}\mathcal{J}_1(z)=1,\\ \mathcal{J}_2(z)=(1-z)^{\min(\beta,1-2\epsilon)},\\ \mathcal{J}_3(z)=(1-z)^{\min(\beta,2-4\epsilon)},\end{cases} \tag{C.13}
$$

$$
\zeta\neq0\colon\ \begin{cases}\mathcal{J}_1(z)=1,\\ \mathcal{J}_2(z)=(1-z)^{1-\beta-\epsilon},\\ \mathcal{J}_3(z)=(1-z)^{1-\beta-\epsilon},\end{cases} \tag{C.14}
$$

where, in the case $\zeta\neq0$, we select $\beta\in(\frac{1}{2},1)$ and $\epsilon<1-\beta$.

C.3 Check of hypothesis (S2)

In this Appendix we prove that hypothesis (S2) of Appendix C.1 is verified for the quadratic operator pencil $S_{\tilde\omega}$, coming from Sturm-Liouville ODEs as in (3.5), where $S_{\tilde{\omega}} = \mathcal{J}^{-1} L_{\tilde{\omega}}$ on $H = L^2((0,1); \mathcal{J}(z)\,\mathrm{d}z)$ and $\mathcal{J}(z)$ is as in in (3.19). We verify that the spectrum of $S_{\tilde{\omega}}$ is real with at most two isolated points in $\mathbb{C} \setminus \mathbb{R}$, displaced symmetrically with respect to the real axis.

Notice that the Green distribution $\mathcal{G}_{\tilde{\omega},\zeta}$ computed in Section 3.3.3 has a branch cut at $\mathrm{Im}[\tilde{\omega}] = 0$ and that, only for certain values of ζ, it can have poles with $\mathrm{Im}[\tilde{\omega}] \neq 0$. As seen in Section 3.3.3, these poles coincide with the zeros of the coefficient $\mathcal{N}_{\tilde{\omega},\zeta}$ in (3.39).

A direct inspection shows that the coefficient $\mathcal{N}_{\tilde{\omega},\zeta}$ has at most isolated zeros and that they are reflection symmetric about the real axis. These zeros can be interpreted as bound state frequencies and they form a set $\mathrm{BS}_\zeta \subset \mathbb{C}$, which can be divided in $\mathrm{BS}_\zeta = \mathrm{BS}_\zeta^+ \cup \overline{\mathrm{BS}_\zeta^+}$ with $\mathrm{Im}[\mathrm{BS}_\zeta^+] > 0$. Therefore, $\sigma(S_{\tilde{\omega}}) = \mathbb{R} \cup \mathrm{BS}_\zeta$. Also notice that, by Appendix C.1, the resolvent $T_{\tilde{\omega}} = S_{\tilde{\omega}}^{-1}$ is analytic on $\rho(S_{\tilde{\omega}}) = \mathbb{C} \setminus \sigma(S_{\tilde{\omega}})$.

The structure of BS_ζ can be inferred by looking at the zeros of $\mathcal{N}_{\tilde{\omega},\zeta}$ from Section 3.3.3, which are defined precisely as the solutions of the transcendental equation[3]

$$\tan(\zeta) = \frac{B}{A} =: \Theta(\tilde{\omega}) \tag{C.15}$$

in the upper half complex plane, $\mathrm{Im}[\tilde{\omega}] > 0$, $\zeta \in [0, \pi)$, together with their complex conjugates.

As we will see, $\mathrm{BS}_\zeta^+ = \emptyset$ or $\mathrm{BS}_\zeta^+ = \{\tilde{\omega}_\zeta\}$, consisting only of a single point. In the case $\zeta = \pi/2$, any $\tilde{\omega}$ at which $\Theta(\tilde{\omega})$ has a pole can be interpreted as a solution of (C.15). The RHS of (C.15) is a ratio of products of gamma functions with $\tilde{\omega}$-dependent parameters. For generic values of these parameters, the function has only simple zeros at $\tilde{\omega}_\pm(n)$ and simple poles at $\tilde{\omega}^\pm(n)$ for $n = 0, 1, 2, \ldots$, where

$$\tilde{\omega}_\pm(n) = \pm\frac{k}{\ell} - k\Omega_H - 2i(n+\beta)\frac{(r_+ \mp r_-)}{\ell^2}\,, \tag{C.16}$$

$$\tilde{\omega}^\pm(n) = \pm\frac{k}{\ell} - k\Omega_H - 2i(n+1-\beta)\frac{(r_+ \mp r_-)}{\ell^2}\,. \tag{C.17}$$

Its asymptotic behaviour for $|\tilde{\omega}| \to \infty$ can be inferred from the Stirling asymptotic formula, which yelds[4]

$$\Theta(\tilde{\omega}) = \frac{\Gamma(\sqrt{\mu^2+1})}{\Gamma(-\sqrt{\mu^2+1})}\left(\frac{\ell^4(-i\tilde{\omega})^2}{4(r_+^2 - r_-^2)}\right)^{-\sqrt{\mu^2+1}}[1 + \mathcal{O}(|\tilde{\omega}|^{-1})]\,. \tag{C.18}$$

The zeros and poles of $\Theta(\tilde{\omega})$ correspond to the explicit solutions of (C.15), for $\zeta = 0$ (Dirichlet) and $\zeta = \pi/2$ (Neumann) boundary conditions, respectively.

[3] See Eq. (3.48). Here A and B are again as in (3.46).

[4] The branch of the power function must agree with the principal branch whenever $-i\tilde{\omega} > 0$. Notice that some of the poles or zeros may merge for special values of the parameters.

At the same time, explicit solutions[5] for a generic value of ζ cannot be obtained, due to the transcendental nature of equation (C.15). Nonetheless, we can make few qualitative remarks.

Firstly, ζ is always real. Therefore $\tilde{\omega} \in \mathbb{C}$ for which $\Theta(\tilde{\omega}) \notin \mathbb{R}$ is never a solution of (C.15). Conversely, when $\Theta(\tilde{\omega})$ is real, equation (C.15) is satisfied for $\zeta = \arctan(\Theta(\tilde{\omega}))$. Hence, given a fixed ζ, the solutions of (C.15) exist and they lie in the regions with real phase $\arg[\Theta(\tilde{\omega})] = 0$ or π.

Physically speaking, one can reach the following conclusions.

In the case $\mu^2 \geqslant 0$, the only allowed boundary condition (see Section 3.2.3) is the case $\zeta = 0$ (Dirichlet), which corresponds to zeros of $\Theta(\tilde{\omega})$. As it can be inferred from (C.16), all of the zeros lie on the lower half complex plane. Therefore, no solutions of (C.15) with $\mathrm{Im}[\tilde{\omega}] > 0$ can be found and there are no bound state frequencies, that is $\mathrm{BS}^+_\zeta = \varnothing$.

In the case $-1 < \mu^2 < 0$, all the poles and zeros lie in the lower half complex plane. The closest pole to the real axis is

$$\tilde{\omega}^+(0) = \frac{k}{\ell} - k\Omega_H - i\left(1 - \sqrt{\mu^2+1}\right)\frac{(r_+ - r_-)}{\ell^2} \; .$$

In this case, the solutions with $\mathrm{Im}[\tilde{\omega}] > 0$ are parametrized by $\zeta \in [\zeta_c, \pi)$ and lie on the single line of real phase stretching from this pole. This phase line crosses the $\mathrm{Im}[\tilde{\omega}] = 0$ axis at $\tilde{\omega} = 0$, where

$$\Theta(0) = \frac{\Gamma(2\beta - 1)\left|\Gamma\left(1 - \beta + i\ell\frac{k}{r_+}\right)\right|^2}{\Gamma(1-2\beta)\left|\Gamma\left(\beta + i\ell\frac{k}{r_+}\right)\right|^2} = \tan(\zeta_c) \; .$$

Since $-1 < \mu^2 < 0$, then $\beta \in (\frac{1}{2}, 1)$, which implies $\zeta_c \in (\frac{\pi}{2}, \pi)$. Furthermore, we notice that the solution $\tilde{\omega} = \tilde{\omega}_\zeta$ is isolated and simple[6]. Therefore, in this case $\mathrm{BS}_\zeta = \{\tilde{\omega}_\zeta, \overline{\tilde{\omega}_\zeta}\}$. In Figure 3.1 we plotted the real and the imaginary parts of the frequency $\tilde{\omega}_\zeta$ as a function of ζ for some sample values of the other involved parameters.

C.4 Check of hypothesis (S3)

In this Appendix we prove that hypothesis (S3) of Appendix C.1 is verified for the quadratic operator pencil $S_{\tilde{\omega}}$, coming from Sturm-Liouville ODEs as in (3.5), where $S_{\tilde{\omega}} = \mathcal{J}^{-1}L_{\tilde{\omega}}$ on $H = L^2((0,1); \mathcal{J}(z)\,\mathrm{d}z)$ and $\mathcal{J}(z)$ is as in

[5] A solution can be found numerically for any value of the parameters μ^2, ℓ, r_+, r_- and k describing the scalar field and the BTZ black hole.

[6] This could be rigorously established by a careful application of the *argument principle*, which we omit for brevity, to the function $f(\tilde{\omega}) = \tan(\zeta) - \Theta(\tilde{\omega})$, which confirms the existence of a single simple zero $\tilde{\omega}_\zeta \in \mathrm{Im}[\tilde{\omega}]$ provided the integrals $\oint \frac{f'(\tilde{\omega})}{f(\tilde{\omega})}\frac{\mathrm{d}\tilde{\omega}}{2\pi i}$ stabilize to the value 1 over a sequence of simple closed and positive contours whose interior exhausts the upper half complex plane.

in (3.19). We show that there exists a spectral measure for the linearised pencil $\mathbf{S}_{\tilde{\omega}}$ in (C.3).

In the following we adopt the notation of Appendix C.1. Therefore, let $\mathcal{K} = (H^2, [-,-])$, be Krein space [97, 98] with bounded inner product $[\mathbf{v}, \mathbf{u}] = (\mathbf{v}, \mathbf{R}\mathbf{u})$. As already stated, the spectral problem of the linear operator pencil $\mathbf{S}_{\tilde{\omega}} = \mathbf{P} + \tilde{\omega}\mathbf{R}$ is equivalent to the spectral problem $-\mathbf{R}^{-1}\mathbf{P} = \tilde{\omega}\mathbf{I}$, where now the operator $\mathbf{A} \doteq -\mathbf{R}^{-1}\mathbf{P}$ is self-adjoint with respect to the inner product $[-,-]$.

Even if no general spectral theorem is available for an arbitrary self-adjoint operator on a Krein space, there are some special cases where it is possible to prove the existence of a spectral measure $\mathbf{E}(\nu)$. In these cases, hypothesis (S3) of Appendix C.1 is verified. As already stated, hypothesis (S3) is verifiable whenever the operator $\mathbf{A}$ is *definitizable*[7], that is, when there exists a polynomial $p(\tilde{\omega})$ of degree k with real coefficients such that

$$[\mathbf{u}, p(\mathbf{A})\mathbf{u}] \geqslant 0 \quad \text{for each } \mathbf{u} \in D(\mathbf{A}^k) \ .$$

In order to prove that the operator $\mathbf{A}$ defined as in Appendices C.1 and Section 3.3.3 is definitizable, we proceed as follows.

Firstly, since

$$[\mathbf{u}, (-\mathbf{A}_0)\mathbf{u}] \geqslant 0 \quad \text{for all } \mathbf{u} \in D(\mathbf{A}_0) \ ,$$

we suppose that there exists a definitizable closed restriction $\mathbf{A}_0$ of $\mathbf{A}$ to a subdomain $D(\mathbf{A}_0) \subset D(\mathbf{A})$. The restricted operator $\mathbf{A}_0$ may no longer be self-adjoint with respect to the Krein inner product, but it is possible to find [101] a Friedrichs self-adjoint extension $\mathbf{A}_1$ satisfying $[\mathbf{u}, (-\mathbf{A}_1)\mathbf{u}]$ on its domain. Secondly, we notice that the operator $\mathbf{A}$ is defined as an ordinary differential operator. Therefore the difference of the resolvents

$$(\mathbf{A}_1 - \tilde{\omega}\mathbf{I})^{-1} - (\mathbf{A} - \tilde{\omega}\mathbf{I}) \tag{C.19}$$

is a finite rank operator[8]. This difference is therefore described by the so-called *Krein resolvent formula* [102, §106].

Thirdly, we observe that [103], when at least one of the Krein self-adjoint operators $\mathbf{A}_1$ or $\mathbf{A}$ is definitizable and the difference of their resolvents (C.19) is of finite rank for at least one $\tilde{\omega} \in \rho(\mathbf{A}_1) \cap \rho(\mathbf{A})$, then both operators are definitizable.

In the case of interest, $H = L^2((0,1); \mathcal{J}(z)\ \mathrm{d}z)$. let us consider

$$\mathbf{u} := [\Psi \quad \Phi]^T \in D(\mathbf{A}_0)$$

[7]The corresponding spectral theorem can be found in [98] and [100].

[8]In order to prove that the ranks is finite, it is sufficient to notice that an ordinary differential operator has a finite dimensional space of solutions.

consisting of smooth functions with compact support. We can then write explicitly $[\mathbf{u}, (-\mathbf{A})\mathbf{u}]$. Integration by parts leads to the following result

$$(\Psi, (-\mathcal{J}^{-1} L_{\tilde{\omega}=0})\Psi) + (\Phi, \mathcal{R}_2 \Phi) = \int_0^1 \mathrm{d}z \left[z \left| \frac{\mathrm{d}\Psi(z)}{\mathrm{d}z} \right|^2 \right.$$
$$\left. + \left(\frac{\ell^2 k^2 (1-z) + r_+^2 \mu^2}{4 r_+^2 (1-z)^2} \right) |\Psi(z)|^2 + \frac{\ell^4 \mathcal{J}(z) |\Phi(z)|^2}{4(r_+^2 - r_-^2)} \right] .$$

Let us analyse the two relevant regimes. When $-1 < \mu^2 < 0$, since the term proportional to k^2 in the integrand is strictly greater than 0, in view of (3.34) and of the results of Appendix C.1, $-\mathcal{J}^{-1} L_{\tilde{\omega}=0}$ is a self-adjoint operator with strictly positive spectrum.. Therefore it is a positive operator and the integral is non-negative When $\mu^2 \geqslant 0$, all terms appearing under the integral are manifestly non-negative, and so is the whole integral. Thus, the restriction of $\mathbf{A}$ to $D(\mathbf{A}_0)$ satisfies the relation

$$[\mathbf{u}, (-\mathbf{A}_0)\mathbf{u}] \geqslant 0 \quad \text{for all } \mathbf{u} \in D(\mathbf{A}_0) \ .$$

This implies that $\mathbf{A}$ is definitizable.

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