Università degli Studi di Pavia

Joint PhD Program in Mathematics Pavia - Milano Bicocca - INdAM

On Shimura Subvarieties of the Torelli Locus and Ramified Prym Maps

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XXXIII Cycle

Jew, Gentile, Black Man, White
We all want to help one another, human beings are like that
We want to live by each other's happiness, not by each other's misery
We don't want to hate and despise one another.
And this world has room for everyone, and the good Earth is rich and can provide for everyone
The way of life can be free and beautiful, but we have lost the way.
...
Let us fight for a new world - a decent world that will give men a chance to work - that will give youth a future and old age a security.
...
Let us fight to free the world - to do away with national barriers -
to do away with greed, with hate and intolerance.
Let us fight for a world of reason,
a world where science and progress will lead to all men's happiness.
...
Let us all unite.

Charlie Chaplin, The Great Dictator.
Humanity against Coronavirus Pandemic, 2020.

La musica è come la vita,
si può fare in un solo modo: insieme.
Sono un uomo con una disabilità evidente
in mezzo a tanti uomini con disabilità che non si vedono.
La musica ci insegna la cosa più importante che esista:
ascoltare.

Ezio Bosso
(*1971-2020*)

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Introduction

The purpose of this thesis is two-fold: to investigate the existence of totally geodesic subvarieties of the moduli space of principally polarized abelian varieties, $\mathcal{A}_g$, contained in the Jacobian locus and to study the geometry of certain positive dimensional fibres of some ramified Prym maps. Totally geodesic subvarieties constitute a useful tool to study the extrinsic geometry of the Jacobian locus inside $\mathcal{A}_g$ and they are involved in the rather famous Coleman-Oort conjecture. Furthermore, they motivate our interest in Prym maps. Indeed it turns out that certain positive dimensional fibres represent a good place to look for totally geodesic subvarieties.

The thesis is thus divided into two parts.

- In **Part I** we study Galois coverings of curves of positive genus $g' \geq 1$ which yield infinitely many new examples of Shimura curves in genus $g \leq 4$. The result obtained in this part can be found in [34].

- In **Part II** we study ramified Prym maps and we give a geometric description of their positive dimensional fibres. The result obtained in this part can be found in [35].

We introduce them separately.

Let $\mathcal{M}_g$ be the moduli space of curves, $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties and let

$$j : \mathcal{M}_g \to \mathcal{A}_g$$

be the period map, usually called Torelli map. It sends each smooth projective curve of genus g, $[C] \in \mathcal{M}_g$, to its Jacobian variety, $[JC, \Theta_C] \in \mathcal{A}_g$, as principally polarized abelian variety. By Torelli Theorem j is injective.

The *Torelli locus* $\mathcal{T}_g$ is the closure of $j(\mathcal{M}_g)$ in $\mathcal{A}_g$. Both $\mathcal{M}_g$ and $\mathcal{A}_g$ are complex orbifold and it is well-known ([78]) that the restriction of j to the set of non-hyperelliptic curves (denoted by $\mathcal{M}_g^*$) is an orbifold immersion.

Recall that $\mathcal{A}_g$ has a natural metric. Indeed, it is the quotient of the Siegel space $\mathfrak{S}_g$, which is an irreducible Hermitian symmetric space of non-compact type, by a proper discontinuous action of $Sp(2g, \mathbb{Z})$. We denote the corresponding metric connection by ∇. Hence $\mathcal{A}_g$ is endowed with a locally symmetric metric, the so-called *Siegel metric*.

Since for $g \geq 4$ the dimension of $\mathcal{M}_g$ is strictly smaller than the dimension of $\mathcal{A}_g$, it makes sense to study the metric properties of $\mathcal{M}_g$ (identified with its image through j) with respect to the Siegel metric.

Very few is known about this topic and here we would like to present some results in this direction. The rough idea behind this investigation is that the Torelli locus is expected to be "very curved" inside $\mathcal{A}_g$.

On one hand, there are results concerning the second fundamental form of the embedding $j : \mathcal{M}_g^* \to \mathcal{A}_g$ giving an upper bound for the possible dimension of a totally geodesic submanifold of $\mathcal{A}_g$ contained in the Torelli locus. We are referring to [21, 22, 37, 40, 41].

On the other hand, one may look at totally geodesic subvarieties of $\mathcal{A}_g$ and ask whether $\mathcal{T}_g$ contains some of them. By definition they are images of totally geodesic submanifold of $\mathfrak{S}_g$, i.e. $Y \subset \mathfrak{S}_g$ such that the second fundamental form of the immersion $Y \hookrightarrow \mathfrak{S}_g$

$$\begin{aligned} \mathrm{II}_Y : TY \times TY &\to N_{\mathfrak{S}_g/Y} \\ (u, v) &\mapsto (\nabla_u v)^\perp \end{aligned}$$

is identically equal to zero. The expectation is that the Torelli locus should contain few totally geodesic subvarieties. The analogous statement for a surface in a 3-space is that the surface should not contain too many lines.

The conjecture is the following:

Conjecture 1. For large genus there does not exist any positive dimensional totally geodesic subvariety of $\mathcal{A}_g$ contained in the Torelli locus.

We point out that this conjecture is a bit stronger than the rather famous Coleman-Oort's conjecture on the non-existence of *special* subvarieties in the Torelli locus for high genus: a *special* or *Shimura subvariety* of $\mathcal{A}_g$ is by definition a Hodge locus for the tautological family of principally polarized abelian varieties on $\mathcal{A}_g$. Hence we can read Conjecture 1 in terms of the following

Conjecture 2 (Coleman-Oort)**.** For large genus there should not exist positive dimensional Shimura subvarieties Z $\subset \mathcal{A}_g$ *generically contained in* $\mathcal{T}_g$, that is $Z \subset \mathcal{T}_g$ and $Z \cap j(\mathcal{M}_g) \neq \emptyset$.

Shimura varieties are totally geodesic. More precisely, by results of Mumford and Moonen ([67]), we have the following:

Theorem 1. *An algebraic totally geodesic subvariety of $\mathcal{A}_g$ is Shimura if and only if it contains a CM point.*

In this way it becomes evident the double nature of Conjecture 2: the notion of CM point is arithmetic while the condition of being totally geodesic refers to the locally symmetric geometry coming from the Siegel space. Important results in this direction are achieved in [17], [24], [45], [46], [60], [61] and [70] is a very good survey.

There are al least *some* (to be precise 32) Shimura subvarieties contained in $\mathcal{T}_g$. All are in low genus ($g \leq 7$) and they are constructed as families of Jacobians of Galois covers of the line (see [23], [70], [69] for abelian groups and [32] for a complete list including also non abelian cases) and of elliptic curves (see [36]).

All these examples of families of Galois covers satisfy a sufficient condition that we briefly explain: take a family of Galois covers $C \to C' = C/G$, where the genera $g(C) = g$, $g(C') = g'$, the number of ramification points r and the monodromy are fixed. Let Z denote the closure in $\mathcal{A}_g$ of the locus described by $[JC]$ for C varying in the family. Then

$$\dim \mathsf{Z} = 3g' - 3 + r.$$

The simple numerical condition

$$N := \dim(S^2(H^0(K_C)))^G = \dim H^0(2K_C)^G (= \dim \mathsf{Z}) \qquad (*)$$

is sufficient to ensure that Z is Shimura (see [32, 36], resp. Theorem 3.9 and 3.7).

Moonen in [69] proved that when $g' = 0$ and the group G is cyclic $(*)$ is also necessary for Z to be Shimura. Mohajer and Zuo [68] extended this to the case where $g' = 0$, G is abelian and the family is 1-dimensional. In both cases, the authors also showed that condition $(*)$ holds only in the known examples. These results are proved using methods from positive characteristic and it seems to be complicated to generalize them in case of any G.

A completely different Hodge theoretic argument was given in [21, Prop. 5.2], but it only works for some of the families of cyclic covers of $\mathbb{P}^1$. It is unknown whether $(*)$ is necessary in general for a family of covers to yield a Shimura subvariety or whether other families exist which satisfy $(*)$.

In [32] the authors gave the complete list of all the families of Galois covers of $\mathbb{P}^1$ of genus $g \leq 9$ satisfying condition $(*)$ and hence yielding Shimura subvarieties of $\mathcal{A}_g$ contained in the Torelli locus. They got 30 examples, recovering those already found in [69].

Later, in [36], other Shimura subvarieties were constructed considering families of Galois covers of elliptic curves satisfying $(*)$. Their data are the following:

(1) $g = 2, G = \mathbb{Z}/2, N = 2.$ (4) $g = 3, G = \mathbb{Z}/4, N = 2.$
(2) $g = 3, G = \mathbb{Z}/2, N = 4.$ (5) $g = 3, G = Q_8, N = 1.$
(3) $g = 3, G = \mathbb{Z}/3, N = 2.$ (6) $g = 4, G = \mathbb{Z}/3, N = 3.$

Families (2) and (6) give two new Shimura subvarieties while the remaining yield Shimura varieties already found by means of Galois coverings of $\mathbb{P}^1$.

Furthermore, in the same paper, the authors studied families of Galois covers over curves of genus $g' > 1$. No example was found, but it was shown that when $(*)$ holds then $g \leq 6g' + 1$.

Our first result is the following:

Theorem 2. *The only positive dimensional families of Galois covers $C \to C' = C/G$ with $g' \geq 1$ and satisfying $(*)$ are the 6 families found in [36]. In particular all of them have $g' = 1$.*

This shows that condition $(*)$ is very strong when $g' > 0$. Of course the moduli image of some family could be a Shimura subvariety even if $(*)$ does not hold.

To prove Theorem 2 we first prove that the assumptions imply $g' \leq 3$. This reduces the problem to the analysis of a finite number of cases. The étale covers are ruled out using some elementary representation theory. The ramified cases are checked by a computer program as in [36].

One of the 2 families of coverings over elliptic curves found in [36], precisely family (6), had been studied by Grushevsky and Möller [44], who got the following remarkable result: the Prym map for this family has 1-dimensional fibres which are totally geodesic. As a consequence, they obtained uncountably many totally geodesic curves generically contained in $\mathcal{T}_4$, countably many of which are Shimura.

This phenomenon motivated our study of all the Shimura subvarieties found in [36]. We construct two fibrations. Indeed, let M be the subvariety of $\mathcal{M}_g$ which parametrizes curves $[C]$ occurring in one of the 6 families above, i.e. curves admitting an effective holomorphic action of G with quotient map $f : C \to C' := C/G$. Then for every family $(1) - ... - (6)$ we consider the following diagram:

$$\begin{tikzcd} & \mathsf{M} \arrow[dl, "\mathcal{P}"'] \arrow[dr, "\varphi"] & \\ \mathcal{A}^{\delta}_{g-g'} & & \mathcal{A}_{g'} \end{tikzcd} \tag{1}$$

where C is sent by $\mathcal{P}$ to the Prym variety associated to the map f and by φ to JC'.

We show the following:

Theorem 3. *Consider a positive dimensional family of Galois covers $C \to C' = C/G$ with $g' \geq 1$ and satisfying $(*)$ (i.e. one of the 6 families in [36]). Every irreducible component of a fibre of the Prym map is a totally geodesic subvariety of $\mathcal{A}_g$ of dimension $g'(g'+1)/2$.*

Theorem 4. *Consider a family of Galois covers $C \to C' = C/G$ with $g' \geq 1$ of dimension $N > 0$ and satisfying $(*)$ (i.e. one of the 6 families in [36]). Every irreducible component of a fibre of the map φ is a totally geodesic subvariety of $\mathcal{A}_g$ of codimension 1.*

Both Theorems thus guarantee the existence of infinitely many new totally geodesic subvarieties generically contained in $\mathcal{T}_2$, $\mathcal{T}_3$ and $\mathcal{T}_4$, countably many of which are Shimura. Indeed we have the following:

Corollary. Families $(1), (2), (3), (4), (6)$ are fibred in totally geodesic curves via their Prym maps and are fibred in totally geodesic subvarieties of codimension 1 via their maps φ. Therefore they contain infinitely many totally geodesic subvarieties and countably many are Shimura subvarieties. The Prym map of (5) is constant.

It is remarkable that the proofs of Theorems 3, 4 introduce a new tool which produces infinitely many new totally geodesic examples at once. A key ingredient is the decomposition, up to isogeny, of the Jacobians of the curves of the families occurring in the fibres. This makes possible the comparison between the fibres and certain known totally geodesic subvarieties of $\mathcal{A}_g$ and hence it allows us to conclude.

The group algebra decomposition plays an important role also in the study of several features of the examples known so far. It is a theory of independent interest which studies a way to use the action of a finite group G on an abelian variety A to decompose A as the product of abelian subvarieties up to isogeny. In particular, as in our situation, the G-action on a smooth projective curve C passes to its Jacobian and it induces a decomposition of JC. Important results in this direction are presented in [58, 15, 14]. Moreover, in the case of Jacobians, thanks to [82] we know the dimension of the terms of the decomposition while thanks to [51] we can recognize them as Jacobians of intermediate quotients.

We use the group algebra decomposition to study families of Jacobians satisfying condition $(*)$ with two different perspectives.

On one hand the analysis of the decomposition of the Jacobians JC, for C occurring in the families yielding the Shimura varieties of [32, 36], gives us all possible inclusions between the families. In this way, we also check which of these families are contained in a fibre of the Prym map $\mathcal{P}$ or of the map φ of one of the 6 families above.

On the other hand it makes evident that most of the examples of [32, 36] turn out to have a completely decomposable Jacobian. Indeed we show the following:

Proposition 5. *For $g = 2, 3, 4$ there are Shimura varieties whose generic point has a totally decomposable Jacobian variety. In particular the following decompositions hold:*

- $\star$ *In $g = 2$ we have families:*
  - $(3) = (5) = (28) = (30)$ *which decomposes as E_1^2;*
  - $(4) = (29)$ *which decomposes as E_1^2;*
  - $(26) = (1e)$ *which decomposes as $E_1 \times E_2$.*
- $\star$ *In $g = 3$ we have families:*
  - $(7) = (23) = (34) = (5e)$ *which decomposes as $E_1 \times E_2^2$;*
  - (22) *which decomposes as $E_1^2 \times E_2$;*
  - $(33) = (35)$ *which decomposes as E_1^3;*
  - $(31) = (3e)$ *which decomposes as $E_1 \times E_2^2$;*
  - $(32) = (4e)$ *which decomposes as $E_1 \times E_2^2$;*
  - (27) *which decomposes as $E_1 \times E_2 \times E_3$.*

$\star$ *In $g = 4$ we have families:*

- $(13) = (24)$ *which decomposes as $E_1 \times E_2^2 \times E_3$;*
- $(25) = (38)$ *which decomposes as $E_1^2 \times E_2^2$;*
- (37) *which decomposes as $E_1 \times E_2^3$.*

This Proposition gives a partial answer to a problem formulated by Moonen and Oort in [70]. There they looked for genus $g \geq 2$ such that there exists a positive dimensional Shimura subvariety Z of $\mathcal{T}_g$ with totally decomposable generic point. We remark that our answer is forced to be partial: first, because we don't know if there exist Shimura varieties in $g \geq 7$ and second, since the group algebra decomposition is not exhaustive, we don't know if the families in $5 \leq g \leq 7$ admit a further (possibly complete) decomposition.

The Proposition above justifies our following

Question 1. Is there any relation between totally geodesic subvarieties of $\mathcal{T}_g$ and loci of totally decomposable Jacobians?

Unfortunately, we prove that there is no clear relation between the two properties. Indeed we exhibit the example of a totally geodesic subvariety of $\mathcal{T}_3$, precisely family (9) of [32], which does not have totally decomposable Jacobians. This is shown using results of [51, 59, 82] and `SAGE` computations (see [9]). On the other hand, we illustrate a specific sublocus of $\mathcal{T}_3$ whose generic Jacobian is totally decomposable and we prove that it cannot be totally geodesic.

This concludes Part I.

In Part I of this thesis we construct infinitely many examples of totally geodesic and of Shimura subvarieties of $\mathcal{A}_g$ generically contained in the Torelli locus as fibres of ramified Prym maps. Moreover, we show that some of the families of Galois covers yielding Shimura varieties are contained in fibres of ramified Prym maps.

It thus seems natural to look for totally geodesic subvarieties at fibres of the Prym maps.

This problem is still open but it has motivated our interest in Prym maps and their fibres, which are examined in Part II. Actually, our work fits in the broader context of the problems concerning the geometry of Prym varieties and Prym maps and it turns out to be interesting by its own.

Prym varieties and Prym maps establish a bridge between the geometry of curves and that of abelian varieties. Indeed, at least in the étale case, they allow studying a bigger class of principally polarized abelian varieties than that of Jacobians using geometric objects: the covers. As such, they have been studied for over 100 years.

The Prym map $\mathcal{P}_{g,r}$ assigns to a degree 2 morphism $\pi : D \longrightarrow C$ of a smooth complex irreducible curve ramified in an even number of points $r \geq 0$, a polarized abelian variety $P(\pi) = P(D, C)$ of dimension $g-1+\frac{r}{2}$, where $g > 0$ is the genus of C. The variety $P(\pi)$ is called the Prym variety of π and is defined as the connected component of the origin of the kernel of the norm map $\mathrm{Nm}_\pi : JD \longrightarrow JC$. Hence, denoting by $\mathcal{R}_{g,r}$ the moduli space of isomorphism classes of the morphisms π, we have maps

$$\mathcal{P}_{g,r} : \mathcal{R}_{g,r} \longrightarrow \mathcal{A}^{\delta}_{g-1+\frac{r}{2}}, \tag{2}$$

to the moduli space of abelian varieties of dimension $g-1+\frac{r}{2}$ with polarization of type $\delta := (1, \dots, 1, 2, \dots, 2)$, with 2 repeated g times if $r > 0$ and $g-1$ times if $r = 0$.

The case $r = 0$ is very classical. Indeed, as already said, Prym varieties of unramified coverings are principally polarized abelian varieties and thus they have been studied for many many years, initially by Wirtinger [90], Schottky and Jung [84] (among others) in the second half of the 19th century from an analytic point of view. They were studied later from an algebraic point of view in the seminal work of Mumford [71] in 1974. We refer to [29, section 1] for a historical account.

Since Mumford's work, a lot of information has been obtained about the unramified (or "classical") Prym map $\mathcal{P}_{g,0}$. This theory is strongly related to the study of the Jacobian locus, Schottky equations and rationality problems among other topics.

By comparing the dimension of the moduli spaces occurring in (2) (for the case $r = 0$) we see that $\dim \mathcal{R}_g \geq \dim \mathcal{A}_{g-1}$ for $g \leq 6$ (to simplify the notation we refer to $\mathcal{P}_{g,0}, \mathcal{R}_{g,0}$ with $\mathcal{P}_g, \mathcal{R}_g$). Hence it is natural to check when it is possible to realize a (general) abelian variety as the Prym variety of a certain cover, i.e. when $\mathcal{P}_g$ is dominant. Wirtinger ([90]) gave a positive answer to this question. Furthermore different approaches of Friedman-Smith ([38]), Kanev ([52]) and Welters ([89]) showed that $\mathcal{P}_g$ is generically injective for $g \geq 7$. The tetragonal construction of Donagi ([25]) shows that it is never injective.

On the other hand, for $g \leq 6$, a detailed study of the structure of the fibres was provided by the works of Verra ([87], for $g = 3$), Recillas ([83] for $g = 4$), Donagi ([25] for $g = 5$) and Donagi and Smith ([26] for $g = 6$). All these results have been summarized under a uniform presentation in the fundamental work of Donagi [25].

Contrary to the unramified case, less was known about the ramified Prym map. Indeed it has deserved less attention in the literature. Although some specific cases were considered previously in [5] and in [74], a systematic study of the properties of the ramified Prym map in full generality started with the work of Marcucci and Pirola [64] (which was published more than 20 years later than the works concerning étale Prym maps!). Combining their results with the main theorems of Marcucci-Naranjo in [63] and Naranjo-Ortega in [75], the generic Torelli theorem is proved for all the cases where the dimension of the source $\mathcal{R}_{g,r}$ is smaller than the dimension of the target $\dim \mathcal{A}^{\delta}_{g-1+\frac{r}{2}}$, i.e. when

$$3g-3+r \leq \frac{1}{2}(g-1+\frac{r}{2})(g+\frac{r}{2}), \tag{3}$$

with the exception of the only equidimensional case, that is when $g = 3$ and $r = 4$, where it was known by [5] and by [74] that the map is dominant of degree 3. Actually, very recently, a global Torelli Theorem has been announced for all g and $r \geq 6$. We are referring to the work of Ikeda ([49]) for $g = 1$ and to that of Naranjo-Ortega ([76]) for all g. It is remarkable to notice that this result says that the situation is very different from that holds for the étale Prym map.

Our aim is to complete the study of the ramified Prym map $\mathcal{P}_{g,r}$ analysing the geometry of the generic fibre when

$$\dim \mathcal{R}_{g,r} > \dim \mathcal{A}^{\delta}_{g-1+\frac{r}{2}}.$$

Inequality (3) shows that this is only possible in the following six cases:

$$\begin{aligned} 1 \leq g \leq 4 \quad & r = 2 \\ 1 \leq g \leq 2 \quad & r = 4. \end{aligned}$$

The case $g = 1, r = 4$ was considered by Barth in his study of abelian surfaces with polarization of type $(1, 2)$ (see [6]).

For the remaining cases we have found (except for the case $g = 4$) direct procedures mainly based on the bigonal construction (see [25]) and the extended trigonal construction (see [57]).

For the case $r = 2$, $g = 4$ we look at the fibre of $\bar{\mathcal{P}}_5$ studied by Donagi in [25] and we intersect it with an appropriate open set in the boundary of $\bar{\mathcal{R}}_5$ given by "admissible" covers with one node (in the sense of Beauville) which are identified with elements of $\mathcal{R}_{4,2}$ by glueing the two ramification points (and, accordingly, the two branch points).

The strategy of identifying the moduli space $\mathcal{R}_{g,2}$ with an open set in the boundary divisor of $\bar{\mathcal{R}}_g$ and then of studying the étale fibre described by the mentioned work of Verra, Recillas and Donagi could have been used for all cases $r = 2$ and $1 \leq g \leq 4$. Unfortunately, for some among the four cases (in particular $g = 2, r = 2$), this procedure becomes challenging. This is the reason why we decide to tackle our problem with a different, and more direct, approach.

We prove the following:

Theorem 6. *Assume that*

$$(g, r) \in \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (4, 2)\},$$

then the ramified Prym map $\mathcal{P}_{g,r}$ is dominant. Moreover, the generic fibre can be described as follows.

a) For a generic elliptic curve E the fibre $\mathcal{P}^{-1}_{1,2}(E)$ is isomorphic to $L_1 \sqcup \ldots \sqcup L_4$, where each L_i is the complement of three points in a projective line.

b) (Barth) Let (A, L) be a generic abelian surface with a polarization of type $(1, 2)$. Then there is a natural polarization L^* of type $(1, 2)$ in the dual abelian variety A^* and the fibre $\mathcal{P}^{-1}_{1,4}(A)$ is canonically isomorphic to the linear system $|L^*|$.

c) *The generic fibre of* $\mathcal{P}_{2,2}$ *is isomorphic to the complement of* 15 *lines in a projective plane.*

d) *The generic fibre of* $\mathcal{P}_{2,4}$ *is isomorphic to the complement of* 15 *points in an elliptic curve.*

e) *Let* X *be a generic quartic plane curve, consider the variety* $\mathsf{G}^1_4(X)$ *of the* g^1_4 *linear series on* X*, and denote by* i *the involution* $L \mapsto \omega_X^{\otimes 2} \otimes L^{-1}$*. Then* $\mathcal{P}^{-1}_{3,2}(JX)$ *is isomorphic to the quotient by* i *of an explicit* i*-invariant open subset of* $\mathsf{G}^1_4(X)$*.*

f) *Let* (V, δ) *be a generic element in* $\mathcal{RC}^+$ *and let* $\Gamma \subset JV$ *be the curve of lines* l *in* V *such that there is a* 2*-plane* Π *containing* l *with* $\Pi \cdot V = l + 2r$*. Then* $\mathcal{P}^{-1}_{4,2}(V, \delta)$ *is isomorphic to the irreducible étale double covering of* Γ *attached to the restriction of* δ *to* Γ*.*

To be more precise in the last statement of our Theorem, we need to recall that Donagi found a birational map

$$\kappa : \mathcal{A}_4 \dashrightarrow \mathcal{RC}^+,$$

where $\mathcal{RC}^+$ is the moduli space of pairs (V, δ), V being a smooth cubic threefold and δ an "even" 2-torsion point in the intermediate Jacobian JV (see [25, section 5]).

Finally we describe some examples of irreducible components of fibres of ramified Prym maps which yield totally geodesic or Shimura subvarieties of $\mathcal{A}_g$.

In particular, the images in $\mathcal{M}_2$ and in $\mathcal{M}_3$ of $\mathcal{R}_{1,2}$, respectively $\mathcal{R}_{1,4}$, are the bielliptic loci and in [36] it is shown that they yield Shimura subvarieties of $\mathcal{A}_2$, and $\mathcal{A}_3$ respectively (they are the families (1) and (2) above). A straightforward application of Theorem 3 shows that the irreducible components of the fibres of the Prym maps $\mathcal{P}_{1,2}$, $\mathcal{P}_{1,4}$ yield totally geodesic curves in $\mathcal{A}_2$ and $\mathcal{A}_3$, countably many of them are Shimura.

At the end, we give a new explicit example of a totally geodesic curve which is an irreducible component of a fibre of the Prym map $\mathcal{P}_{1,2}$.

Structure of the Thesis

This thesis consists of two parts: they are interconnected and self-contained too. Therefore they can be found of independent interest.

The first part is composed of three chapters while the second one by two chapters. Chapters 2, 3 and 5 contain our original results. The reader will find a specific introduction at the beginning of each of them.

In the following we briefly describe the contents of each chapter.

In **Chapter 1** we present general preliminaries on curves, Jacobians and Torelli morphism. We recall several well-known results, most of them without proofs, on group actions on Riemann surfaces, Galois covers and associated monodromy map. Then we explain Riemann Existence Theorem and how it works in families, i.e. how it is possible to start from a numerical datum to produce a family of Galois coverings with fixed genera, number of ramification points and monodromy. Finally, we introduce group representation theory and the definition of Galois orbits of complex irreducible representations to construct rational irreducible representations.

In **Chapter 2** we start with a description of symmetric (Riemannian or not) manifold and of its totally geodesic subvarieties. Then we focus on a particular Hermitian symmetric space, the Siegel space $\mathfrak{S}_g$, and we describe a technique used to compare two different totally geodesic submanifolds of $\mathfrak{S}_g$. Later, in the context of the so-called Coleman-Oort conjecture, we introduce Shimura subvarieties of $\mathcal{A}_g$ as totally geodesic subvarieties with an additional arithmetical property. In particular, we explain condition $(*)$ of [32, 36] and how it has been used to produce 30+2 examples of Shimura subvarieties of $\mathcal{A}_g$ generically contained in the Torelli locus considering Galois covering of $\mathbb{P}^1$ and of elliptic curves. The same condition is used to give a bound on the genus of curves occurring in families satisfying $(*)$. By means of this bound, we "complete" the classification of examples of Shimura subvarieties of $\mathcal{T}_g$ obtained from families of Galois covering satisfying $(*)$. Finally, we show that the 6 families of [36] admit two fibrations in totally geodesic subvarieties. In this way, we show the existence of infinitely many new examples, countably many of which are Shimura.

In **Chapter 3** we study group algebra decomposition and we apply this tool to decompose the Jacobians occurring in the families of the Galois covering satisfying condition $(*)$ found in [32, 36]. In particular, we show that many among these examples have totally decomposable Jacobians. This motivates our comparison between totally geodesic subvarieties of $\mathcal{T}_g$ and subloci whose general element has completely decomposable Jacobian. Unfortunately, we show that there does not exist a definite relation between the two properties.

In **Chapter 4** we generalize the definition of Jacobian variety considering also singular curves and we imitate the construction in case of cubic 3-folds. Then we define Prym varieties and Prym maps associated either to unramified covers or to ramified ones. Moreover, we recall known results concerning when these maps are generically injective, injective and dominant. We also study the codifferential of the Prym map in both cases. It turns out that the study of the generic fibre of the ramified Prym map was still open and this is the reason why we address this problem. Then we recall the partial compactification $\bar{\mathcal{R}}_g$ of $\mathcal{R}_g$ given by Beauville. Indeed, in [7], the author extended the classical Prym map to certain "admissible covers" in such a way that the extended Prym map becomes proper. Finally, we describe polygonal constructions: the bigonal, the trigonal (also in ramified case) and the tetragonal ones.

In **Chapter 5** we focus on the study of the structure of the generic fibre of the ramified Prym map when the dimension of the source is strictly greater than that of the target. This is only possible in six cases that are thus addressed. One among them had been already considered by Barth in [6]. We devote one section for each of them and for completeness also the case of Barth is recalled. The most involved cases are $g = 3$, $r = 2$ and $g = 4$, $r = 3$. By means of the trigonal construction, the case $g = 3$, $r = 2$ is related with tetragonal series on a generic quartic plane curve which does not contain two divisors of type $2p + 2q$. In the case $g = 4$, $r = 2$ we need to take care of the behaviour at the boundary of Donagi's description of the fibre of $\bar{\mathcal{P}}_5$. In particular, we have to study quadrics containing a nodal canonical curve of genus 5 which we consider that can be interesting on its own. Finally, we explain the link between the two parts of this Thesis. Indeed we give some explicit examples of fibres of Prym maps as totally geodesic subvarieties of $\mathcal{T}_g$.

Grazie!

La matematica vista nella giusta luce, possiede non soltanto verità ma anche suprema bellezza; una bellezza fredda e austera. (B. Russell)

Ringrazio di cuore la mia relatrice Paola Frediani per avermi quantomeno posizionata sotto la "giusta luce", con l'impegno, la fatica e il carisma che le ho richiesto in tutti questi anni. Spero con i tuoi insegnamenti di poter imparare a cogliere ed apprezzare ogni volta di più questa bellezza! **Ringrazio** moltissimo il mio co-relatore Carlos Naranjo per aver illuminato strade nuove, nella matematica e in Barcellona. Sempre con estrema pazienza e tanto affetto. È stato come in una giornata di sole! **Agradezco** mucho Anita Rojas y el Departamento de Matemáticas de la UChile por toda la disponibilidad y la calidez con la cual alegraron mi estancia en Santiago. ¡Me recibieron como si fuera parte del grupo universidario chileno! **Ringrazio** la mia "famiglia", in senso lato. Quella che ha senso pratico e non ne ha affatto. Quella delle curve di genere g e quella delle ciambelle con uno o più buchi, vuote o ripiene, con o senza zucchero. Quella che con un silenzio o con una risata scalda il cuore.

I

Shimura Varieties in the Torelli Locus

CHAPTER 1

Basics I

1.1 Complex tori and abelian varieties

Curve and their Jacobians play a key role in this thesis so we start with a brief recall of basic results on complex tori.

Definition 1.1. A *complex torus* $T := {}^{V}\!/_{L}$ of dimension g is the quotient of a complex vector space V of dimension g by a lattice L (i.e. a discrete subgroup of maximal rank $2g$) in V. Using the natural quotient map $\pi : V \to T$ we get that T is a compact (g-dimensional) complex manifold endowed with a group structure.

We can always describe T by using a matrix: take basis $e_1, ..., e_g$ for V and $\lambda_1, ..., \lambda_{2g}$ for L and write $\lambda_j = \sum \lambda_{ij} e_i$. Then we can define

$$\Pi = (\lambda_{ij})$$

the *period matrix* of the torus. This matrix gives algebraic obstructions to let T be a projective variety.

Definition 1.2. An *abelian variety* is a complex torus which admits a projective embedding into some $\mathbb{P}^N$.

Theorem 1.1.1 (Kodaira)**.** *T is an abelian variety if and only if it admits a polarization, i.e. there exists $H : V \times V \to \mathbb{C}$ hermitian form such that:*

1. $H > 0$;
2. $\operatorname{Im} H(L \times L) \subseteq \mathbb{Z}$

$\operatorname{Im} H$ is the first Chern-class of an ample line bundle on T which gives the projective embedding. Since $E := \operatorname{Im} H$ is integer-valued, there exists a basis for L (called symplectic) such that E has associated matrix of type

$$E = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$$

where Δ is the diagonal matrix $\Delta := (d_1, ..., d_g)$. Δ gives the type of the polarization and if $\Delta = (1, ..., 1)$ then T is a *principally polarized abelian variety*, from now on denoted by *ppav*. Usually the zero-locus of the unique section of the associated ample line bundle is denoted by Θ.

Riemann bilinear relations provide necessary and sufficient condition, in terms of Π and E, for T to be an abelian variety. Indeed we have the following (see for instance [11]):

Theorem 1.1.2. *(Riemann Bilinear Relations) A complex torus T is a polarized abelian variety if and only if there exists a basis for V and a symplectic basis for L such that the period matrix becomes*

$$\Pi = \begin{pmatrix} \Delta & Z \end{pmatrix},$$

where Δ is the diagonal-type matrix and Z is symmetric with positive definite imaginary part.

The set

$$\mathfrak{S}_g = \{Z \in M(g, \mathbb{C}) : Z = Z^t \text{ and } \operatorname{Im} Z > 0\}$$

is known as the *Siegel space*. By construction it parametrizes the set of polarized abelian varieties of a given type Δ with a symplectic basis. The symplectic group $Sp(2g, \mathbb{Z})$ acts on $\mathfrak{S}_g$ in the following way:

$$R \cdot Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (A + CZ)(B + DZ)^{-1}.$$

Clearly we have a surjective quotient morphism:

$$\mathfrak{S}_g \to \mathcal{A}_g := \mathfrak{S}_g \Big/ Sp(2g, \mathbb{Z}),$$

where we let $\mathcal{A}_g$ be the moduli space of *ppav* of dimension g.

Obviously every 1-dimensional complex torus is an abelian variety and it is easy to show that there exists a principal polarization. Suppose that $X := \mathbb{C}/L$ has lattice L generated by $1, \tau$, where τ is such that $\operatorname{Im} \tau > 0$. Then taking

$$E(a_1 + \tau b_1, a_2 + \tau b_2) = a_1 b_2 - a_2 b_1 \qquad \forall\, a_i, b_i \in \mathbb{R} \tag{1.1}$$

we get the desired principal polarization. Moreover note that if we consider $H_1(X, \mathbb{Z}) = \langle 1, \tau \rangle$ then (1.1) gives the (geometric) intersection number of the 1-cycles $a_1 + \tau b_1, a_2 + \tau b_2$.
More generally

Definition 1.3. Let us consider X a compact Riemann surface of genus $g \geq 1$. Then its *Jacobian variety* is defined as the quotient

$$JX := H^0(X, \omega_X)^* \Big/ H^1(X, \mathbb{Z})$$

The variety JX is a *ppav*. In fact the injection

$$H_1(X, \mathbb{Z}) \hookrightarrow H^0(X, \omega_X)^*$$
$$[\gamma] \mapsto (\omega \to \int_\gamma \omega)$$

extends the (geometric) intersection number of the lattice $L = H_1(X, \mathbb{Z})$ to the $\mathbb{C}$-vector space $V = H^0(X, \omega_X)^*$. It is easy to see that this induced map determines a canonical principal polarization on JX.

Therefore, letting $\mathcal{M}_g$ be the moduli space of smooth projective curves of genus g, we can introduce the *Torelli map*

$$j : \mathcal{M}_g \to \mathcal{A}_g$$
$$[X] \mapsto [JX, \Theta]$$

Both $\mathcal{M}_g$ and $\mathcal{A}_g$ have a natural structure of quasi-projective variety and j is a regular map. In particular a fundamental result in algebraic geometry asserts the following:

Theorem 1.1.3 (Torelli). *The map j is injective.*

1.2 Group action on Riemann Surfaces

This section is devoted to fundamental results on actions of group on Riemann surfaces. All the details are extracted from [66].

Definition 1.4. Let G be a group and consider X a Riemann surface. An *action* of G on X is a map $G \times X \to X$ which sends the pair (g, p) to the point $g \cdot p$ such that:

- $(gh) \cdot p = g \cdot (h \cdot p) \quad \forall p \in X$ and $g, h \in G$;
- $e \cdot p = p \quad \forall p \in X$, where e is the identity element of the group.

The *orbit* of a point p is the set $G \cdot p := \{g \cdot p : g \in G\}$.

The *stabilizer* a point p is the subgroup $G_p := \{g \in G : g \cdot p = p\}$. It will be useful to note that points in the same orbit have conjugate stabilizers. In fact: $G_{g \cdot p} = gG_pg^{-1}$. Moreover $|G \cdot p||G_p| = |G|$.

The *kernel* of an action is the subgroup $K := \{g \in G : g \cdot p = p \quad \forall p \in X\}$. It's a normal subgroup of G and the quotient G/K acts on X with trivial kernel and identical orbits to that of G. Therefore we usually work with trivial kernel, i.e. with *effective actions*.

The action is *continuous* (resp. *holomorphic*) if the map which sends p to $g \cdot p$ is continuous (resp. holomorphic).

It is possible to show that if the action is holomorphic and effective then every finite stabilizer actually it's a finite cyclic subgroup of G. Furthermore, if G is finite then the set of points with non-trivial stabilizer is discrete.

The *orbit space* is the set of orbits. It is described by the quotient map $\pi : X \to X/G$, which associates each point to its orbit. Clearly choosing for X/G the quotient topology, π is continuous. Moreover the following proposition holds:

Proposition 1.2.1. *Let us consider an effective and holomorphic action of a finite group G on a Riemann surface X. Then $^X/_G$ is a Riemann surface and there exist complex charts which make π holomorphic. Moreover π has degree equal to the cardinality of the group and $mult_p(\pi) = |G_p|$ for any point $p \in X$.*

Remark 1. The construction of the complex structure of $^X/_G$ can also be done requiring that the action is *properly discontinuous*. This means that for each pair of points $(p, q) \in X$ there exist neighborhoods U_p and U_q such that the set $\{g \in G : (g \cdot U_p) \cap U_q\}$ is finite. This forces the quotient space to be Hausdorff. Also the set of point with non-trivial stabilizer is discrete and it is possible to reply an infinite-version of Proposition 1.2.1. Nevertheless from now on we will deal with finite group.

A map $\pi : X \to {}^X/_G$ is called *Galois covering* with Galois group G. Indeed, out of its branch locus (and consequently out of its preimage), π is a topological cover such that for every pair of point $x_0, x_1 \in X$ with $\pi(x_0) = \pi(x_1)$ there exists a covering transformation $g \in G$ such that $g \cdot x_0 = x_1$. When the group is cyclic we refer to π saying *cyclic cover*.

Lemma 1.2.2. *Let G be a finite group acting holomorphically and effectively on X. Consider the quotient map $\pi : X \to {}^X/_G$. Then for every branch point $\overline{p} \in {}^X/_G$ there is an integer $r \geq 1$ such that $\pi^{-1}(\overline{p})$ consists of exactly $|G|/r$ points and each of these points has order r.*

Proof. Take a branch point $\overline{p}$ and let $x_1, ..., x_s$ be the points of X lying above it. Since they are in the same orbit, they all have conjugate stabilizers. Thus each stabilizer is of the same order, call it r. This implies $s = |G|/r$. □

Therefore we have the following formulation of Riemann-Hurwitz's formula:

Corollary 1.2.2.1. *Let G be a finite group which acts holomorphically and effectively on a Riemann surface X with quotient map $\pi : X \to {}^X/_G$. Suppose that there are k branch points $\overline{p}_1, ..., \overline{p}_k \in {}^X/_G$ such that π has order r_i at the $|G|/r_i$ points above $\overline{p}_i$. Then:*

$$2g(X) - 2 = |G|(2g({}^X/_G) - 2) + \sum_{i=1}^{k} \frac{|G|}{r_i}(r_i - 1). \tag{1.2}$$

This Corollary gives a bound on the order of groups G that can act on a Riemann surfaces of genus greater or equal than 2. Indeed the following holds:

Theorem 1.2.3 (Hurwitz)**.** *Let G be a finite group that acts holomorphically and effectively on a Riemann surface X such that $g(X) \geq 2$. Then:*

$$|G| \leq 84(g - 1)$$

Since the full group of $\mathrm{Aut}(X)$ certainly acts holomorphically and effectively on X Hurwit's Theorem always implies

$$|\,\mathrm{Aut}(X)| \leq 84(g(X)-1),$$

i.e. the finiteness of the automorphism group of a Riemann surface of genus $g \geq 2$. It's known that this is not true in case of genus $g = 0$ (since we have the Moebius transformations $x \mapsto (a+bx)/(c+dx)$) and in case of elliptic curves (where the automorphism group is infinite because of the translations). On the other hand the generic curve of genus $g \geq 3$ has no automorphism except for the identity.

1.3 Covers and Monodromy

In this section we introduce the concept of *monodromy* of a holomorphic map between Riemann surfaces and we show how it can be used to recover the map itself. Our attention will be devoted to covering maps.

Let us take V a topological space and fix a base point q. The fundamental group $\pi_1(V,q)$ acts on the universal cover of V and determines a 1-1 correspondence between isomorphism classes of connected coverings of V, i.e. maps $f : U \to V$, and conjugacy classes of subgroups $H \subseteq \pi_1(V,q)$. The degree of the covering is the index of H as a subgroup of the fundamental group of V.

Now take a loop γ based on q and consider its d-lifted paths $\tilde{\gamma}_1, ..., \tilde{\gamma}_d$. Each $\tilde{\gamma}_i$ is the unique lift of γ such that $\tilde{\gamma}_i(0) = x_i$, where we denote $x_i, i = 1, ..., d$ the preimages in $f^{-1}(q)$. Being $\tilde{\gamma}_i$ a lift of γ, its endpoint $\tilde{\gamma}_i(1)$ has to be among $\{x_i, i = 1, ..., d\}$. Put $\tilde{\gamma}_i(1) = x_j = x_{\sigma(i)}$. By construction σ is a permutation of the indices $\{1, ..., d\}$ and it is easy to see that it depends only on the homotopy class of γ.

Definition 1.5. The *monodromy representation* of a covering map $f : U \to V$ of finite degree d is the group homomorphism

$$\begin{aligned} \rho : \pi_1(V,q) &\to S_d \\ [\gamma] &\mapsto \sigma \end{aligned}$$

where S_d is the symmetric group of the permutations of d elements.

Note that since U is connected the image of the monodromy map is a *transitive* subgroup of S_d, i.e. for every pair of indices i, j there exists σ in $\rho(\pi_1(V,q))$ such that $\sigma(i) = j$.

The process can be reversed: suppose to start with a group homomorphism $\rho : \pi_1(V,q) \to S_d$, from the fundamental group of a topological space to S_d, which has transitive image. Consider the subgroup

$$H := \{[\gamma] \in \pi_1(V,q) : \rho([\gamma])(1) = 1\}$$

then H has index d in S_d and thus it induces a connected covering space $f : U \to V$ of degree d. Of course, this covering has associated monodromy map given by the homomorphism ρ with which we start.

Actually, our interest is devoted to holomorphic non-constant map $F : X \to Y$ between compact Riemann surfaces. Call d the degree. Note that because of the ramification, F is not a topological cover. Call $B \subset Y$ the finite set of branch points and let $R = F^{-1}(B)$. Therefore the restriction $F|_{X \smallsetminus R} : X \smallsetminus R \to Y \smallsetminus B$ is a true topological cover.

Now apply Definition 1.5 to get the monodromy homomorphism $\rho : \pi_1(Y \smallsetminus B, q) \to S_d$. Hence, with ρ, we can construct a topological covering $F_\rho : U_\rho \to Y \smallsetminus B$. It is possible to show that plugging all holes given by branch points and their preimages, we get a map $F_\rho : X_\rho \to Y$ between compact Riemann surfaces which has branch points at most in B. Moreover, removing B from Y and its preimage from X_ρ, we get once more the same monodromy morphism ρ associated to F.

This proves the following

Theorem 1.3.1 (Riemann's Existence Theorem). *Let Y be a compact Riemann surface and let B be a finite subset of Y. The following 1-1 correspondence holds:*

$$\left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{holomorphic maps}\\ F : X \to Y\\ \text{of degree } d\\ \text{whose branch points lie in } B\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{group homomorphisms}\\ \rho : \pi_1(Y \smallsetminus B, q) \to S_d\\ \text{with transitive image}\\ \text{up to conjugacy in } S_d\end{array}\right\}$$

1.3.1 Coverings of $\mathbb{P}^1$

Let us specialize Riemann's Existence Theorem to the case of holomorphic coverings of the projective line $\mathbb{P}^1$. Fix r points $t = (t_1, ..., t_r)$ in $\mathbb{P}^1$ and a base point $t_0 \in U_t = \mathbb{P}^1 \smallsetminus \{t_1, ..., t_r\}$.

The fundamental group of U_t is a free group on r generators which satisfy a single relation:

$$\pi_1(U_t, t_0) \cong \Gamma_{0,r} := \langle [\gamma_1], [\gamma_2], ..., [\gamma_r] : \ [\gamma_1][\gamma_2]...[\gamma_r] = 1 \rangle,$$

where $[\gamma_i]$ is the homotopy class of a small loop around t_i.

This means that a group homomorphism $\rho : \pi_1(U_t, t_0) \to S_d$ is determined by the choice of r permutations $\sigma_1, ..., \sigma_r$ such that $\sigma_1 ... \sigma_r = 1$.

The r-tuple $(\sigma_1, ..., \sigma_r)$ is often called *generating vector* of ρ.

Now we focus on the case of $f : X \to \mathbb{P}^1$, Galois cover of $\mathbb{P}^1$ branched on t. Denote, as before, $U_t = \mathbb{P}^1 \smallsetminus \{t_1, ..., t_r\}$ and $V = f^{-1}(U_t)$.

Thus $f|_V : V \to U_t$ is an unramified (hence topological) Galois cover. Let G be its group of deck transformations. Then there is a surjective homomorphism $\pi_1(U_t, t_0) \twoheadrightarrow G$, which is well defined up to composition by inner automorphism of G. Since $\pi_1(U_t, t_0) \cong \Gamma_{0,r}$, we automatically get the epimorphism

$$\theta : \Gamma_{0,r} \twoheadrightarrow G.$$

Putting $m_i = \text{ord}(\theta(\gamma_i))$, the local monodromy around t_i, we obtain the vector $\mathbf{m} = (m_1, ..., m_r)$. Therefore we can give the following

Definition 1.6. A *datum* is a triple $(\mathbf{m}, G, \theta)$, where $\mathbf{m} = (m_1, ..., m_r)$ is an r-tuple of integers $m_i \geq 2$, G is a finite group and $\theta : \Gamma_{0,r} \twoheadrightarrow G$ is an epimorphism such that $\theta(\gamma_i)$ has order m_i for each i.

Thus a Galois cover of $\mathbb{P}^1$ branched on r points gives rise, up to some choices, to a datum. Riemann's Existence Theorem ensures that the process can be reversed: a branch locus t and a datum determine (up to isomorphism) a cover of $\mathbb{P}^1$.

Formula (1.2) with $r_i = m_i$ determines the genus of X.

Cyclic covers of $\mathbb{P}^1$

Here we would like to study the connection between plane curves and cyclic coverings of the projective line.

A m-cyclic cover of the projective line is a curve C which admits a degree m morphism to $\mathbb{P}^1$ such that the associated Galois group is cyclic.

An irreducible cyclic cover of $\mathbb{P}^1$ can be given by a prime ideal

$$(y^m - (x - a_1)^{d_1} \cdot ... \cdot (x - a_r)^{d_r}) \subset \mathbb{C}[x, y].$$

Note that this ideal defines an affine curve in $\mathbb{A}^2$ which possibly has singularities if there are $d_i > 1$. Nevertheless there exists a unique smooth projective curve C birationally equivalent to this plane curve. Moreover C admits a natural projection which, on the affine coordinates (x, y), is the map

$$\begin{aligned} \pi : C &\to \mathbb{P}^1 \\ (x, y) &\mapsto x \end{aligned}$$

which is holomorphic. Hence one obtains the cyclic cover of the smooth curve C onto $\mathbb{P}^1$.

Remark 2. The "cyclicity" of these curves comes from the existence of an automorphism σ: choosing a $m^{th}-$root of the unity ξ we can define the map $\sigma : C \to C$ which sends $(x, y) \mapsto (x, \xi y)$. Notice that $\pi \circ \sigma = \pi$.

Lemma 1.3.2. *Assume that $d_1, ..., d_r < m$. Let C be the (non-singular projective) curve given by*

$$y^m = (x - a_1)^{d_1} \cdot ... \cdot (x - a_r)^{d_r}.$$

Then the Galois group G is $\mathbb{Z}/m\mathbb{Z}$ and the covering $C \to \mathbb{P}^1$ is given by the monodromy map
$$\rho : \pi_1(\mathbb{P}^1 \smallsetminus \{a_1, ..., a_r\}, p) \to \mathbb{Z}/m\mathbb{Z} \qquad [\gamma_i] \mapsto d_i,$$
where γ_i are loops running counterclockwise around exactly one a_i.

The point ∞ is a branch point and
$$\rho(\gamma_\infty) = -\sum_{i=1}^{r} d_i \ \ mod \ m$$
iff m doesn't divide $\sum_{i=1}^{r} d_i$.

Let $m_i = ord_{\mathbb{Z}/m\mathbb{Z}}(d_i)$, the associated datum is $((m_1, ..., m_r), \mathbb{Z}/m\mathbb{Z}, \rho)$.

Here we point out that the condition
$$[\gamma_1][\gamma_2]...[\gamma_r] = 1$$
implies
$$\rho(\gamma_{)}\rho(\gamma_2)...\rho(\gamma_r) = 1,$$
i.e. $\sum_{i=1}^{r} d_i + d_\infty \equiv 0 \; mod \, m$ (using the additional group structure of $\mathbb{Z}/m\mathbb{Z}$).

Corollary 1.3.2.1. *Let $G = \mathbb{Z}/m\mathbb{Z}$, $d \in \mathbb{Z}$ and $[d]_m$ the residue class of d in G. Consider the monodromy description of the local covers given by*
$$y^m = (x - a_1)^{d_1} \cdot ... \cdot (x - a_r)^{d_r}$$
and
$$y^m = (x - a_1)^{[dd_1]_m} \cdot ... \cdot (x - a_r)^{[dd_r]_m}$$
as shown Lemma 1.3.2. Since they coincide, we conclude that the two covers are equivalent.

1.3.2 Coverings of curves of genus $g' \geq 0$

The construction in case of $g' = 0$ can be generalized considering Galois covering $f : X \to Y$, where Y is a curve of genus $g' \geq 0$.

Let, as before, $t = (t_1, ..., t_r)$ be the branch locus of f and $U_t = Y \smallsetminus t$. Choosing $t_0 \in U_t$ we get
$$\pi_1(U_t, t_0) \cong \Gamma_{g',r} := \langle \alpha_1, \beta_1, ..., \alpha_{g'}, \beta_{g'}, \gamma_1, ..., \gamma_r : \prod_{i=}^{r} \gamma_i \prod_{i=1}^{g'} [\alpha_i, \beta_i] = 1 \rangle,$$
where $\alpha_1, \beta_1, ..., \alpha_{g'}, \beta_{g'}$ are loops in U_t which intersect only in t_0 and determine a basis for $H_1(Y, \mathbb{Z})$, while $\gamma_1, ..., \gamma_r$ are constructed as follows: for every i take a path a_i connecting t_0 with a point τ_i in U_t "near" t_i, then take a loop b_i in U_t based on τ_i with winding number one around the branch point t_i. The path $a_i^{-1} b_i a_i$ is the desired loop γ_i (based on t_0).

Now set $V = f^{-1}(U_t)$ and $f|_V : V \to U_t$ the induced topological Galois covering. The isomorphism $\pi_1(U_t, t_0) \cong \Gamma_{g',r}$ produces an epimorphism $\theta : \Gamma_{g',r} \to G$, where G is the Galois group.

Consequently we can reformulate the following:

Definition 1.7. A *datum* is a triple $(\mathbf{m}, G, \theta)$, where $\mathbf{m} = (m_1, ..., m_r)$ is an r-tuple of integers $m_i \geq 2$, G is a finite group and $\theta : \Gamma_{g',r} \twoheadrightarrow G$ is an epimorphism such that $\theta(\gamma_i)$ has order m_i for each i.

Also in this case that the process can be reversed: fixing a curve Y and a branch locus t, a datum $(\mathbf{m}, G, \theta)$ gives rise to a Galois covering $f : X \to Y$ with Galois group G.

1.3.3 Families of Galois coverings

Here we would like to show that the same process can be reversed also in families, namely to any datum it is possible to associate a family of Galois coverings of curves of genus g'. In order to explain how this process works, we need to borrow the formalism of Teichmüller theory, referring to [2][Chap. XV].

Let us fix a compact oriented topological surface Σ of genus g' and a ordered finite subset $P = (p_1, ..., p_r)$ of points of Σ such that $2g' - 2 + r > 0$.

Definition 1.8. Let $(Y, \mathbf{y})$ be a *P-pointed curve*, that is $\mathbf{y} : P \to Y$ is an injective map and $y_i = \mathbf{y}(p_i)$ are the marked points. A *Teichmüller structure* on $(Y, \mathbf{y})$ is the datum of the isotopy class $[f]$ of an orientation-preserving homeomorphism

$$f : (Y, \mathbf{y}) \to (\Sigma, P),$$

where the allowable isotopies are those which map y_i to p_i for every i and for each choice of P.

Definition 1.9. Two marked curves with Teichmüller structure $(Y, \mathbf{y}, [f])$ and $(Y', \mathbf{y}', [f'])$ are equivalent if there exists a biholomorphism of r-pointed curves

$$\varphi : (Y, \mathbf{y}) \to (Y', \mathbf{y}')$$

such that

$$\varphi(y_i) = y_i' \qquad \text{and} \qquad [f] = [f'\varphi].$$

The resulting quotient space is called the *Teichmüller space* $\mathcal{T}_{g',r}$ of a surface of genus g' with $r > 1$ marked points.

Remark 3. The Teichmüller structure rigidifies the r-marked curve $(Y, \mathbf{y})$. This means that

$$\mathrm{Aut}(Y, \mathbf{y}, [f]) = 1.$$

Indeed, in genus 0 the number of marked points is at least 3 and hence $(Y, \mathbf{y})$ is already rigid; in general an automorphism φ of a marked surface such that $[f\varphi] = [\varphi]$ induces the identity in integral cohomology. Hence, one can show that φ is forced to be the identity.

It is exactly this "rigidity" that makes $\mathcal{T}_{g',r}$ a smooth manifold naturally equipped with a universal family.

Now we will define an important group and we will briefly describe its action on the Teichmüller space.

Definition 1.10. *The mapping class group*, also called Teichmüller modular group, is the group of all isotopy classes of orientation-preserving homeomorphism of (Σ, P) into itself. We denote it by $\operatorname{Map}_{g',r}$.

The mapping class group acts naturally on $\mathcal{T}_{g',r}$ in this way:

$$[\gamma] \cdot [Y, \mathbf{y}, [f]] = [Y, \mathbf{y}, [\gamma \circ f]] \tag{1.3}$$

Indeed, if γ and δ are isotopic then the marked structures $[\gamma] \cdot [Y, \mathbf{y}, [f]]$ and $[\delta] \cdot [Y, \mathbf{y}, [f]]$ are equivalent and hence (1.3) is well-defined. It is possible to show that the action is holomorphic, properly discontinuous (so the stabilizers are finite) and, in general, non free.

One can consider the quotient map

$$\mathcal{T}_{g',r} \to \mathcal{T}_{g',r} / \operatorname{Map}_{g',r}$$

which is just the forgetful map $(Y, \mathbf{y}, [f]) \mapsto (Y, \mathbf{y})$. It yields the following:

Theorem 1.3.3. *The orbit space of the Teichmüller space with respect to the mapping class group is the moduli space of r-marked genus g' curves:*

$$\mathcal{M}_{g',r} = \mathcal{T}_{g',r} \Big/ \operatorname{Map}_{g',r}.$$

At least set-theoretically the above equality is easy to see. Indeed, if $x \in \mathcal{T}_{g',r}$ and $[\gamma] \in \operatorname{Map}_{g',r}$, then x and $[\gamma] \cdot x$ map to the same point in $\mathcal{M}_{g',r}$. Conversely, if $[x] = [x']$, where $x = [Y, \mathbf{y}, [\gamma \circ f]]$ and $x' = [Y', \mathbf{y}', [f']]$, then (by definition of Teichmüller space) there exists an isomorphism $\varphi : (Y, \mathbf{y}) \to (Y', \mathbf{y}')$. Therefore $x' = [f' \varphi f^{-1}] \cdot x$ shows that x and x' are in the same orbit.

Moreover, since the Teichmüller space is topologically a ball and the mapping class group is a discrete group acting on it via a properly discontinuous action, the moduli space of curves inherits the structure of topological orbifold:

Definition 1.11. A n-dimensional *orbifold* is the datum of an Haursdorff topological space X together with an orbifold atlas, i.e. a collection $\mathcal{U} = \{(U_i, G_i, \pi_i)\}$ of n-dimensional compatible orbifold charts. An *orbifold chart* is a 3-tuple (U, G, π) such that:

- U is open in $\mathbb{R}^n$;
- G is a finite group of homeomorphisms of U;

- $\circ$ $\pi : U \xrightarrow{q} U/G \xrightarrow{t} X$ is the composition of q quotient map and t which induces a homeomorphism between U/G and an open set $V \subset X$.

An embedding $\lambda : (U_i, G_i, \pi_i) \to (U_j, G_j, \pi_j)$ between two charts is a smooth embedding $\lambda : U_i \to U_j$ such that $\pi_j \circ \lambda = \pi_i$. Two charts are compatible if there exists a chart (W, H, φ) with $\varphi(W) \subset \pi_i(U_i) \cap \pi_j(U_j)$ and two embeddings $\lambda_{i,j} : (W, H, \varphi) \to (U_{i,j}, G_{i,j}, \pi_{i,j})$.

Like a manifold, an orbifold is specified by local conditions; however, instead of being locally modelled on open subsets of $\mathbb{R}^n$, an orbifold is locally modelled on quotients of open subsets of $\mathbb{R}^n$. We can think on it as manifolds with isolated singularities. This allows to work as they were smooth.

Now we are ready to show that Riemann's Existence Theorem works also in families. Indeed, let us start with $(\mathbf{m}, G, \theta)$ a datum. Fix a point $[Y, \mathbf{y} = (y_1, ..., y_r), [f]]$ in $\mathcal{T}_{g',r}$, i.e. Y is a compact Riemann surface of genus g', $\mathbf{y}$ is an r-tuple of points of Y such that $y_i \neq y_j$ for all $i \neq j$. Moreover, choosing a point $p_0 \in \Sigma \smallsetminus P$, we can fix an isomorphism

$$\Gamma_{g',r} \cong \pi_1(\Sigma \smallsetminus P, p_0).$$

Hence, composing with θ, we get an epimorphism

$$\psi : \pi_1(Y \smallsetminus \mathbf{y}, f^{-1}(p_0)) \cong \pi_1(\Sigma \smallsetminus P, p_0) \cong \Gamma_{g',r} \twoheadrightarrow G.$$

ψ is the monodromy map associated to a Galois covering $C_{\mathbf{y}} \to Y$ branched at the points y_i with local monodromy m_i. Riemann-Hurwitz formula allows us to compute the genus of $C_{\mathbf{y}}$. Call it g. We can observe two facts:

- $\star$ The curve $C_{\mathbf{y}}$ comes naturally with an isotopy class of homeomorphism to a fixed branched cover $\Upsilon \to \Sigma$, where Υ has the same role of Σ in case of topological surface of genus g. Thus we obtain a map $\mathcal{T}_{g',r} \to \mathcal{T}_g$.
- $\star$ There exists a monomorphism of G into the mapping class group Map_g. In fact $G < \mathrm{Aut}(C_{\mathbf{y}}) \subseteq \mathrm{Diff}^+(C_{\mathbf{y}}) \twoheadrightarrow \mathrm{Map}_g$ and if $\varphi \in \mathrm{Aut}(C_{\mathbf{y}})$ is such that the induced $\varphi_* : H_1(C_{\mathbf{y}}, \mathbb{Z}) \to H_1(C_{\mathbf{y}}, \mathbb{Z})$ is the identity, then $\varphi = \mathrm{Id}$ (see Remark 3). Notice that the immersion depends on θ. Hence call the image G_θ.

It turns out that the image of $\mathcal{T}_{g',r}$ in $\mathcal{T}_g$ coincides with $\mathcal{T}_g^{G_\theta}$. Indeed the following holds (see [16],[42] for a proof):

Theorem 1.3.4. *$\mathcal{T}_g^{G_\theta}$ is a complex submanifold of $\mathcal{T}_g$ of dimension $3g' - 3 + r$ and it is isomorphic to the Teichmüller space $\mathcal{T}_{g',r}$.*

The isomorphism can be described as follows: if $(C, [h])$ is a curve with a marking such that $(C, [h]) \in \mathcal{T}_g^{G_\theta}$, then the corresponding point in $\mathcal{T}_{g',r}$ is $[(C/G, b = (b_1, ..., b_r), F)]$, where F is the induced marking (see [42]) and b is the branch locus of the projection map $C \to C/G$.

Remark 4. On $\mathcal{T}_g^{G_\theta}$ we have a universal family $\mathcal{C} \to \mathcal{T}_g^{G_\theta}$ of curves with a G-action. It is just the restriction of the universal family on $\mathcal{T}_g$.

We denote $\mathsf{M}(\mathbf{m}, G, \theta)$ the image of $\mathcal{T}_g^{G_\theta}$ in $\mathcal{M}_g$. It is an irreducible algebraic subvariety of the same dimension of $\mathcal{T}_g^{G_\theta} \cong \mathcal{T}_{g',r}$, i.e. $3g' - 3 + r$. As explained in [42, p. 79] there is an intermediate variety $\widetilde{\mathsf{M}}(\mathbf{m}, G, \theta)$ such that the projection factors through

$$\mathcal{T}_g^{G_\theta} \to \widetilde{\mathsf{M}}(\mathbf{m}, G, \theta) \xrightarrow{\nu} \mathsf{M}(\mathbf{m}, G, \theta) \tag{1.4}$$

The variety $\widetilde{\mathsf{M}}(\mathbf{m}, G, \theta)$ is the normalization of $\mathsf{M}(\mathbf{m}, G, \theta)$ and it parametrizes (as desired) Galois coverings of curves of genus g' with datum $(\mathbf{m}, G, \theta)$.

Hurwitz equivalence classes

Different data $(\mathbf{m}, G, \theta)$ and $(\mathbf{m}, G, \theta')$ may give rise to the same subvariety of $\mathcal{M}_g$. This is related to the choice of the isomorphism $\pi_1(\Sigma \smallsetminus P, p_0) \cong \Gamma_{g',r}$. The change from one choice to another can be described using an action of a "geometric" mapping class group which takes into account the differentiable structure of the curve Σ marked in r points.

Definition 1.12. The *(geometric) mapping class group* is the quotient between orientation-preserving diffeomorphisms of $\Sigma \smallsetminus \{p_1, ..., p_r\}$ and the ones isotopic to the identity:

$$\mathrm{Map}_{g',[r]} = \mathrm{Diff}^+(\Sigma \smallsetminus \{p_1, ..., p_r\}) \Big/ \mathrm{Diff}^0(\Sigma \smallsetminus \{p_1, ..., p_r\}) \cdot$$

Hence we can formulate the following:

Definition 1.13. The orbits of the action of the group $\mathrm{Map}_{g',[r]} \times \mathrm{Aut}(G)$ (Hurwitz's moves) are called *Hurwitz equivalence classes*.

To simplify the analysis, let us focus in case of $g' = 0$ and we refer to [12].

Theorem 1.3.5. *The mapping class group* $\mathrm{Map}_{0,[r]}$ *is isomorphic to the Braid group*

$$\mathbf{B_r} := \langle \sigma_1, ..., \sigma_r : \sigma_i\sigma_j = \sigma_j\sigma_i \textit{ for } |i-j| \geq 2,\ \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \rangle.$$

This allows us to describe explicitly Hurwitz's moves. Indeed, there exists a morphism $\varphi : \mathbf{B_r} \to \Gamma_{0,r}$ defined as

$$\varphi(\sigma_i)(\gamma_i) = \gamma_{i+1}, \quad \varphi(\sigma_i)(\gamma_{i+1}) = \gamma_{i+1}^{-1}\gamma_i\gamma_{i+1},$$
$$\varphi(\sigma_i)(\gamma_j) = \gamma_j \ \text{ for } j \neq i, i+1.$$

Therefore we get an action of $\mathbf{B_r}$ on the set of data in this way :

$$\sigma \cdot (\mathbf{m}, G, \theta) := (\sigma(\mathbf{m}), G, \theta \circ \varphi(\sigma^{-1})),$$

where $\sigma(\mathbf{m})$ is the permutation of $\mathbf{m}$ induced by σ. Moreover $\mathrm{Aut}(G)$ acts by

$$\alpha \cdot (\mathbf{m}, G, \theta) := (\mathbf{m}, G, \alpha \circ \theta).$$

Considering both actions, we obtain the aforementioned Hurwitz equivalence classes: data in the same class give rise to the same subvariety $\mathsf{M}(\mathbf{m}, G, \theta)$ of $\mathcal{M}_g$.

1.4 Groups Representation

This section is devoted to the introduction of some basic notions in representation theory. Everything and much more can be found in [85].

Let G be a finite group and V a $\mathbb{K}-$vector space (usually the field will be $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$).

Definition 1.14. A *representation* of G on V is a group homomorphism

$$\rho : G \to \mathrm{GL}(V).$$

The *degree* of ρ is the dimension of the vector space V.

Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ be two representations of G on $\mathbb{K}-$vector spaces V and W.

A linear map $f : V \to W$ is said *G-equivariant* if $f(\rho(g)v) = \tau(g)f(v)$ for all $g \in G$ and $v \in V$. The set of such maps is denoted $\mathrm{Hom}_G(V, W)$.

The representations ρ and τ are *isomorphic* if there exists $f \in \mathrm{Hom}_G(V, W)$ which is also a isomorphism of vector spaces.

Definition 1.15. Let us consider a subspace $W \subset V$. W is *G-invariant* if

$$\rho(g)\, W \subset W \quad \forall g \in G.$$

A representation $\rho : G \to \mathrm{GL}(V)$ is *irreducible* if there are no G-invariant subspaces except for $W = \{0\}$ or $W = V$.

We remark that every representation on a 1-dimensional vector space is automatically irreducible. Moreover it is possible to show that every representation can be decomposed as a (non-unique) direct sum of irreducible representations.

Lemma 1.4.1 (Schur's Lemma). *Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ be two irreducible representations of G on $\mathbb{K}-$vector spaces V and W. Let $f \in \mathrm{Hom}_G(V, W)$. Then:*

- *$f = 0$ or f is an isomorphism;*
- *If $\mathbb{K} = \mathbb{C}$ and $\rho = \tau$ then $f = \lambda\,\mathrm{Id}$. Thus $\mathrm{End}_G(V) = \mathbb{C}$.*

Definition 1.16. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G on $\mathbb{K}-$vector space V. The *character* of the representation ρ is the map

$$\begin{aligned} \chi : G &\to \mathbb{K} \\ g &\mapsto \mathrm{Tr}(\rho(g)), \end{aligned}$$

where $\mathrm{Tr}\rho(g))$ is the trace of the linear map $\rho(g) : V \to V$

Note that the character depends only on the isomorphism class of a representation.

Proposition 1.4.2. *Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ be two irreducible representations of G on $\mathbb{C}$-vector spaces V and W with characters χ_ρ, χ_τ. Then:*

* $\chi_\rho(e) = \dim V$*, where e is the identity element of the group;*
* $\chi_\rho(g) = \chi_\rho(hgh^{-1})$ *for any $g, h \in G$. This implies that χ is constant on the conjugacy classes of G;*
* $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$*;*
* *The representation $\rho \oplus \tau : G \to \mathrm{GL}(V \oplus W)$ has character $\chi_\rho + \chi_\tau$;*
* *The representation $\rho \otimes \tau : G \to \mathrm{GL}(V \otimes W)$ has character $\chi_\rho \cdot \chi_\tau$;*

It is possible to define a Hermitian scalar product $(\cdot, \cdot)$ on the space of $\mathbb{C}$-valued functions. In particular:

Proposition 1.4.3. *Consider $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ two representations of G on V and W with characters χ_ρ, χ_τ. Then*

$$(\chi_\rho, \chi_\tau) := \frac{1}{|G|} \sum_{g \in G} \overline{\chi_\rho(g)} \chi_\tau(g)$$

is a well-defined Hermitian scalar product on the space of characters.

The strength of this scalar product lies in the following:

Theorem 1.4.4. *The irreducible characters of a group G are orthonormal with respect to this scalar product. In fact, taking $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ two complex irreducible representations then:*

$$(\chi_\rho, \chi_\tau) = \dim \mathrm{Hom}_G(V, W) = \begin{cases} 1 & \text{if } \rho \cong \tau \\ 0 & \text{if } \rho \ncong \tau \end{cases}$$

Therefore the characters of irreducible representations are linearly independent. Hence we can state this fundamental result:

Theorem 1.4.5. *The irreducible characters of a finite group G form an orthonormal basis for the complex vector space of the class functions $\alpha : G \to \mathbb{C}$, i.e. maps which are constant on conjugacy classes.*

Moreover, the number of the irreducible representations of G (up to isomorphism) is equal to the number of conjugacy classes of G.

The above results reduce the study of representations to that of their characters. A easy consequence is the following:

Lemma 1.4.6. *Let $\chi_1, ..., \chi_s$ be the irreducible characters of a finite group G and let $\rho_i : G \to \mathrm{GL}(V_i)$ be the corresponding complex irreducible representations. Then every complex representation $\rho : G \to \mathrm{GL}(V)$ decomposes into irreducible representations as:*

$$V \cong V_1^{n_1} \oplus V_2^{n_2} \oplus ... \oplus V_s^{n_s} \qquad \textit{with} \quad n_i = (\chi_\rho, \chi_i).$$

Moreover

$$(\chi_\rho, \chi_\rho) = n_1^2 + n_2^2 + ... + n_s^2$$

Remark 5. This is the *canonical decomposition* we usually have in mind. In this sense one can say that there is *uniqueness* of the decomposition of a representation into irreducible representations.

Corollary 1.4.6.1. *If χ is the character of a representation on V, then (χ, χ) is a positive integer and $(\chi, \chi) = 1$ if and only if V is irreducible.*

Proof. Indeed if $\sum n_i^2 = 1$, where $n_i = \dim V_i$ is the dimension of the ith-irreducible complex representation, then all n_i are equal to zero except for one among them which is equal to 1. Hence V is irreducible.

□

Corollary 1.4.6.2. *Every character χ_ρ admits a decomposition of type:*

$$\chi_\rho = \sum_{i=1,...,s} n_i \chi_i,$$

where $\chi_1, ..., \chi_s$ are the irreducible characters G.

Corollary 1.4.6.3. *If G is an abelian group then $\dim V_i = 1,\ \forall i$.*

Proof. Indeed, let

$$\rho_R : G \to \mathbb{C}(G)$$

be the so-called *regular representation* defined on the group algebra $\mathbb{C}(G)$ as $\rho_R(g)e_h = e_{gh}$. Therefore $(\chi_R, \chi_r) = |G|$ and if $n_i = \dim V_i$ then $\sum n_i^2 = |G|$. This implies $n_i = 1\ \forall i$.

□

In the following we will focus on our case of interest: let C be a compact Riemann surface of genus $g \geq 2$ and G a finite subgroup of $\mathrm{Aut}(C)$ that acts on C. Being $\psi : G \to \mathrm{Aut}(C)$ the group action, we can define the representation of G associated to ψ via pull-back of holomorphic 1-forms in the following way:

$$\begin{aligned} \varphi : G &\to \mathrm{GL}(H^0(C, \omega_C)) \\ g &\mapsto [\omega \mapsto \psi(g^{-1})^* \omega]. \end{aligned}$$

Calling χ_φ the associated character, we would like to study a decomposition as in Corollary 1.4.6.2. Chevalley-Weil formula provides a way to compute integers n_i. We will refer to [18], [30], [31, Section 2] and [73].

Let us denote by $\pi : C \to C/G$ the quotient map and suppose that an element $g \in G$ fixes a point $p \in C$. Let m be the order of g. Then the differential $(dg)_p$ acts on T_pC by multiplication by an mth root of unity which we will call $\zeta_m(p)$. Indeed, the action can be linearised in a neighborhood of p using a local coordinate z, centred on p, in order to have that g acts as $z \mapsto \zeta_m(p)z$.

Denote $\zeta_m = e^{\frac{2\pi i}{m}}$, $I(m) := \{\nu \in \mathbb{Z} : 1 \leq \nu \leq m$ and $\gcd(\nu, m) = 1\}$ and $\mathrm{Fix}_\nu(g) := \{p \in C : g \cdot p = p$ and $\zeta_m(p) = \zeta_m^{-\nu}\}$. Moreover let $\langle x_i \rangle$ be the non trivial stabilizers of order m_i at the branch point t_i's of π. Then we have the following (see [30, Theorem V.2.9])

Theorem 1.4.7 (Eichler trace formula)**.** *Let g be an automorphism of order $m > 1$ acting on C. Then:*

$$\chi_\varphi(g) = \mathrm{Tr}(\varphi(g)) = 1 + \sum_{p \in Fix(g)} \frac{\zeta_m(p)}{1 - \zeta_m(p)},$$

where Fix(g) is the set of fixed points of g.

Collecting terms with equal exponent we can restate the Eichler's formula in the following way:

Corollary 1.4.7.1. *Keeping the same notation as before, we have the following:*

$$\chi_\varphi(g) = 1 + |C_G(g)| \sum_{\nu \in I(m)} \left(\sum_{\substack{1 \geq i \geq r, \\ m | m_i, \\ g \sim x_i^{m_i \nu / m}}} \frac{1}{m_i} \right) \frac{\zeta_m^\nu}{1 - \zeta_m^\nu},$$

where $C_G(g)$ denotes the centralizer of g in G and $\sim$ the equivalence relation given by conjugation in G.

A very important consequence of the Eichler trace formula is the Chevalley-Weil formula. As said before, it gives the multiplicity of the given irreducible representation of G on $H^0(C, \omega_C)$, i.e. the integral coefficients n_i.

Theorem 1.4.8 (Chevalley-Weil formula, [18])**.** *Consider a Galois cover $\pi : C \to C'$ with Galois group G and r branched points. Call φ a representation on $H^0(C, \omega_C)$. Let* $\boldsymbol{m}$*=$(m_1, ..., m_r)$ be the monodromy and $\rho_1, ..., \rho_s$ irreducible representations of G of degree d_j with $j = 1, ..., s$. Moreover denote by $E_{i,\alpha}$ the number of eigenvalues of $\rho_j(x_i)$ equal to $\zeta_{m_i}^\alpha$. Then the following holds:*

$$(\chi_\varphi, \chi_j) = d_j(g(C') - 1) + \sum_{i=1}^{r} \sum_{\alpha=0}^{m_i - 1} \frac{\alpha \cdot E_{i,\alpha}}{m_i} + (\chi_\varphi, \chi_{triv}), \tag{1.5}$$

where χ_{triv} is the trivial character.

1.4.1 Galois action on irreducible complex characters

Let us consider a finite group G and let $\rho_i : G \to GL(V_i), i = 1, ..., r$ be the corresponding complex irreducible representations with characters $\chi_1, ..., \chi_r$. As G is finite, each $g \in G$ has finite order and the same occurs for $\rho_i(g)$. Therefore the eigenvalues of $\rho_i(g)$ are roots of unity and $\chi_i(g)$ is an element of the cyclotomic field $\mathbb{Q}(\xi_N)$. Here we denote with N the cardinality of the group G and with ξ_N a primitive $N-$root of the unity.

$\mathbb{Q}(\xi_N)$ is a Galois extension of $\mathbb{Q}$ with Galois group $Gal_N := \left(\mathbb{Z}/N\mathbb{Z}\right)^*$.

Definition 1.17. The *character field* of ρ_j is

$$K_j := \mathbb{Q}(\chi_j(g))_{g \in G}.$$

It is the field obtained extending the rational numbers by the values of the character χ_j.

As K_j is Galois over $\mathbb{Q}$, we have $k_j := [K_j : \mathbb{Q}] = |Gal(K_J/\mathbb{Q})|$. This Galois group acts on the set of irreducible complex characters of G. For $\sigma \in Gal(K_J/\mathbb{Q})$ we define

$$\sigma(\chi_j) : G \to K_j, \qquad \sigma(\chi_j)(g) := (\chi_j(g))^\sigma.$$

Call

$$\{V_j^\sigma,\ \sigma \in Gal(K_J/\mathbb{Q})\}$$

the *Galois orbit* of V_j.

Remark 6. Using character theory it is possible to show that $\sigma(\chi_j)$ is the character of an irreducible complex representation of the same dimension of V_j.

Definition 1.18. The *Schur index* of V_j is the smallest positive integer s_{V_j} such that there exists a field extension L_{V_j} of K_j of degree s_{V_j} over which V_j can be defined.

One can show that s_{V_j} divides the dimension of V_j. Therefore, in case of abelian groups, Schur indices of irreducible complex representations are always equal to 1. It is possible to show that the same occurs considering Dihedral groups D_n.

Theorem 1.4.9. *Let G be a finite group and consider $\{V_1, ..., V_r\}$ a set constructed by taking one representative from each Galois orbit of all the complex irreducible representations of G. Then for each rational irreducible representation U of G there exist one V_j satisfying:*

$$U \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{i=1}^{s_j} \bigoplus_{\sigma \in Gal(K_J/\mathbb{Q})} V_j^\sigma, \tag{1.6}$$

where s_j denotes the Schur index of V_j. Conversely for every V_j in a Galois orbit, the rhs of (1.6) *is the complexification of a rational irreducible representation of G.*

CHAPTER 2

Infinitely many Shimura Curves in Genus $g \leq 4$

In this chapter, we discuss the existence of infinitely many new examples of Shimura subvarieties contained in the Torelli locus which arise studying the fibres (or, better, their irreducible components) of two fibrations appropriately introduced. Sometimes we will refer to Shimura subvarieties as "special" subvarieties. This is done according to the literature.

We will work in this setting: $\mathcal{M}_g$ is the moduli space of smooth projective curves of genus g, $\mathcal{A}_g$ is the moduli space of principally polarized abelian varieties of dimension g and $j : \mathcal{M}_g \to \mathcal{A}_g$ is the Torelli map. Its image $\mathcal{T}_g^0 := j(\mathcal{M}_g)$ is known as the Jacobian locus. We will look at its closure in $\mathcal{A}_g$ and we will denote it by $\mathcal{T}_g$.

Our goal is to study the geometric properties of this locus with respect to the symmetric structure that $\mathcal{A}_g$ inherits from the Siegel space (of which $\mathcal{A}_g$ is the quotient through the action of the group $Sp(2g, \mathbb{Z})$). One expects the Torelli locus to be "very curved" with respect to this locally symmetric ambient space.

This is the leitmotiv which led to the formulation of:

Conjecture 2.1 (Coleman-Oort)**.** For large g there are no Shimura subvarieties of positive dimension generically contained in $\mathcal{T}_g$.

A subvariety $\mathsf{Z} \subseteq \mathcal{A}_g$ is generically contained in $\mathcal{T}_g$ if

$$\mathsf{Z} \subseteq \mathcal{T}_g \text{ and } \mathsf{Z} \cap \mathcal{T}_g^0 \neq \emptyset.$$

Furthermore, a subvariety $\mathsf{Z} \subseteq \mathcal{A}_g$ is special if it is a totally geodesic subvariety with an extra arithmetic condition that we will explain later.

All the examples of totally geodesic subvarieties known so far are in genus $g \leq 7$ and they are constructed in this way: consider a family of Galois covers $f : C \to C'$, where the genera g, g', the number of ramification points and the monodromy are fixed. Let this family be parametrized by M (as described in Section 1.3.3) and let Z be the closure of $j(\mathsf{M})$ in $\mathcal{A}_g$. Then the numerical condition

$$\dim(\mathrm{Sym}^2 H^0(\omega_C))^G = \dim H^0(\omega_C^{\otimes 2})^G = \dim \mathsf{Z} \qquad (*)$$

is sufficient to make Z a special subvarieties of $\mathcal{A}_g$ generically contained in $\mathcal{T}_g$.

Here we show that condition $(*)$ is very strong when $g' > 0$, indeed we show that there are only 6 families of Galois covers of elliptic curves which satisfy $(*)$ and hence which yield Shimura subvarieties. For $g' > 1$ no examples can exist.

Having these examples in mind, we prove a theorem which allows us to pass from local isogenies (defined between Jacobians of two different subloci of $\mathfrak{S}_g$) to a global isogeny (under some assumption that we will explain). Then we apply this tool to study the fibres of two morphisms defined on M and we show that there exist global isogenies with certain totally geodesic subvarieties of $\mathcal{A}_g$. This yields infinitely many new examples of totally geodesic subvarieties of $\mathcal{A}_g$, countably many of which are Shimura.

The chapter is organized as follows.

In Section 2.1 we give some preliminaries on symmetric spaces and we introduce the concept of totally geodesic subvarieties and we explain why their existence is expected in an ambient space which has a symmetric structure.

In Section 2.2 we focus on a particular symmetric domain, the Siegel space, which we describe accordingly. Moreover, we explain a technique which compares two different totally geodesic submanifolds of two Siegel spaces.

In Section 2.3 we introduce Shimura subvarieties with their geometric and arithmetic properties, in particular we deal with Shimura varieties of PEL type. Then we show how the condition $(*)$ implies that a family of Galois covers yields a special subvariety following [32] and [36]. In particular, we give the complete list of Galois covers of elliptic curves satisfying the condition as presented in [36]. Moreover, we prove that $(*)$ gives a bound on the genus g' of curves which occur in families of coverings which satisfy $(*)$. Finally, we use the six families of [36] to show that they admit two fibrations in totally geodesic subvarieties. Therefore we get infinitely many new examples.

In Section 2.4 we describe the inclusions among families of [32] and of [36]. In order to do so we describe several features of these families and we show that sometimes they occur as fibres of the two fibrations previously introduced.

2.1 Symmetric Spaces

The definition of a general (i.e. not necessarily Riemannian) symmetric space is obtained from the definition of Riemannian symmetric space by dropping the request that the connection is compatible with a Riemannian metric.

In the following, we will consider a differentiable manifold M with an affine connection ∇, i.e. a linear connection on TM. As Riemannian geometry, with the aid of ∇ one can define geodesics, exponential maps, completeness and curvature.

A diffeomorphism $f : M \to M$ is an *affine transformation* if it preserves the connection [53, vol. I, p. 225]. A non-trivial theorem proves that the group $\mathrm{Aff}(M)$ of all affine transformations of M is a Lie group acting smoothly and transitively on M (see e. g. [53, vol. I, p. 229]).

Definition 2.1. (M, ∇) is a *symmetric space* if

* ∇ is symmetric i.e. the torsion tensor T satisfies $T(\nabla) = 0$;
* M is connected;
* M is complete with respect to ∇;
* for each point $x \in M$ there is a *symmetry* at x, i.e. an affine transformation $s_x : M \to M$ such that $s_x(x) = x$ and $(ds_x)_x = -\operatorname{Id}_{T_xM}$.

A Riemannian Manifold $(M, \langle\ ,\ \rangle)$ is *locally symmetric* if for every point $x \in M$ the symmetry s_x is defined on a geodesic ball centred on x. As in the Riemannian case, there is a local version of the last condition of Definition 2.1 for a differentiable manifold (M, ∇). It corresponds to require $\nabla R = 0$, where we denote with R the curvature tensor.

Simple consequences are collected, without proof, in the following:

Proposition 2.1.1. *Take (M, ∇) a symmetric space.*

⊛ *If γ is a geodesic with $\gamma(0) = x$, then $s_x(\gamma(t)) = \gamma(-t)$.*

⊛ *$\forall\, x, y \in M$ there exists a symmetry $\sigma : M \to M$ such that $\sigma(x) = y$.*

⊛ *If M is homogeneous*[1] *and there exists a symmetry at one point then M is symmetric.*

The Definition (2.1), together with the above Proposition, tells us that for every point x in a symmetric space there exists a symmetry s_x which "flips" geodesic starting in x.

Remark 7. Note that is always possible to define the required s_x on a normal ball as $s_p(exp(tv)) = exp(-tv)$. In a symmetric space the condition is stronger since s_x is required to be a symmetry for every x.

Assume that (M, ∇) is a symmetric space and let $G := \operatorname{Aff}_0(M)$ denote the connected component of the identity. Therefore G is the identity component of a Lie group whose action on M is transitive. Thus also its action is transitive on M. This implies the following:

Proposition 2.1.2. *Take a point $x \in M$ and $H := G_x$, its stabilizer. Then the map $G/H \to M$, which sends gH to $g \cdot p$, is a diffeomorphism. Therefore a symmetric space may be written as a homogeneous space G/H.*

[1]A smooth manifold M endowed with a transitive, smooth action by a Lie group G is called *homogeneous space*.

Now fix a point $x \in M$ and define an involutive automorphism of G as:

$$\begin{aligned} \sigma : G &\to G \\ g &\mapsto s_x \circ g \circ s_x \end{aligned}$$

If we set $G^\sigma = \{g \in G : \sigma(g) = g\}$ and we let G_0^σ be its component of the identity, then

$$G_0^\sigma \subseteq H \subseteq G^\sigma.$$

σ is usually called *Cartan involution*.

Conversely, assume that (G, H, σ) is a *symmetric triple*, i.e. G is a connected Lie group, σ is an involutive automorphism of G and H is a closed subgroup lying between G^σ and its component of the identity. The following holds:

Proposition 2.1.3. *$M := G/H$ is a* reductive homogeneous space. *It admits the so-called* canonical connection ∇ *which has $T = 0$ and $\nabla R = 0$. Moreover, it is the unique connection on G/H which remains invariant by the symmetries of M.*

For a proof see [53, vol. II, ch. X].

Every symmetric space gives rise to a symmetric triple of the Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$, where $\mathfrak{g}, \mathfrak{h}$ are the Lie algebras of G and H, respectively, and σ is the automorphism of $\mathfrak{g}$ induced by the homonymous on G.
Since σ is involutive its eigenvalues are +1 and -1 and $\mathfrak{h}$ is the eigenspace of 1. Then if $\mathfrak{m} := \{X \in \mathfrak{g} : d\sigma(X) = -X\}$ we get

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

This is called the *canonical decomposition of* $(\mathfrak{g}, \mathfrak{h}, \sigma)$.

It is possible to show the existence of an isomorphism:

$$\mathfrak{m} \cong T_o M, \quad X \mapsto \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \exp(tX) \cdot o. \tag{2.1}$$

2.1.1 Totally geodesic subvarieties

Suppose to start with (M, ∇), a manifold with an affine connection and consider $M' \subset M$ a submanifold with the induced connection ∇'. This gives us the following exact sequence of tangent bundles:

$$0 \to TM' \to TM|_{M'} \xrightarrow{\pi} N \to 0,$$

where $N := {}^{TM|_{M'}}/_{TM'}$ is the normal bundle.

The second fundamental form is defined as:

$$\begin{aligned} \mathrm{II} : TM' \times TM' &\to N \\ (X, Y) &\mapsto \pi(\nabla_X Y). \end{aligned}$$

Remark 8. If ∇ is symmetric, and this will be the case in our following analysis, then $\mathrm{II} \in \Gamma(\mathrm{Sym}^2 T^*M' \otimes N)$. Therefore its dual map will be defined as:

$$\mathrm{II}^* = \rho : N^* \to \mathrm{Sym}^2 T^*M'.$$

Definition 2.2. A submanifold M' of M is *totally geodesic* if every geodesic in M', with respect to ∇', remains a geodesic in M, with respect to ∇.

Proposition 2.1.4. *The following are equivalent:*

1. (M', ∇') *is totally geodesic in* (M, ∇)*;*
2. $\mathrm{II} \equiv 0$.

Proof. Clearly 2. $\Rightarrow$ 1. To prove that 1. $\Rightarrow$ 2. we need to show that $\mathrm{II}(X,Y) = 0$ for every $X, Y \in \Gamma(TM')$. Take p in M' and choose local coordinates $(x_1, ..., x_m)$ in a neighborhood V of the point such that $U = \{x \in M : x_{m'+1}(x) = ... = x_m(x) = 0\}$ together with $(x_1, ..., x_{m'})$ gives a local chart for $p \in M'$. Using these coordinates we have:

$$X = \sum_{i=1}^{m'} x^i \frac{\partial}{\partial x_i} \qquad Y = \sum_{i=1}^{m'} y^i \frac{\partial}{\partial x_i}.$$

Therefore:

$$\begin{aligned} \nabla_X Y = \sum_{i=1}^{m'} x^i \nabla_{\frac{\partial}{\partial x_i}} \Big(\sum_{j=1}^{m'} y^j \frac{\partial}{\partial x_j} \Big) &= \sum_{i,j=1}^{m'} x^i y^j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + \sum_{i,j=1}^{m'} x^i \frac{\partial y^j}{\partial x_j} \frac{\partial}{\partial x_j} = \\ &= \sum_{i,j=1}^{m'} x^i y^j \Big(\sum_{k=1}^{m} \Gamma_{ij}^k \frac{\partial}{\partial x_k} \Big) + \sum_{i,j=1}^{m'} x^i \frac{\partial y^j}{\partial x_j} \frac{\partial}{\partial x_j} \end{aligned}$$

where Γ_{ij}^k are the Christoffel-symbols of ∇. Using the assumptions we conclude, i.e. we prove that $\Gamma_{ij}^k = 0$ for $1 \le i, j \le m'$ and $m' + 1 \le k \le m$. In fact, if $\gamma : I \to M'$, in local coordinates $t \mapsto (x_1(t), ..., x_{m'}(t))$, is a geodesic for ∇', it satisfies

$$\frac{\partial^2 x^k}{\partial t^2} + \sum_{i,j=1}^{m'} \Gamma_{ij}^k \frac{\partial x^i}{\partial t} \frac{\partial x^j}{\partial t} = 0 \quad \forall\, k = 1, ..., m'. \tag{2.2}$$

Condition 1. says that the curve $t \mapsto (x_1(t), ..., x_{m'}(t), 0, ..., 0)$ automatically satisfies (2.2) for $k = 1, ..., m$. Therefore we get what expected.

□

Note that condition 2. tells us that ∇' is just the restriction of ∇ applied to fields in $\Gamma(TM')$.

In case of a symmetric space we have the following important

Proposition 2.1.5. *Let us take a Riemannian symmetric space* (M, ∇) *and consider a submanifold* M' *such that* $s_x(M') = M'$ *for every* $x \in M$. *Then* M' *is a totally geodesic submanifold of* M *and it is itself symmetric.*

Proof. We would like to show that $\mathrm{II} \equiv 0$. By definition s_x preserves the connection and, since on the tangent space it is equal to $-Id$, it preserves T_xM' and the orthogonal component too. Thus it preserves II. Using the fact that an odd-order s_x-invariant tensor is identically equal to 0 we conclude. By restriction $s_x|_{M'}$ we get its properties as symmetric space.

□

Remark 9. This Proposition tells us that we should expect the existence of a lot of totally geodesic subvarieties contained in Riemannian symmetric spaces. A very enlightening example is given thinking on Euclidean spaces where totally geodesic submanifolds are affine subspaces.

In a symmetric space $M = G/H$ totally geodesic submanifolds are related to subspaces of the Lie algebra $\mathfrak{g}$ of G. Let's try to clarify this characterization.

Definition 2.3. Let (G, H, σ) be a symmetric triple. A *subtriple* is a triple (G', H', σ') with G' a connected Lie subgroup of G invariant by σ, $H' = G' \cap H$ and $\sigma' := \sigma|_{G'}$.

A subtriple is automatically a symmetric triple: the symmetry s_x of M at a point x of M' restricts to a symmetry of M'.

Theorem 2.1.6. *Let* (G, H, σ) *be a symmetric triple and let* (G', H', σ') *be a subtriple. Then:*

1⋇ *the inclusion* $G' \subset G$ *induces a natural embedding*

$$G'/H' \cong M' := G' \cdot o \subset M = G/H$$

and M' *is a totally geodesic submanifold of* M. *Moreover the canonical connection of* M *restricts to the canonical connection of* M'*;*

2⋇ *Conversely set* $o := [H]$ *in* $M = G/H$ *and let* $M' \subset M$ *be a complete and connected totally geodesic submanifold of* M. *Set* $G'' = \{g \in G : g(M') = M'\}$. *Then* G'' *is a Lie subgroup of* G. *Let* G' *denote the identity component of* G''. *Then* G' *is invariant by* σ, *so* $(G', H' := H \cap G', \sigma' := \sigma|_{G'})$ *is a subtriple and* $M' = G'/H'$*;*

3⋇ *If* $M' \subset M$ *is a totally geodesic submanifold through* o, *then via* (2.1) *we have* $T_oM' \cong \mathfrak{m}'$ *for a* Lie triple system $\mathfrak{m}'$, *i.e. for a vector subspace* $\mathfrak{m}' \subset \mathfrak{m}$ *satisfying* $[[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] \subset \mathfrak{m}'$.

See [53, vol. II, p. 234-237] for a proof.

2.2 Siegel Space

This section is devoted to the definition of the Siegel space and to the description of some of its properties.

Set $V := \mathbb{R}^{2g}$ and define

$$\mathcal{C}(V) := \{J \in \operatorname{End} V : J^2 = -\operatorname{Id}_V\}. \tag{2.3}$$

This set is invariant by the adjoint action of $\mathrm{GL}(V)$ on $\operatorname{End} V$.

Let V_J denote the complex vector space with underlying real space V and complex multiplication defined by $i \cdot v := Jv$, with $J \in \mathcal{C}(V)$. As usual, extending J for $\mathbb{C}$-linearity, we get an endomorphism $J : V_J \to V_J$ with eigenvalues $\pm i$. Then we have the well-known decomposition

$$V_J = V_J^{1,0} \oplus V_J^{0,1}.$$

Remark 10. $\mathcal{C}(V)$ is a complex manifold: indeed the map $J \mapsto V_J^{0,1}$ is a diffeomorphism of $\mathcal{C}(V)$ onto the set $\Omega := \{W \in \mathbb{G}(g, V_{\mathbb{C}}) : W \cap \overline{W} = \{0\}\}$, which is an open subset of the Grassmannian in the analytic topology. It is not Zariski open (already in case $g = 1$), since its complement is not analytic. Hence $\mathcal{C}(V)$ is a complex manifold not quasi-projective.

Let J, J' be points of $\mathcal{C}(V)$. Fix bases $\{e_i\}$ and $\{e_i'\}$ of V_J and $V_{J'}$ respectively. Thus

$$V_J = \langle e_1, \dots, e_g, Je_1, \dots, Je_g \rangle$$

and the same holds for

$$V_{J'} = \langle e_1', \dots, e_g', J'e_1', \dots, J'e_g' \rangle.$$

Hence there is a unique map $a \in \mathrm{GL}(V)$ such that

$$a(e_i) = e_i' \quad \text{and} \quad a(Je_i) = J'e_i'.$$

It follows that $aJ = J'a$, i.e. $\operatorname{Ad} a(J) = J'$. This shows that the action of $\mathrm{GL}(V)$ on $\operatorname{End} V$ is transitive. Therefore:

$$\mathcal{C}(V) \cong \mathrm{GL}(V) \Big/ \mathrm{GL}(V_J)^{2}$$

Thus $\mathcal{C}(V)$ is a manifold with two connected components. The connected component containing J is the orbit $\mathrm{GL}^+(V) \cdot J \cong \mathrm{GL}^+(V)/\mathrm{GL}(V_J)$.

Fix $J \in \mathcal{C}(V)$ and consider the automorphism

$$\sigma_J : \mathrm{GL}(V) \to \mathrm{GL}(V), \quad \sigma_J(a) := \operatorname{Ad} J(a) = -JaJ.$$

Since $J^2 = -\operatorname{Id}_V$, σ_J is involutive and $\mathrm{GL}(V_J)$ is its fixed point set. This proves the following:

[2]Here we are using the following $\mathrm{GL}(V)_J = \{a \in \mathrm{GL}(V) : aJ = Ja\} = \mathrm{GL}(V_J)$

Proposition 2.2.1. *The connected component of $\mathcal{C}(V)$ containing J is the symmetric space associated with the symmetric triple $(\mathrm{GL}^+(V), \mathrm{GL}(V_J), \sigma_J)$*

Remark 11. It is possible to replace $\mathrm{GL}^+(V)$ with $\mathrm{SL}(V)$ and (consequently) $\mathrm{GL}(V_J)$ with the stabilizer of J in $\mathrm{SL}(V)$, that is $\mathrm{SL}(V) \cap \mathrm{GL}(V_J)$. Since $\mathrm{SL}(V)$ is simple, Theorem 3.4 of [53, vol. II, p. 232] implies that this space admits a symmetric pseudo-Riemannian metric. Moreover, since the induced stabilizer is non-compact, this symmetric space is not Riemannian.

In the following we will often refer to $\mathcal{C}(V)$ as a symmetric space, even tough this is has been showed to be true only for its connected components.

Now we are ready for the introduction of the main character of this section: the Siegel space.

Definition 2.4. A real $2g-$dimensional *symplectic vector space* is the datum of a vector space V (of dimension $2g$) together with a symplectic form $\omega \in \Lambda^2 V^*$, i.e. a map $\omega : V \times V \to \mathbb{R}$ which is bilinear, alternating and non-degenerate.

Lemma 2.2.2. *Let (V, ω) a symplectic vector space and let $J \in \mathcal{C}(V)$. The following are equivalent:*

1. *$J^*\omega = \omega$, that is $\omega(Jx, Jy) = \omega(x, y) \ \ \forall x, y \in V$;*
2. *$\omega(Jx, y) + \omega(x, Jy) = 0 \ \ \forall x, y \in V$;*
3. *$g_J := \omega(\cdot, J\cdot)$ is symmetric.*

Definition 2.5. The *Siegel space* associated to the symplectic pair (V, ω) is:

$$\mathfrak{S}(V, \omega) := \{J \in \mathcal{C}(V) : J^*\omega = \omega, \ g_J \text{ is positive definite}\}.$$

Remark 12. It is possible to show that there exists a basis for V (known as *symplectic basis*) $V = \langle e_1, ..., e_g, e_{g+1}, ..., e_{2g}\rangle$ such that

$$\omega(e_i, e_{j+g}) = \delta_{ij}, \quad \omega(e_i, e_j) = \omega(e_{i+g}, e_{g+j}) = 0, \quad 1 \leq i, j \leq g.$$

This basis is very useful to show that $\mathfrak{S}(V, \omega) \neq \emptyset$. Indeed if we define:

$$\begin{cases} Je_i = e_{g+i} \\ Je_{g+i} = -e_i \end{cases}$$

we immediately obtain $J \in \mathfrak{S}(V, \omega) \neq \emptyset$.

Proposition 2.2.3. *The Siegel space $\mathfrak{S}(V, \omega)$ is a totally geodesic submanifold of $\mathcal{C}(V)$. It is itself a symmetric space and in fact a Riemannian one.*

Proof. Fix $J \in \mathfrak{S}(V,\omega)$. Then $\mathrm{Sp}(V,\omega) := \{L \in \mathrm{GL}(V) : L^*\omega = \omega\}$ is invariant by σ_J, since $\sigma_J = \mathrm{Ad}\, J$ and $J \in \mathrm{Sp}(V,\omega)$. Set

$$G' := \mathrm{Sp}(V,\omega), \quad H' := G' \cap \mathrm{GL}(V_J) \ \text{ and } \ \sigma'_J := \sigma_J|_{G'}.$$

Then (G', H', σ'_J) is a subtriple of $(\mathrm{GL}^+(V), \mathrm{GL}(V_J) \cap \mathrm{GL}^+(V), \sigma_J)$. Hence, by Theorem 2.1.6, $\mathfrak{S}(V,\omega) = G'/H'$ is totally geodesic submanifold and also a symmetric space itself.

Now consider on V_J the Hermitian product

$$H_J(x,y) := g_J(x,y) - i\omega(x,y).$$

Then H' is the unitary group $\mathrm{U}(V_J, H_J)$. Since this is a compact group the symmetric space $\mathfrak{S}(V,\omega)$ is Riemannian, see [48, p. 209].

□

2.2.1 Families of Isogenous Abelian Varieties

Set $\Lambda := \mathbb{Z}^{2g}$. As usual if $F \subset \mathbb{C}$ is a field we set $\Lambda_F := \Lambda \otimes_{\mathbb{Z}} F$. Therefore $V = \Lambda_{\mathbb{R}}$ and $T := V/\Lambda$ is a real torus of dimension $2g$. Since the tangent bundle to T is trivial, any $J \in \mathcal{C}(V)$ yields a complex structure on T. We denote T_J the complex torus obtained in this way.

Any complex torus of dimension g is isomorphic to T_J for some J. In this sense $\mathcal{C}(V)$ is a parameter space for g-dimensional complex tori.

General theory on complex tori guarantees that if $f : T_J \to T_{J'}$ is an isomorphism, then f lifts to an isomorphism

$$a : V \xrightarrow{\cong} V \ \text{ such that } \ a(\Lambda) = \Lambda \ \text{ and } \ \mathrm{Ad}\, a(J) = J'.$$

Hence:

Proposition 2.2.4. *T_J and $T_{J'}$ are isomorphic if and only if there is $a \in \mathrm{GL}(\Lambda)$ such that* $\mathrm{Ad}\, a(J) = J'$.

More generally let us consider an isogeny $f : T_J \to T_{J'}$, i.e. a surjective morphism with finite kernel. Then f lifts to a linear map $a : V \to V$ of maximal rank, hence invertible, such that $J'a = aJ$ (i.e. f is holomorphic) and $a(\Lambda) \subset \Lambda$. It follows that $a \in \mathrm{GL}(\Lambda_{\mathbb{Q}})$.

Conversely, given $a \in \mathrm{GL}(\Lambda_{\mathbb{Q}})$ such that $J'a = aJ$, multiplying a by an appropriate positive integer m, we get a linear map $m \cdot a : V \to V$ such that $m \cdot a(\Lambda) \subset \Lambda$. This induces a surjective holomorphic morphism $f : T_J \to T_{J'}$, which is an isogeny. Therefore we can say that:

Proposition 2.2.5. *T_J is isogenous to $T_{J'}$ if and only if there is $a \in \mathrm{GL}(\Lambda_{\mathbb{Q}})$ such that* $\mathrm{Ad}\, a(J) = J'$.

Let us consider analytic subset of $\mathcal{C}(V)$. We have the following:

Lemma 2.2.6. *Let $Z_1, Z_2 \subset \mathcal{C}(V)$ be irreducible analytic subsets. Let Ω be a non-empty open subset of Z_1. Assume that for any $J_1 \in \Omega$ there is some $J_2 \in Z_2$ such that T_{J_1} is isogenous to T_{J_2}. Then there is $a \in \mathrm{GL}(\Lambda_{\mathbb{Q}})$ such that $\mathrm{Ad}\, a(Z_1) \subset Z_2$.*

Proof. Given $a \in \mathrm{GL}(\Lambda_{\mathbb{Q}})$, let $\Gamma_a \subset \mathcal{C}(V) \times \mathcal{C}(V)$ denote the graph of $\mathrm{Ad}\, a$:

$$\Gamma_a = \{(J, J') \in \mathcal{C}(V) \times \mathcal{C}(V) : J' = \mathrm{Ad}\, a(J)\}.$$

If $\pi_j : \mathcal{C}(V) \times \mathcal{C}(V) \to \mathcal{C}(V)$ denotes the projection on the j-th factor, the assumption is equivalent to saying that

$$\Omega \subset \bigcup_{a \in \mathrm{GL}(\Lambda_{\mathbb{Q}})} \pi_1(\Gamma_a \cap \pi_2^{-1}(Z_2)).$$

Indeed, if $J_1 \in \Omega$, there is $J_2 \in Z_2$ such that T_{J_1} and T_{J_2} are isogenous, i.e. there is $a \in \mathrm{GL}(\Lambda_{\mathbb{Q}})$ such that $(J_1, J_2) \in \Gamma_a$.

For each $a \in \mathrm{GL}(\Lambda_{\mathbb{Q}})$ the intersection $\Gamma_a \cap \pi_2^{-1}(Z_2)$ is an analytic subset, so we can find a sequence $\{K_{a,i}\}_{i \in \mathbb{N}}$ of compact subsets of $\mathcal{C}(V) \times \mathcal{C}(V)$ such that

$$\Gamma_a \cap \pi_2^{-1}(Z_2) = \bigcup_{i=1}^{\infty} K_{a,i}.$$

Then

$$\Omega = \bigcup_{a,i} \Omega \cap \pi_1(K_{a,i}).$$

Since $K_{a,i}$ is compact, the set $\Omega \cap \pi_1(K_{a,i})$ is closed in Ω. As i and a vary in countable sets, Baire theorem [13, p. 57] implies that there are i and a such that $\pi_1(K_{a,i})$ contains an open subset U of Ω. The set U is clearly open also in Z_1 and satisfies $U \subset \pi_1(\Gamma_a \cap \pi_2^{-1}(Z_2))$. This means that if $J \in U$, there is $J' \in Z_2$ such that $(J, J') \in \Gamma_a$. In other words $\mathrm{Ad}\, a(J) \in Z_2$ for any $J \in U$.
Hence, setting $f = \mathrm{Ad}\, a : \mathcal{C}(V) \to \mathcal{C}(V)$, we have $f(U) \subset Z_2$. Therefore $f^{-1}(Z_2) \cap Z_1$ is an analytic subset of Z_1, which contains the open subset $U \subset Z_1$. By the Identity Lemma [43, p. 167] this implies that $f^{-1}(Z_2) \cap Z_1 = Z_1$ i.e. $f(Z_1) \subset Z_2$.

$\square$

Now we reformulate the previous Lemma at the level of the Siegel space $\mathfrak{S}(V, \omega)$.

Proposition 2.2.7. *Let ω_1, ω_2 be symplectic forms on V. Assume that Z_1 is an irreducible analytic subset of $\mathfrak{S}(V, \omega_1)$ and that Z_2 is a totally geodesic submanifold of $\mathfrak{S}(V, \omega_2)$. Let Ω be a non-empty open subset of Z_1 with the property that for any $J_1 \in \Omega$ there is some $J_2 \in Z_2$ such that T_{J_1} is isogenous to T_{J_2}. Assume moreover that $\dim Z_1 = \dim Z_2$. Then there is $a \in \mathrm{GL}(\Lambda_{\mathbb{Q}})$ such that $\mathrm{Ad}\, a(Z_1) = Z_2$. Moreover Z_1 is a totally geodesic submanifold of $\mathfrak{S}(V, \omega_1)$.*

Proof. By Lemma 2.2.6 there is $a \in \mathrm{GL}(\Lambda_{\mathbb{Q}})$ such that $\mathrm{Ad}\, a(Z_1) \subset Z_2$.

Since Z_2 is irreducible, any proper analytic subset of Z_2 is nowhere dense in Z_2, see e.g. [43, p. 168]. Since $\dim Z_1 = \dim Z_2 = n$ we conclude that $\mathrm{Ad}\, a(Z_1) = Z_2$. This proves the first assertion.

By assumption Z_2 is totally geodesic in $\mathfrak{S}(V, \omega_2)$ and Proposition 2.2.3 asserts the same for $\mathfrak{S}(V, \omega_2)$ in $\mathcal{C}(V)$. Thus Z_2 is totally geodesic in $\mathcal{C}(V)$[3]. Since, by assumption, $Z_1 \subset \mathfrak{S}(V, \omega_1)$ and $\mathrm{Ad}\, a$ is an affine transformation of $\mathcal{C}(V)$, we obtain that Z_1 is in fact a totally geodesic submanifold of $\mathfrak{S}(V, \omega_1)$, as desired.

□

The following definition goes back to Moonen [67].

Definition 2.6. Let ω be a polarization of type D and denote by $\pi : \mathfrak{S}(V, \omega) \to \mathcal{A}_g^D$ the canonical projection. A *totally geodesic subvariety* of $\mathcal{A}_g^D$ is a closed algebraic subvariety $W \subset \mathcal{A}_g^D$, such that $W = \pi(Z)$ for some totally geodesic submanifold $Z \subset \mathfrak{S}(V, \omega)$.

We wish to prove an analogue of Proposition 2.2.7 for subvarieties of $\mathcal{A}_g$ instead of $\mathfrak{S}_g$. A difficulty in passing from $\mathfrak{S}_g$ to $\mathcal{A}_g$ comes from the fact that the map $\pi : \mathfrak{S}_g \to \mathcal{A}_g^D$ is of infinite degree and ramified.

The strategy is to factor π as an unramified covering of infinite degree followed by a finite map. This "descends" Proposition 2.2.7 to $\mathcal{A}_g$.

Lemma 2.2.8. *Let X, Y, Z be reduced complex analytic spaces. Let $p : X \to Y$ be an unramified covering and let $q : Y \to Z$ be a finite Galois covering. If $Z' \subset Z$ is an irreducible analytic subset and X' is an irreducible component of $(qp)^{-1}(Z')$, then $qp(X') = Z'$.*

Proof. Let $\{Y_i\}$ be the irreducible components of $q^{-1}(Z')$. We claim that $q(Y_i) = Z'$ for each i. Since q is a finite map each $q(Y_i)$ is an analytic subset of Z'. Obviously

$$Z' = qq^{-1}(Z') = \cup_i q(Y_i).$$

By Baire's theorem there is some i_0 such that $q(Y_{i_0})$ has non-empty interior, therefore $Z' = q(Y_{i_0})$. Since $q : Y \to Z$ is a Galois cover with finite Galois group G, it follows that $q^{-1}(Z') = G \cdot Y_{i_0}$, so we have

$$Y_i = g \cdot Y_{i_0} \ \forall\, i \text{ and for some } g \in G \ \Rightarrow q(Y_i) = q(Y_{i_0}) = Z'.$$

This proves the claim.

Now fix a point x of X' that does not lie in any other irreducible component of $(qp)^{-1}(Z')$. Since $p(X') \subset q^{-1}(Z')$ and X' is irreducible, $p(X')$ is contained in a unique irreducible component Y' of $q^{-1}(Z')$. By the above $q(Y') = Z'$. To conclude it is enough to show that $p(X') = Y'$.

[3]Indeed suppose (M, g) a Riemannian manifold with $M' \subset M$ a totally geodesic submanifold. If M'' is a submanifold of M', then M'' is totally geodesic in M if and only if it is totally geodesic in M'.

Set $y := p(x) \in Y'$. Let $u : \tilde{Y'} \to Y'$ be the universal cover. Fix $\tilde{y} \in u^{-1}(y)$. By the lifting theorem there is a holomorphic map $\tilde{f} : (\tilde{Y'}, \tilde{y}) \longrightarrow (X, x)$ such that $p\tilde{f} = f := iu$, where $i : Y' \to Y$ denotes the inclusion. Since Y' is irreducible, also $\tilde{Y'}$ is irreducible, hence $\tilde{f}(\tilde{Y'})$ is contained in a unique irreducible component, which is necessarily X' (because $\tilde{f}(\tilde{y}) = x$). So $\tilde{f}(\tilde{Y'}) \subset X'$. It follows that $Y' = iu(\tilde{Y'}) = f(\tilde{Y'}) = p\tilde{f}(\tilde{Y'}) \subset p(X')$.

On the other hand we have $p(X') \subset Y'$ by construction. Hence $p(X') = Y'$, as desired.

□

Proposition 2.2.9. *Let ω be a symplectic form of type D. Denote by $\pi : \mathfrak{S}(V, \omega) \to \mathcal{A}_g^D$ the canonical projection. If $W \subset \mathcal{A}_g^D$ is an irreducible analytic subset, then for any irreducible component Z of $\pi^{-1}(W)$ we have $\pi(Z) = W$.*

Proof. The polarization ω is a non-degenerate alternating form $\omega : \Lambda^2(\Lambda) \to \mathbb{Z}$ of type $D = (d_1, ..., d_g)$, where $d_1|d_2|...|d_g$. Let n be a natural number such that $(d_g, n) = 1$.

Consider a symplectic level n structure, i.e. a symplectic isomorphism of the set of n-torsion points $A[n]$ of $A = V/\Lambda$ with $(\mathbb{Z}/n\mathbb{Z})^{2g}$. Denote by $\Gamma_D(n)$ the subgroup of the automorphisms of the pair (Λ, ω) which induces the trivial action on $\Lambda/n\Lambda$. If n is large enough with respect to the polarisation D, then the quotient

$$\mathfrak{S}(V, \omega) \Big/ \Gamma_D(n) =: \mathcal{A}_g^{D,(n)}$$

is smooth and the map $p : \mathfrak{S}(V, \omega) \to \mathcal{A}_g^{D,(n)}$ is a topological covering (cf. e.g. [39]). Thus π factors through the topological covering $p : \mathfrak{S}(V, \omega) \to \mathcal{A}_g^{D,(n)}$ and a finite Galois covering $q : \mathcal{A}_g^{D,(n)} \to \mathcal{A}_g^D$. The result follows from Lemma 2.2.8.

□

Theorem 2.2.10. *Let D_1 and D_2 be types of g-dimensional abelian varieties. Let $W_1 \subset \mathcal{A}_g^{D_1}$ and $W_2 \subset \mathcal{A}_g^{D_2}$ be irreducible analytic subsets. Assume that there is a non-empty subset U of W_1 such that*

(1) U is open in the complex topology,

(2) any $[A_1] \in U$ is isogenous to some $[A_2] \in W_2$.

Then $\dim W_1 \leq \dim W_2$.

Moreover if $\dim W_1 = \dim W_2$ *and W_2 is a totally geodesic subvariety, then W_1 also is totally geodesic.*

Proof. Denote by $\pi_i : \mathfrak{S}(V, \omega_i) \to \mathcal{A}_g^{D_i}$ the canonical projections.

Let Z_i be an irreducible component of $\pi_i^{-1}(W_i)$. By Proposition 2.2.9 $\pi_i(Z_i) = W_i$. Set $\Omega := Z_1 \cap \pi_1^{-1}(U)$. Clearly for any $J_1 \in \Omega$ there is some $J_2 \in Z_2$ such that T_{J_1} and T_{J_2} are isogenous. By Lemma 2.2.6 there is $a \in \mathrm{GL}(\Lambda_{\mathbb{Q}})$ such that $\mathrm{Ad}\, a(Z_1) \subset Z_2$. Hence

$$\dim W_1 = \dim Z_1 \leq \dim Z_2 = \dim W_2.$$

By Definition 2.6 we can assume that $Z_2 \subset \mathfrak{S}(V, \omega_2)$ is a totally geodesic submanifold. Proposition 2.2.7 implies that Z_1 is totally geodesic, hence the second assertion.

□

2.3 Special Subvarieties of $\mathcal{A}_g$

In this section we discuss the notion of *special subvarieties* or *Shimura subvarieties of* $\mathcal{A}_g$. Since the abstract formalism of Shimura subvarieties is rather cumbersome, we will give a brief introduction functional to our purposes. Then, concretely, we will work with equivalent definitions which make the machinery easier to be understood. In particular we will focus on the geometric characterization of Shimura subvarieties as totally geodesic subvarieties of $\mathcal{A}_g$ admitting a complex multiplication point. Moreover we will deal with a particular class of Shimura subvarieties, *PEL Shimura subvarieties*, that we will describe accordingly.

Special subvarieties are the Hodge loci of certain natural variations of Hodge structures. In the following we will give some sketch about this approach, mainly referring to very good surveys [70] and [40].

Let us start taking $J \in \mathfrak{S}(V, \omega)$ and denoting with A_J the complex torus obtained as V/Λ (where $\Lambda \cong \mathbb{Z}^{2g}$ is such that $\Lambda_{\mathbb{R}} = V$). A_J is a complex torus with complex structure J and polarization ω.

Definition 2.7. A *rational Hodge Structure of weight* k is the datum of:

1. A finite dimensional $\mathbb{Q}-$vector space H;
2. A decomposition of $H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$ such that $H^{q,p} = \overline{H^{p,q}}$.

An equivalent definition is obtained by replacing the direct sum decomposition of H by the *Hodge filtration*, a finite decreasing filtration of H by complex subspaces F^pH such that:

$$\forall p, q \ : \ p + q = k, \qquad F^pH \cap \overline{F^qH} = 0 \quad \text{and} \quad F^pH \oplus \overline{F^qH} = H.$$

In this case

$$H^{p,q} = F^pH \cap \overline{F^qH} \quad \text{and} \quad F^pH = \bigoplus_{i \geq p} H^{i,k-i}.$$

Definition 2.8. A *variation of Hodge structure (VHS)* of weight k on a complex manifold X consists of a locally constant sheaf $\mathcal{H}$ together with a decreasing Hodge filtration F^p on $H = \mathcal{H} \otimes \mathcal{O}_X$, subject to the following two conditions:

1. The filtration induces a Hodge structure of weight k on each stalk $\mathcal{H}_x$ of the sheaf $\mathcal{H}$;
2. (Griffiths transversality) The natural connection on H maps F^p into $F^{p-1} \otimes \Omega^1_X$.

Once the lattice $\Lambda \subset V$ has been fixed there is a natural (and tautological) variation of Hodge structure on $\mathfrak{S}(V,\omega)$: take the constant lattice Λ and for $J \in \mathfrak{S}(V,\omega)$ consider the Hodge structure of weight 1 $(\Lambda, V_j^{1,0})$. This variation of Hodge structure descends to an (orbifold) variation of Hodge structure of $\mathcal{A}_g$ whose fibre over A_j is $H^{1,0}(A_J)$.

Definition 2.9. A *special subvarieties or Shimura subvarieties* of $\mathcal{A}_g$ is a Hodge locus of the natural variation of Hodge structure described above.

The zero dimensional special subvarieties are called *CM-points*: they correspond to abelian varieties endowed with a very rich endomorphism algebra. Indeed, we have the following:

Definition 2.10. $A \in \mathcal{A}_g$ is a *complex multiplication point* if $A = A_1 \times ... \times A_n$ satisfies

$$\mathrm{End}_{\mathbb{Q}}(A) := \mathrm{End}(A) \otimes \mathbb{Q} \supset K_1 \times ... \times K_n,$$

with

$$K_i \subset \mathrm{End}_{\mathbb{Q}}(A_i) \quad \text{such that} \quad [K_i : \mathbb{Q}] \geq 2g_i, \quad \text{where} \quad g_i = \dim(A_i).$$

In case A simple abelian variety, this is the same to ask that $\mathrm{End}_{\mathbb{Q}}(A)$ is a quadratic extension of a totally real field. For instance, in case $g = 1$, abelian varieties whose lattices are $\mathbb{Z}[i\sqrt{n}]$, with $n \in \mathbb{N}$, are CM-points.

We have the following result (see [70] for details):

Proposition 2.3.1. *Special subvarieties of $\mathcal{A}_g$ contain a subset of CM points which is dense for the Zariski topology.*

This arithmetic tool is particularly useful for our purposes: it makes possible the geometric characterization of Shimura varieties that we will use. Indeed, it is proved in [67] that:

Theorem 2.3.2 (Moonen)**.** *An algebraic totally geodesic subvariety of $\mathcal{A}_g$ is special if and only if it is totally geodesic and contains a CM point.*

From now on we will refer to this Theorem as our definition of Shimura subvarieties, i.e. totally geodesic subvarieties of $\mathcal{A}_g$ with an extra arithmetic condition.

2.3.1 Special Subvarieties of $\mathcal{A}_g$ Generically Contained in $\mathcal{T}_g$

In this section we focus on Shimura subvarieties of $\mathcal{A}_g$ which are generically contained in the Jacobian locus. We will shortly explain what we are looking for.

Let us recall from Chapter 1 that the Torelli map $j : \mathcal{M}_g \to \mathcal{A}_g$ associates to the point $[C] \in \mathcal{M}_g$ the moduli point of its Jacobian variety JC together with the principal polarization induced from the intersection form. Set $\mathcal{T}_g^0 := j(\mathcal{M}_g)$ and call it the *open Torelli locus* (sometimes also referred as *Jacobian locus*). The closure of $\mathcal{T}_g^0$ in $\mathcal{A}_g$ is the *Torelli locus* and it is denoted by $\mathcal{T}_g$.

Working over $\mathbb{C}$, both $\mathcal{M}_g$ and $\mathcal{A}_g$ can be provided with the structure of complex analytic orbifolds. We have the following:

Theorem 2.3.3 (Oort-Steenbrink, [78]). *The restriction of j to the set of non-hyperelliptic curves is an (orbifold) immersion.*

Moreover we recall that $\mathfrak{S}_g$ is an irreducible Hermitian symmetric space and hence it induces a locally symmetric geometry on $\mathcal{A}_g$. In particular, if $\mathcal{M}_g^*$ denotes the complement of the hyperelliptic locus, it becomes natural to study the extrinsic geometry of $j(\mathcal{M}_g^*)$ as suborbifold of the Riemannian orbifold $\mathcal{A}_g$.

This study is still largely open. The rough idea behind these results is that $j(\mathcal{M}_g^*)$ should be "very curved" inside $\mathcal{A}_g$. In other words the geometry of the Torelli locus is expected to be "complicated". A way to explain this is the study of totally geodesic subvarieties: while $\mathcal{A}_g$ has a lot of totally geodesic submanifold (since it is a local symmetric domain), we cannot think the same for $j(\mathcal{M}_g^*)$. The analogous statement for a surface in a 3-space is that the surface shouldn't contain too many lines.

This is the geometrical interpretation of:

Conjecture 2.2 (Oort, [77]). For large g (in any case $g \geq 8$), there does not exist a special subvariety $Z \subset \mathcal{A}_g$ with $\dim(Z) \geq 1$ and such that $Z \subseteq \mathcal{T}_g$ and $Z \cap \mathcal{T}_g$ is non-empty (this corresponds to ask Z "generically contained" in $\mathcal{T}_g$).

The arithmetical side of this conjecture refers to

Conjecture 2.3 (Coleman, [20]). For large g there are only finitely many non-singular projective curves C, up to isomorphism, of genus g and such that the Jacobian JC is a CM abelian variety.

We remark that the assumption $g \geq 8$ is due to the existence of counterexamples in case $g \leq 7$.

In this thesis we will address special subvarieties of $\mathcal{T}_g$ which are of PEL type. The name is due to the fact that PEL special subvarieties have a modular interpretation in terms of abelian varieties with a Polarization, given Endomorphisms and a Level structure.

We define PEL subvarieties as follows (see [70, section 3.9]):

Definition 2.11. Fix a point J_0 in $\mathfrak{S}(V, \omega)$ and set

$$D := \mathrm{End}_{\mathbb{Q}}(A_{J_0}) := \{f \in \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) : J_0 f = f J_0\}.$$

The *PEL-type special subvariety* $\mathsf{Z}(D)$ is defined as the image in $\mathcal{A}_g$ of the connected component of the set $\{J \in \mathfrak{S}_g : D \subseteq \mathrm{End}_{\mathbb{Q}}(A_J)\}$ that contains J_0.

The rest of this section is devoted to the description of results of Frediani, Ghigi, Penegini and Porru in [32], [36]. There the authors study a sufficient condition for a family of Galois covers to yield a Shimura subvariety of $\mathcal{A}_g$ of PEL type. We start from some preliminary Lemmas.

Lemma 2.3.4. *Let $G \subseteq Sp(\Lambda, \mathbb{Q})$ be a finite subgroup and denote by $\mathfrak{S}_g^G$ the set of fixed points in $\mathfrak{S}_g$. Then $\mathfrak{S}_g^G$ is a complex connected submanifold of $\mathfrak{S}_g$.*

Set $D_G := \{f \in \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) : Jf = fJ,\ \forall\ J \in \mathfrak{S}_g^G\}$. Then:

Lemma 2.3.5. *If $J \in \mathfrak{S}_g^G$ then $D_G \subseteq \mathrm{End}_{\mathbb{Q}}(A_J)$ and the equality holds for J in a dense subset of $\mathfrak{S}_g^G$.*

Lemma 2.3.6. *The image of $\mathfrak{S}_g^G$ in $\mathcal{A}_g$ coincides with the PEL subvariety $\mathsf{Z}(D_G)$. Moreover if $J \in \mathfrak{S}_g^G$, then $\dim \mathfrak{S}_g^G = \dim \mathsf{Z}(D_G) = \dim(\mathrm{Sym}^2\Lambda_{\mathbb{R}})^G$, where $\Lambda_{\mathbb{R}} \cong V$ is endowed with the complex structure J.*

Finally, we have:

Theorem 2.3.7 ([32] §3.9, [36] §3.7)**.** *Fix a datum $\Delta = (\boldsymbol{m}, G, \theta)$ and put $\mathsf{M}_\Delta := \mathsf{M}((\boldsymbol{m}, G, \theta))$. Moreover denote by*

$$\mathsf{Z}_\Delta := \overline{j(\mathsf{M}_\Delta)}$$

the image in $\mathcal{A}_g$ through Torelli morphism. Thus Z_Δ is an algebraic subvariety of $\mathcal{A}_g$ of dimension $3g' - 3 + r$. Assume moreover that:

$$N := \dim(\mathrm{Sym}^2 H^0(C, \omega_C))^G = 3g' - 3 + r, \tag{$*$}$$

where $[C]$ represents the isomorphism class of a curve C of genus g which admits an effective holomorphic action of G. Denoting with $C \to C' := C/G$ the quotient map (C' has genus g'), then Z_Δ is a special subvariety of PEL type of $\mathcal{A}_g$ that is generically contained in the Torelli locus.

Proof. Firstly notice that the equivalence class of the representation of G on $H^0(C, \omega_C)$ does not change for $[C]$ varying in M_Δ. Hence N is well-defined and only depends on Δ.

Let $\mathcal{C} \to \mathcal{T}_g^G$ be the universal family as in Remark 4. For every point $t \in \mathcal{T}_g^G$, G acts holomorphically on C_t, so it maps injectively into $Sp(\Lambda, Q)$, where $\Lambda = H^1(C_t, \mathbb{Z})$ and Q is the intersection form. Denote by G' the image of G in $Sp(\Lambda, Q)$. It does not depend on t since it is purely topological. Recall that $\mathfrak{S}_g$ parametrizes complex structures on the real torus $\Lambda_{\mathbb{R}}/\Lambda = H^1(C_t, \mathbb{R}) \big/ H^1(C_t, \mathbb{Z})$ which are compatible with the polarization Q. The period map associates to the curve C_t the complex structure J_t on $\Lambda_{\mathbb{R}}$ obtained from the splitting $H^1(C_t, \mathbb{C}) = H^{1,0}(C_t) \oplus H^{0,1}(C_t)$ and the isomorphism $H^1(C_t, \mathbb{R}) \otimes \mathbb{C} = H^1(C_t, \mathbb{C})$. The complex structure J_t is invariant by G', since the group G acts holomorphically on C_t. This shows that $J_t \in \mathfrak{S}_g^G$, so the Jacobian $j(C_t)$ lies in $\mathsf{Z}(D_{G'})$. This shows that $\mathsf{Z}_\Delta \subseteq \mathsf{Z}(D_{G'})$. Since $\mathsf{Z}(D_{G'})$ is irreducible (see e.g. Lemma (2.3.4)), to conclude the proof it is enough to check that they have the same dimension. The dimension of Z_Δ is $3g' - 3 + r$, since it is the image through an injective morphism of an algebraic subvariety of $\mathcal{M}_{g'}$ of dimension $3g' - 3 + r$. By Lemma 2.3.6, if $J \in \mathfrak{S}_g^{G'}$, then $\dim \mathsf{Z}(D_{G'}) = \dim \mathfrak{S}_g^{G'} = \dim(\mathrm{Sym}^2\Lambda_{\mathbb{R}})^{G'}$. If J corresponds to the Jacobian of a curve C in the family, then $(\mathrm{Sym}^2\Lambda_{\mathbb{R}})^{G'}$ is isomorphic to the dual of $(\mathrm{Sym}^2 H^0(C, \omega_C))^G$. Thus $\dim \mathsf{Z}(D_{G'}) = N$ and $(*)$ yields the result.

□

Condition $(*)$ was first introduced by Moonen in [69]. There he proved that when $g' = 0$ and the group G is cyclic $(*)$ is also a necessary for Z_Δ to be Shimura. To show this, Moonen used deep results in positive characteristic which turned out to be difficult to be generalized in case of G non abelian.

A completely different approach was used in [21]. There the authors studied the second fundamental form of the embedding $j : \mathcal{M}_g^* \to \mathcal{A}_g$ and they showed that:

Proposition 2.3.8 ([21], §5.4). *If there are no nonzero quadrics in $I_2(\omega_C)$, which are invariant under the action of the group G, then Z_Δ is totally geodesic.*

Note that Theorem 2.3.7 agrees and strengthens the Proposition above. Indeed condition $(*)$ requires that the multiplication map

$$m : (\mathrm{Sym}^2 H^0(C, \omega_C))^G \to H^0(C, \omega_C^{\otimes 2})^G$$

is an isomorphism. This corresponds to require

$$\ker m = \emptyset,$$

that is there are no nonzero G-invariant quadrics.

The same authors (loc. cit.) show that when the group G is cyclic of order d, then for fixed d there exists only a finite number of families which can be totally geodesic.

Remark 13. In [68] Mohajer and Zuo proved that $(*)$ is also a necessary condition in case of G abelian, $g' = 0$ and $r = 4$. It is still unknown whether $(*)$ is necessary in general for a family of covers to yield a Shimura subvariety or whether other families exist which satisfy $(*)$.

Frediani, Ghigi, Penegini and Porru applied condition $(*)$ to do a systematic search of special subvarieties of PEL-type $\mathsf{Z}(\mathbf{m}, G, \theta)$ obtained as Galois cover of the projective line (see [32]) and then also of elliptic curves (see [36]). Using `MAGMA`, a computer algebra program, they determined all possible families of curves parametrized by $\mathsf{Z}(\mathbf{m}, G, \theta)$ with genus $g \leq 9$ and they computed the number N above. Checking which families satisfy the condition $(*)$ they get the following results:

Theorem 2.3.9 ([32], §1.6). *For $g \leq 9$ there are exactly 40 data $(\boldsymbol{m}, G, \theta)$ with $g' = 0$ such that $N = r - 3$. For these 40 data the image $\mathsf{Z}(\boldsymbol{m}, G, \theta)$ yield a Shimura subvariety of $\mathcal{A}_g$ generically contained in the Torelli locus. All these data occur in $g \leq 7$.*

Notice that among these data there are 20 cyclic data already known in the literature and collected in [69] and 7 abelian but non-cyclic data presented in [70]. As seen in 1.3.3, it can happen that different data give rise to the same subvarieties of $\mathcal{A}_g$. Indeed Theorem 1.9 of [32] asserts that there are only 30 examples of Shimura subvarieties of $\mathcal{A}_g$ obtained as Galois covering of $\mathbb{P}^1$.

Later, considering Galois coverings of elliptic curves, they got:

Theorem 2.3.10 ([36], §1.1). *For all $g \geq 2$ and $g' = 1$ there are exactly 6 positive dimensional families $\mathsf{Z}(\boldsymbol{m}, G, \theta)$ which satisfy condition $(*)$, i.e.*

$$N = r = \dim \mathsf{Z}_{\Delta}.$$

2 of them yield new Shimura subvarieties All of them occur in $g \leq 4$.

The 6 families are the following:

$(1e)$ $g = 2$, $G = \mathbb{Z}/2\mathbb{Z}$, $N = r = 2$.

$(2e)$ $g = 3$, $G = \mathbb{Z}/2\mathbb{Z}$, $N = r = 4$.

$(3e)$ $g = 3$, $G = \mathbb{Z}/3\mathbb{Z}$, $N = r = 2$.

$(4e)$ $g = 3$, $G = \mathbb{Z}/4\mathbb{Z}$, $N = r = 2$.

$(5e)$ $g = 3$, $G = Q_8$, $N = r = 1$.

$(6e)$ $g = 4$, $G = \mathbb{Z}/3\mathbb{Z}$, $N = r = 3$.

Only $(2e)$ and $(6e)$ yield new Shimura subvarieties generically contained in $\mathcal{T}_g$ while the others yield Shimura subvarieties which have already been obtained as families of Galois coverings of $\mathbb{P}^1$. In fact in [36] it was shown that:

- $(1e)$gives the same subvariety as (26) of Table 2 in [32] (this was already found in [70]).
- $(3e)$ gives the same subvariety as (31) of Table 2 in [32].
- $(4e)$ gives the same subvariety as (32) of Table 2 in [32].
- $(5e)$ gives the same subvariety as $(34) = (23) = (7)$ of Table 2 in [32] (see also Table 1 in [32] to see that these families are the same).
- Apart from $(1e)$, none of these families is contained in the hyperelliptic locus.

It is important to notice that family $(6e)$ had been already studied by Pirola [81] to disprove a conjecture of Xiao. This family is also studied in [44] by Grushevsky and Möller: there they proved that it is fibred in totally geodesic curves. We will come back to this fact since one of the central theorem of this thesis concerns a generalization of this result: we will show that the phenomenon described in [44] actually occurs for all the six families of [36].

2.3.2 Bounds on the Genus

In this section, we deal with the natural question of Shimura varieties obtained from families of coverings of higher genus curves. Here we describe the result obtained in [34] where we concluded the classification showing that there exist no more examples of Shimura subvarieties obtained as Galois coverings of curves of genus $g' > 1$ which satisfy condition $(*)$. The problem was addressed for the first time in [36]. There the authors considered coverings of curves of genus $g' \geq 1$. As already mentioned in the previous section they found six examples of families of coverings of elliptic curves which yield Shimura subvarieties. They considered also cases of $g' > 1$. No examples were found and they showed that:

Theorem 2.3.11 ([36], §4.11). *If $g' \geq 1$ and we have a positive dimensional family of Galois coverings $f : C \to C'$ with $g = g(C)$ and $g' = g(C')$ which satisfies condition $(*)$, then*

$$g \leq 6g' + 1 \tag{2.4}$$

Notice that this theorem asserts that for $g \geq 8$ (resp. 14) there do not exist positive dimensional families of Galois coverings with $g' = 1$ (resp. 2) which satisfy $(*)$. Moreover condition (2.4), together with computations done in `MAGMA`, excluded the existence of any other family satisfying $(*)$ in case of $g' = 1$ or $g' = 2$.

The Theorem above gives a first bound for the genera g, g' of curves which possibly occur in families of Galois coverings which yield Shimura subvarieties. In the following we focus on the case where $g' \geq 1$ and we complete the analysis.

Let us recall some notation: fix a datum Δ and a point of $\widetilde{\mathsf{M}}_\Delta$, i.e. the isomorphism class of a curve C of genus g, which admits an effective holomorphic action of G. Denote by $C' := C/G$ the quotient, which has genus g' and by $f : C \to C'$ the quotient map. Since by assumption $g' \geq 1$, we can consider the norm map induced between Jacobian varieties

$$\mathrm{Nm} : JC \to JC'$$

and the Prym variety[4]

$$P(f) := (\ker \mathrm{Nm})^0.$$

Call δ the type of the polarization obtained by restricting the theta divisor of JC to $P(f)$ and denote by $\mathcal{A}^\delta_{g-g'}$ the moduli space of abelian varieties of dimension $g - g'$ with a polarisation of type δ.

Proposition 2.3.12. *G is a group of automorphisms of $P(f)$ as a polarized abelian variety.*

Proof. Indeed, f and Nm are G-invariant. Hence $P(f)$, which lies in the kernel of Nm, is G-invariant. Furthermore the polarization is preserved.

□

[4] Prym varieties and Prym maps are the main characters of the second part of this thesis. For more details see Section 4.2.

Denote by
$$\mathsf{P}_\Delta \subset \mathsf{A}^\delta_{g-g'}$$
the Shimura variety parametrizing abelian varieties with an action of G of the same type as $P(f)$. This variety is constructed as in [32, §3] and, roughly speaking, it can be seen as the image in $\mathcal{A}^\delta_{g-g'}$ of $(\mathfrak{S}^\delta_{g-g'})^G$. Next consider the Prym map:

$$\begin{aligned} \mathcal{P} : \widetilde{\mathsf{M}}_\Delta &\to \mathsf{P}_\Delta \\ [C] &\mapsto [P(f), \Theta|_{P(f)}] \end{aligned}$$

If $g' = 0$, then $\mathcal{P}$ is just the Torelli morphism composed with ν in (1.4), so we are just in the setting of Theorem 2.3.7, which in fact asserts that under condition $(*)$ we have $\overline{j(\mathsf{M}_\Delta)} = \mathsf{P}_\Delta$. Instead, when $g' \geq 1$, the Prym map gives rise to some additional geometry. We take advantage of it to obtain our new bound on genera g, g' of curves which yield Shimura subvarieties. Indeed, we prove the following:

Theorem 2.3.13. *Consider a datum $\Delta = (\boldsymbol{m}, G, \theta)$ with $g' \geq 1$, $3g'+r > 3$ (i.e. $\dim \mathsf{M}_\Delta > 0$) and which satisfies condition $(*)$. Then:*

- $\odot$ *$\mathcal{P}$ is dominant;*
- $\odot$ *$g' \leq 3$.*

Proof. Both $\widetilde{\mathsf{M}}_\Delta$ and P_Δ are complex orbifolds and $\mathcal{P}$ is an orbifold map. We wish to show that $\mathcal{P} : \widetilde{\mathsf{M}}_\Delta \to \mathsf{P}_\Delta$ is generically submersive.

Fix a point $x = [C] \in \widetilde{\mathsf{M}}_\Delta$, denote by $f : C \to C' := C/G$ the covering and set $(A, \Theta) = P(f)$, so that $\mathcal{P}(x) = [A, \Theta]$. We get that

$$T_x\widetilde{\mathsf{M}}_\Delta = H^1(C, T_C)^G \quad \text{and} \quad T_{\mathcal{P}(x)}\mathsf{P}_\Delta = (\mathrm{Sym}^2 H^0(A, \Omega^1_A)^*)^G \tag{2.5}$$

are the orbifold tangent spaces of $\widetilde{\mathsf{M}}_\Delta$ (resp. P_Δ) at x (resp. $\mathcal{P}(x)$). Furthermore we observe that
$$H^0(\omega_C) = H^0(\omega_C)^G \oplus H^0(\omega_C)^- \tag{2.6}$$
where $H^0(\omega_C)^G \cong H^0(C', \omega_{C'})$ and $H^0(\omega_C)^- \cong H^0(A, \Omega^1_A)$ denotes the sum of the non-trivial isotypic components as a representation of G.

Taking the dual tangent spaces of (2.5) we have:

$$T^*_x\widetilde{\mathsf{M}}_\Delta \cong H^0(C, \omega_C^{\otimes 2})^G \quad \text{and} \quad T^*_{\mathcal{P}(x)}\mathsf{P}_\Delta \cong \left(\mathrm{Sym}^2 H^0(C, \omega_C)^-\right)^G. \tag{2.7}$$

So the codifferential (i.e. the dual of the differential) of $\mathcal{P}$ at x is a map

$$d\mathcal{P}^*_x : (\mathrm{Sym}^2 H^0(\omega_C)^-)^G \to H^0(C, \omega_C^{\otimes 2})^G.$$

This map is just the restriction of the multiplication map

$$m : \mathrm{Sym}^2 H^0(\omega_C) \to H^0(C, \omega_C^{\otimes 2}). \tag{2.8}$$

Denote by $\mathcal{HE}_g \subset \mathcal{M}_g$ the hyperelliptic locus. Assume first that M_Δ is not contained in $\mathcal{HE}_g$ and that $x \notin \mathcal{HE}_g$. Then the multiplication map (2.8) is surjective by Noether theorem. Moreover Schur's lemma implies:

$$m((\mathrm{Sym}^2 H^0(\omega_C))^G) = H^0(C, \omega_C^{\otimes 2})^G.$$

Hence condition $(*)$ implies that

$$m|_{(\mathrm{Sym}^2 H^0(\omega_C))^G} : (\mathrm{Sym}^2 H^0(\omega_C))^G \longrightarrow H^0(C, \omega_C^{\otimes 2})^G \tag{2.9}$$

is in fact an isomorphism. But $(\mathrm{Sym}^2 H^0(\omega_C)^-)^G \subset (\mathrm{Sym}^2 H^0(\omega_C))^G$, so we conclude that $d\mathcal{P}_x^*$ is injective. Hence $d\mathcal{P}_x$ is surjective. This shows that $\mathcal{P}$ is submersive at x.

Assume now that $\mathcal{M}_\Delta \subset \mathcal{HE}_g$ and denote by $\sigma : C \to C$ the hyperelliptic involution. Then m maps $\mathrm{Sym}^2 H^0(\omega_C)$ onto $H^0(C, \omega_C^{\otimes 2})^{\langle\sigma\rangle}$. Indeed σ acts as multiplication by -1 on $H^0(C, \omega_C)$ and thus

$$m : \mathrm{Sym}^2 H^0(\omega_C) = (\mathrm{Sym}^2 H^0(\omega_C))^{\langle\sigma\rangle} \twoheadrightarrow H^0(C, \omega_C^{\otimes 2})^{\langle\sigma\rangle}.$$

Now, if $\sigma \in G$, then $H^0(C, \omega_C^{\otimes 2})^G \subset H^0(C, \omega_C^{\otimes 2})^{\langle\sigma\rangle}$. Just as before Schur lemma shows that (2.9) is onto and $(*)$ yields that (2.9) is an isomorphism. It follows that $\mathcal{P}$ is submersive at x.

If, instead, $\sigma \notin G$, denote by $\tilde{G}$ the subgroup of $\mathrm{Aut}(C)$ generated by G and σ. Arguing as above we conclude that the multiplication map $(\mathrm{Sym}^2 H^0(\omega_C))^{\tilde{G}} \to H^0(C, \omega_C^{\otimes 2})^{\tilde{G}}$ is surjective. Since $\mathcal{M}_\Delta \subset \mathcal{HE}_g$, by definition of σ we have

$$(\mathrm{Sym}^2 H^0(\omega_C))^G = (\mathrm{Sym}^2 H^0(\omega_C))^{\tilde{G}} \quad \text{and} \quad H^0(\omega_C^{\otimes 2})^G = H^0(\omega_C^{\otimes 2})^{\tilde{G}}.$$

Therefore also in this case the multiplication map (2.9) is surjective. By $(*)$ it is an isomorphism and thus $d\mathcal{P}_x$ is surjective.

We have proved that in case $\mathcal{M}_\Delta \subset \mathcal{HE}_g$, $\mathcal{P}$ is submersive on $\widetilde{\mathsf{M}}_\Delta$ in the orbifold sense, while in case $\mathcal{M}_\Delta$ is not contained in $\mathcal{HE}_g$, $\mathcal{P}$ is submersive on $\nu^{-1}(\mathcal{M}_\Delta \setminus \mathcal{HE}_g)$. At any case $\mathcal{P}$ is generically submersive (in the orbifold sense) hence it is dominant.

From (2.6) we get

$$(\mathrm{Sym}^2 H^0(\omega_C))^G \cong \mathrm{Sym}^2 H^0(\omega_{C'}) \oplus (\mathrm{Sym}^2 H^0(\omega_C)^-)^G. \tag{2.10}$$

Since the multiplication map (2.9) at a generic point is an isomorphism, its restriction to $\mathrm{Sym}^2 H^0(\omega_{C'})$ is injective. Moreover it maps $\mathrm{Sym}^2 H^0(\omega_{C'})$ to $H^0(\omega_{C'}^{\otimes 2})$ which is included in $H^0(\omega_C^{\otimes 2})^G$. Hence

$$\dim(\mathrm{Sym}^2 H^0(\omega_{C'})) \leq \dim H^0(\omega_{C'}^{\otimes 2}),$$

which yields $g' \leq 3$.

□

This Theorem reduces the problem of the existence of families of curves which give rise to new Shimura subvarieties to the analysis of a finite number of cases. The ramified cases are checked using `MAGMA`. The étale case are ruled out by the following:

Lemma 2.3.14. *There do not exist positive dimensional families of étale coverings $f : C \to C' = C/G$ with $g' = g(C') \geq 2$ satisfying condition $(*)$.*

Proof. Assume that a family of étale coverings is given satisfying $(*)$ and $g' \geq 2$. Then $\dim(\mathrm{Sym}^2 H^0(\omega_C))^G = \dim H^0(\omega_C^{\otimes 2})^G = 3g' - 3$ since $r = 0$. We have

$$H^0(\omega_C) \cong H^0(\omega_{C'}) \oplus V^- \quad \text{with} \quad V^- = \bigoplus_{\chi \in I} \nu_\chi V_\chi.$$

Here I is the set of non-trivial irreducible characters of G. Therefore

$$(\mathrm{Sym}^2 H^0(\omega_C))^G \cong \mathrm{Sym}^2 H^0(\omega_{C'}) \oplus (\mathrm{Sym}^2(V^-))^G$$

and thus

$$\begin{gathered} 3g' - 3 = \dim(\mathrm{Sym}^2 H^0(\omega_C))^G = \\ \dim \mathrm{Sym}^2 H^0(\omega_{C'}) + \dim(\mathrm{Sym}^2(V^-))^G \geq \\ \dim \mathrm{Sym}^2 H^0(\omega_{C'}) = \frac{g'(g'+1)}{2} = 3g' - 3. \end{gathered}$$

The last equality follows since $g' = 2, 3$. Hence $(\mathrm{Sym}^2(V^-))^G = 0$.

By Chevalley-Weil formula (1.5) we have $\nu_\chi = (\dim V_\chi)(g' - 1) > 0$, for all non-trivial irreducible character χ. So for any $\chi \in I$ we have $(\mathrm{Sym}^2 V_\chi)^G = 0$ and for any $\chi, \chi' \in I$, we have $(V_\chi \otimes V_{\chi'})^G = 0$ if $\chi \neq \chi'$. If there is a non-trivial 1-dimensional representation V_χ, this is impossible. In fact let χ' be the character of V_χ^*. If $\chi \neq \chi'$, then $(V_\chi \otimes V_{\chi'})^G \neq 0$. If $\chi = \chi'$, then $0 \neq (V_\chi \otimes V_\chi)^G \cong (\mathrm{Sym}^2(V_\chi))^G$, since $\Lambda^2 V_\chi = 0$ because $\dim V_\chi = 1$.

By Theorem (2.3.13) we know that $g' \leq 3$ and from of [36, Thm. 1.2] that $g \leq 6g'+1$. Denote by $d := |G|$. If $g' = 2$, we have $d = g - 1 \leq 6g' = 12$ by Riemann-Hurwitz, while in case $g' = 3$, we have $g - 1 = 2d$, hence $d = \frac{g-1}{2} \leq 3g' = 9$. So at any case $d \leq 12$ and all groups with $|G| \leq 12$ admit non-trivial 1-dimensional irreducible representations. This gives a contradiction and thus it concludes the proof.

□

The classification of families of Galois coverings which satisfy condition $(*)$ and yield Shimura subvarieties is thus concluded. We resume our result in the following:

Theorem 2.3.15. *The only positive dimensional families of Galois coverings $f : C \to C' = C/G$ with $g' = g(C') \geq 1$ and $g = g(C)$ which satisfy condition $(*)$ have $g' = 1$ and they are the 6 families found in [36].*

Proof. From Theorem (2.3.13) we know that $g' \leq 3$. From Theorem 1.2 of [36] we know that $g \leq 6g' + 1$. From Lemma 2.3.14 we know that the covering has to be ramified and by computer calculations as in [36] we find exactly the 6 families of [36] with $g' = 1$.

□

2.3.3 Infinitely Many New Examples

This is the very central part of this chapter. Here we show the strength of Theorem 2.2.10. In [34] we assert that it provides a very useful tool which exploits condition $(*)$ to construct infinitely many new examples of totally geodesic subvarieties contained in $\mathcal{T}_g$, with $g \leq 4$.

Actually, a similar result is presented in case of genus 4 by Grushevsky and Möller in [44]. In this paper, in fact, the authors study family (6e) and they show that its Prym map is a fibration in curves, which are totally geodesic. As consequence, they obtain uncountably many totally geodesic curves generically contained in $\mathcal{T}_4$, countably many of which are Shimura.

The goal of this section is to prove that this phenomenon occurs for every families (1e)-(6e) and not only for family (6e). Indeed, the following holds:

Theorem 2.3.16. *Consider a datum $\Delta = (\boldsymbol{m}, G, \theta)$ with $g' \geq 1$, $3g'+r > 3$ (i.e. $\dim \mathsf{M}_\Delta > 0$), and which satisfies condition $(*)$. Then for every $y \in \operatorname{Im} \mathcal{P}$ and for every irreducible component F of $\mathcal{P}^{-1}(y)$, the closure $W := \overline{j(F)}$ is a totally geodesic subvariety of $\mathcal{A}_g$ of dimension $\dfrac{g'(g'+1)}{2}$.*

Proof. We start by proving the dimension statement. First we compute the dimension of generic fibres of $\mathcal{P}$.

By $(*)$ $\dim \mathsf{M}_\Delta = \dim(\operatorname{Sym}^2 H^0(\omega_C))^G$. Fix x such that $\mathcal{P}$ is submersive at x. Set $y = \mathcal{P}(x)$. Then, using (2.10) we get

$$\begin{gathered}\dim_x \mathcal{P}^{-1}(y) = \dim \widetilde{\mathsf{M}}_\Delta - \dim \mathsf{P}_\Delta = \\ \dim(\operatorname{Sym}^2 H^0(\omega_C))^G - \dim(\operatorname{Sym}^2 H^0(\omega_C)^-)^G = \\ \dim \operatorname{Sym}^2 H^0(\omega_{C'}) = \frac{g'(g'+1)}{2} \geq 1.\end{gathered} \tag{2.11}$$

Hence the generic fibre of $\mathcal{P}$ has dimension $\dfrac{g'(g'+1)}{2}$. Therefore

$$\dim W = \dim F \geq \frac{g'(g'+1)}{2}.$$

If $y = [A, \Theta] \in \operatorname{Im} \mathcal{P}$, denote by $W' \subset \mathcal{A}_g^D$ the closure of the variety parametrizing abelian varieties isomorphic to products $A \times j(C')$, where $[C'] \in \mathcal{M}_{g'}$ (with D denoting the appropriate product polarization). Observe that W is an irreducible analytic subset of $\mathcal{A}_g$ and W' is an irreducible analytic subset of $\mathcal{A}_g^D$. Clearly the generic point of W is isogenous to some point of W'.

By Theorem 2.3.13 $g' \leq 3$, so W' parametrizes abelian varieties of the form $A \times B$ with $[B] \in \mathcal{A}_{g'}$. Hence W' is a totally geodesic subvariety of $\mathcal{A}_g^D$ of dimension $g'(g'+1)/2$. Indeed, e.g. in [33, Proposition 5.6], it is shown that

- $\mathcal{A}_{g-k} \times \mathcal{A}_k$ is totally geodesic in $\mathcal{A}_g$;

- fixing $A_0 \in \mathcal{A}_k$ and taking the map $h : \mathcal{A}_{g-k} \to \mathcal{A}_k$ which sends $A \mapsto A \times A_0$, the preimage $h^{-1}(Z)$ of a totally geodesic subvariety $Z \subset \mathcal{A}_g$ is totally geodesic in $\mathcal{A}_{g-k}$.

This shows that $\mathcal{A}_{g-k}$ totally geodesic in $\mathcal{A}_g$ and thus, in our case, that W' is totally geodesic in $\mathcal{A}_g^D$.

Theorem 2.2.10 implies that $\dim W \leq \dim W'$, so $\dim W = g'(g'+1)/2$. This proves the dimension statement.

Now, we apply the second part of Theorem 2.2.10, we conclude that W is totally geodesic.

□

Remark 14. Since we know that the data satisfying the assumptions of the Theorem have $g' = 1$, the fibres are 1-dimensional. Thus the Theorem above shows that having at least a family which satisfy the assumptions then there exist infinitely many new examples of totally geodesic curves contained in $\mathcal{T}_2$ and in $\mathcal{T}_3$. Our proof recovers those already found in [44].

Inspired by Theorem (2.3.16), the second part of this section is devoted to the study of a second fibration for our families of curves: we will show that, again, Theorem 2.2.10 lets the fibres yield new totally geodesic subvarieties of $\mathcal{A}_g$ contained in $\mathcal{T}_g$.

Fixing a datum $\Delta = (\mathbf{m}, G, \theta)$ with $g' \geq 1$, $3g' + r > 3$, we consider the map

$$\varphi : \widetilde{\mathsf{M}}_\Delta \to \mathcal{A}_{g'} \tag{2.12}$$

which associates to $[C \to C']$ the Jacobian $[JC']$. We prove the following:

Theorem 2.3.17. *Consider a datum Δ with $g' \geq 1$, $3g' + r > 3$ (i.e. $\dim \mathsf{M}_\Delta > 0$), and which satisfies condition $(*)$. Then for every $y \in j(\mathcal{M}_{g'})$ and for every irreducible component Y of $\varphi^{-1}(y)$, the closure $X := \overline{j(Y)}$ is a totally geodesic subvariety of $\mathcal{A}_g$ of dimension $d := N - \dfrac{g'(g'+1)}{2}$.*

Proof. In the proof of Theorem 2.3.13 we have shown that if $(*)$ holds, for the generic point $x = [C \to C'] \in \widetilde{\mathsf{M}}_\Delta$ the multiplication map

$$m : (\mathrm{Sym}^2 H^0(\omega_C))^G \to H^0(\omega_C^{\otimes 2})^G \tag{2.13}$$

is an isomorphism. Since we can decompose

$$(\mathrm{Sym}^2 H^0(\omega_C))^G \cong \mathrm{Sym}^2 H^0(\omega_{C'}) \oplus (\mathrm{Sym}^2 H^0(\omega_C)^-)^G, \tag{2.14}$$

the isomorphism (2.13) implies that the restriction of m to $\mathrm{Sym}^2 H^0(\omega_{C'})$ is injective. As this is the codifferential of φ at x, we have proved that

$$d\varphi_x : H^1(T_C)^G \to \mathrm{Sym}^2 H^0(\omega_{C'})^*$$

is surjective. Hence for the generic fibre we have:

$$\dim \varphi^{-1}(y) = N - \frac{g'(g'+1)}{2},$$

so

$$\dim(X) = \dim(Y) \geq N - \frac{g'(g'+1)}{2}.$$

Let $y = [J(C'), \Theta] \in j(\mathcal{M}_{g'})$, denote by $X' \subset \mathcal{A}_g^D$ the closure of the variety parametrizing abelian varieties isomorphic to products $j(C') \times B$, where $B \in \mathsf{P}_\Delta$ (with D denoting the appropriate product polarization). As already observed P_Δ is a Shimura subvariety of $\mathsf{A}^\delta_{g-g'}$. Therefore X' is a totally geodesic subvariety of $\mathcal{A}_g^D$ and its dimension equals $\dim \mathsf{P}_\Delta$.

As noted in (2.7), $T^*_{\mathcal{P}(x)}\mathsf{P}_\Delta \cong \left(\mathrm{Sym}^2 H^0(C, \omega_C)^-\right)^G$. By (2.13) and (2.14) we get:

$$\dim \mathsf{P}_\Delta = \dim \left(\mathrm{Sym}^2 H^0(\omega_C)\right)^G - \dim \mathrm{Sym}^2 H^0(\omega_{C'}) = N - \frac{g'(g'+1)}{2}.$$

Since the generic point of X is isogenous to some point of X', Theorem 2.2.10 guarantees $\dim X \leq \dim X'$. Therefore

$$\dim X = N - \frac{g'(g'+1)}{2}.$$

This proves the dimension statement.

Now we can apply the second part of Theorem 2.2.10 to conclude that X is totally geodesic.

□

Remark 15. Since we know that the data satisfying the assumptions of the Theorem have $g' = 1$, we have in fact $d = N - 1 = r - 1$. This means that, having at least a family which satisfy the assumptions, the Theorem above shows the existence of infinitely many totally geodesic subvarieties of dimension 1,2 and 3 which are generically contained in $\mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$.

Remark 16. Theorems 2.3.16 and 2.3.17 deal with fibres of $\mathcal{P}$ and φ. One can apply Theorem 2.2.10 also to $\mathsf{Z} := \overline{j(\mathsf{M}_\Delta)}$ as a whole. In this case, remembering Proposition 2.2.7, one gets an isomorphism $a \in \mathrm{GL}(\Lambda_\mathbb{Q})$ that maps a lifting to Siegel space of Z to an appropriate lifting of $\mathcal{A}_1 \times \mathsf{P}_\Delta$. (By the way note that this proves again that Z is totally geodesic.)

Both Z and $\mathcal{A}_1 \times \mathsf{P}_\Delta$ have a *product structure*: $\mathcal{A}_1 \times \mathsf{P}_\Delta$ has the natural projections π_i on the factors, while Z has the maps φ and $\mathcal{P}$. It is natural to ask whether this $a \in \mathrm{GL}(\Lambda_\mathbb{Q})$ can be chosen in such a way that $\pi_1 \circ a = \varphi$ and $\pi_2 \circ a = \mathcal{P}$. This means that (at the level of Siegel space!) $a(y) = (\varphi(y), \mathcal{P}(y))$. It seems unlikely that such a map can be gotten by methods similar to those of Theorem 2.2.10. Since we are dealing with only 6 families an explicit analysis could in principle answer this question. Yet this is probably non-trivial.

Finally, in light of Theorem 2.3.10, we know that there are only 6 families which satisfy condition $(*)$ with $g' \geq 1$. Therefore, applying Theorems 2.3.16 and 2.3.17 we can announce the following:

Corollary 2.3.17.1. *Families* (1e), (2e), (3e), (4e), (6e) *are fibred in totally geodesic curves via their Prym maps and are fibred in totally geodesic subvarieties of codimension 1 via the map φ. Therefore they contain infinitely many totally geodesic subvarieties and countably many Shimura subvarieties.*

The Prym map of family (5e) *is constant, its image is the square of the elliptic curve $y^2 = x^3 - x$, i.e. the elliptic curve with lattice $\mathbb{Z} + i\mathbb{Z}$.*

Proof. This follows immediately from Theorems 2.3.16 and 2.3.17. Since all these families yield Shimura subvarieties of $\mathcal{A}_g$, they contain countably many CM points, hence the fibres of the two maps $\mathcal{P}$ and φ passing through these points are Shimura subvarieties.

Since family (5e) is one dimensional, it is itself a fibre of its Prym map $\mathcal{P}$, which therefore is constant. The computation of this constant abelian surface is given in the following section in the proof of Proposition 2.4.2.

□

2.4 Explicit Analysis of the Fibrations

In the last part of this chapter we analyse several features of the examples listed in [32] and [36]. In particular we will focus on the two fibrations $\mathcal{P}$ and φ introduced in the previous paragraph.

We describe all the possible inclusions among the families of Galois covers of $\mathbb{P}^1$ or of elliptic curves yielding Shimura subvarieties of $\mathcal{A}_g$ known so far. We show that some of them occur as irreducible components of fibres of the Prym map or are contained in fibres of the map φ of one of the 6 families in [36]. In this way, we detect some of the infinitely many totally geodesic subvarieties of $\mathcal{T}_g$ found in Theorems 2.3.16 and 2.3.17 as fibres of $\mathcal{P}$ and of φ of the families of [36].

We will devote a subsection to every genus. All of them start with a diagram where we collect all the possible inclusions between the families of Galois covering which yield Shimura varieties recorded in [32] and [36]. In the diagrams they are labelled using the same notation of [32] with a marked letter *e* in case of families of [36]. The arrows denote the inclusions and the dimension of the families grows from the bottom to the top of the diagrams.

We will analyse the different families of [32] one at a time. For their description we refer to section 1.3.1. We will give a presentation of the Galois group G and an explicit description of the monodromy map

$$\theta : \Gamma_{0,r} \to G,$$

where $\Gamma_{0,r} = \langle [\gamma_1], [\gamma_2], ..., [\gamma_r] : \gamma_1\gamma_2...\gamma_r = 1 \rangle$. We set $x_i = \theta(\gamma_i)$, $i = 1, ..., r$ and $\mathbf{m} = (m_1, \ldots, m_r)$, where $m_i = o(x_i)$. For simplicity we will just write $\mathbf{x} = (x_1, ..., x_r)$

to describe the monodromy. We will also use the notation of MAGMA for the irreducible representations of the group G.

Notice that for simplicity we will omit the parameters of the families, i.e. we will refer to curves C without any subscript.

2.4.1 Genus $g = 2$

In genus 2 we have the following diagram: in the lowest line we have the 1-dimensional families, family $(1e) = (26)$ has dimension 2, while family (2) has dimension 3 and it is $\mathcal{M}_2$. It is trivial that all our families are contained in (2). Indeed:

$$\begin{tikzcd}
& (2) & \\
& (1e) = (26) \arrow[u] & \\
(3) = (5) = (28) = (30) \arrow[ur] & & (4) = (29) \arrow[ul]
\end{tikzcd}$$

Notice that the bielliptic locus in genus 2 (i.e. family (1e)) has codimension 1 in $\mathcal{M}_2$ and it is totally geodesic in $\mathcal{A}_2$. This only happens in genus 2. In fact in [32] it is shown that if $g \geq 3$ and $Y \subset \mathcal{M}_g$ is an irreducible divisor, then there is no proper totally geodesic subvariety of $\mathcal{A}_g$ containing $j(Y)$.

Proposition 2.4.1. *In genus 2 families (3)=(5)=(28)=(30) and (4) = (29) are not contained in any fibre of $\mathcal{P}$, nor in any fibre of φ of the family (1e).*

Proof. We show that both the curves in family (30) and the ones in family (29) have Jacobians that are isogenous to the self product of an elliptic curve. Thus none of these two families is fibre of the Prym map of $(1e)$, nor of the map φ of $(1e)$.

Let us start from $\mathbf{(3) = (5) = (28) = (30)}$.

$$G = D_6 = \langle x, y : x^6 = y^2 = 1, y^{-1}xy = x^5 \rangle,$$
$$\mathbf{x} = (x^3, x^2y, x^3y, x^4) \quad \mathbf{m} = (2, 2, 2, 3).$$

MAGMA gives us $H^0(C, \omega_C) = V_6$, being V_6 a 2-dimensional $\mathbb{C}$-irreducible representation for G. This implies that $\dim(\mathrm{Sym}^2 H^0(C, \omega_C))^G = 1$. Thus condition $(*)$ effectively holds (since the family has dimension 1).

The group algebra decomposition for the curves C in the family decomposes the Jacobian JC up to isogeny[5]. In this case it gives $JC \sim B_6^2$, where B_6 is 1-dimensional. Since

[5]For details on abelian varieties, group action and their induced decomposition see Section 3.1 and references therein.

the family is non-constant this immediately implies that B_6 varies in family, hence the thesis. In order to check that $(30) \subset (1e)$ let us consider the subgroup $H = \langle y \rangle$ and let us look at the diagram:

$$\begin{tikzcd} C \arrow[dd, "\psi"'] \arrow[dr, "f"] & \\ & C/H \arrow[dl] \\ \mathbb{P}^1 & \end{tikzcd} \tag{2.15}$$

We get that f has only two critical points of order 2 in $\psi^{-1}(z_2)$ and no more in $\psi^{-1}(z_i)$ with $i = 1, 3, 4$. Here $z_1, ..., z_4$ are the critical values of ψ. This implies that $E := C/H$ is an elliptic curve (using Riemann-Hurwitz formula) and, in particular, the desired inclusion (30)⊂(1e).

$(4) = (29)$

$$G = D_4 = \langle x, y : x^4 = y^2 = 1, y^{-1}xy = x^3 \rangle,$$
$$\mathbf{x} = (x^3y, x^2, y, x^3) \quad \mathbf{m} = (2, 2, 2, 4) \quad H^0(C, \omega_C) = V_5.$$

The representation V_5 has dimension 2 and $\dim(\mathrm{Sym}^2 H^0(C, \omega_C))^G = 1$, hence $(*)$ is verified. The group algebra decomposition gives $JC \sim B_5^2$. Therefore, as above, it is impossible for (29) to lie in a fibre of maps $\mathcal{P}, \varphi$ of family $(1e)$.
As before it is sufficient to consider the subgroup $H = \langle y \rangle$ and to study a diagram similar to (2.15) to obtain that C/H is a curve of genus 1. Indeed, in this case $f : C \to C/H$ has two critical points of order 2 in $\psi^{-1}(z_3)$. This guarantees the inclusion $(29) \subset (1e)$.
□

Remark 17. As seen in the proof of this Proposition, the study of the families is done looking at diagrams as the one in (2.15): intermediate quotients are the tools which check the inclusions and the compatibility of the monodromies of the families.

From now on we will omit all the pictures since they are very similar.

2.4.2 Genus $g = 3$

In genus 3 we have the following diagram. In the lowest line we have the 1-dimensional families, in the second one the 2-dimensional ones. Family (27) has dimension 3 while family $(2e)$, the bielliptic locus in genus 3, has dimension 4.

$$\begin{tikzcd}
 & & (2e) & \\
 & & & (27) \arrow[ul] \\
(6) & (8) & (31) = (3e) \arrow[uu] & (32) = (4e) \arrow[u] \\
(9) \arrow[u] \arrow[uuurr, bend left] & (22) \arrow[u] \arrow[uuur] & (33) = (35) \arrow[u] \arrow[ur] & (7) = (23) = (34) = (5e) \arrow[u]
\end{tikzcd}$$

Proposition 2.4.2.

i) *Family* (34) *is a fibre of the Prym map of the bielliptic locus* (2e) *and also a fibre of the Prym map of* (4e).

ii) *Families* (9) *and* (22) *are both contained in fibres of the map* φ *of* (2e) *and are not contained in any fibre of the map* $\mathcal{P}$ *of* (2e).

iii) *Family* (33) =(35) *is not contained in any fibre of* φ *nor in any fibre of* $\mathcal{P}$ *of* (2e),(3e) *or* (4e).

iv) *Families* (31) = (3e) *and* (32) = (4e) *are not contained in fibres of the map* φ *of* (2e).

v) *Family* (27) *is not contained in a fibre of the map* φ *of* (2e).

Proof. We begin from $\mathbf{(5}\boldsymbol{e}\mathbf{) = (7) = (23) = (34)}$

$$\begin{aligned} G = \mathbb{Z}/4 \times \mathbb{Z}/2 \rtimes \mathbb{Z}/2 = \langle g_1, g_2, g_3 : g_1^2 = g_2^2 = g_3^4 = 1, \\ g_2 g_3 = g_3 g_2, g_1^{-1} g_2 g_1 = g_2 g_3^2, g_1^{-1} g_3 g_1 = g_3 \rangle, \\ \mathbf{m} = (2, 2, 2, 4), \quad \mathbf{x} = (g_1, g_1 g_2 g_3^3, g_2 g_3^2, g_3^3). \end{aligned}$$

Using the notation of `MAGMA`, we can decompose $H^0(C, \omega_C) \cong V_6 \oplus V_{10}$, where V_i are irreducible $\mathbb{C}$-representations of G such that $\dim(V_6) = 1$ and $\dim V_{10} = 2$. Moreover $(\mathrm{Sym}^2 H^0(C, \omega_C))^G \cong \mathrm{Sym}^2 V_6$, which has dimension 1, hence condition $(*)$ holds.
The group algebra decomposition gives us a decomposition of the Jacobian JC up to isogeny: $JC \sim B_6 \times B_{10}^2$. Both the B_i's have dimension 1.
Choose $H = \langle g_3^2 \rangle$ and consider the map $f : C \to C/H$. Call $z_1, ..., z_4$ the critical values of the map $\psi : C \to C/G$. One immediately verifies that there are only four critical points of index two for f, all placed in the fibre $\psi^{-1}(z_4)$. Applying Riemann-Hurwitz formula, we obtain $g(C/H) = 1$. Notice that this gives us the inclusion of (34) in (2e).
Denote by $E := C/H$, we get $B_6 \sim E$, and $H^0(E, \omega_E) = H^0(C, \omega_C)^H = V_6$. Hence the curve E moves, and the Prym variety $P(C, E) \sim B_{10}^2$. Thus the Prym variety $P(C, E)$

doesn't move, hence family (34) is a fibre of the Prym map of the family (2e).
Now we consider the normal subgroup $Q_8 = \langle g_3^2, g_2g_3, g_1g_3, g_1g_2g_3^2 \rangle$ which is isomorphic to the quaternion group. The degree 8 map $C \to C/Q_8$ has a single branch point and the quotient C/Q_8 has genus 1. So (34)= (5e).
Now we show that this family is also a fibre of the Prym map of family (4e). In fact, consider the subgroup $H' = \langle g_1g_3 \rangle$ of order 4, which contains H, and the map $f' : C \to C/H'$. One can check that f' has four critical points of order two and two critical values. Therefore C/H' has genus 1. This provides the inclusion (34)⊂(4e). Moreover $H^0(C, \omega_C)^{H'} = V_6$, so the curve $E' = C/H'$ moves and it is isogenous to $E = C/H$ while $P(C, E')$ is fixed and it is isogenous to $P(C, E)$. Hence (34) is also a fibre of the Prym map of (4e).
Finally one can see that B_{10} is obtained as the quotient $C/\langle g_1 \rangle$. It is the elliptic curve with j-invariant equal to 1728 and that has Legendre equation: $y^2 = x(x^2 - 1)$. To check this, consider the commutative diagram

$$\begin{tikzcd}
 & & C/\langle g_1 \rangle = B_{10} \arrow[d, "\varepsilon"] \\
C \arrow[urr, bend left, "\gamma"] \arrow[rr, "\alpha"] \arrow[drr, "\varphi"] \arrow[dd, "\psi"] & & C/\langle g_1, g_3^2 \rangle = \mathbb{P}^1 \arrow[d, "h"] \\
 & & C/\langle g_1, g_3 \rangle = \mathbb{P}^1 \arrow[dll, bend left, "\pi"] \\
C/G = \mathbb{P}^1 & &
\end{tikzcd}$$

Assume that the critical values z_i of ψ are (as usual) $[\lambda, 0, \infty, 1]$. Then we can suppose that the map $\pi : C/\langle g_1, g_3 \rangle \cong \mathbb{P}^1 \to C/G \cong \mathbb{P}^1$ is $z \mapsto z^2$. Hence the critical values of φ are $[\mu, -\mu, 1, -1]$, where $\mu^2 = \lambda$, with induced monodromy $(g_1, g_1g_3^2, g_3, g_3)$. We can assume that $h(z) = \frac{z^2+c}{z^2-c}$, where $c = \frac{1+\mu}{\mu-1}$, so the critical values of α are $[\frac{1+\mu}{\mu-1}, -\frac{1+\mu}{\mu-1}, 1, -1, \infty, 0]$ with monodromy $(g_1, g_1, g_1g_3^2, g_1g_3^2, g_3^2, g_3^2)$. So finally we see that the critical values of the double cover $\varepsilon : B_{10} = C/\langle g_1 \rangle \to C/\langle g_1, g_3^2 \rangle \cong \mathbb{P}^1$ are $1, -1, \infty, 0$, thus B_{10} has equation $y^2 = x(x^2 - 1)$.

(33)=(35).

$$G = S_4 \text{ with } g_1 = (12),\ g_2 = (123),\ g_3 = (13)(24) \text{ and } g_4 = (14)(23).$$
$$\mathbf{x} = (g_1g_2^2, g_3g_4, g_1, g_2^2g_4) \quad \mathbf{m} = (2, 2, 2, 3).$$

Moreover $H^0(C, \omega_C) \cong V_4$ (V_4 is an irreducible representation of dimension 3) and so $(\mathrm{Sym}^2 H^0(C, \omega_C))^G \cong (\mathrm{Sym}^2 V_4)^G$.
Considering the group algebra decomposition of the Jacobians of these curves we obtain $JC \sim B_4^3$, where B_4 is an elliptic curve. Since (35) is a 1-dimensional family, B_4 has to

move. Hence (35) is not a fibre of the Prym map of $(2e)$ nor of $(3e),(4e)$. For the same reason it is not contained in a fibre of the map φ of $(2e),(3e),(4e)$. Let us describe the inclusions.
Consider the subgroup $H_2 = \langle g_3 \rangle$, isomorphic to $\mathbb{Z}/2$, and take the quotient map $f_2 : C \to C/H_2$. f_2 has four critical points of order 2 in $\psi^{-1}(z_2)$. Riemann-Hurwitz formula gives $g(C/H_2) = 1$. This shows the inclusion $(35) \subset (2e)$.
Take now the subgroup $H_3 = \langle g_2 \rangle$, which has order 3, and consider the quotient map $f_3 : C \to C/H_3$. We have two critical points for f_3 in $\psi^{-1}(z_4)$ of multiplicity equal to 3. Hence C/H_3 has genus 1. This shows the inclusion $(35) \subset (3e)$.
Finally, considering $H_4 = \langle g_1 g_4 \rangle \cong \mathbb{Z}/4$ we get $(35) \subset (4e)$. Indeed the quotient map $f_4 : C \to C/H_4$ has four critical points of order 2 in $\psi^{-1}(z_2)$. Hence f_4 has two critical values, as desired by $(4e)$.
This concludes the analysis of the inclusions. We observe that $H^0(C,\omega_C)^{H_i}$ is a 1-dimensional subspaces of V_4 for $i = 2,3,4$. This implies the following isogenies: $C/H_2 \sim C/H_3 \sim C/H_4 \sim B_4$.

(22)

$$G = \mathbb{Z}/2 \times \mathbb{Z}/4 \text{ with } g_1 = (0,1),\ g_2 = (1,0) \text{ and } g_3 = (0,2).$$
$$\mathbf{x} = (g_3, g_2g_3, g_1g_2, g_1g_3) \quad \mathbf{m} = (2,2,4,4).$$

We have $H^0(C,\omega_C) \cong V_3 \oplus V_7 \oplus V_8$ and $(\mathrm{Sym}^2 H^0(C,\omega_C))^G \cong V_3 \otimes V_7$. The group algebra decomposition gives $JC \sim B_3 \times B_8$, with $\dim(B_3) = 2$ and $\dim(B_8) = 1$. Consider the subgroup $H = \langle g_2 g_3 \rangle$. One easily checks that the quotient curve $E = C/H$ has genus one, with $(H^0(C,\omega_C))^H = V_8$. Therefore E is isogenous to B_8 and it remains fixed. This proves that family (22) is contained in a fibre of the map φ of $(2e)$. Hence this fibre has an irreducible component whose image in $\mathcal{A}_3$ is a Shimura subvariety of dimension 3.
We add some details on this family which show that the jacobians of this family are decomposable as products of elliptic curves. Take another subgroup of G: $H' = \langle g_2 \rangle$. The quotient map $C \to C/H'$ is étale so, applying Riemann-Hurwitz, we have $g(C/H') = 2$. As $\dim(V_3^{H'}) = 1 = s_{V_3}$, where s_{V_3} is the Schur index of V_3, and since $\dim(V_8^{H'})=0$ we apply Jiménez' criterion (see Lemma 3.1.4) and we obtain $B_3 \sim JC'$, where we set $C' := C/H'$.
Consider now the degree 4 map $C' \to C'/\langle \overline{g_1} \rangle \cong \mathbb{P}^1$, where $\langle \overline{g_1} \rangle \cong G/H' \cong \mathbb{Z}/4$. One easily sees that the family $C' \to C'/\langle \overline{g_1} \rangle$ coincides with the family $(4) = (29)$. Indeed, $H^0(C',\omega_{C'}) = V_3 \oplus V_7$ and thus $(\mathrm{Sym}^2 H^0(C',\omega_{C'}))^{\langle g_1 \rangle} = V_3 \otimes V_7$, that is it has $N = 1$. The multiplication map

$$m : (\mathrm{Sym}^2 H^0(C',\omega_{C'}))^{\langle g_1 \rangle} \to H^0(C',\omega_{C'}^{\otimes 2})^{\langle g_1 \rangle}$$

sends the generator v_3 of V_3 (resp. v_7 of V_7) to $v_3 \cdot v_7$ and thus it is an isomorphism. This shows that family $C' \to C'/\langle \overline{g_1} \rangle$ is Shimura with induced monodromy $\bar{\mathbf{x}} = (\overline{g_3}, \overline{g_3}, \overline{g_1}, \overline{g_1 g_3})$, i.e. it is family $(4) = (29)$.
As already seen, the group algebra decomposition of the Jacobians for (29) is $JC' \sim F^2$,

where F is an elliptic curve. Thus we conclude $JC \sim E \times F^2$. Since family (29) is non-constant F moves and thus (22) is not contained in a fibre of the Prym map of $(2e)$.

(9)

$$G := \mathbb{Z}/6 = \langle g_1, g_2 : g_1^2 = g_2^3 = 1 \rangle,$$
$$\mathbf{x} = (g_1, g_2^2, g_2^2, g_1 g_2^2), \quad \mathbf{m} = (2, 3, 3, 6),$$
$$H^0(C, \omega_C) \cong V_4 \oplus V_5 \oplus V_6 \quad (\mathrm{Sym}^2 H^0(C, \omega_C))^G \cong V_4 \otimes V_6.$$

Let C be a curve in the family and denote as usual the quotient map by $\psi : C \to C/G$. Now take the subgroup $H := \langle g_1 \rangle$. The corresponding quotient map $C \to C/H$ has three critical points, of index 2, in $\psi^{-1}(z_1)$ and a single one, of the same type, in $\psi^{-1}(z_4)$. By Riemann-Hurwitz formula we see that $E := C/H$ is an elliptic curve with $H^0(E, \omega_E) = V_5$. Firstly we get (9)$\subset$(2e). Moreover observe that E does not move.
The group algebra decomposition gives $JC \sim B_4 \times B_5$. The term B_5 is isogenous to the elliptic curve E, that is fixed, while $B_4 \sim P(C, E)$ has dimension 2 and it moves. This shows that family (9) is contained in a fibre of the map φ of $(2e)$. Hence this fibre of φ has an irreducible component which gives rise to a Shimura subvariety of $\mathcal{A}_3$ of dimension 3.

Now we analyse the families of dimension 2 ($N = 2$).

(31)

$$G = S_3 = \langle g_1, g_2 : g_1^2 = g_2^3 = 1, g_1^{-1} g_2 g_1 = g_2^2 \rangle,$$
$$\mathbf{x} = (g_1 g_2^2, g_1 g_2, g_1, g_1 g_2^2, g_2^2), \quad \mathbf{m} = (2, 2, 2, 2, 3).$$

We have $H^0(C, \omega_C) = V_2 \oplus V_3$, where V_2 has dimension 1, V_3 has dimension 2, and $(\mathrm{Sym}^2 H^0(C, \omega_C))^G = \mathrm{Sym}^2 V_2 \oplus (\mathrm{Sym}^2 V_3)^G$. The group algebra decomposition gives $JC \sim B_2 \times B_3^2$, where both terms have dimension 1.
Denote as usual by $\psi : C \to C/G \cong \mathbb{P}^1$ the quotient map, consider the subgroup $H = \langle g_2 \rangle$ and the map $\alpha : C \to C/H$. We have two critical points of index 3 in $\psi^{-1}(z_5)$, hence $E := C/\langle g_2 \rangle$ has genus 1 and one can show that $H^0(E, \omega_E) = V_2$. Thus $B_2 \sim E$ and we also have shown that $(31) \subset (3e)$. Actually, since both have dimension 2, they induce the same family.
Finally one can easily see that $C/\langle g_1 \rangle$ has genus 1 and $B_3 \sim J(C/\langle g_1 \rangle)$. The quotient map $C \to C/\langle g_1 \rangle$ has four critical values and so $(31) \subset$ (2e).
Notice that both B_2 and B_3 move, so (31) is not contained in a fibre of the map φ. Due to dimension reason (31) cannot be contained in a fibre of the Prym map of families $(2e), (3e), (4e)$. The same will occur for other families analysed in the following paragraphs.

(32)

$$G = D_4 = \langle x, y : x^4 = y^2 = 1, y^{-1} x y = x^3 \rangle,$$
$$\mathbf{x} = (x^3 y, y, x^2, y, xy), \quad \mathbf{m} = (2, 2, 2, 2, 2).$$

Moreover $H^0(C,\omega_C) \cong V_4 \oplus V_5$, where V_4 has dimension 1, V_5 has dimension 2 and $(\mathrm{Sym}^2 H^0(C,\omega_C))^G \cong \mathrm{Sym}^2 V_4 \oplus (\mathrm{Sym}^2 V_5)^G$. The group algebra decomposition yields $JC \sim B_4 \times B_5^2$.
Take the subgroup $H = \langle x \rangle$ and consider the quotient map $\alpha : C \to C/H$. We get four critical points in $\psi^{-1}(z_3)$ of index 2, hence $g(C/H) = 1$. This shows the inclusion $(32) \subset (4e)$. Dimension reasons give the equality.
Consider the subgroup $H' = \langle x^2 \rangle$ of H. We can factor the degree four map α into two maps of degree 2. The map $\alpha' : C \to C/H'$ has four critical points of index 2 in $\psi^{-1}(z_3)$, hence C/H' has genus 1. We remark that it is isogenous to C/H. Indeed, we have $H^{1,0}(C/H) = H^{1,0}(C/H') \cong V_4$, therefore $C/H \sim C/H' \sim B_4$. The map α' shows the inclusion $(32) \subset (2e)$.
Consider the subgroup $K = \langle y \rangle$. One immediately checks that C/K has genus 1 and $H^{1,0}(C/K) \subseteq V_5$, hence $B_5 \sim C/K$. Both B_4 and B_5 move, hence (32) is not contained in a fibre of the map φ.

Before going on with the analysis, we would like to observe that the 2-dimensional families **(6)** and **(8)** do not admit a 2:1 map on an elliptic curve. Hence they are not contained in $(2e)$. The group algebra decomposition does not decompose their Jacobian. Notice that family (8) is hyperelliptic.

Let us now describe the only family of dimension $N = 3$.

(27)

$$
\begin{gathered}
G = \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle g_1, g_2 : g_1^2 = g_2^2 = 1 \rangle, \\
\mathbf{x} = (g_2, g_1g_2, g_1, g_1g_2, g_1, g_2), \quad \mathbf{m} = (2,2,2,2,2,2), \\
H^0(C,\omega_C) \cong V_2 \oplus V_3 \oplus V_4.
\end{gathered}
$$

Every V_i has dimension 1 and moreover $(\mathrm{Sym}^2 H^0(C,\omega_C))^G \cong \mathrm{Sym}^2 V_2 \oplus \mathrm{Sym}^2 V_3 \oplus \mathrm{Sym}^2 V_4$.
The Jacobian decomposes up to isogeny as $JC \sim B_2 \times B_3 \times B_4$, where B_i's are three different elliptic curves.
One easily checks that $B_2 \sim C/H$, $B_3 \sim C/H'$ and finally $B_4 \sim C/H''$, where $H = \langle g_2 \rangle$, $H' = \langle g_1 \rangle$ and $H'' = \langle g_1g_2 \rangle$. Both three quotient maps have degree 2 and they have four critical points of order 2: the first ramifies over $\psi^{-1}(z_1)$ and over $\psi^{-1}(z_6)$, the second over $\psi^{-1}(z_3)$ and $\psi^{-1}(z_5)$, the third on $\psi^{-1}(z_2)$ and $\psi^{-1}(z_4)$ (as usual $z_1, ..., z_6$ are the critical values of $\psi : C \to \mathbb{P}^1$). This gives the inclusion $(27) \subset (2e)$. All the three elliptic curves move, so (27) is not a fibre of the map φ.

This ends up the discussion of inclusions in genus 3.

□

2.4.3 Genus $g = 4$

In genus 4 we have the following diagram of inclusions.

$$
\begin{tikzcd}
 & (10) & & (6e) & & \\
 & (14) \arrow[u] & & & & \\
(11) & (13)=(24) \arrow[u] & (25)=(38) \arrow[ul] \arrow[uur] & (12) \arrow[uu] & (37) \arrow[uul] & (36)
\end{tikzcd}
$$

Families in the lowest line are one-dimensional, (14) has dimension 2, while (10) and $(6e)$ have dimension 3.

Proposition 2.4.3. *Family* (12) *is contained in a fibre of the Prym map of* $(6e)$*, while family* $(25) = (38)$ *is contained in a fibre of the map* φ *of* $(6e)$*. Family* (37) *is not contained in any fibre of the Prym map nor of the map* φ *of* $(6e)$*.*

Proof. Let us start by considering family **(12)**

$$
\begin{gathered}
G = \mathbb{Z}/6 = \langle g_1 g_2 : g_1^2 = g_2^3 = 1 \rangle, \\
\mathbf{x} = (g_1, g_1g_2, g_1g_2, g_1g_2), \quad \mathbf{m} = (2,6,6,6), \\
H^0(C, \omega_C) \cong V_2 \oplus V_3 \oplus 2V_6, \quad (\mathrm{Sym}^2 H^0(C,\omega_C))^G \cong \mathrm{Sym}^2 V_2.
\end{gathered}
$$

The group algebra decomposition gives $JC \sim B_2 \times B_3 \times B_6$, where the first two terms have dimension 1 while the third one has dimension equal to 2. Consider the subgroup $H := \langle g_2 \rangle \cong \mathbb{Z}/3$. Call z_1, z_2, z_3, z_4 the branch points for the map $\psi : C \to C/G$ and consider the quotient map $f : C \to C/H$. There are three critical points for f, of order 3, one for every the preimage $\psi^{-1}(z_i), i = 2,3,4$. Applying Riemann-Hurwitz formula we get that the genus of $E := C/H$ is one and we the inclusion (12)$\subset$(6e). Moreover, since $H^0(C,\omega_C)^H = V_2$, we get $B_2 \sim E$.
Consider $H' := \langle g_1 \rangle \cong \mathbb{Z}/2$. Each critical point for ψ is also critical of order two for the quotient map $C \to C/H'$. Riemann-Hurwitz formula implies that C/H' is an elliptic curve. We have $H^0(C/H', \omega_{C/H'}) = V_3$ and thus $B_3 \sim C/H'$.
Notice that the terms B_3 and B_6 don't move and their product is isogenous to the Prym variety $P(C,E)$. Thus, as already observed in [44], (12) is contained in a fibre of the Prym map of $(6e)$.
On the other hand since E moves, family (12) cannot be contained in a fibre of the map φ of $(6e)$.

(25) = (38)

$$
\begin{gathered}
G = \mathbb{Z}/3 \times S_3 \text{ with } g_1 = ([0]_3, (12)),\ g_2 = ([1]_3, (1)) \text{ and } g_3 = ([0]_3, (123)). \\
\mathbf{x} = (g_1 g_3^2, g_1 g_3, g_2 g_3, g_2^2), \quad \mathbf{m} = (2,2,3,3).
\end{gathered}
$$

We know that $H^0(C,\omega_C) \cong V_3 \oplus V_4 \oplus V_8$ and that $(\mathrm{Sym}^2 H^0(C,\omega_C))^G \cong V_3 \otimes V_4$. The first two V_i's have dimension 1 while V_8 has dimension equal to 2. The Jacobian decomposes as $JC \sim B_3 \times B_8^2$, the first term is 2-dimensional while the second is 1-dimensional.
Set $H := \langle g_2 g_3 \rangle \cong \mathbb{Z}/3$ and consider the quotient map $f : C \to C/H$. We get three critical points of f of order 3 all contained in $\psi^{-1}(z_3)$. Hence $g(C/H) = 1$ and we also see the inclusion $(38) \subset (6e)$. Moreover H fixes a 1-dimensional subspace of V_8, thus we get $C/H \sim B_8$. Note that this term of the decomposition doesn't move. This implies that family (38) is contained in a fibre of φ of $(6e)$. Thus this fibre determines a Shimura subvariety of $\mathcal{A}_4$ of dimension 2.
Now take the quotient for $H' = \langle g_3 \rangle \cong \mathbb{Z}/3$. The correspondent quotient map is étale. Therefore we obtain $g(C/H') = 2$. Due to the fact that $\dim(V_3^{H'})=1=s_{V_3}$ and that $\dim(V_8^{H'})=0$, there exists an isogeny between $J(C/H')$ and B_3 (see [51, Lemma 1]). Since H' is normal in G we can now look at the map $C/H' \to \mathbb{P}^1$. This is a Galois covering with Galois group $G/H' \cong \mathbb{Z}/6$ and $\mathbf{m} = (2,2,3,3)$. Actually we have only one 1-dimensional family with this datum and it corresponds to family (5) of [32]. As already explained in the proof of Proposition (2.4.1) the group algebra decomposition on the Jacobian of this family gives us $B_3 \sim J(C/H') \sim E^2$, where E is an elliptic curve. Since E moves (otherwise both family (38) and (5) would be constant), family (38) is not contained in a fibre of the Prym map of $(6e)$.
Finally let us check the inclusion in family (14). Suppose to consider the quotient $K = \langle g_1, g_2 \rangle \cong \mathbb{Z}/6$. The quotient C/K has genus 0. Indeed the quotient map $C \to C/K$ has 5 critical values: the first two have three ramification points of order 2 while the last three have two ramification points of order 3. This gives an induced monodromy of type $\mathbf{m} = (2,2,3,3,3)$, i.e. that of family (14) (as we will explain at the end of this proof).

(37)

$$G = A_4 \text{ with } g_1 = (123), g_2 = (12)(34) \text{ and } g_3 = (13)(24).$$
$$\mathbf{x} = (g_3, g_1 g_3, g_1, g_1 g_2 g_3), \quad \mathbf{m} = (2,3,3,3).$$

Moreover we have $H^0(C,\omega_C) \cong V_2 \oplus V_4$, where V_2 has dimension 1, V_4 has dimension equal to 3 and $(\mathrm{Sym}^2 H^0(C,\omega_C))^G \cong (\mathrm{Sym}^2 V_4)^G$. The Jacobian decomposes completely as $JC \sim B_2 \times B_4^3$.
Take the quotient $\psi : C \to C/G$ with branch points denoted by $z_i,\ i = 1,2,3,4$ and consider the subgroup generated by $\langle g_1 \rangle$. It is a cyclic group of order 3. Studying the map $C \to C/\langle g_1 \rangle := E$ we get three critical points, of order 3, respectively in the fibres $\psi(z_i)^{-1}, i = 2,3,4$. This implies $g(E) = 1$ and also the inclusion in the family $(6e)$. Moreover $\langle g_1 \rangle$ fixes a 1-dimensional subspace of V_4. Thus $H^0(E,\omega_E) = (V_4)^{\langle g_1 \rangle}$ and so $E \sim B_4$. Note that E moves.
If we consider the subgroup $H := \langle g_2, g_3 \rangle$ and its associated quotient map $f : C \to C/H$, all the critical points in the fibre of z_1 are critical points of order 2 for f. Thanks to Riemann-Hurwitz formula we see that the curve $F := C/H$ is elliptic. Since Fix(H)=V_2 we obtain $F \sim B_2$. Although this curve remains constant for the family, we have that $P(C,E) \sim B_2 \times B_4^2$, hence it moves. Thus (37) is neither contained in a fibre of the Prym

map of (6e) nor in a fibre of the map φ of $(6e)$.

For sake of completeness let us check what occurs for the other families which yield Shimura varieties in genus 4.

Families **(11)** and **(36)** admit (respectively) an action of $\mathbb{Z}/5$ (and Q_8). They both don't have a map of degree 3 on an elliptic curve. Hence they are not contained in $(6e)$. Notice, moreover, that the group algebra doesn't decompose their Jacobian.

Family **(13)** $=$ **(24)** is described as follows: $G = \mathbb{Z}/6 \times \mathbb{Z}/2$, the generators for the monodromy are $((3,1),(0,1),(2,0),(1,0))$, $H^0(C,\omega_C) \cong V_6 \oplus V_7 \oplus V_8 \oplus V_{11}$, all 1-dimensional, and $(\mathrm{Sym}^2 H^0(C,\omega_C))^G = V_7 \otimes V_{11}$. Its Jacobian decomposes as $JC \sim B_6 \times B_7 \times B_8$: B_6 and B_8 have dimension 1 while B_7 has dimension 2. Indeed, $E_6 := C/\langle(3,1)\rangle$ has genus 1 and $B_6 \sim JE_6$, $E_8 := C/\langle(0,1)\rangle$ has genus 1 and $B_6 \sim JE_8$ and, finally, $C' := C/\langle(3,0)\rangle$ has genus 2 and $B_7 \sim JC'$. Since G is abelian the subgroup $\langle(3,0)\rangle$ is normal and hence we can consider the family of degree 6 maps $C' \to \mathbb{P}^1$. It corresponds to the Shimura curve (5). In fact, $H^0(C,\omega_C))^{\langle(3,0)\rangle} = V_7 \oplus V_{11}$ and an easy calculus shows that the induced N is equal to 1. Since (5) is the unique Shimura family which has the same datum of $C' \to \mathbb{P}^1$ we conclude.
Notice that JC decomposes completely. It is not included in $(6e)$ since it does not admit a map 3:1 on an elliptic curve.
The quotient given by subgroup $\langle(1,1)\rangle$ gives the inclusion in (14). Indeed the map $C \to C/\langle(1,1)\rangle \cong \mathbb{P}^1$ has 5 critical values with induced monodromy $((3,1),(3,1),(2,0),(2,0),(2,0))$, i.e. the one desired by (14).

Indeed, family **(14)** has data: $G = \mathbb{Z}/6$, $\mathbf{x} = ([3]_6,[3]_6,[2]_6,[2]_6,[2]_6)$, $H^0(C,\omega_C) = V_4 \oplus V_5 \oplus 2V_6$ and Jacobian decomposed as $JC \sim B_5 \times B_4$. The first term is 1-dimensional and the second has dimension 3. Considering the subgroup $\langle[3]_6\rangle$ we get $C \to E$, where $E := C/\langle[3]_6\rangle$ is in fact an elliptic curve. Moreover, since $H^0(E,\omega_E) = V_5$, we have $E \sim B_5$. On the other hand the quotient by $\langle[4]_6\rangle$ gives a 3:1 map to $\mathbb{P}^1$ with induced monodromy of type $\mathbf{m} = (3,3,3,3,3,3)$. This determines the inclusion in family (10).

Family **(10)** is a 3-dimensional family of curves which have an action of $\mathbb{Z}/3$ with monodromy $\mathbf{m} = (3,3,3,3,3,3)$. It is different from $(6e)$ because it doesn't admit a 3:1 map on elliptic curves. The Jacobians of the curves of this family aren't decomposed by the group algebra.

$\square$

CHAPTER 3

Totally Decomposable vs Totally Geodesic

The main character of this Chapter is a comparison between two interesting types of subvarieties of $\mathcal{A}_g$: subloci of totally decomposable abelian varieties and totally geodesic subvarieties. Here an abelian variety is said completely decomposable if it is isogenous to the product of elliptic curves.

Starting from Poincaré's Reducibility Theorem, many authors have investigated on possible decompositions of abelian varieties as a product of abelian subvarieties. However most authors were mainly interested in elliptic factors of Jacobian varieties. In [58], Lange and Recillas provide a technical tool (nowadays know as the group algebra decomposition) to decompose abelian varieties using group actions of finite groups. The Chilean school of mathematics gave great contributions in this direction founding way to better characterize this decomposition. In particular, in case of Jacobian varieties: thanks to [82] we know the dimensions of the pieces, thanks to [59] we know the induced polarization, thanks to [51] we can recognize terms as Jacobians of intermediate quotients.

Moreover, since the paper of Ekedahl and Serre [27], there has been a big amount of work on completely decomposable abelian varieties, looking for genus in which there exists, or there cannot exist, a curve with totally decomposable Jacobian.

Our interest in this particular subclass of Jacobian varieties starts from the fact that many examples of Shimura subvarieties of $\mathcal{T}_g$ among those obtained as Galois coverings of $\mathbb{P}^1$ and of elliptic curves in [32] and in [36], turned out to have a completely decomposable Jacobian variety. Hence the leitmotiv of this Chapter is the following:

Question 3.1. How is it possible to compare subloci of $\mathcal{A}_g$ of totally decomposable abelian varieties and totally geodesic subvarieties?

The Chapter is organized as follows.

In Section 3.1, which is the first half of the Chapter, we recall how the group algebra decomposition works. Letting G be a finite group acting on an abelian variety A, we describe how $\mathbb{Q}[G]$, the $\mathbb{Q}$-algebra associated to G, decomposes in simple algebras $\mathcal{Q}_i$ and how this decomposition goes down to A, obtaining the so-called isotypical decomposition. Then we explain how it is possible to use minimal left ideals to decompose $\mathcal{Q}_i$

further. In this way we can improve the isotypical decomposition to obtain the group algebra decomposition.

In case of Jacobian varieties, we cite Theorems concerning the terms which appear in this decomposition trying to explain how they can be interpreted as Jacobians of intermediate quotients. Moreover we report a criterion which establishes, in case of abelian surfaces, when the group algebra decomposition is satisfactory, i.e. when the terms of the decomposition cannot be broken again.

Finally, we focus on the case of totally decomposable abelian varieties. We address this particular type of abelian varieties addressing a question formulated by Moonen and Oort in [70]. There the authors ask about a possible decomposition of Jacobians occurring in the Shimura varieties there constructed (we remind that the same are recollected and enlarged in [32]). Indeed we show that a remarkable number of the examples presented in [32] and in [36] actually carry curves with totally decomposable Jacobian.

In Section 3.2 we analyse this question and we compare the two properties: being a totally geodesic subvariety and being a locus of totally decomposable Jacobians. By means of concrete examples we show that, unfortunately, there does not exist a link between them. Indeed we describe in detail a family of genus 3 curves with completely decomposable Jacobians which cannot be totally geodesic and then another family of genus 3 curves which yields a Shimura curve in $\mathcal{A}_3$ and we show that their Jacobians decompose as the product of a fixed elliptic curve and an abelian surface which does not admit any sub elliptic curve.

3.1 Group Algebra Decomposition

In this section we describe how to use the action of a finite group G on an abelian variety A to decompose A up to isogeny. In particular we will focus on the Jacobian case $A = JX$, where X is a smooth projective curve with a G-action. For this section we refer to the seminal work of Lange-Recillas [58] and to [15], [82].

Let us take an abelian variety A with a finite group G which acts on it. The action induces an algebra homomorphism:

$$\varrho : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(A)$$
$$\sum x_g e_g \mapsto \sum x_g \theta_g,$$

where $\mathbb{Q}[G]$ denotes the group algebra of G over $\mathbb{Q}$ and θ_g is the endomorphism of A which sends $a \mapsto g \cdot a$.

In order to obtain proper abelian subvarieties of A it it necessary to choose suitable elements α of $\mathbb{Q}[G]$ and to look at the corresponding $\mathrm{Im}\,\varrho(\alpha)$. For this reason we recall that $\mathbb{Q}[G]$ is a semi-simple $\mathbb{Q}$-algebra and thus it admits a unique decomposition

$$\mathbb{Q}[G] = \mathcal{Q}_1 \times ... \times \mathcal{Q}_r$$

in simple $\mathbb{Q}$-algebras. Hence we immediately get a decomposition of the unit element as:

$$1 = e_1 + ... + e_r.$$

The elements $e_1, ..., e_r$ form a set of orthogonal idempotents of $\mathbb{Q}[G]$. Since they are uniquely determined, they well-define

$$A_1 \times ... \times A_r, \tag{3.1}$$

with $A_i = \mathrm{Im}(e_i)$. The product (3.1) is called *isotypical decomposition* of A, it is unique up to permutation of the terms and the addition map induces an isogeny with A. The terms A_i are called *isotypical components*.

There is a way to compute the idempotents in terms of the representations of the group G. Indeed, letting $V_1, ..., V_s$ be the $\mathbb{C}$-irreducible representations of G with characters $\chi_1, .., \chi_s$, it is possible to define projectors

$$p_j := \frac{\deg \chi_j}{|G|} \sum_{g \in G} \overline{\chi_j(g)} \cdot g \in \mathbb{C}[G]$$

and to show that

$$e_j = p_{j_1} + ... + p_{j_{k_j}},$$

where $V_{j_1}, ..., V_{j_{k_j}}$ are the irreducible representations of G which are Galois-conjugate to V_j. Notice that this proves that the components A_i correspond one to one to the irreducible $\mathbb{Q}$-representations $W_1, ..., W_r$ of G (for details see [58], pp. 137-139).

It is also known that the simple algebras $\mathcal{Q}_i$ can be decomposed into a product of minimal left ideals (all isomorphic) and hence a further decomposition of A is determined. Indeed, the following holds:

Proposition 3.1.1. *Let G be a finite group acting on an abelian variety A. Let $W_1, ..., W_r$ denote the irreducible $\mathbb{Q}$-representations of G and $n_i := \dfrac{d_{V_i}}{s_{V_i}}$ with d_{V_i} is the degree of a complex irreducible representation V_i associated to W_i and s_{V_i} is its Schur index. Then there are abelian subvarieties $B_1, ..., B_r$ of A and an isogeny*

$$A \sim B_1^{n_1} \times ... \times B_r^{n_r}. \tag{3.2}$$

This is called the *group algebra decomposition* of A. Further informations relating the terms can be obtained taking into account the geometric properties of the action.

Remark 18. Some of the varieties B_i's may be of dimension zero for some particular actions. For instance, in the case of A being the Jacobian of a Riemann surface X with G action, the variety B_1 corresponding to the trivial representation W_1 may be taken as the Jacobian of the Riemann surface X/G, whose genus may be equal to zero.

In [82] the author focuses on the Jacobian case $A = JX$ and she studies a technique to evaluate the dimension of each B_i using an explicit generating vector for the action of G on X. Indeed, let G be a group acting on X, $\pi : X \to X/G$ the quotient map branched on r points and $(\mathbf{m}, G, \theta)$ the corresponding datum. Following [82], we call *geometric signature* the tuple $(g', [m_1, C_l], ..., [m_r, C_r])$: g' is the genus of the quotient X/G and C_i is the conjugacy class of $\theta(\gamma_i)$, i.e. the conjugacy class of the non trivial stabilizer around the ith branch point.

Theorem 3.1.2 (Rojas, [82]). *Let G be a finite group acting on a Riemann surface X with geometric signature as above. Then the dimension of any subvariety B_i associated to a non-trivial irreducible $\mathbb{Q}$-representation W_i is given by*

$$\dim B_i = l_i(d_{V_i}(g' - 1)) + \frac{1}{2}\sum_{j=1}^{r}(d_{V_i} - \dim \mathit{Fix}_{\theta(\gamma_j)} V_i),$$

where V_i is a $\mathbb{C}$-irreducible representation associated to W_i and l_i is the degree of the extension $[L_{V_i} : \mathbb{Q}]$.

Intermediate coverings play a key role in identifying elements of the decomposition. We begin with the following:

Proposition 3.1.3 (Carocca-Rodríguez, [15]). *Given a Galois cover with corresponding group algebra decomposition*

$$A \sim B_1^{n_1} \times ... \times B_r^{n_r}.$$

If H is a subgroup of G and $X \to X/H := X_H$ is the corresponding quotient map, then the group algebra decomposition of JX_H is given as

$$JX_H \sim B_1^{n_1^H} \times ... \times B_r^{n_r^H}, \quad \textit{with } n_i^H = \frac{d_{V_i}^H}{s_{V_i}}$$

where $d_{V_i}^H$ denotes the dimension of the vector subspace V_i^H of V_i.

This Proposition provides a criterion to identify if a factor in (3.2) is isogenous to the Jacobian of a quotient of X. This is shown in

Lemma 3.1.4 (Jiménez, [51]). *If $H < G$ is such that $\dim_{\mathbb{C}} V_i^H = s_{V_i}$ and $\dim_{\mathbb{C}} V_j^H = 0$ for all j, $j \neq i$, such that $\dim_{\mathbb{C}} V_j \neq 0$, then*

$$B_i \sim JX_H.$$

It is still an open question whether terms B_i's in the group algebra decomposition are simple or not. Results in this direction are obtained, for instance, in [3], [4]. There the authors relate abelian subvarieties of an abelian varieties A with elements α in the rational Néron-Severi group $NS_{\mathbb{Q}}(A) := (\mathrm{Pic}(A)/\mathrm{Pic}^0(A)) \otimes \mathbb{Q}$. In particular they give numerical characterization for non-simplicity. Here we recall (without proof) the following:

Proposition 3.1.5 (§4.4, [4])**.** *Let (A, L) be a polarized abelian surface of type $(1, d)$ with period matrix $\Pi = \begin{pmatrix} 1 & 0 & \tau_1 & \tau_2 \\ 0 & d & \tau_3 & \tau_4 \end{pmatrix}$. Then A admits a sub elliptic curve if and only if there exists a vector $(a_1, ..., a_6) \in \mathbb{Q}^6$ satisfying:*

$$\begin{cases} -d = da_2 + a_5 \\ 0 = (\tau_1\tau_3 - \tau_2^2)a_1 - da_3\tau_1 + da_2\tau_2 - a_5\tau_2 + a_4\tau_3 + da_6 \\ 0 = a_3a_4 - a_2a_5 + a_1a_6. \end{cases} \tag{3.3}$$

This Proposition, applied to explicit examples (see [4, Section 5]) shows that there are cases where the terms B_i's in the group algebra decomposition admit a further decomposition.

3.1.1 Totally decomposable abelian varieties

Starting from the work of Ekedahl and Serre [27] where the authors find curves up to genus 1297 with totally decomposable Jacobian varieties, there has been much interest in curves with this property. Several authors address this problem developing different techniques but, since the publication of Ekedahl and Serre's list of genera admitting a curve with totally decomposable Jacobian, not many new examples have been found. An important progress is obtained in [80], where the authors used a new approach involving group algebra decomposition of intermediate coverings to get new examples in new genera.

In this section we deal with totally decomposable abelian variety from a different point of view. Indeed, our interest is inspired by the following question asked by Moonen and Oort:

Question 3.2 (§6.7, [70])**.** For which $g \geq 2$ does there exist a positive dimensional Shimura subvariety Z generically contained in $\mathcal{T}_g$ and such that the abelian variety corresponding with the geometric generic point of Z is isogenous to a product of elliptic curves?

As already explained in the previous Chapter, we know only sufficient criteria which yield special subvarieties. Therefore our answer is forced to be partial. It is resumed in the following:

Proposition 3.1.6. *For $g = 2, 3, 4$ there are Shimura varieties whose generic point has a totally decomposable Jacobian variety. They are obtained in [32] and [36] as families of Galois covering of $\mathbb{P}^1$ or of elliptic curves and they satisfy condition $(*)$ (for details see Chapter 2, Section 2.3.1). In particular, using the same notation of [32] and [36], the following holds.*

- $\star$ *In $g = 2$ we have families:*
  - $(3) = (5) = (28) = (30)$ *which decomposes as E_1^2;*

- $(4) = (29)$ *which decomposes as* E_1^2*;*
- $(26) = (1e)$ *which decomposes as* $E_1 \times E_2$*.*

$\star$ *In* $g = 3$ *we have families:*

- $(7) = (23) = (34) = (5e)$ *which decomposes as* $E_1 \times E_2^2$*;*
- (22) *which decomposes as* $E_1^2 \times E_2$*;*
- $(33) = (35)$ *which decomposes as* E_1^3*;*
- $(31) = (3e)$ *which decomposes as* $E_1 \times E_2^2$*;*
- $(32) = (4e)$ *which decomposes as* $E_1 \times E_2^2$*;*
- (27) *which decomposes as* $E_1 \times E_2 \times E_3$*.*

$\star$ *In* $g = 4$ *we have families:*

- $(13) = (24)$ *which decomposes as* $E_1 \times E_2^2 \times E_3$*;*
- $(25) = (38)$ *which decomposes as* $E_1^2 \times E_2^2$*;*
- (37) *which decomposes as* $E_1 \times E_2^3$*.*

Proof. The proof of this list is contained in the analysis of the families given in Section 4 of the previous Chapter. The totally decomposability is obtained using the group algebra decomposition and Lemma 3.1.4. Indeed, this Lemma gives a way to recognize terms in the decomposition as Jacobians of intermediate quotients. In cases of families (22), $(13) = (24)$ and $(25) = (38)$ respectively the 2-dimensional term given by the group algebra decomposition is shown to be the Jacobian of families (4), (3) and (5) respectively. Since these ones are totally decomposable we conclude.

$\square$

We remark that for the remaining Shimura varieties of [32] and [36] the group algebra decomposition does not give interesting information and thus we will not list them here. In particular, since the decomposition furnished by the group algebra is not exhaustive, we cannot answer to Question (3.2) in case of $g \geq 5$.

3.2 The Comparison

Motivated by what is collected in Proposition (3.1.6), this section is devoted to the following:

Question 3.3. Is there any relation between totally geodesic subvarieties and loci of totally decomposable abelian varieties of $\mathcal{A}_g$?

Indeed, we show in Proposition (3.1.6) that, al least in low genus, the group algebra decomposition, applied to the Jacobians of the curves which occur in Galois coverings satisfying $(*)$, gives us very frequently totally decomposable abelian varieties.

Completely decomposable Jacobian varieties play a key role in the study of Shimura subvarieties of $\mathcal{A}_g$ presented in [61]. In this paper the authors focus on specific counterexamples to the Coleman-Oort's conjecture. In particular, they treat the case of Shimura curves which parametrize g-dimensional principally polarized abelian varieties that are isogenous to a g-fold self-product of some elliptic curve. They prove the following:

Theorem 3.2.1 (Lu-Zuo)**.** *For $g > 11$, there does not exist a Shimura curve generically contained in $\mathcal{T}_g$ which parametrizes principally polarized abelian varieties of dimension g isogenous to a g-fold self-product of some elliptic curve.*

Notice that this Theorem gives an answer to Question (3.2) in the 1-dimensional case. By sake of completeness, we quickly recall that the same Theorem partially answers to what asked by Ekedahl and Serre regarding the existence of high genus curves with completely decomposable Jacobian. Indeed, it has the following:

Corollary 3.2.1.1. *For each fixed integer g greater than 11, there exist, up to isomorphism, at most finitely many smooth projective curves of genus g whose Jacobians are isogenous to g-fold self-product of a single elliptic curve with bounded isogenous degrees.*

Justified by these applications, the goal of the remaining part of this Chapter is to address Question (3.3). We will analyse the two implications separately, devoting a section to each of them. Unfortunately, the answer is negative.

3.2.1 Totally decomposable $\nRightarrow$ totally geodesic

Let us start considering a 1-dimensional family of curves with totally decomposable Jacobian varieties. Using the same notation of Section 1.3.3, we let G be a finite group acting on genus g curves with datum $(\mathbf{m}, G, \theta)$. Therefore we denote by $\mathsf{M}(\mathbf{m}, G, \theta)$ the variety in $\mathcal{M}_g$ which parametrizes Galois coverings $C \to C/G$ with datum $\Delta = (\mathbf{m}, G, \theta)$ and by Z the corresponding family of Jacobians in $\mathcal{A}_g$. Here, by means of a concrete example, we show that the total decomposability of the Jacobian varieties JC, does not guarantee Z to be totally geodesic in $\mathcal{T}_g$.

For simplicity we will focus on the case of Jacobians decomposable as

$$JC \sim E_1^k \times E_2^l, \quad k, l \in \mathbb{N} \tag{3.4}$$

with E_1 not isogenous to E_2. Obviously our argument can be used in case of $JC \sim E_1 \times E_2 \times .... \times E_n$, with $n \in \mathbb{N}$.

Remark 19. The assumption E_1 not isogenous to E_2 is necessary. Indeed in case of decomposition of type $JC \sim E^k$ we could apply Theorem (2.2.10) to get a global isogeny between Z and $\underbrace{\mathcal{A}_1 \times ... \times \mathcal{A}_1}_{k \text{ times}}$ which is already known to be totally geodesic in $\mathcal{A}_k$.

We start with two technical facts:

Lemma 3.2.2. *Let $(M, \langle,\rangle)$, $(M', <,>)$ be two Riemannian manifolds and $h : M \to M'$ be a map. Then the graph of h, denoted by Γ_h, is a totally geodesic submanifold of $M \times M'$ iff for every geodesic γ of M then $h(\gamma)$ is a geodesic of M'.*

Proof. First we consider the $\Leftarrow$) implication. Let us take $(x, y) \in \Gamma_h$ and $(u, v) \in T_{(x,y)}\Gamma_h$, i.e. $v = (dh)_x u$ with $(dh)_x : T_xM \to T_yM'$ and $u \in T_xM$. The Existence and Uniqueness Theorem for geodesic guarantees that there exist $\epsilon > 0$ and a geodesic $\gamma^u : (-\epsilon, \epsilon) \to M$ such that:

$$\gamma^u(0) = x \quad \text{and} \quad \dot{\gamma}^u(0) = u.$$

For the same reason there exists a geodesic γ^v passing through y at time 0 with tangent vector v. By uniqueness of the geodesic we have

$$\gamma^v = h \circ \gamma^u,$$

since $h \circ \gamma^u$ has the same initial data of γ^v. This shows that the geodesic (γ^u, γ^v) contained in $M \times M'$ is actually contained in Γ_h. Therefore Γ_h is a totally geodesic submanifold of $M \times M'$.

In order to prove $\Rightarrow$) we suppose Γ_h is totally geodesic in $M \times M'$. As before let us take $(x, y) \in \Gamma_h$ and $(u, v) \in T_{(x,y)}\Gamma_h$, i.e. $y = h(x)$, $v = (dh)_x u$ with $(dh)_x : T_xM \to T_yM'$ and $u \in T_xM$. Moreover let (γ^u, γ^v) be the geodesics in $M \times M'$ such that

$$\begin{matrix} \gamma^u(0) = x \\ \dot{\gamma}^u(0) = u \end{matrix} \quad \text{and} \quad \begin{matrix} \gamma^v(0) = y \\ \dot{\gamma}^v(0) = v. \end{matrix}$$

The assumptions $(u, v) \in T_{(x,y)}\Gamma_h$ and Γ_h totally geodesic in $M \times M'$ force (γ^u, γ^v) to be contained in Γ_h. Therefore we have $\gamma^v = h \circ \gamma^u$. This shows that h sends geodesics to geodesics and hence we conclude.

□

Remark 20. We apply this Lemma letting the Siegel space play the role of M, M' and considering the lifts of subvarieties of type Z at the level of $\mathfrak{S}_g$. Indeed, as explained in Chapter 2, the Siegel space is a Riemannian symmetric space (see for instance Proposition (2.2.3)). Therefore we remark that there exists a geodesic through any two fixed points of $\mathfrak{S}_1$. The same holds also in case of $\mathfrak{S}_1 \times \mathfrak{S}_1$.

Lemma 3.2.3. *Let Z be a irreducible 1-dimensional totally geodesic submanifold of $\mathfrak{S}_1 \times \mathfrak{S}_1$ such that $\pi_1(Z) = \mathfrak{S}_1$, where π_1 is the projection on the first factor. Then*

$$\pi_1^{-1}(*) \cap Z = \{p\},$$

i.e. the preimage of a generic point $*$ in $\mathfrak{S}_1$ is a singleton.

Proof. Suppose by contradiction that $\{p, q\} \subset \pi_1^{-1}(*) \cap \mathrm{Z}$. Then there would exist a geodesic γ in $\pi_1^{-1}(*) = * \times \mathfrak{S}_1 \cong \mathfrak{S}_1$ connecting the two points. The assumption of Z totally geodesic in $\mathfrak{S}_1 \times \mathfrak{S}_1$, together with the fact that $\mathfrak{S}_1$ is totally geodesic in $\mathfrak{S}_1 \times \mathfrak{S}_1$, implies

$$\gamma \subseteq \mathrm{Z}, \quad \text{hence} \quad \gamma \subseteq Z \cap \pi_1^{-1}(*).$$

Therefore $Z \cap \pi_1^{-1}(*)$ is not discrete and thus, by the irreducibility of Z, $Z \subseteq \pi_1^{-1}(*)$. This gives a contradiction since $\pi_1(\mathrm{Z}) = \mathfrak{S}_1$.

□

These two Lemmas have the following consequence:

Proposition 3.2.4. *Let Z be a irreducible 1-dimensional totally geodesic submanifold of $\mathfrak{S}_1 \times \mathfrak{S}_1$ such that $\pi_1(Z) = \mathfrak{S}_1$. Then:*

1. *Z is the graph of a function $h : \mathfrak{S}_1 \to \mathfrak{S}_1$;*
2. *For every geodesic γ of $\mathfrak{S}_1$, the image $h(\gamma)$ is still a geodesic.*

The goal now is to give a concrete example of a 1-dimensional family of curves in $\mathcal{M}_3$ such that the corresponding Z has Jacobians of type (3.4). We will show that such a Z cannot be totally geodesic in $\mathcal{T}_3$.

Using, as usual, the notation of `MAGMA`, the data are the following:

$$\begin{gathered} G = D_6 = \langle x, y :\ x^6 = y^2 = 1,\ yxy = x^{-1} \rangle, \\ \mathbf{x} = (y, x^3, x^4 y, x), \quad \mathbf{m} = (2,2,2,6), \\ H^0(C, \omega_C) \cong V_4 \oplus V_6, \end{gathered} \tag{$\bigstar$}$$

where V_i are irreducible representations of G such that $\dim(V_4) = 1$ and $\dim V_6 = 2$. An easy calculus shows

$$(\mathrm{Sym}^2 H^0(C, \omega_C))^G \cong \mathrm{Sym}^2 V_4 \oplus (\mathrm{Sym}^2 V_6)^G.$$

Since $\dim(\mathrm{Sym}^2 V_6)^G = 1$, we get $N = \dim(\mathrm{Sym}^2 H^0(C, \omega_C))^G = 2$. The family is 1-dimensional because it is ramified over four points (which we can fix as $\{z_1 = 1, z_2 = \lambda, z_3 = 0, z_4 = \infty\}$). Therefore condition $(*)$ does not hold. The group algebra decomposition for all the curves C in the family gives us

$$JC \sim B_4 \times B_6^2,$$

where both the B_i's have dimension one. As desired, this family has totally decomposable Jacobian varieties.

Let us take the subgroup $H := \langle x^3 y, x^4 \rangle$, isomorphic to S_3, and considering the quotient map $f : C \to C/H$ we get only two critical points of order 3 in $\varphi^{-1}(z_4)$. Applying Riemann-Hurwitz we get that $E := C/H$ is an elliptic curve. Moreover, since $H^0(E, \omega_E) = H^0(C, \omega_C))^H = V_4$ we get that $B_4 \sim E$ (see Lemma 3.1.4).

The map $E \to \mathbb{P}^1$ inherits a point of order 2 over all z_i's. Thus we get the equation of E as the plane cubic

$$E_\lambda : \quad y^2 = x(x-1)(x-\lambda). \tag{3.5}$$

Notice that, as expected, it moves: indeed it depends on λ, the parameter of the family.

In order to study B_6, we consider the following diagram:

$$\begin{tikzcd}
C \arrow[r] \arrow[dr, "f'"] \arrow[dd, "\psi"'] & F \arrow[d] \\
& \mathbb{P}^1 \arrow[dl, "\pi"] \\
\mathbb{P}^1 &
\end{tikzcd} \tag{3.6}$$

The map f' is the quotient map which correspond to the subgroup $H' := \langle x^3, y\rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. It has:

- two critical points on $\psi^{-1}(z_1)$ with stabilizer $\langle y\rangle$. Call q_1 their image through f' in $\mathbb{P}^1$ and x_1 that of the other points of $\psi^{-1}(z_1)$.
- Six critical points on $\psi^{-1}(z_2)$ with stabilizer $\langle x^3\rangle$. Call q_2, q_3, q_4 their image through f'.
- Two critical points on $\psi^{-1}(z_3)$ with stabilizer $\langle y\rangle$. Call q_5 their image through φ' and x_2 that of the other points of $\psi^{-1}(z_1)$.
- Two critical points on $\psi^{-1}(z_4)$ with stabilizer $\langle x^3\rangle$. Call q_6 their image in $\mathbb{P}^1$.

This means that the order 3 map $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ has a critical point of order 2 in the fibres $\pi^{-1}(z_1)$ and $\pi^{-1}(z_3)$; moreover it has a critical point of order 3 in $\pi^{-1}(z_4)$. If we fix $x_2 = 0, q_5 = 1, q_6 = \infty$ we obtain

$$\pi(z) = -\frac{27}{4}z^3 + \frac{27}{4}z^2.$$

This implies that $q_1 = -\frac{1}{3}$ and that

$$(z - q_2)(z - q_3)(z - q_4) = -\frac{27}{4}z^3 + \frac{27}{4}z^2 - \lambda.$$

Now consider the subgroup $H'' := \langle y\rangle$ and the quotient map $f'' : C \to C/H''$. Since f' has six critical values in $q_1, ..., q_6$, we get that f'' has only two critical points of order 2 in the preimage $f'^{-1}(q_1)$. This means that $F := C/H''$ is an elliptic curve. As above since $H^0(F, \omega_F) = H^0(C, \omega_C))^{H'} \subset V_6$ we get the isogeny $F \sim B_6$. Studying the map $F \to \mathbb{P}^1$ we obtain

$$F_\lambda : \quad y^2 = (z - q_2)(z - q_3)(z - q_4) = -\frac{27}{4}z^3 + \frac{27}{4}z^2 - \lambda. \tag{3.7}$$

Notice that, again, the equation depends on the parameter so it moves in family.

Proposition 3.2.5. *The curves E_λ and F_λ are not isogenous.*

Proof. It is quite easy to see that the two curves cannot be isogenous. The idea is to study how the j-invariants $j(E_\lambda), j(F_\lambda)$ degenerate moving λ to limit values. Since E is presented in Legendre form, we have that (see e. g. [47, p. 317]):

$$j_1(\lambda) := j(E_\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}. \tag{3.8}$$

On the other hand, using the linear change of variables $x = az + \frac{1}{3}$, where a satisfies $a^3 = -\frac{16}{27}$, we put the equation (3.7) in the standard form

$$y^2 = 4x^3 - g_2 x - g_3 \quad \text{with} \quad g_2 = -\frac{9}{4}a \quad \text{and} \quad g_3 = \lambda - \frac{1}{2}.$$

This implies that

$$j_2(\lambda) := j(F_\lambda) = \frac{16 \cdot 27}{\lambda(1 - \lambda)}.$$

Comparing the j-invariants at limiting conditions, we see that

$$j_1(\lambda) \xrightarrow{\lambda \to \infty} \infty \qquad \text{while} \qquad j_2(\lambda) \xrightarrow{\lambda \to \infty} 0. \tag{3.9}$$

Since they have a different behaviour for λ near ∞ we conclude.

□

Thus we conclude with the following:

Theorem 3.2.6. *The totally decomposable family (★) is not totally geodesic in $\mathcal{A}_3$.*

Proof. By above description we know that for every covering $C \to C/G$ of family (★) the Jacobian decomposes as:

$$JC_\lambda \sim E_\lambda \times F_\lambda^2.$$

Set $\mathbb{C}^{**} := \mathbb{C} \setminus \{0, 1\}$ and consider the following diagram (we refer to [54, Chapter 5]):

$$\begin{tikzcd} \mathfrak{S}_1 \arrow[d, "u"] \arrow[drr, "\pi"] & & \\ \mathbb{C}^{**} = \mathfrak{S}_1 / \Gamma_2 \arrow[rr, "j"] & & \mathfrak{S}_1 / \mathrm{SL}(2, \mathbb{Z}) = \mathbb{C}, \end{tikzcd}$$

The map u is the universal cover $\mathfrak{S}_1 \to \mathfrak{S}_1 / \Gamma_2$, where Γ_2 is the subgroup of $\mathrm{SL}(2, \mathbb{Z})$ acting on $\mathfrak{S}_1$ through automorphisms $\tau \mapsto \dfrac{a\tau + b}{c\tau + d}$, with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathit{mod}\ 2.$$

The map j is defined as in (3.8):

$$j(\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2},$$

i.e. the j-invariant for the curve $E_\lambda:\ y^2 = x(x-1)(x-\lambda)$. Finally π is the quotient map described in Section 1.1.

Let us identify the diagonal of $\mathfrak{S}_1\times\mathfrak{S}_1$ with $\mathfrak{S}_1$ and assume, by contradiction, that our family yields a totally geodesic submanifold $\mathsf{Z} \subset \mathfrak{S}_1 \times \mathfrak{S}_1$. Proposition 3.2.4 guarantees that there exists

$$h : \mathfrak{S}_1 \to \mathfrak{S}_1 \quad \text{s.t.} \quad \mathsf{Z} = \Gamma_h.$$

Let $\lambda_n \in B$ be a sequence such that $\lambda_n \xrightarrow{n\to\infty} \infty$ and take $\tau_n \in \mathfrak{S}_1$ such that $u(\tau_n) = \lambda_n$. Then we have

$$\pi \circ h(\tau_n) = j_1(u(h(\tau_n))) = j_2(\lambda_n)$$

while

$$\pi(\tau_n) = j_1(\lambda_n).$$

In fact, considering the map

$$\begin{aligned} \mathfrak{S}_1 &\to \mathcal{A}_3 \\ \tau &\mapsto [E_{u(\tau)} \times E_{u(h(\tau))} \times E_{u(h(\tau))}], \end{aligned}$$

if $u(\tau) = \lambda$ then we get

$$[F_\lambda] = [E_{u(h(\tau))}] \quad \text{as element in } \mathcal{A}_1.$$

Therefore we obtain

$$j_2(\lambda_n) = j_1(u(h(\tau_n))),$$

with E_λ (and resp. F_λ) as in (3.5) (and resp. (3.7)).

But this is impossible since (3.9) tells us that $j_2(\lambda_n)$ converges while $j_1(\lambda_n)$ diverges. Hence τ_n diverges and the same occurs for $h(\tau_n)$ and for $j_1(u(h(\tau_n)))$.

□

3.2.2 Totally geodesic $\not\Rightarrow$ totally decomposable

In this section, we show that the assumption on Z to be totally geodesic, actually also Shimura since we will deal with a family $\mathsf{M}(\mathbf{m}, G, \theta)$ satisfying condition $(*)$, doesn't guarantee JC to be totally decomposable.

We will address this problem working on a concrete example of a Shimura subvariety of $\mathcal{A}_3$ (family (9) of [32]) and showing that it doesn't carry completely decomposable Jacobians. We refer to the routine developed by Behn, Rodríguez and Rojas in [9]. It works in `SAGE` and its code is available at

http://geometry.uchile.cl.

In the following we paste the outputs of the program and we explain their meanings step by step.

```
In [1]: G=SmallGroup(6,2)
```

```
In [2]: V=find_generator_representatives(G,[2,3,3,6])
        len V

Out[2]: 1

In [3]: V[0]

Out[3]: [(1,4)(2,5)(3,6), (1,3,5)(2,4,6), (1,3,5)(2,4,6), (1,6,5,4,3,2)]
```

These commands give to `SAGE` the data of family (9): the group $(\mathbb{Z}/6)$, the order of the stabilizers ([2,3,3,6]) and the generating vector `V` for the action.

```
In [4]: X=CW(G, V[0])
        N=SerreFormula(G,X)
        N

Out[4]: 1
```

Here the program evaluates `N`$=N = \dim(\mathrm{Sym}^2 H^0(C,\omega_C))^{\mathtt{G}}$ using [32, §2.4]. Notice that family (9) satisfies condition $(*)$, as expected.

```
In [5]: P=Poly(G,V[0])
        P.symplectic_group_generators()

Out[5]:[
        [ 0 -1  0  0  1 -1]  [ 0  0  0  0  1  0]  [ 0  0  0  0  1  0]
        [-1  0  0 -1  0  1]  [-1  0 -1  1 -2  1]  [-1  0 -1  1 -2  1]
        [ 0  0 -1  1 -1  0]  [ 1  0  0  0  1 -1]  [ 1  0  0  0  1 -1]
        [ 0  0  0  0 -1  0]  [ 0 -1  0 -1  1 -1]  [ 0 -1  0 -1  1 -1]
        [ 0  0  0 -1  0  0]  [-1  0  0  0 -1  0]  [-1  0  0  0 -1  0]
        [ 0  0  0  0  0 -1], [ 0  0  1  0  1 -1], [ 0  0  1  0  1 -1],

        [ 0  0  0 -1  0  0]
        [ 0  1  1  1 -1  0]
        [ 0 -1  0 -1  1  0]
        [ 1  0  0  0  1  0]
        [ 0  1  0  1 -1  1]
        [ 0  0 -1  0 -1  1]
        ]
```

As explained in Section 1.1, the group $Sp(2g,\mathbb{Z})$ acts on the Siegel space $\mathfrak{S}_g$ with quotient space $\mathcal{A}_g$. The action of the group `G` on the curves of the family (9), hence on their Jacobians, induces a symplectic representation ρ : `G` $\to Sp(2g,\mathbb{Z})$. Here the program gives the symplectic representation for the set of generators of the action.

```
In [6]: I=P.moebius_invariant_ideal()
        I.dimension()
```

```
Out[6]: 1
```

Using the symplectic representation of `G`, here the program gives the dimension of the space of invariant Riemann matrices under the action of `G`. Notice that it agrees with N.

```
In [7]: decomposing_curves(P, furthe=False)
```

```
Out[7]: [
        [ 0  1  1 -1  0  1]  [   0  1/2    0]
        [ 0  1  0  0  1  0]  [   0  1/2  1/2]
        [ 1  0  0  1 -1 -1]  [   1    0 -1/2]
        [ 0  0  0  1 -1 -1]  [   0    0 -1/2]
        [ 0  0  0  1  1  1]  [   0    0  1/2]
        [ 0  0  1  0  0  0], [   0    0    0], [[1, 2], [2]], [1, 1]
        ]
```

Here it gives the terms of the group algebra decomposition for the Jacobians of the family and it provides the induced polarizations. As already described in the analysis of families of genus 3 given in Section 2.4.2, here we see that $JC \sim S \times E$, where S is an abelian surface of polarization of type $(1\ 2)$ and E is an elliptic curve.

```
In [8]: A=decomposed_action(P)
        A
```

```
Out[8]: [[
        [-1  0  0  0]  [ 0  2  1  0]  [ 0  2  1  0]  [ 0 -2 -1  0]
        [ 0 -1  0  0]  [-1 -1  0 -1]  [-1 -1  0 -1]  [ 1  1  0  1]
        [ 0  0 -1  0]  [ 1  0 -1  2]  [ 1  0 -1  2]  [-1  0  1 -2]
        [ 0  0  0 -1], [ 0 -1 -1  0], [ 0 -1 -1  0], [ 0  1  1  0]
        ],
        [
        [-1  0]  [ 0  2]  [ 0  2]  [ 0 -2]
        [ 0 -1], [-1 -1], [-1 -1], [ 1  1]
        ]]
```

Here the action of each element of the generating vector found above is decomposed in blocks matrices where each block acts on the corresponding term of the group algebra decomposition of JC.

The goal now is to use the induced action on the surface S and to use criteria explained in the first half of this chapter to try to decompose it further.

```
In [9]: i=moebius_invariant(A[0],polarization=[1 2])
        i
```

```
Out[9]: [
          Ideal (0, x0^2 - 1/2*x1^2 + x0 + 2*x1 - 1,
```

```
-x0*x1 + 1/2*x1*x2 + 2*x0 - x2,
-x0^2 + 1/2*x1^2 - x0 - 2*x1 + 1,
-x1^2 + 1/2*x2^2 + 4*x1 + x2 - 2,
x0*x1 - 1/2*x1*x2 - 2*x0 + x2,
x1^2 - 1/2*x2^2 - 4*x1 - x2 + 2)
of Multivariate Polynomial Ring in x0, x1, x2
over Rational Field,

[x0 x1]
[x1 x2]
]
```

With this function the program returns all the matrices with coefficient in the given polynomial ring R whose images in $R/$`i` are invariant under the action of the symplectic matrices collected in `A[0]`. This means that here the program is providing the period matrices of the abelian surfaces S of the family.

```
In [10]: A.<x0,x1,x2>=AffineSpace(QQ,3)

In [11]: V=A.subscheme([x0^2 - 1/2*x1^2 + x0 + 2*x1 - 1, -x0*x1
         + 1/2*x1*x2 + 2*x0 - x2, -x0^2 + 1/2*x1^2 - x0 - 2*x1 + 1,
         -x1^2 + 1/2*x2^2 + 4*x1 + x2 - 2, x0*x1 - 1/2*x1*x2 - 2*x0
         + x2, x1^2 - 1/2*x2^2 - 4*x1 - x2 + 2]);
         V

Out[11]: Closed subscheme of Affine Space of dimension 3
         over Rational Field defined by:
         -x0*x1 + 1/2*x1*x2 + 2*x0 - x2,
         x0^2 - 1/2*x1^2 + x0 + 2*x1 - 1,
         -x1^2 + 1/2*x2^2 + 4*x1 + x2 - 2

In [12]: V.irreducible_components()

Out[12]: Closed subscheme of Affine Space of dimension 3
         over Rational Field defined by:
         x1 - 2,
         2*x0 + x2 + 2,
         x2^2 + 2*x2 + 4,
         Closed subscheme of Affine Space of dimension 3
         over Rational Field defined by:
         2*x0 - x2,
         2*x1^2 - x2^2 - 8*x1 - 2*x2 + 4
```

The first component is 0-dimensional and hence it is not considered. The second one is of dimension 1 and thus it agrees with the fact that the surfaces S depend on the parameter of the family (as we already know from the analysis in Section 2.4.2). Indeed it can be summarized in:

```
In [13]:x0,x1,x2=var('x0,x1,x2')
        solve([2*x0 - x2==0, 2*x1^2 - x2^2 - 8*x1 - 2*x2 + 4==0], x0,x1,x2)

Out[13]:[
        [x0 == r1, x1 == sqrt(2*r1^2 + 2*r1 + 2) + 2, x2 == 2*r1],
        [x0 == r2, x1 == -sqrt(2*r2^2 + 2*r2 + 2) + 2, x2 == 2*r2]
        ]
```

This shows that our surfaces are described by the following 1-parameter period matrices:

$$\begin{pmatrix} 1 & 0 & \mathtt{t} & 2 \pm \sqrt{2\mathtt{t}^2 + 2\mathtt{t} + 2} \\ 0 & 2 & 2 \pm \sqrt{2\mathtt{t}^2 + 2\mathtt{t} + 2} & 2\mathtt{t} \end{pmatrix}.$$

We have the following:

Theorem 3.2.7. *The Shimura family* (9) *is not completely decomposable.*

Proof. It only remains to show that the surfaces $S_{\mathtt{t}}$ which occur in the group algebra decomposition of the Jacobians $JC_{\mathtt{t}}$ of the curves of the family are not further decomposable. Notice that we put the subscript $\mathtt{t}$ to remind that the term S of the decomposition varies in families.

We can apply Proposition 3.1.5 and look for a vector $(a_1, ..., a_6) \in \mathbb{Q}^6$ which satisfies:

$$\begin{cases} 2a_2 + a_5 + 2 = 0 \\ (-2\mathtt{t} \mp 4\varsigma - 6)a_1 - 2\mathtt{t}a_3 + 2(2 \pm \varsigma)a_2 - (2 \pm \varsigma)a_5 + 2\mathtt{t}a_4 + 2a_6 = 0 \\ a_3a_4 - a_2a_5 + a_1a_6 = 0, \end{cases}$$

where $\varsigma(\mathtt{t}) = \sqrt{2\mathtt{t}^2 + 2\mathtt{t} + 2}$.

The second equation can be reformulated as:

$$\varsigma(-4a_1 + 2a_2 - a_5) + \mathtt{t}(-2a_1 - 2a_3 + 2a_4) + (-6a_1 + 4a_2 - 2a_5 + 2a_6) = 0.$$

Notice that if $\tau_2 = 2 - \varsigma$ the situation is the same.

To make the system satisfied for every value of $\mathtt{t}$ we need to impose the following conditions

$$\begin{cases} 2a_2 + a_5 + 2 = 0 \\ -4a_1 + 2a_2 - a_5 = 0 \\ a_1 + a_3 - a_4 = 0 \\ 3a_1 - 2a_2 + a_5 - a_6 = 0 \\ a_3a_4 - a_2a_5 + a_1a_6 = 0, \end{cases}$$

Therefore we ask to SAGE:

```
In[14]: a1,a2,a3,a4,a5,a6=var('a1,a2,a3,a4,a5,a6')
        solve([a5+2*a2+2==0,-4*a1+2*a2-a5==0,a1+a3-a4==0,
        3*a1-2*a2+a5-a6==0,a3*a4-a2*a5+a1*a6==0],a1,a2,a3,a4,a5,a6)
```

```
Out[14]:[
        [a1 == r1, a2 == r1 - 1/2, a3 == -1/2*r1 + 1/2*sqrt(-3*r1^2 + 2),
        a4 == 1/2*r1 + 1/2*sqrt(-3*r1^2 + 2), a5 == -2*r1 - 1, a6 == -r1],
        [a1 == r2, a2 == r2 - 1/2, a3 == -1/2*r2 - 1/2*sqrt(-3*r2^2 + 2),
         a4 == 1/2*r2 - 1/2*sqrt(-3*r2^2 + 2), a5 == -2*r2 - 1, a6 == -r2],
        [a1 == 0, a2 == (-1/2), a3 == 1/2*sqrt(2), a4 == 1/2*sqrt(2),
        a5 == -1, a6 == 0],
        [a1 == 0, a2 == (-1/2), a3 == -1/2*sqrt(2), a4 == -1/2*sqrt(2),
        a5 == -1, a6 == 0]
        ]
```

Since Proposition 3.1.5 requires a vector $(a_1, ..., a_6) \in \mathbb{Q}^6$ we need $\texttt{r1} \in \mathbb{Q}$ (resp. $\texttt{r1} \in \mathbb{Q}$) such that

$$\sqrt{-3\mathrm{r}1^2 + 2} \in \mathbb{Q}, \quad \text{resp.} \quad \sqrt{-3\mathrm{r}2^2 + 2} \in \mathbb{Q},$$

i.e. a rational point (x, y) which satisfies:

$$3x^2 + y^2 = 2.$$

This is the negative answer:

```
In [15]: C=Conic([3,1,-2])
```

```
Out[15]: Projective Conic Curve over Rational Field defined by
         3*x^2 + y^2 - 2*z^2
```

```
In [16]: C.has_rational_point()
```

```
Out[16]: False
```

Hence we conclude.

$\square$

II

Prym Maps and Fibres

CHAPTER 4

Basics II

4.1 Abelian Varieties and Jacobians

As we saw in Section 1.1, to any smooth algebraic curve X of genus g we can associate a principally polarized abelian variety, its Jacobian variety. Here we generalize this construction showing how this works in case of X singular and how an analogous association is possible in case of cubic 3-folds, i.e. hypersurfaces of degree 3 in complex projective 4-space. Finally, we recall some results on dual abelian varieties and dual polarizations.

4.1.1 Generalized Jacobians

Let us take a connected curve X with only ordinary double points as singularities. Let $\nu : X^\nu \to X$ denote the normalization map and let $X^\nu = \cup X_i$ be the irreducible components decomposition. We put

$$\delta = \#\{X_i\} \quad \text{and} \quad \gamma = \#\text{Sing}(X).$$

We have the following exact sequence

$$0 \to \mathcal{O}_X \to \nu_* \mathcal{O}_{X^\nu} \to \bigoplus_{p \in \text{Sing}(X)} \mathbb{C}_p \to 0,$$

which yields the following exact sequence in cohomology:

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X^\nu, \mathcal{O}_{X^\nu}) \to C^\gamma \to H^1(X, \mathcal{O}_X) \to H^1(X^\nu, \mathcal{O}_{X^\nu}) \to 0.$$

It then follows the following formula for the (arithmetic) genus of X:

$$p_a(X) = \sum_{i=1}^{\delta} g_i + \gamma - \delta + 1,$$

where $g_i = h^1(X_i, \mathcal{O}_{X_i})$ is the (geometric) genus of the i-th component of the normalized curve X^ν.

Let us now consider the sheaf of regular, non-vanishing function $\mathcal{O}_X^*$ of X. A cohomology sequence as above gives us:

$$0 \to (\mathbb{C}^*)^{\gamma-\delta+1} \to \operatorname{Pic}(X) \xrightarrow{\nu^*} \operatorname{Pic}(X^\nu) \to 0, \tag{4.1}$$

where we identified $\operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ (resp. $\operatorname{Pic}(X^\nu) = H^1(X^\nu, \mathcal{O}_{X^\nu}^*)$) and ν^* is the pull-back of line bundles.

Remark 21. Serre duality theorem (see for instance Hartshorne [47]) says that there exist on X a dualizing sheaf ω_X. Letting $n_i, i = 1, ..., \gamma$ be the nodes of X and p_i, q_i the two branches of n_i in X^ν, then the following formula holds

$$\nu^*\omega_X = \omega_{X^\nu}(\sum(p_i + q_i)).$$

Definition 4.1. The *generalized Jacobian* of a nodal curve X is the semi-abelian variety (i.e. a commutative group variety which is an extension of an abelian variety by a torus) JX still identified with the moduli space, $\operatorname{Pic}^0(X)$, of line bundles of degree 0 in each component. Thus it is defined by the exact sequence (4.1) taking the degree 0 part, that is:

$$0 \to (\mathbb{C}^*)^{\gamma-\delta+1} \to JX \xrightarrow{\nu^*} JX^\nu = \prod_{i=1}^{\delta} JX_i \to 0. \tag{4.2}$$

4.1.2 Intermediate Jacobians and Fano variety

In [19], Clemens and Griffiths consider an auxiliary variety of a smooth cubic threefold V which is known as the intermediate Jacobian JV. It is a principally polarised abelian variety which plays a role similar to that of the Jacobian to study divisors on curves.

We begin by recalling that in case of a smooth curve X we have:

$$JX \cong H^{1,0}(X)/H^1(X, \mathbb{Z}) \cong \operatorname{Pic}^0(X).$$

Notice that the complex torus which appears in the equation above can be always obtained also in case of X smooth n-dimensional Kähler manifold, using properties coming from Hodge theory. Indeed, the decomposition

$$H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) \quad \text{and} \quad H^{1,0}(X) = \overline{H^{0,1}(X)}$$

and the projection

$$H^1(X, \mathbb{R}) \subset H^1(X, \mathbb{C}) \to H^{0,1}(X)$$

makes $H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{R})$ a lattice in the complex vector space $H^{0,1}(X)$. Thus, by construction, $\operatorname{Pic}^0(X)$ is a complex torus.

Similarly, in case of odd cohomology of higher degree, we have the decomposition

$$H^{2k-1}(X, \mathbb{C}) = F^k H^{2k-1}(X) \oplus \overline{F^k H^{2k-1}(X)}$$

and the projection

$$H^{2k-1}(X,\mathbb{R}) \to H^{2k-1}(X,\mathbb{C})/F^kH^{2k-1}(X).$$

Notice that we are just using Hodge filtration in higher degree as introduced in Section 2.3. Moreover we have

$$\mathrm{rk}\, H^{2k-1}(X,\mathbb{Z}) = \dim\, H^{2k-1}(X,\mathbb{R}).$$

The image

$$L_k := \mathrm{Im}(H^{2k-1}(X,\mathbb{Z}) \to H^{2k-1}(X,\mathbb{C})/F^kH^{2k-1}(X))$$

is a lattice of maximal rank in the complex vector space

$$V_k := H^{2k-1}(X,\mathbb{C})/F^kH^{2k-1}(X).$$

It is natural to give the following

Definition 4.2. The *k-th intermediate Jacobian* is the quotient

$$J^kX := {V_k}\big/{L_k}.$$

Therefore the intermediate Jacobian of a smooth 3-fold V is

$$JV := {H^{1,2}(V) + H^{0,3}(V)}\Big/{H^3(V,\mathbb{Z})}.$$

In case of V cubic 3-fold we have $h^{0,3}(V) = 0$. Moreover Clemens and Griffiths ([19]) prove the existence of a non-degenerate Hermitian form for complex vector space $H^{1,2}(V)$. Thus JV is a 5-dimensional principally polarized abelian variety. We remark that a Torelli-type Theorem holds also in case of principally polarized intermediate Jacobians of cubic 3-folds.

Finally we give the following

Definition 4.3. The *Fano variety of lines* $F(V)$ is the variety which parametrises the lines contained in a smooth n-dimensional cubic hypersurface $V \subset \mathbb{P}^{n+1}$.

It is possible to show that it is a smooth manifold of dimension $2(n-2)$.

Conic Bundles

Here we give a brief recall of Mumford's theory on conic bundles, referring to the Appendix to [19]. His theory gives also a proof that if V is a non-singular cubic 3-fold then JV is not the Jacobian of a curve.

Let V be a non-singular cubic hypersurface in $\mathbb{P}^4$. Let $l \subset V$ be a line in V and let

$$\pi_l : \mathbb{P}^4 \to \mathbb{P}^2$$

be a projection centred along l. Hence, letting $\tilde{V} := \mathrm{Bl}_l V$ be the blow-up of V along l, it is induced a morphism (still denote with the same letter)

$$\pi_l : \tilde{V} \to \mathbb{P}^2.$$

By construction, the fibres of π_l over a generic point $p \in \mathbb{P}^2$ are given by conic curves on V which are coplanar with l.

Definition 4.4. What described above is a *conic bundle*.

The discriminant locus of π_l is a quintic plane curve $Q \subset \mathbb{P}^2$, this means that the fibres are non-singular conics except along Q where the fibre degenerates to the sum of two lines. For $p \in Q$

$$\pi_l^{-1}(p) = r + s,$$

where r, s are lines coplanar with l.

Theorem 4.1.1 (Mumford). *There exists an isomorphism*

$$JV \cong P(\tilde{Q}, Q),^{1}$$

where $\tilde{Q}$ is the double cover of Q given by the two components in $\pi_l^{-1}(p)$, $p \in Q$. In other words, $\tilde{Q}$ parametrizes the lines $l' \subset V$ meeting l in one point. Furthermore $\tilde{Q}$ is smooth and $\tilde{Q} \to Q$ unramified for generic l and admissible in any case.

We conclude this Section with the following

Remark 22. Let Π be a plane of $\mathbb{P}^4$ meeting V in 3 lines l, l', l''. The construction above gives us 3 plane quintics Q, Q', Q'' with respectively double covers $\tilde{Q}, \tilde{Q}', \tilde{Q}''$. The lines l', l'' are sent by π_l to the same point $p \in Q$ and they determine a tetragonal map

$$f : Q \to \mathbb{P}^1,$$

given by $\mathcal{O}_Q(-p)$. Similarly for Q', Q''. It is possible to show that

$$(\tilde{Q}, Q, f), \quad (\tilde{Q}', Q', f'), \quad (\tilde{Q}'', Q'', f''),$$

are tetragonally related (for details on tetragonal construction see Section 4.4.3).

4.1.3 Dual abelian varieties

Here we recall some properties of abelian varieties and their dual varieties referring to [10], [11] and [72].

[1] Prym varieties associated to étale double coverings are defined in Section 4.2.

Let (A, L) be a polarized abelian variety of type $(d_1, ..., d_g)$ and consider the polarization map

$$\begin{aligned} \lambda_L : A &\to A^* \\ a &\mapsto \tau_a^* L \otimes L^{-1}, \end{aligned}$$

where τ_a is the translation by a in A and $A^* = \mathrm{Pic}^0(A)$ is the dual abelian variety. It is remarkable to notice that λ_L is an isogeny. Indeed we have the following results:

Theorem 4.1.2. *Let L be an ample line bundle and $M \in \mathrm{Pic}^0(A)$. Then for some $a \in A$*

$$M \simeq \tau_a^* L \otimes L^{-1}$$

i.e. the map λ_L is surjective.

Theorem 4.1.3. *If L is a polarization of type $(d_1, ..., d_g)$ with $d_i | d_{i+1}$ for all $i = 1, ..., g$, then*

$$K(L) = Ker(\lambda_L) \simeq (\mathbb{Z}/d_1\mathbb{Z} \times ... \times \mathbb{Z}/d_g\mathbb{Z})^2.$$

Therefore $\deg(\lambda_L) = (d_1 \cdot ... \cdot d_g)^2$.

Here we list, without proof, some known facts concerning maps among Picard groups. We refer to [72].

Proposition 4.1.4. *The map λ_L is a homomorphism of groups for every line bundle L.*

We will call *morphism of abelian varieties* a morphism of the underlying algebraic varieties which is also a morphism of groups.

Proposition 4.1.5. *If $f : A \to B$ is a morphism of abelian varieties, the induced map among Picard groups $\mathrm{Pic}(B) \to \mathrm{Pic}(A)$ maps $\mathrm{Pic}^0(B) \to \mathrm{Pic}^0(A)$. Therefore we get a morphism $f^* : B^* \to A^*$.*

Proposition 4.1.6. *If $f : A \to B$ is an isogeny of complex tori then the dual map $f^* : B^* \to A^*$ is also an isogeny and its kernel is isomorphic to $Hom(ker(f), \mathbb{C})$. In particular $deg(f) = deg(f^*)$.*

To conclude this section we recall the following result concerning the duality construction of Birkenhake and Lange: denote by $\mathcal{A}_g^{(d_1,...d_g)}$ the coarse moduli space parametrizing isomorphism classes of g-dimensional polarized abelian varieties of type $(d_1, ...d_g)$. Then the following holds:

Theorem 4.1.7 ([10], Theorem 3.1)**.** *There is a canonical isomorphism of coarse moduli spaces*

$$\begin{aligned} \mathcal{A}_g^{(d_1,...d_g)} &\to \mathcal{A}_g^{(\frac{d_1 d_g}{d_g}, \frac{d_1 d_g}{d_{g-1}}, ..., \frac{d_1 d_g}{d_1})} \\ (A, L) &\mapsto (A^*, L^*) \end{aligned} \tag{4.3}$$

sending a polarized abelian variety to its polarized dual abelian variety.

Here L^* is the polarization on the dual abelian variety A^* which satisfies

$$\lambda_{L^*} \circ \lambda_L = (d_g)_A \quad \text{and} \quad \lambda_L \circ \lambda_{L^*} = (d_g)_{A^*},$$

where $(d_g)_A : A \to A$, $(d_g)_{A^*} : A^* \to A^*$ are the multiplications by d_g. Notice that the dual polarization L^* satisfies $(L^*)^* = L$.

4.2 Prym varieties and Prym maps

The Torelli's map gives an injective way to associate to any smooth curve of genus g a principally polarized abelian variety (its Jacobian). Mumford in [71] shows that an analogous association can be done sending any étale double covering of a smooth projective curve of genus g to an element of $\mathcal{A}_{g-1}$, called the Prym variety of the covering. One advantage of the construction is that it allows one to apply the theory of curves to the study of a wider class of abelian varieties than Jacobians. Indeed the Jacobian locus in $\mathcal{A}_{g-1}$ is contained inside the boundary of the Prym locus $\bar{\mathcal{P}}_g$, i.e. the closure in $\mathcal{A}_{g-1}$ of the locus of Prym varieties associated to étale double coverings. Since such varieties play the role of the main character of the second part of this Thesis, here we recall some constructions and definitions which concern them referring to [1], [26], [29] and [71].

4.2.1 The étale case

Let C be a smooth curve of genus g and let $\eta \in \mathrm{Pic}^0(C) \smallsetminus \mathcal{O}_C$ be a line bundle together with a isomorphism $\eta^2 \cong \mathcal{O}_C$ (a 2-torsion point in JC). Then one can construct an unramified double cover

$$\pi : \tilde{C} \to C$$

where

$$\tilde{C} := \text{Spec}(\mathcal{O}_C \otimes \eta) \quad \text{with} \quad g(\tilde{C}) = 2g - 1.$$

Conversely, every étale double covers of C arises in this way. This means that these coverings are parametrized by the moduli space:

$$\mathcal{R}_g := \{(C, \eta) : \ C \in \mathcal{M}_g, \ \eta \in \mathrm{Pic}^0(C) \smallsetminus \mathcal{O}_C, \ \eta^{\otimes 2} = \mathcal{O}_C\} /_{\cong}.$$

Remark 23. By construction we have

$$\pi_*(\mathcal{O}_{\tilde{C}}) = \mathcal{O}_C \oplus \eta^{-1} = \mathcal{O}_C \oplus \eta,$$

since $\eta^{\otimes 2} = \mathcal{O}_C$. Therefore, by projection formula, for every line bundle $L \in \mathrm{Pic}(C)$ we have

$$H^0(\tilde{C}, \pi^* L) = H^0(C, \pi_* \pi^* L) = H^0(C, L) \oplus H^0(C, L \otimes \eta).$$

The covering map induces a *norm map*

$$\begin{aligned} \mathrm{Nm} : \mathrm{Pic}^r(\tilde{C}) &\to \mathrm{Pic}^r(C) \\ \mathcal{O}_{\tilde{C}}(\sum r_i p_i) &\mapsto \mathcal{O}_C(\sum r_i \pi(p_i)). \end{aligned}$$

among degree r-line bundles. In particular, in case of degree 0, it is defined as a map between the Jacobians:

$$\mathrm{Nm} : J\tilde{C} \to JC.$$

Definition 4.5. The *Prym variety* of the covering $\pi : \tilde{C} \to C$ is

$$P(\pi) = P(C, \eta) := \mathrm{Nm}^{-1}(0)^0,$$

i.e. it is the connected component of the kernel of the Norm map containing the origin.

The Prym variety associated to π is a $(g-1)$-dimensional abelian variety which carries a principal polarization. This is shown by Mumford in the following

Theorem 4.2.1 ([71]). *Let $\omega_C \in \mathrm{Pic}^{2g-2}(C)$ be the canonical line bundle of C and let $\mathrm{Nm} : \mathrm{Pic}^{2g-2}(\tilde{C}) \to \mathrm{Pic}^{2g-2}(C)$ be the Norm map at the level of $(2g-2)$-degree line bundles. Then:*

i) $\mathrm{Nm}^{-1}(\omega_C) = \mathrm{Nm}^{-1}(\omega_C)^{even} \sqcup \mathrm{Nm}^{-1}(\omega_C)^{odd}$*, i.e. it is the disjoint union of two two translates of the same abelian variety, $P(\pi)$, distinguished as*

$$\mathrm{Nm}^{-1}(\omega_C)^{even} = \{L \in \mathrm{Nm}^{-1}(\omega_C) : h^0(L) \equiv 0 \; mod\, 2\}$$

and

$$\mathrm{Nm}^{-1}(\omega_C)^{odd} = \{L \in \mathrm{Nm}^{-1}(\omega_C) : h^0(L) \equiv 1 \; mod\, 2\}.$$

ii) Denoting with $\Theta_{\tilde{C}}$ the Theta-divisor of $J\tilde{C}$, we have

$$\Theta_{\tilde{C}} \cdot \mathcal{P}(\pi) = 2\Xi_C,$$

where Ξ_C is a principal polarization such that

$$\Xi_C := \{L \in \mathrm{Nm}^{-1}(\omega_C) : h^0(L) \geq 2\}.$$

The moduli space $\mathcal{R}_g$ is thus involved in the following diagram:

$$\begin{tikzcd} & \mathcal{R}_g \arrow[dl, "\varphi"'] \arrow[dr, "\mathcal{P}_g"] & \\ \mathcal{M}_g & & \mathcal{A}_{g-1} \end{tikzcd} \tag{4.4}$$

Here we outline an interesting correspondence between the moduli space of curves and the moduli space of principally polarized abelian varieties. On one hand we have the forgetful map φ which sends pairs (C, η) to C. Notice that this map is finite, indeed it has fibres of cardinality $2^{2g} - 1$. On the other hand, we have the unramified (or “classical”) Prym map $\mathcal{P}_g$ which assigns the Prym variety $P(\pi)$ to any double cover $\pi : \tilde{C} \to C$ of $\mathcal{R}_g$. Since Mumford’s work, a lot of information has been obtained about it. This theory is strongly related to the study of the Jacobian locus, Schottky equations and rationality problems among other topics. An easy computation shows that

$$\dim \mathcal{R}_g = 3g-3 \quad \geq \quad \dim \mathcal{A}_g = \frac{g(g-1)}{2} \quad \text{if } g \leq 6. \tag{4.5}$$

It thus makes sense to ask if for low values of g the Prym map is dominant, i.e. if we can realize a (general) principally polarized abelian varieties of dimension less or equal that 6 as the Prym variety of a étale double covering.

Beauville in [7] introduces a partial compactification $\overline{\mathcal{R}}_g$ of $\mathcal{R}_g$ parametrizing admissible double coverings of stable curves of genus g (see Section 4.3 for details). It turns out that the moduli space $\mathcal{R}_g$ is a dense open subset of $\overline{\mathcal{R}}_g$. Moreover he extends $\mathcal{P}_g$ to a proper map

$$\overline{\mathcal{P}}_g : \overline{\mathcal{R}}_g \to \mathcal{A}_{g-1}.$$

The following theorems give an answer to the above question and to others strongly connected to it.

Theorem 4.2.2 (Wirtinger, [90])**.** *The Prym map is dominant if $g \leq 6$.*

Theorem 4.2.3 (Friedman-Smith, [38], Kanev, [52])**.** *The Prym map is generically injective for $g \geq 7$.*

Theorem 4.2.4 (Donagi, [25])**.** *The Prym map is never injective.*

Indeed, in [25], Donagi associates two admissible double covers to an unramified double cover of a smooth tetragonal curve showing that the three coverings have the same Prym variety. This construction, known as *tetragonal construction* (see Section 4.4.3), proves Theorem 4.2.4.

For exhaustive proofs of the three above results we refer the reader to the mentioned papers.

Inequality (4.5), together with Theorem 4.2.2, justifies an analysis of the geometric properties of the generic fibre. Indeed a detailed study of the structure of the fibre is provided by the work of Verra ([87]), Recillas ([83]), Donagi ([25]) and Donagi and Smith ([26]). These are the results:

Theorem 4.2.5 (Verra)**.** *Let S be a generic principally polarized abelian surface. Then $\mathcal{P}_3^{-1}$ is biregular to the Siegel modular quartic threefold*[2].

Theorem 4.2.6 (Recillas)**.** *The fibre of $\mathcal{P}_4$ at a general abelian 3-fold A is birational to the Kummer variety $A/(\pm 1)$.*

Theorem 4.2.7 (Donagi)**.** *The fibre of $\mathcal{P}_5$ over general $A \in \mathcal{A}_4$ is isomorphic to a double cover of the Fano surface of the lines of a certain cubic threefold V.*

Theorem 4.2.8 (Donagi-Smith)**.** *The map $\mathcal{P}_6$ is generically finite and it has degree 27.*

All these results have been summarized under a uniform presentation by Donagi in [25].

[2]The Siegel modular quartic threefold $V \subset \mathbb{P}^4$ is considered by Van de Geer in [86].

The codifferential of the Prym map

In his very famous paper ([7]), Beauville studies the differential of the Prym map computing its codifferential in case $\tilde{C}, C$ smooth.

Keeping in mind that moduli spaces of diagram (4.4) have the structure of complex orbifold we can work as they were smooth computing their orbifold tangent spaces. Therefore, using the fact that $\mathcal{R}_g$ is an unramified cover of $\mathcal{M}_g$ as seen in diagram (4.4), we get:

$$T_{(C,\eta)}\mathcal{R}_g = T_C\mathcal{M}_g = H^1(C, T_C) = H^0(C, \omega_C^{\otimes 2})^*.$$

Moreover

$$T_A\mathcal{A}_{g-1} = \mathrm{Sym}^2 H^0(A, T_A),$$

where A is a $(g-1)$-dimensional abelian variety. In case $A = P(\pi)$ there is a further identification

$$T_A\mathcal{A}_{g-1} = \mathrm{Sym}^2 H^0(A, T_A) = \mathrm{Sym}^2 H^0(C, \omega_C \otimes \eta)^*$$

which follows immediately from the definition of $P(\pi)$ as subvariety of $J\tilde{C}$ and the splitting

$$H^0(\omega_{\tilde{C}}) = H^0(\omega_C) \oplus H^0(\omega_C \otimes \eta).$$

The following holds:

Proposition 4.2.9 ([7])**.** *The codifferential of the Prym map*

$$d\mathcal{P}_g^* : T^*_{P(\pi)}\mathcal{A}_{g-1} \to T^*_{(C,\eta)}\mathcal{R}_g$$

can be identified with the multiplication map

$$m : \mathrm{Sym}^2 H^0(\omega_C \otimes \eta) \to H^0(\omega_C^{\otimes 2}), \tag{4.6}$$

using the isomorphism $\omega_C^{\otimes 2} \otimes \eta^2 \cong \omega_C^{\otimes 2} \otimes \mathcal{O}_C \cong \omega_C^{\otimes 2}$.

In particular, (4.6) shows that $\ker(d\mathcal{P}_g^*)$ is given by quadrics containing the Prym-canonical curve $\psi(C)$, defined as the image of C through the Prym-canonical map

$$\psi : C \to \mathbb{P}^{g-2},$$

associated with the complete linear system $|\omega_C \otimes \eta|$.

Remark 24. In case of admissible covers $C = X/p \sim q$ of types as described in Examples 1, 2, Donagi and Smith (see [26, §3]) show that the kernel of the codifferential of the Prym map can be naturally identified with the space of quadrics containing the canonical curve $\psi(X)$ and the chord connecting $\psi(p)$ to $\psi(q)$.

4.2.2 The ramified case

In this section we look at the analogue of the Prym variety in the case of ramified double covers. We refer to [55], [63], [64], [75] and [76].

Let C be an irreducible smooth complex projective curve of genus g and let $\pi : D \to C$ be a smooth double cover ramified in $r > 0$ points.

Definition 4.6. The *Prym variety* $P(\pi)$ associated to the cover π is the kernel of the Norm homomorphism

$$P(\pi) := \ker \mathrm{Nm} \subset JD.$$

Notice that here, since $r \neq 0$, we are using that the kernel of Nm is connected (see for example [71, Section 3,Lemma]). The Prym variety is an abelian variety of dimension $g - 1 + \frac{r}{2}$ and polarization Ξ, induced by the principal polarization of JD, of type

$$\delta := (1, ..., 1, \underbrace{2, ..., 2}_{g \text{ times}}).$$

Giving a covering $\pi : D \to C$ is equivalent to give a triple (C, B, η), with B a reduced divisor in C of even degree $r > 0$ and η a line bundle over C satisfying $\eta^{\otimes 2} \cong \mathcal{O}_C(B)$. Indeed the projection

$$\pi : D = \mathrm{Spec}(\mathcal{O}_C \oplus \eta^{-1}) \to \mathrm{Spec}(\mathcal{O}_C) = C$$

defines a double cover branched over B. Therefore these coverings are parametrized by the moduli space

$$\mathcal{R}_{g,r} := \{(C, \eta, B) : \ C \in \mathcal{M}_g, \ \eta \in \mathrm{Pic}^{\frac{r}{2}}(C), \ B \, \text{reduced divisor in} \, |\eta^{\otimes 2}|\} /_{\cong}.$$

Denoting with $\mathcal{A}^{\delta}_{g-1+\frac{r}{2}}$ the moduli space of abelian varieties of dimension $g - 1 + \frac{r}{2}$ with polarization of type δ, for any $g > 1$ and $r > 0$ the *ramified Prym map* is the morphism:

$$\begin{aligned} \mathcal{P}_{g,r} : \mathcal{R}_{g,r} &\to \mathcal{A}^{\delta}_{g-1+\frac{r}{2}} \\ [\pi : D \to C] &\mapsto [P(\pi), \Xi], \end{aligned}$$

Proposition 4.2.10 ([55], §4.1)**.** *The codifferential of $\mathcal{P}_{g,r}$ at the generic point $[(C, B, \eta)]$ is given by the multiplication map:*

$$d\mathcal{P}^*_{g,r} : \mathrm{Sym}^2 H^0(\omega_C \otimes \eta) \to H^0(\omega_C^{\otimes 2} \otimes \mathcal{O}_C(B)).$$

Although some specific cases were considered previously in [5] and [74], a systematic study of the properties of the ramified Prym map and of its codifferential in full generality starts with the work of Marcucci and Pirola [64] where firstly appeared evident that the generic Torelli-type problems are plenty of rich geometry. Generic Torelli theorems state that the Prym map is generically injective as soon as

$$\dim \mathcal{R}_{g,r} \leq \dim \mathcal{A}^{\delta}_{g-1+\frac{r}{2}}$$

that is

$$3g - 3 + r \ \leq \ \frac{1}{2}\Big(g - 1 + \frac{r}{2}\Big)\Big(g + \frac{r}{2}\Big).$$

The generic finiteness of $\mathcal{P}_{g,r}$ is shown by Lange and Ortega with the following:

Proposition 4.2.11 ([56]). *If*

$$g \geq 2 \text{ and } r \geq 12, \qquad g \geq 3 \text{ and } r = 8,$$
$$g \geq 5 \text{ and } r = 4,$$

the codifferential of the Prym map is surjective at the generic point (C, B, η) and hence $\mathcal{P}_{g,r}$ is generically finite.

On the other hand, the generic injectivity can be stated combining the result of [64] with the main theorems of [63] and of [75] in the following

Theorem 4.2.12. *A generic Torelli theorem holds for all the cases where the dimension of $\mathcal{R}_{g,r}$ is smaller or equal to the dimension of the target $\mathcal{A}^{\delta}_{g-1+\frac{r}{2}}$, except for the case $g = 3, r = 4$, where the Prym map has degree 3.*

Moreover, very recently, the following result has been obtained by Ikeda

Theorem 4.2.13 ([49]). *If $g = 1$ and $r \geq 6$ then the Prym map is injective.*

This theorem goes, for the first time, in the opposite direction of what expected. Indeed it shows that a modified version of Donagi's result in the étale case (see theorem 4.2.4) should not apply to ramified Prym maps. Inspired by the surprising work of Ikeda, Naranjo and Ortega prove the following

Theorem 4.2.14 ([76]). *The Prym map $\mathcal{P}_{g,r}$ is an embedding for all $r \geq 6$ and all $g > 0$.*

In next chapter we address to the opposite side of the study of the ramified Prym map: the structure of the generic fibre when

$$\dim \mathcal{R}_{g,r} > \dim \mathcal{A}^{\delta}_{g-1+\frac{r}{2}}. \tag{4.7}$$

This is only possible in six cases:

1. $g = 1, r = 2$;
2. $g = 1, r = 4$;
3. $g = 2, r = 2$;
4. $g = 2, r = 4$;
3. $g = 3, r = 2$;

4. $g = 4$, $r = 2$.

We remark that the case $g = 1, r = 4$ is considered by Barth in his study of abelian surfaces with polarization of type $(1, 2)$ (see [6]).

Our first technical result is the following:

Proposition 4.2.15. *Assume that*

$$(g, r) \in \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (4, 2)\},$$

then the ramified Prym map $\mathcal{P}_{g,r}$ is dominant.

Proof. It is enough to show that for a generic (C, η, B) there are no quadrics containing the image of

$$\varphi_{\omega_C \otimes \eta} : C \to \mathbb{P}H^0(C, \omega_C \otimes \eta)^*.$$

A detailed analysis of the six cases leads to the following situations.

- There is nothing to prove in the cases $(g, r) \in \{(1, 2), (2, 2), (1, 4)\}$. Indeed $PH^0(C, \omega_C \otimes \eta)^*$ is restricted to a point for $(g, r) = (1, 2)$, while $PH^0(C, \omega_C \otimes \eta)^* = \mathbb{P}^1$ for $(g, r) \in \{(2, 2), (1, 4)\}$.
- For $(g, r) = (3, 2)$ and $(g, r) = (2, 4)$ the curve $\varphi_{\omega_C \otimes \eta}(C)$ is a plane curve (with nodes) of degree 5 and 4 respectively. Indeed for $(g, r) \in \{(3, 2), (2, 4)\}$ we have $PH^0(C, \omega_C \otimes \eta)^* = \mathbb{P}^2$ and $\deg(\omega_C \otimes \eta) = 5, 4$ respectively.
- For the case $(g, r) = (4, 2)$ we identify elements $(C, \eta, p + q) \in \mathcal{R}_{4,2}$ with coverings $(C^* = C/p \sim q, \eta^*)$ of type $(*)$ in $\overline{\mathcal{R}}_5$ (using the same procedure explained in Example 1) and we refer to [26, Proposition 3.4.1]. There the authors study the codifferential of the extended $\overline{\mathcal{P}}_5$. They prove that the kernel at a point $(D^*, C^*) \in \overline{\mathcal{R}}_5 \backslash \mathcal{R}_5$ can be naturally identified with the space of quadrics containing the canonical curve $\varphi_{\omega_C}(C)$ and the chord $\overline{\varphi_{\omega_C}(p), \varphi_{\omega_C}(q)}$ (as already recalled in Remark 24). Since C has genus 4, it is contained in a unique quadric Q. Being p and q generic, Q doesn't contain the required chord.

Since the Prym-canonical curve $\varphi_{\omega_C \otimes \eta}(C)$ is always contained in no quadrics we can conclude.

$\square$

Corollary 4.2.15.1. *The assumptions of the previous proposition imply that the dimension of the generic fibre $F_{g,r}$ of $\mathcal{P}_{g,r}$ is:*

$$\dim F_{1,2} = 1,\ \dim F_{2,2} = 2,\ \dim F_{3,2} = 2,\ \dim F_{4,2} = 1,$$
$$\dim F_{1,4} = 1,\ \dim F_{2,4} = 1.$$

For a detailed analysis of the geometric structure of the general fibre of these maps, we refer the reader to Chapter 5.

4.3 Admissible coverings and extended Prym maps

The problem of extending Prym maps to possibly singular and ramified covers was introduced by Beauville ([7]) who first gave the definition of admissible double covers and he obtained a proper map which on a dense open set factors through the previously described map $\mathcal{P}_g$. Roughly speaking, he extended the Prym map to

$$\bar{\bar{\mathcal{P}}}_g : \bar{\bar{\mathcal{R}}}_g \to \bar{\mathcal{A}}_{g-1},$$

where $\bar{\bar{\mathcal{R}}}_g$ (resp. $\bar{\mathcal{A}}_{g-1}$) is a suitable compactification of $\mathcal{R}_g$ (resp. $\mathcal{A}_{g-1}$) and then he restricted it to the open subset $\bar{\mathcal{R}}_g \subset \bar{\bar{\mathcal{R}}}_g$ of admissible covers in the sense that their Prym varieties are abelian varieties in $\mathcal{A}_{g-1}$.

Let us recall the definition of admissible double covers and generalized Prym varieties as presented in [7].

Let $\tilde{C}$ be a connected curve with only ordinary double points, $\tilde{\nu} : \tilde{N} \to \tilde{C}$ its normalization and $i : \tilde{C} \to \tilde{C}$ an involution. Assume

($\star$) the fixed points of i are exactly the singular points and at a singular point the two branches are not exchanged under i

and call C the quotient $\tilde{C}/\langle i \rangle$, $\pi : \tilde{C} \to C$ the projection and $\nu : N \to C$ the normalization of C. Consider the Norm map induced among the associated generalized Jacobians as a morphism of algebraic groups $\mathrm{Nm} : J\tilde{C} \to JC$ and let $P^+ := \ker(\mathrm{Nm})^0$ the associated subgroup variety.

Proposition 4.3.1 ([7],§3.5). *P^+ is an abelian variety of dimension $\rho_a(C) - 1$, where $\rho_a(C)$ is the arithmetic genus of C.*

In fact, using the definition of Cartier divisors on singular curves as $\tilde{C}$ and looking at the Norm maps involved in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbb{C}^*)^\gamma & \longrightarrow & J\tilde{C} & \longrightarrow & J\tilde{N} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle Nm} & & \big\downarrow{\scriptstyle Nm} & & \big\downarrow{\scriptstyle Nm} & & \\ 0 & \longrightarrow & (\mathbb{C}^*)^\gamma & \longrightarrow & JC & \longrightarrow & JN & \longrightarrow & 0 \end{array} \tag{4.8}$$

it follows that P^+ is complete and connected. Hence it is an abelian variety, as desired.

Definition 4.7. We call P^+ the *generalized Prym variety* of the covering $\pi : \tilde{C} \to C$. Moreover if $\tilde{C}$ is smooth, we say that $P^+ = P$ is a standard Prym variety.

Section 5 of [7], studies Prym varieties under more general assumptions. Using the same notation as before, let $\tilde{C}$ be a connected curve of genus $2g - 1$ with only ordinary double points and $i : \tilde{C} \to \tilde{C}$ an involution. Let C be the quotient curve, $\mathrm{Nm} : J\tilde{C} \to JC$

the Norm morphism and P the connected component to the identity of its kernel (as before a semi-abelian variety). Assume

$$(\star\star)\qquad \begin{cases} i \text{ is not the identity on any component of } \tilde{C}; \\ \rho_a(C) = g; \\ P \text{ is an abelian variety.} \end{cases}$$

Notice that the assumption $(\star\star)$ is equivalent to $(\star)$ if there are no components neither nodes of $\tilde{C}$ exchanged by i .

Definition 4.8. P is the *generalized Prym variety* of the covering $\pi : \tilde{C} \to C$. As before if $\tilde{C}$ is smooth, we say that P is a standard Prym variety.

More explicitly we can resume the previous condition in the following:

Theorem 4.3.2. *A stable curve $\tilde{C}$ with involution i and quotient map $\pi : \tilde{C} \to C$ is admissible, that is the associated Prym variety is an abelian variety, if and only if all the fixed points of i are nodes of $\tilde{C}$ where the branches are not exchanged and the number of the nodes exchanged under i equals the number of the irreducible components exchanged under i.*

Very clarifying possibilities are the following:

Example 1. Let $\pi : D \to C \in \mathcal{R}_{g,2}$ be a double cover branched in p, q two distinct points in C and consider

$$\tilde{C} := C \big/ p \sim q, \qquad \tilde{D} := D \big/ \tilde{p} \sim \tilde{q},$$

where $\tilde{p}, \tilde{q}$ are the ramification points in D over p, q and $p \sim q$ means that we are identifying the two points of the curve as in Figure 4.1. By definition $\tilde{C}$ is a curve of arithmetic genus $g + 1$ with a node. The covering $\tilde{\pi} : \tilde{D} \to \tilde{C}$ is admissible of type $(\star)$ and hence is an element of $\bar{\mathcal{R}}_{g+1}$.

This example shows that the moduli spaces $\mathcal{R}_{g,2}$ can always be embedded into Beauville's partial compactification $\bar{\mathcal{R}}_{g+1}$ of the moduli space of étale double coverings of curves of genus $g + 1$ by identifying the two branch points of the base curve (and doing the same for the covering curve). The closure of the image of $\mathcal{R}_{g,2}$ is an irreducible divisor in the boundary of $\bar{\mathcal{R}}_{g+1}$ that we denote by Δ^n.

Example 2. Let X be an element in $\mathcal{M}_{g-1}$ with two marked distinct points p, q. Let C_1, C_2 be isomorphic copies of X. Then

$$\tilde{C} := C_1 \sqcup C_2 \big/ p_1 \sim q_2, p_2 \sim q_1$$

together with the involution $i : \tilde{C} \to \tilde{C}$ which exchange C_1 with C_2 and $p_1 \sim q_2$ with $p_2 \sim q_1$ determines a quotient curve $C := \tilde{C}/\langle i \rangle$ with exactly one node q as in Figure 4.2. The covering $\pi : \tilde{C} \to C$ is étale of degree 2 and maps the two nodes of $\tilde{C}$ to q. It is admissible of type $(\star\star)$ and hence it is an element of $\bar{\mathcal{R}}_g$. The normalization of C is the curve X and $P(\pi)$ is isomorphic to JX.

Coverings of such type are called Wirtinger's covers. This construction shows that the Jacobian locus is thus contained in the Prym locus.

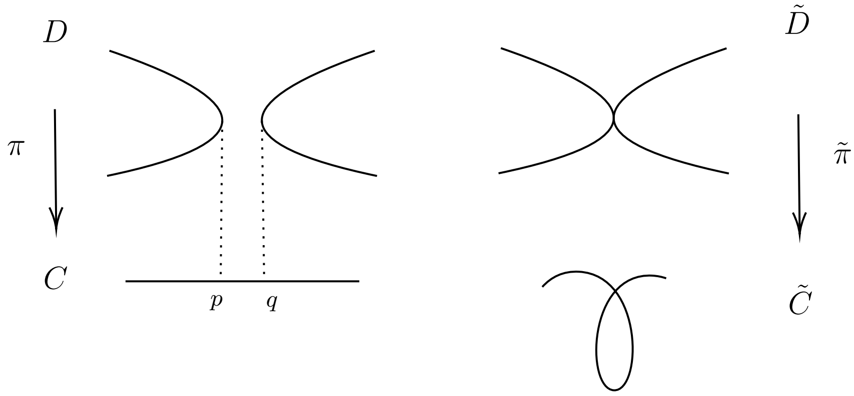

Figure 4.1
$\pi : D \to C$ ramified double cover, $\tilde{\pi} : \tilde{D} \to \tilde{C}$ admissible cover of type $(\star)$.

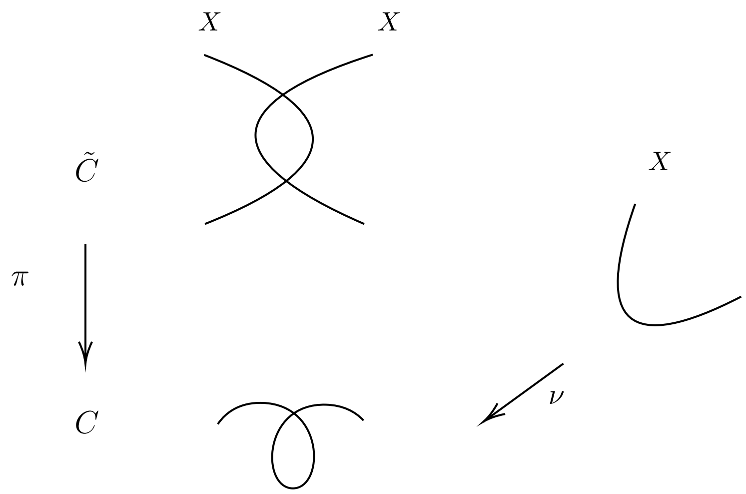

Figure 4.2
$\pi : \tilde{C} \to C$ admissible cover of type $(\star\star)$, $\nu : X \to C$ normalization map.

4.4 Polygonal constructions

This section is devoted to the description of the so-called polygonal construction. They provide a very useful tool which starts from a “tower” of covering maps

$$A \to B \to C$$

and produces new “towers”

$$A' \to B' \to C', \qquad A'' \to B'' \to C'', \qquad \dots$$

determining relations among the Prym varieties. All details of these constructions are borrowed from [25].

Let us consider a curve C of genus g with a map $f : C \to \mathbb{P}^1$ of degree n and a 2-sheeted ramified covering $\pi : D \to C$. Then we can always associate a 2^n- covering

$$D' \to \mathbb{P}^1$$

defined in the following way: the fibre over a point $p \in \mathbb{P}^1$ is given by the 2^n sections s of π over p. This means that:

$$s : f^{-1}(p) \to \pi^{-1} f^{-1}(p) \quad \text{and } \pi \circ s = id. \tag{4.9}$$

D' can be better described inside $D^{(n)}$, where $D^{(n)}$, as usual, represents the n-symmetric product of the curve D and it parametrizes effective divisors of degree n. Indeed it can be described by the following fibre product diagram:

$$\begin{tikzcd} D' \arrow[r, hook] \arrow[d, "2^n:1"] & D^{(n)} \arrow[d, "\pi^{(n)}"] \\ \mathbb{P}^1 \arrow[r, hook] & C^{(n)} \end{tikzcd} \tag{4.10}$$

where $\mathbb{P}^1$ is embedded in $C^{(n)}$ by sending a point p to its fibre $f^{-1}(p)$.

D' carries a natural involution $i' : D' \to D'$ defined as follows:

$$q_1 + \dots + q_n \mapsto i(q_1) + \dots + i(q_n) \tag{4.11}$$

where i is the involution of D which induces the covering π. Moreover we can define an equivalence relation on D' identifying two sections s_i, s_j if they correspond to a even number of changes $q_i \mapsto i(q_i)$. This gives another tower

$$D' \to O \to \mathbb{P}^1$$

where O is the quotient obtained considering this equivalence. It is known as the Orientation cover of $f \circ \pi$.

For n even the involution i' respects the equivalence, thus we have the following sequence of maps:

$$D' \to D'/i' \to O \to \mathbb{P}^1,$$

with, respectively, degrees equal to $2, 2^{n-2}, 2$. On the contrary, for n odd, the equivalence classes are exchanged so we have a diagram of the following type:

$$\begin{tikzcd} & D' \arrow[dl, "2:1"'] \arrow[dr] & \\ D'/i' \arrow[dr] & & O \arrow[dl, "2:1"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

Definition 4.9. D is orientable (over $\mathbb{P}^1$) if the orientation cover $O \to \mathbb{P}^1$ is trivial.

We conclude recalling, without proof, two results shown in [25].

Proposition 4.4.1. *Let $C \to \mathbb{P}^1$ a branched cover and $\pi : \tilde{C} \to C$ an unramified double cover. Then $\tilde{C}$ is orientable over $\mathbb{P}^1$.*

Proposition 4.4.2. *If D is orientable then D' is reducible: $D' = D^0 \cup D^1$.*

- *If n is even then i' acts on each D^j and the quotient has a degree 2^{n-2} map to $\mathbb{P}^1$;*

- *If n is odd then i' exchanges the two branches D^j. Each D^j has a map of degree 2^{n-1} to $\mathbb{P}^1$.*

4.4.1 The bigonal construction

Let us see an application of the polygonal construction described above in case of $n = 2$. Starting from a tower

$$D \xrightarrow{\pi} C \xrightarrow{f} \mathbb{P}^1,$$

where π and f both have degree 2, we get the following diagram:

$$\begin{tikzcd} & D' \arrow[dl, "\pi'"] \arrow[dd, "h"] \arrow[r, hook] & D^{(2)} \arrow[dd, "\pi^{(2)}"] \\ C' \arrow[dr, "f'"] & & \\ & g^1_2 \arrow[r, hook] & C^{(2)} \end{tikzcd} \tag{4.12}$$

The fibre of h over a point $k \in \mathbb{P}^1$ is given by the sections s introduced in (4.9). Call $C' := D'/\langle i' \rangle$, where i' is the induced involution on D' of (4.11). The possible situations over k are the following (see [25], pp. 68-69):

1) If π, f are étale the same are π', f';

2) If f is étale while π is branched at one point of $f^{-1}(k)$, then h inherits two critical points of order 2 in the fibre which are exchanged by i. This means that are π' is étale, while f' is branched;

3) Viceversa if π is étale while f is branched in k, then h has a critical point of order 2 in the fibre and 2 more points which are exchanged by i'. This means that π' has a critical point of order 2 while f' is étale;

4) If π, f are both branched the same are π', f' (in particular h has a single critical point of order 4);

5) If f is étale while π is branched at both points then C' has a node over k and π' becomes an admissible cover which looks like Example 1.

Remark 25. The bigonal construction can be extended by continuity to allow $D \to C$ to degenerate to an admissible cover as in Example 1. Let k be the image in $\mathbb{P}^1$ of the node of C. Then the associated tower $D' \xrightarrow{\pi'} C' \xrightarrow{f'}$ has f' étale while π' is branched at both points of ${f'}^{-1}(k)$.

We call an element $D \to C \to \mathbb{P}^1$ *general* if it avoids situations of type 5) (where the bigonal construction deals with singular admissible coverings).

Proposition 4.4.3. *Assuming $f \circ \pi$ general then $g(D') = r + g - 2$ and $g(C') = \frac{r}{2} - 1$.*

Proof. By generality assumption, a straightforward application of Riemann-Hurwitz formula for h gives:

$$2g(D') - 2 = 4(-2) + 2(r - r') + 6 - r' + 3r',$$

where r is the number of branch points of π and r' the number of branch points of π which are also ramification points for f. Similarly for π', we get:

$$2g(D') - 2 = 2(2g(C') - 2) + 6 - r' + r'.$$

□

Lemma 4.4.4 ([25], Lemma 2.7). *The bigonal construction is symmetric: if it takes $D \xrightarrow{\pi} C \xrightarrow{f} \mathbb{P}^1$ to $D' \xrightarrow{\pi'} C' \xrightarrow{f'} \mathbb{P}^1$ then it takes $D' \xrightarrow{\pi'} C' \xrightarrow{f'} \mathbb{P}^1$ to $D \xrightarrow{\pi} C \xrightarrow{f} \mathbb{P}^1$.*

Moreover the following holds:

Theorem 4.4.5 (Pantazis,[79]). *The Prym varieties $P(D, C)$ and $P(D', C')$ associated to the two bigonally-related covering maps $D \to C$ and $D' \to C'$ are dual each other as polarized abelian varieties.*

4.4.2 The trigonal construction

The polygonal construction in the case $n = 3$ was studied by Recillas in [83]. It is known as *trigonal construction* and it deals with étale double covers of smooth trigonal curves. Here we recall the main steps of the construction referring to [11, Chapter 12].

Denote $\mathcal{R}^{tr}_{g+1}$ the moduli space of 2:1 étale coverings of trigonal curves C of genus $g+1$. Each point in $\mathcal{R}^{tr}_{g+1}$ corresponds to a triple (C, η, M), where $\eta \in \mathrm{Pic}^0(C)$ such that $\eta \neq 0$ and $\eta^2 = \mathcal{O}_C$ gives the double covering and M is the g^1_3 which gives the map to $\mathbb{P}^1$. This means that we consider towers

$$\tilde{C} \xrightarrow{\pi} C \xrightarrow{3:1} \mathbb{P}^1.$$

Now call $\mathcal{M}^{tet}_{g,0}$ the open subspace of $\mathcal{M}_g$ given by tetragonal curves X with the property that above each point of $\mathbb{P}^1$ the associated linear series g^1_4 has at least one étale point. In [83], Recillas showed the existence of a canonical isomorphism:

$$\mathcal{R}^{tr}_{g+1} \to \mathcal{M}^{tet}_{g,0}.$$

As in [11] we actually look at the opposite direction of this arrow.

Let us consider a general tetragonal curve X of genus g. *General* means that X is not hyperelliptic and that the degree 4 map $k : X \to \mathbb{P}^1$ contains at least one étale point on each fibre. Identifying the linear series g^1_4 with a $\mathbb{P}^1$ embedded in $X^{(4)}$ (as already done in (4.10)), we can define

$$\tilde{C} := \{p_1 + p_2 \in X^{(2)} : \, p_1 + p_2 + p_3 + p_4 \in g^1_4 \text{ for some } p_3, p_4 \in X\}.$$

Immediately we obtain a degree 6 map

$$h : \tilde{C} \to \mathbb{P}^1 \subset X^{(4)}$$
$$p_1 + p_2 \mapsto p_1 + p_2 + p_3 + p_4.$$

Indeed if $p \in \mathbb{P}^1$ is such that $k^{-1}(p) = \{p_1, p_2, p_3, p_4\}$ then

$$h^{-1}(p) = \{p_1 + p_2, p_1 + p_3, p_1 + p_4, p_2 + p_3, p_2 + p_4, p_3 + p_4\}$$

and thus h has the claimed degree.

A local analysis guarantees that $\tilde{C}$ is smooth and irreducible and Riemann-Hurwitz formula that it has genus $g+1$. Moreover $\tilde{C}$ carries the natural involution

$$i : \tilde{C} \to \tilde{C}$$
$$p_1 + p_2 \mapsto p_3 + p_4$$

which, by assumptions, is fixed point free. Therefore, since h induces a g^1_3 on $C := \tilde{C}/\langle i \rangle$, $\pi : \tilde{C} \to C$ is the double étale cover which realizes the aforementioned isomorphism. Hence we can state the following

Theorem 4.4.6 (Recillas)**.** *The trigonal construction gives an isomorphism:*

$$T_0 : \mathcal{R}^{tr}_{g+1} \to \mathcal{M}^{tet}_{g,0}$$
$$(C, \eta, M) \mapsto (X, k),$$

Moreover, calling $P(\pi)$ the Prym variety associated to π, we have:

$$P(\pi) \cong JX$$

as isomorphism of principally polarized abelian variety.

Notice that by Proposition 4.4.2 we know that the trigonal construction in case of $\tilde{C} \to C \to \mathbb{P}^1$ orientable produces a reducible curve. The two components are isomorphic tetragonal curves of genus g. We take one of them to define the image (X, k) of (C, η, M) through T_0.

In [25], Donagi showed how the trigonal construction can be extended to admissible double coverings of trigonal curves of genus $g + 1$, whose Prym variety is an abelian variety of dimension g. The map T_0 is extended to a partial compactification $\bar{\mathcal{R}}^{tr}_{g+1}$ and formally the construction remains the same as before. Indeed it is proved the following:

Theorem 4.4.7 ([25], §2.9)**.** *The trigonal construction induces an isomorphism*

$$\bar{T}_0 : \bar{\mathcal{R}}^{tr}_{g+1} \xrightarrow{\cong} \mathcal{M}^{tet}_{g}.$$

Moreover if $\tilde{C}^* \xrightarrow{\pi^*} C^* \xrightarrow{h^*} \mathbb{P}^1$ is sent to the tetragonal curve $k^* : X^* \to \mathbb{P}^1$, then

$$P(\pi^*) \cong J(X^*),$$

where $P(\pi^*)$ denotes the Prym variety associated to the covering $\tilde{C}^* \xrightarrow{\pi^*} C^*$.

Furthermore he shows that the possible situation over a point $p \in \mathbb{P}^1$ are the following:

1) π^*, h^* and k^* are étale;

2) h^* and k^* have a ramification point of order 2 and π^* is étale;

3) h^* and k^* have a ramification point of order 3 and π^* is étale;

4) π^* is admissible, h^* has a simple node and a smooth point and k^* has two ramification points of order 2;

5) π^* is admissible, h^* has a node and it is ramified at exactly one branch and k^* has a ramification point of order 4.

The ramified Trigonal construction

A trigonal construction is valid also in the case of double covers of trigonal curves with two ramification points. This has been proved by Lange and Ortega in [57]. Here we recall their construction using their notation.

Denote by $\mathcal{R}b^{tr}_{g,2}$ the moduli space of ramified double covers $\pi : D \to C$ of smooth trigonal curves C of genus g and suppose that the branch locus of π is disjoint from the ramification locus of the degree 3 map $f : C \to \mathbb{P}^1$. We call an element $D \xrightarrow{\pi} C \xrightarrow{f} \mathbb{P}^1$ *special* if the branch locus of π (given by two points p_1, p_2) is contained in a fibre of f, otherwise we will call it *general*. Let $\mathcal{R}b^{tr}_{g,2,sp}$ be the moduli space of special elements.

Moreover we denote $\mathcal{M}^{tet}_{g,*}$ the moduli space of pairs (X, k) of smooth tetragonal curves with a 4:1 map $k : X \to \mathbb{P}^1$ with at least one étale point on each fibre with the exception of exactly one fibre which consists of two simple ramification points.

Theorem 4.4.8 ([57], §4.3). *The map*

$$\mathcal{R}b^{tr}_{g,2,sp} \to \mathcal{M}^{tet}_{g,*} \tag{4.13}$$

is an isomorphism. Moreover if $D \xrightarrow{\pi} C \xrightarrow{f} \mathbb{P}^1$ is an element of $\mathcal{R}b^{tr}_{g,2,sp}$ and X is the corresponding tetragonal curve, then we have an isomorphism of ppav:

$$P(\pi) \cong JX.$$

This Theorem deals with the boundary of the extended map $\bar{T}_0$ defined by Donagi. Indeed starting from a covering of special type $D \to C \to \mathbb{P}^1$, the identification of the branch points p_1, p_2, and of the corresponding ramification points (as done in Example 1), produces an admissible cover $\pi^* : D^* \to C^*$. Moreover the assumption of "speciality" guarantees C^* trigonal. Hence we obtain an element in the partial compactification $\bar{\mathcal{R}}^{tr}_{g+1}$ (to be precise of type 4)) and thus we can apply $\bar{T}_0$.

4.4.3 The tetragonal construction

Here we focus on the an application of the polygonal construction in case of $n = 4$.

Let us start with a tower

$$D \xrightarrow{\pi} C \xrightarrow{f} \mathbb{P}^1,$$

where f has degree 4 and $D \to C$ is an unramified double cover. The polygonal construction determines a new tower

$$D' \to D'/i \to \tilde{\mathbb{P}}^1 \to \mathbb{P}^1,$$

with maps of degree 2,4,2 (as explained in Section 4.4). By Propositions 4.4.1 and 4.4.2 D is orientable and thus we have splittings:

$$\begin{aligned} D' &= D'_1 \bigsqcup D'_2 \\ D'/i &= C'_1 \bigsqcup C'_2 \\ \tilde{\mathbb{P}}^1 &= \mathbb{P}^1 \bigsqcup \mathbb{P}^1. \end{aligned}$$

Here we recall, without proof, the main properties of this construction.

Proposition 4.4.9. *The tetragonal construction associates to towers such as $D \xrightarrow{\pi} C \xrightarrow{f} \mathbb{P}^1$ two new towers*

$$D_i' \xrightarrow{\pi_i'} C_i' \to \mathbb{P}^1, \quad i = 1, 2$$

of the same type. Furthermore, the tetragonal construction is a triality: starting from $D_1' \xrightarrow{\pi_1'} C_1' \to \mathbb{P}^1$ (resp. $D_2' \xrightarrow{\pi_2'} C_2' \to \mathbb{P}^1$) it returns $D_2' \xrightarrow{\pi_2'} C_2' \to \mathbb{P}^1$ (resp. $D_1' \xrightarrow{\pi_1'} C_1' \to \mathbb{P}^1$) and $D \xrightarrow{\pi} C \xrightarrow{f} \mathbb{P}^1$

Theorem 4.4.10 (Donagi, [25]§2.16). *The tetragonal construction commutes with the Prym map, that is:*

$$P(\pi) \cong P(\pi_1') \cong P(\pi_2')$$

We remark that this Theorem is the main tool used by Donagi to show the non-injectivity of the étale Prym map (see Theorem 4.2.4).

CHAPTER 5

The Fibres of the Ramified Prym Map

This chapter is devoted to the study of the ramified Prym maps, in particular it will concern the study of the geometric properties of the structure of the generic positive dimensional fibre.

As we saw in Chapter 4, the Prym map $\mathcal{P}_{g,r}$ assigns to a degree 2 covering $\pi : D \longrightarrow C$, of a smooth complex irreducible curve ramified in an even number of points $r \geq 0$, a polarized abelian variety $P(\pi) = P(D, C)$ of dimension $g-1+\frac{r}{2}$, where g is the genus of C. Hence, denoting by $\mathcal{R}_{g,r}$ the moduli space of isomorphism classes of the morphisms π, we have maps:

$$\mathcal{P}_{g,r} : \mathcal{R}_{g,r} \longrightarrow \mathcal{A}^{\delta}_{g-1+\frac{r}{2}},$$

to the moduli space of abelian varieties of dimension $g-1+\frac{r}{2}$ with polarization of type $\delta := (1, \ldots, 1, 2, \ldots, 2)$, with 2 repeated g times if $r > 0$ and $g-1$ times if $r = 0$. Notice that in case of $r = 2$ the Prym variety is always principally polarized.

The case $r = 0$ is very classical. Indeed, Prym varieties of unramified coverings are principally polarized abelian varieties and they have been studied for over 100 years, initially by Wirtinger, Schottky and Jung (among others) in the second half of the 19th century from the analytic point of view. They were studied later from an algebraic point of view in the seminal work of Mumford [71] in 1974. This work inspired many papers from different mathematicians. Indeed nowadays almost everything is known about the "classical" Prym map $\mathcal{P}_{g,0}$ (as already said usually denoted $\mathcal{P}_g$).

The goal of this Chapter is to complete the study of the ramified degree 2 Prym maps by means of a systematic analysis of the fibre in low genus.

Indeed, very recently, as recalled in Chapter 4 Section 4.2.2, a global Torelli theorem has been announced for all $g > 0$ and $r \geq 6$ in the work of Ikeda ([49]) and of Naranjo-Ortega ([76]).

Here we address to the opposite side of the study of the ramified Prym map: the structure of the generic fibre when

$$\dim \mathcal{R}_{g,r} = 3g - 3 + r > \dim \mathcal{A}^{\delta}_{g-1+\frac{r}{2}} = \frac{1}{2}(g - 1 + \frac{r}{2})(g + \frac{r}{2}), \tag{5.1}$$

that is when :

$$r = 2 \quad \text{and} \quad 1 \leq g \leq 4;$$
$$r = 4 \quad \text{and} \quad 1 \leq g \leq 2.$$

Proposition 4.2.15 and Corollary 4.2.15.1 determine the dimension of the generic fibre.

We observe that the case $g = 1, r = 4$ was already considered by Barth in his study of abelian surfaces with polarization of type $(1, 2)$ (see [6]).

It is worthy to mention that degree 2 coverings ramified in 2 points, i.e. in four among the six cases listed above, can be seen as the normalization of coverings of nodal curves, that corresponds to the opposite procedure to the one explained in Example 1. In this way, the moduli space $\mathcal{R}_{g-1,2}$ can be identified with an open set of a boundary divisor of $\bar{\mathcal{R}}_g$, where $\bar{\mathcal{R}}_g$ is the Beauville's extension ([7]) of $\mathcal{R}_g$ of "admissible" coverings and it makes the extended Prym map

$$\bar{\mathcal{P}}_g : \bar{\mathcal{R}}_g \to \mathcal{A}_{g-1}$$

proper.

With this strategy the works of Verra, Recillas and Donagi which study the fibre of $\bar{\mathcal{P}}_3, \bar{\mathcal{P}}_4$ and $\bar{\mathcal{P}}_5$ (see Theorems 4.2.5, 4.2.6, 4.2.7) could help to understand the cases $r = 2$ and $2 \leq g \leq 4$. Unfortunately this way becomes cumbersome since the intersection of the generic fibre with the boundary is usually difficult to be described.

For this reason, we will use (except for the case $g = 4$) direct procedures to study the fibre mainly based on the bigonal construction (see [25]) and the extended trigonal construction (see [57]).

The chapter is organized as follows.

In Section 5.1 we study the fibre $\mathcal{P}_{1,2} : \mathcal{R}_{1,2} \to \mathcal{A}_1$ over a generic elliptic curve. In particular, Mumford's diagrams turn out to be the main tool.

In Section 5.2 we study the fibre of $\mathcal{P}_{1,4} : \mathcal{R}_{1,4} \to \mathcal{A}_2^{(1,2)}$ over a generic polarized abelian surface (A, L) of type $(1, 2)$. This is a result of Barth ([6]) that we recall for the sake of completeness.

In Section 5.3 we study the fibre of $\mathcal{P}_{2,2} : \mathcal{R}_{2,2} \to \mathcal{A}_2$ over a generic principally polarized abelian surface. Since $\mathcal{R}_{2,2}$ parametrizes double coverings of genus 2 curves C, we can use the hyperelliptic involution of C to apply the bigonal construction which turns out to be the main tool.

In Section 5.4 we study the fibre of $\mathcal{P}_{2,4} : \mathcal{R}_{2,4} \to \mathcal{A}_3^{(1,2,2)}$ over a generic polarized abelian 3-fold. The situation is very similar to that described previously in case of $r = 2$. Indeed, with the appropriate changes for $r = 4$, we adopt a strategy analogous to the previous case still using the bigonal construction.

In Section 5.5 we study the fibre of $\mathcal{P}_{3,2} : \mathcal{R}_{3,2} \to \mathcal{A}_3$ over a generic principally polarized abelian 3-fold A. Assuming that A is the Jacobian of a general curve X we describe the fibre studying its g^1_4's. In particular, the ramified trigonal construction due to [57] turns out to be the key ingredient. Indeed, it allows to identify pairs $(x, k : X \xrightarrow{4:1} \mathbb{P}^1)$ with elements in $\mathcal{R}_{3,2}$.

In Section 5.6 we study the fibre of $\mathcal{P}_{4,2} : \mathcal{R}_{4,2} \to \mathcal{A}_4$ over a generic principally polarized abelian 4-fold A. Here we take care of the behaviour at the boundary of the rather sophisticated Donagi's description of the fibre of $\bar{\mathcal{P}}_5$ (see [25, section 5]). In particular, we have to study quadrics containing a nodal canonical curve of genus 5, as suggested by a Theorem of Izadi ([50]). We prove a Lemma concerning a parametrization of singular quadrics containing nodal curve which can be interesting on its own.

In Section 5.7 we glue the two parts of this Thesis. Indeed we describe some examples of irreducible components of fibres of ramified Prym maps that yield totally geodesic or Shimura subvarieties of $\mathcal{A}_g$. In particular we read the result presented in Chapter 2 in terms of the analysis of the fibre of the Prym maps done here. Therefore we recall that the irreducible components of the fibres of the Prym maps $\mathcal{P}_{1,2}$ and $\mathcal{P}_{1,4}$ yield infinitely many totally geodesic curves in $\mathcal{A}_2$ and in $\mathcal{A}_3$. Countably many of them are Shimura. Finally we give a new explicit example of a totally geodesic curve which is an irreducible component of a fibre of $\mathcal{P}_{1,2}$.

5.1 Case $g = 1$, $r = 2$

Let us consider $\pi : D \to C$ a double ramified covering in $\mathcal{R}_{1,2}$ and take

$$\mathcal{P}_{1,2} : \mathcal{R}_{1,2} \to \mathcal{A}_1,$$

the corresponding Prym map.

We denote by $b_1 + b_2$ the branch divisor (on C) and $r_1 + r_2$ the ramification divisor (on D). The covering π is determined by the data $(C, \eta, B = b_1 + b_2)$ where

$$\eta \in \mathrm{Pic}^1(C) = C \quad \text{satisfies} \quad \eta^{\otimes 2} = \mathcal{O}_C(b_1 + b_2). \tag{5.2}$$

Since $h^0(C, \mathcal{O}_C(b_1 + b_2)) = 2$, the linear series $|b_1 + b_2|$ gives a map

$$f : C \xrightarrow{2:1} \mathbb{P}^1$$

ramified in four points $p_1, p_2, p_3, p_4 \in C$. Therefore $2p_i \in |b_1 + b_2|$. By construction (i.e. assumption (5.2)), η is one of the sheaves $\mathcal{O}_C(p_i)$.

Calling σ the involution on C attached to f, then

$$\sigma(b_1) = b_2 \quad \text{and} \quad \sigma(p_i) = p_i, \ \ i = 1, ..., 4.$$

Hence σ leaves invariant $b_1 + b_2$ and η. The following is well-known:

Lemma 5.1.1. *Let σ be an involution on a curve C leaving invariant a reduced divisor B and a sheaf $\eta \in \mathrm{Pic}(C)$ such that $\eta^{\otimes 2} \cong \mathcal{O}_C(B)$. Let $\pi : D \to C$ be the double covering attached to (C, η, B), then there exists an involution $\tilde{\sigma}$ on D lifting σ: $\sigma \circ \pi = \pi \circ \tilde{\sigma}$.*

We have the following Cartesian diagram:

$$\begin{tikzcd} & D \arrow[dl] \arrow[d, "\pi"] \arrow[dr] & \\ \mathbb{P}^1 \arrow[dr] & C \arrow[d, "f"] & E \arrow[dl, "f'"] \\ & \mathbb{P}^1 & \end{tikzcd} \tag{5.3}$$

Here E is the quotient of D by $\tilde{\sigma}$ while $\mathbb{P}^1$ is the quotient of D by $\tau\tilde{\sigma}$, where τ is the involution attached to π. Indeed, without loss of generality, assume that π corresponds to the point $(C, \eta \cong \mathcal{O}_C(p_1), b_1 + b_2)$. Then the condition

$$\sigma \circ \pi = \pi \circ \tilde{\sigma},$$

together with Riemann-Hurwitz formula for the quotient map $D \to D/\langle\tilde{\sigma}\rangle$, guarantees that the preimages of p_1 by π are fixed points of $\tilde{\sigma}$ and they are the only ones. On the other hand, $\tau\tilde{\sigma}$ fixes the preimages by π of p_2, p_3, p_4. Therefore E is an elliptic curve, while $D/\langle\tau\tilde{\sigma}\rangle$ is $\mathbb{P}^1$. Notice that if this is not true the contrary holds: $\tau\tilde{\sigma}$ fixes the preimages of p_1 by π, hence $D/\langle\tau\tilde{\sigma}\rangle$ is an elliptic curve, while τ fixes the preimages of p_2, p_3, p_4 by π and thus $D/\langle\tilde{\sigma}\rangle$ is $\mathbb{P}^1$.

Using Mumford's results on hyperelliptic Pryms for the diagram (5.3) ([71, section 7]), we get

$$P(D, C) \cong JE \times J\mathbb{P}^1 \cong E.$$

Calling $a_i = f(p_i)$ and $b = f(b_1) = f(b_2)$, the above description of the ramification locus of the map $D \to E$ implies that the branch locus of $f' : E \to \mathbb{P}^1$ is given by b, a_2, a_3, a_4. Thus we get the following:

Theorem 5.1.2. *Fix a generic elliptic curve $E \in \mathcal{A}_1$. The preimage of E by the ramified Prym map $\mathcal{P}_{1,2}$ is isomorphic to $L_1 \sqcup \ldots \sqcup L_4$, where each L_i is the complement of three points in a projective line.*

Proof. Start with E represented as a double covering of $\mathbb{P}^1$ branched in four points c_1, c_2, c_3, c_4 and put

$$L_i = \mathbb{P}^1 \backslash \{c_1, ..., \hat{c}_i, ..., c_4\}, \quad i = 1, ..., 4.$$

Then for any $q \in L_1$ we get a unique element in $\mathcal{P}_{1,2}^{-1}(E)$ constructed in the following way: let C be the covering of $\mathbb{P}^1$ branched in q, c_2, c_3, c_4 and denote with b_1, b_2 the preimages of c_1 via this covering. Then $D \to C$ in $\mathcal{P}_{1,2}^{-1}(E)$ is determined by

$$B = b_1 + b_2 \quad \text{and by} \quad \eta = \mathcal{O}_C(p),$$

where p is the ramification point in C attached to q. Doing the same for the other $L_i's$, we conclude.

□

5.2 Case $g = 1$, $r = 4$

The case

$$\mathcal{P}_{1,4} : \mathcal{R}_{1,4} \to \mathcal{A}_2^{(1,2)} \tag{5.4}$$

is completely studied in [6]. Here we include the main result without proof by the sake of completeness.

Actually, instead of (5.4), it is easier to study the composition:

$$\mathcal{R}_{1,4} \to \mathcal{A}_2^{(1,2)} \xrightarrow{\cong} \mathcal{A}_2^{(1,2)} \tag{5.5}$$

where the isomorphism sends the Prym variety to its dual (see Section 4.1.3).

Fix a general polarized abelian surface (A, L) of type (1,2). We have that:

Proposition 5.2.1 ([6], pp. 46-48). *The pencil $|L|$ has no fixed component and its base locus consists of four points $e_1, ..., e_4$. The general member $D \in |L|$ is an irreducible smooth curve of genus 3. Moreover L is symmetric and the same occurs for all $D \in |L|$.*

Furthermore if $D \in |L|$ is smooth, the multiplication by -1 has exactly 4 fixed points on it. This means that the quotient $D/\langle -1 \rangle$ is an elliptic curve. A result of general interest is the following:

Proposition 5.2.2. *Let D be a smooth curve of genus 3. Then the following conditions are equivalent:*

- *D admits an elliptic involution, that is D admits a 2:1 map onto an elliptic curve;*
- *D admits an embedding into an abelian surface A.*

The main result of [6] is the following

Theorem 5.2.3 (§1.12, Duality Theorem). *Let D be a smooth curve of genus 3 and let $\pi : D \to E$ a double cover over an elliptic curve E. Moreover let $D \hookrightarrow A$ be the corresponding embedding into an abelian surface A. Then both A and the Prym variety $P(\pi)$ carry a natural polarization of type $(1, 2)$. One is isomorphic to the dual of the other.*

This justifies the study of the fibre of $\mathcal{P}_{1,4}$ through the isomorphism (5.5) and it yields the following

Theorem 5.2.4. *The fibre of the Prym map $\mathcal{P}_{1,4}$ over a general polarized abelian surface (A, L) is parametrized by the linear system $|L^*|$, where L^* is the dual polarization on A^* as defined in Theorem 4.3.*

5.3 Cases $g = 2$ and $r = 2$

This section is devoted to the analysis of the fibres of the Prym map

$$\mathcal{P}_{2,2} : \mathcal{R}_{2,2} \to \mathcal{A}_2.$$

Let us take an element $(C, \eta, B) \in \mathcal{R}_{2,2}$. Since the genus of C is 2, we can apply the bigonal construction using the hyperelliptic involution of C. Thus we pass from towers:

$$D \to C \xrightarrow{2:1} \mathbb{P}^1$$

to towers

$$D' \xrightarrow{\pi'} C' \xrightarrow{f'} \mathbb{P}^1,$$

of degree 2 maps. Proposition 4.4.3 tells us that π' has 6 branch points and that C' is a curve of genus 0 (possibly with one node if it occurs a fibre of type 5) for the tower $D \to C \to \mathbb{P}^1$).

Remark 26. To better understand the nodal case, let us see the map f' as the choice of two different points in the projective line. Thereby, the limit case appears when the two points come together.

We introduce two suitable moduli spaces.

- $\mathcal{M}_{0,6}$ is the moduli space of 6 unordered different points in the projective line.
- $\mathcal{M}_{0,6,2}$ is the moduli space of two collections of points in $\mathbb{P}^1$: 6 unordered different points and 2 unordered different points more (different from the 6 points already chosen).

We observe that a partial compactification $\bar{\mathcal{M}}_{0,6,2}$ of the moduli space $\mathcal{M}_{0,6,2}$ consists in allowing the set of two points being a repeated one.

Section 4.4.1, in particular Lemma 4.4.4, shows that the bigonal construction yields an injective map:

$$b : \mathcal{R}_{2,2} \to \bar{\mathcal{M}}_{0,6,2},$$

which is an isomorphism onto its image $b(\mathcal{R}_{2,2})$. Indeed, as already observed in Remark 25 (according to [25, Section 2.3]), the bigonal map extends to nodal admissible coverings. Thus the inverse map is the bigonal construction again. We denote the image $b(\mathcal{R}_{2,2})$ by $\bar{\mathcal{M}}^0_{0,6,2}$. In order to give precise description of the moduli space $\bar{\mathcal{M}}^0_{0,6,2}$ we first observe what follows.

Remark 27. It is possible to see $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ as a conic via

$$\begin{aligned} \nu_2 : \mathbb{P}^1 &\to \mathbb{P}^2 \\ [x_0 : x_1] &\mapsto [x_0^2 : x_0x_1 : x_1^2], \end{aligned}$$

the Veronese embedding of degree 2. Hence the symmetric product $Sym^2\mathbb{P}^1$ can be identified with a projective plane in the standard way: the pairs of points (possibly equal) correspond to lines and therefore $Sym^2\mathbb{P}^1$ can be identified with $(\mathbb{P}^2)^*$.

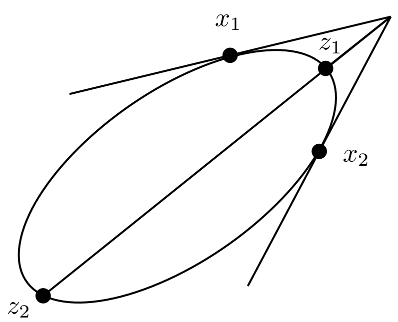

Figure 5.1
Points forming an harmonic ratio.

By definition, every point in $\mathcal{M}_{0,6,2}$ corresponds to the collections of 6 unordered different points $p_1, ..., p_6$ and 2 unordered different points x_1, x_2 more. The way that each pair of different points x_1, x_2 determines a $2:1$ map on the conic is easy: two points z_1, z_2 correspond by the involution $\sigma_{x_1+x_2}$ with fixed points x_1, x_2, if they form an harmonic ratio:

$$|x_1, x_2; z_1, z_2| = -1.$$

Geometrically, viewing the points in the plane, this means that the pole of the line x_1x_2 is aligned with z_1 and z_2 as in Figure 5.1:

We have 6 marked points $p_1, ..., p_6$ in the line and we have to avoid property 5) of Section 4.4.1. Indeed we don't want b^{-1} (which, as said, is the bigonal construction again) recovers towers with nodal curve. This means that we have to eliminate the pairs $x_1 + x_2 \in Sym^2\mathbb{P}^1$ such that $\sigma_{x_1+x_2}(p_i) = p_j$ for some $i \neq j$. That is:

$$\bar{\mathcal{M}}^0_{0,6,2} = \{[(p_1 + \ldots + p_6, x_1 + x_2)] \in \bar{\mathcal{M}}_{0,6,2} \mid |x_1, x_2; p_i, p_j| \neq -1, \forall i \neq j\}.$$

Then we have a commutative diagram:

$$\begin{tikzcd} \mathcal{R}_{2,2} \arrow[r] \arrow[d, "b"] & \mathcal{A}_2 \\ \bar{\mathcal{M}}^0_{0,6,2} \arrow[r, "\varphi"] & \mathcal{M}_{0,6} \arrow[u] \end{tikzcd} \tag{5.6}$$

where φ is the forgetful map and $\mathcal{M}_{0,6} \to \mathcal{A}_2$ is the composition of two maps. Indeed we have the map $\mathcal{M}_{0,6} \to \mathcal{M}_2$, which sends the 6 marked points of $\mathbb{P}^1$ to the corresponding genus 2 curve with hyperelliptic involution branched over there, and the Torelli morphism $\mathcal{M}_2 \to \mathcal{A}_2$. Therefore, studying the fibre of φ, we conclude with the following:

Theorem 5.3.1. *The fibre of the Prym map $\mathcal{P}_{2,2}$ over a general principally polarized abelian surface S is isomorphic to a projective plane minus 15 lines.*

Proof. Let S be a general principally polarized abelian surface, assume that S is the Jacobian of a genus 2 curve H and represent H as an element in $\mathcal{M}_{0,6}$, where the six marked points $p_1, ..., p_6$ are all different and correspond to the branch locus of the hyperelliptic involution. Diagram (5.6) says that we must look at the fibre of φ over H. The harmonic condition $|x_1, x_2; p_i, p_j| = -1$ says that x_1, x_2 and the pole p_{ij} of $p_i p_j$ are in a line. Therefore, looking at the dual, we have to rule out the points of the 15 lines

$$(p_{ij})^* \subset (\mathbb{P}^2)^*.$$

Notice that the limit case $x_1 = x_2$ means that p_{ij} belongs to the tangent line at the point and this is not excluded in the fibre.

□

5.4 Case $g = 2$ and $r = 4$

Here we study the fibres of

$$\mathcal{P}_{2,4} : \mathcal{R}_{2,4} \to \mathcal{A}_3.$$

As in the previous section, since an element $\pi : D \to C$ in $\mathcal{R}_{2,4}$ admits naturally a 2:1 map to $\mathbb{P}^1$ (the hyperelliptic involution of C), we can apply the bigonal construction to produce an injective map:

$$b : \mathcal{R}_{2,4} \to \tilde{\mathcal{R}}_{1,6},$$

where $\tilde{\mathcal{R}}_{1,6}$ is the moduli space of isomorphism classes of pairs (π', f'). On one side π' is a double cover of irreducible curves satisfying the following condition

(*) the involution on D' attached to π' has 6 smooth fixed points, the curve D' has at most one node and, in this case, the point is fixed and the two branches are not exchanged under the involution.

On the other side f' is a g^1_2 on C'.

Let us denote by $\tilde{\mathcal{R}}^0_{1,6}$ the image of b. This is an open set that, as in the previous section, can be described explicitly

$$\tilde{\mathcal{R}}^0_{1,6} = \{(\pi', f') \in \tilde{\mathcal{R}}_{1,6} \mid f'(p_i) \neq f'(p_j)\ \forall i \neq j\}.$$

Again (see Lemma 4.4.4 and Remark 25), the symmetry of the bigonal construction makes b an isomorphism onto its image.

Remark 28. In order to state the next diagram we need first to extend $\mathcal{P}_{1,6}$ to the partial compactification $\bar{\mathcal{R}}_{1,6}$ of the double coverings of curves of (arithmetic) genus 1 satisfying (*). This can be proved imitating Beauville's construction of the extension of the Prym map to admissible coverings (see Section 4.3). This means that first the Prym map $\mathcal{P}_{1,6}$ is extended to a certain compactification $\bar{\bar{\mathcal{R}}}_{1,6}$ of $\mathcal{R}_{1,6}$. Then it is restricted to the subset $\bar{\mathcal{R}}_{1,6}$ which allows double coverings satisfying condition (*) showing that the Prym varieties associated with such coverings are actually abelian varieties.

Similarly to what we saw in the previous Section we have a commutative diagram:

$$\begin{tikzcd} \mathcal{R}_{2,4} \arrow[r] \arrow[d, "b"] & \mathcal{A}_3^{(1,2,2)} \cong \mathcal{A}_3^{(1,1,2)} \\ \tilde{\mathcal{R}}^0_{1,6} \arrow[r, "\varphi"] & \bar{\mathcal{R}}_{1,6} \arrow[u] \end{tikzcd} \tag{5.7}$$

where φ is the forgetful map. Moreover the isomorphism $\mathcal{A}_3^{(1,2,2)} \cong \mathcal{A}_3^{(1,1,2)}$ is given by (4.3) and sends a polarized abelian threefold to its dual (endowed with the dual polarization). The remaining vertical arrow

$$\bar{\mathcal{P}}_{1,6} : \bar{\mathcal{R}}_{1,6} \to \mathcal{A}_3^{(1,1,2)}$$

is the extension of the Prym map $\mathcal{P}_{1,6}$ (as in Remark 28). From the result of Ikeda (Theorem 4.2.13), we know that $\mathcal{P}_{1,6}$ is injective and in fact an embedding (see Theorem 4.2.14). Therefore the extension to $\bar{\mathcal{R}}_{1,6}$ is generically injective. Pantazi's Theorem (see Theorem 4.4.5 on duality between Prym varieties of bigonally related towers) guarantees the commutativity of (5.7). Thus we can conclude with the following:

Theorem 5.4.1. *The fibre of the Prym map $\mathcal{P}_{2,4}$ over a general $A \in \mathcal{A}_3$ is isomorphic to an elliptic curve E minus 15 points.*

Proof. Let us consider a generic polarized abelian threefold in $\mathcal{A}_3^{(1,2,2)}$ and let $(E, \eta, p_1 + \ldots + p_6)$ be its unique preimage in $\mathcal{R}_{1,6}$. Call $B = p_1 + \ldots + p_6$ the branch divisor . Diagram (5.7) says that the fibre over A is isomorphic to the fibre of φ over $(E, \eta, p_1 + \ldots + p_6)$. This means that it is isomorphic to:

$$\mathrm{Pic}^2(E) \smallsetminus \bigcup_{\substack{p_i, p_j \in B, \\ p_i \neq p_j}} \mathcal{O}_E(p_i + p_j).$$

Indeed we have to consider all the possible maps $f' : E \to \mathbb{P}^1$ of degree 2 avoiding those towers $D' \to E \xrightarrow{f'} \mathbb{P}^1$ which are sent by b^{-1} to nodal towers.

The isomorphism $\mathrm{Pic}^2(E) \cong E$ concludes the proof.

□

5.5 Case $g = 3, r = 2$

This section is devoted to the Prym map

$$\mathcal{P}_{3,2} : \mathcal{R}_{3,2} \to \mathcal{A}_3. \tag{5.8}$$

Let us start with $\pi : D \to C$ an element of $\mathcal{R}_{3,2}$, this means that C is a smooth curve of genus 3, and we denote by $B = p_1 + p_2$ the branch divisor of π.

Remark 29. Each non-hyperelliptic curve of genus 3 admits a 1-dimensional space of g^1_3's. In particular, identifying C with its canonical image (a quartic plane curve) and considering the line $l = p_1 + p_2$ passing through p_1 and p_2 we can always get two degree 3 maps: they are defined considering the two different projections from one of the two remaining points x, y of the intersection $C \cdot l$. In fact if we consider the canonical divisor

$$K_C = p_1 + p_2 + x + y$$

we get $h^0(C, \omega_C(-x)) = h^0(C, \omega_C(-y)) = 2$ and we can use the associated linear systems to define the 3:1 maps to $\mathbb{P}^1$. Call them f_x and f_y. Both have, by definition, the two branch points p_1, p_2 on the same fibre and they are the unique trigonal maps on C with this property.

We will use the following diagram to describe the fibres of $\mathcal{P}_{3,2}$:

$$\begin{tikzcd}
 & \mathcal{M}^{tet}_{3,*} \arrow[r, "2:1"] & \mathcal{M}q^{tet}_{3,*} \arrow[dr] & \\
\mathcal{R}b^{tr}_{3,sp} \arrow[ur, "\cong"] \arrow[d, "2:1"] & & & \mathcal{M}_3 \arrow[d, "j"] \\
\mathcal{R}_{3,2} \arrow[rrr, "\mathcal{P}_{3,2}"] & & & \mathcal{A}_3
\end{tikzcd} \tag{5.9}$$

As already seen in Section 4.4.2, we denote by $\mathcal{R}b^{tr}_{3,sp}$ the moduli space of ramified double covers $\pi : D \to C$ of smooth trigonal curves C of genus 3 such that the branch locus of π is contained in a fibre of the degree 3 map to $\mathbb{P}^1$ (towers of "special" type). By above considerations on g^1_3's, the forgetful map

$$\mathcal{R}b^{tr}_{3,sp} \to \mathcal{R}_{3,2}$$

is a 2:1 map. Furthermore the ramified trigonal construction determines the existence of an isomorphism between $\mathcal{R}b^{tr}_{3,sp}$ and $\mathcal{M}^{tet}_{3,*}$ (see Theorem 4.4.8). For convenience of the reader we recall that $\mathcal{M}^{tet}_{3,*}$ is the moduli space of pairs (X, g^1_4) of smooth tetragonal genus 3 curves X with a 4:1 map $X \to \mathbb{P}^1$ with at least one étale point on each fibre with the exception of exactly one fibre which consists of two simple ramification points.

We will study the fibre of $\mathcal{P}_{3,2}$ using the map $\mathcal{M}^{tet}_{3,*} \to \mathcal{M}_3$. Notice that in the above diagram it factors as the composition of two maps:

$$\mathcal{M}^{tet}_{3,*} \to \mathcal{M}q^{tet}_{3,*} \to \mathcal{M}_3.$$

The first one is defined as the quotient map associated to an involution that acts on $\mathcal{M}^{tet}_{3,*}$. We will describe this action later. The second one is the forgetful map.

Finally, j is just the Torelli morphism.

Let us take a general abelian threefold $A \in \mathcal{A}_3$. We can assume that A is the Jacobian of a general curve X of genus 3. In order to study the fibres of $\mathcal{M}^{tet}_{3,*} \to \mathcal{M}_3$, which looks at degree 4 maps on the curve X with fibres of the aforementioned type, we need to recall some facts on g^1_4's on X.

5.5.1 The blow-up

Let $X \subset \mathbb{P}^2 = \mathbb{P}(H^0(X, \omega_X)^*)$ be a non hyperelliptic curve of genus 3 canonically embedded. Let $\mathsf{G}^1_4(X)$ be the variety of all g^1_4 linear series on X, complete or not (see [1], chapter IV). Then by Riemann-Roch

$$\psi : \mathsf{G}^1_4(X) \longrightarrow W^1_4(X) = \mathrm{Pic}^4(X)$$

is a birational surjective map which is an isomorphism out of $W^2_4(X) = \{\omega_X\}$. In fact

$$\mathsf{Supp}(\mathsf{G}^1_4(X)) = \{(L, V) \mid L \in \mathrm{Pic}^4(X), V \in Gr(2, H^0(X, L))\}.$$

Therefore if

$$L \neq \omega_X \quad \text{we have} \quad h^0(X, L) = 2$$

and thus the fibre of ψ over L is just the complete linear series $(L, H^0(X, L))$. Calling E the preimage of the canonical sheaf, then

$$\mathsf{G}^1_4(X) \smallsetminus E \cong \mathrm{Pic}^4(X) \smallsetminus \{\omega_X\}.$$

The set E parametrizes all the non-complete g^1_4 linear series $|V|$ on X which correspond to

$$Gr(2, H^0(X, \omega_X)) \cong \{\text{lines in } \mathbb{P}H^0(X, \omega_X)\} = \mathbb{P}(H^0(X, \omega_X)^*) = \mathbb{P}^2.$$

In other words $\mathsf{G}^1_4(X)$ is the blow-up of $\mathrm{Pic}^4(X)$ at ω_X and the points of the exceptional divisor correspond to points in the plane $\mathbb{P}^2$ where the curve X is canonically embedded. The linear series $|V|$ is the projection from this point (which is the kernel of the projectivization of the map $H^0(X, \omega_X)^* \twoheadrightarrow V^*$). If the point belongs to X itself, then the linear series has a base point.

Since we are interested in the fibre of $\mathcal{P}_{3,2}$ over a general JX, we can assume that X has exactly 28 bitangents, that is that there are not hyperflexes in X (points p such that the tangent line at p intersects X in $4p$). In fact, the curves with hyperflexes define a divisor in $\mathcal{M}_3$. Each bitangent defines a divisor of the form

$$2p_i + 2q_i \in |K_X|$$

Denote by $\mathcal{B} \subset X^{(2)}$ the set $\{p_i + q_i \mid i = 1, \ldots, 28\}$ and let

$$S := \mathrm{Bl}_{\mathcal{B}} X^{(2)}$$

be the surface obtained by blowing-up $X^{(2)}$ at $\mathcal{B}$. By the universal property of the blow-up we have a diagram:

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & \mathsf{G}^1_4(X) \\ \downarrow & & \downarrow \psi \\ X^{(2)} & \xrightarrow{\varphi_0} & \mathrm{Pic}^4(X), \end{array} \tag{5.10}$$

where $\varphi_0(x+y) = \mathcal{O}_X(2x+2y)$.

5.5.2 Geometric description of the complete linear series

In the case of complete linear series g^1_4, we can describe geometrically the divisors in the image of φ: fix two different points $r, s \in X$ such that the line $l = r + s$ intersects X in four different points. Put

$$l \cdot X = r + s + u + v.$$

Denote by t_r, t_s, t_u, t_v the tangent lines to X at the points r, s, u, v respectively. Let us define:

$$\mathcal{F}_{r,s} = \{\text{conics through } u, v \text{ tangent to } t_u, t_v \text{ at } u, v \text{ resp.}\} \cong \mathbb{P}^1.$$

If $Q \in \mathcal{F}_{r,s}$, then $Q \cdot X = 2u+2v+p_1+p_2+p_3+p_4$. All these degree 8 divisors are linearly equivalent on X (they belong to $|2K_X|$ since we are intersecting with a conic). One of these conics is the double line $l^2 \in \mathcal{F}_{r,s}$ which intersects X in the divisor $2u+2v+2r+2s$. Therefore:

$$2u + 2v + 2r + 2s \sim 2u + 2v + p_1 + p_2 + p_3 + p_4,$$

hence

$$2r + 2s \sim p_1 + p_2 + p_3 + p_4.$$

Now the description of the g^1_4 is simple: given a point $p_1 \in X$, there is a unique $Q \in \mathcal{F}_{r,s}$ passing through p_1. Then there is a map

$$f_{r,s} : X \longrightarrow \mathcal{F}_{r,s} \cong \mathbb{P}^1 \quad s.t. \quad p_1 \mapsto Q.$$

The fibre (over Q) is the divisor $p_1 + p_2 + p_3 + p_4$ considered above.

Notice that one of the fibres is $2r + 2s$, hence $f_{r,s}$ is one of the g^1_4 we are looking for.

In the same way taking the pencil $\mathcal{F}_{u,v}$ of conics tangent to t_r (resp. t_s) at r (resp. s), intersecting the conics with X and subtracting the divisor $2r + 2s$ we obtain the linear series $f_{u,v} : X \longrightarrow \mathcal{F}_{u,v} \cong \mathbb{P}^1$.

5.5.3 The curve of g^1_4's with two special fibres

We need to determine the curve on $\varphi(S) \subset \mathsf{G}^1_4(X)$ given by the g^1_4's on X with two fibres of type $2p + 2q$ (more than two is not possible).

In the case of linear series in E (the non-complete linear series), these clearly correspond to points which are in two bitangents.

In the other cases we have to understand when a map $f_{r,s}$ as above has a second fibre of the form $2x + 2y$. Thanks to the description of the previous Section, we know that actually we only have to look at

$$\begin{aligned}\Gamma := \{r + s \in X^{(2)} \mid \exists \text{ a conic } Q \text{ (of rank at least 2)} \\ \text{with } Q \cdot X = 2r + 2s + 2x + 2y\},\end{aligned}$$

i.e. we need to study conics which are tangent to X. This corresponds to find all line bundles $\mathcal{O}(D)$ with $\deg(D) = 4$ and $\mathcal{O}(2D) = \mathcal{O}_X(2)$ (notice that by adjunction formula $\omega_X = \mathcal{O}_X(1)$).

Consider the composition of maps

$$X^{(2)} \times X^{(2)} \xrightarrow{m} X^{(4)} \xrightarrow{s} \mathrm{Pic}^8(X),$$

where m is the addition of divisors and s is the "square" map $\sum p_i \mapsto \mathcal{O}_X(2\sum p_i)$. The map s is surjective since it is the composition of two surjective maps.

We observe that $s^{-1}(\omega_X^{\otimes 2})$ is the disjoint union of $2^{2 \cdot g(X)} = 2^6 = 64$ components. One is isomorphic to a projective plane and it is simply the canonical linear series. This component is rather uninteresting since it gives only double lines l^2. The other 63 components are projective lines corresponding to the paracanonical systems $|\omega_X \otimes \alpha|$, $\alpha \in JX_2 \smallsetminus \{0\}$. A divisor D in one of these lines is thus formed by 4 points not in a line and such that there is a conic intersecting X in $2D$. Define $\Gamma_\alpha := m^{-1}(|\omega_X \otimes \alpha|)$ for a non trivial 2-torsion point α. Then

$$\Gamma = \bigcup_{\alpha \in JX_2 \smallsetminus \{0\}} \Gamma_\alpha.$$

Since Γ does not contain points of $\mathcal{B}$, its preimage in S is isomorphic to Γ hence it is a disjoint union of curves in S that we still denote by Γ.

Call U_X the open set obtained subtracting to S the set Γ and the set of points in the exceptional divisors corresponding to points belonging to two bitangents.

5.5.4 The involution on $\mathsf{G}^1_4(X)$

We want to prove that the natural involution in $\mathcal{R}^{tr}_{3,sp}$, which exchanges the trigonal maps f_x and f_y, corresponds, via the trigonal construction, to the involution

$$\begin{aligned}i : \mathcal{M}^{tet}_{3,*} &\to \mathcal{M}^{tet}_{3,*} \\ (X, L) &\mapsto (X, \omega_X^{\otimes 2} \otimes L^{-1}).\end{aligned}$$

Remark 30. Notice that the involution in $\mathcal{R}b^{tr}_{3,sp}$ does not exchange the covering (it acts only on the trigonal series). Since the Prym variety of the covering is isomorphic to the Jacobian of the associated tetragonal curve we know that the involution i has to leave the curve X invariant.

Fixing the curve X, i acts on $\mathrm{Pic}^4(X)$. Moreover, since $i(\omega_X) = \omega_X$, the involution i lifts to $\mathsf{G}^1_4(X)$ and, by construction, it leaves $\varphi(S)$ invariant.

Proposition 5.5.1. *The involution i on $\mathrm{Pic}^4(X)$ lifts to an involution in $\mathsf{G}^1_4(X)$ and it acts as the identity on the exceptional divisor E.*

Proof. To simplify the notation put $\mathrm{Pic} = \mathrm{Pic}^4(X)$. The involution i has an isolated fixed point at ω_X. In fact, by definition of i, the fixed points are the line bundles L such that $L^{\otimes 2} = \omega_X^{\otimes 2}$. This happens if and only if $L = \omega_X \otimes \eta$, where η is a two torsion point.

The exceptional divisor E is equal to $\mathbb{P}(T_{\omega_X}\mathrm{Pic})$ so the action of i on E is given by the projectivisation of the differential of i at ω_X: di_{ω_X}. We claim that di_{ω_X} is $-Id$, hence it is the identity on $E = \mathbb{P}(T_{\omega_X}\mathrm{Pic})$. In fact by the linearisation theorem of Cartan, there exist local coordinates z in a neighborhood U of ω_X such that

$$i(z) = Az, \quad \text{where } A^2 = \mathrm{Id}\,.$$

Thus the eigenvalues of A are ± 1. But if there is an eigenvalue equal to 1, there would exist a space of fixed points which is positive dimensional. This leads to a contradiction.

□

Let us fix the quartic X as above and the two complete linear series $f_{r,s}$, $f_{u,v}$ such that r, s, u, v are on a line l.

Proposition 5.5.2. *The two linear series $f_{r,s}$, $f_{u,v}$ correspond by the involution i.*

To prove the equality of the involutions it is enough to prove the coincidence of both involutions for these examples since they are the generic elements.

Define (following Recillas, see Section 4.4.2):

$$\tilde{D}_{r,s} = \{a + b \in X^{(2)} \mid f_{r,s}(a) = f_{r,s}(b)\},$$

and the analogous for $\tilde{D}_{u,v}$. Then there are involutions $\sigma_{r,s}$ (and resp. $\sigma_{u,v}$) on the curves $\tilde{D}_{r,s}$ (and resp. $\tilde{D}_{u,v}$) sending each pair of points to the complement in the corresponding linear series. We denote by $\tilde{C}_{r,s}$ and $\tilde{C}_{u,v}$ the quotient (trigonal) curves. Recillas trigonal construction (see (4.4.6)) says that there are isomorphisms of principally polarized abelian varieties:

$$P(\tilde{D}_{r,s}, \tilde{C}_{r,s}) \cong JX \cong P(\tilde{D}_{u,v}, \tilde{C}_{u,v}).$$

The assignment explained with these steps

$$(X, f_{r,s}) \mapsto (\tilde{D}_{r,s}, \tilde{C}_{r,s}, M)$$

is the inverse of the trigonal construction. M is the g^1_3 on $\tilde{C}_{r,s}$ which sends

$$[p_1 + p_2] = [p_3 + p_4] \in \tilde{C}_{r,s}$$

to the corresponding conic in $\mathcal{F}_{r,s}$. Its fibre is of the form

$$\{[p_1 + p_2], [p_1 + p_3], [p_1 + p_4]\}.$$

Notice that here the tetragonal maps $f_{r,s}$ (and resp. $f_{u,v}$) come with two simple ramification points over l^2. This implies that $(\tilde{D}_{r,s}, \tilde{C}_{r,s})$ (and resp. $(\tilde{D}_{u,v}, \tilde{C}_{u,v})$) is a Beauville admissible double cover of type $(\star)$. This is due to the extended trigonal construction $\bar{T}_0$ studied by Donagi (see Theorem 4.4.7 and the correspondence among fibres there recalled).

Call $(D_{r,s}, C_{r,s})$ (and resp. $(D_{u,v}, C_{u,v})$) the normalizations of these coverings.

Proposition 5.5.3.

a) There is a canonical isomorphism $\tilde{D}_{r,s} \xrightarrow{\lambda} \tilde{D}_{u,v}$ compatible with the involutions: $\lambda \circ \sigma_{r,s} = \sigma_{u,v} \circ \lambda$. In particular there is an isomorphism $\tilde{C}_{r,s} \xrightarrow{\bar{\lambda}} \tilde{C}_{u,v}$.

b) Let $b \in \tilde{C}_{r,s}$ be the branch point of $\tilde{D}_{r,s} \xrightarrow{\pi_{r,s}} \tilde{C}_{r,s}$, then $b' := \bar{\lambda}(b)$ is the branch point of $\tilde{D}_{u,v} \longrightarrow \tilde{C}_{u,v}$.

c) There exist points $x \in C_{r,s}$ and $y \in C_{u,v}$ such that $|b_1 + b_2 + x|$ and $|b'_1 + b'_2 + y|$ are the corresponding trigonal series (where b_i and b'_i are the preimages of b and b' in the normalizations of $\tilde{C}_{r,s}$ and $\tilde{C}_{u,v}$).

d) There is an isomorphism $\mathcal{O}_{C_{u,v}}(b'_1 + b'_2 + \bar{\lambda}(x) + y) \cong \omega_{C_{u,v}}$.

Proof. Let $p_1 + p_2 \in \tilde{D}_{r,s}$. By definition $h^0(X, \mathcal{O}_X(2r + 2s - p_1 - p_2)) = 1$. Thus, by Serre duality we have that

$$\begin{aligned} 1 = & h^0(X, \omega_X(p_1 + p_2 - 2r - 2s)) = \\ & h^0(X, \mathcal{O}_X(r + s + u + v + p_1 + p_2 - 2r - 2s)) = \\ & h^0(X, \mathcal{O}_X(u + v + p_1 + p_2 - r - s)). \end{aligned}$$

Let $q_1 + q_2 \in |u + v + p_1 + p_2 - r - s|$. Let us see that $q_1 + q_2 \in \tilde{D}_{u,v}$. Indeed:

$$\begin{aligned} & h^0(X, \mathcal{O}_X(2u + 2v - q_1 - q_2)) = \\ & h^0(X, \mathcal{O}_X(2u + 2v - u - v - p_1 - p_2 + r + s)) = \\ & h^0(X, \mathcal{O}_X(u + v + r + s - p_1 - p_2)) = 1. \end{aligned}$$

Therefore the map $\tilde{D}_{r,s} \xrightarrow{\lambda} \tilde{D}_{u,v}$ given by

$$\lambda(p_1 + p_2) = q_1 + q_2 \sim p_1 + p_2 + u + v - r - s,$$

is well defined and the compatibility with the involutions is an exercise. This proves a). Observe that b) is an obvious consequence once we notice that $\sigma_{r,s}$ has a unique fixed point given by $r + s$. The same occurs in $u + v$ for $\sigma_{u,v}$. From point a) we know that

$\lambda(r+s) = u+v$. Thus, calling b and b' the images of $r+s$ (resp. $u+v$) in $\tilde{C}_{r,s}$ (resp. in $\tilde{C}_{u,v}$), we get

$$\bar{\lambda}(b) = b'.$$

To prove c) we refer to the description of the extended trigonal construction $\bar{T}_0$ given by Donagi. Indeed, we have that the fibre of the 3:1 map $\tilde{C}_{r,s} \to \mathbb{P}^1$ over l^2 consists of a node in b and an additional point $x = \pi_{r,s}(2r) = \pi_{r,s}(2s)$. The normalization of $\tilde{C}_{r,s}$ gives the trigonal series $|b_1 + b_2 + x|$. The same occurs for $\tilde{C}_{u,v}$ calling $y = \pi_{u,v}(2u) = \pi_{u,v}(2v)$.

Finally we conclude with d). First notice that with an abuse of notation we are still calling $\bar{\lambda}$ the isomorphism induced between the normalized curves $C_{r,s} \to C_{u,v}$. Then consider $C_{r,s}$ and $C_{u,v}$ as quartic plane curves and the canonical divisors obtained intersecting $C_{r,s}$ (resp. $C_{u,v}$) with the line $b_1 + b_2$ (resp. $b_1' + b_2'$). Thus we get

$$K_{C_{r,s}} = x + b_1 + b_2 + z \quad \text{and} \quad K_{C_{u,v}} = y + b_1' + b_2' + w.$$

Now we have two possibilities:

$$w = \bar{\lambda}(x) \quad \text{or} \quad w = \bar{\lambda}(z).$$

We claim that $w = \bar{\lambda}(x)$. In fact if $w = \bar{\lambda}(z)$, since by construction $x = \pi_{r,s}(2r) = \pi_{r,s}(2s)$ then we would have $\lambda(2r) = 2u$ or $\lambda(2r) = 2v$. But this contradicts the definition of λ given above. Hence we get $\mathcal{O}_{C_{u,v}}(b_1' + b_2' + \bar{\lambda}(x) + y) \cong \omega_{C_{u,v}}$.

□

Remark 31. The isomorphism of Proposition 5.5.3[d)], gives the compatibility between the two trigonal maps f_x and f_y defined for the general element of $\mathcal{R}b_{3,sp}^{tr}$ and the two trigonal maps obtained on $C_{r,s}$ (resp. $C_{u,v}$) projecting from x or from z (resp. from $\bar{\lambda}(x)$ or from y).

5.5.5 The Fibre

At this point we have all the tools to explain diagram 5.9. Hence we are ready to state the following:

Theorem 5.5.4. *The fibre of $\mathcal{P}_{3,2}$ at a generic JX is isomorphic to the quotient of $\varphi(\mathsf{U}_X) \subset \mathsf{G}_4^1(X)$ by the involution i.*

Proof. Starting with a general 3-dimensional abelian variety, i.e. the Jacobian of a curve X, diagram (5.9) says that the fibre of $\mathcal{P}_{3,2}$ over JX is described by the fibre over X of the map $\mathcal{M}_{3,*}^{tet} \to \mathcal{M}_3$. Thus we need to look for all tetragonal maps $k : X \to \mathbb{P}^1$ which have an étale point on every fibre and only a fibre with exactly two ramification points of order 2.

In order to obtain such k's, we consider the map φ_0 in (5.10) and we look at its image in $\mathrm{Pic}^4(X)$. The blow up S of $X^{(2)}$ at $\mathcal{B}$ recovers all tetragonal maps obtained as projections from points on bitangent lines. Hence, considering the open set U_X, we avoid tetragonal maps which have two fibres of type $2p + 2q$ (which are not allowed by the trigonal construction).

Finally, since $\mathcal{R}b^{tr}_{3,sp}$ has an involution which exchanges the two special trigonal series, we let i act on $\mathcal{M}^{tet}_{3,*}$ to identify the two tetragonal maps on X which correspond (by the isomorphism (4.13)) to the trigonal maps f_x and f_y and we denote by $\mathcal{M}q^{tet}_{3,*}$ the corresponding moduli space. Letting i act on $\varphi(\mathsf{U}_X)$, we obtain the fibre over JX.

□

5.6 Case $g = 4, r = 2$

In this last case we identify $\mathcal{R}_{4,2}$ with $\Delta^{n,0}$, the set of isomorphism classes of irreducible admissible coverings of curves of arithmetic genus 5 with exactly one node. The identification is done through the procedure explained in Example 1. Notice that $\Delta^{n,0}$ is a dense open set of an irreducible divisor Δ^n in the boundary of $\bar{\mathcal{R}}_5$.

Thus we take care of the map

$$\mathcal{P}_{4,2} : \mathcal{R}_{4,2} \to \mathcal{A}_4$$

using Beauville's proper extension

$$\bar{\mathcal{P}}_5 : \bar{\mathcal{R}}_5 \to \mathcal{A}_4$$

and guided by Donagi's description of its generic fibre (see [25, Section 5]).

5.6.1 Donagi's Construction

In [25], the author defines a birational map

$$\kappa : \mathcal{A}_4 \dashrightarrow \mathcal{RC}^+,$$

where $\mathcal{RC}^+$ is the moduli space of pairs (V, δ), where V is a smooth cubic threefold V and $\delta \in JV_2$ a non-zero 2-torsion point in the intermediate Jacobian JV with a "parity" condition that we will explain later. An explicit open set in $\mathcal{A}_4$ where κ is an isomorphism is given in [50].

Let $F(V)$ be the Fano surface of lines in V and, recalling that

$$Pic^0(F(V)) \cong JV,$$

let

$$\tau : \widetilde{F(V)} \to F(V)$$

be the double covering attached to δ.

We have the following:

Theorem 5.6.1 ([25], §5.1). *Let $A \in \mathcal{A}_4$ be a generic abelian 4-fold and let $\widetilde{(V, \delta)}$ be its image through k. Then the fibre of $\kappa \circ \bar{\mathcal{P}}_5$ at (V, δ) is isomorphic to the surface $\widetilde{F(V)}$, which is the unramified double covering of $F(V)$ attached to δ.*

The key point of the proof goes around a triangular diagram ([25], Figure (5.7)). We would like to recall it and to sketch the proof in order to make easier the study of the ramified Prym map which we are interested in.

Let $A \in \mathcal{A}_4$ be a generic abelian fourfold and put $\kappa(A) = (V, \delta)$.

Choose a generic line $l \in F(V)$ and denote by $\pi_l : \widetilde{Q_l} \longrightarrow Q_l$ the admissible double covering attached to the conic bundle structure on V provided by l. Then Q_l is a smooth quintic plane curve and $P(\widetilde{Q_l}, Q_l) \cong JV$ (as seen in 4.1.2).

Let $\sigma \in (JQ_l)_2$ be the 2-torsion point that determines π_l. Then, by the general theory of Prym varieties (see [71, page 332, Corollary 1]), there is an exact sequence

$$0 \longrightarrow \langle\sigma\rangle \longrightarrow \langle\sigma\rangle^{\perp} \longrightarrow P(\widetilde{Q_l}, Q_l)_2 = JV_2 \longrightarrow 0, \tag{5.11}$$

where $\langle\sigma\rangle^{\perp} \subset (JQ_l)_2$ is the orthogonal with respect to the Weil pairing. Denote by ν a preimage of the fixed 2-torsion point δ in JV_2, then there is another preimage

$$\nu' := \nu + \sigma.$$

These three elements define an isotropic subgroup W_l of rank 2 on JQ_l, that is a 2-dimensional $\mathbb{Z}/2$-subspace of $(JQ_l)_2$ on which the intersection pairing $\langle\ ,\ \rangle$ is identically zero.

The parity condition of $(V, \delta) \in \mathcal{RC}^+$ means that

$$h^0(Q_l, \mathcal{O}_{Q_l}(1) \otimes \nu) \quad \text{and} \quad h^0(Q_l, \mathcal{O}_{Q_l}(1) \otimes \nu')$$

are even. Thus there are two curves of genus 5, $\tilde{C}$ and $\tilde{C}'$, such that

$$J\tilde{C} \cong P(Q_l, \nu) \quad \text{and} \quad J\tilde{C}' \cong P(Q_l, \nu').$$

Indeed, this is due to the 1-1 correspondence

$$\{\text{curves C of genus 5}\} \leftrightarrow \left\{ \begin{array}{c} [Q, \alpha] \ \text{ s.t.} \\ \alpha \in \operatorname{Pic}^0(Q) \smallsetminus \mathcal{O}_Q,\ \alpha^{\otimes 2} \cong \mathcal{O}_Q \ \text{ and} \\ h^0(Q, \mathcal{O}_Q(1) \otimes \alpha) \text{ is even} \end{array} \right\},$$

where (Q, α) corresponds to the 2:1 unramified coverings

$$W^1_4(C) \to Q := W^1_4(C)/i \quad \text{with} \quad i(L) = \omega_C \otimes L^{-1}.$$

It is possible to show that Q is a plane quintic and $JC \cong P(Q, \alpha)$.

Looking at Donagi's diagram (Figure 5.2), the two curves $\tilde{C}$ and $\tilde{C}'$ give two lines that intersect in the point Q_l.

Using for $P(Q_l, \nu)$ an exact sequence similar to (5.11), we get:

$$0 \longrightarrow \langle\nu\rangle \longrightarrow \langle\nu\rangle^{\perp} \longrightarrow P(Q_l, \nu)_2 = J\tilde{C}_2 \longrightarrow 0.$$

Therefore the rank 2 subgroup $W_l \subset \langle\nu\rangle^{\perp}$ determines on $J\tilde{C}$ a 2-torsion point μ. Similarly there is a $\mu' \in J\tilde{C}'_2$. Denoting with λ the sheet interchange for the covering τ, Donagi proves that:

$$\lambda(\tilde{C}, \mu) = (\tilde{C}', \mu') \qquad \text{and} \qquad P(\tilde{C}, \mu) \cong P(\tilde{C}', \mu') \cong A.$$

This shows that the preimages of l by τ are the elements $(\tilde{C}, \mu), (\tilde{C}', \mu')$ obtained previously and thus it concludes the proof.

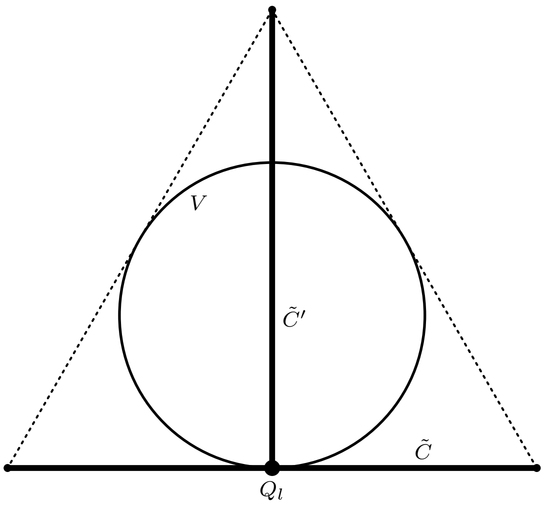

Figure 5.2: Donagi's description of the generic fibre of $\bar{\mathcal{P}}_5$.

5.6.2 Discriminant curves and quadrics

Our aim is to identify which elements of $\widetilde{F(V)}$ correspond to admissible irreducible double coverings of nodal curves. In other words, we want to find the intersection:

$$\widetilde{F(V)} \cap \Delta^{n,0}.$$

In fact, we prove (see Proposition 5.6.6 below) that the image of this intersection by the double covering

$$\tau : \left(\kappa \circ \bar{\mathcal{P}}_5\right)^{-1}(V, \delta) = \widetilde{F(V)} \to F(V)$$

lies in a curve Γ already considered in the literature. It is related to the geometry of the cubic 3-folds and to the following Beauville's result:

Proposition 5.6.2 ([8], §1.2). *Let V be a cubic 3-fold with l a line in V. Let $\pi_l : \tilde{V} \to \mathbb{P}^2$ be a conic bundle and let Q_l be the discriminant curve. Then:*

- *The curve Q_l has at most ordinary double point as singularities.*
- *If p is a regular point in Q_l, then the corresponding conic is formed by two different lines.*
- *If p is a node of Q_l, then the corresponding conic is formed by a double line.*

This justifies the interest in the following set of lines:

$$\Gamma := \{l \in F(V) \mid \exists \text{ a plane } \Pi \text{ and a line } r \in F(V) \text{ with } V \cdot \Pi = l + 2r\},$$

which parametrizes lines l for which the discriminant curve Q_l is singular. We recall the following:

Proposition 5.6.3 ([75]). *Let V be a generic cubic threefold. Then the curve Γ is smooth and irreducible and belongs to the bicanonical system of $F(V)$.*

Proposition 5.6.4 ([8]). *For any $l \in \Gamma$, the discriminant curve Q_l has only one node.*

Let us denote by $\widetilde{\Gamma}$ the curve $\tau^{-1}(\Gamma)$. Then we have the following technical:

Lemma 5.6.5. *The curve $\widetilde{\Gamma}$ is irreducible.*

Proof. Indeed, otherwise it would be the union of two disjoint curves G_1, G_2. Using the projection formula with respect to the 2:1 map τ we get that $G_i^2 = \Gamma^2 = 4 \cdot K_{F(X)}^2 > 0$. By the Index Theorem $G_1^2 > 0$ and $G_1 \cdot G_2 = 0$ imply $G_2^2 \leq 0$, a contradiction.

□

Since the fibre we are looking for is one dimensional, the next result implies that it is an open set of $\widetilde{\Gamma}$:

Proposition 5.6.6. *For a generic cubic threefold V we have that*

$$\tau(\widetilde{F(V)} \cap \Delta^{n,0}) \subset \Gamma.$$

We have two proofs for this fact. The first follows closely Donagi's description of the fibre and concludes that if $l \in F(V) \setminus \Gamma$ then $\tau^{-1}(l)$ is given by the smooth coverings $(\tilde{C}, \mu)$ and $(\tilde{C}', \mu')$. Therefore $\tau^{-1}(l) \subset \mathcal{R}_5$.

The second proof, which takes all the remaining part of this Section, is more constructive and more useful for our purposes. We show directly that for a covering in $\Delta^{n,0}$ the corresponding line l belongs to Γ. This approach relies on the following result of Izadi (see [50, Theorem 6.13]):

Theorem 5.6.7. *Let (V, δ) be a generic smooth cubic threefold and let $\pi^* : D^* \to C^*$ be an admissible covering in the fibre of (V, δ). Assume that $\tau(\pi^*) = l \in F(V)$. Then the discriminant quintic Q_l of the conic bundle structure attached to l parametrizes the set of singular quadrics through the canonical model of C^*.*

By canonical model we mean the image of C^* by the morphism attached to the dualizing sheaf.

Remark 32. The line l attached to π^* is defined in [50] in a different way. However it is proved in 6.30 in loc. cit. that it equals $\tau(\pi^*)$.

Let $\pi : D \longrightarrow C$ be an element in $\mathcal{R}_{4,2}$ and denote by $\pi^* : D^* \longrightarrow C^*$ the corresponding admissible covering in $\Delta^{n,0} \subset \bar{\mathcal{R}}_5$. By definition

$$C^* = C/b_1 \sim b_2$$

is a curve of arithmetic genus 5 with a node in p obtained by glueing the two branch points b_1, b_2 of π (as usual see technique of Example 1).

Theorem 5.6.7 tells us that we have to study the set of singular quadrics through the canonical model of the curve C^*. This leads to the following quite computational Lemma. We warn the reader that the proof is a little bit long and it requires a technical result which is included (see Proposition 5.6.9 below).

Lemma 5.6.8.

a) The quintic plane curve parametrizing the singular quadrics containing the image of the canonical map of C^* is a quintic with exactly one node. In particular $\tau(\pi^*) \in \Gamma$.

b) The quintic plane curve parametrizing the singular quadrics containing the canonical image of an arithmetic genus 5 curve with at least two nodes is a nodal quintic with at least two nodes.

Proof. Observe that it is enough to prove $a)$ assuming that π is a general element of $\mathcal{R}_{4,2}$. In particular we assume that C is not trigonal.

The map $\varphi : C \to \mathbb{P}(H^0(C, \omega_C(b_1 + b_2))^*)$ satisfies $\varphi(b_1) = \varphi(b_2)$ and it is an isomorphism out these two points. Hence $\varphi(C) = C^*$ and φ can be seen as the normalization $n : C \to C^*$ composed with the inclusion $C^* \subset P(H^0(C, \omega_C(b_1 + b_2)))^* = \mathbb{P}^4$.

We have the following exact sequence:

$$0 \to \omega_{C^*} \to n_*(\omega_C(b_1 + b_2)) \to \mathbb{C}_p \to 0, \qquad (5.12)$$

which induces

$$0 \to H^0(C^*, \omega_{C^*}) \to H^0(C, \omega_C(b_1 + b_2)) \xrightarrow{res} \mathbb{C} \to \mathbb{C} \to 0, \qquad (5.13)$$

where res is the map $\omega \mapsto \mathrm{res}_{b_1}\omega + \mathrm{res}_{b_2}\omega$. By the residue theorem it vanishes identically. Therefore

$$H^0(C^*, \omega_{C^*}) \cong H^0(C, \omega_C(b_1 + b_2)).$$

Now let L be a g^1_3 on C and consider bases

$$H^0(C, L) = \langle t_1, t_2 \rangle, \qquad H^0(C, \omega_C \otimes L^{-1}) = \langle s_1, s_2 \rangle.$$

Put

$$\omega_1 = t_1 s_1 \quad \omega_2 = t_2 s_1 \quad \omega_3 = t_1 s_2 \quad \omega_4 = t_2 s_2,$$

to get

$$H^0(C, \omega_C) = \langle \omega_1, \omega_2, \omega_3, \omega_4 \rangle \ \text{ and } \ H^0(C, \omega_C(b_1 + b_2)) = \langle \omega_1, \omega_2, \omega_3, \omega_4, \omega_5 \rangle.$$

We obtain the following diagram:

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[dr, "g"'] & \mathbb{P}^4 \arrow[d, dashed] \\ & \mathbb{P}^3 \end{tikzcd} \qquad (5.14)$$

where g is the canonical map and the vertical rational map is given by dualizing the inclusion $H^0(K_C) \subset H^0(K_C(b_1+b_2))$. It corresponds to the projection from the point p.

Since C is a general curve of genus 4, there exists a unique quadric Q containing its canonical model and it has rank 4, namely:

$$Q = \omega_1 \odot \omega_4 - \omega_2 \odot \omega_3.$$

In particular, in the chosen coordinates, $\varphi(b_i) = p = [0:0:0:0:1]$ $(i=1,2)$ and

$$Q = \{x_1x_4 - x_2x_3 = 0\}.$$

The preimage of Q by the projection is a cone with vertex p which contains C^* and has rank four (and in fact the same equation). We still call it Q.

Using now

$$0 \to \omega_{C^*}^{\otimes 2} \to n_*(\omega_C^{\otimes 2}(2b_1+2b_2)) \to \mathbb{C}_p \to 0 \tag{5.15}$$

and its corresponding long exact sequence in cohomology, we obtain that also in the case of a nodal curve of arithmetic genus 5

$$\dim I_2(\omega_{C^*}) = 3.$$

Taking Q, Q_1, Q_2 as a basis, we would like to show that the discriminant curve Δ of the family of quadrics

$$\mathbb{P}(I_2(K_{C^*})) = \mathbb{P}(\langle Q, Q_1, Q_2\rangle)$$

is nodal. By the above considerations $p \in S(Q) \cap C^*$, where $S(\cdot)$ denote the singular locus.

In the paper [88], Wall studied the discriminant locus of nets of quadrics. In particular [88, Lemma 1.1] ensures that $([1:0:0], p)$ belongs to $S(N)$, where

$$N := \{([\lambda_0 : \lambda_1 : \lambda_2], x) \mid x^t(\lambda_0 A_0 + \lambda_1 A_1 + \lambda_2 A_2)x = 0\} \subset \mathbb{P}(\langle Q, Q_1, Q_2\rangle) \times \mathbb{P}^4$$

is the universal family of the net of quadrics containing C^* ($A_i, i = 0,1,2$ are the matrices associated to Q, Q_1, Q_2). To be more precise we would have to write $Q = Q_{[1:0:0]}$ and the analogous for other Q_i. We will omit the subscript when it will be possible.

Assuming that every point in $S(C^*)$ is tame (we give the definition below), the map

$$S(N) \to S(C^*) \quad \text{defined as} \quad (\lambda = [\lambda_0 : \lambda_1 : \lambda_2], x) \mapsto x,$$

becomes bijective. Since, in our case, $S(C^*) = \{p\}$, we obtain that

$$S(N) = \{([1:0:0], p)\}.$$

Moreover $p \in S(C^*)$ and $([1:0:0], p) \in S(N)$ have the same type of singularity (by [88, Proposition 1.3]).

Proposition 5.6.9. *The map*

$$\rho : N \to \mathbb{P}^2$$
$$(\lambda, x) \mapsto \lambda$$

sends $S(N)$ *to* $S(\Delta)$.

Proof. First observe that since

$$\rho^{-1}(\Delta) = \{(x, \lambda) : x \in Q_\lambda \text{ and } Q_\lambda \text{ is singular}\},$$

we get

$$S(N) \subseteq \rho^{-1}(\Delta).$$

We conclude with the Jacobian criterion. Indeed, using local coordinates for which a singular point of N is $([1:0:0], [0:0:0:0:1])$, we have:

$$Q \leftrightarrow \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; Q_1 \leftrightarrow \begin{pmatrix} & & \\ & * & \\ & & 0 \end{pmatrix}, \; Q_2 \leftrightarrow \begin{pmatrix} & & \\ & * & \\ & & 0 \end{pmatrix},$$

since p is a point in all quadrics of the net and Q is singular in p. Therefore $\lambda_0 A_0 + \lambda_1 A_1 + \lambda_2 A_2$ has a 0 in position $(5,5)$, homogeneous linear polynomials $l = l(\lambda_1, \lambda_2)$ out of the diagonal and $l + a_i$ on the diagonal ($i = 1,2,3,4$).

Put $G(\lambda_0, \lambda_1, \lambda_2) = det(\lambda_0 A_0 + \lambda_1 A_1 + \lambda_2 A_2)$. Then it is possible to write

$$G = f_5 + \lambda_0 f_4 + \lambda_0^2 f_3 + \lambda_0^3 f_2,$$

where f_i are homogeneous polynomials in (λ_1, λ_2) of degree i. Therefore $\partial_\lambda G(p) = 0$, i.e. G is singular and thus $[1:0:0]$ belongs to $S(\Delta)$.

□

Applying [88, Theorem 1.4] for $([1:0:0])$ in $S(\Delta)$ (and resp. $([1:0:0], p) \in S(N)$), we conclude that the discriminant locus of N has a unique nodal point, as claimed.

It only remains to show that p is tame. By definition a point of C^* is *tame* if the tangent planes to C^* at the point span a 2-dimensional vector space. We check that p is tame: call $\pi_i, i = 0,1,2$ the tangent planes of the three quadrics at p. In coordinates:

$$\pi_i : (0:0:0:0:1) A_i \mathbf{y} = 0, \qquad i = 0,1,2.$$

Thus:

$$pA_0\mathbf{y} = (0:0:0:0:1) \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{y} = 0$$

and

$$pA_i\mathbf{y} = 0 \Leftrightarrow (a^i_{5,1}, a^i_{5,2}, a^i_{5,3}, a^i_{5,4}, a^i_{5,5})\mathbf{y} = 0,$$

where $a^i_{5,j}$ are the coefficients of the last row of the matrices $A_i, i = 1, 2$. Call these vectors $\mathbf{a_1}, \mathbf{a_2}$. Then p is tame if

$$\dim\langle \mathbf{a_1}, \mathbf{a_2}\rangle = 2.$$

Suppose, by contradiction, that $\mathbf{a_2} = \mu\mathbf{a_1}$. Then $Q' : \mu A_1 - A_2$ belongs to $I_2(K_C)$, so $Q' = \nu Q$. Therefore

$$0 = \nu Q - Q' = \nu Q - \mu Q_1 + Q_2,$$

which is impossible. This concludes the proof of Lemma 5.6.8, part $a)$.

In order to prove part $b)$, let us start with an admissible double-nodal covering $\pi^{2*} : D^{2*} \to C^{2*}$, i.e. $S(C^*) = \{p_1, p_2\}$. Consider the following partial normalization maps:

$$\begin{tikzcd} N_1 \arrow[r, "\alpha"] \arrow[rr, bend left, "n"] & \tilde{N}_1 \arrow[r, "\beta"] & C^{2*} \end{tikzcd}$$

where n is the normalization, β is the partial normalization of the node in p_2 while α of the one in p_1.

A short exact sequence for $\omega_{\tilde{N}_1}$ similar to (5.12) ensures that

$$\dim I_2(\omega_{\tilde{N}_1}) = \dim I_2(\omega_{N_1}(q_1 + q_1')) = 1,$$

where q_1, q_1' are the two points of N_1 sent to p_1 by α.

We remark that the unique quadric Q containing the image of

$$N_1 \to \tilde{N}_1 \subset \mathbb{P}(H^0(\omega_{N_1}(q_1 + q_1')))$$

cannot be singular in p_1. Otherwise, if we write in local coordinates

$$Q = \sum_{i,j\le 4} a_{ij}x_ix_j \quad \text{and} \quad p_1 = [0 : 0 : 0 : 1],$$

we would get $\partial_i Q(p_1) = a_{i4} = 0$ for every i. But this would imply $Q \in I_2(\omega_{N_1})$, which is impossible since $I_2(\omega_{N_1}) = 0$. Then, dualizing the inclusion

$$H^0(\omega_{\tilde{N}_1}) \subset H^0(\omega_{\tilde{N}_1}(q_2 + q_2')) \quad \text{(points } q_2, q_2' \text{ are identified in } p_2 \in C^{2*}),$$

we obtain a diagram as (5.14) where the rational map $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ is given by the projection from p_2. The preimage of Q is a cone with vertex p_2 which is smooth in p_1 and which contains our curve with two nodes. With an abuse of notation, we still denote it by Q.

The short exact sequence (5.15) for the bicanonical $\omega^{\otimes 2}_{C^{2*}}$ of C^{2*} shows that $\dim I_2(\omega_{C^{2*}}) = 3$. Call, as before, $N \subset \mathbb{P}(I_2(\omega_{C^{2*}})) \times \mathbb{P}^4$ the universal family of the net of quadrics containing C^{2*}. Thus, Q is the point $\lambda = [1 : 0 : 0]$ in $\mathbb{P}^2$.

Since C^{2*} has two singular points, [88, Lemma 1.2] shows that there exists $\mu \in \Delta$ (the discriminant curve of the net $\mathbb{P}(I_2(\omega_{C^{2*}}))$ such that $p_1 \in S(Q_\mu)$. Hence $Q \neq Q_\mu$. This concludes the proof of the case of two nodes: (λ, p_2) and (μ, p_1) belong to $S(N)$. Therefore the map ρ of Proposition 5.6.9 determines two different singular points in Δ.

The cases $\#S(C^*) = 3, 4$ are similar: the partial normalization at one point leads to a curve of arithmetic genus 4 with singular points. Call one of them p_1. As above we find a quadric Q in I_2 which is a cone with vertex p_1 on a quadric which is smooth in at least one of the remaining nodes. Applying Wall's theorems, we know the existence of another quadric which is singular in at least one among the other nodes. This leads to a discriminant curve Δ which has at least two singular points.

□

5.6.3 The fibre

Previous Sections explain how to use Donagi's description of the fibre of $\bar{\mathcal{P}}_5$ over (V, δ) to look for the elements in the boundary. This justifies the study of the curve Γ. Moreover Izadi's description of the discriminant quintics attached to the conic bundle structure provided by V leads to the study of quadrics containing a nodal canonical curve of genus 5 (which, actually, could be interesting on its own). Hence we state the following:

Theorem 5.6.10. *The generic fibre of $\mathcal{P}_{4,2}$ at (V, δ) is isomorphic to $\widetilde{\Gamma}$.*

Proof. Take $\pi : D \to C$ in $\mathcal{R}_{4,2}$ and denote, as above, π^* the corresponding element in $\Delta^{n,0}$. Lemma 5.6.8a) shows that $\tau(\pi^*)$ belongs to Γ. Therefore, in order to show that the generic fibre of $\mathcal{P}_{4,2}$ at (V, δ) is isomorphic to $\widetilde{\Gamma}$ and not only contained as an open subset, it remains to prove that $\widetilde{\Gamma}$ doesn't include elements of the boundary of $\bar{\mathcal{R}}_5$ which are not admissible covering of irreducible curves with exactly one node. Since $A \in \mathcal{A}_4$ is generic, we can suppose A simple and hence, using [7], we can just take into account coverings of irreducible curves. Finally, the inclusion $\Delta^{n,0} \subset \Delta^n$ guarantees that we only have to take care of admissible coverings of irreducible curves with more than one node. Therefore, suppose by contradiction that $\widetilde{\Gamma}$ contains admissible coverings of irreducible curves with (at least) two nodes. Lemma 5.6.8b) gives us a quintic plane curve with at least two nodes. This contradicts the assumption on Γ and thus we can conclude.

□

5.7 Fibres of the Prym map and Shimura varieties

This section creates a link between Part I and Part II of this Thesis. Indeed, here we characterize some irreducible components of certain fibres of the ramified Prym maps as Shimura subvarieties of $\mathcal{A}_g$. To do this we refer to some explicit examples.

Recall that, as treated in Chapter 2, first in [69], [70] and then in [32], [36] examples of Shimura subvarieties of $\mathcal{A}_g$ generically contained in the Torelli locus have been constructed as families of Jacobians of Galois covers of $\mathbb{P}^1$ or of elliptic curves. In these works there have been constructed (to be precise) 32 examples of Shimura subvarieties of $\mathcal{T}_g$. All are collected in [32] and in [36].

Some of them are contained in (or equal to) fibres of ramified Prym maps, as we will see below.

As seen in Part I of this Thesis, in [34] (see Chapter 2, Theorem 2.3.16) and [44] infinitely many examples of totally geodesic and of Shimura varieties generically contained in the Torelli locus have been constructed as fibres of ramified Prym map. In particular, if we focus on degree 2 Prym maps, we refer to $\mathcal{P}_{1,2}$ and to $\mathcal{P}_{1,4}$.

Indeed, the images of $\mathcal{R}_{1,2}$ (and resp. $\mathcal{R}_{1,4}$) in $\mathcal{M}_2$ (and resp. in $\mathcal{M}_3$) are the bielliptic loci in genus 2 (and resp. in genus 3). In [36] they are denoted as families $(1e)$ (and resp. $(2e)$) and it is shown that their images in $\mathcal{A}_2$, resp. $\mathcal{A}_3$, via the Torelli maps, are Shimura subvarieties. In [34] it is proven the following:

Proposition 5.7.1. *The irreducible components of the fibres of the Prym maps $\mathcal{P}_{1,2}$, $\mathcal{P}_{1,4}$ are totally geodesic curves. Moreover, $(1e)$ and $(2e)$ contain a dense subset of CM-points, since are Shimura. Hence countably many of the fibres are Shimura curves.*

For the proof we refer the reader to Corollary 2.3.17.1.

A detailed analysis of the families found in [32], [36] (as done in Chapter 2, Section 2.4) characterizes some of the fibres of the Prym maps described in this Chapter as totally geodesic subvarieties of $\mathcal{A}_g$. Indeed we show the following:

Proposition 5.7.2.

a) *Family $(7) = (23) = (34)$ of [32] is a fibre of the Prym map $\mathcal{P}_{1,4}$, which is a Shimura curve.*

b) *Family (24) of [32] is contained in a fibre of the Prym map $\mathcal{P}_{2,2}$*

Proof. For the proof of part $a)$ we refer to Proposition 2.4.2, part i. Let us see $b)$: (24) is a family of curves D of genus 4 with an action of a group

$$G = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/3 = \langle g_1, g_2, g_3 : \ g_1^2 = g_2^2 = 1, \ g_3^3 = 1 \rangle$$

and quotient

$$\pi : D \to D/G \cong \mathbb{P}^1.$$

The branch locus B consists of 4 distinct points. The monodromy map of the covering is

$$\theta : \pi_1(\mathbb{P}^1 - B) \cong \langle \gamma_1, ..., \gamma_4 \mid \gamma_1\gamma_2\gamma_3\gamma_4 = 1 \rangle \to G$$
$$\theta(\gamma_1) = g_2, \quad \theta(\gamma_2) = g_1 g_2,$$
$$\theta(\gamma_3) = g_3, \quad \theta(\gamma_4) = g_1 g_3^2,$$
$$\text{with } \mathbf{m} = (2, 2, 3, 6).$$

Using the notation of MAGMA, we can decompose

$$H^0(D, \omega_D) \cong V_6 \oplus V_7 \oplus V_8 \oplus V_{11},$$

where V_i are irreducible $\mathbb{C}$-representations of G all 1-dimensional (the group is abelian). Moreover, the group algebra decomposition gives us a decomposition of the Jacobian JC up to isogeny:

$$JC \sim B_6 \times B_7 \times B_8,$$

where B_6 and B_8 are 1-dimensional, while $\dim B_7 = 2$.

One easily checks that the map $D \to D/\langle g_1 \rangle$ is a double covering of a genus 2 curve C, ramified over 2 points. Moreover the Prym variety $P(D, C)$ is isogenous to $E \times E'$ where

$$E = D/\langle g_2 \rangle \sim B_6 \quad \text{and} \quad E' = D/\langle g_1 g_2 \rangle \sim B_8.$$

Indeed

$$H^0(D, \omega_D)^{\langle g_2 \rangle} = V_6 \quad \text{and} \quad H^0(D, \omega_D)^{\langle g_1 g_2 \rangle} = V_8.$$

The curves E and E' do not move, since the Galois covers $E \to E/(G/\langle g_2 \rangle) = D/G = \mathbb{P}^1$ and $E' \to E'/(G/\langle g_1 g_2 \rangle) = D/G = \mathbb{P}^1$ both have only 3 critical values. This shows that the family of covers $D \to C$ is contained in a fibre of the Prym map $\mathcal{P}_{2,2}$.

□

Finally, for the sake of completeness, we present a result related to a Prym map not involved in the study of the fibre done in this Chapter. Let us define $\mathcal{R}_g(d)$ the moduli space of degree d étale coverings $\tilde{C} \to C$ of curves of genus g and let $P(\tilde{C}, C)$ be the associated Prym variety defined, as usual, as the connected component containing the zero of the kernel of the norm map. Let us adopt the same notation as in [56]: let δ be the type of the induced polarization on $P(\tilde{C}, C)$ by $J\tilde{C}$ and let $\mathcal{B}^\delta$ be the the component of the moduli space of abelian varieties of dimension $(d-1)(g-1)$ with polarization of type δ compatible with the action of the group $\mathrm{Aut}(\tilde{C}, C)$. Then

$$\mathcal{P}_g(d) : \mathcal{R}_g(d) \to \mathcal{B}^\delta$$

is the associated Prym map. The following holds:

Theorem 5.7.3 (Lange-Ortega, [56]). *Let the notations be as above. Then the Prym map $\mathcal{P}_g(d)$ is dominant and generically finite exactly in the following cases:*

$$\begin{aligned}(g, d) &= (6, 2),\\ &= (4, 3),\\ &= (2, 7).\end{aligned}$$

Our analysis shows the existence of a “special” positive dimensional fibre for a Prym map not occurring in this list. Indeed we have the following:

Proposition 5.7.4. *Family* $(25) = (38)$ *of is contained in a fibre of the Prym map*

$$\mathcal{P}_3(3) : \mathcal{R}_2(3) \to \mathcal{A}_2,$$

where $\mathcal{R}_2(3)$ *is the moduli space of étale coverings of degree 3.*

Proof. Let us borrow from Chapter 2 the data for this family:

$$G = \mathbb{Z}/3 \times S_3 \text{ with } g_1 = ([0]_3, (12)),\ g_2 = ([1]_3, (1)) \text{ and } g_3 = ([0]_3, (123)).$$
$$\mathbf{x} = (g_1 g_3^2, g_1 g_3, g_2 g_3, g_2^2), \quad \mathbf{m} = (2, 2, 3, 3).$$

We know that $H^0(\tilde{C}, \omega_{\tilde{C}}) \cong V_3 \oplus V_4 \oplus V_8$ and that the Jacobian decomposes as $J\tilde{C} \sim B_3 \times B_8^2$, where the first term is 2-dimensional while the second is 1-dimensional. It is easy to check that $\tilde{C} \to \tilde{C}/\langle g_3 \rangle \cong \mathbb{Z}/3$ is an unramified covering map of degree 3 of a genus 2 curve C. Moreover we have

$$\dim(V_3^{\langle g_3 \rangle}) = 1 = s_{V_3} \quad \text{and} \quad \dim(V_8^{\langle g_3 \rangle}) = 0,$$

hence $JC \sim B_3$ (see [51, Lemma 1]). Therefore the Prym variety $P(\tilde{C}, C)$ is isogenous to B_8^2, where the elliptic curve

$$E := \tilde{C}/\langle g_2 g_3 \rangle \sim B_8$$

remains fixed in family. This shows that the family of covers $\tilde{C} \to C$ is contained in a fibre of the Prym map $\mathcal{P}_3(3)$.

□

5.7.1 A new example

We finish this Chapter giving an explicit new example of a totally geodesic curve which is an irreducible component of a fibre of the Prym map $\mathcal{P}_{1,2}$.

Consider a family of Galois covers $\psi : D \to D/G \cong \mathbb{P}^1$, with

$$g(D) = 11 \quad \text{and} \quad G = (\mathbb{Z}/4 \times \mathbb{Z}/4) \ltimes \mathbb{Z}/2,$$

ramified over

$$B = \{P_1 = \lambda, P_2 = 1, P_3 = 0, P_4 = \infty\}.$$

We use the following presentation of G:

$$G \cong \langle g_1, g_2, g_3, g_4, g_5 \mid g_1^8 = g_2^2 = g_3^4 = g_4^4 = g_5^2 = 1,\ g_1^2 = g_4,$$
$$g_3^2 = g_4^2 = g_5,\ g_1^{-1} g_2 g_1 = g_2 g_3,\ g_1^{-1} g_3 g_1 = g_3 g_5,\ g_2^{-1} g_3 g_2 = g_3 g_5 \rangle.$$

Notice that $G = (\langle g_1 g_2 g_3 \rangle \times \langle g_4 \rangle) \ltimes \langle g_2 \rangle \cong (\mathbb{Z}/4 \times \mathbb{Z}/4) \ltimes \mathbb{Z}/2$. The monodromy of the cover $\theta : \pi_1(\mathbb{P}^1 - B) \cong \langle \gamma_1, ..., \gamma_4 \mid \gamma_1 \gamma_2 \gamma_3 \gamma_4 = 1 \rangle \to G$ is

$$[\theta(\gamma_1) = g_2 g_3 g_5,\ \theta(\gamma_2) = g_3 g_4 g_5,\ \theta(\gamma_3) = g_1 g_2 g_4 g_5,\ \theta(\gamma_4) = g_1 g_3 g_4 g_5]$$

and these elements have orders $\mathbf{m} = (2, 2, 4, 8)$ in G.

Consider the subgroup $H = \langle g_2, g_5 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ of G. By Riemann Hurwitz formula one easily computes the genus of the quotient $C := D/H$ which is 2. Set

$$K := \langle g_2, g_3 g_4 \rangle \cong D_4,$$
$$K_1 := \langle g_2, g_4 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/4, \quad K_2 := \langle g_2, g_3 \rangle \cong D_4.$$

The genus 2 curve C occurs in three distinct double covers:

$$f : C = D/H \to D/K \cong \mathbb{P}^1,$$
$$f_1 : C = D/H \to D/K_1 =: E_1, \quad f_2 : C = D/H \to D/K_2 =: E_2,$$

where E_1 and E_2 are elliptic curves and $J(C)$ is isogenous to $E_1 \times E_2$.

The double covers

$$\pi_1 : E_1 = D/K_1 \to D/\langle g_2, g_3, g_4 \rangle \cong \mathbb{P}^1, \quad \text{and}$$
$$\pi_2 : E_2 = D/K_2 \to D/\langle g_2, g_3, g_4 \rangle \cong \mathbb{P}^1$$

allow to express the elliptic curves E_1 and E_2 in Legendre form:

$$E_1 : y^2 = x(x - \mu)(x^2 - 1), \ \ E_2 : y^2 = x(x^2 - 1),$$

where $\mu^2 = \lambda$.

Thus it becomes evident that the elliptic curve E_2 does not depend on the parameter. So the Prym variety $P(C, E_1)$ is isogenous to the fixed elliptic curve E_2. Therefore we have the following:

Proposition 5.7.5. *The 1-dimensional family of double covers $\pi_1 : C \to E_1$ described above is contained in a fibre of the Prym map $\mathcal{P}_{1,2}$: it has Jacobians isogenous to the product of two elliptic curves, with one of them fixed. Hence it gives a new example of a totally geodesic curve which is an irreducible component of a fibre of $\mathcal{P}_{1,2}$.*

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