Processes and diagrams: an integrated and multidisciplinary approach for the education of quantum information science

A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy

by

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Certificate

It is certified that the work contained in this thesis entitled **“Processes and diagrams: an integrated and multidisciplinary approach for the education of quantum information science”** by **Claudio Sutrini** has been carried out under my supervision and that it has not been submitted elsewhere for a degree.

Prof. Chiara Macchiavello
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Declaration

This is to certify that the thesis titled **"Processes and diagrams: an integrated and multidisciplinary approach for the education of quantum information science"** has been authored by me. It presents the research conducted by me under the supervision of **Prof. Chiara Macchiavello**.

To the best of my knowledge, it is an original work, both in terms of research content and narrative, and has not been submitted elsewhere, in part or in full, for a degree. Further, due credit has been attributed to the relevant state-of-the-art and collaborations with appropriate citations and acknowledgments, in line with established norms and practices.

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Abstract

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The background to this thesis is the παιδέια , the education. To educate is a dialectical process that moves from an abstract line of thought, through scientifically designed techniques, into concrete action; and *vice versa.* We believe that, in general, educating today means enabling teachers first and their students second, to be able to read and interpret the complexity of phenomena, to teach them a model for observing this complexity, describing it, analyzing it and, finally, making it their own. In this thesis, we attempt to make sense of these needs by describing an integrated and multidisciplinary pathway, whose diagrammatic language pushes towards the search for a universal approach to science.

An initial educational contribution is thus made to the understanding of the dialectic between disciplines: theoretical physics, experimental physics, computer science, mathematics and mathematical logic are presented in their mutual influence, in an attempt to clarify the informational viewpoint on modern physics. The search for this dialectic for educational purposes is, in our opinion, the most significant contribution of the present work.

To address this issue, we sought to build a community of practice on the topics of the second quantum revolution. Guided by the Model of Educational Reconstruction (MER), we built a first course for teacher professional development that would enable teachers to be introduced to quantum computation and quantum communication. The emergence and development of quantum technologies provides the impetus for a deep conceptual change: "*a paradigm shift from quantum theory as a theory of microscopic matter to quantum theory as a framework for technological applications and information processing*". This shift is supported, theoretically, by the informational interpretation of the postulates of quantum mechanics: preparation, transformation and measurement are reinterpreted computationally as the encoding, processing and decoding of information; and *vice versa*. In this interpretation, what changes between classical and quantum theory? From a logical point of view, the transition from bit to qubit, from a physical point of view, the laws of composition of systems. We therefore present monoidal categories as a natural theoretical framework for the description of physical systems and processes for quantum and non-quantum computation and communication, demonstrating how this language is suitable for an integrated and multidisciplinary approach.

The cultural impact of the proposal, the fruitful interaction between researchers in physics education and those in the area of theoretical research, and the passion of some teachers made it possible to start a collaboration to build an educational sequence for students. The result of this collaboration is a teaching leaning sequence on quantum technologies for students, led by the MER and based on inquiry-based learning and the modelling-based teaching. Supported by these methodological frameworks, we produced lessons and worksheets all along the way that had the dual task of supporting teachers' work and students' learning. They also made it possible to experimentally verify the positive and critical effects of the proposal. The instructional materials constructed, the data analysis and the constant monitoring with the teachers involved, determined the development of a second course for teacher professional development, inspired by the first, based entirely on research. We hope that this attempt at integrated and multidisciplinary approach for the education of quantum information science, based on the concept of compositionality and the diagrammatic model, can be increased and provide inspiration for future educational paths in other disciplines as well.

Acknowledgements

When you embark on such an adventure in your forties, you need help. So, first of all, I need to thank those who have shared and supported my research project. First of all, thank to Chiara Macchiavello, my tutor, who was able to support my ideas and guide me along a path of training first and research later, which was not easy for either of us. I would like to thank the research area in physics education that I met as a future teacher back in 2013 and have been privileged to work with over the past few years: thanks to Massimiliano Malgieri for welcoming me and sharing the research work in the teaching part; a special thanks to Giacomo Zuccarini for the hundreds of hours passed talking, discussing, thinking, designing and constructing most of the teaching-learning sequence presented in this thesis. Finally, a special thank you to Anna De Ambrosis, whose meeting has profoundly influenced my professional life over the last ten years.

Many thanks to professors and colleagues in the area of theoretical research who helped me in understanding advanced concepts of quantum theory. Foremost among them are Paolo Perinotti and Marco Erba, whose enthusiasm and availability have been invaluable to my work. I would also like to thank Lorenzo Maccone for his help and clarifications on his demonstration of Bell's inequality. Lastly, I cannot fail to mention my two office mates, Stefano Mangini and Leonardo Vaglini: I thank Stefano for introducing me to the fascinating world of machine learning and for solving many computer problems that came my way; Leonardo, for sharing the operational point of view of quantum theory and a world view that goes far beyond physics research.

I would now like to thank those with whom I have been able to collaborate over the past years. First of all, I thank Maria Bondani, who, above all, got me interested in experimental quantum optics: having attended her lectures contributed greatly to the project presented; in addition, Olivia Levrini and Elisa Ercolessi for their peaceful and fruitful collaboration in the activities we carried out together. Of course, I cannot but thank with infinite affection Sara Satanassi and Filippo Pallotta with whom I substantially shared the PhD project on quantum technologies in Italy.

However, the help one needs is not only professional. I thank my colleagues in mathematics and physics: they actually wrote this thesis. I thank them for the passion and professionalism shown in two years of working together. Of all of them, I would like to mention Mariacristina, Silvia, Monica, Roberta and Giacomo who took the proposed topics to their classes. I want to thank my parents and my sister Livia for sharing the first part of my life and contributing to my formation as a person. Lastly, and first of all, I would like to thank my wife Rosa for accompanying and supporting me over these three years together with our two beautiful little girls Sofia and Arianna. Thank you for reminding me every day that the most beautiful *research* is the one that the four of us carry out daily together.

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Abbreviations

MER	Model of Educational Reconstruction
IBL	Inquiry-Based Learning
MBT	Modelling-Based Teaching
CC	Conceptual Change
cc	classical computation
CP	Classical Physics
QP	Quantum Physics
QG	Quantum Gates
QC	Quantum Computation
TLS	Teaching-Learning Sequence
QIS	Quantum Information Science
CSG	Castel San Giovanni
Fi	Firenze
SS	Summer School
REW	Real Equally Weighted
OPT	Operational Probabilistic Theory
FSM	Finite State Machine
CNN	Classical Neural Network

OPT	Operational Probabilistic Theories
CQT	Categorical Quantum Theory
DBR	Designed Based Research
TRQ	Teacher Research Question
PBS	polarizing beam splitter

List of Publications

Publications from Thesis

1. Paper 1 *SUTRINI, C.; MALGIERI, M.; MACCHIAVELLO, C. Quantum technologies: a course for teacher professional development. In: Journal of Physics: Conference Series. IOP Publishing, 2022. p. 012018.*.
2. Paper 2. *SUTRINI,C; ZUCCARINI, G; MALGIERI, M; CERUTI, M; PANCANI, G; MACCHIAVELLO, C. Logic circuits and optical circuits: an educational path of construction of quantum computation for secondary school students. Submitted to Il nuovo cimento*.
3. Paper 3. *SUTRINI, C; ZUCCARINI, G; MALGIERI, M; BONDANI, M; MACCHIAVELLO, C. An educational model of the Deutsch algorithm for secondary school. Submitted to Physics Education*.
4. Paper 4. *BONDANI, M; CHIOFALO, M,L; ERCOLESSI, E; MACCHIAVELLO, C; MALGIERI, M; MICHELINI, M; MISHINA, O; ONORATO, P; PALLOTTA, F; SATANASSI, S; STEFANEL, A; SUTRINI, C; TESTA, I; ZUCCARINI, G. Introducing Quantum Technologies at Secondary School Level: Challenges and Potential Impact of an Online Extracurricular Course. Physics, 2022, 4.4: 1150-1167*
5. Paper 5. *SUTRINI, C; ZUCCARINI, G; MALGIERI, M; MACCHIAVELLO, C. A possible role of the second quantum revolution in Physics Education. In: Carvalho, G.S., Afonso, A.S. & Anastácio, Z. (Eds.). (2022). Fostering scientific citizenship in an uncertain world (Proceedings of ESERA 2021). Braga: CIEC, University of Minho. ISBN 978-972-8952-82-2*

Others

1. Paper 6 *SUTRINI, C.; PASSARO, D.; PALLOTTA, F. The potential of using Jupyter Notebook in physics education: Experimentation for high school students. Il nuovo cimento C, 2022, 45.6: 1-4*.
2. Paper 7 *PALLOTTA, F; PASSARO, D; SUTRINI, C. Le potenzialità di Jupyter nella didattica della fisica Giornale di Fisica, issue 4, year 2021, pp. 445-459 DOI: 10.1393/gdf/i2022-10419-y*.

A Rosa, meraviglia della mia vita,
e alle piccole Sofia e Arianna

Chapter 1

Introduction

I forgot the word that I wanted to say,
And thought, unembodied, returns to the hall of shadows.
O.E. Mandelshtam, The Swallow

In the last few years, the interest in education at all levels on quantum technologies, i.e. technologies revolving on the manipulation and control of individual quantum systems, has increased. Projects such as the National Quantum Initiative [2] in the US and the Quantum Flagship in the EU [3], have also promoted improvements in education and outreach. For this reason, the education research community has performed efforts to expand its traditional goals. New objectives include teaching quantum technologies (i.e., applications in computing, communication, simulation and sensors), and - one step further -, learning how to teach quantum physics through quantum technologies. However, while research on the teaching and learning of quantum physics is a well-developed field within physics education ([4], [5]), Quantum Information Science (QIS) is a novelty from the point of view of education. In particular, it is necessary to build programs and curricula for non-physics students and, at a lower level, for high school students. Recently, 34 QIS experts from both academia and industry signed an open letter [6] calling for an earlier start of QIS education in the academic career and recommending the involvement of education experts in curriculum development. An early introduction of QIS topics was also the subject of a recent educational survey [7] in which interviewed QIS instructors expressed interest in research-based instructional materials, while displaying a remarkably wide range of opinions on the desirable content and prerequisites of future undergraduate QIS courses. In [8], the authors identified the core ideas for QIS courses suitable for computer science students with superposition and entanglement of qubits, quantum computer, quantum algorithms, and quantum cryptography. Research-based course proposals for different targets, ranging from secondary school students [9] to undergraduates with little or no physics background [10], have recently appeared in the educational literature. Here we present a research-based course for teacher professional development and a TLS for secondary school students, that resonates with most of the previously reported indications and advances but also proposes significant innovations. Two elements characterize and differentiate our work from others:

1. the educational reconstruction for instruction is based on the construction of a diagrammatic *model* ([11], [12]) whose theoretical framework (categorical approach) has been developed in recent years in advanced research into both the fundamentals of quantum theory and into several applications such as , for example, quantum machine learning and quantum natural language processing;

2. our approach was to attempt, through a course providing common grounds and motivation, the formation of a community of practice ([13]) motivated to longitudinal and inter-disciplinary curriculum innovation.

Although in the case of high school students, the diagrammatic approach can only be partially developed, it represents the most significant contribution to educational content research for education, and we hope it can be adequately developed in the future to reach a large population of students.

1.1 Informational approach for quantum education

The advent of the second quantum revolution and the new technologies that are emerging has shed light on the possibility of changing the focus of the study on quantum mechanics determining a *paradigm shift from quantum theory as a theory of microscopic matter to quantum theory as a framework for technological applications and information processing*[1]. Those who wish to promote the study of these topics among teachers and students at different levels (from high school to higher education) have to consider the problem of what are, in this perspective, the elements that characterize the physical theory of computation and quantum information, possibly but not necessarily emphasizing by contrast what distinguish them from the classical approach. This is in line with Bennett's pioneering work on the thermodynamics of computation [14], [15] and with the consequent analysis

[1] European Quantum Flagship (2020, February). Strategic Research Agenda. `https://ec.europa.eu/newsroom/dae/document.cfm?doc_id=65402`

of Toffoli and Fredkin on reversible logic [16] and with the problem of the simulation of quantum physics by classic computers posed by Feynman [17]. These discussions laid the problematic basis for formulating the concept of computing machine as D. Deutsch will do a few years later in [18]. In this work D.Deutsch emphasizes that the aspect on which attention has to be focused, when we choose an informational and computational approach to introduce the key concepts of quantum physics, is the close link between physical information, support systems and the theory to describe their use.

In this sense, the first problem around which the design process develops is that of coding, which concerns which properties of the physical support are used to express a certain information by means of the state vector. The introduction of qubits is the consequence. Given the problem of the properties "owned" by a physical system that we describe through the state vector, the question of how it is possible to manipulate the information encoded in the state, that is, how this state evolves over time, is addressed. The evolution of the quantum state exploits features of the theory that have no classical analogue: the superposition principle, the nature of compound systems - in particular the concept of entanglement - and quantum interference are an opportunity to develop algorithms and protocols that are based on a logic of a different nature than the classical ones, allowing us to solve known problems more efficiently and to introduce new ones[2]. Although the information is encoded on a quantum support with the properties that distinguish it, at the end of any protocol the information that we can actually derive backs to be strictly classical expressed by strings of bits: the measurement returns on the one hand to the well-known values 0 and 1 of the classical Boolean logic, but does so through an operation that is intrinsically probabilistic.

In conclusion, the approach that we propose for the study and teaching of quantum technologies involves a change of perspective on quantum mechanics: from an interpretative model of real phenomena to a description of information processes based on properties of quantum objects. The three main moments of this description are the encoding of information, which coincides with the preparation of a state of qubits, the processing of information, which coincides with the unitary transformations with which the state is manipulated, and the reading of the output information, that is the measurement. This approach manifests its algorithmic nature and reveals in a substantially necessary way the traditional axiomatic quantum mechanics reinterpreted in informational terms. The computer is the emblem of this approach, understood as an experiment, a machine whose

[2]Both superposition and interference are actually concepts that have analogues in classical physics as well. To understand the relationship between the classical and quantum approaches from an educational point of view, one can see [19]

operation logic is deeply connected to the physics of the support on which the information is encoded, manipulated, and measured.

1.2 Diagrammatic model: Compositionality and Functors

We have already mentioned that what distinguishes our approach to quantum computation and information is a diagrammatic approach. There are two aspects that this approach tends to emphasize: intrinsically the *compositionality*, extrinsically the interpretation by *functors* between categories. As in [20] "*compositionality describes and quantifies how complex things can be assembled out of simpler parts*", i.e. the logic of interaction. In our work, we present physical processes using diagrams. These diagrams are made up of wires (physical systems) and boxes (transformations). We will see how the concept of sequential and parallel composition of wires and boxes in these diagrams is fundamental to grasp the difference between classical and quantum physics in general and classical and quantum computation in particular. The advantage of this type of approach is that it can be used, potentially, in any discipline: physics, chemistry, music, computer science, neuroscience *et cetera*. In its abstractness, the diagrammatic approach actually offers currently unexplored perspectives on a profoundly interdisciplinary approach to teaching, even at the undergraduated level.

While the compositional aspect concerns each discipline intrinsically, the *functorial* approach allows one diagram to be interpreted in another. This is one of the most important aspects in our work: diagrammatic language is in itself totally abstract, it has a value that we might say is merely syntactic. However, the existence of particular functors makes it possible to interpret it (semantics) in logic, in the physical theory of computation and in the physical theory of physical devices of linear optics. In a certain sense, functors are a map of signification that supports us, on a theoretical level, in the transition between mathematics, logic, theoretical physics and experimental physics. These signification maps are the most innovative theoretical support for educational reconstruction that we present.

1.3 Design project and research questions

There were two goals of the research project as it developed over these three years: the first was to build a course for teacher professional development about the topics of quantum computation and quantum information with a high cultural impact. This course would generate the conditions for shared work between researchers and teachers in order to bring

the topics covered to high school students. The second was to construct the materials for a research-based teaching and learning sequence to create the conditions for presenting some of the topics within curricular pathways.
These objectives were pursued over the three years period according to a subdivided action as in Fig.1.1

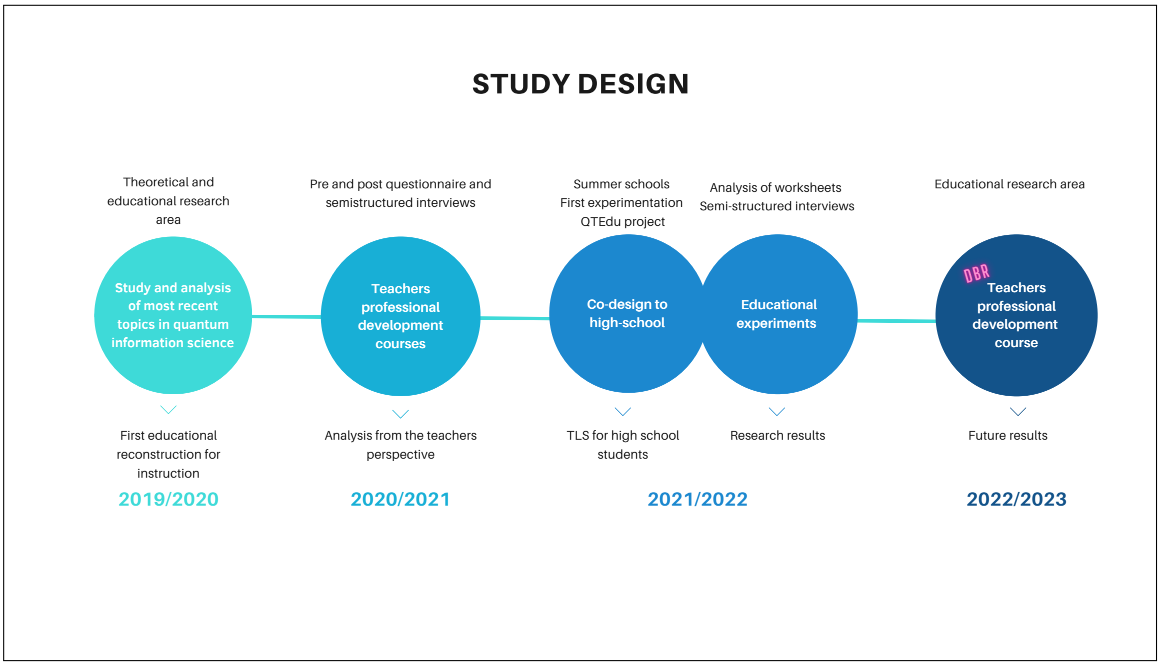


FIGURE 1.1: Study design

As seen from the figure, within the physics department, the theoretical research area on quantum computation and communication and the research area in physics education collaborated on the project. The outcome of the first year's work was a course for teachers on the topics of the second quantum revolution[3]. The data collected from pre-tests and post-tests, exercises and semi-structured interviews provided the first data on which to build subsequent actions. In particular, we took note of the teachers' reflections, needs and requests. Based on these, we organized a follow-up course on the approach to quantum mechanics with polarization based on research in physics education. After doing this and monitoring the teachers involved with a second step of interviews, it was possible to divide the teachers into working groups on various educational paths not only dedicated to the topics traditionally dealt with in the last year. In this way, it was possible to begin

[3]See Alain Aspect introduction in [21]

TABLE 1.1: Research questions

Teachers research questions	**Students research questions**
TRQ1: *How is it possible to construct an adequate content simplification process to present the topics of the second quantum revolution to teachers in a meaningful way from very advanced theoretical aspects?*	**SRQ1**: *How is it possible to construct an adequate content simplification process to present the topics of the second quantum revolution to students in a meaningful way from very advanced theoretical aspects?*
TRQ2: *How to make the contents and themes of the second quantum revolution sufficiently fruitful to teachers to develop a personal commitment to longitudinal, interdisciplinary educational innovation directed towards themes of quantum information and computation?*	**SRQ2**: *How effective is an integrated and multidisciplinary approach in order to enable students to understand some topics of quantum computation and quantum information?*
TRQ3: *What are the most appropriate environment and methods for building a distributed, online community of practice of teachers revolving around the themes of the second quantum revolution?*	**SRQ3**: *Based on findings from the first two research questions, what design principles can be formulated for the development of TLS resources in quantum computation for high school students?*

a process of strict collaboration between researchers and teachers to co-design research-based teaching and learning sequences. In spring 2022, two curricular experiments were carried out directly by two teachers with expert supervision. The material prepared for both teachers and students allowed them to follow a clearly defined teaching trajectory. The same materials, with some initial modifications, were used in September 2022 during the Summer School on Quantum Technologies organised by the Physics Department of the University of Pavia. The materials constructed and the data analysis contributed to the realisation of a second 20-hour course for teachers held between October and December 2022, for which further development is planned in the coming months. These activities, all within our department, were enriched by fruitful collaborations with other universities (in particular, the University of Insubria in Como and the University of Bologna) for the realization of Summer Schools in quantum technologies and two extracurricular online courses on quantum physics concepts and quantum technologies applications for high school for a total of around 500 students throughout Italy.

The results of the work carried out in these years can be summarized as answers to a few general research questions that have emerged in progress and which are set out in the table Tab.1.1.

1.4 Organization of the Thesis

The first three chapters aim to introduce the quantum information theory that will later be used in educational reconstruction. Chapter 2 introduces qubits, quantum logic gates and the measurement problem before entering into the fundamental topic of compound systems. Entanglement analysis is a key theme in the description of Deutsch-Jozsa and Grover's algorithm and, of course, the teleportation protocol. The study of entanglement in algorithms allows us a first approach to topics strongly related to computer science. In fact, we describe a line of research that we have begun to work on, on the use of hypergraph states to solve isomorphism tests for hypergraphs that could be extremely useful for certain aspects of machine learning. Chapter 3, supported by theoretical elements related to category theory, also given in the appendix, is of fundamental importance for understanding the role of diagrammatic representations in our path. We propose two examples in which we use known approaches to describe classical sequential and combinatorial logic in a way not found in the literature: in the first case, we define the concept of a finite-state machine via the Grothendieck construction and use it to introduce universal classical computation. The second example is inspired by a recent work that describes all moments of supervised learning using the same categorical structure and constructs classical computation via the concept of a neural network. Chapter 4 describes classical logic in its natural categorical environment: the ***Set*** category. This category bridges with the physical description of the devices that implement the computation. The diagrammatic interpretation of the physical processes is contextualized in the case of both OPTs and the Categorical Quantum Theory. Finally, we briefly describe how a similar diagrammatic calculation can be introduced to describe the theory of linear optical devices. The dialectic between diagrammatic representation and logical, physical and experimental interpretations is realized using the functor concept, thus supporting, from a theoretical point of view, the choice to use diagrams as a model for the construction of the educational sequence.

The second part of the thesis is dedicated to educational aspects. Chapter 5 introduces the theoretical and methodological frameworks used relating to physics education: the MER, a theoretical framework designed to guide the researcher in the clarification and analysis of science content; the Inquiry-Based Learning (IBL) and the Modelling-Based Teaching (MBT) because in our educational proposal for secondary school students our aim is to help students develop an organized knowledge structure concerning QIS embedded in active and constant engagement in construction and reconstruction knowledge through hands-on interactions. The last two chapters represent the core of the research.

Chapter 6 describes the work done with high school teachers. We describe the professional development course and the subsequent collaborative work. We then propose the data analysis based on pre and post-tests and mainly semi-structured interviews. The research work allows us to answer the three questions in Tab.1.1 and supports the choices made for constructing of the teaching-learning sequence for students that is the subject of Chapter 7. Chapter 7 synthesizes all the work done. In the first part, we propose an important contextualization concerning the design: following the indications of the MER reconstruction, we propose an analysis from the theoretical, teacher and student perspectives leading to the reconstruction of content for instruction. Then we describe each sequence step in detail: for each lesson, we briefly describe the content, learning goals, strategies, instruments and methods; we then describe the lesson in detail, supported by the instructional materials produced. We finish with the data analysis, the answers to the research questions and the design principles obtained *ex post*. The thesis concludes with a summary and the possible future development directions.

With reference to the list of publications, we note that:

1. parts of Chapter 1 are taken from papers 3, 4 and 5;
2. many parts of Chapter 6 are taken from papers 1;
3. Chapter 7 is based on papers 2 for the coding construction part, paper 3 for the elementarization part of the algorithms and 4 in the description of the online extracurricular course.

Chapter 2

Quantum computation and quantum information from the theoretical perspective

In this chapter, we introduce the theory of quantum computation and quantum communication used for educational reconstruction. We address these topics in the traditional way: qubits, logic gates and standard circuit representation. We will attempt to do this both in the theoretical description and in the aspects of the physical devices that implement qubits, logic gate and measurement with linear optics.

2.1 The postulates of quantum mechanics

Before dealing specifically with the computational aspect, we want to introduce the basic elements of quantum mechanics and then address the special case of qubits in detail. For this purpose we briefly introduce and comment on the postulates in a manner similar to that done in [22].

Postulate 1 Associated to any isolated physical system is an Hilbert space on the complex field, called *state space* of the system. The system is completely described by its unit *state vector* $|\psi\rangle$.

This first postulate *sets up the arena in which quantum mechanics take place*[1] and the state of a quantum mechanical system includes all the information you can know about it ([23]).
Given n state vectors $|\psi_i\rangle$, any their linear combination $\sum_i \alpha_i|\psi_i\rangle$ is a *superposition* of the states $|\psi_i\rangle$ with *amplitude* α_i for the state $|\psi_i\rangle$.
The second postulate gives a description of evolution of states:

Postulate 2 The evolution of a closed physical system is described by a *unitary transformation* U so that

$$|\psi'\rangle = U|\psi\rangle$$

if $|\psi\rangle$ is the state of the system at the time t_1 and $|\psi'\rangle$ is the state of the system at the time t_2.

The last postulate is devoted to the interaction of the system with the outside world, which is crucial if we imagine that we want to perform experiments and determine probability distributions relating to the occurrence of events.

Postulate 3 A projective measurement is described by an Hermitian operator on the state space of the system being observed, the *observable*

$$M = \sum_m m P_m$$

where P_m is the projector onto the eigenspace of M with the eigenvalue m. The eigenvalues, m, of the observable are the possible outcomes of the measurement. Moreover, the probability of getting m, is given by

$$p(m) = \langle\psi|P_m|\psi\rangle$$

and the state of quantum system after the result m is

$$\frac{P_m|\psi\rangle}{\sqrt{p(m)}}$$

To conclude this brief exposition, it remains to present the fundamental postulate that describes the composition of systems and represents one of the features that most profoundly distinguishes classical from quantum physics:

[1]See [22].

Postulate 4 The state space of a composite physical system is the tensor product of the state space of the component physical systems.

We will write the compound state as

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes ... \otimes |\psi_n\rangle$$

2.2 Qubit, quantum logic gates and measurements.

As we shall see, the fundamental idea that we want to bring from an educational point of view is that, when we speak of bits, we mean a system that can exist in two distinct information-carrying states. Therefore, when we introduce symbol as **0** and **1** for the classical bit[2], we are considering a physical system associated. We can therefore identify the numerical value, traditionally $\{0, 1\}$, of the bit with the bit itself. Therefore, in classical computation we should say a state of bit (existing in the space state $S = \{\mathbf{0}, \mathbf{1}\}$), but we will use, with justified abuse of term, just the name bit. The only possible reversible operations (gates) in such system are the identity and the not gate. If we associate with **0** and **1** two vectors as follows

$$\mathbf{0} \longrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{1} \longrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

the traditional boolean functions become

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

By analogy with the classical computation, we can introduce a quantum bit - *qubit* - as a two-level quantum system, a 2-dimensional complex Hilbert space. In this space we can use the vector $|0\rangle$ and $|1\rangle$ as *computational (orthonormal) basis*. Thus, a generic state can be written as[3]:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{2.1}$$

where $\alpha, \beta \in \mathbb{C}$. Any vector as 2.1 with the normalization condition

$$|\alpha|^2 + |\beta|^2 = 1 \tag{2.2}$$

[2]Let us directly give a vector construction of bits.
[3]Physically, from the superposition principle!

is called *qubit.* Similarly to what has already been done, we can associate a generic qubit $|\psi\rangle$ the vector whose components are the coefficients of the elements of the computational basis:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

We can introduce a geometrical representation of qubit: the ***Bloch sphere***.
We know that a pure state is an entire equivalent class of vectors. In this way ([24]), the space of pure qubit states is the complex projective space $\mathbb{C}\mathbf{P}^1$. But $\mathbb{C}\mathbf{P}^1 = \mathbf{S}^2$, the *Bloch sphere*.
To express mathematically the link between a generic point on the spherical surface and the corresponding qubit state, we consider that the four real parameters (two for each complex number) actually become two due to the normalization condition and the possibility of ignoring the global phase. In a simple way (see [25] or [26]) one can demonstrate the biunivocal correspondence between a generic point on the Bloch sphere and the reparametrized qubits with two real parameters:

$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\phi}sin\frac{\theta}{2}|1\rangle, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi \tag{2.3}$$

as it's possible to see in Fig. 2.1.

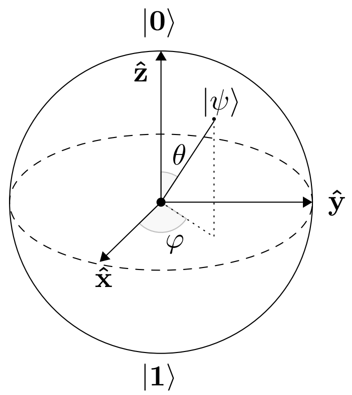


FIGURE 2.1: Bloch sphere qubit representation

As the qubit is defined, it is therefore evident that the information is then all stored in the two complex numbers and it is potentially infinite. But the effect of a measurement in the computational basis is that obtain a classical bit of information with a probability established by the Born rule:

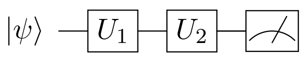
$$p(0) = |\langle 0|\psi\rangle|^2, \quad p(1) = |\langle 1|\psi\rangle|^2 \tag{2.4}$$

The intrinsically probabilistic aspect of measurement is "*another important way a qubit is different from a bit*" ([27]). When a measurement is realized, the state $|\psi\rangle$ collapses in $|0\rangle$ or $|1\rangle$[4].

These considerations about measurement just made remind us (see [25]) that a two-level system can be used in practice as a qubit if it is possible to

1. *prepare* a well-defined state ($|0\rangle$);
2. *transform* any state of qubit into any other by means of unitary transformations;
3. *measure* the qubit state in the computational basis $\{|0\rangle, |1\rangle\}$.

This tripartition suggests a circuit model of computation that we will explore in more detail in the next chapters, but which for clarity of exposition we prefer to anticipate (see [22]): "*a circuit is made of* ***wires*** *and* ***gates****, which carry information around, and perform a simple computational task*". In the case of quantum circuits, it is usual to add a box related to the measurement as well. The above can be found made explicit in 2.5:



(2.5)

In case the computation requires more than one qubit, the corresponding circuit will have as many wires as there are qubit states.

It now remains to address the transformation issue: the single-qubit gates.
As in the classic case, we can develop our quantum computation introducing single-qubit gates, some particular linear unitary transformations[5] that we will use in our work:

$$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad H \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.6}$$

We will refer to these gates as respectively *X-gate* or *Not-gate*, *Y-gate*, *Z-gate* or *Flip-gate* and *Hadamard-gate*. The *Z-gate* is a particular case of the *phase-shift gate* defined as

$$R_z(\gamma) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\gamma} \end{bmatrix} \tag{2.7}$$

[4]In reality, states $(a/|a|)|0\rangle$ and $(b||b|)|1\rangle$ are achieved. But we now that a unitary coefficient can be ignored.

[5]The need for unitarity arises from the necessity to preserve the norm. Moreover, in this way the transformation is reversible

whose effect on a generic qubit is a counterclockwise rotation through an angle γ about the *z-axis* of the Bloch sphere:

$$R_z(\theta)|\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\gamma} \end{bmatrix} \begin{bmatrix} cos\frac{\theta}{2} \\ e^{i\phi}sin\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} cos\frac{\theta}{2} \\ e^{i(\phi+\gamma)}sin\frac{\theta}{2} \end{bmatrix} \tag{2.8}$$

If $\gamma = \pi$, we found the *Z-gate*.
We could demonstrate that any unitary operation on a single qubit can be constructed with Hadamard and phase-shift gates (see [25]). In general, each single-qubit logic gate is a linear combination of the Pauli operators $\{\mathbf{I}, X, Y, Z\}$.

2.3 Compound system: bipartite quantum system

It is possible to grasp the profound difference between classical and quantum computation when we step to consider multi-qubit quantum systems. As Horodecki in [28] "*the "effect" of the replacement of the classical concept of phase space by the abstract Hilbert space makes a gap in the description of composite systems*". If we consider a multipartite system and its n subsystems we have two different ways to see the pure total state: in the classical case the *Cartesian* product ensures that "*the total state is always a product state of the n separate systems*". But if we consider the tensor product between Hilbert spaces, we soon realize that its nature and the superposition principle do not always allow us to perform the same operation.

Consider a system of two qubits. Similarly to the case of one qubit, we can consider an orthonormal basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ of $\mathbf{C}^2 \otimes \mathbf{C}^2$. Any two qubit state can be written, therefore, as complex linear combination

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \tag{2.9}$$

with the obvious normalization condition $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$. Classically we think about this state as compounded by two different states

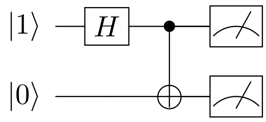
$$|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle) = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle \tag{2.10}$$

However, it is straightforward to verify that there are states, the Bell's states[6] for example, that cannot be written according to the 2.10. States that are not product states are called *entangled states*.

We would now like to understand what particular features entangled states have. To do this, we can consider the *singlet state*

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle)\text{[7]} \tag{2.11}$$

First, we could imagine that the state 2.11 is the quantum state of two qubits shared by Alice and Bob that are now in the same room or in different ones but able to communicate. As we will see in detail in the chapter 4[8], we can represent this situation with the following circuit[9]:



(2.12)

If Alice (first register) measures her qubit by projecting onto the computational basis $\{|0\rangle_A, |1\rangle_A\}$ and Bob (second register) measures his qubit by projecting onto the computational basis $\{|0\rangle_B, |1\rangle_B\}$, they find the outcomes of measurements perfectly anti- correlated. It can be shown that this happens regardless of the choice of measurement basis.

To deeply understand what is hidden in the state 2.11, and entangled states in general, let us introduce a different formalism from the one used so far: *the density matrix.*

2.3.1 The density operator

The traditional axiomatic description of quantum mechanics, which we have essentially implied by introducing quantum computation, is such that "*our axioms characterize the quantum behavior of the entire universe*" ([29]) or, similarly, we assume that we have perfect knowledge of quantum systems in use. However, in most cases, we will be interested in a very small part of the Universe, or for example, we have a quantum system whose preparation is not completely under control [30]. In this case, we can consider the prepared state as a statistical mixture of pure states $\{|\psi\rangle, p_i\}$ weighted with parameters representing

[6]Let us recall Bell's four states:

$$|\psi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),\ |\psi_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),\ |\psi_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),\ |\psi_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

[7]$|0\rangle|0\rangle$ and $|00\rangle$ are different ways to write the same state

[8]Briefly, the wires are the qubits and the boxes are the logic gates.

[9]The second gate is the *CNOT-gate*: $CNOT(|x\rangle|y\rangle = |x\rangle|x \oplus y\rangle)$

the respective probabilities p_i and such that $\sum_i p_i = 1$.

These considerations allow us to show ([25]) how probability in quantum physics takes on two distinct roles. Indeed, given an observable A, the expectation value is

$$\langle A \rangle = \sum_{k=1}^{l} p_k \sum_{i=1}^{n} a_i \langle \psi_k | P_i | \psi_k \rangle = \sum_{k=1}^{l} p_k \langle \psi_k | A | \psi_k \rangle \tag{2.13}$$

where P_i is the projector onto the subspace associated with the eigenvalue a_i of A. In the 2.13 we can recognize two probabilities: the first, the weights p_k is an epistemic probability; the second, characterizing the measurement process, $\langle \psi_k | P_i | \psi_k \rangle$ are intrinsic (non epistemic [31]). Moreover, if we introduce the *density operator*

$$\rho \equiv \sum_{k} p_k |\psi_k\rangle\langle\psi_k| \tag{2.14}$$

we obtain

$$\langle A \rangle = Tr[\rho A] \tag{2.15}$$

and

$$p(i) = Tr(\rho P_i) \tag{2.16}$$

are the probabilities that a measurement of an observable A gives outcomes a_i. The equation 2.16 represents a generalized Born rule.

Therefore we use the density operator 2.14 to formalize the statistical mixture of pure states $\{|\psi\rangle, p_i\}$ so it completely characterizes a system in a mixed-state configuration. In the particular case of only one (pure) state the 2.14 becomes

$$\rho = |\psi\rangle\langle\psi| \tag{2.17}$$

The formalism of the density operator allows us to focus our attention in a subsystem of a compound quantum system. To do this we have to introduce the *reduced density operator* (see [32] and [22]).

Let A and B be two any observable for two different systems. The compound system is described by the density operator ρ^{AB}. We want describe a part, for example A of the first system. In order to do this we want to demonstrate that the expectation value of A is

$$\langle A \rangle = \langle A \otimes \mathbf{I} \rangle \tag{2.18}$$

In fact, if we consider the joint probability of measure a_n in the first system and b_m in the second, and P^A, P^B the corresponding projectors, we have from 2.16

$$p_{A,B}(n,m) = \langle P_n^A \otimes P_m^B \rangle = Tr[(P_n^A \otimes P_m^B)\rho^{AB}] \tag{2.19}$$

The marginal probability to obtain a_n regardless of the outcome of measurement on system two is

$$p_A(n) = \sum_m p_{A,B}(n,m) = \sum_m \langle P_n^A \otimes P_m^B \rangle = \langle P_n^A \otimes \mathbf{I} \rangle \tag{2.20}$$

from which 2.18.
If we define the *partial trace* as

$$Tr_A[A \otimes B] := Tr[A]B \tag{2.21}$$

it follows that

$$\langle A \rangle = Tr[\rho^{AB}(A \otimes \mathbf{I})] = Tr[Tr_B[\rho^{AB}(A \otimes \mathbf{I})]] = Tr[Tr_B[\rho^{AB}]A] = Tr[\rho^A A] \tag{2.22}$$

where

$$\rho^A = Tr_B[\rho^{AB}] \tag{2.23}$$

is the *reduced density operator* (partial trace) or the *marginal state* of the first system of the compound system ρ^{AB}. The partial trace "*is the unique operation which gives rise to the correct description of observable quantities for subsystems of a compound system*" ([22]).

Back to the singlet state 2.11 and using the density operator formalism, we can write the state as

$$\rho_{|\psi\rangle} = \frac{1}{2}(|01\rangle\langle 01\rangle + |10\rangle\langle 10| - |01\rangle\langle 10\rangle - |10\rangle\langle 01\rangle) \tag{2.24}$$

and we immediately achieve the marginal state of the first system

$$\rho^A = Tr_B[\rho_{|\psi\rangle}] = \frac{1}{2}\mathbf{I} \tag{2.25}$$

which is the *maximally mixed state*[10]. This feature tells us that the best possible knowledge of a whole does not necessary include the best possible knowledge of all its parts [33]:

> "*Thus one disposes provisionally until the entanglement is resolved by actual observation of only a common description of the two in that space of higher*

[10] For an equivalent demonstration with matrix calculation, see for example [25]

> *dimension. This is the reason that knowledge of the individual systems can decline to the scantiest, even to zero, while that of the combined system remains continually maximal. The best possible knowledge of a whole does not include the best possible knowledge of its parts—and this is what keeps coming back to haunt us*".

In the case of the singlet, even maximum knowledge of the whole (the state is a pure state), implies minimum knowledge when we ignore one of the two parts (whether measurements have not been made on it or we are unaware of the results of those measurements).
The most important manifestation of the entanglement is the impossibility to reconcile the statistics of joints measurement with *local realism*. We must abandon the idea that a measurement on a system is only the "reading of the pre-existing quantity encoded on the system" ([34]) or abandon relativistic causality. This is the result of Bell's theorem ([35], [36],[37]).

2.3.2 Universal quantum computation

We know that for irreversible classical computation (see [38]) the logic gates *NAND* and *COPY* form a universal set of gates. Moreover the *Toffoli* or *Fredkin* gate are both universal, since it is easily demonstrated that *NAND*-gate and *COPY*-gate can be obtained from these (see [38]).
We have a similar result for quantum computing ([22], [25], [39]):

Theorem 2.1. *Single-qubit gates and CNOT-gate are universal for quantum computation.*

We now have all the elements to describe how it is possible to perform a quantum computation in a real physical device:

1. *prepare* the quantum computer in well-defined initial state $|\psi_i\rangle$, for example the qubit state $|0...00\rangle$;
2. *process* the qubit with any given unitary transormation U[11], leading to $|\psi_f\rangle = U|\psi_i\rangle$;
3. *measure*, at the end of the algorithm, in a computational basis.

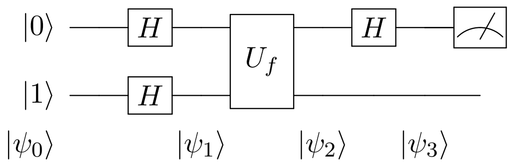
[11] Always decomposable with the universal gate set introduced!

2.4 Quantum algorithms

In this section we will briefly introduce the two algorithms using in TLS for students and presented also in the course for teacher professional development: Deutsch and Grover algorithms. We will highlight its main features and describe the role of entanglement.

Deutsch's algorithm Let us consider a boolean function $f : \{0,1\} \longrightarrow \{0,1\}$. We say that f is *balanced* if $f(0) \neq f(1)$; *constant*, otherwise. We wish to find an algorithm to determine whether the function is balanced or constant. It is clear thet we need to requires two queries of the oracle[12] in a classical computer.

In 1985 D. Deutsch [18] proposes a quantum algorithm solving more efficiently the problem. Here we present the circuit representation as in [40]:



(2.26)

This algorithm combines three features characteristic of quantum computation: quantum parallelism, the role of the phase factor in compound systems and the quantum interference ([22], [25]).

The first part of the circuit (Hadamard gate and Oracle U_f[13]) introduces the quantum parallelism [22]. We use the superposition and the linearity of operator to evaluate the function f for many different values of x simultaneously. The *ancillary* qubit (second qubit) after the Hadamard gate introduces a phase that can be kicked back ([40], [25]) in front of the qubit *target* (first qubit). Finally, the quantum interference allows us to answer the problem implementing the oracle once only.

Formally:

$$|\psi_1\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right] \otimes \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

After the function evaluation in Oracle if we pose $x \in \{0,1\}$ we obtain

$$|\psi_2\rangle = U_f|x\rangle \otimes \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] = (-1)^{f(x)}|x\rangle \otimes \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

[12] The oracle is a black-box evaluating f

[13] $U_f|xy\rangle := |x\rangle|y \oplus f(x)\rangle$

By expressing the values of x we achieve

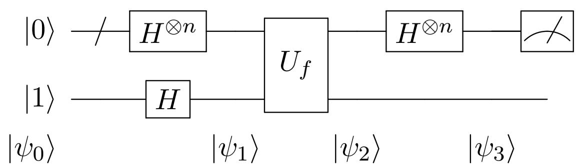
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left[(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle\right] \otimes \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Finally, omitting the now useless second register, we obtain after the last Hadamard gate (see[40])

$$|\psi_3\rangle = (-1)^{f(0)}|f(0) \oplus f(1)\rangle = \begin{cases} \pm|1\rangle & if\ f(0) \neq f(1) \\ \pm|0\rangle & if\ f(0) = f(1) \end{cases}$$

The final measurement of the first qubit gives with probability $p = 1$ the answer to the problem.

Deutsch-Jozsa algorithm In 1992 Deutsch and Jozsa [41] present the generalization of the previous algorithm:



(2.27)

The quantum features used are the same of the previous one.
Formally, after the first Hadamard gates

$$|\psi_1\rangle = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

The operator U_f calculates f in all the values

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle\left[\frac{|f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}}\right]$$

Using the phase kick back

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}(-1)^{f(x)}|x\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

At this point the last qubit may be ignored and we apply the Hadamard transform

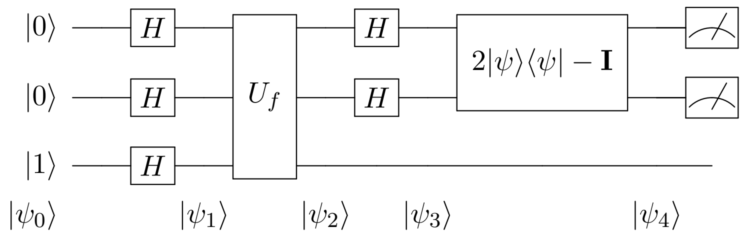
$$|\psi_3\rangle = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}(-1)^{f(x)}\left[\frac{1}{\sqrt{2^n}}\sum_{z=0}^{2^n-1}(-1)^{x\cdot z}|z\rangle\right] = \frac{1}{2^n}\sum_{z=0}^{2^n-1}\left[\sum_{x=0}^{2^n-1}(-1)^{f(x)+x\cdot z}\right]|z\rangle$$

The final measurement gives us

$$\left| \frac{1}{2^n} \sum_{x=0}^{2^n-1} (-1)^{f(x)} \right|^2 = \begin{cases} 1 & if\ f\ is\ constant \\ 0 & if\ f\ is\ balanced \end{cases}$$

for the probability to measure $|0\rangle^{\otimes n}$[14].

Grover's algorithm In 1996 Lov Grover [42] introduces a quantum search algorithm that can reduce the problem of searching for an element x_0 in an unstructured database of 2^n elements, from $O(2^N)$ number of calls of black-box to $O(\sqrt{2^n})$. For $n = 2$ the circuit representation of the algorithm is (see [22], [43])



(2.28)

where

$$f(x) = \begin{cases} 1 & if\ x = x_0 \\ 0 & otherwise \end{cases} \tag{2.29}$$

and $U_f(|x\rangle|y\rangle) = |x\rangle|y \oplus f(x)\rangle$. Since $|x\rangle \longmapsto (-1)^{f(x)}|x\rangle$ We can thus define an *Oracle*

$$\mathbf{U}_f := \mathbf{I} - 2|x_0\rangle\langle x_0| \tag{2.30}$$

In this particular case the algorithm is deterministic and with one implementation of circuit allows us to obtain the correct result [15]. The algorithm has exactly the same tripartition as Deutsch's algorithm: quantum parallelism; the role of the oracle that allows phase kick back $(-1)^{f(x)}$; and, finally, the last part that determines an interference such that only the qubit encoding the searched element is achieved.

We can generalize this very simple case by considering the search for M elements in a database of $N = 2^n > 4$.

For this purpose we call $G := (2|\psi\rangle\langle\psi| - \mathbf{I})\mathbf{U}_f$ the *Grover operator* and distinguish between the elements we are looking for, x_i, and those we are not looking for, x_j.

[14] In this case, f is balanced if it is equal to 1 for exactly half of all the possible x, and 0 for the other half.

[15] For explicit calculations see for example [22]; in matrix form [25]

Defined normalized states as

$$|\alpha\rangle = \frac{1}{\sqrt{N-M}}\sum_i |x_i\rangle, \qquad |\beta\rangle = \frac{1}{\sqrt{M}}\sum_j |x_j\rangle \tag{2.31}$$

the initial state $|\psi_0\rangle$ is

$$|\psi_0\rangle = \sqrt{\frac{N-M}{N}}|\alpha\rangle + \sqrt{\frac{M}{N}}|\beta\rangle \tag{2.32}$$

a vector in the plane defined by orthonormal $|\alpha\rangle$ and $|\beta\rangle$. In this way, we have an immediate geometric interpretation of the algorithm: G is a rotation in the plane of θ (see [22]). Since

$$G^k|\psi\rangle = cos\left(\frac{2k+1}{2}\theta\right)|\alpha\rangle + sin\left(\frac{2k+1}{2}\theta\right)|\beta\rangle \tag{2.33}$$

it is possible to prove ([22]) that the operator must be iterated

$$R \leq \left[\frac{\pi}{4}\sqrt{\frac{N}{M}}\right] \tag{2.34}$$

times.

2.4.1 Entanglement in quantum algorithms: Hypergraph States

In this section we would like to describe the role of entanglement in these two quantum algorithms. To do this, we will first introduce the concept of hypergraph states and show how they appear in both algorithms. We will then briefly show what the role of entanglement is. We will then indicate a possible line of development for the hypergraph isomorphism test by generalising the graph case recently achieved in [44]. The solution of such a test may have useful repercussions in some supervised learning cases such as graph matching.

In the even brief description of the algorithms, we have obtained after the oracle the state

$$|\psi\rangle = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}(-1)^{f(x)}|x\rangle \tag{2.35}$$

where $|x\rangle$ represent the computational basis states of n qubits and f is the boolean function that needs to be evaluated. As in [45], we can call these states Real Equally

Weighted (REW) pure state. The set of these states coincides (see [45]) with the set $G_{\leq n}$ of *hypergraph states* ([46]).

Definition 2.2. Given a mathematical graph $g = \{V, E\}$, where V is the set of vertices and E the set of edges, we can find the *graph quantum state* as follows:

1. assigning to each vertex a qubit $|+\rangle$;
2. performing a controlled-Z operation[16] between any two qubits that are connected by an edge.

The qubit graph state is therefore written

$$|g\rangle = \prod_{\{i_1,i_2\}\in E} C^2 Z_{i_1,i_2} |+\rangle^{\otimes n}$$

where $\{i_1, i_2\} \in E$ means that the 2 vertices are connected by an edge.
Graphically:

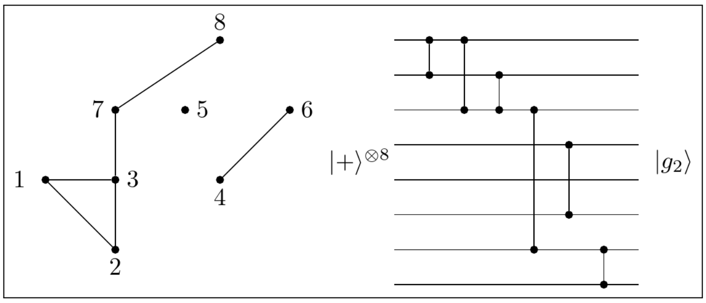


FIGURE 2.2: Example of graph state

We can generalize this definition in case the underlying structure is a hypergraph:

Definition 2.3. Given a mathematical hypergraph $g = \{V, E\}$, where V is the set of vertices and E the set of hyperedges, we can find the *hypergraph quantum state* as follows:

1. assigning to each vertex a qubit $|+\rangle$;
2. performing a controlled-Z operation between all connected by hyperedges qubit.

The qubit hypergraph state is therefore written

$$|g\rangle = \prod_{k=1}^{n} \prod_{\{i_1,\ldots,i_k\}\in E} C^k Z_{i_1,\ldots,i_k} |+\rangle^{\otimes n}$$

[16] CZ-gate is defined by $C^2 Z_{i_1,i_2} = diag(1, 1, 1, -1)$

where $\{i_1..., i_k\} \in E$ means that the k vertices are connected by a k-hyperedge[17]. Graphically:

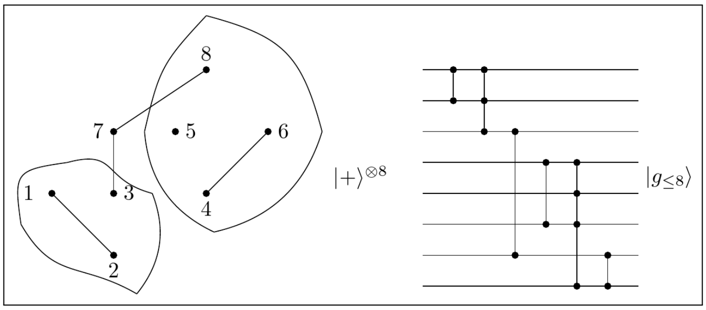


FIGURE 2.3: Example of hypergraph state

In [47], the autors demostrated that, in Deutsch-Jozsa algorithm, if the function f is constant the state 2.35 is separable, whereas in the case where f is balanced the state can be either separable or entangled. Morover, as the number of qubits involved increases, the presence of entangled states increases exponentially. The presence of the multipartite entanglement within the first register "*is needed to accomadate all of possible (balanced) functions*".

More interesting is the analysis of Grover's algorithm[18] (see [48] and [49]). In the first of these works, the authors study the entanglement properties of the state 2.35 as function of the number of qubits for $M = 1$ and $M = 2$ solutions. Referring to the geometric measure of entanglement ([50])

$$E_q(|\psi\rangle) = 1 - max_{|\psi\rangle \in S_q} |\langle\psi|\phi\rangle|^2 \tag{2.36}$$

where S_q represents the set of q-separable states, they demostrated that "*the amount of entanglement decreases for increasing number of qubits*". But ([49]) even for a very large number of qubits n there is a genuine multipartite entanglement. The interpratation of state 2.35 as a hypergraph state, makes this immediately obvious. If we consider the state $|\psi_{M=1}\rangle$ (with a signus minus in front of the component $|1...11\rangle$), it corresponds to the hypergraph with the unique hyperedge of order n.

2.4.1.1 Graph and hypergraph states for quantum machine learning

We briefly present in this section a line of development on the use of hypergraph states that we are currently working on and that fits in with the research work that has used the

[17] We denote by $C^k Z_{i_1,...,i_k}$ the general controlled-Z gate acting on the k qubits.

[18] We discuss here only the case in which $M = 1$ as in the algorithm explained to students.

concepts of graph and hypergraph states for quantum machine learning in recent years. For example in [51], the authors use the weighted hypergraph[19] to generalize the implementation of quantum artificial neuron to accept continuous- instead of discrete-valued input vectors, without increasing the number of qubits," *crucial step to allow for a direct application of gradient descent based learning procedures, which would not be compatible with binary-valued data encoding*". We would like to use hypergraph states to solve another kind of problem, and generalise a part of the work presented in [44]. In this work the authors use the ZX-calculus, equivalent to the traditional approach[20].

Without entering into too much detail, one of the most important problems for supervised machine learning is isomorphism tests for graphs. Supervised learning on molecules has incredible potential to be useful in quantum chemistry, drug discovery, and materials science (see [52] for references). In [52] the authors emphasize the fact that neural network models[21] have already been introduced and the importance of the fact that they are invariant under the symmetries of molecules. They reformulate existing models into a single common framework called Message Passing Neural Networks (MPNN), highly effective class of graph neural networks (see [55]) that iteratively update the representations of each vertex based on their local neighborhoods. As pointed out in [44], "*one well-known limitation of MPNNs is their expressive power which is upper bounded by the 1-dimensional Weisfeiler-Lehman algorithm (1-WL) for graph isomorphism testing*". We briefly analyze the line of reasoning ([56] and [57]).

We can consider a definition of graph whose vertexes are labelled:

Definition 2.4. A graph G is a triplet (V, E, l) where

1. V is the set of vertices;
2. E is the set of edges;
3. given an alphabet Σ, $l : V \longrightarrow \Sigma$

Moreover, we define the *neighbourhood*

$$\mathcal{N}(v) := \{v' \in V | (v, v') \in E\}$$

[19] For definition see [45]

[20] We will introduce the ZX-calculus only in later chapters. Moreover, there is neither conceptual nor practical gain in categorical description here.

[21] The concept of neural network will be introduced in the last chapter. For an introduction see [53], [54]

We say $G = (V, E, l)$ and $G' = (V', E', l')$ are isomorphic if there exist a permutation $\pi \in S_n$ that relabel the vertex of G to produce the graph G'.

$1 - WL$ **algorithm** The $1 - WL$ algorithm is described by the following steps:

1. for $i = 0$, $M_i(v) := l_0 = l(v)$;
2. for $i > 0$, assign a multiset $M_i(v)$ to each vertex in G and G' consisting of the multiset $\{\{l_{i-1}(u)|u \in \mathcal{N}(v)\}\}$;
3. order the elements in $M_i(v)$ in ascending order and we form string $s_i(v)$;
4. add $l_{i-1}(v)$ to $s_i(v)$ and call the resulting string $s_i(v)$;
5. order all of the strings $s_i(v)$ to a new label, using a function $f : \Sigma^* \longrightarrow \Sigma$ such that
$$f(s_i(v)) = f(s_i(w)) \Leftrightarrow s_i(v) = s_i(w)$$
6. $l_i(v) := f(s_i(v))$ for all v in G and G'.

The Weisfeiler-Lehman algorithm terminates if the sets of newly created labels are not identical in G and G'. In this case the graphs are not isomorphic. If the sets are identical after n iterations, the algorithm gives no answer (for a graphical example see [56] p.2548).

The problem of this algorithm is that it fails in some very simple cases, as, for example, the two graphs in Fig. 2.4.

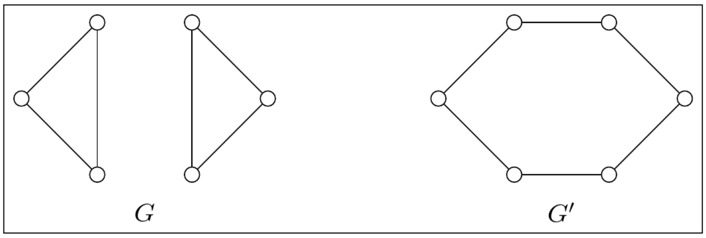


FIGURE 2.4: Two graphs indistinguishable by $1 - WL$

In [44] the authors introduce the concept of *equivariant quantum graph circuit* a simple application of which makes it possible to pass the isomorphism test:

1. associate each vertex with the quantum state $|+\rangle$;
2. apply an *edge layer* with a $C^2Z(\alpha) = diag(1, 1, 1, e^{-i\alpha})$;
3. apply a *vertex layer* with an H gate at each vertex;

4. after a single measurement, measure k vertexes as a $|1\rangle$ state and $6-k$ as $|0\rangle$. For each k, an appropriate aggregator can map this to a different prediction.

In [44] is shown as for $\alpha = \pm\pi$ the distributions of the number of $|1\rangle$s measured do differ, and an accuracy of 0.625 is achievable. Moreover, "*this would naturally get better as we increase the number of qubits used*". It is evident from the first two steps above that the variational circuit introduced is based on an encoding using weighted graph states.

The research proposal we are developing is based on three considerations:

1. in recent years, there has been a growing awareness of the need to use hypergraphs to encode information for machine learning, for example in the case of relational aspects of data to decision-making [58] of social network [59];
2. most research lines in this sense approximate hypergraphs as graphs, and simplifies the problem above to the graph embedding framework (for example [60]);
3. in [61] the authors propose an unified framework (UniGNN) for graph and hypergraph neural networks. In this work they report a generalization of the isomorphism test $1-WL$ for hypergraphs ($1-GWL$) and they prove that message-passing based UniGNNs are at most as powerful as 1-dimensional Generalized Weisfeiler-Leman ($1-GWL$) algorithm in terms of distinguishing non-isomorphic hypergraphs.

We are actually trying to evaluate whether quantum encoding via hypergraph states could bring any advantages.

2.5 Linear optics for quantum computation and quantum information

All that we have discussed up to this point has not posed the problem of a possible experimental realisation. The path we will describe in the chapter 7 has as its characteristic element, the possibility of constructing optical circuits able, at least from an ideal point of view, to realize encodings and quantum logic gates. We introduce in this section some of the main elements of single-photon computation: the short description we present has as references [62], [63], [64], [65], [66], [67], [68], [69]. Subsequently, we will show how some very recent research work is constructing, on the level of optical devices, a diagrammatic language analogous to that used for the logical description of quantum computation ([70], [71], [72]). The functors between categories will ultimately justify our educational choices from an advanced theoretical point of view.

2.5.1 Photonic qubits

There are different ways to encode information in photonic qubits. First we could consider the polarization. The basic idea is to consider the classical description of polarization using the Jones vector ([73]):

$$\vec{J} = \begin{bmatrix} E_{0,H} e^{i\phi_H} \\ E_{0,V} e^{i\phi_V} \end{bmatrix} \tag{2.37}$$

where $E_{0,H}$ and $E_{0,V}$ denote respectively the amplitude of the wave vector in the horizontal and vertical direction, and ϕ_H and ϕ_V the corresponding phases. The operation on states are descripted with the Jones matrix, defined as

$$\vec{J'} = M \vec{J} \tag{2.38}$$

It is common to normalize it to 1 at the starting point of calculation for simplification. In this way this definition is consistent of qubit and the Jones matrix is a single qubit logic gate.

Once the qubit has been encoded in polarization, a generic logic gate can be realized through the haf-wave plates. Indeed, we know that ([74]) the matrix

$$T(\phi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{bmatrix} \tag{2.39}$$

transforms a wave with field components $(E_{0,H}, E_{0,V})$ in $(E_{0,H}, e^{-i\phi} E_{0,V})$ thereby delaying the V component by a phase ϕ while leaving the H component unchanged. The H and V axes are called the fast and slow axes of the retarder, respectively[22]. If the fast axis of the wave plate is oriented along an arbitrary angle θ with respect to the horizontal axis, the transformation matrix can be determined by applying $R(\theta)$:

$$T'(\phi) = R(\phi) T(\phi) R(-\phi) \tag{2.40}$$

[22] We are assuming horizontal polarisation coinciding with the x-axis

where $R(\phi)$ is the rotation matrix.
In the case of half-wave plate ($\phi = \pi$), the equation 2.40 becomes

$$U_{HWP}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \tag{2.41}$$

where overall phases are omitted ([75]). It is straightforward to demonstrate that for $\theta = 0$, $\theta = \pi/4$ and $\theta = \pi/8$ the matrix in 2.42 represents respectively a Z-gate, a X-gate and a H-gate. In this way, HWPs allow linear polarization states to be transformed into other linear polarization states.
In the case of quarter-wave plate ($\phi = \pi/2$), the equation 2.40 becomes

$$U_{QWP}(\theta) = \begin{bmatrix} 1 + i\cos(2\theta) & i\sin(2\theta) \\ i\sin(2\theta) & 1 - i\cos(2\theta) \end{bmatrix} \tag{2.42}$$

QWPs can create circularly polarized light from linearly polarized light.
Any general unitary transformation U can be achieved by using a combination of HWPs and QWPs ([76]):

$$U = U_{QWP}(\theta_1)U_{HWP}(\theta_2)U_{QWP}(\theta_3) \tag{2.43}$$

The second way used in our work to encoded a qubit is with two spatial modes[23] ([67]): $|0\rangle_L = |1\rangle \otimes |0\rangle$ and $|1\rangle_L = |0\rangle \otimes |1\rangle$. The optical device to realize this encoding is the unpolarized beam splitter. A beam splitter is a semi-reflective mirror which splits an incident beam into two parts: a transmitted part and a reflected part. We consider our representation of beam splitter in the Fig. 2.5

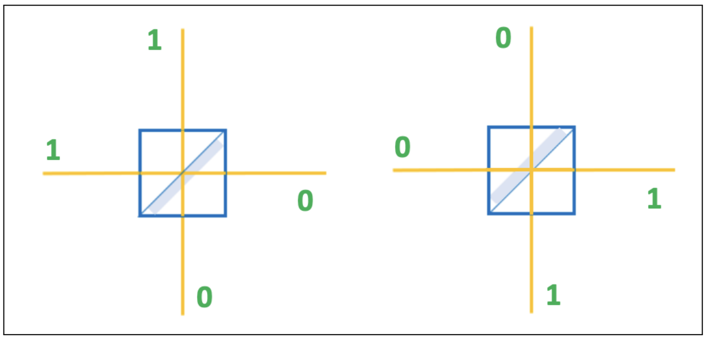


FIGURE 2.5: Qubit encoded with two spatial modes

[23] In our first educational reconstruction, we used the expression dual-rail. In Chapter 7 we will reintroduce this term for consistency with what we have done. But in future worksheets and in 3 we have abandoned this terminology.

where the reflection with phase-shifting is the path 1.

Classically, we can describe the action of the beam splitter with the Jones formalism. We have in input the electromagnetic field

$$\vec{J} = \begin{bmatrix} E_{0,0} \\ E_{0,1} e^{i\phi} \end{bmatrix} \tag{2.44}$$

where the vector components are the electromagnetic components on two paths. For simplicity of notation, we can write

$$\vec{J} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.45}$$

where $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ represent the path 0 and 1 in 2.5, and $a^2 + b^2 = 1$

In this way we can introduce the beam splitter matrix assuming

$$BS = \begin{bmatrix} r & t \\ t & -r \end{bmatrix} \tag{2.46}$$

where r and t are the reflection and transmission coefficients, satisfying the condition $r^2 + t^2 = 1$. The general description of the beam splitter action is obtained as in the figure 2.6:

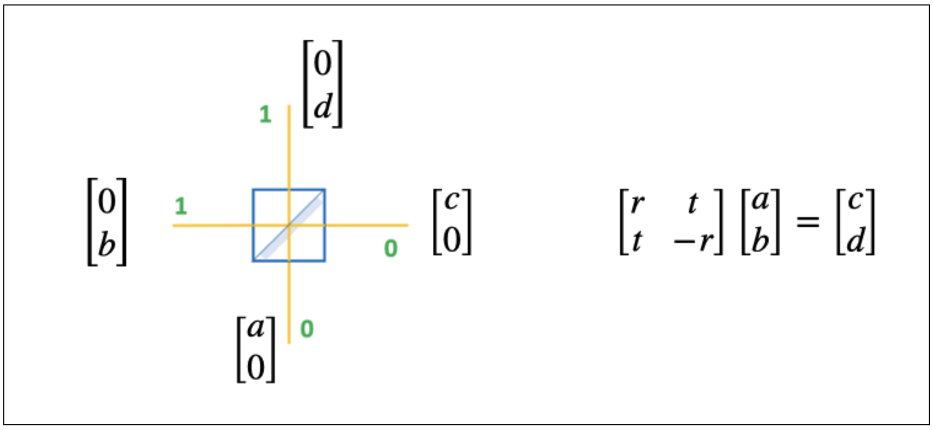


FIGURE 2.6: Lossless general beam splitter

In the particolar case in which $r = t$, we obtain the 50-50 beam splitter . It is straightforward to see that the beam splitter matrix logically represents a H-gate. Indeed, the single-photon interpretation is immediate: if the source is at the bottom, we encode the qubit $|0\rangle_L$; if it is on the left the qubit $|1\rangle_L$. The quantum model of the interaction in the figure 2.6 produces the following Heisenberg evolution of the mode operators (see [64] pag. 104-105)

$$\widehat{a} \longmapsto r\widehat{a} + t\widehat{b}$$

$$\widehat{b} \longmapsto t\widehat{a} - r\widehat{b}$$

with, for $\widehat{a}$, $[\widehat{a}_i, \widehat{a}_j^\dagger] = \delta_{i,j}$; analogously for $\widehat{b}$.
These equations imply the following qubit evolution

$$\begin{aligned} |10\rangle &\longmapsto r|10\rangle + t|01\rangle \\ |01\rangle &\longmapsto t|10\rangle - r|01\rangle \end{aligned} \tag{2.47}$$

The equations 2.47, describe, for $r = t$ the H-gate action on the logical qubits $|0\rangle_L$ and $|1\rangle_L$.
Moreover, a simple phase shift of π to achieve the Z-Gate as well. The X-gate can be considered a simple labelling change (for these two gates see 7).
It should be noted that in our algorithms dual-rail coding always has two H gates: one at the beginning for superposition and the other at the end for interference: this from an experimental point of view leads us to introduce a Mach-Zehnder interferometer, which is a key element of our ideal designs with optical devices (see Fig. 2.7).

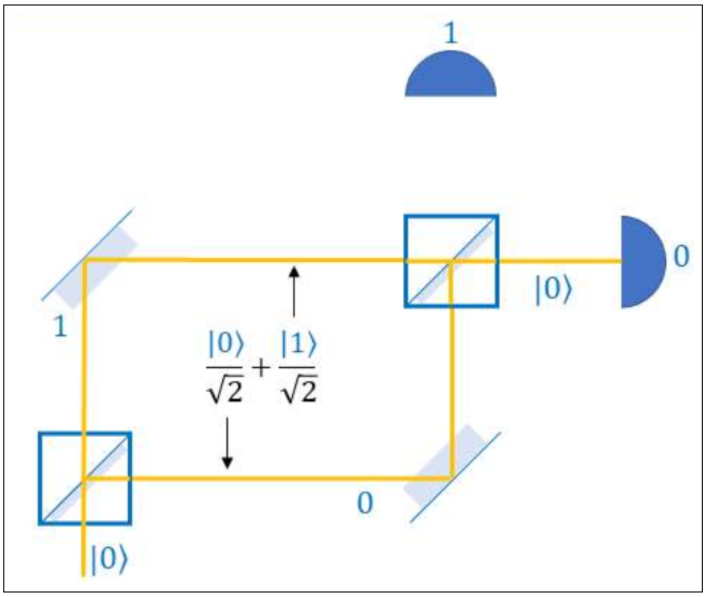


FIGURE 2.7: Mach-Zehnder interferometer

The last device we have to introduce is the polarizing beam splitter (PBS). A PBS splits a beam depending on its polarizations, usually separating an input beam into two modes with orthogonal polarization. Light that is vertically polarized is reflected, whereas horizontally polarized light is transmitted through a PBS. We will use this devise for measurement in polarization.

2.6 Conclusions

We introduced quantum computation and quantum information theory related to the theoretical and experimental elements that will be the object of educational reconstruction for instruction. The approach given in this chapter follows the traditional one. However, we use diagrammatic representations in a deeper way, so much so that we consider them as a *model*. In order to understand the meaning of this approach, the following two chapters present the topics from a categorical point of view.

Chapter 3

Categories for sciences

3.1 Introduction

In this chapter, we introduce the categorical language[1] that aims to unify the several levels of our educational reconstruction for instruction. In particular, we show how category theory is naturally suited to describe computation from both a logical-formal and physical-experimental point of view. For this reason, we present two original examples of computation using finite-state machines and neural networks: we hope in this way to make the role of category theory for the description of sequential and parallel compositional processes more evident. The detailed study of these two examples anticipates the link with the physical theory of computation, whether classical or quantum.

3.2 The role of the category theory for the thesis

When category theory was born some eighty years ago, it represented a useful synthesis of topology and algebraic approaches. It soon began to show its strength within mathematics, first with Grothendieck's approach to cohomology ([77]) and then with Lawvere, who posed the problem of category theory as a foundation for mathematics ([78]):

> "In the mathematical development of recent decades one sees clearly the rise of the conviction that the relevant properties of mathematical objects are those

[1] The complete presentation of the category theory concepts used in the following chapters can be found in Appendix C

> which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were thought to be made of. The question thus naturally arises whether one can give a foundation for mathematics which expresses wholeheartedly this conviction concerning what mathematics is about."

In the early 1980s, J.Lambek showed that the type and programs used in computer science form a specific kind of category. Lambek himself later became one of the most important supporters of the role of categories in the transition from linguistics to physics. This is precisely one of the aspects that have most influenced this thesis from a general point of view: it is evident from the work of the last few years that category theory is a natural environment in which to describe the multidisciplinarity of our path. This approach takes its cue, from an application point of view, from research work on, for instance, categorical quantum mechanics applied to linguistics or machine learning. Lambek himself ([79]) describes the use of compact monoidal categories to link the Chomsky's linguistics to physics. Other works in this direction have been emerging in recent years (e.g. ([80], [81], [82], [83], [84], [85], [86], [87], [88], [89], [90]).
More generally, a line of research related to applied categories is spreading, which aims ([20]) to integrate knowledge across disciplines. As the author points out, in recent years category theory has developed in chemistry, neuroscience, biology, natural language processing, database theory *et cetera*. This is with the intention of

> "use of categorical concepts as a natural part of transferring and integrating knowledge across disciplines. The restructuring employed in applied category theory cuts through jargon, helping to elucidate common themes across disciplines. Indeed, the drive for a common language and comparison of similar structures in algebra and topology is what led to the development category theory in the first place, and recent hints show that this approach is not only useful between mathematical disciplines, but between scientific ones as well."

David Spivak himself in [91] describes the intention of the book as follows:

> "This book is intended to create a bridge between the vast array of mathematical concepts that are used daily by mathematicians to describe all manner of phenomena that arise in our studies and the models and frameworks of scientific disciplines such physics, computation, and neuroscience."

It is precisely in physics, and in fundamental of quantum theory in particular, that category theory has become widespread over the last 15 to 20 years. If this is obvious for example in ([92], [93]), it is also evident in [32] [94] where the categorical framework is only mentioned.

In agreement with the previous considerations the chapter develops along the following sequence: first, in section 3.3, we intuitively introduce the compositional approach typical of category theory in a very concrete case, the preparation of a recipe. Then, in 3.4, we describe in a totally abstract manner the basic mathematical structure that we will use in the following chapters[2]: the monoidal categories and functors between categories. We will interpret these structures from a diagrammatic point of view, emphasising that everything that can be done in theory has a translation into diagrams and *vice versa*. Next, in 3.5, we will give a more rigorous example of how these structures can be used to link linguistics to quantum mechanics. The example begins to show the strength of the language introduced: the possibility of linking similar structures born in different fields. Abstract language is interpreted into specific categories and these are connected by functors. The final two sections, 3.6 and 3.7, we develop with categorical methods known in the literature two original examples of universal computation: the first, finite state machine, related to classical sequential logic, the second, neural network computer, to more traditional Boolean logic.

3.3 A world of processes

In order to understand the diagrammatic approach in an intuitive and non-rigorous manner, we will begin by describing the realisation of a recipe as some teachers proposed during an hour-course on the relationship between logic and physics. During that lesson, the teachers could construct classical logic using the Set category (the same one that allows classical physics to be analysed). The example drew attention to the possibility of representing a process (in that case, the preparation of a recipe) that a diagram could describe. This particular diagrammatic approach is at the heart of the path we are presenting and has its origins in the concept of the monoidal category (see [95] for the foundation and for example [96] for the graphical languages).

During a meeting with the teachers, they were asked to describe the preparation and realisation of a recipe: the teachers proposed *pasta alla carbonara*.

In the description of the recipe, we focused our attention on the fundamental elements

[2]The complete presentation of the category theory concepts used in the following chapters can be found in AppendixC

that, when reread rigorously, allowed us to understand the nature and structure of the monoidal category[3]:

Ingredients The objects of our recipe are, of course, the ingredients that are needed to make it.

Preparation We are not interested in the history of these ingredients (where they were bought, when, etc.), but we know that they are available to the cook in a certain way.

Cooking, actions on ingredients etc. At this point, the cook begins to use different ingredients for different procedures that can be carried out in sequence but also in parallel to be combined at the end.

Final result Ultimately, we will have the *pasta carbonara* dish, and nothing else matters (whether someone will eat it or it will be used as an example for a cooking video).

We show in Fig. 3.1 a useful representation of these processes.

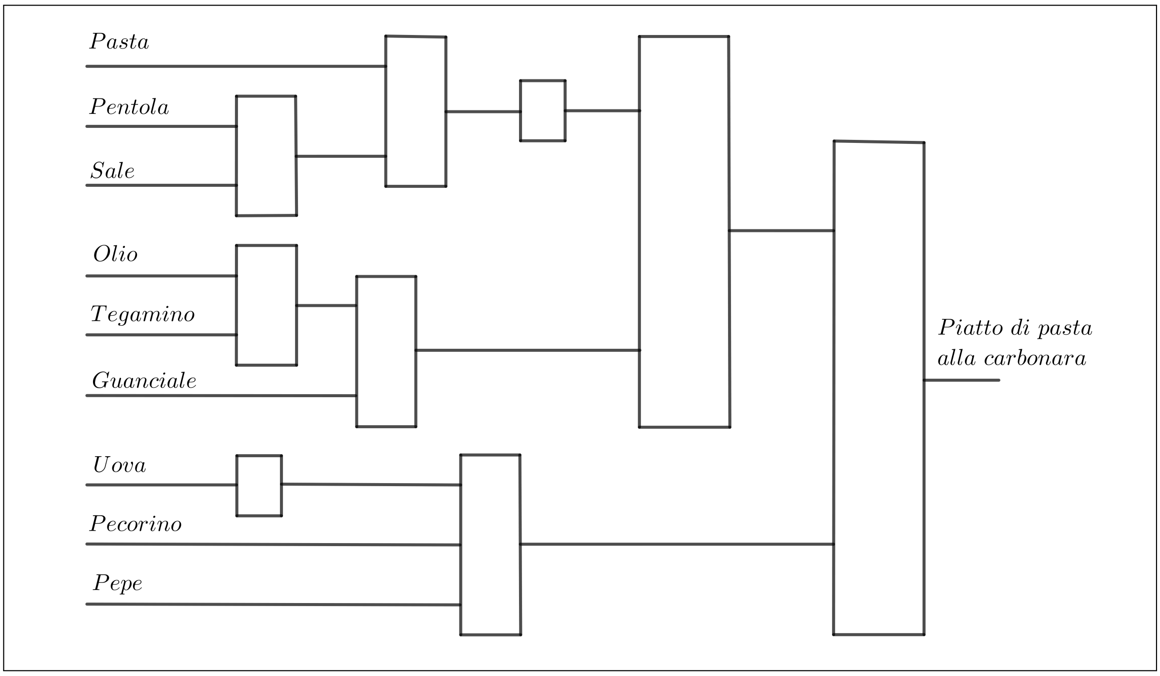


FIGURE 3.1: Recipe of *pasta alla carbonara*

As we have said, the proposed example does not claim to be rigorous or exhaustive. However, it is representative of the approach used to describe the physical processes of

[3]We will give the definition in the next section.

computation and can be extended more generally to any kind of processes. The reason why this is possible lies in the fact that the underlying mathematical structure is totally abstract. And we like to say: "*more abstract, more concrete*".

3.4 The model: symmetric monoidal category and diagrammatic representations

In this section we give the essential definitions that are necessary for a first understanding of the monoidal approach. These definitions will also always be presented in the form of diagrams, diagrammatic representations being fundamental in our work. The works from which we take our definitions and main results are the classic books on category theory and some recent works with a more applied approach (see [97], [98], [99], [100], [93], [101], [102], [91]). The approach will be essentially theoretical, but to make the reading more usable we will use, whenever possible, the example of the recipe and a particular category: ***Set***.

3.4.1 Definition of category and diagrammatic representation

To start modelling our recipe, we need an environment to define the ingredients and possible procedures on them. In addition, we need to consider the possibility of operating several actions in succession: for example we can take the *guanciale*, dice it and then put it in a pan. Mathematically, what we need is objects, transformations (including identity) and sequential composition.

Definition 3.1. A *Category* $\mathfrak{C}$ consists of the fallowing data:

- *Objects*: $A, B, C, ...$, constituting the collection $Ob(\mathfrak{C})$;
- *Arrows* or *morphisms*: $f, g, h, ...$, constituting the collection[4] $Ar(\mathfrak{C})$;
- a pair of mapping dom, $cod : Ar(\mathfrak{C}) \longrightarrow Ob(\mathfrak{C})$ which to each arrow f assign its domain and codomain. If $f : A \longrightarrow B$ we call $A = dom(f)$ and $B = cod(f)$. $\forall A, B \in Ob(\mathfrak{C})$ we define

$$\mathfrak{C}(A, B) := \{f \in Ar(\mathfrak{C}) | f : A \longrightarrow B\}$$

[4]Regarding the need to refer to a collection and not to a set of objects and morphisms see [93] pag. 3 or [101] pag. 6

This set is the ***hom-set***;

- for any object $A \in Ob(\mathfrak{C})$, a ***identity morphism*** $id_A : A \longrightarrow A$ is designated;
- for any pair of morphisms $f \in \mathfrak{C}(A, B)$ and $g \in \mathfrak{C}(B, C)$, there is an arrow $h \in \mathfrak{C}(A, C)$ ***composition*** of f and g:

$$h := g \circ f : A \longrightarrow C$$

These data are required to satisfy the following axioms:

Unit: $\forall\ f \in \mathfrak{C}(A, B),\ f \circ id_A = f = id_B \circ f;$

Associativity: $\forall\ f \in \mathfrak{C}(A, B),\ \forall\ g \in \mathfrak{C}(B, C),\ \forall\ h \in \mathfrak{C}(C, D),$

$$(h \circ g) \circ f = h \circ (g \circ f)$$

A more rigorous example is that of the category ***Set*** whose objects are the sets and morphisms the functions between sets. We can introduce a diagrammatic language to represent a generic category that will be extremely useful below and is a distinctive characteristic of the work we present[5].
Following the definition, we represent the *data* as in Fig. 3.2.

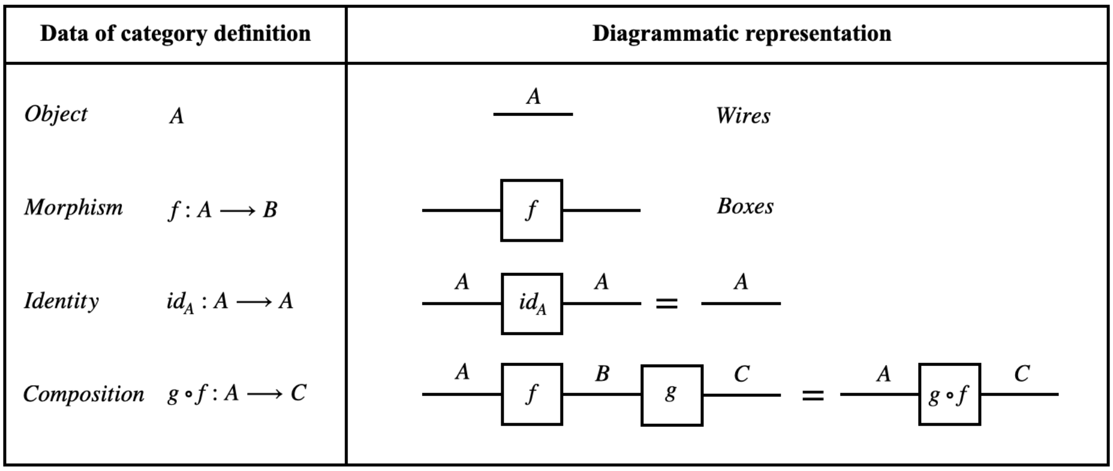


FIGURE 3.2: Diagrammatic representation of *data* in the category $\mathfrak{C}$ definition

Similarly, the axioms in Fig. 3.3.

[5]The abstract representation has immediate interpretation both in our recipe and in the category ***Set***

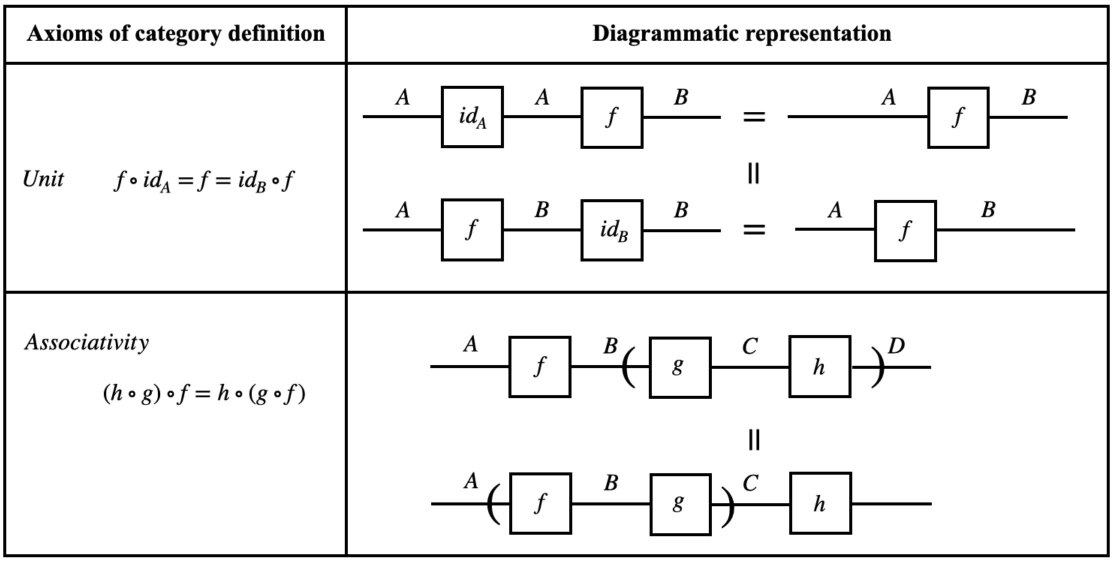


FIGURE 3.3: Diagrammatic representation of *axioms* in the category $\mathfrak{C}$ definition

Let us now introduce the concept of a functor, i.e. a map between categories.

Definition 3.2. A ***functor*** $\mathfrak{F} : \mathfrak{C} \longrightarrow \mathfrak{D}$ between categories $\mathfrak{C}$ and $\mathfrak{D}$, consists in a mapping of object to object and arrows to arrows in such a way that composition and identities are preserved. This means that:

- $\mathfrak{F}(f) : \mathfrak{F}(A) \longrightarrow \mathfrak{F}(B)$, to every $f \in \mathfrak{C}(A, B)$;
- $\mathfrak{F}(id_A) = id_{\mathfrak{F}(A)}$, to every $A \in Ob(\mathfrak{C})$;
- $\mathfrak{F}(g \circ f) = \mathfrak{F}(g) \circ \mathfrak{F}(f)$, to every $f \in \mathfrak{C}(A, B)$ and $g \in \mathfrak{C}(B, C)$.

A functor $\mathfrak{F}$ represents a way of interpreting the category $\mathfrak{C}$ in the category $\mathfrak{D}$. In this regard, one of the most frequently used possibilities is to interpret a category in ***Set***. We shall see in 3.10 the usefulness of this approach.

Let us now continue with our recipe. So far we have focused on a single ingredient, but it is clear that this is extremely limiting in cooking! It is clear that we want to use several ingredients at the same time in our cooking and perform several procedures together. This is the sense of introducing parallel composition on objects (the ingredients) and morphisms (the procedures). In ***Set*** this is equivalent to considering ordered pairs of sets and ordered pairs of functions, i.e. in defining the Cartesian product. Let us now come to the general definition.

In the previous section, we defined a category basically as a quadruple $\mathfrak{C} = (Ob(\mathfrak{C}), Ar(\mathfrak{C}), id, \circ)$. This definition allows for the possibility of composing in morphisms sequentially. However, for our purposes we also need to be able to compose objects and morphisms in parallel. We therefore introduce the concept of ***monoidal category***:

Definition 3.3. A *monoidal category* consists of the fallowing data:

- a category $\mathfrak{C}$;
- a functor $\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$ called ***tensor product***;
- an object $I \in \mathbf{C}$, called ***unit object***;
- a family of natural isomorphisms $\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$ for any triplet of objects $A, B, C \in \mathbf{C}$, called *associators.*
- a family of natural isomorphisms $\lambda_A : I \otimes A \longrightarrow A$ for each $A \in \mathbf{C}$, colled *left unitor* ;
- a family of natural isomorphisms $\rho_A : A \otimes I \longrightarrow A$ for each object $A \in \mathbf{C}$, called *right unitor.*

These data are required to satisfy the triangle equation and the pentagon equation (see Appendix C).

As before, we introduce the corresponding diagrammatic language for *strict*[6] *monoidal categories.* Following the definition, we represent the data as in Fig. 3.4
Similarly, the axioms in Fig. 3.5 and Fig. 3.6
Here we would like to interpret some of the elements introduced in our preparation of a recipe and in the category ***Set***. The unit object is used to indicate a certain preparation, a state in which an ingredient is: we are not interested in the previous history of the ingredient (where it was bought, whether it was put in the fridge and the like), we are only interested in saying that on our kitchen counter we have it. The same goes for the final outcome of our recipe: when it is finished, we are not interested in knowing what will happen to the dish of pasta carbonara: whether it will be served at the table and eaten or used as a demonstration on a television programme.

[6]See AppendixC

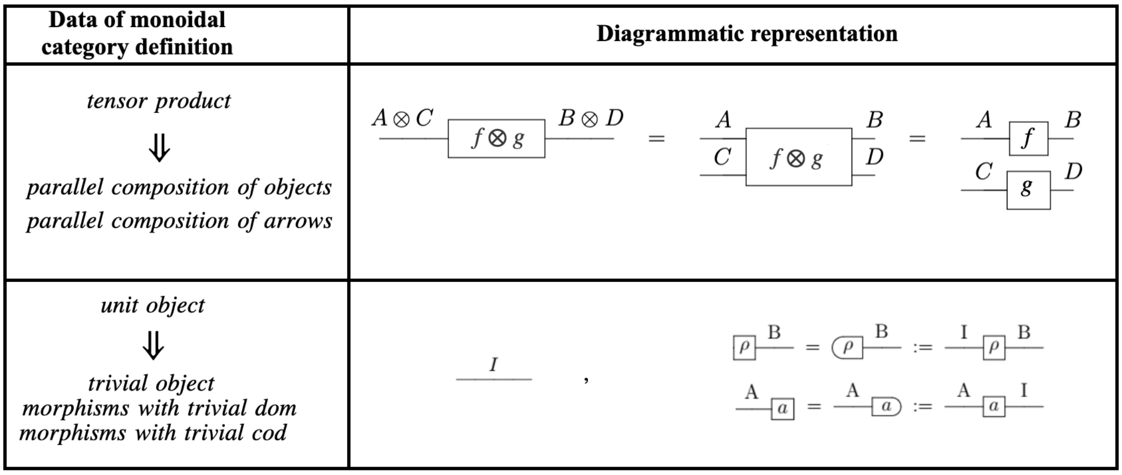


FIGURE 3.4: Data representation in the monoidal category definition

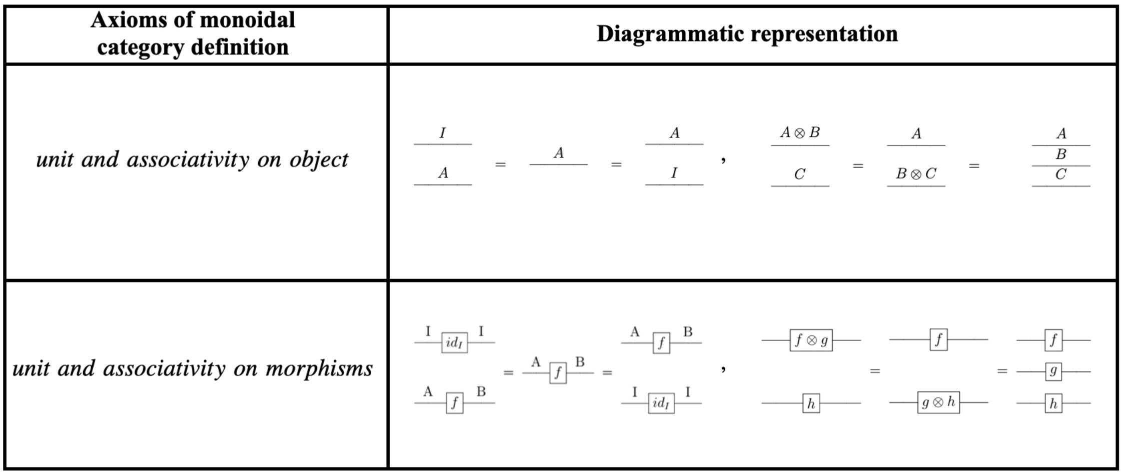


FIGURE 3.5: Axioms representation in the monoidal category definition

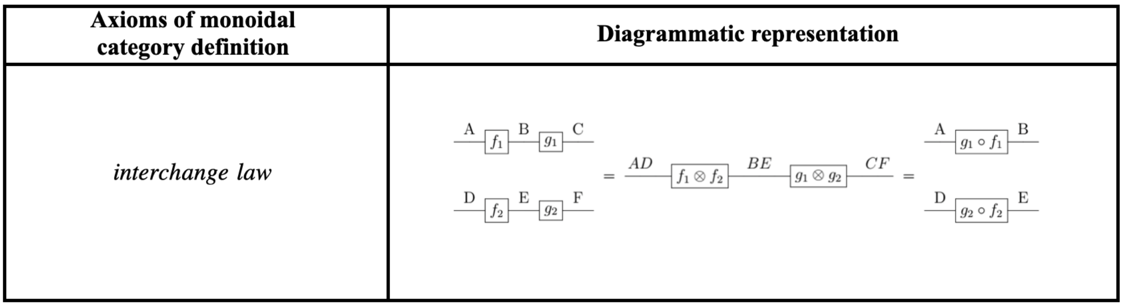


FIGURE 3.6: Interchange law representation in the monoidal category definition

With regard to ***Set***, the unit object is the *singleton* set $\{\bullet\}$, used to indicate an element of a set:

$$\{\bullet\} \longrightarrow A$$

In this way an element is a morphism. We will see the importance of this definition for the introduction of the concepts of states and effects.

To finish the presentation of what will be the main categorical framework, we need to introduce the concept of the monoidal ***symmetric*** category:

Definition 3.4. A *symmetric monoidal category* $\mathfrak{C} = (Ob(\mathfrak{C}), Ar(\mathfrak{C}), id, \circ, \otimes, \; I, \sigma)$ is a monoidal category $\mathfrak{C}$ with a *natural isomorphism* ***SWAP***

$$\sigma : Ob(\mathfrak{C}) \times Ob(\mathfrak{C}) \longrightarrow Ob(\mathfrak{C}) \times Ob(\mathfrak{C})$$

$$A \otimes B \longmapsto B \otimes A$$

satisfying $\forall \; A, B \in Ob(\mathfrak{C})$ the axioms:

- $\sigma_{B,A} \circ \sigma_{A,B} = id_{A \otimes B} \quad\quad \sigma_{A,I} = id_A$;
- $(g \otimes f) \circ \sigma_{A,C} = \sigma_{B,D} \circ (f \otimes g)$;
- $(id_B \otimes \sigma_{A,C}) \circ (\sigma_{A,B} \otimes id_C) = \sigma_{A,B \otimes C}$.

Diagrammatically, the $SWAP$ corresponds to a swap of wires and it is involutory.
In the more general case of isomorphism instead of identity in the definition we speak of monoidal braided category.

Let's go to see the most important result that accounts for the strength of diagrammatic representation in relation to theory:

Theorem 3.5 (Correctness of graphical calculus for braided (symmetric) monoidal categories)**.** *A well-typed equation between morphisms in a braided monoidal category follows from the axioms if and only if it holds in the graphical language up to spatial isotopy (graphical equivalence).*

Observation 3.6. We use the notion of isotopy because we assume the diagrams lie in a cube in the three-dimensional space: the input wires terminate in the left face and the output in the right face. This is also called *spatial isotopy* (We talk about *graphical equivalence* if there is a spacial isotopy and $\sigma_{A,B} = \sigma^{-1}_{B,A}$).

What we have seen so far allows us to introduce diagrammatic representations such as those in Fig. 3.7:

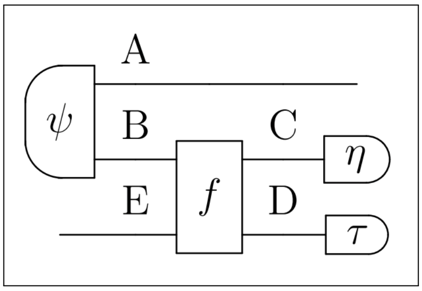


FIGURE 3.7: Circuit diagram with casual structure from left to right

The following examples refer to concepts introduced in the chapter and others that can be found in the Appendix C

3.5 From linguistics to Physics

Let us briefly propose an example taken from linguistics itself that we hope will clarify the approach that formed the background to our work and that will recall some of the elements given in the preceding paragraphs and give an initial concrete interpretation.

In 1956, Chomsky [103] proposed the linguistic theory of types. According to this approach, the words of a language are characterised by their function. It became immediately evident that this approach represented a computational view of the language problem. A few years later, Lambek [104] mathematised Chomsky's work. The idea is to assign as in primitive types:

1. s, the type of sentences;
2. n, the type of names.

"*From the primitive types we form compound types, by the recursive definition: If x and y are types, then so are x/y (read x over y) and $y\backslash x$ (read y under x)*".
In this way, for example, a transitive verb is always preceded by a name (subject) and followed by a name ('object'), whereas an adjective is preceded by a name. According to the above definitions we can, as in [104], express the sentence "John likes fresh milk" in

this way:

$$John\ (likes\ (fresh\ milk))$$

$$n \quad (n/s\backslash n \quad (n/n \quad n))$$

By simplifying algebraically we obtain a grammatically correct sentence:

$$n \quad (n/s\backslash n \quad (n/n \quad n)) \longrightarrow n \quad (n/s\backslash n) \quad n \longrightarrow n \;\; n/s \longrightarrow s$$

The mathematical structure recognised by Lambek forty years later ([105]) is that of the pregroup, which from a categorical point of view is nothing more than a particular compact closed monoidal category. The transition to a diagrammatic representation and the study of natural language (meaning of a sentence) by means of diagrammatic quantum computation would have been immediate ([82], [106]). Let us briefly discuss the idea.

Let us consider an even simpler example than the previous one: "*Claudio loves chocolate*". Our aim is to understand the meaning of this sentence, especially in relation to grammatically correct and equivalent sentences such as "*Cats love water*". First, we move from syntax to semantics for individual words using the *distributional model of meaning*. The idea is to associate a word with a vector, the components of which show how many times that word in a given text is associated with certain words (chosen as a base). The idea is that the meaning of a word depends on its context. From the category of pregroups, we have moved in this way to the category of vector spaces and linears maps thanks to a functor that reinterprets the first category in the second ([107], [20]):

$$sintax \longrightarrow semantics$$

$$Preg(P, \leq, \otimes) \longrightarrow FVec(V(\mathbb{K}), f, \otimes)$$

Of course we cannot use the distributional model of meaning to determine the meaning of a sentence. However, the compositionality aspect typical of the categorical approach helps (see [108]): *the meaning of a (syntactically complex) whole is a function only of the meanings of its (syntactic) parts (object amd morphism) together with the manner in which these parts were combined (parallel composition).*

Using this principle, we can consider language in its two components, syntax and semantics, in the product category $FVec \times P$:

$$\begin{tikzcd} & language \arrow[dl, "semantics"'] \arrow[d] \arrow[dr, "sintax"] & \\ FVec & FVec \times P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & P \end{tikzcd}$$

and thus the meaning of a sentence is a morphism in this category, i.e. "*a sentence is a process that alters the meanings of its words*" ([107]).
If we use the category structure associated with quantum physics, the Choi–Jamiołkowski isomorphism (compactness of category) allows morphisms (verbs) to be understood as states (names). The diagrammatic representation is straightforward (Fig. 3.8).

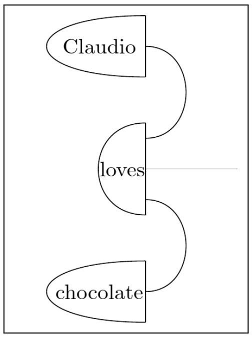


FIGURE 3.8: An example of quantum circuit for natural language processing

If we were interested in the truth value of the sentence, we would carry out a measurement, i.e. we would close the diagram and obtain a number (0 or 1, unless a scalar) (for details see [107]).
To understand why, according to this line of research, it makes sense to describe natural language by means of diagrams of a quantum nature, consider an extremely significant sentence:

> "This interaction logic is a very novel feature that in most sciences has never been identified until recently. The reason being that exact sciences are still highly reductionist, describing things by their make-up, rather than focusing on interactions of systems. In the case of language it is clear that words interact tremendously, as witnessed by the wire structures in our diagrams. Similarly, the same diagrams appear when we represent quantum teleportation. As Schrödinger pointed out, in quantum theory we simply can't ignore the

particular nature of how systems interact, and he even called it the defining feature of quantum theory."

and more:

> "What quantum theory and natural language share at a fundamental level is an interaction structure. This interaction structure, together with the specification of the spaces where the states live, determines the entire structure of processes of a theory. So the fact that quantum theory and natural language also share the use of vector spaces for describing states—albeit for very different reasons—makes those two theories (essentially) coincide."

This interaction logic is expressed in the already cited *compositionality*, which in this case we can summarise with a slogan:

Grammar is all about how word meanings interact.

What we deduce is that the natural environment of language is to be found in a mathematical structure able of accounting for the relationships between elements and the possibility of composing parts of speech and whole sentences. The closed and compact monoidal category structure seems extremely suitable for this purpose. This structure is also used in some approaches to quantum machine learning, as we will see in 3.7.

3.6 Finite state machines

This section aims to introduce the Finite State Machine (FSM) by giving some examples and showing how the categorical language also provides a useful tool for formally defining them. In particular, we will introduce classical logic gates by applying Grothendieck construction to the concept of a finite-state machine. This is the first of two examples of applications of the categorical language for describing classical computation.

As R. Feynman in [39], we can represent a *FSM* as a (black) box with wire in input and output: inside the box we can represent, in analogy to what is done for example in thermodynamics, the transition from a state Q to a state Q':

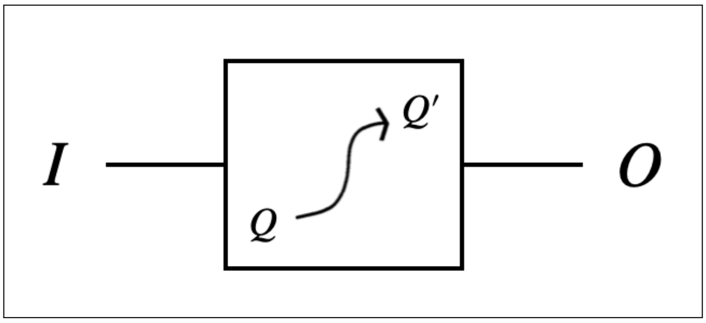


FIGURE 3.9: Finite state machine

We can interpret Fig. 3.9 from an compositional[7] point of view in this way:

- the machine starts off in a certain state, Q;
- it then receives an input I, a bit of information;
- the machine reacts to this input by changing to another state, Q';
- it spits something out - a response O to the input I.

In order to develop classical computation, let us consider the following *FSM*:

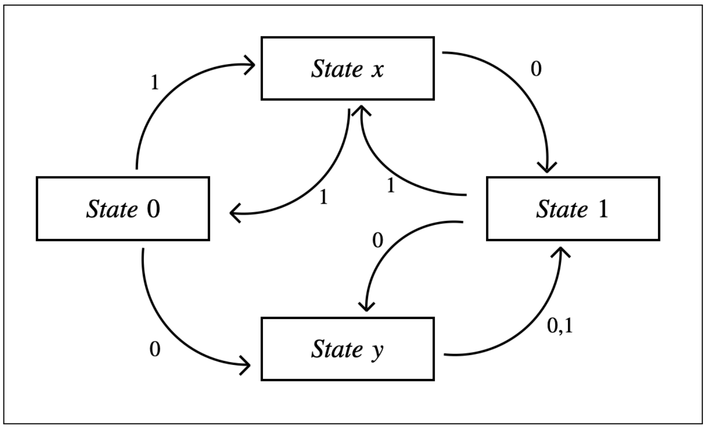


FIGURE 3.10: NAND Finite state machine

implementing a *NAND-gate* in sequential logic[8]. We can describe the *FSM* in Fig. 3.10 with a table, known as the action table (see Tab. 3.1).

[7]We compose more processes in sequence if we think of an input state as a preparation and an output as a observation!

[8]For an introduction to sequential logic see [109] or [110]

TABLE 3.1: Action table of $NAND - FSM$

Action table		
Id	**0**	**1**
State 0	State y	State x
State x	State 1	State 0
State y	State 1	State 1
State 1	State y	State x

In the Tab. 3.1 the first column represent the outcomes of the action of *identity* function (no input), the other two the outcomes of the action of the inputs **0** and **1**. Another way to describe the Fig. 3.10 is as follows (Tab. 3.2)

TABLE 3.2: Truth-table of NAND gate resulting from the *FSM* in Fig. 3.10

Table of Fig. 3.10			
Current state	**Input**	**Input**	**Final state**
State 0	0	0	State 1
State 0	0	1	State 1
State 0	1	0	State 1
State 0	1	1	State 0
State 1	0	0	State 0
State 1	0	1	State 0
State 1	1	0	State 0
State 1	1	1	State 1

which expresses compositionally the truth table of the NAND logic gate (the first line, for example, means that if we apply 0 twice to *State* 0 we get *State* 1).
The table Tab. 3.1 allows us to think about the action of a particular monoid on the set of states $S = \{State\ 0,\ State\ x,\ State\ y,\ State\ 1\}$. The monoid must consider that once we have fixed a state, it must be possible to apply the input 0 or 1, either once or in all their possible combinations. For this reason we can represent the monoid as in Fig. 3.11.

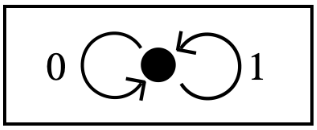


FIGURE 3.11: Free monoid on $\{0, 1\}$

Fig. 3.11 is nothing more than the diagrammatic representation of what we already know (see for example [97],[85]), namely that a monoid is a category with an object.

To better understand the meaning of this representation, let us introduce the definition of *List* of a set:

Definition 3.7. Let I be a set. A *list in* I is a pair (n, f) where $n \in N$ is the leght of the list, and $f : \underline{n} \longrightarrow I$ is a function. In this way a list

$$(n, f) = [f(1), f(2), ..., f(n)]$$

and $List(I)$ is the set of lists in I.

The monoid of Fig. 3.11 is denoted by $List(\{0, 1\})$, if we consider the notations [85].
The action represented in Tab. 3.2 is the action of $List(\{0, 1\})$ on S and this action is equivalent to give a function $\delta : \{0, 1\} \times S \longrightarrow S$.
From this it immediately follows that we can interpret a finite state machine defining the *NAND-gate* as the action of a free monoid on the set $\{0, 1\}$. This according to the definition in [111]:

Definition 3.8. A *finite state machine* over the finite alphabet Σ[9] is a system $M(S, s_0, \delta, F)$ where:

- $S \neq \emptyset$ is the *state set*;
- $\delta : \Sigma \times S \longrightarrow S$ is the *state-transition function*;
- $s_0 \in S$ is the *initial state*;
- $F \subseteq S$ is the *set of final states.*

Since every action of a monoid coincides with a functor from the monoid M to the set category[10]

$$F : M \longrightarrow \mathbf{Set}$$

we can exploit the Grothendieck construction (see [91], [112], [113]) to define finite state machines on a high level of abstraction, and thus, as a special case, classical logic gates.
We recall here the Grothendieck construction, also known as the category of elements:

Definition 3.9. Let M be a category, and let $F : M \longrightarrow \mathbf{Set}$ be a functor. The *category of elements of* F $\int_M F$, is the category in which:

[9] In our example $\Sigma = \{0, 1\}$
[10] The table 3.2 records the functor!

objects $Ob(\int_M F) := \{(m, x) \mid m \in Ob(M), x \in F(M)\}$;

arrows $Hom_{\int_M F}((m, x), (m', x')) := \{f : m \longrightarrow m' \mid F(f)(x) = x'\}$

We can then give the abstract definition of a *finite state machine*:

Definition 3.10. A *finite state machine* is the category of elements of the functor F.

In the case of logic gates, we need only consider $M = List(\{0, 1\})$. In this way the Fig. 3.10 is a representation of the category defining the abstract concept of the *NAND* state machine.

3.6.0.1 Summary and generalisation

If we consider the sequential logic as in [109], a logic gate is a *FSM*. In fact we can introduce a set S, set of states, and two possible inputs in $\Sigma\{0, 1\}$, functions that can be implemented an arbitrary number of times, possibly even none, with any sequence. This suggests considering a monoid structure, precisely the *free* monoid generated by Σ, $M = (List(\Sigma),\ [\ ],\ \circ)$, where $\circ$ in this case is the lists concatenation. A monoid is a category with an object and thus we can represent it by means of Fig. 3.11.
However, this is not enough because we do not know how the monoid acts (how inputs act on states); therefore we introduce the action of the monoid as a functor

$$F : M \longrightarrow \mathbf{Set}$$

which is recorded by the action table 3.1.
A *NAND-FSM* is the functor F recorded by the previous table, i.e. it is the same as the action of a free monoid $M = (List(\Sigma),\ [\ ],\ \circ)$ on $S = \{State\ 0,\ State\ x,\ Statey,\ State1\}$. For a correct representation of such a *FSM*, we make use of the Grothendieck construction, i.e. we represent in the same diagram the objects and arrows of the category of the element of the functor F. In this way, the Action table in Tab. 3.1 become Fig. 3.10.

Observation 3.11. The construction performed - Grothendieck construction - can not only be trivially applied to any logic gate, but can be extended to any category, including monoidal ones (see [114],[91] and [112]).

Observation 3.12. The introduction to the logic gates of compositional logic shows one of the two specific aspects of the categorical approach to computation: *sequential composition*. We shall now show another way of introducing classical logic, which allows, thanks to *parallel composition*, the well-known logic circuits to be obtained.

3.7 Neural network computer

This section aims to introduce *Classical Neural Network (CNN)* by giving in particular the construction of *perceptron* and showing how the categorical language also provides a useful tool for formally defining them. As mentioned above, this example shows the need to extend the category concept to the monoidal category to account for the combinational logic.
In analogy with biological neurons (see [115], [116], [54], [53]), we can introduce a simple artificial neuron. The fundamental unit is the perceptron, the artificial equivalent of biological neurons. Synapses can be matched with weights, so that each input is multiplied by a weight before being sent to the equivalent of the cell body. Here, the weighted signals are summed together to supply a node activation (Fig. 3.12):

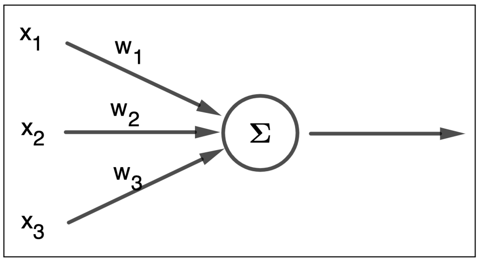


FIGURE 3.12: Simple artificial neuron: perceptron

The neuron's output y - 0 or 1 - is determined by whether the weighted sum is less than or greater than some *threshold value* T:

$$y = \begin{cases} 0, & if \quad \sum_{i=1}^{n} x_i w_i \leq T \\ 1, & if \quad \sum_{i=1}^{n} x_i w_i > T \end{cases}$$

Expressing in the Fig. 3.12 whether the neuron is activated or not, we obtain Fig. 3.13.

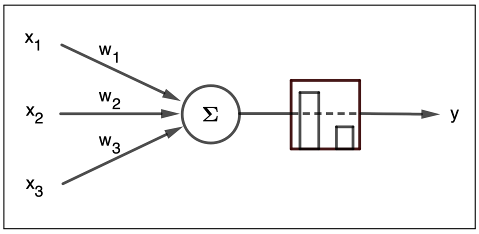


FIGURE 3.13: Perceptron: complete description

Within the node, we can indicate the difficulty of activating it, what is called the *perceptron's bias*: the higher the value, the easier it is for the neuron to be activated.
We observe that we can consider in Fig. 3.12 inputs as nodes with only outputs and thus obtain a representation that for our purposes will be more interesting (Fig. 3.14):

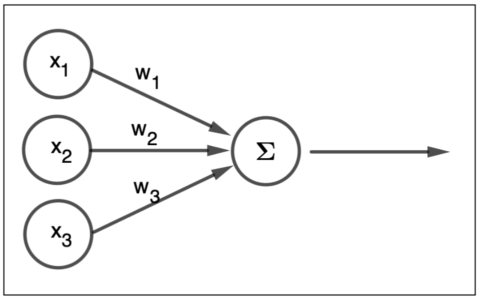


FIGURE 3.14: Perceptron: advanced description

Similarly, we can interpreter outputs as nodes without outputs.
For example, we can consider for example the *NAND*-gate[11] (Fig. 3.15):

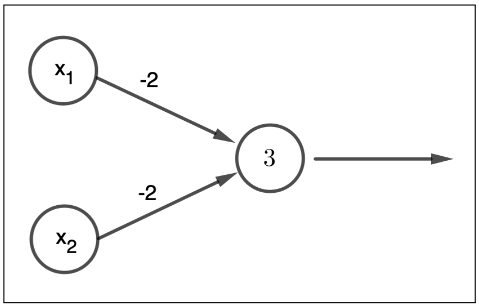


FIGURE 3.15: NAND-gate with perceptron

Because *NAND*-gates are universal for computation, it follows that the perceptrons are also universal for computation. Using the *sequential* and *parallel* composition of the perceptrons, we obtain a neural network, in strict analogy with the biological one, realising the binary sum algorithm.

[11] This example from [53] is very interesting because it shows what R. Feynman did in [39] with strips and pebbles to introduce the binary sum algorithm.

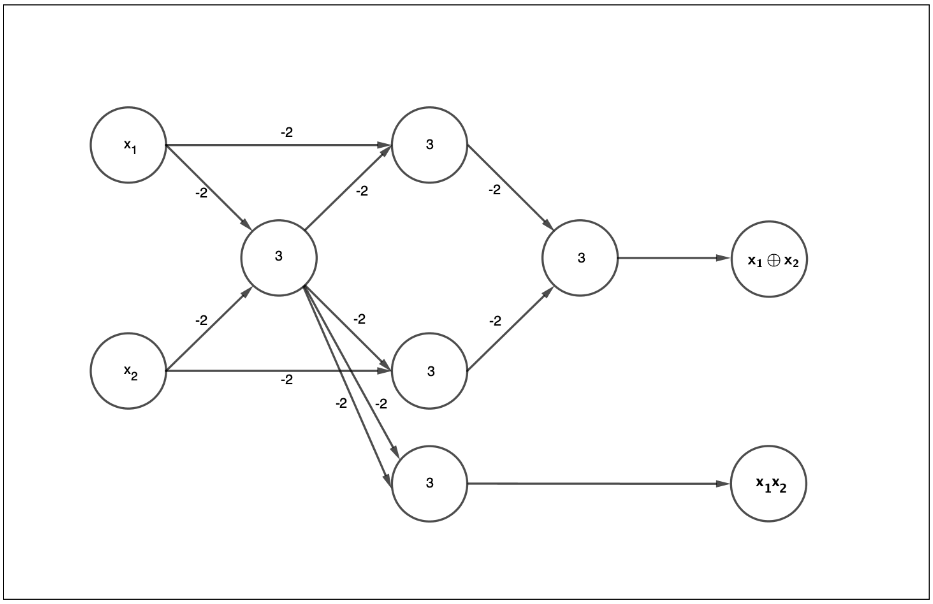


FIGURE 3.16: Binary addition with neural network

The neural network in Fig. 3.16 immediately shows the typical three-partition of any computational process: information encoding, information processing, decodnig information as can be seen in the following picture (Fig. 3.17):

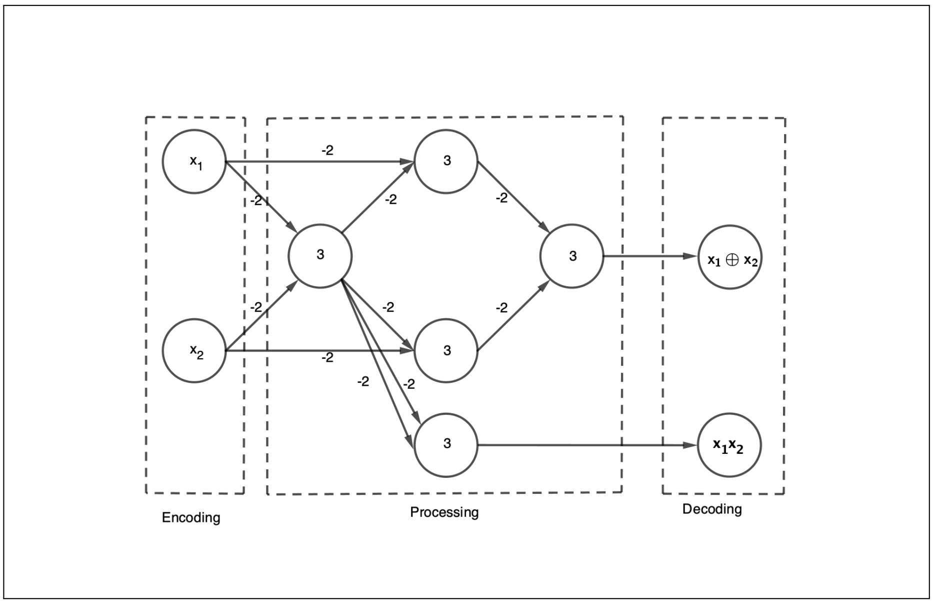


FIGURE 3.17: Three-partition of binary addition

Knowing that each perceptron realises a Nand-gate, we can translate the representation in Fig. 3.16 into the circuit representation in Fig. 3.18: it is easy to verify by means of traditional truth tables that this representation is equivalent to the previous one.

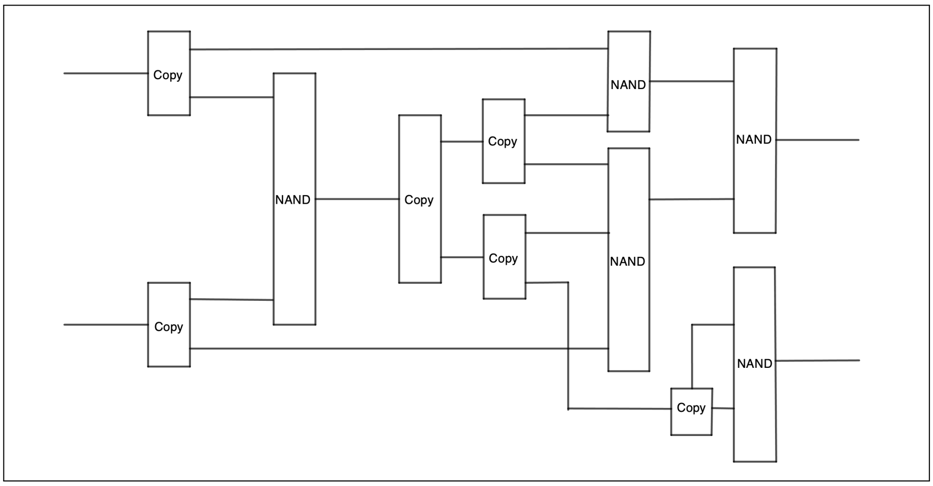


FIGURE 3.18: Circuital representation of binary sum.

As in the previous example, we will use this tool to perform classical computation and show how this can be well described in the categorical language. The approach we follow has proved so useful that an extension of it has made it possible to express the entire supervised deep learning with the same mathematical structure[12] (see [83], [84] ,[86], [87]).

We have seen in Fig. 3.15 the possibility to realise a Nand-gate with a perceptron and in Fig. 3.16 the possibility to realise a computation with a neural network. The basic idea is that artificial neurons can be sequentially composed in series and in parallel. This leads us to introduce a monoidal category, Para(Smooth), the category of neural networks (see [87]) (an artificial neuron is a special case) from a strictly symmetrical monoidal category, the need for which is clear in order to define the concept of reparametrisation. We will introduce this construction diagrammatically and only for computational case[13], leaving at the end the formal construction whose full description can be found in the references already indicated.
First, we translate the perceptron representation into a diagrammatic representation more similar to the one used in this work, which emphasises the parameter as somehow different from the data (input): Fig. 3.19 shows how to construct the particular image from the known representation, with the obvious requirement that $\mathbf{x} \in \{0,1\}^n$.

[12]The extension to the quantum case is close by means of working on monoidal categories that are not necessarily Cartesian [117].
[13]We do not need the Para(Smooth) category entirely, because we do not have to implement the back-propagation.

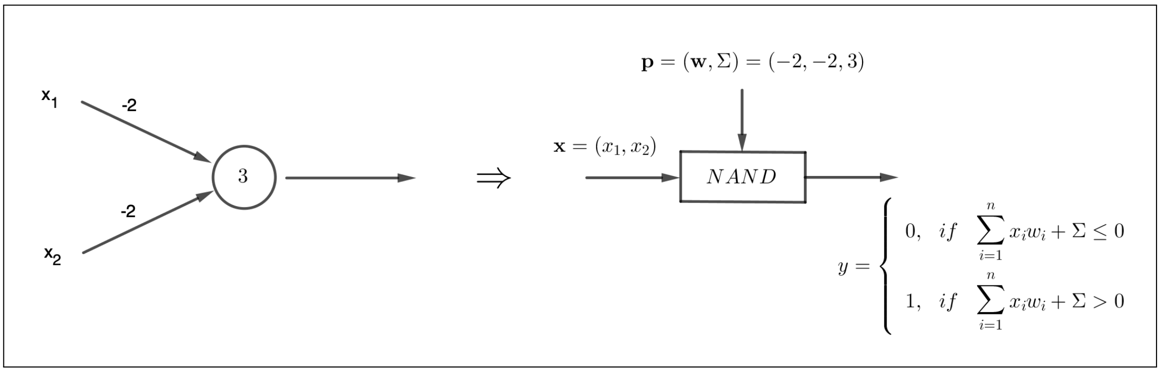


FIGURE 3.19: Diagrammatical representation of Nand-gate with perceptron.

The representation above provides the basis for an obvious generalisation that allows sequential and parallel composition to be introduced graphically in a very simple way as in Fig. 3.20

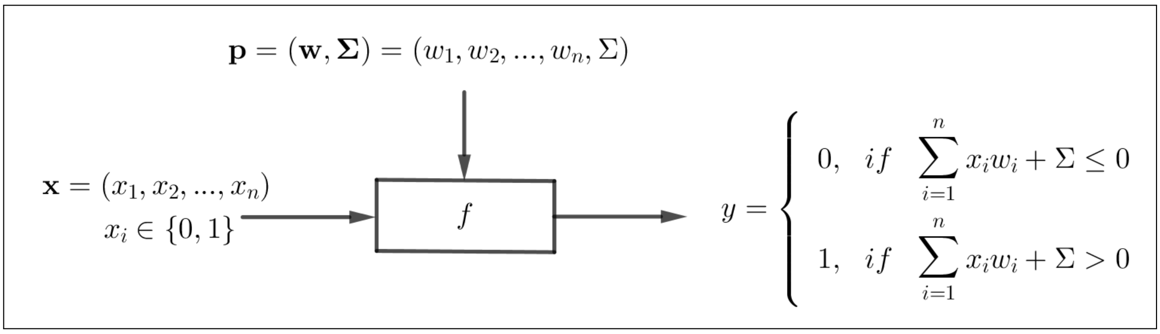


FIGURE 3.20: Diagrammatical representation of a any logic-gate with categorical description of the perceptron.

For example, the sequential composition of *NAND* and *NOT* logic gate becomes as in Fig. 3.21.
Similarly, we can introduce the parallel composition as in Fig. 3.22.
What we have seen can be formalised, in the specific case of Boolean logic, in the following way:

Definition 3.13. Let $\mathfrak{C}$=$(Ob(\mathfrak{C}), Ar(\mathfrak{C}), id, \circ, \otimes, \ I, \sigma)$ be a strict symmetric monoidal category, then we define the category $\mathbf{Para}(\mathfrak{C})$ with:

- $Ob(Para(\mathfrak{C}))$:= $Ob(\mathfrak{C})$
- $Ar(Para(\mathfrak{C}))$:=$\{(P, f)|P \in Ob(\mathfrak{C}), \ f : P \otimes A \longrightarrow B, \ A, B \in Ob(Para(\mathfrak{C}))\}$
- Id_A:=(I, id_A)
- $(P, f) \circ (P', f') := (P \otimes P', f' \circ (id \otimes f))$, with $(P, f) : A \longrightarrow B$ and $f' : B \longrightarrow C$.

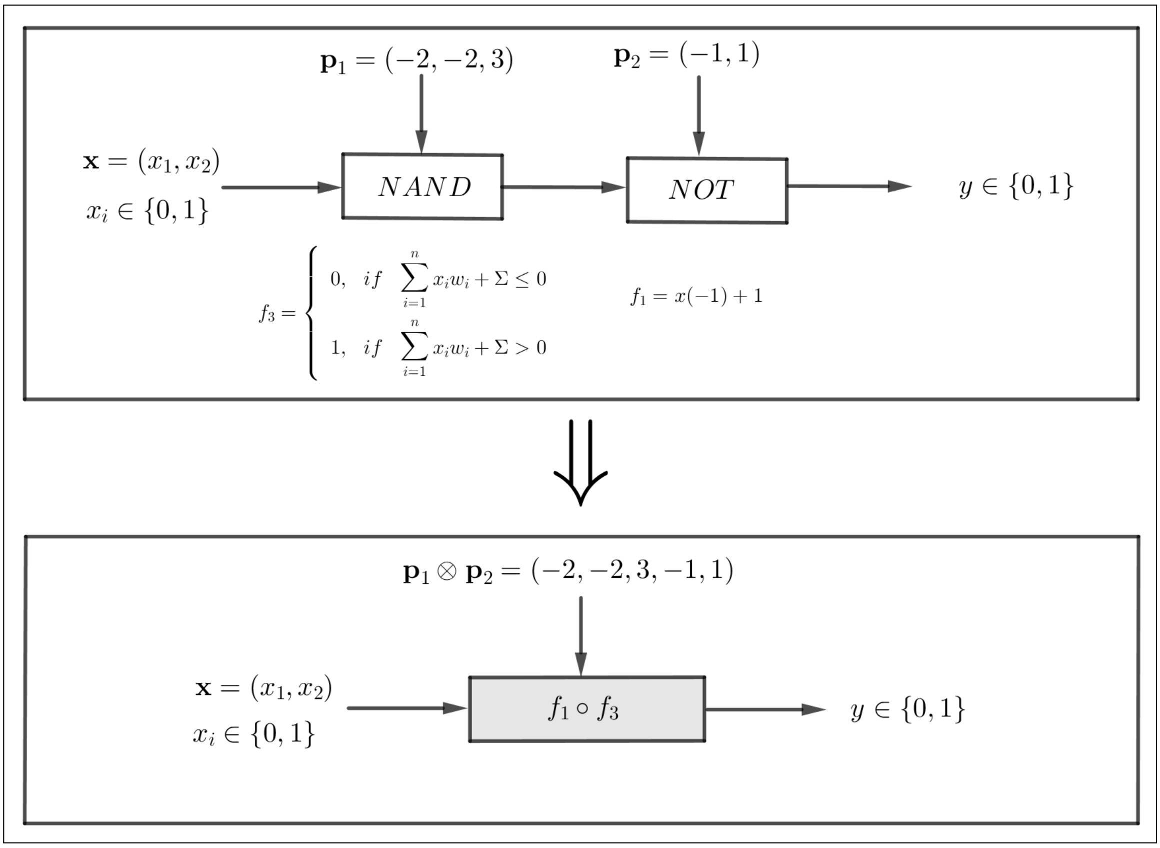


FIGURE 3.21: Diagrammatical representation of a sequential composition of Nand and Not gates

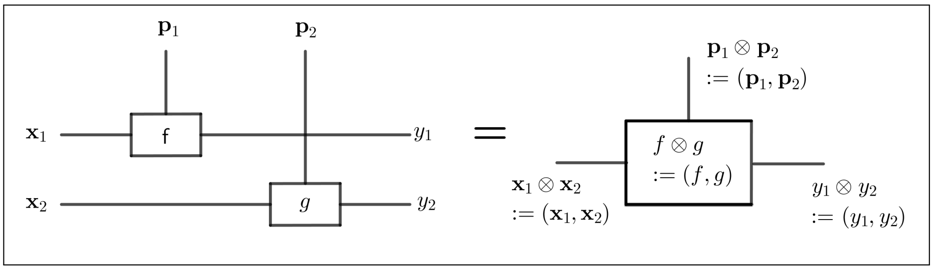


FIGURE 3.22: Diagrammatical representation of a parallel composition of logic gates

Monoidal structure is inherited from $\mathfrak{C}$ (see [86]).

If in Def. 3.13 we use $(\mathfrak{C})$ the category with

- $Ob((\mathfrak{C})) := \mathbb{N}$
- $Ar(\mathfrak{C}) := \{(P, f) | P \in \mathbb{R}^p,\ f : \{0,1\}^n \longrightarrow \{0,1\}^m\}$[14]

[14] In general it is useful to define morphisms es $Ar(\mathfrak{C}) := \{(P, f) | P \in \mathbf{R}^p,\ f : \mathbf{R}^n \longrightarrow \mathbf{R}^m\}$ to be able to use differential calculus for backpropagation.

we obtain the category of the logic gates with their compositions introduced diagrammatically in Fig. 3.20, 3.21 and Fig. 3.22. This allows, equivalently, a diagrammatic representation by string diagram (see [96],[87]).

3.7.0.1 Summary and observations

Considering the traditional combinational logic, we can build the logic gates with a perceptron and realise any computation with a neural network, a sequential and parallel composition of artificial neurons.
We represented the perceptron in Fig. 3.14 and the binary addition with neural network in Fig. 3.16. These representations suggest using the concept of monoidal category to describe the computational approach (input, transformation, output) and a particular parameterization to take into account the weights. The mathematical structure that brings these considerations together is **Para**($\mathfrak{C}$), where $\mathfrak{C}$ is a monoidal category. In this way we can associate for instance to the neural network describing the *XOR* logical gate (Fig. 3.23).

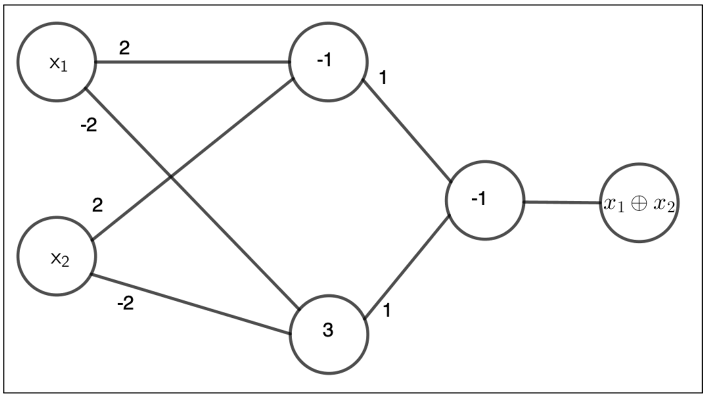


FIGURE 3.23: Neural network computing *XOR* gate as parallel composition of *OR* and *NAND*; sequential composition with *AND* gate.

The corresponding diagrammatic representation based on **Para**($\mathfrak{C}$) (Fig. 3.24)

Observation 3.14. The examples just introduced show once again how proper a compositional introduction to computation can be. It is also advantageous to consider our steps to obtain the most abstract construction possible. First, we introduced a model of an element of nature, the neuron. Then we built a network structure using sequential and parallel compositions of perceptrons from this. We realised that it was then possible to use

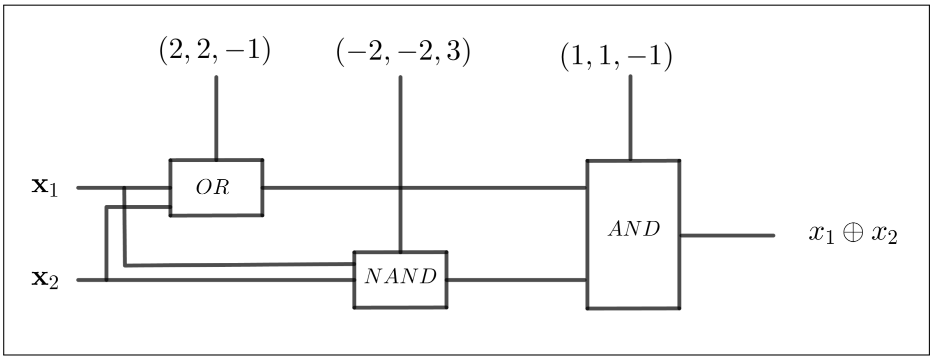


FIGURE 3.24: Diagrammatic representation of Fig. 3.23

more abstract object to describe this process: the category $\mathbf{Para}(\mathfrak{C})$. Finally, we took advantage of diagrammatic representations to restore the pictures to their form throughout this work.

Observation 3.15. The computation introduced through $\mathbf{Para}(\mathfrak{C})$ sheds light on some significant aspects if reread in the physical sphere: the monoidal category presented has as its monoidal product the Cartesian product; this means that the nature of compound systems is separable. Furthermore, the sequential composition is the function composition; this computation is not necessarily invertible. Finally, as a probabilistic structure, we have the trivial one that associates probability 1 of obtaining a specific outcome from the computation, i.e. it certifies the pre-existing value of one particular physical property.

3.8 Conclusions

We have introduced the concept of a monoidal category in an abstract form and provided a suitable diagrammatic representation. Furthermore, the theorem 3.5 ensures that everything that is proved by syntactic rules on diagrams is also proved theoretically and *vice versa*. The abstract language was then interpreted in the natural language and logic-computation field, providing the first original examples of how it can be used in the sciences. The link with physics is currently only touched upon. It is the task of the next chapter to provide a comprehensive picture of the connection between logic, physics (classical and quantum) and diagrammatic language.

Chapter 4

Physics for computation: a compositional approach

4.1 Introduction

This chapter builds on some of the considerations of early researchers in quantum computation to introduce the link between computation and physics according to a compositional perspective, i.e. by exploiting the sequential and parallel composition intrinsic to monoidal categories.
We have already introduced at the end of the previous chapter the concept of finite state machine (3.6) and neural network (see 3.7), and we gave their categorical construction to introduce classical computation. The two special cases are here generalized to the environment best appropriate for the description of classical logic in 4.3, the ***Set*** *topos*. Diagrammatic representations thus achieve an interpretation in the computational framework.
The use of the compositional approach and the introduction of the ***Set*** category make the transition from computation to physics natural: the logical actions of encoding, processing and decoding information become the physical processes of preparation, transformation and measurement of physical state for bit (4.4). In this way, a generic experiment (truth value of a physical proposition) is reinterpreted from a computational point of view and the isomorphism between Boolean algebra and the residues classes *modulo* 2 completes the correspondence. The last sections are dedicated to the interpretation of the diagrammatic mathematical elements introduced in the previous chapter from a physical point of view

in the case of *Operational Probabilistic Theories* (4.5) and ZX-calculus (4.7). Finally, we consider an equivalent diagrammatic language related to linear optics (4.8).

4.2 The physics of computation

The beginning of our discussion can be found by rereading D. Deutsch's famous description of what is meant by a computer [18]:

> "Intuitively, a computing machine is any physical system whose dynamical evolution takes it from one of a set of 'input' states to one of a set of 'output' states. The states are labelled in some canonical way, the machine is prepared in a state with a given input label and then, following some motion, the output state is measured. For a classical deterministic system the measured output label is a definite function f of the prepared input label; moreover the value of that label can in principle be measured by an outside observer (the 'user') and the machine is said to 'compute' the function f. "

and the assessments of Turing's work in T. Toffoli and E. Fredkin's 1982 article on reversible logic [16]:

> "The Turing machine embodies in a heuristic form the axioms of computability theory. From Turing's original discussion (Turing, 1936) it is clear that he intended to capture certain general physical constraints to which all concrete computing processes are subjected, as well as certain general physical mechanisms of which computing processes can undoubtedly avail themselves .At the core of Turing's arguments, or, more generally, of Church's thesis, are the following physical assumptions:
>
> **P1** The speed of propagation of information is bounded. (No "action at a distance": causal effects propagate through local interactions.)
>
> **P2** The amount of information which can be encoded in the state of a finite system is bounded.
>
> **P3** It is possible to construct macroscopic, dissipative physical devices which perform in a recognisable and reliable way the logical functions AND, NOT, and FAN-OUT. "

And again on the role of physics and axioms:

> “This paper deals with conservative logic, a new mathematical model of computation which explicitly reflects in its axioms certain fundamental principles of physics...
> Computation - whether by man or by machine - is a physical activity, and is ultimately governed by physical principles. An important role for mathematical theories of computation is to condense in their axioms, in a stylised way, certain facts about the ultimate physical realisability of computing processes. With this support, the user of the theory will be free to concentrate on the abstract modelling of complex computing processes without having to verify at every step the physical realisability of the model. Thus, for example, a circuit designer can systematically think in terms of Boolean logic (using, say, the AND, Not, and FAN-OUT primitives) with the confidence that any network he designs in this way is immediately translatable into a working circuit requiring only well - understood, readily available components (the ”gates ”, ”inverters”, and ”buffers” of any suitable digital- logic family). It is clear that for most routine applications one need not even be aware of the physical meaning of the axioms. However, in order to break new ground one of the first things to do is find out what aspects of physics are reflected in the axioms. ”

By the physics of computation, we mean precisely the existing tense relationship between the mathematical theory describing computation and the physical theory explaining the nature of the devices to implement it. The mathematical framework for both descriptions will be that of braided monoidal categories with a probabilistic structure: the difference between classical and quantum computation is manifested in particular by the role of parallel composition and probability within the framework. Technically, this results in an extension of the algebraic structure

4.3 From categorical classical computation to physics

After the last two examples described in the previous chapter, we will now deal with the introduction of classical computation following what was done in [93] and [118] with the *Set* category, restricted to the particular case where the sets are Boolean.

We know that classically logic gates are boolean function

$$f : \{0,1\}^n \longrightarrow \{0,1\}^m$$

We can construct the category of Boolean functions in this way:

Definition 4.1. ***Set*** $= \mathfrak{S}$ is the braided monoidal category $\mathfrak{S} = (Ob(\mathfrak{S}), Ar(\mathfrak{S}), id, \circ, \otimes, I, \sigma)$ defined by:

- $Ob(\mathfrak{S})$ are the sets;
- $Ar(\mathfrak{S})$ are the functions $f : A \longrightarrow B$, with $\forall A, B \in \mathfrak{S}$;
- $id : A \longrightarrow A$, $\forall A \in \mathfrak{S}$, is the identity function;
- $\circ : Ar(\mathfrak{S}) \times Ar(\mathfrak{S}) \longrightarrow Ar(\mathfrak{S})$ is the composition of functions.

These elements trivially satisfy the unit and associativity conditions. The parallel composition is defined by the functor $\otimes : \mathfrak{S} \times \mathfrak{S} \longrightarrow \mathfrak{S}$ (the Cartesian product) and the unit object $I = \{*\}$. These elements trivially satisfy the unitality and associativity on objects, the unitality and associativity on arrows and the interchange law. Moreover, these make it possible to construct logic gates. Finally, for braiding (symmetry) we can introduce $\sigma_{A,B} : A \times B \longrightarrow B \times A$ s.t. $(a, b) \sim (b, a)$.

The categorical approach to classical logic links computation as an abstract theory and physics, meaning the study of the systems that implement this computation. For example we can consider this traditional logical circuit:

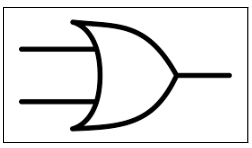
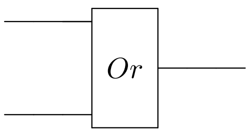


FIGURE 4.1: Circuit representation of the *Or*-gate

In our representation it becomes:



From a strictly logical point of view, we divide the representation into three parts: input, logic gate and output. From a physical point of view, i.e. the device to implement this operation, the diagram is interpreted as follows: the first part refers to the *preparation* of the initial states; the second to the *transformation* of the states; the third to their *measurement*, which in classical deterministic computation is equivalent to the confirmation of the properties possessed by the output states. The mathematical structure introduced allows the introduction of *initial state*, *effects* and *scalars* using the *terminal object* in ***Set***:

States A *state*, a class of bitstring preparation, is a transformation without input $\{*\} \longrightarrow \{0,1\}^n$;

Effects An *effect*, a class of bitstring observation, is a transformation without output $\{0,1\}^m \longrightarrow \{*\}$;

Scalars A scalar is a transformation without any input and output $\{*\} \longrightarrow \{*\}$.

In the physical context, we speak of preparing a state, transforming a state and measuring a state. In this way, we can introduce our diagrammatic representation of classical computation as in Fig. 4.2

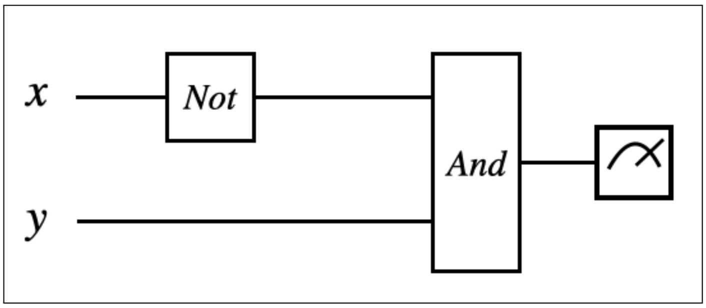


FIGURE 4.2: Diagrammatic representation of diagram for classical computation.

and this diagram immediately implies a physical interpretation that describes a device to implement it: we prepare the physical systems to realise two bits encoding the initial information; therefore we have an experimental setup able to realise physical transformations on the state corresponding to the logical gates; finally we realise a measurement[1]. This approach is illustrated in Fig. 4.3

But with this theoretical set-up, the problem arises that we cannot eliminate a non-trivial notion of measurement. In *Set* $= \mathfrak{S}$ the monoidal unit object is terminal, meaning

[1]We will see how useful this is in implementing quantum logic gates through optical devices.

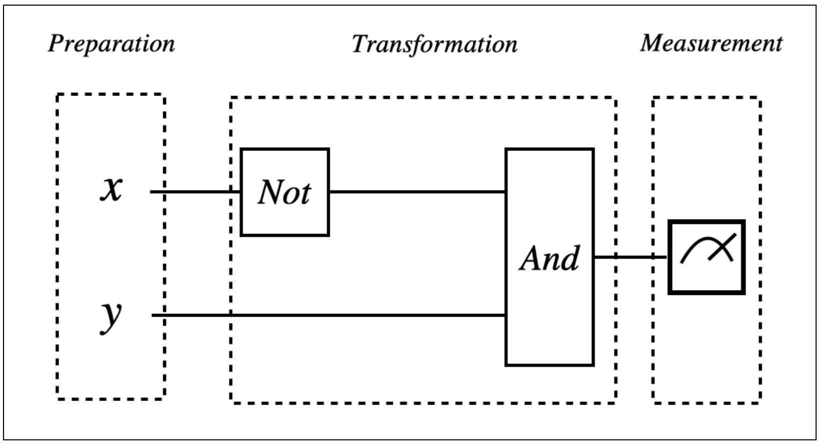


FIGURE 4.3: Physical interpretation of a classical logical circuit

$Hom(A, I)$ has only a single element for any object A. We can say as [92], this computation is *boring*. We can prevent this trouble, using the categorical logic described in [118]. It is impossible to describe the entire construction here, but we will give the main ideas with special attention to the logic of predicates that arises as necessary. We can also emphasise that this approach can also be introduced in the case of quantum calculus, but for reasons that will become clear later, we will not follow this way.

4.3.1 Predicates, tests and measurement instruments

We know that ***Set*** has terminal object $\{*\} = 1$ and *coproduct* the direct sum ([93]). We repeat here the definition already given, but made more explicit in the context we are analysing.

Definition 4.2. Given two objects A e B in a category, a ***product*** is an object $A \times B$ with morphisms $A \times B \xrightarrow{p_A} A$ and $A \times B \xrightarrow{p_B} B$, s.t. if $X \xrightarrow{f} A$ and $X \xrightarrow{g} B$, exists only one morphysm $\binom{f}{g} : X \longrightarrow A \times B$ making both trinagles commute[2]:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "\binom{f}{g}"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B \end{tikzcd}$$

[2] $p_A \circ \binom{f}{g} = f$ and $p_B \circ \binom{f}{g} = g$

A ***coproduct*** is the dual notion.

$$
\begin{tikzcd}
 & X & \\
A \arrow[ur, "f"] \arrow[r, "k_1"'] & A \times B \arrow[u, "\binom{f}{g}"] & B \arrow[ul, "g"'] \arrow[l, "k_2"]
\end{tikzcd}
$$

We can define a $2-test$ on an object X as a map $p : X \longrightarrow 2 \cdot 1 = 1+1 = \{0,1\}$. If we think about the propositional logic this test is called *predicate* p and $Pred(X) = Hom(X, 1+1)$. In particular we can define the *true predicate* as

$$1_X = (X \xrightarrow{!_X} 1 \xrightarrow{k_1} 1+1)$$

and the *false predicate* as

$$1_X = (X \xrightarrow{!_X} 1 \xrightarrow{k_2} 1+1)$$

The predicates $Pred(1)$ on the final object $1 \in$ play a special role and will be called scalars (sometimes probabilities).

Set the *2-test* give a partition on X in 2 subsets $p^{-1}(i) \subseteq X$, with $i \in \{0,1\}$. Obviously in this case a predicate is a characteristic function $X \longrightarrow 2$ and the probabilities are the Booleans $\{0,1\}$.

In this way, following [118], we can describe the whole logical-computational process as the sequence of three foundamental moments[3]:

States The state is a class of preparation $\omega : 1 \longrightarrow \{0,1\}$, called traditionally *bit*;

Computations The computation is a class of transformation $\{0,1\}^n \longrightarrow \{0,1\}^m$

Predicates A predicate is a map $p : \{0,1\}^m \longrightarrow \{0,1\}$ showing the truth value.

The construction we have just seen, however, is not enough: it is used to observe that there may be two different output options, but not which one in 1.

In ***Set*** we can introduce a *measurement instrument* for classical computation

$$instr_p : \{0,1\}^m \longrightarrow 2 \cdot \{0,1\}^m$$

[3]For the role of the classes, we refer to [94]

defined $instr_p(\underline{x}) = k_i\underline{x}$ iff $p(\underline{x}) = i$, making the diagram commute

$$\begin{tikzcd}
\{0,1\}^m \arrow[r, "p"] \arrow[dr, "instr_p"'] & 2\cdot 1 \\
& 2\cdot\{0,1\}^m \arrow[u, "n\cdot !"']
\end{tikzcd}$$

$Instr_p$ allows us to carry out a non-trivial measurement by also providing outcomes.

4.3.2 A second approach: the *topos Set*

What is described in the previous paragraph describes what some might call *categorical classical logic*. This approach arose precisely from the developments in the study of category theory in the field of physics ([95], [119], [118], [120]). Traditionally, there is another approach, again categorical in nature, which describes classical logic by means of a higher-order structure, the topos theory[4]. For the purpose of completeness we also briefly describe this approach[5]:

Definition 4.3. A *topos* is a category $\mathfrak{C}$ such that:

1. $\mathfrak{C}$ has all finite limits;
2. $\mathfrak{C}$ has a subobject classifier;
3. $\mathfrak{C}$ has all exponentials.

Without going into detail, the subobject classifier is the set

$$\Omega_{Set} := \mathbb{B} = \{true, false\}$$

together with the morphism

$$true : 1 \longrightarrow \Omega$$

and a iniective function

$$m : X \longrightarrow Y$$

able to find a characteristic function $\ulcorner m \urcorner : Y \longrightarrow \Omega$ such that

[4]This approach has given rise to other works on the logic of quantum physics. These include that of A. Doring and C. Isham [121]

[5]For the definition see [99]

$$\ulcorner m \urcorner (y) := f(x) = \begin{cases} true & if\ m(x) = y \quad for\ some\ x \in X \\ false & otherwise \end{cases}$$

This approach allow us to obtain the logic gates in a very simple manner (for details see [102] last chapter) and introduce the characteristic function, fundamental as we shall see in the next section[6]. This approach is more similar to the considerations that follow. The cost is the renunciation of classical logic for an intuitionist logic ([118]).

4.4 Physical interpretation

We can translate the previous considerations of 4.3 in physics.
What means to find the truth value of a physical proposition? As Susskind in [122], we have to prepare an experimental setup, prepare initial states, to evolve states and then carry out a measurement to establish the truth value of the predicate. The approach followed in the previous section makes it possible to translate into categorical language what is already known and very clearly set out in [123] and which we outline here:

1. Let us consider a *classical physical system*, for example an object of mass m;
2. the *State space* is $S = \{(x(t), p(t)) \subset R^2\}$;
3. a *physical quantity* is a function $f : S \longrightarrow \mathbf{R}$;
4. we talk about *property* if the physical quantity has a specific value $f(s)$;
5. we can consider a statement about the properties of the system (true or false), i.e. the value of a physical quantity lies in some $\Delta \subseteq \mathbf{R}$, i.e. the system possesses this property;
6. to each proposition we can associate a subset of S, $E := f^{-1}(\Delta) \subseteq S$;
7. E realise a partition of S and with a *characteristic function* χ, we obtain the true value of proposition.

This can be summarised graphically as follows[7]

[6]For more on the differences between the approaches in the last two paragraphs, see [118]
[7]The importance of this figure will be evident in the develop of the educational reconstruction for in service teachers

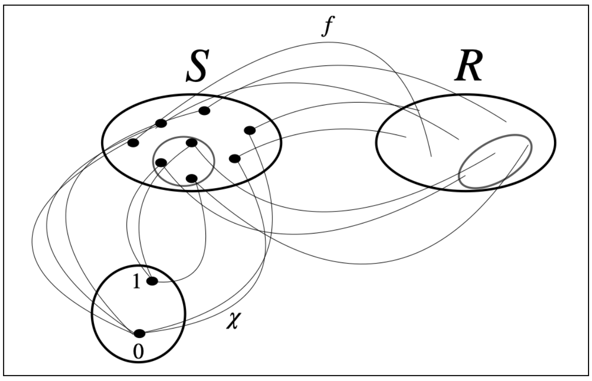


FIGURE 4.4: Representation of the formalisation of the truth value of a physical proposition

Reread in the terms of the categorical language introduced earlier, it becomes:

a. Let us consider classical physical systems; an object A in **Set**;

b. we prepare the initial state coding the information on a physical quantity of a system, i.e. we consider the morphisms from terminal object 1 in the object representing a system $1 \longrightarrow A$;

c. we realise the experiment, i.d. we consider the transformations on the systems, the morphisms in general between two different objects $f : A \longrightarrow B$;

d. we read the result on a measurement device and then we obtain the true value of proposition, i.e. we use the *measurement instrument* $instr_p : X \longrightarrow 2 \cdot X$ to obtain 0 or 1, or the morphism linked to subobject classifier.

The well-known isomorphism between the Boole algebra $(\mathbf{P}(S), \cup, \cap, \emptyset, S, (-)^c)$ and $(\mathbf{Z}_2^n, +, \cdot, 0, 1, (1 - (-)))$ allows you to think every classical physical proposition in computational terms in **Set**:

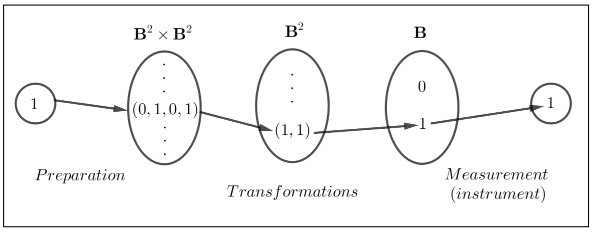


FIGURE 4.5: Representation of a particular implementation of XOR gate in Fig.3.24.

which in the circuit representations we will use later is represented as

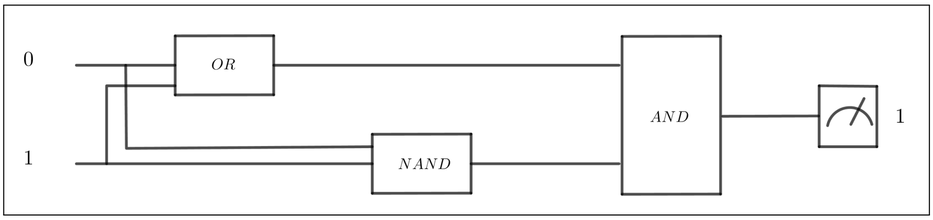


FIGURE 4.6: Circuit of a particular implementation of XOR gate in Fig.4.5

4.5 Monoidal category and probabilities: physical interpretation in Operational Theories

We are finally ready to systematically interpret diagrams and the rules governing them from a physical point of view. The references in this regard are of two kinds: OPT and Categorical Quantum Theory (CQT). The framework common to both are the monoidal braided categories with the addition of a probabilistic structure (which in the case of CQT determines the extension of the categorical structure up to Hypergraph categories[8]). Below we will follow the presentation of OPT given in [94] making explicit and precise the links with the category theory introduced in the previous chapter. This approach is very interesting from the point of view of *operational interpretation*, which suggests the fact that diagrams, which abstractly are elements of a mathematical theory, are constructed as a generalization of a scheme of an experimental protocol [125]. This approach supports, from a theoretical point of view, the choice of designing an educational sequence in which the diagrams will have multiple interpretations: theoretical from a computational point of view, theoretical from a physical point of view, and theoretical from the point of view of the physical devices (ideal and otherwise) used for their realization. These ideas are the basis for the construction of the diagrammatic model, where the concept of model will be discussed and explained in the next chapter.

An *OPT* is a theory that make "*predictions about joint events depending on their reciprocal connections*" ([34]). In order to do this OPT "*is a non-trivial extension of probability theory, which in turn is an extension of logic*" ([34], [126], [127]).

In the following, we will introduce the two elements that constitute the basis of OPT: *the operational structure* and *the probabilistic structure*. The first is strictly related to the concept of the monoidal symmetrical category; the second allows the linear structure of the theory to be introduced.

In full correspondence to the tables C.1 and C.2 we present their physical interpretations of category $\mathfrak{C} = (Ob(\mathfrak{C}), Ar(\mathfrak{C}), id, \circ)$:

[8] For the relations between operational theories and CQT see [124]

Data of category definition	**Diagrammatic representation**	**Experimental setting**
Systems A	$\overset{A}{\text{———}}$ $\quad A, B, C, \ldots \in Sys(\Theta)$	*Physical system in laboratory*
Tests (Events) $f : A \longrightarrow B$	$\overset{A}{\text{—}}\boxed{\mathfrak{F}_x}\overset{B}{\text{—}}$ $\quad T_X^{A\longrightarrow B} \in Test(\Theta),\ T_X^{A\longrightarrow B} = \{\mathfrak{T}_x\}_{x\in X} \subseteq Event(A \longrightarrow B)$	*Physical processes*
Identity test $id_A : A \longrightarrow A$	$\overset{A}{\text{——}} = \overset{A}{\text{—}}\boxed{id_A}\overset{A}{\text{—}}$	*Processes does nothing*
Consecutive tests $g \circ f : A \longrightarrow C$	$\overset{A}{\text{—}}\boxed{\mathfrak{T}_x}\overset{B}{\text{—}}\boxed{\mathfrak{T}'_y}\overset{C}{\text{—}} = \overset{A}{\text{—}}\boxed{\mathfrak{T}'_y \circ \mathfrak{T}_x}\overset{C}{\text{—}}$	*Consecutive occurrence of two physical processes*

FIGURE 4.7: Physical interpretation of category data in OPT

and for the axioms

Axioms of category definition	**Diagrammatic representation**	**Experimental setting**
Unit $f \circ id_A = f = id_B \circ f$	$\overset{A}{\text{—}}\boxed{id_A}\overset{A}{\text{—}}\boxed{\mathfrak{T}_x}\overset{B}{\text{—}} = \overset{A}{\text{——}}\boxed{\mathfrak{T}_x}\overset{B}{\text{—}}$ $=$ $\overset{A}{\text{—}}\boxed{\mathfrak{T}_x}\overset{B}{\text{—}}\boxed{id_B}\overset{B}{\text{—}} = \overset{A}{\text{—}}\boxed{\mathfrak{T}_x}\overset{B}{\text{——}}$	*Equivalence stretching input or output wires*
Associativity $(h \circ g) \circ f = h \circ (g \circ f)$	$\overset{A}{\text{—}}\boxed{\mathfrak{T}_x}\overset{B}{\text{—}}\big(\boxed{\mathfrak{T}'_y}\overset{C}{\text{—}}\boxed{\mathfrak{T}''_z}\big)\overset{D}{\text{—}}$ $=$ $\overset{A}{\text{—}}\big(\boxed{\mathfrak{T}_x}\overset{B}{\text{—}}\boxed{\mathfrak{T}'_y}\big)\overset{C}{\text{—}}\boxed{\mathfrak{T}''_z}\overset{D}{\text{—}}$	*No preferential order*

FIGURE 4.8: Physical interpretation of category axioms in OPT

As you can see from the first table, $Sys(\Theta)$, $Test(\Theta)$ and $Event(\Theta)$ that are respectively the collection of systems, test and events of a theory Θ, are the primitive entities.
Let us pay attention to the fact that in an experiment we see neither systems nor tests, but they are outcomes that are the only real elements. For example we can think about a Stern-Gerlach apparatus ([34]) as in Fig. 4.9: the experiment consists of an oven that produced a beam of neutral atoms, a region of space with an inhomogeneous magnetic field, and a detector for the atoms. We found that the beam was split into two in its passage through the magnetic field. One beam was deflected upwards and one downwards in relation to the direction of the magnetic field gradient.
In this case we have two events associated with the test: the two possible transformations corresponding to the particle passing through the upper or the lower pinhole. Up and down are the two outcomes. The input and output systems are a particle spin. A particular outcome space is the singleton: when you have only one element. Deterministic events are those whose associated test is singleton.

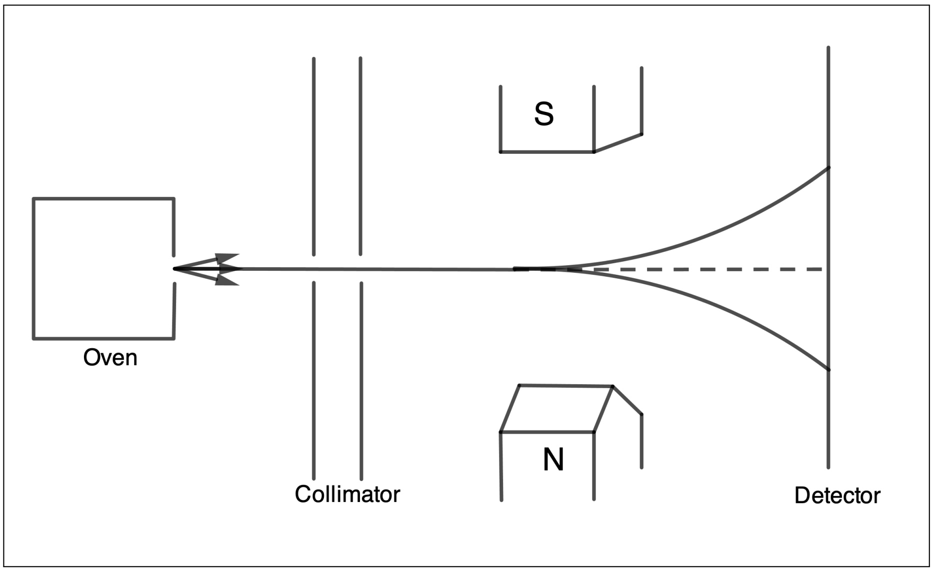


FIGURE 4.9: Stern-Gerlach device to measure the spin component of neutral particles along the z-axis

What has just been introduced accounts for the sequential structure of physical processes, but not for their composition: let us therefore add the monoidal structure (see Fig. 4.10)

$$\mathfrak{C} = (Ob(\mathfrak{C}), Ar(\mathfrak{C}), id, \circ, \otimes,\ I)$$

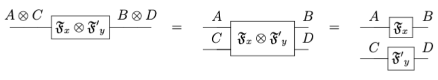
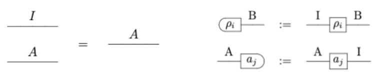
Data of monoidal category definition	**Diagrammatic representation**	**Experimental setting**
monoidal product ⇓ *parallel composition of system* *parallel composition of test*		*Composite systems* *Composite processes*
unit object ⇓ *trivial system* *preparation tests* *observation tests*		*Physical processes in which are discarded input or output* *Preparations* *Observations*

FIGURE 4.10: Physical interpretation of monoidal category data in OPT

and for the axioms as in Fig. 4.12. The simmetry σ in $\mathfrak{C} = (Ob(\mathfrak{C}), Ar(\mathfrak{C}), id, \circ, \otimes,\ I, \sigma)$ has the following physical meaning: two experimenters must be able to exchange their respective systems.

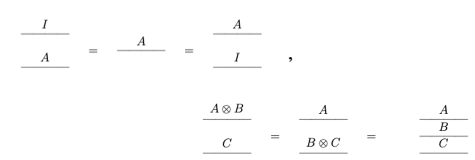
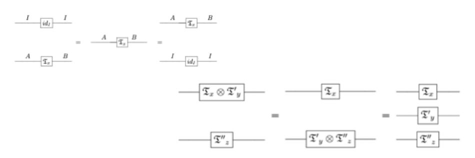
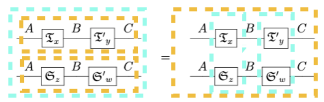
Axioms of monoidal category definition	Diagrammatic representation	Experimental setting
unit and associativity on systems		*If we carry out experiments in more than one laboratory, the order in which we consider them does not matter*
unit and associativity on tests		*If we carry out experiments in more than one laboratory, the order in which we consider them does not matter*
interchange law		*The order is only a theoretical manipulation*

FIGURE 4.11: Physical interpretation of monoidal category axioms in OPT

Let us now introduce the *probabilistic structure*. We will introduce only the role of probability and probabilistic equivalence; for more details [94].

Probabilities We can say that "*the general purpose of an operational theory is that of predicting and accounting for the joint probability of events corresponding to a particular circuit connection*". Following this perspective, the diagrams in an OPT are closed circuits (directed acyclic graphs) that correspond to a probability distribution:

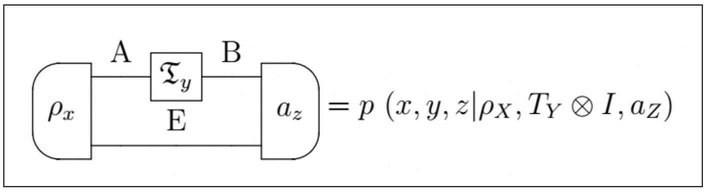


FIGURE 4.12: Closed diagrams: a probabilistic interpretation.

Probabilistic equivalence The previous definition allows us to define an equivalence relation:

Definition 4.4. Let Θ be an OPT.
$\forall\ A \in Sys(\Theta),\ \forall\ B \in Sys(\Theta)$
$\forall\ \mathfrak{T}_1 \in Event(A \longrightarrow B),\ \forall\ \mathfrak{T}_2 \in Event(A \longrightarrow B)$

$$\mathfrak{T}_1 \sim \mathfrak{T}_2$$

if $\forall\ E \in Sys(\Theta),\ \forall\ \rho \in Event(I \longrightarrow AE),\ \ \forall\ a \in Event(BE \longrightarrow I)$

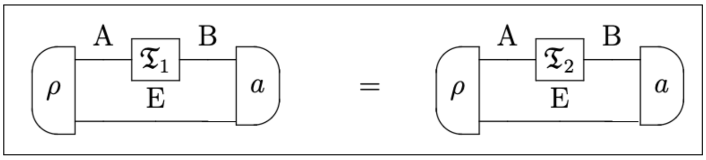


FIGURE 4.13: Probabilistic equivalence

This relation gives us the possibility to introduce the concept of state and effect. Let us see how.

Definition 4.5. We define *transformation* from a system$A$ to a system B a quotient class of events

$$Transf(A \longrightarrow B) := Event(A \longrightarrow B)/\sim$$

Therefore, we can define

state as the member of the quotient class of *preparations*[9]

$$St(A) := Transf(I \longrightarrow A)$$

effects as the member of the quotient class of *observations*

$$Eff(A) := Transf(A \longrightarrow I)$$

The sequential composition of states and effects is a number in $[0, 1]$.
Finally, we define

instruments as the member of the quotient class of *Tests*

$$Instr(\Theta) = Tests(A \longrightarrow B)/\sim$$

An OPT can be defined by specifying:

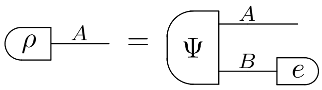
1. the systems $Sys(\Theta)$;
2. a parallel composition rule $\otimes$ for systems and states;
3. the instruments $Instr(\Theta)$ and their parallel composition

[9]The operational definition of state just given clearly implies the definition of a qubit (or a bit). For the teleportation protocol, for example, we can use equivalently the polarization of a photon and the spin of an electron, "*which both correspond to the same quantum system, i.e. the qubit*"([34])

4.6 The purification postulate

An OPT has no classical or quantum interpretation, given its completely abstract nature. It is the postulates inserted later that connote its character. In particular, this section introduces the postulate of purification in a diagrammatic manner. This postulate is what characterises quantum theory concerning classical theory ([128], [129], [130] [34]).

Definition 4.6 (Purification)**.** A pure state $\Psi \in \mathfrak{S}_1(AB)$ is a *purification* of $\rho \in \mathfrak{S}_1(A)$ if[10] $|\rho)_A = (e|_B|\Psi)_{AB}$. Diagrammatically,



(4.1)

where the left-hand side of the equal is the diagrammatic representation of the partial trace.

Without going into too much detail, "*the purification principle states that every state has a purification, unique modulo reversible transformations on the purifying system*" ([130]). This means that when we have a mixed state, our ignorance comes from the fact that the system we are considering A is actually part of a larger AB system of which we have total knowledge. From a foundational point of view, the purification principle is very important because it links information theory to physical theory, making it a full-fledged a physical theory of information ([34]):

> "Information theory would not make sense without the notions of probability and mixed state, for the whole point about information is that there are things that we do not know in advance. But in the world of classical physics of Newton and Laplace, every event is determined and there is no space for information at the fundamental level. In principle, it does not make sense to toss a coin or to play a game of chance, for the outcome is already determined and, with sufficient technology and computational power, can always be predicted. In contrast, purification tells us that "ignorance is physical." Every mixed state can be generated in a single shot by a reliable procedure, which consists in putting two systems in a pure state and discarding one of them. As a result of this procedure, the remaining system will be a physical token of our ignorance."

Moreover, the postulate of purification is what distinguishes classical from quantum theory, as can be seen from some of its fundamental implications: the existence of entangling gates,

[10]This is equivalent to $\rho^A = Tr_B|\Psi\rangle_{AB}\langle\Psi|$.

the *No information without disturbance* theorem and the teleportation protocol. These and other results can be easily derived diagrammatically by using the categorical framework we partially described in the previous chapter (see [128], [129] and [34]).
What we would like to emphasise, while not going into this description, is the importance of the role of entanglement and the nature of compound systems on the one hand; on the other hand, to show once again that the categorical approach allows the main results to be achieved in a simple way thanks to the corresponding diagrammatic representations.

4.7 From circuits to diagrams: categorical approach.

In this section we will introduce the categorical approach, ZX-*caluclus* in particular ([131], [132], [133], [134], [1]), to diagrammatic model of quantum computation. This categorical approach arises from the early work first works realised at the Oxford University ([119], [135], [98]) and has developed in recent years not only in physics (for example [92], [93]) but also and especially in computer science (for example [136], [80], [81], [82], [83], [84], [85], [86], [87], [88], [89], [90]). This interdisciplinarity makes it extremely useful for understanding the philosophy of our work. In this specific case, it will allow us to give a high-level interpretation of the quantum teleportation protocol (see [119], [137] [1]).
The categorical framework for quantum computation in ZX-*calculus* is the concept of ***Prop*** (see [138]).

Definition 4.7. A *Prop* is a symmetric strict monoidal category $(\mathfrak{C}, 0, +)$ for which

1. $Ob(\mathfrak{C}) = \mathbb{N}$, or equivalently the monoid $Ob(\mathfrak{C})$ is spanned by a unique object (1 is the *generating object*);
2. the monoidal unit is $0 \in \mathbb{N}$;
3. the monoidal product is given by the addition

This means that give a *Prop* means to specify (see [102], [132]):

1. a *sequential composition* of morphisms

$$\circ : \mathfrak{C}(m, n) \times \mathfrak{C}(n, p) \longrightarrow \mathfrak{C}(m, p)$$

$$(f : m \longrightarrow n,\ g : n \longrightarrow p) \quad \longmapsto \quad (g \circ f) : m \longrightarrow p$$

satisfying the associativity axiom;

2. a *parallel composition*

$$+ : \mathfrak{C}(m,n) + \mathfrak{C}(p,q) \longrightarrow \mathfrak{C}(m+p, n+q)$$

$$(f : m \longrightarrow n,\ g : p \longrightarrow q) \quad \longmapsto \quad (f+g) : m+p \longrightarrow n+q$$

satisfying the associativity and the interchange axioms;

3. $\forall n \in \mathbf{N}$, the *identity map*

$$id_n : n \longrightarrow n$$

satisfiyng the unit axiom;

4. the *unit object* 0 satisfiyng the unit and associativity axioms;

5. a *symmetry*

$$\sigma_{m,n} : m+n \longrightarrow n+m$$

satisfiyng the relative axioms.

It is extremely intuitive to see how this mathematical structure is appropriate to describe computation, classical and not just quantum.

Classical computation Let A be a set. In $\boldsymbol{Fun}_X$, the *Prop* of sets and functions, the set $\boldsymbol{Fun}_X[n,m]$ is the set of functions from X^n to X^m with the usual sequential composition and the tensor product as parallel composition. If we choose $X = \{0,1\}$, we obtain a framework suitable for the classical computation.

Quantum computation Let $\mathbb{K}$ be a field. We can define the *Prop* $\mathbf{Mat}_{\mathbb{K}^d}$ assuming

$$\mathbf{Mat}_{\mathbb{K}^d}[n,m] := \mathcal{M}_{d^n \times d^m}(\mathbb{K})$$

the set of matrices of size $d^m \times d^n$ over the field $\mathbb{K}$. The composition is the traditional matrix product and the monoidal product is the Kronecker product. If we consider $\mathbb{K} = \mathbb{C}$ and $d = 2$, we obtain the category $\mathbf{Qubits} := \mathbf{Mat}_{\mathbb{C}^2}$.

Similar to what we did abstractly in chapter 2, we can add structure and introduce the concept of *compact closed dagger Prop*. Finally, we can add the concept of Frobenius structure and bialgebra and achieve the complete structure defining the ZX-*calculus*. It is not possible to describe the meaning of the whole construction in this work. However, we show in the following table (Tab. 4.2) the connection between algebraic constructions and diagrammatic representations in relation to quantum theory. Next, we will introduce some definitions and their corresponding diagrammatic representation so that we can discuss the teleportation protocol and show how, by manipulating diagrams alone, it is possible to prove its correctness.

TABLE 4.1: Diagrams

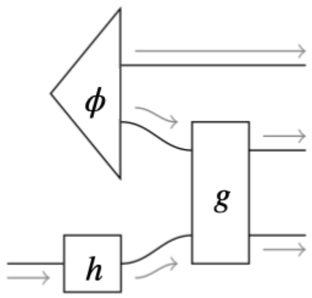
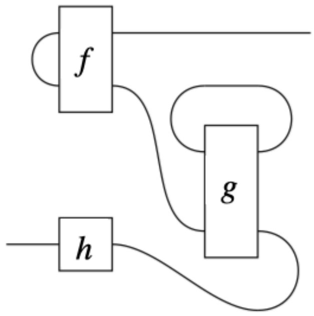
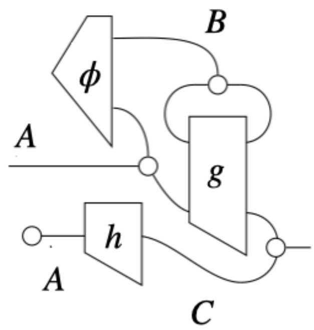
Symmetric monoidal category	**Compact closed †-SMC**	**Hypergraph category**
Circuit diagram	*String diagram*	*Spider diagram*
		
1. Adjoint 2. Isometries and Unitaries 3. Positive processes 4. Born rule	1. Separability 2. Entanglement 3. Process-state duality 4. Traspose	1. Clonable states 2. ONB 3. Observables 4. Measurement (PVM) 5. Strong Complementary

TABLE 4.1: The table highlights respectively the characteristic elements of quantum theory that are characteristic of each additional algebraic structure in the transition from a monoidal category to a hypergraph category.

4.7.1 ZX-calculus

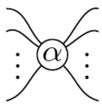
In this section, we introduce *ZX-diagrams* and set out some rules of calculation. We have no way of making a complete description (for a rigorous discussion for example [134], [92], [1]), but the elements we will introduce will make it possible to present an advanced approach to the teleportation protocol that exploits syntactic rules on diagrams to achieve a demonstration without the need to develop calculations in the Dirac formalism nor with matrix algebra.
In its complete definition, the *ZX-calculus* consists of two strongly complementary classical structures of Frobenius in a compact dagger category, on an underlying object ([93]). Below we introduce the fundamental elements for our purposes.

Definition 4.8. Given a Frobenius algebra $(C, \mu, \eta, \delta, \epsilon)$, we call a family of morphisms

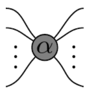
$$S_{n,m} := \mu_n \circ \delta_m : n \longrightarrow m$$

spiders.

We are ready to introduce the two spider generators:

Z-spider  $:= |00\cdots0\rangle\langle 00\cdots0| + e^{i\theta}|11\cdots1\rangle\langle 11\cdots1|$

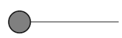
(a) Z-spider

X-spider  $:= |++\cdots+\rangle\langle ++\cdots+|+e^{i\theta}|--\cdots-\rangle\langle --\cdots-|$

(b) X-spider

From these two spiders, it is possible to achieve most of the useful tools for calculation:

Table 4.2: States, effects and operators in ZX-calculus

Algebraic description	**Matrix description**	***ZX-calculus***
$\sqrt{2}\lvert 0\rangle = \lvert +\rangle + \lvert -\rangle$	$\sqrt{2}\begin{bmatrix}1\\0\end{bmatrix}$	

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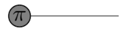
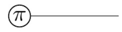
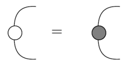
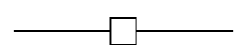
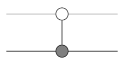
$\sqrt{2}\|+\rangle = \|0\rangle + \|1\rangle$	$\begin{bmatrix}1\\1\end{bmatrix}$	
$\sqrt{2}\|1\rangle = \|+\rangle - \|-\rangle$	$\sqrt{2}\begin{bmatrix}0\\1\end{bmatrix}$	
$\sqrt{2}\|-\rangle = \|0\rangle - \|1\rangle$	$\sqrt{2}\begin{bmatrix}1\\-1\end{bmatrix}$	
$\sqrt{2}\langle 0\| = \langle +\| + \langle -\|$	$\sqrt{2}\begin{bmatrix}1 & 0\end{bmatrix}$	
$\sqrt{2}\langle 1\| = \langle +\| - \langle -\|$	$\sqrt{2}\begin{bmatrix}0 & 1\end{bmatrix}$	
$\|00\rangle + \|11\rangle\ (= \|++\rangle + \|--\rangle)$	$\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$	
$P_Z(\alpha) = \|0\rangle\langle 0\| + e^{i\alpha}\|1\rangle\langle 1\|$	$\begin{bmatrix}1 & 0\\0 & e^{i\alpha}\end{bmatrix}$	
$P_X(\alpha) = \|+\rangle\langle +\| + e^{i\alpha}\|-\rangle\langle -\|$	$\frac{1}{2}\begin{bmatrix}1+e^{i\alpha} & 1-e^{i\alpha}\\1-e^{i\alpha} & 1+e^{i\alpha}\end{bmatrix}$	
$H = e^{-i\frac{\pi}{4}}(P_Z(\frac{\pi}{2}) \circ P_X(\frac{\pi}{2}) \circ P_Z(\frac{\pi}{2}))$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$	
$CNOT =$ $(\|0\rangle\langle 0\| + \|1\rangle\langle 1\| \otimes \|+\rangle\langle ++\| + \|-\rangle\langle --\|)$ $\circ(\|00\rangle\langle 0\| + \|11\rangle\langle 1\| \otimes \|0\rangle\langle 0\| + \|1\rangle\langle 1\|)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{bmatrix}$	

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TABLE 4.2: States, effects and operators in ZX-calculus

Here are the three rules necessary for the teleportation protocol (Table 4.3):

TABLE 4.3: ZX-calculus rules

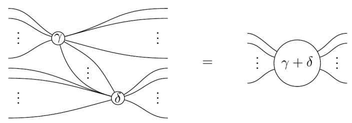
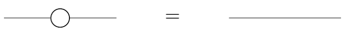
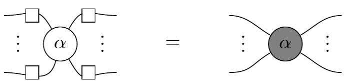
Name	***ZX-calculus***	**Description**
Spider fusion		We can fuse adjacent spiders with the same colour adding the phases
Identity		A $2k\pi$ phase can be removed.
Hadamard colour change		Two spider of different colours are related to each other by Hadamard gates.

TABLE 4.3: ZX-calculus rules

Let us make a few comments on the elements introduced in the previous tables:

Remark 4.9. First we note that with spiders we can also introduce scalars ([139]). Let us consider some examples in Fig. 4.14.

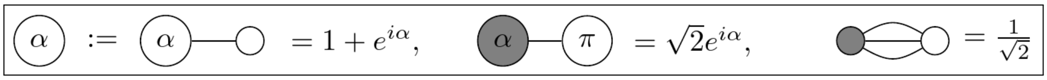


FIGURE 4.14: Some examples of scalar in *ZX-calculus*

In most works on the ZX-calculus, scalars are often eliminated: “A particular case where non-zero scalar factors can be ignored is when dealing with ZX-diagrams representing unitary quantum circuits.” ([1]). If we wanted, for example, to introduce a *CNOT* gate correctly, we could do so in the two equivalent ways shown in Fig. 4.15

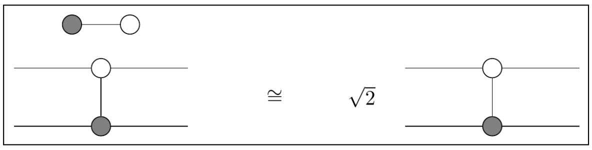


FIGURE 4.15: Two diagrammatically equivalent variants of the *CNOT* gate

Remark 4.10. From the identity rule we obtain a simpler representation of Bell states that we will use in the teleportation protocol (see Fig. 4.16)

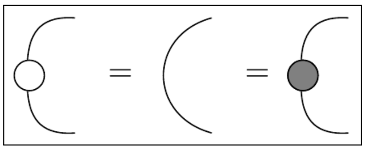


FIGURE 4.16: *Cup* state: Bell state

4.7.1.1 Teleportation protocol

We are now ready to discuss the quantum teleportation protocol and translate the traditional circuit description into a ZX diagram. The calculation rules will allow us to achieve an extremely elegant and meaningful illustration.
Alice and Bob long ago generated and shared a maximally entangled state $|\psi_{00}\rangle$. Now Alice and Bob are far apart and can only communicate via classical channels. Charlie possesses a generic qubit state $|\psi\rangle$ and asks Alice to deliver Charlie's qubit to Bob. The quantum circuit for teleporting a qubit is traditionally as in Fig. 4.17:

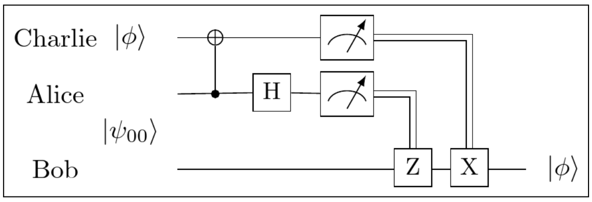


FIGURE 4.17: Traditional circuit representation of teleportation protocol.

We first convert this representation into a ZX-diagram (Fig. 4.18):

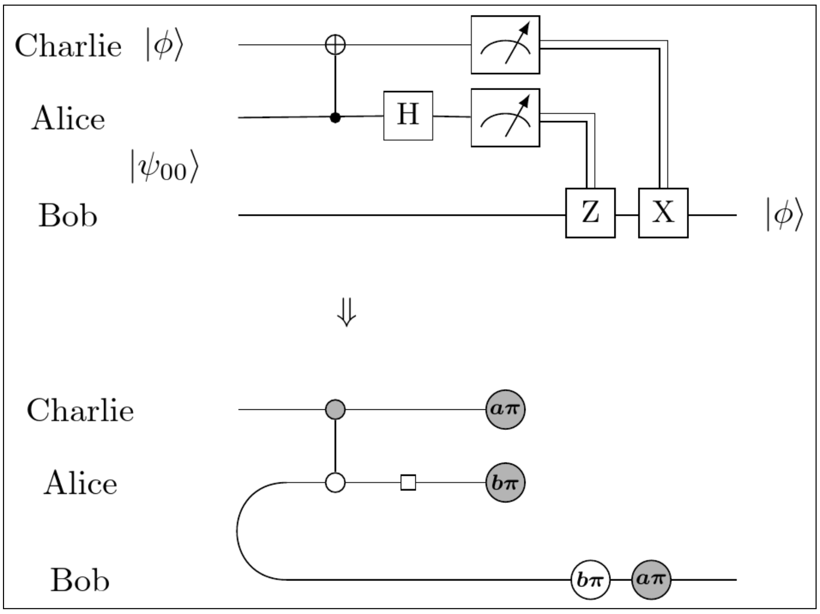


FIGURE 4.18: From quantum circuit for teleporting qubit to ZX diagram

where "*the usage of a variable*[11] *to denote a measurement outcome is a simple 'hack' to deal with classically controlled circuits*". For more details on measuring in ZX-calculus see [134] and [140].

By means of the rules described in Tab. 4.3, we obtain the diagrammatic demonstration (Fig. 4.19).

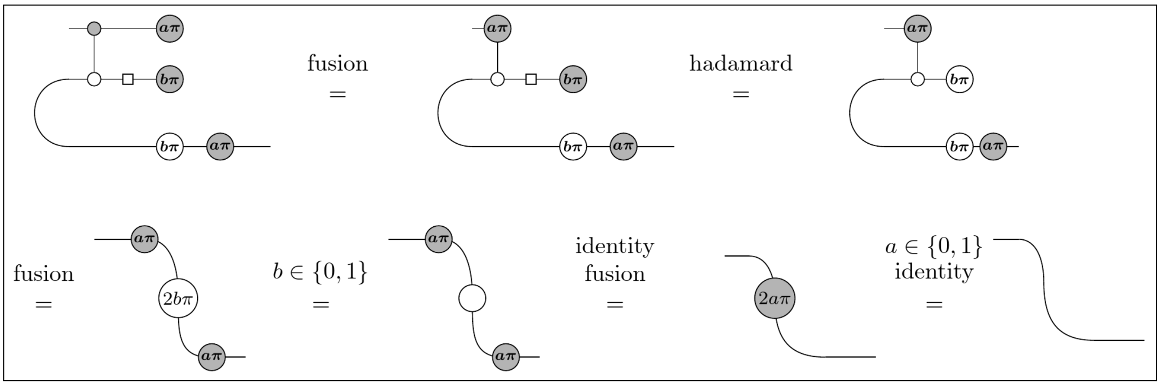


FIGURE 4.19: ZX calculus for quantum teleportation protocol (This figure is a development of demonstration in [1]).

The last diagram in the demonstration shows the fact that information flowed from Alice (Charlie) to Bob without being distorted. What allows this protocol to be successful is to be found in the interpretation of entanglement in compact monoidal categories. The

[11] In the protocol $a, b \in \{0,1\}$

presence of the concept of *name* and *coname* (see chapter 2) makes it possible to introduce state-process duality (Choi–Jamiołkowski isomorphism) and makes the nature of teleportation evident from a diagrammatic point of view (see [119], [141], [135] and [137]) as flow of information. In particular, the information flow refers to the role of bipartite entanglement in the protocol itself. In this sense entanglement becomes synonymous with the possibility of information flow.

4.8 Diagrammatical representation of quantum linear optics

We would like to finish this chapter by briefly introducing the categorical diagrammatic language in the case of linear optics as well. Research work on this approach is extremely recent ([71], [70] and [72]). As previously, the aim of this section is to show the links between abstract diagrammatic representations and their interpretations: in this case, the existence of a link (a functor!) between the ZX-calculus and a similar language defined to describe the theory of optical devices used in quantum linear optics. We have no way of exploring this construction in detail, but we do want to introduce a couple of elements that should explain what is possible using a diagrammatic language to describe linear optics for quantum computation. In summary, the idea is as follows:

1. We can introduce the monoidal category **LO** of optical circuits which are obtained by sequential and parallel composition of the following diagram:

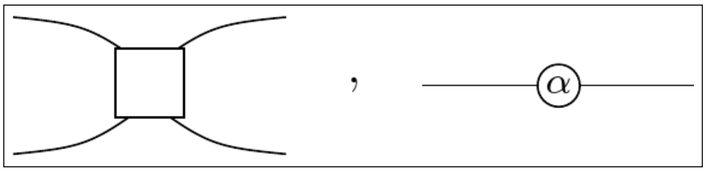


FIGURE 4.20: Beam splitter and phase shift diagrams

depicting the beam splitter and phase shifter devices. In this category $Ob(\mathbf{LO})$ are the optical modes and the transformation are the *beam splitter*

$$BS : a \otimes a \longrightarrow a \otimes a$$

that acts on a pair of optical modes, and the *phase shift*

$$S(\alpha) : a \longrightarrow a$$

that acts on a single mode with a parameter $\alpha \in [0, 2\pi]$.

2. The immediate (classical) interpretation of **LO** category is given in $\mathbf{Mat}_{\oplus}$, the category of matrices over complex numbers, where $\oplus$ is the direct sum of vector spaces. In fact we can consider the functor

$$U : \mathbf{LO} \longrightarrow \mathbf{Mat}_{\oplus} \tag{4.2}$$

defined by posing on objects

$$U(a) = \mathbb{C} \tag{4.3}$$

and on arrows

$$U(S(\alpha)) = (e^{i\alpha}) \tag{4.4}$$

$$U(BS) = \frac{e^{i\phi_0}}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \tag{4.5}$$

In the case of half-silvered mirrors, we obtain the matrix used above ([142])

$$U(BS) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Two generators allow us to interpret the Mach-Zehnder interferometer as in Fig. 4.21

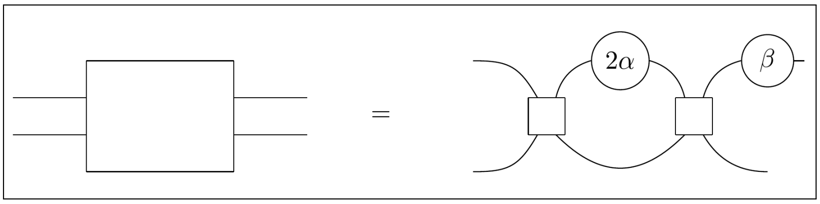


FIGURE 4.21: Mach-Zehnder interferometer diagram in **LO**

The classical interpretation of this diagram is then given by the traditional matrix of beam splitter.

3. Not only we can give a classical interpretation of the linear optical circuits as complex-valued matrices, but we can also give a graph-theoretic interpretation of these circuits introducing **Path**, a Prop category generated by a bialgebra, for counting paths. In this way we can interpret the previous diagrams by means of the functor[12]

$$F : \mathbf{LO} \longrightarrow \mathbf{Path} \tag{4.6}$$

[12] For the diagrams see [71]

In a similar way to the ZX-calculus, we can identify in this category (in the classical case) six generators (see [71]): we thus build a diagrammatic calculation on classical optical circuits (coherent light). It is not important here to go into details, but what is important is that there is a categorical diagrammatic interpretation supported by the existence of functorial correspondences. In fact, the following propositions hold:

Proposition 4.11. *There is a monoidal functor*

$$C : \textbf{\textit{Path}} \longrightarrow \textbf{\textit{Mat}}_{\oplus} \tag{4.7}$$

In this way makes sense the equation 4.6:

Proposition 4.12. *The classical interpretation of linear optics factors through the Path calculus, i.e. the functor F defined above satisfies $U = F \circ C$.*

We can summarize the above by requiring it to commute the following diagram:

$$\begin{tikzcd} \textbf{LO} \arrow[r, "U"] \arrow[dr, "F"'] & \textbf{Mat}_{\oplus} \\ & \textbf{Path} \arrow[u, "C"'] \end{tikzcd} \tag{4.8}$$

Following the representations given in [71] we can diagrammatically translate the Mach-Zehnder interferometer into the **Path**-calculus as in Fig. 4.22

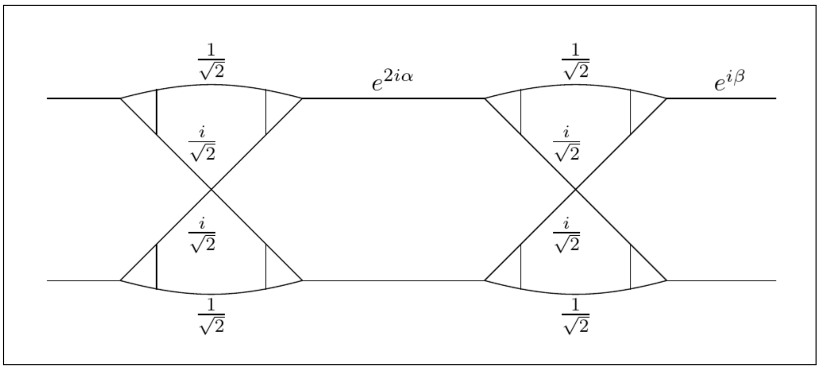


FIGURE 4.22: Mach-Zehnder interferometer diagram in **Path**-calculus

4. It is possible develop a quantized calculus **QPath** introducing the creation and annihilation of particles as generators:

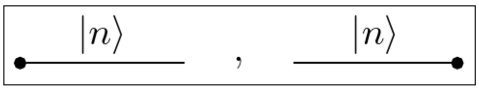


FIGURE 4.23: **QPath** generators in addition to those of **Path**

With these two generators we can build the creation and annihilation operators on a single modes as **QPath** diagrams (The Fig. 4.24 is build by analogy with the example 4.1 in [71])

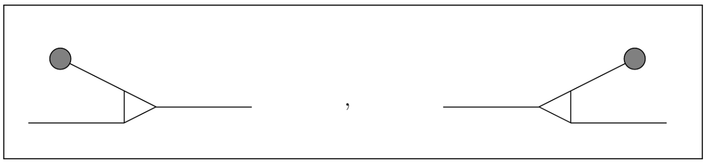


FIGURE 4.24: Creation and annihilation operators on a single modes as **QPath** diagrams

where the black nodes in **QPath** are mapped respectively to $|n\rangle$ and $\langle n|$, indicating that the mode is occupied by n particles. This matching is made possible by the bosonic Fock space functor

$$B : \mathbf{Mat}_{\oplus} \longrightarrow \mathbf{Hilb}_{\otimes} \tag{4.9}$$

where $\mathbf{Hilb}_{\otimes}$ is the monoidal category of Hilbert spaces and bounded linear maps with the tensor product as monoidal product.
In analogy with 4.8 we can describe the above considerations with

$$\begin{tikzcd} \mathbf{LO} \arrow[r, "U"] \arrow[dr, "F"'] & \mathbf{Mat}_{\oplus} \arrow[r, "B"] & \mathbf{Hilb}_{\otimes} \\ & \mathbf{QPath} \arrow[u, "C"'] \arrow[ur, "Q"'] & \end{tikzcd} \tag{4.10}$$

where the functor Q represents the diagrammatic quantum interpretation of the optical circuits.

5. We can interpret the **ZX**-calculus in **QPath** constructing a functor between categories. We can introduce the dual-rail encoding of polarization is the map

$$|H\rangle \longrightarrow |0,1\rangle \quad , \quad |V\rangle \longrightarrow |1,0\rangle$$

with which the dual-rail encoding consists in encoding a polarised mode of light as a pair of spatial modes in **LO**. In this way, for example, the Z basis of dual rail qubit may be expressed as a pair of **QPath** diagrams:

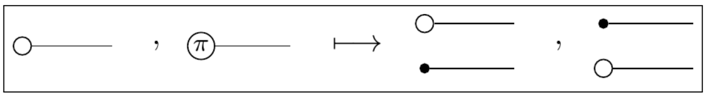


FIGURE 4.25: The Z basis in **QPath**

Similarly, it is possible to construct a whole diagrammatic language for linear optics for computation (see [71]).
Ultimately, we can realize diagrammatic computation even on optical circuits of linear optics in the case of single-photon devices to realize quantum computation.

4.9 Conclusions

This chapter has allowed us to interpret the categorical structures introduced in the previous chapter and in Appendix C from a computational point of view first and a physical point of view later. First, our attention has focused on the classical case, but what we have seen represents a sufficiently developed framework for introducing quantum computation. The second part developed the categorical approach to quantum theory (OPT) and categorical quantum computation (ZX-calculus), both from a theoretical point of view and in terms of its possible implementation by optical devices. The last section, in particular, seems to be able to theoretically support the choice of an equivalent diagrammatic language in terms of theory and experimental device theory linked to linear optics.
The presentation, from a logical point of view, is thus concluded: thanks to the presence of functors between categories, the abstract diagrammatic presentation can be interpreted from a logical, physical-computational and physical-experimental point of view. If from a theoretical point of view this is the correct approach, didactically we have to reverse the construction: it will be our task to design a TLS for students in such a way that the construction and the use of the diagrammatic model is their task. Only in this way can we ensure the correct understanding of the proposed approach.

Chapter 5

Theoretical and methodological framework

This chapter begins the second part of the thesis. We introduces the theoretical and methodological frameworks used relating to physics education: the MER, a theoretical framework designed that guided us in the clarification and analysis of science content, and in the design of educational pathways; the IBL and the MBT because in our educational proposal for secondary school students our aim is to help students develop an organized knowledge structure concerning QIS embedded in active and constant engagement in construction and reconstruction knowledge through hands-on interactions.
The next two chapters will describe the research conducted with teachers and students.

5.1 Introduction

The research questions presented in the introductory chapter are simultaneously posed on two distinct and connected levels: those relating to the work of teachers and that of students. In particular, this work aims to guide an educational reconstruction of the content that ultimately allows the topics of the second quantum revolution to be brought into the curriculum of secondary schools. The point of view expressed is that this reconstruction should take place through the work of teachers and researchers. Appropriate theoretical and methodological tools must support such work from the perspective of answering research questions[1].

The first part of the chapter addresses two fundamental issues: the educational reconstruction of content and the relationship between mathematics and physics. The first one is presented within the framework of the MER[144], a theoretical framework that addresses the problem of reconstructing science content so that it can become instructional content. We describe this framework in section 4.1. The second focuses on the approach related to the physical theory of information, and especially a computational approach such as the one presented here offers insights into the dialectic between mathematics and physics. Some research results concerning these topics are then described, as well as the model of mathematical reasoning in physics in [143] already used in the introductory part of QP introducing our TLS for secondary school students. To these will be added specific considerations on constructivism and category theory, characterising the approach used in this work. In particular, it will be shown how the abstract nature of category theory offers, in practice, an extremely concrete tool for interpreting physical and other processes. The high conceptual value of the proposal, therefore, shifts from the algebraic-formal plane to the conceptual plane of processes and their composition, resulting in a shift of focus from algebraic calculus to structure in a much more modern perspective of mathematics. The structural aspect will merit permitting the exact conceptual representation of both theory and experimental implementations, seen as interpretations of the same mathematical framework through specially constructed diagrammatic representations.

The second part of the chapter is completely dedicated to the presentation of research tools and methods used in both the teacher professional development course and the TLS for students. As far as teachers are concerned, we will focus on the data analysis resulting carried out through qualitative methods and how these aspects influenced the design work for the TLS. As far as students are concerned, we will focus on the teaching strategies adopted and the tools used to implement them: inquiry-based learning and modelling-based teaching. Finally, we will focus on the role of worksheets as working materials and data analysis from the perspective of Designed Based Research (DBR) ([145], [146], [147]).

[1] It should be noted that the decision to include the pathway in continuity with the approach to QP through polarisation (see [143]) also requires a form of continuity in terms of the methodology and instruments used. The part that follows is basically a confirmation of this and an extension of it in terms of the aspects that more appropriately refer to the physical theory of computation and information.

5.2 The model of educational reconstruction

The MER ([144],[148]) is a theoretical framework for research and development in science education. As the authors underline in [144]

> "The key message of the model is that science subject matter content (including concepts and principles as well as conceptions about science and the scientific inquiry processes) may not be presented in a somewhat reduced or simplified manner in science instruction. The science content structure for instruction is somewhat more elementary (from the science point of view) on the one hand but richer, on the other hand, as the elements of science content of a certain topic need to be put into contexts that make sense to the students and may be understood by them."

The model draws on the position of epistemological constructivism on the one hand [149], and on the other hand on the European tradition, German above all, of European *Didaktik* and *Bildung* (formation). The *Bildung* draws its specificity from the need to build the individual in his or her entirety, in this case to transform a student into a citizen; in this way the interdisciplinary nature of the science of education is expressed ([150]).
Two major conceptions of German Didaktik are the *Didaktische Analyse* and the *Elementarisierung*. According to this approach, content should not be regarded as *given*, but will be the result of a reconstruction process that takes into account both of the content to be learnt and the students' cognitive and affective variables [144].

> "The science content is not viewed as "given" but has to undergo certain reconstruction processes. The science content structure (e.g. for the force concept) has to be transformed into a content structure for instruction."

As regards the *Elementarisierung* in MER, we must take into account three issues that help to explain its meaning. The first aspect to consider is the identification of first elements, fundamental entities, of a certain content to have to be addressed for instruction. This search is a function of the objectives and aims of teaching so that they are also clear to the students. The second aspect is the need to reduce the elements of complexity of a particular science so that the topics are accessible to the learners. This involves finding a way to introduce to students those elementary entities described in the first part. The difficulty in this second task consists in finding a strict compromise between scientific rigour and accessibility for students. All this must be done in order not to interpret it in terms of merely simplifying science content. Finally, it is necessary to consider the students' learning processes in relation to teaching methods able to make the transition from the pre-instructional conceptions implicit in each student into scientific concepts.

5.3 Three components of MER

Based on what has been presented in the previous paragraphs, we can now accurately describe the three components that characterise the MER:

Clarification and Analysis of Science Content The aim of this first component is to transform a certain science content structure into a content structure for instruction. This is well illustrated by Fig. 5.1 from [144]

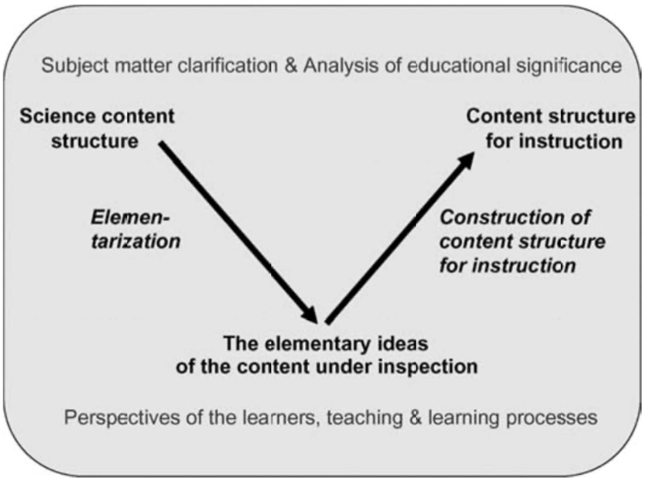


FIGURE 5.1: Steps towards a content structure for instruction

This first component consists of two processes: the elementarization and the construction of the content structure for instruction. The first, which we have already discussed, is therefore a fundamental step for the construction of content structure for instruction. In order to implement this first part of the model, the research methods refer to the qualitative analysis of the manuals and scientific papers, to be conducted from the science education perspective. This aspect is crucial for several reasons. Firstly, textbooks address experts and express knowledge in an abstract and highly condensed manner. Furthermore, working on the most recent research articles can avoid using terms that have fallen into disuse and conceptualisations that have been shown to be wrong later on. There is also an exciting linguistic aspect to be addressed, namely the semantic area of words used in science instead of the common usage of the same. In parallel, content analysis can benefit considerably from research into students' ideas on the specific topic in question (both pre-instruction and post-instruction [144]).

Research on Teaching and Learning The second step identified by the method requires the first step to be based on empirical research on teaching and learning. This research focuses on students' pre-instructional conceptions, affective variables and the role of instructional methods, experiments, etc. Furthermore, it is hugely significant to research teachers' beliefs about scientific concepts, how their students learn and how they can support the learning process. Also of great interest for our work (see [144])

> "However, for a number of new and also traditional topics little to no research at all is available. In these cases, research on teaching and learning and the process of educational reconstruction are closely interrelated. Here qualitative methods like interviews or small scale learning process studies prevail."

Design and Evaluation of Teaching and Learning Environments The third component includes the design of teaching materials, learning development activities and actual teaching-learning sequences. The design of these elements is subordinate to the research on students' perspectives on the one hand and the results of the elementarisation of topics on the other.

The components described above should not be regarded as static, but in their dialectical exchange. This is why the procedure must be repeated step by step recursively ([148]).

5.4 Dialectic between Mathematics and Physics

As we have already emphasised, one of the distinctive aspects of the work we are presenting here is the special relationship between mathematics and physics. This relationship is expressed in the fact that the approach that allows us to introduce the themes of the second quantum revolution is that of the physical theory of information. In this section,

we will first describe some of the results in literature on the role of mathematics in quantum physics. We will focuse on the informational approach to physics. Finally, we will anticipate some considerations regarding the role that a categorical description play in our work in creating a unique model for describing the computational theory, the underlying quantum physics and the operation of physical devices to realise experimental setups.
The reference article in the literature is mainly [143] because the introductory part of the QP[2] was based on it, and [151] whose subject of the semiotic resource system fits well with the particular focus on the problem of language in our work.
In the first perspective we followed the approach of [152] according to which

> "One important reason for the power granted to the physical science might be due to the deep relationship between physics and mathematics. Several historical and philosophical studies show that mathematics and physics are strongly interrelated in a fruitful and multifaceted manner. The description of physical processes by mathematical means is one of the most characteristic traits of physics itself. If analysed more precisely, the role of mathematics in physics has multiple aspects: it serves as a tool (pragmatic perspective), it acts as a language (communicative function) and it provides a way of logical deductive reasoning (structural function)."

In this work, the authors identify two main roles of mathematics in physics: one technical and the other structural, the latter referring to the role of maths in structuring physical entities and situations emerging from the processes of interpretation and formalization. From ad educational perspective, the structural role creates traditionally more difficulties since it consists of both mathematization and interpretation [152]. About the second work [151], the use of diagrammatic representations of categorical origin and the dialectic with other types of representations is one of the most significant aspects not only from a theoretical but also from a educational point of view.
The evolution of the TLS and the diagrammatic representation fit into this educational context and is compared with it in an attempt to extend it.

5.4.1 Mathematics for teaching quantum physics

In [143] the authors review several approaches to the education of quantum physics in secondary schools to examine the respective mathematical structures used to make the

[2] The relevant part in this regard is that of paragraph 4 of [143] concerning an implementation in the context of polarisation

theory's conceptual aspects more comprehensible. This aspect is intrinsic to developing the most recent physical theories and QP among them. However, it is also relevant from an educational point of view: the conceptual difficulty of QP can also be solved didactically to some extent by a formal as well as a conceptual approach[3]. The authors seek a synthesis between the purely formal attitude (see [4]) and the purely logical-conceptual attitude that can be an obstacle to effective learning.
From an educational point of view, many representations are used to represent physical processes, laws or relationships, but the authors note that each representation can only highlight one aspect. As they suggest , the interaction between representations is a possible way forward. In [154] for example, the author highlight some aspects that can benefit from multiple representation:

1. *Multiple representations can support learning by allowing for complementary information or complementary roles.*

2. *Secondly, multiple representations can be used so that one representation constrains interpretations of another one.*

3. *Multiple representations can support the construction of deeper understanding when learners relate those representations to identify what are shared invariant features of a domain and what are properties of individual representations.*

These aspects can all together or partially contribute to a deeper understanding[4].
There are, however, two considerations to be made: the first concerns the fact that, given the advanced tools required to introduce the mathematical model of QP, an elementarization process is necessary, including a conscious transition between the different representations. The second is that the use of representations must not create an excessive cognitive load and must therefore be wisely utilised and carefully developed [151].
The ability to consciously use and combine multiple representations to shed light on the mathematical structures from a physics perspective is defined as "visualisation"by the authors. This visualisation is to be accomplished in a dialectical synthesis of physics and mathematics that acts in three stages as shown in the Fig. 5.2.

This dialectic is made explicit in [143]:

1 physical processes are mapped onto mathematical elements;

[3] An interesting review in this regard can be found in [153]
[4] See also [155] and [156]

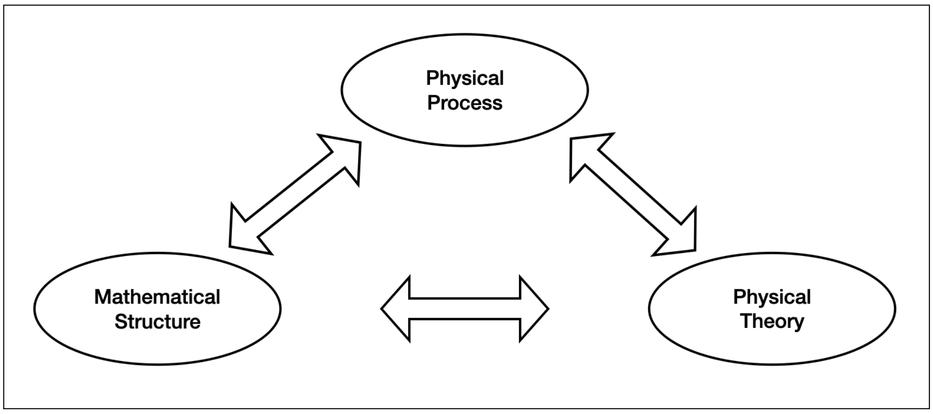


FIGURE 5.2: Relationship between Physics and Mathematics

2 an attempt is made to give a physical interpretation of mathematical operations;

3 The previous two points will find synthesis in physical theory and retroactively explain the previous ones.

Due to the conceptual difficulty of QP, it is necessary to find appropriate representations that reflect the mathematical formalism[5].

5.4.2 Two-State Systems Representations and Its Mathematical Structure

The representation of two-state systems in relation with the mathematical structure deserves special attention for our design. In generale we can consider the various types of representation of two-state systems and their characterisation: experimental, model of experiment, pictorial, symbolic, graphical and algebraic (see [143]). This several representations for the same object[6] have the virtue of supporting the understanding of characteristic features of quantum physics: the superposition, the time evolution and the measuring process, the uncertainty and the probability. We are interested in observing the approach taken by the authors: representations contribute to a *disciplinary discourse* and, if used wisely, can contribute significantly to a comprehensive understanding of these concepts. Some aspects of the educational reconstruction resulting from [143] and [157] will be made

[5] We will see that diagrammatic manipulation rules on circuits and quantum computation rules live in the same mathematical structure: that is their power! Everything that is done in theory can be done in diagrams and vice versa. Proofs are *equi-valent*!

[6] In this case the object is the two-state system. In this thesis the discussion about the term *object*, there will be later.

explicit in the chapter on the construction of the TLS for students.

We would now like to focus on the concept of disciplinary discourse, aiming to problematize the dialectic (morphism) between experiment, physical theory and mathematical model. We want to express this dialectic in a mapping of meaning between the various representations.

5.4.3 Disciplinary discourse: a semiotic approach to Physics

In [151] the authors place themselves in a Foucoltian semiotic perspective of signification (see [158],) when they argue the need to link a discipline - physics in our case - but for them, the discourse is entirely general, to the discourse about the discipline. To be more faithful to what the authors claim, what characterises a discipline is the possibility for a community to share knowledge that is encoded by the system of semiotic resources that are developed to represent this knowledge. As argued by Lemke (see [156]) verbal language as well as mathematics or visual representation (semiotic resources), bring together a typological and a topological aspect. The first concerns category issues, such as categories of processes or relations. In this aspect, the distinction is clear: belonging to a category is defined. Either an entity belongs or does not belong to a given category. However, the second aspect, the topological aspect, is the one that best suits science education, which is why it can be called topological semiotics. In this case, the meaning of an element requires the change by even infinitesimal degrees and, thus, a language capable of expressing quantitative elements. For these purposes, languages of visual representation are much more powerful than natural language. However, science envisages the appropriation of a language that has characteristics of both types ([156]):

> "To characterize material processes and their relationships we need both categorial descriptions and quantitative reasoning, and this fact created a historical pressure that gradually built a bridge between the linguistic and the visual-gestural: the result was mathematics, built out from the linguistic as the algebraic extension of the semantics of natural language in matters of quantity, ratio, and continuous variation, and built out from the visual-gestural side as geometric diagram and eventually Cartesian coordinate graph. The ability in mathematics first to create correspondences between algebraic and geometric representations, and eventually to construct complete equivalences between them provided the missing link that enabled science to reason both verbally

and quantitatively, both typologically and topologically, about material phenomena and processes."

According to Lemke ([156]), scientific concepts are strictly related to experimental actions: on the one hand, therefore, a constellation of multiple signs of natural language, mathematics and visual representation; on the other hand, the actual operations of experimental manipulation of apparatus and measurement (See Fig. 5.3)

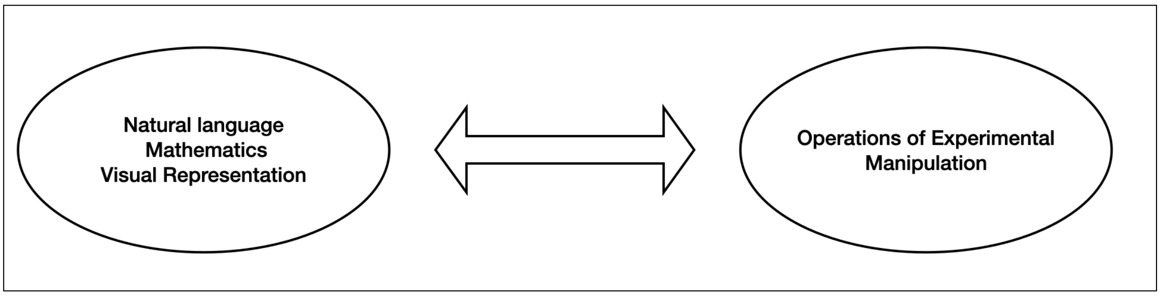


FIGURE 5.3: Dialectic Experiment - Language: *An operational definition of a scientific quantity is not just a material procedure, it is also a meaningful sequence of actions, which is connectable logically to our verbal definition of the quantity and to its mathematical relationships to other quantities in a theory or model for which we can give verbal justifications in relation to the kinds of human problems these quantities and relations are useful in solving.*

Following these kinds of considerations, in [151] the authors introduce the concept of *Disciplinary Discourse*. Disciplinary discourse must take into account words, symbols, gestures, diagrams, formulas etc. of a particular discipline; but also *the artefacts, pieces of apparatus, measuring devices, etc. and the actions, practices and methods.* Thus, there are three components of disciplinary discourse: representations, tools and activities.

Representations This first aspect is what we addressed in our analysis of Lemke's work.

Tools Traditionally the relationship between mathematics and physics is defined by the representations. However, our approach requires, as will become clear later on, that tools should also be included in this perspective. Tools have the merit of being able to achieve a condensation of meaning. In our case, the mathematics and syntactic representation of the theory is the same as that of the theory of experimental devices. Then the mediation of meaning operated by the devices, moves into the mathematical domain and vice versa.

Activities Finally, the last key element is the activities: the laboratory as an operational practice becomes fundamental. Here too, an operational approach emphasises that

> the practice of constructing a particular experimental device becomes itself, once it has been formalised through an algorithmic procedure, the carrier of meaning within the mathematical discourse in physics.

In conclusion, one wants to emphasise the semiotic value of a disciplinary discourse that, if appropriated by students, should result in integrated knowledge, possibly enabling students autonomously to determine further facets of meaning. In particular, this is true in the discourse in mathematics and that between mathematics and physics. In our work, different representations and tools will be placed within the same theory, even if not explicitly for students. Thus we hope to allow a deeper network of meanings to be constructed through the study of transition maps[7].

5.5 Educational strategies

Let us now turn our attention to two fundamental aspects that define the methodologies adopted for the implementation of TLS for students and, consequently, for the thinking work done together with the teachers who implemented them in the classrooms: IBL and MBT.

5.5.1 Inquiry-Based Learning

As we shall see, the TLS for students and the teaching-learning materials constructed for classroom work are deeply inquiry-based.
Like the MER, the IBL is also based on theories related to constructivism, understood as *theory or philosophy about how an individual learns, one in which the student is embedded in active engagement and is constantly constructing and reconstructing knowledge through hands-on interactions* ([159]). This approach in particular differs from others such as behaviourism and subjectivism in that it considers learning to be self-regulated and socially mediated, as the student actively engages, interacts and operates within the boundaries of his or her environment. This means taking a different perspective from that of the mind as a tabula rasa ([160]): students bring their own social, cultural and educational history. This history cannot be ignored; otherwise there is a risk that the student will accept new concepts only for the immediate purpose of evaluation ([161])

[7] We saw at the conclusion of the theoretical chapters how the abstract diagrammatic representation can then be interpreted through functors such as computational logic, underlying physical theory and physical theory of experimental devices.

> "...students come into the classroom with preconceptions about how the world works. If their initial understanding is not engaged, they may fail to grasp the new concepts and information that are taught, or they may learn for the purposes of a test but revert to their preconceptions outside the classroom."

In this regard, a brief history of some constructivist approaches who have brought significant insights into how students learn is given in [159]. Here are the most significant aspects for us:

1. The stimulus from prior knowledge: According to J. Dewey's approach ([162]), posing a significant problem from the perspective of the student's prior knowledge activates the learning process. Here is the entire statement from ([162]):

   > "Education may be conceived either retrospectively or prospectively. That is to say, it may be treated as process of accommodating the future to the past, or as an utilization of the past for a resource in a developing future. The former finds its standards and patterns in what has gone before. The mind may be regarded as a group of contents resulting from having certain things presented. In this case, the earlier presentations constitute the material to which the later are to be assimilated. Emphasis upon the value of the early experiences of immature beings is most important, especially because of the tendency to regard them as of little account. But these experiences do not consist of externally presented material, but of interaction of native activities with the environment which progressively modifies both the activities and the environment."

   It is clear, therefore, that in this perspective, a teaching proposal must be able to start from elements known to the students in order to determine in them an activation that introduces a construction perspective for the future. For the student to accept the effort of modifying his conceptions, he must start from a problem to which his own experience can relate.

2. The transformation of previous conceptions into new ones is what Piaget called *adaptation* ([163]). This adaptation occurs through two processes: assimilation and accommodation. With the first, the learner uses new information and transforms the new knowledge in order to adapt it to existing mental models. The mental models are altered with the second to accept the new knowledge. Adaptation is significant because it occurs when a student encounters phenomena contrary to the knowledge

possessed. According to Piaget, there must be a balance between assimilation and accommodation. If this is not the case, we speak of cognitive dissonance, which, if wisely measured, can be extremely useful from an educational point of view. However, care (see [159])

> "When a new event doesn't fit an individual's presently held belief system, it can possibly be discarded because it doesn't fit with the person's cognitive model of understanding. Assimilation, accommodation, and disequilibrium are the basis for constructivist thinking, with conceptual change constantly at work."

This conceptual change, which we will also discuss later, must arise spontaneously for each individual when the previous model is unsatisfactory with current conceptions.

3. Finally, according to social constructivism, Vygotsky focuses on two fundamental elements in his *Thought and Language* ([164]): language and social interaction. From this point of view, it is not only the interaction with physical objects that are considered but also the social mediation effect in which the learner is involved. Regarding our work, in particular, two elements interest us: the first, more immediate, is the fact that the student needs a teacher to develop the skills in which he or she is deficient. The second, less obvious and more profound, refers to the study that Vygotskian psychology has made of the relationship between thought and speech. What is required of the student is to translate language, the external discourse presented by the teacher, into thought. This thought is mediated by the individual heritage of each student, who then proceeds in the most complex process: the reconstruction of discourse from his or her thought. To do this, the inquiry performs a fundamental function[8].

According to the constructivist approach (see [159])

> "Most traditional teaching is focused more closely on what students can achieve independently, but a constructivist teacher teaches to the upper zone by providing assistance to students' performance through prompts, leading questions, hints and clues, or asking students to clarify their thoughts about the phenomenon being studied."

Summarising in [159], the author describes a constructivist approach as being based on a few fundamental elements:

[8] We will return to these aspects in a section devoted to them

1. new knowledge can be fostered by activating the senses;
2. the acceptance or non-acceptance of new situations is determined by the students' current knowledge;
3. how current processes are interpreted depends on prior knowledge;
4. knowledge is constructed by the learner trying to build connections between old and new knowledge mediated by language;
5. links between knowledge are the basis for the construction of new knowledge and conceptual change;
6. understanding of concepts is determined by the learner through a continuous process of construction and reconstruction;
7. learning is an individual and social process;
8. enquiry is a teaching strategy that allows the learner to test their own theories and scientific knowledge for adherence to new knowledge. This strategy ultimately enables conceptual change to be realised.

These characteristics are summarised in what is called a learning cycle. Actually, there are several in the literature (for example [165], [166]), but their common denominator is expressed by certain fundamental moments that allow the teacher to focus on certain aspects and the student to proceed, supported, in learning:

Engagement The foundations of learning are laid through the teacher's strong engagement. The teacher grabs the students' attention by emphasising the lesson's purpose and discrepant aspects that create cognitive dissonance. This dissonance should immediately activate the students concerning their previous experience.

Exploration The exploration stage allows students to experience hands-on learning and helps to level the class.

Explanation The teacher puts into logic using appropriate language, reconciling prior knowledge and new information, the new concepts being formed in the form of (individual and social) experience.

Elaboration The new meaning, constructed as a synthesis of prior experience and conceptual change, can be reinforced through guided investigations in which students are supported to defend their finding and justify them through logic and concrete evidence.

Evaluation The teacher resumes the entire lesson and summarizes what has been done by asking higher-order questions.

In the stages now presented, a typical feature of enquiry begins by structuring the lesson in such a way as to activate inquiry. In [159] are summarized the four levels of instruction. The table 5.1 represents these levels: As can be deduced from the table, the greater the

Table 5.1: Invitation to Inquiry Grid

	Demonstrated Inquiry	Structured Inquiry	Guided or Teacher-Initiated Inquiry	Self-Directed or Student-Initiated Inquiry
Posing the Question	Teacher	Teacher	Teacher	Student
Planning the Procedure	Teacher	Teacher	Student	Student
Communicating the Results	Teacher	Student	Student	Student

student's activity, the less the teacher's intervention. We describe the four levels below:

Demonstrated Inquiry The purpose of this type of investigation is to elicit further questions in order to extend the initial investigation. The teacher asks questions, explains the procedure, and informs the students of the results. It is usually extremely interesting, especially if the conclusions are counterintuitive. For this reason, it can be used to introduce a topic and activate student engagement through the astonishment that is realised.

Structured Inquiry This time it which the students have to give an explanation based on the evidence collected and shown by the teacher. In this case, the responsibility for designing tables with the data, analysing them and determining the consequences is given to the students. This is traditionally used in high school workshops where the teacher provides an initial question and gives the materials and directions on how to use them. Then the students do the work, collect the data, draw conclusions and communicate them. This activity usually acts as a bridge to gaining confidence in activities where the degree of responsibility increases. However, if, in doing so, the students discover discrepancies with what they expect, then this type of work can become an inquiry in its own right. Furthermore, the data design aspect is crucial and reveals the autonomous characteristic of this investigation.

Guided or Teacher-Initiated Inquiry The investigation design is also left to the students. The teacher is left with the task of providing the research questions or hypotheses and actively participating in the student's work to organise their work, showing any inconsistencies with the questions asked. In this third case, the responsibility shifts from the teacher to the student. This is normally followed by a structured investigation that lays the foundations for a higher-level investigation that would otherwise be seen as excessively difficult.

Self-Directed or Student-Initiated Inquiry It is the highest level of investigation. The teacher only has the role of facilitating the investigation and nothing else. To do this, he or she goes around the desks, asking ancillary questions to allow the students to obtain answers independently and find reliable sources from which to obtain any information. At the moment of communication, the teacher plays the role of the organiser by facilitating the discussion. The three moments of the investigation are all the student's responsibility, including the research questions and the choice of hypotheses to be tested.

We will see how the building of TLS and materials fits into this classification and how, for the most part, these will be guided or structured investigations.

5.5.1.1 Conceptual Change

One of the most significant aspects of the learning process is that of conceptual change:

> "When a new situation arises that is inconsistent with a child's present schema (such as the data from the mass experiment), the student may either disregard the new information because it doesn't fit with the presently held notion, or he or she may change or give up the previously held notion and accept a new notion based on new evidence."

This occurs through a continuous alternation of assimilation and accommodation: in the first moment, students become aware of stimuli, concepts and elements of the external world about existing models, while the second is the modification and adaptation of cognitive structures to new situations ([159]) This way, the cognitive balance is achieved without conceptual change developing. This happens in teaching practice just as it has happened in the history of science. In the particular case of the transition to quantum physics, this aspect was dealt with by Malgieri and Zuccarini in ([19]): we report here only the most significant aspects of this work while emphasising that the whole introductory part of QM presented to students is based on these aspects.

First of all, it must be emphasised that the Conceptual Change (CC), in the case of quantum theory, is extremely special since it involves several domains of knowledge: while it first concerned physics and chemistry, today, with the physical theory of information, it poses the problem of conceptual changes in logic, computer science, information and communication theory. All this, if it is to be taught through CC,

> “requires the gradual elaboration and revision of complex knowledge systems consisting of many interrelated elements. It is a difficult process whose promotion requires the interplay of multiple instructional strategies. In addition, it may involve not only changes in learners’ cognition, but also in metacognition, epistemic beliefs, beliefs about learning and other factors (e.g., interest, attitudes). Finally, an extensive sociocultural support is needed for achieving all these kinds of change.”

According to the authors, change occurs on three distinct levels: “quasi-qualitative”, “mathematical” and “visual”:

1. The first of these levels requires a revision of the basic terms of classical theory through discipline-specific language. The changes involve two aspects: the definitional (physical quantity, measurement, state) and the metaphorical. Studies on both have been addressed in the literature in recent years (see [167]).

2. Classical mechanics, indeed, uses an advanced mathematical language that students are familiar with by the fifth year. They are basically expected to be able to master mathematics and interpret it consistently from a physical point of view. The changes between Classical Physics (CP) and QP are identified in four types: conceptualisation of a construct and its referent; the role of its constituents; notable instances of the construct and the physical situations of its interest; and the structural role of the construct. These aspects are significant in constructing the introductory path to QP and will also be contextualised concerning the mathematical objects used in our TLS on computation.

3. Finally, and this is undoubtedly one of the most relevant aspects for us, the authors identify three elements that can be visualised in QP: experimental setups, simulations and mathematical constructs[9].

5.5.1.2 Thought and Word

Concluding the section on *Inquiry*, let us focus on one of the main aspects of constructivism: the role that language, and in particular the relationship between language and thought, plays in learning. Vygotsky's work in this regard remains fundamental today in several aspects. We will only address those that seem to us most relevant for the continuation of the work.

The structuralism's perspective emphasizes a dialectical relationship between speech and thought. Vygotsky ([164]) laments that previous studies have considered the two elements separately and linked them either in functional relations or from a structural point of view. However, the two starting objects, thought and speech, are perceived as independent and isolated in each case. The approach of constructivism is to consider them in their dialectical. This can be clearly seen when the question of meaning arises. As Vygotsky suggests, the word never refers to a single object, but to an entire class of objects. Hence it represents a form of generalisation. Meaning, inherent in the word itself, is thus a process of generalisation that occurs from sensation to thought. So it would seem that thought is the leading actor in this process. However meaning is not separable from the word: it lives with it. However, then does the word mean speech or thought? Vigotsky's answer is clear: dynamism.

> "It is both at one and the same time; it is a unit of verbal thinking. It is obvious, then, that our method must be that of semantic analysis. Our method must rely on the analysis of the meaningful aspect of speech; it must be a method for studying verbal meaning."

However, speech plays a fundamental role in the discussion because of a less obvious but equally interesting aspect: it is a means of social communication. Such social interaction is impossible not only without a common alphabet of signs but also without a shared meaning. Verbal meaning becomes how humans communicate; dialectically, communication

[9] Our path will allow a unified approach to the first and third.

develops shared meaning and reflects on the world in a generalised way. It is, therefore, simultaneously a means of understanding and communication:

> "The word is almost always ready when the concept is. Therefore, it may be appropriate to view word meaning not only as a unity of thinking and speech but as a unity of generalization and social interaction, a unity of thinking and communication."

The inevitable conclusion of these considerations is that understanding discourse begins with understanding speech and thought and that even this is not enough if one does not grasp the motivational aspects for which such thoughts are expressed. In total agreement with these considerations, we designed the worksheets that accompanied the entire TLS for students and the subsequent course of the lessons.

5.5.2 Modelling-Based Teaching

We have introduced the MER, and described the inquiry-based learning; it remains to describe the MBT tool that will be used for students to construct the quantum model of the light and the models for polarization and dual-rail implementations. The ultimate model, however, will be the diagrammatic representation in all its meaning (computational logical and physical-experimental)

The reference work for the MBT can be found in [168]. In the first part, the authors emphasise one aspect that characterises our project and underpins the relationship between science, teachers and students. Indeed, it seems that the development of cultural material on science depends on the level of mental activity of individuals on the subjects. However, this high level leads to a high emotional engagement conditioned critically by the teacher's leadership ([169]):

> "Attitudes and achievement among students can be improved through frequent use of student-centred teaching methods and degraded through frequent use of teacher-centred methods... in spite of extensive data to the contrary, teachers continue to implement teacher- centred practice in their science classes."

It is therefore essential in the first instance to provide teacher training based on three main elements:

1. awareness of the difference between disciplinary competences and teaching tasks;
2. the need for pedagogic content knowledge;
3. critical review of their personal beliefs based on own educational experience.

According to the authors, modelling favours the possibility of teachers going in the directions described above and, at the same time, should help students to scientific literacy and raise their awareness of scientific research's nature and socio-cultural importance.

5.5.2.1 From models to modelling

In order to fully understand the meaning of modelling, we must briefly highlight in what sense the concept of model is used both from a cognitive psychology point of view and from philosophy born as a critique of the semantic approach to the concept of model ([168]). From the perspective of cognitive psychology, a model (mental model) makes it possible to explain and make predictions about a phenomenon and solve problems involving it. What is interesting is the construction phase of the model and the subsequent utilization step. More extensive and detailed is the contribution of philosophy. We will only focus on the aspects that interest us that are related to the work of Knuuttila (see [11], [12]).

The concept of the epistemic artefact described in these works follows some fundamental characteristics:

1. models turn out to be concretely constructed objects so that this construction enables and constrains scientific reasoning. This approach of external aid is closely linked to that of scaffolding: they represent a support for identifying the most significant aspects of the object of study and thus making use of them;
2. models have an evolutionary nature. This implies that their epistemic value is defined by our interaction with them. As a synthesis of all parts of its development, the model synthesises empirical, theoretical and conceptual components;
3. the artefactual approach is linked to instrumentalism and operationalism: the epistemic value refers to the functional, not the representational aspect of the model itself;
4. models are artefacts created to solve scientific problems in practice.

These few considerations are enough to make the process, called modelling-based teaching, fundamental and which in the light of the considerations made we are now going to describe.

According to the authors, modelling is a cyclic process of knowledge construction that takes place in four mutually forming and supporting phases. In the first phase, the *creation* of the proto-model, three elements are identified: purposes, experiences and sources.

Concerning purposes, the modeller must define an initial purpose for the model; then he or she acquires from the experience, the study of related literature or the analysis of empirical data the elements to support the creation of the proto-model; finally, he or she draws on the mathematical tools used to connect the elements of experience to support and develop it. The second moment is the *expression* of the proto-model: in any mode of representation or combination of these (visual, virtual, gestural, mathematical, verbal, etc.). The choice of mode of representation is fundamental and is guided by four elements: the purpose of the process, the nature of the elements to be modelled, epistemic practices and, finally, the target. At this time or later, it is also necessary to define the representation codes, i.e. the meaning of the artefact details. The third stage is the *testing* of the model[10]. The model is accepted and acceptable if it can pass tests certifying its validity. If there are problems in the validations, the model can be modified or replaced as a last resort. Finally, in the *evaluation* of the model, one must understand the scope and limits of the model itself. Such reflection can lead back to the initial phase, and the cycle can resume.

5.6 Instruments and Methods: the role of worksheets

The worksheets that are used in our work have multiple uses. First and foremost, they are designed to get students to work independently to become personally active in constructing knowledge. The possibility of peer discussion inherent in this type of tool also makes it possible to overcome any difficulties. The worksheets are in total agreement with the work of the University of Washington([171]). A common trait is the constant presence of questions that force students to reason independently and justify their reasoning. In particular (see [171])

> "The structure is provided by tutorial worksheets that have been designed to help students confront and resolve specific difficulties. The worksheets contain

[10] A test can also be carried out by means of a qualitative exploration or thought experiment, as the overall aim is to develop and refine a scientific explanation in the form of a model ([170]).

> questions that try to break the reasoning process into steps of just the right size for students to become actively involved. If the steps are too small, little thinking may be necessary. If the steps are too large, the students may become lost unless an instructor is by their side. The tutorial homework assignments help students reinforce and extend what they have learned."

Since the worksheets are carried out in the classroom, it is the teacher's task to support the work, and their use also allows teachers to understand the difficulties their students may be having. In particular, the micro-steps in which the worksheets are structured allow them to grasp specifically where the significant difficulties lie. The third use of these worksheets is closely linked to the data collection and analysis we propose in this research. Thanks to the collection of the worksheets and their analysis, it is possible to monitor the students' learning, propose changes to the worksheets themselves and possibly modify one or more parts of the TLS. Finally, they constitute a useful working tool in teachers' professional development activities ([172]).

The importance of these tools combined with the inquiry method has also been emphasised in recent works: for example [173] and [174] . Particularly in the first of these works, it is emphasised that learning is more practical and time-efficient. Furthermore, the activities in the worksheet maximize understanding and are necessary for the learning process. As in [174]

> "It shows that the availability of student worksheet is one of the factors that can improve students' understanding and skills so that they can improve their competence. "

If this is then combined with the inquiry stages, it becomes a very powerful tool: "It aims to make students active in building knowledge and developing attitudes and skills through direct activities that their do".
We will see that the entire TLS is built with worksheets that also take on different purposes at various times. Furthermore, the answers to the research questions on the student path will be generated and obtained from the analysis of these tools.
In the chapter on teacher course of professional development, we will introduce the qualitative analysis methods used.

Chapter 6

Quantum technologies: course(s) for teacher professional development

In this chapter, we describe the teacher professional development that arises from the four steps designed for teachers:

1. A teacher professional development course on quantum technologies;
2. the follow-up course about QP in the context of linear polarization of the photon
3. the co-design and action research projects;
4. the second course.

First, we provide in Section 6.1 an overview of current research on teaching quantum technologies and quantum information science, a highly active research area, and discuss the connections of such research with our own. Next, we report on a professional development course based on this longitudinal and interdisciplinary approach that was organized in the context of the Italian PLS-Piano Lauree Scentifiche (Plan for Science Degrees) and the education section of the Quantum Flagship. The course was structured into three parts: a first sequence of 20 hours on the fundamental topics of quantum computation and communication, whose outcomes are discussed in Section 6.3.2; a follow-up part of 10 hours, attended only by teachers interested in starting action research projects, focusing on the educational challenges in teaching-learning quantum technologies, the design of active-learning strategies for overcoming these challenges, and the introduction of a physical context suitable to describe quantum systems and devices that can encode and process information (Section 6.3.3); finally, the ongoing co-design sessions and action research projects with teachers are discussed in Section 6.4. Being aware that asking teachers to discuss quantum information and communication topics directly in the regular curriculum would require a profound cultural revision and a steep learning trajectory, we instead directed their efforts on the design of teaching-learning sequences based on an exploration of the connections between physics concepts and the problem of computation at different school grade levels. Due to the restrictions imposed in response to the COVID-19 pandemic, all lessons, although initially planned as a traditional classroom course, were performed in synchronous distance learning. Interaction between teachers was limited, and the means of delivery were approximately 80% frontal lesson, 10% group discussion and 10% of teachers performing individual activities such as exercises or answering questions, which were discussed immediately afterwards. Finally, the last part (Section 6.6) is addressed to a brief description of the course for teachers proposed between October and December 2022 and implemented in light of the educational experiments carried out with students.

6.1 Previous research on teaching-learning quantum technologies and information science

In recent works, a number of authors have proposed courses, tools and strategies in an effort to advance the scope of education to quantum mechanics (QM) in secondary school to include topics related to the "second quantum revolution" [3]. For example, Walsh et al. [175] have designed and tested a one year high school course on quantum computing based on classical wave optics, with a focus on hands-on experiments and simulation activities adopting an inquiry-based approach, and the contextual introduction of new topics and competencies (such as the matrix formalism, or Python programming skills) when needed for the completion of students' inquiry projects. Satanassi et al. [9] developed a quantum computing course for high school students based on the general idea of leading students to follow the evolution of computational thinking in human history, from the most primitive computing machines, and ending with quantum computers and algorithms. The final part of their course uses a spin first approach, with the re-interpretation of Stern-Gerlach experiments in terms of information input (the state preparation), information processing (the state evolution) and information output (the measurement) playing a central role as a bridge from basic QM to quantum computation. Pospiech [176] proposes a course on QM for the German high school, in which quantum computing and quantum cryptography are introduced as rich technological contexts in which the fundamental concepts of quantum theory (e.g. superposition, entanglement, incompatibility, measurement) find their full development and application. According to the author, teaching QM in the context of quantum technologies has positive reflexes on conceptual understanding, on students' ability to construct consistent mental models, and on the epistemological acceptance of QM as an ordinary physical theory. Research-based course proposals based on a hands-on approach for different targets, ranging from secondary school students [177] to undergraduates with little or no physics background [10] have very recently appeared in the educational literature. However, while research on the teaching and the learning of quantum physics is a well-developed field within physics education [4],[5] and student difficulties at different levels, both in general and in connection with different teaching approaches, have undergone significant clarification, quantum technology and information science represents still a largely uncharted territory. There is a need to build effective programs and to design curricula for diverse student populations and educational levels, identifying goals and challenges according to the context at hand. Recently, quantum computation experts from both academia and industry signed an open letter [6] calling for an earlier start of education in quantum computer science in the academic career and recommending the involvement of education experts in curriculum development. An early introduction of such topics was also the subject of a recent educational survey [7] in which interviewed instructors in quantum information science expressed interest in research-based instructional materials, while displaying a remarkably wide range of opinions on the desirable content and prerequisites of future undergraduate courses. In [8], the authors identified the core ideas for quantum computing courses suitable for computer science students with superposition and entanglement of qubits, quantum computer, quantum algorithm, and quantum cryptography. The course for teacher professional development represents part of a larger program, which will be pursued in future works, aimed at defining the contours of a possible educational reconstruction of quantum computation and communication topics suitable for secondary school students. Since we believe that any such project is doomed to failure if it is not grounded on a community of motivated teachers, our work has started from the attempt to build such community, and most importantly, listen to teachers' perspectives and needs concerning the content. Although there are obvious similarities and parallelisms with some of the proposals in the literature, in particular [9], [176], our work, as described in [178] originated mainly for a deep analysis of the subject matter, the search for an historical-epistemological route to make the content in principle accessible to secondary school students, and the attempt to exploit as much as possible the educational connections of the topic, both in a longitudinal sense, within the physics curriculum (e.g. connecting the debate on the second principle of thermodynamics, through the Szilard analysis of a single-molecule heat engine [179] to the discovery of Landauer's principle) and in an interdisciplinary sense, mainly, but not exclusively, with the mathematics curriculum on themes related to logic and probability ([126],[127]).

6.2 Teachers professional development course

The first course, realized between October 2020 and March 2021, took place in light of a research activity on the topics of the second quantum revolution. We first identified topics characteristic of quantum computation and communication that could be presented to teachers and then taken to their students (For an in-depth description see 7.2.1). The work carried out at a distance due to the special conditions associated with the dissemination of Covid-19 had two fundamental goals that we can identify in the following three research questions (Teacher Research Question (TRQ)):

TRQ1: *How is it possible to construct an adequate content simplification process to present the topics of the second quantum revolution to teachers in a meaningful way from very advanced theoretical aspects?*

TRQ2: *How to make the contents and themes of the second quantum revolution sufficiently fruitful to teachers to develop a personal commitment to longitudinal, interdisciplinary educational innovation directed towards themes of quantum information and computation?*

TRQ3: *What are the most appropriate environments and methods for building a distributed, online community of practice of teachers revolving around the themes of the second quantum revolution?*

Here are some brief comments on these three questions:

TRQ1 According to the MER, the process of elementarization starts with the analysis of scientific content, which, of course, cannot be directly transferred to teachers or students. The advanced texts present a language that is probably too laborious for the teachers. Therefore, a first revision of the contents was necessary to present them to the course participants partly from the perspective of possible applicability in the classroom. This first adaptation process will be modified and improved from the teachers' observations and experience. The subsequent study process on teaching and learning sequences (TLS) can rely on educational paths already present in the literature, in particularly on the spin-first approach. It is more difficult to find TLS about topics such as entanglement, quantum information, and computation theory, which are still only minimally introduced in secondary schools (often only as extracurricular interventions).

TRQ2 The new topics would stimulate teachers to a high-level reorganization of the physics content ([180]), different grouping topics in the physics and mathematics curriculum under the common perspective of representing instances of overlap and interplay between the discourse of physics and the problem of computation and information. In this way, we thought that teachers could be motivated to introduce quantum information and computation topics, not as one different subject of the physics curriculum, but as the culmination point of a longitudinal and interdisciplinary path they could have developed through the course of several years.

TRQ3 One of our final goals is to construct a community of practice ([13]) whose common purpose is to perform curriculum innovation towards topics related to the second quantum revolution. The shared content is meant to be elaborated by the teachers and researchers within the community of practice into several different types of educational intervention, distributed along the curriculum, also depending on the classes each of the participant teachers currently works in. A central aim of the course is to have teachers develop some degree of personal commitment to the objectives of curriculum innovation.

6.2.1 Structure of the educational path

The Educational path[1] has a total duration of about 20 hours and is structured according to the following steps summarized in Tab. 6.1: 1) introduction to physics problem of classical computation; 2) building the quantum logical language and the origin of quantum algorithms; 3) introduction to entanglement and development of quantum protocols

[1]The pdf of the lessons can be found in `http://www-5.unipv.it/dida-pls/Materiali.htm`. The lessons refer to the first implementation of the course. Many integrations have been made in the second implementation, which we will briefly describe at the end of the chapter.

TABLE 6.1: Structure of the educational path.

Introduction	**Building**	**Development**
Physics problem of computation	QP with Stern-Gerlach device	Entanglement
	From bit to qubit - Quantum circuits	Bell's inequalities - No-cloning theorem
	Quantum algorithms	Quantum protocols: dense coding and teleportation
		Cryptography

6.2.1.1 Physics problem of computation

The first reflections on the thermodynamics of computation, related to Bennett's works [14], allow to introduce and develop reversible logic and reflect on the relationship between logic gates and entropy. Bennett's plan [181] requires to demonstrate the thermodynamic reversibility of the calculation by following a very precise path, which we have retraced during the lesson:

> "A proof of the thermodynamic reversibility of computation requires not only showing that logically irreversible operations can be avoided, but also showing that, once the computation has been rendered into the logically reversible format, some actual hardware, or some physically reasonable theoretical model, can perform the resulting chain of logically reversible operations in a thermodynamically reversible fashion. "

From the start, logical operations can be described in terms of physical systems, as Feynman did in [39] by representing the "and" and "or" operations in terms of the rules of binary addition on rows of pebbles. The relationship between physics, logic and computation, discovered through this simple example, allows us to introduce the classical logic gates, along with a new circuital language that will be used in all the following lessons (see Fig. 6.1), while maintaining a connection between logical operations and physical systems.

The next step in reinforcing the connection between physics and computation is the demonstration that reversible logic does not require a necessary theoretical minimum of energy

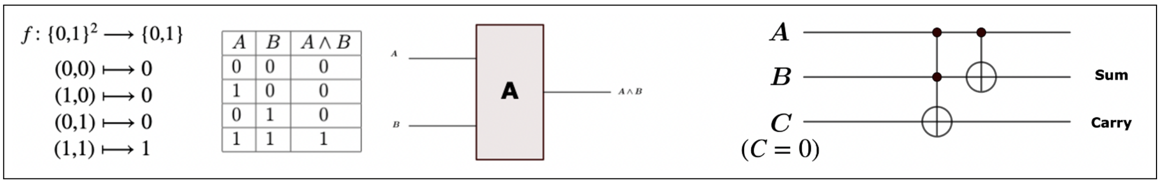


Figure 6.1: (Left) Boolean function, truth table and circuital representation of AND logic gate. (Right) Reversible circuit of binary addition.

dissipation, since theoretically the only thermodynamically irreversible operation is the erasure of information (Landauer's principle). The birth of reversible logic and the corresponding operators [16] is the first formal step towards an extension, both necessary and intuitive, from classical logic to what we may improperly call "quantum logic". This passage, although touching less known authors and topics in physics and computation, is historically well founded, essential to the consistency of the teaching sequence, and implies a profound reflection on the link between computation and the physical support used to perform it. Through the introduction of reversible logic it can be shown how the division into preparation, transformation and measurement begins to manifest itself, and while classically its consequences are not deep, it will become fundamental in the quantum field. Bennett's demonstration of the in principle reversibility of the process of copying a bit using the example of a one-domain ferromagnet [14] is also very interesting, both (as it was originally presented) as a concrete example of application of Landauer's principle, and for the possibility to link it to the quantum no-cloning theorem. At the end of this first meeting, teachers should begin to become aware of the possibility to establish a close relationship between physics, logic and computation [123], and perhaps of the educational possibilities implied by problematizing and exploiting such link. In fact, the topics discussed can be an interesting starting point for reflecting on educational paths in physics and mathematics to be implemented in the first three years of high school (insights into classical logic, thermodynamics, computation, matrix algebra).

6.2.1.2 Quantum Physics with Stern-Gerlach device

The second meeting is organized in collaboration with the educational research area of Insubria University (see [182]) whom we worked with for a Summer School on quantum computation for high school students. With the aid of the Quvis simulations [183] of the Stern-Gerlach device, it was possible to bring the concept of preparation of states, evolution and measurement into the quantum context by means of the appropriate formal language, both in the Dirac notation (which is introduced contextually to the analysis of two level systems) and in matrix computation. This aspect is a very important tool for the subsequent use in the computational and informational field. In particular, the use of two-level systems [23] – which implies the adoption of a spin first approach well established in PER – allows to perform the transition from the concept of a bit to a quantum bit - the qubit - and to introduce logical operators acting on them. In the last part of the lesson, it was possible to explore more in depth the role of probability in quantum measurement, the issue of incompatible observables, and the concept of quantum interference with the analogy of the Stern-Gerlach device and the double-slit experiment with photons.

6.2.1.3 From bit to qubit - Quantum circuits

This third lesson allows us to extend the physical problem of computation to quantum systems. The initial motivation arises from R. Feynman's considerations about the simulation of physical systems [17]: "What kind of computer are we going to use to simulate physics?"and even before that "what kind of physics are we going to imitate?". The route we explore with teachers, is that if we want to imitate quantum physics, the natural choice is to renounce both classical logic and classical probability. To understand this fundamental aspect, we have first described using the language of set theory the basic structure of classical physics, and then shown how propositional logic and probability can be constructed as theories concerning subsets of the set S of possible states of classical physics [123]. Thus, in this sense, classical physics carries in itself a necessary structure for the image in Fig. 6.2 can be read equivalently in the three disciplines with simple terminological substitutions: a proposition, in fact, is a true or false (1 or 0) statement about a certain property, i.e. whether or not the value of a physical quantity satisfies certain conditions; similarly, a random variable on event space can take one of the experimental outcomes. It should be evident how much this description can allow for multidisciplinary educational paths well before the fifth year of study. Awareness of this unit emphasizes even more the fact that the spin properties and the double-slit photons experiment need a critical review of both logical connectives and probability theory. The existence of incompatible properties in quantum physics raises for example, the problem of the truth value of the conjunction of propositions about the spin values for a single quantum object on different axes, and quantum interference experiments suggest the addition of an interference term in the computation of probabilities for events which can realize through mutually exclusive paths, which cannot be explained by classical probability theory. Furthermore, one possible educational advantage in using this approach, is that based on our previous experience [184] the sudden realization by students that quantum mechanics may be at

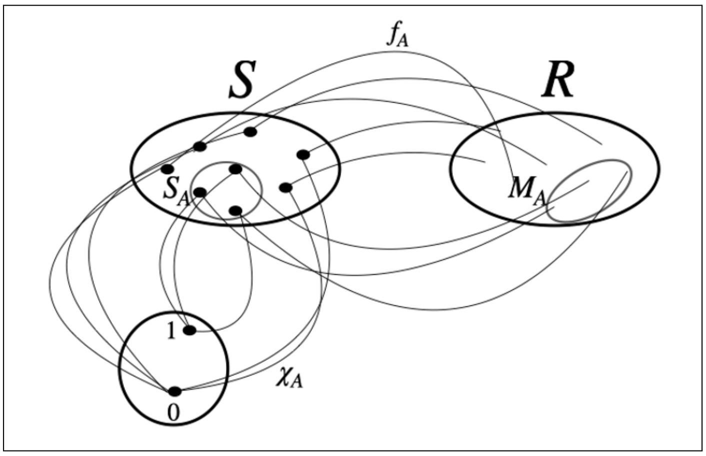


FIGURE 6.2: In physics, a proposition is a true or false statement about a certain property, i.e. whether or not the value of a quantity (a function f_A from the phase space S to $\mathbf{R}$) satisfies certain conditions. Similarly, in probability theory, a proposition is a true or false statement about the value of a random variable, and the possibility of constructing probabilities is granted by the existence of a measure on S. The link with propositional logic is thus immediate if we consider the characteristic function χ_A able to establish the truth values.

odds with classical proposition logic and ordinary probability theory may cause them to reject the theory as absurd and wrong, since they typically perceive proposition logic to be hierarchically superior to physics. However, the gradual construction of an intertwined link between classical mechanics, proposition logic and probability, puts these theories at least on equal grounds from an epistemological point of view, and may help students more easily accept the consequences of adopting quantum mechanics as a fundamental physical theory.

By using two-level systems, we can then generalize the concept of the classic bit to quantum bit - a qubit - and study some logical operators that act on them (see Fig. 6.3). The generalization allows us to establish a correspondence between Boolean functions, describing classical connectives, and unitary operators describing the evolution of a system in quantum physics. All the objects involved in the formalism used also have a simple circuit representation, which may be seen as one of the distinctive features of our approach. The possibility of working on new and strongly decontextualized symbolic representations allows the development of an autonomous and complete language that we believe can be of great interest and help for the more in-depth exploration characterizing the second part of the educational path.

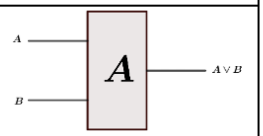
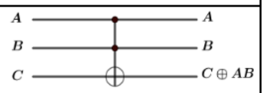
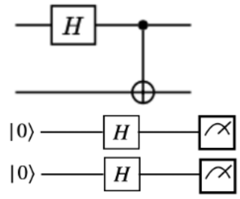
	Information coding (preparation)	Logic gate (transformation)	Decoding of information (measurement)	Circuit representation **Examples**
Classical logic	Bit 0, 1	Boolean functions $f:\{0,1\}^n \longrightarrow \{0,1\}^m$ $Not, And, Or\ldots$	Confirms the value of the bit on which it acts. We obtain a bit of classical information	
Reversible classical logic	Bit 0,1	Invertible Boolean functions $f:\{0,1\}^{n+m} \longrightarrow \{0,1\}^{n+m}$ $Toffoli, Fredkin$	Confirms the value of the bit on which it acts. We obtain a bit of classical information	
Classical computation	Classical state (potential)	Diodes, transistor	Measurement instrument (circuit element) of the potential	
Quantum logic	Qubit $\|\psi\rangle = \alpha\|0\rangle + \beta\|1\rangle$ $\|\alpha\|^2 + \|\beta\|^2 = 1$ $\alpha, \beta \in C$	Unitary operators $X, Or, And, H, Z\ldots$	It randomly chooses one of the two eigenvalues associated with the basis qubit states. We get a bit of classical information	
Quantum computation	Quantum state Spin $\|\uparrow\rangle, \|\downarrow\rangle$ Polarization $\|V\rangle, \|H\rangle$	Different experimental realizations	Measurement devices (intrinsically probabilistic)	

FIGURE 6.3: From classical to quantum computation.

The probabilistic interpretation of qubits makes it possible to introduce composite systems and the tensor product fairly and introduce multi-qubit logic gates. The action of logic gates has been described both in Dirac notation and with matrices. An in-depth understanding of the correspondence between circuit element, the Dirac and the matrix formalism, and the physical systems that implement them, is an essential part of learning and grasping the concepts introduced. Some exercises were left for the teachers to familiarize with the new language introduced. The teachers' answers were put into the prepared folder and corrected in the next lesson.

6.2.1.4 Quantum algorithms

The last lesson of the first part is devoted to the introduction of two quantum algorithms: Deutsch' and Grover's algorithm [18], [42]. The algorithm concept in our educational path is used not only as a mere symbolic manipulation but also in the close connection it may have with the physical world [185]. This allows the proposed algorithms to be studied on three decreasing levels of abstractness: the circuit-representational level, the formal algebraic level and, finally, their ultimate interpretation on the level of physical theory (in Fig. 6.4 we can see a representation combining the first and last levels above described).

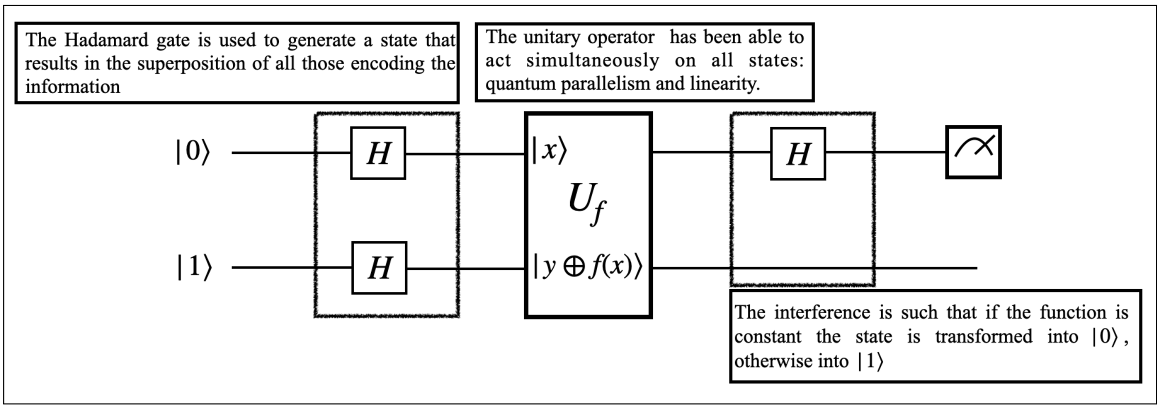


FIGURE 6.4: Deutsch algorithm circuit

The generalization of the Deutsch-Jozsa algorithm has been presented in the form of a "Bank Robbery" problem by translating the example described in the book *"Q is for Quantum"* of Terry Rudolph (see `https://www.qisforquantum.org`) into formal and circuit language. This last meeting concludes the first part of the course. The two algorithms contain most of the concepts covered and allow the use of quantum superposition and quantum interference so that the advantage in quantum computation becomes evident.

6.2.1.5 Entanglement - No cloning theorem

The second part begins with the analysis of entanglement, which, as J. Preskill indicated in [29], "The deep ways that quantum information differs from classical information involve the properties, implications, and uses of quantum entanglement". These considerations led us to shift the emphasis from entanglement as a problem (see Schroedinger in [33]) to entanglement as an opportunity.

Formally, the difference between tensor and Cartesian products is what characterizes quantum systems compared to classical systems. With Horodecki in [28]

> "According to the classical description the total (pure) state space of the system is the Cartesian product of the n subsystem spaces, implying that the total state is always a product state of the n separate systems; in contrast, according to the quantum formalism, the total Hilbert space H is a tensor product of the subsystem spaces $H = \otimes_{l=1}^{n} H_l$. Then the superposition principle allows us to write the total state of the system in the form
>
> $$|\psi\rangle = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} |i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_n\rangle$$
>
> which cannot in general be described as a product of states of individual subsystems $|\psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle$."

The focus was, therefore, on the concept of separable states and the distinction between classical and quantum correlation.
Following Ghirardi's arguments in [186] and in [187], we can introduce classical correlations due to our ignorance about the system (see Fig. 6.5)

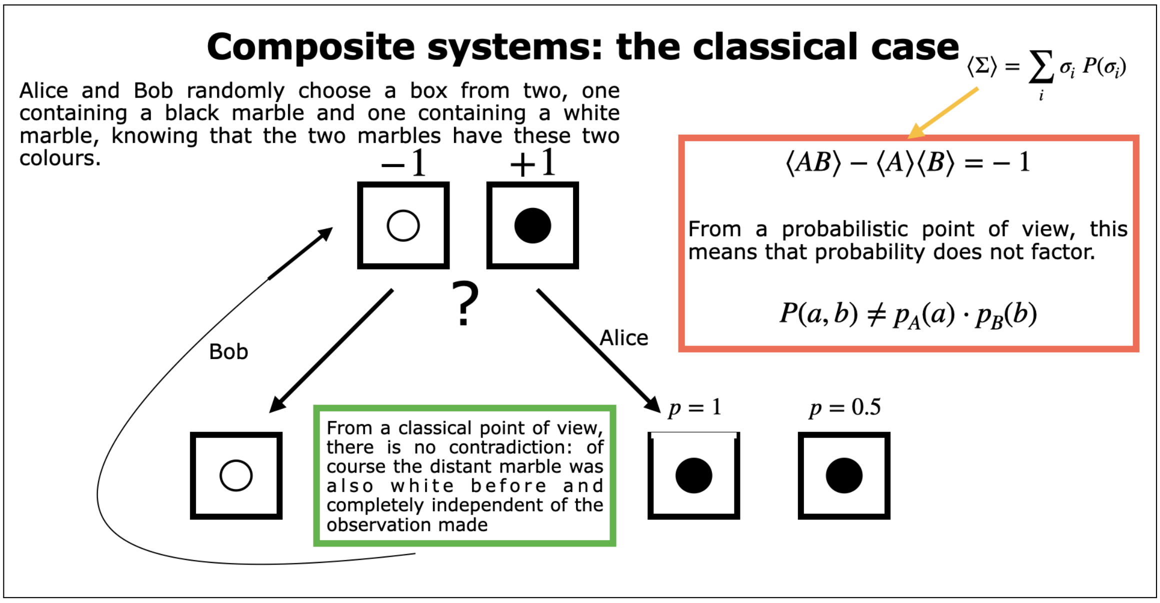


FIGURE 6.5: Slide of the course about the classical composite systems

But as J. Bell wonderfully explains in [36], these correlations are of a deeply different nature from quantum ones

> "The philosopher in the street, who has not suffered a course in quantum mechanics, is quite unimpressed by Einstein—Podolsky—Rosen correlations'. He can point to many examples of similar correlations in everyday life. The case of Bertlmann's socks is often cited. Dr. Bertlmann likes to wear two socks of different colours. Which colour he will have on a given foot on a given day is quite unpredictable. But when you see that the first sock is pink you can be already sure that the second sock will not be pink. Observation of the first, and experience of Bertlmann, gives immediate information about the second. There is no accounting for tastes, but apart from that there is no mystery here. And is not the EPR business just the same?"

The answer is of course, no!
The following description focuses on analyzing of pure and mixed classical and quantum states. This is to distinguish between *determinism*, *epistemic probability* and *non-epistemic probability* [187].
The considerations made about spin allow for a discussion of both the case of separability and entangled states (see Fig. 6.6). Similar considerations have been made using polarization.

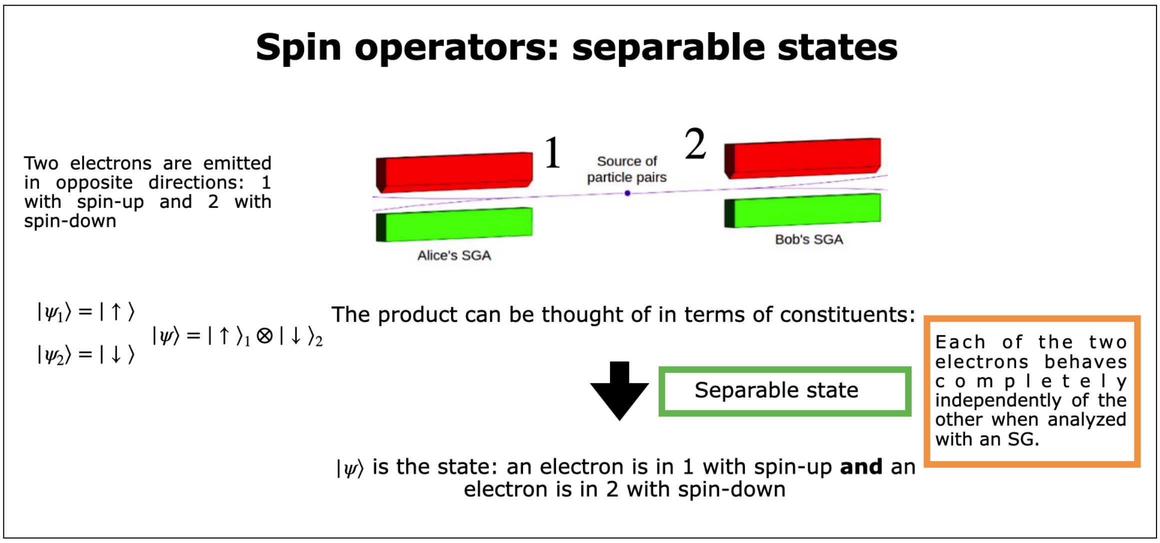


FIGURE 6.6: Slide of the course about separable states

If a source emits two identical particles whose initial state[2] is $|\psi\rangle = |0\rangle_1 \otimes |1\rangle_2$, Alice and Bob are certain that they can operate on the two particles without their action changing the state of the other's particle in any way. This means that each of the two qubits still

[2]In this case we are associating with the state $|0\rangle$ the state $|\uparrow\rangle$ and the state $|1\rangle$ the state $|\downarrow\rangle$.

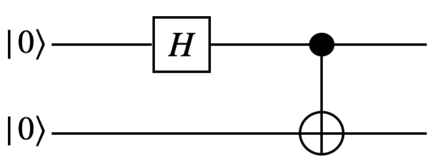
possesses at least one defined property, and, from a probabilistic point of view, correlations between measurements on the two systems characterise independent events.
We could even admit that Alice and Bob prepared their qubits separately in two sufficiently distant laboratories. It follows, in general, that the generic state prepared by Alice and Bob is of the type

$$|\psi\rangle_{AB} = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle \tag{6.1}$$

if Alice prepares $|\psi\rangle_A = \alpha|0\rangle + \beta|1\rangle$ and Bob $|\psi\rangle_B = \gamma|0\rangle + \delta|1\rangle$.
However, theory tells us that there could also be other states since, in general,

$$|\psi\rangle_{AB} = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle, \quad a^*a + b^*b + c^*c + d^*d = 1 \tag{6.2}$$

The possibility of introducing entangled states becomes immediate thanks to circuit representations and is an immediate consequence of quantum computation. In fact, Bell's states can quickly be introduced with the following circuit representation (Fig. 6.7)



$$|\psi_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|00\rangle \mapsto |\psi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$
$$|01\rangle \mapsto |\psi_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$
$$|10\rangle \mapsto |\psi_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
$$|11\rangle \mapsto |\psi_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

FIGURE 6.7: Sequence of logic gate to obtain the Bell's states

However, what really differs an entangled state from a generic classical state?
If we consider an electron-emitting source in the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2)$ the probability distribution is

$$P(0,0) = P(1,1) = 0 \quad P(0,1) = P(1,0) = \frac{1}{2} \tag{6.3}$$

This distribution is also fully compatible with the mixed-state hypothesis (like the Dr. Bertlmann' socks). But if that were the case, we could not get

$$P(+,+) = P(-,-) = 0 \quad P(+,-) = P(-,+) = \frac{1}{2} \tag{6.4}$$

as obtained experimentally and as theory predicts by admitting the state $|\psi\rangle$.
What we have seen leads to the possibility of explaining, through simple matrix accounts of operators, the characteristic that Schroedinger already identified as the most problematic

of entangled states in [33]. In fact, if we consider the pure state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2)$, the expectation values on all three spin components are zero

$$\langle Z \rangle = \langle X \rangle = \langle Y \rangle = 0 \tag{6.5}$$

This means that although we have the most information about the compound system, we have no information about its components ([28], [188]).
After some historical considerations on the extent to which entanglement has created disputes within the physics community, we have provide a rigorous and accessible proof of Bell's theorem following [37]:

Theorem 6.1. *Quantum mechanics cannot be both locally and counterfactual-definite.*

To prove this theorem, Bell provided an inequality (referring to correlations of measurement results) that is satisfied by all local and counterfactual-defined theories. He then showed that quantum mechanics violates this inequality and thus cannot be both local and counterfactual-defined.
In [37] the Bell inequality is demonstrated using the area to represent probabilities
To conclude, we have introduced three quantum states that do not satisfy the inequality.

6.2.1.6 No-cloning theorem

As G. Ghirardi in [31]

> "As already anticipated, after the clear cut proof by J.S. Bell of the fundamentally nonlocal nature of physical processes involving far away constituents in an entangled state, many proposals have been put forward, either in private correspondence or in scientific papers, suggesting how to put into evidence superluminal effects. We will begin by reviewing a series of proposal whose rebuttal did require only to resort to the standard formalism or to well established facts, such as those put into evidence by the Wigner-Araki-Yanase theorems."

Before addressing the topic, let us recall the statement of the theorem:

Theorem 6.2. *It is impossible to build a machine operating unitary transformations and being able to clone the generic state of a qubit.*

The story of the non-cloning theorem was an opportunity to discuss some aspects of the role of science and scientific publications that might be of interest when brought to young students. Indeed, re-reading the words of N. Herbert in [189]

> "The theorem of Bell guarantees that two quantum systems which have interacted in the past can no longer be regarded as independent systems. 1) The mathematical inseparability of the quantum theoretical representation is an essential part of nature, not a mere accident of the formalism. These once interacting systems - which in general may be space-like separated, hence truly isolated according to special relativity - remain in some sense connected in a manner unmediated, unmitigated, and immediate. If this instant quantum connection were directly observable - rather than indirectly verified via Bell's argument - it would put quantum mechanics into conflict with special relativity by permitting faster-than-light signaling."

and the comment by G. Ghirardi in [31]

> "The FLASH paper[3] was sent for refereeing to A. Peres and to me. Peres' answer was rather peculiar: I recommended to the editor that this paper should be published. I wrote that it was obviously wrong, but I expected that it would elicit considerable interest and that finding the error would lead to significant progress in our understanding of physics. I also was rather worried for various reasons. I was not an expert on Lasers and I was informed that A. Gozzini and R. Peierls were trying to disprove Herbert's conclusion by invoking the unavoidable noise affecting the Laser which would inhibit its desired functioning. On the other hand, I was convinced that quantum theory in its general formulation and not due to limitations of practical nature would make unviable Herbert's proposal. After worrying for some days about this problem I got the general answer: while it is possible to devise an ideal apparatus which clones two orthogonal states with 100% efficiency, the same apparatus, if the linear quantum theory governs its functioning, cannot clone states which are linear combination of the previous ones. Here is my argument, on the basis of which I recommended rejection of Herbert's paper."

The Herbert's paper will be published, and the simplicity of the proposed demonstration of theorem shows how difficult it was to understand the deeper meaning of Bell's theorem

[3] In this work Hebert had proposed an ideal polarization-based experimental apparatus capable, according to the American physicist, of realizing superluminal communications ([189]).

and the implications it implied[4] (see Fig. 6.8).

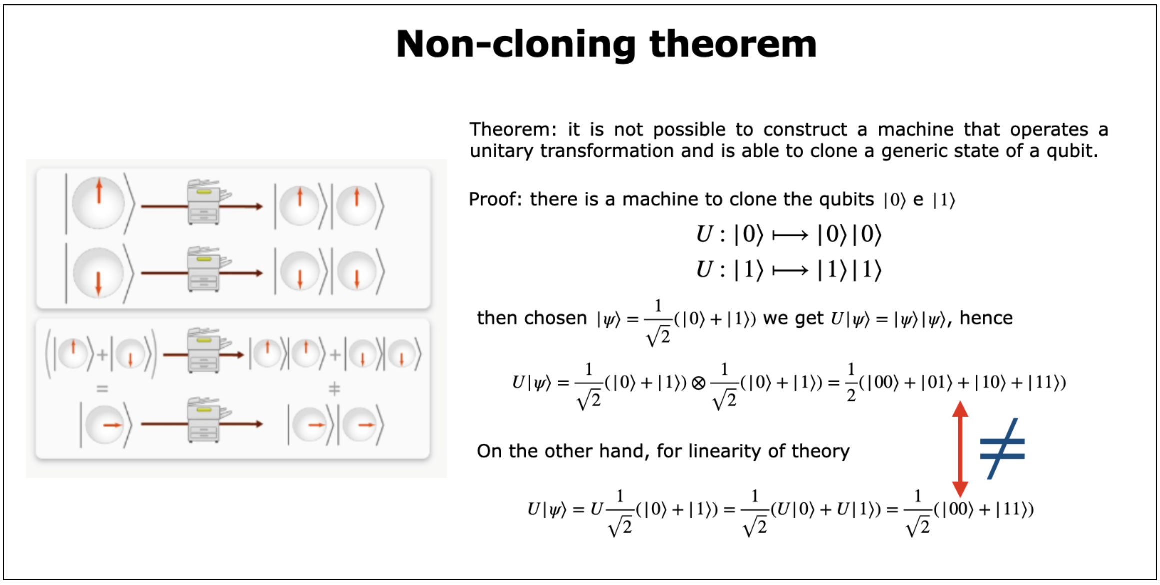


FIGURE 6.8: Proof of the non-cloning theorem presented in a slide of course

6.2.1.7 Entanglement for quantum information protocols

The two whole meeting required to deal with entanglement in-depth clarify because "*the deep ways that quantum information differs from classical information involve the properties, implications, and uses of quantum entanglement*" [29].

6.2.1.8 Dense-coding protocol

The proposed protocol allows two bits of classical information to be transmitted via a single qubit with a quantum channel. The circuit representation introduced makes it possible to effectively explain the protocol by translating the hypothetical physical actions performed by Alice and Bob into immediately clear logic gates (see Fig. 6.9)
The protocol is straightforward from a formal point of view, making it possible to emphasize certain aspects that make it highly meaningful:

1. the dense coding is not possible in classical physics, since a classical bit also has a well-defined value prior to its measurement;

[4] See [190] for the picture in Fig.6.8

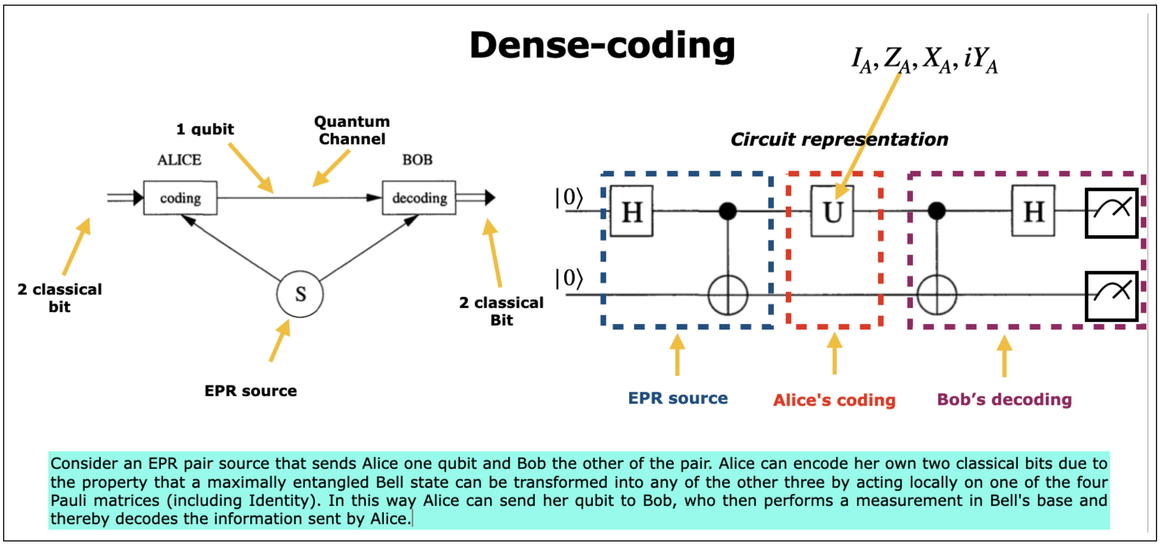


FIGURE 6.9: Explanation of dense-coding protocol in a slide of the course

2. the message is highly confidential; " *The transmitted qubit has density matrix* $\rho_A = \frac{1}{2}I_A$*, and so carries no information at all.*" ([29]);

3. Alice could send the first qubit to Bob long before she knew what his message would be.

Already from this first protocol, the change of perspective from entanglement as a problem to entanglement and non-locality as a resource should be clear [191]:

> "Entanglement and non-locality are now understood to figure prominently in the microphysical world, a realm into in which technology is rapidly hurtling."

6.2.1.9 Quantum teleportation protocol

In the case of teleportation, too, the circuit approach allowed us to develop the protocol in a conceptually and formally rigorous way[5]. In this case, however, it was preferred to complete the calculations only at the end, focusing initially only on the most significant aspects (Fig. 6.10).
Further considerations were made at the end of the demonstration:

1. Classical communication ensures that the protocol does not involve superluminal communication;

[5] Essentially following the approach given in [9]

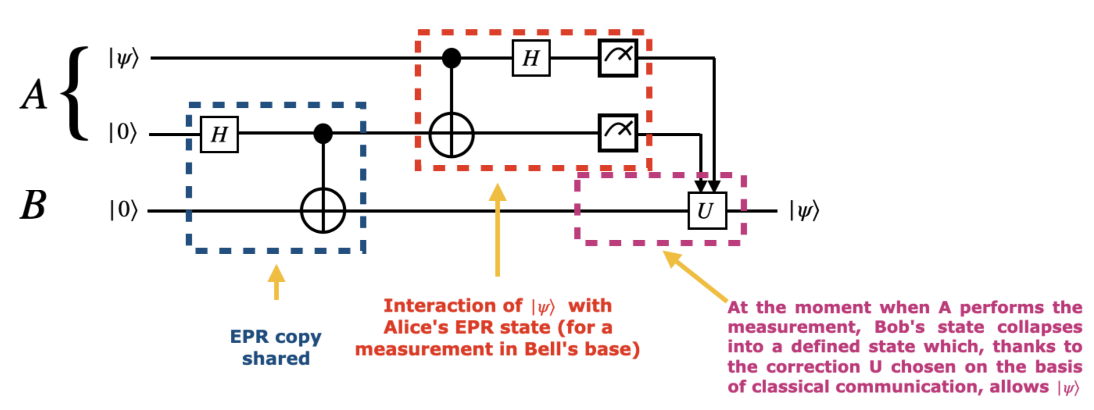


FIGURE 6.10: Significant conceptual elements of teleportation protocol

2. the information is teleported, not the physical system whose state it is encoded on;
3. there is no violation of the non-cloning theorem since the state $|\psi\rangle$ possessed by Alice disappears in the act of measurement to reappear after the correction made by Bob.

The words in [29] p.164 exemplify the singularity of the protocol.

6.2.1.10 Cryptography

The discussion on cryptographic protocols closes the track. It was left for last because it is possible to introduce it either by talking about entanglement or without it. The ideal construction of a secure cryptographic key is undoubtedly one of the most exciting aspects of the properties of quantum states and is also one of the most significant fields of research at present. For this reason, we have presented a historical path showing the three significant moments in the encryption theory: the linguistic approach, the mathematical approach and the physical approach.
We have described the BB84, BBM92 and E91 protocols concerning spin; the first two using St. Andrews simulations (see Fig. 6.11).

6.3 Quantum technologies course: first implementations, follow-up course. Context, data and results

We organized the course in the context of the Italian PLS-Piano Lauree Scentifiche (Plan for Science Degrees) and the education section of the Quantum Flagship. We divided

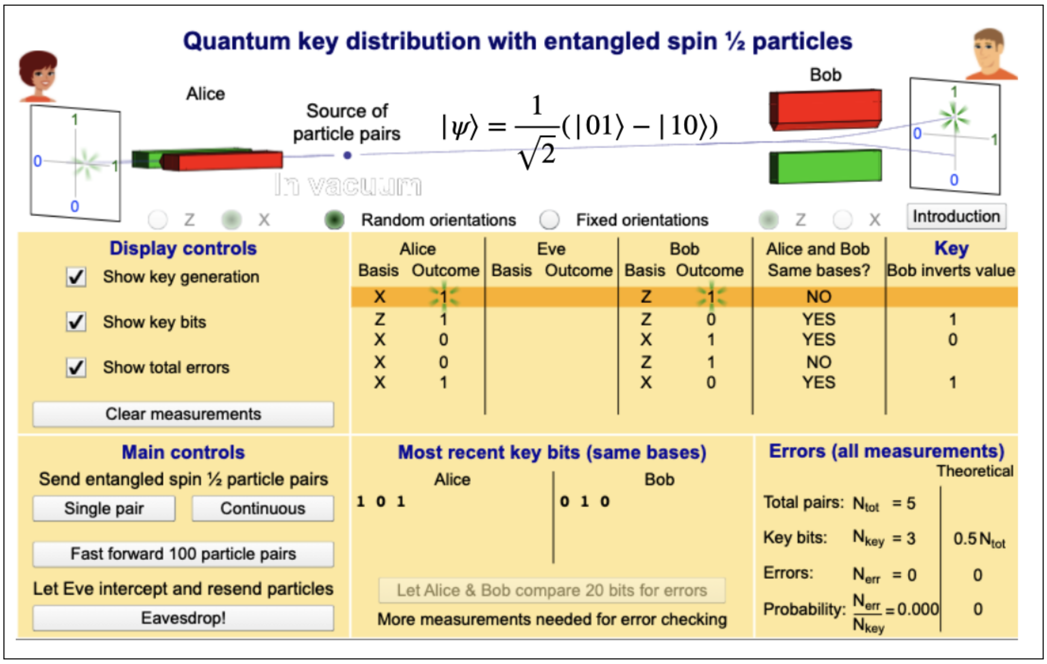


FIGURE 6.11: The BBM92 protocol with St. Andrews simulation

the twenty hours planned into two parts: the first 10 hours in October and November and the second in February and March. We preferred to divide the course in two to allow teachers to complete their study on the first lessons before addressing entanglement. Due to the restrictions imposed in response to the COVID-19 pandemic, all lessons, although initially planned as a traditional classroom course, were performed in synchronous distance learning. Interaction between teachers was limited, and the means of delivery were approximately 80% frontal lesson, 10% group discussion, and 10% teachers performing individual activities such as exercises or answering questions, which were discussed immediately afterwards.

6.3.1 Initial context

The teachers are equally divided between graduates in physics and mathematics; almost all of them work in science-oriented high schools, and in their schools, students are required to develop basic computer skills at most. Almost all of them teach following the textbook approach: photons and the photoelectric effect, the Compton effect, Bohr's model of the atom and interpretation of atomic spectra, De Broglie's hypothesis and wave-corpuscle dualism, wave-particle dualism, Schrödinger's equation, energy levels, wave functions and probability waves, Heisenberg's uncertainty principle; this can also be seen in part from the topics addressed (See Fig. 6.12).

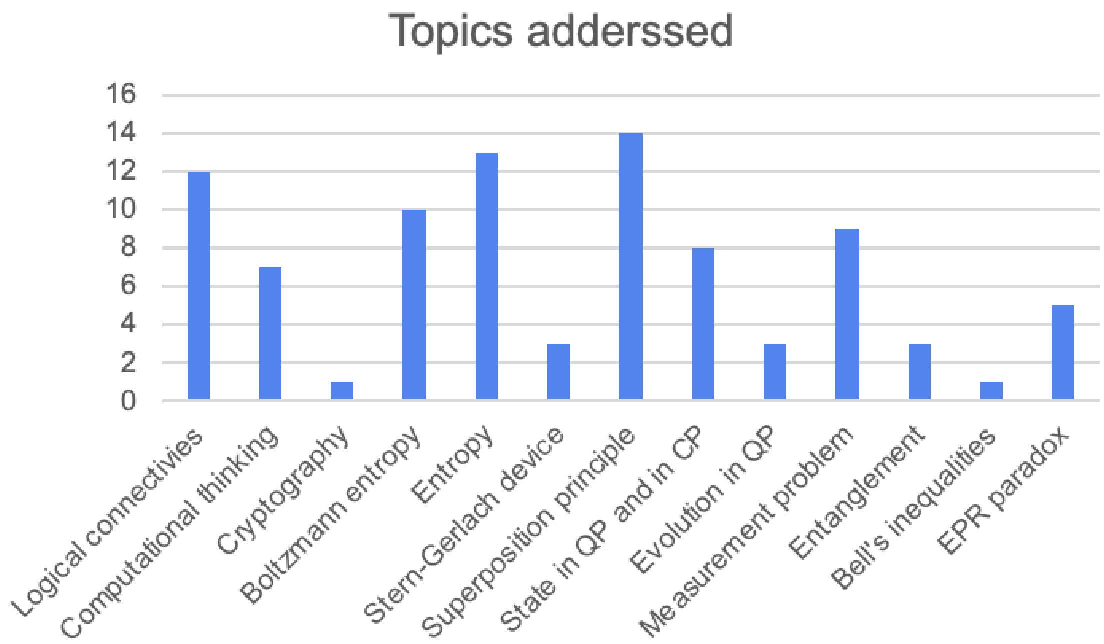


FIGURE 6.12: Topics traditionally treated by teachers in their school curriculum

Moreover, they had almost no knowledge of topics such as the thermodynamics of computation, quantum computers, algorithms, cryptography and quantum teleportation. Even the questions about the photon's polarization showed little prior knowledge.
On the other hand, it was evident that teachers considered the topics that would be covered to be very significant for their professional development (Fig. 6.13[6]),

[6]From here on in each image we consider 1 not at all; 2 very little; 3 a little; 4 quite a lot; 5 much; 6 very much

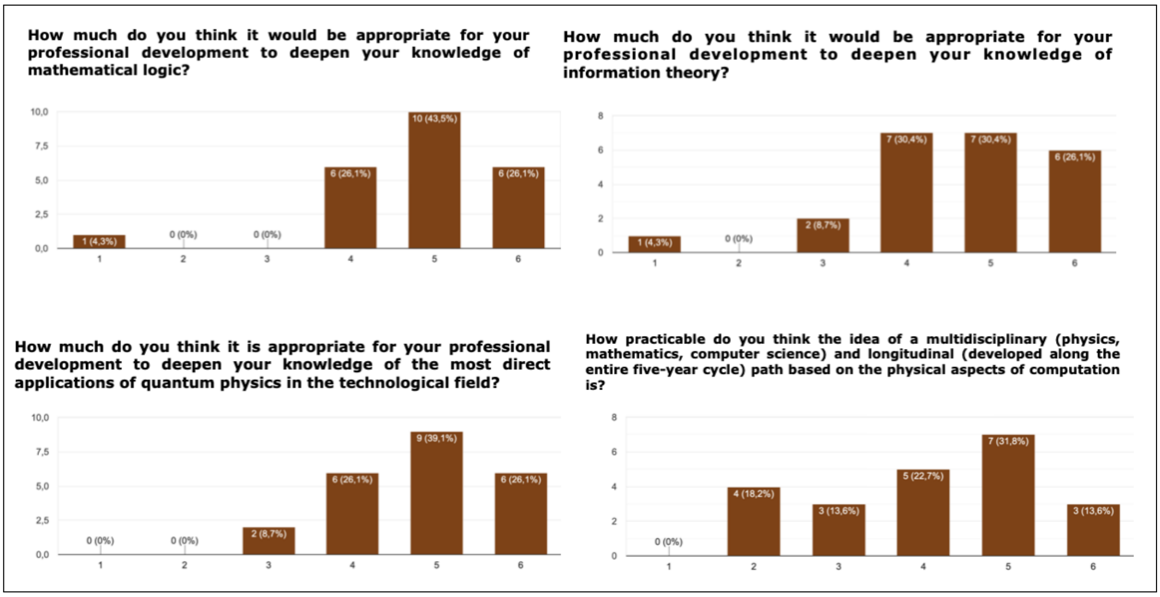


FIGURE 6.13: Answers on the professional development of teachers in the pre-test

but they are much more doubtful about whether it will be of interest to students (Fig. 6.14).

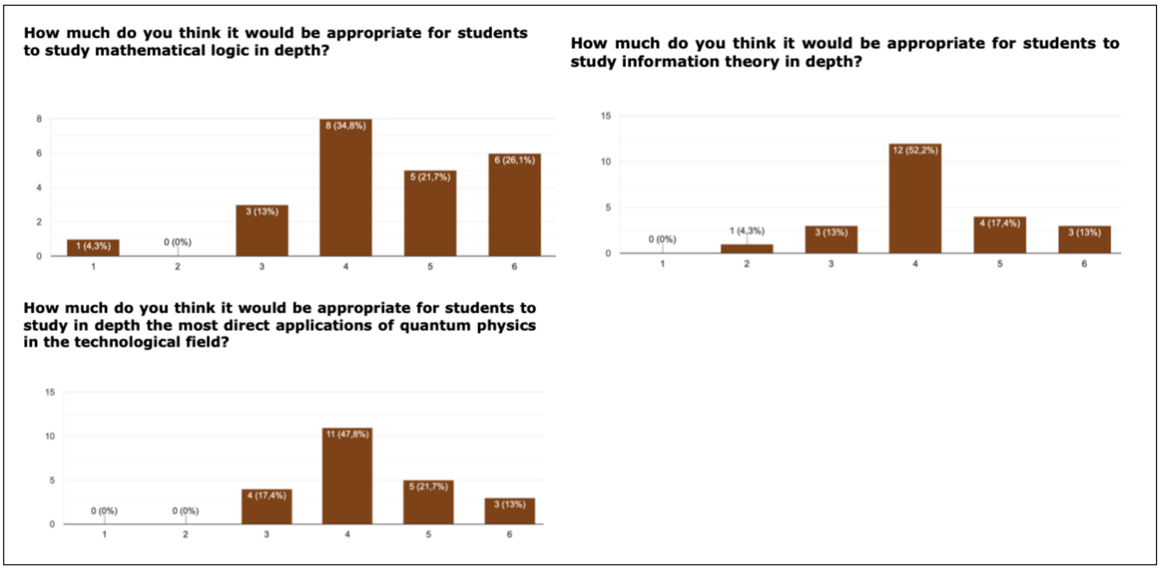


FIGURE 6.14: Answers on the possible relevance for their students of the topics addressed

6.3.2 Data analysis

At the end, we collected data by several means:

1. 23 pre-questionnaires and 14 post-questionnaire completed touching both disciplinary aspects and items related to personal engagement and involvement;

2. Semi-structured interviews with volunteering teachers.

6.3.2.1 Post-course questionnaire

As shown in Fig. 6.15, in general, teachers displayed a strong appreciation for the topics covered, particularly on the second part about entanglement, which using the formalism of logic gates, can be treated in a formally rigorous and conceptually meaningful way.

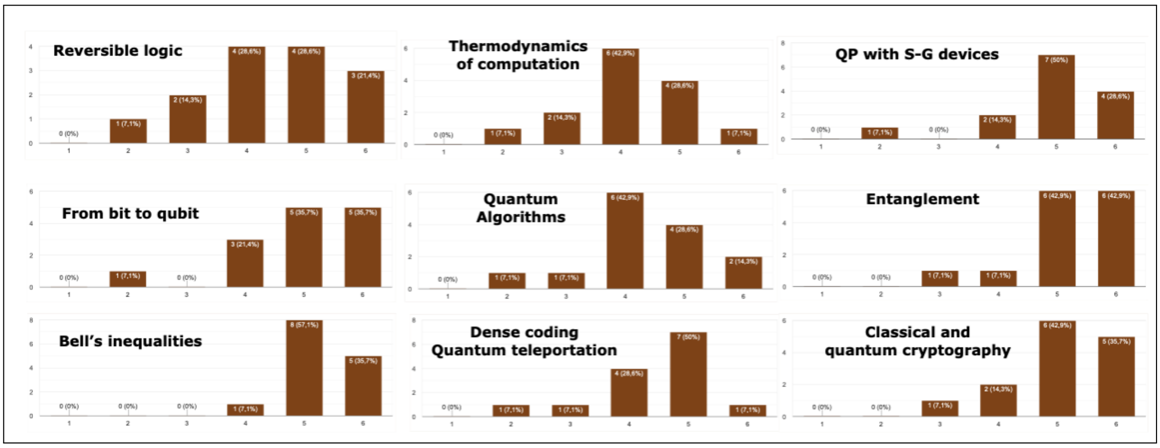


FIGURE 6.15: Interest level about topics introduced

We note positive answers about having dealt with the course topics and the importance they could potentially have for the students[7]. On the other hand, we also note a significant degree of scepticism about introducing them into the curriculum, either just in principle or adopting a longitudinal and multidisciplinary perspective (See Fig. 6.16 and 6.17).

6.3.2.2 Semi-structured interviews during and after the course

We created a matrix for mapping interview questions onto research questions [192]. After the initial questions, we divided the interview protocol matrix into four parts:

- specific questions about the topics;
- didactic feasibility of the proposed topics, multidisciplinarity and cultural impact;

[7] This is extremely interesting especially in relation to the pre-test where there was more scepticism about it.

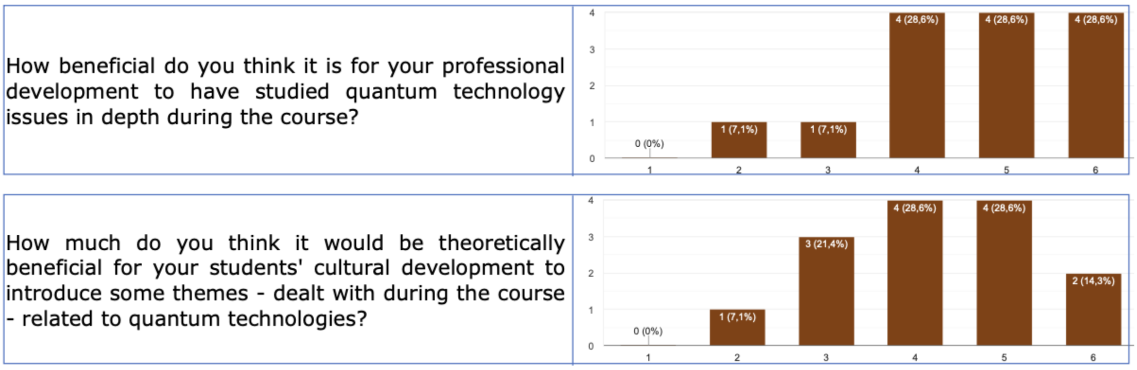


Figure 6.16: Teachers' opinion on whether themselves and students can benefit from instruction in quantum technologies

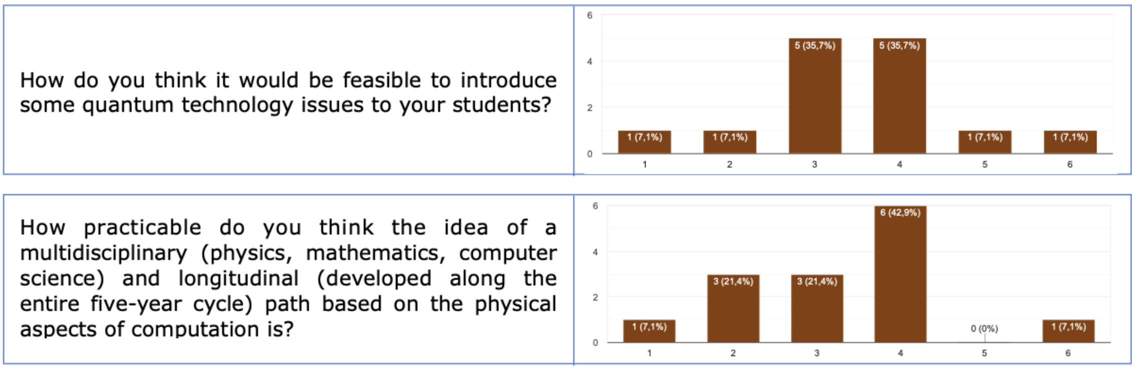


Figure 6.17: Propensity to introduce the topics of the second quantum revolution into the curriculum in general, and from the perspective of a longitudinal and multidisciplinary approach

- questions about the formalism used and circuital language;
- questions related to lessons and materials;
- questions related to the school context.

Each section has general questions to arrive at key questions that interested these first interviews. The transcription of interviews has been included in the matrix for analysis and comparison. A priori coding scheme [193] is developed out of course goals and two of our assumptions: the topics covered would have been interesting, especially for mathematics graduates; the possibility of introducing the topics before the final year would have made them more practically usable in teaching. As the interview ended, we allowed the participant to raise any issues not addressed.

Specific questions about the topics treated Regarding mathematical logic, everyone acknowledges that it is not taught in great depth and is only linked to set theory and propositional calculus in the first year. Teachers after the course believe it would be appropriate to expand the weight of the topic, but this seems not easy in a school context that seems to progressively assign lesser and lesser importance to it. The cultural importance of introducing non- classical logics is recognized. The strong link proposed between logic and the thermodynamics of computation has caused surprise and difficulty, in particular to the two teachers with a degree in mathematics. All teachers recognize the effectiveness of the introduction to QP through the Stern- Gerlach device, but some would prefer to use polarization because it is a topic already known to students. The formalism used for the introduction to quantum computation is considered suitable for high school students if appropriately trained, and it is seen to be very positive especially by the two mathematics graduates. The abstractness of language, however, raises the problem of immediate physical interpretation.

> "*I always struggle (with physical interpretation) but it (circuit language) is an absolutely interesting tool to use.*"

This aspect is even more evident in the study of the two proposed algorithms, whose computational aspect seem, in part because of their complexity, to be prevailing on its physical aspects.

Didactic feasibility, multidisciplinarity and cultural impact The interviewees generally agree on the high cultural value of the proposal, and the importance of a multidisciplinary approach for the education of future scientists. However, they recognize the introduction of these topics into the traditional curriculum as problematic, unless some of the content is deeply revised from the early years. The possibility of introducing topics linked to very recent technological developments is viewed as a likely source of students' engagement. Particular importance was attached to the cultural impact of the course, and that the two teachers with a degree in mathematics, in their free final remarks, stressed the great impact that the topics covered had had on their desire to study and explore QP and on a new vision of the world arising from it. Here are the words of a teacher:

> "*I was pleased to have attended the course because it allowed me to see a new way of thinking that I did not know, and it made me aware of the need for me to be trained in this regard, and that the students also need to be stimulated because contemporary physics is working on these things. It made me realize how much I don't know and that I need to be trained in this area. This unfortunately comes in a year when there are so many*

> *other problems. I don't think I can do that in my class. I'm sorry. For the future, however, I think it's something I should definitely consider.*"

Formalism and circuit language Teachers were able to see the concepts introduced with both matrix algebra and Dirac notation. They showed no difficulty in accepting both and in thinking that they can also be introduced to their students from the early years of high school. Finally, the value of circuit representations is recognized, although the references to physics are not always explicit. Doubts remain as to how long it will take for all students to achieve satisfactory results.

> "*It seemed to me a very interesting way of setting up the problem. Simple formalism with matrices even without Dirac notation. The idea of being able to use acceptable formalism even to do difficult topics from a formal point of view (entanglement) is very interesting.*"

However, some teachers also point out the aspects of difficulty that the mathematical formalism of the course implies:

> "*Interesting, it opened me up to a world. It made me want to pick up on some topics and go deeper into the computation part. Tiring course. The students should be used to concepts and not formalism. Or rather, the students should be made to understand how formalism arises. Very formal approach that I found very difficult: I have a more critical view today.*"

Lessons and materials The interviewees complained about the difficulty of following two and a half hours in the afternoon after teaching in distance learning in the morning. The particular condition due to the pandemic seems to make the course more difficult to follow. Nevertheless, the demands are adequate for a course that certainly has the claim of not being superficial.

Context In general, teachers complain of the difficulty in implementing serious shared teaching design for both individual reasons of individual colleagues and contextual reasons.

> "*Multidisciplinarity is difficult because of the connection between colleagues and the very restricted curriculum. We are monads as teachers. We have no co-presence hours. Between colleagues there is also appreciation, but in the end there are no opportunities: in most cases the links between topics are all left to the students.*"

6.3.3 Follow-up course

After the initial sequence, we asked teachers who were available and willing to continue their professional development how work could be most profitably organized in the coming months with the aim of bringing some of the topics covered in class. The teachers' requests we report were the basis for the subsequent actions taken by the research team to continue the collaboration.

1. Teachers requested time to review and study the materials introduced, and the organization of further meetings devoted to questions, clarifications and additional information.

   > "*I would need to review everything from the beginning to try to understand what is proposed. Meetings would be useful to understand and dissolve doubts.*"

2. All teachers asked for further help in the reconstruction of the content for instruction based on preliminary teaching-learning sequences already designed by our group, but so far tested only with self-selected, motivated secondary school students.

   > "*Some meeting to share the didactic experiences carried out after the course could be useful*"

3. Teachers were interested in preparing simplified materials trying to identify at least minimal learning paths to propose to students, taking into account the heterogeneity of classes.

   > "*It would be interesting to prepare simplified materials trying to identify an at least minimal path to propose to students, taking into account heterogeneous classes in which there are potentially few students interested regardless*"

6.3.3.1 Organization of the follow-up course

The follow-up part of the course was organized into five meetings and with some specific aims:

1. clarifying the educational issues and challenges behind the design of a teaching-learning sequence, in particular those related to the progressive acquisition, during

the secondary school curriculum, of the prerequisites needed for quantum technologies;

2. exemplifying the implementation of these principles of design in terms of active-learning strategies that are feasible at high school level;

3. developing more in detail a physical context suitable to describe quantum systems and devices that can possibly encode and process information (photon polarization).

In the first meeting, we discussed the results of research on student understanding of quantum physics, highlighting three different kinds of learning challenges that need to be taken into account: interpretive difficulties, conceptual fragmentation, and epistemic challenges. In theory change from classical to quantum physics, basic terms of the former, such as 'measurement' and 'state', undergo a shift in meaning, giving rise to interpretive difficulties (see [194] for difficulties with quantum measurement). In addition, students struggle to overcome knowledge fragmentation on the quantum model, as attested by the strong context-dependence of their reasoning even after long periods of instruction on the topic [195]. Last, learning quantum physics requires students to renounce a set of basic beliefs about nature at a time, depriving them of important resources in building a plausible mental model of quantum systems and processes [196].

We explained to teachers that an awareness of research findings about domain-specific learning challenges can provide them with reliable guidelines in the design of instructional materials on the topic. In the second and third meeting, we presented a teaching-learning sequence designed to help students overcome these different challenges. For this purpose, we revised an educational path presented in Pospiech et al., Section 4 [176]. The sequence is set in the context of the linear polarization of light, involving a repeated alternation of empirical explorations of the phenomenon and related devices (polarizing filters, birefringent crystals, etc.) at a macroscopic level, and model building activities at the level of single photons. We illustrated how the challenges were addressed by means of various kind of active-learning strategies grouped as knowledge revision activities, knowledge integration activities, and exploration of epistemic practices. The first set involves the revision of basic terms of classical physics such as physical quantity, measurement, state, vector, superposition, interference, whose quantum versions represent the conceptual and mathematical tools needed to cope with a course on quantum technology. The second introduces and develops the framework of the 'relations between properties', i.e., the rules that determine the acquisition, the loss and the retention of definite values of observables in the measurement process. The knowledge of these relations allows students to analyze measurement not only in the context of photon polarization, but also in other physical situations (e.g., the hydrogen atom), by means of already known instruments, in order to help students overcome fragmentation without introducing sophisticated mathematical constructs. The third is the operationalization of historically significant practices of the theoretical physicist - e.g., thought experiments, interpretation of known laws within new models, etc. - in terms of inquiry-based activities. These activities are used for helping students accept the quantum description of the world as a plausible and reliable product of their own inquiry.In these meetings, we showed several examples of ways in which basic design needs can be translated into concrete activities that can be experienced in the classroom. In the last two meetings, we put the work done in the context of photon polarization at the service of quantum technologies, describing how to use already known experimental tools (birefringent crystals) and new ones (phase shifters) to build logic gates acting on a polarization encoded qubit (fourth lesson). In the fifth lesson, we introduced the last device (non-polarizing beam-splitter), which allowed us to discuss dual-rail encoding. Finally, we showed how this set of tools can be used to realize two-qubit gates, circuits and algorithms such as Deutsch's and Grover's, and, as a result, how mathematics, physics and the concrete realization of technological networks can be integrated into an interdisciplinary perspective.

6.3.3.2 Significance and role of the follow-up course

The five meetings following the course were essential to enable teachers to revise the concepts learned in terms of teaching methods that can be implemented in the classroom. The following is an interesting commentary by one of the teachers:

> "*The first course was a very significant, complex course: a course for teachers that was held at a higher level so that we could then reconstruct the concepts for the students. Translating into a educational sequence requires confronting with others, reflecting and concretely preparing the activities: you can't do it alone.*"

Teachers also underlined that a decisive step for them was to start reflecting autonomously on possible educational paths to bring in class part of the topics discussed. Synthetically, some distinctive features of the second part of the course which contributed to its perceived productivity were as follows:

1. Each meeting was attended by 5 or 6 teachers, and based each time on their needs and requests;
2. since topics had already been introduced it was possible to focus on more specific parts making the approach more accurate;
3. the topics introduced were presented in the light of an educational reconstruction for teaching;
4. each topic was treated with the continual prospect of evolving into a teaching experiment the following year.

These features led to more participation during the meetings, more questions and more observations made by the teachers even though they were still carried out in distance learning. All this allowed a qualitative leap in the relationship between the researchers and the teachers, as a prelude to the activities in action research projects.

6.3.3.3 Semi-structured interviews after the follow-up course

Similar to what we did during the course, at the end of the follow-up course we conducted some semi-structured interviews. We have divided the interview protocol matrix into three parts:

- specific course questions;
- questions on the reasons for the future experimentation;
- questions related to the school context.

We present the most significant aspects that emerged accompanied by some sentences from the teachers.

Specific course questions The in-depth course was not seen as necessary for the clarification of topics nor for the decision to present them in classes (except in one case). However, all the teachers recognize the importance of having clarified possible teaching strategies for presenting topics seen as fundamental but complex:

> "*Starting with knowledge of a topic then requires a reworking for the students a channel between the topic, what I have understood and what key*

> *concepts need to be proposed that can be meaningful to the students. Translating this into a teaching proposal requires discussion, reflection and actually preparing the activities: it cannot be done alone.*"

and the importance of the more experimental approach:

> "*I would say that the discussion on polarization was very important. The experimental part is always crucial.*"

Questions on the reasons for the future experimentation The reasons given by the teachers for choosing to bring specific topics into the classroom reflect the cultural value of the proposal. In particular, they emphasize the possibility of introducing very recent topics rigorously, enabling everyone to benefit from them:

> "*The topics are very topical. The world around them can be explained by this course. They can glimpse possible future professions, new horizons for the children to see.*"

> "*Important path and I hope one day it will become part of the standard proposal within our classrooms. It is part of today and our future. The children must get to know it.*"

> "*It will not reach everyone in the same way, but I hope that for some, it will create passions and curiosity that can influence future choices.*"

Questions related to the school context Teachers feel involved in the school's activities but highlight that their proposals for in-depth studies or experiments are often not followed up. This is mainly due to the numerous administrative commitments and the fact that sometimes directors, families and colleagues get in the way of such activities.

> "*In the first years, there was restraint, especially in colleagues, while families did not. Management tends not to leak anything.*"

> "*Colleagues do not have the time and therefore tend to pull down. The level has to be the same, so better to pull down. However, only because they do not have the time, they have family situations, et cetera. It is difficult to find time for self-training.*"

> "*I don't know if mathematicians will be willing to follow this path. It is exhausting. They have a reverential fear of quantum physics. You can*

> *see the link between physical mathematics and computer science very well here. I would have to do training within the school on this path. They would hardly be able to do training on the course because we are very busy with school administration.*"

Further on the relationship between context and support from academics:

> "*School context is very important (this makes the support of the university world even more important), although I left because I thought things were right for my students to do. If you don't have support, nothing gets done. There is no opposition, but no support either.*"

6.4 Co-design TLSs and implementation in classroom

At the end of the follow-up course, those teachers who intended to start autonomous experimentations based on the course materials were divided into thematic groups and started discussing between themselves and with tutors in the perspective of planning and performing didactic proposals in the context of their classrooms. Considerations related to the dynamics of the various class groups and the teachers' level of interest and appropriation of the course materials guided each teacher in choosing the target classroom and the general theme for their first experimentation. From the start, participants were encouraged to work within a proper action research perspective, i.e. the well-known phases of planning, execution, observation, reflection and evaluation [197]; and were introduced to the basics of the Model of Educational Reconstruction (MER) [148] as a general guideline for the design of teaching-learning sequences. It is, however, emphasized that, from the perspective of action research, the primary goal is not the production of new knowledge, not even knowledge in education research (although it may be gained as a byproduct in some cases), but self-development and the improvement of one's educational practice [198].

Three groups were formed to work on proposals concerning respectively a) quantum computation and communication; b) the thermodynamics of computation and the Landauer principle; and c) the connection between classical logic, probability and experimental outcomes. Such proposals are intended for the fifth, third, and first year of high school, respectively. In the preparatory phase for educational planning, the guiding role of researchers is arranging on the table the various elements contributing to the design (historical and epistemological analysis of the science content, relevant research literature, possible difficulties which students could encounter) and the instruments for evaluation of the sequence

was still relevant with a progressively higher level of autonomy by the teachers.
We allowed a great deal of autonomy in the pathways concerning topics at least partly traditionally addressed in the curriculum. On the other hand, more intervention took place in the design of the computation and quantum information teaching-learning sequences: in the latter case, the design and realization of lecture materials were conducted in collaboration with the teachers but realised by the researchers[8].

6.4.1 Quantum algorithms and quantum teleportation

The two teachers involved in the experiments with their classes expressed a desire to follow two paths: one on quantum algorithms and the other on entanglement and teleportation. This choice was due to two particularities of their classes: the first teacher had an applied science high school class in which the students did two hours a week of computer science for five years; the second teacher, on the other hand, had a traditional science high school class and preferred to bring them some cultural aspects of the debate on EPR states.
As already mentioned, in the next chapter, we will clearly describe the entire teaching-learning sequence; here, we will include how the work with the teachers contributed to the design.

The two teachers wanted an introductory course on polarization that would enable students to introduce the main elements of QM through simple experiments with poor materials (filters and calcite crystals); furthermore, both favoured introducing polarisation encodings and paths for the description of algorithms or quantum protocols. This necessitated a restructuring of the original QM path so that the elements chosen were consistent for the presentation of subsequent topics. For these reasons, we have assumed a part common to both paths and one that differs, as shown in the table Tab. 6.2. We produced the materials in close cooperation with the teachers who contributed by defining the areas that the students had already done in previous years, the elements that should have been focused on more than expected, and any parts that were not entirely clear among those prepared by the researchers. In addition, when studying the lessons for preparation, they could point out some aspects that were unclear in their training and needed to be improved in explanation.
The reference to the Inquiry-Based Learning and Modelling-Based Teaching model was evident and continuous during this phase of material construction. This was with the dual

[8] We report on each aspect of this specifically in the next chapter. In the following paragraphs we will only give a brief description

Table 6.2: Structure of the educational experiments

Introduction to QP	**Quantum computation and quantum information**	**Development**
The quantum state and its vector	From classical to quantum computation - Logical circuits	Quantum algorithms
Quantum superposition	Polarisation encoding - Optical circuits	Quantum teleportation
Propagation and entanglement	Dual-rail encoding - Optical circuits	
Introduction to quantum measurement and observables	Computation with two qubits - Optical implementation circuits	

intent of training the teachers involved in constructing teaching paths and, simultaneously, of seeing these practices developed directly in the teaching materials created.
The two teachers carried out both teaching experiments with only the non-active presence of the researcher who was writing.
In the end, we conducted two interviews to reflect on the entire training experience and, precisely, on the educational experimentation in the classrooms.

6.4.1.1 Final interviews

We have divided the interview protocol matrix into six parts:

1. questions about the perception of their students;
2. question about the lessons and materials used;
3. questions about their role as teachers and how they prepared for the lessons;
4. questions about the entire professional development course, including the experimentation work;
5. questions about the role of the researchers in relation to the experimentation;
6. questions about the context.

Here are the characteristics and common traits resulting from the interviews.

1) The teachers are generally satisfied with the participation of their students. They highlight the effort put in during the course. However, they highlight that some students were much more active than usual.

> "*From some, I would not have expected involvement; instead, they followed very closely, and I think it was excellent, especially since they usually had difficulties studying physics.*"

And more

> "*The students all showed interest, even those who did not want to continue scientific studies. Even those who participate little usually participated more than usual.*"

2) Conflicting aspects emerge about lessons and worksheets: Concerning the worksheets, they would have the merit of activating the students, but also the constraint of resulting in a strict lecture; this aspect is further intensified by the lecture delivered with slides.

> "*I am not used to handling lectures via slides; it was arduous. I would have preferred to write on the blackboard the way I am used to. Constructing the individual conceptual or calculation steps with them. I found it a bit difficult that way. Perhaps a second time I will be more confident and free.*"

> "*For the worksheets, they have the merit of activating many more students even if they were struggling a lot.*"

And more

> "*I appreciate this kind of work: I sometimes try it myself. I think it is beneficial for the activation of the students. Sometimes I think the rigidity of the worksheet is to the detriment of the rhythm that has to be adapted to the student's ability to follow. It is a suitable method that should be made more flexible.*"

> "*For a very advanced proposal, there needs to be very definite material to support the lesson. The teacher's creative process will come in a few years. These are concepts not used daily and are part of research beyond what I studied at university. The probability of saying wrong things is minimized.*"

3) Teachers emphasise the importance of the theoretical lessons carried out in the first teacher professional development course, especially the formal aspect of theory concerning calculation:

> "*I re-read the lecture notes taken during the meetings and studied the slides before the lectures...*
> *The calculus part was crucial: reviewing the lectures was the part I needed most. Even a deeper calculus than the one brought to the students.*"

However, the aspect of collaboration with researchers is also emphasised already in these questions:

> "*Collaboration is crucial. On my own, I couldn't have done it. The meetings we had after deciding that we would do the experiment were crucial. I reviewed all the lectures, reworked the calculations, and attended the meetings asking for a teaching proposal, and from there, with the help of the researchers, we were able to prepare the course.*"

And most of all

> "*Until you get to prepare to present them in class, you don't fully understand everything. When you have to share with your students, you have to go deep: how do I explain it to them, how do I let them know. Without the help of those who do research in education, you might do something halfway, something not so meaningful.*"

About the teacher's role in experiments, enthusiasm in presenting new topics is highlighted and supported by prepared learning materials:

> "*The fact that I have new topics to present is, for me, a source of enthusiasm and not of difficulty.*"

> "*Sometimes a little insecurity, but no big difference. I was looking for your (researcher present) approval. She was happy when you were present because she was more relieved. If you weren't there, I was afraid I wouldn't be able to answer some questions, which didn't happen. When I did Deutsch without you, I was excited: I got into the lesson and went straight through, also thanks to the learning materials provided.*"

4) One of the two teachers reflects on the whole experience and how the increased understanding of the topic coincided with the awareness and desire to share it with their students:

> "*I enrolled for the in-depth study on Stern-Gerlach to make a meaningful modern physics proposal. Then I realized it was something very different. Effort: I remember it at first. But it was something I didn't know and interested me more. I had to understand. I studied: once I understood, I became convinced of its importance and the need to bring it into the classroom. It did not seem easy, but now I know it can be done. It takes much work, but it is worth it.*"

Both agreed that the key moment of their educational path was to bring the topics into the classroom:

> "*A key moment is to have brought it into the classroom to understand some of the critical issues in how I brought certain aspects into the classroom, the time spent, et cetera.*"

5) Regarding the role of researchers, both teachers emphasise the importance of working in collaboration with those carrying out research in physics education, in particular for the official nature of the teaching proposal:

> "*Important to make the educational pathway official: in collaboration with the university. Research pathway in physics education that justifies and supports the choice. If I did it again, I would ask for it again. Everything went as I thought. Your presence gave me security. It's nice working together.*"

> "*It is important to stimulate the students, to make them see that it is not just something of their teacher but that there is a larger structure behind it: the university, researchers, etc. It would have made their approach more serious. It is not just teaching but there is experimental work behind it.*"

6) The last questions were once again about context. The importance of the work carried out in collaboration with university researchers emerges from the two teachers' considerations:

> "*It justifies having an institution behind it to take away criticism from students, colleagues, parents, etc. If a university does not support you, it is less acceptable.*"

The perspective on the influence of the work performed on colleagues is different[9]:

[9] As we will explain further, in fact in the schools where the researchers went there was in fact an involvement of other teachers who are now contributing to the formation of pathways in the first classes.

"*I missed the support of my colleagues and their understanding. I expected them to be more present. The computer science colleague was absent, and the maths colleague had other things to think about.It could have been interesting to do it more transversally.*"

"*In my opinion, the perception has changed in the school because I have seen teachers who normally do not want to innovate in physics didactic but prefer in pedagogy etc., participate. They participated constructively. Then the fact that others want to experiment with logic, physics and probability is different from what happened in previous years. I hope it was a way of changing things. I would like to have a uniquely dedicated school to these activities.*"

6.4.2 Maxwell's demon and the second principle of thermodynamics

A group of five teachers worked for several months on the possibility of introducing some aspects of the thermodynamics of computation into their classrooms. The researchers presented an approach to thermodynamics that they had dealt with before and that was particularly adapted to computational development (see [199], [200]). Then, together with the teachers, they studied some work on the solution of Maxwell's demon paradox, up to its solution linked to information theory (see [201]). The outcome of this shared work was a path that ended with Landauer's solution exposed in computational terms. Without entering into detail, what is important to emphasize is that some of the teachers shared the idea of anticipating the relationship between physics and computation following one of the goals of our research approach.

6.4.3 Physics, logic and probability

In June 2022, at the explicit request of some teachers, three two-hour meetings were held on a possible introduction to the link between physics, logic and probability in the first year of high school. Five teachers who had already attended the first training course and some of their colleagues interested in the topics discussed for possible educational experimentation attended the meetings. In these meetings, the diagrammatic approach was explained in depth. The fundamental topic was the possibility of introducing the circuit language concerning Boolean logic and the physics of computation from the first year. The teacher of the educational experiment in Castel San Giovanni (which we will discuss in the next chapter) and one of her colleagues designed an 18-hour course for their students. The course involves the students' construction of a water computer and progressive abstraction at the circuit level up to the link between propositional and Boolean logic. The goal for the fifth year is to introduce quantum algorithms so that their students have been on a path to the diagrammatic model since the first year.

6.5 Results and research questions

From the questionnaires, the interviews and the work done with the teachers, we drew many indications that not only made it possible to answer the initial research questions but also contributed (as will be seen in the next chapter) to the design of the pathways for the students. The outcome of all this is the second training course, which we will briefly discuss in the next section.

TRQ1 *How is it possible to construct an adequate content simplification process to present the topics of the second quantum revolution to teachers in a meaningful way from very advanced theoretical aspects?*

Given the innovation of the topics proposed, a process of elementarization must consider the need to present the topics with great depth and precision; this from both a formal and conceptual point of view. The formalism used is considered appropriate for both teachers and their students. This simplification must constantly live up to two aspects: the presentation of the educational materials and the continuous comparison with the experimental aspects. The diagrammatic model as a common framework for computational and physics topics seems to be a useful tool.

TRQ2 *How make the contents and themes of the second quantum revolution sufficiently fruitful to teachers to develop a personal commitment to longitudinal, interdisciplinary educational innovation directed towards themes of quantum information and computation?*

The teachers immediately agreed on the strong cultural impact for themselves and their students of teaching proposal. The main doubts revolved around the possibility of presenting the topics in their classrooms. However, the results of the follow-up course, of the shared design, and in general of the path shared between researchers and teachers, was to make concrete the intention of some teachers to present these topics to their students. Moreover, to do so not only in teaching experiments in the fifth year, where it is more natural to deal with these topics, but also in the first, third and fourth classes. The interdisciplinary and integrated approach with a high cultural impact, the record of lessons and the materials for lessons, all supported by continuous work and discussions with researchers, have activated some teachers to develop a personal commitment to longitudinal, interdisciplinary educational innovation directed towards themes of quantum information and computation. Some criticalities emerge from the use of worksheets linked to slides; however, for a first explanation, they also have the merit of reducing possible errors to a minimum.

TRQ3 *What are the most appropriate environment and methods for building a distributed, online community of practice of teachers revolving around the themes of the second quantum revolution?*

This seemed the most critical aspect at the beginning of the path with the teachers. Despite the activation of a forum to collect projects, questions and curiosity, we seemed unable to create the conditions for building a distributed, online community of practice of teachers. However, two aspects seem to have changed this situation:

1. Some teachers are also attending meetings of the new vocational training course. These also include a couple of teachers who carried out the experimentations in their classrooms last year. Their presence is now constant in the training activities we offer on these topics, and some teachers have started to share experiences and materials: they are beginning to become a reference.
2. However, a second aspect must be emphasized. During the experiments, one of the researchers was often present in the respective schools, also conducting meetings for all mathematics and physics in-service teachers. This made it possible to get to know the environment, including school directors and to activate a shared and extensive collaboration. The result is the participation of other teachers from the same school in the training meetings. In this way, a reference figure, a kind of expert teacher, who coordinates, supported by the researchers, small working groups on the topics of the second quantum revolution has been established in the individual schools. The aim is to share these experiences in specially organized meetings.

Not being able to count on in-person participation (at least in recent years) seems to require time and direct intervention in schools to generate a community of practice.

In particular, the presence in schools seems to allow for overcoming specific contextual difficulties that emerged in the interviews. We are not able to give further indications at the moment.

6.6 Quantum technologies course: second implementation.

Considering the results of the research carried out during the first course of teacher professional development and the educational experiments with students (see 7), a second course was designed and implemented. Here are the main new elements.

1. An extra meeting was devoted to an introduction to quantum mechanics with polarization, which was necessary in order to be able to use the teaching materials during the lectures.
2. The lessons were revised to allow an initial approach to the worksheets used during the teaching experiments. In particular, the elementarization elements of the algorithms and the teleportation protocol, and the modelling of polarization and dual-rail models, were carefully addressed.
3. Great importance was given to implementations with optical devices.
4. The importance of the diagrammatic model was posed early on. The possibility of interpreting diagrams logically and physically was the constant interpretative key. Examples from the worksheets designed for the students supported the meetings on coding, logic gates, quantum algorithms and teleportation.
5. An advanced course on educational materials and data analysis of experiments is planned for teachers who would like to introduce a course in their classrooms.

This second course is the high point of the whole project. We will examine what impact it will have in the coming months.

Chapter 7

Quantum technologies for students: design, implementations and results

This chapter describes the TLS for students that we designed, realized and implemented, based on the considerations made in the previous chapters. Unlike chapter six, where we wanted to give a chronological dimension of the work done with teachers, here we aim to express the work in its logical perspective[1]. We will consider the following line of development. First, we will briefly return to some general considerations outlined in the first chapter, representing the historical-cultural and conceptual framework we intend to move. We will then describe the general aims of the proposed work by making explicit the research questions. Moreover, we will develop the fundamental moments of the MER: first, to build the content structure for instruction, we will approach the analysis from a theoretical point of view. By the general learning aims, we present an analysis of the theory exposed in Chapters 3 and 4. In the second step, we will focus on the perspectives of students and teachers. In the end, we will present the reconstruction of the key ideas and development of physical information theory into a content structure for instruction.
In order to understand whether the designed reconstruction is congruent with the mentioned general goals, we will carefully describe a hypothetical learning trajectory in such a way as to bring to light the succession of tasks and activities. This will be the most substantial part as we will have to describe in detail the materials built in relation to the

[1] To do this we will briefly return to some of the concepts expressed in previous chapters. Although they are already present in the thesis, we prefer to bring them back into this chapter to support understanding and argumentation.

goals and methodology exposed in Chapter 5. The final part, then, will be spent on data analysis and evaluation with regard to the design hypothesis and the research questions: at the conclusion, it will then be possible to identify the design principles that, ex-post, will define the TLS.

7.1 Introduction

Industrial policies in the EU and US in the past six years have fostered a fast and increasing interest in quantum mechanics as a pivotal area for the development of future technological and societal advancements [202]. In particular, quantum mechanics is at the heart of innovations that include intelligent sensors, networking, communication, computing hardware, algorithms, and other facilitating technologies. Such interest has led to the launch of far-reaching institutional programs such as the National Quantum Initiative Act [203], [2] in the US and the Quantum Flagship in the EU [3]. Also in the perspective of these projects, there has been a rise of worldwide interest in expanding education at all levels on technological applications revolving on the manipulation and control of individual quantum systems, the so called second quantum revolution. However, this interest vividly clashes with physics education research evidence according to which quantum mechanics is perceived as a difficult and demanding subject area, whose concepts are considered too abstract and difficult ([204],[205],[206],[207],[208],[209]). Many studies also show that students hold a variety of misconceptions in quantum mechanics ([195],[4],[5]) due, for instance, about the need to reconsider the key concepts of classical physics and to bridge physics and chemistry concepts ([210],[211],[212]). Although there have been attempts to use quantum computation as a context for the initial introduction of quantum physics [8-10], these are intended mostly for the undergraduate level: proposals for secondary school have used a more traditional sequential approach, that starts with an introduction of quantum physics as a preliminary step to address quantum computation ([10],[213],[214]). To make the pathway feasible at the curricular level, we prefer to follow the second solution. This is also to highlight any traits of the pathway that can be brought forward to the years prior to the fifth, with a view to redesigning the curriculum from the first year.
But beyond the cultural-historical motivations related to the technological development of the present, the work we propose is also based on motivations intrinsic to the disciplines involved. Indeed, we believe that the proposed interdisciplinary approach can profoundly bring out the dialectic between physics, mathematics and computer science. In this way we seek to enhance topics that are too often isolated (think, for example, of the role of probability calculus and propositional logic in secondary school, as described by teachers

themselves) or undervalued in their interaction (think, for example, of new math textbooks in which physics is used as exercises on a particular topic). Despite the little research on these topics, especially at the curricular level, the considerations led us to refer to the MER for pathway design. We have, in addition, exploited, where possible, the existing literature, the work done with teachers and exposed in the previous chapter, the first on-line experiments carried out with the Galilei high school in Voghera, the Summer schools and PCTO work done at a distance learning in collaboration mainly with the University of Insubria and Bologna.

So the purpose of this chapter is to describe and propose an educational reconstruction of the themes of the second quantum revolution based on the structure of science content and the relevant published literature on teaching and learning about it, supplemented by our empirical results. The educational reconstruction for instruction that we present attempts to answer the following research questions:

SRQ1 : *How is it possible to construct an adequate content simplification process to present the topics of the second quantum revolution to students in a meaningful way from very advanced theoretical aspects?*

SRQ2 : *How effective is an integrated and multidisciplinary approach in order to enable students to understand some topics of quantum computation and quantum information?*

SRQ3 *Based on findings from the first two research questions, what design principles can be formulated for the development of TLS resources in quantum computation for high school students?*

We wonder, then, how we can support students toward the deepest possible understanding of both the computational aspects understood from a logical-formal point of view and the characteristics of quantum physics that underlie them and enable experimental implementations, at least from an ideal point of view. For clarity, we anticipate that the proposed TLS consists of three separate steps:

1. introduction to QP;

2. from classical to quantum computation;

3. applications: quantum algorithms and teleportation protocol[2]

7.2 Educational reconstruction

7.2.1 Analysis from the Theoretical Perspective

In this section, we present an analysis of the theory from the perspective of the general goal of our research and respect to national indications for high schools in Italy. We will emphasize some aspects of indications especially relevant to our educational path. We will underline any critical issues with respect to established teaching practice. These critical issues emerge from interviews conducted with teachers and experiences with students in the project already mentioned.
We will differentiate, like the guidelines, between general lines and competences on the one hand, and specific learning goals on the other. We will do this for the three main disciplines of the TLS: physics, mathematics, computer science[3].
Remaining, for the moment, in the general context of the scientific domain, the student is asked to know:

1. *understand the specific formal language of mathematics, know how to use the typical procedures of mathematical thinking, know the fundamental contents of the theories underlying the mathematical description of reality*;

2. *master the fundamental contents of the physics and natural sciences (chemistry, biology, earth sciences, astronomy), mastering their own procedures and methods of investigation, also in order to be able to orientate themselves in the field of applied sciences*;

3. *to be able to use informatics and telematics tools critically in study and research activities; to understand the methodological value of informatics in the formalization and modelling of complex processes and in the identification of solution procedures.*

By analysing these three points in their mutual influence, we can find the overall aim of our work: to enable students to grasp the dialectic between physics, mathematics and computer science.

[2] We will speak about the quantum cryptography, but we don't introduce in the two experiments carried out with students.

[3] Computer science is not separate from mathematics in the traditional address; for clarity we will separate the two subjects.

From a university teaching perspective, these issues have been recently addressed by 34 QIS experts from both academia and industry who signed an open letter providing indications on a range of aspects ([6]). In particular, they called for an earlier start of QIS education in the academic career, recommending to establish introductory-level science and engineering classes that introduce students to the foundations of the subject-matter. Such courses should be designed for non-physicists and have pre-requisites commensurate with students from any technical field. Education in QIS should also promote the development of high-level experimental skills including, e.g., experimental and engineering design or modelling of experiments, which are challenging to attain in existing traditional programs. For all these purposes, they argue for the early involvement of education experts in curriculum development.

The general aim outlined above also naturally arises in the light of the historical development that led to the first formulations of the physical theory of quantum information. Let's think of the pioneering work of C. Bennett (see [14], [181]) and that of Toffoli and Fredkin [16]. They show precisely the connection between computation and thermodynamics. Linking this to the problem of the reversibility of classical logic gates, the role of the physical problem of computation becomes evident. It is no coincidence that R. Feynman, between 1983 and 1986, gave a course at the California Institute of Technology that was published posthumously in the famous Lectures on computation ([39]), which were strongly influenced by these works. In this sense, what we would like to emphasise is that the need to bring computation back to a physical (thermodynamic) dimension was as far from the horizon of scientists in the 1970s as it probably is from secondary school students (and as we have seen from their teachers as well). However, it represents a fundamental point in the birth of the theory of quantum computation. It is the light in which many statements by early scholars that highlight their views on the nascent discipline should be seen. The first is certainly that in [16], which links Turing's work to certain implied physical assumptions according to the authors (see the beginning of chapter 3) and also those from Deutsch [18] and Ekert [215], which we quote below:

> "Intuitively, a computing machine is any physical system whose dynamical evolution takes it from one of a set of 'input' states to one of a set of 'output' states. The states are labelled in some canonical way, the machine is prepared in a state with a given input label and then, following some motion, the output state is measured. For a classical deterministic system the measured output label is a definite function f of the prepared input label; moreover the value of

> that label can in principle be measured by an outside observer (the 'user') and the machine is said to 'compute' the function f."
>
> "Today, one can briefly define cryptography as a mathematical system of transforming information so that it is unintelligible and therefore useless to those who are not meant to have access to it. However, as the computational process associated with transforming the information is always performed by physical means, one cannot separate the mathematical structure from the underlying laws of physics that govern the process of computation."

Here, then, our TLS can find in the history of computation and quantum information the first conceptually relevant element[4]: the semantic shift (extension) of the word computation from the area of logic-mathematics to that of physics. Or rather: what is brought to light is, we might say, the need to consistently problematize, when talking about computation, whether we are referring to hardware or software, to physics or logic. But following Vygotsky's teaching in [164], we want to consider the two elements not as separate but to study their dialectic: only the study of reciprocal influences, of the transition of the first into the second and *vice versa*, both from a formal and conceptual point of view, will allow them to define themselves more clearly and allow us to achieve an integrated model of computation. Therefore, we need a language that can do this as deeply as possible. And to do it in the same way whether we are dealing with classical or quantum computation: the diagrammatic language described in Chapters 2 and 3 serves this purpose[5].

In some of the more recent axiomatic formulations of quantum theory, the use of category theory and its possible diagrammatic circuit representations is deeply embedded even when not explicitly decried ([217], [32], [92], [94], [93]). We also find its use in several more application-oriented works, especially, but not only, in computer sciences ([80], [81], [82], [83], [84], [85], [86], [87], [88], [89], [90]), but also in the field of physical of computation ([70], [71], [72]). What these works show is the possibility of using appropriate monoidal categories to describe processes: be they physical processes, linguistic (texts), musical (compositions), chemical or other. The attempt is to construct a unifying language even when, as in the machine learning we saw in Chapter 4, what we are studying is characterised by apparently very different processes. The unifying attempt of these works translates in our research into the use of a language able to create a unified model for logic,

[4]The importance of the history and philosophy of physics for the construction of routes is well known ([148]and [216].

[5]It should be noted that, in principle, it allows generalisation far beyond simple physics

the physical theory of computation, and the corresponding experimental realizations in the quantum case using optical devices ([70], [71], [72]). Therefore, we have introduced useful categorical tools appropriate for defining a language bringing together computational theory, physical theory and implementation using optical devices[6]. This integrated and multidisciplinary approach to the theory of computation and information, first mediated and then described through circuit representation, leads us to ask what conceptual changes are necessary and possible for high school students. Consider summarise these changes in the table 7.1:

TABLE 7.1: Conceptual changes

Changes			Key features
Perspective change	from quantum theory as a theory of microscopic matter	to quantum physics as a framework for technological applications and information processing	Coding, processing, decoding information
Physics change	from classical physics	to quantum physics	Randomness, uncertainty and entanglement
Logic change	from classical logic	to quantum processing	Quantum parallelism, linearity of Oracle, nature of compound system, computational interference
Probability change	From epistemic, classical or axiomatic	to intrinsic, Bayesian	Nature of State and Measurement. Probability as extension of logic

For clarity, we consider briefly, as in Chapter 2, how the elements introduced in the table

[6] The operational approach in this sense is a great help ([32], [94]) because it builds theory precisely from the idea that transformations are equivalence classes of operations performed in a laboratory.

arise from considerations intrinsic to quantum algorithms and quantum communication protocols such as teleportation and key generation and sharing in quantum cryptography.

Quantum algorithms We first analysed quantum algorithms, having to make the choice of presenting only the simplest but no less significant ones. The algorithms of Deutsch, Deutsch-Jozsa and Grover ([18], [41], [42]) were the main objects of our study. It was clear from the beginning that some distinctive elements of quantum physics allowed for a deeply different kind of computation: the superposition principle, the linearity of the unitary operators, the particular nature of compound systems expressed mathematically by the tensor product, and quantum interference are the elements underlying the advantage that Quantum Computation (QC) offers over classical computation (cc). In particular, the superposition principle makes it possible to switch from an exponential number of coding bits to a polynomial number of qubits; through the linearity of Quantum Gates (QG) the classical functions act simultaneously on all coding states. The presence of an ancilla makes it clear how the phase combined with the nature of the tensor product allows for special encodings even on signs that can be exploited to create interference and achieve the solution to a problem posed more efficiently. However, it has become clear from later work ([40], [47]) that entanglement plays a crucial role in both algorithms: in the first case, "*multipartite entanglement within the first register is needed to accommodate all possible (balanced) functions*"; in the second, the authors shown that "*the entanglement properties of the initial state of the first register depend on the number of solutions, and we have demonstrated that such a state is also typically multipartite entangled when a small number of items is searched for*". The circuit representation using Hypergraph category (ZX calculus) allows these algorithms to be reinterpreted from the point of view of information flow and the topological structure of quantum algorithms ([218], [219]). This approach introduces a higher-level interpretative key that is extremely useful in the case of quantum teleportation.

References to the physical aspects of quantum computation are clarified by studying some experimental realisations of algorithms based on optical devices and linear optics. The concern in this study was not so much about the actual exploitation of such devices from a quantum technology perspective. However, they have two fundamental merits: they are based on linear optics, and thus not too complex to implement; moreover, the strict link between circuit description and the experimental setup is evident from this work ([220], [221], [69], [222], [223], [224]). The importance of the possibility of an integrated perspective of computation capable of constructing a model that takes into account the physical theory of computation and its realisation in experimental setups strongly conditioned by the diagrammatic approach is immediately apparent. As a definitive confirmation of the latter, we cite the three papers published in spring 2022 that shed light on the link between high-level (software) and lower-level (hardware) circuit representations ([70], [71], [72]).

Quantum teleportation From the point of view of quantum communication, the teleportation protocol ([225]), which was just before Grover's algorithm, is of extreme relevance. Here the role of entanglement as a resource is emphasised: "Maximal entanglement is necessary and sufficient for faithful teleportation". The discussion of the role of EPR states is an important historical element, not so much from the point of view of the current approach to quantum communication, but above all to highlight the difficulty that a physicist like Einstein had in accepting certain consequences, at the time only theoretical, of quantum theory. The change of perspective from entanglement seen as a problem to a resource is one of the most significant aspects of the path that ideally linked the German scientist's 1935 article ([226]) with the recently awarded Nobel Prize in Physics. The change of perspective offered by Bell's theorem ([35]), had the merit of moving a possible demonstration from the theoretical to the experimental point of view. John Clauser, Alain Aspect and Anton Zeilinger were recently awarded the Nobel Prize for their work. And these works have achieved definitively observed violations of Bell's inequality precisely through studying entangled photons and optical devices. Back to the protocol, it is evident that it highlights one of the most significant features of quantum behaviour: non-local correlations. It further emphasizes the need to refer to measurement outcomes as correlated and not to quantum systems in conceptual terms. The approach using circuit representations has the merit of making the introduction of entangled states extremely simple from a logical point of view. The case of a possible experimental setup involving entangled photon emission by means of parametric-down conversion is different. But once again, developing a higher diagrammatic register makes it possible, as we have seen, to achieve the protocol. Using simple syntactic manipulations of diagrams at the end, we will obtain a single line connecting Alice and Bob: the information has been transported ([137],[1])!

Quantum cryptography One of the aspects that most determines the study and funding of quantum technologies is the distribution of keys for encrypting and decrypting a message. Since the publication of Shor's algorithm in 1999 ([227]), the classical RSA encryption protocol was potentially much more attackable. It was based on the difficulty of factoring prime numbers, and the quantum algorithm undermined its reliability. First in 1984 and then in 1991 ([228], [215]), two ideally secure key creation and distribution protocols were introduced that were much more straightforward than the attackable RSA. Whereas the RSA protocol was based on the difficulty of inverting modular functions, the two quantum cryptographic protocols are based on constituent elements of the reality of quantum systems: the superposition state in the first case and entanglement in the second. The possibility of replacing an articulate mathematical protocol with a simple physical one is the most culturally significant aspect of this last part. But not only for this, we want to point it out. It is emblematic of the informational approach to quantum physics. For this reason, although it was not actually used in the paths we will discuss in this chapter, it seemed appropriate to include it in our discussion. We will make no further comments on this.

Lastly, it should be noted that the proposed topics were chosen primarily based on their analysis and not with reference to any particular indications. In this regard, it should be noted that there is no uniformity of opinion as to what topics are necessary for proper training in this respect. It is also interesting to note as they do in [229] that "*each course is intended by its instructor to tell a slightly different story about QIS even if the course topics are ostensibly similar*". However, the importance is emphazised that any canonical course does not come at the expense of the interdisciplinary approach: "*Yet it is important to make sure that canonicalization does not come at the expense of the discipline's interdisciplinary perspective, which is arguably among the greatest strengths of QIS as a research area*". If this is true at university level, the considerations can be extended to high school education.

7.2.2 Analysis from the students perspective

Only in the last years, TLS have begun to be suggested for secondary school students on the topics of the second quantum revolution and quantum technologies[7] ([175], [176], [9], [230]). We focus in some detail on [230] to which we actively contributed.

Introducing quantum technologies: Online Extracurricular Course The activity, carried out in spring 2021, was the result of the joint efforts of the Italian communities of researchers. We decided to use the context of quantum technologies to convey the concepts of quantum mechanics. Our basic assumption is that quantum technologies may reduce the students' perceived abstractness of quantum mechanics,

[7] See also section 6.1

which often comes from limited access to suitable experimental and mathematical literacies.

The basic idea of the educational path is to describe the logic of quantum physics by establishing a parallelism with the logic circuits of information theory ([9], [231]). The axioms of quantum mechanics describe the preparation of a state, its evolution/manipulation, and its measurement, which can be interpreted as information input, information processing, and information output, respectively. This parallelism makes it possible to introduce the fundamental properties of quantum states (superposition and entanglement) and to introduce the "qubit," the quantum extension of the classical "bit," and the elementary transformation of the qubit in terms of quantum gates. Simulations and descriptions of experiments with spin and polarization are used to discuss the physical implementation of qubits. The radical novelty introduced by quantum theory becomes clear from the analysis of the superposition state, the meaning of probability, and the role of measurement.

The activities of our learning path were structured in the following steps: four introductory lectures (one hour each), an in-depth course of three lectures (one and a half hours each) on specific aspects of quantum technologies, and a closing lecture (one and a half hour); see Fig.7.1[8] This study has a twofold aim: (1) to evaluate whether

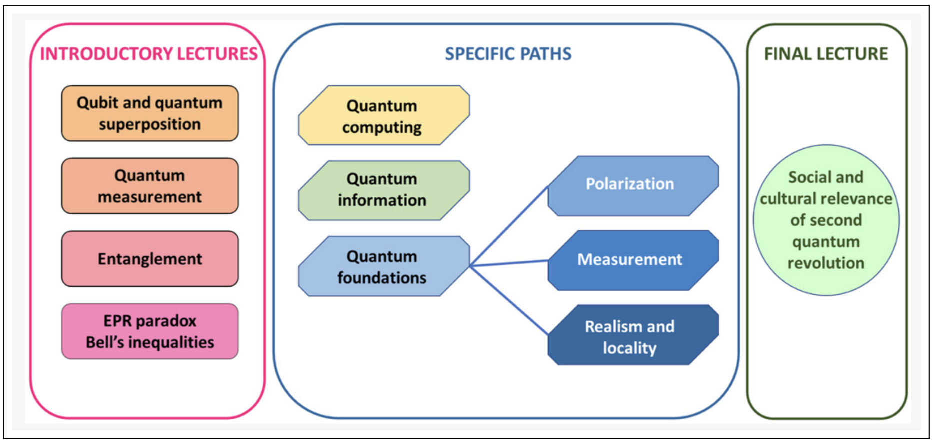


FIGURE 7.1: Overview of the educational path activities.

the designed path helped the students to grasp a basic knowledge of fundamental quantum physics; (2) to evaluate whether the designed didactical path improved students' views about quantum technologies. Thus, we posit the following research questions:

[8]For details of meetings see [230]

RQ1 To what extent was the educational path on quantum technologies effective in improving the students' knowledge about fundamental quantum mechanics concepts?

RQ2 To what extent was the educational path on quantum technologies effective in improving the students' views about quantum technologies?

The study was carried out in the context of the Italian plan called Paths for Transversal Competencies and Orientation (PCTO). The PCTO activities are mandatory for students and include either career orientation or vocational practice. The sample consisted of N = 279 Italian high-school students (females: 24.4%; males: 73.8%; prefer not to say: 1.8%) from 16 different schools distributed across Italy. The course was restricted to students attending the 12th (N = 101, average age: 18.0 ± 0.4 s.d. (standard deviation)) and 13th grades (N = 178; average age: 19.0 ± 0.5 s.d.). The great majority (82%) of the students attended the Liceo Scientifico (mathematically-oriented high school), about 12% attended an applied science course (natural sciences-oriented high school), and about 6% attended a technical school.

For the present study, we developed a short questionnaire (QTI) featuring eight items on the topics addressed during the educational path. The questionnaire (see [230] appendix B) was built on our prior studies ([232],[233],[234],[235]). The table in Fig. 7.2 summarizes the concepts and topics addressed in the questionnaire Five

Concept	Sub-Topic	Item
State	Bra-Ket Formalism	Q3 Q4 Q7
	State and eigenstate in quantum physics	Q1 Q2 Q3
	Logical gate	Q1
	Qubit	Q3
Superposition		Q1 Q7
Measurement	Statistical nature of the measurement	Q2 Q6 Q8
	Wave function collapse	Q7
Entanglement	Formalism	Q4
	measurement on entangled states	Q5

FIGURE 7.2: The main topics addressed in the quantum technologies inventory (QTI).

elements were used to measure students' views about quantum technologies (VAQT):

1. Assuming to use ideal measuring instruments, in physics I must describe the results of measurements probabilistically only if I have incomplete information about the system;

2. It is possible for physicists to carefully perform the same experiment and get two very different results that are both correct;
3. Quantum computers will never work, because it is impossible to build an hardware that is accurate enough;
4. Scientists say that quantum communication makes it possible to teleport a particle from one place to another;
5. Scientists say that quantum communication does not make possible the teleportation of a particle from one place to another, but only the transfer of its characteristics.

Overall, 176 students responded to the QTI after the didactical path, while 162 completed the VAQT instrument before and after the out-of-school activities.
The distribution of the students' scores is shown is in Fig.7.3 The average pre-

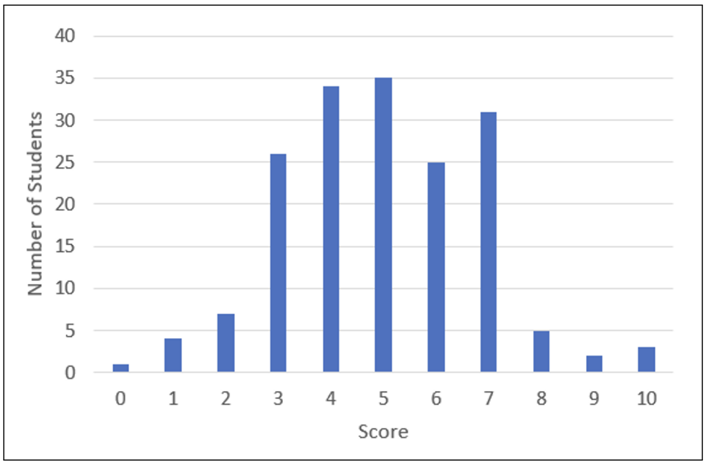


FIGURE 7.3: Distribution of students' scores in the quantum technologies inventory (QTI).

instruction score in the VAQT items about general epistemic aspects was 3.04 ±.85 s.d., while the average post-instruction score was 3.40 ± 0.91 s.d. To better understand this trend, we divided the sample into four groups according to their performance and then calculated the effect size of the difference between the post and pre-test for the VAQT items for each group. Fig.7.4 and Fig.7.5 show the average pre-instruction and post-instruction scores for each group.
The analysis of the students' answers to the QTI shows that, on average, the educational path was useful to familiarize students with fundamental aspects of quantum mechanics. From the analysis of the students' responses to the VAQT, it emerges that, on average, the proposed path was effective in letting students acquire more informed views about general epistemic aspects of quantum mechanics and quantum

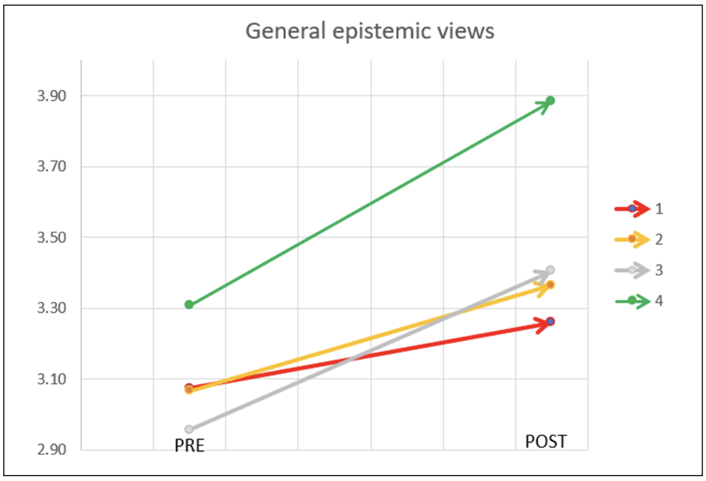


FIGURE 7.4: Pre-instruction vs. post-instruction average scores of the views about quantum technologies (VAQT) instrument (general epistemic views) according to the performance in the QTI for four groups.

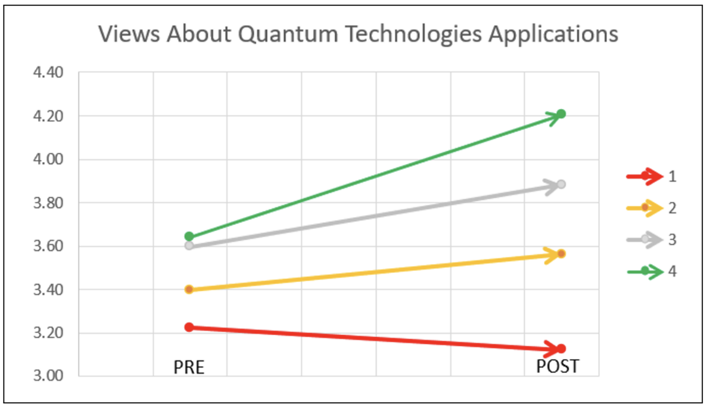


FIGURE 7.5: Pre-instruction vs. post-instruction average scores of the VAQT instrument (applications of quantum technologies) according to the performance in the QTI for four groups.

technology applications. Limitations of the study include the involvement of a small sample and the remote distance modality of the didactical activities. Results are also limited by the use of instruments not yet validated.

On the other hand, the literature, especially concerning courses designed and implemented in curricular contexts, is extremely scarce. This is why we consider it most appropriate to once again begin with a few selected objectives within the national indications and, this time, broken down by the individual subjects included in the constructed multidisciplinary pathway:

Physics, general guidelines: the student, at the end of the course, must have learnt the fundamental concepts in their historical and philosophical context. In terms of skills, among others, he/she must be able to construct and validate models and understand the scientific and technological choices affecting the society in which he/she lives.

Physics, specific learning goals: in the guidelines on quantum physics, an essentially quasi-historical framework is outlined, strongly linked to the macroscopic-microscopic relationship. Furthermore, it is left to the student to investigate topics of interest to him/her on the relationship between science and technology.

Mathematics, general guidelines: at the end of the course, the student must be familiar with concepts and methods of the discipline both in their intrinsic development and relevant to the description and prediction of physical phenomena. The mathematization of the physical world refers to the infinitesimal calculus resulting from the scientific revolution of the 17th century. He must also be able to construct a mathematical model of a set of phenomena.

Mathematics, specific learning goals: in the first two years, the student is expected to study linear algebra at least in its simplest forms related to the concept of vector, matrix and the operations between them. In the fifth year, the student must learn the characteristics of discrete and continuous probability distributions. Furthermore, he/she will deepen the concept of mathematical model and must be able to construct and analyse examples of them.

Computer science, specific learning goals: according to the guidelines, one of the elements to be developed is the concept of the algorithm and the elaboration of algorithmic problem-solving strategies for simple and easily modelled problems; in addition, the concept of a calculable function and calculable problems will be given with examples.

Bringing these considerations together, it seems evident that there is a hope that students will actually be able to build an integrated model of the three disciplines. Such a wish is undoubtedly commendable, but as is evident from informal interviews with students, from personal experience as a teacher and above all from the interviews carried out in the professional development course for teachers, such indications do not find their way into teaching practice. Moreover, an analysis of the most used manual in Italy, shows how they are not yet ready for an integrated proposal, except in the most trivial manner of the word. The construction of physical exercises in the mathematics book and a few worksheets with

Geogebra or Python to carry out simple projects are not concretely included in constructing a teaching-learning sequence. What is needed in the field of quantum physics, but not only there, is an approach that aims to develop a sense of the transition between the single disciplines, to show how they mutually support and justify each other despite their fundamental peculiarities: only in this way can one actually speak of an integrated perspective. In accordance with the categorical approach we have seen in the previous chapters, we could say that an integrated perspective must focus on the morphisms between disciplines in a much deeper way.

Consider now proceed to some critical issues that need to be addressed for an effective educational reconstruction for instruction.
Firstly, as can be deduced from the pretest data on 23 teachers on the course, half introduce logical connectives and do so in a traditional manner linked to propositional logic, and only seven introduce the problem of algorithmic translation of a problem. The interviews then show that the logical aspects almost always remain isolated to the single course in the first year, are no longer used later, and above all, are seldom contextualized to the physics of the devices used to realize them. Even from the pretest of the course in Voghera, of the 11 students, only three had studied logical connectives, and they were all from the course of study with additional computing hours. In any case, a mainly logical-formal perspective of the TLS in its part relating to algorithms, teleportation and cryptography needed a significant redesign for at least a couple of solid reasons. The first is about all the aspects that research in education and physics education has pointed out concerning the role of experiments, which we have described in chapter 5. The second was the evidence of several students from the Voghera course who, despite having demonstrated a very high level of understanding of the topics proposed in the final test, complained of too much abstractness and too little attention to the experimental aspects. This was because the teaching content had not yet been definitively reconstructed and to objective problems related to the fact that that course was taught at a distance, effectively limiting the possibility of being able to act with students in the laboratory.
Possible solutions to the abstractness of mathematical logic and its algorithmic calculation are to be sought in an interdisciplinary approach that integrates the most significant aspects of the relationship between physics, mathematics and computer science into a coherent path. We can achieve this from the perspective of the previous section, with particular attention to the construction of the diagrammatic model from the classical case. And we have to do this with the main idea already expressed that the information encoded in a bit corresponds, from a physical point of view, to a vector representing the state of

a property, classical or quantum, of a given physical entity. Awareness of this dialectic between disciplines and a presentation of QP based on two-state systems should naturally lead to the question of what happens if we change the encoding of information, i.e. if we encode information with the state of a quantum system. The construction of the concept of the quantum state and its main properties within the theory becomes fundamental at this point. We know how many and what problems this element has. In fact the very concept of quantum state gives rise to learning difficulties in the context of photon polarization, where students show a reluctance to think about the polarization states of a photon as a two-state system ([5]) or interpret the state vector as the mathematical representation of a physical quantity ([176]). In the context of spin-half particles, instead, the main challenge is discriminating between entities of the lab space and of the state space, with students mixing-up features of both spaces ([5]). In order to promote a deeper understanding of the information encoding procedure, there is a need to help students overcome difficulties with the concept of state and state space. Students must therefore be supported to encode information in different physical properties, examining how they are linked to the vectors of the abstract two-dimensional Hilbert space according to the property at hand, with the aim to build a global knowledge structure on the relation between physical properties and state space in the encoding procedure [19].
The analyses made on the integrated perspective also remain the same for the applications such as quantum algorithms, teleportation and cryptography on which the literature concerning the conceptual understanding of students in curricular paths is almost totally absent. Applications can thus become a confirmation and test of the constructed model. Such a confirmation makes it clear what advantages such a conceptual change can bring from both a formal logical and experimental perspective.

7.2.3 Analysis from the teachers perspective

In our discussion, defining teachers' perspectives for an educational reconstruction of content is also useful. In Italy, mathematics and physics teachers can have a degree in mathematics or physics. In both cases, however, it is infrequent for them to know about the topics of the second quantum revolution. In this case, therefore, we refer specifically to the data collected in the first course of professional development.

pre-test most of the teachers teach both mathematics and physics; the knowledge required for students graduating from their institution on computer science is basic or less; almost all of them follow a quasi-historical reconstruction of quantum physics education regarding textbooks; none of the teachers involved had even sufficient knowledge of quantum computation and quantum information;

post-test the topics proposed were highly appreciated; they found it difficult to propose them in the classroom; they found a multidisciplinary pathway hardly feasible; in general they found the circuit approach very useful to show the dialectic between the disciplines involved;

interviews teachers either do not deal with logic at all or only address the topic of propositional logic; the link between logic and thermodynamics also caused difficulties for teachers; preference on the part of teachers for an approach with polarisation (because many do not know spin, and because it is a topic that is at least classically dealt with); they consider formalism suitable for students, but some pose the problem of an adequate physical interpretation; all recognise the high cultural value of the proposal, but emphasise the difficulty of implementing it in the curriculum in practice.

To conclude, the teachers found many interested stimuli; but equally critical seems to be the position regarding topics that are basically not addressed in the curriculum, and the risk that it remains too theoretical and difficult to interpret on a physical level.
From the even partial data reported, it seems evident that there is a need to build a polarisation-based TLS that can offer a concrete integrated and disciplinary perspective that is able to express not only the logic of quantum physics (expressed in algorithms or communication protocols), but also how it can be concretely implemented at least in ideal experimental setups, obviously based on optical devices.

7.2.4 Reconstructing content for secondary school

We have identified the most significant conceptual changes inherent in quantum computation and information theory and incorporated them into the Tab. 7.1 In light of these, we have researched several topics that we consider particularly illustrative that can be introduced at the high school level. Furthermore, we have analyzed the challenges that such a reconstruction requires from the perspective of the existing literature and data from our own experiences. We now show how these considerations and their mutual relationship guided our transformation of the science content structure on the physical theory of computation and quantum information into a content structure for education following the MER framework. This reformulation of content for education is thus at the heart of the student learning environment (and a large part of the second course for teacher professional development).

We, therefore, aim to build a synthesis between content and design requirements to achieve a successful educational reconstruction. To do this, in accordance with [236], we propose a reconstruction that takes into account both content and design characteristics. We, therefore, include in the table Tab. 7.2 the learning goals supplemented by some design hypotheses[9]:

TABLE 7.2: Learning goals in relation to content

Content	**Learning goals**
Introduction to QP	Introducing quantum physical quantity, measurement, state, vector, superposition, interference; develop the framework of the 'relations between properties', i.e., the rules that determine the acquisition, the loss and the retention of definite values of observables in the measurement process
Computational approach to problems: classical computation	Interpreting a problem and its solution from a logic-computational point of view. Linking logical to physical aspects (software to hardware).
From bit to qubit (1): one qubit computation	Introducing and developing quantum computation from the new perspectives characteristic of quantum systems: quantum state, superposition and phase. Using Dirac's vector formalism and its geometric interpretation for new single-qubit computation.
From wave model to Single-photon model for computation (1): encode information.	Describing the transition from the known wave model of polarization via Jones vectors and use it to build the polarization qubit.

(Continued on the next page)

[9]As already mentioned, we have replaced the expression dual-rail with spatial mode.

(Continued from the previous page)

Single-photon model for computation (2): preparation, transformation and measurement to encode, process and decode information. Optical circuit representation and ideal experimental setup.	Building the single-photon model of polarization-encoded computation.
Dual-rail model: preparation, transformation and measurement to encode, process and decode information. Optical circuit representation and ideal experimental setup.	Building the single-photon model based on the dual-rail in an interferometer.
From bit to qubit (2): two qubit computation.	Introducing and developing quantum computation from the new perspectives characteristic of quantum systems: quantum state, superposition, phase and entanglement. Using the Dirac vector formalism for new two-qubit computation. Differentiating separable states from entangled states.
Two-qubit computation: complete model. Logic and optical circuits.	Correctly solve logic circuits and, transformed into optical circuits, propose correct ideal experimental setups.
Deutsch Algorithm.	Using the quantum computational model to solve a particular problem. Using the model to understand quantum advantage by recognising which quantum properties determine it.
Grover Algorithm.	Using the quantum computational model to solve a particular problem. Using the model to understand quantum advantage by recognising which quantum properties determine it.
Teleportation Protocol.	Using the quantum computational model to solve a particular problem. Using the model to understand quantum advantage by recognising which quantum properties determine it.

TABLE 7.2: Learning goals in relation to content

The learning goals are intended to specify step by step better the general aim that we have set out and seen emerge from the national indications. The design hypotheses serve to guide the teaching and learning of the theory of quantum information physics:

DH1 Students can master mathematical formalism if supported by multiple representations (algebraic, geometric, diagrammatic).

DH2 Constantly explaining the relationship between classical and quantum elements helps to exceed the classical approach and grasp the quantum characteristics proper.

DH3 Students, if properly guided through specially designed materials, can construct the computational model using optical devices (half wave plates, phase shifters, beam splitters, polarising beam splitters).

DH4 The presentation of algorithms and protocols focused on a concrete problem to solve, engaging students and inviting even less competent students to comprehension.

DH5 References to a concrete problem in the algorithms and protocols enable the advantages of quantum computation to be grasped.

DH6 The diagrammatic model can appears to the students in its entirety.

7.2.5 Instruments and Methods

We refer to what has already been said in 5.6.

7.3 Learning path outline

In this section we will describe in deep the sequence of learning with respect to table 7.2. We will do this reasonably schematically to allow the reader to focus on the significant aspects following the design logic. Further, necessary comments will be added from time to time for the clarifications we think necessary. References to the theoretical and methodological framework will also be indicated but not overemphasised, as further analysis is deemed unnecessary, having already been carried out in Chapter 5. However, we will always try to clarify which aspects of the framework are present.

7.3.1 Introduction to QP

It is not the task of this thesis to describe a pathway that is already present in the literature ([143], [19]). We only briefly mention the main elements and bibliographical references[10]. To introduce quantum mechanics, we adopted and revised a teaching/learning sequence presented in Pospiech et al., section 4 [143]. The basic features of the quantum description and its mathematical representation in terms of ket vectors emerge in a modelling activity starting with an exploration of the interaction of macroscopic light beams with polarizing filters and calcite crystals followed by the discussion of related quantitative experiments and laws on a purely empirical basis. After examining evidence on the detection and polarization of single light quanta, the development of a photon model of the physical situation takes place within an idealized environment for thought experiments and computer simulations including sources of photons on demand in a known polarization mode, active filters, non-absorbing birefringent crystals, and ideal detectors. Students are led to revise basic terms of classical physics such as physical quantity, measurement, state, vector, superposition, interference (Fig. 7.6), for developing an understanding of their quantum counterparts [19].

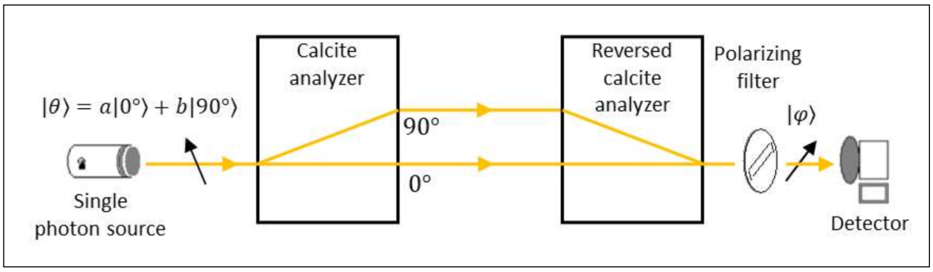


FIGURE 7.6: Interference (only ket vectors): the probability that a photon prepared in the polarization state $|\theta\rangle$ passes the filter with axis at ϕ, thus changing its state to $|\phi\rangle$ to be collected by the detector, is equal to the probability that the photon reaches the detector after passing through the crystals with only the 0° path open $(a|0°\rangle \cdot |\phi\rangle)^2$ plus the probability that it reaches the detector after passing through the crystals with only the 90° path open $(b|90°\rangle \cdot |\phi\rangle)^2$ plus an interference term: $2a(|0°\rangle \cdot |\phi\rangle)b(|90°\rangle \cdot |\phi\rangle)$, with $a, b \in \mathbb{R}$.

7.3.2 The computational approach to problems: classical computation

Consider the main features of this first part of TLS:

[10] For further consideration, we have already discussed this briefly in 6.3.3

Content Introduction to classical logic as a method to allow a computer to solve a problem.

Learning Goals Interpreting a problem and its solution from a logic-computational point of view. Beginning to link this logical approach to physical aspects of computation in the same circuit representation (software - hardware). Understanding the role of the database. Understanding the role of status in encoding information.

Changes Change of perspective. From the theory of computation as mathematical theory to the physical theory of computation.

Strategies We first adopt a strategy to leave the ingenuous relationship with the physical world; having done so, we adopt another to return to the real world in a more precise and rigorous manner:

1. strategy to progressively shift attention from an object to one of its properties and finally to the state that encodes the information related to that property, also in the classical case;
2. strategy to return from an apparently abstract concept of state (bit) to the preparation of the physical system in that state related to one of its properties.

Instruments We do not use any particular instrument in this first meeting.

Methods Inquiry-Based learning. Structured Inquiry.

Description In the first part of the lesson (*Engagement*), the teacher places a previously prepared coin on the desk and asks the students to define a procedure to determine whether the coin is genuine (two different images on the two sides) or counterfeit. The teacher supports the discussion by understanding that the procedure should be rough as follows: I observe the first side and note the image; I turn the coin over and note the image; if they are different, the coin is genuine; otherwise, it is counterfeit. After doing so, the teacher introduces a similar problem, but one that shifts the focus from the coin object to a database containing information about coins. Below there is the text of the *Coins problem* we constructed:

> *A mint has a machine that produces a unique series of coins, with one silver (A-side) and one gold (B-side) face and engraves on each face an image that can be a head (H) or cross (C). The machine has a software that, on output stores of each coin:*
>
> 1. *a number (e.g. coin 130 is expressed by the string (1000001) (if we start with all zeros for the number 1!!!));*

> 2. *a pair whose first element expresses the image printed on side A, the second on side B.*
>
> *If a coin is correctly made, the machine stores a pair of the type (H,C) or (C,H) at the output; if, on the other hand, there has been some manufacturing defect, a pair of the type (C,C) or (H,H). The mint needs to eliminate counterfeit (defective) coins and asks a programmer to create an algorithm that interrogates the available database to recognise the counterfeit coins that can then be eliminated.*

Presented and explained the problem, the teacher supports the students through the mathematization of the problem and its solution (*Exploration*). He continuously underlines the links to what has been done before to verify the coin's authenticity on the desk. In particular, he emphasises the role of the database and the function that carries the information. In the end, he presents a first classical diagrammatic form of solution as in Fig. 7.7:

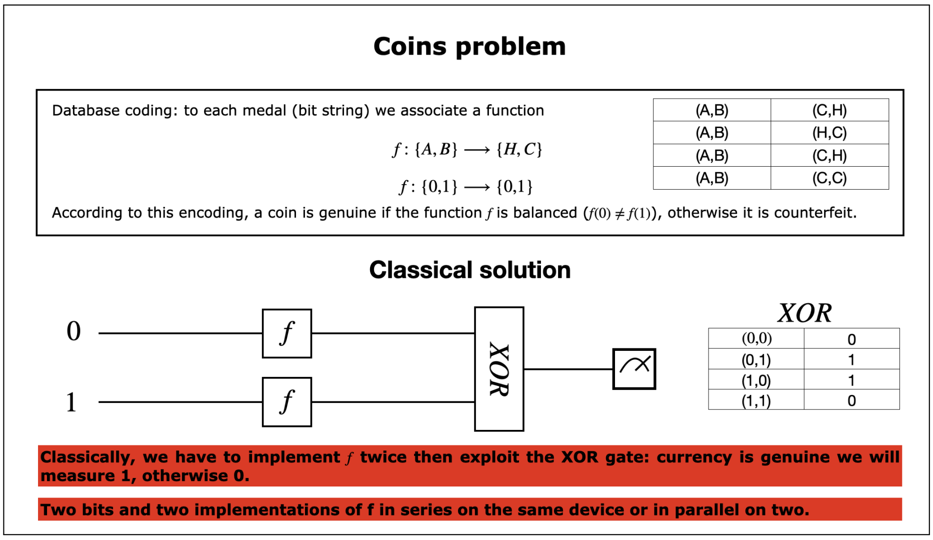


FIGURE 7.7: The circuit representation of the *coins problem* classical solution

After resuming the logic of solving the problem (Explanation), the teacher asks a question for the second part of the lesson, the part that will allow us to return, now more aware, to the physics of the systems required to implement the calculation: *What does it mean to solve a problem for a computing machine?*.

The teacher stimulates the students' answers and leads them to identify three key actions for a problem to be solved in machine language (second *Exploration*):

1. coding the information related to the problem;

2. process the information;
3. answer (yes - no) to the problem (Decoding).

The instructor uses the remainder of the lesson to support students in moving from an approach to computation from a mathematical, if not strictly Boolean (software), point of view to one closely linked to physically realisable implementations. At this point, it is crucial to show that a bit is not a particular classical two-valued state, but a class. This semantic shift is achieved by supporting the translation of actions and elements from software to hardware as shown in the following table[11]:

	Actions	Elements
Software	Coding, Processing, Decoding	Bit, States of bit, Gates, Readout
Hardware	Preparation, Transformation , Measurement	Systems, States, Physical Devices, Measurement Instruments

These considerations therefore become an opportunity to introduce a set of universal Boolean logic gates, beyond $CNOT$, including their circuit representation. What was previously developed is then revised by interpreting the circuits from both from a logical and a physical point of view (*explanation*) Fig. 7.8.
In the end (*Elaboration and Evaluation*), the teacher asks the students to read the logic circuit relating to the solution of the coin problem and interpret the computational approach to *coins problem* step by step from, both a logical and a physical point of view (concerning a possible calculating machine among those seen during the lesson: traditional computer, water computer, abacus). At the end of this first part, it should be natural to use a language that semantically takes into account both the logical and physical aspects: preparing, processing and measuring bits.

Conclusions The first part of our TLS is designed to be presented at the level of education *structured inquiry* and, although still at an introductory level, it takes us through the significant stages of Inquiry. The students come to a new awareness: when talking about calculation, information, or communication, it will always be necessary to reflect on the state of the physical system on which the related information is encoded. However, it will be necessary to abstract from the particular

[11] We also wanted to pose the aspect of measurement from a classical point of view, even though, in the deterministic case, at least, it is redundant.

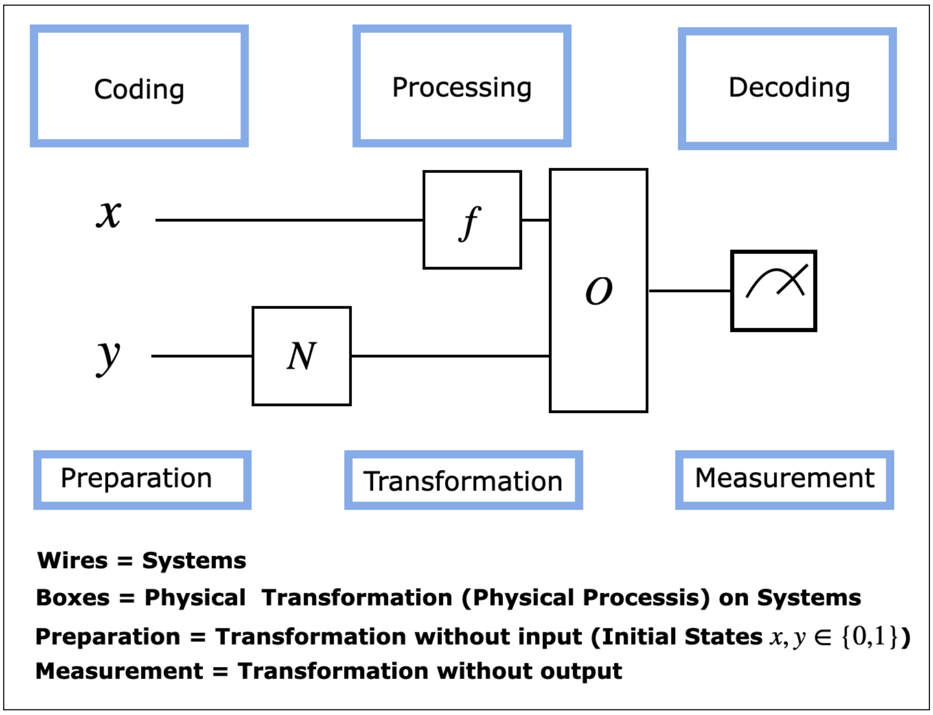


FIGURE 7.8: The circuit representation of classical computation: this circuit can be interpreted both from a logical and physical point of view.

physical system used, as it is at best only an expression of a particular implementation and not conceptually significant. The tool that embodies this need is the circuit representation. At this stage, it is an intuitive representation of the formal logical and physical processes in a computer at the software and hardware level.

Observation 7.1. By designing this TLS directly for the last year of high school, we are forced to introduce circuit representation and interpret it directly. But the work that some teachers are planning in their first classes, which we have discussed in Chapter 6 , allows us to realise how important a longitudinal perspective on these topics is for us.

7.3.3 From bit to qubit. From classical to quantum computation

Consider the main features of this second and fifth part of TLS[12]:

Content From *classical bit* to *quantum bit*: qubit, one-qubit and two-qubit quantum gates and quantum measurement for computation. Compound states: separable states and entangled states. Entangling gates. Circuit representations.

[12]We prefer to put them together for the sake of homogeneity, but in the TLS, two-qubit computation is introduced only after the single-qubit model has been fully developed.

Learning Goals Introducing and developing quantum physics from the new perspectives of quantum bit: quantum state, superposition, phase, entanglement. Using Dirac's vector formalism and its geometric interpretation for a new one- and two-qubit computation. Distinguishing separable from entangled states. Interpreting the results obtained, using circuits, from the point of view of the encoding, processing and decoding information.

Changes Change of perspective, change of physics, and change of the role of probability. The problem of compound quantum systems.

Strategies Make the classical construction of the physical theory of computation and, by conceptual and representational parallelism, build the quantum one:

1. the diagrammatic representation is the same (wires and boxes and double interpretation): states, logic gates and measurements will generally have to be redefined;
2. just as the bit (state of bit) is constructed from a classical state (an equivalent class) relative to classical physical quantities (electric potential state, water device state), the state of qubit is defined from the polarisation state;
3. new operators that process qubits are introduced on the basis of physical necessity;
4. decoding is affected by the quantum measurement problem.

Instruments We do not use any particular instrument in this first meeting.

Methods Inquiry-Based learning. Structured Inquiry. Exercises in small groups.

Description First, the teacher draws on the link between computation and classical physics presented in the first part of TLS. He then asks a question that is the key passage [13]:

Classical computation is based on the principles of classical physics. What happens if we change physics? What happens if the systems whose states we use to encode information have non-classical behaviour?

The discussion and the answer to the question (If physics changes, computation changes!) allow us to introduce a fundamental aspect: the diagrammatic representation introduced does not change. We will again use the formalism of quantum physics (see Fig. 7.9). The answer given by the students allows the teacher to

[13] He may also refer to Feynman's 1981 lecture to introduce culturally significant elements

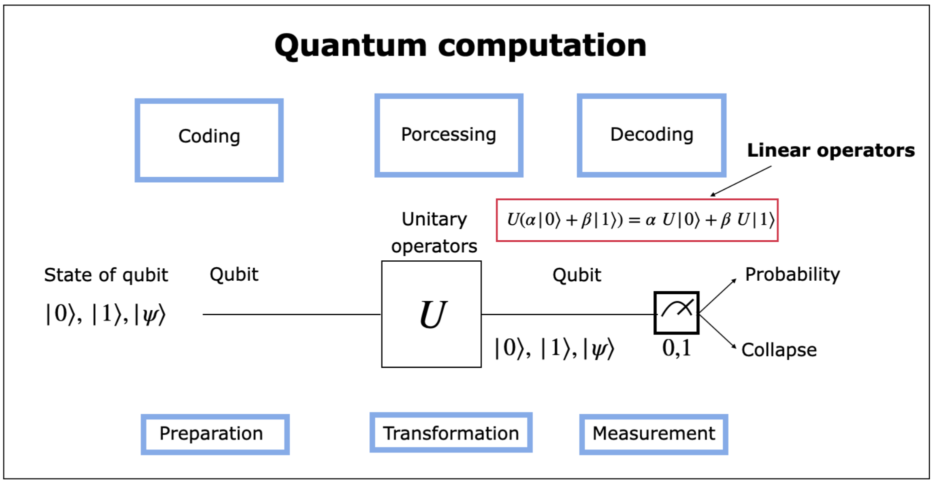


FIGURE 7.9: The circuit representation of quantum computation: this circuit can be interpreted both from a logical and physical point of view.

introduce the qubits of the computational basis linked to the corresponding polarisation states. However, we can speak of a generic qubit thanks to the superposition principle. The quantum logic gates are introduced from the physical characteristics seen in the introductory course: quantum superposition justifies the introduction of the Hadamard gate H; phase that of the Z. Using only some logic gates (X, H, Z) we can introduce superpositions states to real values and to give a clear geometric meaning to the qubits and logic gates: the former are seen as orthogonal vectors in a plane, the latter as axial symmetries (see Fig. 7.10).

LOGIC GATE	GEOMETRICAL INTERPRETATION: AXIAL SYMMETRY	TRUTH TABLE	CIRCUIT REPRESENTATION
I IDENTITY		$|0\rangle \mapsto |0\rangle$; $|1\rangle \mapsto |1\rangle$	
X NOT	$\alpha = \pi/4$	$|0\rangle \mapsto |1\rangle$; $|1\rangle \mapsto |0\rangle$	X
H HADAMARD	$\alpha = \pi/8$	$|0\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$; $|1\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$	H
Z FLIP	$\alpha = 0$	$|0\rangle \mapsto |0\rangle$; $|1\rangle \mapsto -|1\rangle$	Z

FIGURE 7.10: Summary table: one-qubit logic gates in geometric, algebraic and circuit description

Students, to the teacher's questions, spontaneously identify symmetry as a transformation associated with the proposed quantum operators. They also identify the specific symmetries for each logic gate. At the end of this first part of the one-qubit computation, the teacher assigned some logic circuits to be solved in small groups and then corrected them. As we will specify more in the section on exercises, it is very important that the teacher during group work and corrections constantly emphasises the reference to the language used. In particular, distinguishing between the logical algebraic approach, the geometric approach and the physical approach. As we will specify more in the exercise section, the teacher must constantly emphazise the reference to the language used during group work and corrections. In particular, distinguishing the logical algebraic approach from the geometric one, never forget the connection to the physics of the physical systems involved (polarization) (see Fig. 7.11).

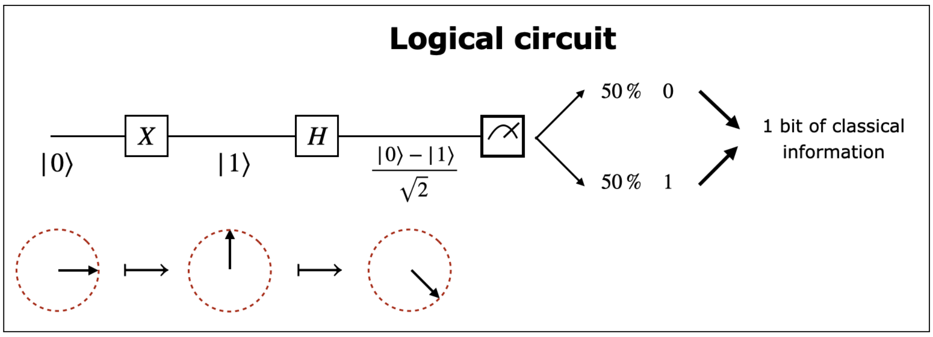


FIGURE 7.11: An example of single qubit logical circuit

The fifth part of TLS introduces multi-qubit computation.
The teacher anticipates that precisely in the multi-qubit composition lies the heart of the opportunities offered by quantum computation. This opportunity is due to the profound difference between the nature of classical and quantum compound systems. Students know from the classical case that a multi-bit computation (a computation that takes parallel composition into account) is based on the Cartesian product because n-tuples of bits are used. The link now established with physical systems allows the following question to be asked:

What kind of states do we get from compound systems related to multiple photons?

The state construction is taken by formal extension from single-qubit computation as in Fig. 7.12

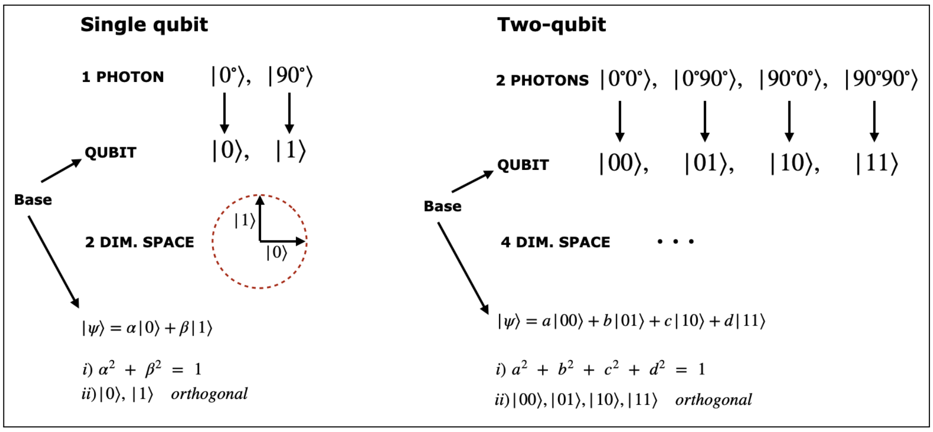


FIGURE 7.12: Introduction to two-qubit computation

The extension is not perfectly natural, especially since students cannot justify the orthonormality of vectors from a mathematical point of view. However, the classification of observables made in the introductory course allows us to explain the orthogonality of base qubits from a physical point of view: mutually exclusive states. The unitary norm condition creates no problems, given the established probabilistic interpretation. Once this generalisation has been achieved, the teacher must clearly express the fact that from this point on, what is realised from a informational point of view does not depend on the particular experimental realisation (i.e. it does not depend on the particular physical system under consideration: all two-state quantum systems will behave in this way).
However, it is also clear from Fig. 7.12. that the state of two 0°-polarised photons is introduced. The teacher follows a very strict line of reasoning here, which can be made explicit by the following question:

What means polarisation state of two photons polarised at 0°?

We can help students give two possible answers:

1. Alice and Bob are in two separate laboratories and separately prepare two photons with polarisation states $|0°\rangle$;
2. Alice and Bob are in the same laboratory and carry out a preparation emitting two photons whose compound system is a polarisation state $|0°0°\rangle$.

Reasoning classically, there is no difference between the two answers. Thanks to an integrated model between physics and algebra of state spaces, it is possible to verify the validity of the classical approach in a simple and clear way (see Fig. 7.13). With a simple circuit example, the characteristic aspect of separable states is emphasised:

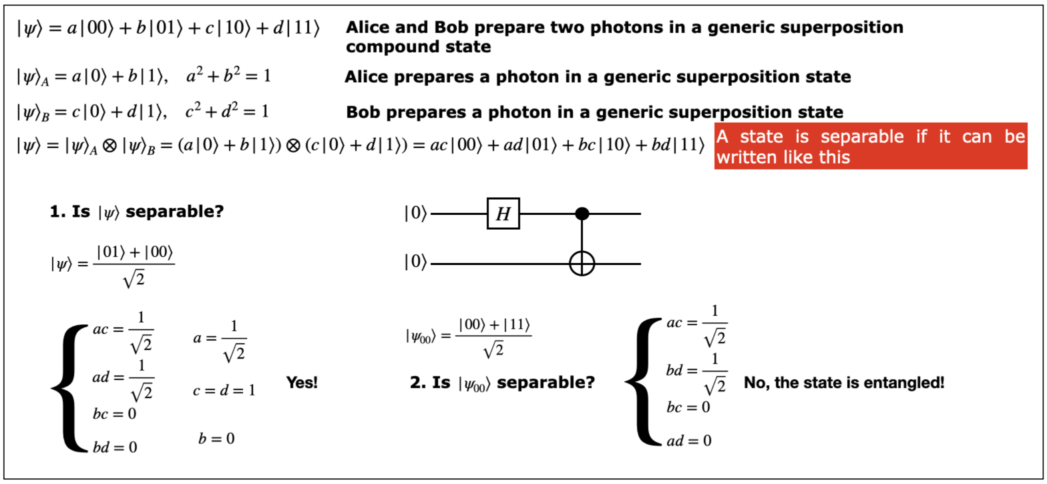


FIGURE 7.13: Introduction to compound systems: entangled states

the evolution of the compound system is the composition of the evolutions of the individual systems. This is also expressed formally with an algebraic identity that is its direct translation (see Fig. 7.14)

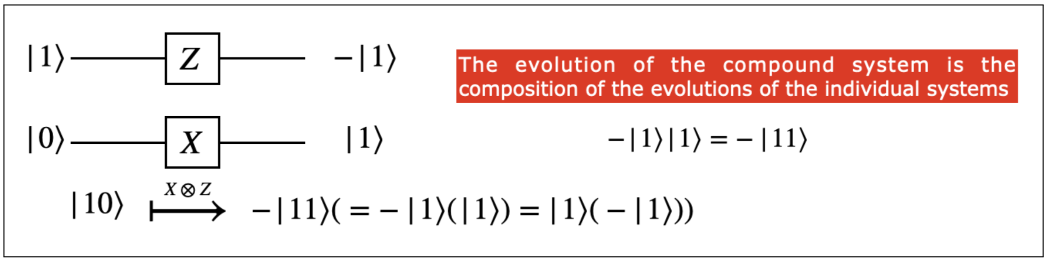


FIGURE 7.14: Evolution of separable states

Conclusions At the end of the construction of quantum computation, turning the discussion on its head, the teacher will have to explain that the informational approach is a way of describing physical processes in which one does not dwell on the particular process relating to the specific system. The qubit is an abstract state that, from an informational point of view, carries information about a probability distribution of the outcomes of class of experiments. But never mind the particular experiment performed or the specific physical system. Thanks to the reference to photon physics, the students were able to support the construction of the qubit concept and study its characteristics. Now, they are ready to abandon their particular implementation of them. However, in doing so, the educational risk lies in the purely syntactic learning of quantum computation. For this reason, we devise a concrete implementation of qubits and their logic gates and measurement in two properties of the photon: polarisation state and spatial state.

7.3.4 "Ideal "physical devices for quantum computation

From a formal point of view, the construction of the algebraic language is sufficient to describe quantum algorithms and protocols and to understand their significant aspects. However, we have introduced the circuit representation with its double interpretation to have another reference to support the students' understanding: the experimental element. How important this is for conceptual change has already been indicated in Chapter 5. Furthermore, following the description of the concept of model and modelling given in the same chapter, the third and fourth parts of the TLS are aimed precisely at the final construction of the quantum computational model. The worksheets can be found in Appendix B.

Content Polarization qubit. Dual-rail qubit. Quantum gate and circuit with one polarization qubit and one dual-rail qubit.

Learning goals Describe the transition from the known wave model of polarisation via Jones vectors and use it to construct the polarisation qubit. Construct the single-photon model of computation. Construct the single-photon model based on the paths in an interferometer. Correctly interpret logic circuits and, transforming them into optical circuits, propose correct ideal experimental setups.

Strategies Using the students' prior knowledge of plane electromagnetic waves to present phase shifting devices, which are the fundamental tools needed to build polarization-based logic gates by means of already familiar materials (i.e., birefringent crystals). Furthermore, we construct a reasonable simplification that allows us to introduce the notation with Jones vectors. The formal analogy with a generic polarisation state, and hence qubits, is discussed in detail. We focus in particular on the relation between physical properties and state vectors, and of the features of the corresponding spaces. We then use the optical devices described in the previous parts to have students construct an ideal physical model of computation. Similarly for the dual-rail computation.

Instruments Summary table and worksheets.

Methods Inquiry-Based learning: guided inquiry. Modelling-Based teaching.

Description At the beginnig, the goal must be set: to construct a computation based on two properties of the photon. These two properties will be the polarisation property and the spatial property (electric field modes).

Polarization: In the unit on quantum physics, light polarization has been examined in purely empirical terms, without any reference to the classical wave interpretation of the phenomenon. Now, in order to present phase shifting devices, which are the fundamental tools needed to build polarization-based logic gates by means of already familiar materials (i.e., birefringent crystals), we introduce the electromagnetic description of light in a simple form. For this purpose, we limit ourselves to plane electromagnetic waves, assuming to work only with waves of the same chosen frequency for the rest of the course. Since the direction of the linear polarization of light is identified by the electric field vector, we focus only on its mathematical expression. After recalling the concepts of global phase, of phase difference and its role in wave interference, we present students with linear isotropic dielectrics, i.e. for our purpose, phase shifting materials that do not change the direction of the polarization. In the course, we only work with real numbers. Therefore, the refractive index and the thickness of the material is chosen to obtain a phase shift of π (see Fig. 7.15).

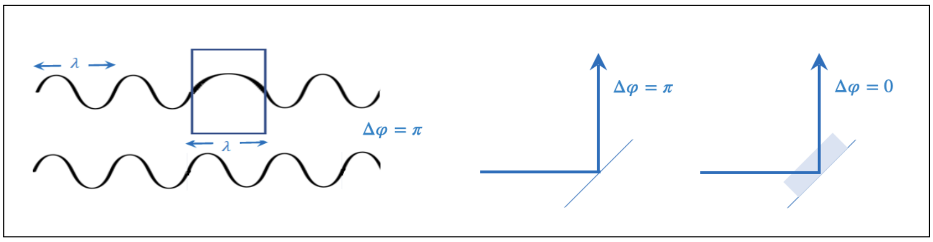


FIGURE 7.15: Representation of the phase shift, the mirror and the phase-free mirror

The following step is expressing the electric field vector in a form analogous to a polarization ket vector. Since we are interested only in the direction of linear polarization and the relative phase of the orthogonal components of the wave, we use a representation in terms of Jones vectors i.e. we omit the spatiotemporal elements from the cosine, normalize the amplitude of the vector and set the global phase to zero. For a field oscillating in an arbitrary direction, the result is a normalized Jones vector (see Fig. 7.16)

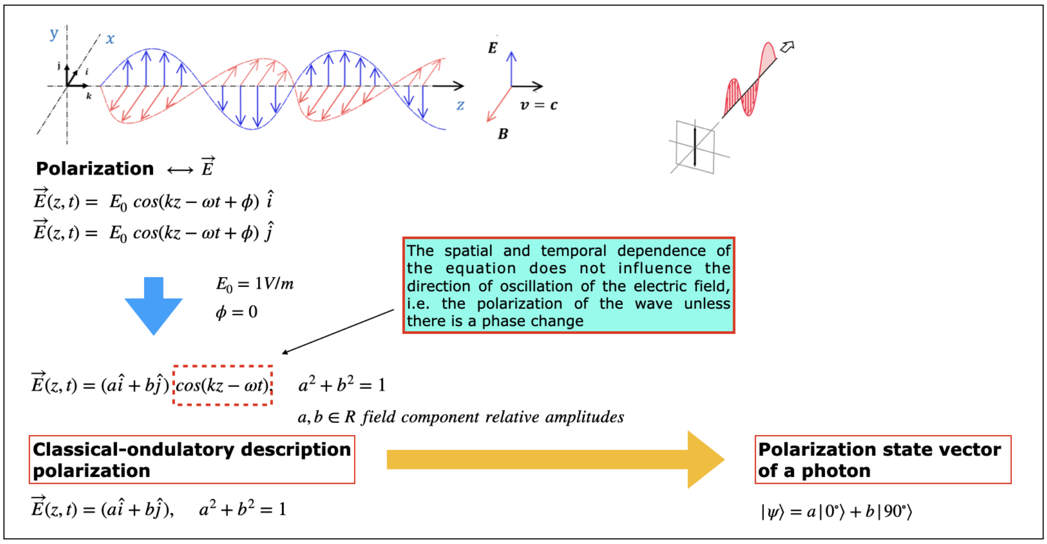


FIGURE 7.16: From the classical description of polarisation to the state vector of a photon.

Since we restrict us to linear polarization, the coefficients of the Jones vector are real; if the value of only one coefficient is negative, this corresponds to a phase difference of π between the two components. The mathematical expression is identical to that of a generic quantum state of linear polarization of a photon. However, their physical interpretation is hardly the same. Although we previously in the course (in the introductory unit on quantum mechanics,) stressed the differences between classical vectors representing physical quantities and quantum state vectors (as well as between their spaces), we seize the occasion offered by the mathematical identity between a classical Jones vector and the quantum vector of polarization for reinforcing this process of revision (see Tab. 7.3). The plane formed by all possible directions of the linear polarization of a classical light wave can be seen as a plane in ordinary physical space. The coefficients of the Jones vector describe the amplitudes of the components of a normalized physical quantity - the electric field vector - and their phase difference, while their square is proportional to the fraction of energy associated to each component. The state plane of the linear polarization of a photon is instead an abstract vector space. The squares of the coefficients represent the probabilities that, in a polarization measurement, the polarization of the photon is measured in one of two perpendicular directions. We also specify that only in the case of linear polarization the angles between two polarization vectors and the corresponding state vectors are the same (students will directly experience a different situation in the next unit on dual-rail encoding). In order to enhance student engagement in this task, the discussion has been converted into interpretive activities.

Table 7.3: Comparison of classical-ondulatory and quantum description

	Polarisation of the classical plane electromagnetic wave $\mathbf{E} = a\mathbf{i} + b\mathbf{j}$	Photon polarization $\|\psi\rangle = a\|0°\rangle + b\|90°\rangle$
Physical interpretation and unit of measure of the vector in the left-hand side of the equation		
Space to which vector belongs		
Interpretation of coefficients and their square		
Physical interpretation of the superposition sign		

After introducing polarisation coding, two aspects remain to be addressed for the model to be complete: evolution and measurement. However, at this point, all the conceptual instruments required to build logic gates acting on one polarization-encoded qubit are available. We accompany the students' construction with a worksheet and some oral questions to be asked by the teacher. The role of modelling-based teaching and inquiry-based learning express their full value here. The ideal physical implementation of the gates is quite simple. If we adopt the convention of encoding the horizontal state of polarization as $|0\rangle$ and the vertical one as $|1\rangle$, we only need to insert a phase shifter in the extraordinary ray of the two-crystal system already shown in Fig. 7.17 to obtain a Z logic gate

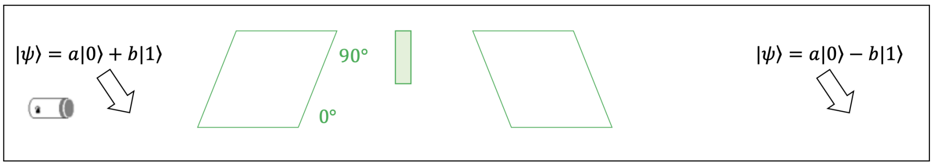


Figure 7.17: Ideal implementation of Z gate on a polarization qubit

Actually, this setup can be used for implementing an infinite number of gates. As a matter of fact, by rotating a birefringent crystal around its ordinary axis, we obtain a beam separation on different couples of perpendicular directions of polarization. It follows that every gate which can be described as an axial symmetry of the state plane is realizable in this way by rotating the system at an appropriate angle relative to the photon propagation line. In particular, if $\theta = 45°$, we obtain a X gate; if $\theta = 22,5°$, a $Hadamard$ gate. We now describe the pathway designed for the students in relation to the specially constructed worksheet:

1. The first question requires to determine the action of a phase shifter of π on the state vector of photons prepared in $|1\rangle$ and $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$. An explanation of the answer is also requested. The student should retrace the conceptual path from the classical polarisation to the qubit in the following way: a phase change of π of the classical wave is equivalent to affixing a minus sign to the field vector. Given the formal identity between the relative amplitudes of the electric field and the coefficients of the photon's polarization state vector, inserting a phase shifter π in the path of the photon is equivalent to affixing a minus sign to its state vector.

2. At this point, there is a question in which the students, in small groups, try to construct the Z port with the optical devices introduced (Fig. 7.18)

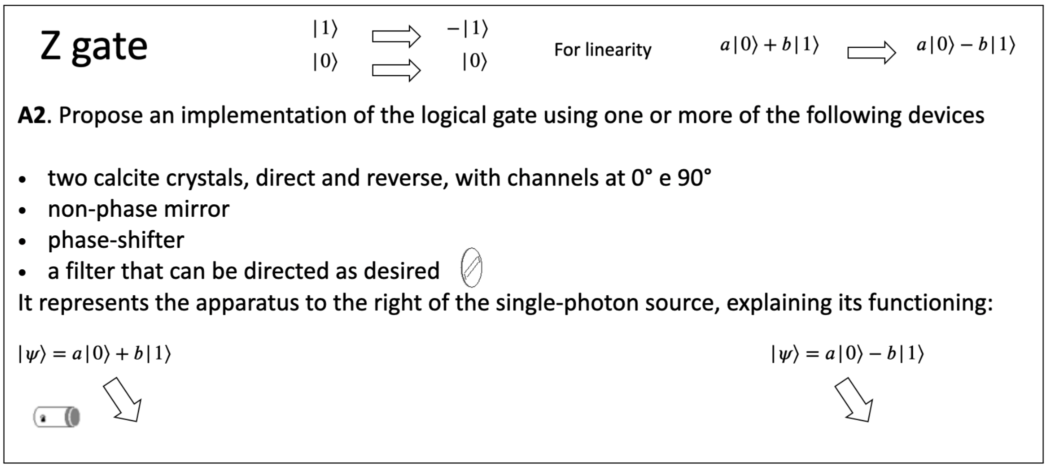


FIGURE 7.18: Worksheet question about the implementation of the Z-gate

3. The question is solved and commented on. The construction of the other logic gates follows from the correspondence between the geometric interpretation of the logic gates and the role of the ordinary propagation path. Students are asked which of the two paths corresponds to the axis of symmetry of the Z-gate. Identified the ordinary path as the physical correspondent of the symmetry axis, students are asked to design the X and H gates. Now, the teacher present students with half-wave plates, a more realistic device producing the same transformation which can also be interpreted as an axial symmetry around the slow axis. The circuit representation given is of a box indicating the angle

of rotation of half-wave plate. The red color identifies the fact that they are devices that act on the state of polarization.

4. Polarization measurements are performed by using an additional tool, that is already available to students since the introductory unit on quantum mechanics: the calcite crystal. However, in order to reconcile its visualization with that of a beam-splitter - which represents the basic device used for implementing logic gates with a dual-rail-encoded qubit - we slightly modify the setup by adding a mirror coated with a π phase shifting material. In this way, the two components of a classical light beam (or the entangled components of the photon state in the spatial and polarization modes) propagate in perpendicular directions. The setup of Fig. 7.19 plays a role analogous to that of a polarizing beam-splitter.

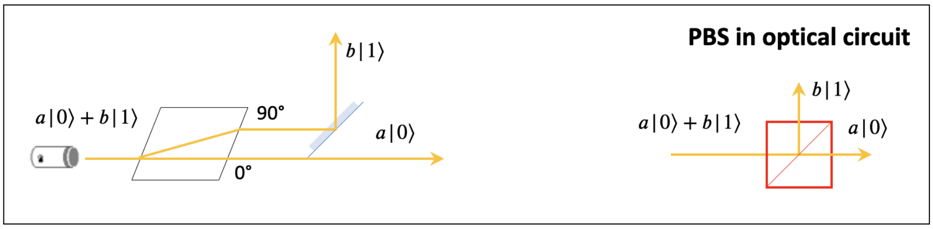


FIGURE 7.19: Idealized realization of a polarizing beam-splitter and its circuit representation

The computation realised in polarisation is finally expressed through its complete optical circuit representation (Fig. 7.20)

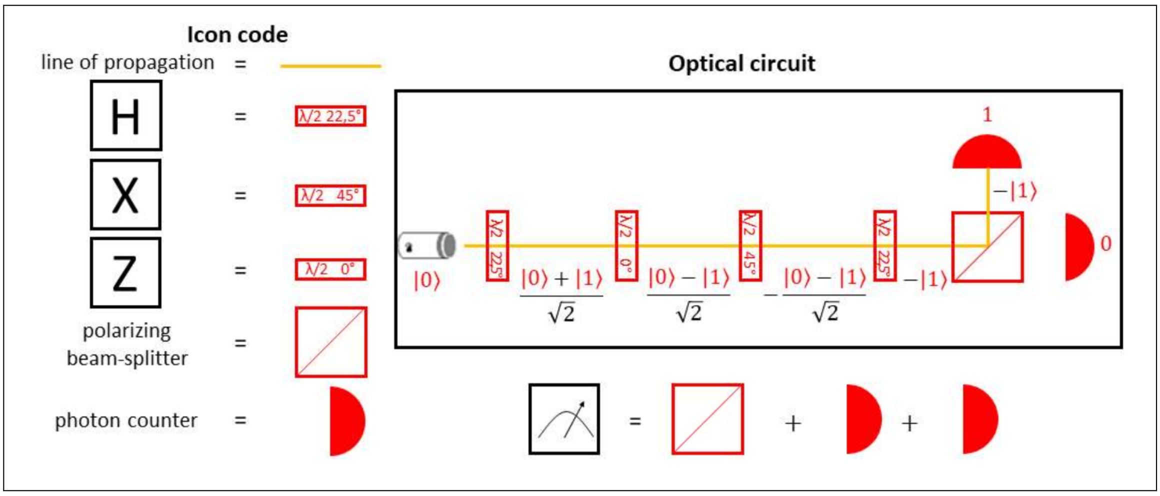


FIGURE 7.20: Idealized implementation of the circuit with one polarization encoded qubit on an optical bench. We denote as "optical circuit" the visual representation of the experimental design of a circuit diagram by encoding properties of one photon.

Dual-rail: The basic device that is needed to prepare a qubit and act as logic gate in a dual-rail encoding is a non-polarizing beam-splitter, that we present in an idealized simple form. We describe a cube beam splitter as made of two triangular glass prisms glued together, either applying a dielectric and a adhesive semi-reflective coating on one of their diagonal faces before gluing. One wishes:

1. incident rays from the adhesive side and incident rays on the dielectric side are reflected exclusively by it;
2. ensure alignment between incident and transmitted rays and phase relationship between emitting rays: same phase for the same path unless there are phase shifts in reflection.

Respectively, the beam-splitter is made so that:

1. internal anti-reflection coatings are added to the interface to prevent unwanted reflections;
2. adhesive, dielectric and coatings of marginal thickness in relation to the λ of the wave and/or materials with n, n_1, n_2 indices of close values.

The corresponding iconography used with the students follows (Fig. 7.21):

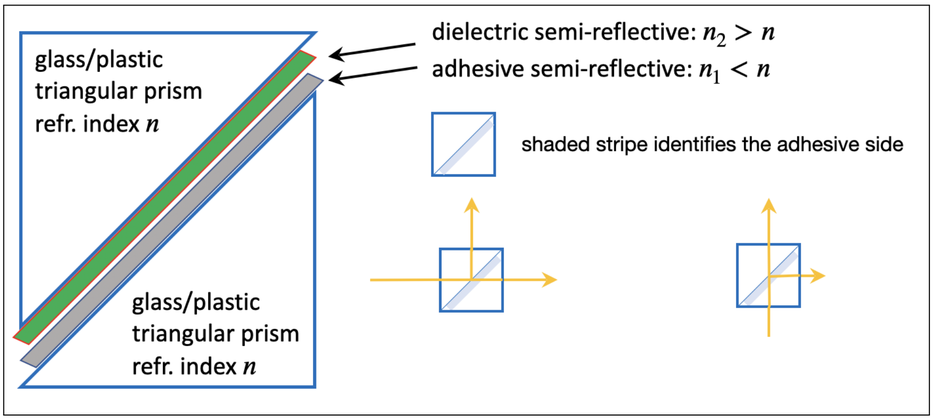


FIGURE 7.21: Cubic beam-splitter with dielectric and adhesive.

The encoding of the paths may be performed so that those two corresponding to a reflection without phase shift are labeled as 0 and the other two as 1. The beam-splitter can be rotated to invert the position of the two prisms and, as a result, the encoding of the paths. This flexibility will allow us to implement various logic circuits with generalized Mach-Zehnder setups without resorting to waveguides, only by choosing the orientation of the beam- splitters. In the color code, elements, devices and vectors pertaining to the dual-rail encoding will be colored in blue (see Fig. 7.22).

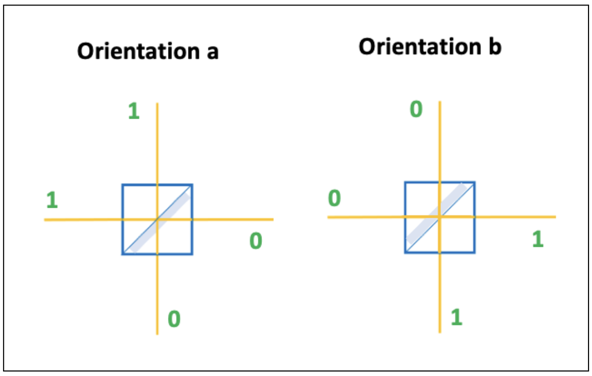


FIGURE 7.22: Beam-splitters: phase shift and encoding of the paths depending on the orientation of the prisms.

As a first step, students are asked to identify which side of the interface the glue is on and which side the dielectric is on and report on the corresponding ray whether or not reflection occurs (see Fig. 7.23).

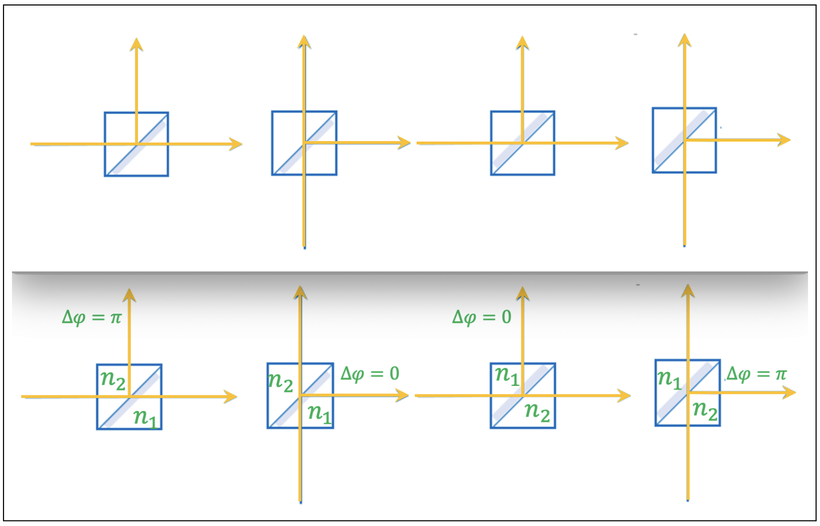


FIGURE 7.23: First question on dual-rail coding

The analysis of the action of a beam-splitter on a classical light beam starts with a 50:50 device (half of the light is transmitted, half reflected). Since we are interested only in the fraction of amplitude of the two outgoing beams and in their relative phases, we simplify the expression of the field vector similarly to what is done in the phasor representation of electromagnetic waves [237]: in such simplified representation we keep the amplitude of the field, normalized with respect to the amplitude of the input beam, and the relative phase between the two outgoing beams (which in our case can only be 0 or π). These quantities could be represented by the two components of a column vector whose rows correspond to the two different paths. Since in a 50:50 beam-splitter energy is equally divided between the two paths, we obtain the results shown in Fig. 7.24, on the left. Of course, after the beam-splitter the relative phase of the two components will, in general, depend also on their difference in path length, but since in the dual-rail case we are always considering beams whose optical paths have the same physical length, we ignore such consideration with students at this point.

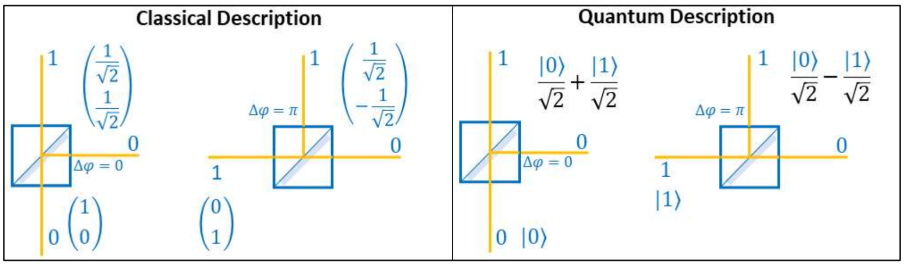


FIGURE 7.24: The action of a 50:50 beam-splitter: mathematical analogy between the classical and the quantum description.

As in the case of polarization, the transition to a quantum description involves a mathematical analogy and strong differences in the physical interpretation, which are explained to students, by leveraging the knowledge of quantum superposition. Again, students are asked to complete a special table (Tab. 7.4). The teacher provides a shared correction comparing the different student answers. The two-component phasor-like description of the field vector still represents a physical quantity propagating on the optical bench of the lab, and the coefficients of the two paths are the relative amplitudes of the outgoing beams. The dual-rail state of a photon, instead, is an abstract vector whose components describe the probability that, placing a photon counter on each path, the photon is collected on the path labeled as 0 or 1. One advantage however is that while in the case of polarization students may confuse polarization and state vectors, especially since the angle formed by two polarization vectors is equal to the one formed by the corresponding states, for the encoding of which-path information in the dual-rail, the choice of representing the classical amplitudes associated to different paths as orthogonal components of a vector is only due to mathematical convenience (it is a synthetic representation of two different vectors originating at different points in space), thus facilitating the differentiation between vectors representing physical quantities and state vectors. In

Table 7.4: Comparison of classical-ondulatory and quantum description of dual rail

	Description of the relative amplitude and phase of the field on the two paths (a, b)	Description of the spatial state of the photon on the two paths $\|\psi\rangle = a\|0\rangle + b\|1\rangle$
Interpretation of coefficients and their square		
Physical interpretation of the superposition sign		
Is the angle between the components fixed? If yes, specify its physical interpretation, if no, explain why		
Does it make sense to talk about superposition components? If yes, specify its meaning, if no, explain why		

the quantum case, the choice of $|0\rangle$ and $|1\rangle$ as orthogonal basis states becomes instead necessary, and can be motivated by the fact that they correspond to mutually exclusive outcomes of measurement.

In the dual-rail context, the construction of the qubit is not as immediate as in the case of polarization. As a matter of fact, identifying physical properties that can correspond to the states $|0\rangle$ and $|1\rangle$ is a necessary but not sufficient condition to encode information in a qubit. We must be capable of preparing arbitrary superpositions of the basis states on which devices implementing logic gates can act. In our unit on polarization-encoded qubits, the problem has been solved in advance, as we assume to use sources emitting photons on demand in an arbitrary state of polarization. Here, this trick is not available. The key to the solution is preparing quantum states by means of a custom-designed beam-splitter, with transmission and reflection coefficients chosen in accordance with the goals of the designer. In this case, the sign of the superposition can be established in two ways: either by choosing the incoming path (0 or 1), or by placing a phase shifter in one outgoing path. This part is constructed with a series of oral questions that students answer during the lesson.

The construction of a model is only complete if it is such that it can be used in other contexts. Thus if the polarization is a suitable context for promoting an understanding of fundamental features of the quantum picture of the world and an intuitive situation for the initial construction of a qubit, only the addition of a different context makes it possible for students to build a sophisticated understanding of the demands involved in a quantum encoding procedure and of the variety of relations that can be established between physical properties and state space in this process. On the one hand, therefore, we are defining the last degree of construction of the photon computation model. On the other, we are contributing to the formation of a more general model, which is the circuit model.

As the polarization, after introducing dual-rail coding, two aspects remain to be addressed for the model to be complete: evolution and measurement.
The Hadamard gate is immediately introduced by the points raised about the 50-50 beam-splitter. The Z-gate is introduced by the students in the form of a design question as seen in Fig. 7.25.

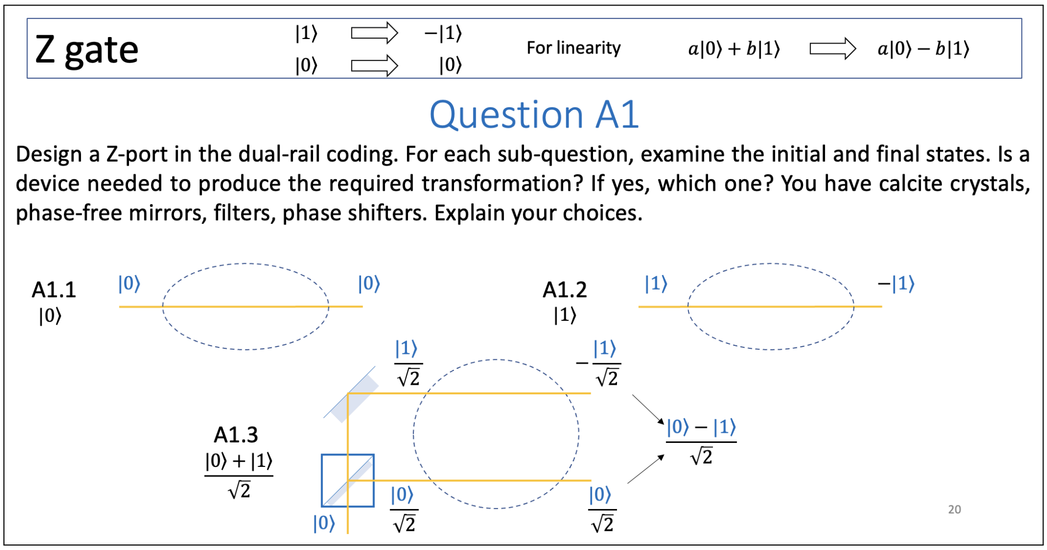


FIGURE 7.25: An example of question in worksheets: Z-gate in dual-rail

The X gate is conceptually more sophisticated: it corresponds to swapping the labels of the two paths from a certain point forward (Fig. 7.26). Since putting a label on a path is a conventional choice, its change is not associated *per sé* to any physical process. This apparent paradox is resolved by observing that the addition of a X gate affects the configuration of the devices placed beyond the gate. You may need to change the orientation of a beam-splitter or - as we will see in Grover's algorithm - the realization of a CX gate. Again, we have constructed a special question (Fig.

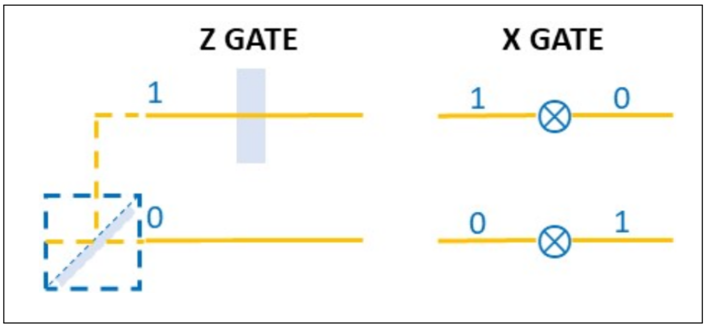


FIGURE 7.26: Idealized design of Z and X gates on a dual-rail-encoded qubit. In order to avoid a confusion between this representation and a two-qubit circuit - where the parallel lines stand for different registers -, in the first example we added the beam-splitter and two outgoing beams (dashed lines).

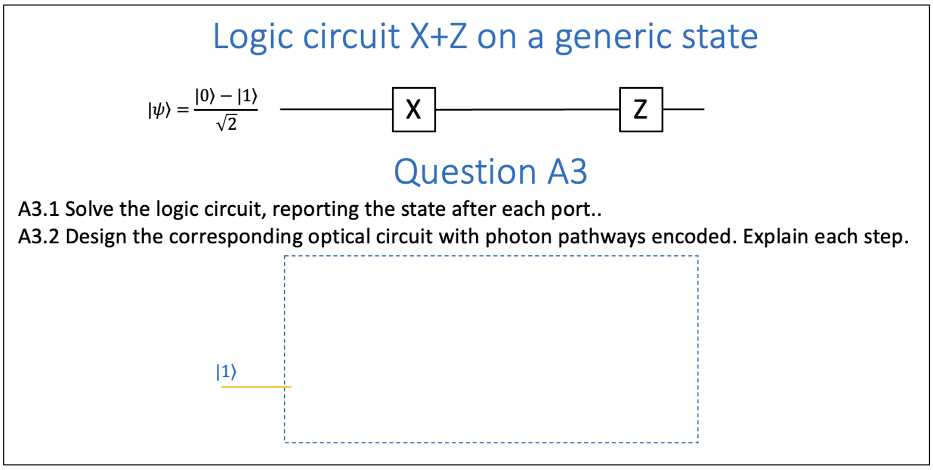


FIGURE 7.27: The use of X-gate in the circuits.

7.27): A circuit formed by two H gates and a measurement device corresponds to the basic setup of a Mach-Zehnder interferometer (Fig. 7.28): a source (omitted in the figure), two 50:50 beam-splitters, two mirrors with no phase shift and photon counters. As for polarization, students are asked to represent the implementation of single-qubit circuits in a dual-rail encoding. In accordance with current lines of research on quantum information processing based on linear optics [238], in the rest of the course we present students with generalized versions of the Mach-Zehnder interferometer as the setup for implementing algorithms and protocols.

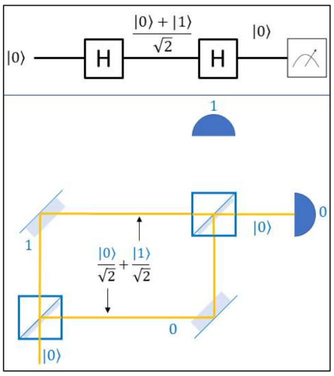


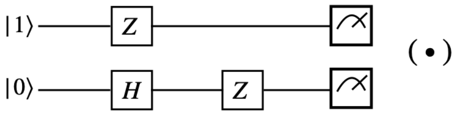
FIGURE 7.28: An example of logic circuit and the corresponding optical circuit using dual-rail encoding.

Conclusion The students' construction of a physical model of computation ends with these observations. Furthermore, this ends three of the four phases of the modelling we analyzed in Chapter 5. The circuit model now appears in almost all its strength as an epistemic artefact. The students built their model through a qualitative description where there was already dual theoretical and experimental value. Then they focused on the formal logical aspect and finally constructed an implementation based on ideal optical devices. At this point, we identified two characteristics that would effectively complete the building of the model: the first was the possibility of working on exercises that would allow the model to be defined in all its extension. These exercises focused the student's attention on transitioning from one interpretation of circuits to another: maps rather than objects. Above all, it was missing the concrete realization of an experimental setup that could also introduce those aspects of body language that are so important in learning and building a model. The first enables a meta-reflection on what the students have done so far. The second, with the work on algorithms and protocols, will allow a final evaluation of the model.

Exercises Consider as an example one of the proposed exercises[14] (Fig. 7.29) and give some indications on the aspects that need to be followed when conducting it in class with students. We present the worksheet designed for the students and teachers with comments. The teacher formulates the aim of the exercise: to use the model to make explicit the links between the computational and physical aspects. The first three items serve to take up the formal description: be it algebraic or geometric; at single separate qubits or considering the compound system (Fig. 7.30). One must

[14] For the complete materials, in Italian, see `https://drive.google.com/drive/u/0/folders/1Lg3_YmxUkTX2HvbI3PLx0995Dz6CZogo`

Consider the following circuit:



1. Develop the two registers individually in Dirac notation, explaining, when measuring, which classical bits are obtained on each register, and with what probability (Insert, above the arrow, the corresponding logic gate. In the case of measurement, put the corresponding icon).

   $|1\rangle \longmapsto$ ...... $\longmapsto$ .... , with p= ......; ...... with p= ......

   $|0\rangle \longmapsto$ ...... $\longmapsto$ ...... $\longmapsto$ .... , with p= ......, ... with p= ......

   Overall, at the end of the circuit we will obtain the classic bit pair(s)

   (,) with p= ......

   (,) with p= ......

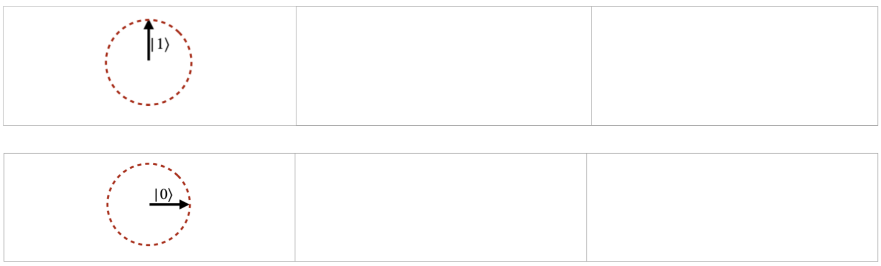
   Explain

   ____

   ____

FIGURE 7.29: Worksheet designed as an introduction to solving recapitulation exercises on the circuit model: first item

2. For each register, represent in the plan the state vector after each logic gate, explaining which transformation is performed.



3. Consider the compound system $|10\rangle$ and evolve it in Dirac notation along the circuit. Imagine we make a measurement on the final state: what classical bits would we obtain? With what probability? Explain how the results were obtained.

   $|10\rangle \xrightarrow{Z\otimes H}$ ... ... $\xrightarrow{I\otimes Z}$ ... ... $\longrightarrow$ (,) with p = ... e (,) with p=

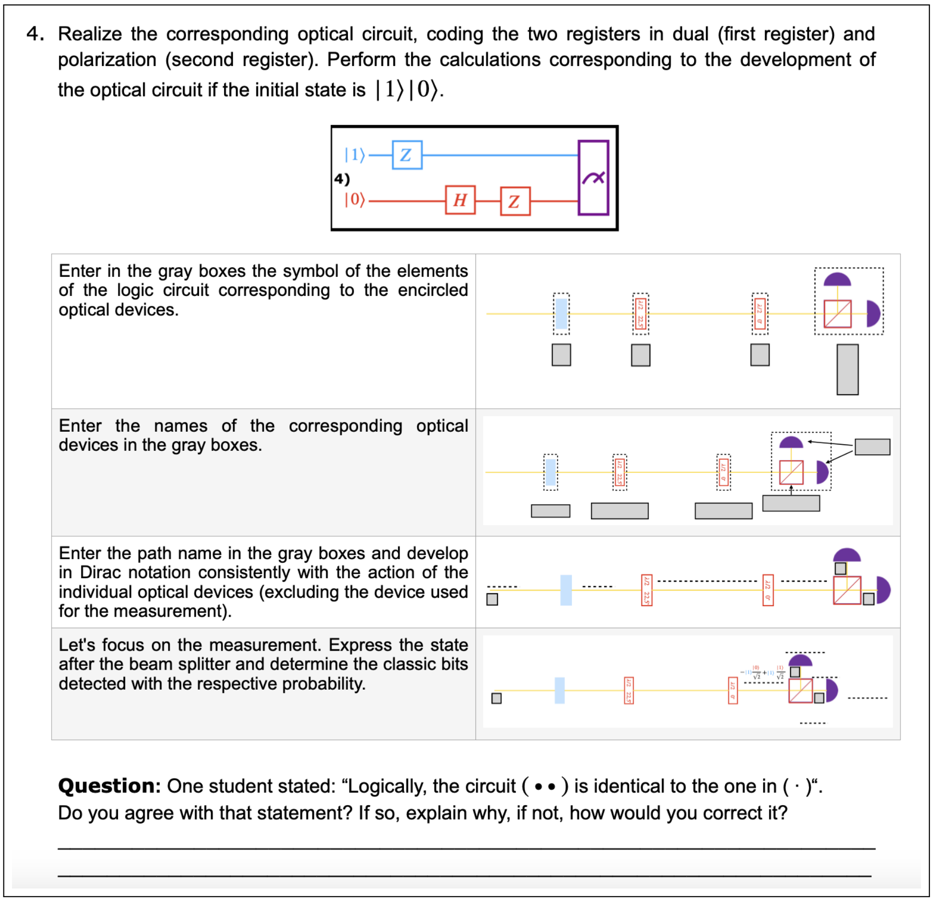
   ____

   ____

FIGURE 7.30: Worksheet designed as an introduction to solving recapitulation exercises on the circuit model: second and third items

be very careful in the third item to use the language of compound systems to distinguish it with the first. Moreover, it would be important first to use an exclusively mathematical language and then extend the semantics beyond algebra or geometry: prepare, transform and measure a qubit. The semantic extension allows the bridge to be built to the second part, which relates to optical circuits.

The second part is dedicated to optical circuits in which four possible implementations are analyzed that take into account the possible two encodings (first dual-rail register and second in polarization or vice versa) and the order of the logic gates[15]. Below are two images showing the difference between the setups in the case of different order and coding choices. The items are the same, proposed in the work sheet in all cases analyzed (Fig. 7.31 and Fig. 7.32).


4. Realize the corresponding optical circuit, coding the two registers in dual (first register) and polarization (second register). Perform the calculations corresponding to the development of the optical circuit if the initial state is $|1\rangle|0\rangle$.

Enter in the gray boxes the symbol of the elements of the logic circuit corresponding to the encircled optical devices.	
Enter the names of the corresponding optical devices in the gray boxes.	
Enter the path name in the gray boxes and develop in Dirac notation consistently with the action of the individual optical devices (excluding the device used for the measurement).	
Let's focus on the measurement. Express the state after the beam splitter and determine the classic bits detected with the respective probability.	

Question: One student stated: "Logically, the circuit (• •) is identical to the one in (·)". Do you agree with that statement? If so, explain why, if not, how would you correct it?

FIGURE 7.31: Worksheet designed as an introduction to solving recapitulation exercises on the circuit model: optical circuits

[15] This is an extremely interesting aspect that the teacher must point out: from a logical-formal point of view, the order does not matter; when we interpret the circuit as a possible experimental setup, even if ideal, the order of implementation can change and even a lot the setup.

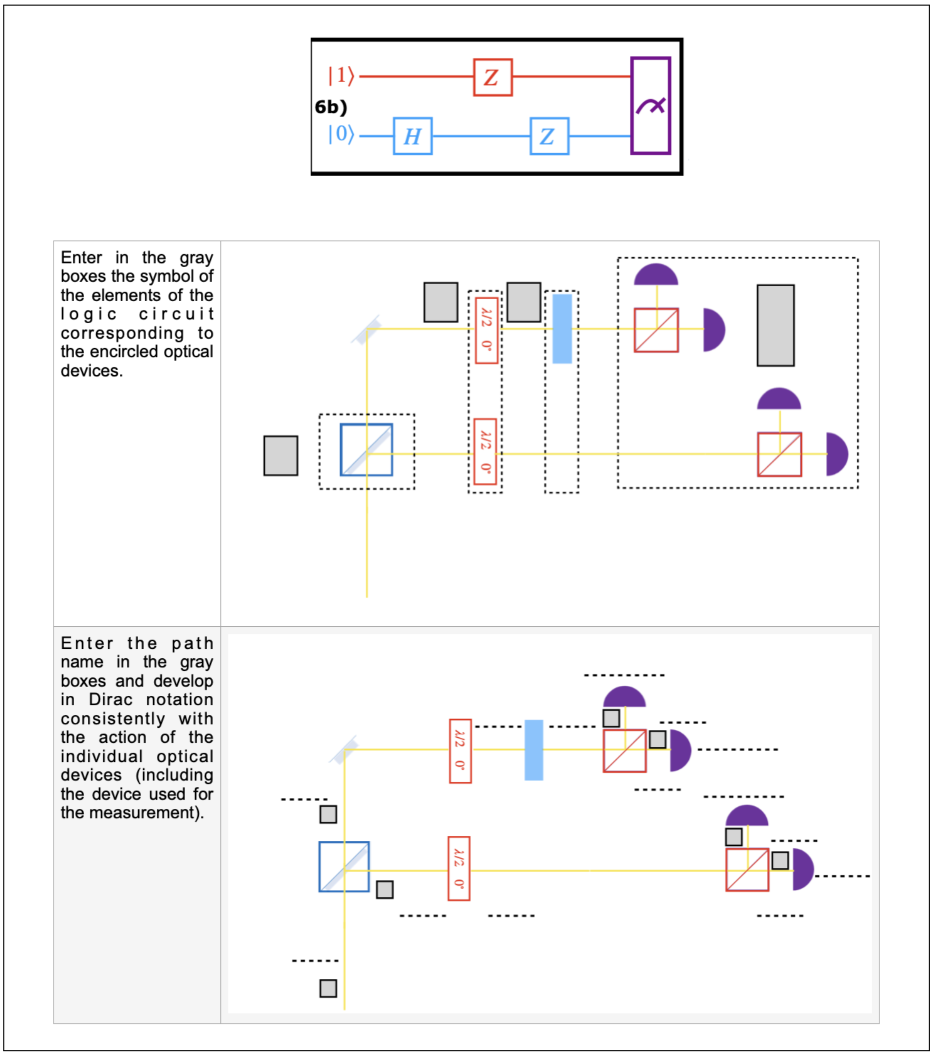


FIGURE 7.32: Worksheet designed as an introduction to solving recapitulation exercises on the circuit model: optical circuits with different encoding.

The guided exercise greatly simplifies the task of students who have the opportunity, guided by the teacher, to capture all the most important aspects of the two codes before. Furthermore, developing algebraic calculus parallel to the appearance of optical devices is fundamental for the work that students will have to carry out in the algorithms and protocol of teleportation: connecting the algebra of Hilbert spaces, information processing and physical characteristics underlying that special transformation. The exercises proposed are the occasion for a final comparison between the implementation of separable logic gates and non-separable gates (see Fig. 7.33 and Fig. 7.34)

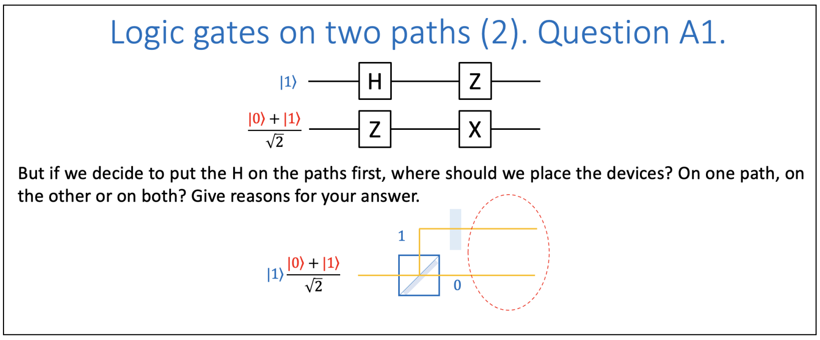


FIGURE 7.33: Oral questions on devices in the case of polarization separable gates in an interferometer

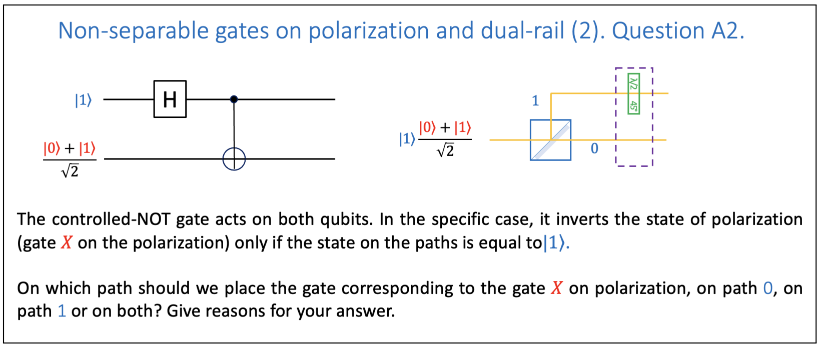


FIGURE 7.34: Oral questions on devices in the case of polarization non-separable gates in an interferometer. The correct answer is present in the circuit on the right.

7.3.5 Quantum algorithms

Students should at this point be able to approach the study of quantum algorithms from a formal, conceptual point of view and in relation to an ideal physical implementation. We propose a reconstruction that focuses on a concrete problem to be solved. Then an elementarization of information processing and finally a physical implementation with ideal devices (and more!). As before, we guide the students with specially designed worksheets (see Appendix B). Given the complexity of the topic, we have constructed the worksheet for the students following a detailed design sheet that we comment on.

Content Quantum algorithms: Deutsch and Grover algorithm.

Learning goals Using the quantum computational model to solve a particular problem. Use the model to understand quantum advantage by recognizing which own quantum properties determine it. Completing the cycle of circuit model building.

Strategies First, the coin problem already described in the introduction to classical computation is reintroduced. Students construct their understanding of the algorithm by means of an elementarization scheme described by three processes: enabling of parallelism, transfer of the whole information to the target and enabling of interference.

Instruments Highly structured worksheet

Methods Inquiry-Based learning: guided inquiry.

Description For the work sheet in its entirety, please refer to the Appendix B. Following are the main aspects of the design sheet with reference to the proposed questions and the development of the algorithm.

Items A and B: quantum parallelism The concept of quantum parallelism is introduced with questions A1 and A2 to one qubit and then extending the considerations to two or more qubits. Furthermore, it is shown to the students that the linearity of quantum operators allows simultaneous process over all states of the computational basis (B1).

Learning goals Deducing the possibility of implementing f in parallel on all base states due to the superposition principle and linearity of operators. Knowing how to interpret domain coding (side A and side B) in the case of coins

Prerequisites Understanding classical (bits) and quantum (qubits) coding, classical logic gates (Boolean functions) and quantum (linear operators); superposition principle; coin problem: classical coding; algorithm circuit visualisation.

Activities and Questions The teacher again introduces the coin problem by repeating the classical solution. He then introduces the Deutsch algorithm circuit, anticipating that the properties of quantum computation will allow the problem to be solved more efficiently.

First, quantum parallelism is introduced from a formal point of view. Then students are asked to interpret it in relation to the coin problem by focusing on the encoding of a piece of information (in this case the coin side). To finish this first step, question B1 introduces a thought that it is only the co-presence of the superposition principle and the linearity of operators that makes it possible to benefit fully from coding on quantum states. The generalisation to the case of multiple qubits is needed both for the extension of the algorithm to the Deutsch-Jozsa algorithm and for the encoding necessary for the Grover algorithm.

Item C: oracle and compound systems To encode the output information exclusively on the target register, associating each side of the coin (silver and gold) with the corresponding image (heads or tails) we use the sign: + heads, - tails. To do this we introduce an auxiliary register (qubit ancilla) and define the oracle. We support students in understanding the coding on signs and states through a special case. It is then up to the teacher to derive the general case.

Learning goals Recognise the operator U as CNOT in the particular case examined; recognise the property of compound quantum systems (or, in terms of formal representation, the tensor product) that allows the transfer of information from the ancilla to the target: the possibility of moving a coefficient (-1) from one component state to another; interpreting codomain coding (Head and Tails) on the first register in the case of the coin; predicting information from a hypothetical measurement.

Prerequisites Classical and quantum compound systems (Cartesian product and tensor product); CNOT truth table (classical and quantum); f Boolean function of the coin problem; algorithm circuit visualisation.

Activities and Questions The teacher explains the need to introduce an auxiliary qubit and describes the behaviour of the oracle. The students are then asked to determine the behaviour of the oracle in a particular case (C1). It then becomes clear that the whole information encoded in the Boolean function is transferred from the ancilla to the target in the form of a positive or negative sign attached to each basis vector. Questions C2 and C3 are used to recognize the properties of compound systems exploited here. Question C4 relates the considerations made to the concrete problem posed and thus to the information process. The teacher then shows the students that the oracle's output state includes all the coin information. However, we are not yet able to use it (C5). The impossibility of reconstructing the state through measurement and the role of probability are emphasised.

Items D and E: computational interference Before the third part, the teacher must perform the calculations necessary to express the generic output state of the oracle in a compact form: $\frac{1}{\sqrt{2}}[(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle]$. This shows that the sign depends on the action of f. The last part describes the third process peculiar to the quantum algorithms presented: enabling of interference.

Learning goals Understanding the role of interference in the general case.

Prerequisites Quantum computation

Activities and Questions After generalising the output state of the oracle, students must compile a table (D1) expressing the state after the last Hadamard gate and the corresponding classical information bit. In this way they have all the information to interpret the outcome of the measurement according to the coin problem (D2). The last question (E1), asks students to reflect on the advantage of the quantum solution to the problem. The final part (F1 and F2) asks the students to interpret the ideal experimental setup in order to realise the algorithm according to the logical development: the circuit language is finally a complete model able to express the deep links between all languages introduced in the TLS: logical-algebraic, informational and physical. The quantum advantage is then described in the case of the generalised Deutsch-Jozsa algorithm.

Conclusions The developed worksheet is designed to follow the hypothetical learning trajectory in all its parts. Students study Deutsch's algorithm through an elementarization that divides information processing into three fundamental processes: enabling of parallelism, transfer of the complete information to the target and enabling of interference. Understanding this first algorithm should allow them to approach Grover's algorithm with the proper awareness. In addition, the construction of the circuit model is almost final: what is still missing is an actual experimental realisation and comparison with what has been done from the perspective of logic and optical circuits. Below is a picture of a purpose-built experimental setup for the Summer School organised in September 2022 (Fig. 7.35) and the relationship with the logic and optical circuit[16] (Fig. 7.36).

[16] For further details see paper 3

FIGURE 7.35: Experimental setup for the implementation of Deutsch's algorithm

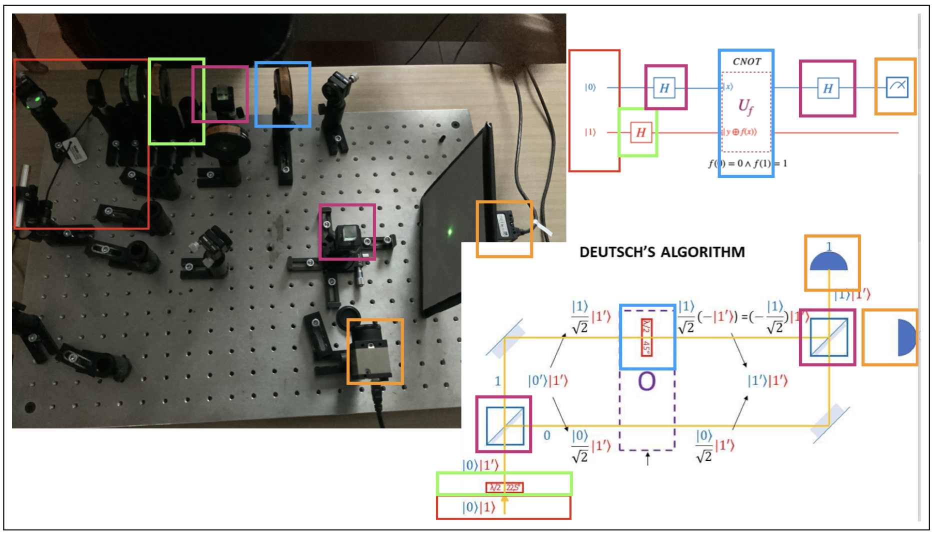


FIGURE 7.36: Circuit model: the dialectic between theory and experiment is identified by the coloured boxes. Students can finally gain an integrated perspective on the quantum computing problem.

7.3.5.1 Grover's algorithm

The lesson on Grover's algorithm was designed in a very similar way to that for Deutsch's algorithm. We only propose the problem to the students here, and refer to Appendix B for the worksheet. We will only make a few additional remarks.

Bank robbery Inspired by "*Q is for Quantum*" of T. Rudolph[17], we propose to the students a bank robbery based on the idea of protecting the *caveau* with a quantum computer:

> "*A famous bank has just equipped itself with a new control system that uses quantum computers to check the authenticity of gold bars in the national reserve. The new control system is very expensive, but the board of directors justified the purchase by saying that this way it is possible to carry out checks on the vaults at a speed and accuracy that cannot be compared to other systems.*
> *Moreover, a surveillance microchip has been inserted into some bars to deter possible thieves. For additional security, the insertion of the national reserve bars into the caveau is fully computerised thanks to state-of-the-art quantum computers that insert the bars by exploiting the randomness of random number generators (non-first numbers bars without microchip coding 0, first numbers with microchip coding 1). Anyone stealing a microchipped bars would be traceable within a short time.*
>
> *What appeared to be an unbreakable system soon turns out to have a serious flaw: the gang boss is in fact an expert in quantum computing and knows how to exploit the bank's quantum computers to gain access (remotely) in a short time to the information needed to avoid stealing the microchipped bars and being traced.*"

Classically, we have to implement f on average $2^n/2$ times to determine whether or not a bar has the microchip. With the students, at first, we deal with the case where there are four bars and the third is the one with the microchip (this is only for clarity in the calculations).

The line of development is similar to that of Deutch's algorithm as can be understood from the worksheet in the Appendix B.

Remark 7.2. As we can see, we gave more weight to the implementation part with optical devices. This was for two reasons: on the one hand, to allow the students to think more about the design of an ideal experimental setup; on the other hand, it was possible to

[17] `https://www.qisforquantum.org`

gather further data on the students' actual ability to design optical setups implement specific sequences of logic gates.

7.3.6 Quantum teleportation

Students should at this point be able to approach the study of quantum teleportation protocol from a formal, conceptual point of view and in relation to an ideal physical implementation. As before, we propose an elementarization to grasp the role of the entangled qubits shared by Alice and Bob. The designed worksheet can be found in Appendix B.

Content Teleportation protocol.

Learning goals Using the quantum computational model to solve a particular problem. Use the model to understand quantum advantage by recognizing which own quantum properties determine it. Completing the cycle of circuit model building.

Strategies We begin the explanation with a preparatory activity in case Alice does not share entangled photons. By comparing the protocol with this first part, we support the students in reflecting on the role of the entangled photon pair shared by Alice and Bob. Given the difficulty of the calculation, we support the entire explanation with the help of optical devices.

Instruments Highly structured worksheet

Methods Inquiry-Based learning: guided inquiry

Description First, it is possible to introduce the problem from a narrative point of view, e.g. with a spy-story similar to what they do in [22]. Then by means of the worksheet he develops the teleportation protocol (see Appendix B).

Item A: Preparatory activity: Charlie and Alice The first questions introduce the problem where Alice's qubit is not entangled with Bob's qubit. Students have the opportunity to carry out calculations, make measurements and implement an experimental setup in the simplest case of separate states. The role of the specific questions becomes clear with the development of the next part.

Learning goals Solve the circuit with respect to a generic state and recognise whether the final state is separable or entangled. Construct the respective optical circuit.

Prerequisites Separable and entangled states. Preparation of general state in dual-rail (generic beam-splitter device).

Activities and Questions The teacher presents the preparatory circuit and poses the first question A1. The students develop the calculation and the teacher corrects all the steps. Once this is done, the students answer questions A2 and A3 in order to identify the state as separable and obtain the bits of classical information obtained by measuring. Then, they design the corresponding optical circuit.

Item B: Alice and Bob share an entangled state The teacher takes up the protocol circuit and makes it clear that Alice and Bob share two entangled qubits. The questions allow students to understand the role of this entangled state in relation to the previous case.

Learning goals Understand what is the effect of Alice and Bob sharing two entangled qubits.

Prerequisites Compound systems: separable and entangled states.

Activities and Questions Questions B1 and B2 allow through formal parallelism with the previous part to show that this time the qubit states shared between Alice and Charlie are entangled. This consideration opens up question B2' which shows the weirder side of the protocol as well described by J.Preskill in [29]

> *"Initially, Bob's qubit is completely unentangled with the unknown qubit C , but Alice's Bell measurement establishes a correlation between A and C"*

Items C and D: measurement and error correction The last part of the protocol shows how by knowing the outcome of the measurements Bob can reconstruct the initial state owned by Charlie. This part is left almost entirely for the students to carry out and only corrected at the end by the teacher.

Learning goals Understanding that what is actually being teleported is information and not the system on whose state that information is encoded. Understand that the laws of Einsteinian relativity are not violated.

Prerequisites Role of measurement in quantum computation and state collapse.

Activities and Questions The teacher, at his or her discretion, proposes the calculation required to obtain the state before measurement on the first two registers. In each case the state is given to the students and the tables in C1 and D1 allow the students to reconstruct Charlie's initial state. A question D2 is then posed that is of deep importance from a conceptual and historical perspective. In fact, there have been quite a few testimonies to an interpretation of the protocol as synonymous with superluminal communication.

Finally, the ideal setup, which encodes Charlie and Bob's qubit on two different properties of two different systems, is aimed at making it easier to answer the E2 question and understand how it is that information is not teleported.

Conclusions The developed worksheet is designed to follow the hypothetical learning trajectory in all its parts. Two main aspects characterize this work: elementarization and the interpretive support provided by optical circuits. Thanks to the former in fact, we try to highlight the role of the entangled pair of qubits shared by Alice and Bob in giving rise to the further entanglement between Alice's qubit and Charlie's qubit. The second should clarify without doubt that it is information and not a physical system that is being teleported.

7.4 The educational experiments: context, data and results

7.4.1 Data analysis

We carried out a qualitative analysis of the worksheets proposed during the TLS. We divided the answers into correct, deviant (incorrect) and blank or insignificant. A detailed analysis of the explanations follows this. The analysis is carried out for the dual purpose of assessing the impact on student learning and evaluating any changes needed to improve the results. What we report here are the possible lines of thought of the students and the most significant elements that emerged from studying the answers given by the students. We use the answers of the summer school students only to support particular lines of interpretation on deviations from the hypothetical learning trajectory. In Appendix A we have included the worksheets (except for **S1** and **S4**[18]) labelled with **S**.

S1 (CSG): Comparison of classical-ondulatory and quantum description

This sheet (see 7.3) was assigned at home and only 9 students did the homework. The first interesting aspect that arises seems to be the overlap between the mathematical plane and the physical plane of the laboratory. No one refers to the plane of the physical space of the laboratory but speaks generically of the real plane, "*the real Cartesian plane*". It is possible, but not certain, that the word *real* refers to the fact that it is not abstract as in the quantum case[19]. Only one student answers correctly from the quantum point of view. For three others, there is no difference between the two cases.

There are many incomplete answers regarding the physical interpretation of the coefficients and their squares. In the classical case, the role of the coefficients is identified, but there is seldom any reference to the energy in relation to the squares. In the quantum case, conversely, squares are correctly interpreted as probabilities, but there is never any reference to probability amplitudes.

Regarding the physical interpretation of the superposition sign, three students identified the change of direction of the electric field. Three spoke generically of phase change without specifying whether global or local phase. It is unclear whether this last answer is due to the clarity of the question. No one refers to polarization, but only to electric field considerations. Finally, one student answers by remaining linguistically in the strictly mathematical sphere and does not physically interpret the

[18]**S1** and **S4** are, respectively, the tables 7.3 and 7.4 and are not included in the appendix.

[19]The words real and Cartesian form an oxymoron, the words real and Cartesian form an oxymoron, unless one means *real* because it is formed by the straight lines of real numbers

considerations made. Concerning the change of sign of the superposition in the state vector, four students answered as fallows: "*with the change of sign the result does not change but the process to reach the result changes*" although it is not explained what is meant by a *process*, it seems that students grasp the point that the state is different but upon measurement provides the same probability distribution.

S1 (Fi): Comparison of classical-ondulatory and quantum description

This sheet (see 7.3) was assigned at home and 15 students did the homework.

About the Physical interpretation and unit of measure of the vector in the left-hand side of the equation, all students answered correctly in the classical case. In contrast, only a few responded entirely in the quantum case. Several students referred to the state vector without making explicit the abstract nature and consequent absence of units. Note that some students confused the word *dimension* with *units*, attributing no dimension to the state vector.

About the plane to which vector belongs, almost all students referred to the physical space of the laboratory; some used a sentence such as "*maths laboratory space*".Two students spoke of mathematical space from a quantum perspective, probably referring to abstractness. Half referred to Hilbert space or state space without further specification about the units of measure.

About the interpretation of coefficients and their square, half answered the first part of the question correctly, but only one student referred to energy in the classical case. About the quantum case, most identified the role of probabilities, but none distinguished between probability amplitude and probability. Usually, only the role of the squares of the coefficients is discussed. In 13 out of 15 answers the word *probability* appeared. Three students emphasized the formal analogy between $a^2 + b^2 = E^2$ and $a^2 + b^2 = 1$

Regarding the last item, half spoke about the direction on the electric field. But most of all, referred to the fact that in the act of measurement, the sign is not measurable: "*the sign in quantum is not observable*"; "*It does not influence because it is not observable*".

S1 (SS) We observe that even in the case of the self-selected Summer School students, few specify that state vectors are abstract and, therefore, without units. Furthermore, in the third item, no one referred to the probability amplitudes of the observables but only to the role of the coefficient squares. Finally, the answers given in the last item show the misunderstanding between the global and local phase[20] (as already

[20] Despite a change made between the first two versions and this one.

seen in the two curricular experiments):

> "*The minus sign indicates a π shift by changing the sign to both a and b*"

and more

> "*The sign indicates the direction of displacement from the equilibrium position. If it changes, crests and troughs exchange*"

S1: Observations and first proposals for new design There are three elements we would like to focus on. First, it is evident that students struggle to discuss the physical interpretation aspect. This leads students, for example, to recognize Hilbert's space of states but not to feel the need to emphasise its abstractness with respect to laboratory space. Secondly, it is very difficult to distinguish between coefficients and their squares in the quantum case. Finally, there is often confusion between the global and local phase. It would probably be helpful to divide the questions into two to force students to think about each part of the question. In addition, it should be better emphasised that a formal analogy should be discussed in depth and critically. Leaving the worksheets to be done at home in the case of the two curricular experiments was a necessity due to the time available. It would be appropriate to carry out the worksheet at school under the teacher's guide. We also have to consider that some students who had answered similar questions correctly in the introductory course made mistakes in this form. This could be due to the short time spent on it or the limited time students spend reviewing the topics introduced in the classroom.

S2 (CSG): Polarization gates At this meeting, 22 students were present. We consider each item on the worksheet.

A1 14 students answered the entire question correctly. There are a variety of explanations. We show in the following table some examples of these explanations in which algebraic, geometric and CP words appear in each answer in order to examine the field of explanations (Tab. 7.5):

Table 7.5: Example of A1 item explanations (CSG)

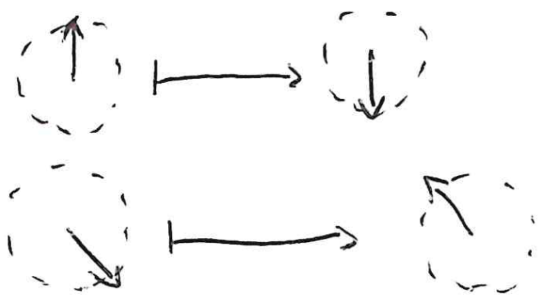
CP	**Algebra**	**Geometry**	**Explanation**
x	x	x	"The maxima become the minima of the wave: it is equivalent to changing signs in the state vectors". Adds geometric explanation on the unit circle and the algebraic results.
x	x	x	"Minus sign of the field vector: makes a half turn". Adds the algebraic results.
	x	x	"A minus sign is added". Adds the geometric and algebraic results
	x		"If you shift by π the result is always the opposite." Adds the algebraic results.
	x	x	 "A phase shifter of π puts a minus sign at the value."

Table 7.5: Example of A1 item explanations (CSG)

The full analysis shows that: 10 integrated a geometric explanation with a geometric one; 2 explicitly referred to the strictly physical context of the wave or electromagnetic field. 6 students described the transformation in the plane of states as a rotation of π.

A2 10 students correctly represented the device. 6 try to give an explicit explanation. We show in the following table some examples of representations and explanations (Tab. 7.6):

TABLE 7.6: Example of A2 item representations and explanations (CSG)

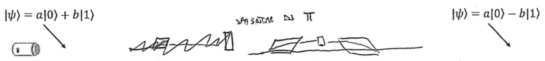
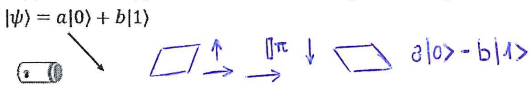
Devices representation	**Explanations**
	"The first crystal splits the photons into states $\|0°\rangle$ and $\|90°\rangle$ the phase shifter transforms state $\|1\rangle$ into $-\|1\rangle$; the inverse crystal recombines the two states"
	"Considering $\|0\rangle + \|1\rangle$ and $\|0\rangle - \|1\rangle$ respectively with $\|0°\rangle + \|90°\rangle$ and $\|0°\rangle - \|90°\rangle$ and since a phase shifter applies a negative sign, it should suffice to apply a phase shifter on the 90° polarized channel the phase shifter beyond the calcite crystal and use another crystal"
	"The first crystal divides the photon in $\|0\rangle$ and $\|1\rangle$. The phase shifter acts on $\|1\rangle$ and makes it $-\|1\rangle$. The second crystal brings the two states together."

TABLE 7.6: Example of A2 item representations and explanations (CSG)

It should be noted that half of those who explain the device indicated the split of the photon or its state.

A3 6 students answered substantially correct. The other answers are very vague with no apparent logical line of development. Most, however, seem to realise that the same devices are needed, without giving further explanation. Three of them thought that the solution to the problem in the role of the phase shifter should change from the previous case.

S2 (Fi): Polarization gates At this meeting, 18 students were present. We consider each item on the worksheet.

A1 16 students answered the entire question correctly. The full analysis shows that: 4 integrated a algebraic explanation with a geometric one; 5 only algebraic and 5 only geometric. 1 explicitly referred to the classical physical context of electromagnetic field. 6 students described the transformation in the plane of states as a rotation of π.

We present a table (Tab. 7.7) similar to the previous one.

TABLE 7.7: Example of A1 item explanations (CSG)

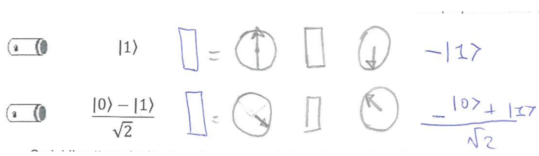
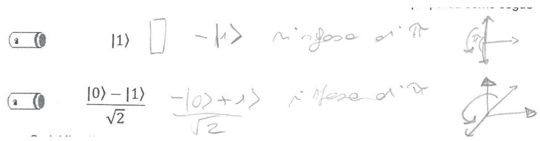
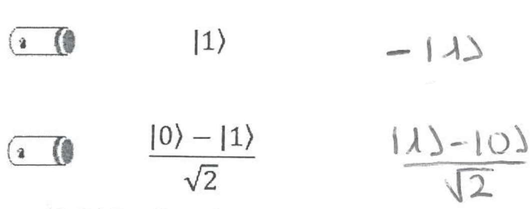
<table>
<tr><th>CP</th><th>Algebra</th><th>Geometry</th><th>Explanation</th></tr>
<tr><td></td><td>x</td><td>x</td><td>
“The phase shifter leads to a change of sign of the initial state vector.”</td></tr>
<tr><td></td><td></td><td>x</td><td>
“Phase shift of π”</td></tr>
<tr><td>x</td><td>x</td><td></td><td>
“The phase shifter π inverts sign to the field vector”</td></tr>
</table>

Table 7.7: Example of A1 item explanations (CSG)

A2 17 students correctly represented the device. 10 tried to give an explicit explanation. We show in the following table some examples of representations and explanations (Tab. 7.8):

Table 7.8: Example of A2 item representations and explanations (Fi)

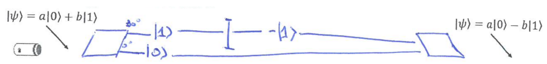
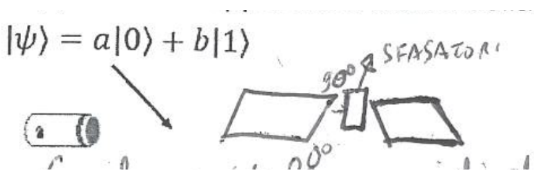
Devices representation	**Explanations**
	No explanation
	"With the crystal, the photon is divided at 0° and 90°; at this point by 90°, the phase shift is made and results $a\lvert 0\rangle - b\lvert 1\rangle$."

Table 7.8: Example of A2 item representations and explanations (Fi)

Very few students represented classical trajectories. 6 students described the effect of the first crystal with a separation (division) either from a classical point of view (beam) or even of the single photon. The second crystal then has the task of bringing the beams together. This problematic aspect, however, allowed students, at least those who explain, to grasp well the link between ordinary and extraordinary channels, polarization and encoding with qubits.

A3 Although 10 students answered and justified question A2, only 4 are able to design logic gates X and H. Three students used the rotated crystals, but do not inserted the phase shifter. Two other students used only the first rotated crystal. Two students inserted a filter between the two crystals with polarization angle 45° and $22,5°$.

S2 (SS): Polarization gates The self-selected students all answered question A1 correctly. Interestingly, again, many identified the phase shift as a rotation of π (some even justified it with supplementary arcs). But it is above all question A2 that points to the issue of the separation of the beams. The analysis of these students' answers allows us to identify some possible lines of interpretation. We classify them in the following table (Tab. 7.9) whose columns are organized by keyword:

TABLE 7.9: Classification of A2 item (SS).

Ray	**Flow**	**Light beam**	**Photon**
"The first crystal divides the ray in two; we then place a phase shifter only for the 90° ray to phase it and bring it to the end with a negative sign as required."	"The first crystal separates the flows: on flow $\|1\rangle$ we put a phase shifter so that $\|1\rangle$ becomes $-\|1\rangle$."	"With calcite (first) we have a 0°-polarized beam that remains unchanged and a 90°-polarized beam (extraordinary) modified by the phase shifter; with a last calcite the beams come together."	"The crystal temporarily separating the photons at 0° and 90° makes it possible to phase only the photons at 90° by π.

TABLE 7.9: Classification of A2 item (SS).

It is thus clear that there are two kind of problems. The first concerns an essentially classical approach: there is no difference in the description of light as a electromagnetic field (despite the construction work and continuous comparison between the classical and quantum cases). The second concerns the confirmation of the interpretation of the state $a|0\rangle + b|1\rangle$ not as a superposition but as a mixture.
Here is one of the answers considered correct

> "*I use two calcite crystals, direct and reverse, and at the 90°-channel I would place a phase shifter to change only $b|1\rangle$ and leaving $a|0\rangle$ the same.*"

Only half of the students manage to design X and H gates. However, for the most part it is clear that a rotation of the two crystals is necessary.

S2: Observations and first proposals for new design Consider focusing on some elements that emerged from the analysis of Sheet **S2**, some of which are also evident from the images and answers we highlighted above.

The answers to item A1, and also in many cases A2, show that the possibility of describing the system's evolution using multiple representations can be advantageous. The algebraic and geometric approach included in the representations of physical devices shows how it is possible to combine visual representation with more formal aspects. However, it is evident that most students have a deeply classical vision, and this manifests itself when asked for an explanation. It is possible that students are building the model and that it is still too constrained by the language of classical physics. This can possibly be evaluated as the TLS continues. Somewhat different is the case with the interpretation of the state as mixing and not superposition. In this case, one actually focuses on the single photon and not on light as a wave, but once again, the classical interpretation regarding the trajectory seems evident.

A more detailed discussion is needed on the fact that few students, far fewer even in the case of SS, can construct the logic gates X and H once they recognize the correspondence between the ordinary propagation path and the axis of symmetry in the plane of states. The answer to this apparent inconsistency could be derived from the many explicit indications that the phase shifter realises a rotation in question A1. The whole effect of the crystal and phase-shifting system on the extraordinary path is to realize symmetry in the space of states. This aspect probably needs to be emphasised by a further question, also oral, in which students are asked to explain the link between the geometric interpretation of the Z-gate and the set of devices used for its realisation. The possibility of confusing the role of the two geometric transformations introduced could be the cause of the failure to answer item A3 correctly. A simple algebraic demonstration might also be helpful.

If, as in the case of the Summer School, access to university laboratories was possible, the process of modelling polarization computation could be concluded realizing of special worksheets to translate ideal optical circuits on a real optical bench. In this way, it could become evident how optical circuits represent an operational description of what is to be done during an experiment.

Learning goals With reference to the supposed learning goals, we can say that the polarization coding only partially achieved them. The greatest difficulties have been in the interpretation of the qubit in polarization rather than in its use. Furthermore, the construction of the logic gates as an extension of the Z gate was not as straightforward as we had hypothesized.

S3 (CSG): Dual-rail encoding At this meeting, 21 students were present. We consider each item on the worksheet[21].

B1 16 students answered the entire question correctly; 3, partially; 2 blanks.

B2.1 14 students answered the entire question correctly of which 10 indicate the refractive index; 1 blank; two incomplete.

B2.2 15 students answeres that does not depend on the sense in which the light beam travels. Below are some of the explanations:

> "*The value does not change depending on the sense of travel but only on the orientation of N and n.*"

And more

> "*No, because they have the same phase shift since they reflect on the same surface*"

The 11 students who give an explanation, give it substantially correctly.

S3 (Fi): Dual-rail encoding At this meeting, 18 students were present. We consider each item on the worksheet[22].

B1 16 students answered the entire question correctly; 2, partially.

B2.1 15 students answered the entire question correctly and indicate the refractive index; two partially correct.

B2.2 14 students answered that does not depend on the sense in which the light beam travels. Below are some of the explanations:

> "*No because it always affects the same crystal.*"

And more

> "*No. It depends on the prism and not the sense.*"

The 6 students who give an explanation, give it substantially correctly.

[21] In this teaching experiment, we modelled the beam splitters in a different way than previously described. In order to explain the presence or the lack of a phase shift between the incident and the reflected beam, we present to students an even more stylized physical situation in which we ignore the presence of the interposed film and consider only two prisms, one of which has a higher refractive index (N) and the other a lower one (n). At the diagonal interface of such an object, partial transmission and partial reflection will still happen, although control of their respective weight would not be practicable. In this setting, Fresnel laws prescribe that the reflection of beams that travel from the prism with lower refractive index to the other involves a phase shift of π. While the reflection of beams travelling in the opposite direction does not produce a phase shift. The encoding of the paths may be performed so that those two corresponding to a reflection without phase shift are labeled as 0 and the other two as 1.

[22] The same considerations made for CSG apply to this educational experiment

S3 (SS): Dual-rail encoding In this case we have used the worksheet presented in 7.3.4.

B1 10 students answered the entire question correctly; 2, partially.

B2.1 10 students answered the entire question correctly of which 4 indicate the refractive index.

B2.2 9 students answered that does not depend on the sense in which the light beam travels. Below is one of the incorrect explanations:

> "*It only depends on whether they are reflected/transmitted by the adhesive/dielectric*"

And one of the correct

> "*The ray will be reflected by the same material with the same index of refraction.*"

S3: Observations and first proposals for new design This part concerning dual-rail coding did not create any substantial problems. The impact of the beam-splitters more in-depth representation of the process remains to be evaluated: having created some difficulties in the SS, it could give more in curricular experiments.
If, as in the case of the Summer School, access to university laboratories was possible, the process of modelling dual-rail computation could be concluded realizing of special worksheets to translate ideal optical circuits on a real optical bench. In this way, it could become evident how optical circuits represent an operational description of what is to be done during an experiment.

S4 (CSG): Comparison of classical-ondulatory and quantum description for dual rail. This sheet (see Tab. 7.4) was assigned at home and 20 students did the homework. First, we observe a significant increase in homework, demonstrating a gradual improvement in attention to the proposed educational path.We then discuss the specific questions. About the classical interpretation in the first question (Interpretation of coefficients and their square), we observe that only two students answered correctly and completely even though they identified the reflection and transmission coefficients with a and b and not with their squares. Many more students recognize (7 students) the role of the square of coefficients in the quantum case. Regarding the sign of the superposition, as in **S1**, many students spoke of global phase. Precisely because of this, we modified the question. In the CSG experiment, the question was probably less clear: "*Physical interpretation of the sign*

of the superposition (coefficients a and b of opposite sign[23]*)*". 7 students, moreover, spoke of probability in the classical case and of these five also in the quantum case, without explaining how the minus sign should be interpreted from a probabilistic point of view. The third question asked whether the angle between the components was fixed. The answers in the classical case are of extreme interest: 7 students, indeed, stated that the components are fixed since the directions have an angle of 90°. 8 students answered correctly in the quantum case.
Regarding the possibility of speaking of superposition in the two cases classical and quantum, only four students answered correctly in the classical case, but without adequate justification. Approximately half answered correctly in the quantum case.

S4 (Fi): Comparison of classical-ondulatory and quantum description for dual rail. This sheet (see Tab. 7.4) was assigned at home and 17 students did the homework. About the classical interpretation in the first question (Interpretation of coefficients and their square), we observe that 9 students answered correctly, at least partially, in the classical case; only 6 in the quantum case. However, it should be noted that most students refer to the probability of finding a photon on a path, but without further considerations. Regarding the sign of the superposition, as in CSG, many students talked about global phase, but in the case of this class there is a significant aspect: 5 students felt the need to add that sign-related aspects are not measurable. Even when it is evident from the answer that it is the local phase. About the possibility of speaking of superposition in the two cases classical and quantum, we prefer not to analyse the answers as many identical answers can be seen.

S4 (SS): Comparison of classical-ondulatory and quantum description for dual rail. This sheet (see Tab. 7.4) was assigned at home and all students did the homework. We would like to emphasize only one aspect: regarding the sign in the superposition, despite the change in the questions, half of the students seem to have referred to the global phase.

Observations and first proposal for new design In general, similar considerations apply to those made in comment on the worksheet **S1**. However, one begins to see an increase in the ability to motivate answers. About the answers, we would like to emphasize two more aspects: manifestly, the difference in the vector on the paths whose components identify fields acting at different points in space, while the one

[23] We will see in the next analysis that despite this change there were some students who misunderstood the question.

in polarization has components relating to the same field, is complicated to grasp; this tells us how difficult it is to understand the concept of superposition (classical or quantum) in its totality. But above all, we want to emphasize how classic notions that students should know adequately are instead full of errors, misunderstandings and overlaps. In a TLS based on the conceptual change that refers to prior knowledge, this aspect is fundamental.

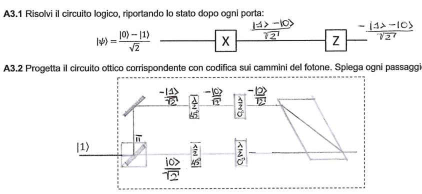
S5(CSG): Dual-rail gates At this meeting, 17 students were present. We consider each item on the worksheet.

A1 11 students answered question correctly. Almost no one feels the need to explain their choices. Tree students insert tools already used in the case of polarisation, such as plates or birefringent crystals. Some also set the phase shifter to $|0\rangle$.

A2 8 students answered question correctly. A student indicates that paths should be renamed without indicating any symbols. 3 question blank.

A3 8 students answered two sub-questions correctly. 5 students leave blank at least one of the two sub-questions. 2 students realize the ideal optical device correctly, but but they do not answer the first question. Below is a table (Tab. 7.10) with some examples and some considerations (Larger pictures can be found in the Appendix D).

TABLE 7.10: Examples of A3 item representations and explanations (CSG).

Answers	**Observations**
	The logic circuit is carried out correctly. The student uses logic gates in polarisation even where encoding was required to be in dual-rail. At the end it inserts a birefringent crystal to bring the two paths together. Draw the beam-splitter 50-50 correctly.

(Continued on the next page)

(Continued from the previous page)

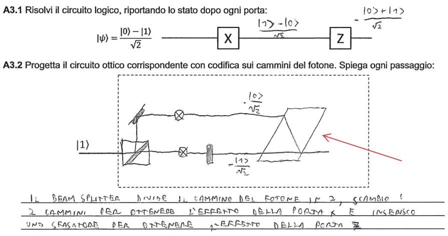
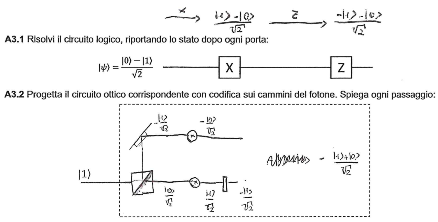
	The logic circuit is carried out correctly. The representation of the beam-splitter seems to be correct. Uses dual-rail logic gates correctly. Does not develop step-by-step algebraic calculations in relation to experimental devices. At the end it inserts a birefringent crystal to bring the two paths together. He is the only one to give an explanation: "*The BS splits the path of the photon into two; I exchange the two paths to achieve the effect of an X and insert a phase shifter to achieve the effect of a Z gate*".
	Answers both sub-questions correctly.

TABLE 7.10: Examples of A3 item representations and explanations (CSG).

A4 5 question blank. The others, correct.

S5(Fi): Dual-rail gates At this meeting, 18 students were present. We consider each item on the worksheet.

A1 All students answered question correctly.

A2 All students answered question correctly.

A3 In the case (see Fig. 7.37) of the educational experiment in Florence, this third question was slightly different to prepare them for the teleportation protocol: 15 students answered two sub-questions basically correctly. Of these, 6 do not

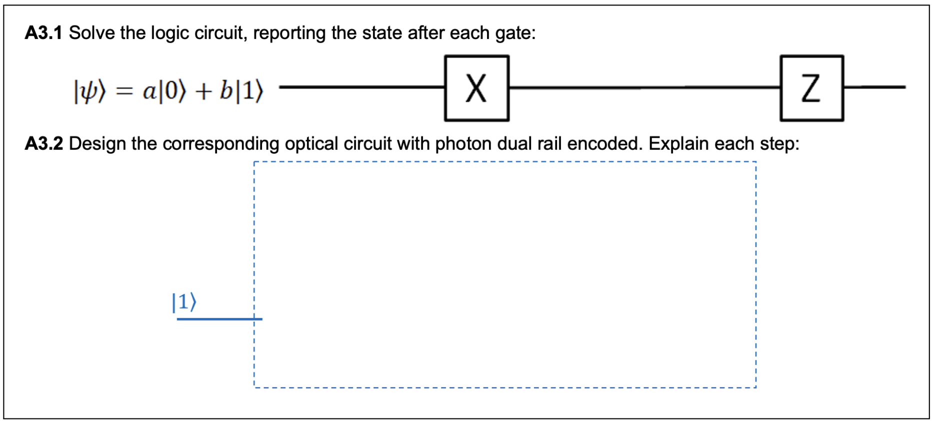


FIGURE 7.37: Item A3 of educational experiment in Florence

make the preparatory beam splitter explicit. All students answer correctly at the first sub-section. Below is a table (Tab. 7.11) with some examples and some considerations (Larger pictures can be found in the Appendix D).

TABLE 7.11: Examples of A3 item representations and explanations (Fi).

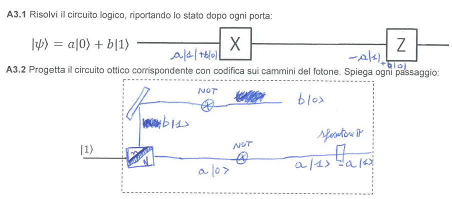
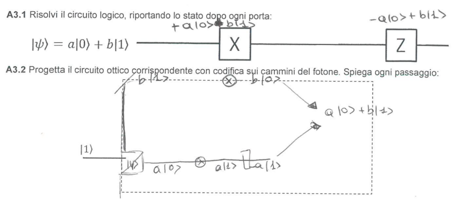
Answers	**Observations**
	One of the typical errors in this design was to consider the iconography of a 50-50 beam-splitter. Furthermore, positioned as in the image it should have created a phase shift of π at $\|1\rangle$.
	The output of the not-gate is not made explicit in the algebraic part. The mirror representation is not accurate. Otherwise all correct.

TABLE 7.11: Examples of A3 item representations and explanations (Fi).

A4 5 question blank. The others, correct.

S5(SS): Dual-rail gates We analyze each item on the worksheet.

A1 All but one of the students answered question correctly. Many include explicit explanations in contrast to the two previous situations. For example Fig. 7.38, 7.39 and 7.40:

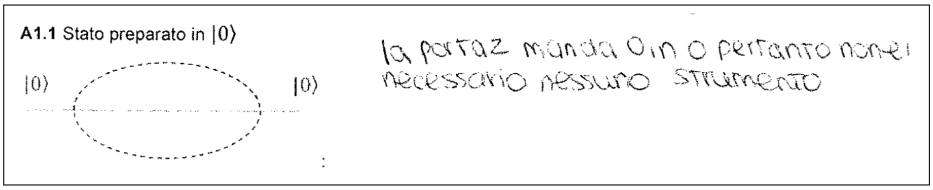


FIGURE 7.38: "*Z gate sends 0 in 0 so no instrument is needed*".

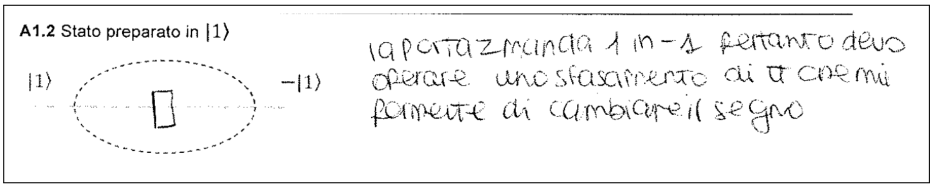


FIGURE 7.39: "*Z gate sends 1 to -1 so I have to operate a phase shift of* π *which allows me to change the sign*".

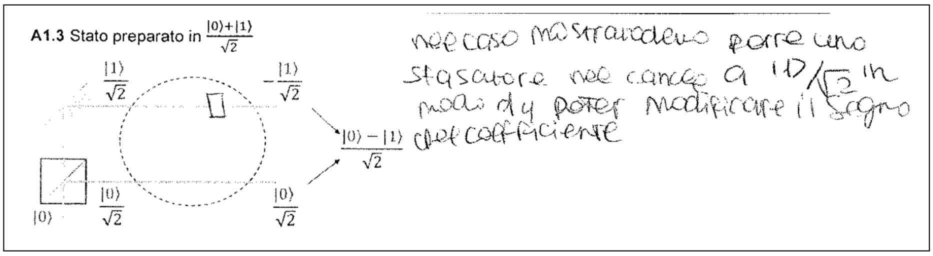


FIGURE 7.40: "*In the case shown, I must place a phase shifter in the channel* $1/\sqrt{2}$ *so that I can change the sign of the coefficient*".

A2 All students answered question correctly.

A3 8 students answered two sub-questions basically correctly. 11 only at the first. Here, too, there are a few cases in which designs feature elements of polarization coding. Below is a table (Tab. 7.12) with some examples and some considerations (Larger pictures can be found in the Appendix D).

Table 7.12: Examples of A3 item representations and explanations (SS).

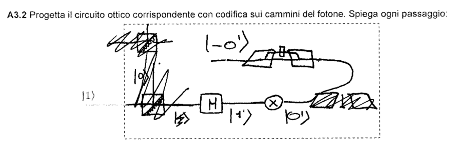
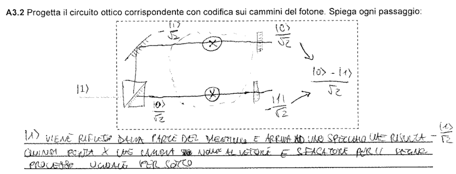
Answers	**Observations**
	The overlapping of linguistic registers can be seen in this picture: in the same image there is a logical representation of a Hadamard gate, a device in polarization and one in dual-rail.
	In this case, a precise explanation is given: "$\|1\rangle$ *is reflected from the dielectric side and arrives at a mirror that is* $-\|1\rangle/\sqrt{2}$ *then the* X*-gate changes the name of the vector and the phase shifter changes its sign. Similar process below.* In this case, we realise that the phase shift is attributed to the mirror and not to the beam-splitter; moreover, the phase shifter is erroneously placed on both paths. Finally, there is some confusion in the linguistic registers.

Table 7.12: Examples of A3 item representations and explanations (SS).

A4 5 question blank. The others, correct.

S5: Observations and first proposals for new design The construction of dual-rail logic gates was also tackled with good results. The critical aspect is partly the design of the optical circuits. This, from now on, will prove to be the most problematic aspect of the computational model. The exercises carried out together to support the design of optical circuits have only partly helped the students.

Learning goals With reference to the supposed learning goals, we can say that dual-rail coding has largely achieved them.

S6(CSG): Deutsch algorithm At this meeting, 16 students were present. We analyze each item on the worksheet. In this case, given the length and complexity of the worksheet on Deutsch's algorithm, we prefer to report the various items directly below to facilitate the reading of the data analysis. The complete worksheet can be found in the Appendix B.

Quantum parallelism:

A1: *In relation to the coin problem: which side(s) of the coin is/are encoded in the output state of the first Hadamard? (Remember that according to the database encoding silver face = 0; gold face = 1).*

15 students answered correctly and 7 made explicit reference to the superposition. The explanations refer either to superposition or to the role of probability. Here are some examples:

Superposition :

> "*There is superposition and there is information on both*"

Probability :

> "*They are coded with a probability of 0.5 for both*"

Superposition and Probability :

> "50% *gold* $|0\rangle$, 50% *silver* $|1\rangle$, H = *superposition therefore equiprobable.*"

A2: *Establish whether we gain an advantage over the classical case by using the H gate. If yes, in what does it consist? If not, why?*

10 students answered as we would have expected that there is an advantage in double coding that has no classical correspondent. But as many as three students say that this is not an advantage because you have a 50% of probability of achieving one of the two. Consider give an example:

> "*In this case, no; we have to verify the properties of the coin so we have to be sure of the measured bit.*"

B1 *What property of the operators ensures that the quantum advantage of being able to act simultaneously on both qubits of the computational base can actually be exploited? Justify your answer.*

We have not analysed the answers because the teacher anticipates it during the explanation.

Oracle and compoud systems:

C1: *The operator U is a logical gate with two inputs and two outputs. Writing the truth table (the behaviour of the operator on* $|00\rangle, |01\rangle, |10\rangle, |11\rangle$*) of the logical gate in case* $f(0) = 0 \wedge f(1) = 1$.
Are there any logic gates of your knowledge that operate in this way? If yes, please specify which. If no, explain why.

13 students completed the table correctly and 11 recognized the $CNOT$ gate.

C2: *The sign minus can be transferred from the ancilla to the target. What feature of quantum physical systems is exploited?*

Only four students recognized the role of the tensor product. 9 referred to superposition. Some spoke generically of a *translation* of the minus sign and mention phase.

C3[24]: *To which state vector does the minus sign in the compound state belong? Explain.*

9 students answered correctly. 7 said that the sign belongs only to the ancilla.

C4: *What image(s) on the face of the coin tell us about the status of the target coming out of the oracle? Explain.*

14 students answered correctly many of whom justified their answer with the superposition.

C5: *If we implemented the circuit a large number of times in the same initial condition and measured the target, could we know whether the coin was genuine or counterfeit?*

7 students answered correctly and explained the role of the minus sign in the measurement:

[24] The two questions C2 and C3 were proposed together and corrected only when the students had answered both of them.

"*No, we could not distinguish the superposition sign.*"

4 students justified the negative answer by linking it to probabilistic aspects of equiprobability between the two classical bits obtained by measurement:

"*No because doing the measurement I would still have a* 50% *chance.*"

Interference and final conclusions:

D1: *Knowing that the final state before the last Hadamard is*

$$|\psi\rangle = \frac{1}{\sqrt{2}}((-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle)$$

complete the table

Boolean function	State after the Oracle	State after the last Hadamard	Classic bit after measurement and probability
$f(0) = 0 \wedge f(1) = 0$	$\|\psi\rangle =$ ______ = ____	$\|\psi\rangle_{fin} =$ ______	____ , p= ____
$f(0) = 0 \wedge f(1) = 1$	$\|\psi\rangle =$ ______ = ____	$\|\psi\rangle_{fin} =$ ______	____ , p= ____
$f(0) = 1 \wedge f(1) = 0$	$\|\psi\rangle =$ ______ = ____	$\|\psi\rangle_{fin} =$ ______	____ , p= ____
$f(0) = 1 \wedge f(1) = 1$	$\|\psi\rangle =$ ______ = ____	$\|\psi\rangle_{fin} =$ ______	____ , p= ____

FIGURE 7.41: D1 item table

D2: *By observing the table above, establish what relationship there is between the authenticity of the coin and the outcome of the measurement. Explain.*

Most students (13) correctly completed the table and identified (12) the correlation between output bits and coin authenticity.
Some explanations still show difficulty in distinguishing between the state and outcome of a measurement:

"*Every time we measure* $\pm|1\rangle$ *out, we find a genuine coin*"

E1: *How many times does the quantum operator* U_f *have to be implemented to determine whether a coin among those in the database is genuine or counterfeit? What is the advantage over classical computation?*

12 students answered correctly. We observe that three students referred to the advantage from a timing point of view and not from a logical point of view.

There was no time to carry out the last part of the worksheet, the one on implementations with optical devices. Four students did it at home and handed it in at a later date (see Fig. 7.43 for the optical circuit.). Fig. 7.42 shows one of the answers. The algebraic part in relation to optical devices is carried out correctly and the student also uses colour codes appropriately. More difficulties are noted in the description of the devices, in particular on the function of the half-wave plate at $22,5°$. The connection between the development of the logic of the algorithm and the role of the devices is obtained only with the comparison of the logic and optical circuit. There are no comments on the description of the coin problem given during the lesson.

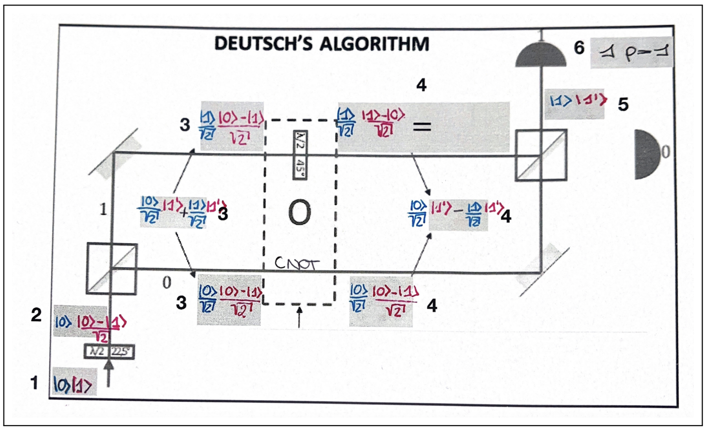


FIGURE 7.42: Worksheet submitted by a student: "*At the beginning we find a half-wave plate at* $22,5°$ *which acts on polarization. Then a 50:50 beam splitter without phase shift which acts on the path and divides it into two paths. On path 1 there is a half-wave plate at* $45°$ *which is equivalent to an* X*-gate that inverts the labels. Finally a new 50:50 beam splitter which realises the Hadamard gate on the path. Finally a photon detector.*"

S6(SS): Deutsch algorithm : Two elements should be emphazised that differentiate the work on Deutsch's algorithm done during the SS: firstly, it was possible to carry out the board in its entirety; furthermore, three lab hours were dedicated to the study of real devices for the implementation of logic gates and Deutsch's algorithm with linear optics. Consider analyze only the item on the worksheet which show significative differences from CSG or allow some misleading lines of reasoning to be better determined.

Quantum parallelism:

A1: *In relation to the coin problem: which side(s) of the coin is/are encoded in the output state of the first Hadamard? (Remember that according to the database encoding silver face = 0; gold face = 1).*

Again, there are some (7) students who referred to the superposition principle, others only to probability, not allowing one to understand whether there is underlying misunderstanding with mixed states. Here are some examples:

Probability :

> "50% *probability of the golden face coming out e* 50% *of the silver face coming out*"

Superposition and Probability :

> "*Both are detected with* $p = 50\%$*. Superposition state.*"

A2: *Establish whether we gain an advantage over the classical case by using the H gate. If yes, in what does it consist? If not, why?*

An interesting aspect is that some students explained by anticipating the fact the next question:

> "*Yes, because it is as if we could make calculations on two states at the same time.*"

B1 *What property of the operators ensures that the quantum advantage of being able to act simultaneously on both qubits of the computational base can actually be exploited? Justify your answer.*

11 students answered correctly; some justify their response:

> "*The advantage is to operate on one and the other at the same time.*"

Oracle and compound systems:

C2: *The sign minus can be transferred from the ancilla to the target. What feature of quantum physical systems is exploited?*

10 students answered correctly; 5 refer to tensor product and 4 refer to quantum compound systems; one wrote, not entirely correctly, the algebraic steps that allow the sign to be transferred[25].

[25] $|1\rangle|-1\rangle = -|1\rangle|1\rangle = |-1\rangle|1\rangle$

C3[26]: *To which state vector does the minus sign in the compound state belong? Explain.*

10 students answered correctly. Only one say that the sign belongs only to the ancilla.

C5: *If we implemented the circuit a large number of times in the same initial condition and measured the target, could we know whether the coin was genuine or counterfeit?*

Only 4 students referred to the problem of sign. Most justify by saying that the measurement is stochastic.

F1: *Describe the optical devices used to realize the optical circuit of Deutsch's algorithm (in the case under consideration described by the figure in the table below) in the order in which they are encountered in the circuit. Explain their function in relation to the logical development of the algorithm.*

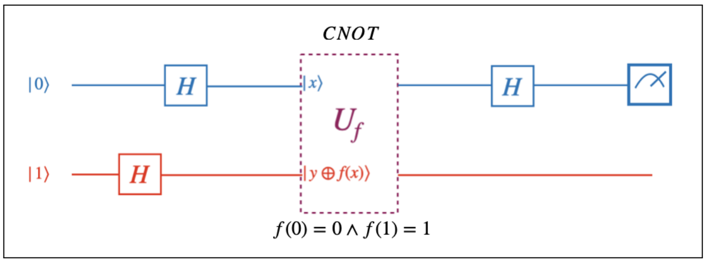


FIGURE 7.43: Optical circuit of Deutsch algorithm.

F2: *Enter in the grey boxes the corresponding state in Dirac notation consistent with the action of the individual optical devices (including the device used for measurement). Also express the classical information bit obtained and its probability*[27].

Half of the students completed question F2 completely correctly; some do not consider only the sign changing after the half-wave plate $\lambda/2$ at 45°. Almost all tried to give a description of the link between logic and optical devices. They always did this by comparing logic and optical circuits; they never referred to the coin problem described during the lesson. Here is an example in which, apart from an

[26] The two questions C2 and C3 were proposed together and corrected only when the students had answered both of them.
[27] See Fig. 7.42

error on the phase shift, the approach of total reconciliation between the two circuit representations, the logical and the optical, is evident:

> "*The first Hadamard acts on the polarisation which is represented (in the optical circuit) by an* $\lambda/2$ *at* 22.5°*, the axis of symmetry. We therefore have a 50-50 beam splitter with a phase shift on the paths: the transmitted path is shifted by* π *and this corresponds to the second Hadamard gate. The oracle in the logic circuit is realised with an* $\lambda/2$ *at* 45° *which acts as a CNOT. The last beamsplitter acts on the paths and represents the last Hadamard on the paths. We therefore have the measurement made by two detectors.*"

S6: Observation and first proposal for new design Two critical factors emerge from the analysis of the item responses: the role of superposition in respect to the mixture of states (a problem that has emerged previously); the nature of compound systems, in particular the role of the minus sign in the compound states.
To be precise, answers that explicitly use the word probability in items **A1** and **A2** do not always make the mistake between superposition and mixture of states obvious. However, previous worksheets have already pointed this out and it is reasonable to assume that the same problem exists here. It would be appropriate to construct a small task to solve with an algorithm and show what changes in the two cases. For example, we could exploit the coin flipping game ([239]) to describe the state in the classical case and in the quantum one. Imagine that Alice and Bob are playing as in [239]. Imagine that Charlie has access to the moves of both. In this case the classical state would be deterministic and the introduction of probability would only be due to an epistemic problem. In the quantum case Charlie would describe the state after the Hadamard used by Alice as $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ which intrinsically introduces probability. A circuit translation exercise of the game with Charlie's introduction might be interesting. By playing a one hand at time (single bit or qubit), it may become clearer to students what it means in relation to an encoding problem such as the coin in Deutsch's algorithm. We summarize in a table (Tab. 7.13) the description of the state after Alice's preparation according to Alice Bob and Charlie in the classical case and in the quantum case:

TABLE 7.13: Coin flipping game after Alice preparation.

	Classical computer	**Quantum computer**
Alice	Identity: 0	Hadamard: $\frac{1}{\sqrt{2}}(\vert 0\rangle + \vert 1\rangle)$
Bob	0 or 1 with $p = 0.5$	?
Charlie	0	$\frac{1}{\sqrt{2}}(\vert 0\rangle + \vert 1\rangle)$

TABLE 7.13: Coin flipping game.

If we imagine that Charlie has access to information about the transformations made by Alice and Bob, in the classical case this is equivalent to having a deterministic state, in the quantum case it remains a superposition. Here, then, Charlie shows the difference between epistemic (Bob state) and intrinsic probability.
The problem of the tensor product algebra, on the other hand, is more tricky. In effect, we have imposed the properties of the tensor product. It might make it clearer to you the vector construction of the tensor product with which you could actually show that

$$-(|\psi\rangle \otimes |\phi\rangle) = (-|\psi\rangle) \otimes |\phi\rangle = |\psi\rangle \otimes (-|\phi\rangle)$$

However, students should learn the tensor product between vectors axiomatically. It is certainly a real possibility if linear algebra has been covered in the school curriculum.

S7(CSG): Grover algorithm At this meeting, 18 students were present. We analyze each item on the worksheet. In this case, given the length and complexity of the worksheet of Grover's algorithm, we prefer to report the various items directly below to facilitate the reading of the data analysis. The complete worksheet can be found in the Appendix B.

Quantum parallelism:

A1: *In relation to the microchip bar problem: which bar(s) is/are encoded in the* *output state by the two Hadamards on the target register? Explain.*

11 students answered correctly and 4 worksheets are blank.

A2: *Coming out of the Hadamard gate on the ancilla the state is* $|y\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$ *Denoted* $|y \oplus f(x)\rangle$ *the output status of the oracle on the ancilla, complete the following table (see Fig. 7.44)*

Bar	Function	Output state
Without microchip	$f(x) = \ldots\ldots$	$\lvert y \oplus f(x)\rangle = \ldots\ldots\ldots$
With microchip	$f(x) = \ldots\ldots$	$\lvert y \oplus f(x)\rangle = \ldots\ldots\ldots = \ldots\ldots\ldots$

FIGURE 7.44: A2 item table in the Grover's Algorithm.

All students completed table correctly.

A3: *In the oracle step, how does the status of the ancilla corresponding to* $|10\rangle$ *(qubit related to the bar with microchip)?*
16 students answered correctly to question A3 (3 spoke of phase shifting).

A4.1: *How many times is it necessary to implement the oracle to complete the previous table? Explain.*
A4.2: *Do we gain an advantage over the classical case? Explain.*

17 students answered correctly to A1 justifying in a personal way:

> "*Once, because thanks to the superposition property it analyses all 4 cases.*"

> "*Once because it sees all the states of the computational base.*"

13 students recognized the advantage but in general they talked about fewer steps without specifying.

Oracle and compound systems:
B1: *What feature of quantum systems allows us to go from*
$\frac{1}{2}(|00\rangle\otimes|1'\rangle+|01\rangle\otimes|1'\rangle+|10\rangle\otimes(-|1'\rangle)+|11\rangle\otimes|1'\rangle)$ *to* $\frac{1}{2}(|00\rangle+|01\rangle-|10\rangle+|11\rangle)\otimes|1'\rangle$*?*
B2: *To which register does the minus sign in the compound state belong?*

7 students indicated the properties of the tensor product. Still 4 students indicated the superposition; others the linearity. Almost everyone recognised that the minus sign belongs to the compound state $target \otimes ancilla$.

B3: *Which bar(s) in the caveau tells us the status of the target register coming out of the oracle? What information does it give us? Explain.*
B4: *If we implemented the circuit a large number of times in the same initial condition and measured the target, could we know which bar is microchipped? Explain.*

Most students recognized that the minus sign indicates the microchip bar, but not everyone answered the question specifically about the others bars. 14 students answered correctly, providing different justifications:

> "*The measurement is still affected by the probability coming out of the oracle (0.5).*"
>
> "*You cannot distinguish the sign of superposition.*"

Interference:
C1: *By inserting the detectors at the end of the circuit, what pair of classical bits would we obtain? How does this allow us to solve the problem of determining the bar with microchips? Explain.*
C2: *How many times does the quantum operator have to be implemented to find the microchip bar among those in the database? What is the advantage over classical computation?*

11 correct answers and 4 blank. 4 students specified that the output bit pair is found with probability $p = 1$. Almost everyone recognized the quantum advantage.

Implementation with optical devices:
D1: Half of the students design the first two Hadamard gates completely correctly. Of the remainder, almost all of them design at least partially correctly. Below are some pictures and comments (Tab. 7.16):

TABLE 7.14: Examples of D1 item representations (CSG).

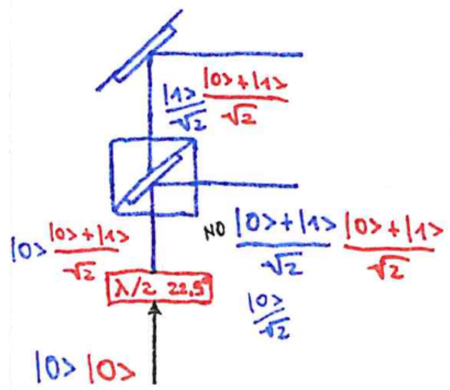
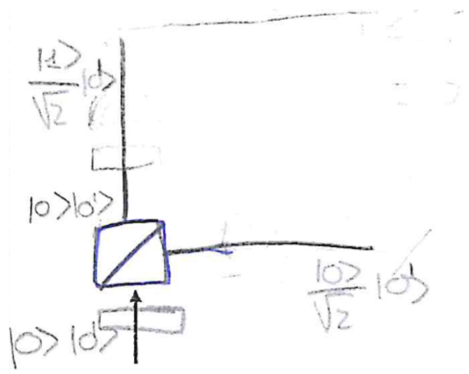
Designs	**Observations**
	In this image, we can see how the student has begun to correctly master the colour codes, the attention shown in designing the beam splitter and the state correction to algebraically identify the part on path 1 and the part on path 0.
	In this case, however, the student inserted the two devices correctly, but did not indicate the angle of the half-wave plate, nor did he specify the orientation of the beam splitter. The development of the algebraic calculation is correct.

TABLE 7.14: Examples of D1 item representations (CSG).

D2: 8 students answered correctly. Many others with some errors but showing some confidence with optical circuits (Tab. 7.15):

TABLE 7.15: Examples of D2 item representations (CSG).

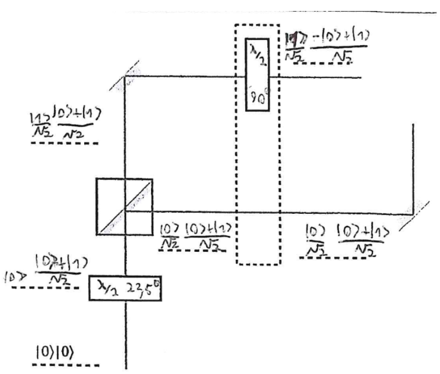
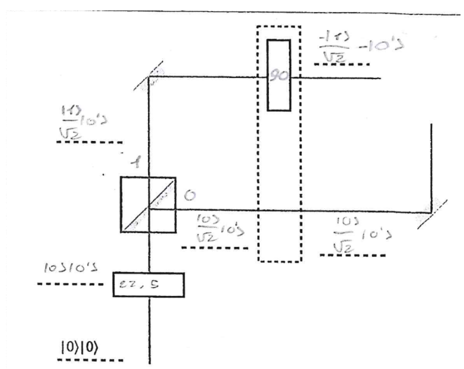
Designs	**Observations**
	Although the student did not use colours, he demonstrated a good understanding of the relationship between the logic and optical circuit. He recognised the unknown logic gate and its effect was well evidenced by the fact that he has made the state explicit after the first half-wave plate.
	In this case, however, we can see a mistake that some students made: as they did not make the state $\|0'\rangle$ explicit, they were unable to write the effect of the second half-wave plate correctly.

TABLE 7.15: Examples of D2 item representations (CSG).

D3: This question was asked in the last minutes of the lesson. 8 sheets are blank. Respondents also often inverted the role of half-wave plates in relation to paths, as can be seen in Fig. 7.45.

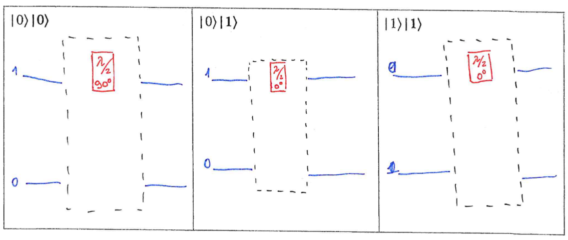


FIGURE 7.45: Example of D3 item representation (CSG).

S7(SS): Grover algorithm We will focus below on just a few items of particular significance in relation to the CSG pathway and with a view to a possible revision of the educational sequence.

Basically, all students answered the items in part A on quantum parallelism correctly and explained with attention.

Oracle and compound systems:
B1: *What feature of quantum systems allows us to go from*
$\frac{1}{2}(|00\rangle \otimes |1'\rangle + |01\rangle \otimes |1'\rangle + |10\rangle \otimes (-|1'\rangle) + |11\rangle \otimes |1'\rangle)$ *to* $\frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle) \otimes |1'\rangle$*?*

Half of the students correctly stated the tensor product or the nature of quantum compound systems, but half stated linearity as the motivation.

Interference and final conclusions
C2: *How many times does the quantum operator have to be implemented to find the microchip bar among those in the database? What is the advantage over classical computation?*

10 students (only 4 in Deutsch) explicitly mentioned the sign in their answer. The remainder only referred to the probabilistic aspect.

Implementation with optical devices:
In this case we also have information about items D3 and D4 which are performed correctly by essentially all students.

Observation and first proposal for new design Despite the undoubted improvement in the data compared to Deutsch's algorithm, the phase problem in compound systems remains widespread, especially in curricular experimentation (see observation in Deutsch algorithm analysis).
Most students, however, manage to follow the reasoning during the sequences of the worksheet and show mastery of both the algebraic and implementation aspects with optical devices. These are most problematic when students are asked to design them *ex novo*, while the ability to grasp the link between logic circuit elements and ideal devices is evident when the circuit is already shown to them. However, several students were able to design correctly by also including the related algebraic part. The

link between concrete problem, algorithmic solution and ideal experimental realization, although with some difficulties, seems to have achieved most of the students in the case of the curricular experimentation, almost all in the case of the Summer School students.

Learning goals With respect to the learning goals, we feel it is appropriate to emphasize that at the end of the two algorithms, it emerges from the data that a large proportion of the students (half in the curricular experimentation) know how to move between the registers proposed by the diagrammatic model: in particular, the ability to link the logical-formal aspect to the informational process relating to specific problems seems consolidated; furthermore, although to a less consistent degree, the students begin to know how to translate logical circuits into optical circuits by developing the corresponding algebraic calculation. Although there is still a lack of an adequate part in the TLS concerning laboratories with optical devices, the diagrammatic model seems to have been consolidated. Naturally not for all the students, but, given the difficulty and the innovation of the proposed project, we consider this part significant.

S8(Fi): Teleportation protocol At this meeting, 20 students were present. We analyze each item on the worksheet. In this case, given the length and complexity of the worksheet of quantum teleportation, we prefer to report the various items directly below to facilitate the reading of the data analysis. The complete worksheet can be found in the Appendix B.

Preparatory activity: Alice and Charlie:

A1: *Develop in the box below both registers together in Dirac notation (preserving the colours as in the figure) showing that*

$$|0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \longmapsto \ \dots \ \longmapsto \frac{(a+b)|0\rangle + (a-b)|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

17 students performed the calculations correctly. Many show an advanced use of colour codes as can be seen in Fig. 7.46

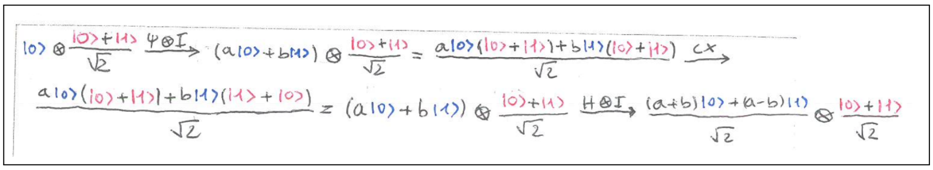


FIGURE 7.46: Example of A1 item calculation (Fi).

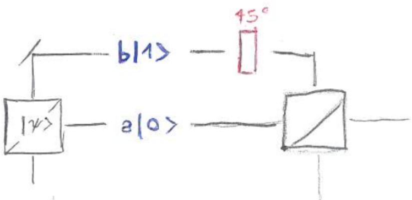
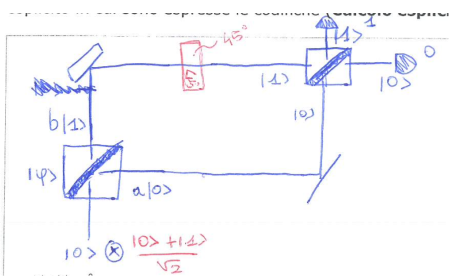
A2: *Establish whether the state composed immediately before the Hadamard is separable or entangled.*
A3: *Establish, when measuring, which classical bits are obtained on each register, and with what probability.*

Almost all students answered correctly and many also added correct reasons and explicit calculations.

A4: *Design the corresponding optical circuit starting from the left, with particular attention to the orientation of any beam-splitters. Help yourself with the explicit calculation in which the encodings are expressed.*

On this design question, students showed much more difficulty. Here are some examples with their considerations in the Tab. 7.16 (Larger pictures can be found in the Appendix D).

Table 7.16: Examples of D4 item representations (Fi).

Designs	**Observations**
	The picture shows many errors and imprecisions: the two arms of the interferometer have different lengths, the algebraic calculation is not properly introduced, the orientation of the second beam splitter is not specified and the measurement part is completely missing.
	The design is much more accurate than the previous one, but it seems that the only relevant aspect is the dual-rail encoding: the measuring apparatus only detects the bit corresponding to that encoding consistent with the partial algebraic development.

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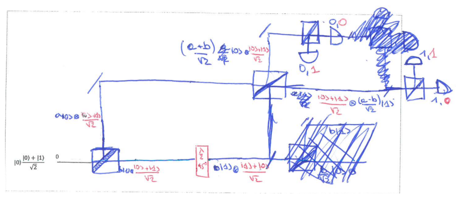
	In this latter case, the progression on the two registers up to the measurement is more rigorous. Missing is the orientation of the second beam splitter and comments on the different nature of the beam splitter in the measurement.

TABLE 7.16: Examples of D4 item representations (Fi).

In general, the most common errors were the insertion of the half-wave plate on the wrong arm, the absence of a precise orientation of the second beam splitter and the absence of measuring equipment or its partial use.

Entanglement: Alice and Bob share an entangled state:
B1: *The difference in the preparation is that the state* $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ *held by Alice is entangled with Bob's. Complete the second row of the table by highlighting the formal analogy with the case discussed above in the first row.*
B2: *Is the obtained state a separable state? If yes, write it as a state product. If not, explain why.*
B2': *So what is the effect of Alice and Bob sharing two entangled qubits?*

12 students answered correctly and presented the factorized state. Many have correctly justified the fact that the state is entangled. 13 correctly understood and explained the effect of Alice and Bob sharing an entangled state; the others left the answer blank.

B3: *Insert the optical device that generates* $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ *the entangled state used by Alice and Bob in the teleportation protocol*
Almost all students inserted a half-wave plate, but only 7 specified the angle.

Measurement:
C1: *The state immediately before the two detectors (after even the last Hadamard*

gate) is therefore

$$|\psi_1\rangle = \frac{1}{2}\left[|00\rangle\ (a|0\rangle + b|1\rangle) + |01\rangle\ (a|1\rangle + b|0\rangle) + |10\rangle\ (a|0\rangle - b|1\rangle) + |11\rangle\ (a|1\rangle - b|0\rangle)\right]$$

Complete the following table (Fig. 7.47)

Alice's Qubit before measurement	Measurement outcome	Probability outcome	Bob's Qubit after measurement

FIGURE 7.47: table of C1 item

17 students have completed the table correctly.

Correction:

D1: *The result of Alice's measurement is to collapse Bob's state into one of the four states in the last column of the table above. At this point, Alice communicates the measurement result to Bob via a classical communication channel. Depending on Alice's communication, Bob can correct his qubit to reconstruct the initial state shared by Charlie with Alice. Complete the following table (Fig. 7.48).*

Charlie's Initial Qubit	Alice's bit pair after measurement	Bob's Qubit before correction U	Transformation(s) required for correction (U)

FIGURE 7.48: table of D1 item

D2: *A student in another class said: "Bob can get information about $|\psi\rangle$ instantly". Do you agree? Explain why.*

18 students complete the table correctly. But above all, 15 students explain the reasons why superluminal communication is not possible. Many make explicit reference to the limit in classical communication between Alice and Bob. Others impose the

limits of relativity without contextualising them in the specific case.

Implementation with optical devices:

E1: *In each of the four cases (Fig. 7.49), express the state possessed by Bob after Alice's measurement and insert, if necessary, one or more devices to realize the corrections (U)*

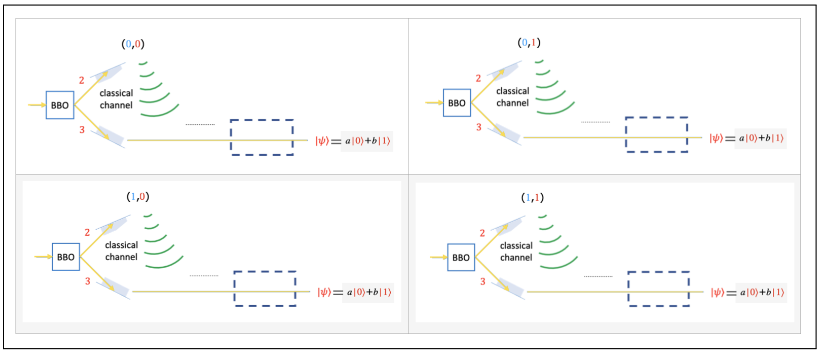


FIGURE 7.49: Picture of E1 item

E2: *Bob's reconstructed state is encoded in polarization while Charlie's original state was encoded in dual-rail. Do we deduce from this is that he was teleported? Explain.*

15 students answered correctly at item E1; 3 reverse the order of the logic gates in the last correction; 2 blank.

For reasons of time, the last question was asked orally. The students seem to understand that it is the information and not the qubit that is being teleported.

S8(SS): Teleportation protocol There are two most obvious differences from the data collected in the curricular experiment: more errors in item A1 and greater precision in the design of item A4. Actually, the calculation errors were not confirmed in item B1, which was performed largely correctly. It is remarkable, from this point of view, that a lot of time was spent on calculation during the Florence experiment, as confirmed by the interview with the teacher. In the summer school, the work done in the laboratory and the focus on the design of optical circuits seems to have favoured design skills. The other items were carried out correctly with very high percentages and correct justifications. It is noted that it is more difficult to identify the role of entanglement in the protocol (item B2').

S8: Observation and first proposal for new design In view of the results obtained, it seems important to be able to introduce the experimental part into the TLS. This aspect is crucial for the final construction of the diagrammatic model. The difficulties and possible solutions will be discussed in the conclusions.

Learning goals With respect to the learning goals, many students comprehensively understand the role of entanglement and classical communication in the protocol. The algebraic aspect, if properly supported, is more easily internalised than implementations with optical devices. This makes it all the more important to introduce real experimental practice into TLS.

7.5 Evaluation of design hypotheses

We comment, in the light of the data analysis carried out, the interviews with the teachers involved in the experiments, the exercises solved in classroom and the notes taken by the researcher, on the design hypotheses. We report each design hypothesis, the items used for the evaluation and the final evaluation of each.

DH1 *Students can master mathematical formalism if supported by multiple representations (algebraic, geometric, diagrammatic).*

It is evident that most students manage to master the formalism with the help of one or more representations (see **S2** item A1, **S5** item A3.1, **S6** items C1 and D1, **S7** item A2, **S8** items A1, A3, B1, C1, D1). In general, the aspect of mathematical formalism was the one that caused the least difficulty.

DH2 *Constantly explaining the relationship between classical and quantum elements helps to exceed the classical approach and grasp the quantum characteristics proper.*

The transition from classical to quantum was extremely difficult when building the model (see Tab. 7.3 and Tab. 7.4). When it came to applying the model in the algorithms and the teleportation protocol, however, the classical-quantum dialectic enabled the students to grasp the quantum advantages (see DH5). This leads us to two evaluations, one intrinsic to TLS and one of a general nature: the first concerns the need for more time to be devoted to certain phases of construction: in particular those of polarisation and dual rail coding in relation to the concept of superposition first of all. Secondly, students are often not adequately supported by previous knowledge of classical physics.

DH3 *Students, if properly guided through specially designed materials, can construct the computational model using optical devices (half wave plates, phase shifters, beam splitters, polarising beam splitters).*

Considering the difficulty and the innovation of the proposed educational pathway, we find the results of the construction of optical circuits encouraging. In simpler designs, students operate well (see **S5** item A3.2, **S7** items D1 and D2, **S8** items B3 and E1); they struggle more if the optical circuit to be realized involves many devices, implements two registers and the students are not guided (see **D7** item D3, **S8** item A4). We think that the possibility of proposing laboratory activities, or possibly the construction of a specially designed simulation, can greatly facilitate these results.

DH4 *The presentation of algorithms and protocols focused on a concrete problem to solve, engaging students and inviting even less competent students to comprehension.*

Interviews with the teachers of the two classes and notes taken by the researcher confirm the engagement of students especially in lessons on algorithms and teleportation protocol. In addition, one teacher emphasized the involvement of students who normally have little interest in the subject.

DH5 *References to a concrete problem in the algorithms and protocols enable the advantages of quantum computation to be grasped.*

In general, students responded with satisfactory percentages to questions requiring them to explain the quantum advantage (see **S6** items A2, C4, D2 and E1, **S7** items A4.1, A4.2 and C2, **S8** items B2' and D2).

DH6 *The diagrammatic model appears to the students in its entirety.*

This is, to all intents and purposes, the hypothesis that we do not feel able to confirm. The proposed experiments still lack a strong experimental approach in the laboratory for the model to be complete. SS was the first example where this aspect entered into teaching, but our research is focused on curricular paths. However, what emerges from the two experiments is that about half the class is able to interpret and design diagrams both logically and experimentally and to connect their meaning, thanks to the designed worksheets, to the solution of real problems (see **S7** and **S8**).

7.6 Research questions and design principles

SRQ1 : *How is it possible to construct an adequate content simplification process to present the topics of the second quantum revolution to students in a meaningful way from very advanced theoretical aspects?*

Based on the design hypotheses and the data from the experiments, we conclude that to construct an adequate content simplification process it is needed:

1. greatly support the transition from classical to quantum without taking the correctness of prior knowledge for granted;
2. introduce a minimal quantum mechanical formalism (Dirac notation) appropriate for addressing qubits, quantum logic gates and measurements;
3. allow students to continuously grasp the dialectic between logical-formal and physical-experimental aspects in relation to real problems;
4. support the students' diagrammatic modelling process by clarifying at all times the interpretation being given (computational, physical, experimental);
5. support the understanding of algorithms and protocols through the elementaryization process. The case of the tripartition of information processing in the case of algorithms is emblematic of this approach.

The polarization-based approach seems adequate to support all these requests because we are able to construct a coherent and comprehensive educational pathway.

SRQ2 : *How effective is an integrated and multidisciplinary approach in order to enable students to understand some topics of quantum computation and quantum information?*

The data that emerged from the worksheets concerning algorithms and teleportation protocol seem encouraging. Despite various critical issues that emerged during the construction of the model, the concluding worksheets (those relating to Grover's algorithm and the teleportation protocol) show that a large number of the students obtained an adequate grasp of the topics of quantum computation and information, and that this grasp is often linked to the possibility of reading the problems with an integrated and multidisciplinary approach.

SRQ3 *Based on findings the first two research questions, what design principles can be formulated for the development of TLS resources in quantum computation for high school students?*

In the light of all the work that has been done and the answers to the first two research questions, we can finally state three design principles that we also believe should be foundational to TLS:

1. **Integrated and multidisciplinary approach**: aware that instruction on theoretical topics such as non-classical logic and circuits is an indispensable requirement for the acquisition of a functional understanding of related algorithms and protocols, however, there is a need to offer integrated educational paths allowing students to make a connection between abstract mathematical content stemming from a seemingly counterintuitive physical theory and the description of systems and networks that can possibly encode and process information. As a result, we choose to develop a learning trajectory where the study of logic/computation and the physical implementation of gates and algorithms progress in parallel, engaging students in the design of realistic circuits in which state preparation, transformation and measurement are performed by physically realistic devices. The climax of the learning trajectory is the diagrammatic model.
2. **Quantum physics and information processing**: the second principle for the design of our course concerns the understanding of the core content, which may be identified with the conceptual junction between quantum physics and information processing. This junction is represented by the physical and informational interpretation of a state vector lying in a two-dimensional Hilbert space equipped with linear dynamics. This leads us to our design principle: guiding students to encode information in different physical properties, examining how they are linked to the vectors of the abstract two-dimensional Hilbert space according to the property at hand, with the aim to build a global knowledge structure on the relation between physical properties and state space in the encoding procedure.
3. **Elementarization** About algorithms, we suggest to decompose the structure of the information processing phase into three sequential processes: (1) the enabling of parallelism by means of Hadamard gates on target register and ancilla qubit to generate an equal superposition of all the states of the computational

basis on which the oracle can act at once; (2) the transfer of the whole information encoded in the oracle function to the target register(s) in the form of a positive or negative sign attached to each basis vector, possibly establishing an entanglement between the qubits; (3) the enabling of interference by means of a network of logic gates to produce - often in concurrence with the exploitation of entanglement - the desired state on which measurement can be performed. About the teleportation protocol, we suggest that the case in which Alice does not share an entangled qubit with Bob be put before the discussion of the actual protocol. The particular feature of the protocol (the correlations between Alice's qubits and Bob's) will then emerge with greater emphasis directly from considerations made by the students than in the case examined in the first part.

Chapter 8

Summary, limits and future perspectives

We presented a research project that had as its first goal to build a course for the teacher professional development on the topics of quantum computation and communication with high cultural impact. Our research relied on the MER for the clarification and analysis of the scientific content, and in the design of educational pathways. An initial course conducted between October 2020 and March 2021 was able to collect research data on teachers' perspectives on the topics covered; it was also possible to create an initial group of teachers engaged in close collaboration with the researchers to co-design teaching paths for students. This resulted in a TLS for students and a series of research-based instructional materials. In particular, we have seen how the TLS and the worksheets were constructed on the basis of two well-defined theoretical frameworks: the IBL and the MBT to help students to develop an organized knowledge structure concerning QIS embedded in active and constant engagement in construction and reconstruction knowledge through hands-on interactions. Moreover, it was possible to start designing courses for first-year students on the physics of classical computation and third-year students on the problems of the thermodynamics of computation.
Although we have clearly divided the work done with teachers and the work done with students into two separate chapters, it has in fact been built up over time thanks to the continuous interactions between the two aspects. This was necessary because not enough educational research literature can be found on the subject of quantum technologies; but it is also our belief that only the active presence of teachers can create effectively realizable

pathways in the curriculum. The analysis of the research data, obtained from tests relating to the teacher training course, from the semi-structured interviews carried out several times during the training course, from the worksheets used in the two curricular experiments at Castel San Giovanni and Florence and those of the Summer School on Quantum Technologies, made it possible to answer at least in part some significant research questions both relating to the work carried out with teachers and to the work dedicated to students.
Specifically with regard to teachers, we observed that:

1. a process of elementarization of the proposed contents is necessary in view of the fact that they are substantially new to teachers; however, this process can only be supported by great formal and conceptual rigour. Dirac formalism and matrix algebra are considered adequate. Furthermore, the simplification must constantly live up to two aspects: the presentation of the educational materials and the continuous comparison with the experimental aspects. The diagrammatic model as a common framework for computational and physics topics seems to be a useful tool;

2. the interdisciplinary and integrated approach with a high cultural impact, the record of lessons and the materials for lessons, all supported by continuous work and discussions with researchers, have activated some teachers to develop a personal commitment to longitudinal, interdisciplinary educational innovation directed towards themes of quantum information and computation;

3. the constant presence of the researchers in the co-design work, the presence of one of them during the teaching experiments, the realization of training meetings for the teachers in service in the schools involved made it possible to get to know the environment, including school directors and to activate a shared and extensive collaboration. The result is the participation of other teachers from the same school in the training meetings. In this way, a reference figure, a kind of expert teacher, who coordinates, supported by the researchers, small working groups on the topics of the second quantum revolution has been established in the individual schools. We have not yet been able to establish a full-fledged community of practice, but we hope to be able to do so in the future given the latest developments.

In summary, the aspects of disciplinary content, the continuous co-design work and the presence of the researchers in the context of the schools helped to create an environment that could educate and activate some teachers on topics that were not only difficult but also almost completely unknown and traditionally not covered in the school curriculum, showing, if need be, how interesting and stimulating the topics could be.

The work carried out together with teachers produced a TLS on the topics of quantum computation and quantum communication that can answer three further research questions from whose answers we derive that:

1. to construct an adequate content simplification process it is needed:
   (a) greatly support the transition from classical to quantum without taking the correctness of prior knowledge for granted;
   (b) introduce a minimal quantum mechanical formalism (Dirac notation) appropriate for addressing qubits, quantum logic gates and measurements;
   (c) allow students to continuously grasp the dialectic between logical-formal and physical-experimental aspects in relation to real problems;
   (d) support the students' diagrammatic modelling process by clarifying at all times the interpretation being given (computational, physical, experimental);
   (e) support the understanding of algorithms and protocols through the elementaryization process. The case of the tripartition of information processing in the case of algorithms is emblematic of this approach.
2. a large number of the students obtained an adequate grasp of the topics of quantum computation and information, and that this grasp is often linked to the possibility of reading the problems with an integrated and multidisciplinary approach;
3. it is possible to identify three design principles that guide the TLS:
   (a) present an integrated and multidisciplary approach;
   (b) guiding students to encode information in different physical properties;
   (c) propose an elementarization for intruction of quantum algorithms and quantum protocols.

It is worth emphasizing that the approach of the proposed educational paths is ultimately based on the possibility of constructing a diagrammatic model whose strength lies in its theoretical abstractness linked to the theory of categories: it is precisely this abstractness that allows diagrams to be interpreted in the three aspects that are significant for us: logical, physical-theoretical and experimental.

8.1 Limits and future perspectives

There are some objective limitations to the work we have presented. The first and probably the most important one is that it is not yet able to present a pathway that introduces quantum physics from the informational approach in a strong way. We preferred to build a course that would fit into the tradition of quantum physics education (polarization approach) and go further in continuity with it in order to fully realize the integrated and interdisciplinary perspective. However, we are convinced that after a few implementations of the TLS, the time will come to review the entire sequence. However, the possibility of working simultaneously on the logical-computational, physical-theoretical and experimental aspects (albeit relating to ideal optical devices in the two curricular experiments) we believe is an extremely significant achievement and marks an absolute novelty in the panorama of educational research on these topics.
We report two further limitations, which, however, will also be the starting point for the main possible future developments:

1. the fact that only in the Summer School was it possible to work with real optical devices. The diagrammatic model only materializes when it also becomes experimental practice: thus the latter phase, which could also become the former, is absolutely fundamental and is missing from the two experiments presented. To a large extent, this was also due to the restrictions on the use of laboratories during the Covid pandemic19;

2. the diagrammatic model does not show its full potential in high school pathways. In fact, the possibility of working syntactically on diagrams is far from being exploited in a pathway for secondary school students.

These observations allow us to introduce what we see as the most promising future developments:

1. We believe that closer collaboration with the experimental area is necessary for the realization of experimental setups appropriate to the proposed course. Furthermore, it might be interesting to develop simulations to allow students to build experimental setups at least in a virtual laboratory;

2. we think that the TLS can be developed in such a way that it can also become suitable for university paths for non-physicists: the categorical approach, for example, could

fit very well for computer scientists. Furthermore, the possibility of describing non-physical processes by means of diagrams also makes the pathway interesting for science faculties, such as chemistry, which are only apparently less involved.

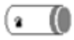
Appendix A

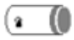
Worksheets: introduction to quantum computation

S2 Polarization gates

A. Implementation of a qubit on the linear polarization of the photon

A.1 Determine the action of a phase shifter of π on the state vector of prepared photons as follows

 $|1\rangle$

 $\dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$

Write the resulting vector in the space to the right of the phase shifter symbol and explain:

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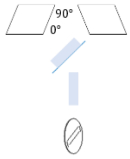
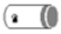
A.2 Propose an implementation of Z logical gate using one or more of the following devices:



- two calcite crystals, direct and reverse, with channels at 0° e 90°
- non-phase mirror
- a phase-shifter
- a filter can be directed as desired

Represent the apparatus to the right of the single-photon source, explaining its functioning:

$|\psi\rangle = a|0\rangle + b|1\rangle$ $|\psi\rangle = a|0\rangle - b|1\rangle$



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A.3 Propose a realization of the X and H gates on the polarization of a photon, specifying the devices used and the conditions of their use.

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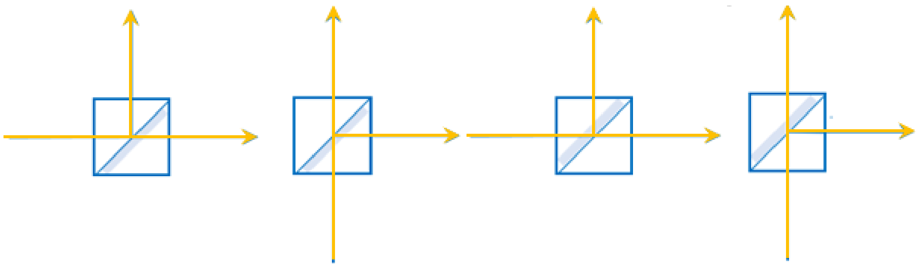
S4 – Propagation and coding on dual-ail

A1. For each of the figures

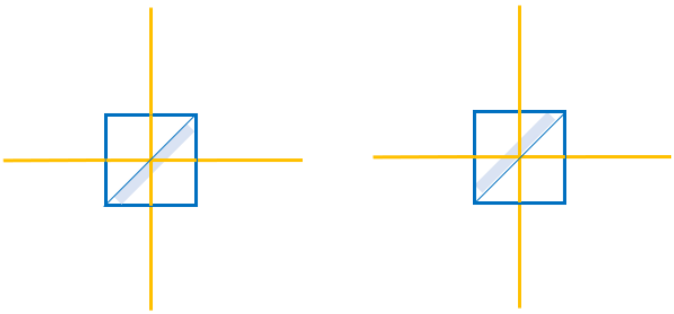


- identifies which side of the interface the adhesive is on ($n_1 < n$) and on which side the dielectric ($n_2 > n$), writing the refractive index of each on the corresponding side of the interface;
- determines whether the reflection is without phase shift ($\Delta\varphi = 0$) or with phase shift ($\Delta\varphi = \pi$) and report it on the reflected beam.

A2.1 For each of the two figures, label the incoming and outgoing arms according to the phase of the reflected rays

- 0: branches corresponding to reflection without phase shift;
- 1: branches corresponding to reflection with phase shift.

If it is helpful, help yourself by indicating the refractive index value on each prism



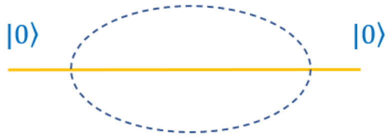
A2.2 Consider two arms with the same label. Does the value assigned depend on the direction in which the light beam passes through these arms? If yes, explain how it varies depending on the direction, if no, explain why.

S5 – Gates and optical circuits in dual rail

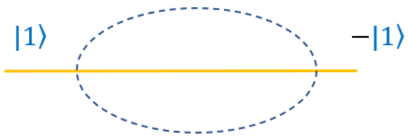
A. Implementation of a qubit on the two possible paths of a photon

A1. Design a Z-gate in dual rail coding. For each sub-question, examine the initial and final states. Is a tool needed to produce the required transformation? If yes, which one? You have calcite crystals, phase-free mirrors, filters, phase shifters. Explain your choices.

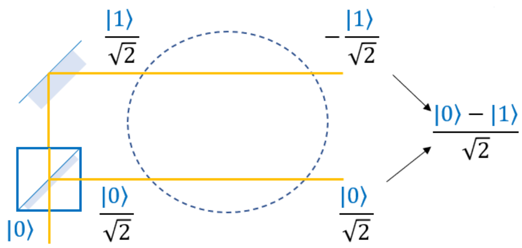
A1.1 State prepared in $|0\rangle$



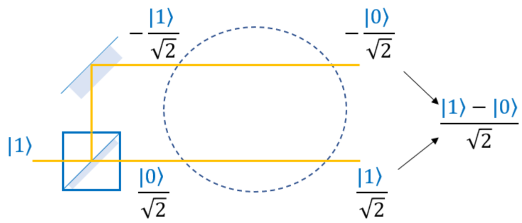
A1.2 State prepared in $|1\rangle$



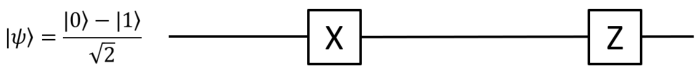
A1.3 State prepared in $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$



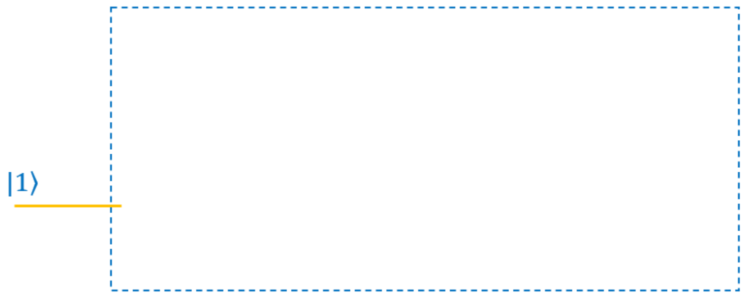
A2. Identifies the correct symbolic representation of the X gate on a state prepared in $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$ and explain



A3.1 Solve the logic circuit, reporting the state after each gate:



A3.2 Design the corresponding optical circuit with encoding dual rail. Explain each step:



A4. A colleague of yours last year made the following statement: «Saying that the state is prepared in $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ or in $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$ is equivalent to saying that it was initially prepared in $|0\rangle$ or in $|1\rangle$ and passed an H-gate ». Do you agree with your colleague? Justify your answer.

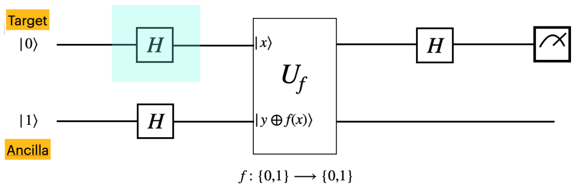
Appendix B

Worksheets: quantum algorithms and quantum teleportation

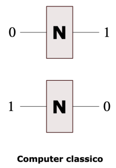
S6 - DEUTSCH ALGORITHM

Quantum parallelism

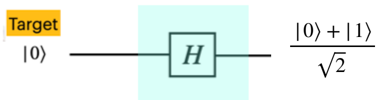
Let us consider the first part of the circuit.



Classically, if we want to know the result of applying a Boolean function $f : \{0{,}1\} \longrightarrow \{0{,}1\}$ on a certain number x two classical calculations are required to find the result: for x = 0 and for x = 1. For example, the logic gate Not:



A. (Input encoding) Let us now consider the first part of the algorithm circuit:



Question A1. In relation to the coin problem: which side(s) of the coin is/are encoded in the output state of the first Hadamard? (Remember that according to the database encoding silver face = 0; gold face = 1)

1. Silver face
2. Golden face
3. Both

Explain why:

Question A2. Establish whether we gain an advantage over the classical case by using the H gate. If yes, in what does it consist? If no, why?

B. (Oracle) From the considerations made in previous meetings, we know that a quantum logic gate acts on a generic qubit as in the figure:

$$a|0\rangle + b|1\rangle \; — \boxed{O} — \; aO(|0\rangle) + bO(|1\rangle)$$

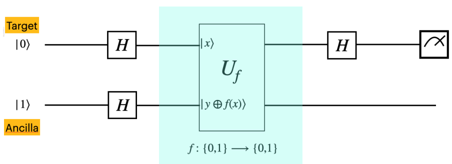
In the case of the quantum logic gate Not (classical example on the previous page), therefore, we obtain

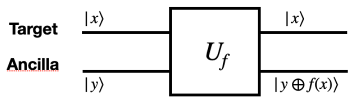
$$|1\rangle \; — \boxed{H} \xrightarrow{\frac{|0\rangle - |1\rangle}{\sqrt{2}}} \boxed{X} — \; \frac{|1\rangle - |0\rangle}{\sqrt{2}}$$

Question B1. What property of the operators ensures that the quantum advantage of being able to act simultaneously on both qubits of the computational base can actually be exploited? Justify your answer.

Oracle and compound systems:

To encode the output information exclusively on the target register, associating each side of the coin (silver and gold) with the corresponding image (heads or tails) we use the sign: + heads, - tails.
To do this we introduce an auxiliary register (qubit ancilla) and define the oracle as follows:





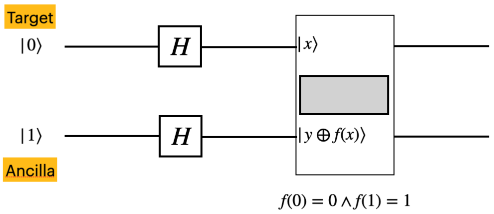
As can be seen on the ancilla, the action of the operator U depends on the value of f on the target. The target remains unchanged.

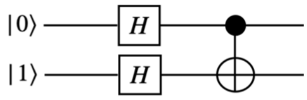
C1. The operator U is a logical gate with two inputs and two outputs. Writing the truth table (the behaviour of the operator on $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$) of the logical gate in case $f(0) = 0 \wedge f(1) = 1$.

Input	**Output**
$\|00\rangle$	
$\|01\rangle$	
$\|10\rangle$	
$\|11\rangle$	

Are there any logic gates of your knowledge that operate in this way? If yes, please specify which. If no, explain why.

So in this case, the corresponding circuit is (insert name in the grey box)





Performing the calculations we obtain:

$$\frac{|0\rangle}{\sqrt{2}}|1'\rangle + \frac{|1\rangle}{\sqrt{2}}(-|1'\rangle)$$

The minus sign (which we remind we want to use to encode the image on the two sides of the coin) can be transferred from the ancilla to the target, the register on which the measurement will be performed to obtain the output needed to answer the problem.

$$\frac{|0\rangle}{\sqrt{2}}|1'\rangle - \frac{|1\rangle}{\sqrt{2}}(|1'\rangle) = (\frac{|0\rangle - |1\rangle}{\sqrt{2}})|1'\rangle$$

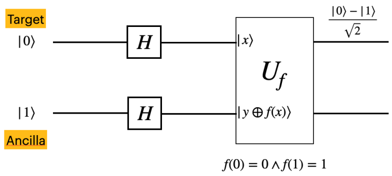
C2. What feature of quantum physical systems is exploited?

C3. To which state vector does the minus sign in the compound state belong?

1. only to the target
2. only to the ancilla
3. target ⊗ ancilla

Explain:

Let us look again at the coin problem from which we started.
On the way out of the oracle, thanks to the property of compound systems, on the target register we find the state we have just obtained

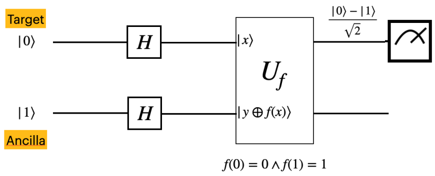


C4. What image(s) on the face of the coin tell us about the status of the target coming out of the oracle?

1. Heads
2. Tails
3. Heads and Tails

Explain:

This state stores as much information as possible about the coin to determine whether it is genuine or counterfeit: the qubits encode the sides of the coin, the respective sign the image contained within it.



C5 If we implemented the circuit a large number of times in the same initial condition and measured the target, could we know whether the coin was genuine or counterfeit?

Generalization

We can generalize what we have seen in the specific case where $f(0) = 0 \wedge f(1) = 1$ e and obtain as an output state from the oracle $\frac{1}{\sqrt{2}}((-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle)$

The meaning of which is: the sign depends on how it acts f!

It is f to determine which image is present on each side of the coin.

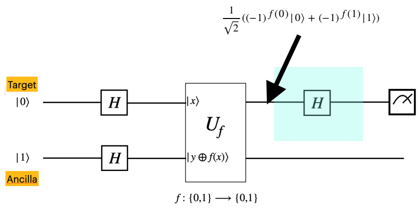
In particular if f is balanced ($f(0) \neq f(1)$) (authentic coin!) the signs will be opposite, otherwise ($f(0) = f(1)$) concordant (counterfeit coin!).

It follows that the qubits coming out of the oracle will be
1) $\pm|1'\rangle$ whether the function is balanced and thus whether the coin is authentic in the coding described for the database;

2) $\pm|0'\rangle$ if the function is constant and thus if the coin is not authentic in the coding described for the database.

As we have seen, this is not enough to determine whether the coin is genuine or counterfeit. However, we can exploit a further characteristic of quantum systems: interference!

Interference



We know in fact that the Hadamard gate on a superposition state creates interference phenomena such that $|0'\rangle \longmapsto |0\rangle$ e $|1'\rangle \longmapsto |1\rangle$.
Knowing that the final state before the last Hadamard is $|\psi\rangle = \frac{1}{\sqrt{2}}((-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle)$,

D1. Complete the table

Boolean function	State after the Oracle	State after the last Hadamard	Classic bit after measurement and probability
$f(0) = 0 \wedge f(1) = 0$	$|\psi\rangle =$ ________ = _____	$|\psi\rangle_{fin} =$ _________	____ , p= ___
$f(0) = 0 \wedge f(1) = 1$	$|\psi\rangle =$ ________ = _____	$|\psi\rangle_{fin} =$ _________	____ , p= ___
$f(0) = 1 \wedge f(1) = 0$	$|\psi\rangle =$ ________ = _____	$|\psi\rangle_{fin} =$ _________	____ , p= ___
$f(0) = 1 \wedge f(1) = 1$	$|\psi\rangle =$ ________ = _____	$|\psi\rangle_{fin} =$ _________	____ , p= ___

D2. By observing the table above, establish what relationship there is between the authenticity of the coin and the outcome of the measurement. Explain.

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Final conclusions

E1. How many times does the quantum operator U_f have to be implemented to determine whether a coin among those in the database is genuine or counterfeit? What is the advantage over classical computation?

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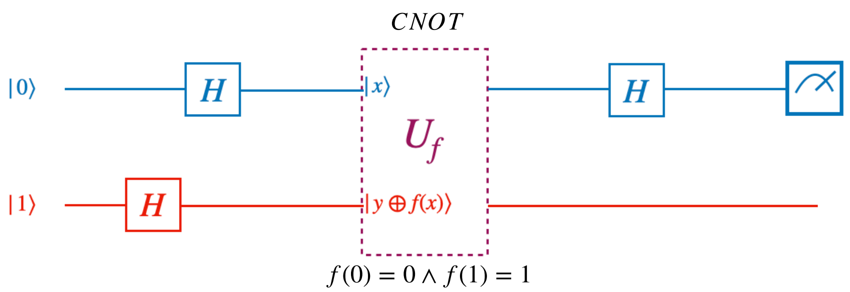
Implementation with optical devices



F1. Describe the optical devices used to realize the optical circuit of Deutsch's algorithm (in the case under consideration described by the figure in the table below) in the order in which they are encountered in the circuit. Explain their function in relation to the logical development of the algorithm.

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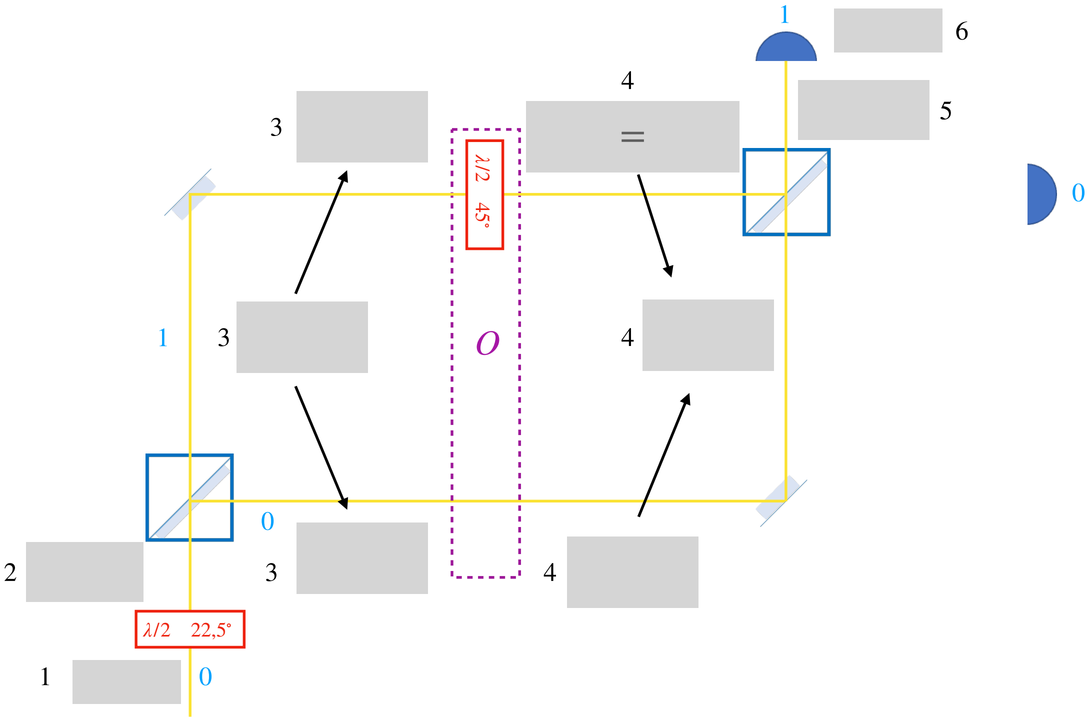
F2. Enter in the grey boxes the corresponding state in Dirac notation consistent with the action of the individual optical devices (including the device used for measurement). Also express the classical information bit obtained and its probability.



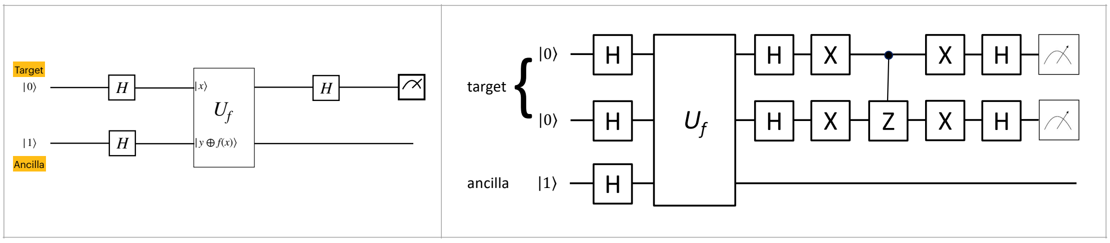
S7 - GROVER'S ALGORITHM

The goal of the algorithm is, in this case, to determine the microchipped bar from among the four in the bank's caveau.
It should also be reminded that the $f : \{0,1\}^2 \longrightarrow \{0,1\}$

$$f(x) = \begin{cases} 1 & if\ x = x_0 \\ 0 & otherwise \end{cases}$$

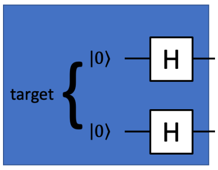
is defined on the codes for the four bars at values in $\{0,1\}$ and identifies, in the database, the only bar with a microchip by returning 1 (which in the discussion of our case is for simplicity's sake the one corresponding to the coding $x_0 = (1,0)$).

In order to deeply understand the behaviour of Grover's search algorithm, we shall use what was done in the analysis of Deutsch's algorithm; the processes, in fact, remain the same.
First, we present the two circuits related to Deutsch's and Grover's algorithm:



Quantum Parallelism

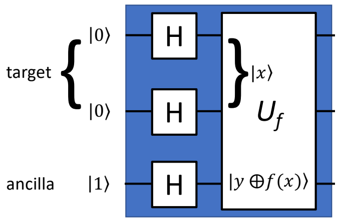
Let us therefore consider the first part of the circuit relating to the target register:



$$|00\rangle \longmapsto \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

Question A1: In relation to the microchip bar problem: which bar(s) is/are encoded in the output state by the two Hadamards on the target register? Explain

To encode the output information exclusively on the target register, we use the sign: + without microchip, - with microchip.
To do this, we introduce an auxiliary register (qubit ancilla) and define the oracle as follows:



Question A2: Coming out of the Hadamard gate on the ancilla the state is $|y\rangle = \dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$.

Denoted $|y \oplus f(x)\rangle$ the output status of the oracle on the ancilla, complete the following table:

Bar	Function	Output state
Without microchip	$f(x) = \ldots\ldots$	$|y \oplus f(x)\rangle = \ldots\ldots\ldots$
With microchip	$f(x) = \ldots\ldots$	$|y \oplus f(x)\rangle = \ldots\ldots\ldots = \ldots\ldots\ldots$

Question A3: In the oracle step, how does the status of the ancilla corresponding to $|10\rangle$ (qubit related to the bar with microchip)?

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Question A4.1: How many times is it necessary to implement the oracle to complete the previous table? Explain.

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Question A4.2: Do we gain an advantage over the classical case? Explain.

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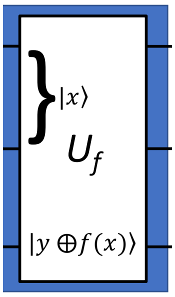
Oracle and compound systems

From the above considerations, the following state will arise out of the oracle



$$\frac{1}{2}(|00\rangle \otimes |1'\rangle + |01\rangle \otimes |1'\rangle + |10\rangle \otimes (-|1'\rangle) + |11\rangle \otimes |1'\rangle) \quad (1)$$

that we can immediately write

$$\frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle) \otimes |1'\rangle \quad (2)$$

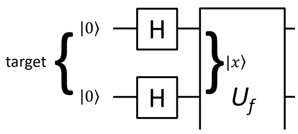
Question B1: What feature of quantum systems allows us to go from (1) to (2)?

Question B2: To which register does the minus sign in the compound state belong?

1. only to the target
2. only to the ancilla
3. target $\otimes$ ancilla

Explain:

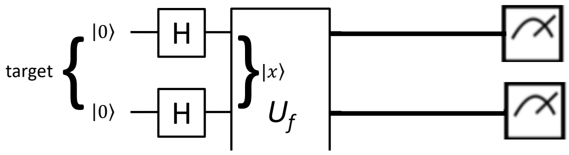
Let us return to the bar problem from which we started.
On exit from the oracle, thanks to the property of compound systems, on the target register we find the state we have just obtained



$$\frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$$

Question B3: Which bar(s) in the *caveau* tells us the status of the target register coming out of the oracle? What information does it give us? Explain.

This state therefore contains all possible information about the bars to determine whether they are without or with a microchip: the qubits encode the bars, the respective sign the presence or absence of the microchip.



Question B4: If we implemented the circuit a large number of times in the same initial condition and measured the target, could we know which bar is microchipped? Explain

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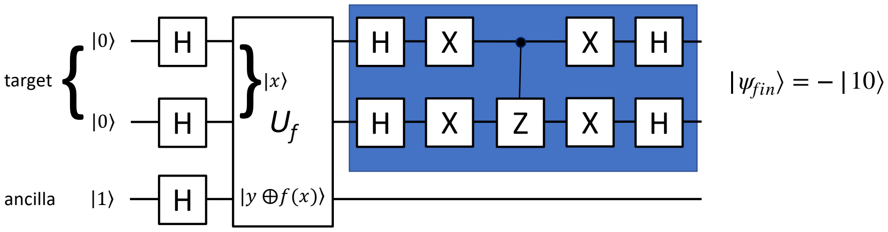
Interference



The remaining part of the circuit is used to create interference so as to have a state whose measurement allows the problem to be solved.

Question C1: By inserting the detectors at the end of the circuit, what pair of classical bits would we obtain? How does this allow us to solve the problem of determining the bar with microchips? Explain.

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Final conclusions

Question C2: How many times does the quantum operator U_f have to be implemented to find the microchip bar among those in the database? What is the advantage over classical computation?

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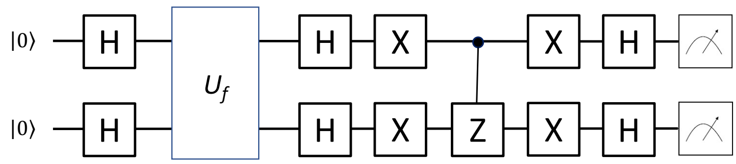
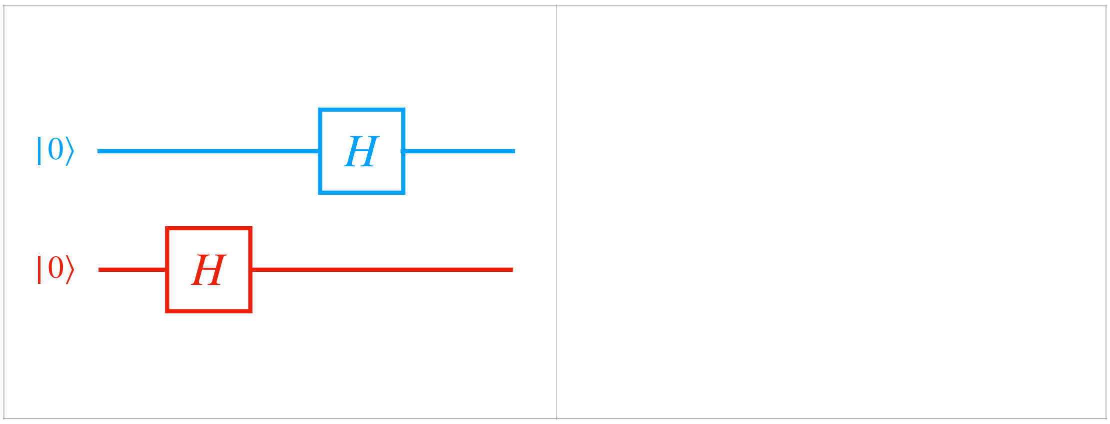
Implementation with optical devices

In the optical device realization of the algorithm, we will not use the qubit ancilla. This is to avoid the need for an additional encoding (3 qubits!). This means that the oracle will have to be made in such a way that it directly shifts to the state corresponding to π the microchip bar (in general, therefore, we will have 4 different oracles, but we will always assume that the qubit corresponding to the microchip bar is $|10\rangle$).

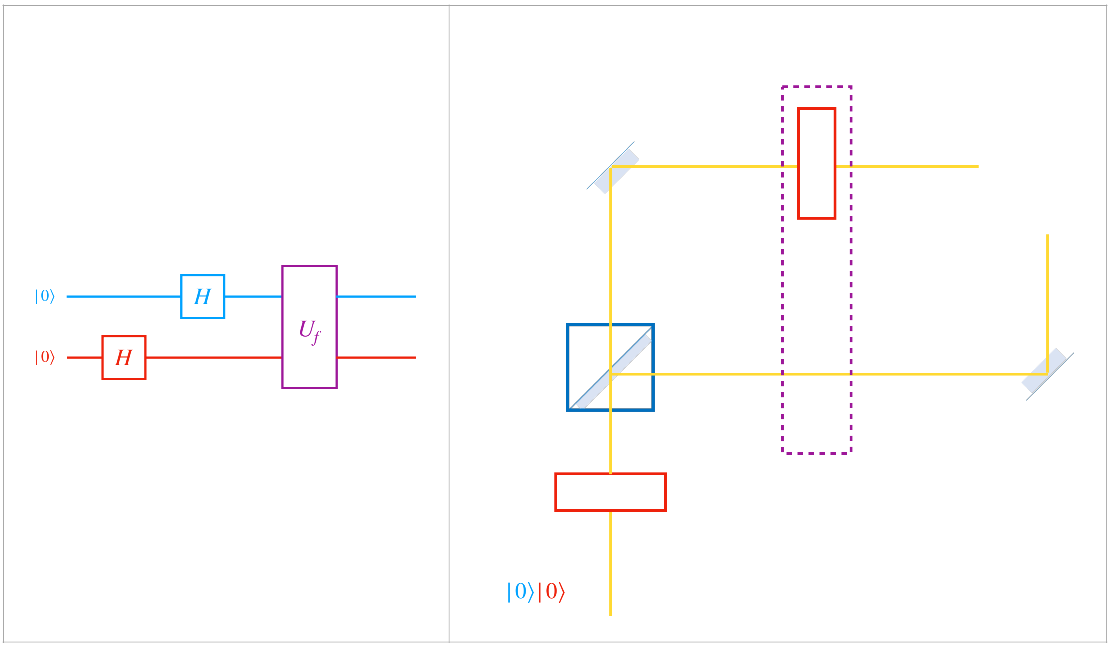


Question D1: Build an optical device able to realize the first part of the circuit, according to known colour conventions (blue=paths - red=polarization). Then insert along the paths the corresponding states in Dirac notation.



Explain:

Question D2: The second part of the circuit refers to the oracle. Consider the case where the bar with the microchip is the one corresponding to the coding $|10\rangle$ and complete the circuit in the figure by inserting the missing parts (Hint: be careful, because the plate acting on the polarization must only change sign to that component if it is equal to $|0\rangle$. What is the axis of symmetry that performs this process?).

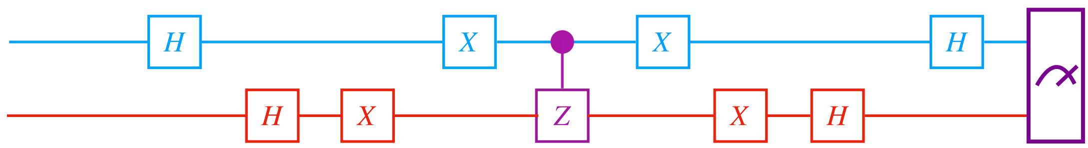


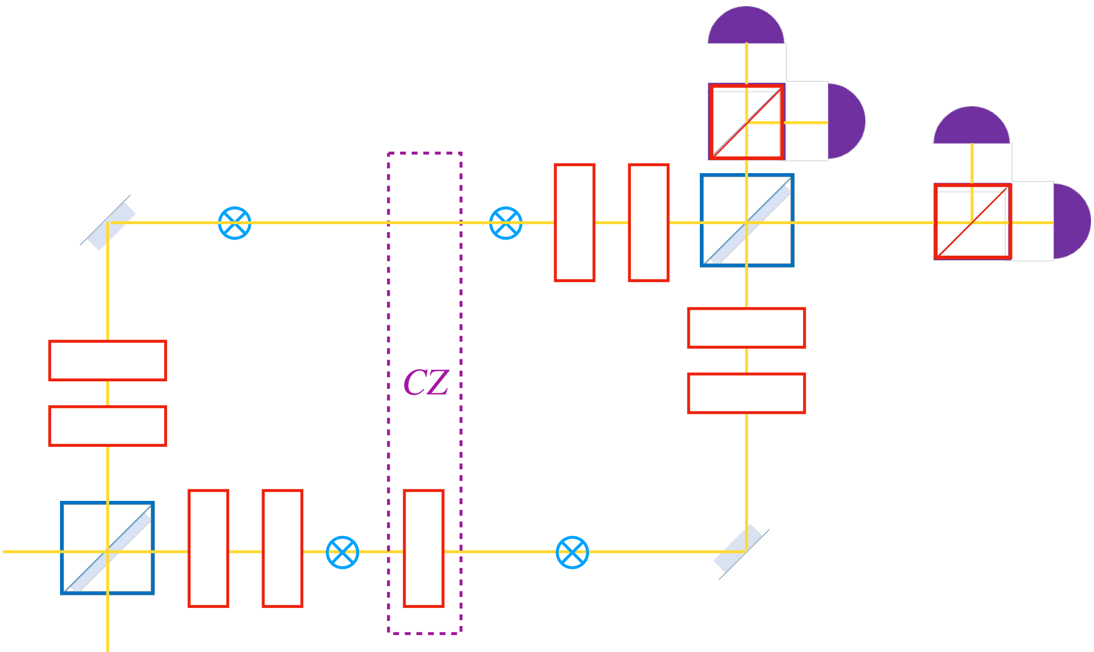
Explain the oracle design:

Question D3: Building the oracle in the situation where the qubit to which the sign is to be changed is the one indicated in each individual cell

$\|0\rangle\|0\rangle$	$\|0\rangle\|1\rangle$	$\|1\rangle\|1\rangle$

Question D4: The last part of the circuit corresponds to interference and measurement. Complete (enter the angles of the plates, write next to the physical devices their role with respect to the logic circuit, indicate next to the corresponding detector the classical bit pair (1,0) detected) the circuit in the figure corresponding only to the part after the oracle (Hint: remember that the CZ gate changes sign only at the qubit $|11\rangle$).

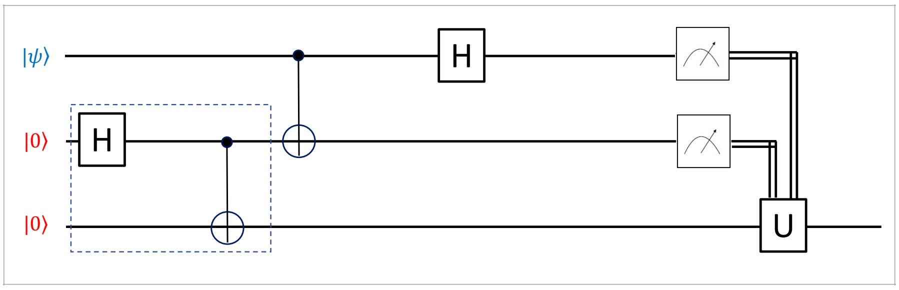




Explain

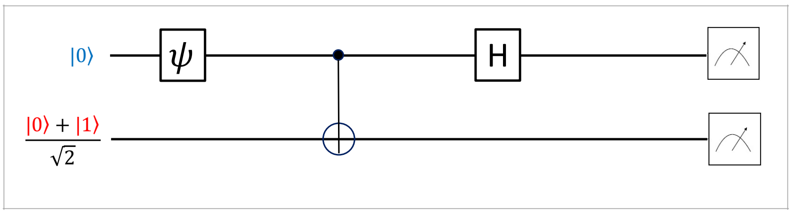
S8 - QUANTUM TELEPORTATION

The goal of the protocol is to teleport the information stored in Charlie's qubit (First register) from Alice (Second register) to Bob (Third register) using classical communication.



Preparatory activity: Charlie and Alice

Let us therefore consider the first two registers, those corresponding to Charlie and Alice. Charlie prepares a qubit in the generic state and delivers it to Alice (the state is unknown to Alice!); Alice has a qubit in superposition as in the following image:



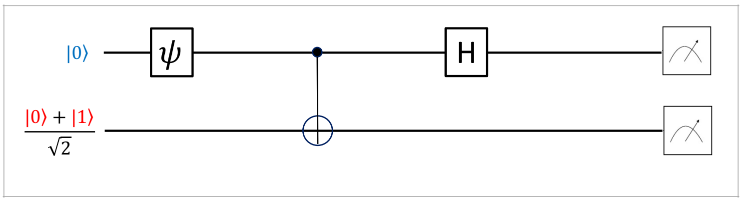
Question A1: Develop in the box below both registers together in Dirac notation (preserving the colours as in the figure) showing that

$$|0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \cdots \quad \longrightarrow \quad \cdots \quad \frac{(a+b)|0\rangle + (a-b)|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Question A2: Establish whether the state composed immediately before the Hadamard is separable or entangled.

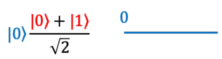
Question A3: Establish, when measuring, which classical bits are obtained on each register, and with what probability.

If we encode Charlie's information on dual-rail and Alice's in polarization, we then obtain the optical circuit



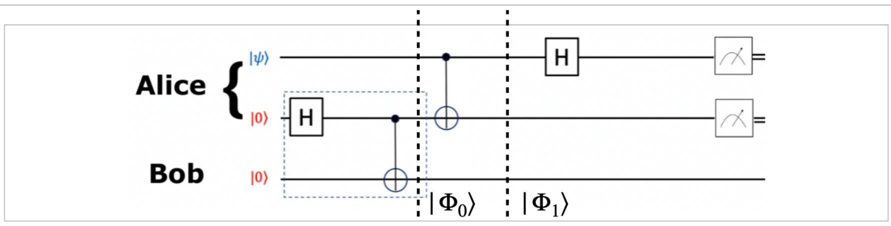
And the state $\frac{(a+b)|0\rangle+(a-b)|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ before measurement:

Question A4: Design the corresponding optical circuit starting from the left, with particular attention to the orientation of any beam-splitters. Help yourself with the explicit calculation in which the encodings are expressed.



Entanglement: Alice and Bob share an entangled state

Let us now consider the entire circuit with the three registers.
Compared to the previous case, the difference in the preparation is that the state held by Alice is entangled with Bob's, as can be seen in the dashed line; we will assume already prepared the state $|\psi\rangle$ (prepared by Charlie and delivered to Alice):



The corresponding entangled state is $\frac{|0\rangle|0\rangle+|1\rangle|1\rangle}{\sqrt{2}}$

We can see that the formal development is similar to the previous one

Question B1: Complete the second row of the table by highlighting the formal analogy with the case discussed above in the first row.

$\vert\Phi_0\rangle$	$\vert\Phi_1\rangle$
$(a\vert 0\rangle+b\vert 1\rangle)\otimes\frac{\vert 0\rangle+\vert 1\rangle}{\sqrt{2}}=\frac{a\vert 0\rangle(\vert 0\rangle+\vert 1\rangle)+b\vert 1\rangle(\vert 0\rangle+\vert 1\rangle)}{\sqrt{2}}$	$\frac{a\vert 0\rangle(\vert 0\rangle+\vert 1\rangle)+b\vert 1\rangle(\vert 1\rangle+\vert 0\rangle)}{\sqrt{2}}=(a\vert 0\rangle+b\vert 1\rangle)\otimes\frac{\vert 0\rangle+\vert 1\rangle}{\sqrt{2}}$

Question B2: Is the obtained state $\frac{a|0\rangle(|00\rangle+|11\rangle)+b|1\rangle(|10\rangle+|01\rangle)}{\sqrt{2}}$ a separable state?

If yes, write it as a state product. If not, explain why.

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Question B2': So what is the effect of Alice and Bob sharing two entangled qubits?

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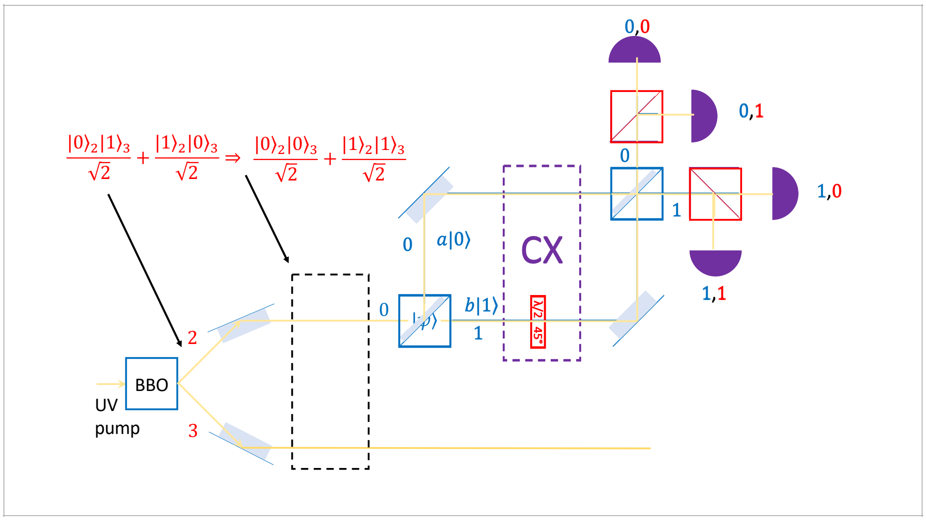
To achieve an entangled state, non-linear crystals are used that emit

entangled photon pairs in the state $\frac{|0\rangle|1\rangle + |1\rangle|0\rangle}{\sqrt{2}}$

Question B3: Insert the optical device that generates $\frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}}$

the entangled state used by Alice and Bob in the teleportation protocol



Measurement

Alice has both the qubit delivered by Charlie (first register) and the entangled one (second register) shared with Bob. Thus, the state controlled by Alice is the one corresponding to the first two registers; the third is related to Bob.

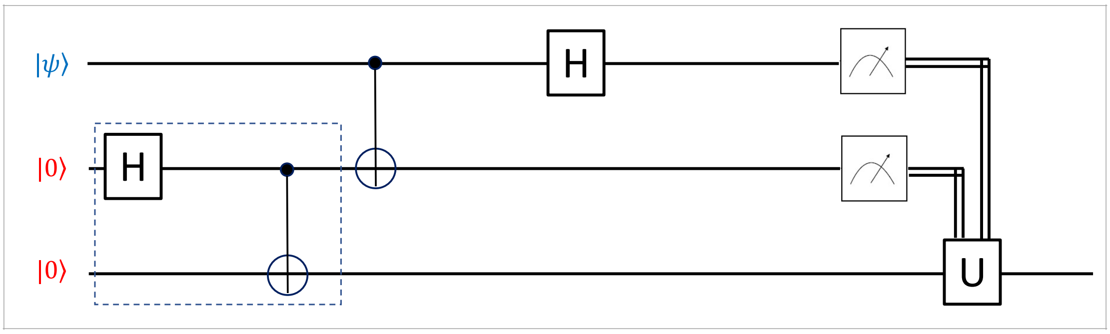
The state immediately before the two detectors (after even the last Hadamard gate) is therefore

$$1/2[|0\rangle|0\rangle(a|0\rangle + b|1\rangle) + |0\rangle|1\rangle(a|1\rangle + b|0\rangle) + |1\rangle|0\rangle(a|0\rangle - b|1\rangle) + |0\rangle|1\rangle(a|1\rangle - b|0\rangle)]$$

Question C1: Complete the following table

Alice's Qubit before measurement	Measurement outcome	Probability outcome	Bob's Qubit after measurement

Correction



The result of Alice's measurement is to collapse Bob's state into one of the four states in the last column of the table above.
At this point, Alice communicates the measurement result to Bob via a classical communication channel.
Depending on Alice's communication, Bob can correct his qubit to reconstruct the initial state shared by Charlie with Alice.

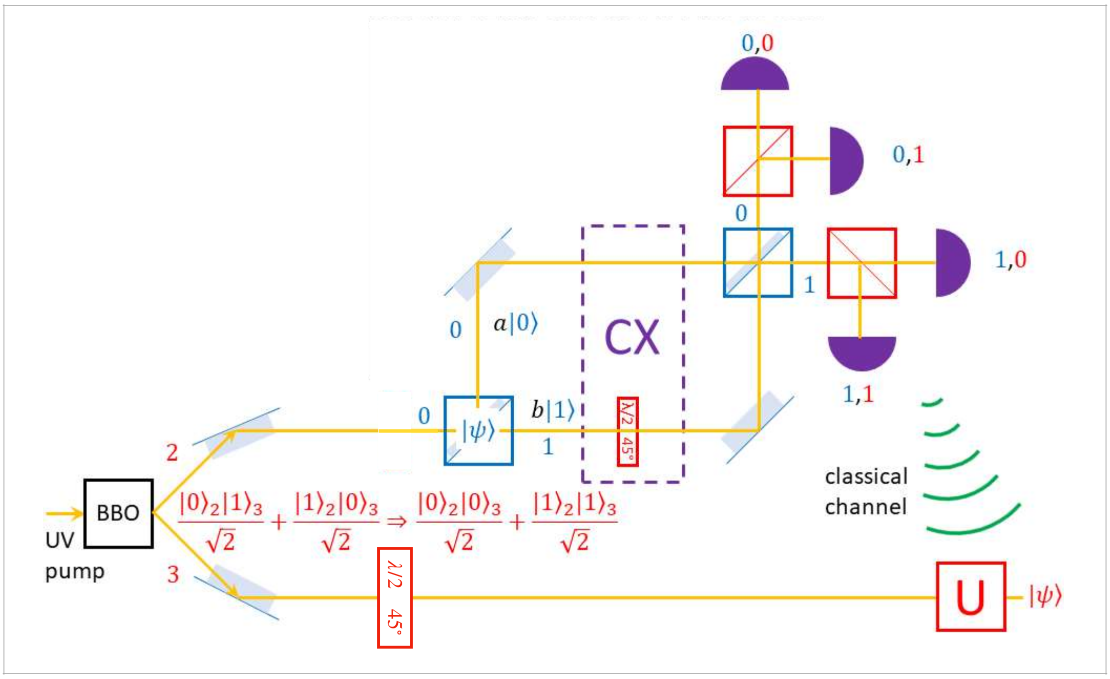
Question D1: Complete the following table

Charlie's Initial Qubit	Alice's bit pair after measurement	Bob's Qubit before correction U	Transformation(s) required for correction (U)

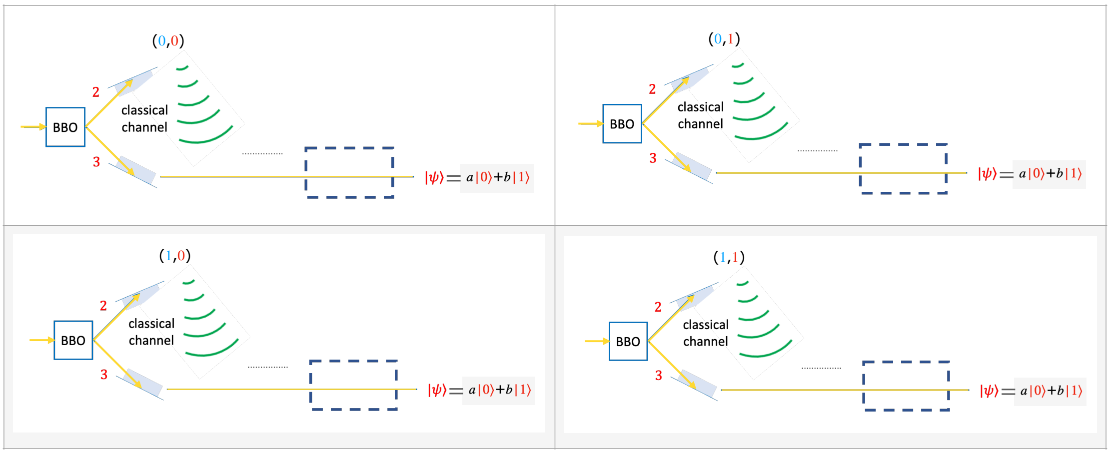
In this way Bob obtained the qubit prepared by Charlie.

Question D2: A student in another class said: "Bob can get information about $|\psi\rangle$ instantly". Do you agree? Explain why.

Implementation with optical devices



Question E1: In each of the four cases, express the state possessed by Bob after Alice's measurement and insert, if necessary, one or more devices to realize the corrections (U).



Question E2: Bob's reconstructed state is encoded in polarization while Charlie's original state was encoded in dual-rail. Do we deduce from this is that he was teleported? Explain.

Appendix C

Category theory for quantum computation

C.0.1 Definition of category and diagrammatic representation

In this appendix we present the basic notions in category theory which appear in the continuation of the thesis. The reference to the recipe is in chapter 3. To start modelling our recipe, we need an environment to define the ingredients and possible procedures on them. In addition, we need to consider the possibility of operating several actions in succession: for example we can take the *guanciale*, dice it and then put it in a pan. Mathematically, what we need is objects, transformations (including identity) and sequential composition.

Definition C.1. A *Category* $\mathfrak{C}$ consists of the fallowing data:

- *Objects*: $A, B, C, ...$, constituting the collection $Ob(\mathfrak{C})$;
- *Arrows* or *morphisms*: $f, g, h, ...$, constituting the collection[1] $Ar(\mathfrak{C})$;
- a pair of mapping *dom*, *cod* : $Ar(\mathfrak{C}) \longrightarrow Ob(\mathfrak{C})$ which to each arrow f assign its domain and codomain. If $f : A \longrightarrow B$ we call $A = dom(f)$ and $B = cod(f)$.
  $\forall A, B \in Ob(\mathfrak{C})$ we define

  $$\mathfrak{C}(A, B) := \{f \in Ar(\mathfrak{C}) | f : A \longrightarrow B\}$$

  This set is the ***hom-set***;

[1] Regarding the need to refer to a collection and not to a set of objects and morphisms see [93] pag. 3 or [101] pag. 6

- for any object $A \in Ob(\mathfrak{C})$, a ***identity morphism*** $id_A : A \longrightarrow A$ is designated;
- for any pair of morphisms $f \in \mathfrak{C}(A, B)$ and $g \in \mathfrak{C}(B, C)$, there is an arrow $h \in \mathfrak{C}(A, C)$ ***composition*** of f and g:

$$h := g \circ f : A \longrightarrow C$$

These data are required to satisfy the following axioms:

Unit: $\forall\ f \in \mathfrak{C}(A, B),\ \ f \circ id_A = f = id_B \circ f;$

Associativity: $\forall\ f \in \mathfrak{C}(A, B),\ \forall\ g \in \mathfrak{C}(B, C),\ \forall\ h \in \mathfrak{C}(C, D),$

$$(h \circ g) \circ f = h \circ (g \circ f)$$

A more rigorous example is that of the category ***Set*** whose objects are the sets and morphisms the functions between sets. We can introduce a diagrammatic language to represent a generic category that will be extremely useful below and is a distinctive characteristic of the work we present[2].
Following the definition, we represent the *data* as in Fig.C.1. Similarly, the axioms in

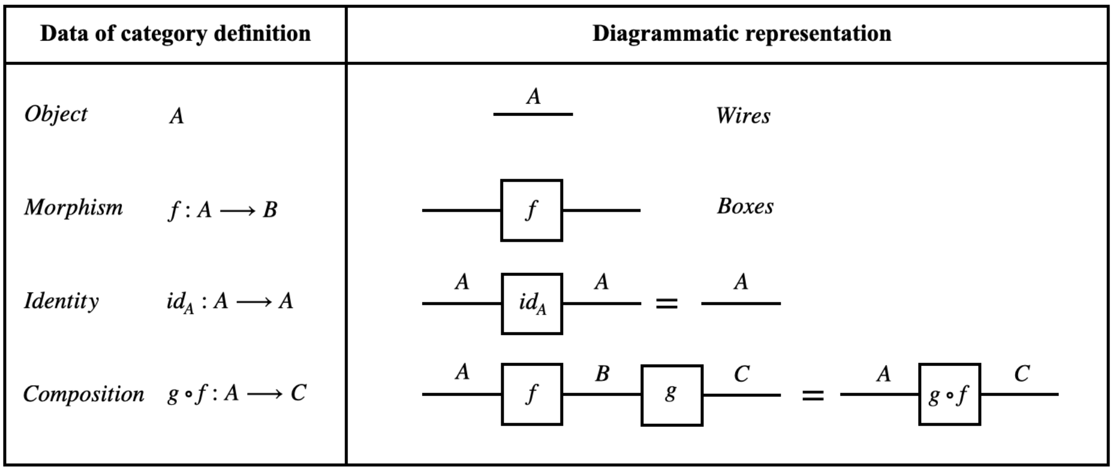


FIGURE C.1: Diagrammatic representation of *data* in the category $\mathfrak{C}$ definition

Fig.C.2.

[2]The abstract representation has immediate interpretation both in our recipe and in the category ***Set***

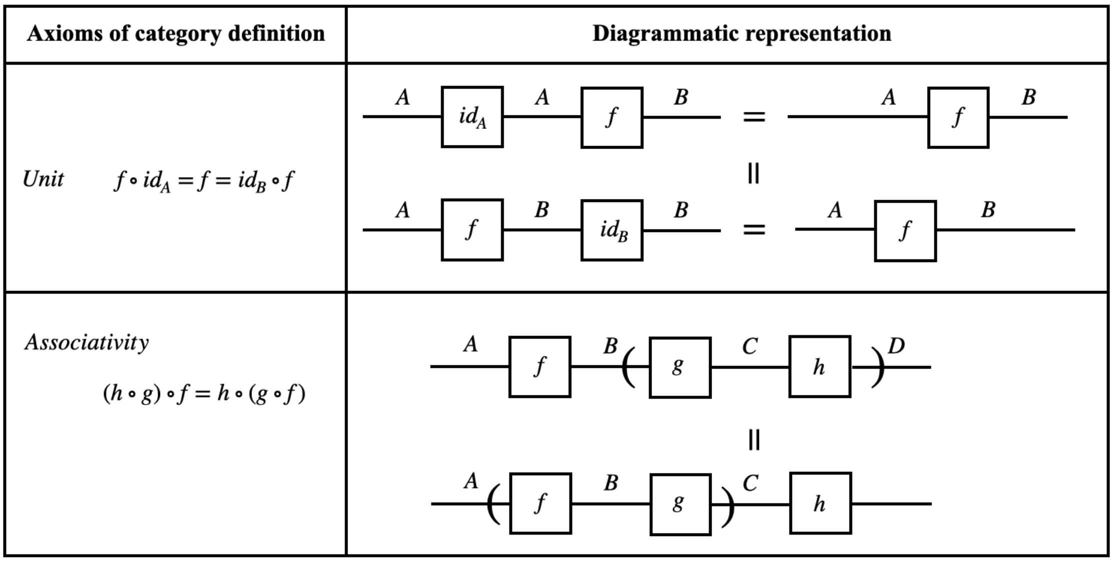


FIGURE C.2: Diagrammatic representation of *axioms* in the category $\mathfrak{C}$ definition

C.0.1.1 Useful notions

In this section we will introduce the fundamental definition that will be used with regard to the notion of category. Despite their significance in ***Set*** is straightforward, their use in the following chapters will clarify their meaning.

Definition C.2. An **isomorphism** in a category $\mathfrak{C}$ is a morphism $f : A \longrightarrow B$ for which there exists a morphism $g : B \longrightarrow A$ satisfying:

$$g \circ f = id_A, \quad and \quad f \circ g = id_B$$

Since the inverse is unique, we write $g = f^{-1}$. Moreover, we will write $A \cong B$, if there exist the isomorphism f.

An interesting aspect is that it is possible to reverse the directionality of arrows within a category $\mathfrak{C}$ without breaking the conditions of being a category:

Definition C.3. Given a category $\mathfrak{C}$, the **opposite category** $\mathfrak{C}^{op}$ is defined such that

- $Ob(\mathfrak{C}^{op}) = Ob(\mathfrak{C})$
- $\mathfrak{C}^{op}(B, A) := \mathfrak{C}(A, B)$

Composition and identities are received from $\mathfrak{C}$. The axioms are trivially satisfied.

Another construction that will later be used to extend the category concept to the monoidal case is that of ***Product Category***:

Definition C.4. Given two categories $\mathfrak{C}$ and $\mathfrak{D}$, their *product* is the category $\mathfrak{C} \otimes \mathfrak{D}$, for which:

- $Ob(\mathfrak{C} \otimes \mathfrak{D})$ is a collection of pairs (A, B) of object $A \in Ob(\mathfrak{C})$ and $B \in \mathfrak{D}$;
- $Ar(\mathfrak{C} \otimes \mathfrak{D})$ is a collection of pairs (f, g) of morphisms with $f \in \mathfrak{C}(A, C)$ and $g \in \mathfrak{D}(B, D)$

Some basic construction will be extremely useful later. To begin with, let us consider the notions of ***initial*** and ***terminal objects***:

Definition C.5. An object X in a category $\mathfrak{C}$ is

- *initial* if for every object $A \in Ob(\mathfrak{C})$ there is exactly one morphism $X \longrightarrow A$;
- *terminal* if, dually, for every object $A \in Ob(\mathfrak{C})$ there is exactly one morphism $A \longrightarrow X$.

If a category has these structures, these are unique up to isomorphism.
In ***Set*** any 1-element set is a terminal object, and the empty set is the initial object.

Definition C.6. Given two objects A and B in a category, a ***product*** is an object P with morphisms $P \xrightarrow{p_A} A$ and $P \xrightarrow{p_B} B$, s.t. if $X \xrightarrow{f} A$ and $X \xrightarrow{g} B$, there exists only one morphysm $u : X \longrightarrow P$ making both triangles commute[3]:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "u"] \arrow[dr, "g"] & \\ A & P \arrow[l, "p_A"] \arrow[r, "p_B"'] & B \end{tikzcd}$$

A ***coproduct*** is the dual notion.

$$\begin{tikzcd} & X & \\ A \arrow[ur, "f"] \arrow[r, "k_1"'] & P \arrow[u, "u"] & B \arrow[ul, "g"'] \arrow[l, "k_2"] \end{tikzcd}$$

[3] $u \circ p_A = f$ and $u \circ p_B = g$

In ***Set*** *products* are given by the Cartesian product with p_a and p_B the projections, and *coproducts* by the disjoint union. We shall see the importance of this definition for the possibility of determining the truth value in classical logic (and not only!) in 4.3.
We can then give the very important definition of a ***Cartesian Category***:

Definition C.7. A category $\mathfrak{C}$ is *Cartesian* if it has a terminal object and product of any pair of object[4].

One of the mottos of category theory is that *morphisms are more important than objects.* A specific name for a morphism between categories is therefore to be expected.

Definition C.8. A ***functor*** $\mathfrak{F} : \mathfrak{C} \longrightarrow \mathfrak{D}$ between categories $\mathfrak{C}$ and $\mathfrak{D}$, consists in a mapping of object to object and arrows to arrows in such a way that composition and identities are preserved. This means that:

- $\mathfrak{F}(f) : \mathfrak{F}(A) \longrightarrow \mathfrak{F}(B)$, to every $f \in \mathfrak{C}(A, B)$;
- $\mathfrak{F}(id_A) = id_{\mathfrak{F}(A)}$, to every $A \in Ob(\mathfrak{C})$;
- $\mathfrak{F}(g \circ f) = \mathfrak{F}(g) \circ \mathfrak{F}(f)$, to every $f \in \mathfrak{C}(A, B)$ and $g \in \mathfrak{C}(B, C)$.

A functor $\mathfrak{F}$ represents a way of interpreting the category $\mathfrak{C}$ in the category $\mathfrak{D}$. In this regard, one of the most frequently used possibilities is to interpret a category in ***Set***. We shall see in 3.10 the usefulness of this approach.
Just as there are morphisms between categories, the functors, so there are morphisms between functors, the ***natural transformations***:

Definition C.9. Let $\mathfrak{F}, \mathfrak{G} : \mathfrak{C} \longrightarrow \mathfrak{D}$ be functors.
A *natural transformation* $\alpha : \mathfrak{F} \Rightarrow \mathfrak{G}$ between them, denoted

$$\begin{tikzcd} \mathfrak{C} \arrow[r, bend left=50, "\mathfrak{F}"] \arrow[r, bend right=50, "\mathfrak{G}"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathfrak{D} \end{tikzcd}$$

is a family of arrows in $\mathfrak{D}$ indexed by objects $A \in \mathfrak{C}$,

$$\{\alpha_A : \mathfrak{F}(A) \longrightarrow \mathfrak{G}(A)\}$$

[4]This definition is very important from a computational point of view because whereas the ***Set*** category with which we describe classical systems and computation is Cartesian, ***Hilb*** in which we describe quantum systems and quantum computation is not. This has the implication that classically there is a uniform copying system of states, from the quantum point of view, there is not.

in $\mathfrak{D}$ such that for any morphism $f : A \longrightarrow B$ in $\mathfrak{C}$ the following diagram commutes

$$\begin{tikzcd} \mathfrak{F}(A) \arrow[r, "\alpha_A"] \arrow[d, "\mathfrak{F}(f)"'] & \mathfrak{G}(A) \arrow[d, "\mathfrak{G}(f)"] \\ \mathfrak{F}(B) \arrow[r, "\alpha_B"'] & \mathfrak{G}(B) \end{tikzcd}$$

C.0.1.2 Monoidal category: definition and diagrammatic representation

Let us now continue with our recipe. So far we have focused on a single ingredient, but it is clear that this is extremely limiting in cooking! It is clear that we want to use several ingredients at the same time in our cooking and perform several procedures together. This is the sense of introducing parallel composition on objects (the ingredients) and morphisms (the procedures). In ***Set*** this is equivalent to considering ordered pairs of sets and ordered pairs of functions, i.e. in defining the Cartesian product. Let us now come to the general definition.

In the previous section, we defined a category basically as a quadruple $\mathfrak{C} = (Ob(\mathfrak{C}), Ar(\mathfrak{C}), id, \circ)$. This definition allows for the possibility of composing in morphisms sequentially. However, for our purposes we also need to be able to compose objects and morphisms in parallel. We therefore introduce the concept of ***monoidal category***:

Definition C.10. A *monoidal category* consists of the fallowing data:

- a category $\mathfrak{C}$;
- a functor $\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$ called ***tensor product***;
- an object $I \in \mathbf{C}$, called ***unit object***;
- a family of natural isomorphisms $\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$ for any triplet of objects $A, B, C \in \mathbf{C}$, called *associators.*
- a family of natural isomorphisms $\lambda_A : I \otimes A \longrightarrow A$ for each $A \in \mathbf{C}$, colled *left unitor* ;
- a family of natural isomorphisms $\rho_A : A \otimes I \longrightarrow A$ for each object $A \in \mathbf{C}$, colled *right unitor.*

These data are required to satisfy the following axioms for any $A, B, C, D \in Ob(\mathfrak{C})$:

triangle equation

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (I \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

Pentagon equation

$$\begin{tikzcd}
& (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A,B\otimes C,D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[dr, "id_A \otimes \alpha_{B,C,D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A,B,C} \otimes id_D"] \arrow[dr, "\alpha_{A\otimes B,C,D}"'] & & & A \otimes (B \otimes (C \otimes D)) \\
& (A \otimes B) \otimes (C \otimes D) \arrow[urr, "\alpha_{A,B,C\otimes D}"'] & &
\end{tikzcd}$$

Equivalently, but more conveniently for us, we can write

Definition C.11. A strict monoidal category is a sextuple $\mathfrak{C} = (Ob(\mathfrak{C}), Ar(\mathfrak{C}), id, \circ, \otimes,\ I)$ consisting of the fallowing data:

- a category $\mathfrak{C}$;
- a functor $\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$ called ***tensor product***;
- an object $I \in \mathbf{C}$, called ***unit object***;

These data are required to satisfy the following axioms:

unit and associativity on objects $\forall\ A, B, C \in \mathfrak{C}$

$$I \otimes A \cong A \cong A \otimes I, \quad (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

unit and associativity on morphisms $\forall\ f \in \mathfrak{C}(A, B),\ \forall\ g \in \mathfrak{C}(B, C),\ \forall\ h \in \mathfrak{C}(C, D)$

$$id_I \otimes f \cong f \cong f \otimes id_I, \quad (f \otimes g) \otimes h \cong f \otimes (g \otimes h)$$

interchange law $\forall\ f_1 \in \mathfrak{C}(A, B),\ \forall\ g_1 \in \mathfrak{C}(B, C), \forall\ f_2 \in \mathfrak{C}(D, E),\ \forall\ g_2 \in \mathfrak{C}(E, F)$

$$(g_1 \otimes g_2) \circ (f_1 \otimes f_2) = (g_1 \circ f_1) \otimes (g_2 \circ f_2)$$

If the natural isomorphisms in definition are all identities, we say that the monoidal category is *strict*.

As before, we introduce the corresponding diagrammatic language for *strict monoidal categories*. Following the definition, we represent the data as in Fig.C.3

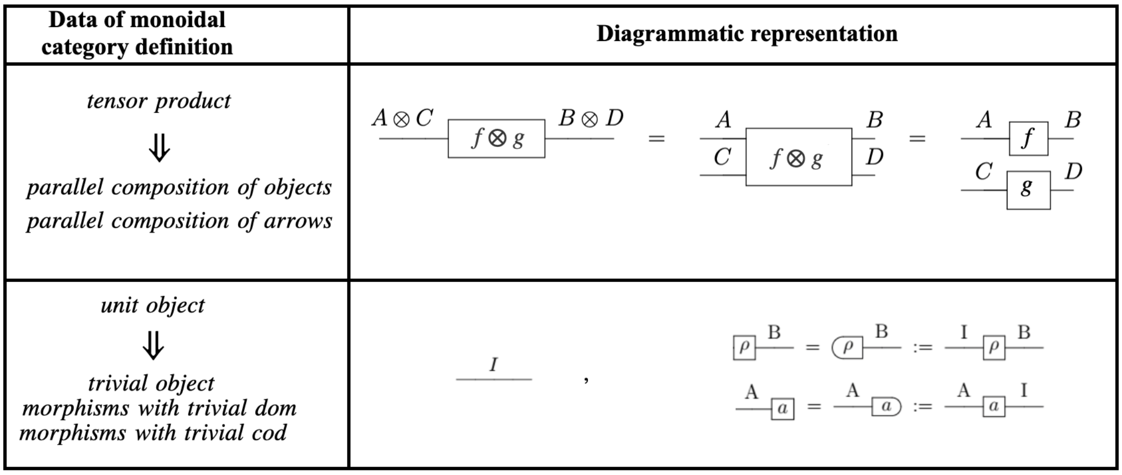


FIGURE C.3: Data representation in the monoidal category definition

Similarly, the axioms in Fig.C.4 and Fig.C.5

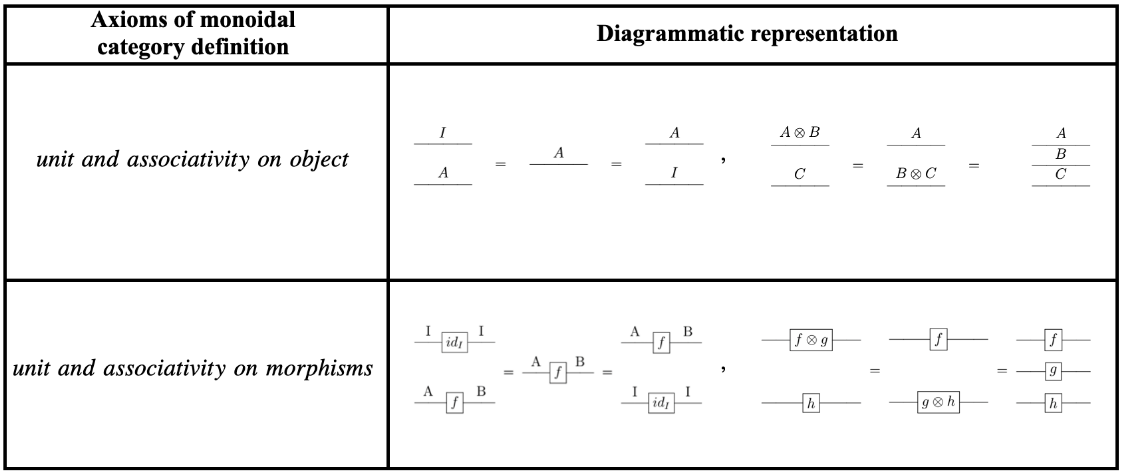


FIGURE C.4: Axioms representation in the monoidal category definition

Here we would like to interpret some of the elements introduced in our preparation of a recipe and in the category ***Set***. The unit object is used to indicate a certain preparation, a state in which an ingredient is: we are not interested in the previous history of the ingredient (where it was bought, whether it was put in the fridge and the like), we are only interested in saying that on our kitchen counter we have it. The same goes for the final outcome of our recipe: when it is finished, we are not interested in knowing what will happen to the dish of pasta carbonara: whether it will be served at the table and eaten

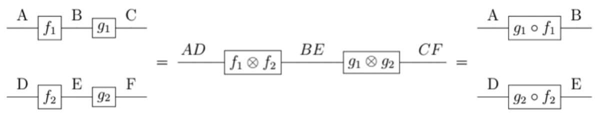
Axioms of monoidal category definition	**Diagrammatic representation**
interchange law	

FIGURE C.5: Interchange law representation in the monoidal category definition

or used as a demonstration on a television programme.
With regard to ***Set***, the unit object is the *singleton* set $\{\bullet\}$, used to indicate an element of a set:

$$\{\bullet\} \longrightarrow A$$

In this way an element is a morphism. We will see the importance of this definition for the introduction of the concepts of states and effects.

To finish the presentation of what will be the main categorical framework, we need to introduce the concept of the monoidal ***symmetric*** category:

Definition C.12. A *symmetric monoidal category* $\mathfrak{C} = (Ob(\mathfrak{C}), Ar(\mathfrak{C}), id, \circ, \otimes, \ I, \sigma)$ is a monoidal category $\mathfrak{C}$ with a *natural isomorphism* $\boldsymbol{SWAP}$

$$\sigma : Ob(\mathfrak{C}) \times Ob(\mathfrak{C}) \longrightarrow Ob(\mathfrak{C}) \times Ob(\mathfrak{C})$$

$$A \otimes B \longmapsto B \otimes A$$

satisfying $\forall\ A, B \in Ob(\mathfrak{C})$ the axioms:

- $\sigma_{B,A} \circ \sigma_{A,B} = id_{A \otimes B} \qquad \sigma_{A,I} = id_A$;
- $(g \otimes f) \circ \sigma_{A,C} = \sigma_{B,D} \circ (f \otimes g)$;
- $(id_B \otimes \sigma_{A,C}) \circ (\sigma_{A,B} \otimes id_C) = \sigma_{A,B \otimes C}$.

Diagrammatically, the $SWAP$ corresponds to a swap of wires and it is involutory.
In the more general case of isomorphism instead of identity in the definition we speak of monoidal braided category.

Let's go to see the most important result that accounts for the strength of diagrammatic representation in relation to theory:

Theorem C.13 (Correctness of graphical calculus for braided (symmetric) monoidal categories)**.** *A well-typed equation between morphisms in a braided monoidal category follows from the axioms if and only if it holds in the graphical language up to spatial isotopy (graphical equivalence).*

Remark C.14. We use the notion of isotopy because we assume the diagrams lie in a cube in the three-dimensional space: the input wires terminate in the left face and the output in the right face. This is also called *spatial isotopy* (We talk about *graphical equivalence* if there is a spacial isotopy and $\sigma_{A,B} = \sigma^{-1}_{B,A}$).

What we have seen so far allows us to introduce diagrammatic representations such as those in Fig.C.6:

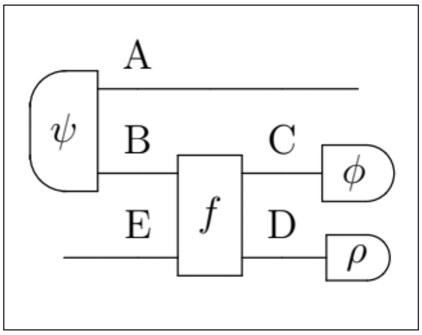


FIGURE C.6: Circuit diagram with casual structure from left to right

As in the case of categories, we can also introduce functors between monoidal categories:

Definition C.15. A *monoidal functor* between the two monoidal categories $\mathfrak{C} = (Ob(\mathfrak{C}), Ar(\mathfrak{C}), id_{\mathfrak{C}}, \circ, \otimes_{\mathfrak{C}},\ I_{\mathfrak{C}})$ and $\mathfrak{D} = (Ob(\mathfrak{D}), Ar(\mathfrak{D}), id_{\mathfrak{D}}, \circ, \otimes_{\mathfrak{D}},\ I_{\mathfrak{D}})$ consists of the fallowing data:

- a functor $\mathfrak{F} : \mathfrak{C} \longrightarrow \mathfrak{D}$;
- a morphism $\epsilon\ :\ id_{\mathfrak{D}} \longrightarrow \mathfrak{F}(id_{\mathfrak{C}})$
- a set of natural transformations $\alpha_{A,B}\ :\ \mathfrak{F}(x) \otimes_{\mathfrak{D}} \mathfrak{F}(y) \longrightarrow \mathfrak{F}(x \otimes_{\mathfrak{C}} y)$, for all $A, B \in \mathfrak{C}$

These data are required to satisfy the following axioms:

unit For all $A \in \mathfrak{C}$ the following diagrams commute:

$$\begin{tikzcd}
\mathfrak{F}(A) \otimes_{\mathfrak{D}} I_{\mathfrak{D}} \arrow[r, "\rho^{\mathfrak{D}}_{\mathfrak{C}(A)}"] \arrow[d, "id_{\mathfrak{F}(A)} \otimes_{\mathfrak{D}} \epsilon"'] & \mathfrak{F}(A) \arrow[d, "\mathfrak{F}(\rho_A^{-1})"] \\
\mathfrak{F}(A) \otimes_{\mathfrak{D}} \mathfrak{F}(I_{\mathfrak{C}}) \arrow[r, "\alpha_{A, I_{\mathfrak{C}}}"'] & \mathfrak{F}(A \otimes_{\mathfrak{C}} I_{\mathfrak{C}})
\end{tikzcd}
\qquad
\begin{tikzcd}
I_{\mathfrak{D}} \otimes_{\mathfrak{D}} \mathfrak{F}(A) \arrow[r, "\lambda^{\mathfrak{D}}_{\mathfrak{F}(A)}"] \arrow[d, "\epsilon \otimes_D id_{\mathfrak{F}(A)}"'] & \mathfrak{F}(A) \arrow[d, "\mathfrak{F}(\lambda_A^{-1})"] \\
\mathfrak{F}(I_{\mathfrak{C}}) \otimes_{\mathfrak{D}} \mathfrak{F}(A) \arrow[r, "\alpha_{I_{\mathfrak{C}}, A}"'] & \mathfrak{F}(I_{\mathfrak{C}} \otimes_{\mathfrak{C}} A)
\end{tikzcd}$$

associativity For all objects $A, B, C \in \mathfrak{C}$ the following diagram commutes:

$$\begin{tikzcd}
(\mathfrak{F}(A) \otimes_{\mathfrak{D}} \mathfrak{F}(B)) \otimes_{\mathfrak{D}} \mathfrak{F}(C) \arrow[r, "\alpha^{\mathfrak{D}}_{\mathfrak{F}(A),\mathfrak{F}(B),\mathfrak{F}(C)}"] \arrow[d, "\alpha_{A,B} \otimes_{\mathfrak{D}} id_{\mathfrak{F}(C)}"'] & \mathfrak{F}(A) \otimes_{\mathfrak{D}} (\mathfrak{F}(B) \otimes_{\mathfrak{D}} \mathfrak{F}(C)) \arrow[d, "id_{\mathfrak{F}(A)} \otimes_{\mathfrak{D}} \alpha_{B,C}"] \\
\mathfrak{F}(A \otimes_{\mathfrak{C}} B) \otimes_{\mathfrak{D}} \mathfrak{F}(C) \arrow[d, "\alpha_{A \otimes_{\mathfrak{C}} B, C}"'] & \mathfrak{F}(A) \otimes_{\mathfrak{D}} \mathfrak{F}(B \otimes_{\mathfrak{C}} C) \arrow[d, "\alpha_{A, B \otimes_{\mathfrak{C}} C}"] \\
\mathfrak{F}((A \otimes_{\mathfrak{C}} B) \otimes_{\mathfrak{C}} C) \arrow[r, "\mathfrak{F}(\alpha^{\mathfrak{C}}_{(A,B,C)})"'] & \mathfrak{F}(A \otimes_{\mathfrak{C}} (B \otimes_{\mathfrak{C}} C))
\end{tikzcd}$$

Definition C.16. A *braided monoidal functor* is a monoidal functor $\mathfrak{F} : \mathfrak{C} \longrightarrow \mathfrak{D}$ between braided monoidal categories making the following diagram commute:

$$\begin{tikzcd}
\mathfrak{F}(A) \otimes_{\mathfrak{D}} \mathfrak{F}(B) \arrow[r, "\sigma^{\mathfrak{D}}_{\mathfrak{F}(A),\mathfrak{F}(B)}"] \arrow[d, "\alpha_{A,B}"'] & \mathfrak{F}(B) \otimes_{\mathfrak{D}} \mathfrak{F}(A) \arrow[d, "\alpha_{B,A}"] \\
\mathfrak{F}(A \otimes_C B) \arrow[r, "\mathfrak{F}(\sigma_{A,B})"'] & \mathfrak{F}(A \otimes_C B)
\end{tikzcd}$$

Let us now more briefly introduce three extensions that will be fundamental to the understanding of the examples we will give: the ZX-calculus in the quantum domain and, more generally, the role of hypergraph categories. The corresponding diagrammatic parts will not be discussed in detail[5].

[5]What we have seen so far, as we shall see in the next chapter, represents the theoretical framework for beginning to interpret diagrams both from a physical (not only, but in our case this is what interests us) and computational point of view. Only the frameworks of the next paragraphs will be able to give rise to

C.0.1.3 From monoidal categories to hypergraph categories

In this section we will introduce some developments that allow monoidal theory to be interpreted as quantum theory at least in the case of pure states. We will not go into the details of these constructions especially from a diagrammatic point of view. For this see [96], [92], [93].

First, briefly, the idea is that in certain situations, frequently used in computer science, we may be interested not only in considering flows of information that go back and result in following operations. One thinks, for example[6], of backpropagation in the case of machine learning. In this case we can consider a map n describing a neural network and its update $R(n)$ (see [87]) that allows parameters to change during learning. Its diagrammatic representation will be of the type:

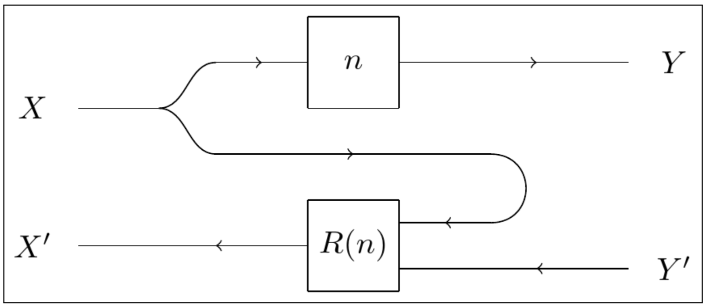


FIGURE C.7: Map and backwardsmap for backpropagation in machine learning

The concept of a compact closed monoidal category is what we need (dagger for needs arising from physics!).

Definition C.17. A *dagger monoidal category* is a monoidal category $\mathfrak{C}$ together a functor $\dagger : \mathfrak{C} \longrightarrow \mathfrak{C}$ such that:

1. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$;
2. $id_H^\dagger = id_H$
3. $(f^\dagger)^\dagger = f$

true quantum computation based on intrinsically diagrammatic syntactic rules (as we shall see in Chapter 4 when we introduce the teleportation protocol).

[6]This example will be used to introduce the neural network computer in chapter3

We therefore say that it is an involutive, contravariant functor, identity on object. Furthermore, the dagger structure is compatible with the monoidal structure:

a $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, for all f, g;

b the canonical isomorphysm $\alpha_{A,B,C}, \lambda_A, \rho_A$ of the monoidal structure is unitary ($f^\dagger \circ f = id_A$ and $f \circ f^\dagger = id_B$)

The presence of the dagger functor makes it possible to introduce the concept of *adjoint, unitary, isometry, self-adjoint* and *positive* maps[7].

The second step we take is to define the concept of ***dagger compact closed monoidal*** category, but first we need to define the concept of a ***dual object***[8]:

Definition C.18. Let be $\mathfrak{C}$ a monoidal category:
$L \in Ob(\mathfrak{C})$ is ***left-dual*** to an object $R \in Ob(\mathfrak{C})$
$R \in Ob(\mathfrak{C})$ is ***right-dual*** to an object $L \in Ob(\mathfrak{C})$
if the is a ***unit*** morphism

$$\eta : I \longrightarrow R \otimes L$$

and there is a ***counit*** morphism

$$\epsilon : L \otimes R \longrightarrow I$$

making the following diagrams commute:

$$\begin{tikzcd}
R \arrow[r, "\rho_R"] \arrow[d, "id_R"'] & R \otimes I \arrow[r, "id_R \otimes \eta"] & R \otimes (L \otimes A) \arrow[d, "\alpha_{R,L,R}"] \\
R & I \otimes R \arrow[l, "\lambda_A^{-1}"] & (R \otimes L) \otimes R \arrow[l, "\epsilon_R \otimes id_R"]
\end{tikzcd}$$

$$\begin{tikzcd}
L \arrow[r, "\lambda_L"] \arrow[d, "id_L"'] & I \otimes L \arrow[r, "\eta_A \otimes id_L"] & (L \otimes R) \otimes L \arrow[d, "\alpha_{L,R,L}^{-1}"] \\
L & L \otimes I \arrow[l, "\rho_L^{-1}"] & L \otimes (R \otimes L) \arrow[l, "id_L \otimes \epsilon"]
\end{tikzcd}$$

[7]In the detail of quantum theory, we can introduce *bra-ket* and *ket-bra* as a composition of morphisms. A dagger gives a correspondence between *states* and *effects*. Furthermore, the introduction of Born's rule to tie the measurements to probabilities is straightforward.

[8]This definition is the basis for a simple diagrammatic representation of Bell's states.

We can then give the follow definition:

Definition C.19. A strict **compact**[9] closed category $\mathfrak{C}$ is symmetric monoidal category such that $\forall A \in Ob(\mathfrak{C})$ exist:

1. the dual object $A^* \in Ob(\mathfrak{C})$;
2. the pair (unit and counit) $\eta_A, \epsilon_A \in Ar(\mathfrak{C})$

$$\eta_A : I \longrightarrow A^* \otimes A \quad \epsilon_A : A \otimes A^* \longrightarrow I$$

which are such that the following two diagrams commute:

$$\begin{tikzcd} A \arrow[d, "id_A \otimes \eta_A"'] \arrow[dr, "id_A"] & \\ A \otimes A^* \otimes A \arrow[r, "\epsilon_A \otimes id_A"'] & A \end{tikzcd} \qquad \begin{tikzcd} A^* \arrow[r, "\eta_A \otimes id_{A^*}"] \arrow[dr, "id_{A^*}"'] & A^* \otimes A \otimes A^* \arrow[d, "id_{A^*} \otimes \epsilon_A"] \\ & A^* \end{tikzcd}$$

A very important concept we can introduce in compact closed category is ***name*** and ***coname***:

Definition C.20. Le $\mathfrak{C}$ be a compact closed category and $f : A \longrightarrow B$ a morphism. We define:

1. *name* $\ulcorner f \urcorner : I \longrightarrow A^* \otimes B$;
2. *coname* $\llcorner f \lrcorner : A \otimes B^* \longrightarrow I$

which are such that the following two diagrams commute:

$$\begin{tikzcd} I \arrow[d, "\eta_A"'] \arrow[dr, "\ulcorner f \urcorner"] & \\ A^* \otimes A \arrow[r, "id_{A^*} \otimes f"'] & A^* \otimes B \end{tikzcd} \qquad \begin{tikzcd} A \otimes B^* \arrow[r, "f \otimes id_{B^*}"] \arrow[dr, "\llcorner f \lrcorner"'] & B \otimes A^* \arrow[d, "\epsilon_B"] \\ & I \end{tikzcd}$$

This second step allows us to introduce the *string diagrams*[10] as in Fig.C.8

[9] The adjective *closed* allows us to introduce the Choi-Jamiolkowsky isomorphism. In fact, for every two object A and B in $\mathfrak{C}$, there is a special objects $[A \longrightarrow B]$ whose states encode morphism in $\mathfrak{C}$(A,B). In particular *compact closed* means that $[A \longrightarrow B] := A^* \otimes B$. The subsequent definition of name and coname concludes the introduction to state-process duality (see [92]).

[10] Let us make a few observations regarding Fig.C.8: with the semicircle before f, we have highlighted the diagrammatic representation of Bell's states; with the closed wire around g, the concept of partial trace in categorical quantum theory.

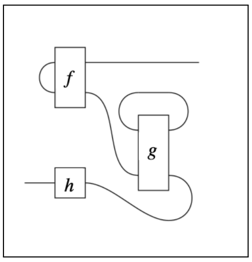


FIGURE C.8: String diagrams with the possibility to connect all to all.

The last step we need to take is to introduce the ***hypergraph categories***,whose string diagrams are hypergraphs. To do this we need to introduce a *Frobenius structure* and the *complementarity.*

We start introducing the concept of ***internal comonoid***:

Definition C.21. Let $\mathfrak{C}$ be a monoidal category. An *internal comonoid* is an object $C \in Ob(\mathfrak{C})$ together with a pair of morphims

$$C \otimes C \xleftarrow{\delta} C \xrightarrow{\epsilon} I,$$

where δ is the *comoltiplication* and ϵ the *comultiplicative unit* (counit), which are such that

$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[d, "\delta"'] & C\otimes C \arrow[d, "id_C\otimes\delta"] \\ C\otimes C \arrow[r, "\delta\otimes id_C"'] & C\otimes C\otimes C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & C \arrow[dl, "\simeq"'] \arrow[d, "\delta"] \arrow[dr, "\simeq"] & \\ I\otimes C & C\otimes C \arrow[l, "\epsilon\otimes id_C"] \arrow[r, "id_C\otimes\epsilon"'] & C\otimes I \end{tikzcd}$$

commute.

The notion of internal monoid is dual to the notion of *internal comonoid.*
We have what we need to define the ***Frobenius structure***[11]:

Definition C.22. Let $\mathfrak{C}$ a symmetric monoidal category. A *Frobenius structure* in $\mathfrak{C}$ is a quintuple $(C, \mu, \eta, \delta, \epsilon)$ such that:

1. (C, μ, η) is an internal monoid;

2. (C, δ, ϵ) is an internal comonoid;

[11]The concept of comonoid and the Frobenius laws allows us to introduce a *observable structure* in a dagger symmtric monoidal category

3. the Frobenius laws hold: $(id \otimes \mu) \circ (\delta \otimes id) = \delta \circ \mu = (\mu \otimes id) \circ (id \otimes \delta)$

We can therefore define the concept of ***hypergraph category***([102])

Definition C.23. Le be $\mathfrak{C}$ a symmetric monoidal category. $\mathfrak{C}$ is *hypergraph* if each object C is equipped with a Frobenius structure $(C, \mu, \eta, \delta, \epsilon)$ such that the Frobenius algebra structure of any tensor product $C \otimes D$ is induced in the canonical way from those of C and D.

The last concept we need introduce in this introduction to category theory is the concept of Z^****-algebra***:

Definition C.24. A symmetric monoidal category $\mathfrak{C}$ is a *bialgebra* if $\forall C \in Ob(\mathfrak{C})$ exist:

1. a monoid (C, μ, η);
2. a comonoid (C, δ, ϵ);

which are such that the following diagrams commute:

$$\begin{tikzcd}
C \otimes C \arrow[r, "\eta"] \arrow[d, "\delta\otimes\delta"'] & C \arrow[r, "\delta"] & C \otimes C \\
C \otimes C \otimes C \otimes C \arrow[rr, "id\otimes\sigma\otimes id"] & & C \otimes C \otimes C \otimes C \arrow[u, "\mu\otimes\mu"']
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes I \arrow[d, "\eta"'] \arrow[dr, "\eta\otimes\eta"] & \\
C \arrow[r, "\delta"'] & C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes C \arrow[r, "\mu"] \arrow[dr, "\epsilon\otimes\epsilon"'] & C \arrow[d, "\epsilon"] \\
& I \otimes I
\end{tikzcd}$$

and $\epsilon \circ \eta = id_I$.

Definition C.25. A Z^*-*algebra* is formed by two Frobenius algebrsa such that:

1. the first diagram in the previous definition commute;
2. two algebras are compatible, i.e. $\mu \circ \delta = \eta \circ \epsilon$.

In the particular case the two classical structure form a scaled[12] bialgebra each other, they define the ***strong complementary***[13]:

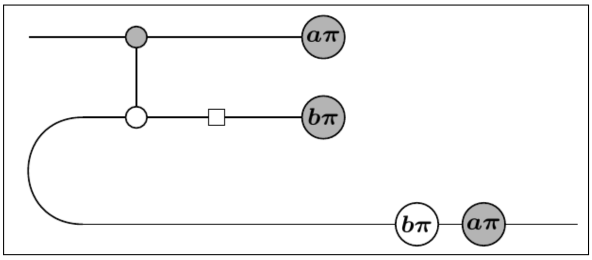


FIGURE C.9: Spider diagrams for teleportation protocol

[12]For the difference with our definition see [132].

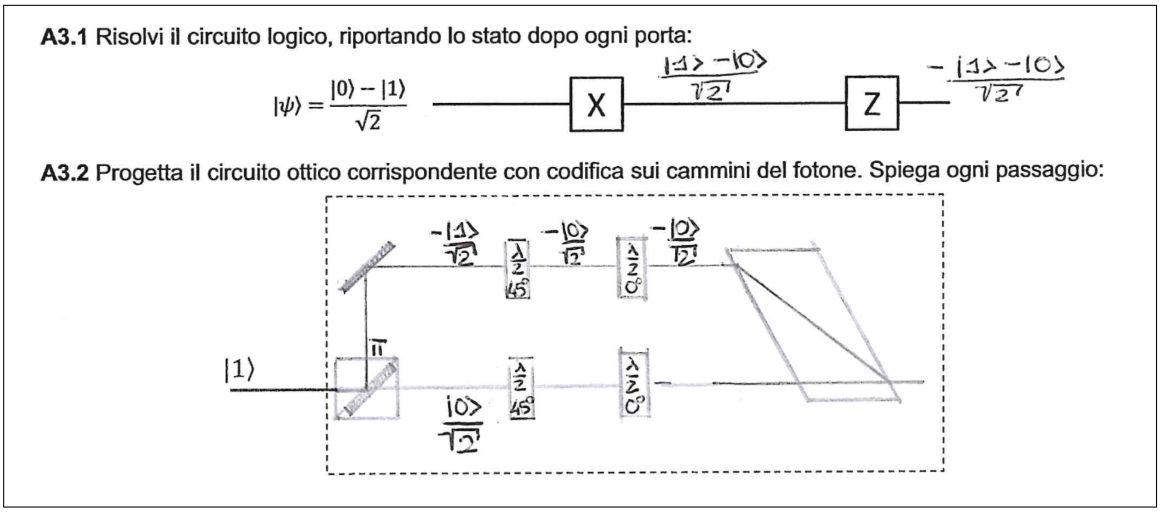
[13]The last of these constructions allows the introduction of the concept of quantum measurement with respect to two strongly complementary observables. All these elements form the basis of the ZX calculus, with which we will show in Chapter 4 how the teleportation protocol can be demonstrated with simple syntactic rules on diagrams.

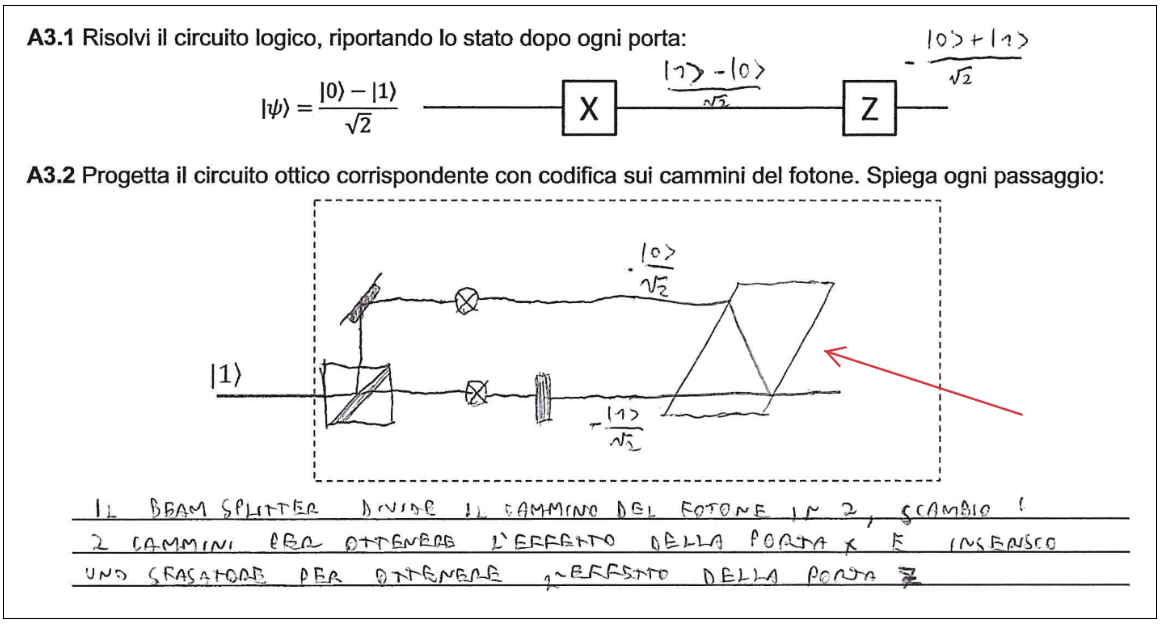
Appendix D

Worksheets: pictures of the tables in Chapter 7

We include in this appendix some pictures from the Chapter 7 that require larger sizes to facilitate reading.

Tab. 7.10



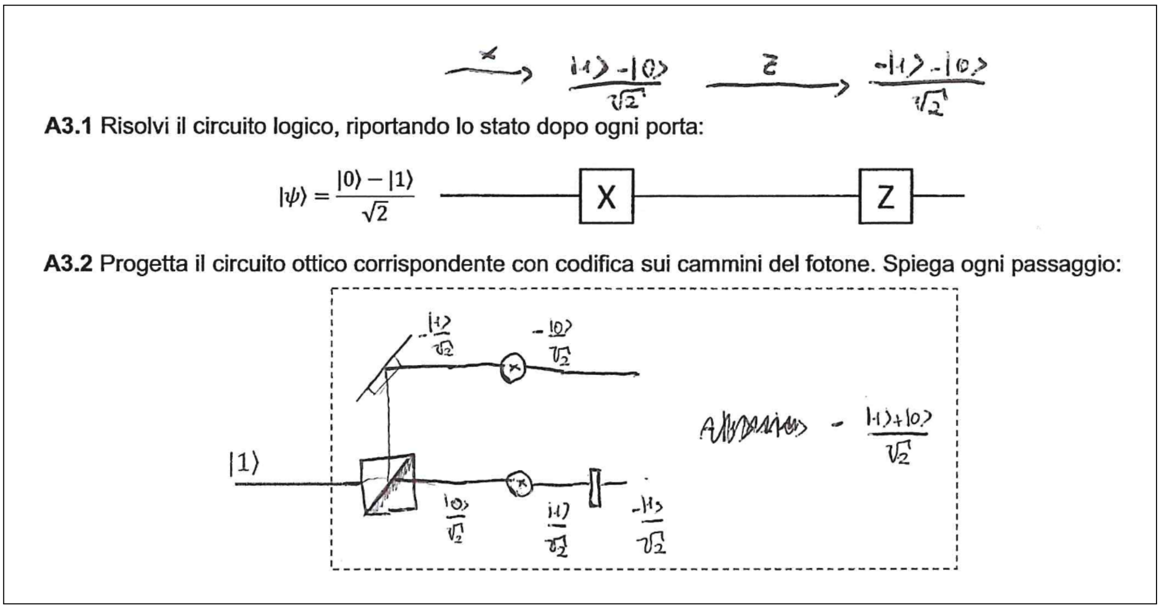


A3.1 Risolvi il circuito logico, riportando lo stato dopo ogni porta:

$$|\psi\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \xrightarrow{X} \frac{|1\rangle - |0\rangle}{\sqrt{2}} \xrightarrow{Z} -\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

A3.2 Progetta il circuito ottico corrispondente con codifica sui cammini del fotone. Spiega ogni passaggio:

IL BEAM SPLITTER DIVIDE IL CAMMINO DEL FOTONE IN 2, SCAMBIO I 2 CAMMINI PER OTTENERE L'EFFETTO DELLA PORTA X E INSERISCO UNO SFASATORE PER OTTENERE L'EFFETTO DELLA PORTA Z



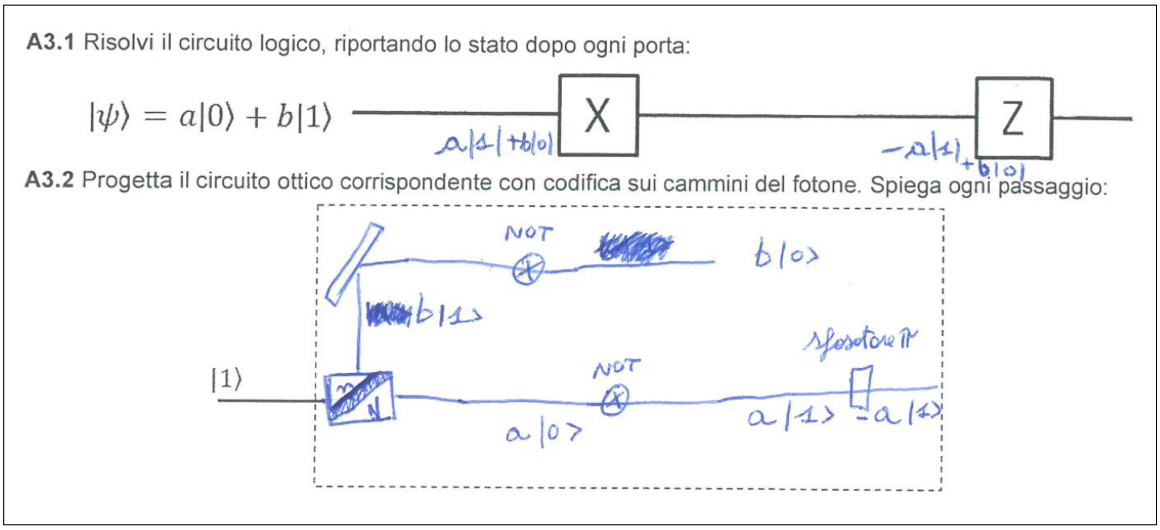
$\xrightarrow{X} \frac{|1\rangle - |0\rangle}{\sqrt{2}} \xrightarrow{Z} \frac{-|1\rangle - |0\rangle}{\sqrt{2}}$

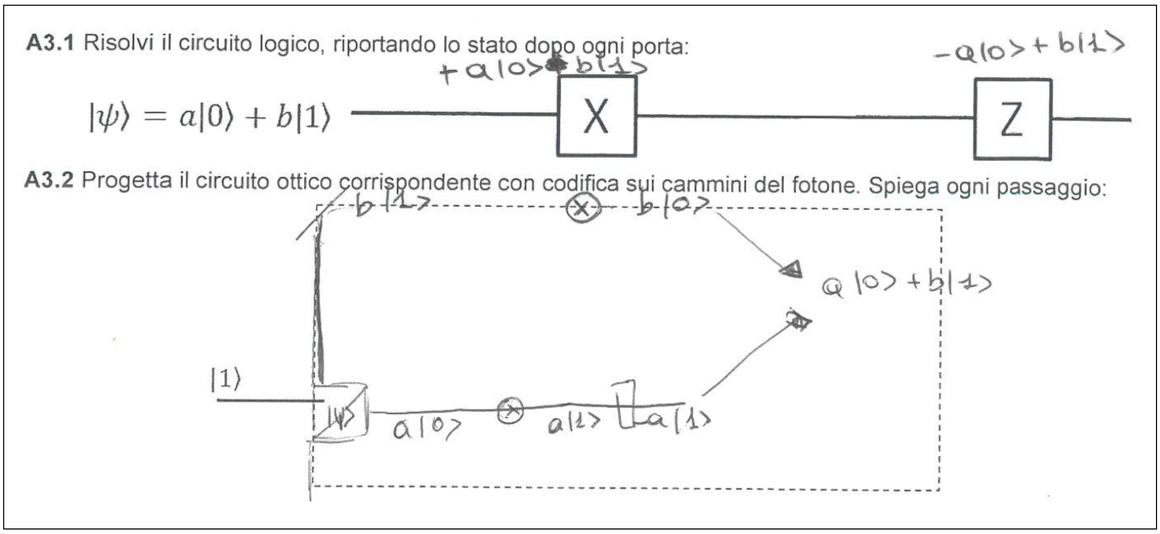
A3.1 Risolvi il circuito logico, riportando lo stato dopo ogni porta:

$$|\psi\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \quad \boxed{X} \quad \boxed{Z}$$

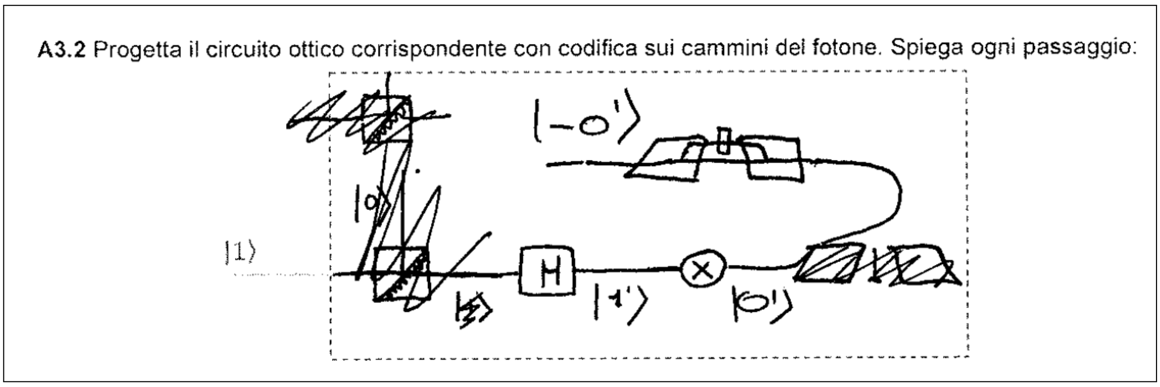
A3.2 Progetta il circuito ottico corrispondente con codifica sui cammini del fotone. Spiega ogni passaggio:

Tab. 7.11

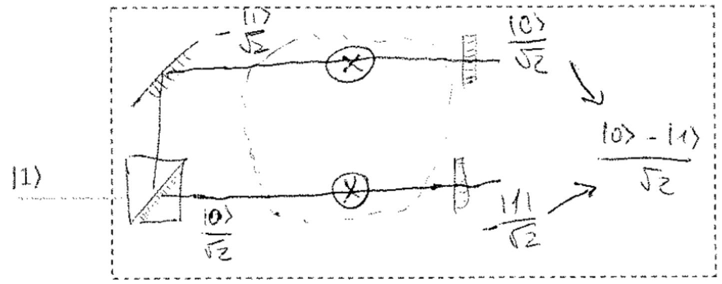

A3.1 Risolvi il circuito logico, riportando lo stato dopo ogni porta:
|ψ⟩ = a|0⟩ + b|1⟩
X
Z
A3.2 Progetta il circuito ottico corrispondente con codifica sui cammini del fotone. Spiega ogni passaggio:
|1⟩
NOT
b|0>
b|1>
sfasatore π
a|0>
a|1>
-a|1>


A3.1 Risolvi il circuito logico, riportando lo stato dopo ogni porta:
-a|0>+b|1>
|ψ⟩ = a|0⟩ + b|1⟩
X
Z
A3.2 Progetta il circuito ottico corrispondente con codifica sui cammini del fotone. Spiega ogni passaggio:
b|1>
b|0>
a|0>+b|1>
|1⟩
|ψ>
a|0>
a|1>
-a|1>

Tab. 7.12

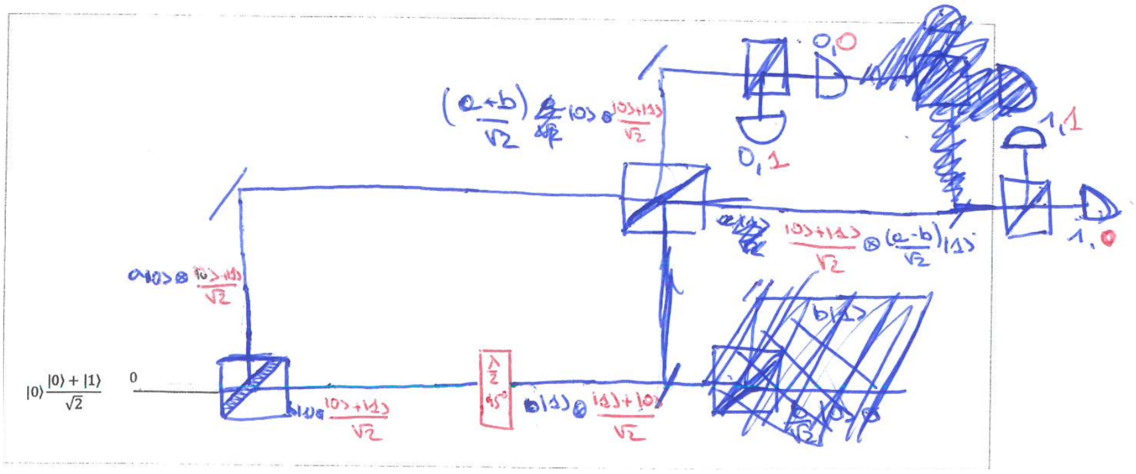

A3.2 Progetta il circuito ottico corrispondente con codifica sui cammini del fotone. Spiega ogni passaggio:
|-0'⟩
|0⟩
|1⟩
H
|1'⟩
|0'⟩

A3.2 Progetta il circuito ottico corrispondente con codifica sui cammini del fotone. Spiega ogni passaggio:



$|1\rangle$ viene riflesso dalla parte del [illegible] e arriva ad uno specchio che risulta $-\frac{|1\rangle}{\sqrt{2}}$ quindi porta X che cambia [illegible] al vettore e sfasatore per il fotone procede uguale per sotto

Tab. 7.16



Bibliography

[1] J. van de Wetering, "Zx-calculus for the working quantum computer scientist," *arXiv preprint arXiv:2012.13966*, 2020.

[2] M. G. Raymer and C. Monroe, "The us national quantum initiative," *Quantum Science and Technology*, vol. 4, no. 2, p. 020504, 2019.

[3] M. Riedel, M. Kovacs, P. Zoller, J. Mlynek, and T. Calarco, "Europe's quantum flagship initiative," *Quantum Science and Technology*, vol. 4, no. 2, p. 020501, 2019.

[4] K. Krijtenburg-Lewerissa, H. J. Pol, A. Brinkman, and W. Van Joolingen, "Insights into teaching quantum mechanics in secondary and lower undergraduate education," *Physical review physics education research*, vol. 13, no. 1, p. 010109, 2017.

[5] C. Singh and E. Marshman, "Review of student difficulties in upper-level quantum mechanics," *Physical Review Special Topics-Physics Education Research*, vol. 11, no. 2, p. 020117, 2015.

[6] C. D. Aiello, D. D. Awschalom, H. Bernien, T. Brower, K. R. Brown, T. A. Brun, J. R. Caram, E. Chitambar, R. Di Felice, K. M. Edmonds, *et al.*, "Achieving a quantum smart workforce," *Quantum Science and Technology*, vol. 6, no. 3, p. 030501, 2021.

[7] J. Meyer, G. Passante, S. J. Pollock, and B. R. Wilcox, "The interdisciplinary quantum information classroom: Themes from a survey of quantum information science instructors," *arXiv preprint arXiv:2202.05944*, 2022.

[8] S. Seegerer, T. Michaeli, and R. Romeike, "Quantum computing as a topic in computer science education," in *The 16th Workshop in Primary and Secondary Computing Education*, pp. 1–6, 2021.

[9] S. Satanassi, E. Ercolessi, and O. Levrini, "Designing and implementing materials on quantum computing for secondary school students: The case of teleportation," *Physical Review Physics Education Research*, vol. 18, no. 1, p. 010122, 2022.

[10] Ö. Salehi, Z. Seskir, and İ. Tepe, "A computer science-oriented approach to introduce quantum computing to a new audience," *IEEE Transactions on Education*, vol. 65, no. 1, pp. 1–8, 2022.

[11] T. Knuuttila, "Modelling and representing: An artefactual approach to model-based representation," *Studies in History and Philosophy of Science Part A*, vol. 42, no. 2, pp. 262–271, 2011.

[12] T. Knuuttila and M. Boon, "How do models give us knowledge? the case of carnot's ideal heat engine," *European journal for philosophy of science*, vol. 1, no. 3, pp. 309–334, 2011.

[13] E. Wenger, *Communities of practice: Learning, meaning, and identity.* Cambridge university press, 1999.

[14] C. H. Bennett, "The thermodynamics of computation—a review," *International Journal of Theoretical Physics*, vol. 21, no. 12, pp. 905–940, 1982.

[15] C. H. Bennett, "Notes on the history of reversible computation," *ibm Journal of Research and Development*, vol. 32, no. 1, pp. 16–23, 1988.

[16] E. Fredkin and T. Toffoli, "Conservative logic," *International Journal of theoretical physics*, vol. 21, no. 3, pp. 219–253, 1982.

[17] R. P. Feynman, "Simulating physics with computers," in *Feynman and computation*, pp. 133–153, CRC Press, 2018.

[18] D. Deutsch, "Quantum theory, the church–turing principle and the universal quantum computer," *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, vol. 400, no. 1818, pp. 97–117, 1985.

[19] G. Zuccarini and M. Malgieri, "Modeling and representing conceptual change in the learning of successive theories: The case of the classical-quantum transition," *arXiv preprint arXiv:2108.06919*, 2021.

[20] T.-D. Bradley, "What is applied category theory?," *arXiv preprint arXiv:1809.05923*, 2018.

[21] J. S. Bell and J. S. Bell, *Speakable and unspeakable in quantum mechanics: Collected papers on quantum philosophy.* Cambridge university press, 2004.

[22] M. A. Nielsen and I. Chuang, "Quantum computation and quantum information," 2002.

[23] D. McIntyre, C. Manogue, and J. T.-Q. Mechanics, "A paradigms approach," 2014.

[24] I. Bengtsson and K. Życzkowski, *Geometry of quantum states: an introduction to quantum entanglement.* Cambridge university press, 2017.

[25] G. Benenti, G. Casati, and G. Strini, *Principles of quantum computation and information-volume I: Basic concepts.* World scientific, 2004.

[26] A. Di Pierro, "Quantum computing," *University of Verona, Tech. Rep*, 2010.

[27] J. Preskill, "Quantum computing 40 years later," *arXiv preprint arXiv:2106.10522*, 2021.

[28] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, "Quantum entanglement," *Reviews of modern physics*, vol. 81, no. 2, p. 865, 2009.

[29] J. Preskill, "Lecture notes for physics 229: Quantum information and computation," *California Institute of Technology*, vol. 16, no. 1, pp. 1–8, 1998.

[30] M. G. Paris, "The modern tools of quantum mechanics," *The European Physical Journal Special Topics*, vol. 203, no. 1, pp. 61–86, 2012.

[31] G. Ghirardi, "Entanglement, nonlocality, superluminal signaling and cloning," *arXiv preprint arXiv:1305.2305*, 2013.

[32] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Quantum from principles," in *Quantum theory: informational foundations and foils*, pp. 171–221, Springer, 2016.

[33] E. Schrödinger, "The present status of quantum mechanics," *Die Naturwissenschaften*, vol. 23, no. 48, pp. 1–26, 1935.

[34] G. M. D'Ariano, G. Chiribella, and P. Perinotti, *Quantum theory from first principles: an informational approach.* Cambridge University Press, 2017.

[35] J. S. Bell, "On the einstein podolsky rosen paradox," *Physics Physique Fizika*, vol. 1, no. 3, p. 195, 1964.

[36] J. S. Bell, "Speakable and unspeakable in quantum mechanics," 1989.

[37] L. Maccone, "A simple proof of bell's inequality," *American Journal of Physics*, vol. 81, no. 11, pp. 854–859, 2013.

[38] J. Stolze and D. Suter, *Quantum computing: a short course from theory to experiment.* John Wiley & Sons, 2008.

[39] R. P. Feynman, T. Hey, and R. W. Allen, *Feynman lectures on computation.* CRC Press, 2018.

[40] R. Cleve, A. Ekert, C. Macchiavello, and M. Mosca, "Quantum algorithms revisited," *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, vol. 454, no. 1969, pp. 339–354, 1998.

[41] D. Deutsch and R. Jozsa, "Rapid solution of problems by quantum computation," *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences*, vol. 439, no. 1907, pp. 553–558, 1992.

[42] L. K. Grover, "A fast quantum mechanical algorithm for estimating the median," *arXiv preprint quant-ph/9607024*, 1996.

[43] C. Lavor, L. Manssur, and R. Portugal, "Grover's algorithm: Quantum database search," *arXiv preprint quant-ph/0301079*, 2003.

[44] P. Mernyei, K. Meichanetzidis, and I. I. Ceylan, "Equivariant quantum graph circuits," in *International Conference on Machine Learning*, pp. 15401–15420, PMLR, 2022.

[45] M. Rossi, M. Huber, D. Bruß, and C. Macchiavello, "Quantum hypergraph states," *New Journal of Physics*, vol. 15, no. 11, p. 113022, 2013.

[46] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. Nest, and H.-J. Briegel, "Entanglement in graph states and its applications," *arXiv preprint quant-ph/0602096*, 2006.

[47] D. Bruß and C. Macchiavello, "Multipartite entanglement in quantum algorithms," *Physical Review A*, vol. 83, no. 5, p. 052313, 2011.

[48] M. Rossi, D. Bruß, and C. Macchiavello, "Hypergraph states in grover's quantum search algorithm," *Physica Scripta*, vol. 2014, no. T160, p. 014036, 2014.

[49] M. Rossi, D. Bruß, and C. Macchiavello, "Scale invariance of entanglement dynamics in grover's quantum search algorithm," *Physical Review A*, vol. 87, no. 2, p. 022331, 2013.

[50] T.-C. Wei and P. M. Goldbart, "Geometric measure of entanglement and applications to bipartite and multipartite quantum states," *Physical Review A*, vol. 68, no. 4, p. 042307, 2003.

[51] S. Mangini, F. Tacchino, D. Gerace, C. Macchiavello, and D. Bajoni, "Quantum computing model of an artificial neuron with continuously valued input data," *Machine Learning: Science and Technology*, vol. 1, no. 4, p. 045008, 2020.

[52] J. Gilmer, S. S. Schoenholz, P. F. Riley, O. Vinyals, and G. E. Dahl, "Neural message passing for quantum chemistry," in *International conference on machine learning*, pp. 1263–1272, PMLR, 2017.

[53] M. A. Nielsen, *Neural networks and deep learning*, vol. 25. Determination press San Francisco, CA, USA, 2015.

[54] K. Gurney, *An introduction to neural networks*. CRC press, 2018.

[55] J. Zhou, G. Cui, S. Hu, Z. Zhang, C. Yang, Z. Liu, L. Wang, C. Li, and M. Sun, "Graph neural networks: A review of methods and applications," *AI Open*, vol. 1, pp. 57–81, 2020.

[56] N. Shervashidze, P. Schweitzer, E. J. Van Leeuwen, K. Mehlhorn, and K. M. Borgwardt, "Weisfeiler-lehman graph kernels.," *Journal of Machine Learning Research*, vol. 12, no. 9, 2011.

[57] N. T. Huang and S. Villar, "A short tutorial on the weisfeiler-lehman test and its variants," in *ICASSP 2021-2021 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pp. 8533–8537, IEEE, 2021.

[58] M. M. Wolf, A. M. Klinvex, and D. M. Dunlavy, "Advantages to modeling relational data using hypergraphs versus graphs," in *2016 IEEE High Performance Extreme Computing Conference (HPEC)*, pp. 1–7, IEEE, 2016.

[59] Y. Gao, Y. Feng, S. Ji, and R. Ji, "Hgnn$^+$: General hypergraph neural networks," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 2022.

[60] Y. Feng, H. You, Z. Zhang, R. Ji, and Y. Gao, "Hypergraph neural networks," in *Proceedings of the AAAI conference on artificial intelligence*, vol. 33, pp. 3558–3565, 2019.

[61] J. Huang and J. Yang, "Unignn: a unified framework for graph and hypergraph neural networks," *arXiv preprint arXiv:2105.00956*, 2021.

[62] J. L. O'brien, "Optical quantum computing," *Science*, vol. 318, no. 5856, pp. 1567–1570, 2007.

[63] T. C. Ralph and G. J. Pryde, "Optical quantum computation," in *Progress in optics*, vol. 54, pp. 209–269, Elsevier, 2010.

[64] H.-A. Bachor and T. C. Ralph, *A guide to experiments in quantum optics*. John Wiley & Sons, 2019.

[65] C. Adami and N. J. Cerf, "Quantum computation with linear optics," in *NASA International Conference on Quantum Computing and Quantum Communications*, pp. 391–401, Springer, 1998.

[66] E. Knill, R. Laflamme, and G. J. Milburn, "A scheme for efficient quantum computation with linear optics," *nature*, vol. 409, no. 6816, pp. 46–52, 2001.

[67] P. Kok, W. J. Munro, K. Nemoto, T. C. Ralph, J. P. Dowling, and G. J. Milburn, "Linear optical quantum computing with photonic qubits," *Reviews of modern physics*, vol. 79, no. 1, p. 135, 2007.

[68] G. Weihs and A. Zeilinger, "Photon statistics at beam-splitters: an essential tool in quantum information and teleportation," *Coherence and Statistics of Photons and Atoms*, pp. 262–288, 2001.

[69] T. B. Pittman, B. C. Jacobs, and J. D. Franson, "Quantum computing using linear optics," *arXiv preprint quant-ph/0406192*, 2004.

[70] A. Clément, N. Heurtel, S. Mansfield, S. Perdrix, and B. Valiron, "Lov-calculus: A graphical language for linear optical quantum circuits," *arXiv preprint arXiv:2204.11787*, 2022.

[71] G. De Felice and B. Coecke, "Quantum linear optics via string diagrams," *arXiv preprint arXiv:2204.12985*, 2022.

[72] P. McCloud, "The category of linear optical quantum computing," *arXiv preprint arXiv:2203.05958*, 2022.

[73] R. C. Jones, "A new calculus for the treatment of optical systemsi. description and discussion of the calculus," *Josa*, vol. 31, no. 7, pp. 488–493, 1941.

[74] S. BEA and M. Teich, "Fundamentals of photonics," *Wiley*, p. 313, 1991.

[75] S. Barz, "Quantum computing with photons: introduction to the circuit model, the one-way quantum computer, and the fundamental principles of photonic experiments," *Journal of Physics B: Atomic, Molecular and Optical Physics*, vol. 48, no. 8, p. 083001, 2015.

[76] N. K. Langford, *Encoding, manipulating and measuring quantum information in optics.* PhD thesis, Citeseer, 2007.

[77] A. Grothendieck, "Sur quelques points d'algèbre homologique," *Tohoku Mathematical Journal, Second Series*, vol. 9, no. 2, pp. 119–183, 1957.

[78] F. W. Lawvere, "The category of categories as a foundation for mathematics," in *Proceedings of the conference on categorical algebra*, pp. 1–20, Springer, 1966.

[79] J. Lambek, "Compact monoidal categories from linguistics to physics," in *New structures for physics*, pp. 467–487, Springer, 2010.

[80] B. Coecke, G. d. Felice, K. Meichanetzidis, and A. Toumi, "How to make qubits speak," in *Quantum Computing in the Arts and Humanities*, pp. 277–297, Springer, 2022.

[81] K. R. Kaushik and A. E. Martin, "A mathematical neural process model of language comprehension, from syllable to sentence," 2022.

[82] B. Coecke, G. de Felice, K. Meichanetzidis, and A. Toumi, "Foundations for near-term quantum natural language processing," *arXiv preprint arXiv:2012.03755*, 2020.

[83] D. Shiebler, B. Gavranović, and P. Wilson, "Category theory in machine learning," *arXiv preprint arXiv:2106.07032*, 2021.

[84] B. Fong, D. Spivak, and R. Tuyéras, "Backprop as functor: A compositional perspective on supervised learning," in *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pp. 1–13, IEEE, 2019.

[85] B. Fong and D. I. Spivak, *An invitation to applied category theory: seven sketches in compositionality.* Cambridge University Press, 2019.

[86] B. Gavranović, "Compositional deep learning," *arXiv preprint arXiv:1907.08292*, 2019.

[87] G. S. Cruttwell, B. Gavranović, N. Ghani, P. Wilson, and F. Zanasi, "Categorical foundations of gradient-based learning," in *European Symposium on Programming*, pp. 1–28, Springer, Cham, 2022.

[88] C. Zhao and X.-S. Gao, "Analyzing the barren plateau phenomenon in training quantum neural networks with the zx-calculus," *Quantum*, vol. 5, p. 466, 2021.

[89] R. Yeung, "Diagrammatic design and study of ans\"{a} tze for quantum machine learning," *arXiv preprint arXiv:2011.11073*, 2020.

[90] A. Toumi, R. Yeung, and G. de Felice, "Diagrammatic differentiation for quantum machine learning," *arXiv preprint arXiv:2103.07960*, 2021.

[91] D. I. Spivak, *Category theory for the sciences.* MIT Press, 2014.

[92] B. Coecke and A. Kissinger, "Picturing quantum processes," in *International Conference on Theory and Application of Diagrams*, pp. 28–31, Springer, 2018.

[93] C. Heunen and J. Vicary, *Categories for Quantum Theory: an introduction.* Oxford University Press, 2019.

[94] G. M. D'Ariano, M. Erba, and P. Perinotti, "Classicality without local discriminability: Decoupling entanglement and complementarity," *Physical Review A*, vol. 102, no. 5, p. 052216, 2020.

[95] A. Joyal and R. Street, "The geometry of tensor calculus, i," *Advances in mathematics*, vol. 88, no. 1, pp. 55–112, 1991.

[96] P. Selinger, "A survey of graphical languages for monoidal categories," in *New structures for physics*, pp. 289–355, Springer, 2010.

[97] S. Mac Lane, *Categories for the working mathematician*, vol. 5. Springer Science & Business Media, 2013.

[98] S. Abramsky and N. Tzevelekos, "Introduction to categories and categorical logic," in *New structures for physics*, pp. 3–94, Springer, 2010.

[99] S. Awodey, *Category theory.* Oxford university press, 2010.

[100] B. Coecke and E. O. Paquette, "Categories for the practising physicist," in *New structures for physics*, pp. 173–286, Springer, 2010.

[101] E. Riehl, *Category theory in context.* Courier Dover Publications, 2017.

[102] B. Fong and D. I. Spivak, "Seven sketches in compositionality: An invitation to applied category theory," *arXiv preprint arXiv:1803.05316*, 2018.

[103] N. Chomsky, "Three models for the description of language," *IRE Transactions on information theory*, vol. 2, no. 3, pp. 113–124, 1956.

[104] J. Lambek, "The mathematics of sentence structure," *The American Mathematical Monthly*, vol. 65, no. 3, pp. 154–170, 1958.

[105] J. Lambek, "Type grammar revisited," in *International conference on logical aspects of computational linguistics*, pp. 1–27, Springer, 1997.

[106] B. Coecke, "The mathematics of text structure," in *Joachim Lambek: The Interplay of Mathematics, Logic, and Linguistics*, pp. 181–217, Springer, 2021.

[107] B. Coecke, M. Sadrzadeh, and S. Clark, "Mathematical foundations for a compositional distributional model of meaning," *arXiv preprint arXiv:1003.4394*, 2010.

[108] F. J. Pelletier, "The principle of semantic compositionality," *Topoi*, vol. 13, no. 1, pp. 11–24, 1994.

[109] R. H. Katz and G. Borriello, "Contemporary logic design," 2005.

[110] J. Cavanagh, *Sequential logic: analysis and synthesis.* CRC Press, 2018.

[111] M. O. Rabin and D. Scott, "Finite automata and their decision problems," *IBM journal of research and development*, vol. 3, no. 2, pp. 114–125, 1959.

[112] J. Moeller and C. Vasilakopoulou, "Monoidal grothendieck construction," *arXiv preprint arXiv:1809.00727*, 2018.

[113] D. I. Spivak, "Database queries and constraints via lifting problems," *Mathematical structures in computer science*, vol. 24, no. 6, 2014.

[114] D. I. Spivak, "Category theory for scientists," *arXiv preprint arXiv:1302.6946*, 2013.

[115] W. S. McCulloch and W. Pitts, "A logical calculus of the ideas immanent in nervous activity," *The bulletin of mathematical biophysics*, vol. 5, no. 4, pp. 115–133, 1943.

[116] F. Rosenblatt, "Principles of neurodynamics. perceptrons and the theory of brain mechanisms," tech. rep., Cornell Aeronautical Lab Inc Buffalo NY, 1961.

[117] G. Cruttwell, J. Gallagher, J.-S. P. Lemay, and D. Pronk, "Monoidal reverse differential categories," *arXiv preprint arXiv:2203.12478*, 2022.

[118] B. Jacobs, "New directions in categorical logic, for classical, probabilistic and quantum logic," *Logical Methods in Computer Science*, vol. 11, 2015.

[119] S. Abramsky and B. Coecke, "A categorical semantics of quantum protocols," in *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 2004.*, pp. 415–425, IEEE, 2004.

[120] B. Coecke, D. J. Moore, and S. Smets, "Logic of dynamics and dynamics of logic: some paradigm examples," in *Logic, Epistemology, and the Unity of Science*, pp. 527–555, Springer, 2009.

[121] A. Döring and C. Isham, ""what is a thing?": Topos theory in the foundations of physics," in *New structures for physics*, pp. 753–937, Springer, 2010.

[122] L. Susskind, "The theoretical minimum: quantum mechanics," 2012.

[123] C. J. Isham, *Lectures on quantum theory Mathematical and structural foundations*. Allied Publishers, 2001.

[124] S. Abramsky and C. Heunen, "Operational theories and categorical quantum mechanics," *Logic and algebraic structures in quantum computing and information,, Lecture Notes in Logic*, vol. 45, pp. 88–122, 2016.

[125] P. W. Bridgman, P. W. Bridgman, P. W. Bridgman, and P. W. Bridgman, *The logic of modern physics*, vol. 3. Macmillan New York, 1927.

[126] R. T. Cox, "The algebra of probable inference," *American Journal of Physics*, vol. 31, no. 1, pp. 66–67, 1963.

[127] E. T. Jaynes, *Probability theory: The logic of science*. Cambridge university press, 2003.

[128] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Probabilistic theories with purification," *Physical Review A*, vol. 81, no. 6, p. 062348, 2010.

[129] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Informational derivation of quantum theory," *Physical Review A*, vol. 84, no. 1, p. 012311, 2011.

[130] G. M. D'Ariano and P. Perinotti, "Quantum theory is an information theory," *Foundations of Physics*, vol. 46, no. 3, pp. 269–281, 2016.

[131] A. Hadzihasanovic, K. F. Ng, and Q. Wang, "Two complete axiomatisations of pure-state qubit quantum computing," in *Proceedings of the 33rd annual ACM/IEEE symposium on logic in computer science*, pp. 502–511, 2018.

[132] T. Carette and E. Jeandel, "On a recipe for quantum graphical languages," *arXiv preprint arXiv:2008.04193*, 2020.

[133] K. F. Ng and Q. Wang, "A universal completion of the zx-calculus," *arXiv preprint arXiv:1706.09877*, 2017.

[134] B. Coecke and R. Duncan, "Interacting quantum observables: categorical algebra and diagrammatics," *New Journal of Physics*, vol. 13, no. 4, p. 043016, 2011.

[135] S. Abramsky and B. Coecke, "Categorical quantum mechanics," *Handbook of quantum logic and quantum structures*, vol. 2, pp. 261–325, 2009.

[136] S. Abramsky and B. Coecke, "Physics from computer science," 2007.

[137] B. Coecke, "The logic of entanglement," in *Horizons of the Mind. A Tribute to Prakash Panangaden*, pp. 250–267, Springer, 2014.

[138] S. MacLane, "Categorical algebra," *Bulletin of the American Mathematical Society*, vol. 71, no. 1, pp. 40–106, 1965.

[139] C. Comfort, "The zx&-calculus: A complete graphical calculus for classical circuits using spiders," *arXiv preprint arXiv:2004.05287*, 2020.

[140] A. Kissinger and J. van de Wetering, "Universal mbqc with generalised parity-phase interactions and pauli measurements," *Quantum*, vol. 3, p. 134, 2019.

[141] S. Abramsky, "No-cloning in categorical quantum mechanics," *Semantic Techniques in Quantum Computation*, pp. 1–28, 2009.

[142] F. Hénault, "Quantum physics and the beam splitter mystery," in *The Nature of Light: What are Photons? VI*, vol. 9570, pp. 199–213, SPIE, 2015.

[143] G. Pospiech, A. Merzel, G. Zuccarini, E. Weissman, N. Katz, I. Galili, L. Santi, and M. Michelini, "The role of mathematics in teaching quantum physics at high school," in *Teaching-Learning Contemporary Physics*, pp. 47–70, Springer, 2021.

[144] R. Duit, H. Gropengießer, U. Kattmann, M. Komorek, and I. Parchmann, "The model of educational reconstruction–a framework for improving teaching and learning science," in *Science education research and practice in Europe*, pp. 13–37, Brill, 2012.

[145] P. Reimann, "Design-based research," in *Methodological choice and design*, pp. 37–50, Springer, 2011.

[146] M. W. Easterday, D. R. Lewis, and E. M. Gerber, "Design-based research process: Problems, phases, and applications," Boulder, CO: International Society of the Learning Sciences, 2014.

[147] A. Bakker and D. v. Eerde, "An introduction to design-based research with an example from statistics education," in *Approaches to qualitative research in mathematics education*, pp. 429–466, Springer, 2015.

[148] U. Kattmann, R. Duit, H. Gropengießer, and M. Komorek, "Educational reconstruction–bringing together issues of scientific clarification and students' conceptions," in *Annual meeting of the national association of research in science teaching (NARST), St. Louis*, pp. 1–20, 1996.

[149] D. C. Phillips, *Constructivism in Education: Opinions and Second Opinions on Controversial Issues. Ninety-Ninth Yearbook of the National Society for the Study of Education.* ERIC, 2000.

[150] R. Duit, D. I. F. T. A. UST, and A. Widodo, "Teaching science for conceptual change: Theory and practice," in *International handbook of research on conceptual change*, pp. 499–515, Routledge, 2013.

[151] J. Airey and C. Linder, "A disciplinary discourse perspective on university science learning: Achieving fluency in a critical constellation of modes," *Journal of Research in Science Teaching: The Official Journal of the National Association for Research in Science Teaching*, vol. 46, no. 1, pp. 27–49, 2009.

[152] O. Uhden, R. Karam, M. Pietrocola, and G. Pospiech, "Modelling mathematical reasoning in physics education," *Science & Education*, vol. 21, no. 4, pp. 485–506, 2012.

[153] M. Michelini and A. Stefanel, "Approaches on t/l quantum physics from per literature," in *Teaching-Learning Contemporary Physics*, pp. 3–17, Springer, 2021.

[154] S. Ainsworth, "The educational value of multiple-representations when learning complex scientific concepts," in *Visualization: Theory and practice in science education*, pp. 191–208, Springer, 2008.

[155] M.-A. Geyer and W. Kuske-Janßen, "Mathematical representations in physics lessons," in *Mathematics in physics education*, pp. 75–102, Springer, 2019.

[156] J. L. Lemke, "Teaching all the languages of science: Words, symbols, images, and actions," in *Conference on science education in Barcelona*, 1998.

[157] G. Zuccarini, "Physics student ideas on quantum state and its formal representations," *Il nuovo cimento C*, vol. 37, no. 4, pp. 273–283, 2014.

[158] M. Foucault, *Archaeology of knowledge.* routledge, 2013.

[159] D. Llewellyn, *Teaching high school science through inquiry: A case study approach.* Corwin Press, 2005.

[160] J. Locke, *An essay concerning human understanding.* Kay & Troutman, 1847.

[161] N. R. Council *et al.*, *How people learn: Brain, mind, experience, and school: Expanded edition.* National Academies Press, 2000.

[162] J. Dewey, "Democracy and education by john dewey," *Project Gutenberg*, 1916.

[163] J. Piaget, "Science of education and the psychology of the child. trans. d. coltman.," 1970.

[164] L. S. Vygotsky, *Thought and language.* MIT press, 2012.

[165] R. W. Bybee, J. A. Taylor, A. Gardner, P. Van Scotter, J. C. Powell, A. Westbrook, and N. Landes, "The bscs 5e instructional model: Origins and effectiveness," *Colorado Springs, Co: BSCS*, vol. 5, pp. 88–98, 2006.

[166] M. Pedaste, M. Mäeots, L. A. Siiman, T. De Jong, S. A. Van Riesen, E. T. Kamp, C. C. Manoli, Z. C. Zacharia, and E. Tsourlidaki, "Phases of inquiry-based learning: Definitions and the inquiry cycle," *Educational research review*, vol. 14, pp. 47–61, 2015.

[167] T. G. Amin, "Conceptual metaphor and the study of conceptual change: Research synthesis and future directions," *International Journal of Science Education*, vol. 37, no. 5-6, pp. 966–991, 2015.

[168] J. K. Gilbert and R. Justi, *Modelling-based teaching in science education*, vol. 9. Springer, 2016.

[169] A. L. Odom, E. R. Stoddard, and S. M. LaNasa, "Teacher practices and middle-school science achievements," *International Journal of Science Education*, vol. 29, no. 11, pp. 1329–1346, 2007.

[170] M. Windschitl, J. Thompson, and M. Braaten, "Beyond the scientific method: Model-based inquiry as a new paradigm of preference for school science investigations," *Science education*, vol. 92, no. 5, pp. 941–967, 2008.

[171] L. C. McDermott, "Improving the teaching of science through discipline-based education research: An example from physics," *European Journal of Science and Mathematics Education*, vol. 1, no. 1, pp. 1–12, 2013.

[172] M. Michelini, L. Santi, and A. Stefanel, "Worksheets for pupils involvement in learning quantum mechanics," *Frontiers of Physics Education*, pp. 102–111, 2008.

[173] M. Havid, T. Septiani, *et al.*, "Preliminary analysis to develop student worksheet based on inquiry based learning model for 21st century physics learning," in *Journal of Physics: Conference Series*, vol. 1481, p. 012057, IOP Publishing, 2020.

[174] Y. Yulkifli, M. V. Ningrum, and W. Indrasari, "The validity of student worksheet using inquiry-based learning model with science process skill approach for physics learning of high school," *Jurnal Penelitian & Pengembangan Pendidikan Fisika*, vol. 5, no. 2, pp. 155–162, 2019.

[175] J. A. Walsh, M. Fenech, D. L. Tucker, C. Riegle-Crumb, and B. R. La Cour, "Piloting a full-year, optics-based high school course on quantum computing," *Physics Education*, vol. 57, no. 2, p. 025010, 2021.

[176] G. Pospiech, "Quantum cryptography as an approach for teaching quantum physics," in *Teaching-Learning Contemporary Physics*, pp. 19–31, Springer, 2021.

[177] P. P. Angara, U. Stege, and A. MacLean, "Quantum computing for high-school students an experience report," in *2020 IEEE International Conference on Quantum Computing and Engineering (QCE)*, pp. 323–329, IEEE, 2020.

[178] C. Sutrini, M. Malgieri, and C. Macchiavello, "Quantum technologies: a course for teacher professional development.," in *Journal of Physics: Conference Series*, vol. 2297, p. 012018, IOP Publishing, 2022.

[179] H. S. Leff and A. F. Rex, "Maxwell's demon," *Entropy, information, computing*, 1990.

[180] A. Van Heuvelen, "Learning to think like a physicist: A review of research-based instructional strategies," *American Journal of physics*, vol. 59, no. 10, pp. 891–897, 1991.

[181] C. H. Bennett, "Notes on the history of reversible computation," *IBM Journal of research and development*, vol. 44, no. 1/2, p. 270, 2000.

[182] F. Pallotta, "Bringing the second quantum revolution into high school," *arXiv preprint arXiv:2206.15264*, 2022.

[183] A. Kohnle, C. Benfield, D. Cassettari, T. Edwards, A. Fomins, A. Gillies, G. Hähner, C. Hooley, N. Korolkova, J. Llama, *et al.*, "Quvis: The quantum mechanics visualization project," 2018.

[184] M. Malgieri, P. Onorato, and A. De Ambrosis, "Test on the effectiveness of the sum over paths approach in favoring the construction of an integrated knowledge of quantum physics in high school," *Physical Review Physics Education Research*, vol. 13, no. 1, p. 010101, 2017.

[185] R. K. Hill, "What an algorithm is. philosophy & technology 29 (1): 35–59," 2016.

[186] G. Ghirardi, *Sneaking a look at god's cards: unraveling the mysteries of quantum mechanics-revised edition.* Princeton University Press, 2007.

[187] G. Ghirardi, L. Marinatto, and T. Weber, "Entanglement and properties of composite quantum systems: a conceptual and mathematical analysis," *Journal of statistical Physics*, vol. 108, no. 1, pp. 49–122, 2002.

[188] G. M. D'Ariano, "It from qubit," in *It From Bit or Bit From It?*, pp. 25–35, Springer, 2015.

[189] N. Herbert, "Flash—a superluminal communicator based upon a new kind of quantum measurement," *Foundations of Physics*, vol. 12, no. 12, pp. 1171–1179, 1982.

[190] W. Dür and S. Heusler, "What we can learn about quantum physics from a single qubit," *arXiv preprint arXiv:1312.1463*, 2013.

[191] G. Jaeger, *Entanglement, information, and the interpretation of quantum mechanics.* Springer Science & Business Media, 2009.

[192] M. Castillo-Montoya, "Preparing for interview research: The interview protocol refinement framework," *The qualitative report*, vol. 21, no. 5, pp. 811–831, 2016.

[193] V. K. Otero and D. B. Harlow, "Getting started in qualitative physics education research," *Reviews in PER*, vol. 2, no. 1, 2009.

[194] G. Zhu and C. Singh, "Improving students' understanding of quantum measurement. i. investigation of difficulties," *Physical Review Special Topics-Physics Education Research*, vol. 8, no. 1, p. 010117, 2012.

[195] E. Marshman and C. Singh, "Framework for understanding the patterns of student difficulties in quantum mechanics," *Physical Review Special Topics-Physics Education Research*, vol. 11, no. 2, p. 020119, 2015.

[196] G. Ravaioli and O. Levrini, "Accepting quantum physics: analysis of secondary school students' cognitive needs," in *Conference proceedings, ESERA Conference*, 2017.

[197] P. Checkland and S. Holwell, “Action research: its nature and validity,” *Systemic practice and action research*, vol. 11, no. 1, pp. 9–21, 1998.

[198] A. Feldman, “Enhancing the practice of physics teachers: Mechanisms for the generation and sharing of knowledge and understanding in collaborative action research,” *Journal of Research in Science Teaching: The Official Journal of the National Association for Research in Science Teaching*, vol. 33, no. 5, pp. 513–540, 1996.

[199] M. Malgieri, P. Onorato, A. Valentini, and A. De Ambrosis, “Improving the connection between the microscopic and macroscopic approaches to thermodynamics in high school,” *Physics Education*, vol. 51, no. 6, p. 065010, 2016.

[200] M. Malgieri[1], C. Sutrini, M. Di Savino, P. Onorato, and A. De Ambrosis, “A teaching-learning sequence to bridge the gap from micro to macro in thermodynamics,” *Teaching-Learning Sequences as Innovations for Science Teaching and Learning*, p. 711.

[201] H. Leff and A. F. Rex, *Maxwell’s Demon 2 Entropy, Classical and Quantum Information, Computing.* CRC Press, 2002.

[202] M. F. Fox, B. M. Zwickl, and H. Lewandowski, “Preparing for the quantum revolution: What is the role of higher education?,” *Physical Review Physics Education Research*, vol. 16, no. 2, p. 020131, 2020.

[203] “National quantum initiative act.,” `https://www.congress.gov/bill/115th-congress/house-bill/6227`.

[204] M. Ayene, J. Kriek, and B. Damtie, “Wave-particle duality and uncertainty principle: Phenomenographic categories of description of tertiary physics students’ depictions,” *Physical Review Special Topics-Physics Education Research*, vol. 7, no. 2, p. 020113, 2011.

[205] J.-L. Ke, M. Monk, and R. Duschl, “Learning introductory quantum physics: Sensori-motor experiences and mental models,” *International Journal of Science Education*, vol. 27, no. 13, pp. 1571–1594, 2005.

[206] I. Johnston, K. Crawford, and P. Fletcher, “Student difficulties in learning quantum mechanics,” *International Journal of Science Education*, vol. 20, no. 4, pp. 427–446, 1998.

[207] K. Mannila, I. T. Koponen, and J. A. Niskanen, “Building a picture of students’ conceptions of wave-and particle-like properties of quantum entities,” *European journal of physics*, vol. 23, no. 1, p. 45, 2001.

[208] G. TSAPARLIS and G. PAPAPHOTIS, “Quantum-chemical concepts: Are they suitable for secondary students?,” *Chemistry Education Research and Practice*, vol. 3, no. 2, pp. 129–144, 2002.

[209] G. Tsaparlis and G. Papaphotis, “High-school students’ conceptual difficulties and attempts at conceptual change: The case of basic quantum chemical concepts,” *International Journal of Science Education*, vol. 31, no. 7, pp. 895–930, 2009.

[210] G. Kalkanis, P. Hadzidaki, and D. Stavrou, “An instructional model for a radical conceptual change towards quantum mechanics concepts,” *Science education*, vol. 87, no. 2, pp. 257–280, 2003.

[211] T. Bouchée, L. de Putter-Smits, M. Thurlings, and B. Pepin, “Towards a better understanding of conceptual difficulties in introductory quantum physics courses,” *Studies in Science Education*, vol. 58, no. 2, pp. 183–202, 2022.

[212] C. Stefani and G. Tsaparlis, “Students’ levels of explanations, models, and misconceptions in basic quantum chemistry: A phenomenographic study,” *Journal of Research in Science Teaching: The Official Journal of the National Association for Research in Science Teaching*, vol. 46, no. 5, pp. 520–536, 2009.

[213] G. Carrascal, A. A. Del Barrio, and G. Botella, “First experiences of teaching quantum computing,” *The Journal of Supercomputing*, vol. 77, no. 3, pp. 2770–2799, 2021.

[214] J. R. Wootton, F. Harkins, N. T. Bronn, A. C. Vazquez, A. Phan, and A. T. Asfaw, “Teaching quantum computing with an interactive textbook,” in *2021 IEEE International Conference on Quantum Computing and Engineering (QCE)*, pp. 385–391, IEEE, 2021.

[215] A. K. Ekert, “Quantum cryptography and bell’s theorem,” in *Quantum Measurements in Optics*, pp. 413–418, Springer, 1992.

[216] O. Levrini, “The role of history and philosophy in research on teaching and learning of relativity,” in *International handbook of research in history, philosophy and science teaching*, pp. 157–181, Springer, 2014.

[217] L. Hardy, "A formalism-local framework for general probabilistic theories, including quantum theory," *Mathematical Structures in Computer Science*, vol. 23, no. 2, pp. 399–440, 2013.

[218] J. Vicary, "Topological structure of quantum algorithms," in *2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science*, pp. 93–102, IEEE, 2013.

[219] W. Zeng, "The abstract structure of quantum algorithms," *arXiv preprint arXiv:1512.08062*, 2015.

[220] P. Kwiat, J. Mitchell, P. Schwindt, and A. White, "Grover's search algorithm: an optical approach," *Journal of Modern Optics*, vol. 47, no. 2-3, pp. 257–266, 2000.

[221] S. Takeuchi, "Experimental demonstration of a three-qubit quantum computation algorithm using a single photon and linear optics," *Physical review A*, vol. 62, no. 3, p. 032301, 2000.

[222] A. De Oliveira, S. Walborn, and C. Monken, "Implementing the deutsch algorithm with polarization and transverse spatial modes," *Journal of Optics B: Quantum and Semiclassical Optics*, vol. 7, no. 9, p. 288, 2005.

[223] P. Zhang, R.-F. Liu, Y.-F. Huang, H. Gao, and F.-L. Li, "Demonstration of deutsch's algorithm on a stable linear optical quantum computer," *Physical Review A*, vol. 82, no. 6, p. 064302, 2010.

[224] B. Perez-Garcia, R. I. Hernandez-Aranda, A. Forbes, and T. Konrad, "The first iteration of grover's algorithm using classical light with orbital angular momentum," *Journal of Modern Optics*, vol. 65, no. 16, pp. 1942–1948, 2018.

[225] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, "Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels," *Physical review letters*, vol. 70, no. 13, p. 1895, 1993.

[226] A. Einstein, B. Podolsky, and N. Rosen, "Can quantum-mechanical description of physical reality be considered complete?," *Physical review*, vol. 47, no. 10, p. 777, 1935.

[227] P. W. Shor, "Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer," *SIAM review*, vol. 41, no. 2, pp. 303–332, 1999.

[228] C. H. Bennett and G. Brassard, "An update on quantum cryptography," in *Workshop on the theory and application of cryptographic techniques*, pp. 475–480, Springer, 1984.

[229] J. C. Meyer, G. Passante, S. J. Pollock, and B. R. Wilcox, "Today's interdisciplinary quantum information classroom: Themes from a survey of quantum information science instructors," *Physical Review Physics Education Research*, vol. 18, no. 1, p. 010150, 2022.

[230] M. Bondani, M. L. Chiofalo, E. Ercolessi, C. Macchiavello, M. Malgieri, M. Michelini, O. Mishina, P. Onorato, F. Pallotta, S. Satanassi, *et al.*, "Introducing quantum technologies at secondary school level: Challenges and potential impact of an online extracurricular course," *Physics*, vol. 4, no. 4, pp. 1150–1167, 2022.

[231] N. D. Mermin, *Quantum computer science: an introduction.* Cambridge University Press, 2007.

[232] M. Michelini and G. Zuccarini, "University students' reasoning on physical information encoded in quantum state at a point in time," *2014 PERC Proceedings*, pp. 187–190, 2014.

[233] M. Michelini and A. Stefanel, "A path to build basic quantum mechanics ideas in the context of light polarization and learning outcomes of secondary students," in *Journal of Physics: Conference Series*, vol. 1929, p. 012052, IOP Publishing, 2021.

[234] U. S. di Uccio, A. Colantonio, S. Galano, I. Marzoli, F. Trani, and I. Testa, "Design and validation of a two-tier questionnaire on basic aspects in quantum mechanics," *Physical Review Physics Education Research*, vol. 15, no. 1, p. 010137, 2019.

[235] G. Zuccarini, "Analyzing the structure of basic quantum knowledge for instruction," *American Journal of Physics*, vol. 88, no. 5, pp. 385–394, 2020.

[236] M. Kersting, E. K. Henriksen, M. V. Bøe, and C. Angell, "General relativity in upper secondary school: design and evaluation of an online learning environment using the model of educational reconstruction," *Physical Review Physics Education Research*, vol. 14, no. 1, p. 010130, 2018.

[237] J. Walker, R. Resnick, and D. Halliday, *Halliday and resnick fundamentals of physics.* Wiley, 2014.

[238] S.-H. Tan and P. P. Rohde, "The resurgence of the linear optics quantum interferometer—recent advances & applications," *Reviews in Physics*, vol. 4, p. 100030, 2019.

[239] R. Müller and F. Greinert, "Playing with a quantum computer," *arXiv preprint arXiv:2108.06271*, 2021.

[240] M. L. Dalla Chiara, R. Giuntini, R. Leporini, E. Negri, and G. Sergioli, "Quantum information, cognition, and music," *Frontiers in Psychology*, vol. 6, p. 1583, 2015.

[241] M. L. Dalla Chiara, R. Giuntini, R. Leporini, and G. Sergioli, *Quantum Computation and Logic. How Quantum Computers Have Inspired Logical Investigations.* Springer, 2018.

[242] Y. Maruyama, "Category theory and foundations of life science: A structuralist perspective on cognition," *Biosystems*, vol. 203, p. 104376, 2021.

[243] E. F. Redish, "Problem solving and the use of math in physics courses," *arXiv preprint physics/0608268*, 2006.

[244] A. Sfard, "On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin," *Educational studies in mathematics*, vol. 22, no. 1, pp. 1–36, 1991.