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On the study of multistage stochastic vector quasi-variational problems

Elena Molho[1] · **Domenico Scopelliti**[2]

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Abstract
This paper focuses on the study of multistage stochastic vector generalized quasi-variational inequalities with a variable ordering structure. The proposed multistage stochastic vector quasi-variational problems are defined in a suitable functional setting relative to a finite set of final possible states and certain information fields; these formulations are a multicriteria extension of the multistage stochastic variational inequalities. A relevant aspect of these problems is the presence of the nonanticipativity constraints on the variables of the problem; stage by stage, these constraints impose the measurability with respect to the information field at that stage. Without requiring any assumption of monotonicity, we prove some existence results by using a nonlinear scalarization technique. On this basis, we analyze multistage stochastic vector Nash equilibrium problems: as an example, we focus on a suitable multistage stochastic bicriteria Cournot oligopolistic model.

Keywords Multistage stochastic vector variational inequality · Nonlinear scalarization function · Nonanticipativity · Variable ordering structure · Generalized Nash-Cournot game

1 Introduction

Introduced by Stampacchia and Fichera in 1964, variational inequalities theory has its origins in the calculus of variations. Thereafter, it was soon recognized its role in the formulation of the quantitative and qualitative analysis of several mathematical problems such as optimization problems, equilibrium problems, complementary and fixed point problems. Nowadays,

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✉ Elena Molho
elena.molho@unipv.it

Domenico Scopelliti
domenico.scopelliti@unibs.it

1 Department of Economics and Management, University of Pavia, Via San Felice 5, 27100 Pavia, Italy

2 Department of Economics and Management, Contrada S. Chiara 50, 25122 Brescia, Italy

the variational inequalities theory unifies a large range of applications arising in economics, finance, management, game theory, control theory, operations research, and several branches of engineering sciences. In 1980, Giannessi [18] introduced the concept of a vector variational inequality in a finite-dimensional space. In the last decades, vector variational inequalities have been intensively studied by several authors in different settings in relation to vector optimization, vector equilibrium, and the duality theory; see, e.g., [2, 13, 32] and references therein.

However, although some practical problems involve only deterministic data, in most real-world applications there are many important cases where problem data contain some uncertainty and randomness. Moreover, in many real-life applications, a decision-maker has to make sequential decisions, motivating the interest in stochastic variational problems of multistage nature as a natural extension of deterministic variational inequalities. Indeed, to capture the dynamics that are essential to stochastic decision processes in response to an increasing level of information, Rockafellar and Wets [27] introduced a multistage stochastic variational problem. This formulation provides innovative and flexible tools to study real-life problems complicated by time, uncertainty, risk and allows us to capture the role of information in the recursive decision processes; see, e.g., [15, 20, 22, 29]. The key concept of this new formulation turns out to be the nonanticipativity: some constraints have to be included in the formulation of the problem to take into account the partial information structure progressively revealed. To our knowledge, however, a multicriteria counterpart of a multistage stochastic variational problem is not present in the literature; in many real-life applications, such an extension would come in handy and/or necessary. Indeed, it is easy to imagine real-life situations in which a decision-maker, stage by stage, has to make opportune choices under several different criteria, uncertain conditions, and partial information. Motivated by this fact, the aim of this paper is to deal with the analysis of such a multicriteria generalization that could pave the way to a wide range of possible applications with an approach that is closer to the real-life mechanisms. Following this spirit, we introduce the study of multistage stochastic Nash equilibrium problems with vector-valued payoff functions and variable ordering structures as a suitable multicriteria extension of the multistage stochastic Nash equilibrium problems considered in [27]. In doing this, we are inspired by [1, 3]: in the quoted papers, the authors propose the study of deterministic vector Nash equilibrium problems with different approaches and assumptions on the structure of the problem. In addition, as a practical application, we analyze a suitable multistage stochastic bicriteria generalized Nash-Cournot game. We recall that the Cournot oligopolistic model is one of the most widely used modelizations to study strategic interactions among non-cooperative firms: classical real-world applications regards the study of electricity markets problems, gas markets problems, etc; see, e.g., [24, 25] and the references therein. Moreover, the use of a vector-valued payoff function could allow us to take into account the well-known approach of the managerial theory of the firm, where the objectives of the management do not always coincide with the aims of the shareholders; see, e.g., [6].

We focus on the study of multistage stochastic vector generalized quasi-variational inequalities with a variable ordering structure. Formally, a variable ordering structure is a set-valued map with conic values, hence each element of the domain is associated with an order. The idea of variable ordering structures was firstly introduced by Yu [31] in terms of domination structures: it is a generalization of the fixed domination structure in multicriteria decision-making problems. Thereafter, this concept attracted an increasing interest leading to its theoretical development in different settings; see, e.g., [13, 14, 19] and references therein. This is due to the fact that, in real-life problems, a variable ordering structure better reflects the features of a problem than a corresponding formulation with a constant ordering structure:

important applications arise in economics, decision theory, portfolio management, medicine, etc; see, e.g, [30] and references therein. However, it is important to avoid the requirement of unduly restrictive conditions on such set-valued maps with conic values.

A powerful approach used for analyzing a vector formulation is to reduce it to an easier scalar problem. Several scalarization methods have been introduced in the literature; see, e.g., [2]. Here, we focus on nonlinear scalarization methods. The original version of the nonlinear scalarization function used in this work is due to Gerstewitz [17]. Thereafter, several versions have been proposed in the literature to generalize the original formulation; see, e.g., [13, 30]. This is motivated by the fact that nonlinear scalarization functions allow us to analyze various nonconvex problems with multiple objectives. In this paper, we adapt to our framework the scalarization function introduced in [12] to study multistage stochastic vector quasi-variational problems without requiring any monotonicity condition on the principal operator.

The paper is organized as follows. Section 2 is devoted to the introduction of some preliminary notions and the multistage-functional framework in which we operate. This allows us, in Sect. 3, to generalize, in a multicriteria setting and with variable ordering structures, the multistage stochastic variational inequalities, in basic and extensive form, introduced by Rockafellar and Wets [27]. In particular, the variable ordering structure $\mathcal{C}(\cdot)$, built on the functional space $\mathcal{L}_P$, where the extensive form of the problem is embedded, is transformed into a corresponding variable ordering structure $\mathcal{D}(\mathcal{C}(\cdot))$ on $\mathbb{R}^P$, used in the corresponding basic form. Under suitable assumptions, the solutions of a vector variational formulation in the extensive form are linked with those of the corresponding problem in the basic form. Subsequently, in Sect. 4, a nonlinear scalarization function is introduced in our functional setting with a variable ordering structure. In particular, to reduce the requirements on the variable ordering structures, we make use of the concept of cosmically upper continuity rather than the classical upper semicontinuity. On this basis, a multistage stochastic vector quasi-variational problem is opportunely studied by the scalarization procedure introduced. To support our results, in Sect. 5, multistage stochastic Nash equilibrium problems with vector-valued payoff functions and variable ordering structures are considered; moreover, as an application of interest, a suitable multistage stochastic bicriteria Cournot oligopolistic model is analyzed. Finally, a section with the conclusions is given.

2 Preliminary notions

This paper aims to generalize the multistage stochastic variational formulations introduced in [27] to deal with multicriteria problems. To this goal, for the convenience of the reader, some preliminary notations, definitions, and tools are recalled.

Let X be a finite-dimensional Hilbert space, $\langle\cdot,\cdot\rangle$ be the corresponding inner product, and $B(0,1)$ be the closed unit ball of X. We denote by $\mathrm{int}A$, $\mathrm{cl}A$ and $\mathrm{conv}A$, respectively, the interior, the clousure, and the convex hull of a set $A \subseteq X$. Let $T \subset X$ be a proper, convex, and closed cone with $\mathrm{int}T \neq \emptyset$; then, T induces a partial ordering in X so that, for any $x, y \in X$, the following holds:

$$x <_T y \quad \Leftrightarrow \quad y - x \in \mathrm{int}\, T \qquad \text{and} \qquad x \not<_T y \quad \Leftrightarrow \quad y - x \notin \mathrm{int}\, T.$$

Let X and Y be finite-dimensional Hilbert spaces and $\Gamma : X \rightrightarrows Y$ be a set-valued map. We recall some classical definitions that we will use in the sequel; for further details, the interested reader can refer to [2, 7, 14] and the references therein. A set-valued map $\Gamma : X \rightrightarrows Y$ is said to be

- *Lower semicontinuous* at $x \in X$ if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, with $x_n \to x$, and for any $y \in \Gamma(x)$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset Y$, with $y_n \in \Gamma(x_n)$ for any $n \in \mathbb{N}$ and $y_n \to y$.
- *Upper semicontinuous* at $x \in X$ if for any open set V in Y containing $\Gamma(x)$, there exists an open neighborhood U of x in X such that $\Gamma(\hat{x}) \subseteq V$ for all $\hat{x} \in U$.
- *Closed* at $x \in X$ if for any sequences $\{x_n\}_{n\in\mathbb{N}} \subset X$, $\{y_n\}_{n\in\mathbb{N}} \subset Y$, with $x_n \to x$, $y_n \in \Gamma(x_n)$ and $y_n \to y$, then $y \in \Gamma(x)$.

We point out that the condition of upper semicontinuity is unduly restrictive for set-valued maps with conic values; see, e.g., [9, 14]. To overcome this problem, the following concepts were introduced and analyzed in [21]: $\Gamma : X \rightrightarrows Y$ is said to be

- *Cosmically upper continuous* at $x \in X$ if the mapping $x \to \Gamma(x) \cap B(0, 1)$ is upper semicontinuous at $x \in X$.
- *Cosmically closed* at $x \in X$ if the mapping $x \to \Gamma(x) \cap B(0, 1)$ is closed at $x \in X$.

From (vi) of Proposition 2.1 in [21], Γ is cosmically closed at $x \in X$ if and only if it is closed at $x \in X$. In general, instead, the cosmical upper continuity of Γ does not imply its upper continuity; for further details, the interested reader can refer to [14, 21].

2.1 The scalar case: time-uncertainty-information structure and functional space

To study multicriteria multistage problems under uncertainty and partial information, we preliminarily introduce a suitable multistage-functional framework in which we operate; for further details, the interested reader can refer to [22, 27].

Let $\mathcal{T} := \{1, \ldots, t, \ldots, T\}$ and $\mathcal{T}_0 := \{0\} \cup \mathcal{T}$ be a finite sets of stages, respectively, without and with the initial stage. At each stage $t \in \mathcal{T}$, $\Xi_t := \left\{\xi_t^1, \ldots, \xi_t^{k_t}\right\} = \left\{\xi_t^{j_t}\right\}_{j_t=1,\ldots,k_t}$ denotes a finite set of all uncertain situations that could occur, while ξ_0 represents the unique initial situation. Let us consider the following sample space:

$$\Omega := \{\xi_0\} \times \Xi_1 \times \cdots \times \Xi_t \times \cdots \times \Xi_T \text{ such that}$$
$$\omega := (\xi_0, \xi_1^{j_1}, \ldots, \xi_t^{j_t}, \ldots, \xi_T^{j_T}) \in \{\xi_0\} \times \Xi_1 \times \cdots \times \Xi_t \times \cdots \times \Xi_T,$$

where Ω is the finite set of all *scenarios*, that is, the possible occurrences on the entire history. Let $\mathbb{P} = (\pi(\omega))_{\omega\in\Omega}$ be a probability measure on Ω such that each ω has an assigned strictly positive probability $\pi(\omega)$. Let

$$x := (x_0, x_1, \ldots, x_t, \ldots, x_T) \in \mathbb{R}^{G_0} \times \mathbb{R}^{G_1} \times \cdots \times \mathbb{R}^{G_t} \times \cdots \mathbb{R}^{G_T} = \mathbb{R}^G,$$

where $G := G_0 + G_1 + \ldots + G_t + \ldots + G_T$. Then, the following $T + 1$-stage pattern is considered

$$(\xi_0,\ x_0),\ (\xi_1^{j_1},\ x_1),\ \ldots,\ (\xi_t^{j_t},\ x_t),\ \ldots,\ (\xi_T^{j_T},\ x_T),$$

where $\xi_t \in \Xi_t$ stands for the information revealed at the t-th stage when the decision x_t has to be made. To opportunely formalize the recursive nature of such a decision process, the introduction of suitable information fields is needed.

Definition 1 A family of information-partitions of Ω is $\mathcal{P} := \{F_t\}_{t=0,\ldots,T}$ where, for all $t \in \mathcal{T}_0$, $F_t := \left\{F_t^1, \ldots, F_t^{k_t}\right\} = \left\{F_t^{j_t}\right\}_{j_t=1,\ldots,k_t}$ is a partition of Ω such that

(i) $F_0 = \{\Omega\}$;
(ii) for all $t = 1, \ldots, T$, $F_{t+1} \subset F_t$, that is: if $F_{t+1}^{j_{t+1}} \in F_{t+1} \Rightarrow F_{t+1}^{j_{t+1}} \subset F_t^{j_t}$ for some $F_t^{j_t} \in F_t$;
(iii) $F_T = \Omega$.

For all $t \in \mathcal{T}_0$, the set $F_t^{j_t}$ is called *elementary event* and the partition F_t is called *event*.

If two scenarios $\omega_s, \omega_c \in \Omega$ are in the same set $F_t^{j_t} \in F_t$, then they are indistinguishable at stage t on the basis of available information: since they share the same path up to stage t, the known information is the same, that is

$$\omega_s = (\xi_0, \xi_1^{j_1}, \ldots, \xi_t^{j_t}, \xi_{t+1}^s, \ldots, \xi_T^s) \quad \text{and} \quad \omega_c = (\xi_0, \xi_1^{j_1}, \ldots, \xi_t^{j_t}, \xi_{t+1}^c, \ldots, \xi_T^c).$$

To study this time-uncertainty-information structure, in [27] the authors consider the following linear functional space

$$\begin{aligned} \mathcal{L}_G(\Omega, \mathbb{P}) := \mathcal{L}_G &= \left\{ \text{the collection of all functions} \quad x : \Omega \to \mathbb{R}^G \right\} \\ &= \prod_{t \in \mathcal{T}_0} \mathcal{L}_{G_t} = \prod_{t \in \mathcal{T}_0} \left\{ \text{the collection of all functions } \ x_t : \Omega \to \mathbb{R}^{G_t} \right\} \end{aligned}$$

equipped with the following *expectational inner product* and the associated norm

$$\langle\langle x, y \rangle\rangle := \mathbb{E}[\langle x, y \rangle] = \sum_{\omega \in \Omega} \pi(\omega) \langle x(\omega), y(\omega) \rangle, \qquad \|x\| := (\mathbb{E}[\langle x, x \rangle])^{\frac{1}{2}}, \tag{1}$$

where, for all $\omega \in \Omega$, $x(\omega) = (x_t(\omega))_{t \in \mathcal{T}_0}$, $y(\omega) = (y_t(\omega))_{t \in \mathcal{T}_0}$, and $\langle \cdot, \cdot \rangle$ identifies the classical inner product in the G-dimensional Euclidean space; hence, such a structure makes $\mathcal{L}_G$ a finite-dimensional Hilbert space.

Definition 2 Given the information-partitions $\mathcal{P} = \{F_t\}_{t \in \mathcal{T}_0}$ of Ω, let $F_{\bar{t}} \in \mathcal{P}$. We say that $x \in \mathcal{L}_G$ is $F_{\bar{t}}$*-measurable* with respect to $\mathcal{P}$ if, for all $j_{\bar{t}} = 1, \ldots, k_{\bar{t}}$, it holds:

$$\forall \omega_s, \omega_c \in F_{\bar{t}}^j \qquad x_t(\omega_s) = x_t(\omega_c) \qquad \forall t = 0, \ldots, \bar{t}.$$

We say that $x \in \mathcal{L}_G$ is *measurable* if it is F_t-measurable for all $F_t \in \mathcal{P}$ and $t \in \mathcal{T}_0$.

On the basis of Definitions 1 and 2, the following subspace is introduced

$$\mathcal{N}_G := \{x \in \mathcal{L}_G : x_t \ is \quad F_t - measurable \quad \forall t \in \mathcal{T}_0\}.$$

It is called *nonanticipativity* constraints subspace and it will play a central role in the multistage stochastic variational formulation that we are going to study. Moreover, let

$$\mathcal{M}_G := \{\rho \in \mathcal{L}_G : \langle\langle \rho, y \rangle\rangle = 0 \ \ \forall y \in \mathcal{N}_G\} = (\mathcal{N}_G)^{\perp}$$

be the *nonanticipativity multipliers subspace*; then, according to the Riesz orthogonal decomposition, it results

$$\mathcal{L}_G = \mathcal{N}_G + (\mathcal{N}_G)^{\perp} = \mathcal{N}_G + \mathcal{M}_G.$$

From now on, we pose $\mathcal{K} := \{x \in \mathcal{L}_G : x(\omega) \in K(\omega) \, \forall \omega \in \Omega\}$, with $K(\omega) \subseteq \mathbb{R}^G$ nonempty, convex, and closed for each $\omega \in \Omega$. So, in this multistage-functional framework, Rockafellar and Wets introduced in [27] the following stochastic variational problems.

Definition 3 Let $\Phi : \mathcal{L}_G \to \mathcal{L}_G$ such that, for each $x \in \mathcal{L}_G$ and $\omega \in \Omega$, it results $\Phi(\omega, x(\omega)) \in \mathbb{R}^G$. A multistage stochastic variational inequality in *basic form*, associated with Φ and $\mathcal{K}$, is the following problem:

$$\text{find } \bar{x} \in \mathcal{K} \cap \mathcal{N}_G \text{ such that } \langle\langle \Phi(\bar{x}), x - \bar{x} \rangle\rangle \geq 0 \quad \forall x \in \mathcal{K} \cap \mathcal{N}_G; \tag{2}$$

whereas, a multistage stochastic variational inequality in *extensive form*, associated with Φ and $\mathcal{K}$, is the following problem:

$$\text{find } \bar{x} \in \mathcal{N}_G \text{ such that } \bar{\rho} \in \mathcal{M}_G \text{ so that}$$

$$\forall \omega \in \Omega \quad \langle \Phi(\omega, \bar{x}(\omega)) + \bar{\rho}(\omega), x(\omega) - \bar{x}(\omega) \rangle \geq 0 \quad \forall x(\omega) \in K(\omega). \tag{3}$$

Under suitable conditions, a vector is a solution of the basic formulation (2) if and only if it is a solution of the extensive problem (3); see Theorem 3.2 in [27]. In particular, the multistage stochastic variational inequality (3) turns out to be relevant from an applicative point of view, in terms of computation of the solution; see, e.g., [28].

3 Multistage stochastic vector quasi-variational problems

In this section, we aim to formalize a multicriteria extension of the multistage stochastic variational problems introduced in Definition 3. In particular, since such an extension is proposed to study applicative problems as close as possible to the dynamics leading the choices in the real-world, we focus on the study of multistage stochastic vector variational problems with variable ordering structure. Formally, a variable ordering structure is a set-valued map with conic values such that each element of its domain is associated with an order: in real-life problems (such as, for instance, in portfolio management, location problems, medical image registration, etc.), a variable ordering structure better reflects the features of a problem than a corresponding formulation with a constant ordering structure; see, e.g., [8, 14, 30].

3.1 Variable ordering structures

Let $P := \{1, \ldots, p, \ldots, P\}$ be a finite set of criteria. With respect to the functional space $\mathcal{L}_G$, we introduce the variable ordering structures $\mathcal{C} : \mathcal{L}_G \rightrightarrows \mathcal{L}_P$ as follows:

$$\forall x \in \mathcal{L}_G\,, \qquad \mathcal{C}(x) := \{z \in \mathcal{L}_P \,:\, z(\omega) \in C(x(\omega)) \;\; \forall \omega \in \Omega\}\,, \tag{4}$$

where $C(x(\omega)) \subset \mathbb{R}^P$ is a proper, convex, and closed cone with int $C(x(\omega)) \neq \emptyset$. Hence, for each $x \in \mathcal{L}_G$ and $\omega \in \Omega$, $(\mathbb{R}^P, C(x(\omega)))$ is a quasi-ordered space. In this way, for each $x \in \mathcal{L}_G$, it follows that $(\mathcal{L}_P, \mathcal{C}(x))$ is a quasi-ordered space.

In the sequel, we are going to work with expected values in $\mathbb{R}^P$: starting from $\mathcal{C}$, we are going to build a variable ordering structure that is suitable to formulate the basic form of a multistage stochastic vector quasi-variational problem by means of a vector of expectational inner products in $\mathbb{R}^P$. Hence, starting from the original variable ordering structure $\mathcal{C} : \mathcal{L}_G \rightrightarrows \mathcal{L}_P$ with values in the functional space $\mathcal{L}_P$, we consider a coherent variable ordering structure $\mathcal{D}(\mathcal{C})$ in the form of a set-valued map from $\mathcal{L}_G$ to $\mathbb{R}^P$. To this aim, we consider the transformation

$$\mathcal{D} : \mathcal{L}_P \to \mathbb{R}^P \quad \text{such that} \quad z \;\to\; \mathcal{D}(z) := \hat{z} = \sum_{\omega \in \Omega} \pi(\omega) z(\omega). \tag{5}$$

Then, we introduce the following set-valued map

$$\mathcal{D}(\mathcal{C}) : \mathcal{L}_G \rightrightarrows \mathbb{R}^P \quad \text{such that} \quad x \to \mathcal{D}(\mathcal{C}(x)), \tag{6}$$

where $\mathcal{D}(\mathcal{C}(x)) \subset \mathbb{R}^P$ is a proper, convex, and closed cone with int $\mathcal{D}(\mathcal{C}(x)) \neq \emptyset$. In particular, for each $x \in \mathcal{L}_G$, $(\mathbb{R}^P, \mathcal{D}(\mathcal{C}(x)))$ is a quasi-ordered space. Hence, in order to work in a framework of expected values, we opportunely switch from $(\mathcal{L}_P, \mathcal{C}(x))$ to $(\mathbb{R}^P, \mathcal{D}(\mathcal{C}(x)))$ by means of the linear transformation (5).

Moreover, let $\mathcal{A} \subseteq \mathcal{L}_P$ be nonempty. We introduce the following set:

$$\widetilde{\mathcal{A}} := \{z \in \mathcal{A} : \sum_{\omega \in \Omega} \pi(\omega) z(\omega) = 0_P\}. \tag{7}$$

We point out that, if $\mathcal{A} = \mathcal{L}_P$, it follows that $\widetilde{\mathcal{A}} = \ker \mathcal{D}$.

Lemma 1 *For any $x \in \mathcal{L}_G$, let $(\mathcal{L}_P, \mathcal{C}(x))$ and $(\mathbb{R}^P, \mathcal{D}(\mathcal{C}(x)))$ be two quasi-ordered spaces defined, respectively, throughout (4) and (6). Then, for each $z \in \mathcal{A}$, it holds:*

$$z <_{\mathcal{C}(x)} 0_P \quad \Rightarrow \quad \sum_{\omega \in \Omega} \pi(\omega) z(\omega) <_{\mathcal{D}(\mathcal{C}(x))} 0_P.$$

The converse implication holds, if $\widetilde{\mathcal{A}} \subseteq \mathcal{C}(x)$.

Proof Firstly, we observe that

$$z <_{\mathcal{C}(x)} 0_P \quad \Leftrightarrow \quad \forall \omega \in \Omega\,, \quad z(\omega) <_{C(x(\omega))} 0_P.$$

For each $x \in \mathcal{L}_G$, since int $\mathcal{C}(x) \neq \emptyset$, it results that int $\mathcal{C}(x) =$ relint $\mathcal{C}(x)$; see, e.g., [26]. Then, from Lemma 2.1 (iii) in [10], it follows that

$$\begin{aligned} \forall \omega \in \Omega\,, \quad -z(\omega) \in \operatorname{int} C(x(\omega)) \;&\Rightarrow\; -\sum_{\omega \in \Omega} \pi(\omega) z(\omega) \in \mathcal{D}(\operatorname{int} \mathcal{C}(x)) \\ &\Rightarrow\; -\sum_{\omega \in \Omega} \pi(\omega) z(\omega) \in \operatorname{int} \mathcal{D}(\mathcal{C}(x)), \end{aligned}$$

that is, $\sum_{\omega \in \Omega} \pi(\omega) z(\omega) <_{\mathcal{D}(\mathcal{C}(x))} 0_P$. Conversely, let us suppose that $-\sum_{\omega \in \Omega} \pi(\omega) z(\omega) \in$ int $\mathcal{D}(\mathcal{C}(x))$; since $\widetilde{\mathcal{A}} \subseteq \mathcal{C}(x)$, it follows that

$$-z \in \operatorname{int} \mathcal{C}(x) + \widetilde{\mathcal{A}} \subseteq \operatorname{int} \mathcal{C}(x) + \mathcal{C}(x) \subseteq \operatorname{int} \mathcal{C}(x);$$

hence, $z <_{\mathcal{C}(x)} 0_p$. □

3.2 Vector variational formulations

Starting from Definition 3, we use the set-valued map (6) in order to formally introduce the basic and the extensive formulation of a multistage stochastic vector generalized quasi-variational inequality: each formulation corresponds to a multicriteria extension of the corresponding multistage stochastic variational problem in which the principal operator is a set-valued map and the constraint set is subject to modifications depending on the considered point.

Preliminarily, let $L(\mathcal{L}_G, \mathcal{L}_P)$ be the space of all continuous linear operators from $\mathcal{L}_G$ to $\mathcal{L}_P$, we introduce

$$\Phi := (\Phi_1, \ldots, \Phi_p, \ldots, \Phi_P) : \mathcal{L}_G \rightrightarrows L(\mathcal{L}_G, \mathcal{L}_P), \tag{8}$$

where $\Phi_p : \mathcal{L}_G \rightrightarrows \mathcal{L}_G$ for each $p \in P$. In particular, we can rewrite the set-valued map (8) as

$$\Phi : \Omega \times \mathbb{R}^G \rightrightarrows L(\mathbb{R}^G, \mathbb{R}^P) \quad \text{so that} \quad \Phi(\omega, \cdot) : \mathbb{R}^G \rightrightarrows L(\mathbb{R}^G, \mathbb{R}^P) \quad \text{and}$$
$$\Phi(\cdot, x(\cdot)) : \Omega \rightrightarrows L(\mathbb{R}^G, \mathbb{R}^P);$$

hence, for each $\varphi(\omega) \in \Phi(\omega, x(\omega))$, it follows that $\varphi(\omega)$ is an $P \times G$ matrix. Moreover, we consider $\mathcal{S} : \mathcal{K} \rightrightarrows \mathcal{K}$ as the set-valued map such that, for any $x \in \mathcal{K}$, $\mathcal{S}(x)$ is defined as

$$\mathcal{S}(x) : \Omega \rightrightarrows \mathbb{R}^G \quad \text{such that} \quad \omega \to \mathcal{S}(x)(\omega) := \mathcal{S}(\omega, x(\omega)) \subseteq K(\omega).$$

Let $\langle\langle \cdot, \cdot \rangle\rangle_P$ be a P-dimensional vector such that each component is an expectational inner product defined as in (1); then, we can now formally introduce a multicriteria extension of Definition 3.

Definition 4 *(Basic Form)* A multistage stochastic vector generalized quasi-variational inequality in *basic form*, associated with Φ and $\mathcal{S}$, is the following problem:

$$\text{find } \bar{x} \in \mathcal{S}(\bar{x}) \cap \mathcal{N}_G \text{ such that } \exists \varphi \in \Phi(\bar{x}) \text{ and } \langle\langle \varphi, x - \bar{x} \rangle\rangle_P \notin - \operatorname{int} \mathcal{D}(\mathcal{C}(\bar{x})) \quad \forall x \in \mathcal{S}(\bar{x}) \cap \mathcal{N}_G . \tag{9}$$

In particular, since $\varphi = (\varphi_1, \ldots, \varphi_p, \ldots, \varphi_P) \in \Phi(\bar{x})$, it results $\varphi_p \in \Phi_p(\bar{x})$ for each $p \in P$. Then, we can write the expectational inner product in (9) in the following explicit way:

$$\begin{aligned} \langle\langle \varphi, x - \bar{x} \rangle\rangle_P &:= \big(\langle\langle \varphi_1, x - \bar{x} \rangle\rangle, \ldots, \langle\langle \varphi_p, x - \bar{x} \rangle\rangle, \ldots, \langle\langle \varphi_P, x - \bar{x} \rangle\rangle\big) \\ &= \left(\sum_{\omega \in \Omega} \pi(\omega) \langle \varphi_p(\omega), x(\omega) - \bar{x}(\omega) \rangle \right)_{p \in P}. \end{aligned}$$

If $\mathcal{S}(x) \cap \mathcal{N}_G = \mathcal{K} \cap \mathcal{N}_G$ for each $x \in \mathcal{K}$, problem (9) reduces to a *multistage stochastic vector generalized variational inequality*; if Φ is single-valued, problem (9) reduces to a *multistage stochastic vector quasi-variational inequality*; if both $\Phi(x)$ is a singleton and $\mathcal{S}(x) \cap \mathcal{N}_G = \mathcal{K} \cap \mathcal{N}_G$, for each $x \in \mathcal{K}$, problem (9) reduces to a *multistage stochastic vector variational inequality*.

Furthermore, we introduce the extensive formulation of a multistage stochastic vector quasi-variational problem, which may be useful for computational purposes. Such a formulation enables the decomposition of the basic form problem (9) into a separate problem for each scenario.

Definition 5 *(Extensive Form)* A multistage stochastic vector generalized quasi-variational inequality in *extensive form*, associated with Φ and $\mathcal{S}$, is the following problem:

$$\text{find } \bar{x} \in \mathcal{S}(\bar{x}) \cap \mathcal{N}_G \text{ such that } \exists \varphi \in \Phi(\bar{x}) \text{ and } \bar{\rho} \in \mathcal{M}_G \text{ so that}$$

$$\forall \omega \in \Omega \quad \langle \varphi(\omega) + \bar{\bar{\rho}}(\omega), x(\omega) - \bar{x}(\omega) \rangle_P \notin - \operatorname{int} C(\bar{x}(\omega)) \quad \forall x(\omega) \in \mathcal{S}(\omega, \bar{x}(\omega)), \tag{10}$$

where $\bar{\bar{\rho}}(\omega)$ is a $P \times G$ matrix whose rows are all copies of $\bar{\rho}(\omega)$.

We point out that $\bar{\rho} \in \mathcal{M}_G$ plays the role of a *Lagrangian multiplier for the nonanticipativity constraints*: for each $\omega \in \Omega$, the multiplier $\bar{\rho}(\omega)$ is a G-dimensional vector. However, in the vector variational formulation (10), $\varphi(\omega)$ is an $P \times G$ matrix; hence, for mathematical convenience, we introduce $\bar{\bar{\rho}}(\omega)$ as the $P \times G$ matrix in which all the p rows are copies of $\bar{\rho}(\omega)$.

Remark 1 If $P = 1$, $C(\bar{x}(\omega))$ is the non-negative ray for each $\omega \in \Omega$, $\Phi(x)$ is a singleton and $\mathcal{S}(x) \cap \mathcal{N}_G = \mathcal{K} \cap \mathcal{N}_G$ for each $x \in \mathcal{K}$, then the vector variational formulations (9) and (10) reduce, respectively, to the scalar variational problems (2) and (3).

Now, our aim is to investigate the relationships among the solutions of a vector variational formulation in the extensive form with those of the corresponding problem in the basic form. A solution to (9) may not be a solution to (10) even in the simplest case of a constant ordering structure, as it is shown in the following example.

Example 1 We pose $\mathcal{T}_0 = \{0, 1\}$, $\Omega = \{\omega_1, \omega_2\}$, and $\pi(\omega_1) = \pi(\omega_2) = \frac{1}{2}$. Let $\bar{x} = 0_2$, we consider:

- for all $x \in \mathcal{L}_2$ and $\omega \in \Omega$, $C(x(\omega)) = \mathbb{R}^2_+$;
- $\Phi : \mathcal{L}_2 \to L(\mathcal{L}_2, \mathcal{L}_2)$ such that

$$\Phi(\omega_1, \bar{x}(\omega_1)) := (\Phi_1(\omega_1, \bar{x}(\omega_1)), \Phi_2(\omega_1, \bar{x}(\omega_1))) = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \qquad \text{and}$$
$$\Phi(\omega_2, \bar{x}(\omega_2)) := (\Phi_1(\omega_2, \bar{x}(\omega_2)), \Phi_2(\omega_2, \bar{x}(\omega_2))) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix};$$

- for all $x \in \mathcal{L}_2$, $\mathcal{S}(x) \cap \mathcal{N}_2 = \mathcal{K} \cap \mathcal{N}_2 = \mathcal{K} = \big\{x \in \mathcal{N}_2 : x(\omega) = (x_0(\omega), x_1(\omega)) \in K(\omega)$ $\forall \omega \in \Omega\big\}$ such that

$$\begin{aligned} &K(\omega_1) := \{x(\omega_1) \in \mathbb{R}^2 \;:\; x_0(\omega_1) \geq 0\,, \quad x_0(\omega_1) = x_0 \\ &x_1(\omega_1) \;\geq\; 0\,, \quad \frac{x_0}{2} \leq x_1(\omega_1) \leq x_0\} \\ &K(\omega_2) := \big\{x(\omega_2) \in \mathbb{R}^2 \;:\; x_0(\omega_2) \geq 0\,, \quad x_0(\omega_2) = x_0\,, \quad x_1(\omega_2) \geq 0\big\}\,; \end{aligned}$$

then, it follows that $\bar{x} \in \mathcal{K}$ is a solution of the following multistage stochastic vector variational inequality in basic form

$$\begin{aligned} \langle\langle \Phi(\bar{x}), x - \bar{x}\rangle\rangle_2 &= \left(\sum_{\omega\in\Omega} \pi(\omega)\langle \Phi_1(\omega, \bar{x}(\omega)), x(\omega) - \bar{x}(\omega)\rangle, \sum_{\omega\in\Omega} \pi(\omega)\langle \Phi_2(\omega, \bar{x}(\omega)), x(\omega) - \bar{x}(\omega)\rangle\right) \\ &= (x_1(\omega_1) + x_1(\omega_2), x_0 - x_1(\omega_1) + x_1(\omega_2)) \notin -\operatorname{int} \mathcal{D}(\mathbb{R}^2_+) \qquad \forall x \in \mathcal{K} \end{aligned}$$

being that $x_1(\omega_1) + x_1(\omega_2) \geq 0$ for all $x \in \mathcal{K}$. However, $\bar{x} \in \mathcal{K}$ is not a solution of the corresponding multistage stochastic vector variational inequality in extensive form: for $\omega_1 \in \Omega$ and $\bar{\rho} \in \mathcal{M}_2$, it results

$$\begin{aligned} &\langle \Phi(\omega_1, \bar{x}(\omega_1)) + \bar{\bar{\rho}}(\omega_1), x(\omega_1) - \bar{x}(\omega_1)\rangle_2 \\ &= (\langle \Phi_1(\omega_1, \bar{x}(\omega_1)) + \bar{\rho}(\omega_1), x(\omega_1) - \bar{x}(\omega_1)\rangle, \langle \Phi_2(\omega_1, \bar{x}(\omega_1)) + \bar{\rho}(\omega_1), x(\omega_1) - \bar{x}(\omega_1)\rangle) \\ &= (-x_0 + x_1(\omega_1), x_0 - 2x_1(\omega_1)) \in -\operatorname{int} \mathbb{R}^2_+ \qquad \forall x(\omega_1) \in \operatorname{int} K(\omega_1), \end{aligned}$$

where $\langle\langle \bar{\rho}, x\rangle\rangle = 0$ for all $x \in \mathcal{K}$.

Under some additional assumptions, we prove that a solution to (10) is a solution to (9). In particular, let $(\bar{x}, \varphi) \in \mathcal{L}_G \times \Phi(\bar{x})$ and $\mathcal{A}(\bar{x}) := \big\{\langle \varphi(\cdot) + \bar{\bar{\rho}}(\cdot), x(\cdot) - \bar{x}(\cdot)\rangle_P \in \mathcal{L}_P \;:\; x \in \mathcal{S}(\bar{x})\big\}$; then, the set in (7) is now defined as follows

$$\widetilde{\mathcal{A}}(\bar{x}) := \big\{\langle \varphi(\cdot) + \bar{\bar{\rho}}(\cdot), x(\cdot) - \bar{x}(\cdot)\rangle_P \in \mathcal{A}(\bar{x}) \;:\; \langle\langle \varphi + \bar{\bar{\rho}}, x - \bar{x}\rangle\rangle_P = 0_P\big\}. \qquad (11)$$

Theorem 1 *For any $x \in \mathcal{L}_G$, let $(\mathcal{L}_P, \mathcal{C}(x))$ and $(\mathbb{R}^P, \mathcal{D}(\mathcal{C}(x)))$ be two quasi-ordered spaces defined, respectively, throughout (4) and (6). Let $\widetilde{\mathcal{A}}(\bar{x})$ be defined as in (11). If $\bar{x} \in \mathcal{L}_G$ solves (10) and $\widetilde{\mathcal{A}}(\bar{x}) \subseteq \mathcal{C}(\bar{x})$, then $\bar{x}$ solves (9).*

Proof Let $\bar{x} \in \mathcal{S}(\bar{x}) \cap \mathcal{N}_G$ be a solution to (10), with $\bar{\rho} \in \mathcal{M}_G$ the relative nonanticipativity multiplier. We suppose that there exists $\hat{x} \in \mathcal{S}(\bar{x}) \cap \mathcal{N}_G$ such that

$$\langle\langle \varphi, \hat{x} - \bar{x} \rangle\rangle_P \in -\operatorname{int} \mathcal{D}(\mathcal{C}(\bar{x})). \tag{12}$$

Since $\hat{x} \in \mathcal{S}(\bar{x}) \cap \mathcal{N}_G$, it follows that $\langle\langle \bar{\bar{\rho}}, \hat{x} - \bar{x} \rangle\rangle_P = 0_P$; hence, from (12), it holds

$$\langle\langle \varphi + \bar{\bar{\rho}}, \hat{x} - \bar{x} \rangle\rangle_P <_{\mathcal{D}(\mathcal{C}(\bar{x}))} 0_P. \tag{13}$$

Thanks to Lemma 1, from (13) it results

$$\forall \omega \in \Omega \qquad \langle \varphi(\omega) + \bar{\bar{\rho}}(\omega), \hat{x}(\omega) - \bar{x}(\omega) \rangle_P \in -\operatorname{int} C(\bar{x}(\omega)). \tag{14}$$

However, (14) contradicts the fact that $\bar{x}$ is a solution to (10); hence, we conclude that $\bar{x}$ solves (9). □

From now on, our focus will be on the study of multistage stochastic vector generalized quasi-variational inequalities in basic form (9). From a theoretical and modeling point of view, it represents the novelty of our study due to the time-uncertainty-information structure, the functional space, and the variable ordering structure (6) involved. Moreover, for simplicity of notation, we denote the multistage stochastic vector generalized quasi-variational inequalities in basic form (9) by $SVGQVI(\Phi, \mathcal{S}_{\mathcal{N}})$, where $\mathcal{S}_{\mathcal{N}} : \mathcal{K} \rightrightarrows \mathcal{K}$ is such that $\mathcal{S}_{\mathcal{N}}(x) = \mathcal{S}(x) \cap \mathcal{N}_G$.

4 Nonlinear scalarization approach

In this section, we analyze a suitable scalarization method and we use it to prove the existence of solutions to a multistage stochastic vector quasi-variational problem. A fundamental role will be played by a generalization of the well-known nonlinear scalarization function introduced by Gerstewitz [17] in the framework of vector optimization. However, in order to study the multistage stochastic vector quasi-variational problems introduced in the previous section, we point out that we cannot directly apply the nonlinear scalarization technique introduced in [12]. Indeed, in order to fit with our functional setting and with the ordering structure (6), we set

$$\xi : \mathcal{L}_G \times \mathbb{R}^P \to \mathbb{R} \quad \text{such that} \quad \xi(x, \hat{z}) := \inf \left\{ \lambda \in \mathbb{R} : \hat{z} \in \lambda \hat{e}(x) - \mathcal{D}(\mathcal{C}(x)) \right\}, \tag{15}$$

where $\hat{e} : \mathcal{L}_G \to \mathbb{R}^P$ is defined as

$$\hat{e}(x) = \sum_{\omega \in \Omega} \pi(\omega) e(x(\omega)) \in \operatorname{int} \mathcal{D}(\mathcal{C}(x)), \tag{16}$$

with $e : \mathcal{L}_G \to \mathcal{L}_P$ such that, for all $x \in \mathcal{L}_G$, $e(x(\omega)) \in \operatorname{int} C(x(\omega))$ for all $\omega \in \Omega$.

In particular, throughout the particular construction of $\hat{e}$ in (16) and the structure of $\mathcal{D}(\mathcal{C})$, relation in (15) allows us to opportunely work in an expected values framework, that is, ξ perfectly fits with the time-uncertainty-information structure and the functional space in which we operate.

As proved in Proposition 2.1 in [12], ξ is a well-defined function and its minimum is attained.

Remark 2 For all $x \in \mathcal{L}_G$, if $\hat{e}(x) = \hat{e}$ and $\mathcal{D}(\mathcal{C}(x)) = T$, then (15) reduces to

$$\xi_{\hat{e}}(\hat{z}) = \inf \left\{ \lambda \in \mathbb{R} : \hat{z} \in \lambda\hat{e} - T \right\}, \tag{17}$$

that is, the original version of nonlinear scalarization function due to Gerstewitz [17].

Moreover, in our framework, we introduce some important properties on ξ.

Proposition 1 *Let ξ be defined as in (15). The following statements hold:*

(i) *for any $\eta \in \mathbb{R}$ and $(x, \hat{z}) \in \mathcal{L}_G \times \mathbb{R}^P$, it results*

$$\xi(x, \hat{z}) \geq \eta \quad \Leftrightarrow \quad \hat{z} \notin \eta\hat{e}(x) - \operatorname{int} \mathcal{D}(\mathcal{C}(x)),$$
$$\xi(x, \hat{z}) \leq \eta \quad \Leftrightarrow \quad \hat{z} \in \eta\hat{e}(x) - \mathcal{D}(\mathcal{C}(x));$$

(ii) *for any $x \in \mathcal{L}_G$, $\xi(x, \cdot)$ is positively homogenous;*
(iii) *for any $x \in \mathcal{L}_G$ and $\hat{z}_1, \hat{z}_2 \in \mathbb{R}^P$, it results*

$$\xi(x, \hat{z}_1 + \hat{z}_2) \leq \xi(x, \hat{z}_1) + \xi(x, \hat{z}_2) \quad \textit{and} \quad \xi(x, \hat{z}_1 - \hat{z}_2) \geq \xi(x, \hat{z}_1) - \xi(x, \hat{z}_2).$$

Proof The claims follow from Proposition 2.3, 2.4, and 2.5 in [12]. □

At this point, we opportunely introduce the set-valued map $W : \mathcal{L}_G \rightrightarrows \mathbb{R}^P$ such that $W(x) = \mathbb{R}^P \backslash \operatorname{int} \mathcal{D}(\mathcal{C}(x))$. From now on, we require the following assumptions on the variable ordering structure.

Assumptions A

(A.1) $\operatorname{int} \mathcal{D}(\mathcal{C})^{-1}(\hat{r}) = \left\{x \in \mathcal{L}_G \;,\; \hat{r} \in \operatorname{int} \mathcal{D}(\mathcal{C}(x))\right\}$ is an open set;
(A.2) $\mathcal{D}(\mathcal{C})$ is cosmically upper continuous;
(A.3) W is cosmically upper continuous.

Remark 3 For each $x \in \mathcal{L}_G$, if $\mathcal{D}(\mathcal{C}(x))$ is a constant cone, then Assumptions A.2 and A.3 are clearly satisfied.

Since $\mathcal{D}(\mathcal{C})$ and W are set-valued maps with conic values, we make use of the cosmically upper continuity requirement in place of the classic upper semicontinuity assumption. In particular, in Example 2.3 of [9], the authors observed that the concept of upper semicontinuity is not appropriate to capture the behavior of set-valued maps with conic values. Recently, the concept of cosmically upper continuity was used in the study of vector optimization problems with variable ordering structure; see, e.g., [14].

Proposition 2 *Let ξ be defined as in (15). The following statements hold:*

(i) *for any $x \in \mathcal{L}_G$, $\xi(x, \cdot)$ is convex;*
(ii) *if Assumptions A.1 and A.2 are satisfied, $\xi(\cdot, \cdot)$ is lower semicontinuous;*
(iii) *if Assumptions A.1 and A.3 are satisfied, $\xi(\cdot, \cdot)$ is upper semicontinuous.*

Proof (i) $\xi(x, \cdot)$ *is convex*. Let $\hat{z}_1, \hat{z}_2 \in \mathbb{R}^P$ and $\tau \in (0, 1)$. If one considers $\xi(x, \tau\hat{z}_1 + (1 - \tau)\hat{z}_2)$, then from (iii) of Proposition 1 it follows that

$$\xi(x, \tau\hat{z}_1 + (1 - \tau)\hat{z}_2) \leq \xi(x, \tau\hat{z}_1) + \xi(x, (1 - \tau)\hat{z}_2)$$

and, since $\xi(x, \cdot)$ is positively homogenous, it results

$$\xi(x, \tau\hat{z}_1) + \xi(x, (1 - \tau)\hat{z}_2) = \tau\xi(x, \hat{z}_1) + (1 - \tau)\xi(x, \hat{z}_2);$$

hence, $\xi(x, \cdot)$ is convex.
(ii) $\xi(\cdot, \cdot)$ *is lower semicontinuous.*

For any $\eta \in \mathbb{R}$, let $\widetilde{B} := \left\{(x, \hat{z}) \in \mathcal{L}_G \times \mathbb{R}^P : \xi(x, \hat{z}) \leq \eta\right\}$. To prove that $\xi(\cdot, \cdot)$ is lower semicontinuous, it is sufficient to verify that $\widetilde{B}$ is closed: let $\left\{(x_n, \hat{z}_n)\right\}_{n\in\mathbb{N}} \subseteq \widetilde{B}$ and $(x, \hat{z}) \in \mathcal{L}_G \times \mathbb{R}^P$ such that $x_n \to x$ and $\hat{z}_n \to \hat{z}$, we have to prove that $(x, \hat{z}) \in \widetilde{B}$. From (i) of Proposition 1, it results

$$\forall n \in \mathbb{N} \quad \eta\hat{e}(x_n) - \hat{z}_n \in \mathcal{D}(\mathcal{C}(x_n)).$$

For each $x \in \mathcal{L}_G$, since $\mathcal{D}(\mathcal{C}(x))$ is convex, $\operatorname{int}\mathcal{D}(\mathcal{C}(x)) \neq \emptyset$, and Assumption A.1 holds, it follows that $\operatorname{int}\mathcal{D}(\mathcal{C})(\cdot)$ has a continuous selection $\hat{e}(\cdot)$ according to Browder continuous selection Theorem; hence, as $x_n \to x$, it results that $\hat{e}(x_n) \to \hat{e}(x)$. Moreover, we pose

$$\zeta : \mathcal{L}_G \rightrightarrows \mathbb{R}^P \quad \text{such that} \quad x \to \zeta(x) = \mathcal{D}(\mathcal{C}(x)) \cap B(0, 1);$$

since $\mathcal{D}(\mathcal{C}(x))$ is closed for each $x \in \mathcal{L}_G$ and the intersection of closed sets is a closed set, then ζ has closed values. Furthermore, from Assumption A.2, it results that ζ is upper semicontinuous; thus, thanks to Proposition 7 in [4], it follows that ζ is closed, that is, $\mathcal{D}(\mathcal{C})$ is cosmically closed. From (vi) of Proposition 2.1 in [21], it follows that $\mathcal{D}(\mathcal{C})$ is closed; hence, as $x_n \to x$ and $\hat{z}_n \to \hat{z}$, it results

$$\left(\eta\hat{e}(x_n) - \hat{z}_n \in \mathcal{D}(\mathcal{C}(x_n))\right) \to \left(\eta\hat{e}(x) - \hat{z} \in \mathcal{D}(\mathcal{C}(x))\right),$$

that is, $(x, \hat{z}) \in \widetilde{B}$.
(iii) $\xi(\cdot, \cdot)$ *is upper semicontinuous.*

For any $\eta \in \mathbb{R}$, let $\widehat{B} := \left\{(x, \hat{z}) \in \mathcal{L}_G \times \mathbb{R}^P : \xi(x, \hat{z}) \geq \eta\right\}$. To prove that $\xi(\cdot, \cdot)$ is upper semicontinuous, it is sufficient to verify that $\widehat{B}$ is closed: let $\left\{(x_n, \hat{z}_n)\right\}_{n\in\mathbb{N}} \subseteq \widehat{B}$ and $(x, \hat{z}) \in \mathcal{L}_G \times \mathbb{R}^P$ such that $x_n \to x$ and $\hat{z}_n \to \hat{z}$, we have to prove that $(x, \hat{z}) \in \widehat{B}$. From (i) of Proposition 1, it results

$$\forall n \in \mathbb{N} \quad \eta\hat{e}(x_n) - \hat{z}_n \in \mathbb{R}^P \setminus \operatorname{int}\mathcal{D}(\mathcal{C}(x_n)).$$

In (ii) we already proved that, as $x_n \to x$, $\hat{e}(x_n) \to \hat{e}(x)$. Moreover, we pose

$$\widehat{\zeta} : \mathcal{L}_G \rightrightarrows \mathbb{R}^P \quad \text{such that} \quad x \to \widehat{\zeta}(x) = W(x) \cap B(0, 1);$$

hence, since $W(x) = \mathbb{R}^P \setminus \operatorname{int}\mathcal{D}(\mathcal{C}(x))$ is closed for each $x \in \mathcal{L}_G$, with similar arguments to the one used in (ii) and thanks to Assumption A.3, it follows that W is closed. Then, as $x_n \to x$ and $\hat{z}_n \to \hat{z}$, it results

$$\left(\eta\hat{e}(x_n) - \hat{z}_n \in \mathbb{R}^P \setminus \operatorname{int}\mathcal{D}(\mathcal{C}(x_n))\right) \to \left(\eta\hat{e}(x) - \hat{z} \in \mathbb{R}^P \setminus \operatorname{int}\mathcal{D}(\mathcal{C}(x))\right),$$

that is, $(x, \hat{z}) \in \widehat{B}$. □

At this point, we exploit a particular composition of ξ that will play a central role in our results. For any $(x, \varphi) \in \mathcal{L}_G \times L(\mathcal{L}_G, \mathcal{L}_P)$, we consider the following function:

$$\begin{aligned} \xi(x, \langle\langle\varphi, \cdot\rangle\rangle_P) \ : \ & \mathcal{L}_G \to \mathbb{R} \\ & y \ \to \xi(x, \langle\langle\varphi, y\rangle\rangle_P). \end{aligned} \tag{18}$$

We point out that all the results obtained in Proposition 2 are still valid for this particular composition of ξ, thanks to the continuity of the inner product. In particular, under Assumptions A, $\xi(\cdot, \cdot)$ is continuous and, from the continuity of the inner product, also $\xi(\cdot, \langle\langle\cdot, \cdot\rangle\rangle_P)$ is continuous.

In the multistage-functional framework in which we operate, we characterize the solutions of $SVGQVI(\Phi, \mathcal{S}_{\mathcal{N}})$ in terms of a suitable optimization problem involving the nonlinear scalarization function ξ.

Theorem 2 *For any $x \in \mathcal{L}_G$, let $(\mathbb{R}^P, \mathcal{D}(\mathcal{C}(x)))$ be a quasi-ordered space defined as in (6). Let ξ be defined as in (15). Then, $\bar{x}$ is a solution to (9) if and only if it holds*

$$\min_{x \in \mathcal{S}_{\mathcal{N}}(\bar{x})} \xi(\bar{x}, \langle\langle \varphi, x - \bar{x} \rangle\rangle_P) = 0. \tag{19}$$

Proof Let $\bar{x}$ be a solution to (9). From (i) of Proposition 1, it follows that

$$\langle\langle \varphi, x - \bar{x} \rangle\rangle_P \notin -\operatorname{int} \mathcal{D}(\mathcal{C}(\bar{x})) \quad \forall x \in \mathcal{S}_{\mathcal{N}}(\bar{x}) \Leftrightarrow \xi(\bar{x}, \langle\langle \varphi, x - \bar{x} \rangle\rangle_P) \geq 0 \quad \forall x \in \mathcal{S}_{\mathcal{N}}(\bar{x}). \tag{20}$$

Let $x = \bar{x}$, it follows that $\langle\langle \varphi, \bar{x} - \bar{x} \rangle\rangle_P = 0_P$; hence, it holds

$$\xi(\bar{x}, 0_P) = \min \left\{ \lambda \in \mathbb{R} : 0_P \in \lambda \hat{e}(x) - \mathcal{D}(\mathcal{C}(\bar{x})) \right\} = 0.$$

From (20), condition (19) follows. Conversely, if $\bar{x}$ is a solution to (19), from (i) in Proposition 1, the claim follows. □

Without requiring any monotonicity assumption on the principal operator of the variational formulation, we are going to prove the existence of solutions of the multistage stochastic vector quasi-variational problem (9). The lack of monotonicity requirements could be a crucial aspect in the study of some applicative problems. Indeed, as we will study in Sect. 5, some equilibrium-type problems have a decomposable structure, that is, they can be formulated in terms of variational problems over the Cartesian product of spaces. However, this particular structure does not always preserve certain monotonicity assumptions. In our results, as we do not require any monotonicity assumption, the applicability is not a priori limited for such problems.

Theorem 3 *For any $x \in \mathcal{L}_G$, let $(\mathbb{R}^P, \mathcal{D}(\mathcal{C}(x)))$ be a quasi-ordered space defined as in (6) and Assumptions A be satisfied. Let ξ be defined as in (15) and $\mathcal{K} \subseteq \mathcal{L}_G$ be nonempty, convex, and compact. Let $\Phi : \mathcal{K} \rightrightarrows L(\mathcal{L}_G, \mathcal{L}_P)$ be upper semicontinuous with nonempty, convex, and compact values. Let $\mathcal{S}_{\mathcal{N}}$ be closed, lower semicontinuous with nonempty and convex values. Then, the $SVGQVI(\Phi, \mathcal{S}_{\mathcal{N}})$ admits at least a solution.*

Proof For any $(\tilde{x}, \tilde{\varphi}) \in \mathcal{L}_G \times L(\mathcal{L}_G, \mathcal{L}_P)$, we consider $\xi(\tilde{x}, \langle\langle \tilde{\varphi}, \cdot \rangle\rangle_P)$ as in (18) and we introduce the set-valued map $\Lambda : \mathcal{K} \times L(\mathcal{L}_G, \mathcal{L}_P) \rightrightarrows \mathcal{K}$ defined as

$$\Lambda(\tilde{x}, \tilde{\varphi}) := \left\{ \hat{x} \in \mathcal{S}_{\mathcal{N}}(\tilde{x}) : \min_{x \in \mathcal{S}_{\mathcal{N}}(\tilde{x})} \xi(\tilde{x}, \langle\langle \tilde{\varphi}, x \rangle\rangle_P) = \xi(\tilde{x}, \langle\langle \tilde{\varphi}, \hat{x} \rangle\rangle_P) \right\}.$$

Since $\mathcal{S}_{\mathcal{N}}$ is closed, it has closed vales; for any $x \in \mathcal{L}_G$, as $\mathcal{S}_{\mathcal{N}}(x) \subseteq \mathcal{K}$, hence $\mathcal{S}_{\mathcal{N}}$ has compact values. Moreover, since $\mathcal{S}_{\mathcal{N}}$ is closed and $\mathcal{K}$ is compact, it follows that $\mathcal{S}_{\mathcal{N}}$ is upper semicontinuous according to closed graph theorem; hence, as $\mathcal{S}_{\mathcal{N}}$ is lower semicontinuous, it follows that $\mathcal{S}_{\mathcal{N}}$ is continuous. We recall that the minimum value of $\xi(\tilde{x}, \langle\langle \tilde{\varphi}, \cdot \rangle\rangle_P)$ is the maximum value of $-\xi(\tilde{x}, \langle\langle \tilde{\varphi}, \cdot \rangle\rangle_P)$. Thanks to Proposition 23 in [4], Λ is upper semicontinuous. Moreover, $\Lambda(\tilde{x}, \tilde{\varphi})$ is the set of all minimizers of $\xi(\tilde{x}, \langle\langle \tilde{\varphi}, \cdot \rangle\rangle_P)$ on $\mathcal{S}_{\mathcal{N}}(\tilde{x})$; hence, from (i) of Proposition 2, it follows that $\Lambda(\tilde{x}, \tilde{\varphi})$ is a convex set, for any $(\tilde{x}, \tilde{\varphi}) \in \mathcal{K} \times L(\mathcal{L}_G, \mathcal{L}_P)$. Therefore, Λ has nonempty, convex, and compact values. Furthermore, according to Theorem 3 in [7], $\Phi(\mathcal{K})$ is compact; hence, we consider the compact set $D := \operatorname{conv}\Phi(\mathcal{K})$ and we pose

$$\Gamma : \mathcal{K} \times D \rightrightarrows \mathcal{K} \times D \quad \text{such that} \quad (\tilde{x}, \tilde{\varphi}) \to \Gamma(\tilde{x}, \tilde{\varphi}) := \Lambda(\tilde{x}, \tilde{\varphi}) \times \Phi(\tilde{x}).$$

Since Λ is upper semicontinuous with nonempty, convex, and compact values, Γ is upper semicontinuous with nonempty, convex, and compact values; hence, as $\mathcal{K} \times D$ is nonempty, convex and compact, according to Kakutani fixed point Theorem, it follows that Γ has a fixed point, that is

$$(\bar{x}, \bar{\varphi}) \in \Gamma(\bar{x}, \bar{\varphi}) \quad \text{so that} \quad \bar{x} \in \Lambda(\bar{x}, \bar{\varphi}) \quad \text{and} \quad \bar{\varphi} \in \Phi(\bar{x}).$$

In particular, since $\bar{x} \in \Lambda(\bar{x}, \bar{\varphi})$, it follows that

$$\xi(\bar{x}, \langle\langle \bar{\varphi}, x \rangle\rangle_P) - \xi(\bar{x}, \langle\langle \bar{\varphi}, \bar{x} \rangle\rangle_P) \geq 0 \qquad \forall x \in \mathcal{S}_{\mathcal{N}}(\bar{x});$$

then, according to (iii) of Proposition 1, from the last inequality it results

$$\xi(\bar{x}, \langle\langle \bar{\varphi}, x \rangle\rangle_P - \langle\langle \bar{\varphi}, \bar{x} \rangle\rangle_P) \geq 0 \quad \forall x \in \mathcal{S}_{\mathcal{N}}(\bar{x})$$

so that, it holds:

$$\xi(\bar{x}, \langle\langle \bar{\varphi}, x - \bar{x} \rangle\rangle_P) \geq 0 \quad \forall x \in \mathcal{S}_{\mathcal{N}}(\bar{x}) \quad \Rightarrow \quad \langle\langle \bar{\varphi}, x - \bar{x} \rangle\rangle_P \notin -\operatorname{int} \mathcal{D}(\mathcal{C}(\bar{x})) \;\; \forall x \in \mathcal{S}_{\mathcal{N}}(\bar{x}),$$

that is, $SVGQVI(\Phi, \mathcal{S}_{\mathcal{N}})$ admits at least a solution. □

In the special case of a constant ordering structure, Assumptions A are trivially satisfied. Hence, the following existence result holds.

Corollary 1 *Let $(\mathbb{R}^P, T)$ be a quasi-ordered space defined as in (6), where $T = \mathcal{C}(x)$ for all $x \in \mathcal{L}_G$. Let ξ be defined as in (17) and $\mathcal{K} \subseteq \mathcal{L}_G$ be nonempty, convex, and compact. Let $\Phi : \mathcal{K} \rightrightarrows L(\mathcal{L}_G, \mathcal{L}_P)$ be upper semicontinuous with nonempty, convex, and compact values. Let $\mathcal{S}_{\mathcal{N}}$ be closed, lower semicontinuous with nonempty and convex values. Then, the $SVGQVI(\Phi, \mathcal{S}_{\mathcal{N}})$ admits at least a solution.*

Moreover, the results used to prove Theorem 3 (Corollary 1) are not limited to the multistage-functional framework in which we operate: indeed, in a finite-dimensional setting with a variable ordering structure, such results can be quickly adapted also for the study of deterministic vector quasi-variational problems. So, let $T : \mathbb{R}^G \rightrightarrows \mathbb{R}^P$ be such that, for all $x \in \mathbb{R}^G$, $T(x)$ is a proper, convex, and closed cone with $\operatorname{int} T(x) \neq \emptyset$ and let $\widetilde{\Phi} : \mathbb{R}^G \rightrightarrows L(\mathbb{R}^G, \mathbb{R}^P)$ and $\widetilde{\mathcal{S}} : K \rightrightarrows K$ be two set-valued maps, with $K \subseteq \mathbb{R}^G$ nonempty, convex, and compact. Then, we introduce the following vector quasi-variational problem:

$$\text{find} \quad \bar{x} \in \widetilde{\mathcal{S}}(\bar{x}) \quad \text{such that} \quad \exists h \in \widetilde{\Phi}(\bar{x}) \quad \text{with} \quad \langle h, x - \bar{x} \rangle_P \notin -\operatorname{int} T(\bar{x}) \qquad \forall x \in \widetilde{\mathcal{S}}(\bar{x}). \tag{21}$$

Corollary 2 *For any $x \in \mathbb{R}^G$, let $(\mathbb{R}^P, T(x))$ be a quasi-ordered space, where T satisfies Assumptions A. Let $\widetilde{\Phi} : K \rightrightarrows L(\mathbb{R}^G, \mathbb{R}^P)$ be upper semicontinuous with nonempty, convex, and compact values. Let $\widetilde{\mathcal{S}}$ be closed, lower semicontinuous with nonempty and convex values. Then, the vector variational problem (21) admits at least a solution.*

Proof From Theorem 3, the thesis follows by setting:

$$\mathcal{T}_0 = \{0\} \quad \text{so that} \quad \omega = \xi_0 \quad \text{and} \quad \Omega = \{\omega\} \quad \text{with} \quad \pi(\omega) = 1. \tag{22}$$

Indeed, from (22), it follows that $\mathcal{L}_G$ reduces to $\mathbb{R}^G$; hence, $\mathcal{D}(\mathcal{C}(x)) = T(x)$ for all $x \in \mathbb{R}^G$, $\Phi = \widetilde{\Phi}$ and $\mathcal{S}_{\mathcal{N}} = \widetilde{\mathcal{S}}$. □

Remark 4 We point out that, with respect to Theorem 3.2 in [12], our results hold under different assumptions on the structure of the problem. Moreover, adapting opportunely the notations to our setting, the authors require that $\mathcal{D}(\mathcal{C})$ and W are both upper semicontinuous while, in Theorem 3, we just require the cosmically upper continuity.

5 Multistage stochastic vector Nash equilibrium problems

In [27], the authors underlay that multistage stochastic variational inequalities allow characterizing solutions to problems of stochastic optimization and equilibrium when initial decisions are followed by recursive decisions in later stages. In this section, in order to support the proposed multicriteria extension, we focus on the study of multistage stochastic Nash equilibrium problems with vector-valued payoff functions and variable ordering structures. We recall that incomplete preferences can be represented by vector-valued functions; see, e.g., [23]. Hence, besides their interest in specific applications, Nash games with vector-valued payoff functions can be used also to study models of strategic interactions without any completeness assumption on the preferences of the players; see, e.g., [5]. Here, we are able to capture equilibrium conditions for Nash games of multistage stochastic nature in which each player, knowing the other players' strategies, stage by stage chooses its optimal strategy under several criteria, uncertain conditions, and partial information.

Let $\mathcal{I} := \{1, \dots, i, \dots, I\}$ be a finite set of players: each $i \in \mathcal{I}$ has control over a strategy $x_i \in \mathcal{L}_{G_i}$, where

$$x_i := (x_{i0}, x_{i1}, \dots, x_{it}, \dots, x_{iT}) \in \mathcal{L}_{G_{i0}} \times \mathcal{L}_{G_{i1}} \times \cdots \times \mathcal{L}_{G_{it}} \times \cdots \times \mathcal{L}_{G_{iT}} := \mathcal{L}_{G_i}.$$

In the time-uncertainty-information structure considered, through the availability of recursive decisions as information develops, the players interact repeatedly over time by means of the choices on their strategies. We denote by $x \in \mathcal{L}_G$ the vector of all players' strategies:

$$x := (x_i)_{i\in\mathcal{I}} = (x_i, x_{-i}), \quad \text{where} \quad x_{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_I) \in \mathcal{L}_{G_{-i}};$$

the notation $-i$ represents all the players in $\mathcal{I}$ except i and $G := \sum_{i\in\mathcal{I}} G_i = G_{-i} + G_i$, where $G_{-i} = \sum_{\hat{i}\neq i} G_{\hat{i}}$. For each $i \in \mathcal{I}$, we consider the vector-valued payoff function $u_i : \mathcal{L}_G \to \mathbb{R}^P$ so that

$$u_i := (u_{i1}, \dots, u_{ip}, \dots, u_{iP}) : \Omega \times \mathbb{R}^G \to \mathbb{R}^P, \quad \text{with} \quad u_{ip} : \Omega \times \mathbb{R}^G \to \mathbb{R}.$$

Then, we introduce the following vector-valued expected payoff function

$$\mathcal{U}_i := (\mathcal{U}_{i1}, \dots, \mathcal{U}_{ip}, \dots, \mathcal{U}_{iP}) : \mathcal{L}_G \to \mathbb{R}^P, \quad \text{where} \quad \mathcal{U}_{ip} : \mathcal{L}_G \to \mathbb{R};$$

in particular, for any $x \in \mathcal{L}_G$, $\mathcal{U}_i$ is defined as

$$\mathcal{U}_i(x) := \mathbb{E}\left[u_i(\omega, x(\omega))\right] = \left(\sum_{\omega\in\Omega} \pi(\omega) u_{ip}(\omega, x(\omega))\right)_{p\in P},$$

where

$$\mathcal{U}_i(x) \equiv \mathcal{U}_i(x_i, x_{-i}) \quad \text{and} \quad u_{ip}(\omega, x(\omega)) \equiv u_{ip}(\omega, x_i(\omega), x_{-i}(\omega)). \tag{23}$$

From now on, let $\mathcal{K}_i := \left\{x_i \in \mathcal{L}_{G_i} : x_i(\omega) \in K_i(\omega), \forall \omega \in \Omega\right\}$ be the feasible region of each player $i \in \mathcal{I}$, with $K_i(\omega) \subseteq \mathbb{R}^{G_i}$ nonempty and convex. We pose $\mathcal{K} := \prod_{i\in\mathcal{I}} \mathcal{K}_i = (\mathcal{K}_{-i} \times \mathcal{K}_i) \subseteq \mathcal{L}_G$, with $\mathcal{K}_{-i} = \prod_{\hat{i}\neq i} \mathcal{K}_{\hat{i}}$, such that $K(\omega) := \prod_{i\in\mathcal{I}} K_i(\omega) \subseteq \mathbb{R}^G$ for all $\omega \in \Omega$. Then, we introduce the following constraint set-valued map

$$\mathcal{S} : \mathcal{K} \rightrightarrows \mathcal{K} \quad \text{such that} \quad \mathcal{S}(x) := \prod_{i\in\mathcal{I}} \mathcal{S}_i(x_{-i}), \quad \text{where} \quad \mathcal{S}_i(x_{-i}) \subseteq \mathcal{K}_i;$$

in particular, $\mathcal{S}_i : \mathcal{K}_{-i} \rightrightarrows \mathcal{K}_i$ is defined as $\mathcal{S}_i(x_{-i}) := \left\{x_i \in \mathcal{K}_i : x_i(\omega) \in S_i(\omega, x_{-i}(\omega)), \forall \omega \in \Omega\right\}$. Moreover, we introduce the set-valued map $\mathcal{D}(\mathcal{C}_i) : \mathcal{L}_G \rightrightarrows \mathbb{R}^P$

so that, for any $x_{-i} \in \mathcal{L}_{G_{-i}}$, each player i has the following variable ordering structure

$$\mathcal{D}(\mathcal{C}_i(\cdot, x_{-i})) : \mathcal{L}_{G_i} \rightrightarrows \mathbb{R}^P \quad \text{such that} \quad x_i \to \mathcal{D}(\mathcal{C}_i(x_i, x_{-i})),$$

where $\mathcal{D}(\mathcal{C}_i(\cdot, x_{-i}))$ is defined as in (6) by taking all the other players' strategies $x_{-i} \in \mathcal{L}_{G_{-i}}$ given. The aim of each player i, known the other players' strategies $x_{-i} \in \mathcal{L}_{G_{-i}}$, is to find a strategy $\bar{x}_i \in \mathcal{S}_i(x_{-i}) \cap \mathcal{N}_{G_i}$ that minimizes the vector-valued expected payoff function $\mathcal{U}_i(\cdot, x_{-i})$ with respect to the partial ordering induced by $\mathcal{D}(\mathcal{C}_i(\cdot, x_{-i}))$:

$$\min_{\mathcal{D}(\mathcal{C}_i(x_i, x_{-i}))} \mathcal{U}_i(x_i, x_{-i}) \quad \text{such that} \quad x_i \in \mathcal{S}_i(x_{-i}) \cap \mathcal{N}_{G_i}, \tag{24}$$

that is, $\bar{x}_i \in \mathcal{S}_i(x_{-i}) \cap \mathcal{N}_{G_i}$ is a *weakly efficient strategy* such that

$$\mathcal{U}_i(x_i, x_{-i}) - \mathcal{U}_i(\bar{x}_i, x_{-i}) \notin -\text{int}\, \mathcal{D}(\mathcal{C}_i(\bar{x}_i, x_{-i})) \quad \forall x_i \in \mathcal{S}_i(x_{-i}) \cap \mathcal{N}_{G_i}.$$

At this point, if we denote by

$$\mathcal{E} := ((\Omega, \mathbb{P}), \mathcal{P}, (u_i)_{i \in \mathcal{I}}, (\mathcal{D}(\mathcal{C}_i))_{i \in \mathcal{I}})$$

the time-uncertainty-information structure, the vector-valued payoff functions, and the ordering structures of the game described up to now, then we can formalize the following multistage stochastic vector Nash equilibrium problem.

Definition 6 A vector $\bar{x} \in \mathcal{S}(\bar{x}) \cap \mathcal{N}_G$ is a *multistage stochastic vector generalized Nash equilibrium* for $\mathcal{E}$ if

$$\forall i \in \mathcal{I} \qquad \mathcal{U}_i(x_i, \bar{x}_{-i}) - \mathcal{U}_i(\bar{x}_i, \bar{x}_{-i}) \notin -\text{int}\, \mathcal{D}(\mathcal{C}_i(\bar{x}_i, \bar{x}_{-i})) \quad \forall x_i \in \mathcal{S}_i(\bar{x}_{-i}) \cap \mathcal{N}_{G_i}. \tag{25}$$

If $\mathcal{S}(x) \cap \mathcal{N}_G = \mathcal{K} \cap \mathcal{N}_G$ for each $x \in \mathcal{K}$, the equilibrium problem in Definition 6 reduces to a *multistage stochastic vector Nash equilibrium* for $\mathcal{E}$; in addition, if $P = 1$ and $C_i(\bar{x}_i(\omega), \bar{x}_{-i}(\omega))$ is the non-negative ray for each $\omega \in \Omega$ and $i \in \mathcal{I}$, the equilibrium problem in Definition 6 reduces to the *multistage stochastic Nash equilibrium* introduced in [27].

In force of the results obtained in the previous sections, we study the multistage stochastic vector generalized Nash equilibrium problem (25) by using a suitable multistage stochastic vector quasi-variational problem. To this aim, we preliminarily recall the concept of Gâteaux differentiability. Let X and Y be two Hilbert spaces, $f : X \to Y$ is said to be Gâteaux differentiable at $x \in X$ if the limit $\langle Df(x), d\rangle = \lim_{\eta \to 0} \frac{1}{\eta}[f(x + \eta d) - f(x)]$ exists for all $d \in X$. The Gâteaux derivative of f at x, $Df(x)$, is a continuous linear map from X to Y; see, e.g, [2]. In what follows, for each $i \in \mathcal{I}$, we require the Gâteaux differentiability of $\mathcal{U}_i$. In particular, for any $x_{-i} \in \mathcal{L}_{G_{-i}}$, $\mathcal{U}_i(\cdot, x_{-i})$ is Gâteaux differentiable at (x_i, x_{-i}) with respect to x_i if there exists

$$\begin{aligned} \lim_{\eta \to 0} \frac{1}{\eta} \left[\mathcal{U}_i(x_i + \eta d_i, x_{-i}) - \mathcal{U}_i(x_i, x_{-i})\right] &= \langle\langle Du_i(x_i, x_{-i}), d_i\rangle\rangle_P \\ &= \sum_{\omega \in \Omega} \pi(\omega) \langle Du_i(\omega, x_i(\omega), x_{-i}(\omega)), d_i(\omega)\rangle_P \end{aligned}$$

for all $d_i \in \mathcal{L}_{G_i}$, where $D\mathcal{U}_i(\cdot, x_{-i}) : \mathcal{L}_{G_i} \to L(\mathcal{L}_{G_i}, \mathcal{L}_P)$, for all $x_i \in \mathcal{L}_{G_i}$, is defined as

$$\begin{aligned} D\mathcal{U}_i(x_i, x_{-i}) : \Omega \to L(\mathbb{R}^{G_i}, \mathbb{R}^P) \quad \text{such that} \\ \omega \to D\mathcal{U}_i(x_i, x_{-i})(\omega) = Du_i(\omega, x_i(\omega), x_{-i}(\omega)) \end{aligned}$$

by using similar arguments to the ones in [27]. We say that $\mathcal{U}_i$ is Gâteaux differentiable if it is Gâteaux differentiable with respect to all the i-variables, with $i \in \mathcal{I}$; see, e.g, [1]. Furthermore, if $\mathcal{U}_i(\cdot, x_{-i})$ is Gâteaux differentiable and, for any $x_i \in \mathcal{L}_{G_i}$, $\mathcal{D}(\mathcal{C}_i(x_i, x_{-i})) \subset \mathbb{R}^P$ is a proper, convex, closed, and pointed cone with int $\mathcal{D}(\mathcal{C}_i(x_i, x_{-i})) \neq \emptyset$, we say that $\mathcal{U}_i(x_i, x_{-i})$ is weakly $\mathcal{D}(\mathcal{C}_i(x_i, x_{-i}))$-pseudoconvex at (x_i, x_{-i}) with respect to x_i if, for all $y_i \in \mathcal{L}_{G_i}$,

$$\langle\langle Du_i(x_i, x_{-i}), y_i - x_i\rangle\rangle_P \notin -\text{int}\,\mathcal{D}(\mathcal{C}_i(x_i, x_{-i})) \quad \text{implies}$$

$$\mathcal{U}_i(y_i, x_{-i}) - \mathcal{U}_i(x_i, x_{-i}) \notin -\text{int}\,\mathcal{D}(\mathcal{C}_i(x_i, x_{-i})); \tag{26}$$

see, e.g., [2]. Then, we introduce the multistage stochastic vector quasi-variational problem:

$$\text{find } \bar{x} \in \mathcal{S}(\bar{x}) \cap \mathcal{N}_G \text{ such that}$$

$$\forall i \in \mathcal{I} \quad \langle\langle Du_i(\bar{x}_i, \bar{x}_{-i}), x_i - \bar{x}_i\rangle\rangle_P \notin -\text{int}\,\mathcal{D}(\mathcal{C}_i(\bar{x}_i, \bar{x}_{-i})) \quad \forall x_i \in \mathcal{S}_i(\bar{x}_{-i}) \cap \mathcal{N}_{G_i}. \tag{27}$$

From now on, for each $x \in \mathcal{L}_G$ and $i \in \mathcal{I}$, $\mathcal{D}(\mathcal{C}_i(x_i, x_{-i})) \subset \mathbb{R}^P$ is a proper, convex, closed, and pointed cone, with int $\mathcal{D}(\mathcal{C}_i(x_i, x_{-i})) \neq \emptyset$. The following result allows us to characterize the equilibrium problem (25) in terms of a multistage stochastic vector quasi-variational inequality: the proof is inspired by Theorem 5.24 in [2].

Theorem 4 *For any $i \in \mathcal{I}$ and $x_{-i} \in \mathcal{L}_{G_{-i}}$, let $\mathcal{U}_i(\cdot, x_{-i})$ be Gâteaux differentiable and $\mathcal{S}_i(x_{-i}) \cap \mathcal{N}_{G_i} \neq \emptyset$ such that, for all $\omega \in \Omega$, $S_i(\omega, x_{-i}(\omega))$ is convex. If $\bar{x} \in \mathcal{L}_G$ is a solution to (25), then it solves problem (27). Conversely, if $\bar{x} \in \mathcal{L}_G$ is a solution to (27), then it solves problem (25) if, for any $i \in \mathcal{I}$ and $x_{-i} \in \mathcal{L}_{G_{-i}}$, $\mathcal{U}_i(\cdot, x_{-i})$ is weakly $\mathcal{D}(\mathcal{C}_i(\cdot, x_{-i}))$-pseudoconvex.*

Proof Let $\bar{x} \in \mathcal{L}_G$ be solution to (25). For all $i \in \mathcal{I}$, since $\mathcal{S}_i(\bar{x}_{-i})$ is convex and $\mathcal{N}_{G_i}$ is convex as subspace of $\mathcal{L}_{G_i}$, it follows that $\mathcal{S}_i(\bar{x}_{-i}) \cap \mathcal{N}_{G_i}$ is convex; hence, for any $x_i \in \mathcal{S}_i(\bar{x}_{-i}) \cap \mathcal{N}_{G_i}$ and $\eta \in [0, 1]$, we consider $\bar{x}_i + \eta(x_i - \bar{x}_i) \in \mathcal{S}_i(\bar{x}_{-i}) \cap \mathcal{N}_{G_i}$ so that

$$\mathcal{U}_i(\bar{x}_i + \eta(x_i - \bar{x}_i), \bar{x}_{-i}) - \mathcal{U}_i(\bar{x}_i, \bar{x}_{-i}) \in \mathbb{R}^P \setminus -\text{int}\,\mathcal{D}(\mathcal{C}_i(\bar{x}_i, \bar{x}_{-i})).$$

Being that $\mathbb{R}^P \setminus -$ int $\mathcal{D}(\mathcal{C}_i(\bar{x}_i, \bar{x}_{-i}))$ is a closed cone, it results that

$$\forall i \in \mathcal{I} \quad \lim_{\eta \to 0^+} \frac{1}{\eta}\left[\mathcal{U}_i(\bar{x}_i + \eta(x_i - \bar{x}_i), \bar{x}_{-i}) - \mathcal{U}_i(\bar{x}_i, \bar{x}_{-i})\right] \in \mathbb{R}^P \setminus -\text{int}\,\mathcal{D}(\mathcal{C}_i(\bar{x}_i, \bar{x}_{-i}));$$

hence, since $\mathcal{U}_i$ is Gâteaux differentiable at $(\bar{x}_i, \bar{x}_{-i})$ with respect to $\bar{x}_i$, it follows that

$$\forall i \in \mathcal{I} \quad \langle\langle Du_i(\bar{x}_i, \bar{x}_{-i}), x_i - \bar{x}_i\rangle\rangle_P \notin -\text{int}\,\mathcal{D}(\mathcal{C}_i(\bar{x}_i, \bar{x}_{-i})).$$

Conversely, let $\bar{x} \in \mathcal{L}_G$ be solution to (27); then, it is solution to (25) being that, for all $i \in \mathcal{I}$, relation (26) holds. □

For any $i \in \mathcal{I}$, let $W_i : \mathcal{L}_G \rightrightarrows \mathbb{R}^P$ such that $W_i(x) = \mathbb{R}^P \setminus \text{int}\,\mathcal{D}(\mathcal{C}_i(x))$, we opportunely replicate the Assumptions A introduced in Sect. 4 in order to fit with the study of the equilibrium problem (25).

Assumptions A$_i$

(A$_i$.1) int $\mathcal{D}(\mathcal{C}_i)^{-1}(\hat{r}) = \left\{(x_i, x_{-i}) \in \mathcal{L}_G,\ \hat{r} \in \text{int}\,\mathcal{D}(\mathcal{C}_i(x_i, x_{-i}))\right\}$ is an open set;
(A$_i$.2) $\mathcal{D}(\mathcal{C}_i)$ is cosmically upper continuous;
(A$_i$.3) W_i is cosmically upper continuous.

For any $i \in \mathcal{I}$, we define $\xi_i : \mathcal{L}_G \times \mathbb{R}^P \to \mathbb{R}$ as in (15) relatively to $\mathcal{D}(\mathcal{C}_i)$, that is

$$\xi_i(x, \hat{z}_i) = \xi_i((x_i, x_{-i}), \hat{z}_i) := \inf\left\{\lambda_i \in \mathbb{R} : \hat{z}_i \in \lambda_i \hat{e}_i(x_i, x_{-i}) - \mathcal{D}(\mathcal{C}_i(x_i, x_{-i}))\right\}. \quad (28)$$

We point out that, for each $i \in \mathcal{I}$, $\mathcal{S}_i$ is independent from player i's strategy; hence, for mathematical convenience and simplicity of notation, we could also consider $\mathcal{S}_{\mathcal{N},i} : \mathcal{K} \rightrightarrows \mathcal{K}_i$ such that $\mathcal{S}_{\mathcal{N},i}(x) = \mathcal{S}_i(x_{-i}) \cap \mathcal{N}_{G_i}$.

Theorem 5 *For any $i \in \mathcal{I}$, let Assumptions A_i be satisfied, ξ_i be defined as in (28), $\mathcal{U}_i$ be Gâteaux differentiable, and $\mathcal{K}_i \subseteq \mathcal{L}_{G_i}$ be nonempty, convex, and compact. For any $x_{-i} \in \mathcal{K}_{-i}$, let $\mathcal{U}_i(\cdot, x_{-i})$ be weakly $\mathcal{D}(\mathcal{C}_i(\cdot, x_{-i}))$-pseudoconvex on $\mathcal{K}_i$. Let $\mathcal{S}_{\mathcal{N},i}$ be closed, lower semicontinuous with nonempty and convex values. Then, the multistage stochastic vector generalized Nash equilibrium (25) admits at least a solution.*

Proof According to Theorem 4, it is sufficient to prove that the multistage stochastic vector quasi-variational inequality (27) has solutions. Firstly, we pose $\Phi := D\mathcal{U}$, with $D\mathcal{U} := \prod_{i\in\mathcal{I}} D\mathcal{U}_i$ continuous. Moreover, if we set $\mathcal{S}_{\mathcal{N}} := \prod_{i\in\mathcal{I}} \mathcal{S}_{\mathcal{N},i}$, it is closed, lower semicontinuous with nonempty and convex values; see, e.g., [7]. Then, thanks to Theorem 3, the thesis follows. □

Remark 5 In Theorem 3, we do not require monotonicity assumptions on the multistage stochastic vector quasi-variational problem: thanks to this fact, in Theorem 5, we can assume that $\mathcal{U}_i(\cdot, x_{-i})$ is weakly $\mathcal{D}(\mathcal{C}_i(\cdot, x_{-i}))$-pseudoconvex only with respect to the i-th variable.

5.1 Multistage stochastic bicriteria Cournot oligopolistic model

Cournot oligopolistic model is one of the most widely used modelizations to study strategic interactions among non-cooperative firms. In the classical model, each firm produces a finite number of homogeneous goods and aims at maximizing its profit by making a quantity decision for each of them, knowing the quantity produced by the other firms; at the same time, the price of each good depends on the aggregate output of all the firms; see, e.g., [16]. Then, the resulting problem is formulated as a suitable generalized Nash equilibrium problem: in the literature, it is commonly denoted as *generalized Nash-Cournot game* and it is used to the study of several real-world applications such as electricity markets, gas markets, etc; see, e.g., [25] and the references therein. In [24], the authors propose the study of a generalized Nash-Cournot game in a stochastic framework, of single stage nature, by using a suitable stochastic variational inequality approach; in [11], following the managerial theory of the firm in which the profit maximization is not the only objective, the authors propose the study of a deterministic bicriteria Nash-Cournot game, where the *revenue* (total amount of income generated by the sale of the goods) and the *profit* (revenue minus cost incurred in the production) are the two quoted criteria for each firm. Here, inspired by these works and thanks to the results obtained in this paper, we aim to introduce the study of a suitable multistage stochastic bicriteria generalized Nash-Cournot game as an application of the analyzed multistage stochastic Nash equilibrium problem (25). In this way, we are able to capture the evolutionary aspects of the problem and how the non-cooperative firms, stage by stage, interact repeatedly through their quantity decisions as information develops. The uncertainty characterizing the stochastic nature of the model can derive from different sources: geopolitical aspects, economic-financial aspects, availability of raw materials, market demands, etc.

In the time-uncertainty-information structure introduced in Sect. 2, we set a Cournot oligopolistic model in which a finite number of non-cooperative firms, all with the same information, produces a finite number of homogeneous goods. Let $\mathcal{I} := \{1, \dots, i, \dots, I\}$ be the finite set of the non-cooperative firms and $\mathcal{H} := \{1, \dots, h, \dots, H\}$ be the finite set of the goods. By using the same notation introduced at the beginning of the section, each firm i has control over the strategy $x_i := (x_{it})_{t\in\mathcal{T}_0} \in \mathcal{L}_{H(T+1)}$, where $x_{it} := (x_{it}^h)_{h\in\mathcal{H}} \in \mathcal{L}_H$ for all $t \in \mathcal{T}_0$: for each $\omega \in \Omega$, $x_{it}^h(\omega) \in \mathbb{R}_+$ represents the quantity of the good h that firm i choose to produce at stage t if ω occurs. In the same way, $x := (x_i)_{i\in\mathcal{I}} \in \mathcal{L}_{IH(T+1)}$ is the vector of all firms' strategies such that $x := (x_i)_{i\in\mathcal{I}} = (x_i, x_{-i})$ and $x_{-i} \in \mathcal{L}_{(I-1)H(T+1)}$. So, we pose $\mathcal{K} := \prod_{i\in\mathcal{I}} \mathcal{K}_i$ the feasible region of all firms' strategies such that, for each $i \in \mathcal{I}$ and $x_{-i} \in \mathcal{K}_{-i}$, $\mathcal{S}_i(x_{-i}) \subseteq \mathcal{K}_i$ is the strategy set of firm i if x_{-i} is the quantity produced by the other firms.

For any $h \in \mathcal{H}$, we denote by $c_i^h : \mathcal{L}_{T+1} \to \mathcal{L}_{T+1}$ the function such that, for any x_i^h, $t \in \mathcal{T}_0$ and $\omega \in \Omega$, $c_{it}^h(\omega, x_{it}^h(\omega)) \in \mathbb{R}_+$ represents the cost incurred by firm i in the production of the quantity $x_{it}^h(\omega)$ of the good h at stage t if ω occurs; we pose $c_i := (c_i^h)_{h\in\mathcal{H}} : \mathcal{L}_{H(T+1)} \to \mathcal{L}_{H(T+1)}$. Moreover, we denote by $p^h : \mathcal{L}_{I(T+1)} \to \mathcal{L}_{T+1}$ the function such that, for any x^h, $t \in \mathcal{T}_0$ and $\omega \in \Omega$, $p_t^h(\omega, x_t^h(\omega)) \in \mathbb{R}$ represents the price associated with the aggregate output $x_t^h(\omega) \in \mathbb{R}_+^I$ of the good h at stage t if ω occurs; we pose $p := (p^h)_{h\in\mathcal{H}} : \mathcal{L}_{IH(T+1)} \to \mathcal{L}_{H(T+1)}$. For each $i \in \mathcal{I}$, we consider the *bicriteria* function

$$u_i := (u_{i1}, u_{i2}) : \mathcal{L}_{IH(T+1)} \to \mathbb{R}^2,$$

where

- $u_{i1} : \Omega \times \mathbb{R}^{IH(T+1)} \to \mathbb{R}$ represents a *revenue function* such that, for any $(\omega, x(\omega))$, we have

$$u_{i1}(\omega, x(\omega)) := u_{i1}(\omega, x_i(\omega), p(\omega, x_i(\omega), x_{-i}(\omega)));$$

- $u_{i2} : \Omega \times \mathbb{R}^{IH(T+1)} \to \mathbb{R}$ represents a *profit function* such that, for any $(\omega, x(\omega))$, we have

$$u_{i2}(\omega, x(\omega)) := u_{i2}(\omega, x_i(\omega), p(\omega, x_i(\omega), x_{-i}(\omega)), c_i(\omega, x_i(\omega))).$$

In the literature, the price and the cost functions are defined in various ways and with different mathematical properties; see, e.g., [11, 25] and the references therein. However, in this example, we consider general revenue and profit functions, without explicitly specifying their structures, and p and c_i in order to avoid details that are beyond the scope of this work. At the same time, we point out that now we consider a bicriteria function only for simplicity. Indeed, we could extend all to a *multicriteria* problem for each $i \in \mathcal{I}$: for instance, we could opportunely add a risk function as an additional criterion.

Anyway, for each $i \in \mathcal{I}$, $\mathcal{U}_i := (\mathcal{U}_{i1}, \mathcal{U}_{i2}) : \mathcal{L}_{IH(T+1)} \to \mathbb{R}^2$ is the corresponding expected bicriteria function, defined as in (23); moreover, $\mathcal{D}(\mathcal{C}_i) : \mathcal{L}_{IH(T+1)} \rightrightarrows \mathbb{R}^2$ is considered variable ordering structure defined as in (6). Then, we formally introduce the following equilibrium problem.

Definition 7 A vector $\bar{x} \in \mathcal{S}(\bar{x}) \cap \mathcal{N}_{IH(T+1)}$ is a solution of the *multistage stochastic bicriteria generalized Nash-Cournot game* up to now described if

$$\forall i \in \mathcal{I} \qquad \mathcal{U}_i(x_i, \bar{x}_{-i}) - \mathcal{U}_i(\bar{x}_i, \bar{x}_{-i}) \notin \operatorname{int} \mathcal{D}(\mathcal{C}_i(\bar{x}_i, \bar{x}_{-i})) \qquad \forall x_i \in \mathcal{S}_i(\bar{x}_{-i}) \cap \mathcal{N}_{H(T+1)}. \tag{29}$$

According to Theorems 4 and 5, under suitable assumptions, the existence of an equilibrium solution of the problem (29) follows as a natural consequence.

Corollary 3 *Under the assumptions of Theorem 5, the multistage stochastic bicriteria generalized Nash-Cournot game (29) admits at least a solution.*

We point out that the use of the variable ordering structure may be of interest for the considered Nash-Cournot game. Indeed, a model in which the ordering structure of each firm depends on the production level could be useful to represent situations where the trade-off ratio between expected profit and revenue changes if a given threshold of production is reached: for instance, due to a change in the fiscal incentives.

As an example, we can consider the case in which $|\mathcal{H}| = 1$, $\mathcal{T}_0 = \{0, 1\}$, and the variable ordering structure, for any $i \in \mathcal{I}$ and $x \in \mathcal{L}_{I2}$, is defined as

$$\mathcal{D}(\mathcal{C}_i(x_i, x_{-i})) = \begin{cases} \mathbb{R}^2_+ & \text{if } x_{i0} + x_{i1}(\omega) \leq \alpha_i \ \ \forall \omega \in \Omega \\ T_i & \text{otherwise} \end{cases}, \tag{30}$$

where $\alpha_i > 0$ and

$$T_i := \left\{ (y_1, y_2) \in \mathbb{R}^2_+ : y_2 \geq \frac{1}{q_i} y_1, \text{ with } q_i > 0 \right\}.$$

In particular, y_1 refers to a value of the expected revenue function $\mathcal{U}_{i1}$, while y_2 refers to a value of the expected profit function $\mathcal{U}_{i2}$. In this way, for any $i \in \mathcal{I}$ and whenever a given production threshold α_i is exceeded, the variable ordering structure (30) at (x_i, x_{-i}) represents a situation where the preferences of the firm i are such that the trade-off ratio between the value of the expected profit function and the value of the expected revenue function is larger than $\frac{1}{q_i}$.

6 Conclusions

In this paper, we make use of a suitable nonlinear scalarization approach to study a multi-criteria generalization of the multistage stochastic variational inequalities. The multicriteria problems under analysis are complicated by the presence of variable ordering structures and by the dependence of the constraint set on the current value of the variable. The generalizations on the formulation of the problem are considered to capture the dynamics and the issues that can arise from the study of real-world phenomena where time, uncertainty, and an increasing level of information are widespread features. We use a generalization of the well-known nonlinear scalarization technique introduced in [17] to the study of such multistage stochastic vector quasi-variational problems. In order to support the practical applicability of the proposed multicriteria generalization of the multistage stochastic variational inequalities, we introduce the study of multistage stochastic vector Nash equilibrium problems: as an example, we analyze a suitable multistage stochastic bicriteria Cournot oligopolistic model.

This paper is a first approach to the problem. Hence, it is the starting point of our future researches and developments in more general spaces and under weaker requirements on the ordering structure, on Φ and $\mathcal{S}$; in addition, we aim to weaken the assumptions on the vector-valued payoff functions of the multistage stochastic vector Nash equilibrium problem (25). From an applicative point of view, our results pave the way for a wide range of possible future investigations in terms of energy market models, network equilibrium problems, economic equilibrium problems, portfolio management problems, etc, with an approach that is closer to the real-life mechanisms. Motivated by this fact, our future developments will be on the study of computational methods to solve numerically the problem and provide numerical implementations in support of the approach proposed in this paper. In the literature, several computational methods are available to solve vector problems with variable ordering structures; see, e.g., [14, 30] and the references therein. In a stochastic framework, in [32], the authors proposed a *Sample Average Approximation* method for solving a class of stochastic vector variational inequalities. Moreover, for the scalar case, the well-known *Progressive Hedging algorithm* is used to study multistage stochastic variational inequalities; see, e.g., [15, 22, 28, 29]. In particular, it works on the extensive formulation of the multistage stochastic variational inequality by solving in parallel a problem for each possible scenario. Then, we aim to investigate how to extend/combine these computational methodologies to approach numerically the study of multistage stochastic vector quasi-variational problems.

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