Università degli Studi di Pavia
Dipartimento di Matematica "Felice Casorati"

Università degli Studi di Milano Bicocca
Dipartimento di Matematica e Applicazioni

*PhD in Mathematics*

---

# Invariant Measures in Two-Dimensional Fluid Dynamics

PhD Candidate:
**Matteo Ferrari**

Supervisor:
**Prof. Enrico Priola**

September 2024

# *Abstract*

The present thesis is an integrated compilation of the two main works that defined my PhD research at the University of Pavia, University of Milano-Bicocca, and University of York, where I completed a four-month Erasmus Traineeship Programme.

The two principal works are presented as the main chapters of this thesis. They both address the two-dimensional stochastic Navier-Stokes Equations (sNSEs) for homogeneous incompressible fluids and the study of invariant probability measures. The first work [Fer24], supervised by Professor Enrico Priola from the University of Pavia, introduces a novel *a priori* estimate for the sNSEs, set in a bounded domain and with additive noise, which leads to an intriguing application regarding the uniqueness and ergodic properties of its invariant measure. The second work [BF24], co-authored with Professor Zdzisław Brzeźniak from the University of York, explores the inviscid limit for the hyperviscous sNSEs, set in $\mathbb{R}^2$ with additive noise. This investigation results in the proof of the existence, along with some moment estimates, of an invariant measure for the deterministic Euler Equations.

# Contents

# Introduction

The study of fluid dynamics is essential for understanding the behaviour of fluids in motion, with applications spanning from meteorology to engineering. At the heart of this discipline lie the Navier-Stokes and Euler equations, which model incompressible fluid flows. These equations are fundamental in describing the time evolution of velocity and pressure fields in a fluid, and they serve as a cornerstone for theoretical and applied fluid mechanics.

The Navier-Stokes equations for incompressible, homogeneous fluids are

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u = f - \frac{1}{\rho} \nabla p, \\ \operatorname{div} u = 0, \end{cases}$$

where $u$ represents the flow velocity vector field, $p$ the pressure scalar field, $\nu > 0$ is the kinematic viscosity, and $\rho > 0$ is the density of the fluid. The term $f$ serves as an external force, per unit of mass, acting on the system, which may be either deterministic or stochastic. In this system, viscosity ($\nu > 0$) plays a critical role by introducing diffusion that smoothens out velocity fluctuations, thereby modelling viscous fluids like water.

When viscosity vanishes ($\nu = 0$), the Euler equations are recovered:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = f - \frac{1}{\rho} \nabla p, \\ \operatorname{div} u = 0, \end{cases}$$

which describe the motion of inviscid fluids, such as ideal gases. The Euler system differs fundamentally from the Navier-Stokes equations because it lacks the dissipative term $-\nu \Delta u$, meaning that the dynamics are purely advective, without the smoothing effect of viscosity.

A central theme in the analysis of these systems is the study of their long-term behaviour. In particular, understanding the statistical properties of solutions as time tends to infinity is a challenging open problem, especially in the stochastic setting. Invariant measures provide a natural way to capture the statistical steady states of fluid flows. Formally, an invariant measure is a probability distribution over the possible states of the system that remains unchanged as the system evolves over time. Invariant measures are crucial for studying ergodic properties and the long-term statistical behaviour of solutions, especially in turbulent flows where direct analysis of individual trajectories is difficult.

This thesis is dedicated to the study of invariant measures for the $2D$ Navier-Stokes and Euler equations, particularly in the stochastic setting, which is of significant interest in statistical hydrodynamics. Stochastic forcing in these equations models unpredictable external influences, such as thermal fluctuations or turbulent background noise, and leads to the formulation of stochastic partial differential equations (SPDEs).

A preliminary chapter lays the groundwork by introducing the necessary mathematical notations that will be used throughout the thesis.

In the first main chapter, based on [Fer24], we focus on the stochastic Navier-Stokes equations (sNSEs), where an additive Wiener process models the external force $f$. When set in the abstract setting of square-integrable vector fields, the $2D$ SNSEs take the form

$$\mathrm{d}X_t + [AX_t + B(X_t)]\,\mathrm{d}t = G\,\mathrm{d}W_t, \qquad \textit{for } t > 0,$$

where $A$ is the Stokes operator, $B$ represents the non-linearity, and $G\,\mathrm{d}W_t$ models the stochastic forcing. The first contribution of this work is the existence and uniqueness of solutions to the sNSEs, along with their regularity properties under specific conditions on the noise operator $G$.

We further establish the ergodic properties of these solutions by proving the existence of a unique invariant measure, which describes the statistical equilibrium state of the system. This invariant measure is ergodic, meaning that the time averages of observables converge to ensemble averages with respect to this measure. The ergodicity of invariant measures provides a rigorous foundation for the statistical analysis of fluid flows, particularly in two-dimensional turbulence, where direct numerical simulation of the equations is often intractable.

In the second part of the thesis, based on a co-authored work with Professor Zdzisław Brzeźniak, from the University of York, UK, we turn to the Eulerian limits of the $2D$ sNSEs. By considering the limit as $\nu$ tends to 0, we examine the convergence of the stochastic solutions of the Navier-Stokes system to those of the Euler system. This passage from viscous to inviscid fluid dynamics is non-trivial, particularly in unbounded domains like $\mathbb{R}^2$, where the compact embeddings for Sobolev-type spaces are absent, and new techniques are required to manage the lack of compactness. We employ weak topologies and advanced stochastic methods to address these challenges, ultimately deriving invariant measures for the Euler equations as the limit of those from the Navier-Stokes system.

Specifically, we study the stochastic hyperviscous Navier-Stokes equations as an intermediate model:

$$\mathrm{d}X_t + \left[\nu A^\alpha X_t + B(X_t)\right]\mathrm{d}t = \sqrt{\nu}\,\mathrm{d}W_t, \qquad \textit{for } t > 0,$$

where $\alpha > 1$ introduces additional dissipation, regularizing the solutions and allowing us to derive a priori estimates. The choice of the noise scaling factor $\sqrt{\nu}$ plays a crucial role in ensuring the convergence of solutions to the inviscid limit. Using a version of the Krylov-Bogoliubov method, adapted for weak topologies, we establish the existence of invariant measures in this setting and show that these measures converge as the system transitions from the Navier-Stokes to the Euler equations.

# Notations and preliminaries

**Notation 0.1.** Let $\mathcal{X}, \mathcal{Y}$ be topological spaces. We denote by $C(\mathcal{X}; \mathcal{Y})$ the vector space of continuous functions $f : \mathcal{X} \to \mathcal{Y}$. If $\mathcal{Y} = \mathbb{R}$, we simply write $C(\mathcal{X})$ instead of $C(\mathcal{X}; \mathbb{R})$. The subspace of $C(\mathcal{X})$ consisting of bounded functions is denoted by $C_b(\mathcal{X})$. A function $f : \mathcal{X} \to \mathcal{Y}$ is sequentially continuous, and we write $f \in SC(\mathcal{X}; \mathcal{Y})$, if for any $x \in \mathcal{X}$ and any sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$ convergent to $x$ in $\mathcal{X}$, the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$ in $\mathcal{Y}$. If $\mathcal{Y} = \mathbb{R}$ we write $SC(\mathcal{X})$ instead of $SC(\mathcal{X}; \mathbb{R})$.

If $\mathcal{X}$ is compact and $(\mathcal{Y}, \|\cdot\|)$ is a normed vector space, then $C(\mathcal{X}; \mathcal{Y})$ is endowed with the usual supremum norm, *i.e.* $\|f\|_{C(\mathcal{X};\mathcal{Y})} := \sup_{x\in\mathcal{X}} \|f(x)\|$, for $f \in C(\mathcal{X}; \mathcal{Y})$.

*Remark* 0.2. If a function $f : \mathcal{X} \to \mathcal{Y}$ between topological spaces is continuous, then it is sequentially continuous, while the converse is generally false, *i.e.* $C(\mathcal{X}; \mathcal{Y}) \subsetneq SC(\mathcal{X}; \mathcal{Y})$, see [AVA90, Example 1, Chapter 3].

A topological space $\mathcal{X}$ is said to be sequential if sequentially closed subsets are topologically closed. Therefore, under the additional assumption of $\mathcal{X}$ being sequential, a function $f : \mathcal{X} \to Y$ is sequentially continuous if and only if it is continuous, see [AVA90, Proposition 3, Chapter 3]. As a matter of fact, this universal property characterises sequential spaces: a topological space $\mathcal{X}$ is sequential if and only if for any topological space $\mathcal{Y}$, we have $C(\mathcal{X}; \mathcal{Y}) = SC(\mathcal{X}; \mathcal{Y})$. Let us recall, see *e.g.* [AVA90, Chapters 1, 2], that metrizable spaces are sequential spaces.

**Notation 0.3.** Assume that $\mathcal{X}$ is a topological space. We denote by $\mathscr{B}_\mathcal{X}$ its Borel $\sigma$-algebra and by $\mathscr{P}(\mathcal{X})$ the family of probability measures on $(\mathcal{X}, \mathscr{B}_\mathcal{X})$. Moreover, $\mathcal{B}_b(\mathcal{X})$ will denote the Banach space of measurable and bounded functions $\varphi : (\mathcal{X}, \mathscr{B}_\mathcal{X}) \to (\mathbb{R}, \mathscr{B}_\mathbb{R})$, endowed with the sup norm, *i.e.* $\sup\limits_{x\in\mathcal{X}} |\varphi(x)|$, for $\varphi \in \mathcal{B}_b(\mathcal{X})$.

Two probability measures $\mu, \nu$ on $(\mathcal{X}, \mathscr{B}_\mathcal{X})$ are equal if and only if

$$\int_\mathcal{X} \varphi \, \mathrm{d}\mu = \int_\mathcal{X} \varphi \, \mathrm{d}\nu, \qquad \forall \, \varphi \in \mathcal{B}_b(\mathcal{X}).$$

Let $\mu, \mu_n \in \mathscr{P}(\mathcal{X})$, $n \in \mathbb{N}$. We say that $\mu_n \longrightarrow \mu$ in the weak sense in $\mathscr{P}(\mathcal{X})$, or simply in $\mathscr{P}(\mathcal{X})$, if

$$\lim_{n\to\infty} \int_\mathcal{X} f \, \mathrm{d}\mu_n = \int_\mathcal{X} f \, \mathrm{d}\mu, \qquad \forall \, f \in C_b(\mathcal{X}).$$

Assume that $\mu \in \mathscr{P}(\mathcal{X})$ and that $\mathcal{Y}$ is another topological space. If $\Phi : \mathcal{X} \to \mathcal{Y}$ is a Borel measurable function, we denote the pushforward measure of $\mu$ via $\Phi$ by

$$\Phi_*\mu : \mathscr{B}_\mathcal{Y} \ni E \mapsto \mu\big(\Phi^{-1}(E)\big) \in [0, 1],$$

which is a probability measure on $(\mathcal{Y}, \mathscr{B}_\mathcal{Y})$.

**Notation 0.4.** A filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be augmented or to satisfy the usual conditions if it is right-continuous and complete, namely if

- $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$, for any $t \geq 0$,
- $\mathbb{P}$-null sets belong to $\mathcal{F}_t$, for every $t \geq 0$, *i.e.* if $E \subset \Omega$ is such that there exists $N \in \mathcal{F}$ such that $\mathbb{P}(N) = 0$ and $E \subseteq N$, then $E \in \mathcal{F}_t$ for all $t \geq 0$.

**Notation 0.5.** Assume that $\mathcal{X}$, $\mathcal{Y}$ are topological spaces. We say that $\mathcal{X}$ is continuously embedded into $\mathcal{Y}$, and we write $\mathcal{X} \hookrightarrow \mathcal{Y}$, if $\mathcal{X} \subset \mathcal{Y}$ and the map $\iota : \mathcal{X} \ni x \mapsto x \in \mathcal{Y}$ is continuous. The map $\iota$ will be referred to as the natural embedding.

Assume now that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. We say that $\mathcal{X}$ is compactly embedded into $\mathcal{Y}$, and we write $\mathcal{X} \overset{c}{\hookrightarrow} \mathcal{Y}$, if $\mathcal{X} \hookrightarrow \mathcal{Y}$ and the natural embedding is a compact linear operator, *i.e.* if closed balls in $\mathcal{X}$ are compact in $\mathcal{Y}$.

*Remark* 0.6. If $\mathcal{X}, \mathcal{Y}$ are Polish spaces such that $\mathcal{X} \hookrightarrow \mathcal{Y}$, then

$$\mathscr{B}_{\mathcal{X}} = \mathscr{B}_{\mathcal{Y}} \cap \mathcal{X}, \tag{0.1}$$

where $\mathscr{B}_{\mathcal{Y}} \cap \mathcal{X} := \{U \cap \mathcal{X} \ : \ U \in \mathscr{B}_{\mathcal{Y}}\}$. Indeed, if we denote by $\iota : \mathcal{X} \to \mathcal{Y}$ the natural embedding, Kuratowski's Theorem [Kur66, Theorem 1, Section V, Chapter 39] implies that $\iota(\mathcal{X}) \in \mathscr{B}_{\mathcal{Y}}$, see also [Kec95, Theorem 15.1]. Then $\mathcal{X} \in \mathscr{B}_{\mathcal{Y}}$ because $\iota(\mathcal{X}) = \mathcal{X}$ by direct inspection. Then [Łoj88, Theorem 4.3.3] yields $\mathscr{B}_{\mathcal{X}} = \mathscr{B}_{\mathcal{Y}} \cap \mathcal{X}$.

In particular,

$$\mathscr{B}_{\mathcal{X}} \subset \mathscr{B}_{\mathcal{Y}}.$$

Indeed, if $E \in \mathscr{B}_{\mathcal{X}}$, then, by (0.1), there exists $F \in \mathscr{B}_{\mathcal{Y}}$ such that $E = F \cap \mathcal{X}$. Since $\mathcal{X} \in \mathscr{B}_{\mathcal{Y}}$ as discussed, we infer that $E \in \mathscr{B}_{\mathcal{Y}}$.

**Notation 0.7.** Assume that $(\mathcal{X}, \|\cdot\|)$ is a Banach space. We denote by $\mathcal{L}(\mathcal{X})$ the Banach space of linear and bounded operators $T : \mathcal{X} \to \mathcal{X}$, endowed with the operator norm

$$\|T\|_{\mathcal{L}(\mathcal{X})} := \sup_{x \in \mathcal{X}\setminus\{0\}} \frac{\|Tx\|}{\|x\|}, \qquad \forall\, T \in \mathcal{L}(\mathcal{X}).$$

Assume that $\big(\mathcal{H}, \langle\cdot\,,\cdot\rangle\big)$ is a separable Hilbert space, with induced norm $\|\cdot\|$. The symbol $\mathcal{L}_1(\mathcal{H})$ will denote the subspace of $\mathcal{L}(\mathcal{H})$ consisting of trace-class operators, see *e.g.* [Mor18, Section 4.4]. If $T \in \mathcal{L}_1(\mathcal{H})$, then its trace $\mathrm{Tr}[T] \in \mathbb{R}$ is defined as follows. If $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal complete system for $\mathcal{H}$, then

$$\mathrm{Tr}[T] := \sum_{n=1}^{\infty} \langle Te_n, e_n\rangle,$$

where the definition can be shown to be independent of the orthonormal complete system.

*Remark* 0.8. Assuming $\mathcal{H}$ to be a separable Hilbert space, if $S \in \mathcal{L}(\mathcal{H})$ and $T \in \mathcal{L}_1(\mathcal{H})$, then both $ST$ and $TS$ belong to $\mathcal{L}_1(\mathcal{H})$. Moreover, the functional $\mathrm{Tr} : \mathcal{L}_1(\mathcal{H}) \ni T \mapsto \mathrm{Tr}[T] \in \mathbb{R}$ is continuous and satisfies the properties

$$\begin{aligned} &\mathrm{Tr}[TS] = \mathrm{Tr}[ST], && \forall\, S \in \mathcal{L}(\mathcal{H}),\ T \in \mathcal{L}_1(\mathcal{H}), \\ &\mathrm{Tr}[TS] \leq \|S\|_{\mathcal{L}(\mathcal{H})}\, \mathrm{Tr}[T], && \forall\, S \in \mathcal{L}(\mathcal{H}),\ T \in \mathcal{L}_1(\mathcal{H}),\ T \geq 0. \end{aligned}$$

We refer to [Mor18] for a complete dissertation on the topic.

**Notation 0.9.** Assume that $\mathcal{X}$ is a Banach space. We denote by $\mathcal{X}_w$ the topological space $(\mathcal{X}, \tau_{\mathcal{X}}^w)$, where the weak topology $\tau_{\mathcal{X}}^w$ is the smallest topology on $\mathcal{X}$ with respect to which every linear $f : \mathcal{X} \to \mathbb{R}$ is continuous. If $\tau_{\mathcal{X}}^s$ denotes the strong topology on $\mathcal{X}$ (*i.e.*, the natural topology induced by the norm), we will, with a slight abuse of notation, simply denote the topological space $(\mathcal{X}, \tau_{\mathcal{X}}^s)$ by $\mathcal{X}$. We denote by $\mathcal{X}_{bw}$ the topological space $(\mathcal{X}, \tau_{\mathcal{X}}^{bw})$, where the bounded weak topology $\tau_{\mathcal{X}}^{bw}$ is the largest topology on $\mathcal{X}$ that coincides with the weak topology on norm-bounded sets, *i.e.* $C \subset \mathcal{X}_{bw}$ is closed if and only if $C \cap \bar{B}^{\mathcal{X}}$ is closed in $\mathcal{X}_w$ for every closed ball $\bar{B}^{\mathcal{X}} \subset \mathcal{X}$. We refer to [Day73, Section II.5] for details about the bounded weak topology.

*Remark* 0.10. If $\mathcal{X}$ is an infinite-dimensional Banach space, then the topological space $\mathcal{X}_w$ is not sequential, see [GKP16, Theorem 1.5], in particular, it is not metrizable.

*Remark* 0.11. Assuming $\mathcal{X}$ to be a Banach space, we have

$$\tau_{\mathcal{X}}^w \subset \tau_{\mathcal{X}}^{bw} \subset \tau_{\mathcal{X}}^s. \tag{0.2}$$

Hence, for any topological space $\mathcal{Y}$

$$C(\mathcal{X}_w; \mathcal{Y}) \subset C(\mathcal{X}_{bw}; \mathcal{Y}) \subset C(\mathcal{X}; \mathcal{Y}).$$

We show the first inclusion in (0.2). Fix $O \in \tau_{\mathcal{X}}^w$, and a closed ball $\bar{B}^{\mathcal{X}}$ in $\mathcal{X}$. $\bar{B}^{\mathcal{X}}$ is closed in $\mathcal{X}_w$ by [Bre10, Theorem 3.7]. Moreover, $O^c$ is closed in $\mathcal{X}_w$ because $O \in \tau_{\mathcal{X}}^w$. Therefore, $O^c \cap \bar{B}^{\mathcal{X}}$ is the intersection of closed sets in $\mathcal{X}_w$, hence it is closed in $\mathcal{X}_w$. By arbitrariness of $\bar{B}^{\mathcal{X}}$ and definition of bounded weak topology, this implies that $O^c$ is closed in $\mathcal{X}_{bw}$, hence $O \in \tau_{\mathcal{X}}^{bw}$.

We now prove the second inclusion in (0.2). Fix $O \in \tau_{\mathcal{X}}^{bw}$, then we will show that $O^c$ is sequentially closed in $\mathcal{X}$, which implies that $O \in \tau_{\mathcal{X}}^s$ because $\mathcal{X}$ is sequential. Fix $x \in \mathcal{X}$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in $O^c$ convergent in $\mathcal{X}$ to $x$. Since the strong topology is larger than the weak topology, the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to $x$ in $\mathcal{X}_w$. Moreover, assuming $\|\cdot\|$ to be the norm on $\mathcal{X}$, and letting $R := \sup_{n\in\mathbb{N}} \|x_n\|$, we consider the strongly closed ball $\bar{B}_R^{\mathcal{X}} := \{y \in \mathcal{X} \ : \ \|y\| \le R\}$. By direct inspection, $x_n \in O^c \cap \bar{B}_R^{\mathcal{X}}$ for all $n \in \mathbb{N}$ and, by definition of bounded weak topology, $O^c \cap \bar{B}_R^{\mathcal{X}}$ is closed in $\mathcal{X}_w$. Therefore, $x \in \left(O^c \cap \bar{B}_R^{\mathcal{X}}\right) \subset O^c$.

*Remark* 0.12. If $\mathcal{X}$ is a Banach space, then compact sets in $\mathcal{X}_w$ are compact in $\mathcal{X}_{bw}$.

Fix a compact set $K$ in $\mathcal{X}_w$ and a collection $\mathcal{C}$ of open sets in $\mathcal{X}_{bw}$ that cover $K$. $K$ is bounded in $\mathcal{X}$ by [Bre10, Corollary 2.4 and Exercise 3.1], hence there exists a strongly closed ball $\bar{B}^{\mathcal{X}} \subset \mathcal{X}$ such that $K \subset \bar{B}^{\mathcal{X}}$. For any $O \in \mathcal{C}$, we know that $O^c \cap \bar{B}^{\mathcal{X}}$ is closed in $\mathcal{X}_w$ by definition of bounded weak topology, hence $\left(O^c \cap \bar{B}^{\mathcal{X}}\right)^c = O \cup \left(\bar{B}^{\mathcal{X}}\right)^c$ is open in $\mathcal{X}_w$. Moreover, the collection $\left\{O \cup \left(\bar{B}^{\mathcal{X}}\right)^c\right\}_{O\in\mathcal{C}}$ covers $K$ because $O \subseteq O \cup \left(\bar{B}^{\mathcal{X}}\right)^c$ for any $O \in \mathcal{C}$ and $K$ is covered by $\mathcal{C}$. Therefore, by definition of compact set in $\mathcal{X}_w$, we can extract $O_1, \dots, O_N \in \mathcal{C}$ such that $\left\{O_n \cup \left(\bar{B}^{\mathcal{X}}\right)^c\right\}_{n=1}^N$ still covers $K$. However, $K \subset \bar{B}^{\mathcal{X}}$, hence

$$K = \left(\bar{B}^{\mathcal{X}}\right)^c \cap K \subseteq \left(\bar{B}^{\mathcal{X}}\right)^c \cap \bigcup_{n=1}^N \left(O_n \cup \left(\bar{B}^{\mathcal{X}}\right)^c\right) = \bigcup_{n=1}^N O_n,$$

which proves that $K$ is compact in $\mathcal{X}_{bw}$.

*Remark* 0.13. If $\mathcal{H}$ is a separable Hilbert space, then for any topological space $\mathcal{Y}$

$$SC(\mathcal{H}_w;\mathcal{Y}) = C(\mathcal{H}_{bw};\mathcal{Y}).$$

Let $f \in C(\mathcal{H}_{bw};\mathcal{Y})$ and fix a convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}_w$ to some limit $x \in \mathcal{H}$. Then, denoting $\|\cdot\|$ the norm in $\mathcal{H}$, there exists $k \in \mathbb{N}$ such that the strongly closed ball $\bar{B} := \{y \in \mathcal{H} \;:\; \|y\| \le k\}$ contains both $x$ and $x_n$ for all $n \in \mathbb{N}$. If $C$ is a closed set in $\mathcal{Y}$, then its inverse image $f^{-1}(C)$ is closed in $\mathcal{H}_{bw}$, hence $f^{-1}(C) \cap \bar{B}$ is weakly closed by definition of bounded-weak topology. However $f^{-1}(C) \cap \bar{B}$ is the inverse image of $C$ via the restriction $f\big|_{\bar{B}}$ of $f$ to $\bar{B}$. Thus we proved that $f\big|_{\bar{B}} \in C(\bar{B}_w;\mathcal{Y})$. In particular, $f(x_n) = f\big|_{\bar{B}}(x_n)$ tends to $f\big|_{\bar{B}}(x) = f(x)$ in $\mathcal{Y}$ as $n \to \infty$. Therefore, $f \in SC(\mathcal{H}_w;\mathcal{Y})$.

On the other hand, fix $f \in SC(\mathcal{H}_w;\mathcal{Y})$, consider a strongly closed ball $\bar{B} \subset \mathcal{H}$ and the restriction $f\big|_{\bar{B}}$. If $\{x_n\}_{n\in\mathbb{N}} \subset \bar{B}$ is a weakly convergent sequence, then its limit belongs to $\bar{B}$ by [Bre10, Theorem 3.7], thus $f\big|_{\bar{B}} \in SC(\bar{B}_w;\mathcal{Y})$. However, $\bar{B}_w$ is metrizable by the Banach-Alaoglu Theorem, hence sequential. Therefore, $f\big|_{\bar{B}} \in C(\bar{B}_w;\mathcal{Y})$. This implies that, assuming $C$ to be a closed set in $\mathcal{Y}$, the inverse image $\big(f\big|_{\bar{B}}\big)^{-1}(C) = f^{-1}(C) \cap \bar{B}$ is weakly closed, thus $f^{-1}(C)$ is bounded-weakly closed. Hence $f \in C(\mathcal{H}_{bw};\mathcal{Y})$.

*Remark* 0.14. Let $T > 0$ and $\mathcal{H}$ be an abstract real separable Hilbert space, with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Notice that the space $C\big([0,T];\mathcal{H}\big)$ is strictly contained in $C\big([0,T];\mathcal{H}_w\big)$, whereas $L^2(0,T;\mathcal{H})$ has the same elements as $L^2_w(0,T;\mathcal{H})$.

The fact that $L^2(0,T;\mathcal{H})$ and $L^2_w(0,T;\mathcal{H})$ denote the same vector space, despite being equipped with different topologies, is merely a notational convenience.

It is also clear that, if $f \in C\big([0,T];\mathcal{H}\big)$, then $f \in C\big([0,T];\mathcal{H}_w\big)$. Since $[0,T]$ is sequential, it is indeed sufficient to verify the sequential continuity. Let $t \in [0,T]$ and $\{t_n\}_{n\in\mathbb{N}} \subset [0,T]$ be convergent to $t$, then for all $x \in \mathcal{H}$

$$|\langle f(t_n) - f(t), x\rangle| \le \|f(t_n) - f(t)\|\|x\| \longrightarrow 0, \qquad \textit{as } n \to \infty.$$

The opposite inclusion is false and we exhibit a counterexample. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal complete system for $\mathcal{H}$. We consider a strictly decreasing sequence $\{s_n\}_{n\in\mathbb{N}} \subset [0,T]$ converging to 0, such as $s_n = 1/n$, $n \in \mathbb{N}$, and define $f : [0,T] \to \mathcal{H}$ as follows, for $t \in [0,T]$:

$$f(t) := \sum_{n\in\mathbb{N}} \mathbb{1}_{[s_{n+1},s_n)}(t)\left[\cos\left(\frac{\pi}{2}\frac{t-s_n}{s_{n+1}-s_n}\right)e_n + \sin\left(\frac{\pi}{2}\frac{t-s_n}{s_{n+1}-s_n}\right)e_{n+1}\right].$$

Roughly speaking, $f$ is a curve in $\mathcal{H}$, whose support consists of a sequence of arcs of unit circles in the planes spanned by $\{e_n, e_{n+1}\}$, for $n \in \mathbb{N}$. These arcs are traversed faster as time approaches 0. This function $f$ belongs to $C\big([0,T];H_w\big) \cap C\big((0,T];H\big)$, and is not strongly continuous in 0. Strong continuity at every $t \in (0,T] \setminus \{s_n\}_{n\in\mathbb{N}}$ is trivial. Strong continuity at $s_n$, for any $n \in \mathbb{N}$, can be easily proved by showing left and right-continuity:

$$\lim_{t\to s_n^+} f(t) = \lim_{t\to s_n^+} \mathbb{1}_{[s_n,s_{n-1})}(t)\left[\cos\left(\frac{\pi}{2}\frac{t-s_{n-1}}{s_n-s_{n-1}}\right)e_{n-1} + \sin\left(\frac{\pi}{2}\frac{t-s_{n-1}}{s_n-s_{n-1}}\right)e_n\right] = e_n,$$

$$\lim_{t\to s_n^-} f(t) = \lim_{t\to s_n^-} \mathbb{1}_{[s_{n+1},s_n)}(t)\left[\cos\left(\frac{\pi}{2}\frac{t-s_n}{s_{n+1}-s_n}\right)e_n + \sin\left(\frac{\pi}{2}\frac{t-s_n}{s_{n+1}-s_n}\right)e_{n+1}\right] = e_n,$$

$$f(s_n) = \mathbb{1}_{[s_n,s_{n-1})}(s_n)\left[\cos\left(\frac{\pi}{2}\frac{s_n-s_{n-1}}{s_n-s_{n-1}}\right)e_{n-1} + \sin\left(\frac{\pi}{2}\frac{s_n-s_{n-1}}{s_n-s_{n-1}}\right)e_n\right] = e_n.$$

As for $t = 0$, we have by inspection that $f(0) = 0$, which prevents strong continuity because $\|f(t)\| = 1$ for all $t \in (0,T]$. It only remains to be shown that $f$ is weakly continuous at $t = 0$. Let $x \in \mathcal{H}$, then

$$\begin{aligned}
0 &\le \limsup_{t\to 0^+} |\langle f(t), x\rangle| \\
&\le \limsup_{n\to\infty} \sup_{t\in[s_{n+1},s_n)} \left[\cos\left(\frac{\pi}{2}\frac{t-s_n}{s_{n+1}-s_n}\right)|\langle e_n, x\rangle| + \sin\left(\frac{\pi}{2}\frac{t-s_n}{s_{n+1}-s_n}\right)|\langle e_{n+1}, x\rangle|\right] \\
&\le \limsup_{n\to\infty} \big(|\langle e_n, x\rangle| + |\langle e_{n+1}, x\rangle|\big) = 0,
\end{aligned}$$

by the well-known fact $|\langle e_n, x\rangle| \longrightarrow 0$, as $n \to \infty$.

We remark that, by appropriately modifying $f$, one gets an example of a continuous function $[0,T] \to \mathcal{H}_w$, yet everywhere strongly discontinuous.

*Remark* 0.15. If $\mathcal{X}$ is a separable Banach space, then the Borel sigma algebras generated by the strong, bounded weak, and weak topologies coincide:

$$\mathscr{B}_{\mathcal{X}_w} = \mathscr{B}_{\mathcal{X}_{bw}} = \mathscr{B}_{\mathcal{X}}.$$

We refer to [Ziz03, Theorem 7.19] or [Edg79, Corollary 2.4] for the proof of this statement. See also [MS01, Introduction].

In particular, assuming $\mathcal{H}$ to be a separable Hilbert space, if a function $f : \mathcal{H}_w \to \mathbb{R}$ is sequentially continuous, then it is $\mathscr{B}_{\mathcal{H}}$-measurable. Indeed if $f \in SC(\mathcal{H}_w)$, then $f \in C(\mathcal{H}_{bw})$ by Remark 0.13, then $f$ is $\mathscr{B}_{\mathcal{H}_{bw}}$-measurable, however $\mathscr{B}_{\mathcal{H}_{bw}} = \mathscr{B}_{\mathcal{H}}$.

*Remark* 0.16. Assume that $\mathcal{H}$ is a separable Hilbert space and let $\mathcal{Y}$ be a Banach space. If $\mathcal{H} \overset{c}{\hookrightarrow} \mathcal{Y}$, then $\mathcal{H}_{bw} \hookrightarrow \mathcal{Y}$.

If indeed $\iota : \mathcal{H} \to \mathcal{Y}$ denotes the natural embedding, the assertion is equivalent to proving that $\iota \in C(\mathcal{H}_{bw}; \mathcal{Y})$, which is again equivalent to showing that $\iota \in SC(\mathcal{H}_w; \mathcal{Y})$ by Remark 0.13. Therefore, let us fix a weakly convergent sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$, then the sequence is convergent in $\mathcal{Y}$ because of the compact embedding $\mathcal{H} \overset{c}{\hookrightarrow} \mathcal{Y}$, and the assertion follows.

**Notation 0.17.** Assume that $(\mathcal{H}, \langle\cdot\,,\cdot\rangle_{\mathcal{H}})$ is a separable real Hilbert space. If $d \in \mathbb{N}$, we denote the Fourier transform of a Bochner-integrable function $f : \mathbb{R}^d \to \mathcal{H}$ by

$$\mathscr{F}[f] : \mathbb{R}^d \ni \xi \mapsto \mathscr{F}[f](\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}} e^{-i\xi\cdot x} f(x)\, \mathrm{d}x \in \mathcal{H}.$$

The resulting function $\mathscr{F}[f] : \mathbb{R}^d \to \mathcal{H}$ is bounded and continuous. By the Plancherel Theorem, the linear operator $L^1(\mathbb{R}^d; \mathcal{H}) \cap L^2(\mathbb{R}^d; \mathcal{H}) \ni f \mapsto \mathscr{F}[f] \in L^2(\mathbb{R}^d; \mathcal{H})$ is a linear isometry with respect to the $L^2(\mathbb{R}^d;\mathcal{H})$-norm. By a density argument, this operator admits a unique extension to a unitary operator

$$\mathscr{F} : L^2(\mathbb{R}^d;\mathcal{H}) \to L^2(\mathbb{R}^d;\mathcal{H}).$$

In particular, for any $f, g \in L^2(\mathbb{R}^d;\mathcal{H})$ we have

$$\|f\|_{L^2(\mathbb{R}^d;\mathcal{H})} = \left\|\mathscr{F}[f]\right\|_{L^2(\mathbb{R}^d;\mathcal{H})}, \tag{0.3}$$

$$\int_{\mathbb{R}^d} \big\langle f(x), g(x)\big\rangle_{\mathcal{H}}\,\mathrm{d}x = \int_{\mathbb{R}^d} \big\langle \mathscr{F}[f](\xi), \mathscr{F}[g](\xi)\big\rangle_{\mathcal{H}}\,\mathrm{d}\xi. \tag{0.4}$$

In addition, $\mathscr{F}$ is a continuous endomorphism when restricted to the Schwartz space $\mathcal{S}(\mathbb{R}^d;\mathcal{H})$, endowed with its canonical LF topology. Therefore, its unique transpose is well-defined and it is a continuous endomorphism, still denoted by $\mathscr{F}$, acting on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d;\mathcal{H})$, where $\mathcal{H}$ is identified with its topological dual by the usual Riesz identification. Moreover, by denoting ${}_{\mathcal{H}'}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{H}}$ the duality product, we have

$$\mathscr{F}\Big[\,{}_{\mathcal{H}'}\big\langle x, f\big\rangle_{\mathcal{H}}\Big] = {}_{\mathcal{H}'}\big\langle x, \mathscr{F}[f]\big\rangle_{\mathcal{H}}, \qquad \forall\, f \in L^2(\mathbb{R}^d;\mathcal{H}),\ \forall\, x \in \mathcal{H}'. \tag{0.5}$$

# Chapter I

# New *a Priori* Estimate for Stochastic 2D Navier-Stokes Equations with Applications to Invariant Measure

## I.1 Introduction

The Navier-Stokes equations provide a complete characterization of the motion of a viscous Newtonian fluid. For incompressible and homogeneous fluids, the N-S equations take the following form:

$$\begin{cases} \partial_t u - \nu\Delta u + (Du)u = -\frac{1}{\rho}\nabla p + f \\ \operatorname{div} u = 0 \end{cases}, \tag{I.1.1}$$

where the kinematic viscosity $\nu > 0$ and the density $\rho > 0$ are given constants, while $f$ denotes a known external force acting on the system. Here, the unknowns are $p$ and $u$, which represent respectively the pressure scalar field and the flow velocity vector field of the fluid. We set the equation in a non-empty bounded and connected open set $\mathcal{D} \subset \mathbb{R}^2$ with Lipschitz boundary. Consequently, we require $u, f : [0, +\infty) \times \overline{\mathcal{D}} \to \mathbb{R}^2$ and $p : [0, +\infty) \times \overline{\mathcal{D}} \to \mathbb{R}$. Moreover, we choose the units of measurement such that $\nu = 1$. We associate equation (I.1.2) with the Dirichlet boundary condition and an initial datum

$$u = 0 \quad \text{in } [0, +\infty) \times \partial\mathcal{D} \qquad\qquad u(0, \cdot) = u_0(\cdot) \quad \text{in } \overline{\mathcal{D}}.$$

The choice of the domain offers a more general and physically relevant framework compared to the unbounded domain or torus with periodic boundary conditions. Nonetheless, we believe that the novel technique introduced in this chapter can also be extended to these alternative cases. In particular, it may yield stronger results, especially for periodic boundary conditions; see Remark I.1.1 for further details.

In order to study equation (I.1.1), we introduce a basic Hilbert space $H \subset L^2(\mathcal{D}; \mathbb{R}^2)$, see Section I.2.1, which incorporates both the boundary condition and the divergence-free condition. In Section I.2.2, we define the Leray projector $\Pi$ onto $H$, the Stokes operator $A = -\Pi\Delta$, and the Navier-Stokes non-linearity $B$. For a detailed treat, see [Tem01; VF88; Tem95] and the references therein. In the literature the case when the external force $f$ is random is extensively investigated starting from the seminal paper [BT73]. Here we analyze the case when $f$ is formally the time derivative of a cylindrical Wiener process $W$ in $H$, see Section I.2.3, appropriately regularized by an injective and bounded linear operator $G : H \to H$. These modifications turn the equation into a stochastic differential equation in the infinite dimensional Hilbert space $H$

$$\begin{cases} \mathrm{d}u + [Au + B(u)]\,\mathrm{d}t = G\,\mathrm{d}W_t, & \text{for } t > 0, \\ u(0) = x \in H. \end{cases} \tag{I.1.2}$$

We point out that two special types of noise have been investigated in the literature. On one hand, on a more physical ground, most of the current research on additive noise goes in the direction of studying highly degenerate noises, where only few modes are excited, *cf.* [GMR17]. On the other hand, sits the technical question of reaching the white noise case, namely $G = \mathbb{I}_H$, *cf.* [Fla94; FM95; Fer97]. We discuss more on this topic in Remark I.1.1. Even if the techniques employed in this work do not allow to reach the white noise case, the present paper aims in the second direction and concentrates on a case which is quite close to the cylindrical one, in the sense that $\mathrm{Ran}(G) \subset D(A^{\frac{1}{4}+\varepsilon})$ for some $\varepsilon > 0$.

Interesting topics related to equation (I.1.2) include defining a suitable notion of solution, see Definition I.2.6, that assures both its existence and pathwise uniqueness. Once the solution is proven to be a Markov process, one associates with it a Markov semigroup, see Section I.2.5, which formally describes the mean behaviour in time of the solution given the starting point. This leads to other lines of research, including the regularity of the semigroup and the existence, uniqueness, and strong mixing property of the invariant measure $\mu$. The strong mixing property says that the law of the solution with a random starting point converges for long times to $\mu$ in the total variation norm, see Theorem I.4.4. This in particular implies the ergodicity of the invariant measure, *cf.* [DZ14; Sei97], which means that $\mu$ is the equilibrium measure over the phase space $H$. This principle, known as the ergodic principle, forms the basis of the statistical approach to fluid dynamics.

Under the basic conditions made throughout the paper, namely $\mathrm{Ran}(G) \subset D(A^{\frac{1}{4}+\varepsilon})$ for some $\varepsilon > 0$, and $x \in H$, Theorem I.2.7 gathers the results found in the literature about the existence, uniqueness and path regularities of solution. Stricter conditions have also been considered, and are recalled in detail in Remark I.2.9. However, to the best of the author's knowledge, no results have been established for more degenerate noise, *i.e.* if $D(A^{\frac{1}{4}}) \subset \mathrm{Ran}(G)$, at least for the Dirichlet boundary value problem under investigation. The periodic case is briefly discussed in Remark I.1.1.

We improve the existing results as follows, see Section I.3. If $D(A^{\frac{1}{2}}) \subset \mathrm{Ran}(G) \subset D(A^{\frac{1}{4}+\varepsilon})$ for some $\varepsilon \in (0, 1/4]$, and $x \in D(A^{\gamma})$ for some $\gamma \in [0, 1/4 + \varepsilon)$, then the trajectories of the generalized solution are continuous with values in $D(A^{\gamma})$, see Theorem I.3.1. The approach to the study of the regularity of path is classical and employs the splitting of the solution $u = v + z$ where $z$ is an Ornstein-Uhlenbeck process and $v$ solves a partial differential equation (PDE) with random coefficients, see the discussion after Theorem I.3.1. The question of the regularity of the paths is then deferred to the analysis of this random PDE. The main novelty of this work is to show that the solution of this random PDE has continuous paths in regular spaces, under loose regularity assumptions on $z$. This is obtained through a new *a* *priori* estimate inspired by a technique introduced by Sobolevskiĭ in [Sob59] and later studied by Kato, Fujita and Giga, *cf.* [KF62; FK64; Gig81; Gig83; GM85; Gig86], see Lemma I.3.5. We refer to the introduction of [Kie80] for a clever presentation of the Sobolevskiĭ approach, which is based on a semigroup approach that circumvents the appearance of high order norms of $z$ when estimating the nonlinearity in the classical higher order energy estimates. Comparing the semigroup approach initiated by Sobolevskiĭ to the classical energy estimate approach, we can say that, while the energy estimate for solutions is fundamental to prove that there is a global weak solution, every method has advantages and disadvantages. If we discuss the existence of a unique local strong solution, the semigroup method seems to be more powerful than the energy estimates and allows to achieve more regularity. To the best of our knowledge, previous papers on the stochastic N-S equations do not use this semigroup method to establish further regularity of the solution.

In Section I.4 we use the new path regularity result to study the problem of uniqueness and ergodicity of the invariant measure, which is known to exist if $\mathrm{Ran}(G) \subset D(A^{\frac{1}{4}+\varepsilon})$ for any $\varepsilon > 0$, *cf.* [Fla94]. To this aim, we also prove irreducibility and strong Feller properties for the Markov semigroup. We blend and adapt the reasoning followed in [FM95] and [Fer97] (see Remark I.4.2 *(ii)*), which managed to prove uniqueness of invariant measure and related ergodic properties under the stronger assumption $D(A^{\frac{1}{2}}) \subset \mathrm{Ran}(G) \subset D(A^{\frac{3}{8}+\varepsilon})$, for $\varepsilon \in (0, 1/8]$. This upper bound $\mathrm{Ran}(G) \subset D(A^{\frac{3}{8}+\varepsilon})$ appears also in more recent works, see for example [DX11; GM05]. We obtain uniqueness, ergodicity and a strong mixing property for the invariant measure in the hypothesis $D(A^{\frac{1}{2}}) \subset \mathrm{Ran}(G) \subset D(A^{\frac{1}{4}+\varepsilon})$ for $\varepsilon \in (0, 1/4]$. The case $\mathrm{Ran}(G) \subset D(A^{\frac{1}{2}})$ has been studied in [Fer99].

Concerning the main difficulties in proving existence and uniqueness of the invariant measure beyond the threshold $\mathrm{Ran}(G) \subset D(A^{\frac{1}{4}+\varepsilon})$, $\varepsilon > 0$, new regularity results for the solutions are undoubtedly required, see, for instance, the role such regularities play in Lemmas I.4.5, I.4.11, and I.4.12.

*Remark* I.1.1. To generalize the results presented in this paper to a more degenerate noise, with the goal of approaching the white-noise case, it may be worth to couple equation (I.1.1) with periodic boundary conditions. These conditions help isolate and highlight the limitations of the techniques currently available. The Cauchy problem in equation (I.1.2) can then be formulated within a functional framework analogous to the one described in section I.2.1, but adapted to incorporate the periodic boundary conditions. As a matter of fact, on the torus, much more can be achieved using complex Fourier analysis to explicitly characterize the eigenvalues and eigenfunctions of the Laplacian operator. This allows for more robust estimates on the Ornstein-Uhlenbeck process, which are essential for establishing results concerning the solution. For instance, uniqueness of solution holds with space-time white noise, see [Fer06, Theorem 10.1] (with the weak assumptions $x \in H$ and $G = A^{-\varepsilon}$, for any $\varepsilon > 0$), or the recent work [HR24], as well as the strong Feller property, see [ZZ17].

It is worth noting that [DD02; AF04] explore the case of an infinite-dimensional cylindrical white noise on the torus, while [FO19] considers a cylindrical fractional Wiener noise. In particular, we recall that [AF04] treats the stochastic N-S equations with space-time Gaussian white noise, having a Gaussian infinitesimal invariant measure $\mu_\nu$ whose covariance is given in terms of the enstrophy. Pathwise uniqueness for $\mu_\nu - a.e.$ initial velocity is proven for solutions having $\mu_\nu$ as invariant measure.

Eventually, sticking to the periodic setting, [HM06; Kup12] focus on the case where the operator $G$ has a finite-dimensional range, leading to a finite-dimensional stochastic noise, whereas [HM11] sets the 2D stochastic Navier-Stokes Equations on a sphere, with a finite-dimensional Brownian noise. We believe that the novel powerful approach by [HM06], based on the concept of asymptotic strong Feller property, could be adapted also to our situation, thanks to the new regularity results in Section I.3. This would allow us to obtain uniqueness and ergodicity of the invariant measure. However, once we have proved Theorem I.3.1, it seems more straightforward to follow the arguments in [FM95] and [Fer97] to derive results on the invariant measure, as shown in Theorem I.4.4. It should also be noted that the method in [HM06] does not yield mutual equivalence of the solution laws, nor the strong mixing property that we obtained.

## I.2 Preliminaries

### I.2.1 Functional setting

Let $\mathcal{D} \subset \mathbb{R}^2$ be a non-empty, bounded and connected open set with Lipschitz boundary, then we denote by $C_c^\infty(\mathcal{D}; \mathbb{R}^2)$ the linear space of smooth vector fields with compact support and define the following spaces:

$$\begin{aligned} H &:= \overline{\{u \in C_c^\infty(\mathcal{D}; \mathbb{R}^2) \mid \operatorname{div} u = 0\}}^{L^2(\mathcal{D};\mathbb{R}^2)}, \\ H^\alpha &:= \overline{\{u \in C_c^\infty(\mathcal{D}; \mathbb{R}^2) \mid \operatorname{div} u = 0\}}^{H^\alpha(\mathcal{D};\mathbb{R}^2)}, \qquad \forall\, \alpha > 0, \\ V &:= H^1. \end{aligned}$$

They inherit the Hilbert and embedding properties respectively from $L^2(\mathcal{D}; \mathbb{R}^2)$ and from the classical fractional Sobolev spaces $H_0^\alpha(\mathcal{D}; \mathbb{R}^2)$, see for instance [DPV12]. We identify $H$ with its topological dual $H'$ and introduce

$$H^{-\alpha} := (H^\alpha)', \qquad \forall\, \alpha > 0.$$

As a consequence, we have the following compact and dense embeddings for all $\alpha \in (0, 1)$ and $\beta > 1$:

$$H^\beta \hookrightarrow V = H^1 \hookrightarrow H^\alpha \hookrightarrow H = H' \hookrightarrow H^{-\alpha} \hookrightarrow V' = H^{-1} \hookrightarrow H^{-\beta}.$$

We denote the inner product in $H$ by

$$\langle x, y\rangle := \int_{\mathcal{D}} x \cdot y \,\mathrm{d}\mathscr{L}^2 = \sum_{i=1}^{2} \int_{\mathcal{D}} x_i y_i \,\mathrm{d}\mathscr{L}^2 \qquad \forall\, x, y \in H,$$

where we integrate with respect to the two-dimensional Lebesgue measure $\mathscr{L}^2$, we denote by $x_i, y_i$ for $i = 1, 2$ the scalar components of $x, y \in H \subset L^2(\mathcal{D}; \mathbb{R}^2)$ and by $\cdot$ the euclidean inner product in $\mathbb{R}^2$. The norm induced by $\langle\,\cdot\,,\,\cdot\,\rangle$ is denoted by $\|\cdot\|$, while we reserve the symbol $|\cdot|$ for the euclidean norm in $\mathbb{R}^2$. We use the symbol ${}_{V'}\langle\,\cdot\,,\,\cdot\,\rangle_V$ for the duality pairing between $V'$ and $V$. We denote the Hilbert norm on $V$ by

$$\|x\|_V^2 := \sum_{j=1}^{2} \|\partial_j x\|^2 = \sum_{i,j=1}^{2} \int_{\mathcal{D}} (\partial_j x_i)^2 \,\mathrm{d}\mathscr{L}^2 \qquad \forall\, x \in V,$$

where $\partial_j$ is the partial derivative (in the weak sense) with respect to the $j$-th variable in $\mathcal{D} \subset \mathbb{R}^2$.

### I.2.2 Operators

Since $H$ is a Hilbert subspace of $L^2(\mathcal{D};\mathbb{R}^2)$ we can construct an orthogonal projector operator, commonly called Leray projector and denoted by $\Pi : L^2(\mathcal{D};\mathbb{R}^2) \to H$. We define the Stokes operator as

$$A : H^2 \to H : x \mapsto \Pi \begin{pmatrix} -\Delta x_1 \\ -\Delta x_2 \end{pmatrix},$$

where $\Delta$ is the weak Laplacian operator on $H_0^2(\mathcal{D})$. Here $A$ is a linear, bounded, positive-definite, self-adjoint, invertible operator. Its inverse $A^{-1}$ is a compact operator on $H$, therefore there exists an orthonormal complete system $\{e_k\}_{k\in\mathbb{N}}$ in $H$ made of eigenvectors of $A$ and a strictly increasing and diverging sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}} \subset (0,+\infty)$ such that $Ae_k = \lambda_k e_k$ for all $k \in \mathbb{N}$. If we endow the vector space $H^2 = D(A)$ with the norm induced from $H$, then $-A : D(A) \subset H \to H$ is an unbounded, closed, densely defined and self-adjoint operator (see [FM77, Theorem 3]) which generates a one-parameter analytic semigroup of linear bounded operators $\{e^{-tA}\}_{t\geq 0}$ (see [Gig81, Theorem 2]).

If $\alpha > 0$, then we define the injective and bounded linear operator on $H$

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} e^{-tA} \,\mathrm{d}t,$$

where the integral converges in $\mathcal{L}(H)$, and $\Gamma$ is the usual Gamma function.

It is proved in [Mét78; Cae98] that, in 2-space dimensions, there exists $c > 0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{N}$, of $A$ are asymptotic to $ck$, as $k \to \infty$. Therefore, $A^{-\alpha}$ is positive and bounded for any $\alpha > 0$, Hilbert-Schmidt if $\alpha > 1/2$ and trace-class if $\alpha > 1$.

In addition, for $\alpha > 0$, we denote by $D(A^\alpha) \subset H$ the range of $A^{-\alpha}$ and by $A^\alpha : D(A^\alpha) \subset H \to H$ the unbounded inverse operator of $A^{-\alpha} : H \to D(A^\alpha)$, see [Paz83, Chapters 1, 2]. Then, [FM70, Theorem 1.1] shows that

$$D(A^\alpha) = H^{2\alpha}.$$

Moreover, the complete norm on $D(A^\alpha)$ given by

$$\|x\|_{D(A^\alpha)} := \|A^\alpha x\|, \qquad \forall\, x \in D(A^\alpha),$$

is equivalent to the usual Sobolev norm on $H_0^{2\alpha}(\mathcal{D};\mathbb{R}^2)$. Since $A^{-\alpha}$ is bounded and positive, it is also self-adjoint, hence its inverse $A^\alpha : D(A^\alpha) \subset H \to H$ is a self-adjoint unbounded operator, see [Tay96, Proposition 8.2]. Moreover $A^\alpha e^{-tA}$ is bounded for all $t > 0$ and the following property holds (see [FK64, Lemma 2.10])

$$\|A^\alpha e^{-tA}\|_{\mathcal{L}(H)} \leq \left(\frac{\alpha}{e}\right)^\alpha t^{-\alpha} \qquad \forall\, t > 0,\ \forall\, \alpha > 0. \tag{I.2.1}$$

From the properties of $A$ and the Hölder inequality, we derive the following lemma.

**Lemma I.2.1** (Interpolation inequality)**.** *For all $0 \leq p < q < +\infty$ and for all $u \in D(A^q)$ it holds*

$$\|A^r u\| \leq \|A^p u\|^\lambda \|A^q u\|^{1-\lambda} \qquad \forall\, \lambda \in (0,1),\ r := \lambda p + (1-\lambda) q.$$

Eventually, we define the function

$$B : C_c^\infty(\mathcal{D};\mathbb{R}^2) \to H : x \mapsto \Pi\big[(Dx)x\big] = \Pi\begin{pmatrix} x\cdot\nabla x_1 \\ x\cdot\nabla x_2\end{pmatrix},$$

which is commonly known as the Navier-Stokes non-linearity. Similarly, we introduce the trilinear bounded operator

$$b : V\times V\times V \to \mathbb{R} : (x,y,z) \mapsto \int_{\mathcal{D}} \begin{pmatrix} x\cdot\nabla y_1 \\ x\cdot\nabla y_2\end{pmatrix}\cdot z \,\mathrm{d}\mathscr{L}^2 = \sum_{i,j=1}^{2}\int_{\mathcal{D}} z_j\, x_i\, \partial_i y_j \,\mathrm{d}\mathscr{L}^2,$$

which is antisymmetric upon exchange of the second and third entries, thanks to integration by parts and the fact that vector fields in $V$ have vanishing divergence. The following result is taken from [Gig83, Lemma 3.2] and characterizes the extensions of $b$ and $B$.

**Lemma I.2.2.** *For all $\delta\in[0,1)$ and $\theta,\rho>0$ such that $\rho+\theta+\delta\geq 1$ and $\rho+\delta>1/2$, $b$ can be uniquely extended to a bounded trilinear operator and $B$ to a continuous function as follows*

$$b : D(A^\theta)\times D(A^\rho)\times D(A^\delta)\to\mathbb{R},$$
$$B : D(A^{\rho\vee\theta}) \to H^{-2\delta}.$$

*Moreover, there exists a constant $c_0 = c_0(\theta,\rho,\delta)>0$ such that*

$$|b(x,y,z)| \leq c_0\|A^\theta x\|\|A^\rho y\|\|A^\delta z\|, \qquad \|A^{-\delta}B(x)\| \leq c_0\|A^\rho x\|\|A^\theta x\|.$$

*Eventually, as soon as both members make sense, it holds $b(x,y,z) = -b(x,z,y)$.*

We observe in particular that $B$ allows to rewrite $b(x,x,z) = {}_{V'}\langle B(x), z\rangle_V$ for all $x\in D(A^{\frac14})$ and $z\in V$.

### I.2.3 Brownian noise

We consider an augmented filtered probability space $\big(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},\mathbb{P}\big)$, a sequence of mutually independent $\{\mathcal{F}_t\}_{t\geq0}$-adapted real Brownian motions $\{w^k\}_{k\in\mathbb{N}}$ and we introduce a coloured Wiener noise as

$$GW_t := \sum_{k\in\mathbb{N}} w_t^k G e_k \qquad \forall\, t\geq 0, \tag{I.2.2}$$

where $\{e_k\}_{k\in\mathbb{N}}$ are the eigenvectors of $A$ and $G\in\mathcal{L}(H)$ is an injective and bounded linear operator. We observe that if $G$ is Hilbert-Schmidt, $GW$ is an $H$-valued Wiener process, thus the series in equation (I.2.2) converges for all $t\geq 0$ in $L^2(\Omega;H)$ or $\mathbb{P}-a.s.$ in $C\big([0,T];H\big)$ for all $T>0$ (see [DZ14, Theorem 4.5]). However, our primary interest lies in more general noises: if $G$ is not Hilbert-Schmidt, $GW$ is defined as a generalized Wiener process (*cf.* [DZ14, Section 4.1.2]). The main assumptions on $G$ that will be used throughout the paper are the following

$$G\in\mathcal{L}(H),\quad G \text{ injective},\quad \exists\,\varepsilon>0 \quad \text{s.t.}\quad \mathrm{Ran}(G)\subset D(A^{\frac14+\varepsilon}), \tag{$H_0$}$$

$$G\in\mathcal{L}(H),\quad G \text{ injective},\quad \exists\,\varepsilon\in(0,1/4] \quad \text{s.t.}\quad V\subset\mathrm{Ran}(G)\subset D(A^{\frac14+\varepsilon}). \tag{$H_1$}$$

Clearly ($H_1$)$\Longrightarrow$($H_0$). Hypothesis ($H_0$) is the basic assumption found in literature (*cf.* [FM95; Fer97]) that guarantees uniqueness of generalized solution and existence of invariant measure for our equation (see Sections I.2.4 and I.2.5). Conversely, ($H_1$) is the required hypothesis for our setting.

We remark that $\langle GW_t, \phi\rangle = \sum_{k\in\mathbb{N}} w_t^k \langle Ge_k, \phi\rangle$ for all $t \geq 0$ and $\phi \in D(A)$, where the series converges for all $t \geq 0$ in $L^2(\Omega)$ or almost surely in $C([0,T])$ for all $T > 0$.

The following theorem defines and characterizes a stochastic process $\{Z_t\}_{t\geq 0}$, which will be often referred to as stochastic convolution or Ornstein-Uhlenbeck process starting at $0 \in H$.

**Theorem I.2.3.** *Let $\varepsilon, G$ satisfy assumption ($H_0$). There is a pathwise unique predictable process $\{Z_t\}_{t\geq 0}$ with $\mathbb{P}-a.s.$ trajectories $z \in L^1(0,T;H)$ for all $T > 0$ such that, for all $\phi \in D(A)$*

$$\langle z(t), \phi\rangle + \int_0^t \langle z(s), A\phi\rangle \,\mathrm{d}s = \langle GW_t, \phi\rangle \qquad \forall\, t \geq 0,\ \mathbb{P}-a.s.$$

*Moreover $z \in C^\beta\big([0,T]; D(A^\gamma)\big)$ for any $\beta > 0$ and $\gamma \geq 0$ such that $\beta + \gamma < (1/4 + \varepsilon) \wedge 1/2$, for all $T > 0$ and $\mathbb{P}-a.s.$*

*Proof.* For the existence and uniqueness result we refer to [DZ14, Theorem 5.4]. The second statement for $\varepsilon \in (0, 1/4]$ descends from [DZ14, Theorem 5.15], once checked that for any $\alpha < 1/4 + \varepsilon$, it holds

$$\int_0^1 t^{-2\alpha} \|e^{-tA}G\|^2_{\mathcal{L}_2(H)} \,\mathrm{d}t = \int_0^1 t^{-2\alpha}\, \mathrm{Tr}\left[e^{-tA}GG^* e^{-tA}\right] \mathrm{d}t < +\infty.$$

Let $p = 1/4 + \varepsilon$; we have

$$\begin{aligned}\int_0^1 t^{-2\alpha} \|e^{-tA}G\|^2_{\mathcal{L}_2(H)} \,\mathrm{d}t &= \int_0^1 t^{-2\alpha} \|e^{-tA}A^{-p}A^pG\|^2_{\mathcal{L}_2(H)} \,\mathrm{d}t \\ &\leq \|A^pG\|^2_{\mathcal{L}(H)} \int_0^1 t^{-2\alpha} \|e^{-tA}A^{-p}\|^2_{\mathcal{L}_2(H)} \,\mathrm{d}t,\end{aligned}$$

where we used the properties of the Hilbert-Schmidt norm and the fact that $A^pG \in \mathcal{L}(H)$, thanks to hypothesis ($H_0$). In order to compute the Hilbert-Schmidt norm of $e^{-tA}A^{-p}$, we resort to the orthonormal complete system $\{e_k\}_{k\in\mathbb{N}}$ for $H$ made of eigenvectors of $A$

$$\begin{aligned}\int_0^1 t^{-2\alpha} \|e^{-tA}A^{-p}\|^2_{\mathcal{L}_2(H)} \,\mathrm{d}t &= \sum_{k=1}^\infty \lambda_k^{-2p} \int_0^1 t^{-2\alpha} e^{-2t\lambda_k} \,\mathrm{d}t \\ &\leq \sum_{k=1}^\infty \lambda_k^{-2p-1+2\alpha} \int_0^{+\infty} e^{-2s} s^{-2\alpha} \,\mathrm{d}s.\end{aligned}$$

The time integral converges for all $\alpha < 1/2$. It is known (*cf.* [Mét78]) that the Stokes eigenvalues $\lambda_k$ in our two-dimensional setting are asymptotic to $ck$ for some constant $c > 0$, as $k \to \infty$. Consequently the series converges for all $\alpha < p = 1/4 + \varepsilon$. An application of [DZ14, Theorem 5.15] gives $Z$ path regularity $C^\beta\big([0,T]; D(A^\gamma)\big)$ for any $\beta + \gamma < \alpha$ for any $\alpha < 1/4 + \varepsilon$, thus also for any $\beta + \gamma < 1/4 + \varepsilon$. Finally, [DZ14, Section 5.4.2] shows that we can't choose $\gamma = 1/2$, thus preventing the continuity of the trajectories in $V = D(A^{\frac{1}{2}})$, even if $\varepsilon > 1/4$. □

*Remark* I.2.4. Theorem I.2.3 underscores the significance of the hypothesis ($H_0$) on the range of $G$. It sets our minimal regularity for the noise, thus affecting the solution of the Navier-Stokes Equations, as will be discussed in the next section. For $\mathrm{Ran}(G) \subset D(A^{\frac{1}{4}+\varepsilon})$ with $\varepsilon$ arbitrarily close to 0, we reach continuity in time with values in $D(A^{\frac{1}{4}})$. On the other hand, for $\mathrm{Ran}(G) \subset D(A^{\frac{1}{2}})$, $Z$ achieves the maximal regularity as a $\mathbb{P}-a.s.$ continuous process with values in $D(A^{\gamma})$ for $\gamma > 0$ arbitrarily close to $1/2$.

**Example I.2.5.** An example of operator $G$ that satisfies our assumptions ($H_1$) is given by

$$G = A^{-\gamma} L,$$

with $\gamma \in (1/4, 1/2]$ and $L$ an isomorphism in $H$. Another example is

$$Gx = \sum_{k\in\mathbb{N}} \frac{\langle x, e_k\rangle}{\sigma_k} e_k \qquad \forall\, x \in H,$$

where $ak^{1/4+\varepsilon} \le \sigma_k \le bk^{1/2}$ for some constants $a, b > 0$ and all $k \in \mathbb{N}$. Recall indeed that the eigenvalues $\lambda_k$ of the Stokes operator $A$ behave like $ck$, for some $c > 0$, as $k \to \infty$ (*cf.* [Mét78]).

### I.2.4 Abstract equation

We study the abstract stochastic 2D Navier-Stokes Equations in $H$, for some starting point $x \in H$ and some bounded linear operator $G$ satisfying assumption ($H_0$)

$$\begin{cases} \mathrm{d}X_t^x + [AX_t^x + B(X_t^x)]\,\mathrm{d}t = G\,\mathrm{d}W_t & t > 0,\ \mathbb{P}-a.s. \\ X_0^x = x & \mathbb{P}-a.s. \end{cases} . \tag{I.2.3}$$

We state the definition of solution we employ in the paper, which is taken from [Fla94].

**Definition I.2.6.** Given $x \in H$ and $G$ as in hypothesis ($H_0$), a generalized solution to equation (I.2.3) is a progressively measurable process $X^x$ in $H$ with path regularity

$$X^x(\omega) = u \in C\big([0,T]; H\big) \cap L^2\big(0,T; D(A^{\frac{1}{4}})\big) \qquad \forall\, T > 0,\ \mathbb{P}-a.s.\ \omega \in \Omega$$

such that

$$\langle u(t), \phi\rangle + \int_0^t \langle u(s), A\phi\rangle\,\mathrm{d}s = \int_0^t b\big(u(s), \phi, u(s)\big)\,\mathrm{d}s + \langle x, \phi\rangle + \langle GW_t, \phi\rangle$$
$$\forall\, t \ge 0,\ \forall\, \phi \in D(A),\ \mathbb{P}-a.s.$$

The following theorem summarizes the results in the literature regarding the existence, uniqueness, and path regularity of the generalized solution.

**Theorem I.2.7.** *For every $x \in H$ and $\varepsilon, G$ as in hypothesis ($H_0$), there exists a pathwise unique generalized solution to equation (I.2.3) (cf. Definition I.2.6) with trajectories $u$ that satisfy $\mathbb{P}-a.s.$ $u - z \in L^2(0,T;V)$ for all $T > 0$, where $z$ is given by Theorem I.2.3. Moreover this solution is a Markov process in $H$ and satisfies $\mathbb{P}-a.s.$ the following additional path regularities: $u \in L^2\big(0,T; D(A^{(\frac{1}{4}+\varepsilon)\wedge\frac{1}{2}}\big) \cap L^4\big(0,T; D(A^{\frac{1}{4}})\big)$ for all $T > 0$.*

*Proof.* The first assertions follow by [Fla94, Theorem 3.1]. In order to show the additional path regularity, we simply observe that the space $L^2(0,T;V)\cap C\big([0,T];H\big)$ is continuously embedded into $L^4\big(0,T;D(A^{\frac{1}{4}})\big)$. Indeed we have for a generic $v \in L^2(0,T;V)\cap C\big([0,T];H\big)$:

$$\begin{aligned}\int_0^T \|A^{\frac{1}{4}}v(t)\|^4 \,\mathrm{d}t &= \int_0^T \langle A^{\frac{1}{2}}v(t), v(t)\rangle^2 \,\mathrm{d}t \\ &\le \int_0^T \|v(t)\|_V^2\|v(t)\|^2 \,\mathrm{d}t \\ &\le \|v\|^2_{C([0,T];H)}\|v\|^2_{L^2(0,T;V)}.\end{aligned}$$

Therefore if $u$ is the unique solution by [Fla94, Theorem 3.1], then $u-z \in L^4\big(0,T;D(A^{\frac{1}{4}})\big)$ and consequently $u=(u-z)+z \in L^4\big(0,T;D(A^{\frac{1}{4}})\big)$, thanks to Theorem I.2.3. □

*Remark* I.2.8. (*i*) It is proved in [Fer03], that, under the assumption $(H_0)$, a wider class that assures a pathwise unique generalized solution is $C\big([0,T];H\big)\cap L^4\big(0,T;D(A^{\frac{1}{4}})\big)$, to which belongs our unique solution from Theorem I.2.7. The class of uniqueness from [Fer03] is indeed larger, because the author does not require the additional condition $u-z \in L^2(0,T;V)$, which is instead used in [Fla94].

(*ii*) The technique employed in [Fla94] to construct a generalized solution combines the general approach to stochastic partial differential equations with additive noise (see, for instance, [DZ14, Chapter 7]) with the classical energy estimates on the Galerkin approximations used for the deterministic case (see, for instance, [Tem01, Chapter 3]).

(*iii*) While for injective $G \in \mathcal{L}(H)$ with $\mathrm{Ran}(G) \subset D(A^{\frac{1}{2}+\varepsilon})$ for some $\varepsilon > 0$, the maximal regularity in space $u \in L^2(0,T;V)$ follows from the deterministic N-S equations, see [Tem01]; if $G$ is more degenerate, in the sense that $\mathrm{Ran}(G) \subset D(A^{\frac{1}{4}+\varepsilon})$ with $\varepsilon \in (0,1/4]$ (as in assumption $(H_0)$), then the space regularity $u \in L^2\big(0,T;D(A^{\frac{1}{4}+\varepsilon})\big)$ is inherited from the stochastic convolution $Z$, see Theorem II.2.17.
In other words, the wider is the range of $G$, the less regular are the trajectories of $Z$, thus resulting in worse path regularities for the generalized solution. Conversely, when the range of $G$ narrows (but remains not smaller than $V = D(A^{\frac{1}{2}})$), both $Z$ and $X$ exhibit better path regularities. Ultimately, if the range of $G$ is contained within $V = D(A^{\frac{1}{2}})$, the generalized solution achieves maximal path regularity.
Incidentally, under the stronger assumptions $\mathrm{Ran}(G) \subset D(A^{\frac{3}{8}+\varepsilon})$ with $\varepsilon > 0$ and $x \in D(A^{\frac{1}{4}})$, [FM95, Theorem 2.1] claims that $u \in C\big([0,T];D(A^{\frac{1}{4}})\big)\cap L^2\big(0,T;D(A^{\frac{3}{8}})\big)$.

*Remark* I.2.9. We give a complete summary of the problems of existence, uniqueness and additional regularities of generalized solutions, see Definition I.2.6, that we have found in the literature. There are essentially two kinds of approach.

On the one hand, under the hypotheses $x \in H$ and $\mathrm{Ran}(G) \subset D(A^{\frac{1}{4}+\varepsilon})$ for some $\varepsilon > 0$, [Fer03, Theorem 4.1] proves that a generalized solution exists and that it is pathwise unique in the class $C\big([0,T];H\big)\cap L^2\big(0,T;D(A^{(\frac{1}{4}+\varepsilon)\wedge\frac{1}{2}})\big)\cap L^4\big(0,T;D(A^{\frac{1}{4}})\big)$. Moreover, if $\alpha \in (0,1/2)$, $x \in D(A^\alpha)$, and $\mathrm{Ran}(G) \subset D(A^{\frac{1}{4}+\frac{\alpha}{2}+\varepsilon})$, for some $\varepsilon > 0$, then the solution enjoys the path regularity $C\big([0,T];D(A^\alpha)\big)\cap L^{\frac{4}{1-2\alpha}}\big(0,T;D(A^{\frac{1}{4}+\frac{\alpha}{2}})\big)$, *cf.* [Fer03, Theorem 4.2].

On the other hand, under the starting assumptions $x \in H$ and $\mathrm{Ran}(G) \subset D(A^{\frac{1}{4}+\varepsilon})$ for some $\varepsilon > 0$, [Fla94, Theorem 3.1 $(i), (ii)$], [FM95, Theorem 2.1 $(i), (ii)$] and [Fer97, Proposition 3.1], state the existence and uniqueness of a pathwise generalized solution satisfying the additional condition $u - z \in L^2(0, T; V)$, for $T > 0$, where $z$ is the Ornstein-Uhlenbeck process associated to the noise $G\,\mathrm{d}W$, see Section I.2.3. This kind of uniqueness property will be used throughout the paper as well. Moreover, according to [Fla94, Theorem 3.1 $(iv)$], if, in addition, $\mathrm{Ran}(G) \subset D(A^{\frac{1}{4}+\epsilon_0})$ for some $\epsilon_0 > 0$, and $x \in D(A^{\theta})$ for some $\theta \in (0, 2\epsilon_0) \cap (0, 1/2]$, then the trajectories $u$ of the generalized solution satisfy $u \in C\big([0,T]; D(A^{(\frac{1}{4}+\varepsilon)\wedge\theta})\big) \cap L^2\big(0,T; D(A^{(\frac{1}{4}+\varepsilon)\wedge(\frac{1}{2}+\theta)})\big)$, for all $\varepsilon < \epsilon_0$ and all $T > 0$. A particular case of this last assertion, for $\epsilon_0 > 1/8$, can be found in [FM95, Theorem 2.1], that we already recalled at the end of Remark I.2.8 *(iii)*. Eventually, if $\alpha \in [1/4, 1/2)$, $x \in D(A^{\alpha})$, and $D(A^{2\alpha}) \subset \mathrm{Ran}(G) \subset D(A^{\frac{1}{4}+\frac{\alpha}{2}+\varepsilon})$, for some $\varepsilon > 0$, then the solution has trajectories in $C\big([0,T]; D(A^{\alpha})\big) \cap L^{\frac{4}{1-2\alpha}}\big(0,T; D(A^{\frac{1}{4}+\frac{\alpha}{2}})\big)$, *cf.* [Fer97, Theorem 4.1].

### I.2.5 Invariant measure

Let $\mathcal{B}_b(H)$ be the linear space of all Borel and bounded functions $\varphi : H \to \mathbb{R}$, endowed with the complete norm $\|\varphi\|_\infty := \sup_{y\in H} |\varphi(y)|$. According to Theorem I.2.7, we can define the Markov semigroup $P = \{P_t\}_{t\geq 0}$ associated to the generalized solution of equation (I.2.3) as follows

$$P_t\varphi(x) := \mathbb{E}\,\varphi(X_t^x) \qquad \forall\, t \geq 0,\ \forall\, \varphi \in \mathcal{B}_b(H),\ \forall\, x \in H.$$

Let $\mathscr{P}(H)$ be the set of all probability measures over the Borel sigma-algebra $\mathscr{B}_H$ on $H$, then we can define

$$P_t^*\nu(U) := \int_H \mathbb{P}\big(X_t^x \in U\big)\,\nu(\mathrm{d}x) \qquad \forall\, t \geq 0,\ \forall\, \nu \in \mathscr{P}(H),\ \forall\, U \in \mathscr{B}_H,$$

and it is readily verified that $P_t^*\nu$ is again a probability measure on $\mathscr{B}_H$.

A probability measure $\mu \in \mathscr{P}(H)$ is said to be an invariant measure for the semigroup $\{P_t\}_{t\geq 0}$ if $P_t^*\mu = \mu$ for all times $t \geq 0$ (see also [BKP22] and the references therein). The following theorem collects known results from literature regarding existence and uniqueness of the invariant measure (see [Fla94, Theorem 3.3], [FM95, Theorem 3.1] and [Fer97, Corollary 4.1]).

**Theorem I.2.10.** *Under hypothesis ($H_0$) there exists an invariant measure $\mu$ for the Markov semigroup $\{P_t\}_{t\geq 0}$ associated with the generalized solution to equation (I.2.3). Moreover, under the stronger hypotheses*

$$G \in \mathcal{L}(H), \quad G \textit{ injective}, \quad \exists\, \varepsilon \in (0, 1/8] \quad \textit{s.t.} \quad V \subset \mathrm{Ran}(G) \subset D(A^{\frac{3}{8}+\varepsilon}), \tag{$H_2$}$$

*$\mu$ is known to be unique in the set $\mathscr{P}(H)$ and concentrated on the Borel set $D(A^{\frac{1}{4}}) \subset H$.*

Note that ($H_2$)$\Longrightarrow$($H_1$)$\Longrightarrow$($H_0$). In Section I.4 we will generalize the uniqueness and concentration results of this theorem to the weaker assumption ($H_1$).

## I.3 Main result

This section is devoted to the following main result regarding the additional path regularity we obtained for the unique generalized solution.

**Theorem I.3.1.** *Let $\varepsilon, G$ be as in hypothesis ($H_1$). For every $\gamma \in [0, 1/4+\varepsilon)$ and every starting point $x \in D(A^\gamma)$ the unique generalized solution to equation (I.2.3) from Theorem I.2.7 has the additional regularity*

$$u \in C\big([0,T]; D(A^\gamma)\big) \qquad \forall T > 0, \; \mathbb{P}-a.s.$$

*Moreover, if $x \in H$ then $u \in C\big([t_0,T]; D(A^\gamma)\big)$ for all $0 < t_0 < T$ and $\mathbb{P}-a.s.$*

The proof of this theorem employs the Sobolevskiĭ-Kato-Fujita method, which involves the mild formulation to the Navier-Stokes Equations (*cf.* [Sob59; KF62]). We outline the approach to be followed in the subsequent subsections.

1. First, we utilize a known technique to investigate abstract stochastic partial differential equations with additive noise (see, for instance, [DZ14, Chapter 7]). This involves fixing $\mathbb{P}-a.s.$ $\omega \in \Omega$ and formally introducing the equation satisfied by $v = u - z$, where $z$ is the fixed trajectory of the Ornstein-Uhlenbeck process (*cf.* Theorem I.2.3):

$$\begin{cases} v' + Av + B(v+z) = 0 & t > 0 \\ v(0) = x \end{cases} .$$

   We rigorously study this equation by means of finite dimensional approximations $v_n$ and obtain some *a priori* estimates (see Lemma I.3.3), similar to those found in [Fer97].

2. We rewrite $v_n$ through the mild formulation and obtain a new *a priori* bound in $L^\infty\big(0,T; D(A^{\frac{1}{4}})\big)$, arguing as in the Sobolevskiĭ-Kato-Fujita approach (slightly more spatial regularity will be obtained, see Lemma I.3.5).

3. This new estimate allows us to establish the continuity of $v$ through the Arzelà-Ascoli theorem. Subsequently, we define $u_n := v_n + z$ and infer convergence to the unique $u$ given in Theorem I.2.7 in appropriate function spaces (see Theorem I.3.6).

### I.3.1 Finite dimensional approximations for $v = u - z$

For all $n \in \mathbb{N}$ let $\Pi_n$ be the projector onto the finite dimensional subspace of $H$ generated by the first $n$ vectors of the complete orthonormal system $\{e_k\}_{k\in\mathbb{N}}$ (seee Section I.2.2). We denote $H_n = \Pi_n H$, $B_n : H_n \to H_n : x \mapsto \Pi_n B(x)$ and $x_n = \Pi_n x$, for any $x \in H$. Let $z$ be a $\mathbb{P}-a.s.$ fixed trajectory of the stochastic convolution (see Theorem I.2.3), then we study the following equation in finite dimensions over the time interval $[0,T]$ for an arbitrarily fixed $T > 0$:

$$\begin{cases} v_n' + Av_n + B_n\big(v_n + z\big) = 0 & t \in (0,T] \\ v_n(0) = x_n \end{cases} . \tag{I.3.1}$$

We know by the theory of ODEs that there exists a pathwise unique $v_n \in C\big([0,T];H_n\big)$ such that

$$v_n(t) + \int_0^t Av_n(s)\,\mathrm{d}s + \int_0^t B_n\big(v_n(s)+z(s)\big)\,\mathrm{d}s = x_n \qquad \forall\, t\in[0,T],\ \mathbb{P}-a.s. \quad \text{(I.3.2)}$$

We observe that in finite dimensions all the norms on $H_n$ are equivalent, thus $v_n(t) \in D(A^\alpha)$ for all $n\in\mathbb{N}$, $\alpha \geq 0$, $t\geq 0$ and $\mathbb{P}-a.s.$ Moreover, the process $v_n$ has almost surely smooth paths. Therefore the equations in system (I.3.1) are satisfied almost surely in probability and pointwise in time.

*Remark* I.3.2. Inspired by the classical reasoning in [Tem01], it is proved in [Fla94] that, under the assumption ($H_0$), and for any $x\in H$ and $T>0$, there exists a sub-sequence of $v_n$ converging weakly* in $L^\infty(0,T;H)$, weakly in $L^2(0,T;V)$ and strongly in $L^2(0,T;H)$ to a function $v\in C\big([0,T];H\big)\cap L^2(0,T;V)$ which is the pathwise unique solution to equation

$$\begin{cases} v' + Av + B(v+z) = 0 & t\in(0,T] \\ v(0) = x \end{cases} \tag{I.3.3}$$

in the following generalized sense $\mathbb{P}-a.s.$

$$\begin{gathered}\langle v(t),\phi\rangle + \int_0^t \langle v(s), A\phi\rangle\,\mathrm{d}s = \int_0^t b\big(v(s)+z(s),\phi,v(s)+z(s)\big)\,\mathrm{d}s + \langle x,\phi\rangle \\ \forall\,\phi\in D(A),\ \forall\, t\in[0,T].\end{gathered} \tag{I.3.4}$$

We recall that this result does not depend on the stochastic properties of the Ornstein-Uhlenbeck process, but only on its path regularity $z\in C\big([0,T];D(A^{\frac14})\big)$ for all $T>0$.

We henceforth replace $v_n$ with its converging subsequence, that we still denote as $v_n$. We are going to obtain some *a priori* estimates by adapting the calculations provided in the proof of [Fer97, Proposition 4.1].

**Lemma I.3.3.** *Let $\varepsilon, G$ be as in hypothesis ($H_1$). For all $T>0$ and $p\in\Big[4,\frac{4}{1-2\varepsilon}\Big)$ there exists a constant $c_1>0$ that depends on $\varepsilon, G, p, T,$ and $z$ such that*

$$\sup_{n\in\mathbb{N}} \int_0^T \|A^{1/4}v_n(t)\|^p\,\mathrm{d}t \leq c_1\big(1+\|A^\varepsilon x\|^{p-2}\big) \qquad \forall\, x\in D(A^\varepsilon),\ \mathbb{P}-a.s.$$

*Proof.* We rename $2\alpha = 1-4/p$ and we observe that the bounds on $p$ translate into the bounds $0\leq\alpha<\varepsilon$. Let us take the ODE in equation (I.3.1), which is satisfied almost surely in probability and pointwise in time, and take the scalar product in $H$ with $A^{2\alpha}v_n(t)$. By the sake of brevity we omit the dependence on $t$.

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|A^\alpha v_n\|^2 + \|A^{\alpha+\frac12}v_n\|^2 &= \langle v_n', A^{2\alpha}v_n\rangle + \langle Av_n, A^{2\alpha}v_n\rangle \\ &= -\langle A^{\alpha-\frac12}B_n(v_n+z), A^{\alpha+\frac12}v_n\rangle \\ &\leq c_0\|A^{\frac14+\frac{\alpha}{2}}(v_n+z)\|^2\|A^{\alpha+\frac12}v_n\| \\ &\leq 2c_0\Big(\|A^{\frac12}v_n\|\|A^\alpha v_n\| + \|A^{\frac14+\frac{\alpha}{2}}z\|^2\Big)\|A^{\alpha+\frac12}v_n\| \\ &\leq 4c_0^2\|A^\alpha v_n\|^2\|A^{\frac12}v_n\|^2 + 4c_0^2\|A^{\frac14+\frac{\alpha}{2}}z\|^4 + \frac12\|A^{\alpha+\frac12}v_n\|^2\end{aligned}$$

To perform the estimate, we first applied the Cauchy-Schwarz inequality together with Lemma I.2.2, with the choices $\delta = 1/2 - \alpha$, $\theta = \rho = 1/4 + \alpha/2$. Next we used the Young inequality and the interpolation inequality (*cf.* Lemma I.2.1) with the choices $q = 1/2, \lambda = 1/2, r = 1/4 + \alpha/2$. Finally, we applied the Young inequality again.

If we rewrite the first and last member we reach

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|A^\alpha v_n\|^2 &\leq \frac{\mathrm{d}}{\mathrm{d}t}\|A^\alpha v_n\|^2 + \|A^{\alpha+\frac{1}{2}}v_n\|^2 \\ &\leq 8c_0^2\|A^\alpha v_n\|^2\|A^{\frac{1}{2}}v_n\|^2 + 8c_0^2\|A^{\frac{1}{4}+\frac{\alpha}{2}}z\|^4.\end{aligned}$$

We integrate over the time interval $[0, t]$

$$\|A^\alpha v_n(t)\|^2 \leq \|A^\alpha x_n\|^2 + 8c_0^2\|A^{\frac{1}{4}+\frac{\alpha}{2}}z\|^4_{L^4(0,T;H)} + 8c_0^2\int_0^t \|A^\alpha v_n(s)\|^2\|A^{\frac{1}{2}}v_n(s)\|^2\,\mathrm{d}s,$$

and we apply Grönwall's lemma:

$$\|A^\alpha v_n(t)\|^2 \leq \Big(\|A^\alpha x_n\|^2 + 8c_0^2\|A^{\frac{1}{4}+\frac{\alpha}{2}}z\|^4_{L^4(0,T;H)}\Big)\exp\left[8c_0^2\int_0^t \|A^{\frac{1}{2}}v_n(s)\|^2\,\mathrm{d}s\right]. \tag{I.3.5}$$

Since $\alpha < \varepsilon$, as discussed at the beginning of the proof, we have $\|A^\alpha x_n\| = \|\Pi_n A^\alpha x\| \leq c\|A^\varepsilon x\|$. Moreover, Theorem I.2.3 implies that $\|A^{\frac{1}{4}+\frac{\alpha}{2}}z\|_{L^4(0,T;H)}$ is almost surely bounded. Eventually, we know that $v_n$ converges in $L^2(0,T;V)$ as $n$ approaches infinity to the function $v$ defined by equation (I.3.4) (*cf.* Remark I.3.2). Therefore $A^{\frac{1}{2}}v_n$ is uniformly bounded in $L^2(0,T;V)$. To sum up, estimate (I.3.5) results in the following *a priori* bound for $\|A^\alpha v_n(t)\|$, uniform both in $n$ and $t$:

$$\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}\|A^\alpha v_n(t)\| \leq C\big(1 + \|A^\varepsilon x\|\big) \qquad \mathbb{P}-a.s.$$

By means of the interpolation inequality (*cf.* Lemma I.2.1) with coefficients $q = 1/2$, $\lambda = 1 - 2/p$, $r = \lambda\alpha + (1-\lambda)q = 1/2 - 4/p^2$ we have

$$\begin{aligned}\|A^{\frac{1}{4}}v_n(t)\| &\leq c\|A^r v_n(t)\| \\ &\leq c\|A^\alpha v_n(t)\|^\lambda\|A^{\frac{1}{2}}v_n(t)\|^{1-\lambda} \\ &\leq K\big(1 + \|A^\varepsilon x\|\big)^\lambda\|A^{\frac{1}{2}}v_n(t)\|^{1-\lambda}.\end{aligned}$$

By raising to the power of $p = 2/(1-\lambda)$ and integrating in time we reach the sought *a priori* estimate, for a constant $c_1 > 0$ depending only on $\varepsilon, G, p$ and $T$. For all $n \in \mathbb{N}$ and $\mathbb{P} - a.s.$

$$\begin{aligned}\int_0^T \|A^{\frac{1}{4}}v_n(t)\|^p\,\mathrm{d}t &\leq K^p\big(1 + \|A^\varepsilon x\|\big)^{p-2}\int_0^T \|A^{\frac{1}{2}}v_n(t)\|^2\,\mathrm{d}t \\ &\leq c_1\big(1 + \|A^\varepsilon x\|^{p-2}\big).\end{aligned}$$

□

### I.3.2 Mild formulation

We now shift to the mild formulation, thanks to the following standard lemma.

**Lemma I.3.4.** *For any $n \in \mathbb{N}$ the stochastic process $v_n$ satisfies the integral formulation in equation (I.3.2) if and only if it satisfies the following mild formulation*

$$v_n(t) + \int_0^t e^{-(t-s)A} B_n\big(v_n(s) + z(s)\big)\,\mathrm{d}s = e^{-tA} x_n \qquad \forall\, t \geq 0,\ \mathbb{P} - a.s. \tag{I.3.6}$$

Note that the mild formulation has been already used to study the stochastic Navier-Stokes Equations (see, for instance, [DZ96, Chapter 15]). However, it seems that the Sobolevskiĭ-Kato-Fujita technique has not been applied before to the stochastic case. The following crucial lemma provides the new *a priori* estimate for $v_n$ using a method inspired by [Sob59] (see the introduction to [Kie80]).

**Lemma I.3.5.** *Let $\varepsilon, G$ be as in hypothesis ($H_1$). For all $T > 0$, $\gamma \in [1/4, 1/4 + \varepsilon)$ and $x \in D(A^\gamma)$ there exists a constant $c_2 > 0$ that depends on $\varepsilon, G, \gamma, \|A^\gamma x\|, T$ and $z$ such that*

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|A^\gamma v_n(t)\| \leq c_2 \qquad \mathbb{P} - a.s.$$

*The estimate is uniform for $x$ in bounded sets of $D(A^\gamma)$.*

*Proof. Step* 1. We start by estimating the $D(A^{\frac{1}{8}})$-norm of the integral term in equation (I.3.6), thanks to Lemma I.2.2 with the choices $\delta = 5/8, \rho = 1/4, \theta = 1/8$. We obtain, for a constant $C > 0$ depending only on $T$ and on the choices of $\varepsilon, G$

$$\begin{aligned} J &:= \left\| A^{\frac{1}{8}} \int_0^t e^{-(t-s)A} B_n\big(v_n(s) + z(s)\big)\,\mathrm{d}s \right\| \\ &= \left\| \int_0^t A^{\frac{6}{8}} e^{-(t-s)A} A^{-\frac{5}{8}} B_n\big(v_n(s) + z(s)\big)\,\mathrm{d}s \right\| \\ &\leq \int_0^t \left(\frac{3}{4e}\right)^{3/4} (t-s)^{-3/4} c_0 \big\|A^{\frac{1}{4}}\big(v_n(s)+z(s)\big)\big\| \big\|A^{\frac{1}{8}}\big(v_n(s)+z(s)\big)\big\|\,\mathrm{d}s \\ &\leq C \int_0^t (t-s)^{-3/4} \Big(\|A^{\frac{1}{4}} v_n(s)\| + 1\Big)\Big(\|A^{\frac{1}{8}} v_n(s)\| + 1\Big)\,\mathrm{d}s, \end{aligned}$$

where we used equation (I.2.1) and controlled uniformly in time $z$ thanks to Theorem I.2.3. We now use the Hölder inequality with exponent $p > 4$ and the respective $q = (1 - 1/p)^{-1} < 4/3$. Constants $C > 0$ hereafter may vary from line to line, yet they depend only on $\varepsilon, G, T, p$.

$$\begin{aligned} J &\leq C \left[\int_0^t (t-s)^{-3q/4}\,\mathrm{d}s\right]^{1/q} \left[\int_0^t \Big(\|A^{\frac{1}{4}} v_n(s)\| + 1\Big)^p \Big(\|A^{\frac{1}{8}} v_n(s)\| + 1\Big)^p \mathrm{d}s\right]^{1/p} \\ &\leq C \left[1 + \|A^\varepsilon x\|^{p-2} + \int_0^t \|A^{\frac{1}{8}} v_n(s)\|^p \Big(\|A^{\frac{1}{4}} v_n(s)\|^p + 1\Big)\,\mathrm{d}s\right]^{1/p}, \end{aligned}$$

where the first integral is finite because $-3q/4 > -1$ and we controlled the $L^p(0, T; H)$-norm of $A^{\frac{1}{4}} v_n$ thanks to Lemma I.3.3 (thus $p$ must be chosen such that $p < 4/(1 -$

$2\varepsilon)$). We now use this estimate into the mild formulation for $v_n$ (*cf.* equation (I.3.6)) to control $\|A^{\frac{1}{8}}v_n(t)\|$ $\mathbb{P}-a.s.$ and for every $t\in[0,T]$:

$$\begin{aligned}\|A^{\frac{1}{8}}v_n(t)\|^p &\leq 2^{p-1}\Big(\|e^{-tA}\Pi_n A^{\frac{1}{8}}x\|^p + J^p\Big)\\ &\leq C\Big[1+\|A^{\frac{1}{4}}x\|^{p-2}+\int_0^t \|A^{\frac{1}{8}}v_n(s)\|^p\Big(\|A^{\frac{1}{4}}v_n(s)\|^p+1\Big)\,\mathrm{d}s\Big].\end{aligned}$$

The Grönwall Lemma applied $\mathbb{P}-a.s.$ entails

$$\|A^{\frac{1}{8}}v_n(t)\|^p \leq C\big(1+\|A^{\frac{1}{4}}x\|^{p-2}\big)\exp\Big[c_0\int_0^T \|A^{\frac{1}{4}}v_n(s)\|^p\,\mathrm{d}s\Big].$$

We can apply again Lemma I.3.3 and we obtain the uniform bound in $L^\infty\big(0,T;D(A^{\frac{1}{8}})\big)$

$$\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}\|A^{\frac{1}{8}}v_n(t)\| \leq C\big(1+\|A^{\frac{1}{4}}x\|^{1-2/p}\big)\exp\big[C\|A^{\frac{1}{4}}x\|^{p-2}\big] \qquad \mathbb{P}-a.s. \tag{I.3.7}$$

*Step* 2. We now emulate the first step, but with the exponent $\gamma\in[1/4,1/4+\varepsilon)$ instead of $1/8$. We can apply Lemma I.2.2 with the choices $\delta=7/8-\gamma,\rho=\gamma,\theta=1/8$ for every $t\in[0,T]$ and $\mathbb{P}-a.s.$ to obtain

$$\begin{aligned}\|A^\gamma v_n(t)\| &\leq \|e^{-tA}\Pi_n A^\gamma x\| + \Big\|A^\gamma\int_0^t e^{-(t-s)A}B_n\big(v_n(s)+z(s)\big)\,\mathrm{d}s\Big\|\\ &\leq \|A^\gamma x\| + \Big\|\int_0^t A^{\frac{7}{8}}e^{-(t-s)A}A^{\gamma-\frac{7}{8}}B_n\big(v_n(s)+z(s)\big)\,\mathrm{d}s\Big\|\\ &\leq \|A^\gamma x\| + C\int_0^t (t-s)^{-7/8}\big\|A^\gamma\big(v_n(s)+z(s)\big)\big\|\,\big\|A^{\frac{1}{8}}\big(v_n(s)+z(s)\big)\big\|\,\mathrm{d}s\\ &\leq C\Big(1+\|A^\gamma x\|\exp\big[C\|A^\gamma x\|^{p-2}\big]\Big)\Big[1+\int_0^t (t-s)^{-7/8}\|A^\gamma v_n(s)\|\,\mathrm{d}s\Big],\end{aligned}$$

where in the last line we employed equation (I.3.7) and Theorem I.2.3. We resort now to the modified version of Grönwall's lemma A.3 to obtain

$$\sup_{t\in[0,T]}\|A^\gamma v_n(t)\| \leq C\Big(1+\|A^\gamma x\|\exp\big[C\|A^\gamma x\|^{p-2}\big]\Big) \qquad \forall n\in\mathbb{N},\ \mathbb{P}-a.s.$$

The assertion follows easily. □

### I.3.3 New path regularity

We can use Lemma I.3.5 to prove that the generalized solution to equation (I.2.3) characterized by Theorem I.2.7 has trajectories with higher regularity in space.

**Theorem I.3.6.** *Let $\varepsilon, G$ be as in hypothesis $(H_1)$. Let us take $\gamma\in[0,1/4+\varepsilon)$ and $x\in D(A^\gamma)$. Then the generalized solution $X^x$ to equation (I.2.3) from Theorem I.2.7 has $\mathbb{P}-a.s.$ paths in $C\big([0,T];D(A^\gamma)\big)$ for all $T>0$.*

*Proof. Step* 1. We take $\varepsilon, G, \gamma, x, T$ as in the hypotheses and fix $\omega\in\Omega$ $\mathbb{P}-a.s.$ We prove that a subsequence of $A^\gamma v_n$ (*cf.* equation (I.3.6)) converges in $C\big([0,T];H\big)$. As for the term involving the initial datum, we directly have that $e^{\cdot A}x_n$ converges to $e^{-\cdot A}x$ in $C\big([0,T];D(A^\gamma)\big)$ as $n\to\infty$.

As for the term with the non-linearity, we intend to apply the Arzelà-Ascoli theorem. If we rename $h_n : [0,T] \to H : t \mapsto \int_0^t e^{-(t-s)A} B_n\big(v_n(s) + z(s)\big)\, \mathrm{d}s$, we prove that $h_n$ is uniformly bounded in $C^{1/2-\gamma}\big([0,T]; D(A^\gamma)\big)$, which gives equicontinuity. In [Lun95, Section 2.2.1], the Banach space $D_A(\alpha, 1)$ is introduced for any $\alpha \in (0,1)$[1]. The result [Lun95, Proposition 2.2.15] proves that $D_A(\alpha,1)$ is continuously embedded into $D(A^\alpha)$, which directly implies that for any $\beta \in (0,1)$ the space of $\beta$-Hölder continuous functions $C^\beta\big([0,T]; D_A(\alpha,1)\big)$ is continuously embedded into $C^\beta\big([0,T]; D(A^\alpha)\big)$. We apply these results for $\alpha = 1/2 + \gamma \in [1/2, 3/4+\varepsilon) \subset (0,1)$. Moreover, [Lun95, Proposition 4.2.1] gives

$$\|A^{-\frac{1}{2}} h_n\|_{C^{1-\alpha}([0,T];D_A(\alpha,1))} \leq c \sup_{t\in[0,T]} \big\|A^{-\frac{1}{2}} B_n\big(v_n(t) + z(t)\big)\big\|,$$

for a constant $c > 0$ that depends only on $\alpha$ and $T$. This considerations lead to the following estimates, where $C = C(\varepsilon, \gamma, T) > 0$ possibly varies from line to line

$$\begin{aligned}
\|h_n\|_{C^{1/2-\gamma}([0,T];D(A^\gamma))} &= \|A^{-\frac{1}{2}} h_n\|_{C^{1-\alpha}([0,T];D(A^\alpha))} \\
&\leq C \|A^{-\frac{1}{2}} h_n\|_{C^{1-\alpha}([0,T];D_A(\alpha,1))} \\
&\leq C \sup_{t\in[0,T]} \big\|A^{-\frac{1}{2}} B_n\big(v_n(t) + z(t)\big)\big\| \\
&\leq C \Big[ \|A^{\frac{1}{4}} z\|^2_{C([0,T]:H)} + \sup_{t\in[0,T]} \|A^{\frac{1}{4}} v_n(t)\|^2 \Big]
\end{aligned}$$

where we used almost surely Lemma I.2.2 with $\delta = 1/2$ and $\rho = \theta = 1/4$. Finally Lemma I.3.5 gives the uniform estimate in $n$.

By arbitrariness of $\gamma$, we can consider $\gamma < \gamma' < 1/4 + \varepsilon$ and apply Lemma I.3.5 with $\gamma'$ instead of $\gamma$. Since $D(A^{\gamma'})$ is compactly embedded into $D(A^\gamma)$, we have that $h_n(t) = e^{-tA} x_n - v_n(t)$, at any fixed $t \in [0,T]$, lies in a compact set of $D(A^\gamma)$.

We thus apply the Arzelà-Ascoli theorem to the sequence $h_n$ and infer the existence of a sub-sequence converging in $C\big([0,T]; D(A^\gamma)\big)$ to a certain $h \in C\big([0,T]; D(A^\gamma)\big)$. Let us define $\underline{v} := e^{-\cdot A} x - h$, then a subsequence of $v_n$ converges to $\underline{v}$ in $C\big([0,T]; D(A^\gamma)\big)$. We already discussed in Remark I.3.2 that $v_n$ converges weakly* to $v$ in $L^\infty(0,T;H)$, thus $v = \underline{v} \in C\big([0,T]; D(A^\gamma)\big)$.

*Step* 2. We now define $u_n := v_n + z$ for all $n \in \mathbb{N}$ and $\mathbb{P} - a.s.$ By the previous step we obtain uniform convergence in time for $u_n$ to the function $u := v + z \in C\big([0,T]; D(A^\gamma)\big)$. This function satisfies our definition of generalized solution (*cf.* Definition I.2.6). Indeed, by recalling the equation (I.3.4) satisfied by $v$, we have for

[1] It consists of all $x \in H$ such that $s \mapsto s^{-\alpha}\|Ae^{-sA}x\|$ is in $L^1(0,1)$ and it is endowed with the complete norm $\|x\|_{D_A(\alpha,1)} := \|x\| + \int_0^1 s^{-\alpha}\|Ae^{-sA}x\|\, \mathrm{d}s$.

all $\phi \in D(A)$, for all times $t \geq 0$ and almost surely in probability

$$\begin{aligned}\langle u(t), \phi\rangle &= \langle v(t), \phi\rangle + \langle z(t), \phi\rangle \\ &= -\int_0^t \langle v(s), A\phi\rangle \,\mathrm{d}s + \int_0^t b\big(v(s)+z(s), \phi, v(s)+z(s)\big)\,\mathrm{d}s + \langle x, \phi\rangle \\ &\qquad + \langle z(t), \phi\rangle \\ &= -\int_0^t \langle u(s), A\phi\rangle \,\mathrm{d}s + \int_0^t \langle z(s), A\phi \,\mathrm{d}s\rangle + \int_0^t b\big(u(s), \phi, u(s)\big)\,\mathrm{d}s + \langle x, \phi\rangle \\ &\qquad + \langle z(t), \phi\rangle \\ &= -\int_0^t \langle u(s), A\phi\rangle \,\mathrm{d}s + \int_0^t b\big(u(s), \phi, u(s)\big)\,\mathrm{d}s + \langle x, \phi\rangle + \langle GW_t, \phi\rangle,\end{aligned}$$

where in the last line we employed Theorem I.2.3. By the uniqueness result expressed in Theorem I.2.7, it follows that the function $u$ we constructed by finite dimensional approximations coincides with the one in Theorem I.2.7. □

*Remark* I.3.7. It is worth noting that we only used so far the regularities for the trajectories $z$ of the stochastic convolution (*cf.* Theorem I.2.3), reasoning at $\omega \in \Omega$ fixed almost surely. Therefore the theses in Theorem I.3.6 still hold if we replace the stochastic convolution with a generic continuous deterministic function. More in detail: given $\gamma \in [0, 1/2)$, $x \in D(A^\gamma)$ and $z \in C\big([0,T]; D(A^\gamma)\big)$ such that $z(0) = 0$, the unique solution $v$ to equation (I.3.4) has regularity $C\big([0,T]; D(A^\gamma)\big)$.

**Proposition I.3.8.** *Let $\varepsilon, G$ be as in hypothesis ($H_1$). Given $x \in H$, the generalized solution to equation* (I.2.3) *from Theorem I.2.7 has $\mathbb{P} - a.s.$ trajectories in $C\big([t_0,T]; D(A^\gamma)\big)$ for all $0 < t_0 < T$ and all $\gamma \in [0, 1/4 + \varepsilon)$.*

*Proof.* Let us take $x \in H$ and $\varepsilon, G$ as in the hypotheses, then we know by Theorem I.2.7 that there exists a pathwise unique stochastic process $X^x$ with almost surely trajectories $u \in L^2\big(0,T; D(A^{\frac{1}{4}+\varepsilon})\big)$ for all $T > 0$. We take $\gamma \in [0, 1/4+\varepsilon)$, thus $u(t) \in D(A^\gamma)$ for almost every $t > 0$ and $\mathbb{P} - a.s.$ This allows us, once fixed $0 < t_0 < T$, to choose $t_1 \in (0, t_0)$ such that $u(t_1) \in D(A^\gamma)$ $\mathbb{P} - a.s.$, which can be chosen as a more regular starting point for equation (I.2.3). Its unique generalized solution was proved to have paths in $C\big([0,T]; D(A^\gamma)\big)$ almost surely: let us fix one of these trajectories, denoted by $w$, and its respective $\omega \in \Omega$. We now define for that $\omega \in \Omega$

$$\tilde{u}(t) = \begin{cases} u(t) & \text{if } t \in [0, t_1) \\ w(t - t_1) & \text{if } t \geq t_1 \end{cases},$$

then $\tilde{u}$ satisfies the equation and the regularities in Definition I.2.6 for that $\omega \in \Omega$ and with starting point $x \in H$, therefore it must coincide with the trajectory $u$ of $X^x$ by the uniqueness result in Theorem I.2.7. In particular we deduce that $t \mapsto u(t) = \tilde{u}(t) = w(t - t_1)$ is continuous and $D(A^\gamma)$-valued at every $t \geq t_1$, thus also $u \in C\big([t_0,T]; D(A^\gamma)\big)$. We conclude by arbitrariness of $\omega \in \Omega$ $\mathbb{P} - a.s.$ □

## I.4 Application to invariant measure

In this section we use the regularity result obtained in Section I.3 to prove the uniqueness and the related ergodic properties of the invariant measure $\mu \in \mathscr{P}(H)$

provided by Theorem I.2.10. To this purpose we will also prove the strong Feller and irreducibility properties for the Markov semigroup $\{P_t\}_{t\geq 0}$ (see Section I.2.5). We will always assume that the stochastic noise in equation (I.2.3) is given by a cylindrical Wiener process $W$ in $H$ regularized by a linear operator $G$ that satisfies hypothesis ($H_1$). We adapt the reasoning from [FM95; Fer97], where however the uniqueness of the invariant measure is proved only under the stronger hypothesis ($H_2$).

We start by recalling two main properties for the Markov semigroup associated to equation (I.2.3), the first of which is classical and the second was introduced in [FM95].

**Definition I.4.1.** Assume that $\alpha \geq 0$. The Markov semigroup $\{P_t\}_{t\geq 0}$ introduced in Section I.2.5

- is irreducible on $D(A^\alpha)$ if for every time $t > 0$, every point $x \in D(A^\alpha)$ and all open non-empty sets $U \subset D(A^\alpha)$
$$P_t^* \delta_x(U) > 0;$$
- enjoys the *(SF)* property on $D(A^\alpha) \hookrightarrow H$ if the following property holds. If $t > 0$, $\varphi \in \mathcal{B}_b\big(D(A^\alpha)\big)$, $x \in D(A^\alpha)$ and $\{x_n\}_{n\in\mathbb{N}} \subset D(A^\alpha)$ is a bounded sequence in $D(A^\alpha)$ converging to $x \in D(A^\alpha)$ with respect to the norm of $H$, then
$$\lim_{n\to\infty} P_t\varphi(x_n) = P_t\varphi(x).$$

*Remark* I.4.2. (*i*) First of all, observe that the $\sigma$-algebra on $D(A^\alpha)$, $\alpha > 0$, generated by the norm $\|\cdot\|_{D(A^\alpha)} = \|A^\alpha \cdot\|$ and denoted by $\mathscr{B}_{D(A^\alpha)}$ coincides with the one induced from $H$ and denoted by $\mathscr{B}_H \cap D(A^\alpha)$. In particular, we have that all Borel subsets of $D(A^\alpha)$ are Borel subsets of $H$. A proof of this statement can be found in appendix (*cf.* Lemma B.4).

(*ii*) A classical approach to proving uniqueness of the invariant measure for a Markov semigroup involves showing that the semigroup is both irreducible and strong Feller (*cf.* [DZ14, Section 11.2]). This methodology was applied in [Fer97] to our equation, with the stronger hypothesis ($H_2$), to establish uniqueness of $\mu$ within the set of probability measures over $\mathscr{B}_{D(A^{1/4})}$.
On the other hand, in [FM95] the authors introduced two modified versions of irreducibility and strong Feller property, denoted respectively by *(I)* and *(SF)*, which granted uniqueness of the invariant measure in $\mathscr{P}(H)$ under the hypothesis ($H_2$).
We have blended the two notions in Definition I.4.1, by making use of the classical irreducibility and the *(SF)* property from [FM95]. We will prove that they are sufficient, under the more general assumption ($H_1$) and thanks to the results in Section I.3, to establish the uniqueness of the invariant measure within probabilities on $H$ and its concentration on a suitable Borel subset.

(*iii*) It is straightforward that the *(SF)* property on $D(A^\alpha) \hookrightarrow H$ is stronger then the usual strong Feller property on $D(A^\alpha)$, which would prevent us from obtaining uniqueness of the invariant measure over all probabilities on $H$. It is worth mentioning that the *(SF)* property on $D(A^\alpha) \hookrightarrow H$ is however weaker then requiring the continuity of $P_t\varphi$ with respect to the norm of $H$ for all $t > 0$ and $\varphi \in \mathcal{B}_b\big(D(A^\alpha)\big)$. This would actually be sufficient for our goals, but we were not able to prove it.

We first present a lemma, adapted from [FM95, Theorem 4.1], which is auxiliary for our main result. This lemma does not depend on the fact that the Markov semigroup generates from our equation (I.2.3), thus we state it in a general form as follows.

**Lemma I.4.3.** *If a Markov semigroup $\{P_t\}_{t\geq 0}$ on $H$ is both irreducible in $D(A^\alpha)$ and enjoys the (SF) property on $D(A^\alpha) \hookrightarrow H$, for some $\alpha > 0$, then the transition probabilities $P_t^*\delta_x$ for $t > 0$ and $x \in D(A^\alpha)$ are mutually equivalent on $\mathscr{B}_{D(A^\alpha)}$.*

*Proof.* Let us assume that $P_t^*\delta_{x_0}(U) > 0$ for some $t > 0$, $x_0 \in D(A^\alpha)$ and $U \in \mathscr{B}_{D(A^\alpha)}$. We arbitrarily fix $x \in D(A^\alpha)$ and $s > 0$, then the thesis is proved once we show that $P_s^*\delta_x(U) > 0$.

First we suppose $s > t$. The *(SF)* property on $D(A^\alpha) \hookrightarrow H$ applied at time $s - t$ and at point $x_0$ assures that there are $\epsilon, M > 0$ such that $P_{s-t}\mathbb{1}_U(y) > 0$ for all $y \in B^H_\epsilon(x_0) \cap B^{D(A^\alpha)}_M := \{y \in D(A^\alpha) \ : \ \|A^\alpha y\| < M, \ \|y - x_0\| < \epsilon\}$. Moreover $P_t^*\delta_x > 0$ on open non-empty Borel sets in $D(A^\alpha)$ thanks to irreducibility in $D(A^\alpha)$. Resort to the Chapman-Kolmogorov equation to conclude:

$$P_s^*\delta_x(U) = \int_H P_{s-t}\mathbb{1}_U(y)\, P_t^*\delta_x(\mathrm{d}y) \geq \int\limits_{B^H_\epsilon(x_0)\cap B^{D(A^\alpha)}_M} P_{s-t}\mathbb{1}_U(y)\, P_t^*\delta_x(\mathrm{d}y) > 0.$$

Assume now $s \in (0, t]$ and choose $h \in (t - s, t)$ to write

$$0 < P_t^*\delta_{x_0}(U) = \int_H P_{t-h}\mathbb{1}_U(y)\, P_h^*\delta_{x_0}(\mathrm{d}y).$$

Therefore, by density of $D(A^\alpha)$ in $H$, there exists $y_0 \in D(A^\alpha)$ such that $P_{t-h}\mathbb{1}_U(y_0) > 0$ and so we can repeat word for word the first step of the proof by substituting $x_0$ with $y_0$ and $t$ with $t - h < s$. We conclude once again that $P_s^*\delta_x(U) > 0$, which completes the proof. □

We now state the main theorem of the section, which improves Theorem I.2.10.

**Theorem I.4.4.** *Let $P = \{P_t\}_{t\geq 0}$ denote the Markov semigroup in $H$ associated to the solution of equation (I.2.3) (cf. Section I.2.5) under the hypothesis ($H_1$). Let $\mu$ be the invariant measure of Theorem I.2.10. If $P$ is both irreducible on $D(A^{\frac{1}{4}})$ and enjoys the (SF) property on $D(A^{\frac{1}{4}}) \hookrightarrow H$, then*

*(i) $\mu$ is unique in the set $\mathscr{P}(H)$ and is concentrated on $D(A^\gamma)$ for all $\gamma < 1/4 + \varepsilon$,*

*(ii) $\mu$ is ergodic, i.e. for all $\varphi \in \mathcal{B}_b(H)$, it holds*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T P_t\varphi\,\mathrm{d}t = \int_H \varphi\,\mathrm{d}\mu,$$

*(iii) $\mu$ is equivalent to $P_t^*\delta_x$ for all $t > 0$ and $x \in H$,*

*(iv) for all $U \in \mathscr{B}_H$ and $x \in H$ it holds $\lim\limits_{t\to+\infty} P_t^*\delta_x(U) = \mu(U)$,*

*(v) $\mu$ is strongly mixing, i.e. for every $\nu \in \mathscr{P}(H)$ it holds $\lim\limits_{t\to+\infty} \|P_t^*\nu - \mu\|_{TV} = 0$.*

*Proof.* *Step* 1. We first apply the auxiliary Lemma I.4.3 with the choice $\alpha = 1/4$ to obtain that the transition probabilities $P_t^*\delta_x$ for $t > 0$ and $x \in D(A^{\frac{1}{4}})$ are mutually equivalent on $\mathscr{B}_{D(A^{1/4})}$.

*Step* 2. We now prove the equivalence on $\mathscr{B}_H$ of the transition probabilities $P_t^*\delta_x$ for $t > 0$ and $x \in H$. This reasoning is inspired by [FM95, Lemma 4.1], which however required the stronger assumption ($H_2$).

Let $t, s > 0$, $x_0, x \in H$ and $U \in \mathscr{B}_H$ and let us suppose that $P_t^*\delta_{x_0}(U) > 0$, then the thesis is proved if we show that $P_s^*\delta_x(U) > 0$. We use the Chapman-Kolmogorov equation with intermediate time $h \in (0, s \wedge t)$:

$$\begin{aligned} 0 < P_t^*\delta_{x_0}(U) &= \int_H P_{t-h}\mathbb{1}_U(y)\,\mathrm{d}\big(P_h^*\delta_{x_0}\big)(y) \\ &= \int_{D(A^{1/4})} P_{t-h}\mathbb{1}_U(y)\,\mathrm{d}\big(P_h^*\delta_{x_0}\big)(y) + \int_{H\setminus D(A^{1/4})} P_{t-h}\mathbb{1}_U(y)\,\mathrm{d}\big(P_h^*\delta_{x_0}\big)(y) \\ &= \int_{D(A^{1/4})} P_{t-h}\mathbb{1}_U(y)\,\mathrm{d}\big(P_h^*\delta_{x_0}\big)(y). \end{aligned} \tag{I.4.1}$$

The last equality holds because we know by Proposition I.3.8 that $X_h^{x_0} \in D(A^{\frac{1}{4}})$ $\mathbb{P}$-a.s., thus

$$\begin{aligned} 0 &\le \int_{H\setminus D(A^{1/4})} P_{t-h}\mathbb{1}_U(y)\,P_h^*\delta_{x_0}(\mathrm{d}y) \\ &\le P_h^*\delta_{x_0}\big(H \setminus D(A^{\frac{1}{4}})\big) \\ &= \mathbb{P}\big(X_h^{x_0} \in H \setminus D(A^{\frac{1}{4}})\big) = 0. \end{aligned}$$

We deduce by the strict inequality in equation (I.4.1) that there exists $y_0 \in D(A^{\frac{1}{4}})$ such that

$$\begin{aligned} 0 < P_{t-h}\mathbb{1}_U(y_0) &= \mathbb{P}\big(X_{t-h}^{y_0} \in U \cap D(A^{\frac{1}{4}})\big) + \mathbb{P}\big(X_{t-h}^{y_0} \in U \setminus D(A^{\frac{1}{4}})\big) \\ &= P_{t-h}^*\delta_{y_0}\big(U \cap D(A^{\frac{1}{4}})\big), \end{aligned}$$

where we used that $X_{t-h}^{y_0} \in D(A^{\frac{1}{4}})$ almost surely. We now resort to the hypothesis on the equivalence of transition probabilities over $\mathscr{B}_{D(A^{1/4})} = \mathscr{B}_H \cap D(A^{\frac{1}{4}})$ to infer that

$$P_{s-h}^*\delta_y\big(U \cap D(A^{\frac{1}{4}})\big) > 0 \qquad \forall\, y \in D(A^{\frac{1}{4}}). \tag{I.4.2}$$

Therefore, by the fact that $X_s^x, X_h^x \in D(A^{\frac{1}{4}})$ almost surely and the Chapman-Kolmogorov equation, we conclude that

$$\begin{aligned} P_s^*\delta_x(U) &= \mathbb{P}\big(X_s^x \in U \cap D(A^{\frac{1}{4}})\big) + \mathbb{P}\big(X_s^x \in U \setminus D(A^{\frac{1}{4}})\big) \\ &= P_s^*\delta_x\big(U \cap D(A^{\frac{1}{4}})\big) \\ &= \int_H P_{s-h}\mathbb{1}_{U\cap D(A^{1/4})}(y)\,P_h^*\delta_x(\mathrm{d}y) \\ &= \int_{D(A^{1/4})\cup\left(H\setminus D(A^{1/4})\right)} P_{s-h}\mathbb{1}_{U\cap D(A^{1/4})}(y)\,P_h^*\delta_x(\mathrm{d}y) \\ &= \int_{D(A^{1/4})} P_{s-h}\mathbb{1}_{U\cap D(A^{1/4})}(y)\,P_h^*\delta_x(\mathrm{d}y) > 0. \end{aligned}$$

If indeed the last strict inequality were false than we would have $P_{s-h}\mathbb{1}_{U\cap D(A^{1/4})}(y) = 0$ for almost every $y \in D(A^{\frac{1}{4}})$, which contradicts equation (I.4.2).

*Step* 3. Thanks to the previous step, Doob's Theorem implies *(ii)*, *(iii)*, *(iv)* and the uniqueness part in *(i)* (*cf.* [DZ14, Theorem 11.14]). Concentration of $\mu$ in $D(A^\gamma)$ for $\gamma > 0$ close to $1/4 + \varepsilon$ follows by *(iii)*, thanks to Proposition I.3.8. Part *(v)* is due to Seidler (*cf.* [Sei97, Proposition 2.5]). □

In what follows, it is proved that the Markov semigroup associated with the solution of equation (I.2.3) (see Section I.2.5) does indeed satisfy the hypotheses of Theorem I.4.4, thus reaching all the uniqueness and ergodic properties there listed.

### I.4.1 Irreducibility

If $E$ is a normed vector space, we denote with $C_0\big([0,T];E\big)$ the closed subspace of $C\big([0,T];E\big)$, with the induced norm, of all functions $z$ such that $z(0) = 0$. For every $x \in H$ and continuous $z : [0,+\infty) \to H$ with $z(0) = 0$, we study the deterministic abstract equation in $H$ given in system (I.3.3). Its solution is intended in the generalized sense, namely as a function $v \in C\big([0,T];H\big) \cap L^2(0,T;V)$ for all $T > 0$ that satisfies equation (I.3.4) in the deterministic frame.

We adapt the proof of [FM95, Lemma 5.3] thanks to the regularity we obtained in Theorem I.3.1.

**Lemma I.4.5.** *For any $\gamma \in (1/4, 1/2)$, $x \in D(A^\gamma)$ and every $T > 0$ the following map $\Phi = \Phi^{x,T,\gamma}$ is well-defined*

$$\Phi : C_0\big([0,T];D(A^\gamma)\big) \to C\big([0,T];D(A^{\frac{1}{4}})\big) : z \mapsto v + z,$$

*where $v$ is the unique solution in $C\big([0,T];H\big) \cap L^2(0,T;V)$ to system (I.3.3) in the generalized sense expressed by equation (I.3.4). Moreover:*

*(a) The map $\Phi$ is continuous with respect to the assigned topologies.*

*(b) For every $y \in D(A^\gamma)$ there exists $\bar{z} \in C_0\big([0,T];D(A^\gamma)\big)$ such that $\Phi(\bar{z})(T) = y$.*

*Proof.* For fixed $\gamma \in (1/4, 1/2)$, $x \in D(A^\gamma)$ and $T > 0$, the well-posedness of $\Phi$ follows from Remarks I.3.2 and I.3.7. As far as part (a) is concerned, we take an arbitrary $z_0 \in C_0\big([0,T];D(A^\gamma)\big)$ and prove continuity of $\Phi$ at $z_0$. We choose $z$ close to $z_0$ and we show that $\Phi(z)$ is close to $\Phi(z_0)$, in the respective norms.

We denote $u_0 = \Phi(z_0), v_0 = \Phi(z_0) - z_0$ and $u = \Phi(z), v = \Phi(z) - z$. We know by Remark I.3.7 that $u_0, v_0, u, v \in C\big([0,T];D(A^\gamma)\big)$, which guarantees that $B(u_0), B(u) \in C\big([0,T];V'\big)$ by Lemma I.2.2. We introduce for brevity $Z = z_0 - z \in C_0\big([0,T];D(A^\gamma)\big)$, $Y = v_0 - v \in L^2(0,T;V) \cap C\big([0,T];D(A^\gamma)\big)$ and $U = Y + Z = u_0 - u \in C\big([0,T];D(A^\gamma)\big)$. Then $Y$ satisfies for all $\phi \in D(A)$ and $t \geq 0$

$$\langle Y(t), \phi\rangle + \int_0^t \langle Y(s), A\phi\rangle \,\mathrm{d}s = \int_0^t b\big(u_0(s), \phi, u_0(s)\big)\,\mathrm{d}s - \int_0^t b\big(u(s), \phi, u(s)\big)\,\mathrm{d}s,$$

which implies, by the regularity of the terms involved

$$_{V'}\langle Y(t), \phi\rangle_V + \int_0^t {}_{V'}\langle AY(s), \phi\rangle_V \,\mathrm{d}s = \int_0^t {}_{V'}\big\langle B(u(s)) - B(u_0(s)), \phi\big\rangle_V \,\mathrm{d}s. \quad \text{(I.4.3)}$$

By density of $D(A)$ in $V$ and continuity of the duality pairing, equation (I.4.3) can be written for all $\phi \in V$, which implies by a standard result in functional analysis (*cf.* [Tem01, Chapter 3, Lemma 1.1]) that $Y \in H^1(0,T;V')$ and

$$Y'(t) + AY(t) = B(u(t)) - B(u_0(t)) \qquad a.e.\, t > 0,$$

the equality holding in $V'$. We take *a.e.* in time the pairing in $V'/V$ with $Y$ to obtain

$${}_{V'}\langle Y'(t), Y(t)\rangle_V + {}_{V'}\langle AY(t), Y(t)\rangle_V = {}_{V'}\big\langle B(u(t)) - B(u_0(t)), Y(t)\big\rangle_V \qquad a.e.\ t > 0. \tag{I.4.4}$$

We recall that (see [Tem01, Chapter 3, Lemma 1.2])

$$\int_0^t {}_{V'}\langle Y'(s), Y(s)\rangle_V \,\mathrm{d}s = \frac{1}{2}\|Y(t)\|^2 \qquad \forall\, t \geq 0$$

and that ${}_{V'}\langle AY(t), Y(t)\rangle_V = \|Y(t)\|_V^2$ for almost every $t > 0$. Moreover almost everywhere in time it holds (see Lemma I.2.2)

$${}_{V'}\big\langle B(u) - B(u_0), Y\big\rangle_V = b(u_0, Y, Y) - b(u_0, Z, Y) - b(U, u, Y).$$

The three terms in the right-hand side can be controlled as follows for some $\epsilon > 0$, thanks to the uniform bounds in time on $u, u_0$

$$\begin{aligned}
\big|b(u_0, Y, Y)\big| &\leq c_0\|A^{\frac14}u_0\|\|A^{\frac14}Y\|\|Y\|_V \leq C\|Y\|^{1/2}\|Y\|_V^{1/2}\|Y\|_V \leq \epsilon\|Y\|_V^2 + C_\epsilon\|Y\|^2\\
\big|b(u_0, Z, Y)\big| &\leq c_0\|A^{\frac14}u_0\|\|A^{\frac14}Z\|\|Y\|_V \leq \epsilon\|Y\|_V^2 + C_\epsilon\|A^\gamma Z\|^2\\
\big|b(U, u, Y)\big| &\leq c_0\|A^{\frac14}(Y+Z)\|\|A^{\frac14}u\|\|Y\|_V \leq C\big(\|Y\|_V^{1/2}\|Y\|_V^{1/2} + \|A^{\frac14}Z\|\big)\|Y\|_V\\
&\leq \epsilon\|Y\|_V^2 + C_\epsilon||Y||^2 + C_\epsilon\|A^\gamma Z\|^2.
\end{aligned}$$

Therefore if we integrate equation (I.4.4) on the time interval $[0,t]$, we obtain for all times

$$\frac{1}{2}\|Y(t)\|^2 + (1-3\epsilon)\int_0^t \|Y(s)\|_V^2 \,\mathrm{d}s \leq C_\epsilon\Big[\|Z\|^2_{C([0,T];D(A^\gamma))} + \int_0^t \|Y(s)\|^2 \,\mathrm{d}s\Big].$$

After choosing in the last equation $1 - 3\epsilon > 0$, Grönwall's lemma entails

$$\|Y(t)\| \leq C\|Z\|_{C([0,T];D(A^\gamma))}.$$

By means of the interpolation inequality (*cf.* Lemma I.2.1) one obtains, for $\lambda := 1 - \frac{1}{4\gamma}$

$$\|A^{\frac14}Y(t)\| \leq \|Y(t)\|^\lambda \, \|A^\gamma Y(t)\|^{1-\lambda} \leq C\|Z\|^\lambda_{C([0,T];D(A^\gamma))}.$$

We finally proved continuity at $z_0$ for $\Phi$:

$$\begin{aligned}
\|\Phi(z) - \Phi(z_0)\|_{C([0,T];D(A^{1/4}))} &= \sup_{t\in[0,T]} \|A^{\frac14}Y(t) + A^{\frac14}Z(t)\|\\
&\leq C\Big(\|z - z_0\|^\lambda_{C([0,T];D(A^\gamma))} + \|z - z_0\|_{C([0,T];D(A^\gamma))}\Big).
\end{aligned}$$

As far as part *(b)* is concerned, we follow the approach by [Fer97]. For fixed $T > 0$, starting point $x \in D(A^\gamma)$ and terminal point $y \in D(A^\gamma)$ we construct the following function $\bar{u}$, by choosing $0 < t_0 < t_1 < T$

$$\bar{u}(t) = \begin{cases} e^{-tA}x & \text{if } t \in [0, t_0]\\ e^{-(T-t)A}y & \text{if } t \in [t_1, T]\\ \bar{u}(t_0) + \dfrac{t - t_0}{t_1 - t_0}\big(\bar{u}(t_1) - \bar{u}(t_0)\big) & \text{if } t \in (t_0, t_1) \end{cases}.$$

By direct inspection, thanks to equation (I.2.1) (with $\alpha = 1/2 - \gamma$) and the bound $\gamma > 1/4$, we have $\bar{u} \in C\big([0,T]; D(A^\gamma)\big) \cap L^4(0,T; D(A^{\frac{1}{2}}))$. This in turn implies, by Lemma I.2.2 (choosing $\delta = 1/2 - \gamma$ and $\rho = \theta = 1/2$) and the bound $\gamma < 1/2$, that $B(\bar{u}) \in L^2\big(0,T; H^{2\gamma-1}\big)$. Define now $\bar{v}$ as the weak solution of the following linear differential equation, set in $D(A^\gamma)$

$$\begin{cases} \bar{v}' + A\bar{v} = -B(\bar{u}) \\ \bar{v}(0) = x \end{cases} .$$

We know by a standard result in linear parabolic PDEs (*cf.* [RR04, Section 11.1.2]) that there exists a unique $\bar{v} \in H^1\big([0,T]; H^{2\gamma-1}\big) \cap L^2\big(0,T; D(A^{\frac{1}{2}+\gamma})\big)$, hence $\bar{v} \in C\big([0,T]; D(A^\gamma)\big)$, such that $\bar{v}(0) = x$ and

$$\langle\!\langle \bar{v}'(t), \phi \rangle\!\rangle + \langle\!\langle A\bar{v}(t), \phi \rangle\!\rangle = -\big\langle\!\big\langle B(\bar{u}(t)), \phi \big\rangle\!\big\rangle \qquad \forall\, \phi \in D(A^{\frac{1}{2}-\gamma}),\ a.e.\ t > 0,$$

where we denote by $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$ the duality pairing between $H^{2\gamma-1}$ and $D(A^{\frac{1}{2}-\gamma})$. By limiting test functions to $\phi \in D(A) \subset D(A^{\frac{1}{2}-\gamma})$ and integrating over $[0,t] \subset [0,T]$, we reach for all $\phi \in D(A)$ and $t \in [0,T]$

$$\langle \bar{v}(t), \phi \rangle + \int_0^t \langle \bar{v}(s), A\phi \rangle \,\mathrm{d}s \tag{I.4.5}$$

Then we conclude that $\bar{z} := \bar{u} - \bar{v}$ satisfies the thesis of the theorem. Indeed equation (I.4.5) tells us that $\bar{v}$ satisfies in the generalized sense the equation

$$\begin{cases} v' + Av + B(v + \bar{z}) = 0 \\ v(0) = x \end{cases} ,$$

which is known to have a unique generalized solution as discussed at the beginning of the proof. Therefore we have $\Phi(\bar{z}) = \bar{z} + \bar{v} = \bar{u}$, which entails the thesis. $\square$

**Lemma I.4.6.** *Let $\varepsilon, G$ be as in hypothesis ($H_1$). For all $\gamma \in [0, 1/4 + \varepsilon)$, the law of $Z$ as a random variable*[2] *in $C_0\big([0,T]; D(A^\gamma)\big)$ is full, i.e. $\mathbb{P}(Z \in U) > 0$ for every open non-empty subset $U$ of $C_0\big([0,T]; D(A^\gamma)\big)$.*

*Proof.* This lemma is a direct application of [Mas93, Lemma 2.6] once checked that its hypotheses (denoted by $(A1)$ and $(X2)$) are satisfied in our case. Adopting the notations used in the cited reference, we choose $D(A^\gamma)$ as the space $E$, so that $\hat{E} = C_0\big([0,T]; D(A^\gamma)\big)$. Moreover $Q^0$ is the law of the random variable $Z : \Omega \to C_0\big([0,T]; D(A^\gamma)\big)$ and condition $(A1)$ (about $Z$ having a continuous version in $E$) is satisfied thanks to our Theorem I.2.3. Eventually the technical condition $(X2)$ is met thanks to [Mas93, Proposition 2.7], where $Q^{1/2} = G$ is a linear bounded and invertible operator from $H$ to $\mathrm{Ran}(G)$, which is densely and continuously embedded into $D(A^\gamma)$. The hypotheses of [Mas93, Lemma 2.6] are thus satisfied and the thesis states that the closure of $\mathrm{supp}(Q^0)$ in the topology of $\hat{E}$ coincides with $\hat{E}$. Let now $U$ be an open non-empty subset of $C_0\big([0,T]; D(A^\gamma)\big) = \hat{E}$, then the intersection

[2] The stochastic process $\{Z_t\}_{t\in[0,T]}$ can be seen as a random variable $\Omega \to C_0\big([0,T]; D(A^\gamma)\big)$ thanks to [DZ14, Proposition 3.18] and our Theorem I.2.3.

between $U$ and $S := \operatorname{supp}(Q^0)$ is non-empty [3]. Eventually by definition of support of a probability distribution [4] we have $\mathbb{P}(Z \in U) = Q^0(U) \geq Q^0(U \cap S) > 0$. □

**Proposition I.4.7.** *Let $\varepsilon, G$ be as in hypothesis ($H_1$). The Markov semigroup $\{P_t\}_{t\geq 0}$ (cf. Section I.2.5) is irreducible in $D(A^{\frac{1}{4}})$.*

*Proof.* Let us fix $T > 0$, $x \in D(A^{\frac{1}{4}})$ and $U$ open non-empty set in $D(A^{\frac{1}{4}})$, we prove that $P_T^* \delta_x(U) > 0$. Fix $\varepsilon$ and $G$ as in the hypothesis and take $\gamma \in (1/4, 1/4 + \varepsilon)$. By density of $D(A^\gamma)$ in $D(A^{\frac{1}{4}})$, we find $y \in D(A^\gamma)$ and $\delta > 0$ such that the open ball $B_\delta^{D(A^{1/4})}(y) := \{w \in D(A^{\frac{1}{4}}) : \|w - y\|_{D(A^{1/4})} < \delta\}$ is included in $U$. For that $y \in D(A^\gamma)$, part *(b)* of Lemma I.4.5 returns a $\bar{z} \in C_0\big([0,T]; D(A^\gamma)\big)$ such that $\Phi(\bar{z})(T) = y$ and $\mathbb{P} - a.s.$

$$\|u(T) - y\|_{D(A^{1/4})} = \|\Phi(Z)(T) - \Phi(\bar{z})(T)\|_{D(A^{1/4})} \leq \|\Phi(Z) - \Phi(\bar{z})\|_{C([0,T];D(A^{1/4}))}.$$

For the above $\delta > 0$, continuity of $\Phi$ at $\bar{z}$ (see part *(a)* of Lemma I.4.5) returns $\rho > 0$ such that, whenever $\|Z - \bar{z}\|_{C([0,T];D(A^\gamma))} < \rho$, then $\|\Phi(Z) - \Phi(\bar{z})\|_{C([0,T];D(A^{1/4}))} < \delta$. We conclude by monotonicity of probability and Lemma I.4.6

$$P_T^* \delta_x(U) \geq \mathbb{P}\big(\|u(T) - y\|_{D(A^{1/4})} < \delta\big) \geq \mathbb{P}\big(\|Z - \bar{z}\|_{C([0,T];D(A^\gamma))} < \rho\big) > 0.$$

□

*Remark* I.4.8. Irreducibility in $D(A^{\frac{1}{4}})$ implies irreducibility in $H$ for our semigroup. Let indeed $U$ be an open set in $H$, then $U \cap D(A^{\frac{1}{4}})$ is open in $D(A^{\frac{1}{4}})$, as discussed in Remark I.4.2 *(i)*. Moreover, if $x \in H$, then we proved in Proposition I.3.8 that for all $t > 0$ we have $X_t^x \in D(A^{\frac{1}{4}})$ almost surely. Therefore, given irreducibility in $D(A^{\frac{1}{4}})$, $x \in H$ and $t > 0$, we find $t_0 \in (0, t)$ such that $X_{t_0}^x = x_0 \in D(A^{\frac{1}{4}})$ almost surely, thus

$$P_t^* \delta_x(U) \geq P_t^* \delta_x\big(U \cap D(A^{\frac{1}{4}})\big) = P_{t-t_0}^* \delta_{x_0}\big(U \cap D(A^{\frac{1}{4}})\big) > 0.$$

### I.4.2 *(SF)* property

As far as the *(SF)* property is concerned, we proceed by finite dimensional approximations of a suitably modified version of our 2D stochastic Navier-Stokes Equation (I.2.3), similarly to [FM95; Fer97].

For any $n \in \mathbb{N}$ let $\Pi_n$ be the projector onto the subspace of $H$ generated by the eigenvectors $e_k$ of $A$ for $k \in \{1, \ldots, n\}$. For every $R \in (0, +\infty)$ let $\Theta_R : [0, +\infty) \to [0, 1]$ be a smooth function with compact support $[0, R+1]$ and equal to 1 on the interval $[0, R]$.

We study the following modified and approximated version of the stochastic 2D Navier-Stokes Equations, in which the function $B$ is smoothly truncated to 0 whenever the norm in $D(A^{\frac{1}{4}})$ of the solution exceeds the parameter $R$. For every positive

[3] If it were empty, we would reach a contradiction: since $U \cap S = \varnothing \iff S \subset U^c$, then $\hat{E} = \overline{S} \subset \overline{U^c} = U^c \iff U \subset \hat{E}^c = \varnothing$.

[4] The support of a probability distribution $Q^0$ on a topological space is the largest Borel set $S$ with the following property: if $U$ is an open set with non-empty intersection with $S$, then $Q^0(U \cap S) > 0$.

integer $n \in \mathbb{N}$ we denote with $W^n := \Pi_n W = \sum_{k=1}^n w^k e_k$ a finite dimensional standard Brownian motion in $H_n = \Pi_n H$, then for every real parameter $R > 0$, every injective $G \in \mathcal{L}(H)$ and every $x \in H_n$ we consider

$$\begin{cases} \mathrm{d}X_t^{R,n} + AX_t^{R,n}\,\mathrm{d}t + \Theta_R\big(\|A^{\frac{1}{4}}X_t^{R,n}\|^2\big)B_n\big(X_t^{R,n}\big)\,\mathrm{d}t = \Pi_n G\,\mathrm{d}W_t^n & t > 0, \\ X_0^{R,n} = x. \end{cases} \tag{I.4.6}$$

This is an autonomous finite dimensional stochastic differential equation, with an additive noise and a drift which is the sum of a linear term and a globally Lipschitz non-linearity. Therefore, according to the standard existence and uniqueness theorem for solutions to SDEs under regular and Lipschitz coefficients (refer to, for example, [Bal17, Theorems 9.2, 9.6]), there exists a pathwise unique Markov process $X^{R,n}$, which is a solution to the modified and approximated system (I.4.6). When we want to highlight the dependence on the initial datum, we use the notation $X^{R,n}(x)$. Let $P^{R,n}$ denote the Markov semigroup associated with this solution. It is known by classic theory (*cf.* [Cer01, Chapter 1]) that the function $H_n \to H_n : x \mapsto X_s^{R,n}(x)$ is mean-square differentiable, therefore, by means of the Bismut-Elworthy formula (*cf.* [EL94; Bis81]), we know that $P_t^{R,n}\varphi$ is differentiable at all times $t > 0$ for all $\varphi \in C_b(H_n)$ and that, for $x, h \in H_n$

$$\nabla P_t^{R,n}\varphi(x)\cdot h = \frac{1}{t}\mathbb{E}\left[\varphi\big(X_t^{R,n}(x)\big)\int_0^t (\Pi_n G G^*\Pi_n)^{-1/2} D_x\big[X_s^{R,n}(x)\big]h\cdot \mathrm{d}W_s^n\right]. \tag{I.4.7}$$

We denoted with $D_x[X_s^{R,n}(x)]h$ the row-by-column multiplication between the Jacobian matrix evaluated at $x$ of the function $H_n \to H_n : x \mapsto X_s^{R,n}(x)$ and the vector $h \in H_n$.

**Lemma I.4.9.** *For any $G$ as in hypothesis ($H_1$) and for all $R, t > 0$ there exists a constant $C_R(t) > 0$ such that, for all $n \in \mathbb{N}$*

$$\big|P_t^{R,n}\varphi(x) - P_t^{R,n}\varphi(y)\big| \leq C_R(t)\|\varphi\|_\infty\|x-y\| \qquad \forall\,\varphi \in C_b(H_n),\ \forall\,x,y \in H_n.$$

*Proof.* Let us fix $R, t > 0$, $n \in \mathbb{N}$ and take $\varphi : H_n \to \mathbb{R}$ bounded and continuous, then $P_t^{R,n}\varphi$ is differentiable at all times $t > 0$ by the Bismut-Elworthy formula. By the mean value theorem we have, for all $x, y \in H_n$ and $t > 0$

$$\begin{aligned} &\big|P_t^{R,n}\varphi(x) - P_t^{R,n}\varphi(y)\big| \\ &\leq \sup_{z\in H_n}\big|\nabla P_t^{R,n}\varphi(z)\cdot(x-y)\big| \\ &\leq \frac{1}{t}\|\varphi\|_\infty \sup_{z\in H_n}\left(\mathbb{E}\int_0^t \big\|(\Pi_n G G^*\Pi_n)^{-1/2} D_z\big[X_s^{R,n}(z)\big](x-y)\big\|^2\,\mathrm{d}s\right)^{1/2}, \end{aligned}$$

where we employed equation (I.4.7) with $h = x - y$, controlled the first factor inside the expectation via the boundedness of $\varphi$, and Itô's isometry. Since $V \subset \mathrm{Ran}(G)$ and by adapting known estimates from [FM95] (details are provided in the appendix, see Propositions B.1 and B.2), we get

$$\begin{aligned}
&\mathbb{E}\int_0^t \big\|(\Pi_n GG^*\Pi_n)^{-1/2} D_z\big[X_s^{R,n}(z)\big](x-y)\big\|^2\,\mathrm{d}s\\
&\le \mathbb{E}\int_0^t \big\|D_z\big[X_s^{R,n}(z)\big](x-y)\big\|_V^2\,\mathrm{d}s\\
&\le C_R(t)\|x-y\|^2,
\end{aligned}$$

for a certain positive constant depending only on $R$ and $t$. By inserting this result back into equation (I.4.2) we reach the thesis of the lemma for a possibly different constant $C_R(t)$. □

We study the following modified version of the stochastic Navier-Stokes Equations for fixed $R > 0$ and $x \in H$

$$\begin{cases} \mathrm{d}X_t^R + AX_t^R\,\mathrm{d}t + \Theta_R\big(\|A^{\frac14}X_t^R\|^2\big)B\big(X_t^R\big)\,\mathrm{d}t = G\,\mathrm{d}W_t & t > 0,\ \mathbb{P}-a.s.\\ X_0^R = x \qquad \mathbb{P}-a.s. \end{cases} \tag{I.4.8}$$

where the non linearity is continuously truncated to 0 as soon as the $D(A^{\frac14})$-norm of the solution exceeds the parameter $R$.

**Lemma I.4.10.** *The results presented in Section I.3 still hold true if we replace equation (I.2.3) with its modified version (I.4.8). In particular for all $G$ as in hypothesis ($H_1$) and for all $x \in D(A^{\frac14})$ we have $X^R \in C\big([0,T]; D(A^{\frac14})\big)$ for all $R, T > 0$ and $\mathbb{P}-a.s.$ Moreover a sub-sequence of $X^{R,n}$ converges to $X^R$ in $C\big([0,T]; D(A^{\frac14})\big)$ for all $T > 0$, $\mathbb{P}-a.s.$ and uniformly both in $R$ and in $x$, for $x$ in bounded sets of $D(A^{\frac14})$.*

*Proof.* We remark that the previous known results gathered in Theorem I.2.7 maintain their validity for the modified version of the Navier-Stokes Equations (see the appendix in [FM95]). In order to adapt the new results in Section I.3, we proceed by the finite dimensional approximations in equation (I.4.6). The two differences from Section I.3 lie in the presence of the truncating function $\Theta_R$ and in the finite approximation of the Wiener noise.

We observe that the proofs of Lemmas I.3.3 and I.3.5 are not modified if we add $\Theta_R$ in front of the non-linearity $B_n$ and replace $z$ by its finite dimensional approximation $z_n = \Pi_n z$. Indeed we can always employ the obvious estimates $\|\Theta_R\|_\infty = 1$ and $\|A^\alpha z_n(t)\| \le \|A^\alpha z(t)\|$ for all $t, \alpha \ge 0$ and $\mathbb{P}-a.s.$ Thus we obtain analogous *a priori* estimates for $v_n^R := X^{R,n} - z_n$.

In the first step from the proof of Theorem I.3.6, estimates for $v_n^R$ do not change if we replace $B_n$ with $\Theta_R B_n$ and $z$ with $z_n$. We conclude that a sub-sequence of $v_n^R$ converges in $C\big([0,T]; D(A^\gamma)\big)$.

As for the second step from the proof of Theorem I.3.6: we have $X^{R,n} = v_n^R + z_n$ and we need to prove that $z_n$ converges to $z$ in $C\big([0,T]; D(A^\gamma)\big)$. This fact is true for a sub-sequence, that we extract almost certainly through the Arzelà-Ascoli theorem. Indeed, the equi-continuity of the sequence derives from the regularities expressed in Theorem I.2.3 and the following estimate for any $\beta > 0$ with $\beta + \gamma < 1/4 + \varepsilon$

$$\|z_n(t) - z_n(s)\|_{D(A^\gamma)} \le \|z(t) - z(s)\|_{D(A^\gamma)} \le \|z\|_{C^\beta([0,T];D(A^\gamma))}|t-s|^\beta.$$

As for the relative compactness, we use the compact embedding $D(A^\alpha) \hookrightarrow D(A^\gamma)$ for $\gamma < \alpha < 1/4 + \varepsilon$ and the arbitrariness of $\gamma < 1/4 + \varepsilon$. □

Furthermore, it is classical to show that the solution to equation (I.4.8) is a Markov process (*cf.* [DZ96]), thus its Markov semigroup will be denoted by $\{P_t^R\}_{t\geq 0}$.

**Lemma I.4.11.** *For any $G$ as in hypothesis ($H_1$) and for all $R, t > 0$, there exists a constant $C_R(t)$ that satisfies*

$$\left|P_t^R\varphi(x) - P_t^R\varphi(y)\right| \leq C_R(t)\|\varphi\|_\infty\|x-y\| \qquad \forall\,\varphi \in C_b\big(D(A^{\frac{1}{4}})\big),\ \forall\, x, y \in D(A^{\frac{1}{4}}).$$

*Proof.* We fix $R > 0$ and $x, y \in D(A^{\frac{1}{4}})$, we denote by $X^{R,n}(x), X^{R,n}(y)$ for all $n \in \mathbb{N}$ and by $X^R(x), X^R(y)$ the solutions to equations (I.4.6) and (I.4.8) respectively, with starting point $x$ or $y$ respectively. By Lemma I.4.10 we have for all $\varphi \in C_b\big(D(A^{\frac{1}{4}})\big)$ and all $t \geq 0$

$$\varphi\big(X_t^{R,n}(x)\big) \longrightarrow \varphi\big(X_t^R(x)\big), \quad \varphi\big(X_t^{R,n}(y)\big) \longrightarrow \varphi\big(X_t^R(y)\big) \qquad \mathbb{P}-a.s. \text{ as } n \to \infty.$$

Therefore by the Dominated Convergence Theorem we have

$$\begin{cases} P_t^{R,n}\varphi(x) = \mathbb{E}\,\varphi\big(X^{R,n}(x)\big) \longrightarrow \mathbb{E}\,\varphi\big(X^R(x)\big) = P_t^R\varphi(x) \\ P_t^{R,n}\varphi(y) = \mathbb{E}\,\varphi\big(X^{R,n}(y)\big) \longrightarrow \mathbb{E}\,\varphi\big(X^R(y)\big) = P_t^R\varphi(y) \end{cases} \qquad \forall\, t \geq 0, \text{ as } n \to \infty.$$

Thus, for all $t \geq 0$

$$\begin{aligned} \left|P_t^R\varphi(x) - P_t^R\varphi(y)\right| &\leq \left|P_t^R\varphi(x) - P_t^{R,n}\varphi(x)\right| + \left|P_t^{R,n}\varphi(x) - P_t^{R,n}\varphi(y)\right| \\ &\qquad + \left|P_t^{R,n}\varphi(y) - P_t^R\varphi(y)\right| \\ &\leq \left|P_t^{R,n}\varphi(x) - P_t^{R,n}\varphi(y)\right| + o(1), \text{ as } n \to \infty. \end{aligned}$$

By use of the uniform estimate in $n$ provided by Lemma I.4.9, and taking the limit inferior as $n$ goes to infinity, we obtain

$$\left|P_t^R\varphi(x) - P_t^R\varphi(y)\right| \leq C_R(t)\|\varphi\|_\infty\|x-y\| \qquad \forall\, t \geq 0.$$

□

**Lemma I.4.12.** *For every $G$ as in hypothesis ($H_1$), for every bounded and countable set $U \subset D(A^{\frac{1}{4}})$ and for all times $t > 0$ it holds*

$$\lim_{R\to\infty} \sup_{x\in U} \|P_t^{R^*}\delta_x - P_t^*\delta_x\|_{TV} = 0.$$

*Proof.* Let us fix $G, U$ as in the hypothesis and take $x \in U$, $t > 0$ and $\varphi \in C_b\big(D(A^{\frac{1}{4}})\big)$ with $\|\varphi\|_\infty \leq 1$, then we need to prove that

$$\lim_{R\to\infty} |P_t^R\varphi(x) - P_t\varphi(x)| = 0$$

uniformly with respect to $x$ and $\varphi$. We define

$$\Lambda := \left\{\sup_{x\in U}\sup_{t\in[0,T]} \|A^{\frac{1}{4}}X_t^R(x) - A^{\frac{1}{4}}X_t(x)\|^2 > 0\right\} \subset \left\{\sup_{x\in U}\sup_{t\in[0,T]} \|A^{\frac{1}{4}}X_t^R(x)\|^2 > R\right\},$$

where the inclusion holds by definition of $\Theta_R$ and uniqueness of equation (I.4.8). Observe that $\Lambda$ is measurable in $\Omega$, thanks to the continuity of $X(x), X^R(x) : [0,T] \to$

$D(A^{\frac{1}{4}})$ and the fact that $U$ is countable. We also have by direct inspection that $\mathbb{1}_{\Lambda^c} X_t^R(x) = \mathbb{1}_{\Lambda^c} X_t(x)$ for all times and all $x \in U$. Therefore

$$\begin{aligned}
\sup_{x\in U} |P_t^R \varphi(x) - P_t\varphi(x)| &\le \sup_{x\in U} \mathbb{E}\Big[\big|\varphi(X_t^R(x)) - \varphi(X_t(x))\big|(\mathbb{1}_\Lambda + \mathbb{1}_{\Lambda^c})\Big] \\
&= \sup_{x\in U} \mathbb{E}\Big[\big|\varphi(X_t^R(x)) - \varphi(X_t(x))\big|\mathbb{1}_\Lambda\Big] \\
&\le 2\,\mathbb{P}(\Lambda) \\
&\le 2\,\mathbb{E}\left[\mathbb{1}_{(R,+\infty)}\left(\sup_{x\in U}\sup_{t\in[0,T]} \|A^{\frac{1}{4}} X_t^R(x)\|^2\right)\right].
\end{aligned}$$

We know by Lemma I.4.10 that $\|A^{\frac{1}{4}} X_t^R(x)\|$ is $\mathbb{P}-a.s.$ uniformly bounded both in time, in $R$ and in $x$ in bounded sets of $D(A^{\frac{1}{4}})$, thus we conclude by the Dominated Convergence Theorem

$$\sup_{x\in U} \|P_t^{R*}\delta_x - P_t^*\delta_x\|_{TV} \longrightarrow 0 \qquad \text{as } R\to\infty.$$

□

**Theorem I.4.13.** *For any $G$ as in hypothesis ($H_1$), the Markov semigroup $\{P_t\}_{t\ge 0}$ associated to the solution of equation* (I.2.3) *(cf. Section I.2.5) enjoys the (SF) property on $D(A^{\frac{1}{4}}) \hookrightarrow H$.*

*Proof.* By Lemma I.4.11 we have for all $R>0$, $t>0$ and $x,y\in D(A^{\frac{1}{4}})$:

$$\begin{aligned}
\big\|P_t^{R*}\delta_x - P_t^{R*}\delta_y\big\|_{TV} &:= \sup\left\{|P_t\varphi(x) - P_t^R\varphi(y)| : \varphi\in\mathcal{B}_b\big(D(A^{\frac{1}{4}})\big), \|\varphi\|_\infty \le 1\right\} \\
&= \sup\left\{|P_t\varphi(x) - P_t^R\varphi(y)| : \varphi\in C_b\big(D(A^{\frac{1}{4}})\big), \|\varphi\|_\infty \le 1\right\} \\
&\le C_R(t)\|x-y\|,
\end{aligned}$$

where we used the fact that the total variation norm can be equivalently defined through Borel or continuous bounded functions.

We now fix $t>0$, $x\in D(A^{\frac{1}{4}})$ and take an arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}$ bounded in $D(A^{\frac{1}{4}})$ and converging to $x\in D(A^{\frac{1}{4}})$ with respect to the norm of $H$. For any $\epsilon>0$, Lemma I.4.12 returns $R>0$ (which depends both on $\epsilon$ and on $\sup_{n\in\mathbb{N}}\|A^{\frac{1}{4}}x_n\|$) large enough such that

$$\begin{aligned}
|P_t\varphi(x) - P_t\varphi(x_n)| &\le |P_t\varphi(x) - P_t^R\varphi(x)| + |P_t^R\varphi(x) - P_t^R\varphi(x_n)| \\
&\qquad + |P_t^R\varphi(x_n) - P_t\varphi(x_n)| \\
&\le \Big[\|P_t^*\delta_x - P_t^{R*}\delta_x\|_{TV} + \|P_t^{R*}\delta_x - P_t^{R*}\delta_{x_n}\|_{TV} \\
&\qquad + \|P_t^*\delta_{x_n} - P_t^{R*}\delta_{x_n}\|_{TV}\Big]\|\varphi\|_\infty \\
&\le C_R(t)\|\varphi\|_\infty\|x-x_n\| + 2\|\varphi\|_\infty\epsilon.
\end{aligned}$$

By taking the limit inferior as $n$ goes to infinity we find out that

$$0 \le \liminf_{n\to\infty} |P_t\varphi(x) - P_t\varphi(x_n)| \le 2\|\varphi\|_\infty\epsilon,$$

which gives the thesis by arbitrariness of $\epsilon>0$. □

# Chapter II

# Inviscid Limit of the Stochastic Hyperviscous Navier-Stokes Equations and Invariant Measures for the Euler Equations in $\mathbb{R}^2$

## II.1 Introduction

The most famous equations in fluid dynamics, at least from a mathematical point of view, are the Navier-Stokes equations for homogeneous incompressible fluids

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u = f - \frac{1}{\rho}\nabla p, \\ \operatorname{div} u = 0. \end{cases}$$

They describe time and space evolution of the flow-velocity vector field $u$ and the pressure scalar field $p$ of any fluid, given constant parameters $\nu, \rho > 0$ and a vector field $f$. The parameter $\nu$ is the kinematic viscosity, $\rho$ is the density of the fluid, while $f$ is interpreted as an external force per unit of mass acting on the fluid and can be either deterministic or stochastic.

By formally setting the kinematic viscosity $\nu$ to zero, one obtains the Euler equations for homogeneous incompressible fluids

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = f - \frac{1}{\rho}\nabla p, \\ \operatorname{div} u = 0. \end{cases} \tag{II.1.1}$$

The main physical difference between the two systems is the following. The Navier-Stokes equations describe the motion of viscous Newtonian fluids, while the Euler equations predict the behaviour of inviscid fluids.

The problems of the well-posedness, the existence and the uniqueness of solutions for both equations have been widely investigated in literature and are completely solved in the two-dimensional (2D) case, see, for instance, [Wol33; Lad69; Tem01; Jud63; MB02]. Further interesting properties, both from a mathematical viewpoint and for the practical implications in understanding the long-term behaviour and statistical properties of turbulent flows, concern the invariant measures for the two equations. Formally speaking, invariant measures are spatial distributions of the fluid, which remain stationary as time evolves. Rigorous definitions will be given in Sections II.2 and II.5.

A classical method for proving the existence of invariant measures for partial differential equations, both in the deterministic and stochastic setting, is the Krylov-Bogoliubov method; see, for instance, [DZ14; DZ96; BG99]. It has been successfully applied to the stochastic Navier-Stokes equations, starting from the celebrated paper [Fla94]. Instead, concerning the deterministic Euler equations, earlier works have identified invariant measures, in the case of bounded domains and periodic boundary conditions, often employing the Gibbs measure [ARH79; AH89; AC90; Cip99; Bir06]. Kuksin's influential studies have demonstrated that stationary solutions for the Navier-Stokes equations with stochastic forcing converge to non-trivial stationary solutions of the deterministic Euler equations as viscosity vanishes and the noise vanishes as well [Kuk04; Kuk06; Kuk07; Kuk08].

The approach in this paper builds upon foundational works, including [Fer23] and [Lat23], which investigate the Eulerian limits and their connection to the stationary solutions of the 2D hypoviscous and hyperviscous Navier-Stokes equations as viscosity vanishes, in the case of bounded domains with periodic boundary conditions. Instead, we focus on the case of the unbounded domain $\mathbb{R}^2$, inspired by recent advances in stochastic partial differential equations (SPDEs) on unbounded domains, which involve the continuity of the solution flow with respect to weak topologies, a crucial aspect for handling the lack of compact embeddings in unbounded domains.

Specifically, we start from the stochastic hyperviscous 2D Navier-Stokes equations on the domain $\mathbb{R}^2$

$$\mathrm{d}X_t + \left[\nu A^\alpha X_t + B(X_t)\right] \mathrm{d}t = \beta_\nu \, \mathrm{d}W_t, \qquad \textit{for } t > 0, \tag{II.1.2}$$

where $\alpha > 1$, $W$ is a cylindrical Wiener noise, and $\beta_\nu > 0$ is a coefficient to be appropriately chosen. The higher-order dissipation terms given by the power $\alpha > 1$ of the Stokes operator regularise the solutions, and allow us to derive powerful *a priori* estimates. The scaling factor $\beta_\nu$ in front of the external forcing is of paramount importance. The following theorem gathers known results from the literature which clarify the role this factor plays in the deterministic setting on bounded domains.

**Theorem II.1.1** ([Tem95, Theorem 10.1]). *Assume that $\mathcal{D} \subset \mathbb{R}^2$ is a bounded domain with periodic or Dirichlet boundary conditions and let $V$ denote the space of divergence-free vector fields in $H^1(\mathcal{D};\mathbb{R}^2)$. If $\nu > 0$ and $f \in V'$, then there exists $u \in V$ such that*

$$\nu \, {}_{V'}\langle Au, v\rangle_V + b(u,u,v) = {}_{V'}\langle f, v\rangle_V. \tag{II.1.3}$$

*Moreover, if there exists $c > 0$ depending only on $\mathcal{D}$, such that*

$$\|f\|_{V'} < c\nu^2,$$

*then the $u \in V$ from above is unique.*

*Remark* II.1.2. The last theorem motivates the following claim: if the coefficient $\beta_\nu$ in equation (II.1.2) converges sufficiently rapidly to 0, as $\nu \to 0$, then the solutions of the Navier-Stokes equations vanish in the limit $\nu \to 0$ and converge to zero solution to the Euler equations. We will make this claim more precise.

Let us consider the setting of Theorem II.1.1. Assume that $\tilde{f} \in V'$. For all $\nu > 0$, let $\beta_\nu > 0$, and let $u^\nu \in V$ satisfy equation (II.1.3) with $f = \beta_\nu \tilde{f}$. Then, in the limit as $\nu \to 0$, the following statement hold.

- If $\beta_\nu = o(1)$, then $\nu u^\nu \longrightarrow 0$ in $V$. In particular $B(u^\nu) \longrightarrow 0$ in $V'$.
- If $\beta_\nu = o(\nu)$, then $u^\nu \longrightarrow 0$ in $V$.

Indeed, if we test the equation (II.1.3) with $f = \beta_\nu \tilde{f}$, against $u^\nu \in V$, by recalling that $b(u^\nu, u^\nu, u^\nu) = 0$,

$$\nu \|u^\nu\|_V^2 = \beta_\nu \, {}_{V'}\langle \tilde{f}, u^\nu \rangle_V \leq \beta_\nu \|\tilde{f}\|_{V'} \|u^\nu\|_V,$$

which implies

$$\nu \|u^\nu\|_V \leq \beta_\nu \|\tilde{f}\|_{V'},$$

from which the assertions follow.

Therefore, if we want to construct an invariant measure for the Euler equations by studying the inviscid limit of the Navier-Stokes equations, then we need $\beta_\nu$ to converge to 0 less rapidly than $\nu$.

*Remark* II.1.3. Let us now consider the setting of our equation (II.1.2) with $\alpha > 1$, and hint at the reason for our choice of the constant $\beta_\nu$, see Theorem II.5.12 for the details. Assume that $(H, \|\cdot\|)$ is the space of square-integrable divergence-free vector fields on $\mathbb{R}^2$. If we suppose that the $H$-valued process $X^\nu$ is a solution to the equation (II.1.2), then the Itô formula applied to the process $\|X^\nu\|^2$ gives

$$\mathbb{E}\|X_t^\nu\|^2 + 2\nu \mathbb{E} \int_0^t \|A^{\frac{\alpha}{2}} X_s^\nu\|^2 \, \mathrm{d}s = \mathbb{E}\|X_0^\nu\|^2 + \beta_\nu^2 C t, \qquad \forall\, t \geq 0,$$

for a constant $C > 0$ that depends on the Wiener process. If, in addition, $X^\nu$ is stationary, *i.e.* its law is an invariant measure, hence constant in time, then the first terms in both sides of the equality cancel out, and the integral in time is easily computed. We are left with

$$\nu\, t \left( 2\mathbb{E}\|A^{\frac{\alpha}{2}} X_0^\nu\|^2 - \frac{\beta_\nu^2}{\nu} C \right) = 0, \qquad \forall\, t \geq 0.$$

The last equality, together with the stationarity of the law, implies

$$\mathbb{E}\big\|A^{\frac{\alpha}{2}} X_s^\nu\big\|^2 = \mathbb{E}\big\|A^{\frac{\alpha}{2}} X_0^\nu\big\|^2 = \frac{\beta_\nu^2}{\nu} C, \qquad \forall\, s \geq 0,$$

which becomes a uniform estimate with respect to the kinematic viscosity $\nu$ if $\beta_\nu = \sqrt{\nu}$, and thus gives a powerful property to study the inviscid limit.

The last remark, inspired by [Lat23, Proposition 4.2], motivates the choice of the factor $\beta_\nu = \sqrt{\nu}$ in our equation (II.1.2).

Once the well-posedness of the equation (II.1.2) with $\beta_\nu := \sqrt{\nu}$ has been investigated, we establish the existence of invariant measures, by employing a version of the Krylov–Bogoliubov method tailored to weak topologies. This technique was introduced in [MS99], and successfully applied to the stochastic non-linear beam and wave equations [BOS16], the Navier-Stokes equations in unbounded domains [BMO17; BF19], the stochastic Landau-Lifshitz-Bloch equation [BGL20], the stochastic damped Euler Equation [BF20], and the stochastic non-linear and damped Schrödinger equation [BFZ24]. The lack of compactness has been addressed also in [BL06; BL04], where the authors established and Li established the existence of a compact absorbing set for the 2D stochastic Navier-Stokes equations with additive noise in a certain class of unbounded domains.

Later, we use the found invariant measure to construct stationary solutions to the stochastic hyperviscous 2D Navier-Stokes equations, and finally perform the inviscid limit by means of the Jakubowski's version [Jak97, Theorem 2] of the Skorokhod Theorem. To this end, we find a sufficiently large space in which the laws of the viscous sequence are tight. This space has also to be small enough so that, after using the Jakubowski theorem, the convergence in that space is strong enough for the sequences of viscous solutions, as well as some auxiliary processes, to be convergent. A similar technique has been applied for finite-dimensional approximations to SPDEs in [BO13; BM13; BMO17].

Finally, the process resulting from the inviscid limit is proved to be stationary and its law is the sought invariant measure for the Euler equations, which inherits the moment estimates valid for the approximating sequence.

The structure of the paper is as follows. In Section II.2, we present some notations and preliminaries on frequently used functional spaces, operators and stochastic processes. Section II.3 gathers the main results of the paper. Section II.4 formulates and solves the stochastic hyperviscous 2D Navier-Stokes equations, whose Markov property and invariant measure are later studied in Section II.5. In Section II.6, we rigorously pass the stochastic Navier-Stokes equations to the inviscid limit. Hence, we derive an invariant measure for the deterministic Euler equations. Finally, the appendices gather some ancillary results and technical lemmas used throughout the paper.

## II.2 Preliminaries

### II.2.1 Functional setting

**Notation II.2.1.** For $s \in \mathbb{R}$, we recall the definition of the Sobolev space

$$H^s(\mathbb{R}^d;\mathcal{H}) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^d;\mathcal{H}) \ : \ \big[x \mapsto (1+|x|^2)^{s/2}\mathscr{F}[f](x)\big] \in L^2(\mathbb{R}^d;\mathcal{H}) \Big\},$$

which is a Hilbert space if endowed with the inner product

$$\int_{\mathbb{R}^d} (1+|x|^2)^s \big\langle \mathscr{F}[f](x), \mathscr{F}[g](x) \big\rangle_{\mathcal{H}} \,\mathrm{d}x, \qquad \forall\, f, g \in H^s(\mathbb{R}^d;\mathcal{H}),$$

and with the induced norm $\|\cdot\|_{H^s(\mathbb{R}^d;\mathcal{H})}$.

A variant of the classical Sobolev spaces, tailored to deal with the equations of incompressible fluid dynamics, is given in the following definition.

**Definition II.2.2.** For $s \in \mathbb{R}$, we define the Sobolev space of divergence-free vector fields

$$H^s := \Big\{ u \in \mathcal{S}'(\mathbb{R}^2;\mathbb{R}^2) \ : \ \big[x \mapsto (1+|x|^2)^{s/2}\mathscr{F}[u](x)\big] \in L^2(\mathbb{R}^2;\mathbb{R}^2), \ \operatorname{div} u = 0 \Big\},$$

where $|x|$ and $x \cdot y$, for $x, y \in \mathbb{R}^2$, denote the standard Euclidean norm and inner product in $\mathbb{R}^2$. We recall that, for $u \in \mathcal{S}'(\mathbb{R}^2;\mathbb{R}^2)$, the distribution $\operatorname{div} u \in \mathcal{S}'(\mathbb{R}^2;\mathbb{R}^2)$ is defined as

$$\operatorname{div} u := i\mathscr{F}^{-1}\big[x \cdot \mathscr{F}[u](x)\big], \qquad \textit{for } x \in \mathbb{R}^2,$$

where $x \cdot \mathscr{F}[u](x)$ is the real-valued distribution given by the euclidean product in $\mathbb{R}^2$ between the smooth function $\mathbb{R}^2 \ni x \mapsto x \in \mathbb{R}^2$ and the distribution $\mathscr{F}[u] \in \mathcal{S}'(\mathbb{R}^2;\mathbb{R}^2)$. The linear space $H^s$ is endowed with the inner product

$$\langle u, v\rangle_{H^s} := \int_{\mathbb{R}^2} (1+|x|^2)^s \mathscr{F}[u](x) \cdot \mathscr{F}[v](x)\,\mathrm{d}x, \qquad \forall\, u, v \in H^s,$$

which makes $H^s$ a separable Hilbert space. The induced norm is denoted as $\|\cdot\|_{H^s}$.

**Notation II.2.3.** If $s > 0$, the topological dual space of $H^s$ is $H^{-s}$. The duality product is denoted by

$$_{H^{-s}}\langle u, v\rangle_{H^s} = \int_{\mathbb{R}^2} \mathscr{F}[u](x) \cdot \mathscr{F}[v](x)\,\mathrm{d}x, \qquad \forall\, u \in H^{-s},\ v \in H^s.$$

If $s = 0$ we use the following notation:

$$\begin{aligned} H &:= H^0, \\ \|u\| := \|u\|_{H^0} &= \left(\int_{\mathbb{R}^2} |u(x)|^2\,\mathrm{d}x\right)^{1/2}, \qquad \forall\, u \in H, \\ \langle u, v\rangle := \langle u, v\rangle_{H^0} &= \int_{\mathbb{R}^2} u(x) \cdot v(x)\,\mathrm{d}x, \qquad \forall\, u, v \in H, \end{aligned}$$

where the equalities hold thanks to the properties (0.3) and (0.4).

*Remark* II.2.4. If $u \in H$ and $v \in H^s$ for some $s > 0$, then, by direct inspection, since $H^s \hookrightarrow H \hookrightarrow H^{-s}$,

$$\langle u, v\rangle = {}_{H^{-s}}\langle u, v\rangle_{H^s}.$$

**Definition II.2.5.** The Hilbert space $H$ is a closed subspace of $H^0(\mathbb{R}^2;\mathbb{R}^2) = L^2(\mathbb{R}^2;\mathbb{R}^2)$, thus an orthogonal projection

$$\Pi : L^2(\mathbb{R}^2;\mathbb{R}^2) \to H,$$

called Leray projector, is well-defined, see *e.g.* [Bre10, Proposition 5.3, Corollary 5.4]. It is known that the operator $\Pi$ can be extended in such a way that, for any $s \in \mathbb{R}$, $\Pi\big(H^s(\mathbb{R}^2;\mathbb{R}^2)\big) = H^s$.

We derive the following trivial lemma from the definition of the divergence-free Sobolev spaces and the Hölder inequality.

**Lemma II.2.6** (Interpolation inequality)**.** *Assume that $p, q \in \mathbb{R}$ with $p < q$ and $\lambda \in (0,1)$. If $u \in H^q$, then*

$$\|u\|_{H^r} \le \|u\|_{H^p}^{\lambda} \|u\|_{H^q}^{1-\lambda}, \qquad r := \lambda p + (1-\lambda) q.$$

### II.2.2 Operators

The Laplacian on the Sobolev spaces of divergence-free vector fields is a paramount operator in fluid dynamics that has been extensively studied in the literature. We give here the definition of its powers, that will allow us to study the hyperviscous Navier-Stokes equations.

**Definition II.2.7.** For $\alpha \geq 1$, we define

$$A^\alpha u := \mathscr{F}^{-1}\big[x \mapsto (1+|x|^2)^\alpha \mathscr{F}[u]\big], \qquad \forall\, u \in \mathcal{S}'(\mathbb{R}^2;\mathbb{R}^2),$$

where $x \mapsto (1+|x|^2)^\alpha \mathscr{F}[u]$ is the $\mathbb{R}^2$-valued distribution given by the product between the smooth function $\mathbb{R}^2 \ni x \mapsto (1+|x|^2)^\alpha \in \mathbb{R}$ and the distribution $\mathscr{F}[u] \in \mathcal{S}'(\mathbb{R}^2;\mathbb{R}^2)$. If $s \in \mathbb{R}$, the operator $A := A^1 : H^{2+s} \to H^s$ is commonly called the Stokes operator. Whereas, for $\alpha > 1$, the operator $A^\alpha : H^{2\alpha+s} \to H^s$, for $s \in \mathbb{R}$, will be referred to as the hyperviscous Stokes operator.

*Remark* II.2.8. Assume that $s \in \mathbb{R}$ and $\alpha \geq 1$. We will show that the linear operator $A^\alpha : H^{2\alpha+s} \to H^s$ is well-defined.

Let $u \in H^{2\alpha+s}$, then $A^\alpha u \in H^s(\mathbb{R}^2;\mathbb{R}^2)$ by direct inspection. We only need to prove that $\operatorname{div}(A^\alpha u) = 0$ in the distributional sense. Indeed, we have in $\mathcal{S}'(\mathbb{R}^2;\mathbb{R}^2)$

$$\operatorname{div}(A^\alpha u) = i\mathscr{F}^{-1}\Big[x \cdot \mathscr{F}\big[A^\alpha u\big](x)\Big] = i\mathscr{F}^{-1}\Big[(1+|x|^2)^\alpha x \cdot \mathscr{F}[u](x)\Big],$$

where we used Definitions II.2.2, II.2.7. The right-hand side vanishes because $u \in H^{2\alpha+s}$ implies $\operatorname{div} u = 0$, which entails $x \cdot \mathscr{F}[u](x) = 0$ in the distributional sense in $\mathcal{S}'(\mathbb{R}^2;\mathbb{R}^2)$.

*Remark* II.2.9. For $s \in \mathbb{R}$ and $\alpha \geq 1$, the linear operator $A^\alpha : H^{2\alpha+s} \to H^s$ is positive definite, bounded, self-adjoint in $H^s$ and has spectrum $[1, +\infty)$. Moreover, given $u \in \mathcal{S}'(\mathbb{R}^2;\mathbb{R}^2)$, then

$$u \in H^{2\alpha+s} \qquad \Longleftrightarrow \qquad A^\alpha u \in H^s,$$

and,

$$\begin{aligned} &\|u\|_{H^{2\alpha+s}} = \|A^\alpha u\|_{H^s}, && \forall\, u \in H^{2\alpha+s}, \\ &\langle u, v\rangle_{H^{2\alpha+s}} = \langle A^\alpha u, A^\alpha v\rangle_{H^s}, && \forall\, u, v \in H^{2\alpha+s}. \end{aligned}$$

In addition, if $u \in H^{2+s}$, then by the properties of the Fourier transform,

$$Au = u - \Delta u \in H^s.$$

Eventually, if we introduce the normed subspace $D_{H^s}(A^\alpha) := \big(H^{2\alpha+s}, \|\cdot\|_{H^s}\big) \subset H^s$, then, assuming $\nu > 0$, the operator

$$-\nu A^\alpha : D_{H^s}(A^\alpha) \subset H^s \to H^s,$$

generates a contraction analytic semigroup $\big\{e^{-\nu t A^\alpha}\big\}_{t\geq 0}$ of linear bounded operators in $H^s$.

Eventually, one needs to formally introduce the non-linear term that appears in both the Navier-Stokes and the Euler equations.

**Lemma II.2.10.** *Assume that $p, q, r \geq 0$ satisfy the assumption*

$$\begin{cases} p+q+r \geq 1, & \textit{if } p,q,r \neq 1, \\ p+q+r > 1, & \textit{otherwise}, \end{cases}$$

*then the trilinear form*

$$b : H^p \times H^{q+1} \times H^r \ni (u, v, w) \mapsto \int_{\mathbb{R}^2} \begin{pmatrix} u \cdot \nabla v_1 \\ u \cdot \nabla v_2 \end{pmatrix} \cdot w \, \mathrm{d}\mathscr{L}^2 = \sum_{i,j=1}^{2} \int_{\mathbb{R}^2} w_j \, u_i \, \partial_i v_j \, \mathrm{d}\mathscr{L}^2 \in \mathbb{R},$$

*is well-defined and continuous, where we denoted by $\mathscr{L}^2$ the 2-dimensional Lebesgue measure on $\mathbb{R}^2$. Moreover, whenever the expressions make sense,*

$$b(u, v, w) = b(u, w, v), \tag{II.2.1}$$

$$b(u, v, v) = 0, \tag{II.2.2}$$

$$b(u, u, Au) = 0. \tag{II.2.3}$$

*Proof.* The well-posedness of the trilinear form comes from [Tem95, Lemma 2.1]. The two properties in equations (II.2.1), (II.2.2) follow by a density argument from [Tem95, equations (2.33), (2.34)].

The last property (II.2.3) is proved in [Tem95, Lemma 3.1] in the case of periodic boundary conditions, however, its proof can be adapted also to the case of the unbounded domain $\mathbb{R}^2$ as follows. Let us suppose that $u \in C_c^\infty(\mathbb{R}^2; \mathbb{R}^2)$, with $\operatorname{div} u = 0$, then by (II.2.2) and integration by parts

$$\begin{aligned} b(u, u, Au) &= b(u, u, u) - \sum_{i,j,k=1}^{2} \int_{\mathbb{R}^2} (\partial_k^2 u_j) u_i (\partial_i u_j) \, \mathrm{d}\mathscr{L}^2 \\ &= \sum_{i,j,k=1}^{2} \int_{\mathbb{R}^2} (\partial_k u_j)(\partial_k u_i)(\partial_i u_j) \, \mathrm{d}\mathscr{L}^2 + \sum_{i,j,k=1}^{2} \int_{\mathbb{R}^2} (\partial_k u_j) u_i \big(\partial_{ik}^2 u_j\big) \, \mathrm{d}\mathscr{L}^2. \end{aligned}$$

The first sum vanishes because a simple calculation shows that

$$\sum_{i,j,k=1}^{2} (\partial_k u_j)(\partial_k u_i)(\partial_i u_j) = (\operatorname{div} u) \left( \sum_{i,j=1}^{2} (\partial_i u_j)^2 \right) = 0.$$

The second sum vanishes because $(\partial_k u_j)\big(\partial_{ik}^2 u_j\big) = \partial_i\big((\partial_k u_j)^2\big)/2$, hence, by integration by parts

$$\begin{aligned} \sum_{i,j,k=1}^{2} \int_{\mathbb{R}^2} (\partial_k u_j)(\partial_k u_i)(\partial_i u_j) \, \mathrm{d}\mathscr{L}^2 &= \frac{1}{2} \sum_{i,j,k=1}^{2} \int_{\mathbb{R}^2} u_i \partial_i \big((\partial_k u_j)^2\big) \, \mathrm{d}\mathscr{L}^2 \\ &= -\frac{1}{2} \int_{\mathbb{R}^2} (\operatorname{div} u) \left( \sum_{j,k=1}^{2} (\partial_k u_j)^2 \right) \mathrm{d}\mathscr{L}^2 = 0. \end{aligned}$$

The general case is obtained by recalling that compactly supported smooth vector fields are dense in $H^\sigma$, for any $\sigma \in \mathbb{R}$. $\square$

**Definition II.2.11.** Let $p, q, r$ satisfy the assumption of Lemma II.2.10, then we define the bilinear continuous operator

$$B : H^p \times H^{q+1} \ni (u, v) \mapsto \Pi \begin{pmatrix} u \cdot \nabla v_1 \\ u \cdot \nabla v_2 \end{pmatrix} = \Pi \operatorname{div}(u \otimes v) \in H^{-r}.$$

With a slight abuse of notation, by $B$ we will also denote the corresponding quadratic map

$$B : H^{p \vee (q+1)} \ni u \mapsto B(u) := B(u, u) \in H^{-r},$$

called Navier-Stokes non-linearity.

In the following lemma we summarise some frequently used estimates on the operators $b$ and $B$.

**Lemma II.2.12.** *Assume that $\sigma > 2$ and $\varepsilon > 0$. There exists a finite constant $c > 0$ such that*

$$\begin{aligned}
&\|B(u,v)\|_{H^{\sigma-1}} \le c\|u\|_{H^{\sigma-1}}\|v\|_{H^\sigma}, && \forall\, u \in H^{\sigma-1},\ v \in H^\sigma, && \text{(II.2.4)}\\
&\|B(u,v)\| \le c\|u\|_{H^1}\|v\|_{H^{1+\varepsilon}}, && \forall\, u \in H^1,\ v \in H^{1+\varepsilon}, && \text{(II.2.5)}\\
&\|B(u,v)\|_{H^{-\varepsilon}} \le c\|u\|_{H^1}\|v\|_{H^1}, && \forall\, u \in H^1,\ v \in H^1, && \text{(II.2.6)}\\
&|b(u,v,w)| \le c\|u\|_{H^1}\|v\|_{H^1}\|w\|_{H^\varepsilon}, && \forall\, u \in H^1,\ v \in H^1,\ w \in H^\varepsilon. && \text{(II.2.7)}
\end{aligned}$$

*In particular, $B \in C(H^\sigma; H^{\sigma-1})$.*

*Proof.* The first estimate is taken from [KP86, Lemma 1.3]. Then the continuity of $B : H^\sigma \to H^{\sigma-1}$ follows directly from

$$\begin{aligned}
\|B(u) - B(y)\|_{H^{\sigma-1}} &= \|B(u-v,u) + B(v,u-v)\|_{H^{\sigma-1}}\\
&\le c\|u-v\|_{H^{\sigma-1}}\|u\|_{H^\sigma} + c\|v\|_{H^{\sigma-1}}\|u-v\|_{H^\sigma}\\
&\le c\big(\|u\|_{H^\sigma} + \|v\|_{H^\sigma}\big)\|u-v\|_{H^\sigma}.
\end{aligned}$$

In particular, $B : H^\sigma \to H^{\sigma-1}$ is Lipschitz continuous on bounded sets of $H^\sigma$. The second estimate comes from the continuity of the bilinear operator $B$, see Definition II.2.11 with $p = 1$, $q = \varepsilon$, $r = 0$. The third and fourth estimates follow from the continuity of the operators $b$ and $B$, see Definition II.2.11 and Lemma II.2.10 with $p = 1$, $q = 0$, $r = \varepsilon$. □

### II.2.3 The 2D incompressible Euler Equation

We project the Euler equations (II.1.1) on the space $H$ of divergence-free square-integrable vector fields, by means of the Leray operator, see II.2.5. This allows us to incorporate the incompressibility condition in the functional setting and to cancel out the gradient of the pressure. The equations thus reduce to the following (2D incompressible) Euler Equation

$$u' + B(u) = 0, \qquad \textit{on } (0, +\infty). \tag{$EE$}$$

We give the definition of a solution to this equation and recall an existence and uniqueness result from the literature.

**Definition II.2.13.** Assume that $\sigma > 2$. We say that a function $u \in C(\mathbb{R}_+; H^\sigma)$ is a solution to the Euler Equation ($EE$) if in $H^{\sigma-1}$

$$u(t) + \int_0^t B\big(u(s)\big)\,\mathrm{d}s = u(0), \qquad \forall\, t \ge 0. \tag{II.2.8}$$

*Remark* II.2.14. If $t > 0$ and $u \in C(\mathbb{R}_+; H^\sigma)$, then the integral in equation (II.2.8) converges in $H^{\sigma-1}$ by Lemma II.2.12, hence the equation (II.2.8) holds in $H^{\sigma-1}$. However, since both $u(t)$ and $u(0)$ belong to $H^\sigma$, the equation defines the integral in $H^\sigma$ and the equality is thus true in $H^\sigma$.

**Theorem II.2.15** ([KP86, Theorem III], [KP87, Theorem II]). *Assume that $\sigma > 2$. If $x \in H^\sigma$, then there exists a unique solution $u^x$ to the Euler Equation (EE) such that $u^x(0) = x$. Moreover, the Euler flow map*

$$\Phi : \mathbb{R}_+ \times H^\sigma \ni (t, x) \mapsto \Phi(t, x) := u^x(t) \in H^\sigma,$$

*is continuous.*

We aim to prove the existence of an invariant measure for the Euler Equation, whose definition is given below.

**Definition II.2.16.** Assume that $\sigma > 2$. We say that a probability measure $\mu \in \mathscr{P}(H^\sigma)$ is an invariant measure for the Euler Equation (EE) if

$$(\Phi_t)_*\mu = \mu \in \mathscr{P}(H^\sigma), \qquad \forall\, t \geq 0,$$

where, for $t \geq 0$,

$$\Phi_t : H^\sigma \ni x \mapsto \Phi_t(x) := \Phi(t, x) \in H^\sigma.$$

### II.2.4 The Ornstein-Uhlenbeck process

As explained in the introduction, the invariant measure for the Euler Equation (EE) will be obtained as the law of stationary solutions, which are constructed as the limit of stationary solutions for appropriate Navier-Stokes equations, as the kinematic viscosity tends to 0. In order to obtain good estimates for the invariant measures of the Navier-Stokes equation, *i.e.*, uniform with respect to the kinematic viscosity, we add a white noise to the equation that vanishes in the inviscid limit together with the diffusion term.

A usual tool to study the solution to stochastic partial differential equations with additive noise exploits the Ornstein-Uhlenbeck process, as briefly explained in Remark II.4.4, see [DZ14, Chapter 5] for a complete dissertation. Therefore, we now consider the equation that defines this stochastic process, adapted to our setting, and recall its main properties.

**Theorem II.2.17.** *Assume that $\beta \geq 0$ and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be an augmented filtered probability space with an adapted $H^\beta$-valued Wiener process $W$. Assume also that $\alpha > 1$ and $\nu > 0$. Then, there exists a pathwise unique $H^\beta$-valued predictable process $Z$ with regularity*

$$Z \in L^2\big(\Omega \times [0, T]; H^{\beta+\alpha}\big) \cap L^2\big(\Omega; C\big([0, T]; H^\beta\big)\big), \qquad \forall\, T > 0, \tag{II.2.9}$$

*that satisfies the weak formulation for the equation*

$$\begin{cases} \mathrm{d}Z_t + \nu A^\alpha Z_t \,\mathrm{d}t = \sqrt{\nu}\,\mathrm{d}W_t, & \text{for } t > 0, \\ Z_0 = 0, \end{cases} \tag{II.2.10}$$

*i.e. if $\phi \in H^{\beta+2\alpha}$, then $\mathbb{P} - a.s.$*

$$\langle Z_t, \phi\rangle_{H^\beta} + \nu \int_0^t \langle Z_s, A^\alpha \phi\rangle_{H^\beta} \,\mathrm{d}s = \sqrt{\nu}\langle W_t, \phi\rangle_{H^\beta}, \qquad \forall\, t \geq 0.$$

*Furthermore, $Z$ satisfies the mild formulation for equation (II.2.10), i.e. $\mathbb{P}-a.s.$ in $H^\beta$*

$$Z_t = \sqrt{\nu}\int_0^t e^{-\nu(t-s)A^\alpha}\,\mathrm{d}W_s, \qquad \forall\, t \geq 0, \tag{II.2.11}$$

*and the strong formulation, i.e. $\mathbb{P}-a.s.$ in $H^{\beta-\alpha}$*

$$Z_t + \nu\int_0^t A^\alpha Z_s\,\mathrm{d}s = \sqrt{\nu}\,W_t, \qquad \forall\, t \geq 0, \tag{II.2.12}$$

*where a continuous modification for both integral processes in equations (II.2.11), (II.2.12) is employed.*

*Proof.* All the assertions follow from the general theory for linear equations with additive noise from [DZ14, Chapter 5].

Adapting the notations from [DZ14, Section 5.1.1], we see that the abstract Hilbert spaces $U$ and $H$ in the reference are both supposed to be the Sobolev-type space $H^\beta$, and that the operator $B$ is, in our setting, simply $\sqrt{\nu}\,\mathbb{I}_{H^\beta}$. In addition, the abstract linear operator $A$ is replaced by the hyperviscous Stokes operator $-\nu A^\alpha$, which generates a contraction analytic semigroup $\left\{e^{-\nu t A^\alpha}\right\}_{t\geq 0}$ of linear bounded operators in $H^\beta$. The deterministic function $f$, and the initial random variable $\xi$ are both chosen to be 0. Finally, we work under the assumption of the $H^\beta$-valued Wiener process $W$, whose reproducing kernel Hilbert space and covariance operator will be denoted by $U_0$ and $Q_\beta$ respectively. We recall that $Q_\beta \in \mathcal{L}_1(H^\beta)$, and that $U_0 = Q_\beta^{1/2}(H^\beta)$, hence $U_0$ is Hilbert-Schmidt embedded into $H^\beta$.

The pathwise uniqueness result for the weak formulation and the validity of the mild formulation (II.2.11), $\mathbb{P}-a.s.$ in $H^\beta$, follow from [DZ14, Theorem 5.4], while the pathwise continuity in $H^\beta$ is stated in [DZ14, Theorem 5.11]. In order to apply these theorems, the following condition has to be verified: there exists $\gamma \in (0, 1/2)$ such that

$$\int_0^1 t^{-2\gamma}\,\mathrm{Tr}\left[e^{-2\nu t A^\alpha} Q_\beta\right]\mathrm{d}t < +\infty.$$

Indeed, it is sufficient to notice that

$$\mathrm{Tr}\left[e^{-2\nu t A^\alpha} Q_\beta\right] \leq \left\|e^{-2\nu t A^\alpha}\right\|_{\mathcal{L}(H^\beta)} \mathrm{Tr}[Q_\beta] \leq \mathrm{Tr}[Q_\beta] < +\infty,$$

because $Q_\beta \in \mathcal{L}_1(H^\beta)$ and the semigroup is contractive.

As far as the strong formulation in equation (II.2.12) is concerned, we refer to [DZ14, Section 5.6]. We make the following remarks to show that our situation can be reduced to that of [DZ14, Section 5.6]. If $\iota : H^\beta \to H^{-2\alpha}$ denotes the natural embedding under the Riesz identification $H = H'$, then $\iota W$ is an $H^{-2\alpha}$-valued Wiener process with covariance operator $Q := \iota Q_\beta \iota^* \in \mathcal{L}_1\left(H^{-2\alpha}\right)$ and with the same reproducing kernel Hilbert space $Q^{1/2}\left(H^{-2\alpha}\right) = Q_\beta^{1/2}(H^\beta) =: U_0$ as $W$. Hence, the abstract Hilbert space $H$ from [DZ14, Section 5.6] coincides with the Sobolev-type Hilbert space $H^{-2\alpha}$. We also need to set the hyperviscous Stokes operator on $H^{-2\alpha}$, that is to say

$$A^\alpha : D_{H^{-2\alpha}}(A^\alpha) = H \subset H^{-2\alpha} \to H^{-2\alpha}.$$

With these conventions, we seek to apply [DZ14, Theorem 5.38] in space $H^{-2\alpha}$. Indeed, we show that the hypotheses of the cited theorem are satisfied. First, we need to check that $Q^{1/2}(H^{-2\alpha}) = U_0$ is contained into $D_{H^{-2\alpha}}(A^\alpha) = H$: this is true because $U_0$ is embedded into $H^\beta$, which is again embedded into $H$. Moreover, since $U_0$ is Hilbert-Schmidt embedded into $H^\beta$, we also have the Hilbert-Schmidt embedding $\tilde{\imath} : U_0 \hookrightarrow H$ and thus the Hilbert-Schmidt property for the operator $A^\alpha \tilde{\imath} Q^{1/2} : H^{-2\alpha} \to H^{-2\alpha}$. Eventually, the validity of equation (II.2.12) in $H^{-2\alpha}$ follows from [DZ14, Theorem 5.38]. In addition, the $\mathbb{P}-a.s.$ regularity $Z \in C\big([0,T]; H^\beta\big)$ discussed above, allows us to conclude that the equation (II.2.12) is satisfied in $H^{\beta-\alpha}$.

We will now show the regularity in equation (II.2.9). It follows from the Itô formula, see [DZ14, Theorem 4.32] or [Par79, Theorem 1.2], applied to the $H^\beta$-valued Itô process $Z$ and to the function $F : H^\beta \ni x \mapsto \|x\|^2_{H^\beta} \in \mathbb{R}_+$. This $F$ is twice Fréchet differentiable, its differential at $x \in H^\beta$ is the vector $F'(x) = 2x \in H^\beta$ and its second derivative at $x \in H^\beta$ is the operator $F''(x) = 2\mathbb{I}_{H^\beta} \in \mathcal{L}(H^\beta)$. In particular, $F \in C^2(H^\beta)$. This allows us to apply the Itô formula to the $H^\beta$-valued Itô process $Z$ and obtain, $\mathbb{P}-a.s.$

$$\|Z_t\|^2_{H^\beta} = -2\nu \int_0^t \big\langle Z_s, A^\alpha Z_s\big\rangle_{H^\beta} \,\mathrm{d}s + 2\sqrt{\nu}\int_0^t \big\langle Z_s, \,\mathrm{d}W_s\big\rangle_{H^\beta} + \nu\,\mathrm{Tr}[Q_\beta]t, \qquad \forall\, t \geq 0.$$

Assume that $T > 0$ and take the supremum for $t \in [0,T]$ to both members of the last equality, then compute the expectation. We reach

$$\mathbb{E}\sup_{t\in[0,T]} \|Z_t\|^2_{H^\beta} = -2\nu\mathbb{E}\int_0^T \big\|Z_s\big\|^2_{H^{\beta+\alpha}} \,\mathrm{d}s + 2\sqrt{\nu}\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t \big\langle Z_s, \,\mathrm{d}W_s\big\rangle_{H^\beta} + \nu\,\mathrm{Tr}[Q_\beta]T. \tag{II.2.13}$$

We estimate the middle term on the right-hand side, thanks to the Burkholder-Davis-Gundy inequality, see [DZ14, Section 6.4] or [Hyt+16, Theorem 4.2.25, Proposition 4.2.14], and the Young inequality. For a constant $c > 0$ that depends on $W$

$$\begin{aligned} 2\sqrt{\nu}\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t \langle Z_s, \,\mathrm{d}W_s\rangle_{H^\beta} &\leq 2c\sqrt{\nu}\,\mathbb{E}\bigg(\int_0^T \|Z_s\|^2_{H^\beta}\,\mathrm{d}s\bigg)^{1/2} \\ &\leq 2c\sqrt{\nu T}\,\mathbb{E}\sup_{t\in[0,T]}\|Z_t\|_{H^\beta} \\ &\leq 2c^2\nu T + \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|Z_t\|^2_{H^\beta}. \end{aligned}$$

We insert this estimate back into equation (II.2.13) and obtain

$$\frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|Z_t\|^2_{H^\beta} + 2\nu\mathbb{E}\int_0^T \big\|Z_s\big\|^2_{H^{\beta+\alpha}}\,\mathrm{d}s \leq \big(2c^2 + \mathrm{Tr}[Q_\beta]\big)\nu T.$$

The sought regularity in equation (II.2.9) follows from this last estimate and the pathwise continuity, which we already discussed. □

**Example II.2.18.** Assume $\beta \geq 0$. We here show an example of trace-class operator $Q_\beta : H^\beta \to H^\beta$, that generates an $H^\beta$-valued Wiener process $W$ as in Theorem II.2.17.

Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal complete system for $H$. For instance, one could construct such sequence by means of tensor product of Hermite functions on $L^2(\mathbb{R})$, and by projecting them into $H$ through the Leray operator $\Pi$, see Definition II.2.5.

Then, the sequence $\{A^{-\beta/2}e_n\}_{n\in\mathbb{N}}$ is an orthonormal complete system for $H^\beta$. We can therefore define the operator $Q^\beta$ as follows:

$$Q_\beta A^{-\beta/2}e_n := q_n A^{-\beta/2}e_n, \qquad \forall\, n \in \mathbb{N},$$

where $\{q_n\}_{n\in\mathbb{N}}$ is any sequence of real numbers such that the series $\sum_{n=1}^\infty |q_n|$ converges in $\mathbb{R}$.

## II.3 Main result

**Theorem II.3.1.** *Assume that $\sigma > 2$. There exists an invariant measure $\mu \in \mathscr{P}(H^\sigma)$ for the Euler Equation (EE), see Definition II.2.16, that satisfies the following properties. There exists a finite constant $C_\sigma > 0$ such that*

$$\begin{aligned} &\int_{H^\sigma} \|x\|^2_{H^\sigma}\,\mathrm{d}\mu(x) = C_\sigma, \\ &\int_{H^\sigma} \|x\|^{2n}_{H^1}\,\mathrm{d}\mu(x) \le (2n-1)!!\, C_\sigma^n, \qquad \forall\, n \in \mathbb{N}. \end{aligned}$$

*Moreover, assuming that $0 < \beta < (2C_\sigma)^{-1}$,*

$$\int_{H^\sigma} e^{\beta\|x\|^2_{H^\sigma}}\,\mathrm{d}\mu(x) \le 2e^{\frac{2\beta C_\sigma}{1-2\beta C_\sigma}}.$$

## II.4 The stochastic hyperviscous Navier-Stokes Equation

This section is devoted to the study of the 2D hyperviscous Navier-Stokes equations, projected onto the space of square-integrable divergence-free vector fields, with an additive white noise, and coupled with an initial condition that can be either deterministic or random. First, we give the definition of solution for the equation and specify its uniqueness property. Next, we construct a solution that satisfies these properties and obtain some further regularities.

**Definition II.4.1.** Assume that $\nu > 0$ and $\alpha > 1$. We say that the stochastic hyperviscous Navier-Stokes Equation

$$\mathrm{d}X_t + \nu A^\alpha X_t\,\mathrm{d}t + B(X_t)\,\mathrm{d}t = \sqrt{\nu}\,\mathrm{d}W_t, \qquad \textit{for } t > 0, \tag{$HNS_{\nu,\alpha}$}$$

has a solution if there exists an augmented filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ with an adapted $H^{2\alpha}$-valued Wiener process $W$, and a predictable process $X : \mathbb{R}_+ \times \Omega \to H^{2\alpha}$ such that

- $X$ has $\mathbb{P}-a.s.$ trajectories in $C(\mathbb{R}_+; H^{2\alpha}) \cap L^2_{loc}(\mathbb{R}_+; H^{3\alpha})$,
- the following identity holds $\mathbb{P}-a.s.$ in $H^\alpha$

$$X_t + \nu\int_0^t A^\alpha X_s\,\mathrm{d}s + \int_0^t B(X_s)\,\mathrm{d}s = X_0 + \sqrt{\nu}\,W_t, \qquad \forall\, t \ge 0. \tag{II.4.1}$$

In this case, we say that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P}; W; X)$ is a solution to the equation ($HNS_{\nu,\alpha}$) (with kinematic viscosity $\nu$ and hyperviscous power $\alpha$).

The definition of solution for the stochastic hyperviscous Navier-Stokes equation is given in a weak probabilistic sense, *i.e.* with a non-fixed filtered probability space or Wiener process, in anticipation of Sections II.5.1 and II.5.2, where we will need to work with different probability spaces.

**Definition II.4.2.** Assume that $\nu > 0$ and $\alpha > 1$. We say that the solutions to the equation ($HNS_{\nu,\alpha}$) are pathwise unique if, given two solutions $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X^i)$, $i = 1, 2$, such that $\mathbb{P}(X_0^1 = X_0^2) = 1$, then

$$\mathbb{P}\left(\bigcap_{t\geq 0} \{X_t^1 = X_t^2\}\right) = 1.$$

**Definition II.4.3.** Assume that $\nu > 0$ and $\alpha > 1$. We say that the solutions to the equation ($HNS_{\nu,\alpha}$) are unique in law if, given two solutions $(\Omega^i, \mathcal{F}^i, \{\mathcal{F}_t^i\}_{t\geq 0}, \mathbb{P}^i; W^i; X^i)$, $i = 1, 2$, such that $(X_0^1)_*\mathbb{P}^1 = (X_0^2)_*\mathbb{P}^2$ on $(H^{2\alpha}, \mathscr{B}_{H^{2\alpha}})$, then for all $T > 0$

$$(X^1)_*\mathbb{P}^1 = (X^2)_*\mathbb{P}^2 \in \mathscr{P}\big(C([0,T]; H^{2\alpha})\big).$$

*Remark* II.4.4. Fix $\nu > 0$ and $\alpha > 1$. In order to study the solvability of the equation ($HNS_{\nu,\alpha}$), we formally differentiate in time the process $V := X - Z$, where $Z$ is defined in Theorem II.2.17 as the solution of the stochastic equation (II.2.10). We have, without any claim of rigour:

$$\begin{aligned} \mathrm{d}V_t &= \mathrm{d}X_t - \mathrm{d}Z_t \\ &= -\nu A^\alpha X_t \,\mathrm{d}t - B(X_t)\,\mathrm{d}t + \sqrt{\nu}\,\mathrm{d}W_t - \big(\nu A^\alpha Z_t\,\mathrm{d}t + \sqrt{\nu}\,\mathrm{d}W_t\big) \\ &= -\nu A^\alpha V_t\,\mathrm{d}t - B(V_t + Z_t)\,\mathrm{d}t, \\ V_0 &= X_0 - Z_0 = X_0. \end{aligned}$$

In this new equation, the stochastic term is reduced to the term $Z$ inside the non-linearity and does not explicitly appear in the dynamics for $V$. Therefore, we are led to consider the following deterministic equation, for some $x \in H^{2\alpha}$,

$$\begin{cases} v' + \nu A^\alpha v + B(v + z) = 0, & \textit{in } (0, +\infty), \\ v(0) = x, \end{cases} \tag{II.4.2}$$

where $z$ is a deterministic function which plays the role of a trajectory of the Ornstein-Uhlenbeck process, hence, it is supposed to have its regularity. Once the existence and uniqueness for the solution to this deterministic equation will have been proved, a map $\mathcal{V} : z \mapsto v$ will be properly defined, and the process $X$ will be recovered as $X := \mathcal{V}(Z) + Z$.

Before proceeding with the technique described in the above remark, we present a simple lemma which clarifies the definition of solution for the deterministic problem (II.4.2).

**Lemma II.4.5.** *Assume that $\nu > 0$, $\alpha > 1$, $x \in H^{2\alpha}$ and $z \in C(\mathbb{R}_+; H^{2\alpha})$. A function $v \in C(\mathbb{R}_+; H^{2\alpha}) \cap L^2_{loc}(\mathbb{R}_+; H^{3\alpha})$ satisfies the mild formulation in $H^{2\alpha}$ for the problem* (II.4.2)*, i.e.,*

$$v(t) + \int_0^t e^{-\nu(t-s)A^\alpha} B\big(v(s) + z(s)\big)\,\mathrm{d}s = e^{-\nu t A^\alpha} x, \qquad \forall t \geq 0, \tag{II.4.3}$$

*if and only if it satisfies in $H^\alpha$ the strong formulation, i.e.,*

$$v(t) + \nu \int_0^t A^\alpha v(s)\,\mathrm{d}s + \int_0^t B(v(s)+z(s))\,\mathrm{d}s = x, \qquad \forall\, t \geq 0. \tag{II.4.4}$$

*Proof.* The proof of the lemma is classical, see, for instance, [RR04, Theorems 12.14, 12.16].

Let us suppose that $v \in C(\mathbb{R}_+; H^{2\alpha}) \cap L^2_{loc}(\mathbb{R}_+; H^{3\alpha})$ satisfies the mild formulation in $H^{2\alpha}$. First we notice that $v(0) = x$ and we recall that $e^{-\nu\,\cdot\,A^\alpha}x \in C^1((0,+\infty); H^\alpha)$ by the analyticity of the semigroup.

As for the integral term, let us momentarily denote $T := \{(s,t) \in \mathbb{R}^2 \;:\; t > 0,\ 0 < s < t\}$, $\bar{T}$ its closure, and $f : \bar{T} \ni (s,t) \mapsto e^{-\nu(t-s)A^\alpha} B(v(s)+z(s)) \in H^\alpha$. We notice that $f \in C(\bar{T}; H^\alpha)$ by composition of the continuous functions $B(v+z) \in C(\mathbb{R}_+; H^{2\alpha-1})$ by Lemma II.2.12, $e^{-\nu r A^\alpha} \in \mathcal{L}(H^\alpha)$ for any $r \geq 0$, and $e^{-\nu\,\cdot\,A^\alpha} y \in C(\mathbb{R}_+; H^\alpha)$ for any $y \in H^\alpha$. Moreover, the functions $(s,+\infty) \ni t \mapsto f(s,t) \in H^\alpha$ are well-defined for all $s \geq 0$, and continuously differentiable by the analyticity of the semigroup, hence $\partial_t f \in C(T; H^\alpha)$. These fact allow to apply the Leibniz Integral Rule, see [Fol99, Theorem 2.27], and infer that $v = e^{-\nu\,\cdot\,A^\alpha}x - \int_0^\cdot f(s,\cdot)\,\mathrm{d}s \in C^1((0,+\infty); H^\alpha)$ with derivative given at all times $t > 0$ by

$$\begin{aligned}
v'(t) &= -\nu A^\alpha e^{-\nu t A^\alpha} x + \int_0^t \nu A^\alpha e^{-\nu(t-s)A^\alpha} B(v(s)+z(s))\,\mathrm{d}s - B(v(t)+z(t)) \\
&= -\nu A^\alpha \left( e^{-\nu t A^\alpha} x - \int_0^t e^{-\nu(t-s)A^\alpha} B(v(s)+z(s))\,\mathrm{d}s \right) - B(v(t)+z(t)) \\
&= -\nu A^\alpha v(t) - B(v(t)+z(t)).
\end{aligned}$$

The Fundamental Theorem of Calculus then gives equation (II.4.4) in $H^\alpha$.

Conversely, if $v \in C(\mathbb{R}_+; H^{2\alpha}) \cap L^2_{loc}(\mathbb{R}_+; H^{3\alpha})$ satisfies the strong formulation in $H^\alpha$, then we first notice that all the terms in (II.4.4) make sense in $H^\alpha$, indeed $A^\alpha v \in L^2_{loc}(\mathbb{R}_+; H^\alpha)$ and $B(v+z) \in C(\mathbb{R}_+; H^{2\alpha-1})$ by Lemma II.2.12. Also, $v(0) = x$ by direct inspection. Furthermore, $A^\alpha v \in C(\mathbb{R}_+; H)$, thus $v - x = -\nu \int_0^\cdot A^\alpha v(s)\,\mathrm{d}s - \int_0^\cdot B(v(s)+z(s))\,\mathrm{d}s \in C^1(\mathbb{R}_+; H)$. By differentiating in time the equation (II.4.4), in the sense of $H$-valued functions on $[0,+\infty)$, we obtain in $H$

$$v'(t) + \nu A^\alpha v(t) + B(v(t)+z(t)) = 0, \qquad \forall\, t \geq 0. \tag{II.4.5}$$

In order to obtain the equation (II.4.3) from this last identity, we fix $t \geq 0$ and compute the derivative of $(0,t) \ni s \mapsto e^{-\nu(t-s)A^\alpha} v(s) \in H$, thanks to the differentiability of both$(0,+\infty) \ni r \mapsto e^{-\nu r A^\alpha} \in \mathcal{L}(H)$ and $v : \mathbb{R}_+ \to H$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} e^{-\nu(t-s)A^\alpha} v(s) &= \nu A^\alpha e^{-\nu A^\alpha (t-s)} v(s) + e^{-\nu(t-s)A^\alpha} v'(s) \\
&= e^{-\nu A^\alpha (t-s)} (\nu A^\alpha) v(s) + e^{-\nu(t-s)A^\alpha} [-\nu A^\alpha v(s) - B(v(s)+z(s))] \\
&= -e^{-\nu A^\alpha (t-s)} B(v(s)+z(s)).
\end{aligned}$$

For the last equalities, we used the fact that $A^\alpha v(s) \in H$ for all $s \in (0,t)$ to exchange the semigroup with the operator $A^\alpha$, and implemented the equation (II.4.5). The Fundamental Theorem of Calculus gives the equation (II.4.3), whose terms, however, make all sense in $H^{2\alpha}$. $\square$

We now state and prove the main theorem that concerns the existence, uniqueness and dependence on parameters, for the deterministic problem (II.4.2).

**Theorem II.4.6.** *Assume that $\nu > 0$, $\alpha > 1$, $x \in H^{2\alpha}$, and $z \in C(\mathbb{R}_+; H^{2\alpha})$. Then, there exists a unique function*

$$v \in C(\mathbb{R}_+; H^{2\alpha}) \cap L^2_{loc}(\mathbb{R}_+; H^{3\alpha}),$$

*that satisfies the problem (II.4.2) in the mild sense in $H^{2\alpha}$, or strong sense in $H^{\alpha}$, see Lemma II.4.5. Furthermore, the map*

$$\mathcal{V}_{\nu,\alpha} = \mathcal{V} : H^{2\alpha} \times C(\mathbb{R}_+; H^{2\alpha}) \ni (x, z) \mapsto v \in C(\mathbb{R}_+; H^{2\alpha}) \cap L^2_{loc}(\mathbb{R}_+; H^{3\alpha})$$

*is such that $\mathcal{V}(0,0) = 0$ and satisfies the following estimate. For every $T > 0$ and every $R > 0$ there exists $C = C(T, R, \nu, \alpha) > 0$ such that if $\|x_i\|_{H^{2\alpha}} \le R$ and $\sup_{t\in[0,T]} \|z_i(t)\|_{H^{2\alpha}} \le R$, $i = 1, 2$, then*

$$\begin{aligned}\sup_{t\in[0,T]} \|\mathcal{V}(x_1, z_1)(t) - \mathcal{V}(x_2, z_2)(t)\|^2_{H^{2\alpha}} + \int_0^T \|\mathcal{V}(x_1, z_1)(t) - \mathcal{V}(x_2, z_2)(t)\|^2_{H^{3\alpha}}\, \mathrm{d}t \\ \le C\Big(\|x_1 - x_2\|^2_{H^{2\alpha}} + \sup_{t\in[0,T]} \|z_1(t) - z_2(t)\|^2_{H^{2\alpha}}\Big).\end{aligned} \tag{II.4.6}$$

*Proof.* ***Step 1.*** Let us fix $\nu > 0$ and $\alpha > 1$. We introduce

$$\Gamma : H^{2\alpha} \times L^2(\mathbb{R}_+; H^{\alpha}) \to L^2(\mathbb{R}_+; H^{3\alpha}) \cap C_b(\mathbb{R}_+; H^{2\alpha}),$$

such that, for $x \in H^{2\alpha}$ and $f \in L^2(\mathbb{R}_+; H^{\alpha})$,

$$\Gamma(x, f)(t) := e^{-\nu t A^{\alpha}} x - \int_0^t e^{-\nu(t-s)A^{\alpha}} f(s)\, \mathrm{d}s, \qquad \forall\, t \ge 0.$$

We prove that $\Gamma$ is a well-defined bilinear operator and that it satisfies, for a constant $C_\nu > 0$ and for all $(x, f) \in H^{2\alpha} \times L^2(\mathbb{R}_+; H^{\alpha})$

$$\int_0^{+\infty} \|\Gamma(x, f)(t)\|^2_{H^{3\alpha}}\, \mathrm{d}t + \sup_{t\ge 0} \|\Gamma(x, f)(t)\|^2_{H^{2\alpha}} \le C_\nu \|x\|_{H^{2\alpha}} + C_\nu \int_0^{+\infty} \|f(t)\|^2_{H^{\alpha}}\, \mathrm{d}t. \tag{II.4.7}$$

This directly implies that, for any $x \in H^{2\alpha}$ and $T > 0$, the restriction

$$\Gamma^x_T : L^2(0, T; H^{\alpha}) \ni f \mapsto \Gamma^x_T f := \big(\Gamma(x, \tilde{f}\,)\big)\Big|_{[0,T]} \in C\big([0, T]; H^{2\alpha}\big) \cap L^2(0, T; H^{3\alpha})$$

is linear and bounded, where $\tilde{f} \in L^2(\mathbb{R}_+; H^{\alpha})$ is the trivial extension of $f \in L^2(0, T; H^{\alpha})$, *i.e.* $\tilde{f}(t) = f(t)$ for $t \in [0, T]$ and $\tilde{f}(t) = 0$ for $t > T$.

To this end, let us fix $x \in H^{2\alpha}$ and $f \in L^2(\mathbb{R}_+; H^{\alpha})$. For *a.e.* $t \ge 0$, we calculate the Fourier transform of $\Gamma(x, f)(t)$, denoting it by $\hat{u}(t, \cdot)$, then for all $\xi \in \mathbb{R}^2$ we have

$$\begin{aligned}\hat{u}(t, \xi) &= e^{-\nu t(1+|\xi|^2)^{\alpha}} \mathscr{F}[x](\xi) - \int_0^t e^{-\nu(t-s)(1+|\xi|^2)^{\alpha}} \mathscr{F}[f(s)](\xi)\, \mathrm{d}s \\ &=: K_\xi(t)\mathscr{F}[x](\xi) - \Big(\big(\mathbb{1}_{\mathbb{R}_+} K_\xi\big) * \big(\mathbb{1}_{\mathbb{R}_+} \mathscr{F}[f(\,\cdot\,)](\xi)\big)\Big)(t).\end{aligned}$$

In the last line, we defined $K_\xi : \mathbb{R}_+ \ni t \mapsto e^{-\nu t(1+|\xi|^2)^\alpha} \in \mathbb{R}$ and rewrote the second integral as a convolution. We use a straightforward generalisation of the Young convolution inequality, see [Bog07, Theorem 3.9.4]: for measurable functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}^2$

$$\|g * h\|_{L^p(\mathbb{R};\mathbb{R}^2)} \leq \|g\|_{L^q(\mathbb{R})} \|h\|_{L^r(\mathbb{R};\mathbb{R}^2)}, \qquad \forall\, p, q, r \in [1, +\infty] \;\; s.t. \;\; 1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}.$$

We employ this estimate with the choice of parameters $p = r = 2$ and $q = 1$, or $p = \infty$ and $q = r = 2$. We obtain for all $\xi \in \mathbb{R}^2$

$$\begin{aligned}
\|\hat{u}(\,\cdot\,,\xi)\|^2_{L^2(\mathbb{R}_+;\mathbb{R}^2)} &\leq 2\int_0^{+\infty} e^{-2\nu t(1+|\xi|^2)^\alpha} \big|\mathscr{F}[x](\xi)\big|^2 \,\mathrm{d}t \\
&\quad + 2\big\|\mathbb{1}_{\mathbb{R}_+} K_\xi\big\|^2_{L^1(\mathbb{R})} \big\|\mathbb{1}_{\mathbb{R}_+} \mathscr{F}[f(\,\cdot\,)](\xi)\big\|^2_{L^2(\mathbb{R};\mathbb{R}^2)} \\
&= \frac{1}{\nu}(1+|\xi|^2)^{-\alpha}\big|\mathscr{F}[x](\xi)\big|^2 + 2\|K_\xi\|^2_{L^1(\mathbb{R}_+)} \big\|\mathscr{F}[f(\,\cdot\,)](\xi)\big\|^2_{L^2(\mathbb{R}_+;\mathbb{R}^2)},
\end{aligned}$$

and similarly

$$\|\hat{u}(\,\cdot\,,\xi)\|^2_{L^\infty(\mathbb{R}_+;\mathbb{R}^2)} \leq 2\big|\mathscr{F}[x](\xi)\big|^2 + 2\|K_\xi\|^2_{L^2(\mathbb{R}_+)} \big\|\mathscr{F}[f(\,\cdot\,)](\xi)\big\|^2_{L^2(\mathbb{R}_+;\mathbb{R}^2)}.$$

In particular, we have

$$\begin{aligned}
\|K_\xi\|^2_{L^1(\mathbb{R}_+)} &= \left(\int_0^{+\infty} e^{-\nu r(1+|\xi|^2)^\alpha} \,\mathrm{d}r\right)^2 = \frac{1}{\nu^2}(1+|\xi|^2)^{-2\alpha}, \\
\|K_\xi\|^2_{L^2(\mathbb{R}_+)} &= \int_0^{+\infty} e^{-2\nu r(1+|\xi|^2)^\alpha} \,\mathrm{d}r = \frac{1}{2\nu}(1+|\xi|^2)^{-\alpha}.
\end{aligned}$$

By the Fubini Theorem and the last calculations, we infer that

$$\begin{aligned}
&\|\Gamma(x,f)\|^2_{L^2(\mathbb{R}_+;H^{3\alpha})} + \|\Gamma(x,f)\|^2_{L^\infty(\mathbb{R}_+;H^{2\alpha})} \\
&= \int_0^{+\infty}\int_{\mathbb{R}^2} (1+|\xi|^2)^{3\alpha} |\hat{u}(t,\xi)|^2 \,\mathrm{d}\xi\,\mathrm{d}t + \operatorname*{ess\,sup}_{t\geq 0} \int_{\mathbb{R}^2} (1+|\xi|^2)^{2\alpha} |\hat{u}(t,\xi)|^2 \,\mathrm{d}\xi \\
&\leq \int_{\mathbb{R}^2} (1+|\xi|^2)^{3\alpha} \|\hat{u}(\,\cdot\,,\xi)\|^2_{L^2(\mathbb{R}_+;\mathbb{R}^2)} \,\mathrm{d}\xi + \int_{\mathbb{R}^2} (1+|\xi|^2)^{2\alpha} \|\hat{u}(\,\cdot\,,\xi)\|^2_{L^\infty(\mathbb{R}_+;\mathbb{R}^2)} \,\mathrm{d}\xi \\
&\leq \left(2 + \frac{1}{\nu}\right) \int_{\mathbb{R}^2} (1+|\xi|^2)^{2\alpha} \big|\mathscr{F}[x](\xi)\big|^2 \,\mathrm{d}\xi \\
&\quad + \left(\frac{1}{\nu} + \frac{2}{\nu^2}\right) \int_{\mathbb{R}^2} (1+|\xi|^2)^{\alpha} \big\|\mathscr{F}[f(\,\cdot\,)](\xi)\big\|^2_{L^2(\mathbb{R}_+;\mathbb{R}^2)} \,\mathrm{d}\xi \\
&= \left(2 + \frac{1}{\nu}\right) \|x\|^2_{H^{2\alpha}} + \frac{1}{\nu}\left(1 + \frac{2}{\nu}\right) \|f\|^2_{L^2(\mathbb{R}_+;H^\alpha)}.
\end{aligned}$$

This is the sought estimate (II.4.7), once proved that $\Gamma(x,f)$ is continuous in time.

As for the continuity of $\Gamma(x,f)$, first observe that $\hat{u}(\,\cdot\,,\xi)$ is continuous in time for *a.e.* $\xi \in \mathbb{R}^2$, then take a sequence of times $\{t_n\}_{n\in\mathbb{N}}$ convergent to a fixed $t \geq 0$. The above estimate allows to apply the Dominated Convergence Theorem and conclude that

$$\lim_{n\to\infty} \|\Gamma(x,f)(t_n) - \Gamma(x,f)(t)\|^2_{H^{2\alpha}} = \lim_{n\to\infty} \int_{\mathbb{R}^2} (1+|\xi|^2)^{2\alpha} \big|\hat{u}(t_n,\xi) - \hat{u}(t,\xi)\big|^2 \,\mathrm{d}\xi = 0.$$

***Step* 2.** For fixed $T>0$, we introduce the Banach space $\big(\mathcal{Y}_T, \|\cdot\|_{\mathcal{Y}_T}\big)$ given by

$$\begin{aligned}\mathcal{Y}_T &:= L^2(0,T;H^{3\alpha}) \cap C\big([0,T];H^{2\alpha}\big),\\ \|v\|^2_{\mathcal{Y}_T} &:= \|v\|^2_{L^2(0,T;H^{3\alpha})} + \|v\|^2_{C([0,T];H^{2\alpha})}, \qquad \forall\, v \in \mathcal{Y}_T.\end{aligned}$$

Fix $z \in C(\mathbb{R}_+;H^{2\alpha})$. We prove that the function

$$\Lambda_T : \mathcal{Y}_T \ni v \mapsto \Lambda_T(v) := \Gamma_T^x\big(B(v+z)\big) \in \mathcal{Y}_T$$

is a well-defined contraction on bounded sets of $\mathcal{Y}_T$.

First of all, we recall that for sufficiently regular $x_1, x_2, y \in H$

$$B(x_1+y) - B(x_2+y) = B(x_1-x_2, x_1+y) + B(x_2+y, x_1-x_2).$$

Therefore, thanks to the estimate (II.2.4) in Lemma II.2.12 (with $\sigma = \alpha+1 > 2$), and to the Sobolev embeddings, we have for $x_1, x_2, y \in H^{2\alpha}$

$$\begin{aligned}\|B(x_1+y) - B(x_2+y)\|_{H^\alpha} &\le \|B(x_1-x_2, x_1+y)\|_{H^\alpha} + \|B(x_2+y, x_1-x_2)\|_{H^\alpha}\\ &\le c\|x_1-x_2\|_{H^\alpha}\|x_1+y\|_{H^{\alpha+1}}\\ &\quad + c\|x_2+y\|_{H^\alpha}\|x_1-x_2\|_{H^{\alpha+1}}\\ &\le c\|x_1-x_2\|_{H^{2\alpha}}\Big[\|x_1\|_{H^{2\alpha}} + \|x_2\|_{H^{2\alpha}} + 2\|y\|_{H^{2\alpha}}\Big].\end{aligned}$$

If we use this estimate together with the estimate (II.4.7) from the previous step, we obtain for all $v_1, v_2 \in \mathcal{Y}_T$

$$\begin{aligned}\|\Lambda_T(v_1) - \Lambda_T(v_2)\|^2_{\mathcal{Y}_T} &\le C_\nu \|B(v_1+z) - B(v_2+z)\|^2_{L^2(0,T;H^\alpha)}\\ &\le 2c^2C_\nu \int_0^T \|v_1(t) - v_2(t)\|^2_{H^{2\alpha}}\Big[\|v_1(t)\|_{H^{2\alpha}}\\ &\qquad\qquad + \|v_2(t)\|_{H^{2\alpha}} + 2\|z(t)\|_{H^{2\alpha}}\Big]^2 \mathrm{d}t\\ &\le 2c^2C_\nu T\Big[\|v_1\|_{\mathcal{Y}_T} + \|v_2\|_{\mathcal{Y}_T} + 2\|z\|_{C([0,T];H^{2\alpha})}\Big]^2 \|v_1-v_2\|^2_{\mathcal{Y}_T}.\end{aligned}$$

Similarly, we have for all $v \in \mathcal{Y}_T$

$$\begin{aligned}\|\Lambda_T(v)\|^2_{\mathcal{Y}_T} &\le C_\nu\|x\|^2_{H^{2\alpha}} + C_\nu\|B(v+z)\|^2_{L^2(0,T;H^\alpha)}\\ &\le C_\nu\|x\|^2_{H^{2\alpha}} + 2c^2C_\nu T\Big[\|v\|_{\mathcal{Y}_T} + \|z\|_{C([0,T];H^{2\alpha})}\Big]^4.\end{aligned}$$

Therefore, given $\varepsilon \in (0,1)$, we can choose $R>0$ and $T>0$ such that

$$\begin{cases} 2c^2C_\nu T\Big[2R + 2\sup\limits_{t\in[0,T]}\|z(t)\|_{H^{2\alpha}}\Big]^2 \le \varepsilon^2\\ C_\nu\|x\|^2_{H^{2\alpha}} + 2c^2C_\nu T\Big[R + \|z\|_{C([0,T];H^{2\alpha})}\Big]^4 \le R^2 \end{cases},$$

which in turn means that $\Lambda_T$ is an $\varepsilon$-contraction on the closed ball $\{v \in \mathcal{Y}_T : \|v\|_{\mathcal{Y}_T} \le R\}$. Therefore, for a sufficiently small $T>0$, there exists a unique fixed point $v \in \mathcal{Y}_T$ for $\Lambda_T$, which is the unique mild solution to equation (II.4.2) on the interval $[0,T]$.

***Step* 3.** We prove a local uniqueness result for the solution.

Let us take $x_1, x_2 \in H^{2\alpha}$ and $z_1, z_2 \in C\big(\mathbb{R}_+; H^{2\alpha}\big)$. In the previous step we proved the existence of times $T_i > 0$, for $i = 1, 2$, and mild solutions $v_i \in C([0, T_i]; H^{2\alpha}) \cap L^2(0, T_i; H^{3\alpha})$, to the equation (II.4.2) with parameters $x_i, z_i$, respectively. We restrict the functions to the time interval $[0, T]$, where $T := T_1 \wedge T_2$.

On the interval $[0, T]$, we denote $w = v_1 - v_2$, $\zeta = z_1 - z_2$ and $u_i = v_i + z_i$, for $i = 1, 2$. Then $w \in C([0, T]; H^{2\alpha}) \cap L^2(0, T; H^{3\alpha})$, $u_i \in C([0, T]; H^{2\alpha})$, for $i = 1, 2$, and $w$ satisfies in $H^{2\alpha}$ the equation

$$w(t) + \nu \int_0^t A^\alpha w(s)\,\mathrm{d}s + \int_0^t B(u_1(s)) - B(u_2(s))\,\mathrm{d}s = x_1 - x_2, \qquad \forall\, t \in [0, T].$$

If we compute the scalar distributional derivative in time, we obtain

$$w' + \nu A^\alpha w + B(u_1(s)) - B(u_2(s)) = 0,$$

thus we also have $w' \in L^2(0, T; H^\alpha)$. We now take the duality product of this last equation with $A^{2\alpha} w(s) \in H^{-\alpha}$, for *a.e.* $s \in [0, T]$, and integrate over the time interval $[0, t] \subseteq [0, T]$:

$$\begin{aligned} \int_0^t {}_{H^{-\alpha}}\big\langle A^{2\alpha} w(s), w'(s)\big\rangle_{H^\alpha}\,\mathrm{d}s + \nu \int_0^t {}_{H^{-\alpha}}\big\langle A^{2\alpha} w(s), A^\alpha w(s)\big\rangle_{H^\alpha}\,\mathrm{d}s \\ = \int_0^t {}_{H^{-\alpha}}\big\langle A^{2\alpha} w(s), B(u_2(s)) - B(u_1(s))\big\rangle_{H^\alpha}\,\mathrm{d}s. \end{aligned} \tag{II.4.8}$$

We recall from [Tem01, Chapter 3, Lemma 1.2] that the first integral equals $\big(\|w(t)\|^2_{H^{2\alpha}} - \|w(0)\|^2_{H^{2\alpha}}\big)/2$, while the integrand of the second term is simply $\|w(s)\|^2_{H^{3\alpha}}$ for almost any $s \in [0, t]$. Moreover, almost everywhere in time,

$$\begin{aligned} &{}_{H^{-\alpha}}\big\langle A^{2\alpha} w, B(u_2) - B(u_1)\big\rangle_{H^\alpha} \\ &= {}_{H^{-\alpha}}\big\langle A^{2\alpha} w, B(u_2, u_2 - u_1) + B(u_2 - u_1, u_1)\big\rangle_{H^\alpha} \\ &= - {}_{H^{-\alpha}}\big\langle A^{2\alpha} w, B(u_1, w + \zeta)\big\rangle_{H^\alpha} - {}_{H^{-\alpha}}\big\langle A^{2\alpha} w, B(w + \zeta, u_2)\big\rangle_{H^\alpha}. \end{aligned}$$

These last two terms can be controlled with the estimate (II.2.4) in Lemma II.2.12 (with $\sigma = \alpha + 1 > 2$), the Sobolev embeddings and the Young inequality:

$$\begin{aligned} \Big| {}_{H^{-\alpha}}\big\langle A^{2\alpha} w, B(u_1, w + \zeta)\big\rangle_{H^\alpha} \Big| &\leq c \|u_1\|_{H^\alpha} \|w + \zeta\|_{H^{\alpha+1}} \|A^{2\alpha} w\|_{H^{-\alpha}} \\ &\leq \frac{\nu}{4} \|w\|^2_{H^{3\alpha}} + \frac{c}{\nu} \|u_1\|^2_{H^{2\alpha}} \|w + \zeta\|^2_{H^{2\alpha}}. \end{aligned}$$

Analogously

$$\Big| {}_{H^{-\alpha}}\big\langle A^{2\alpha} w, B(w + \zeta, u_2)\big\rangle_{H^\alpha} \Big| \leq \frac{\nu}{4} \|w\|^2_{H^{3\alpha}} + \frac{c}{\nu} \|u_2\|^2_{H^{2\alpha}} \|w + \zeta\|^2_{H^{2\alpha}}.$$

Therefore, rewriting equation (II.4.8) as discussed and implementing these estimates, we obtain for all $t \in [0, T]$

$$\begin{aligned} &\frac{1}{2} \|w(t)\|^2_{H^{2\alpha}} + \frac{\nu}{2} \int_0^t \|w(s)\|^2_{H^{3\alpha}}\,\mathrm{d}s \\ &\leq \|x_1 - x_2\|^2_{H^{2\alpha}} + \frac{c}{\nu} T^2 \Big[\|u_1\|^2_{C([0,T]; H^{2\alpha})} + \|u_2\|^2_{C([0,T]; H^{2\alpha})}\Big] \|\zeta\|^2_{C([0,T]; H^{2\alpha})} \\ &\qquad + \frac{c}{\nu} T \Big[\|u_1\|^2_{C([0,T]; H^{2\alpha})} + \|u_2\|^2_{C([0,T]; H^{2\alpha})}\Big] \int_0^t \|w(s)\|^2_{H^{2\alpha}}\,\mathrm{d}s. \end{aligned} \tag{II.4.9}$$

Grönwall's lemma yields for some finite constant $C = C(T, \nu, u_1, u_2) > 0$

$$\|v_1(t) - v_2(t)\|^2_{H^{2\alpha}} \le C\big(\|x_1 - x_2\|^2_{H^{2\alpha}} + \|z_1 - z_2\|^2_{C([0,T];H^{2\alpha})}\big), \qquad \forall\, t \in [0, T]. \tag{II.4.10}$$

If we insert this estimate back into equation (II.4.9), we also obtain

$$\int_0^T \|v_1(t) - v_2(t)\|^2_{H^{3\alpha}} \,\mathrm{d}t \le C\big(\|x_1 - x_2\|^2_{H^{2\alpha}} + \|z_1 - z_2\|^2_{C([0,T];H^{2\alpha})}\big). \tag{II.4.11}$$

The sought local uniqueness result can be directly inferred from the estimates (II.4.10) and (II.4.11). Specifically, if $x_1 = x_2$ and $z_1 = z_2$, then $v_1 = v_2$ in $Y_T$.

***Step 4.*** We prove *a priori* estimates.

Assume that $T > 0$ and that $v \in \mathcal{Y}_T$ satisfies equation (II.4.3) on $[0, T]$, then we can differentiate in the distributional sense and obtain the differential equation in (II.4.2). We observe that $A^\alpha v$ and $B(v + z)$ belong to $L^2(0, T; H^\alpha)$, thus also $v' \in L^2(0, T; H^\alpha)$. In particular, the differential equation in (II.4.2) is to be interpreted in $H^\alpha$ almost everywhere in time.

For almost every time $t \in [0, T]$, we take the product in $H$ between the differential equation in (II.4.2) and $Av(t) \in H^{3\alpha-2}$. We suppress for simplicity the dependence on $t$. Since $b(x, x, Ax) = 0$ for all $x \in H^2$,

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{H^1} + \nu\|v\|^2_{H^{\alpha+1}} &\le |b(v + z, v + z, Av)| \\
&= |b(v, v, Av) + b(z, v, Av) + b(v + z, z, Av)| \\
&= |b(z, v, Av) + b(v + z, z, Av)| \\
&\le c\|z\|_{H^1}\big(2\|v\|_{H^1} + \|z\|_{H^1}\big)\|Av\|_{H^{\alpha-1}} \\
&\le \frac{\nu}{2}\|v\|^2_{H^{\alpha+1}} + \frac{4c^2}{\nu}\|z\|^2_{H^1}\|v\|^2_{H^1} + \frac{c^2}{\nu}\|z\|^4_{H^1}.
\end{aligned}$$

We used the estimate (II.2.7) in Lemma II.2.12 (with $\varepsilon = \alpha - 1 > 0$) and the Young inequality. We multiply by 2 the previous estimate and rename the constant $c > 0$. By integrating over the time interval $[0, t] \subseteq [0, T]$, we have for all $t \in [0, T]$

$$\begin{aligned}
\|v(t)\|^2_{H^1} + \nu\int_0^t \|v(s)\|^2_{H^{\alpha+1}} \,\mathrm{d}s \le \|x\|^2_{H^1} &+ \frac{c}{\nu}\|z\|^2_{C([0,T];H^1)}\int_0^t \|v(s)\|^2_{H^1} \,\mathrm{d}s \\
&+ \frac{c}{\nu}T\|z\|^4_{C([0,T];H^1)}.
\end{aligned} \tag{II.4.12}$$

By Grönwall's lemma we obtain for all $t \in [0, T]$

$$\|v(t)\|^2_{H^1} \le \Big[\|x\|^2_{H^1} + \frac{c}{\nu}T\|z\|^4_{C([0,T];H^1)}\Big]\exp\Big[\frac{c}{\nu}T\|z\|^2_{C([0,T];H^1)}\Big] =: f(T), \tag{II.4.13}$$

where $f : [0, +\infty) \to [0, +\infty)$ is a continuous function depending only on $x, z, \nu$. By inserting this estimate in the right-hand side of equation (II.4.12) and dividing by $\nu$

$$\int_0^t \|v(s)\|^2_{H^{\alpha+1}} \,\mathrm{d}s \le \frac{1}{\nu}\|x\|^2_{H^1} + \frac{c}{\nu^2}T\|z\|^2_{C([0,T]H^1)}\big[f(T) + \|z\|^2_{C([0,T];H^1)}\big]. \tag{II.4.14}$$

For almost every time $t \in [0, T]$, we take the duality product between the differential equation in (II.4.2) and $A^{2\alpha}v(t) \in H^{-\alpha}$. We suppress for simplicity the dependence on $t$.

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{H^{2\alpha}}+\nu\|v\|^2_{H^{3\alpha}} &\leq \Big|{}_{H^{-\alpha}}\big\langle A^{2\alpha}v, B(v+z)\big\rangle_{H^\alpha}\Big| \\ &= \Big|{}_{H^{-\alpha}}\big\langle A^{2\alpha}v, B(v)+B(z,v)+B(v,z)+B(z)\big\rangle_{H^\alpha}\Big| \\ &\leq c\big(\|v\|^2_{H^{\alpha+1}}+2\|z\|_{H^{\alpha+1}}\|v\|_{H^{\alpha+1}}+\|z\|^2_{H^{\alpha+1}}\big)\|A^{2\alpha}v\|_{H^{-\alpha}} \\ &\leq 2c\big(\|v\|^2_{H^{\alpha+1}}+\|z\|^2_{H^{\alpha+1}}\big)\|v\|_{H^{3\alpha}} \\ &\leq \frac{\nu}{2}\|v\|^2_{H^{3\alpha}}+\frac{4c^2}{\nu}\|z\|^4_{H^{2\alpha}}+\frac{4c^2}{\nu}\|v\|^2_{H^{2\alpha}}\|v\|^2_{H^{\alpha+1}}.\end{aligned}$$

We used the estimate (II.2.4) in Lemma II.2.12 (with $\sigma=\alpha+1>2$), the Sobolev embeddings and the Young inequality. We multiply by 2 and rename the constant $c>0$. By integrating over the time interval $[0,t]\subseteq[0,T]$, we have for all $t\in[0,T]$

$$\begin{aligned}\|v(t)\|^2_{H^{2\alpha}}+\nu\int_0^t\|v(s)\|^2_{H^{3\alpha}}\,\mathrm{d}s \leq \|x\|^2_{H^{2\alpha}}+\frac{c}{\nu}T\|z\|^4_{C([0,T];H^{2\alpha})} \\ +\frac{c}{\nu}\int_0^t\|v(s)\|^2_{H^{2\alpha}}\|v(s)\|^2_{H^{\alpha+1}}\,\mathrm{d}s.\end{aligned} \tag{II.4.15}$$

By Grönwall's lemma we obtain

$$\|v(t)\|^2_{H^{2\alpha}}\leq\Big[\|x\|^2_{H^{2\alpha}}+\frac{c}{\nu}T\|z\|^4_{C([0,T];H^{2\alpha})}\Big]\exp\left[\frac{c}{\nu}\int_0^T\|v(s)\|^2_{H^{\alpha+1}}\,\mathrm{d}s\right]=:g(T), \tag{II.4.16}$$

where $g:[0,+\infty)\to[0,+\infty)$ is a continuous well-defined function, thanks to estimate (II.4.14), and it depends only on $x,z,\nu$. If we insert this estimate in the right-hand side of equation (II.4.15) and later divide by $\nu$, we have

$$\int_0^t\|v(s)\|^2_{H^{3\alpha}}\,\mathrm{d}s\leq\frac{1}{\nu}\|x\|^2_{H^{2\alpha}}+\frac{c}{\nu^2}T\|z\|^2_{C([0,T]H^{2\alpha})}\big[\|z\|^2_{C([0,T];H^{2\alpha})}+g^2(T)\big]=:h(T), \tag{II.4.17}$$

where also $h:[0,+\infty)\to[0,+\infty)$ is a continuous function depending on $x,z,\nu$.

***Step* 5.** We prove that the solution is global. Let us define

$$T_*:=\sup\big\{T>0\mid\exists\,v\in\mathcal{Y}_T\ :\ v=\Lambda_T(v)\big\}. \tag{II.4.18}$$

By *Step* 1, $T_*$ is well-defined and $T_*>0$. Moreover, by the local uniqueness result proved in *Step* 3, we can concatenate functions $v=\Lambda_T(v)\in\mathcal{Y}_T$, for $T\in(0,T_*)$ to get a unique function $v_*\in C\big([0,T_*);H^{2\alpha}\big)\cap L^2_{loc}\big([0,T_*);H^{3\alpha}\big)$ such that $v_*(t)=\Lambda_T\Big(v_*\big|_{[0,T]}\Big)(t)$ for all $0\leq t\leq T<T_*$.

We will prove by contradiction that $T_*=+\infty$.

If $T_*<+\infty$, then take a sequence of times $T_n\geq0$, $n\in\mathbb{N}$, convergent to $T_*$ from below. Applying the *a priori* estimates (II.4.16), (II.4.17) from the previous step to $v_*$ on $[0,T_n]$, yield

$$\begin{aligned}\int_0^{T_*}\|v_*(t)\|^2_{H^{3\alpha}}\,\mathrm{d}t+\sup_{t\in[0,T_*)}\|v_*(t)\|^2_{H^{2\alpha}} &= \lim_{n\to\infty}\Big\|v_*\big|_{[0,T_n]}\Big\|^2_{Y_{T_n}} \\ &\leq \lim_{n\to\infty}\big[g(T_n)+h(T_n)\big] \\ &= g(T_*)+h(T_*).\end{aligned}$$

This proves that $v_* \in L^\infty(0,T_*;H^{2\alpha}) \cap L^2(0,T_*;H^{3\alpha})$. Additionally, by *Step* 1, $\Gamma^x_{T_*}\big(B(v_*+z)\big) \in Y_{T_*}$, hence we can define $v_*(T_*) := \Gamma^x_{T_*}\big(B(v_*+z)\big)(T_*)$ and obtain $v_* = \Lambda_{T_*}(v_*) \in Y_{T_*}$. In particular, $T_*$, as defined in equation (II.4.18), is a maximum.

Nevertheless, if $T_*$ is a maximum, we can study the Cauchy problem (II.4.2) with starting point $v_*(T_*)$. The first step then gives $T' > 0$ and a solution $\bar{v} \in Y_{T'}$. If we let

$$v(t) := \begin{cases} v_*(t) & \textit{if } \ t \in [0,T_*] \\ \bar{v}(t-T_*) & \textit{if } \ t \in [T_*, T_*+T'] \end{cases},$$

then $v$ is well-defined, and one easily verifies that $v = \Lambda_{T_*+T'}v$. This fact contradicts the definition of $T_*$ in equation (II.4.18). We conclude by contradiction that $T_* = +\infty$, namely the unique local solution $v_*$ is global.

***Step* 6.** In the previous step we proved the existence of a unique solution $v \in C(\mathbb{R}_+;H^{2\alpha}) \cap L^2_{loc}(\mathbb{R}_+;H^{3\alpha})$, thus ensuring the well-posedness of the function $\mathcal{V}$ as in the statement of the theorem. It only remains to prove the additional estimate (II.4.6).

Assume that $T > 0$ and $(x_i, z_i) \in H^{2\alpha} \times C\big([0,T];H^{2\alpha}\big)$, $i = 1,2$, are such that $\|x_i\|_{H^{2\alpha}} \leq R$ and $\sup_{t\in[0,T]} \|z_i(t)\|_{H^{2\alpha}} \leq R$ for $i = 1,2$. Let us denote $v_i := \mathcal{V}(x_i, z_i)$, then estimates (II.4.16), (II.4.14) and (II.4.13) imply the existence of a constant $K > 0$ depending on $R, T, \nu > 0$ and $\alpha > 1$ such that

$$\|v_i + z_i\|_{C([0,T];H^{2\alpha})} \leq \|v_i\|_{C([0,T];H^{2\alpha})} + \|z_i\|_{C([0,T];H^{2\alpha})} \leq K.$$

We use this estimate in inequality (II.4.9): for all $t \in [0,T]$

$$\frac{1}{2}\|v_1(t) - v_2(t)\|^2_{H^{2\alpha}} + \frac{\nu}{2}\int_0^t \|v_1(s) - v_2(s)\|^2_{H^{3\alpha}}\,\mathrm{d}s \tag{II.4.19}$$
$$\leq \|x_1 - x_2\|^2_{H^{2\alpha}} + \frac{2c}{\nu}T^2K^2\|z_1 - z_2\|^2_{C([0,T];H^{2\alpha})} + \frac{2c}{\nu}TK^2\int_0^t \|v_1(s) - v_2(s)\|^2_{H^{2\alpha}}\,\mathrm{d}s.$$

Grönwall's Lemma gives an estimate for $\sup_{t\in[0,T]} \|v_1(t) - v_2(t)\|^2_{H^{2\alpha}}$. We insert this estimate back in (II.4.19) and find the sought inequality (II.4.6) for a new constant $C > 0$ depending on $R, T, \nu > 0$ and $\alpha > 1$. □

Before adapting the deterministic results of the last theorem to the stochastic hyperviscous Navier-Stokes Equation, as discussed in Remark II.4.4, we present a version of Itô's Lemma tailored to the equation ($HNS_{\nu,\alpha}$).

**Lemma II.4.7.** *Assume that $\beta \geq 0$ and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be an augmented filtered probability space with an adapted $H^\beta$-valued Wiener process $W$. Let $Q_\beta \in \mathcal{L}_1(H^\beta)$ denote the covariance operator of $W_1$. Assume that $\gamma \in [0,\beta]$, $\nu > 0$, $\alpha > 1$ and that there exists an $H^\gamma$-valued process $X$ satisfying $\mathbb{P} - a.s.$ in $H^\gamma$*

$$X_t + \nu\int_0^t A^\alpha X_s\,\mathrm{d}s + \int_0^t B(X_s)\,\mathrm{d}s = X_0 + \sqrt{\nu}W_t, \qquad \forall\, t \geq 0.$$

*Let $\iota : H^\beta \to H^\gamma$ be the Sobolev embedding and let $Q_\gamma := \iota Q_\beta \iota^* \in \mathcal{L}_1(H^\gamma)$. Assume* *that* $h \in C^2(\mathbb{R}_+)$*, then* $\mathbb{P} - a.s.$*, for all* $t \geq 0$

$$
\begin{aligned}
h\big(\|X_t\|^2_{H^\gamma}\big) = h\big(\|X_0\|^2_{H^\gamma}\big) &- 2\nu \int_0^t \|X_s\|^2_{H^{\gamma+\alpha}} h'\big(\|X_s\|^2_{H^\gamma}\big)\,\mathrm{d}s \\
&- 2\int_0^t \big\langle B(X_s), X_s\big\rangle_{H^\gamma} h'\big(\|X_s\|^2_{H^\gamma}\big)\,\mathrm{d}s \\
&+ 2\sqrt{\nu}\int_0^t h'\big(\|X_s\|^2_{H^\gamma}\big)\langle X_s, \,\mathrm{d}W_s\rangle_{H^\gamma} \\
&+ 2\nu\int_0^t \big\|Q_\gamma^{1/2} X_s\big\|^2_{H^\gamma} h''\big(\|X_s\|^2_{H^\gamma}\big)\,\mathrm{d}s \\
&+ \nu \,\mathrm{Tr}[Q_\gamma]\int_0^t h'\big(\|X_s\|^2_{H^\gamma}\big)\,\mathrm{d}s.
\end{aligned}
\tag{II.4.20}
$$

*Proof.* The thesis follows from the Itô formula, see [DZ14, Theorem 4.32] or [Par79, Theorem 1.2], applied to the Itô process $X$ and to the function

$$
F : H^\gamma \in x \mapsto F(x) := h\big(\|x\|^2_{H^\gamma}\big) \in \mathbb{R}_+.
$$

$F$ is Fréchet differentiable and its differential at $x \in H^\gamma$ is the vector $F'(x) \in H^\gamma$ such that

$$
F'(x) = 2h'\big(\|x\|^2_{H^\gamma}\big)x.
$$

The function $F' : H^\gamma \ni x \mapsto F'(x) \in H^\gamma$ is again Fréchet differentiable and its derivative at $x \in H^\gamma$ is the operator $F''(x) \in \mathcal{L}(H^\gamma)$ such that for all $y \in H^\gamma$

$$
\begin{aligned}
F''(x)y &= 4h''\big(\|x\|^2_{H^\gamma}\big)\langle x, y\rangle_{H^\gamma} + 2h'\big(\|x\|^2_{H^\gamma}\big)y \\
&= \Big[4h''\big(\|x\|^2_{H^\gamma}\big)\|x\|^2_{H^\gamma}\Pi_x + 2h'\big(\|x\|^2_{H^\gamma}\big)\mathbb{I}_{H^\gamma}\Big]y,
\end{aligned}
$$

where $\mathbb{I}_{H^\gamma}$ is the identity operator in $H^\gamma$ and $\Pi_x$ denotes the projection onto the subspace generated by $x$, *i.e.*

$$
\Pi_x : H^\gamma \to H^\gamma, \qquad \Pi_x y := \begin{cases} \dfrac{\langle x, y\rangle_{H^\gamma}}{\|x\|^2_{H^\gamma}}, & \textit{if } x \in H^\gamma \setminus \{0\} \\ 0 & \textit{if } x = 0 \end{cases}.
$$

Hence, the function $F'' : H^\gamma \ni x \mapsto F''(x) \in \mathcal{L}(H^\gamma)$ is continuous, in particular, $F \in C^2(H^\gamma)$. This allows us to apply the Itô formula and obtain, $\mathbb{P} - a.s.$ for all $t \geq 0$ in $\mathbb{R}$

$$
\begin{aligned}
F(X_t) = F(X_0) &- \int_0^t \big\langle F'(X_s), \nu A^\alpha X_s + B(X_s)\big\rangle_{H^\gamma}\,\mathrm{d}s \\
&+ \sqrt{\nu}\int_0^t \big\langle F'(X_s), \,\mathrm{d}W_s\big\rangle_{H^\gamma} \\
&+ \frac{\nu}{2}\int_0^t \mathrm{Tr}\left[F''(X_s)Q_\gamma\right]\mathrm{d}s,
\end{aligned}
$$

where $Q_\gamma := \iota Q_\beta \iota^* \in \mathcal{L}_1(H^\gamma)$ and $\iota : H^\beta \to H^\gamma$ is the Sobolev embedding. The thesis follows after expanding $F, F'$ and $F''$ and observing that $\|X_s\|^2_{H^\gamma}\mathrm{Tr}[\Pi_{X_s}Q_\gamma] =$

$\|Q_\gamma^{1/2} X_s\|_{H^\gamma}^2$. Indeed, if $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal system for the separable Hilbert space $H^\gamma$, then by the self-adjointness of $\Pi_x$ and $Q_\gamma$ and by the usual properties of Hilbert spaces, we have for all $x \in H^\gamma$

$$\begin{aligned}
\|x\|_{H^\gamma}^2 \operatorname{Tr}[\Pi_x Q_\gamma] &= \|x\|_{H^\gamma}^2 \sum_{n\in\mathbb{N}} \langle \Pi_x Q_\gamma e_n, e_n\rangle_{H^\gamma} \\
&= \|x\|_{H^\gamma}^2 \sum_{n\in\mathbb{N}} \langle Q_\gamma e_n, \Pi_x e_n\rangle_{H^\gamma} \\
&= \sum_{n\in\mathbb{N}} \langle x, e_n\rangle_{H^\gamma} \langle e_n, Q_\gamma x\rangle_{H^\gamma} \\
&= \langle x, Q_\gamma x\rangle_{H^\gamma} \\
&= \|Q_\gamma^{1/2} x\|_{H^\gamma}^2 .
\end{aligned}$$

□

**Theorem II.4.8.** *Assume that $\alpha > 1$.*

*(i)* *For any $\nu > 0$, the solutions to the equation ($HNS_{\nu,\alpha}$) are both pathwise unique and unique in law, see Definitions II.4.2 and II.4.3.*

*(ii)* *Assume that $\nu > 0$ and that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is an augmented filtered probability space with an adapted $H^{2\alpha}$-valued Wiener process $W$. Then we have the following properties.*

*(ii.a)* *If $\xi : \Omega \to H^{2\alpha}$ is an $\mathcal{F}_0$-measurable random variable, and if $X^\xi$ is defined $\mathbb{P}-a.s.$ by*

$$X_t^\xi := \mathcal{V}(\xi, Z)(t) + Z_t, \qquad \forall\, t \geq 0,$$

*where $Z$, $\mathcal{V}$ were introduced in Theorems II.2.17 and II.4.6, respectively, then $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X^\xi)$ is a solution to the equation ($HNS_{\nu,\alpha}$), see Definition II.4.1, such that $\mathbb{P}(X_0^\xi = \xi) = 1$.*

*(ii.b)* *For any $x \in H^{2\alpha}$, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X^x)$ be the pathwise unique solution to the equation ($HNS_{\nu,\alpha}$) such that $\mathbb{P}(X_0^x = x) = 1$. Then, for any $T > 0$, the function*

$$H^{2\alpha} \ni x \mapsto X^x \in C([0,T]; H^{2\alpha}) \cap L^2(0,T; H^{3\alpha}),$$

*is $\mathbb{P}-a.s.$ continuous.*

*(iii)* *If $\nu > 0$ and if $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X)$ is a solution to the equation ($HNS_{\nu,\alpha}$) such that*

$$X_0 \in \left( \bigcap_{p\geq 1} L^p(\Omega; H^1) \right) \cap L^4(\Omega; H^\alpha) \cap L^2(\Omega; H^{2\alpha}),$$

*then for all $T > 0$*

$$\begin{aligned}
X \in \left( \bigcap_{p\geq 1} L^p\big(\Omega; C\big([0,T]; H^1\big)\big) \right) \cap L^4\big(\Omega; C\big([0,T]; H^\alpha\big)\big) \\
\cap L^2\big(\Omega; C\big([0,T]; H^{2\alpha}\big)\big) \cap L^2\big(\Omega \times [0,T]; H^{3\alpha}\big).
\end{aligned} \tag{II.4.21}$$

*(iv) For any $p \geq 2$, there exists a finite constant $C_p > 0$, such that, if $\nu > 0$, and if $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X)$ is a solution to the equation ($HNS_{\nu,\alpha}$), then the following estimates hold true for any $T > 0$:*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_{H^1}^p + \nu \int_0^T \|X_t\|_{H^{\alpha+1}}^2 \|X_t\|_{H^1}^{p-2}\,\mathrm{d}t\right] \leq C_p\big(\mathbb{E}\|X_0\|_{H^1}^p + (T\nu)^{p/2}\big), \tag{II.4.22}$$

$$\mathbb{E}\big\|X_T\big\|_{H^1}^p \leq e^{-(p-1)\nu T}\mathbb{E}\|X_0\|_{H^1}^p + C_p. \tag{II.4.23}$$

*Moreover, there exists a finite constant $C > 0$ such that, for any $T > 0$*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_{H^\alpha}^2 + \nu \int_0^T \|X_t\|_{H^{2\alpha}}^2\,\mathrm{d}t\right] \leq C\bigg(\mathbb{E}\|X_0\|_{H^\alpha}^2 + \frac{1}{\nu^2}\mathbb{E}\|X_0\|_{H^1}^2 + \frac{T}{\nu} + T\nu\bigg), \tag{II.4.24}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_{H^\alpha}^4 + \nu \int_0^T \|X_t\|_{H^{2\alpha}}^2\|X_t\|_{H^\alpha}^2\,\mathrm{d}t\right] \leq C\bigg(\mathbb{E}\|X_0\|_{H^\alpha}^4 + \frac{T}{\nu^3}\mathbb{E}\|X_0\|_{H^1}^8 + T^2\nu(T^3+\nu)\bigg) \tag{II.4.25}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_{H^{2\alpha}}^2 + \nu \int_0^T \|X_t\|_{H^{3\alpha}}^2\,\mathrm{d}t\right] \leq C\bigg(\mathbb{E}\|X_0\|_{H^{2\alpha}}^2 + \frac{1}{\nu^2}\mathbb{E}\|X_0\|_{H^\alpha}^4 + \frac{T}{\nu^5}\mathbb{E}\|X_0\|_{H^1}^8 + \frac{T^5}{\nu} + T^2 + T\nu\bigg) \tag{II.4.26}$$

$$\mathbb{E}\big\|X_T\big\|_{H^\alpha}^4 \leq e^{-\nu T}\mathbb{E}\|X_0\|_{H^\alpha}^4 + C\bigg(\frac{1}{\nu^3}\mathbb{E}\|X_0\|_{H^1}^8 + T^4 + 1\bigg), \tag{II.4.27}$$

$$\mathbb{E}\big\|X_T\big\|_{H^{2\alpha}}^2 \leq e^{-\nu T}\mathbb{E}\|X_0\|_{H^{2\alpha}}^2 + C\bigg(\frac{1}{\nu^2}\mathbb{E}\|X_0\|_{H^\alpha}^4 + \frac{T}{\nu^5}\mathbb{E}\|X_0\|_{H^1}^8 + \frac{T^5}{\nu} + T^2 + 1\bigg). \tag{II.4.28}$$

*Proof of **(i)***. Let $\nu > 0$ and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X^i)$, $i = 1,2$ be two solutions to the equation ($HNS_{\nu,\alpha}$) such that the event $\Omega_2 := \{X_0^1 = X_0^2\}$ has probability $\mathbb{P}(\Omega_2) = 1$. Let us also choose $\Omega_1 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_1) = 1$ and for every $\omega \in \Omega_1$ the corresponding trajectory $Z(\omega)$ of the Ornstein-Uhlenbeck process $Z$ from Theorem II.2.17 (with $\beta := 2\alpha$) has the regularity $L^2_{loc}(\mathbb{R}_+; H^{3\alpha}) \cap C(\mathbb{R}_+; H^{2\alpha})$ and satisfies the equality (II.2.12) in $H^{2\alpha}$. If $\omega \in \Omega_1 \cap \Omega_2$, we denote $x := X_0^1(\omega) = X_0^2(\omega)$ and $v_i := X^i(\omega) - Z(\omega)$, for $i = 1, 2$, then, recalling Definition II.4.1, we have the following identity in $H^\alpha$

$$v_i(t) + \nu \int_0^t A^\alpha v_i(s)\,\mathrm{d}s + \int_0^t B\big(v_i(s) + Z_s(\omega)\big)\,\mathrm{d}s = x, \qquad \forall\, t \geq 0.$$

The uniqueness result stated in Theorem II.4.6 implies that $v_1 = v_2 = \mathcal{V}\big(x, Z(\omega)\big)$

in $C(\mathbb{R}_+; H^{2\alpha})$. In particular,

$$1 = \mathbb{P}(\Omega_1 \cap \Omega_2) \leq \mathbb{P}\Big(\bigcap_{t\geq 0}\{v_1(t) = v_2(t)\}\Big) = \mathbb{P}\Big(\bigcap_{t\geq 0}\{X_t^1 = X_t^2\}\Big).$$

The uniqueness in law property descends from [BM13, Corollary 7.7], see also [Ond04, Theorem 2]. □

*Proof of **(ii.a)**.* Let us fix $\nu > 0$ and an augmented filtered probability space, that we denote by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, with an adapted $H^{2\alpha}$-valued Wiener process and an $\mathcal{F}_0$-measurable, $H^{2\alpha}$-valued random variable $\xi$. Let $\Omega_1 \in \mathcal{F}$ be as above, *i.e.*, such that $\mathbb{P}(\Omega_1) = 1$ and $Z(\omega) \in C(\mathbb{R}_+; H^{2\alpha}) \cap L^2_{loc}(\mathbb{R}_+; H^{3\alpha})$ for all $\omega \in \Omega_1$.

We define the process $X^\xi$ as follows

$$X^\xi : \mathbb{R}_+ \times \Omega \ni (t,\omega) \mapsto X_t(\omega) := \mathbb{1}_{\Omega_1}(\omega)\big[\mathcal{V}\big(\xi(\omega), Z(\omega)\big)(t) + Z_t(\omega)\big] \in H^{2\alpha},$$

where $\mathcal{V}$ and $Z$ were defined in Theorems II.4.6 and II.2.17, respectively. By direct inspection, $\mathbb{P}(X_0^\xi = \xi) \geq \mathbb{P}(\Omega_1) = 1$. We now show that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X^\xi)$ is a solution to the equation ($HNS_{\nu,\alpha}$), according to Definition II.4.1.

First of all, the $\mathcal{F}_0$-measurability of $\xi$, the predictability of $Z$, see Theorem II.2.17, the continuity of the function $\mathcal{V}$ introduced in Theorem II.4.6 and the continuity of the evaluation map $\iota_t : C(\mathbb{R}_+; H^{2\alpha}) \ni u \to u(t) \in H^{2\alpha}$, for $t \geq 0$, imply the predictability for $X^\xi$ with values in $H^{2\alpha}$.

If $\omega \in \Omega_1$, then $Z_t(\omega)$ and $\mathcal{V}\big(\xi(\omega), Z(\omega)\big)$ belong to $C(\mathbb{R}_+; H^{2\alpha}) \cap L^2_{loc}(\mathbb{R}_+; H^{3\alpha})$ by definition of $\Omega_1$ and by Theorem II.4.6, respectively. Hence

$$\mathbb{P}\big(X^\xi \in C(\mathbb{R}_+; H^{2\alpha}) \cap L^2_{loc}(\mathbb{R}_+; H^{3\alpha})\big) \geq \mathbb{P}(\Omega_1) = 1.$$

Finally, if $\omega \in \Omega_1$, we denote $z := Z(\omega)$ and $v := \mathcal{V}\big(\xi(\omega), Z(\omega)\big)$. Then $X^\xi(\omega)$ satisfies the following identity in $H^\alpha$ for all $t \geq 0$

$$\begin{aligned}
X_t^\xi(\omega) &= v(t) + z(t)\\
&= -\nu\int_0^t A^\alpha v(s)\,\mathrm{d}s - \int_0^t B\big(v(s) + z(s)\big)\,\mathrm{d}s + \xi(\omega)\\
&\quad - \nu\int_0^t A^\alpha z(s)\,\mathrm{d}s + \sqrt{\nu}\,W_t(\omega)\\
&= -\nu\int_0^t A^\alpha X_s^\xi(\omega)\,\mathrm{d}s - \int_0^t B\big(X_s^\xi(\omega)\big)\,\mathrm{d}s + \xi(\omega) + \sqrt{\nu}\,W_t(\omega).
\end{aligned}$$

In particular, equation (II.4.1) for $X^\xi$ is satisfied $\mathbb{P} - a.s.$ in $H^\alpha$. □

*Proof of **(ii.b)**.* Recall from *(ii.a)* that, given $\nu > 0$ and an augmented filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with an adapted $H^{2\alpha}$-valued Wiener process $W$ and given $x \in H^{2\alpha}$, the solution $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X^x)$ is given by

$$X^x : \mathbb{R}_+ \times \Omega \ni (t,\omega) \mapsto X_t^x(\omega) := \mathbb{1}_{\Omega_1}(\omega)\big[\mathcal{V}\big(x, Z(\omega)\big)(t) + Z_t(\omega)\big] \in H^{2\alpha},$$

where $\Omega_1 \in \mathcal{F}$ is such that $\mathbb{P}(\Omega_1) = 1$ and $Z(\omega) \in C(\mathbb{R}_+; H^{2\alpha}) \cap L^2_{loc}(\mathbb{R}_+; H^{3\alpha})$ for any $\omega \in \Omega_1$. Then, the continuity of the map $\mathcal{V}$ in Theorem II.4.6 gives the additional continuity of the map

$$H^{2\alpha} \ni x \mapsto X^x(\omega) \in C\big([0,T]; H^{2\alpha}\big) \cap L^2(0,T; H^{3\alpha}),$$

for any $T > 0$ and any $\omega \in \Omega_1$.

The solution $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X^x)$ is pathwise unique by *(i)*. Therefore, $X^x$ is indistinguishable from any other process $X$ such that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X)$ is a solution to the equation ($HNS_{\nu,\alpha}$) with $\mathbb{P}(X_0 = x) = 1$. □

*Proof of **(iii)**.* The regularities stated in equation (II.4.21) trivially derive from the estimates (II.4.22), (II.4.25), (II.4.26), and from the path regularities for solutions to the equation ($HNS_{\nu,\alpha}$), see Definition II.4.1. □

*Proof of **(iv)**.* Let us fix $\nu > 0$ and $T > 0$. Assume that $p \geq 2$. We apply seven times the Itô formula in Lemma II.4.7 to the process $X$, twice to the function $h : \mathbb{R}_+ \ni r \mapsto r^{p/2} \in \mathbb{R}_+$ and with $\gamma = 1$, once to the function $h = \mathbb{I}_{\mathbb{R}_+} : \mathbb{R}_+ \ni r \mapsto r \in \mathbb{R}_+$ and with $\gamma = \alpha$, twice to the function $h : \mathbb{R}_+ \ni r \mapsto r^2 \in \mathbb{R}_+$ and with $\gamma = \alpha$, and twice to $h = \mathbb{I}_{\mathbb{R}_+} : \mathbb{R}_+ \ni r \mapsto r \in \mathbb{R}_+$ and $\gamma = 2\alpha$.

***Estimate* (II.4.22).** For $\gamma = 1$, the non-linearity in equation (II.4.20) vanishes because of the well-known property $\langle B(x), x\rangle_{H^1} = \langle B(x), Ax\rangle = 0$, for all $x \in H^2$. Moreover, by simple properties of bounded operators and traces

$$\|Q_1^{1/2}x\|_{H^1}^2 \leq \|Q_1^{1/2}\|_{\mathcal{L}(H^1)}^2\|x\|_{H^1}^2 = \|Q_1\|_{\mathcal{L}(H^1)}\|x\|_{H^1}^2 \leq \mathrm{Tr}[Q_1]\|x\|_{H^1}^2, \qquad \forall\, x \in H^1. \tag{II.4.29}$$

In addition, for $h(r) = r^{p/2}$, $r \geq 0$, we have $h'(r) = pr^{(p-2)/2}/2$ and $h''(r) = p(p-2)r^{(p-4)/2}/4$. We compute the supremum in time over the interval $[0, T]$ on both sides of equation (II.4.20), then take the expectation and finally use the estimate (II.4.29). We obtain

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^1}^p + p\nu\mathbb{E}\int_0^T \|X_s\|_{H^{\alpha+1}}^2\|X_s\|_{H^1}^{p-2}\,\mathrm{d}s \\
&\leq \mathbb{E}\|X_0\|_{H^1}^p + p\sqrt{\nu}\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t \|X_s\|_{H^1}^{p-2}\langle X_s,\, \mathrm{d}W_s\rangle_{H^1} \\
&\qquad + \frac{p(p-1)\nu}{2}\,\mathrm{Tr}[Q_1]\,\mathbb{E}\int_0^T \|X_s\|_{H^1}^{p-2}\,\mathrm{d}s.
\end{aligned} \tag{II.4.30}$$

We control the second term on the right-hand side, thanks to the Burkholder-Davis-Gundy inequality, see [DZ14, Section 6.4] or [Hyt+16, Theorem 4.2.25, Proposition 4.2.14], and by the Schwarz inequality. For a constant $c > 0$ that depends on $W$

$$\begin{aligned}
&p\sqrt{\nu}\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t \|X_s\|_{H^1}^{p-2}\langle X_s,\, \mathrm{d}W_s\rangle_{H^1} \\
&\leq cp\sqrt{\nu}\,\mathbb{E}\left(\int_0^T \|X_s\|_{H^1}^{2p-2}\,\mathrm{d}s\right)^{1/2} \\
&\leq cp\sqrt{\nu}\,\mathbb{E}\left[\sup_{t\in[0,T]}\|X_t\|_{H^1}^{p/2}\left(\int_0^T \|X_t\|_{H^1}^{p-2}\,\mathrm{d}t\right)^{1/2}\right] \\
&\leq \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^1}^p + \frac{c^2p^2\nu}{2}\mathbb{E}\int_0^T \|X_t\|_{H^1}^{p-2}\,\mathrm{d}t.
\end{aligned}$$

By plugging this last estimate into equation (II.4.30), we obtain

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^1}^p + p\nu\mathbb{E}\int_0^T \|X_t\|_{H^{\alpha+1}}^2\|X_t\|_{H^1}^{p-2}\,\mathrm{d}t \\
&\le \mathbb{E}\|X_0\|_{H^1}^p + \frac{p\nu}{2}\big(c^2p + (p-1)\operatorname{Tr}[Q_1]\big)\mathbb{E}\int_0^T \|X_t\|_{H^1}^{p-2}\,\mathrm{d}t.
\end{aligned} \tag{II.4.31}$$

If $p > 2$, the last term in the right-hand side of equation (II.4.31) can be controlled by the Young inequality with conjugate exponents $p/2$ and $p/(p-2)$. For a finite constant $C > 0$, depending on $p, W$, that may vary from line to line

$$\begin{aligned}
\frac{p\nu}{2}\big(c^2p + (p-1)\operatorname{Tr}[Q_1]\big)\mathbb{E}\int_0^T \|X_t\|_{H^1}^{p-2}\,\mathrm{d}t &\le CT\nu\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^1}^{p-2} \\
&\le C(T\nu)^{p/2} + \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^1}^p.
\end{aligned}$$

This inequality remains trivially true for $p = 2$. We then insert this last estimate into equation (II.4.31), and rearrange the terms. We find a new constant $C_p > 0$ depending on $p$ and $W$ such that

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^1}^p + \nu\mathbb{E}\int_0^T \|X_t\|_{H^{\alpha+1}}^2\|X_t\|_{H^1}^{p-2}\,\mathrm{d}t \le C_p\big(\mathbb{E}\|X_0\|_{H^1}^p + (T\nu)^{p/2}\big),$$

which gives the sought estimate (II.4.22).

***Estimate* (II.4.24).** The last but one term in equation (II.4.20), with $\gamma = \alpha$ and $h = \mathbb{1}_{\mathbb{R}_+}$, is equal to 0, and the last term is deterministic. We compute the supremum in time over the interval $[0, T]$ on both sides of equation (II.4.20), then take the expectation. We obtain

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^\alpha}^2 + 2\nu\mathbb{E}\int_0^T \|X_s\|_{H^{2\alpha}}^2\,\mathrm{d}s \le \mathbb{E}\|X_0\|_{H^\alpha}^2 &- 2\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t \big\langle B(X_s), X_s\big\rangle_{H^\alpha}\,\mathrm{d}s \\
&+ 2\sqrt{\nu}\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t \langle X_s, \mathrm{d}W_s\rangle_{H^\alpha} \\
&+ \nu\operatorname{Tr}[Q_\alpha]T.
\end{aligned} \tag{II.4.32}$$

The non-linear term no longer vanishes and has to be controlled by estimate (II.2.5) in Lemma II.2.12 (with $\varepsilon = \alpha - 1 > 0$), the Young inequality, and by the estimate (II.4.22) proved above (with $p = 2$):

$$\begin{aligned}
&\left|2\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t \big\langle B(X_s), X_s\big\rangle_{H^\alpha}\,\mathrm{d}s\right| \\
&\le 2\,\mathbb{E}\int_0^T \big|\big\langle B(X_s), A^\alpha X_s\big\rangle\big|\,\mathrm{d}s \\
&\le 2\,\mathbb{E}\int_0^T \|X_s\|_{H^1}\|X_s\|_{H^\alpha}\|A^\alpha X_s\|\,\mathrm{d}s \\
&\le \frac{1}{\nu}\mathbb{E}\int_0^T \|X_s\|_{H^1}^2\|X_s\|_{H^{\alpha+1}}^2\,\mathrm{d}s + \nu\,\mathbb{E}\int_0^T \|X_s\|_{H^{2\alpha}}^2\,\mathrm{d}s \\
&\le \frac{C_2}{\nu^2}\big(\mathbb{E}\|X_0\|_{H^1}^2 + T\nu\big) + \nu\,\mathbb{E}\int_0^T \|X_s\|_{H^{2\alpha}}^2\,\mathrm{d}s.
\end{aligned} \tag{II.4.33}$$

We control the third term on the right-hand side of the inequality (II.4.32), thanks to the Burkholder-Davis-Gundy inequality, see [DZ14, Section 6.4] or [Hyt+16, Theorem 4.2.25, Proposition 4.2.14], and the Schwarz inequality. For a constant $c > 0$ that depends on $W$

$$\begin{aligned} 2\sqrt{\nu}\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t \langle X_s,\, \mathrm{d}W_s\rangle_{H^\alpha} &\leq 2c\sqrt{\nu}\,\mathbb{E}\left[\left(\int_0^T \|X_s\|_{H^\alpha}^2\,\mathrm{d}s\right)^{1/2}\right] \\ &\leq 2c\sqrt{\nu T}\,\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^\alpha} \\ &\leq 2c^2\nu T + \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^\alpha}^2. \end{aligned} \tag{II.4.34}$$

We plug the estimates (II.4.33) and (II.4.34) into equation (II.4.32), and rearrange the terms:

$$\frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^\alpha}^2 + \nu\mathbb{E}\int_0^T \|X_t\|_{H^{2\alpha}}^2\,\mathrm{d}t \leq \mathbb{E}\|X_0\|_{H^\alpha}^2 + \frac{C}{\nu^2}\mathbb{E}\|X_0\|_{H^1}^2 + \frac{T}{\nu} + 2c^2\nu T,$$

which is the sought estimate (II.4.24), after renaming the constants.

***Estimate* (II.4.25).** By simple properties of bounded operators and traces

$$\left\|Q_\alpha^{1/2}x\right\|_{H^\alpha}^2 \leq \left\|Q_\alpha^{1/2}\right\|_{\mathcal{L}(H^\alpha)}^2\|x\|_{H^\alpha}^2 = \|Q_\alpha\|_{\mathcal{L}(H^\alpha)}\|x\|_{H^\alpha}^2 \leq \mathrm{Tr}[Q_\alpha]\|x\|_{H^\alpha}^2, \quad \forall\, x \in H^\alpha.$$

In addition, if $h(r) = r^2$, $r \geq 0$, we have $h'(r) = 2r$ and $h''(r) = 2$. We employ these considerations in equation (II.4.20) with $\gamma = \alpha$. We compute the supremum in time over the interval $[0, T]$ on both sides of equation (II.4.20), then take the expectation and finally use the estimate (II.4.29). We obtain

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^\alpha}^4 + 4\nu\mathbb{E}\int_0^T \|X_s\|_{H^{2\alpha}}^2\|X_s\|_{H^\alpha}^2\,\mathrm{d}s \\ &\leq \mathbb{E}\|X_0\|_{H^\alpha}^4 - 4\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t \big\langle B(X_s), X_s\big\rangle_{H^\alpha}\|X_s\|_{H^\alpha}^2\,\mathrm{d}s \\ &\quad + 4\sqrt{\nu}\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t \|X_s\|_{H^\alpha}^2\langle X_s,\, \mathrm{d}W_s\rangle_{H^\alpha} \\ &\quad + 6\nu\,\mathrm{Tr}[Q_\alpha]\,\mathbb{E}\int_0^T \|X_s\|_{H^\alpha}^2\,\mathrm{d}s. \end{aligned} \tag{II.4.35}$$

The non-linear term no longer vanishes and has to be controlled by estimate (II.2.5) in Lemma II.2.12 for an appropriate $\varepsilon > 0$ to be determined, and the Young inequality:

$$\begin{aligned} &\left|4\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t \big\langle B(X_s), X_s\big\rangle_{H^\alpha}\|X_s\|_{H^\alpha}^2\,\mathrm{d}s\right| \\ &\leq 4\,\mathbb{E}\int_0^T \big|\langle B(X_s), A^\alpha X_s\rangle\big|\|X_s\|_{H^\alpha}^2\,\mathrm{d}s \\ &\leq 4\,\mathbb{E}\int_0^T \|X_s\|_{H^1}\|X_s\|_{H^{1+\varepsilon}}\|A^\alpha X_s\|\|X_s\|_{H^\alpha}^2\,\mathrm{d}s \\ &= 4\,\mathbb{E}\int_0^T \|X_s\|_{H^1}\|X_s\|_{H^{1+\varepsilon}}\|X_s\|_{H^\alpha}\|X_s\|_{H^{2\alpha}}\|X_s\|_{H^\alpha}\,\mathrm{d}s \\ &\leq \frac{2}{\nu}\mathbb{E}\int_0^T \|X_s\|_{H^1}^2\|X_s\|_{H^{1+\varepsilon}}^2\|X_s\|_{H^\alpha}^2\,\mathrm{d}s + 2\nu\,\mathbb{E}\int_0^T \|X_s\|_{H^{2\alpha}}^2\|X_s\|_{H^\alpha}^2\,\mathrm{d}s. \end{aligned} \tag{II.4.36}$$

We now employ the interpolation inequality in Lemma II.2.6 with the choices $r = 1+\varepsilon$, $q=\alpha$, $\lambda = 1/2$, and the Sobolev embeddings

$$\|X_s\|^2_{H^{1+\varepsilon}} \le \|X_s\|_{H^{2(1+\varepsilon)-\alpha}} \|X_s\|_{H^{\alpha+1}} \le \|X_s\|_{H^1} \|X_s\|_{H^{\alpha+1}}, \tag{II.4.37}$$

where the last inequality holds as soon as

$$2(1+\varepsilon) - \alpha \le 1 \quad \Longleftrightarrow \quad \varepsilon \le \frac{\alpha-1}{2}.$$

We insert the estimate (II.4.37) into the first integral in the last side of the chain of inequalities (II.4.36), and conclude by the Young and the Hölder inequality. We get, for a finite constant $C>0$, independent of $\nu$ or $T$

$$\begin{aligned}
&\frac{2}{\nu}\mathbb{E}\int_0^T \|X_s\|^2_{H^1}\|X_s\|^2_{H^{1+\varepsilon}}\|X_s\|^2_{H^\alpha}\,\mathrm{d}s \\
&\le \frac{2}{\nu}\mathbb{E}\int_0^T \|X_s\|^3_{H^1}\|X_s\|_{H^{\alpha+1}}\|X_s\|^2_{H^\alpha}\,\mathrm{d}s \\
&\le \mathbb{E}\Big[\sup_{t\in[0,T]}\|X_t\|^2_{H^\alpha}\frac{2}{\nu}\int_0^T \|X_s\|^3_{H^1}\|X_s\|_{H^{\alpha+1}}\,\mathrm{d}s\Big] \\
&\le \frac{1}{4}\mathbb{E}\sup_{t\in[0,T]}\|X_t\|^4_{H^\alpha} + \frac{C}{\nu^2}\mathbb{E}\Big(\int_0^T \|X_s\|^3_{H^1}\|X_s\|_{H^{\alpha+1}}\,\mathrm{d}s\Big)^2 \\
&\le \frac{1}{4}\mathbb{E}\sup_{t\in[0,T]}\|X_t\|^4_{H^\alpha} + \frac{C}{\nu^2}T\,\mathbb{E}\int_0^T \|X_s\|^6_{H^1}\|X_s\|^2_{H^{\alpha+1}}\,\mathrm{d}s \\
&\le \frac{1}{4}\mathbb{E}\sup_{t\in[0,T]}\|X_t\|^4_{H^\alpha} + \frac{C}{\nu^3}TC_8\big(\mathbb{E}\|X_0\|^8_{H^1} + T^4\nu^4\big),
\end{aligned}$$

where in the last line we used the estimate (II.4.22) proved above, with $p=8$. If we insert this last estimate into the last line of equation (II.4.36), and rename the constant $C>0$, we obtain

$$\begin{aligned}
&\Big|4\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t \langle B(X_s),X_s\rangle_{H^\alpha}\|X_s\|^2_{H^\alpha}\,\mathrm{d}s\Big| \\
&\le \frac{1}{4}\mathbb{E}\sup_{t\in[0,T]}\|X_t\|^4_{H^\alpha} + C\Big(\frac{T}{\nu^3}\mathbb{E}\|X_0\|^8_{H^1} + T^5\nu\Big) \\
&\quad + 2\nu\,\mathbb{E}\int_0^T \|X_s\|^2_{H^{2\alpha}}\|X_s\|^2_{H^\alpha}\,\mathrm{d}s
\end{aligned} \tag{II.4.38}$$

We control the third term on the right-hand side of the inequality (II.4.35), thanks to the Burkholder-Davis-Gundy inequality, see [DZ14, Section 6.4] or [Hyt+16, Theorem 4.2.25, Proposition 4.2.14], and the Schwarz inequality. For a constant $c>0$

that depends on $W$

$$
\begin{aligned}
&4\sqrt{\nu}\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t \|X_s\|_{H^\alpha}^2\langle X_s,\,\mathrm{d}W_s\rangle_{H^\alpha}\\
&\le 4c\sqrt{\nu}\,\mathbb{E}\left[\left(\int_0^T \|X_s\|_{H^\alpha}^6\,\mathrm{d}s\right)^{1/2}\right]\\
&\le 4c\sqrt{\nu}\,\mathbb{E}\left[\sup_{t\in[0,T]}\|X_t\|_{H^\alpha}^2\left(\int_0^T \|X_t\|_{H^\alpha}^2\,\mathrm{d}t\right)^{1/2}\right]\\
&\le \frac{1}{4}\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^\alpha}^4+16c^2\nu\mathbb{E}\int_0^T\|X_t\|_{H^\alpha}^2\,\mathrm{d}t.
\end{aligned}
\tag{II.4.39}
$$

By plugging the estimates (II.4.38) and (II.4.39) into equation (II.4.35), and rearranging the terms, we obtain

$$
\begin{aligned}
&\frac{3}{4}\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^\alpha}^4+2\nu\mathbb{E}\int_0^T\|X_t\|_{H^{2\alpha}}^2\|X_t\|_{H^\alpha}^2\,\mathrm{d}t\\
&\le \mathbb{E}\|X_0\|_{H^\alpha}^4+C\left(\frac{T}{\nu^3}\mathbb{E}\|X_0\|_{H^1}^8+T^5\nu\right)+2\nu\big(3\operatorname{Tr}[Q_\alpha]+8c^2\big)\mathbb{E}\int_0^T\|X_t\|_{H^\alpha}^2\,\mathrm{d}t\\
&\le \mathbb{E}\|X_0\|_{H^\alpha}^4+C\left(\frac{T}{\nu^3}\mathbb{E}\|X_0\|_{H^1}^8+T^5\nu\right)+2\nu\big(3\operatorname{Tr}[Q_\alpha]+8c^2\big)T\,\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^\alpha}^2,\\
&\le \mathbb{E}\|X_0\|_{H^\alpha}^4+C\left(\frac{T}{\nu^3}\mathbb{E}\|X_0\|_{H^1}^8+T^5\nu\right)+CT^2\nu^2+\frac{1}{4}\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^\alpha}^4,
\end{aligned}
$$

where in the last line we applied the Hölder inequality followed by the Young inequality.

After rearranging the terms and renaming the constants, we reach the sought estimate (II.4.25).

***Estimate* (II.4.26).** We apply again the Itô formula in Lemma II.4.7, with $h=\mathbb{I}_{\mathbb{R}_+}$ and $\gamma=2\alpha$. The last but one term in equation (II.4.20) is equal to 0 and the last term is deterministic. We take the supremum in time and the expectation to both sides.

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^{2\alpha}}^2+2\nu\mathbb{E}\int_0^T\|X_s\|_{H^{3\alpha}}^2\,\mathrm{d}s\\
&=\mathbb{E}\|X_0\|_{H^{2\alpha}}^2-2\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t {}_{H^{-\alpha}}\big\langle A^{2\alpha}X_s,B(X_s)\big\rangle_{H^\alpha}\,\mathrm{d}s\\
&\qquad+2\sqrt{\nu}\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t\langle X_s,\,\mathrm{d}W_s\rangle_{H^{2\alpha}}\\
&\qquad+\nu\operatorname{Tr}[Q_{2\alpha}]T.
\end{aligned}
\tag{II.4.40}
$$

The non-linear term can be controlled by estimate (II.2.4) (with $\sigma=\alpha+1>2$), by the Young inequality, the Sobolev embeddings, and by the estimate (II.4.25):

$$
\begin{aligned}
&\left|2\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t {}_{H^{-\alpha}}\langle A^{2\alpha}X_s, B(X_s)\rangle_{H^\alpha}\,\mathrm{d}s\right| \\
&\leq 2c\,\mathbb{E}\int_0^T \|X_s\|_{H^\alpha}\|X_s\|_{H^{\alpha+1}}\|A^{2\alpha}X_s\|_{H^{-\alpha}}\,\mathrm{d}s \\
&\leq \nu\mathbb{E}\int_0^T \|X_s\|^2_{H^{3\alpha}}\,\mathrm{d}s + \frac{c^2}{\nu}\mathbb{E}\int_0^T \|X_s\|^2_{H^{\alpha+1}}\|X_s\|^2_{H^\alpha}\,\mathrm{d}s \\
&\leq \nu\mathbb{E}\int_0^T \|X_s\|^2_{H^{3\alpha}}\,\mathrm{d}s + \frac{c^2}{\nu}\mathbb{E}\int_0^T \|X_s\|^2_{H^{2\alpha}}\|X_s\|^2_{H^\alpha}\,\mathrm{d}s \\
&\leq \nu\mathbb{E}\int_0^T \|X_s\|^2_{H^{3\alpha}}\,\mathrm{d}s + C\left(\frac{1}{\nu^2}\mathbb{E}\|X_0\|^4_{H^\alpha} + \frac{T}{\nu^5}\mathbb{E}\|X_0\|^8_{H^1} + \frac{T^5}{\nu} + T^2\right).
\end{aligned}
\tag{II.4.41}
$$

We control the third term on the right-hand side of equation (II.4.40), thanks to Burkholder-Davis-Gundy inequality, see [DZ14, Section 6.4] or [Hyt+16, Theorem 4.2.25, Proposition 4.2.14], and the Young inequality: for a constant $c > 0$ that depends on $W$

$$
\begin{aligned}
2\sqrt{\nu}\,\mathbb{E}\sup_{t\in[0,T]}\int_0^t \langle X_s, \mathrm{d}W_s\rangle_{H^{2\alpha}} &\leq 2c\sqrt{\nu}\mathbb{E}\left(\int_0^T \|X_s\|^2_{H^{2\alpha}}\,\mathrm{d}s\right)^{1/2} \\
&\leq 2c\sqrt{\nu T}\,\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_{H^{2\alpha}} \\
&\leq 2c^2\nu T + \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|X_t\|^2_{H^{2\alpha}}.
\end{aligned}
\tag{II.4.42}
$$

We use estimates (II.4.41) and (II.4.42) to bound from above the right-hand side of equation (II.4.40). After rearranging the terms, we find a finite constant $C > 0$, that depends only on $W$, such that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\|X_t\|^2_{H^{2\alpha}} + \nu\int_0^T \|X_t\|^2_{3\alpha}\,\mathrm{d}t\right] \leq C\Big(&\mathbb{E}\|X_0\|^2_{H^{2\alpha}} + \frac{1}{\nu^2}\mathbb{E}\|X_0\|^4_{H^\alpha} \\
&+ \frac{T}{\nu^5}\mathbb{E}\|X_0\|^8_{H^1} + \frac{T^5}{\nu} + T^2 + T\nu\Big).
\end{aligned}
$$

Hence, we proved the sought estimate (II.4.26).

***Estimate* (II.4.23).** As far as the fourth estimate is concerned, we fix $s \in [0, T]$ and apply the Itô formula in Lemma II.4.7 with $\gamma = 1$, $h(r) = r^{q/2}$, for $r \geq 0$, and to the process $\{X_{t+s}\}_{t\geq 0}$, which is an Itô process adapted to the filtration $\{\mathcal{F}_{t+s}\}_{t\geq 0}$

and with $H^{2\alpha}$-valued Wiener process $\{W_{t+s}\}_{t\geq 0}$. We reach, $\mathbb{P}-a.s.$ for all $t\geq 0$

$$
\begin{aligned}
\|X_{t+s}\|_{H^1}^q = \|X_s\|_{H^1}^q &- q\nu \int_0^t \langle A^\alpha X_{u+s}, X_{u+s}\rangle_{H^1} \|X_{u+s}\|_{H^1}^{q-2}\,\mathrm{d}u \\
&- q\int_0^t \langle B(X_{u+s}), X_{u+s}\rangle_{H^1} \|X_{u+s}\|_{H^1}^{q-2}\,\mathrm{d}u \\
&+ q\sqrt{\nu}\int_0^t \|X_{u+s}\|_{H^1}^{q-2}\langle X_{u+s},\,\mathrm{d}W_{u+s}\rangle_{H^1} \\
&+ \frac{q(q-2)\nu}{2}\int_0^t \big\|Q_1^{1/2}X_{u+s}\big\|_{H^1}^2 \|X_{u+s}\|_{H^1}^{q-4}\,\mathrm{d}u \\
&+ \frac{q\nu}{2}\operatorname{Tr}[Q_1]\int_0^t \|X_{u+s}\|_{H^1}^{q-2}\,\mathrm{d}u.
\end{aligned}
$$

The non-linear term vanishes because of the known property $\langle B(x),x\rangle_{H^1} = \langle B(x),Ax\rangle = 0$ for $x\in H^2$. In addition, we evaluate the above expression at the time $t=T-s$ and perform the change of variable $u\mapsto u+s$ in the integrals. Next, we take the expectation to both sides, which cancels the Itô integral. We are left with

$$
\begin{aligned}
\mathbb{E}\|X_T\|_{H^1}^q = \mathbb{E}\|X_s\|_{H^1}^q &- q\nu\,\mathbb{E}\int_s^T \|X_u\|_{H^{\alpha+1}}^2 \|X_u\|_{H^1}^{q-2}\,\mathrm{d}u \\
&+ \frac{q(q-2)\nu}{2}\mathbb{E}\int_s^T \big\|Q_1^{1/2}X_u\big\|_{H^1}^2 \|X_u\|_{H^1}^{q-4}\,\mathrm{d}u \\
&+ \frac{q\nu}{2}\operatorname{Tr}[Q_1]\int_s^T \mathbb{E}\|X_u\|_{H^1}^{q-2}\,\mathrm{d}u.
\end{aligned}
$$

We estimate the right-hand side from above thanks to $\|x\|_{H^{\alpha+1}} \geq \|x\|_{H^1}$ and to $\big\|Q_1^{1/2}x\big\|_{H^1} \leq \operatorname{Tr}[Q_1]\|x\|_{H^1}$ for $x\in H^1$, see equation (II.4.29). We reach

$$
\mathbb{E}\|X_T\|_{H^1}^q \leq \mathbb{E}\|X_s\|_{H^1}^q - q\nu\,\mathbb{E}\int_s^T \|X_u\|_{H^1}^q\,\mathrm{d}u + \frac{q(q-1)\nu}{2}\operatorname{Tr}[Q_1]\int_s^T \mathbb{E}\|X_u\|_{H^1}^{q-2}\,\mathrm{d}u.
$$

When $q>2$, the last integral can be controlled with the Hölder inequality with conjugate exponents $q/2$ and $q/(q-2)$ as follows

$$
\frac{q(q-1)\nu}{2}\operatorname{Tr}[Q_1]\int_s^T \mathbb{E}\|X_u\|_{H^1}^{q-2}\,\mathrm{d}u \leq \nu\, C_q(T-s) + \nu\int_s^T \mathbb{E}\|X_u\|_{H^1}^q\,\mathrm{d}u,
$$

where we introduced a constant $C_q>0$ that depends only on $q$ and $\operatorname{Tr}[Q_1]$. The inequality remains true also for $p=2$. Therefore,

$$
\mathbb{E}\|X_T\|_{H^1}^q \leq \mathbb{E}\|X_s\|_{H^1}^q - (q-1)\nu\int_s^T \mathbb{E}\|X_u\|_{H^1}^q\,\mathrm{d}u + \nu\, C_q(T-s).
$$

By resorting to the version of Grönwall's Lemma A.2, we obtain

$$
\mathbb{E}\big\|X_T\big\|_{H^1}^q \leq \frac{C_q}{q-1} + e^{-(q-1)\nu T}\mathbb{E}\|X_0\|_{H^1}^q,
$$

which gives the sought estimate (II.4.23), after appropriately renaming the constant $C_q>0$.

***Estimate* (II.4.27).** As for the fifth estimate, we start from the Itô formula in Lemma II.4.7 with $h : \mathbb{R}_+ \ni r \mapsto r^2 \in \mathbb{R}_+$ and $\gamma = \alpha$. We take the expectation to both members of equation (II.4.20), which cancels out the Itô integral, and use Fubini's Theorem. We reach for all $t \in [0, T]$

$$\begin{aligned}\mathbb{E}\|X_t\|^4_{H^\alpha} = \mathbb{E}\|X_0\|^4_{H^\alpha} &- 4\nu \int_0^t \mathbb{E}\Big[\|X_s\|^2_{H^{2\alpha}}\|X_s\|^2_{H^\alpha}\Big]\,\mathrm{d}s\\ &- 4\int_0^t \mathbb{E}\Big[\langle B(X_s), X_s\rangle_{H^\alpha}\|X_s\|^2_{H^\alpha}\Big]\,\mathrm{d}s\\ &+ 4\nu\int_0^t \mathbb{E}\Big[\big\|Q_\alpha^{1/2}X_s\big\|^2_{H^\alpha}\Big]\,\mathrm{d}s\\ &+ 2\nu\,\mathrm{Tr}[Q_\alpha]\int_0^t \mathbb{E}\|X_s\|^2_{H^\alpha}\,\mathrm{d}s.\end{aligned}$$

We can compute the distributional derivative with respect to time and later estimate the right-hand side by

$$\big\|Q_\alpha^{1/2}x\big\|^2_{H^\alpha} \le \big\|Q_\alpha^{1/2}\big\|^2_{\mathcal{L}(H^\alpha)}\|x\|^2_{H^\alpha} = \|Q_\alpha\|_{\mathcal{L}(H^\alpha)}\|x\|^2_{H^\alpha} \le \mathrm{Tr}[Q_\alpha]\|x\|^2_{H^\alpha}, \qquad \forall\, x \in H^\alpha.$$

and by

$$|\langle B(x), x\rangle_{H^\alpha}| \le \|B(x)\|\|A^\alpha x\| \le \|x\|_{H^1}\|x\|_{H^{1+\varepsilon}}\|x\|_{H^{2\alpha}}, \qquad \forall\, x \in H^{2\alpha}.$$

We reach, after rearranging the terms, and by means of the Young inequality:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|X_t\|^4_{H^\alpha} + 4\nu\mathbb{E}\Big[\|X_t\|^2_{H^{2\alpha}}\|X_t\|^2_{H^\alpha}\Big]\\ &\le 4\,\mathbb{E}\Big[\|X_s\|_{H^1}\|X_s\|_{H^{1+\varepsilon}}\|X_s\|_{H^\alpha}\|X_s\|_{H^{2\alpha}}\|X_s\|_{H^\alpha}\Big] + 6\nu\,\mathrm{Tr}[Q_\alpha]\mathbb{E}\|X_t\|^2_{H^\alpha}\\ &\le \frac{2}{\nu}\mathbb{E}\Big[\|X_s\|^2_{H^1}\|X_s\|^2_{H^{1+\varepsilon}}\|X_s\|^2_{H^\alpha}\Big] + 2\nu\mathbb{E}\Big[\|X_s\|^2_{H^{2\alpha}}\|X_s\|^2_{H^\alpha}\Big] + \frac{1}{2\nu}\big(6\nu\,\mathrm{Tr}[Q_\alpha]\big)^2\\ &\qquad + \frac{\nu}{2}\mathbb{E}\|X_t\|^4_{H^\alpha}\end{aligned}$$

We rearrange the terms once again, and use the estimate in equation (II.4.37), followed by Young inequality

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|X_t\|^4_{H^\alpha} + 2\nu\mathbb{E}\Big[\|X_s\|^2_{H^{2\alpha}}\|X_s\|^2_{H^\alpha}\Big]\\ &\le \frac{2}{\nu}\mathbb{E}\Big[\|X_s\|^3_{H^1}\|X_s\|_{H^{\alpha+1}}\|X_s\|^2_{H^\alpha}\Big] + \frac{1}{2\nu}\big(6\nu\,\mathrm{Tr}[Q_\alpha]\big)^2 + \frac{\nu}{2}\mathbb{E}\|X_t\|^4_{H^\alpha}\\ &\le \frac{2}{\nu^3}\mathbb{E}\Big[\|X_s\|^6_{H^1}\|X_s\|^2_{H^{\alpha+1}}\Big] + C\nu + \nu\mathbb{E}\|X_t\|^4_{H^\alpha}.\end{aligned}$$

This finally leads, by the Sobolev embedding $\|x\|^4_{H^\alpha} \le \|x\|^2_{H^\alpha}\|x\|^2_{H^{\alpha+1}}$, for $x \in H^{\alpha+1}$, to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|X_t\|^4_{H^\alpha} + \nu\mathbb{E}\|X_t\|^4_{H^\alpha} \le \frac{2}{\nu^3}\mathbb{E}\Big[\|X_s\|^6_{H^1}\|X_s\|^2_{H^{\alpha+1}}\Big] + C\nu$$

The last side is locally integrable in time thanks to (II.4.22). We apply the differential form of Grönwall's Lemma A.1 and obtain for all $t \in [0, T]$

$$\begin{aligned}
\mathbb{E}\|X_t\|_{H^\alpha}^4 &\le e^{-\nu t}\Big(\mathbb{E}\|X_0\|_{H^\alpha}^4 + \int_0^t e^{\nu s}\Big(\frac{2}{\nu^3}\mathbb{E}\Big[\|X_s\|_{H^1}^6\|X_s\|_{H^{\alpha+1}}^2\Big] + C\nu\Big)\,\mathrm{d}s\Big) \\
&\le e^{-\nu t}\mathbb{E}\|X_0\|_{H^\alpha}^4 + \frac{2}{\nu^3}\mathbb{E}\int_0^t \|X_s\|_{H^1}^6\|X_s\|_{H^{\alpha+1}}^2\,\mathrm{d}s + C \\
&\le e^{-\nu t}\mathbb{E}\|X_0\|_{H^\alpha}^4 + \frac{2}{\nu^4}C_8\Big(\mathbb{E}\|X_0\|_{H^1}^8 + (t\nu)^4\Big) + C.
\end{aligned}$$

We used the linearity of the integral for the second inequality, computing the second part and estimating $e^{\nu s} \le e^{\nu t}$ for the first part. We used estimate (II.4.22), with $p = 8$, for the last inequality. The wanted estimate (II.4.27) follows after renaming the constants.

***Estimate* (II.4.28).** As for the last estimate, we start from the Itô formula in Lemma II.4.7 with $h = \mathbb{I}_{\mathbb{R}_+}$ and $\gamma = 2\alpha$. The last but one term in equation (II.4.20) vanishes, while the last is deterministic. Next, we take the expectation to both members, which cancels out the Itô integral. We reach for all $t \in [0, T]$

$$\begin{aligned}
\mathbb{E}\|X_t\|_{H^{2\alpha}}^2 = \mathbb{E}\|X_0\|_{H^{2\alpha}}^2 - 2\nu\int_0^t \mathbb{E}\|X_s\|_{H^{3\alpha}}^2\,\mathrm{d}s + 2\int_0^t \mathbb{E}\ {}_{H^{-\alpha}}\big\langle A^{2\alpha}X_s, B\big(X_s\big)\big\rangle_{H^\alpha}\,\mathrm{d}s \\
+ \nu\,\mathrm{Tr}[Q_{2\alpha}]\,t.
\end{aligned}$$

We can compute the distributional derivative with respect to time and later estimate the right-hand by (II.2.4) (with $\sigma = \alpha + 1 > 2$)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|X_t\|_{H^{2\alpha}}^2 &= -2\nu\mathbb{E}\|X_t\|_{H^{3\alpha}}^2 + 2\mathbb{E}\ {}_{H^{-\alpha}}\big\langle A^{2\alpha}X_s, B\big(X_s\big)\big\rangle_{H^\alpha} + \nu\,\mathrm{Tr}[Q_{2\alpha}] \\
&\le -2\nu\mathbb{E}\|X_t\|_{H^{3\alpha}}^2 + \nu\mathbb{E}\,\|A^{2\alpha}X_t\|_{H^{-\alpha}}^2 + \frac{1}{\nu}\mathbb{E}\big[\|X_t\|_{H^\alpha}^2\|X_t\|_{H^{\alpha+1}}^2\big] \\
&\qquad + \nu\,\mathrm{Tr}[Q_{2\alpha}] \\
&\le -\nu\mathbb{E}\|X_t\|_{H^{2\alpha}}^2 + \frac{1}{\nu}\mathbb{E}\big[\|X_t\|_{H^\alpha}^2\|X_t\|_{H^{\alpha+1}}^2\big] + \nu\,\mathrm{Tr}[Q_\alpha].
\end{aligned}$$

We used the Sobolev embedding $\|x\|_{H^{3\alpha}} \ge \|x\|_{H^{2\alpha}}$, for $x \in H^{3\alpha}$, in the last inequality. The right-hand side is locally integrable in time thanks to (II.4.25). We apply the differential form of Grönwall's Lemma A.1 and obtain for all $t \in [0, T]$

$$\begin{aligned}
\mathbb{E}\|X_t\|_{H^{2\alpha}}^2 &\le e^{-\nu t}\Big(\mathbb{E}\|X_0\|_{H^{2\alpha}}^2 + \int_0^t e^{\nu s}\Big(\frac{1}{\nu}\mathbb{E}\big[\|X_s\|_{H^\alpha}^2\|X_s\|_{H^{\alpha+1}}^2\big] + \nu\,\mathrm{Tr}[Q_{2\alpha}]\Big)\,\mathrm{d}s\Big) \\
&\le e^{-\nu t}\mathbb{E}\|X_0\|_{H^{2\alpha}}^2 + \frac{1}{\nu}\mathbb{E}\int_0^t \|X_s\|_{H^\alpha}^2\|X_s\|_{H^{2\alpha}}^2\,\mathrm{d}s + \mathrm{Tr}[Q_{2\alpha}] \\
&\le e^{-\nu t}\mathbb{E}\|X_0\|_{H^{2\alpha}}^2 + \frac{1}{\nu^2}C\Big(\mathbb{E}\|X_0\|_{H^\alpha}^4 + \frac{T}{\nu^3}\mathbb{E}\|X_0\|_{H^1}^8 + T^2\nu(T^3+\nu)\Big) \\
&\qquad + \mathrm{Tr}[Q_{2\alpha}].
\end{aligned}$$

We used the linearity of the integral for the second inequality, computing the second part and estimating $e^{\nu s} \le e^{\nu t}$ for the first part. We used estimate (II.4.25), for the last inequality. The wanted estimate (II.4.28) follows after renaming the constants. □

## II.5 Stationary solutions

In this section, we first prove that the solutions to the stochastic hyperviscous Navier-Stokes Equation, see Definition II.4.1 and Theorem II.4.8, enjoy the Markov property. Hence, we can define a Markov semigroup $P$ associated to the equation and prove the existence of an invariant measure for the semigroup $P$, together with some moment estimates. Next, given a fixed invariant measure $\mu_\nu$, we are able to construct a stationary solution to the equation ($HNS_{\nu,\alpha}$) distributed as $\mu_\nu$ for all times. The stationarity allows us to obtain further moment estimates for the invariant measure, that are also uniform with respect to the kinematic viscosity $\nu$.

### II.5.1 Markov property

We start by recalling the definition of the usual semigroup associated with a solution to a stochastic equation.

**Definition II.5.1.** Assume that $\nu > 0$ and $\alpha > 1$. We define $P : \mathbb{R}_+ \ni t \mapsto P_t \in \mathcal{L}\big(\mathcal{B}_b(H^{2\alpha})\big)$ as follows. If $x \in H^{2\alpha}$ and if $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X^x)$ is a solution to the equation ($HNS_{\nu,\alpha}$) such that $\mathbb{P}(X_0^x = x) = 1$, see Theorem II.4.8 *(ii.b)*, then for all $\varphi \in \mathcal{B}_b(H^{2\alpha})$ and $t \geq 0$ we let

$$P_t\varphi : H^{2\alpha} \ni x \mapsto \mathbb{E}\big[\varphi(X_t^x)\big] \in \mathbb{R}. \tag{II.5.1}$$

*Remark* II.5.2. Assume that $t \geq 0$. For any $\varphi \in \mathcal{B}_b(H^{2\alpha})$, the definition of $P_t\varphi : H^{2\alpha} \to \mathbb{R}$ is well-posed by Theorem II.4.8 *(ii.b)*. Moreover, $P_t\varphi \in \mathcal{B}(H^{2\alpha})$ by [Ond05, Corollary 23], and it is bounded by simple calculations. Moreover, by direct inspection, the function $P_t : \mathcal{B}_b(H^{2\alpha}) \ni \varphi \mapsto P_t\varphi \in \mathcal{B}_b(H^{2\alpha})$ is linear and bounded.

Finally, the definition depends only on the law of the solution, that is to say, if $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}}; \tilde{W}; \tilde{X}^x)$ is another solution to the equation ($HNS_{\nu,\alpha}$) such that $\tilde{\mathbb{P}}(\tilde{X}_0^x = x) = 1$, then by the uniqueness in law property, see Theorem II.4.8 *(i)*,

$$\begin{aligned}
\mathbb{E}\big[\varphi(X_t^x)\big] &= \int_\Omega \varphi(X_t^x(\omega))\,\mathrm{d}\mathbb{P}(\omega) \\
&= \int_{H^{2\alpha}} \varphi(x)\,\mathrm{d}\big((X_t^x)_*\mathbb{P}\big)(x) \\
&= \int_{H^{2\alpha}} \varphi(x)\,\mathrm{d}\big((\tilde{X}_t^x)_*\tilde{\mathbb{P}}\big)(x) \\
&= \int_{\tilde{\Omega}} \varphi(\tilde{X}_t^x(\omega))\,\mathrm{d}\tilde{\mathbb{P}}(\omega) \\
&= \tilde{\mathbb{E}}\big[\varphi(\tilde{X}_t^x)\big].
\end{aligned}$$

We will now prove that the semigroup $P$ is the Markov semigroup associated to the solution of the stochastic hyperviscous Navier-Stokes equation.

**Theorem II.5.3.** *Assume that $\nu > 0$ and $\alpha > 1$. The solutions to the equation ($HNS_{\nu,\alpha}$) enjoy the Markov property with semigroup $P$. Namely, if $x \in H^{2\alpha}$ and if $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X^x)$ is a solution to the equation ($HNS_{\nu,\alpha}$) such that $\mathbb{P}(X_0^x = x) = 1$, then for all $t \geq 0$, $h \geq 0$ and $\varphi \in \mathcal{B}_b(H^{2\alpha})$*

$$\mathbb{E}\big[\varphi(X_{t+h}^x)\big|\sigma\big(X_s^x\ :\ s \in [0,t]\big)\big] = P_h\varphi(X_t^x), \qquad \mathbb{P}-a.s.$$

*Proof.* Assume the hypotheses and let $t \geq 0$, $h \geq 0$, $x \in H^{2\alpha}$ and $\varphi \in \mathcal{B}_b(H^{2\alpha})$ be fixed. For any $y \in H^{2\alpha}$, we consider a solution $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X^y)$ to the equation ($HNS_{\nu,\alpha}$) such that $\mathbb{P}(X_0^x = y) = 1$.

First of all, since $P_h\varphi \in \mathcal{B}(H^{2\alpha})$ by Definition II.5.1, the function $\Omega \ni \omega \mapsto P_h\varphi\big(X_t^x(\omega)\big) \in \mathbb{R}$ is $\sigma\big(X_t^x\big)$-measurable, hence also $\sigma\big(X_s^x \ : \ s \in [0,t]\big)$-measurable. We need to show that, if $E \in \sigma\big(X_s^x \ : \ s \in [0,t]\big)$, then

$$\mathbb{E}\big[P_h\varphi(X_t^x)\mathbb{1}_E\big] = \mathbb{E}\big[\varphi(X_{t+h}^x)\mathbb{1}_E\big].$$

According to [Wil91, Theorem 9.2], since the family

$$\left\{\bigcap_{i=1}^n \big\{X_{s_i}^x \in \Gamma_i\big\} \ : \ n \in \mathbb{N}, \ \Gamma_1, \dots, \Gamma_n \in \mathscr{B}_{H^{2\alpha}}, \ s_1, \dots, s_n \in [0,t]\right\}$$

is a $\pi$-system that generates the $\sigma$-algebra $\sigma\big(X_s^x \ : \ s \in [0,t]\big)$, it is sufficient to prove that, for arbitrarily fixed $n \in \mathbb{N}$, $\Gamma_1, \dots, \Gamma_n \in \mathscr{B}_{H^{2\alpha}}$ and $s_1, \dots, s_n \in [0,t]$, we have

$$\mathbb{E}\left[P_h\varphi(X_t^x)\prod_{i=1}^n \mathbb{1}_{\Gamma_i}(X_{s_i}^x)\right] = \mathbb{E}\left[\varphi(X_{t+h}^x)\prod_{i=1}^n \mathbb{1}_{\Gamma_i}(X_{s_i}^x)\right]. \tag{II.5.2}$$

We start from the left-hand side and use the definition of semigroup in equation (II.5.1):

$$\begin{aligned}\mathbb{E}\left[P_h\varphi(X_t^x)\prod_{i=1}^n \mathbb{1}_{\Gamma_i}(X_{s_i}^x)\right] &= \int_\Omega P_h\varphi(X_t^x(\omega))\prod_{i=1}^n \mathbb{1}_{\Gamma_i}(X_{s_i}^x(\omega))\,\mathrm{d}\mathbb{P}(\omega)\\ &= \int_\Omega\int_\Omega \varphi\Big(X_h^{X_t^x(\omega)}(\omega')\Big)\prod_{i=1}^n \mathbb{1}_{\Gamma_i}(X_{s_i}^x(\omega))\,\mathrm{d}\mathbb{P}(\omega')\,\mathrm{d}\mathbb{P}(\omega).\end{aligned} \tag{II.5.3}$$

Let us define, for $s \geq 0$,

$$\tilde{X}_s : \Omega \times \Omega \ni (\omega, \omega') \mapsto \tilde{X}_s(\omega, \omega') := \begin{cases} X_s^x(\omega), & \textit{if } s \in [0,t], \\ X_{s-t}^{X_t^x(\omega)}(\omega'), & \textit{if } s > t, \end{cases}$$

$$\tilde{W}_s : \Omega \times \Omega \ni (\omega, \omega') \mapsto \tilde{W}_s(\omega, \omega') := \begin{cases} W_s(\omega), & \textit{if } s \in [0,t], \\ W_{s-t}(\omega') + W_t(\omega), & \textit{if } s > t, \end{cases}$$

and consider the filtered probability space $(\Omega \times \Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_s\}_{s\geq 0}, \tilde{\mathbb{P}})$ augmented from $(\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, \{\mathcal{F}_s \otimes \mathcal{F}_s\}_{s\geq 0}, \mathbb{P} \otimes \mathbb{P})$. We denote $\tilde{\Omega} := \Omega \times \Omega$. In particular, by the Lévy Martingale Characterization Theorem, see [DZ14, Theorem 4.6], $\tilde{W}$ is an $\{\tilde{\mathcal{F}}_s\}_{s\geq 0}$-adapted $H^{2\alpha}$-valued Wiener process. See [DHV16, Lemma 4.8].

We will show now that $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}}; \tilde{W}; \tilde{X})$ is a solution to the equation ($HNS_{\nu,\alpha}$) such that $\tilde{\mathbb{P}}(\tilde{X}_0 = x) = 1$. The path regularities required in Definition II.4.1 and the initial condition are trivially satisfied. We only need to show the validity of the equality (II.4.1) for $\tilde{W}$ and $\tilde{X}$, $\tilde{\mathbb{P}} - a.s.$ in $H^\alpha$. If $s \in [0,t]$, then $\tilde{X}_s(\omega, \omega') = X_s^x(\omega)$ and $\tilde{W}_s(\omega, \omega') = W_s(\omega)$ for all $(\omega, \omega') \in \tilde{\Omega}$, hence there is nothing to prove. If $s > t$, for $\tilde{\mathbb{P}} - a.e.$ $(\omega, \omega') \in \tilde{\Omega}$, by a change of variable in time and by the definition of $X^x$ in equation (II.4.1)

$$\begin{aligned}
&\tilde{X}_s(\omega,\omega')\\
&= X^{X^x_t(\omega)}_{s-t}(\omega')\\
&= -\nu\int_0^{s-t} A^\alpha X^{X^x_t(\omega)}_r(\omega')\,\mathrm{d}r - \int_0^{s-t} B\Big(X^{X^x_t(\omega)}_r(\omega')\Big)\,\mathrm{d}r + \sqrt{\nu}\,W_{s-t}(\omega') + X^x_t(\omega)\\
&= -\nu\int_t^{s} A^\alpha X^{X^x_t(\omega)}_{r-t}(\omega')\,\mathrm{d}r - \int_t^{s} B\Big(X^{X^x_t(\omega)}_{r-t}(\omega')\Big)\,\mathrm{d}r + \sqrt{\nu}\,W_{s-t}(\omega')\\
&\quad -\nu\int_0^{t} A^\alpha X^x_r(\omega)\,\mathrm{d}r - \int_0^{t} B(X^x_r(\omega))\,\mathrm{d}r + \sqrt{\nu}\,W_t(\omega) + x\\
&= -\nu\int_0^{s} A^\alpha \tilde{X}_r(\omega,\omega')\,\mathrm{d}r - \int_0^{s} B(\tilde{X}_r(\omega,\omega'))\,\mathrm{d}r + \sqrt{\nu}\,\tilde{W}_s(\omega,\omega') + x.
\end{aligned}$$

The uniqueness in law from Theorem II.4.8 *(i)* yields $(\tilde{X})_*\tilde{\mathbb{P}} = (X^x)_*\mathbb{P}$ on $\mathscr{P}\big(C([0,T];H^{2\alpha})\big)$ for any $T>0$. Therefore, we can conclude the chain of equalities (II.5.3) by means of Fubini's Theorem and thanks to the fact that $\tilde{X}$ and $X^x$ share the same finite-dimensional laws on $\mathscr{B}_{H^{2\alpha}}$

$$\begin{aligned}
\mathbb{E}\Big[P_h\varphi(X^x_t)\prod_{i=1}^n \mathbb{1}_{\Gamma_i}(X^x_{s_i})\Big] &= \int_{\Omega\times\Omega}\varphi\Big(X^{X^x_t(\omega)}_h(\omega')\Big)\prod_{i=1}^n \mathbb{1}_{\Gamma_i}(X^x_{s_i}(\omega))\,\mathrm{d}(\mathbb{P}\otimes\mathbb{P})(\omega,\omega')\\
&= \int_{\tilde{\Omega}}\varphi\big(\tilde{X}_{t+h}(\omega,\omega')\big)\prod_{i=1}^n \mathbb{1}_{\Gamma_i}\big(\tilde{X}_{s_i}(\omega,\omega')\big)\,\mathrm{d}\tilde{\mathbb{P}}(\omega,\omega')\\
&= \int_{\Omega}\varphi\big(X^x_{t+h}(\omega)\big)\prod_{i=1}^n \mathbb{1}_{\Gamma_i}\big(X^x_{s_i}(\omega)\big)\,\mathrm{d}\mathbb{P}(\omega)\\
&= \mathbb{E}\Big[\varphi(X^x_{t+h})\prod_{i=1}^n \mathbb{1}_{\Gamma_i}(X^x_{s_i})\Big].
\end{aligned}$$

We obtained the sought equality (II.5.2) and the thesis follows. □

We now introduce the adjoint $P^*$ to the Markov semigroup $P$, which will be later used to define the invariant measure for the stochastic hyperviscous Navier-Stokes equations.

**Definition II.5.4.** Assume that $\nu>0$ and $\alpha>1$. We define the adjoint semigroup $P^* := \{P^*_t\}_{t\geq 0}$ of $P$, see Definition II.5.1, as follows

$$P^*_t : \mathscr{P}(H^{2\alpha}) \ni \mu \mapsto P^*_t\mu \in \mathscr{P}(H^{2\alpha}), \qquad \forall\, t\geq 0,$$

where, for all $t\geq 0$,

$$(P^*_t\mu)(U) := \int_{H^{2\alpha}} P_t\mathbb{1}_U\,\mathrm{d}\mu, \qquad \forall\, U\in\mathscr{B}(H^{2\alpha}).$$

*Remark* II.5.5. The definition of $P^*$ is well-posed, for any choice of parameters $\nu>0$ and $\alpha>1$. Namely, for any $\mu\in\mathscr{P}(H^{2\alpha})$ and any $t\geq 0$, one can easily verify that $P^*_t\mu$ is again a probability measure on $(H^{2\alpha},\mathscr{B}_{H^{2\alpha}})$, thanks to the definition of $P_t$, see equation (II.5.1), to Theorem II.4.8 *(ii.b)*, and to the continuity of $P_t$.

Indeed, if $x \in H^{2\alpha}$, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X^x)$ be a solution to the equation ($HNS_{\nu,\alpha}$) such that $\mathbb{P}(X_0^x = x) = 1$. Then we have for all $t \geq 0$

$$\big(P_t^* \mu\big)(U) = \int_{H^{2\alpha}} P_t \mathbb{1}_U \,\mathrm{d}\mu = \int_{H^{2\alpha}} \mathbb{P}(X_t^x \in U)\,\mathrm{d}\mu(x) \geq 0, \qquad \forall\, U \in \mathscr{B}_{H^{2\alpha}},$$

$$\big(P_t^* \mu\big)(H^{2\alpha}) = \int_{H^{2\alpha}} \mathbb{P}\big(X_t^x \in H^{2\alpha}\big)\,\mathrm{d}\mu(x) = \mu(H^{2\alpha}) = 1,$$

and for any countable family of disjoint sets $U_i \in \mathscr{B}_{H^{2\alpha}}$, $i \in \mathbb{N}$,

$$\big(P_t^* \mu\big)\left(\bigcup_{i=1}^{\infty} U_i\right) = \int_{H^{2\alpha}} P_t\left(\sum_{i=1}^{\infty} \mathbb{1}_{U_i}\right)\mathrm{d}\mu = \int_{H^{2\alpha}} \sum_{i=1}^{\infty} P_t \mathbb{1}_{U_i}\,\mathrm{d}\mu = \sum_{i=1}^{\infty} \big(P_t^* \mu\big)(U_i),$$

where we used, in order, the fact that the sets are disjoint, the linearity and continuity of $P_t$, and the Monotone Convergence Theorem.

*Remark* II.5.6. One can prove, by a standard machine from measure theory, that for any $t \geq 0$ and any probability measure $\mu \in \mathscr{P}(H^{2\alpha})$

$$\int_{H^{2\alpha}} \varphi \,\mathrm{d}\big(P_t^* \mu\big) = \int_{H^{2\alpha}} P_t \varphi \,\mathrm{d}\mu, \qquad \forall\, \varphi \in \mathcal{B}_b(H^{2\alpha}). \tag{II.5.4}$$

Indeed, if $\varphi = \mathbb{1}_U$ for some $U \in \mathscr{B}_{H^{2\alpha}}$, then the property (II.5.4) is trivial.

If $\varphi$ is simple, *i.e.* there exist $N \in \mathbb{N}$ and $\{U_i\}_{i=1}^N \subset \mathscr{B}_{H^{2\alpha}}$ such that $\varphi = \sum_{i=1}^N \mathbb{1}_{U_i}$, then the property (II.5.4) follows by linearity of the integral and the previous step.

If $\varphi \in \mathcal{B}_b(H^{2\alpha})$, then there exists a sequence of simple functions $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_b(H^{2\alpha})$ that converges $P_t^*\mu - a.s.$ to $\varphi$. In this general case, the property (II.5.4) follows by the Dominated Convergence Theorem and the previous step.

**Definition II.5.7.** Assume that $\nu > 0$ and $\alpha > 1$. A probability measure $\mu \in \mathscr{P}(H^{2\alpha})$ is an invariant measure for the semigroup $P$ if

$$P_t^* \mu = \mu, \qquad \forall\, t \geq 0.$$

A standard tool to study the existence of invariant measure for stochastic partial differential equations is the Krylov-Bogoliubov Theorem, see, for instance, [DZ96, Section 3.1]. This theorem requires the Feller property for the semigroup $P$ and the tightness for the sequence of probability measures $\frac{1}{n}\int_0^n P_t^* \delta_x \,\mathrm{d}t$, $n \in \mathbb{N}$, where $x$ is any point in $H^{2\alpha}$.

In our scenario, the Feller property is satisfied thanks to Theorem II.4.8 *(ii.b)*. Were the equation set in a bounded domain $D \subset \mathbb{R}^2$, the tightness property could be easily proved by showing some uniform estimates in time for the $p-$moment of the solution, where $p > 1$, with respect to the norm of some space $H^\sigma(D; \mathbb{R}^2)$, where $\sigma < 2\alpha$. Indeed, $H^\sigma(D; \mathbb{R}^2) \hookrightarrow H^{2\alpha}(D; \mathbb{R}^2)$ if $D$ is bounded and $\sigma < 2\alpha$. Hence closed balls in $H^\sigma(D; \mathbb{R}^2)$ are compact in $H^{2\alpha}(D; \mathbb{R}^2)$, and can be thus utilized to verify the tightness of the sequence. However, in our study case, the equation is set in the unbounded domain $\mathbb{R}^2$. In particular, the Sobolev embeddings are still continuous, yet not compact.

In order to address the problem, we resort to a different version of the Krylov-Bogoliubov Theorem, see [MS99, Section 3], which weakens the requirement of tightness and strengthens the Feller property. Specifically, the compact set required by the tightness property can be replaced by a bounded set, provided the ambient space is endowed with the weak topology. Consequently, the Feller property has to be replaced by a sequentially weak version.

**Theorem II.5.8** ([MS99, Proposition 3.1])**.** *Assume that $\nu > 0$ and $\alpha > 1$. Further, suppose that the semigroup $P$, see Definition II.5.1, satisfies the following two hypotheses.*

*(i) (Sequentially weak Feller property) If $\varphi \in C_b(H^{2\alpha}_w)$, $x \in H^{2\alpha}$ and $\{x_n\}_{n\in\mathbb{N}} \subset H^{2\alpha}$ is a sequence weakly convergent to $x$ in $H^{2\alpha}$, then*

$$\lim_{n\to\infty} P_t\varphi(x_n) = P_t\varphi(x), \qquad \forall\, t \geq 0.$$

*(ii) For any $\varepsilon > 0$ there exists $R > 0$ such that*

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T (P_t^*\delta_x)\Big(H^{2\alpha} \setminus \bar{B}_R^{H^{2\alpha}}\Big)\,\mathrm{d}t \leq \varepsilon,$$

*where $\bar{B}_R^{H^{2\alpha}} := \big\{y \in H^{2\alpha} \ : \ \|y\|_{H^{2\alpha}} \leq R\big\}$.*

*Then there exists an invariant measure for the semigroup $P$.*

An ancillary result is required to prove the sequentially weak Feller property for the semigroup.

**Lemma II.5.9.** *Assume that $\nu > 0$, $\alpha > 1$ and $z \in C(\mathbb{R}_+; H^{2\alpha})$. Then the function*

$$H^{2\alpha}_w \ni x \mapsto \mathcal{V}(x,z) \in C\big([0,T]; H^{2\alpha}_w\big),$$

*see Theorem II.4.6, is sequentially continuous. Namely, the following property is satisfied. For any $x \in H^{2\alpha}$, any sequence $\{x_n\}_{n\in\mathbb{N}}$ in $H^{2\alpha}$ weakly convergent to $x \in H^{2\alpha}$ and any $T > 0$*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \Big|\, {}_{H^{-2\alpha}}\big\langle y, \mathcal{V}(x_n,z)(t) - \mathcal{V}(x,z)(t)\big\rangle_{H^{2\alpha}} \Big| = 0, \qquad \forall\, y \in H^{-2\alpha}.$$

*Proof.* ***Step* 1**. Let us fix $T > 0$, $x \in H^{2\alpha}$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ weakly convergent to $x$ in $H^{2\alpha}$. To ease the notation we denote for all $n \in \mathbb{N}$

$$\begin{aligned} v_n &:= \mathcal{V}(x_n,z)\big|_{[0,T]} : [0,T] \to H^{2\alpha},\\ \tilde{v}_n &:= \mathcal{V}(x_n,z)\mathbb{1}_{[0,T]} : \mathbb{R} \to H^{2\alpha},\\ \tilde{f}_n &:= -\nu A^\alpha \tilde{v}_n - B(\tilde{v}_n + z)\mathbb{1}_{[0,T]} : \mathbb{R} \to H^{2\alpha}. \end{aligned}$$

We first prove that there exists $\gamma > 0$ such that the sequence $\{\tilde{v}_n\}_{n\in\mathbb{N}}$ is bounded in $H^\gamma(\mathbb{R}; H^{2\alpha})$.

The sequence

$$\{x_n\}_{n\in\mathbb{N}} \quad \text{is bounded in} \quad H^{2\alpha}, \tag{II.5.5}$$

because it is weakly convergent in $H^{2\alpha}$. Hence the estimate (II.4.6) in Theorem II.4.6 implies that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $C\big([0,T]; H^{2\alpha}\big) \cap L^2(0,T; H^{3\alpha})$. In particular, the sequence

$$\{\tilde{v}_n\}_{n\in\mathbb{N}} \quad \text{is bounded in} \quad L^\infty(\mathbb{R}; H^{2\alpha}) \cap L^2(\mathbb{R}; H^{3\alpha}). \tag{II.5.6}$$

Moreover, for all $n \in \mathbb{N}$, $\tilde{f}_n$ has compact support $[0, T]$, and

$$\begin{aligned}
\left(\int_{\mathbb{R}} \|\tilde{f}_n(t)\|_{H^\alpha} \,\mathrm{d}t\right)^2 &\leq T\int_0^T \|\tilde{f}_n(t)\|_{H^\alpha}^2 \,\mathrm{d}t \\
&\leq 2T\int_0^T \nu^2 \|A^\alpha v_n(t)\|_{H^\alpha}^2 + \|B\big(v_n(t){+}z(t)\big)\|_{H^\alpha}^2 \,\mathrm{d}t \\
&\leq 2T\int_0^T \nu^2 \|v_n(t)\|_{H^{3\alpha}}^2 + c\|v_n(t){+}z(t)\|_{H^\alpha}\|v_n(t){+}z(t)\|_{H^{\alpha+1}} \,\mathrm{d}t \\
&\leq 2\nu^2 T\int_0^T \|v_n(t)\|_{H^{3\alpha}}^2 \,\mathrm{d}t + 4cT^2 \sup_{t\in[0,T]} \big[\|v_n(t)\|_{H^{2\alpha}}^2 + \|z(t)\|_{H^{2\alpha}}^2\big].
\end{aligned}$$

We used the Jensen inequality thanks to the compact support $[0, T]$ of $\tilde{f}_n$, the Young inequality, the estimate (II.2.4) from Lemma II.2.12 (with $\sigma = \alpha + 1 > 2$), and the Sobolev embeddings. The last side of the chain of inequalities is uniformly bounded in $n$ thanks to the estimate (II.4.6) with $z_1 := z$, $x_1 := x_n$ and $x_2 = z_2 := 0$ and to the boundedness of the sequence $\{x_n\}_{n\in\mathbb{N}}$ in $H^{2\alpha}$. This means that $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ is bounded in $L^1(\mathbb{R}; H^\alpha)$, in particular

$$\big\{\mathscr{F}[\tilde{f}_n]\big\}_{n\in\mathbb{N}} \in \big(C(\mathbb{R}; H^\alpha)\big)^{\mathbb{N}} \quad \text{is bounded in} \quad L^\infty(\mathbb{R}; H^\alpha). \tag{II.5.7}$$

Let us now fix $n \in \mathbb{N}$. By direct inspection, the function $\tilde{v}_n$ satisfies in $H^\alpha$

$$\begin{aligned}
\tilde{v}_n(t) &= \begin{cases} \displaystyle\int_0^t \tilde{f}_n(s)\,\mathrm{d}s + x_n, & \forall\, t \in [0, T], \\ 0, & \forall\, t \in \mathbb{R} \setminus [0, T], \end{cases} \\
&= \mathbb{1}_{[0,T]}(t)\bigg(\int_0^t \tilde{f}_n(s)\,\mathrm{d}s + x_n\bigg).
\end{aligned} \tag{II.5.8}$$

Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal complete system for $H^{3\alpha}$, in particular $\{A^{2\alpha}e_k\}_{k\in\mathbb{N}}$ becomes an orthonormal complete system for $H^{-\alpha}$. Since $\tilde{v}_n \in L^2(\mathbb{R}; H^{3\alpha})$, there exists $\{\psi_k\}_{k\in\mathbb{N}} \subset L^2(\mathbb{R})$ such that

$$\begin{aligned}
\tilde{v}_n(t) &= \sum_{k=1}^\infty \psi_k(t) e_k, \qquad a.e.\ t \in \mathbb{R}, \\
\mathscr{F}[\tilde{v}_n](t) &= \sum_{k=1}^\infty \mathscr{F}[\psi_k](t) e_k, \qquad \forall\, t \in \mathbb{R},
\end{aligned}$$

where the series converge in $H^{3\alpha}$. We take the duality product in $H^{-\alpha}/H^\alpha$ of $A^{2\alpha}e_k$ with equation (II.5.8), for any $k \in \mathbb{N}$, and later derive in the distributional sense with respect to time:

$$\begin{aligned}
\Big({}_{H^{-\alpha}}\big\langle A^{2\alpha}e_k, \tilde{v}_n\big\rangle_{H^\alpha}\Big)' = {}_{H^{-\alpha}}\big\langle A^{2\alpha}e_k, \tilde{f}_n\big\rangle_{H^\alpha} &+ {}_{H^{-\alpha}}\big\langle A^{2\alpha}e_k, x_n\big\rangle_{H^\alpha}\delta_0 \\
&- {}_{H^{-\alpha}}\big\langle A^{2\alpha}e_k, \tilde{v}_n(T)\big\rangle_{H^\alpha}\delta_T.
\end{aligned}$$

We apply the distributional Fourier transform in time and resort to the property (0.5)

$$it \; {}_{H^{-\alpha}}\langle A^{2\alpha}e_k, \mathscr{F}[\tilde{v}_n](t)\rangle_{H^\alpha} = {}_{H^{-\alpha}}\langle A^{2\alpha}e_k, \mathscr{F}[\tilde{f}_n](t)\rangle_{H^\alpha} + \frac{1}{\sqrt{2\pi}}\, {}_{H^{-\alpha}}\langle A^{2\alpha}e_k, x_n\rangle_{H^\alpha} - \frac{1}{\sqrt{2\pi}}\, {}_{H^{-\alpha}}\langle A^{2\alpha}e_k, \tilde{v}_n(T)e^{-itT}\rangle_{H^\alpha}.$$

We multiply both sides of the last equality by $\mathscr{F}[\psi_k](t)$ and sum over $k$. We obtain for all $t \in \mathbb{R}$

$$it\big\|\mathscr{F}[\tilde{v}_n](t)\big\|^2_{H^{2\alpha}} = {}_{H^{-\alpha}}\langle A^{2\alpha}\mathscr{F}[\tilde{v}_n](t), \mathscr{F}[\tilde{f}_n](t)\rangle_{H^\alpha} + \frac{1}{\sqrt{2\pi}}\, {}_{H^{-\alpha}}\langle A^{2\alpha}\mathscr{F}[\tilde{v}_n](t), x_n\rangle_{H^\alpha} - \frac{1}{\sqrt{2\pi}}\, {}_{H^{-\alpha}}\langle A^{2\alpha}\mathscr{F}[\tilde{v}_n](t)\tilde{v}_n(T)e^{-itT}\rangle_{H^\alpha}.$$

We take the complex absolute value of both members and estimate the right-hand side

$$\begin{aligned} |t|\big\|\mathscr{F}[\tilde{v}_n](t)\big\|^2_{H^{2\alpha}} &\leq \big\|\mathscr{F}[\tilde{v}_n](t)\big\|_{H^{3\alpha}}\Big[\sup_{t\in\mathbb{R}}\big\|\mathscr{F}[\tilde{f}_n](t)\big\|^2_{H^\alpha} \\ &\qquad + \frac{1}{\sqrt{2\pi}}\big(\|x_n\|_{H^\alpha} + \|\tilde{v}_n(T)\|_{H^\alpha}\big)\Big] \\ &\leq c_1\big\|\mathscr{F}[\tilde{v}_n](t)\big\|_{H^{3\alpha}}, \end{aligned} \tag{II.5.9}$$

where we introduced a finite constant $c_1 > 0$ independent of $n$, thanks to (II.5.7), (II.5.5) and to (II.5.6). A simple real-analysis exercise shows that

$$|t|^{2\gamma} \leq 2\frac{1+|t|}{1+|t|^{1-2\gamma}}, \qquad \forall\, t\in\mathbb{R},\ \forall\,\gamma\in(0,1/2). \tag{II.5.10}$$

Therefore we have

$$\begin{aligned} \int_{\mathbb{R}} |t|^{2\gamma}\big\|\mathscr{F}[\tilde{v}_n](t)\big\|^2_{H^{2\alpha}}\,\mathrm{d}t &\leq 2\int_{\mathbb{R}} \frac{\big\|\mathscr{F}[\tilde{v}_n](t)\big\|^2_{H^{2\alpha}}}{1+|t|^{1-2\gamma}} + \frac{|t|\big\|\mathscr{F}[\tilde{v}_n](t)\big\|^2_{H^{2\alpha}}}{1+|t|^{1-2\gamma}}\,\mathrm{d}t \\ &\leq 2\int_{\mathbb{R}} \big\|\mathscr{F}[\tilde{v}_n](t)\big\|^2_{H^{3\alpha}} + c_1\int_{\mathbb{R}} \frac{\big\|\mathscr{F}[\tilde{v}_n](t)\big\|_{H^{3\alpha}}}{1+|t|^{1-2\gamma}}\,\mathrm{d}t, \end{aligned} \tag{II.5.11}$$

where we employed equation (II.5.10) for the first inequality, and estimate (II.5.9) for the second, together with the embedding $H^{3\alpha} \hookrightarrow H^{2\alpha}$ and with $1+|t|^{1-2\gamma} \geq 1$. We control the first integral by the Plancherel Theorem and the uniform bound in (II.5.6):

$$\int_{\mathbb{R}} \big\|\mathscr{F}[\tilde{v}_n](t)\big\|^2_{H^{3\alpha}}\,\mathrm{d}t = \|\tilde{v}_n\|^2_{L^2(\mathbb{R};H^{3\alpha})} \leq C, \tag{II.5.12}$$

where $C > 0$ is a finite constant independent of $n$. Similarly, thanks to the Hölder inequality

$$\int_{\mathbb{R}} \frac{\big\|\mathscr{F}[\tilde{v}_n](t)\big\|_{H^{3\alpha}}}{1+|t|^{1-2\gamma}}\,\mathrm{d}t \leq \bigg(\int_{\mathbb{R}} \frac{1}{(1+|t|^{1-2\gamma})^2}\,\mathrm{d}t\bigg)^{1/2}\bigg(\int_{\mathbb{R}} \big\|\mathscr{F}[\tilde{v}_n](t)\big\|^2_{H^{3\alpha}}\,\mathrm{d}t\bigg)^{1/2} \leq C', \tag{II.5.13}$$

where $C' > 0$ is a finite constant independent of $n$ as soon as $\gamma < 1/4$. Indeed, the second factor is uniformly bounded by equation (II.5.12), while the first is finite for $\gamma < 1/4$. By plugging estimates (II.5.12) and (II.5.13) into (II.5.11) we conclude that

$$\{\tilde{v}_n\}_{n\in\mathbb{N}} \quad \text{is bounded in} \quad H^\gamma(\mathbb{R}; H^{2\alpha}).$$

***Step* 2**. We proved in *Step* 1 that the sequence $\{\tilde{v}_n\}_{n\in\mathbb{N}}$ is bounded in $H^\gamma(\mathbb{R}; H^{2\alpha}) \cap L^\infty(\mathbb{R}; H^{2\alpha}) \cap L^2(\mathbb{R}; H^{3\alpha})$ for any $\gamma \in (0, 1/4)$. Moreover $H^{2\alpha}$ and $H^{3\alpha}$ are continuously embedded in $H^{2\alpha}_{loc}$, $H^{3\alpha}_{loc}$, respectively. In addition, [Tem01, Theorem 2.2, Chapter 3] states that $H^\gamma(\mathbb{R}; H^{2\alpha}_{loc}) \cap L^2(\mathbb{R}; H^{3\alpha}_{loc})$ is compactly embedded into $L^2(\mathbb{R}; H^{2\alpha}_{loc})$. Therefore,

$$\{\tilde{v}_n\}_{n\in\mathbb{N}} \quad \text{is bounded in} \quad H^\gamma(\mathbb{R}; H^{2\alpha}) \cap L^\infty(\mathbb{R}; H^{2\alpha}) \cap L^2(\mathbb{R}; H^{3\alpha}) \hookrightarrow L^2(\mathbb{R}; H^{2\alpha}_{loc}).$$

Consequently, we find $\tilde{v} \in H^\gamma(\mathbb{R}; H^{2\alpha}) \cap L^\infty(\mathbb{R}; H^{2\alpha}) \cap L^2(\mathbb{R}; H^{3\alpha})$ and a subsequence $\{\tilde{v}_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\tilde{v}_{n_k} \longrightarrow \tilde{v}, \qquad \begin{cases} \textit{weakly* in } L^\infty(\mathbb{R}; H^{2\alpha}), \\ \textit{weakly in } L^2(\mathbb{R}; H^{3\alpha}), \\ \textit{strongly in } L^2(\mathbb{R}; H^{2\alpha}_{loc}), \end{cases} \qquad \textit{as } k \to \infty.$$

If we restrict to the time interval $[0, T]$ we find

$$v_{n_k} = \tilde{v}_{n_k}\big|_{[0,T]} \longrightarrow \tilde{v}\big|_{[0,T]}, \qquad \begin{cases} \textit{weakly* in } L^\infty(0, T; H^{2\alpha}), \\ \textit{weakly in } L^2(0, T; H^{3\alpha}), \\ \textit{strongly in } L^2(0, T; H^{2\alpha}_{loc}), \end{cases} \qquad \textit{as } k \to \infty. \tag{II.5.14}$$

***Step* 3**. We now prove that $\tilde{v}\big|_{[0,T]} = \mathcal{V}(x, z)\big|_{[0,T]}$ and that this is the limit of the whole sequence $\{v_n\}_{n\in\mathbb{N}}$, in the sense specified in (II.5.14).

With a slight abuse of notation we henceforth denote by $\tilde{v}$ the restriction on the interval $[0, T]$ of the function $\tilde{v}$ constructed in the previous step. We also denote by $v$ the restriction to $[0, T]$ of $\mathcal{V}(x, z)$. We know from Theorem II.4.6 that, for every $k \in \mathbb{N}$, the function $v_{n_k}$ satisfies the following identity in $H^\alpha$

$$v_{n_k}(t) + \nu \int_0^t A^\alpha v_{n_k}(s)\,\mathrm{d}s + \int_0^t B\big(v_{n_k}(s) + z(s)\big)\,\mathrm{d}s = x_{n_k}, \qquad \forall\, t \in [0, T].$$

For fixed $t \in [0, T]$, we pass to the limit, for $k \to \infty$, each term in the previous equality thanks to the convergences in (II.5.14).

For all $y \in H^{-2\alpha}$, thanks to $v_{n_k} = \mathcal{V}(x_{n_k}, z) \in C\big([0, T]; H^{2\alpha}\big)$ and to its weak*-convergence in $L^\infty(0, T; H^{2\alpha})$ we have

$${}_{H^{-2\alpha}}\big\langle y, v_{n_k}(t)\big\rangle_{H^{2\alpha}} \longrightarrow {}_{H^{-2\alpha}}\big\langle y, \tilde{v}(t)\big\rangle_{H^{2\alpha}}, \qquad \textit{in } C\big([0, T]\big).$$

Since $v_{n_k}(0) = x_{n_k} \longrightarrow x$ weakly in $H^{2\alpha}$, we also deduce that $\tilde{v}(0) = x$.

Analogously, $A^\alpha v_{n_k} \longrightarrow A^\alpha \tilde{v}$ weakly in $L^2(0, T; H^\alpha)$, thanks to the weak convergence in $L^2(0, T; H^{3\alpha})$ of $v_{n_k}$. Hence, for all $y \in H^{-\alpha}$

$$\nu \int_0^t {}_{H^{-\alpha}}\big\langle y, A^\alpha v_{n_k}(s)\big\rangle_{H^\alpha}\,\mathrm{d}s \longrightarrow \nu \int_0^t {}_{H^{-\alpha}}\big\langle y, A^\alpha \tilde{v}(s)\big\rangle_{H^\alpha}\,\mathrm{d}s, \qquad \textit{in } C\big([0, T]\big).$$

For the integral of the non-linearity we apply Lemma B.3 (with $\beta = \alpha$) to the sequence $\{v_{n_k} + z\}_{n\in\mathbb{N}}$, which satisfies the hypotheses of the lemma because the strong convergence in $L^2(0,T;H^{2\alpha}_{loc})$ implies strong convergence in $L^2(0,T;H^{\alpha+1}_{loc})$ and the weak*-convergence in $L^\infty(0,T;H^{2\alpha})$ implies boundedness in $L^\infty(0,T;H^{2\alpha})$, which yields boundedness in $L^2(0,T;H^{\alpha+1})$.

We thus find, for all $t \in [0,T]$

$$\tilde{v}(t) + \nu \int_0^t A^\alpha \tilde{v}(s)\,\mathrm{d}s + \int_0^t B\big(\tilde{v}(s) + z(s)\big)\,\mathrm{d}s = x,$$

the equality holding in $H^\alpha$.

We invoke the uniqueness result in Theorem II.4.6 to infer

$$\tilde{v} = v \in C\big([0,T];H^{2\alpha}\big) \cap L^2(0,T;H^{3\alpha}).$$

We conclude this step by observing that the whole sequence $\{v_n\}_{n\in\mathbb{N}}$ converges to $v$ in the sense specified in (II.5.14). Indeed, let us choose arbitrarily a strictly increasing sequence of indices $\{n_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$. Then, by *Step 3*, we can extract a subsequence $\{n_{i_k}\}_{k\in\mathbb{N}}$ such that $\big\{v_{n_{i_k}}\big\}_{k\in\mathbb{N}}$ converges to some limit in the sense specified in (II.5.14). We proved in the present step that the limit is the function $v$ from Theorem II.4.6, in particular, it is independent of the sequence of indices $\{n_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$. Since the weak*-convergence in $L^\infty(0,T;H^{2\alpha})$, the weak-convergence in $L^2(0,T;H^{3\alpha})$ and the strong convergence in $L^2(0,T;H^{2\alpha}_{loc})$ are all metrizable on bounded sets, we obtain by the usual contradiction argument that the whole sequence $\{v_n\}_{n\in\mathbb{N}}$ converges to $v$ in the specified topologies.

***Step* 4**. Let us fix $y \in H^{-2\alpha}$, then from the previous steps ${}_{H^{-2\alpha}}\langle y, v_n - v\rangle_{H^{2\alpha}} \longrightarrow 0$ in $L^\infty(0,T)$ as $n \to \infty$ and also ${}_{H^{-2\alpha}}\langle y, v_n - v\rangle_{H^{2\alpha}} \in C\big([0,T]\big)$ for all $n \in \mathbb{N}$. Hence, we reach

$${}_{H^{-2\alpha}}\langle y, v_n - v\rangle_{H^{2\alpha}} \longrightarrow 0 \qquad \textit{in } C\big([0,T]\big),\ \textit{as } n \to \infty,$$

which is the sought thesis. ☐

**Theorem II.5.10.** *Assume that $\nu > 0$ and $\alpha > 1$. There exists an invariant measure for the semigroup $P$, see Definition II.5.1. Moreover, if $\mu_\nu \in \mathscr{P}(H^{2\alpha})$ is an invariant measure for the semigroup $P$, then*

$$\int_{H^{2\alpha}} \|x\|^p_{H^1}\,\mathrm{d}\mu_\nu(x) < +\infty, \qquad \forall\, p \in (2,+\infty), \tag{II.5.15}$$

$$\int_{H^{2\alpha}} \|x\|^4_{H^\alpha}\,\mathrm{d}\mu_\nu(x) < +\infty, \tag{II.5.16}$$

$$\int_{H^{2\alpha}} \|x\|^2_{H^{2\alpha}}\,\mathrm{d}\mu_\nu(x) < +\infty. \tag{II.5.17}$$

*Proof.* ***Step* 1**. The existence result follows from Theorem II.5.8, whose hypotheses we now show to be satisfied.

As for the sequentially weak Feller property, let us take $x \in H^{2\alpha}$ and a sequence $\{x_n\}_{n\in\mathbb{N}} \subset H^{2\alpha}$ weakly convergent to $x$ in $H^{2\alpha}$. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P};W;X^{x_n})$, $n \in \mathbb{N}$, and $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P};W;X^{x})$ be solutions to the equation ($HNS_{\nu,\alpha}$) such that $\mathbb{P}(X_0^{x_n} = x_n) = \mathbb{P}(X_0^x = x) = 1$. It follows from Theorem II.4.8 *(ii.a)* and Lemma II.5.9 that $\mathbb{P} - a.s.$

$$\lim_{n\to\infty} \varphi(X_t^{x_n}) = \lim_{n\to\infty} \varphi\big(\mathcal{V}(x_n,Z)(t) + Z_t\big) = \lim_{n\to\infty} \varphi\big(\mathcal{V}(x,Z)(t) + Z_t\big) = \varphi(X_t^x),\ \forall\, t \geq 0.$$

Thanks to the boundedness of $\varphi$, the Dominated Convergence Theorem gives

$$\lim_{n\to\infty} (P_t\varphi)(x_n) = \lim_{n\to\infty} \mathbb{E}\big[\varphi(X_t^{x_n})\big] = \mathbb{E}\big[\varphi(X_t^x)\big] = (P_t\varphi)(x), \qquad \forall\, t \geq 0.$$

As far as the condition $(ii)$ in Theorem II.5.8 is concerned, we fix $\varepsilon > 0$, then for all $T, R > 0$, by the Chebyshev inequality and estimate (II.4.24)

$$\begin{aligned}
\frac{1}{T}\int_0^T (P_t^*\delta_x)\Big(H^{2\alpha} \setminus \bar{B}_R^{H^{2\alpha}}\Big)\,\mathrm{d}t &= \frac{1}{T}\int_0^T \mathbb{P}\big(\|X_t^x\|_{H^{2\alpha}} > R\big)\,\mathrm{d}t \\
&\leq \frac{1}{TR^2}\int_0^T \mathbb{E}\|X_t^x\|_{H^{2\alpha}}^2\,\mathrm{d}t \\
&= \frac{1}{TR^2}\mathbb{E}\int_0^T \|X_t^x\|_{H^{2\alpha}}^2\,\mathrm{d}t \\
&\leq \frac{C}{\nu TR^2}\bigg(\|x\|_{H^\alpha}^2 + \frac{1}{\nu^2}\|x\|_{H^1}^2 + \frac{T}{\nu} + T\nu\bigg),
\end{aligned}$$

where the constant $C > 0$ is independent of $T$ and $\nu$. The hypothesis is verified if we choose $R > 0$ such that

$$\frac{C}{\nu R^2}\bigg(\frac{1}{\nu} + \nu\bigg) \leq \varepsilon.$$

***Step 2***. Let us now prove the moment estimates for an invariant measure $\mu_\nu \in \mathscr{P}(H^{2\alpha})$ for the semigroup $P$.

Assume that $p > 2$ and let, for any $n \in \mathbb{N}$, $\varphi_n : H^{2\alpha} \ni x \mapsto \|x\|_{H^1}^p \wedge n$. Further assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X^x)$ is a solution to the equation (*$HNS_{\nu,\alpha}$*) such that $\mathbb{P}(X_0^x = x) = 1$. For all $n \in \mathbb{N}$, since $\varphi_n \in \mathcal{B}_b(H^{2\alpha})$, we have by the invariance property of $\mu_\nu$ and the property (II.5.4) for $P_1^*$

$$\begin{aligned}
\int_{H^{2\alpha}} \|x\|_{H^1}^p \wedge n\,\mathrm{d}\mu_\nu(x) &= \int_{H^{2\alpha}} \varphi_n(x)\,\mathrm{d}(P_1^*\mu_\nu)(x) \\
&= \int_{H^{2\alpha}} (P_1\varphi_n)(x)\,\mathrm{d}\mu_\nu(x) \\
&= \int_{H^{2\alpha}} \mathbb{E}\big[\|X_1^x\|_{H^1}^p \wedge n\big]\,\mathrm{d}\mu_\nu(x).
\end{aligned}$$

In the last equality we expanded the definition of $P_1$. Both the first and last members of this chain of equalities converge by the Monotone Convergence Theorem as $n \to \infty$. By uniqueness of the limit we have

$$\begin{aligned}
\int_{H^{2\alpha}} \|x\|_{H^1}^p\,\mathrm{d}\mu_\nu(x) &= \int_{H^{2\alpha}} \mathbb{E}\big[\|X_1^x\|_{H^1}^p\big]\,\mathrm{d}\mu_\nu(x) \\
&\leq \int_{H^{2\alpha}} C_p + e^{-(p-1)\nu}\|x\|_{H^1}^p\,\mathrm{d}\mu_\nu(x) \\
&= C_p + e^{-(p-1)\nu}\int_{H^{2\alpha}} \|x\|_{H^1}^p\,\mathrm{d}\mu_\nu(x),
\end{aligned}$$

where we used the estimate (II.4.23) in Theorem II.4.8 *(iv)*. After rearranging the terms, we reach

$$\int_{H^{2\alpha}} \|x\|_{H^1}^p\,\mathrm{d}\mu_\nu(x) \leq \frac{C_p}{1 - e^{-(p-1)\nu}} =: C_{p,\nu} < +\infty. \tag{II.5.18}$$

We argue *mutatis mutandis* to get the other estimates. With a similar argument as before we achieve

$$\begin{aligned}\int_{H^{2\alpha}} \|x\|_{H^\alpha}^4 \,\mathrm{d}\mu_\nu(x) &= \int_{H^{2\alpha}} \mathbb{E}\big[\|X_1^x\|_{H^\alpha}^4\big] \,\mathrm{d}\mu_\nu(x)\\ &\le \int_{H^{2\alpha}} \left(e^{-\nu}\|x\|_{H^\alpha}^4 + \frac{C}{\nu^3}\|x\|_{H^1}^8 + 2C\right) \mathrm{d}\mu_\nu(x)\\ &\le e^{-\nu}\int_{H^{2\alpha}} \|x\|_{H^\alpha}^4 \,\mathrm{d}\mu_\nu(x) + \frac{C}{\nu^3}C_{8,\nu} + 2C,\end{aligned}$$

where we used the estimate (II.4.27) from Theorem II.4.8 *(iv)*, and the finite constant $C_{p,\nu}$ from equation (II.5.18). The thesis follows by rearranging the terms. Analogously, making use of the estimate (II.4.28) from Theorem II.4.8, we obtain

$$\begin{aligned}\int_{H^{2\alpha}} \|x\|_{H^{2\alpha}}^2 \,\mathrm{d}\mu_\nu(x) &= \int_{H^{2\alpha}} \mathbb{E}\big[\|X_1^x\|_{H^{2\alpha}}^2\big] \,\mathrm{d}\mu_\nu(x)\\ &\le e^{-\nu}\int_{H^{2\alpha}} \|x\|_{H^{2\alpha}}^2 \,\mathrm{d}\mu_\nu(x) + \frac{C}{\nu^2}\int_{H^{2\alpha}} \|x\|_{H^\alpha}^4 \,\mathrm{d}\mu_\nu(x)\\ &\qquad + \frac{C}{\nu^5}\int_{H^{2\alpha}} \|x\|_{H^1}^8 \,\mathrm{d}\mu_\nu(x) + \frac{C}{\nu} + 2C,\end{aligned}$$

which leads to the thesis by the estimates (II.5.15), and (II.5.16) just proved and by rearranging the terms. □

### II.5.2 Construction of stationary solutions

Once proved the existence of invariant measures for the stochastic hyperviscous Navier-Stokes Equation, we need to construct a solution whose law is a given invariant measure. This result is indeed essential to later study the inviscid limit to the Eulerian case, which will be performed by passing to the limit the stationary solutions in appropriate trajectory spaces.

**Proposition II.5.11.** *Assume that* $\alpha > 1$. *There exists an augmented filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$, *with an adapted* $H^{2\alpha}$*-valued Wiener process* $W$, *that satisfies the following property. If* $\nu > 0$ *and if* $\mu_\nu \in \mathscr{P}(H^{2\alpha})$ *is an invariant measure for the equation* ($HNS_{\nu,\alpha}$), *see Definition II.5.7, then there exists a process* $X^\nu : \mathbb{R}_+ \times \Omega \to H^{2\alpha}$ *such that* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P}; W; X^\nu)$ *is a solution to the equation* ($HNS_{\nu,\alpha}$), *see Definition II.4.1, such that for all* $T > 0$

$$\begin{aligned}X^\nu \in &\left(\bigcap_{p\ge1} L^p\big(\Omega; C\big([0,T]; H^1\big)\big)\right) \cap L^4\big(\Omega; C\big([0,T]; H^\alpha\big)\big)\\ &\cap L^2\big(\Omega; C\big([0,T]; H^{2\alpha}\big)\big) \cap L^2\big(\Omega\times[0,T]; H^{3\alpha}\big),\end{aligned} \tag{II.5.19}$$

*and such that* $(X_0^\nu)_*\mathbb{P} = \mu_\nu \in \mathscr{P}(H^{2\alpha})$. *In particular,* $X^\nu$ *is stationary, i.e.*

$$(X_t^\nu)_*\mathbb{P} = \mu_\nu \in \mathscr{P}(H^{2\alpha}), \qquad \forall\, t \ge 0.$$

*Proof.* ***Step 1.*** For fixed $\alpha > 1$, we will construct a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ that satisfies the usual conditions and enjoys the following properties.

- An $H^{2\alpha}$-valued Wiener process $W$ defined and adapted on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ exists.
- For any $\nu > 0$, if $\mu_\nu \in \mathscr{P}(H^{2\alpha})$ is an invariant measure for the equation ($HNS_{\nu,\alpha}$) (which exists thanks to Theorem II.5.10), then there exists a random variable $\xi_\nu : \Omega \to H^{2\alpha}$ distributed as $\mu_\nu$.

First, we define $\Omega^1 := (H^{2\alpha})^{(0,+\infty)}$, we fix, for any $\nu > 0$, an invariant measure $\mu_\nu \in \mathscr{P}(H^{2\alpha})$ for the equation ($HNS_{\nu,\alpha}$), and we let $\bar{\xi}_\nu : \Omega^1 \ni \omega \mapsto \omega(\nu) \in H^{2\alpha}$, for all $\nu > 0$. Then, [Coh13, Exercise 2, Section 10.6] states the existence of a $\sigma$-algebra $\mathcal{F}^1$ on $\Omega^1$ and a probability measure $\mathbb{P}^1$ on $(\Omega^1, \mathcal{F}^1)$ such that $\bar{\xi}_\nu$ is a random variable on $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$ distributed as $\mu_\nu$, for any $\nu > 0$. Next, we endow $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$ with an arbitrary filtration $\{\mathcal{F}^1_t\}_{t\geq 0}$. In addition, we consider a filtered probability space $\big(\Omega^2, \mathcal{F}^2, \{\mathcal{F}^2_t\}_{t\geq 0}, \mathbb{P}^2\big)$ with an adapted $H^{2\alpha}$-valued Wiener process $\bar{W}$.

Finally, we define $(\Omega, \mathcal{F}, \mathbb{P})$ as the completion of $(\Omega^1 \times \Omega^2, \mathcal{F}^1 \otimes \mathcal{F}^2, \mathbb{P}^1 \times \mathbb{P}^2)$ and, for all $t \geq 0$, we let $\mathcal{F}_t$ be the completion with respect to $\mathbb{P}$ of $\bigcap_{s>t} \mathcal{F}^1_s \otimes \mathcal{F}^2_s$. The so constructed filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfies the usual conditions.

Eventually, recalling that $\Omega := \Omega^1 \times \Omega^2$, we let

$$
\begin{aligned}
W_t : \Omega \ni (\omega_1, \omega_2) \mapsto \bar{W}_t(\omega_2) \in H^{2\alpha}, \qquad \forall\, t \geq 0,\\
\xi_\nu : \Omega \ni (\omega_1, \omega_2) \mapsto \bar{\xi}_\nu(\omega_1) \in H^{2\alpha}, \qquad \forall\, \nu > 0,
\end{aligned}
$$

which satisfy the required properties by direct inspection.

***Step 2.*** Let us now fix $\nu > 0$. Theorem II.4.8 *(ii.a)* states the existence of a process $X^{\xi_\nu} : \mathbb{R}_+ \times \Omega \to H^{2\alpha}$, that we hereby denote simply by $X^\nu$, such that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X^\nu)$ is a solution to the equation ($HNS_{\nu,\alpha}$) with $\mathbb{P}\big(X^\nu_0 = \xi_\nu\big) = 1$. In addition, since $(\xi_\nu)_*\mathbb{P} = (X^\nu_0)_*\mathbb{P} = \mu_\nu$, the moment estimates (II.5.15), (II.5.16), (II.5.17) imply $X^\nu_0 \in L^2(\Omega; H^{2\alpha}) \cap L^4(\Omega; H^\alpha) \cap L^p(\Omega; H^1)$ for all $p \geq 1$. Hence, by Theorem II.4.8 *(iii)*, the process $X^\nu$ has the sought regularities.

It only remains to show that $X^\nu$ is stationary. Let us fix $t \geq 0$, then $(X^\nu_t)_*\mathbb{P} = \mu_\nu \in \mathscr{P}(H^{2\alpha})$ if and only if

$$
\int_{H^{2\alpha}} \varphi \,\mathrm{d}\mu_\nu = \int_{H^{2\alpha}} \varphi \,\mathrm{d}\big((X^\nu_t)_*\mathbb{P}\big), \qquad \forall\, \varphi \in \mathcal{B}_b(H^{2\alpha}). \tag{II.5.20}
$$

Let us fix $\varphi \in \mathcal{B}_b(H^{2\alpha})$. The following equalities come from the invariance property of $\mu_\nu$, see Definition II.5.7, the property (II.5.4) for $P_t$, the Change of Variable Theorem and the definition of $P_t$ in equation (II.5.1):

$$
\begin{aligned}
\int_{H^{2\alpha}} \varphi \,\mathrm{d}\mu_\nu &= \int_{H^{2\alpha}} \varphi \,\mathrm{d}\big(P^*_t \mu_\nu\big)\\
&= \int_{H^{2\alpha}} P_t\varphi \,\mathrm{d}\mu_\nu\\
&= \int_{H^{2\alpha}} P_t\varphi \,\mathrm{d}\big((\xi_\nu)_*\mathbb{P}\big)\\
&= \int_{\Omega} (P_t\varphi)\big((\xi_\nu)(\omega)\big) \,\mathrm{d}\mathbb{P}(\omega)\\
&= \int_{\Omega}\int_{\Omega} \varphi\Big(X^{\xi_\nu(\omega)}_t(\omega')\Big) \,\mathrm{d}\mathbb{P}(\omega') \,\mathrm{d}\mathbb{P}(\omega).
\end{aligned}
$$

Let us define

$$\tilde{X}_s : \Omega \times \Omega \ni (\omega, \omega') \mapsto \tilde{X}_s(\omega, \omega') := X_s^{\xi_\nu(\omega)}(\omega') \in H^{2\alpha}, \qquad \forall\, s \geq 0,$$
$$\tilde{W}_s : \Omega \times \Omega \ni (\omega, \omega') \mapsto \tilde{W}_s(\omega, \omega') := W_s(\omega') \in H^{2\alpha}, \qquad \forall\, s \geq 0,$$

and consider the augmented filtered probability space $(\Omega \times \Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_s\}_{s\geq 0}, \tilde{\mathbb{P}})$ constructed, as already outlined in *Step* 1, from $(\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, \{\mathcal{F}_s \otimes \mathcal{F}_s\}_{s\geq 0}, \mathbb{P} \otimes \mathbb{P})$. We denote $\tilde{\Omega} := \Omega \times \Omega$. In particular, $\tilde{W}$ is an adapted $H^{2\alpha}$-valued Wiener process. Moreover, by direct inspection, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_s\}_{s\geq 0}, \tilde{\mathbb{P}}; \tilde{W}; \tilde{X})$ is a solution to ($HNS_{\nu,\alpha}$) such that $\tilde{\mathbb{P}}(\tilde{X}_0 = \xi_\nu) = 1$. In particular, $(\tilde{X}_0)_*\tilde{\mathbb{P}} = (\xi_\nu)_*\mathbb{P}$, thus the uniqueness in law for the solution, see Theorem II.4.8 *(i)*, yields

$$(\tilde{X}_s)_*\tilde{\mathbb{P}} = (X_s^\nu)_*\mathbb{P} \in \mathscr{P}(H^{2\alpha}), \qquad \forall\, s \geq 0.$$

Therefore, we can conclude the chain of equalities above with

$$\begin{aligned}\int_{H^{2\alpha}} \varphi \, \mathrm{d}\mu_\nu &= \int_{\tilde{\Omega}} \varphi\big(\tilde{X}_t(\omega, \omega')\big) \, \mathrm{d}\tilde{\mathbb{P}}(\omega, \omega') \\ &= \int_{\tilde{\Omega}} \varphi \, \mathrm{d}\big((\tilde{X}_t)_*\tilde{\mathbb{P}}\big) \\ &= \int_{\Omega} \varphi \, \mathrm{d}\big((X_t^\nu)_*\mathbb{P}\big).\end{aligned}$$

We obtained the sought equality (II.5.20) and the claim follows. □

Eventually, the stationary solutions provided in the previous theorem, also allow us to obtain further estimates for the invariant measure, as shown in the following result.

**Theorem II.5.12.** *Assume that $\alpha > 1$. There exists a finite constant $C_\alpha > 0$ such that, assuming $\nu > 0$, if $\mu_\nu \in \mathscr{P}(H^{2\alpha})$ is an invariant measure for the equation ($HNS_{\nu,\alpha}$), see Definition II.5.7, then*

$$\int_{H^{2\alpha}} \|x\|^2_{H^{\alpha+1}} \, \mathrm{d}\mu_\nu(x) = C_\alpha, \tag{II.5.21}$$
$$\int_{H^{2\alpha}} \|x\|^{2n}_{H^1} \, \mathrm{d}\mu_\nu(x) \leq (2n-1)!!\, C_\alpha^n, \qquad \forall\, n \in \mathbb{N}. \tag{II.5.22}$$

*Moreover, if $\beta > 0$ is such that $2\beta C_\alpha < 1$, then*

$$\int_{H^{2\alpha}} e^{\beta\|x\|^2_{H^{\alpha+1}}} \, \mathrm{d}\mu_\nu(x) \leq 2e^{\frac{2\beta C_\alpha}{1-2\beta C_\alpha}}. \tag{II.5.23}$$

*Proof.* Fix $\nu > 0$ and $\alpha > 1$. If $\mu_\nu \in \mathscr{P}(H^{2\alpha})$ is an invariant measure for the equation ($HNS_{\nu,\alpha}$), let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; W; X^\nu)$ be the solution to the equation ($HNS_{\nu,\alpha}$) from Proposition II.5.11 that satisfies

$$(X_t^\nu)_*\mathbb{P} = (X_0^\nu)_*\mathbb{P} = \mu_\nu \in \mathscr{P}(H^{2\alpha}), \qquad \forall\, t \geq 0.$$

***Equation* (II.5.21).** We apply the Itô formula in Lemma II.4.7 to the stationary process $X^\nu$ and with the choice $\gamma = 1$ and $h = \mathbb{I}_{\mathbb{R}_+}$. The last but one term in equation (II.4.20) vanishes, while the last is deterministic. Also, the non-linearity vanishes because of the well-known property $\langle B(x), x\rangle_{H^1} = \langle B(x), Ax\rangle = 0$, for all $x \in H^2$. Next, we take the expectation on both members of the equality and obtain

$$\mathbb{E}\|X_t^\nu\|_{H^1}^2 + 2\nu\mathbb{E}\int_0^t \|X_s^\nu\|_{H^{\alpha+1}}^2 \,\mathrm{d}s = \mathbb{E}\|X_0^\nu\|_{H^1}^2 + \nu\,\mathrm{Tr}[Q_1]t, \qquad \forall\, t \geq 0,$$

where, we recall, $Q_1 := \iota Q \iota^* \in \mathcal{L}_1(H^1)$, while $Q \in \mathcal{L}_1(H^{2\alpha})$ is the covariance operator of $W_1$, and $\iota : H^{2\alpha} \to H^1$ is the Sobolev embedding. Since $X^\nu$ is stationary, the first terms on both sides are equal and cancel out. We apply Fubini's Theorem, divide by $2\nu$ and rewrite

$$\int_0^t \left(\mathbb{E}\|X_s^\nu\|_{H^{\alpha+1}}^2 - \frac{\mathrm{Tr}[Q_1]}{2}\right) \mathrm{d}s = 0, \qquad \forall\, t \geq 0.$$

This last equality implies

$$\int_{H^{2\alpha}} \|x\|_{H^{\alpha+1}}^2 \,\mathrm{d}\mu_\nu(x) = \mathbb{E}\|X_t^\nu\|_{H^{\alpha+1}}^2 = \frac{\mathrm{Tr}[Q_1]}{2}, \qquad \forall t \geq 0.$$

The claim is proved with $C_\alpha := \mathrm{Tr}[Q_1]/2$.

***Equation* (II.5.22).** We prove the estimate (II.5.22) by induction on $n \in \mathbb{N}$.

The base case follows from the previous part and the Sobolev embedding $\|x\|_{H^1} \leq \|x\|_{H^{\alpha+1}}$ for $x \in H^{\alpha+1}$. Let us prove the inductive step. We apply the Itô formula in Lemma II.4.7 to to the stationary process $X^\nu$ and with the choices $\gamma = 1$ and $h : \mathbb{R}_+ \ni r \mapsto r^{n+1} \in \mathbb{R}_+$. The non-linearity vanishes because of the well-known property $\langle B(x), x\rangle_{H^1} = \langle B(x), Ax\rangle = 0$, for all $x \in H^2$. Next, we take the expectation on both members of the equation (II.4.20) and obtain for all $t \geq 0$

$$\begin{aligned}
&\mathbb{E}\|X_t^\nu\|_{H^1}^{2n+2} + 2(n+1)\nu\,\mathbb{E}\int_0^t \|X_s^\nu\|_{H^{\alpha+1}}^2 \|X_s^\nu\|_{H^1}^{2n} \,\mathrm{d}s \\
&= \mathbb{E}\|X_0^\nu\|_{H^1}^{2n+2} + 2n(n+1)\nu\,\mathbb{E}\int_0^t \big\|Q_1^{1/2} X_s^\nu\big\|_{H^1}^2 \|X_s^\nu\|_{H^1}^{2(n-1)} \,\mathrm{d}s \\
&\qquad + (n+1)\nu\,\mathrm{Tr}[Q_1]\mathbb{E}\int_0^t \|X_s^\nu\|_{H^1}^{2n} \,\mathrm{d}s \\
&\leq \mathbb{E}\|X_0^\nu\|_{H^1}^{2n+2} + (n+1)(2n+1)\nu\,\mathrm{Tr}[Q_1]\mathbb{E}\int_0^t \|X_s^\nu\|_{H^1}^{2n} \,\mathrm{d}s,
\end{aligned}$$

where we used $\|Q_1^{1/2}x\| \leq \mathrm{Tr}[Q_1]\|x\|_{H^1}$ for all $x \in H^1$, see equation (II.4.29). Since $X^\nu$ is stationary, the first terms on both sides of the inequality are equal and cancel out. We apply Fubini's Theorem, divide by $2(n+1)\nu$ and rewrite

$$\int_0^t \left(\mathbb{E}\big[\|X_s^\nu\|_{H^{\alpha+1}}^2 \|X_s^\nu\|_{H^1}^{2n}\big] - (2n+1)\frac{\mathrm{Tr}[Q_1]}{2}\mathbb{E}\|X_s^\nu\|_{H^1}^{2n}\right) \mathrm{d}s \leq 0, \qquad \forall\, t \geq 0.$$

This last inequality implies, recalling that $(X_s^\nu)_*\mathbb{P} = \mu_\nu \in \mathscr{P}(H^{2\alpha})$ for all $s \geq 0$ and the Sobolev embedding $\|x\|^2_{H^1} \le \|x\|^2_{H^{\alpha+1}}$ for $x \in H^{\alpha+1}$

$$\begin{aligned}\int_{H^{2+\delta}} \|x\|^{2n+2}_{H^1}\,\mathrm{d}\mu_\nu(x) &\le \int_{H^{2\alpha}} \|x\|^2_{H^{\alpha+1}}\|x\|^{2n}_{H^1}\,\mathrm{d}\mu_\nu(x)\\ &= \mathbb{E}\Big[\|X^\nu_s\|^2_{H^{\alpha+1}}\|X^\nu_s\|^{2n}_{H^1}\Big]\\ &\le (2n+1)\frac{\mathrm{Tr}[Q_1]}{2}\mathbb{E}\|X^\nu_s\|^{2n}_{H^1}\\ &= (2n+1)\frac{\mathrm{Tr}[Q_1]}{2}\int_{H^{2+\delta}}\|x\|^{2n}_{H^1}\,\mathrm{d}\mu_\nu(x)\\ &\le (2n+1)!!C_\alpha^{n+1},\end{aligned}$$

where we used the inductive hypothesis in the last step and recalled, from the first part of the proof, that $C_\alpha := \mathrm{Tr}[Q_1]/2$.

***Equation* (II.5.23).** Fix now $\beta > 0$, and apply the Itô formula in Lemma II.4.7 with the choices $\gamma = 1$, $h : \mathbb{R}_+ \ni r \mapsto h(r) = e^{\beta r}$ and to the stationary process $X^\nu$. After taking the expectation to both sides of equation (II.4.20), we obtain for all $t \ge 0$

$$\begin{aligned}\mathbb{E}\, e^{\beta\|X^\nu_t\|^2_{H^1}} = \mathbb{E}\, e^{\beta\|X^\nu_0\|^2_{H^1}} &- 2\beta\nu\,\mathbb{E}\int_0^t e^{\beta\|X^\nu_s\|^2_{H^1}}\|X^\nu_s\|^2_{H^{\alpha+1}}\,\mathrm{d}s\\ &- 2\beta\,\mathbb{E}\int_0^t e^{\beta\|X^\nu_s\|^2_{H^1}}\langle B(X^\nu_s), X^\nu_s\rangle_{H^1}\,\mathrm{d}s\\ &+ 2\beta\sqrt{\nu}\,\mathbb{E}\int_0^t e^{\beta\|X^\nu_s\|^2_{H^1}}\langle X^\nu_s, \mathrm{d}W_s\rangle_{H^1}\\ &+ 2\beta^2\nu\mathbb{E}\int_0^t \|Q_1^{1/2}X^\nu_s\|^2_{H^1}e^{\beta\|X^\nu_s\|^2_{H^1}}\,\mathrm{d}s\\ &+ \beta\nu\,\mathrm{Tr}[Q_1]\mathbb{E}\int_0^t e^{\beta\|X^\nu_s\|^2_{H^1}}\,\mathrm{d}s.\end{aligned}$$

Due to the stationarity of the process, the left-hand side and the first term on the right-hand side are equal. The integral of the non-linearity vanishes, because of the property $\langle B(x), x\rangle_{H^1} = \langle B(x), Ax\rangle = 0$ for all $x \in H^2$. The expectation of the Itô integral is 0 by the properties of Itô integration. We employ Fubini's Theorem and the stationarity of the process to compute the time integral in all the other terms. We have for all $t \ge 0$

$$\begin{aligned}0 = -2\beta\nu\, t\mathbb{E}\big[e^{\beta\|X^\nu_0\|^2_{H^1}}\|X^\nu_0\|^2_{H^{\alpha+1}}\big] + 2\beta^2\nu\, t\mathbb{E}\Big[\big\|Q_1^{1/2}X^\nu_0\big\|^2_{H^1}e^{\beta\|X^\nu_0\|^2_{H^1}}\Big]&\\ + \beta\nu\,\mathrm{Tr}[Q_1]\, t\mathbb{E}\big[e^{\beta\|X^\nu_0\|^2_{H^1}}\big]&.\end{aligned}$$

Let us divide by $\beta\nu t$, for $t > 0$, and rewrite as follows

$$\begin{aligned}0 &= \mathbb{E}\Big[e^{\beta\|X^\nu_0\|^2_{H^1}}\big(\mathrm{Tr}[Q_1] + 2\beta\big\|Q_1^{1/2}X^\nu_0\big\|^2 - 2\|X^\nu_0\|^2_{H^{\alpha+1}}\big)\Big]\\ &\le \mathbb{E}\Big[e^{\beta\|X^\nu_0\|^2_{H^1}}\big(\mathrm{Tr}[Q_1] + 2\beta\,\mathrm{Tr}[Q_1]\|X^\nu_0\|^2_{H^1} - 2\|X^\nu_0\|^2_{H^{\alpha+1}}\big)\Big]\\ &\le \mathbb{E}\Big[e^{\beta\|X^\nu_0\|^2_{H^{\alpha+1}}}\Big(\mathrm{Tr}[Q_1] + 2\big(\beta\,\mathrm{Tr}[Q_1] - 1\big)\|X^\nu_0\|^2_{H^{\alpha+1}}\Big)\Big]\\ &= 2\,\mathbb{E}\Big[e^{\beta\|X^\nu_0\|^2_{H^{\alpha+1}}}\Big(\mathrm{Tr}[Q_1] - \big(1-\beta\,\mathrm{Tr}[Q_1]\big)\|X^\nu_0\|^2_{H^{\alpha+1}}\Big)\Big] - \mathrm{Tr}[Q_1]\mathbb{E}e^{\beta\|X^\nu_0\|^2_{H^{\alpha+1}}},\end{aligned} \tag{II.5.24}$$

where for the first inequality we used $\|Q_1^{1/2}x\|_{H^1}^2 \le \|x\|_{H^1}^2 \operatorname{Tr}[\Pi_x Q_1]$ for all $x \in H^1$, see equation (II.4.29), while for the second we used the Sobolev embedding $H^1 \hookrightarrow H^{\alpha+1}$. Let us baptise the real random variable in the big round brackets as

$$J := \operatorname{Tr}[Q_1] - \big(1 - \beta \operatorname{Tr}[Q_1]\big)\|X_0^\nu\|_{H^{\alpha+1}}^2,$$

and assume that $1 - \beta \operatorname{Tr}[Q_1] > 0$ (*i.e.* $2\beta C_\alpha < 1$), so that $\mathbb{P}(J < \operatorname{Tr}[Q_1]) = 1$. We reach from (II.5.24)

$$\begin{aligned}
\operatorname{Tr}[Q_1]\mathbb{E}e^{\beta\|X_0^\nu\|_{H^{\alpha+1}}^2} &\le 2\mathbb{E}\Big[e^{\beta\|X_0^\nu\|_{H^{\alpha+1}}^2} J\Big] \\
&\le 2\mathbb{E}\Big[e^{\beta\|X_0^\nu\|_{H^{\alpha+1}}^2} J\mathbb{1}_{\{J\ge 0\}}\Big] \\
&\le 2\operatorname{Tr}[Q_1]\mathbb{E}\Big[e^{\beta\|X_0^\nu\|_{H^{\alpha+1}}^2}\mathbb{1}_{\{J\ge 0\}}\Big],
\end{aligned}$$

where we used $J = J\mathbb{1}_{\{J\ge 0\}} + J\mathbb{1}_{\{J<0\}} \le J\mathbb{1}_{\{J\ge 0\}} \le \operatorname{Tr}[Q_1]\mathbb{1}_{\{J\ge 0\}}$, $\mathbb{P}-a.s.$ Observe that

$$J \ge 0 \quad \Longleftrightarrow \quad \|X_0^\nu\|_{H^{\alpha+1}}^2 \le \frac{\operatorname{Tr}[Q_1]}{1-\beta\operatorname{Tr}[Q_1]},$$

hence

$$\operatorname{Tr}[Q_1]\mathbb{E}e^{\beta\|X_0^\nu\|_{H^{\alpha+1}}^2} \le 2\operatorname{Tr}[Q_1]\mathbb{E}\Big[e^{\beta\|X_0^\nu\|_{H^{\alpha+1}}^2}\mathbb{1}_{\{J\ge 0\}}\Big] \le 2\operatorname{Tr}[Q_1]e^{\beta\frac{\operatorname{Tr}[Q_1]}{1-\beta\operatorname{Tr}[Q_1]}}\mathbb{P}(J\ge 0).$$

We divide by $\operatorname{Tr}[Q_1]$ and use a change of variable, recalling that $X_0^\nu$ is distributed as $\mu_\nu$

$$\int_{H^{2\alpha}} e^{\beta\|x\|_{H^{\alpha+1}}^2}\,\mathrm{d}\mu_\nu(x) = \mathbb{E}e^{\beta\|X_0^\nu\|_{H^{\alpha+1}}^2} \le 2e^{\beta\frac{\operatorname{Tr}[Q_1]}{1-\beta\operatorname{Tr}[Q_1]}}.$$

The sought estimate (II.5.23) follows. □

## II.6 Invariant measure for the deterministic Euler Equation

In this section, we study the inviscid limit of the stochastic hyperviscous Navier-Stokes equations. The passage to the limit, as $\nu \to 0^+$, for the stationary solutions to equation ($HNS_{\nu,\alpha}$), see Proposition II.5.11, will be conducted in the spirit of the Skorokhod theorem. Hence, it is necessary to demonstrate some tightness property. However, the usual trajectory spaces for the solution have topologies that are too large to prove tightness. A suitable trajectory space, which allows for the proof of a tightness result and the application of Jakubowski's generalization of the Skorokhod theorem, is introduced in the next definition.

**Definition II.6.1.** Assume that $\alpha > 1$ and let $W$ be an $H^{2\alpha}$-valued Wiener process. Assuming $T > 0$, we define

$$\mathcal{Z}_T := L^2_w(0,T;H^{\alpha+1}) \cap L^2(0,T;H^\alpha_{loc}) \cap C\big([0,T];H^\alpha_w\big) \cap C\big([0,T];U'\big),$$

where

- $U$ is the reproducing kernel of $W$. In particular, $U$ is compactly embedded into $H^{2\alpha}$ and simultaneously $H^{-2\alpha}$ is compactly embedded into $U'$:

$$U \hookrightarrow H^{2\alpha} \hookrightarrow H = H' \hookrightarrow H^{-2\alpha} \hookrightarrow U'.$$

- $H^\alpha_{loc}$ is the vector space $H^\alpha$ endowed with the Fréchet topology generated by the family of seminorms, for $x \in H^\alpha$,

$$[\![x]\!]_{H^\alpha_n} := \inf\big\{\|\varphi x\|_{H^\alpha} \;:\; \varphi \in C^\infty_c(\mathbb{R}^2),\ \operatorname{supp}\varphi \supset \bar{B}_n,\ \varphi|_{\bar{B}_n} = 1,\ \nabla\varphi\cdot x = 0\big\}, \tag{II.6.1}$$

where $\bar{B}_n := \{\xi \in \mathbb{R}^2 \;:\; |\xi| \le n\}$, for $n \in \mathbb{N}$.

- $L^2(0,T;H^\alpha_{loc})$ is the linear space of all measurable functions $v : [0,T] \to H^\alpha$ such that the functions $[\![v]\!]_{H^\alpha_n} : [0,T] \ni t \mapsto [\![v(t)]\!]_{H^\alpha_n} \in [0,+\infty)$, $n \in \mathbb{N}$, belong to $L^2(0,T)$. It is endowed with the Fréchet topology generated by the family of seminorms

$$L^2(0,T;H^\alpha_{loc}) \ni v \mapsto [\![v]\!]_{L^2(0,T;H^\alpha_n)} := \left(\int_0^T [\![v(t)]\!]^2_{H^\alpha_n}\,\mathrm{d}t\right)^{1/2}, \qquad \forall\, n \in \mathbb{N}. \tag{II.6.2}$$

- $L^2_w(0,T;H^{\alpha+1})$ is the linear space $L^2(0,T;H^{\alpha+1})$ endowed with the weak topology, namely the smallest topology that makes all the linear maps $L^2_w(0,T;H^{\alpha+1}) \to \mathbb{R}$ continuous. In particular, a sequence of functions $\{v_n\}_{n\in\mathbb{N}} \subset L^2(0,T;H^{\alpha+1})$ converges in $L^2_w(0,T;H^{\alpha+1})$ if there exists $v \in L^2(0,T;H^{\alpha+1})$ such that

$$\lim_{n\to\infty}\int_0^T {}_{H^{-\alpha-1}}\big\langle \phi(t), v_n(t) - v(t)\big\rangle_{H^{\alpha+1}}\,\mathrm{d}t = 0, \qquad \forall\, \phi \in L^2(0,T;H^{-\alpha-1}).$$

- $C\big([0,T];H^\alpha_w\big)$ is the linear space of continuous functions $v : [0,T] \to H^\alpha_w$, where $H^\alpha_w$ is the space $H^\alpha$ endowed with weak topology, see Notation 0.9. Since the domain $[0,T]$ is a sequential space, the notion of continuity is equivalent to that of sequential continuity, see Notation 0.1. Namely, a function $v : [0,T] \to H^\alpha$ belongs to $C\big([0,T];H^\alpha_w\big)$ if for any sequence of times $\{t_n\}_{n\in\mathbb{N}} \subset [0,T]$ convergent to some $t \in [0,T]$

$$\lim_{n\to\infty} {}_{H^{-\alpha}}\big\langle y, v(t_n) - v(t)\big\rangle_{H^\alpha} = 0, \qquad \forall\, y \in H^{-\alpha}.$$

  This space is endowed with the compact-open topology, that is the smallest topology that contains all the sets $\big\{f \in C\big([0,T];H^\alpha_w\big) \;:\; f(K) \subset V\big\}$, where $K$ varies among compact subsets of $[0,T]$, and $V$ among open subsets of $H^\alpha_w$.

We endow $\mathcal{Z}_T$ with the smallest topology that makes the natural embeddings from $\mathcal{Z}_T$ into each of its four components, continuous (*i.e.* the smallest topology which contains the topologies of all the four components).

**Lemma II.6.2.** *Assume that $\alpha > 1$ and let $W$ be an $H^{2\alpha}$-valued Wiener process. If $\beta \ge -2\alpha$ and $T > 0$, then $C\big([0,T];H^\beta\big) \cap \mathcal{Z}_T \in \mathscr{B}_{\mathcal{Z}_T}$.*

*Proof.* From the embeddings $H^\beta \hookrightarrow H^{-2\alpha} \hookrightarrow U'$, we infer that that $C\big([0,T];H^\beta\big)$ is continuously embedded into $C\big([0,T];U'\big)$. Therefore, by Kuratowski's theorem [Kur66, Theorem 1, Section V, Chapter 39], see also [Kec95, Theorem 15.1], the space $C\big([0,T];H^\beta\big)$ is a Borel subset of $C\big([0,T];U'\big)$. Eventually, the intersection $C\big([0,T];H^\beta\big) \cap \mathcal{Z}_T$ is a Borel subset of $C\big([0,T];U'\big) \cap \mathcal{Z}_T = \mathcal{Z}_T$. □

The following tightness criterion and the subsequent lemma are taken from [BM13, Section 3.2], where the results are stated in a general abstract setting. The abstract Hilbert spaces $U$, $V$ and $H$ in the reference are replaced in our setting by $U$, $H^{\alpha+1}$ and $H^\alpha$, respectively.

**Theorem II.6.3** (Tightness criterion)**.** *Assume that $\alpha > 1$ and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be an augmented filtered probability space with an adapted $H^{2\alpha}$-valued Wiener process $W$. Let $U$ denote the reproducing kernel of $W$. Let also $T > 0$ and $X^n : [0,T] \times \Omega \to U'$, $n \in \mathbb{N}$, be adapted and pathwise continuous stochastic processes. Assume that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[ \sup_{t\in[0,T]} \|X^n_t\|^2_{H^\alpha} + \int_0^T \|X^n_t\|^2_{H^{\alpha+1}}\,\mathrm{d}t \right] < +\infty. \tag{II.6.3}$$

*Assume also that the sequence $\{X^n\}_{n\in\mathbb{N}}$ satisfies the Aldous condition in $U'$, i.e., for every $\varepsilon > 0$ and $\eta > 0$, there exists $\theta \in (0,T]$ such that, for every sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of $[0,T]$-valued stopping times, we have*

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,\theta]} \mathbb{P}\big(\|X^n_{t+\tau_n} - X^n_{\tau_n}\|_{U'} \geq \eta\big) \leq \varepsilon.$$

*Then the laws on $(\mathcal{Z}_T, \mathscr{B}_{\mathcal{Z}_T})$ of the stochastic processes form a tight sequence, i.e. for any $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \mathcal{Z}_T$ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{P}(X^n \in K_\varepsilon) \geq 1 - \varepsilon.$$

The space $\mathcal{Z}_T$ is constructed *ad hoc* to guarantee the thesis of Theorem II.6.3. This space must be equipped with a topology that is sufficiently strong to facilitate the convergence of sequences utilized in the subsequent analysis. Simultaneously, the topology must be weak enough to ensure the tightness result, meaning that there exists a compact set within this topology that satisfies the criteria for tightness.

The Aldous property mentioned in Theorem II.6.3 is quite intricate to be verified. Hence we present, in the following lemma, a sufficient condition to guarantee the Aldous property. We refer to [BM13, Section 3.2] for the proof of this result.

**Lemma II.6.4.** *Assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is an augmented filtered probability space. Let $U'$ be an Hilbert space and $T > 0$. Let $X^n : [0,T] \times \Omega \to U'$, $n \in \mathbb{N}$, be pathwise continuous stochastic processes. Assume that there exist $C > 0$ and $\gamma \in (0,1]$ such that for every $t \in (0,T]$ and any sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of $[0,T]$-valued stopping times*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\|X^n_{\tau_n+t} - X^n_{\tau_n}\|_{U'} \leq C t^\gamma, \tag{II.6.4}$$

*then the sequence $\{X^n\}_{n\in\mathbb{N}}$ satisfies the Aldous condition in $U'$, see Theorem II.6.3.*

We recall the Jakubowski's version [Jak97, Theorem 2] of the Skorokhod theorem, see also [BO13], and deduce a simple corollary, that will later come in handy.

**Theorem II.6.5** (Jakubowski Theorem)**.** *Let $\mathcal{X}$ be a topological space such that there exists a continuous and injective function $F : \mathcal{X} \to \mathbb{R}^{\mathbb{N}}$. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{X}$-valued Borel random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that the sequence of laws $\{(X_n)_*\mathbb{P}\}_{n\in\mathbb{N}}$ is tight in $(\mathcal{X}, \mathscr{B}_{\mathcal{X}})$. Then there exists a subsequence $\{X_{n_k}\}_{k\in\mathbb{N}}$, a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a Borel random variable $Y : (\tilde{\Omega}, \tilde{\mathcal{F}}) \to \mathcal{X}$ and a sequence $\{Y_k\}_{k\in\mathbb{N}}$ of $\mathcal{X}$-valued Borel random variables defined on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ such that*

$$(X_{n_k})_*\mathbb{P} = (Y_k)_*\tilde{\mathbb{P}} \in \mathscr{P}(\mathcal{X}), \qquad \forall\, k \in \mathbb{N}, \tag{II.6.5}$$

$$Y_k \longrightarrow Y, \qquad \tilde{\mathbb{P}} - a.s. \text{ in } \mathcal{X}, \text{ as } k \to \infty. \tag{II.6.6}$$

**Corollary II.6.6.** *Let $\mathcal{X}$ be a topological space such that there exists a continuous and injective function $F : \mathcal{X} \to \mathbb{R}^{\mathbb{N}}$. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a tight sequence of probability measures on $(\mathcal{X}, \mathscr{B}_{\mathcal{X}})$. Then there exists a subsequence $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ and a probability measure $\mu$ on $(\mathcal{X}, \mathscr{B}_{\mathcal{X}})$ such that*

$$\mu_{n_k} \longrightarrow \mu, \qquad \text{in } \mathscr{P}(\mathcal{X}), \text{ as } k \to \infty.$$

*Proof.* Assume the hypotheses. Proposition 10.6.1 in [Coh13] states the existence of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (which is the countable product of the probability spaces $(\mathcal{X}, \mathscr{B}_{\mathcal{X}}, \mu_n)$, $n \in \mathbb{N}$) and of $\mathcal{X}$-valued random variables $X_n$, $n \in \mathbb{N}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, such that $(X_n)_*\mathbb{P} = \mu_n$ on $(\mathcal{X}, \mathscr{B}_{\mathcal{X}})$ for every $n \in \mathbb{N}$.

The sequence $\{X_n\}_{n\in\mathbb{N}}$ satisfies the assumptions of Jakubowski's Theorem II.6.5. Consequently there exists a subsequence $\{X_{n_k}\}_{k\in\mathbb{N}}$, a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a Borel random variable $Y : (\tilde{\Omega}, \tilde{\mathcal{F}}) \to \mathcal{X}$ and a sequence $\{Y_k\}_{k\in\mathbb{N}}$ of $\mathcal{X}$-valued Borel random variables defined on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ such that the properties in (II.6.5) and (II.6.6) hold true. In particular, $\tilde{\mathbb{P}} - a.s.$ convergence implies convergence in law, *i.e.*

$$\mu_{n_k} = (X_{n_k})_*\mathbb{P} = (Y_k)_*\tilde{\mathbb{P}} \longrightarrow Y_*\tilde{\mathbb{P}} = \mu, \qquad \text{as } k \to \infty,$$

where the convergence is meant in the weak sense in $\mathscr{P}(\mathcal{X})$, see Notation 0.3. □

**Lemma II.6.7.** *Assume that $\alpha > 1$ and let $W$ be an $H^{2\alpha}$-valued Wiener process. If $\beta \in \mathbb{R}$, both the spaces $H^\beta_{bw}$, and $\mathcal{Z}_T \times C\big([0,T]; H^\beta\big)$ with the product topology, satisfy the assumption of Theorem II.6.5 and Corollary II.6.6.*

*Proof.* Let us momentarily say that a topological space $\mathcal{X}$ has the $S$ property if there exists an injective and continuous function $F : \mathcal{X} \to \mathbb{R}^{\mathbb{N}}$.

As for the topological space $H^\beta_{bw}$, we consider a complete orthonormal system $\{e_n\}_{n\in\mathbb{N}}$ for the separable Hilbert space $H^\beta$ and define the function

$$F : H^\beta \ni x \mapsto F(x) := \big\{\langle x, e_n\rangle_{H^\beta}\big\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}.$$

Then $F$ is linear, thus $F \in C\big(H^\beta_w; \mathbb{R}^{\mathbb{N}}\big)$ by definition of weak topology, see Notation 0.9, hence $F \in C\big(H^\beta_{bw}; \mathbb{R}^{\mathbb{N}}\big)$ by Remark 0.11. Also, it is injective. Let indeed $x, x' \in H^\beta$ be such that $F(x) = F(x')$, then $F(x - x') = 0$ by linearity, hence $x = x'$ thanks to

$$\|x - x'\|^2_{H^\beta} = \sum_{n=1}^{\infty} \langle x - x', e_n\rangle^2_{H^\beta} = 0.$$

The spaces $C\big([0,T]; H^\beta\big)$ and $C\big([0,T]; U'\big)$ have the $S$ property because they are Polish, see [Bad70, Exposè 8, page 124, Remark $(b)$].

As for the space $\mathcal{Z}_T$, we reason as follows. Let $\iota : \mathcal{Z}_T \to C\big([0,T]; U'\big)$ be the natural embedding, in particular, $\iota$ is injective and continuous. Let $F : C\big([0,T]; U'\big) \to \mathbb{R}^{\mathbb{N}}$ be an injective and continuous map, which exists because $C\big([0,T]; U'\big)$ has the $S$ property. Then $F \circ \iota : \mathcal{Z}_T \to \mathbb{R}^{\mathbb{N}}$ is continuous and injective. Hence $\mathcal{Z}_T$ has the $S$ property.

It only remains to prove that, if two topological spaces $\mathcal{X}$, $\mathcal{Y}$ have the $S$ property, so does the space $\mathcal{X} \times \mathcal{Y}$ with the product topology. Let $F : \mathcal{X} \to \mathbb{R}^{\mathbb{N}}$ and $G : \mathcal{Y} \to \mathbb{R}^{\mathbb{N}}$ be two injective and continuous functions, then, for every $n \in \mathbb{N}$, we denote by $f_n$

and $g_n$ the $n$-th component of $F$ and $G$, respectively. We define $H : \mathcal{X}\times\mathcal{Y} \ni (x,y) \mapsto H(x,y) := \{h_n(x,y)\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, where, for all $n \in \mathbb{N}$ and $(x,y) \in \mathcal{X}\times\mathcal{Y}$

$$h_{2n-1}(x,y) := f_n(x), \qquad\qquad h_{2n}(x,y) := g_n(y).$$

By direct inspection, $h_n$, $n \in \mathbb{N}$, are continuous with respect to the product topology on $\mathcal{X}\times\mathcal{Y}$, thus so is $H$. Finally, $H$ is injective. Let indeed $(x,y),(x',y') \in \mathcal{X}\times\mathcal{Y}$ be such that $H(x,y) = H(x',y')$. Then $f_n(x) = h_{2n-1}(x,y) = h_{2n-1}(x',y') = f_n(x')$ for all $n \in \mathbb{N}$, in particular, $F(x) = F(x')$, thus $x = x'$ by injectivity of $F$. Analogously, $y = y'$ since $G$ is injective and $g_n(y) = h_{2n}(x,y) = h_{2n}(x',y') = g_n(y')$ for all $n \in \mathbb{N}$. Therefore $(x,y) = (x',y')$. □

### II.6.1 The inviscid limit

**Lemma II.6.8.** *Assume that $\alpha > 1$. The following objects exist:*

- *an infinitesimal sequence $\{\nu_j\}_{j\in\mathbb{N}} \subset (0,+\infty)$,*
- *solutions $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P};W;X^j)$ to the equation ($HNS_{\nu,\alpha}$) with kinematic viscosity $\nu_j$, for all $j\in\mathbb{N}$,*
- *an augmented filtered probability space $(\tilde\Omega,\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_t\}_{t\geq0},\tilde{\mathbb{P}})$ with an adapted $H^{2\alpha}$-valued Wiener process $\tilde W$,*
- *$\{\tilde{\mathcal{F}}_t\}_{t\geq0}$-adapted stochastic processes $\tilde X : \mathbb{R}_+\times\tilde\Omega \to H^{\alpha+1}$, and $\tilde X^j : \mathbb{R}_+\times\tilde\Omega \to H^{2\alpha}$, $j\in\mathbb{N}$,*
- *probability measures $\mu \in \mathscr{P}(H^{\alpha+1})$, and $\mu_j \in \mathscr{P}(H^{2\alpha})$, $j \in \mathbb{N}$,*

*and they enjoy the following properties:*

(i) $(\tilde X^j_t)_*\tilde{\mathbb{P}} = (X^j_t)_*\mathbb{P} = \mu_j \in \mathscr{P}(H^{2\alpha})$, *for all* $t \geq 0$ *and* $j\in\mathbb{N}$,

(ii) $(\tilde X_t)_*\tilde{\mathbb{P}} = \mu \in \mathscr{P}(H^{\alpha+1})$ *for all* $t\geq 0$,

(iii) *for all* $T>0$, $\tilde X \in \mathcal{Z}_T$, $\tilde{\mathbb{P}}-a.s.$,

(iv) *for all* $T>0$ *and* $j\in\mathbb{N}$, $\tilde X^j \in C\big([0,T];H^{2\alpha}\big) \hookrightarrow \mathcal{Z}_T$, $\tilde{\mathbb{P}}-a.s.$,

(v) $\tilde X^j \longrightarrow \tilde X$, $\tilde{\mathbb{P}}-a.s.$ *in* $\mathcal{Z}_T$, *as* $j\to\infty$,

(vi) $\tilde X \in \bigcap_{p\geq1} L^p\big(\tilde\Omega; C\big([0,T];H^1\big)\big) \cap L^2\big(\tilde\Omega\times[0,T];H^{\alpha+1}\big)$,

*Proof.* Let us fix $\alpha>1$ and an augmented filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},\mathbb{P})$ with the adapted $H^{2\alpha}$-valued Wiener process $W$ from Proposition II.5.11. Let us consider a sequence $\{\nu_n\}_{n\in\mathbb{N}} \subset (0,+\infty)$ convergent to 0. For every $n\in\mathbb{N}$, let $\mu_n \in \mathscr{P}(H^{2\alpha})$ be an invariant measure for the equation ($HNS_{\nu,\alpha}$) with kinematic viscosity $\nu_n$ instead of $\nu$ (which exists thanks to Theorem II.5.10). Let us also denote by $X^n$ the process from Proposition II.5.11, which satisfies the following properties: $X^n$ has the regularities listed in equation (II.5.19), $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},\mathbb{P};W;X^n)$ is a solution to the equation ($HNS_{\nu,\alpha}$) with kinematic viscosity $\nu_n$, $X^n$ is stationary with law $\mu_n = (X^n_t)_*\mathbb{P} \in \mathscr{P}(H^{2\alpha})$ at every time $t\geq 0$.

***Step 1.*** Let us fix an arbitrary $T > 0$. We intend to apply Theorem II.6.3 to $X^n : [0,T] \times \Omega \to H^{2\alpha}$, $n \in \mathbb{N}$, in order to prove that their laws, on the space $(\mathcal{Z}_T, \mathscr{B}_{\mathcal{Z}_T})$, form a tight sequence.

First, observe that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of $H^{2\alpha}$-valued processes with continuous trajectories, see Definition II.4.1. In particular, they are also $U'$-valued pathwise continuous processes, which allows us to apply Theorem II.6.3.

In order to verify the hypothesis (II.6.3) of Theorem II.6.3, we fix $n \in \mathbb{N}$, and resort to the estimate (II.4.22) in Theorem II.4.8 *(iv)*, with $p = 2$, and to the stationarity of $X^n$. For a constant $C > 0$ dependent only on $W$,

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t^n\|_{H^1}^2 \leq C\big(\mathbb{E}\|X_0^n\|_{H^1}^2 + T\nu_n\big) \leq C \int_{H^{2\alpha}} \|x\|_{H^{\alpha+1}}^2 \,\mathrm{d}\mu_n(x) + CT\nu_n,$$

and

$$\mathbb{E} \int_0^T \|X_t^n\|_{H^{\alpha+1}}^2 \,\mathrm{d}t \leq T\mathbb{E}\|X_t^n\|_{H^{\alpha+1}}^2 = T \int_{H^{2\alpha}} \|x\|_{H^{\alpha+1}}^2 \,\mathrm{d}\mu_n(x).$$

Equation (II.5.21) in Theorem II.5.12, and the boundedness of the sequence $\{\nu_n\}_{n\in\mathbb{N}}$, give the uniform bound in $n$.

As for the second hypothesis of Theorem II.6.3, that is the Aldous condition, we use Lemma II.6.4 applied to $\{X^n\}_{n\in\mathbb{N}}$. By Definition II.4.1 we have for all $n \in \mathbb{N}$, $\mathbb{P} - a.s.$, for all $t \geq 0$, in $H^\alpha$

$$\begin{aligned} X_t^n &= X_0^n - \nu_n \int_0^t A^\alpha X_s^n \,\mathrm{d}s - \int_0^t B\big(X_s^n\big) \,\mathrm{d}s + \sqrt{\nu_n}\, W_t \\ &=: J_1^n + J_2^n(t) + J_3^n(t) + J_4^n(t). \end{aligned}$$

Assume that $\{\tau_n\}_{n\in\mathbb{N}}$ is a sequence of stopping times in $[0,T]$ and fix $n \in \mathbb{N}$ and $t \in [0,T]$. The term $J_1^n := X_0^n$ satisfies hypothesis (II.6.4) for any $C, \gamma > 0$ because it is constant in time. As far as $J_2^n$ is concerned:

$$\begin{aligned} \mathbb{E}\|J_2^n(t+\tau_n) - J_2^n(\tau_n)\|_{U'} &\leq \nu_n \mathbb{E} \int_{\tau_n}^{t+\tau_n} \|A^\alpha X_s^n\|_{H^{1-\alpha}} \,\mathrm{d}s \\ &\leq \nu_n \sqrt{t} \left( \mathbb{E} \int_{\tau_n}^{t+\tau_n} \|X_s^n\|_{H^{\alpha+1}}^2 \,\mathrm{d}s \right)^{1/2} \\ &\leq \sqrt{\nu_n t} \left( \nu_n \mathbb{E} \int_0^{2T} \|X_s^n\|_{H^{\alpha+1}}^2 \,\mathrm{d}s \right)^{1/2} \\ &\leq \sqrt{\nu_n t}\, C \big(\mathbb{E}\|X_0^n\|_{H^1}^2 + \nu_n T\big)^{1/2} \\ &= \sqrt{\nu_n t}\, C \left( \int_{H^{2\alpha}} \|x\|_{H^1}^2 \,\mathrm{d}\mu_n(x) + \nu_n T \right)^{1/2}. \end{aligned}$$

We first used the embedding $H^{1-\alpha} \hookrightarrow H^{-2\alpha} \hookrightarrow U'$, then the Hölder inequality, and at the end the estimate (II.4.22) for $p = 2$, with the fact that $X_0^n$ is distributed as $\mu_n$ on $(H^{2\alpha}, \mathscr{B}_{H^{2\alpha}})$. Theorem II.5.12 and the boundedness of the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ give the uniform bound in $n$.

For the non-linear part, we follow a similar reasoning and employ estimate (II.2.6) in Lemma II.2.12 (with $\varepsilon = \alpha - 1 > 0$):

$$\begin{aligned}\mathbb{E}\|J_3^n(t+\tau_n) - J_3^n(\tau_n)\|_{U'} &\leq \mathbb{E}\int_{\tau_n}^{t+\tau_n} \|B(X_s^n)\|_{H^{1-\alpha}}\,\mathrm{d}s \\ &\leq 2ct\,\mathbb{E}\sup_{t\in[0,2T]}\|X_s^n\|_{H^1}^2 \\ &\leq Ct\big(\mathbb{E}\|X_0^n\|_{H^1}^2 + \nu_n T\big) \\ &= Ct\bigg(\int_{H^{2\alpha}}\|x\|_{H^1}^2\,\mathrm{d}\mu_n(x) + \nu_n T\bigg).\end{aligned}$$

The last term is uniformly bounded in $n$, as already discussed.

As for $J_4^n$, we use the embedding $H^{2\alpha} \hookrightarrow U'$, Hölder's inequality and the covariance operator $Q \in \mathcal{L}_1(H^{2\alpha})$ of $W_1$

$$\mathbb{E}\|J_4^n(t+\tau_n) - J_4^n(\tau_n)\|_{U'} \leq \sqrt{\nu_n}\Big[\mathbb{E}\|W_{t+\tau_n} - W_{\tau_n}\|_{H^{2\alpha}}^2\Big]^{1/2} = \sqrt{\nu_n t}\,\big(\operatorname{Tr}[Q]\big)^{1/2}.$$

The last term is uniformly bounded in $n$ because $\nu_n \longrightarrow 0$ as $n \to \infty$.

By Lemma II.6.4 and Theorem II.6.3 the laws of $X^n$, $n \in \mathbb{N}$, form a tight sequence on $(\mathcal{Z}_T, \mathscr{B}_{\mathcal{Z}_T})$.

***Step* 2.** Assume again that $T > 0$. We apply Theorem II.6.5 to the sequence $\{(X^n, W)\}_{n\in\mathbb{N}}$ of $\mathcal{Z}_T \times C\big([0,T]; H^{2\alpha}\big)$-valued random variables, thanks to Lemma II.6.7.

Observe that $W_*\mathbb{P} \in \mathscr{P}\big(C\big([0,T]; H^{2\alpha}\big)\big)$ is tight because the space is separable and complete, see [Bil99, Theorem 1.3]. Furthermore, *Step* 1 assures that the sequence of laws $\{(X^n)_*\mathbb{P}\}_{n\in\mathbb{N}} \subset \mathscr{P}(\mathcal{Z}_T)$ is tight. These two facts imply that $\{(X^n, W)_*\mathbb{P}\}_{n\in\mathbb{N}}$ is tight in $\big(\mathcal{Z}_T \times C([0,T]; H^{2\alpha}), \mathscr{B}_{\mathcal{Z}_T\times C([0,T];H^{2\alpha})}\big)$. Therefore, Theorem II.6.5 returns a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a subsequence indexed by $\{n_k\}_{k\in\mathbb{N}}$, and $\mathcal{Z}_T \times C\big([0,T]; H^{2\alpha}\big)$-valued random variables $(\tilde{X}, \tilde{W})$, $(\tilde{X}^k, \tilde{W}^k)$, $k \in \mathbb{N}$, such that

$$\begin{aligned}&(\tilde{X}^k, \tilde{W}^k)_*\tilde{\mathbb{P}} = (X^{n_k}, W)_*\mathbb{P} \in \mathscr{P}\big(\mathcal{Z}_T \times C\big([0,T]; H^{2\alpha}\big)\big), \qquad \forall\, k \in \mathbb{N},\\ &(\tilde{X}^k, \tilde{W}^k) \longrightarrow (\tilde{X}, \tilde{W}), \qquad \textit{in } \mathcal{Z}_T \times C([0,T]; H^{2\alpha}),\ \tilde{\mathbb{P}} - \textit{a.s.},\ \textit{as } k \to \infty.\end{aligned}$$

Recall that $\mathscr{B}_{\mathcal{Z}_T\times C([0,T];H^{2\alpha})} = \mathscr{B}_{\mathcal{Z}_T} \times \mathscr{B}_{C([0,T];H^{2\alpha})}$, see [Bog07, Lemma 6.4.2, Volume II]. Therefore, the functions $\tilde{X}, \tilde{X}^k : \tilde{\Omega} \to \mathcal{Z}_T$, $k \in \mathbb{N}$, are $\tilde{\mathcal{F}}/\mathscr{B}_{\mathcal{Z}_T}$-measurable random variables, while $\tilde{W}, \tilde{W}^k : \Omega \to C\big([0,T]; H^{2\alpha}\big)$, $k \in \mathbb{N}$, are $\tilde{\mathcal{F}}/\mathscr{B}_{C([0,T];H^{2\alpha})}$-measurable. Moreover,

$$\begin{aligned}&(\tilde{X}^k)_*\tilde{\mathbb{P}} = (X^{n_k})_*\mathbb{P} \in \mathscr{P}(\mathcal{Z}_T), \qquad \forall\, k \in \mathbb{N},\\ &(\tilde{W}^k)_*\tilde{\mathbb{P}} = \tilde{W}_*\tilde{\mathbb{P}} = W_*\mathbb{P} \in \mathscr{P}\big(C\big([0,T]; H^{2\alpha}\big)\big), \qquad \forall\, k \in \mathbb{N}.\end{aligned} \tag{II.6.7}$$

In particular, it follows easily from the Lévy Martingale Characterization Theorem, that $\tilde{W}, \tilde{W}^k$, $k \in \mathbb{N}$ are $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$-adapted $H^{2\alpha}$-valued Wiener processes, see, for instance, [DHV16, Lemma 4.8]. Eventually,

$$\begin{aligned}&\tilde{X}^k \longrightarrow \tilde{X}, \qquad \textit{in } \mathcal{Z}_T,\ \tilde{\mathbb{P}} - \textit{a.s.},\ \textit{as } k \to \infty,\\ &\tilde{W}^k \longrightarrow \tilde{W}, \qquad \textit{in } C\big([0,T]; H^{2\alpha}\big),\ \tilde{\mathbb{P}} - \textit{a.s.},\ \textit{as } k \to \infty.\end{aligned} \tag{II.6.8}$$

***Step* 3.** We prove further regularities for the stochastic processes $\tilde{X}, \tilde{X}^k : \mathbb{R}_+ \times \tilde{\Omega} \to U'$, $k \in \mathbb{N}$.

Fix $k \in \mathbb{N}$ and $T > 0$. First, recall that $C\big([0,T]; H^{2\alpha}\big) \cap \mathscr{Z}_T \in \mathscr{B}_{\mathscr{Z}_T}$ from Lemma II.6.2. This fact, together with $X^{n_k} \in C\big([0,T]; H^{2\alpha}\big)$, $\mathbb{P} - a.s.$, see Definition II.4.1, and by the property in (II.6.7), yields

$$\tilde{X}^k \in C\big([0,T]; H^{2\alpha}\big), \qquad \tilde{\mathbb{P}} - a.s.$$

Additionally, by (II.6.7), recalling the estimate (II.4.22) from Theorem II.4.8 *(iv)* and the stationarity of the process $\{X_t^{n_k}\}_{t\geq 0}$, we have for any $p \geq 2$ and $k \in \mathbb{N}$

$$\begin{aligned}
&\tilde{\mathbb{E}}\|\tilde{X}^k\|^p_{C([0,T];H^1)} + \tilde{\mathbb{E}}\|\tilde{X}^k\|^2_{L^2(0,T;H^{\alpha+1})}\\
&= \int_{\mathscr{Z}_T} \|x\|^p_{C([0,T];H^1)} + \|x\|^2_{L^2(0,T;H^{\alpha+1})} \,\mathrm{d}\big((\tilde{X}^k)_*\tilde{\mathbb{P}}\big)(x)\\
&= \int_{\mathscr{Z}_T} \|x\|^p_{C([0,T];H^1)} + \|x\|^2_{L^2(0,T;H^{\alpha+1})} \,\mathrm{d}\big((X^{n_k})_*\mathbb{P}\big)(x)\\
&= \mathbb{E} \sup_{t\in[0,T]} \|X_t^{n_k}\|^p_{H^1} + \mathbb{E}\int_0^T \|X_t^{n_k}\|^2_{H^{\alpha+1}} \,\mathrm{d}t\\
&\leq C_p\big(\mathbb{E}\|X_0^{n_k}\|^p_{H^1} + T\nu_{n_k}^{p/2}\big) + T\,\mathbb{E}\|X_0^{n_k}\|^2_{H^{\alpha+1}}\\
&= C_p\bigg(\int_{H^{2\alpha}} \|x\|^p_{H^1}\,\mathrm{d}\mu_{n_k} + T\nu_{n_k}^{p/2}\bigg) + T\int_{H^{2\alpha}} \|x\|^2_{H^{\alpha+1}}\,\mathrm{d}\mu_{n_k},
\end{aligned}$$

where the last term is uniformly bounded in $k$, thanks to Theorem II.5.12 and to the boundedness of $\{\nu_n\}_{n\in\mathbb{N}}$. Consequently, there exists a subsequence of $\{\tilde{X}^k\}_{k\in\mathbb{N}}$ weakly convergent in $L^p\big(\tilde{\Omega}; C\big([0,T]; H^1\big)\big)$ and in $L^2\big(\tilde{\Omega}; L^2(0,T; H^{\alpha+1})\big)$. We can extract a further subsequence convergent $\tilde{\mathbb{P}} - a.s.$ in $L^2(0,T;H^{\alpha+1})$, thus also in $L^2_w(0,T;H^{\alpha+1})$. However, we know from (II.6.8) that the whole sequence $\{\tilde{X}^k\}_{k\in\mathbb{N}}$ converges $\tilde{\mathbb{P}} - a.s.$ to $\tilde{X}$ in $\mathscr{Z}_T$, thus also in $L^2_w(0,T;H^{\alpha+1})$, see Definition II.6.1. By uniqueness of the limit, we get

$$\tilde{X} \in \bigcap_{p\geq 1} L^p\big(\tilde{\Omega}; C\big([0,T]; H^1\big)\big) \cap L^2\big(\tilde{\Omega} \times [0,T]; H^{\alpha+1}\big).$$

***Step* 4.** We show that the process $\tilde{X} : \mathbb{R}_+ \times \tilde{\Omega} \to U'$ is stationary and that it is $H^{\alpha+1}$-valued.

By recalling that $\mathscr{B}_{H^{\alpha+1}} \cap H^{2\alpha} = \mathscr{B}_{H^{2\alpha}}$ from Remark 0.6, we can trivially extend the measures $\mu_{n_k}$, $k \in \mathbb{N}$, to the measurable space $\big(H^{\alpha+1}, \mathscr{B}_{H^{\alpha+1}}\big)$, by defining

$$\tilde{\mu}_{n_k} : \mathscr{B}_{H^{\alpha+1}} \ni E \mapsto \tilde{\mu}_{n_k}(E) := \mu_{n_k}(E \cap H^{2\alpha}), \qquad \forall\, k \in \mathbb{N}.$$

Moreover, since the measurable spaces $\big(H^{\alpha+1}, \mathscr{B}_{H^{\alpha+1}}\big)$ and $\big(H^{\alpha+1}_w, \mathscr{B}_{H^{\alpha+1}_w}\big)$ coincide, see 0.15, we infer $\mathscr{P}(H^{\alpha+1}) = \mathscr{P}\big(H^{\alpha+1}_w\big)$. If $R > 0$, let us denote $\bar{B}_R^{H^{\alpha+1}} :=$

$\{x \in H^{\alpha+1} : \|x\|_{H^{\alpha+1}} \le R\}$, then we have for any $k \in \mathbb{N}$

$$\begin{aligned}\tilde{\mu}_{n_k}\big(H^{\alpha+1} \setminus \bar{B}_R^{H^{\alpha+1}}\big) &= \frac{1}{R^2}\int_{H^{\alpha+1}\setminus \bar{B}_R^{H^{\alpha+1}}} R^2 \,\mathrm{d}\tilde{\mu}_{n_k} \\ &\le \frac{1}{R^2}\int_{H^{\alpha+1}\setminus \bar{B}_R^{H^{\alpha+1}}} \|x\|^2_{H^{\alpha+1}} \,\mathrm{d}\tilde{\mu}_{n_k}(x) \\ &\le \frac{1}{R^2}\int_{H^{2\alpha}} \|x\|^2_{H^{\alpha+1}} \,\mathrm{d}\mu_{n_k}(x) \\ &= \frac{C_\alpha}{R^2},\end{aligned}$$

where we used Theorem II.5.12 in the last line. These calculations imply that, for any $\varepsilon > 0$, there exists $R > 0$ such that $\tilde{\mu}_{n_k}\big(H^{\alpha+1} \setminus \bar{B}_R^{H^{\alpha+1}}\big) \le \varepsilon$ for all $k \in \mathbb{N}$. Closed balls in $H^{\alpha+1}$ are compact in $H_w^{\alpha+1}$ by the Banach-Alaoglu Theorem, thus they are compact also in $H_{bw}^{\alpha+1}$, as discussed in Remark 0.12. Therefore, we showed tightness on $H_{bw}^{\alpha+1}$ for the sequence of laws $\{\tilde{\mu}_{n_k}\}_{k\in\mathbb{N}} \subset \mathscr{P}\big(H_{bw}^{\alpha+1}\big)$. By Lemma II.6.7 and Corollary II.6.6 of Jakubowski's Theorem, there exists a family of indices $\{k_j : j \in \mathbb{N}\} \subset \mathbb{N}$, and a probability measure $\mu \in \mathscr{P}(H^{\alpha+1})$ such that

$$\tilde{\mu}_{n_{k_j}} \longrightarrow \mu, \qquad \textit{in } \mathscr{P}(H_{bw}^{\alpha+1}),\ \textit{as } j \to \infty. \tag{II.6.9}$$

Since the natural embedding $\iota : H^{\alpha+1} \to U'$ is measurable, we can trivially extend each $\tilde{\mu}_{n_{k_j}}$, $j \in \mathbb{N}$, and $\mu$ to the measurable space $(U', \mathscr{B}_{U'})$:

$$\begin{aligned}&\bar{\mu}_j : \mathscr{B}_{U'} \ni E \mapsto \bar{\mu}_j(E) := \tilde{\mu}_{n_{k_j}}(E \cap H^{\alpha+1}) = \mu_{n_{k_j}}(E \cap H^{2\alpha}), \qquad \forall j \in \mathbb{N},\\ &\bar{\mu}\ : \mathscr{B}_{U'} \ni E \mapsto \bar{\mu}(E) := \mu(E \cap H^{\alpha+1}).\end{aligned}$$

Since $H^{\alpha+1} \hookrightarrow H^{-2\alpha}$, and since $H^{-2\alpha}$ is compactly embedded into $U'$, the space $H_w^{\alpha+1}$ is continuously embedded in $U'$. In particular, for any $f \in C_b(U')$, denoting $\iota : H_w^{\alpha+1} \to U'$ the natural embedding, we have $f \circ \iota \in C_b(H_w^{\alpha+1})$. Hence, we infer $\bar{\mu}_j \longrightarrow \bar{\mu}$ in $\mathscr{P}(U')$.

We now show that the convergence in (II.6.9) implies

$$\bar{\mu}_{n_{k_j}} \longrightarrow \bar{\mu}, \qquad \textit{in } \mathscr{P}(U'),\ \textit{as } j \to \infty. \tag{II.6.10}$$

Indeed, since $H^{\alpha+1}$ is compactly embedded into $U'$, we infer that $\iota \in C(H_{bw}^{\alpha+1}; U')$ from Remark 0.16. Hence, if $f \in C_b(U')$, then $f \circ \iota \in C_b(H_{bw}^{\alpha+1})$. Therefore, by the change of variable theorem and the convergence in (II.6.9), we have

$$\lim_{j\to\infty} \int_{U'} f \,\mathrm{d}\bar{\mu}_{n_{k_j}} = \lim_{j\to\infty} \int_{H^{\alpha+1}} f \circ \iota \,\mathrm{d}\tilde{\mu}_{n_{k_j}} = \int_{H^{\alpha+1}} f \circ \iota \,\mathrm{d}\mu = \int_{U'} f \,\mathrm{d}\bar{\mu},$$

which proves the convergence in (II.6.10)

By the convergence in (II.6.8) we know that $\tilde{X}^k \longrightarrow \tilde{X}$, $\tilde{\mathbb{P}} - a.s.$ in $C\big([0,T]; U'\big)$, in particular, for all $t \in [0,T]$, $(\tilde{X}_t^k)_*\tilde{\mathbb{P}} \longrightarrow (\tilde{X}_t)_*\tilde{\mathbb{P}}$ in $\mathscr{P}(U')$. We obtain

$$\begin{array}{c}(\tilde{X}_t)_*\tilde{\mathbb{P}} \longleftarrow (\tilde{X}_t^{k_j})_*\tilde{\mathbb{P}} = \bar{\mu}_j = (\tilde{X}_0^{k_j})_*\tilde{\mathbb{P}} \longrightarrow (\tilde{X}_0)_*\tilde{\mathbb{P}},\\ \qquad\qquad\qquad\quad\ \big\downarrow \\ \qquad\qquad\qquad\quad\ \bar{\mu}\end{array}$$

where the convergences are all intended in $\mathscr{P}(U')$, as $j \to \infty$. We conclude that $\tilde{X}_t$ and $\tilde{X}_0$ have the same distribution $\bar{\mu}$ on $(U', \mathscr{B}_{U'})$, where $\bar{\mu}(H^{\alpha+1}) = \mu(H^{\alpha+1}) = 1$. In particular, the process $\tilde{X}$ is $H^{\alpha+1}$-valued and stationary. $\square$

**Theorem II.6.9.** *Assume that $\sigma > 2$. There exists an augmented filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}})$, and an adapted $H^\sigma$-valued stochastic process $\tilde{X}$ (the same from Lemma II.6.8 with $\alpha := \sigma - 1$), such that*

$$\tilde{X} \in \bigcap_{p\geq 1} L^p\big(\tilde{\Omega}; C([0,T]; H^1)\big) \cap L^2\big(\tilde{\Omega} \times [0,T]; H^\sigma\big), \qquad \forall T \geq 0,$$

*and such that, for $\tilde{\mathbb{P}} - a.e.$ $\omega \in \tilde{\Omega}$, the function $\tilde{X}(\omega) : \mathbb{R}_+ \to H^\sigma$ is a solution to the Euler Equation (EE). Moreover, this process is stationary, i.e. $(\tilde{X}_t)_*\tilde{\mathbb{P}} = (\tilde{X}_0)_*\tilde{\mathbb{P}} \in \mathscr{P}(H^\sigma)$, for $t \geq 0$. In particular, the probability measure $(\tilde{X}_0)_*\tilde{\mathbb{P}} \in \mathscr{P}(H^\sigma)$ is an invariant measure for the Euler Equation (EE), see Definition II.2.16.*

*Proof.* ***Step 1.*** For a fixed $\sigma > 2$, let us define $\alpha := \sigma - 1$ and for that $\alpha$ we take the objects constructed in Lemma II.6.8. For any $j \in \mathbb{N}$, let us define $\tilde{\mathbb{P}} - a.s.$ for all $t \in [0,T]$

$$\begin{aligned}
\tilde{M}^j_t &:= \tilde{X}^j_t + \nu_j \int_0^t A^\alpha \tilde{X}^j_s \,\mathrm{d}s + \int_0^t B\big(\tilde{X}^j_s\big)\,\mathrm{d}s - \tilde{X}^j_0, \qquad \textit{in } H^{1-\alpha},\\
M^j_t &:= X^j_t + \nu_j \int_0^t A^\alpha X^j_s \,\mathrm{d}s + \int_0^t B\big(X^j_s\big)\,\mathrm{d}s - X^j_0, \qquad \textit{in } H^{\alpha},\\
\tilde{M}_t &:= \tilde{X}_t + \int_0^t B\big(\tilde{X}_s\big)\,\mathrm{d}s - \tilde{X}_0, \qquad \textit{in } H^{\alpha}.
\end{aligned}$$

Assume that $y \in H^{\alpha-1}$ and fix $t \in [0,T]$ and $j \in \mathbb{N}$. We show that the real-valued random variables ${}_{H^{1-\alpha}}\langle \tilde{M}^j_t, y\rangle_{H^{\alpha-1}}$ and $\langle M^j_t, y\rangle$ have the same distribution on $(\mathbb{R}; \mathscr{B}_\mathbb{R})$.

First of all, let $\iota_3$ be the natural embedding $\mathcal{Z}_T \hookrightarrow C\big([0,T]; H^\alpha_w\big)$. Let also $\iota_t : C\big([0,T]; H^\alpha_w\big) \ni u \mapsto u(t) \in H^\alpha_w$, and $p_y : H^\alpha_w \ni x \mapsto \langle y, x\rangle \in \mathbb{R}$. Then both the evaluation map $\iota_t$ and the functional $p_y$ are continuous with respect to the specified topologies. Therefore

$$\langle \tilde{X}^j_t, y\rangle_*\tilde{\mathbb{P}} = \big(p_y\, \iota_t\, \iota_3\, \tilde{X}^j\big)_*\tilde{\mathbb{P}} = \big(p_y\, \iota_t\, \iota_3\, X^j\big)_*\mathbb{P} = \big\langle X^j_t, y\big\rangle_*\mathbb{P} \in \mathscr{P}(\mathbb{R}).$$

Let $\iota_1$ be the natural embedding $\mathcal{Z}_t \hookrightarrow L^2_w(0,t; H^{\alpha+1})$. Define the linear operator $l_y : L^2_w(0,t; H^{1-\alpha}) \ni u \mapsto \int_0^t {}_{H^{1-\alpha}}\big\langle u(s), y\big\rangle_{H^{\alpha-1}}\,\mathrm{d}s \in \mathbb{R}$, which is sequentially continuous with the specified topology, thus measurable by Remark 0.15. Observe also that $A^\alpha : L^2_w(0,t; H^{\alpha+1}) \to L^2_w(0,t; H^{1-\alpha})$ is continuous. Therefore

$$\begin{aligned}
\left(\int_0^t {}_{H^{1-\alpha}}\big\langle A^\alpha \tilde{X}^j_s, y\big\rangle_{H^{\alpha-1}}\,\mathrm{d}s\right)_* \tilde{\mathbb{P}} &= \big(l_y\, A^\alpha\, \iota_1\, \tilde{X}^j\big)_*\tilde{\mathbb{P}}\\
&= \big(l_y\, A^\alpha\, \iota_1\, X^j\big)_*\mathbb{P}\\
&= \left(\int_0^t \langle A^\alpha X^j_s, y\rangle\,\mathrm{d}s\right)_* \mathbb{P} \in \mathscr{P}(\mathbb{R}).
\end{aligned}$$

We know from Lemma B.3 (with $\beta = \alpha - 1 > 0$) that the function $L^2(0,t; H^\alpha_{loc}) \ni u \mapsto B \circ u \in L^1(0,t; H^{1-\alpha}_w)$ is sequentially continuous on bounded sets of $L^2(0,t; H^\alpha)$. Observe also that if $u_n \longrightarrow u$ in $\mathcal{Z}_t$, then it converges in $L^2(0,t; H^\alpha_{loc})$ and in $L^2_w(0,t; H^{\alpha+1})$, in particular it is bounded in $L^2(0,t; H^{\alpha+1})$, and so in $L^2(0,t; H^\alpha)$. Therefore the function $\tilde{B} : \mathcal{Z}_t \ni u \mapsto \tilde{B}(u) := B \circ u \in L^1(0,t; H^{1-\alpha}_w)$ is sequentially continuous. Moreover, define the linear operator $r_y : L^1(0,t; H^{1-\alpha}_w) \ni u \mapsto$

$\int_0^t {}_{H^{1-\alpha}}\langle u(s), y\rangle_{H^{\alpha-1}}\,\mathrm{d}s \in \mathbb{R}$, which is sequentially continuous with the specified topology. Therefore the map $r_y\tilde{B} : \mathcal{Z}_t \to \mathbb{R}$ is also sequentially continuous, thus measurable by Remark 0.15. Then

$$\begin{aligned}\left(\int_0^t \langle B(\tilde{X}^j_s), y\rangle\,\mathrm{d}s\right)_*\tilde{\mathbb{P}} &= \big(r_y\tilde{B}(\tilde{X}^j)\big)_*\tilde{\mathbb{P}}\\ &= \big(r_y\tilde{B}(X^j)\big)_*\mathbb{P}\\ &= \left(\int_0^t \langle B(X^j_s), y\rangle\,\mathrm{d}s\right)_*\mathbb{P} \in \mathscr{P}(\mathbb{R}).\end{aligned}$$

***Step* 2**. Assume that $y \in H^{\alpha-1}$ and $t \in [0, T]$, then we show that ${}_{H^{1-\alpha}}\langle \tilde{M}^j_t, y\rangle_{H^{\alpha-1}} \longrightarrow \langle \tilde{M}_t, y\rangle$ in $L^1(\tilde{\Omega})$, as $j \to \infty$, by the Vitali Convergence Theorem.

We proceed to study the $\tilde{\mathbb{P}} - a.s.$ convergence of each term separately, which in turn implies convergence in $\tilde{\mathbb{P}}$-measure. As for the non-linear term, we infer from Lemma II.6.8 $(v)$ that the sequence $\{\tilde{X}^j\}_{j\in\mathbb{N}}$ is $\tilde{\mathbb{P}}-a.s.$ convergent in $L^2_w(0, T; H^{\alpha+1})$, hence bounded in $L^2(0, T; H^{\alpha+1})$, hence bounded in $L^2(0, T; H^{\alpha})$. It also converges $\tilde{\mathbb{P}} - a.s.$ in $L^2(0, T; H^{\alpha}_{loc})$ to $\tilde{X}$. Therefore, by Lemma B.3 (with $\beta = \alpha > 1$), we have $\tilde{\mathbb{P}} - a.s.$

$$\lim_{k\to\infty}\int_0^t \langle B(\tilde{X}^j_s) - B(\tilde{X}_s), y\rangle\,\mathrm{d}s = 0.$$

We proceed similarly for the linear term. The convergence in Lemma II.6.8 $(v)$ implies that $\{\tilde{X}^j\}_{j\in\mathbb{N}}$ converges $\tilde{\mathbb{P}} - a.s.$ to $\tilde{X}$ in $L^2(0, T; H^{\alpha+1}_w)$. Thus, $\tilde{\mathbb{P}} - a.s.$

$$\lim_{j\to\infty}\int_0^t {}_{H^{1-\alpha}}\langle A^\alpha\tilde{X}^j_s - A^\alpha\tilde{X}_s, y\rangle_{H^{\alpha-1}}\,\mathrm{d}s = \lim_{j\to\infty}\int_0^t {}_{H^{-\alpha-1}}\langle A^\alpha y, \tilde{X}^j_s - \tilde{X}_s\rangle_{H^{\alpha+1}}\,\mathrm{d}s = 0.$$

Finally, $\tilde{X}^j \longrightarrow \tilde{X}$ $\tilde{\mathbb{P}} - a.s.$ in $C\big([0, T]; H^{\alpha}_w\big)$ by Lemma II.6.8 $(v)$, hence $\tilde{\mathbb{P}} - a.s.$

$$\lim_{j\to\infty}\langle \tilde{X}^j_t - \tilde{X}_t, y\rangle = 0.$$

Moreover, we know from *Step* 1 that ${}_{H^{1-\alpha}}\langle \tilde{M}^j_t, y\rangle_{H^{\alpha-1}}$ has the same law in $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ as $\langle M^j_t, y\rangle$, for all $j \in \mathbb{N}$. In addition, $M^j = \sqrt{\nu_j}\,W$, $\mathbb{P} - a.s.$, in the trajectory space $C([0, T]; H^{\alpha})$, hence

$$\left({}_{H^{1-\alpha}}\langle \tilde{M}^j_t, y\rangle_{H^{\alpha-1}}\right)_*\tilde{\mathbb{P}} = \langle \sqrt{\nu_j}\,W, y\rangle_*\mathbb{P} \in \mathscr{P}(\mathbb{R}).$$

This fact gives a uniform bound in $j$ in the space $L^2(\tilde{\Omega})$, thanks to

$$\tilde{\mathbb{E}}\left|{}_{H^{1-\alpha}}\langle \tilde{M}^j_t, y\rangle_{H^{\alpha-1}}\right|^2 = \mathbb{E}\big|\langle \sqrt{\nu_j}\,W, y\rangle\big|^2 \le \nu_j\,\|y\|^2\mathbb{E}\|W_t\|^2, \tag{II.6.11}$$

where the last term is bounded in $j$ because the sequence $\{\nu_j\}_{j\in\mathbb{N}}$ is infinitesimal. The uniform bound in $L^2(\tilde{\Omega})$ implies uniform integrability.

To sum up, we obtained the sought uniform integrability and convergence in $\tilde{\mathbb{P}}$-measure for the sequence $\{{}_{H^{1-\alpha}}\langle \tilde{M}^j_t, y\rangle_{H^{\alpha-1}}\}_{j\in\mathbb{N}}$ of real random variables. This allows us to apply Vitali's Convergence Theorem to infer that

$${}_{H^{1-\alpha}}\langle \tilde{M}^j_t, y\rangle_{H^{\alpha-1}} \longrightarrow \langle \tilde{X}_t, y\rangle + \int_0^t \langle B(\tilde{X}_s), y\rangle\,\mathrm{d}s - \langle \tilde{X}_0, y\rangle = \langle \tilde{M}_t, y\rangle,$$

the convergence being in $L^1(\tilde{\Omega})$, as $j \to \infty$.

***Step* 3**. We conclude the reasoning by proving that, for $\tilde{\mathbb{P}} - a.e.\ \omega \in \tilde{\Omega}$, $\tilde{X}(\omega)$ is a solution to the Euler Equation (*EE*).

Assume that $y \in H^{\alpha-1}$ and $t \in [0,T]$. We proved in *Step 2* that ${}_{H^{1-\alpha}}\langle \tilde{M}^j_t, y\rangle_{H^{\alpha-1}} \longrightarrow \langle \tilde{M}_t, y\rangle$ in $L^1(\tilde{\Omega})$, as $j \to \infty$. We see from estimate (II.6.11) that ${}_{H^{1-\alpha}}\langle \tilde{M}^j_t, y\rangle_{H^{\alpha-1}} \longrightarrow 0$ in $L^2(\tilde{\Omega})$, as $j \to \infty$. We infer by the uniqueness of the limit that $\langle \tilde{M}_t, y\rangle = 0$, $\tilde{\mathbb{P}} - a.s.$ This fact implies, by arbitrariness of $y \in H^{\alpha-1}$, that $\tilde{\mathbb{P}} - a.s.$ in $H^{1-\alpha}$

$$\tilde{X}_t + \int_0^t B(\tilde{X}_s)\,\mathrm{d}s = \tilde{X}_0. \tag{II.6.12}$$

Lemma II.6.8 states that $\tilde{X}$ is $H^{\alpha+1}$-valued, thus the identity holds $\tilde{\mathbb{P}} - a.s.$ in $H^{\alpha+1}$. Additionally, we infer from Lemma II.6.8 *(vi)* that $\tilde{\mathbb{P}}\big(\tilde{X} \in L^2(0,T;H^{\alpha+1})\big) = 1$, thus also $\tilde{X}_0 - \int_0^\cdot B(\tilde{X}_s)\,\mathrm{d}s \in C\big([0,T];H^\alpha\big)$, $\tilde{\mathbb{P}} - a.s.$ In particular $\tilde{\mathbb{P}}\big(\tilde{X} \in C(\mathbb{R}_+;H^\alpha)\big) = 1$, by arbitrariness of $T > 0$. It only remains to show that for $\tilde{\mathbb{P}} - a.e.\ \omega \in \tilde{\Omega}$, $\tilde{X}(\omega) \in C(\mathbb{R}_+;H^{\alpha+1})$.

By recalling from Theorem II.2.15 the definition of the Euler flow map $\Phi$, and since $\tilde{X}_0$ is $H^{\alpha+1}$-valued, we define the $H^{\alpha+1}$-valued stochastic process

$$Y : \mathbb{R}_+ \times \tilde{\Omega} \ni (t,\omega) \mapsto Y_t(\omega) := \Phi\big(t, \tilde{X}_0(\omega)\big).$$

Then, by Theorem II.2.15, $\tilde{\mathbb{P}}\big(Y \in C(\mathbb{R}_+;H^{\alpha+1})\big) = 1$, and $\tilde{\mathbb{P}} - a.s.$

$$Y_t - \int_0^t B(Y_s)\,\mathrm{d}s = \tilde{X}_0, \qquad \forall\, t \geq 0. \tag{II.6.13}$$

We compare equations (II.6.12) and (II.6.13) by means of estimate II.2.4 in Lemma II.2.12 (with $\sigma = \alpha + 1 > 2$), and infer that $\tilde{\mathbb{P}} - a.s.$

$$\|\tilde{X}_t - Y_t\|_{H^\alpha} \leq \int_0^t \|\tilde{X}_s - Y_s\|_{H^{\alpha+1}} \|\tilde{X}_s - Y_s\|_{H^\alpha}\,\mathrm{d}s, \qquad \forall t\, \geq 0.$$

This implies by the Grönwall Lemma and the fact that both $\tilde{X}$, and $Y$ belong $\tilde{\mathbb{P}} - a.s.$ to $L^1(0,T;H^{\alpha+1})$ for any $T > 0$, that $\tilde{\mathbb{P}} - a.s.$

$$\tilde{X}_t = Y_t = \Phi\big(t, \tilde{X}_0\big), \qquad \textit{in } H^\alpha,\ \forall\, t \geq 0.$$

Since, for $\tilde{\mathbb{P}} - a.e.\ \omega \in \tilde{\Omega}$, both functions $Y(\omega)$, and $\tilde{X}(\omega)$ are $H^{\alpha+1}$-valued, and since $Y(\omega) \in C(\mathbb{R}_+;H^{\alpha+1})$, we infer that $\tilde{\mathbb{P}}\big(\tilde{X} \in C(\mathbb{R}_+;H^{\alpha+1})\big) = 1$. In particular, for $\tilde{\mathbb{P}} - a.e.\ \omega \in \tilde{\Omega}$, $\tilde{X}(\omega)$ is a solution to the Euler Equation (*EE*).

***Step* 4**. Let us denote $\mu := (\tilde{X}_0)_* \tilde{\mathbb{P}} \in \mathscr{P}(H^\sigma)$. Then we can prove that $\mu$ is an invariant measure for the Euler Equation (*EE*). If indeed $F \in \mathscr{B}_{H^\sigma}$, and $t \geq 0$, we have

$$\begin{aligned}\mu(F) &= \tilde{\mathbb{P}}(\tilde{X}_0 \in F) = \tilde{\mathbb{P}}(\tilde{X}_t \in F) = \tilde{\mathbb{P}}\big(\Phi(t,\tilde{X}_0) \in F\big)\\ &= \int_{H^\sigma} \mathbb{1}_F(\Phi_t)\,\mathrm{d}\mu = \mu\big(\Phi_t^{-1}(F)\big) = (\Phi_t)_*\mu(F).\end{aligned}$$

The second equality is due to the stationarity of $\tilde{X}$, see Lemma II.6.8 *(ii)*, while the third equality comes from the previous step. The chain of equalities implies $\mu = (\Phi_t)_*\mu \in \mathscr{P}(H^\sigma)$. □

### II.6.2 Moment estimates

This last part is devoted to providing some moment estimates to the invariant measure for the Euler Equation constructed above.

**Theorem II.6.10.** *Assume that $\sigma > 2$. There exists a finite constant $C_\sigma > 0$ such that, if $0 < \beta < 2C_\sigma$, and $\mu \in \mathscr{P}(H^\sigma)$ is the invariant measure for the Euler Equations (EE) constructed in Theorem II.6.9, then*

$$\int_{H^\sigma} \|x\|_{H^\sigma}^2 \,\mathrm{d}\mu(x) = C_\sigma,$$
$$\int_{H^\sigma} \|x\|_{H^1}^{2n} \,\mathrm{d}\mu(x) \leq (2n-1)!!\, C_\sigma^n, \qquad \forall\, n \in \mathbb{N},$$
$$\int_{H^\sigma} e^{\beta\|x\|_{H^\sigma}^2} \,\mathrm{d}\mu(x) \leq 2e^{\frac{2\beta C_\sigma}{1-2\beta C_\sigma}}.$$

*Proof.* Let us fix $\sigma > 2$ and consider the objects introduced in Lemma II.6.8 for $\alpha := \sigma - 1$. In particular, we know from Theorem II.6.9 that $\mu = (\tilde{X}_0)_*\tilde{\mathbb{P}} \in \mathscr{P}(H^\sigma)$ is an invariant measure for the deterministic Euler Equations , see Definition II.2.16.

We denote the following Hilbert spaces

$$E_1 = E_3 := H^\sigma, \quad E_2 := H^1,$$

with the respective norms $\|\cdot\|_{E_i}$, $i = 1, 2, 3$, and observe that each $E_i$ is compactly embedded into $U'$, where $U$ is the reproducing kernel of the Wiener process $W$. Let us fix $n \in \mathbb{N}$ and $\beta > 0$, then we define $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, 3$ as follows, for all $r \geq 0$

$$\varphi_1(r) := r^2, \quad \varphi_2(r) := r^{2n}, \quad \varphi_3(r) := e^{\beta r^2}.$$

We observe that each $\varphi_i$ is strictly increasing. For any $i = 1, 2, 3$ we define

$$\eta_i : U' \ni x \mapsto \begin{cases} \varphi_i\big(\|x\|_{E_i}\big) & \forall\, x \in E_i \\ +\infty & \forall\, x \in U' \setminus E_i \end{cases}.$$

Fix $i \in \{1, 2, 3\}$. Example 2.12 in [BS13] proves that $\eta_i : U' \to [0, +\infty]$ is lower semicontinuous, see [Bre10, Section 1.3] and the references therein for the definition and the main properties. In particular, if $x \in U'$ and if $\{x_n\}_{n\in\mathbb{N}} \subset U'$ is a sequence convergent to $x$ in $U'$, then $\eta_i(x) \leq \liminf_{n\to\infty} \eta_i(x_n)$.

Since, by Lemma II.6.8 (v), $\tilde{X}^j \longrightarrow \tilde{X}$, $\tilde{\mathbb{P}} - a.s.$ in $\mathcal{Z}_T$, and since $\mathcal{Z}_T \hookrightarrow C\big([0,T]; U'\big)$, if $\iota : H^{\alpha+1} \to U'$, and $\tilde{\iota} : H^{2\alpha} \to U'$ denote the natural embedding, then

$$\tilde{\iota}\tilde{X}_0^j \longrightarrow \iota\tilde{X}_0, \qquad \text{in } U',\ \tilde{\mathbb{P}} - a.s.,\ \text{as } j \to \infty.$$

Therefore, thanks to the above discussion,

$$\eta_i(\iota\tilde{X}_0) \leq \liminf_{j\to\infty} \eta_i(\tilde{\iota}\tilde{X}_0^j), \qquad \tilde{\mathbb{P}} - a.s.$$

Let us take the expectation with respect to $\tilde{\mathbb{P}}$, that we denote by $\tilde{\mathbb{E}}$, to both members of this inequality and use Fatou's Lemma. We obtain

$$\tilde{\mathbb{E}}\big[\eta_i(\iota\tilde{X}_0)\big] \leq \tilde{\mathbb{E}}\Big[\liminf_{j\to\infty} \eta_i(\tilde{\iota}\tilde{X}_0^j)\Big] \leq \liminf_{j\to\infty} \tilde{\mathbb{E}}\big[\eta_i(\tilde{\iota}\tilde{X}_0^j)\big].$$

We denote by $\bar{\mu}$, and $\bar{\mu}_j$, $j \in \mathbb{N}$, the trivial extensions to $(U', \mathscr{B}_{U'})$ of the probability measures $\mu \in \mathscr{P}(H^{\alpha+1})$, and $\mu_j \in \mathscr{P}(H^{2\alpha})$ from Lemma II.6.8, respectively. Namely,

$$\begin{aligned}
&\bar{\mu} : \mathscr{B}_{U'} \ni F \mapsto \bar{\mu}(F) := \mu(F \cap H^{\alpha+1}) \\
&\bar{\mu}_j : \mathscr{B}_{U'} \ni F \mapsto \bar{\mu}_j(F) := \mu_j(F \cap H^{2\alpha}), \qquad \forall\, j \in \mathbb{N}.
\end{aligned}$$

We infer from Lemma II.6.8 *(i)*, *(ii)*, that

$$\begin{aligned}
&(\iota \tilde{X}_0)_* \tilde{\mathbb{P}} = \bar{\mu} \in \mathscr{P}(U') \\
&(\tilde{\iota} \tilde{X}_0^j)_* \tilde{\mathbb{P}} = \bar{\mu}_j \in \mathscr{P}(U'), \qquad \forall\, j \in \mathbb{N}.
\end{aligned}$$

We conclude the reasoning as follows

$$\begin{aligned}
\int_{H^{\alpha+1}} \varphi_i\big(\|x\|_{E_i}\big)\, \mathrm{d}\mu(x) &= \int_{U'} \eta_i(x)\, \mathrm{d}\bar{\mu}(x) \\
&= \tilde{\mathbb{E}}\big[\eta_i(\iota \tilde{X}_0)\big] \\
&\leq \liminf_{j \to \infty} \tilde{\mathbb{E}}\big[\eta_i(\tilde{\iota} \tilde{X}_0^j)\big] \\
&= \liminf_{j \to \infty} \int_{U'} \eta_i(x)\, \mathrm{d}\bar{\mu}_j(x) \\
&= \liminf_{j \to \infty} \int_{H^{2\alpha}} \varphi_i\big(\|x\|_{E_i}\big)\, \mathrm{d}\mu_j(x).
\end{aligned}$$

The theses follow from the last inequality, thanks to the uniform estimates in Theorem II.5.12. □

# Appendix A

# Grönwall Lemmas

We recall a weak version of the differential form of the classical Grönwall Lemma, which does not require the differentiability of functions.

**Lemma A.1.** *Assume that $f : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function and that there exist $a, b \in L^1_{loc}(\mathbb{R}_+)$ such that, in the distributional sense in $\mathbb{R}^*_+ := (0, +\infty)$*

$$f' \le af + b. \tag{A.1}$$

*Then,*

$$f(t) \le e^{A(t)}\left(f(0) + \int_0^t b(s)e^{-A(s)}\,\mathrm{d}s\right), \qquad \forall\, t \ge 0, \tag{A.2}$$

*where*

$$A(t) := \exp \int_0^t a(s)\,\mathrm{d}s, \qquad \forall\, t \ge 0.$$

*Proof.* Let us define $g : \mathbb{R}_+ \to \mathbb{R}$ by

$$g(t) := f(t)e^{-A(t)} - \int_0^t b(s)e^{-A(s)}\,\mathrm{d}s, \qquad \forall\, t \ge 0.$$

Then the distributional derivative of $g$ on the interval $\mathbb{R}^*_+ := (0, +\infty)$, satisfies

$$g' = f'e^{-A} - afe^{-A} - be^{-A} \le 0, \tag{A.3}$$

where we used the hypothesis in equation (A.1) for the inequality. Let $\rho \in C^\infty_c(\mathbb{R}^*_+)$ be such that $\rho \ge 0$ and $\int_0^{+\infty} \rho(s)\,\mathrm{d}s = 1$. For every $n \in \mathbb{N}$ define $\rho_n(s) := n\rho(ns)$, $s > 0$, then

$$\rho_n \in C^\infty_c(\mathbb{R}^*_+), \qquad \int_0^{+\infty} \rho_n(s)\,\mathrm{d}s = 1, \qquad \forall\, n \in \mathbb{N}.$$

Let $\varphi$ be a generic non-negative function in $C^\infty_c(\mathbb{R}^*_+)$ and fix $n \in \mathbb{N}$. Let us define

$$\psi_n := -\varphi + \rho_n \int_0^{+\infty} \varphi(s)\,\mathrm{d}s.$$

By direct inspection we have $\psi_n \in C^\infty_c(\mathbb{R}^*_+)$ and $\int_0^{+\infty} \psi_n(s)\,\mathrm{d}s = 0$, in particular, its primitive $\Psi_n(t) := \int_0^t \psi_n(s)\,\mathrm{d}s$, $t > 0$, satisfies again $\Psi_n \in C^\infty_c(\mathbb{R}^*_+)$. Therefore, if we test the inequality (A.3) againts $\Psi_n$ we reach

$$\begin{aligned}0 &\geq {}_{\mathcal{D}'(\mathbb{R}_+^*)}\big\langle g', \Psi_n\big\rangle_{C_c^\infty(\mathbb{R}_+^*)}\\ &= -\,{}_{\mathcal{D}'(\mathbb{R}_+^*)}\big\langle g, \psi_n\big\rangle_{C_c^\infty(\mathbb{R}_+^*)}\\ &= \int_0^{+\infty} g(t)\varphi(t)\,\mathrm{d}t - \int_0^{+\infty} g(s)\rho_n(s)\,\mathrm{d}s \int_0^{+\infty} \varphi(t)\,\mathrm{d}t\\ &= \int_0^{+\infty} \varphi(t)\Big(g(t) - \int_0^{+\infty} g(s)\rho_n(s)\,\mathrm{d}s\Big)\,\mathrm{d}t.\end{aligned}$$

This implies, by arbitrariness of $\varphi \in C_c^\infty(\mathbb{R}_+^*; \mathbb{R}_+)$ and continuity of $g$, that for all $t \geq 0$

$$g(t) \leq \int_0^{+\infty} g(s)\rho_n(s)\,\mathrm{d}s = \int_0^{+\infty} g(r/n)\rho(r)\,\mathrm{d}r \longrightarrow g(0), \qquad \text{as } n \to \infty,$$

where we resorted to the Dominated Convergence Theorem to pass to the limit. The sought estimate (A.2) is derived from $g(t) \leq g(0)$, $t \geq 0$, by recalling the definition of $g$, and by multiplying by $e^A$. □

We also recall a simple version of the integral Grönwall's Lemma that suits our necessities. A different proof can be found in [FR01, proof of Theorem 5, step 2].

**Lemma A.2.** *Assume that $f : \mathbb{R}_+ \to \mathbb{R}$ is a non-negative continuous function and that there exist $C_1, C_2 > 0$ such that*

$$f(t) \leq f(s) - C_1 \int_s^t f(r)\,\mathrm{d}r + C_2(t-s), \qquad \forall\, t \geq s \geq 0. \tag{A.4}$$

*Then*

$$f(t) \leq \frac{C_2}{C_1} + e^{-C_1 t} f(0), \qquad \forall\, t \geq 0.$$

*Proof.* Equation (A.4) directly implies that $f$ is Lipschitz continuous, in particular, it is absolutely continuous. Therefore, there exists $f' \in L^1_{loc}(\mathbb{R}_+)$ such that

$$f(t) - f(s) = \int_s^t f'(r)\,\mathrm{d}r, \qquad \forall\, t \geq s \geq 0.$$

This consideration allows to rewrite the inequality (A.4) in the following way

$$\int_s^t \big[f'(r) + C_1 f(r) - C_2\big]\,\mathrm{d}r \leq 0, \qquad \forall\, t \geq s \geq 0.$$

This implies that, for almost any $r \geq 0$

$$f'(r) \leq -C_1 f(r) + C_2.$$

We reach the thesis by applying the differential form of Grönwall's Lemma A.1:

$$\begin{aligned}f(t) &\leq e^{-C_1 t}\Big(f(0) + \int_0^t C_2 e^{C_1 s}\,\mathrm{d}s\Big)\\ &= e^{-C_1 t} f(0) + \frac{C_2}{C_1} e^{-C_1 t}\big(e^{C_1 t} - 1\big)\\ &\leq \frac{C_2}{C_1} + e^{-C_1 t} f(0).\end{aligned}$$

□

Eventually, we provide a modified version of the classical integral Grönwall Lemma. We refer to [Hen81, Section 1.2.1] for the proof.

**Lemma A.3.** *Assume that $a, b \geq 0$, $T > 0$ and $\alpha, \beta \in [0, 1)$. There exists a finite constant $M > 0$ so that for any integrable function $u : (0, T) \to \mathbb{R}$ satisfying*

$$0 \leq u(t) \leq at^{-\alpha} + b \int_0^t (t-s)^{-\beta} u(s)\, \mathrm{d}s, \qquad \textit{for a.e. } t \in (0, T),$$

*we have*

$$0 \leq u(t) \leq aMt^{-\alpha}, \qquad \textit{for a.e. } t \in (0, T).$$

# Appendix B

# Ancillary results

**Proposition B.1.** *Assume that $G \in \mathcal{L}(H)$ is injective and satisfies $V \subset \mathrm{Ran}(G)$. There exists $C > 0$ such that, if $n \in \mathbb{N}$, then*

$$\left\|(\Pi_n GG^*\Pi_n)^{-1/2}x\right\| \le C\|x\|_V, \qquad \forall\, x \in H_n.$$

*Proof.* The proof is inspired by [FM95, Section 6] and we present it for the sake of completeness. Let us fix $n \in \mathbb{N}$.

*Step* 1. We show that the thesis makes sense.

First, $\Pi_n GG^*\Pi_n : H \to H$ is bounded because it is the composition of bounded operators in $H$. In addition, it is non-negative because

$$\langle \Pi_n GG^*\Pi_n x, x\rangle = \|G^*\Pi_n x\|^2 \ge 0, \qquad \forall\, x \in H.$$

Therefore, it is self-adjoint and its square root $(\Pi_n GG^*\Pi_n)^{1/2} : H \to H$ is well-defined and bounded.

Moreover, $\mathrm{Ker}(\Pi_n GG^*\Pi_n) = H_n^\perp$. On one hand, we have $H_n^\perp = \mathrm{Ker}\,\Pi_n \subset \mathrm{Ker}(\Pi_n GG^*\Pi_n)$. On the other hand, we first notice that $G^* : H \to H$ is injective because $\mathrm{Ker}(G^*) = [\mathrm{Ran}(G)]^\perp \subset V^\perp = \{0\}$. Then, if we let $x \in \mathrm{Ker}(\Pi_n GG^*\Pi_n)$, we have $0 = \langle \Pi_n GG^*\Pi_n x, x\rangle = \|G^*\Pi_n x\|^2$, hence $\Pi_n x = 0$ by the injectivity of $G^*$, namely $x \in H_n^\perp$.

Eventually, from $\left[\mathrm{Ran}(\Pi_n GG^*\Pi_n)\right]^\perp = \mathrm{Ker}(\Pi_n GG^*\Pi_n)^* = \mathrm{Ker}(\Pi_n GG^*\Pi_n) = H_n^\perp$, we infer that $\overline{\mathrm{Ran}(\Pi_n GG^*\Pi_n)} = \overline{H_n} = H_n$, hence $\mathrm{Ran}(\Pi_n GG^*\Pi_n) = H_n$ because $H_n$ is finite-dimensional.

Therefore, we conclude that the restriction $\Pi_n GG^*\Pi_n : H_n \to H_n$ is bijective, bounded and positive, hence self-adjoint. Its square root $(\Pi_n GG^*\Pi_n)^{1/2} : H_n \to H_n$ is well-defined and inherits the same properties. Moreover, the inverse operator of $(\Pi_n GG^*\Pi_n)^{1/2}$ is the same as the square root of $(\Pi_n GG^*\Pi_n)^{-1}$, which is simply denoted by $(\Pi_n GG^*\Pi_n)^{-1/2}$, and it is a bounded, positive, bijective and self-adjoint operator in $H_n$.

*Step* 2. The thesis follows from the following claim: there exists $C > 0$, independent of $n$, such that

$$\left\langle \left(A^{\frac{1}{2}}\Pi_n GG^*\Pi_n A^{\frac{1}{2}}\right)^{-1} y, y\right\rangle \le C\|y\|^2, \qquad \forall\, y \in H_n.$$

Let indeed $x \in H_n$, then we define $y := A^{1/2}x$ and observe that $y \in H_n$ because

$H_n = \Pi_n H = \Pi_n V$. Hence, we get

$$\begin{aligned}\big\|(\Pi_n GG^*\Pi_n)^{-1/2}x\big\|^2 &= \big\langle(\Pi_n GG^*\Pi_n)^{-1}x, x\big\rangle \\ &= \big\langle A^{-\frac{1}{2}}(\Pi_n GG^*\Pi_n)^{-1}A^{-\frac{1}{2}}y, y\big\rangle \\ &= \big\langle\big(A^{\frac{1}{2}}\Pi_n GG^*\Pi_n A^{\frac{1}{2}}\big)^{-1}y, y\big\rangle \\ &\le C\|y\|^2 \\ &= C\|x\|_V^2.\end{aligned}$$

The thesis follows after renaming the constant $C$. In the previous chain of inequalities we implicitely used the fact that $A^{\frac{1}{2}}\Pi_n GG^*\Pi_n A^{\frac{1}{2}} : H_n \to H_n$ is bijective and satisfies $(A^{\frac{1}{2}}\Pi_n GG^*\Pi_n A^{\frac{1}{2}})^{-1} = A^{-\frac{1}{2}}(\Pi_n GG^*\Pi_n)^{-1}A^{-\frac{1}{2}}$. This property easily descends from *Step* 1 and from the bijectivity of the self-adjoint bounded operator $A^{\frac{1}{2}}$ on the finite-dimensional space $H_n$.

*Step* 3. Denoting by $G^{-1} : \mathrm{Ran}(G) \subset H \to H$ the unbounded inverse of $G \in \mathcal{L}(H)$, we will show that the operator $G^{-1}A^{-\frac{1}{2}}$ is well-defined and bounded in $H$.

First, notice that $G^{-1}A^{-\frac{1}{2}} : H \to H$ is well-defined because $A^{-\frac{1}{2}}x \in V$ for any $x \in H$ and $V \subset \mathrm{Ran}(G)$. Then we proceed to show that $G^{-1}A^{-\frac{1}{2}} : H \to H$ is bounded by the Closed Graph Theorem. Let us denote by $G_V^{-1} : V \to H : x \mapsto G^{-1}x$ the restriction of $G^{-1}$ to $V$ and assume that $\{x_n\}_{n\in\mathbb{N}} \subset V$ is convergent in $V$ to $x$ and that $\{G_V^{-1}x_n\}_{n\in\mathbb{N}}$ is convergent in $H$ to $y$. Thanks to the inclusion $V \subset \mathrm{Ran}(G)$, for any $n \in \mathbb{N}$ there exists $z_n \in H$ such that $x_n = Gz_n$. Therefore, $Gz_n = x_n$ converges as $n \to \infty$ to $x$ in $V$ and to $Gy$ in $H$ by continuity of $G$. Since $V \hookrightarrow H$ we infer that $x = Gy \in V$, hence $G_V^{-1}x = G^{-1}Gy = y$, *i.e.* $G_V^{-1}$ is closed. By the Closed Graph Theorem, $G_V^{-1} : V \to H$ is bounded. Since $A^{-\frac{1}{2}} : H \to V$ is bounded, we infer, by the composition of bounded operators, that $G^{-1}A^{-\frac{1}{2}} = G_V^{-1}A^{-\frac{1}{2}} \in \mathcal{L}(H)$.

*Step* 4. Assume that $x \in V$, then for all $y \in H$

$$\begin{aligned}\big\langle\big(G^{-1}A^{-\frac{1}{2}}\big)^*G^*A^{\frac{1}{2}}x, y\big\rangle &= \big\langle G^*A^{\frac{1}{2}}x, G^{-1}A^{-\frac{1}{2}}y\big\rangle \\ &= \big\langle A^{\frac{1}{2}}x, GG^{-1}A^{-\frac{1}{2}}y\big\rangle \\ &= \big\langle A^{\frac{1}{2}}x, A^{-\frac{1}{2}}y\big\rangle \\ &= \langle x, y\rangle.\end{aligned}$$

By choosing $y = \big(G^{-1}A^{-\frac{1}{2}}\big)^*G^*A^{\frac{1}{2}}x - x \in H$, we get

$$\big(G^{-1}A^{-\frac{1}{2}}\big)^*G^*A^{\frac{1}{2}}x = x,$$

which implies, by the boundedness of $G^{-1}A^{-\frac{1}{2}} : H \to H$, see *Step* 2,

$$\|x\|^2 \le \big\|\big(G^{-1}A^{-\frac{1}{2}}\big)^*\big\|_{\mathcal{L}(H)}^2\big\|G^*A^{\frac{1}{2}}x\big\|^2 = \big\|G^{-1}A^{-\frac{1}{2}}\big\|_{\mathcal{L}(H)}^2\big\langle A^{\frac{1}{2}}GG^*A^{\frac{1}{2}}x, x\big\rangle.$$

*Step* 5. Let us now prove the claim in *Step* 2. Assume that $y \in H_n$ and let $x := \big(A^{\frac{1}{2}}\Pi_n GG^*\Pi_n A^{\frac{1}{2}}\big)^{-1/2}y \in H_n$, then $x \in V$ because $H_n = \Pi_n V \subset V$, and $y = \big(A^{\frac{1}{2}}\Pi_n GG^*\Pi_n A^{\frac{1}{2}}\big)^{1/2}x$. Therefore, using the estimate from the previous step and denoting $C := \big\|G^{-1}A^{-\frac{1}{2}}\big\|^2_{\mathcal{L}(H)}$,

$$\begin{aligned}
\big\langle\big(A^{\frac{1}{2}}\Pi_n GG^*\Pi_n A^{\frac{1}{2}}\big)^{-1}y,y\big\rangle &= \big\|\big(A^{\frac{1}{2}}\Pi_n GG^*\Pi_n A^{\frac{1}{2}}\big)^{-1/2}y\big\|^2\\
&= \|x\|^2\\
&\le C\big\langle A^{\frac{1}{2}}GG^*A^{\frac{1}{2}}x,x\big\rangle\\
&= C\big\langle A^{\frac{1}{2}}\Pi_n GG^*\Pi_n A^{\frac{1}{2}}x,x\big\rangle\\
&= C\big\|\big(A^{\frac{1}{2}}\Pi_n GG^*\Pi_n A^{\frac{1}{2}}\big)^{1/2}x\big\|^2\\
&= C\|y\|^2.
\end{aligned}$$

□

**Proposition B.2.** *For any $R>0$ and $t>0$ there exists a constant $C_R(t)>0$ such that for all $G$ as in hypothesis ($H_1$) it holds*

$$\int_0^t \big\|D_x[X_s^{R,n}(x)]h\big\|_V^2\,\mathrm{d}s \le C_R(t)\|h\|^2 \qquad \forall\, n\in\mathbb{N},\ \forall\, h,x\in H_n,\ \mathbb{P}-a.s.$$

*Proof.* The proof is inspired by the appendix in [FM95] and we present it for the sake of completeness. We fix $\mathbb{P}-a.s.$ $\omega\in\Omega$ and parameters $n\in\mathbb{N}$, $R>0$. For all starting points $x\in H_n$ and times $t\ge 0$ we denote for simplicity $u(t)=X_t^{R,n}(x)(\omega)$ and we recall its definition:

$$u(t)+\int_0^t Au(s)\,\mathrm{d}s+\int_0^t \Theta_R\big(\|A^{\frac{1}{4}}u(s)\|^2\big)B_n(u(s))\,\mathrm{d}s = x+\Pi_n GW_t^n.$$

Given the fact that this equation is set in the finite dimensional space $H_n$, spanned by the first $n$ eigenvectors of $A$, we take its components with respect to $\{e_k\}_{k=1}^n$ and consider it in $\mathbb{R}^n$. Therefore we perform the Jacobian matrix $D_x$ of $u=(u_1,\dots,u_n)^T$ with respect to the starting point $x=(x_1,\dots,x_n)^T\in H_n$ and apply it to a vector $h=(h_1,\dots,h_n)^T\in H_n$:

$$\begin{aligned}
D_x[u(t)]h+\int_0^t AD_x[u(s)]h\,\mathrm{d}s+\int_0^t \Theta_R'\big(\|A^{\frac{1}{4}}u(s)\|^2\big)h\cdot\nabla_x\|A^{\frac{1}{4}}u(s)\|^2\,B_n\big(u(s)\big)\,\mathrm{d}s&\\
+\int_0^t \Theta_R\big(\|A^{\frac{1}{4}}u(s)\|^2\big)D_x\big[B_n\big(u(s)\big)\big]h\,\mathrm{d}s = h.&
\end{aligned} \tag{B.1}$$

We denote for brevity $U(t)=D_x[u(t)]h=\sum_{j=1}^n h_j\partial_{x_j}u(t)$, suppress the dependence on $t$, derive in time equation (B.1), then take its product in $H_n$ with $U$:

$$\begin{aligned}
\Big\langle\frac{\mathrm{d}}{\mathrm{d}t}U,U\Big\rangle+\langle AU,U\rangle+\Big\langle\Theta_R'\big(\|A^{\frac{1}{4}}u\|^2\big)h\cdot\nabla_x\|A^{\frac{1}{4}}u\|^2\,B_n(u),U\Big\rangle&\\
+\Big\langle D_x[B_n(u)]h,\Theta_R\big(\|A^{\frac{1}{4}}u\|^2\big)U\Big\rangle = 0.&
\end{aligned} \tag{B.2}$$

The first term in the left-hand side of equation (B.2) equals the time derivative of $\|U\|^2/2$, while the second is equal to $\|U\|_V^2$. As for the third one, we start by evaluating the components of the gradient vector. For $j=1,\dots,n$ it holds

$$\partial_{x_j}\|A^{\frac{1}{4}}u\|^2=\int_{\mathcal{D}}\partial_{x_j}|A^{1/4}u|^2\,\mathrm{d}\mathscr{L}^2=2\int_{\mathcal{D}}A^{\frac{1}{4}}u\cdot A^{\frac{1}{4}}\partial_{x_j}u\,\mathrm{d}\mathscr{L}^2=2\big\langle A^{\frac{1}{4}}u,A^{\frac{1}{4}}\partial_{x_j}u\big\rangle,$$

that implies
$$h \cdot \nabla_x \|A^{\frac14}u\|^2 = 2\langle A^{\frac14}u, A^{\frac14}U\rangle,$$
thus, by the Cauchy-Schwarz inequality and Lemma I.2.2
$$\begin{aligned}\Big|\Big\langle \Theta_R'\big(\|A^{\frac14}u\|^2\big)h\cdot\nabla_x\|A^{\frac14}u\|^2\, B_n(u), U\Big\rangle\Big| &\le 2\|\Theta_R'\|_\infty\big|\langle A^{\frac14}u, A^{\frac14}U\rangle\big|\,\big|\langle B_n(u), U\rangle\big| \\ &\le 2c_0\|\Theta_R'\|_\infty\|A^{\frac14}u\|^3\|A^{\frac14}U\|\|U\|_V.\end{aligned}$$
Analogously
$$\begin{aligned}&\Big\langle D_x[B_n(u)]h, \Theta_R\big(\|A^{\frac14}u\|^2\big)U\Big\rangle \\ &= \sum_{j=1}^n h_j\big\langle \partial_{x_j}B_n(u), \Theta_R\big(\|A^{\frac14}u\|^2\big)U\big\rangle \\ &= \sum_{j=1}^n h_j b_n\big(\partial_{x_j}u, u, \Theta_R\big(\|A^{\frac14}u\|^2\big)U\big) + b_n\big(u, \partial_{x_j}u, \Theta_R\big(\|A^{\frac14}u\|^2\big)U\big) \\ &= b_n\big(U, u, \Theta_R\big(\|A^{\frac14}u\|^2\big)U\big) + b_n\big(u, U, \Theta_R\big(\|A^{\frac14}u\|^2\big)U\big),\end{aligned}$$
which entails, by Lemma I.2.2 and $\|\Theta_R\|_\infty = 1$
$$\Big|\Big\langle D_x\big[B_n(u)\big]h, \Theta_R\big(\|A^{\frac14}u\|^2\big)U\Big\rangle\Big| \le 2c_0\|A^{\frac14}U\|\|A^{\frac14}u\|\|U\|_V.$$
By plugging all terms inside equation (B.1), we reach the following estimate
$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 + \|U\|_V^2 &\le 2c_0\big(1\vee\|\Theta_R'\|_\infty\big)\|A^{\frac14}U\|\|U\|_V\big(\|A^{\frac14}u\|^3 + \|A^{\frac14}u\|\big) \\ &\le K_R\|U\|^{1/2}\,\|U\|_V^{1/2}\|U\|_V \\ &\le \frac14 K_R^4\|U\|^2 + \frac34\|U\|_V^2,\end{aligned}$$
where we employed the interpolation inequality $\|A^{\frac14}U\| \le \|U\|^{1/2}\,\|U\|_V^{1/2}$ (see Lemma I.2.1), introduced the function
$$K_R(t) := 2c_0\big(1\vee\|\Theta_R'\|_\infty\big)\Big(\sup_{n\in\mathbb{N}}\|A^{\frac14}u(t)\|^3 + \sup_{n\in\mathbb{N}}\|A^{\frac14}u(t)\|\Big),$$
and iterated the Young inequality. We integrate over the interval $[0,t]$ recalling that $U(0) = D_x[u(0)]h = h$
$$\|U(t)\|^2 + \frac12\int_0^t \|U(s)\|_V^2\,\mathrm{d}s \le \|h\|^2 + \int_0^t K_R^4(s)\|U(s)\|^2\,\mathrm{d}s, \tag{B.3}$$
apply Grönwall's lemma and obtain
$$\|U(t)\|^2 \le \|h\|^2 e^{K_R^4(t)}.$$
Inserting back this estimate in equation (B.3), we reach the thesis for a constant $C_R(t) > 0$
$$\int_0^t \|U(s)\|_V^2\,\mathrm{d}s \le 2\|h\|^2\Big[1 + \int_0^t K_R^4(s)e^{K_R^4(s)}\,\mathrm{d}s\Big] = \|h\|^2 C_R(t).$$
□

**Lemma B.3.** *Assume that $\beta > 0$. Let $u \in L^2(0,T;H^{\beta+1})$ and let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^2(0,T;H^{\beta+1})$ convergent to $u$ in $L^2(0,T;H^{\beta+1}_{loc})$.*

- *If $\beta > 1$, then for all $y \in H^{-\beta}$*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \int_0^t {}_{H^{-\beta}}\big\langle y, B(u_n(s)) - B(u(s))\big\rangle_{H^\beta} \, \mathrm{d}s \right| = 0. \tag{B.4}$$

- *If $\beta \in (0,1]$, then for all $y \in H^\beta$*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \int_0^t {}_{H^{-\beta}}\big\langle B(u_n(s)) - B(u(s)), y\big\rangle_{H^\beta} \, \mathrm{d}s \right| = 0.$$

*Proof.* We give the proof only in the case $\beta > 1$, the complementary being analogous.

***Step* 1**. Assume first that $y \in C_c(\mathbb{R}^2;\mathbb{R}^2) \cap H^{-\beta}$, in particular, there exists $R \in \mathbb{N}$ such that the support of $y$ is contained in $\bar{B}_R := \{\xi \in \mathbb{R}^2 \,:\, |\xi| \le R\}$. Assume also that $x, z \in H^{\beta+1}$. Let us fix $\varphi, \psi \in C_c^\infty(\mathbb{R}^2)$ such that $\bar{B}_R \subset \operatorname{supp}\varphi \cap \operatorname{supp}\psi$, $\varphi(\xi) = \psi(\xi) = 1$ for $\xi \in \bar{B}_R$, and $\nabla\varphi \cdot x = \nabla\psi \cdot z = 0$. Observe in particular that $B(x,z)$, $B(\varphi x, \psi z) \in H \subset L^2(\mathbb{R}^2;\mathbb{R}^2)$, then, thanks to the Plancherel Theorem, and to the bounded support of $y$

$$\begin{aligned}
{}_{H^{-\beta}}\big\langle y, B(x,z)\big\rangle_{H^\beta} &= \int_{\mathbb{R}^2} \mathscr{F}\big[B(x,z)\big] \cdot \mathscr{F}[y] \, \mathrm{d}\mathscr{L}^2 \\
&= \int_{\mathbb{R}^2} \begin{pmatrix} x \cdot \nabla z_1 \\ x \cdot \nabla z_2 \end{pmatrix} \cdot y \, \mathrm{d}\mathscr{L}^2 \\
&= \int_{\bar{B}_R} \begin{pmatrix} x \cdot \nabla z_1 \\ x \cdot \nabla z_2 \end{pmatrix} \cdot y \, \mathrm{d}\mathscr{L}^2 \\
&= \int_{\bar{B}_R} \begin{pmatrix} \varphi x \cdot \nabla(\psi z_1) \\ \varphi x \cdot \nabla(\psi z_2) \end{pmatrix} \cdot y \, \mathrm{d}\mathscr{L}^2 \\
&= \int_{\mathbb{R}^2} \begin{pmatrix} \varphi x \cdot \nabla(\psi z_1) \\ \varphi x \cdot \nabla(\psi z_2) \end{pmatrix} \cdot y \, \mathrm{d}\mathscr{L}^2 \\
&= \int_{\mathbb{R}^2} \mathscr{F}\big[B(\varphi x, \psi z)\big] \cdot \mathscr{F}[y] \, \mathrm{d}\mathscr{L}^2 \\
&= {}_{H^{-\beta}}\big\langle y, B(\varphi x, \psi z)\big\rangle_{H^\beta}.
\end{aligned}$$

By estimate (II.2.4) in Lemma II.2.12 (with $\sigma = \beta + 1 > 2$), there exists a finite constant $c > 0$, depending on $\beta$, such that

$$\begin{aligned}
\left| {}_{H^{-\beta}}\big\langle y, B(x,z)\big\rangle_{H^\beta} \right| &= \left| {}_{H^{-\beta}}\big\langle y, B(\varphi x, \psi z)\big\rangle_{H^\beta} \right| \\
&\le c \|\varphi x\|_{H^\beta} \|\psi z\|_{H^{\beta+1}} \|y\|_{H^{-\beta}} \\
&\le c \|\varphi x\|_{H^{\beta+1}} \|\psi z\|_{H^{\beta+1}} \|y\|_{H^{-\beta}}.
\end{aligned}$$

By taking the infimum to the first and last side of the last chain of inequalities, over all functions $\varphi$, $\psi$ satisfying the previously stated conditions, we get

$$\left| {}_{H^{-\beta}}\big\langle y, B(x,z)\big\rangle_{H^\beta} \right| \le c [\![x]\!]_{H_R^{\beta+1}} [\![z]\!]_{H_R^{\beta+1}} \|y\|_{H^{-\beta}},$$

where we resorted to the seminorm on $H^{\beta+1}_{loc}$ analogous to the one defined in equation (II.6.1).

***Step* 2**. For every $n \in \mathbb{N}$ and *a.e.* $s \in [0,T]$, we use the bilinearity of $B$ with the estimate from the previous step:

$$
\begin{aligned}
&\Big|{}_{H^{-\beta}}\big\langle y, B(u_n(s)) - B(u(s))\big\rangle_{H^\beta}\Big| \\
&= \Big|{}_{H^{-\beta}}\big\langle y, B\big(u_n(s)-u(s), u_n(s)\big) - B\big(u(s), u_n(s)-u(s)\big)\big\rangle_{H^\beta}\Big| \\
&\le c[\![u_n(s)-u(s)]\!]_{H^{\beta+1}_R}\Big([\![u_n(s)]\!]_{H^{\beta+1}_R} + [\![u(s)]\!]_{H^{\beta+1}_R}\Big)\|y\|_{H^{-\beta}} \\
&\le c[\![u_n(s)-u(s)]\!]_{H^{\beta+1}_R}\big(\|u_n(s)\|_{H^{\beta+1}} + \|u(s)\|_{H^{\beta+1}}\big)\|y\|_{H^{-\beta}}.
\end{aligned}
$$

We integrate in time for $s \in [0,t] \subseteq [0,T]$ and resort to the Hölder inequality and to the seminorm on $L^2(0,T;H^{\beta+1}_{loc})$ analogous to the one defined in equation (II.6.2):

$$
\begin{aligned}
&\sup_{t\in[0,T]}\left|\int_0^t {}_{H^{-\beta}}\big\langle y, B(u_n(s)) - B(u(s))\big\rangle_{H^\beta}\,\mathrm{d}s\right| \\
&\le c[\![u_n-u]\!]_{L^2(0,T;H^{\beta+1}_R)}\Big(\|u_n\|_{L^2(0,T;H^{\beta+1})} + \|u\|_{L^2(0,T;H^{\beta+1})}\Big)\|y\|_{H^{-\beta}},
\end{aligned}
$$

By taking the limit inferior as $n \to \infty$ to both members of this last estimate, and by recalling the hypotheses of the lemma, we get the validity of equation (B.4) for $y \in C_c(\mathbb{R}^2;\mathbb{R}^2) \cap H^{-\beta}$.

***Step* 3**. Assume now $y \in H^{-\beta}$ and fix $\varepsilon > 0$. By density, there exists $y_\varepsilon \in C_c(\mathbb{R}^2;\mathbb{R}^2) \cap H^{-\beta}$ such that $\|y - y_\varepsilon\|_{H^{-\beta}} \le \varepsilon$. We use once again the estimate from *Step* 1 for all $n \in \mathbb{N}$ and *a.e.* $s \in [0,T]$:

$$
\begin{aligned}
&{}_{H^{-\beta}}\big\langle y, B(u_n(s)) - B(u(s))\big\rangle_{H^\beta} \\
&= {}_{H^{-\beta}}\big\langle y-y_\varepsilon, B(u_n(s)) - B(u(s))\big\rangle_{H^\beta} + {}_{H^{-\beta}}\big\langle y_\varepsilon, B(u_n(s)) - B(u(s))\big\rangle_{H^\beta} \\
&\le c[\![u_n(s)-u(s)]\!]_{H^{\beta+1}_R}\big(\|u_n(s)\|_{H^{\beta+1}} + \|u(s)\|_{H^{\beta+1}}\big)\|y-y_\varepsilon\|_{H^{-\beta}} \\
&\quad + {}_{H^{-\beta}}\big\langle y_\varepsilon, B(u_n(s)) - B(u(s))\big\rangle_{H^\beta}.
\end{aligned}
$$

We integrate over $s \in [0,t] \subseteq [0,T]$, estimate by the Hölder inequality and take the supremum for $t \in [0,T]$:

$$
\begin{aligned}
&\sup_{t\in[0,T]}\left|\int_0^t {}_{H^{-\beta}}\big\langle y, B(u_n(s)) - B(u(s))\big\rangle_{H^\beta}\,\mathrm{d}s\right| \\
&\le c[\![u_n-u]\!]_{L^2(0,T;H^{\beta+1}_R)}\big(\|u_n\|_{L^2(0,T;H^{\beta+1})} + \|u\|_{L^2(0,T;H^{\beta+1})}\big)\|y\|_{H^{-\beta}} \\
&\quad + \sup_{t\in[0,T]}\left|\int_0^t {}_{H^{-\beta}}\big\langle y_\varepsilon, B(u_n(s)) - B(u(s))\big\rangle_{H^\beta}\,\mathrm{d}s\right|.
\end{aligned}
$$

Both terms in the right-hand side are infinitesimal: the first thanks to the hypotheses, while the second by *Step* 2. Hence, the result follows by taking the limit inferior as $n \to \infty$. □

**Lemma B.4.** *For all $\alpha > 0$ the $\sigma$-algebras $\mathscr{B}_{D(A^\alpha)}$ and $\mathscr{B}_H \cap D(A^\alpha)$ coincide on $D(A^\alpha)$.*

*Proof.* Let $\iota : D(A^\alpha) \to H$ denote the natural embedding and $\tau_{D(A^\alpha)}, \tau_H$ denote the topologies on $D(A^\alpha), H$ induced by the norms $\|\cdot\|_{D(A^\alpha)}$ and $\|\cdot\|$, respectively. By continuity of $\iota$, we have $\iota^{-1}(\tau_H) \subset \tau_{D(A^\alpha)}$, which means that the topology $\tau_{D(A^\alpha)}$ on $D(A^\alpha)$ is finer then the topology induced from $H$ via $\iota$. As a consequence

$$\mathscr{B}_{D(A^\alpha)} = \sigma\big(\tau_{D(A^\alpha)}\big) \supset \sigma\big(\iota^{-1}(\tau_H)\big) = \iota^{-1}\big(\sigma(\tau_H)\big) = \iota^{-1}(\mathscr{B}_H) = \mathscr{B}_H \cap D(A^\alpha),$$

to wit the $\sigma$-algebra induced from $H$ and denoted by $\mathscr{B}_H \cap D(A^\alpha)$ is a sub-$\sigma$-algebra of $\mathscr{B}_{D(A^\alpha)}$.

In order to invert the above inclusion we can not proceed in the same way. Should indeed the two topologies $\tau_{D(A^\alpha)}$,$\tau_H$ coincide, then the respective norms would be equivalent, which is false. Nevertheless, if we were able to prove that the open sets in $D(A^\alpha)$ are Borel sets in $H$, then we would easily reach the thesis by

$$\mathscr{B}_{D(A^\alpha)} = \sigma\big(\tau_{D(A^\alpha)}\big) \subset \sigma\big(\mathscr{B}_H \cap D(A^\alpha)\big) = \mathscr{B}_H \cap D(A^\alpha).$$

In order to show that $\tau_{D(A^\alpha)} \subset \mathscr{B}_H \cap D(A^\alpha)$, it is sufficient to prove that the $\|\cdot\|_{D(A^\alpha)}$-norm on $D(A^\alpha)$ is a $\mathscr{B}_H$-measurable function. Let $\Gamma \subset D(A^\alpha)$ be countable and dense in $H$ (such collection exists because $H$ is separable and $D(A^\alpha)$ is dense in $H$). We have for all $x \in D(A^\alpha)$

$$\|x\|_{D(A^\alpha)} = \|A^\alpha x\| = \sup_{y \in H \setminus \{0\}} \frac{|\langle A^\alpha x, y\rangle|}{\|y\|} = \sup_{y \in \Gamma \setminus \{0\}} \frac{|\langle A^\alpha x, y\rangle|}{\|y\|} = \sup_{y \in \Gamma \setminus \{0\}} \frac{|\langle x, A^\alpha y\rangle|}{\|y\|},$$

where the second equality follows by the characterization of the norm in Hilbert spaces, the third by the density of $\Gamma$ in $H$ and the continuity in $H \setminus \{0\}$ of $y \mapsto |\langle A^\alpha x, y\rangle|/\|y\|$, while the last equality follows by self-adjointness of $A^\alpha$ (see Section I.2.2). For every $y \in \Gamma \setminus \{0\}$ we define the function

$$p_y : H \to \mathbb{R} : x \mapsto p_y(x) = \frac{\langle x, A^\alpha y\rangle}{\|y\|},$$

and we observe that it is a linear and bounded functional, thus continuous, thus $\mathscr{B}_H$-measurable. We therefore conclude that $\|\cdot\|_{D(A^\alpha)} = \sup_{y \in \Gamma \setminus \{0\}} |p_y(\cdot)|$ is $\mathscr{B}_H$-measurable as it is the supremum of a countable family of $\mathscr{B}_H$-measurable functions. □

# Bibliography


[AVA90] L. S. Pontryagin (eds.) A. V. Arkhangel'skiĭ V. V. Fedorchuk (auth.) *General Topology I: Basic Concepts and Constructions Dimension Theory*. 1st ed. Encyclopaedia of Mathematical Sciences 17. Springer-Verlag Berlin Heidelberg, 1990. ISBN: 978-3-642-64767-3.

[AF04] S. Albeverio and B. Ferrario. "Uniqueness of solutions of the stochastic Navier-Stokes equation with invariant measure given by the enstrophy." In: *Ann. Probab.* 32.2 (2004), pp. 1632–1649. ISSN: 0091-1798,2168-894X. DOI: 10.1214/009117904000000379.

[AC90] Sergio Albeverio and Ana-Bela Cruzeiro. "Global flows with invariant (Gibbs) measures for Euler and Navier-Stokes two dimensional fluids." In: *Communications in mathematical physics* 129.3 (1990), pp. 431–444.

[AH89] Sergio Albeverio and Raphael Høegh-Krohn. "Stochastic flows with stationary distribution for two-dimensional inviscid fluids." In: *Stochastic Process. Appl.* 31.1 (1989), pp. 1–31. ISSN: 0304-4149,1879-209X. DOI: 10.1016/0304-4149(89)90100-2.

[ARH79] Sergio Albeverio, M Ribeiro de Faria, and Raphael Høegh-Krohn. "Stationary measures for the periodic Euler flow in two dimensions." In: *Journal of Statistical Physics* 20 (1979), pp. 585–595.

[Bad70] Albert Badrikian. *Séminaire sur les Fonctions Aléatoires Linéaires et les Mesures Cylindriques*. Vol. 139. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1970, pp. vii+221.

[Bal17] Paolo Baldi. *Stochastic Calculus*. Universitext. An Introduction Through Theory and Exercises. Springer, Cham, 2017, pp. xiv+627. ISBN: 978-3-319-62225-5. DOI: 10.1007/978-3-319-62226-2.

[BT73] A. Bensoussan and R. Temam. "Équations Stochastiques du Type Navier-Stokes." In: *J. Functional Analysis* 13 (1973), pp. 195–222. ISSN: 0022-1236. DOI: 10.1016/0022-1236(73)90045-1.

[BF20] Hakima Bessaih and Benedetta Ferrario. "Invariant measures for stochastic damped 2D Euler equations." In: *Comm. Math. Phys.* 377.1 (2020), pp. 531–549. ISSN: 0010-3616,1432-0916. DOI: 10.1007/s00220-020-03714-3.

[Bil99] Patrick Billingsley. *Convergence of probability measures*. Second. Wiley Series in Probability and Statistics: Probability and Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999, pp. x+277. ISBN: 0-471-19745-9. DOI: 10.1002/9780470316962.

[Bir06] Andrei Biryuk. “On invariant measures of the 2D Euler equation.” In: *J. Stat. Phys.* 122.4 (2006), pp. 597–616. ISSN: 0022-4715,1572-9613. DOI: 10.1007/s10955-005-8011-0.

[Bis81] Jean-Michel Bismut. “Martingales, the Malliavin Calculus and Hypoellipticity Under General Hörmander's Conditions.” In: *Z. Wahrsch. Verw. Gebiete* 56.4 (1981), pp. 469–505. ISSN: 0044-3719. DOI: 10.1007/BF00531428.

[Bog07] V. I. Bogachev. *Measure theory. Vol. I, II.* Springer-Verlag, Berlin, 2007, Vol. I: xviii+500 pp., Vol. II: xiv+575. ISBN: 978-3-540-34513-8. DOI: 10.1007/978-3-540-34514-5.

[Bre10] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer Science & Business Media, 2010.

[BF24] Zdzisław Brzeźniak and Matteo Ferrari. “Inviscid Limit of the Stochastic Hyperviscous Navier-Stokes Equations and Invariant Measures for the Euler Equations in $\mathbb{R}^2$.” In: *arXiv* (2024). DOI: https://arxiv.org/abs/2409.17697.

[BF19] Zdzisław Brzeźniak and Benedetta Ferrario. “Stationary Solutions for Stochastic Damped Navier-Stokes Equations in $\mathbb{R}^d$.” In: *Indiana Univ. Math. J.* 68.1 (2019), pp. 105–138. ISSN: 0022-2518,1943-5258. DOI: 10.1512/iumj.2019.68.7551.

[BFZ24] Zdzisław Brzeźniak, Benedetta Ferrario, and Margherita Zanella. “Invariant measures for a stochastic nonlinear and damped 2D Schrödinger equation.” In: *Nonlinearity* 37.1 (2024), Paper No. 015001, 66. ISSN: 0951-7715,1361-6544. DOI: 10.1088/1361-6544/ad0f3a.

[BG99] Zdzisław Brzeźniak and Dariusz Gątarek. “Martingale solutions and invariant measures for stochastic evolution equations in Banach spaces.” In: *Stochastic Process. Appl.* 84.2 (1999), pp. 187–225. ISSN: 0304-4149,1879-209X. DOI: 10.1016/S0304-4149(99)00034-4.

[BGL20] Zdzislaw Brzeźniak, Beniamin Goldys, and Kim Ngan Le. “Existence of a unique solution and invariant measures for the stochastic Landau-Lifshitz-Bloch equation.” In: *J. Differential Equations* 269.11 (2020), pp. 9471–9507. ISSN: 0022-0396,1090-2732. DOI: 10.1016/j.jde.2020.06.061.

[BKP22] Zdzisław Brzeźniak, Tomasz Komorowski, and Szymon Peszat. “Ergodicity for stochastic equations of Navier-Stokes type.” In: *Electron. Commun. Probab.* 27 (2022), Paper No. 4, 10. ISSN: 1083-589X. DOI: 10.1214/21-ecp443.

[BL04] Zdzisław Brzeźniak and Yu-Hong Li. “Asymptotic behaviour of solutions to the 2D stochastic Navier-Stokes equations in unbounded domains—new developments.” In: *Recent developments in stochastic analysis and related topics.* World Sci. Publ., Hackensack, NJ, 2004, pp. 78–111. ISBN: 981-256-104-8.

[BL06] Zdzisław Brzeźniak and Yuhong Li. “Asymptotic Compactness and Absorbing Sets for 2D Stochastic Navier-Stokes Equations on Some Unbounded Domains.” In: *Trans. Amer. Math. Soc.* 358.12 (2006), pp. 5587–5629. ISSN: 0002-9947,1088-6850. DOI: 10.1090/S0002-9947-06-03923-7.

[BM13] Zdzisław Brzeźniak and Elżbieta Motyl. "Existence of a martingale solution of the stochastic Navier-Stokes equations in unbounded 2D and 3D domains." In: *J. Differential Equations* 254.4 (2013), pp. 1627–1685. ISSN: 0022-0396,1090-2732. DOI: 10.1016/j.jde.2012.10.009.

[BMO17] Zdzisław Brzeźniak, Elżbieta Motyl, and Martin Ondrejat. "Invariant Measure for the Stochastic Navier-Stokes Equations in Unbounded 2D Domains." In: *Ann. Probab.* 45.5 (2017), pp. 3145–3201. ISSN: 0091-1798,2168-894X. DOI: 10.1214/16-AOP1133.

[BO13] Zdzisław Brzeźniak and Martin Ondreját. "Stochastic Geometric Wave Equations with Values in Compact Riemannian Homogeneous Spaces." In: *Ann. Probab.* 41.3B (2013), pp. 1938–1977. ISSN: 0091-1798,2168-894X. DOI: 10.1214/11-AOP690.

[BOS16] Zdzisław Brzeźniak, Martin Ondreját, and Jan Seidler. "Invariant measures for stochastic nonlinear beam and wave equations." In: *J. Differential Equations* 260.5 (2016), pp. 4157–4179. ISSN: 0022-0396,1090-2732. DOI: 10.1016/j.jde.2015.11.007.

[BS13] Zdzisław Brzeźniak and Rafael Serrano. "Optimal Relaxed Control of Dissipative Stochastic Partial Differential Equations in Banach Spaces." In: *SIAM J. Control Optim.* 51.3 (2013), pp. 2664–2703. ISSN: 0363-0129,1095-7138. DOI: 10.1137/100788574.

[Cae98] António M. Caetano. "Eigenvalue Asymptotics of the Stokes Operator for Fractal Domains." In: *Proc. London Math. Soc. (3)* 76.3 (1998), pp. 579–602. ISSN: 0024-6115,1460-244X. DOI: 10.1112/S0024611598000331.

[Cer01] Sandra Cerrai. *Second order PDE's in finite and infinite dimension*. Vol. 1762. Lecture Notes in Mathematics. A probabilistic approach. Springer-Verlag, Berlin, 2001, pp. x+330. ISBN: 3-540-42136-X. DOI: 10.1007/b80743.

[Cip99] Fernanda Cipriano. "The two-dimensional Euler equation: a statistical study." In: *Comm. Math. Phys.* 201.1 (1999), pp. 139–154. ISSN: 0010-3616,1432-0916. DOI: 10.1007/s002200050552.

[Coh13] Donald L. Cohn. *Measure Theory*. Second. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, New York, 2013, pp. xxi+457. ISBN: 978-1-4614-6956-8. DOI: 10.1007/978-1-4614-6956-8.

[DZ96] G. Da Prato and J. Zabczyk. *Ergodicity for Infinite Dimensional Systems*. Vol. 229. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1996, pp. xii+339. ISBN: 0-521-57900-7. DOI: 10.1017/CBO9780511662829.

[DD02] Giuseppe Da Prato and Arnaud Debussche. "Two-dimensional Navier-Stokes equations driven by a space-time white noise." In: *J. Funct. Anal.* 196.1 (2002), pp. 180–210. ISSN: 0022-1236,1096-0783. DOI: 10.1006/jfan.2002.3919.

[DZ14] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic Equations in Infinite Dimensions*. Second. Vol. 152. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2014, pp. xviii+493. ISBN: 978-1-107-05584-1. DOI: 10.1017/CBO9781107295513.

[Day73] Mahlon M. Day. *Normed linear spaces*. Third. Vol. Band 21. Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]. Springer-Verlag, New York-Heidelberg, 1973, pp. viii+211.

[DHV16] Arnaud Debussche, Martina Hofmanová, and Julien Vovelle. "Degenerate parabolic stochastic partial differential equations: quasilinear case." In: *Ann. Probab.* 44.3 (2016), pp. 1916–1955. ISSN: 0091-1798,2168-894X. DOI: 10.1214/15-AOP1013.

[DPV12] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. "Hitchhiker's guide to the fractional Sobolev spaces." In: *Bull. Sci. Math.* 136.5 (2012), pp. 521–573. ISSN: 0007-4497,1952-4773. DOI: 10.1016/j.bulsci.2011.12.004.

[DX11] Zhao Dong and Yingchao Xie. "Ergodicity of stochastic 2D Navier-Stokes equation with Lévy noise." In: *J. Differential Equations* 251.1 (2011), pp. 196–222. ISSN: 0022-0396,1090-2732. DOI: 10.1016/j.jde.2011.03.015.

[Edg79] G. A. Edgar. "Measurability in a Banach Space. II." In: *Indiana Univ. Math. J.* 28.4 (1979), pp. 559–579. ISSN: 0022-2518,1943-5258. DOI: 10.1512/iumj.1979.28.28039.

[EL94] K. D. Elworthy and X.-M. Li. "Formulae for the Derivatives of Heat Semigroups." In: *J. Funct. Anal.* 125.1 (1994), pp. 252–286. ISSN: 0022-1236,1096-0783. DOI: 10.1006/jfan.1994.1124.

[Fer24] Matteo Ferrari. "New a Priori Estimate for Stochastic 2D Navier–Stokes Equations with Applications to Invariant Measure." In: *Annali di Matematica Pura ed Applicata (1923-)* (2024), pp. 1–33. DOI: 10.1007/s10231-024-01517-0.

[Fer97] Benedetta Ferrario. "Ergodic Results for Stochastic Navier-Stokes Equation." In: *Stochastics Stochastics Rep.* 60.3-4 (1997), pp. 271–288. ISSN: 1045-1129. DOI: 10.1080/17442509708834110.

[Fer99] Benedetta Ferrario. "Stochastic Navier-Stokes Equations: Analysis of the Noise to Have a Unique Invariant Measure." In: *Ann. Mat. Pura Appl. (4)* 177 (1999), pp. 331–347. ISSN: 0003-4622. DOI: 10.1007/BF02505916.

[Fer03] Benedetta Ferrario. "Uniqueness Result for the 2D Navier-Stokes Equation with Additive Noise." In: *Stoch. Stoch. Rep.* 75.6 (2003), pp. 435–442. ISSN: 1045-1129,1029-0346. DOI: 10.1080/10451120310001644485.

[Fer06] Benedetta Ferrario. "On Some Problems of Regularity in Two-Dimensional Stochastic Hydrodynamics." In: *Stochastic partial differential equations and applications—VII*. Vol. 245. Lect. Notes Pure Appl. Math. Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. 97–103. ISBN: 978-0-8247-0027-0. DOI: 10.1201/9781420028720.ch10.

[Fer23] Benedetta Ferrario. "On 2D Eulerian limits à la Kuksin." In: *J. Differential Equations* 342 (2023), pp. 1–20. ISSN: 0022-0396. DOI: 10.1016/j.jde.2022.09.034.

[FO19] Benedetta Ferrario and Christian Olivera. "2D Navier-Stokes equation with cylindrical fractional Brownian noise." In: *Ann. Mat. Pura Appl. (4)* 198.3 (2019), pp. 1041–1067. ISSN: 0373-3114,1618-1891. DOI: 10.1007/s10231-018-0809-x.

[Fla94] Franco Flandoli. "Dissipativity and invariant measures for stochastic Navier-Stokes equations." In: *NoDEA Nonlinear Differential Equations Appl.* 1.4 (1994), pp. 403–423. ISSN: 1021-9722,1420-9004. DOI: 10.1007/BF01194988.

[FM95] Franco Flandoli and Bohdan Maslowski. "Ergodicity of the 2-D Navier-Stokes Equation Under Random Perturbations." In: *Comm. Math. Phys.* 172.1 (1995), pp. 119–141. ISSN: 0010-3616,1432-0916.

[FR01] Franco Flandoli and Marco Romito. "Statistically stationary solutions to the 3-D Navier-Stokes equation do not show singularities." In: *Electron. J. Probab.* 6 (2001), no. 5, 15. ISSN: 1083-6489. DOI: 10.1214/EJP.v6-78.

[Fol99] Gerald B Folland. *Real analysis: modern techniques and their applications*. Vol. 40. John Wiley & Sons, 1999.

[FK64] Hiroshi Fujita and Tosio Kato. "On the Navier-Stokes Initial Value Problem. I." In: *Arch. Rational Mech. Anal.* 16 (1964), pp. 269–315. ISSN: 0003-9527. DOI: 10.1007/BF00276188.

[FM70] Hiroshi Fujita and Hiroko Morimoto. "On Fractional Powers of the Stokes Operator." In: *Proc. Japan Acad.* 46 (1970), pp. 1141–1143. ISSN: 0021-4280.

[FM77] Daisuke Fujiwara and Hiroko Morimoto. "An $L_r$-theorem of the Helmholtz decomposition of vector fields." In: *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 24.3 (1977), pp. 685–700. ISSN: 0040-8980.

[GKP16] S. Gabriyelyan, J. Kąkol, and G. Plebanek. "The Ascoli property for function spaces and the weak topology of Banach and Fréchet spaces." In: *Studia Math.* 233.2 (2016), pp. 119–139. ISSN: 0039-3223,1730-6337. DOI: 10.4064/sm8289-4-2016.

[Gig81] Yoshikazu Giga. "Analyticity of the Semigroup Generated by the Stokes Operator in $L_r$ Spaces." In: *Math. Z.* 178.3 (1981), pp. 297–329. ISSN: 0025-5874,1432-1823. DOI: 10.1007/BF01214869.

[Gig83] Yoshikazu Giga. "Weak and Strong Solutions of the Navier-Stokes Initial Value Problem." In: *Publ. Res. Inst. Math. Sci.* 19.3 (1983), pp. 887–910. ISSN: 0034-5318,1663-4926. DOI: 10.2977/prims/1195182014.

[Gig86] Yoshikazu Giga. "Solutions for Semilinear Parabolic Equations in $L^p$ and Regularity of Weak Solutions of the Navier-Stokes System." In: *J. Differential Equations* 62.2 (1986), pp. 186–212. ISSN: 0022-0396,1090-2732. DOI: 10.1016/0022-0396(86)90096-3.

[GM85] Yoshikazu Giga and Tetsuro Miyakawa. "Solutions in $L_r$ of the Navier-Stokes Initial Value Problem." In: *Arch. Rational Mech. Anal.* 89.3 (1985), pp. 267–281. ISSN: 0003-9527. DOI: 10.1007/BF00276875.

[GMR17] Nathan Glatt-Holtz, Jonathan C. Mattingly, and Geordie Richards. "On unique ergodicity in nonlinear stochastic partial differential equations." In: *J. Stat. Phys.* 166.3-4 (2017), pp. 618–649. ISSN: 0022-4715,1572-9613. DOI: 10.1007/s10955-016-1605-x.

[GM05] B. Goldys and B. Maslowski. “Exponential ergodicity for stochastic Burgers and 2D Navier-Stokes equations.” In: *J. Funct. Anal.* 226.1 (2005), pp. 230–255. ISSN: 0022-1236,1096-0783. DOI: 10.1016/j.jfa.2004.12.009.

[HM06] Martin Hairer and Jonathan C. Mattingly. “Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing.” In: *Ann. of Math. (2)* 164.3 (2006), pp. 993–1032. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2006.164.993.

[HM11] Martin Hairer and Jonathan C. Mattingly. “A Theory of Hypoellipticity and Unique Ergodicity for Semilinear Stochastic PDEs.” In: *Electron. J. Probab.* 16 (2011), no. 23, 658–738. ISSN: 1083-6489. DOI: 10.1214/EJP.v16-875.

[HR24] Martin Hairer and Tommaso Rosati. “Global existence for perturbations of the 2D stochastic Navier-Stokes equations with space-time white noise.” In: *Ann. PDE* 10.1 (2024), Paper No. 3, 46. ISSN: 2524-5317,2199-2576. DOI: 10.1007/s40818-023-00165-6.

[Hen81] Daniel Henry. *Geometric Theory of Semilinear Parabolic Equations.* Vol. 840. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1981, pp. iv+348. ISBN: 3-540-10557-3.

[Hyt+16] Tuomas Hytönen et al. *Analysis in Banach spaces. Vol. I. Martingales and Littlewood-Paley theory.* Vol. 63. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer, Cham, 2016, pp. xvi+614. ISBN: 978-3-319-48519-5.

[Jak97] A. Jakubowski. “The Almost Sure Skorokhod Representation for Subsequences in Nonmetric Spaces.” In: *Teor. Veroyatnost. i Primenen.* 42.1 (1997), pp. 209–216. ISSN: 0040-361X. DOI: 10.1137/S0040585X97976052.

[Jud63] V. I. Judovič. “Non-stationary flows of an ideal incompressible fluid.” In: *Ž. Vyčisl. Mat i Mat. Fiz.* 3 (1963), pp. 1032–1066. ISSN: 0044-4669.

[KF62] Tosio Kato and Hiroshi Fujita. “On the Nonstationary Navier-Stokes System.” In: *Rend. Sem. Mat. Univ. Padova* 32 (1962), pp. 243–260. ISSN: 0041-8994.

[KP86] Tosio Kato and Gustavo Ponce. “Well-Posedness of the Euler and Navier-Stokes Equations in the Lebesgue Spaces $L^p_s(\mathbb{R}^2)$.” In: *Rev. Mat. Iberoamericana* 2.1-2 (1986), pp. 73–88. ISSN: 0213-2230. DOI: 10.4171/RMI/26.

[KP87] Tosio Kato and Gustavo Ponce. “On Nonstationary Flows of Viscous and Ideal Fluids in $L^p_s(\mathbf{R}^2)$.” In: *Duke Math. J.* 55.3 (1987), pp. 487–499. ISSN: 0012-7094,1547-7398. DOI: 10.1215/S0012-7094-87-05526-8.

[Kec95] Alexander S. Kechris. *Classical descriptive set theory.* Vol. 156. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995, pp. xviii+402. ISBN: 0-387-94374-9. DOI: 10.1007/978-1-4612-4190-4.

[Kie80] Hansjörg Kielhöfer. "Global Solutions of Semilinear Evolution Equations Satisfying an Energy Inequality." In: *J. Differential Equations* 36.2 (1980), pp. 188–222. ISSN: 0022-0396,1090-2732. DOI: 10.1016/0022-0396(80)90063-7.

[Kuk04] Sergei B. Kuksin. "The Eulerian limit for 2D statistical hydrodynamics." In: *J. Statist. Phys.* 115.1-2 (2004), pp. 469–492. ISSN: 0022-4715,1572-9613. DOI: 10.1023/B:JOSS.0000019830.64243.a2.

[Kuk06] Sergei B. Kuksin. *Randomly forced nonlinear PDEs and statistical hydrodynamics in 2 space dimensions.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2006, pp. x+93. ISBN: 3-03719-021-3. DOI: 10.4171/021.

[Kuk07] Sergei B. Kuksin. "Rigorous results and conjectures on stationary space-periodic 2D turbulence." In: *Séminaire: Équations aux Dérivées Partielles. 2006–2007.* Sémin. Équ. Dériv. Partielles. École Polytech., Palaiseau, 2007, Exp. No. VII, 18. ISBN: 978-2-7302-1414-8.

[Kuk08] Sergei B. Kuksin. "On distribution of energy and vorticity for solutions of 2D Navier-Stokes equation with small viscosity." In: *Comm. Math. Phys.* 284.2 (2008), pp. 407–424. ISSN: 0010-3616,1432-0916. DOI: 10.1007/s00220-008-0577-3.

[Kup12] Antti Kupiainen. "ERGODICITY OF TWO DIMENSIONAL TURBULENCE." English. In: *Astérisque* 339 (2012), pp. 137–156. ISSN: 0303-1179.

[Kur66] K. Kuratowski. *Topology. Vol. I.* New. Translated from the French by J. Jaworowski. Academic Press, New York-London; Państwowe Wydawnictwo Naukowe [Polish Scientific Publishers], Warsaw, 1966, pp. xx+560.

[Lad69] Olga Aleksandrovna Ladyzhenskaya. "The Mathematical Theory of Viscous Incompressible Flows." In: *Mathematics and its Applications.* 2 (1969).

[Lat23] Mickaël Latocca. "Construction of high regularity invariant measures for the 2D Euler equations and remarks on the growth of the solutions." In: *Comm. Partial Differential Equations* 48.1 (2023), pp. 22–53. ISSN: 0360-5302,1532-4133. DOI: 10.1080/03605302.2022.2137679.

[Łoj88] Stanisław Łojasiewicz. *An Introduction to the Theory of Real Functions.* Ed. by A. V. Ferreira. Third. A Wiley-Interscience Publication. With contributions by M. Kosiek, W. Mlak and Z. Opial, Translated from the Polish by G. H. Lawden. John Wiley & Sons, Ltd., Chichester, 1988, pp. x+230. ISBN: 0-471-91414-2.

[Lun95] Alessandra Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems.* Modern Birkhäuser Classics. [2013 reprint of the 1995 original] [MR1329547]. Birkhäuser/Springer Basel AG, Basel, 1995, pp. xviii+424. ISBN: 978-3-0348-0556-8.

[MB02] Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and incompressible flow.* Vol. 27. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2002, pp. xii+545. ISBN: 0-521-63057-6.

[Mas93] Bohdan Maslowski. “On Probability Distributions of Solutions of Semilinear Stochastic Evolution Equations.” In: *Stochastics Stochastics Rep.* 45.1-2 (1993), pp. 17–44. ISSN: 1045-1129. DOI: 10.1080/17442509308833854.

[MS99] Bohdan Maslowski and Jan Seidler. “On sequentially weakly Feller solutions to SPDE’s.” In: *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* 10.2 (1999), pp. 69–78. ISSN: 1120-6330,1720-0768.

[MS01] Bohdan Maslowski and Jan Seidler. “Strong Feller Solutions to SPDE’s are Strong Feller in the Weak Topology.” In: *Studia Math.* 148.2 (2001), pp. 111–129. ISSN: 0039-3223,1730-6337. DOI: 10.4064/sm148-2-2.

[Mét78] Guy Métivier. “Valeurs propres d’opérateurs définis par la restriction de systèmes variationnels à des sous-espaces.” In: *J. Math. Pures Appl. (9)* 57.2 (1978), pp. 133–156. ISSN: 0021-7824,1776-3371.

[Mor18] Valter Moretti. *Spectral Theory and Quantum Mechanics: Mathematical Foundations of Quantum Theories, Symmetries and Introduction to the Algebraic Formulation*. Vol. 110. Springer, 2018.

[Ond04] Martin Ondreját. “Uniqueness for stochastic evolution equations in Banach spaces.” In: *Dissertationes Math. (Rozprawy Mat.)* 426 (2004), p. 63. ISSN: 0012-3862,1730-6310. DOI: 10.4064/dm426-0-1.

[Ond05] Martin Ondreját. “Brownian representations of cylindrical local martingales, martingale problem and strong Markov property of weak solutions of SPDEs in Banach spaces.” In: *Czechoslovak Math. J.* 55(130).4 (2005), pp. 1003–1039. ISSN: 0011-4642,1572-9141. DOI: 10.1007/s10587-005-0084-z.

[Par79] E. Pardoux. “Stochastic Partial Differential Equations and Filtering of Diffusion Processes.” In: *Stochastics* 3.2 (1979), pp. 127–167. ISSN: 0090-9491. DOI: 10.1080/17442507908833142.

[Paz83] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Vol. 44. Applied Mathematical Sciences. Springer-Verlag, New York, 1983, pp. viii+279. ISBN: 0-387-90845-5. DOI: 10.1007/978-1-4612-5561-1.

[RR04] Michael Renardy and Robert C. Rogers. *An Introduction to Partial Differential Equations*. Second. Vol. 13. Texts in Applied Mathematics. Springer-Verlag, New York, 2004, pp. xiv+434. ISBN: 0-387-00444-0.

[Sei97] Jan Seidler. “Ergodic Behaviour of Stochastic Parabolic Equations.” In: *Czechoslovak Math. J.* 47(122).2 (1997), pp. 277–316. ISSN: 0011-4642,1572-9141. DOI: 10.1023/A:1022821729545.

[Sob59] P. E. Sobolevskiĭ. “Non-Stationary Equations of Viscous Fluid Dynamics.” In: *Dokl. Akad. Nauk SSSR* 128 (1959), pp. 45–48. ISSN: 0002-3264.

[Tay96] Michael E. Taylor. *Partial Differential Equations. I.* Vol. 115. Applied Mathematical Sciences. Basic Theory. Springer-Verlag, New York, 1996, pp. xxiv+563. ISBN: 0-387-94653-5. DOI: 10.1007/978-1-4684-9320-7.

[Tem95] Roger Temam. *Navier-Stokes Equations and Nonlinear Functional Analysis*. Second. Vol. 66. CBMS-NSF Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995, pp. xiv+141. ISBN: 0-89871-340-4. DOI: 10.1137/1.9781611970050.

[Tem01] Roger Temam. *Navier-Stokes equations*. Theory and numerical analysis, Reprint of the 1984 edition. AMS Chelsea Publishing, Providence, RI, 2001, pp. xiv+408. ISBN: 0-8218-2737-5. DOI: 10.1090/chel/343.

[VF88] M. J. Vishik and A. V. Fursikov. *Mathematical Problems of Statistical Hydromechanics*. Vol. 9. Mathematics and its Applications (Soviet Series). Translated from the 1980 Russian original [MR0591678] by D. A. Leites. Kluwer Academic Publishers Group, Dordrecht, 1988, pp. vii+576. ISBN: 978-94-010-7137-6. DOI: 10.1007/978-94-009-1423-0.

[Wil91] David Williams. *Probability with Martingales*. Cambridge University Press, 1991.

[Wol33] W. Wolibner. "Un theorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long." In: *Math. Z.* 37.1 (1933), pp. 698–726. ISSN: 0025-5874,1432-1823. DOI: 10.1007/BF01474610.

[ZZ17] Rongchan Zhu and Xiangchan Zhu. "Strong-Feller property for Navier-Stokes equations driven by space-time white noise." In: *arXiv preprint arXiv:1709.09306* (2017).

[Ziz03] Václav Zizler. "Nonseparable Banach Spaces." In: *Handbook of the geometry of Banach spaces, Vol. 2*. North-Holland, Amsterdam, 2003, pp. 1743–1816. ISBN: 0-444-51305-1. DOI: 10.1016/S1874-5849(03)80048-7.

# Acknowledgments

I express my gratitude to my supervisor Professor Enrico Priola for his insightful contributions and stimulating discussions on the subject. A special thank goes to Professor Zdzisław Brzeźniak, who hosted me for an Erasmus Traineeship programme at the Department of Mathematics of the University of York, England, and motivated the research that led to Chapter II of this thesis.

The research activity was carried out as part of the PRIN 2022 project "Noise in fluid dynamics and related models". The author is also member of the "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA)", which is part of the "Istituto Nazionale di Alta Matematica (INdAM)"