Joint Ph.D. Program in Mathematics

UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA
UNIVERSITÀ DEGLI STUDI DI PAVIA
ISTITUTO NAZIONALE DI ALTA MATEMATICA

Diffeological Symplectic Frobenius Reciprocity and the Co-Moment Map Portrait of Quantum Hydrodynamics

Supervisor: Prof. Mauro Spera

Ph.D. Thesis of: Gabriele Barbieri
Ciclo XXXVIII

A.A. 2024/2025

Abstract

This thesis addresses two problems in symplectic geometry and mathematical physics, connected by the common framework of Hamiltonian spaces.

The first concerns a symplectic analogue of Frobenius reciprocity, which is realized as a bijection between certain symplectically reduced spaces that are not necessarily manifolds. To address the singularities arising in such quotients, we employ diffeology, a generalization of classical differential geometry that provides a natural setting for treating non-smooth spaces. Motivated by a conjecture in [R22], we show that the symplectic Frobenius reciprocity holds as a diffeomorphism between diffeological spaces, preserving the reduced forms they may carry. This raises the foundational question of when such forms exist. We provide new sufficient conditions, showing that local freeness, or strictness, or properness of the group action guarantees their existence. Parallel results are obtained for prequantum spaces.

The second problem originates in quantum hydrodynamics. Recent work has shown that the Madelung transform defines a moment map correspondence between wave functions and the cotangent bundle of densities [K18, K19]. This correspondence, however, breaks down on nodal lines, where the probability density vanishes and the phase is undefined. We seek analogous relations that overcome this limitation by building on the Clebsch geometry of the probability current developed in [S23], resulting in a network of symplectic manifolds naturally associated with both quantum-mechanical and hydrodynamical structures. These manifolds are connected by moment maps, and their relations remain valid when restricted to the nodal line, thereby extending the geometric interplay between quantum mechanics and fluid dynamics into this singular regime.

Contents

Introduction and overview

This thesis investigates two problems in symplectic geometry and mathematical physics. The first concerns a symplectic formulation of the Frobenius reciprocity and the existence of reduced forms in the context of diffeological spaces. The second addresses the geometric structures underlying quantum hydrodynamics, with a focus on the role of (co-)moment maps and nodal lines. Although these two topics arise in very different contexts, one rooted in representation theory and the other in quantum fluid dynamics, they share a common geometric framework centered on Hamiltonian spaces, which are symplectic manifolds equipped with Lie group actions and equivariant moment maps. The key aspects of this setting manifest differently in each problem: symplectic reduction plays a fundamental role in the statement and development of the first, while moment maps relate the relevant geometric structures in the second. Despite this shared background, the two problems evolve independently and require distinct methods. For this reason, the thesis is divided into two parts, each (almost completely) self-contained and dedicated to one line of inquiry.

The motivations of the first part arise from the correspondence between representation theory and symplectic geometry, exemplified by the Kirillov orbit method [K62] and the Borel-Weil theorem [S54]. This correspondence leads to geometric constructions, realized via symplectic reduction [M74], that mirror the representation-theoretic space of intertwiners and induced representation [K78, W78, G82]. As these objects are classically related by Frobenius reciprocity, a "symplectic Frobenius reciprocity" has also been introduced and studied over the years: in particular, [G83] proved it for coadjoint orbits and [R22] extended it to any Hamiltonian space. In this context, the reciprocity manifests as a bijection between symplectically reduced spaces which, only in favorable cases, are manifolds. This inspired the conjecture formulated in [R22], which claims the bijection to be a diffeomorphism between diffeological spaces, a category that naturally accommodates quotients and singular spaces, while supporting a definition of differential forms. We prove this conjecture, establishing the symplectic Frobenius reciprocity as a diffeological diffeomorphism between reduced spaces which preserves the reduced forms they may carry. We say “may” because it is not known, in general, when a reduced diffeological space carries a reduced diffeological form. This issue is then addressed and it is shown that the group action suffices to be locally free, proper or strict for the reduced form to exist. Analogous constructions and results are defined and proved for prequantum spaces.

The second problem stems from quantum hydrodynamics. The Madelung transform casts the Schrödinger equation for a massive spinless particle into hydrodynamical form, establishing a bridge between quantum mechanics and fluid dynamics [M27, B52a, B52b]. In this framework, we focus on geometric and topological issues concerning the zero set of the wave function, assuming it forms a knot in 3-space. Along this set the probability density vanishes, the phase is undefined, and the standard Madelung velocity breaks down, giving rise to physical and mathematical subtleties. In order to handle this singular regime, we turn to the Clebsch geometry of the probability current, developed in [S23]. This leads to consider a collection of closely related symplectic manifolds, and their restriction to the nodal line [P00], arising from the projective Hilbert space approach to wavefunctions [K18, K19], from coadjoint orbits in hydrodynamics [M83, P92] and Clebsch variables themselves [P92]. The existing relations among these structures are summarized in a diagram whose arrows denote moment maps corresponding to Rasetti-Regge currents (co-moment maps) [R75, P89, P00, P98].

Structure

This thesis is divided into two parts, each with its own introduction and conclusion. Part I consists of three chapters, and Part II of two. In both parts, the first chapter provides background material, while the remaining chapters present original results. The parts are largely self-contained, except for certain foundational definitions on Hamiltonian spaces and Lie group actions given in Chapter 1, which are also needed in Part II.

Chapter 1 introduces the essential definitions and results concerning diffeology, Lie groups and their actions, Hamiltonian and prequantum spaces, and their respective reduction procedures. It also clarifies the correspondence between representation theory and symplectic geometry by presenting the Borel-Weil theorem in Kostant’s formulation. This theorem explains the setting and the reason to describe the geometric (symplectic and prequantum) analogues of the induced representation and the space of intertwiners. The chapter ends with the symplectic and prequantum Frobenius reciprocity and the conjecture of [R22].

Chapter 2 proves this conjecture in both the symplectic and prequantum settings, establishing the symplectic Frobenius reciprocity as a diffeomorphism between diffeological spaces. It is also proved that when one of the spaces carries a diffeological 2-form, the other does as well, and the diffeomorphism preserves these forms.

Chapter 3 addresses the existence of reduced diffeological forms, proving new sufficient conditions: the group action need only be locally free, strict, or proper. Moreover, the strictness result is applied to cotangent bundle reduction.

Chapter 4 develops the geometric and mechanical tools required for Part II. It begins with a range of topics: Donaldson's moment map for the action of volume-preserving diffeomorphisms, the coadjoint orbit description of rotational perfect fluids and the hydrodynamical formulation of the Schrödinger equation. These strands converge in a symplectic-geometric interpretation of quantum hydrodynamics, providing the framework for introducing Clebsch variables, the nodal line and the results of [S23].

Chapter 5 contains the main result of Part II. It introduces several geometric structures arising from wave functions and quantum hydrodynamics and studies their restriction to the nodal line, assumed to be a knot in $\mathbb{R}^3$. Using the tools from Chapter 4, these structures are related and organized into a diagrammatic form that encapsulates the interplay between symplectic geometry, quantum hydrodynamics, and vortex-line geometry.

Results

The first part of this thesis proves the conjecture formulated in [R22], establishing a symplectic version of Frobenius reciprocity as a diffeomorphism between symplectically reduced spaces, viewed within the framework of diffeological spaces, that preserves any reduced 2-forms they may possess. It further addresses the problem of when such reduced forms exist, providing new sufficient conditions.

The second part investigates the symplectic structures emerging from quantum hydrodynamics, develops their interrelations, summarized in diagrammatic form, and analyzes their restriction to the nodal line.

Part I

The content of this part is based on a paper co-authored by the author of the present thesis with Jordan Watts and François Ziegler [B25].

The preliminary chapter (**Chapter 1**) introduces the fundamental definitions and results on diffeology and Hamiltonian spaces. These threads come together in Subsection 1.4.1, where the components of the symplectic reduction theorem are reinterpreted from a diffeological perspective: using the machinery developed in Sections 1.1 and 1.2, the zero level set of the moment map and the corresponding reduced space are equipped with the subset and subquotient diffeologies, respectively, and diffeological reduced forms are defined. The geometric counterparts of the induced representation and the space of intertwiners are also considered in this "diffeological reduction" framework, thereby clarifying the statement of the conjecture of [R22], in Remark 1.6.31. Prequantum spaces are likewise treated.

Before stating the results of this part, we remark that all the groups and manifolds therein are finite-dimensional.

Chapter 2

In this chapter the conjecture of [R22] is proved, along with its prequantum analogue. The following theorem addresses the symplectic case:

Theorem 2.1.1 *Let G be a Lie group, $H \subset G$ a closed subgroup, X a Hamiltonian G-space and Y a Hamiltonian H-space. Then we have an isomorphism of reduced spaces*

$$\mathrm{Hom}_G(X, \mathrm{Ind}_H^G Y) = \mathrm{Hom}_H(\mathrm{Res}_H^G X, Y),$$

i.e. there is a diffeological diffeomorphism t from left to right. Moreover, if one side carries a reduced 2-form, then so does the other, and t relates the 2-forms.

The objects appearing in the theorem are the following:

- The *induced Hamiltonian space* (Definition 1.6.16) is:

  $$\mathrm{Ind}_H^G Y := (T^*G \times Y) /\!\!/ H,$$

  where H, G and Y are as in the theorem and the double slash denotes the symplectically reduced space (Theorem 1.4.8). By Remark 1.6.18, it is a Hamiltonian G-space carrying a reduced form.

- The *symplectic space of intertwiners* (Definition 1.6.21) is:

  $$\mathrm{Hom}_G(X, Y) := (X^- \times Y) /\!\!/ G,$$

  where X and Y are Hamiltonian G-spaces and X^- denotes the Hamiltonian space X, but endowed with opposite symplectic form and moment map. In general, this symplectic reduction does not yield a manifold and thus the Hom-spaces in the theorem are regarded as diffeological spaces equipped with the subquotient diffeology (Definition 1.1.22).

- The restriction functor from Hamiltonian G-spaces to H-spaces is denoted by Res_H^G.

This result presents a geometric interpretation of the symplectic Frobenius reciprocity, established just as a bijection in [R22], and motivates the investigation of reduced 2-forms, pursued in Chapter 3.

The prequantum analogue of the previous theorem is:

Theorem 2.2.1 *Let G be a Lie group, $H \subset G$ a closed subgroup, $\tilde{X}$ a prequantum G-space and $\tilde{Y}$ a prequantum H-space. Then we have an isomorphism of reduced spaces*

$$\mathrm{Hom}_G(\tilde{X}, \mathrm{Ind}_H^G \tilde{Y}) = \mathrm{Hom}_H(\mathrm{Res}_H^G \tilde{X}, \tilde{Y}),$$

i.e. there is a diffeological diffeomorphism t from left to right. Moreover, if one side carries a reduced 1-form, then so does the other, and t relates the 1-forms.

The objects appearing in the theorem are the prequantum versions of the symplectic space of intertwiners and the induced Hamiltonian space: see Definitions 1.6.32 and 1.6.40. Moreover, just like the symplectic one, this result presents a geometric interpretation of the prequantum Frobenius reciprocity and motivates the analysis on the existence of reduced 1-forms.

Chapter 3

In this chapter, we study the existence of reduced forms in both the symplectic and prequantum settings, within the diffeological framework used for the Frobenius reciprocity. We show that local freeness or properness alone of Lie group actions is sufficient for the existence of reduced forms; we also introduce a third sufficient condition, namely *strict actions*.

The proofs of these results are based on Souriau's criterion, which characterizes when a diffeological form descends to the quotient:

Theorem 1.2.9 (Souriau's criterion) *Let X and Y be two diffeological spaces. Let $\pi : X \to Y$ be a subduction and let α be a diffeological k-form on X. Then α is the pullback of a k-form β defined on Y if and only if, for any P, Q plots of X such that $\pi \circ P = \pi \circ Q$, then $P^*\alpha = Q^*\alpha$.*

We apply this theorem to symplectic and prequantum reduction: we take $X = C$, the zero level set of the (Hamiltonian or prequantum) moment map Φ, $Y = C/G$, the reduced space under the action of the Lie group G, π

the projection on the quotient, α the restriction to C of the symplectic (or prequantum) form and β the corresponding reduced form. Thus the idea of the proofs is to verify that, for any P, Q plots of C such that $\pi \circ P = \pi \circ Q$, then $P^*\alpha = Q^*\alpha$ (pullback defined in Definition/Proposition 1.2.6), ensuring the existence of the reduced form.

With this common proof strategy in place, we now turn to the specific cases, starting with strict actions (Definition 3.1.2):

Theorem 3.1.5 *Consider a Hamiltonian G-space X (resp. a prequantum G-space $\tilde{X}$) and suppose that the G-action on the zero level set of the moment map (resp. of the prequantum moment map) is strict. Then the reduced space $X/\!\!/G$ carries a reduced 2-form (resp. $\tilde{X}/\!\!/G$ carries a reduced 1-form).*

Here strictness of the action is just what is needed for a straightforward application of Souriau's criterion. This result is then applied to the existence of reduced forms on induced Hamiltonian and prequantum spaces (see below).

Now let us consider locally free actions:

Theorem 3.2.2 *Consider a Hamiltonian G-space X (resp. a prequantum G-space $\tilde{X}$) and suppose that G is connected and the G-action on the zero level set of the moment map (resp. of the prequantum moment map) is locally free. Then the reduced space $X/\!\!/G$ carries a reduced 2-form (resp. $\tilde{X}/\!\!/G$ carries a reduced 1-form).*

The proof relies on [H11, Theorem 3.5]: a result that extends the notion of integral invariant to regular foliations that may not be sectionable. The group G is assumed to be connected in order to apply this theorem.

Finally, we have the existence result for proper group actions:

Theorem 3.3.16 *Consider a Hamiltonian G-space X (resp. a prequantum G-space $\tilde{X}$) and suppose the G-action on X (resp. on $\tilde{X}$) is proper. Then $X/\!\!/G$ carries a reduced 2-form (resp. $\tilde{X}/\!\!/G$ carries a reduced 1-form).*

Here the Sjamaar-Lerman-Bates theory of stratified symplectic spaces [S91, B97], combined with the approach developed in [W12, Chapter 3], allows the application of Souriau's criterion. Moreover, the resulting 2-form restricts to the Sjamaar-Lerman-Bates 2-form on each reduced piece (Corollary 3.3.28).

This chapter ends with an application of the strict action existence result. As noted in Remarks 1.6.18 and 1.6.34, if $H \subset G$ is a closed subgroup, the induced Hamiltonian space and the induced prequantum space are symplectic, or prequantum, manifolds, respectively. When H is not closed these are no longer manifolds, but they can be regarded as diffeological space and still carry reduced forms:

Theorem 3.4.1 *The induced Hamiltonian (resp. prequantum) space $\mathrm{Ind}_H^G Y$ carries a reduced 2-form (resp. a 1-form), even when H is not closed.*

Although strictness does not appear explicitly in the statement, it is essential: the H-action on T^*G lifts to a principal, hence strict, action on the relevant zero level sets of the moment maps (both Hamiltonian and prequantum). The quotients of these sets yield the induced Hamiltonian and prequantum space (Lemma 3.4.3), and thus Theorem 3.1.5 guarantees the existence of reduced forms.

As an illustration, consider the induced Hamiltonian space of the trivial coadjoint orbit of H:

$$\mathrm{Ind}_H^G\{0\} = T^*G /\!\!/ H.$$

When H is closed G/H is a manifold and $T^*G /\!\!/ H$ is symplectomorphic to $T^*(G/H)$ [M07, Theorem 2.2.2]. When H is not closed, G/H is no longer a manifold, but in the diffeological setting we can still define its cotangent space, equipped with the canonical 2-form $d\,\mathrm{Liouv}$ and a moment map (Definitions 1.2.12 ff.). The following result shows that, if H is dense in G, this identification persists:

Theorem 3.4.8 *Let G be a Lie group and H any dense subgroup. Then the induced space $\mathrm{Ind}_H^G\{0\} = T^*G /\!\!/ H$ with its 2-form (Theorem 3.4.1) and the cotangent space $T^*(G/H)$ with its 2-form $d\,\mathrm{Liouv}$ are isomorphic as parasymplectic Hamiltonian G-spaces.*

The term "parasymplectic" originates in the study of symplectic diffeology [I10, I13, I16], where the precise notion of "symplectic" is still unsettled (see [I16, p.,1310]). Following [I16], we therefore call *parasymplectic space* a diffeological space equipped with a closed 2-form; moreover, since the diffeological spaces considered in the previous theorem also carry a moment map (Definition 1.2.21), we refer to them as *parasymplectic Hamiltonian spaces.*

This result applies, for instance, to the irrational torus G/H, where $G = S^1 \times S^1$ and H is an irrational winding.

Part II

The content of this part is based on a paper co-authored by the author of the present thesis with Mauro Spera [B24].

The preliminary chapter (**Chapter 4**) introduces the fundamental definitions and tools in symplectic geometry and mechanics, which provide a symplectic-geometric description of quantum hydrodynamics in terms of moment maps and coadjoint orbits. Equipped with this machinery, we can revise the results of [S23] on the nodal line evolution and the Clebsch geometry of the probability current, which are central in the treatment of Chapter 5.

Chapter 5

In this chapter, we develop the geometric structures arising in quantum mechanics and fluid dynamics which are related by the main result of this part. These structures are then restricted to the nodal line and related through the Clebsch framework. The resulting picture, summarized in a commutative diagram, highlights the central role of moment maps.

The main theorem is the following:

Theorem 5.5

(i) We have a commutative diagram:

$$\begin{tikzcd}
 & \mathcal{M} \supset \mathcal{O}_{[\psi]} \arrow[r, "\mathfrak{j}"] \arrow[dl, "\mathcal{J}"'] \arrow[d, "\hat{\mathfrak{j}}"] \arrow[dr, "\mathcal{K}"] & \mathcal{O}_w \arrow[d, "\ell"] \\
\mathcal{N} \supset \mathcal{J}(\mathcal{O}_{[\psi]}) \arrow[r, "\nu"] & \mathcal{O}_{\hat{w}} \arrow[r, "\hat{\ell}"] & \mathcal{O}_{w_K}
\end{tikzcd}$$

where all maps other than $\mathcal{J}$ are G-equivariant moment maps and $\mathcal{K}$, ℓ, $\hat{\ell}$ all "localize" to K.

(ii) The overall diagram is compatible with the dynamics, provided we replace the Schrödinger Hamiltonian H by the time dependent Hamiltonian $\tilde{H}$ (4.6.12)*, thus switching to a perfect fluid flow induced by the gradient of the solid angle function, thereby keeping track of the vortex line evolution only.*

(iii) The complex structures on $\mathcal{M}$, $\mathcal{N}$, $\mathcal{O}_{w_K}$ are compatible, at fixed time.

The elements of the diagram are the following:

- The Lie group of volume-preserving diffeomorphisms of $\mathbb{R}^3$ rapidly approaching the identity at infinity is denoted by G. It acts on the function spaces considered here by reparametrization.
- The orbits of G for both the reparametrization action and coadjoint action are denoted by $\mathcal{O}$.
- The elements of $\mathfrak{g}^*$ are regarded as vorticities (Remark 4.2.10).
- The projectivization of the space of compactly supported maps from $\mathbb{R}^3 \to \mathbb{C}$ is denoted by $\mathcal{M}$. It is endowed with the Fubini-Study Kähler

form, which is invariant under the G-action and the elements of the Rasetti-Regge current algebra provide co-moment maps thereon. The corresponding G-equivariant moment map is denoted by $\mathfrak{j}$ and it associates to each wavefunction the corresponding vorticity w (Theorem 4.4.15).

- The maps $\mathbf{n} : S^3 \to S^2$ give rise to a symplectic Kähler manifold $\widetilde{\mathcal{N}}$ [P92], whose projective version is denoted by $\mathcal{N}$. There exists a moment map on the latter, denoted by ν, which associates to each element the corresponding vorticity, denoted by $\hat{w}$ (Section 5.3).
- A variant of the moment map $\mathfrak{j}$, denoted by $\hat{\mathfrak{j}}$, maps $[\psi]$ to the vorticity of the corresponding $[\mathbf{n}]$.
- The zero level set of $[\psi] \in \mathcal{M}$ is assumed to be a knot in $\mathbb{R}^3$; it is referred to as *nodal line* and is denoted by K. Its Poincaré dual is a vorticity w_K concentrated on K; the map $\mathcal{K}$ associates $[\psi] \mapsto w_K$ and it is, in a suitable sense (Subsection 5.2.1), a G-equivariant moment map.
- The maps ℓ and $\hat{\ell}$ localize vorticities on the nodal line (Subsection 5.2.2).
- The map $\mathcal{I} : \mathcal{M} \to \mathcal{N}$ is given by the inverse of the stereographic projection.

The relations between the diagram's nodes are established throughout the chapter and brought together in Section 5.4, using the co-moment map formalism based on Rasetti-Regge currents.

Part I

Geometry of Frobenius reciprocity

Introduction to Part I

Let G be a finite-dimensional Lie group and $H \subset G$ a closed subgroup. A well known correspondence exists between the representation theory of G and the geometry of (finite-dimensional) Hamiltonian G-spaces, as exemplified by the Borel-Weil theorem for compact Lie groups [S54] and by the Kirillov-Bernat theory for exponential Lie groups [K62]. This correspondence, and the extensions developed in [W81], motivate the construction of geometric counterparts to representation-theoretic objects, such as the induced representation Ind_H^G and the space of intertwiners Hom_G. These are defined through the symplectic reduction procedure in the works [K78, W78, G82, G83], and are recalled in Definitions 1.6.16 and 1.6.21. Moreover, they are expected to satisfy analogues of representation-theoretic results: most notably, the Frobenius reciprocity.

A geometric form of Frobenius reciprocity was first formulated in [G83] for coadjoint orbits, and later generalized in [R22] to arbitrary Hamiltonian spaces. There, the Frobenius reciprocity is established as a bijection:

$$t : \operatorname{Hom}_G(X, \operatorname{Ind}_H^G Y) \to \operatorname{Hom}_H(\operatorname{Res}_H^G X, Y), \tag{0.0.1}$$

where X is a Hamiltonian G-space, Y is a Hamiltonian H-space and Res_H^G denotes the restriction from a G-space to an H-space. This bijection, however, is only a first step. In favorable cases, both sides of the correspondence are smooth symplectic manifolds, and one naturally asks whether t is a symplectomorphism. In general, however, these spaces are singular, since they arise from symplectic reduction, and thus fall outside the category of smooth manifolds.

To study such singular reduced spaces in a rigorous differential-geometric setting, we adopt the framework of *diffeology*. Introduced by Souriau in the 1980s, and developed by his students, mainly Iglesias-Zemmour, diffeology offers a flexible notion of smooth structure that extends beyond manifolds to include e.g. quotients, orbit spaces, and even infinite-dimensional spaces. A diffeological space carries a specified set of smooth parametrizations (called plots), enabling one to define smooth maps without requiring charts or even a topology. Moreover, diffeological differential forms can be defined and a diffeological version of Cartan-de Rham calculus naturally holds [S85, S88, I13]. Notably, the category of diffeological spaces is closed under constructions such as quotients and subsets, making it especially suitable for dealing with singular spaces arising from symplectic reduction. In this setting, reduced spaces inherit canonical "subquotient" diffeologies, along with a well defined calculus of differential forms.

Within this diffeological framework, the authors of [R22] conjectured two natural enhancements of the Frobenius reciprocity bijection:

- the map t is a diffeomorphism (in diffeological sense);
- if the reduced spaces carry reduced 2-forms, then t preserves them.

We emphasize that these forms are not assumed to exist and, in general, it is not known whether reduced (diffeological) spaces carry reduced forms.

Chapter 2 of this thesis is devoted to proving the conjecture above, closely following [B25, Part I]. In Theorem 2.1.1, we establish that the map t is indeed a diffeological diffeomorphism. In Theorem 2.2.1, we prove an analogous result in the prequantum setting. Our approach, at first, does not question the existence of reduced forms: rather than assume their presence, we show that if a reduced 2-form exists on one side of (0.0.1), then the same holds on the other, and the map t preserves them.

This clarifies the geometric meaning of Frobenius reciprocity in singular settings and illustrates the effectiveness of diffeology in extending symplectic geometry beyond the smooth category.

Even though the proof of the conjecture glosses over the existence of reduced forms, this naturally leads to ask under what conditions symplectic (or prequantum) reduced forms exist. Chapter 3, which follows [B25, Part II], addresses this question.

The existence of reduced forms in the diffeological setting was first investigated in [W12, K16]. There, the existence was proved under the assumption that the group action is both proper and locally free. We show that either assumption, properness *or* local freeness, is sufficient on its own. In addition, we introduce a third, diffeological in nature, condition: the *strictness* of the group action. In more detail:

- Strict actions (Theorem 3.1.5) are involved in the theory of diffeological bundles [I13, Chapter 8], and allow a direct application of Souriau's criterion for invariant forms (Theorem 1.2.9, [S85, 2.5c]).

- Locally free actions (Theorem 3.2.2) yield reduced forms when G is connected. This follows from results on foliations due to Hector, Macías-Virgós, and Sanmartín-Carbón [H11], with a more recent extension in [M23, Theorem 5.10].
- Proper actions (Theorem 3.3.16) are treated using the theory of stratified symplectic reduction due to Sjamaar, Lerman, and Bates [S91, B97]. We show that the diffeological reduced forms exists globally, and the 2-form restricts to the Sjamaar-Lerman-Bates form on each stratum.

We conclude Part I with an applications section: the existence of reduced forms for induced spaces is proved even when H is not closed. This yields a diffeological version of the Kummer-Marsden-Satzer isomorphism: $(T^*G)/\!\!/H = T^*(G/H)$, whenever H is dense in G, as in the case of an irrational winding in a torus (Theorem 3.4.8).

Before delving into these results, Chapter 1 lays the necessary foundations. We begin with an overview of diffeology, mainly following [I13]. We then review the definitions of moment map and Hamiltonian space, state the symplectic reduction theorem [M74], and regard its elements from a diffeological point of view. Analogous definitions and results for prequantum spaces are also presented. This sets up the framework for Part I.

Then we clarify the nature of the "correspondence" between symplectic geometry and representation theory via a version of the Borel-Weil theorem due to Kostant [G82, §3]. As we said, this motivates the introduction of the symplectic counterparts of the induced representation and the space of intertwiners, leading to the "symplectic Frobenius reciprocity" and the conjecture of [R22]. We also introduce their prequantum analogues and formulate the corresponding prequantum Frobenius reciprocity.

Chapter 1

Tools in diffeology and symplectic geometry

In this preliminary chapter we collect the fundamental definitions and results that will be used throughout Part I of this thesis. The main topics are diffeology and Cartan-de Rham calculus on diffeological spaces, together with the notions of Hamiltonian, prequantum, and diffeological reduction. We also introduce the correspondence between representation theory and symplectic geometry, from which arise the concepts of induced Hamiltonian space and symplectic space of intertwiners.

These constructions are naturally related through a symplectic analogue of Frobenius reciprocity. In parallel, we define the corresponding induced prequantum spaces and prequantum spaces of intertwiners, which are connected by a prequantum version of the Frobenius reciprocity theorem.

Further details on each of these topics are postponed to the respective sections.

1.1 Diffeology

Diffeology is a framework for differential geometry introduced by Souriau [S80, S84] and further developed by his students, mainly Iglesias-Zemmour [I13]. It can be viewed as a variant of the theory of differentiable spaces originally formulated by Chen [C77], from which it primarily differs in motivation: while Chen's theory was influenced by algebraic geometry, diffeology is grounded in the needs of differential geometry [I13, p. xix].

The central idea of diffeology is to specify which maps from $\mathbb{R}^n$ into a set X are to be considered smooth, for all $n \in \mathbb{N}$, and the collection of such maps is called a *diffeology* on X. This structure is refined by requiring the set of smooth maps to satisfy three axioms, thereby endowing X with a generalized smooth structure. A notable feature of this approach is that plots are not required to be injective, making diffeology particularly effective in analyzing spaces that are far from resembling manifolds, such as singular quotients.

Alternative approaches to generalized smooth spaces include those of Sikorski [S72] and Frölicher [F82], but these often fail to capture important examples of singular quotients, such as the irrational torus, which degenerate to trivial structures in those settings [I13, p. xix].

Diffeology also accommodates infinite-dimensional spaces of functions, by equipping them with the functional diffeology, a diffeological structure tailored to their description. Finally, it extends classical differential geometry: every smooth manifold admits a natural diffeology, and many standard results can be recovered in this broader context.

In summary, diffeology offers a flexible and robust framework that unifies the treatment of smooth manifolds, singular spaces, and infinite-dimensional function spaces, combining formal rigor with intuitive accessibility.

In this section, we review the basics of diffeology following [I13, Chapter 1].

Definition 1.1.1. An *n-domain* is an open subset of $\mathbb{R}^n$ with respect to the standard topology. A *domain* is an open subset of $\mathbb{R}^p$, for some $p \in \mathbb{N}$. A *parametrization* on a non-empty set X is a map $P : U \to X$, where U is a domain. The map P is called n-parametrization if U is an n-domain.

Definition 1.1.2. Let X be a non empty set. A *diffeology* on X is a set $\mathcal{D}$ of parametrizations of X, such that the following axioms are satisfied:

1. The set $\mathcal{D}$ contains the constant parametrizations $P : U \to X, P(r) = x, \forall r \in U$, where $x \in X$, $U \subseteq \mathbb{R}^p, p \in \mathbb{N}$.
2. Let $P : U \to X, U \subseteq \mathbb{R}^p, p \in \mathbb{N}$ be a parametrization. If for any $r \in U$ there exists an open neighbourhood V of r such that $P|_V \in \mathcal{D}$, then $P \in \mathcal{D}$.
3. For any $P : U \to X$ in $\mathcal{D}$, for any $V \subseteq \mathbb{R}^p, p \in \mathbb{N}$ and for any $F \in \mathcal{C}^\infty(V, U)$, $P \circ F \in \mathcal{D}$.

The set X equipped with the diffeology $\mathcal{D}$ is called *diffeological space* and is denoted by the pair $(X, \mathcal{D})$. The elements of $\mathcal{D}$ are called *plots* of the space X.

Definition 1.1.3. Let $(X, \mathcal{D})$ and $(X', \mathcal{D}')$ be diffeological spaces. A map $f : X \to X'$ is said to be *smooth* if, for any plot $P \in \mathcal{D}$, $f \circ P \in \mathcal{D}'$. A map $f : X \to X'$ is a *diffeomorphism* if it is bijective and both f and f^{-1} are smooth.

Example 1.1.4. Let M be a finite-dimensional manifold. The *standard manifold diffeology* on M is the set of all smooth (in an ordinary sense) maps from open subsets of $\mathbb{R}^n$ to M, with $n \in \mathbb{N}$. It can be checked that this is actually a diffeology and that it is defined by any atlas of M ([I13, 4.3]). Moreover, any smooth map between manifolds is diffeologically smooth with respect to the standard manifold diffeologies.

Definition 1.1.5. Let $(X, \mathcal{D})$ and $(X', \mathcal{D}')$ be two diffeological spaces. The diffeology $\mathcal{D}$ is said to be *finer* than $\mathcal{D}'$ if $\mathcal{D} \subset \mathcal{D}'$. In this case, $\mathcal{D}'$ is said to be *coarser* than $\mathcal{D}$.

Definition/Proposition 1.1.6. Let $f : X \to X'$ be a map between a set X and diffeological space $(X', \mathcal{D}')$. There exists a coarsest diffeology on X such that f is smooth; it is called *pullback diffeology* by f and it is denoted by $f_*(\mathcal{D}')$. The plots of the pullback diffeology by f are characterised as follows: $P \in f^*(\mathcal{D}')$ if and only if $f \circ P \in \mathcal{D}'$.

Proof. See [I13, 1.26]. $\square$

Definition 1.1.7. Let $f : X \to X'$ be a map between the diffeological spaces $(X, \mathcal{D})$ and $(X', \mathcal{D}')$. The function f is said to be an *induction* if it is injective and $f^*(\mathcal{D}') = \mathcal{D}$, i.e. for any plot P' of X' with values in $f(X)$, $f^{-1} \circ P'$ is a plot of X.

Proposition 1.1.8 (Universal property of inductions)**.** *Let X, X', X'' be diffeological spaces, $f : X \to X'$ a map and $j : X' \to X''$ an induction. Then f is smooth if and only if $j \circ f$ is smooth. Moreover, f is an induction if and only if $j \circ f$ is an induction.*

Proof. See [I13, 1.34]. $\square$

Definition 1.1.9. Let X be a diffeological space and let $S \subset X$. The *subset diffeology* on S is the pullback diffeology of the one on X by the natural inclusion i. A parametrization $P : U \to S$ is a plot of the subset diffeology if and only if $i \circ P$ is a plot of X. The situation is represented in the diagram.

$$\begin{tikzcd}
 & X \\
U \arrow[ur, "i \circ P"] \arrow[r, "P"] & S \arrow[u, hook, "i"']
\end{tikzcd}$$

Remark 1.1.10. Let X be a diffeological space and $S \subset X$, endowed with the subset diffeology. The inclusion $j : S \hookrightarrow X$ is an induction.

Definition/Proposition 1.1.11. Let $f : X \to X'$ be a map between a diffeological space $(X, \mathcal{D})$ and a set X'. There exists a finest diffeology on X' such that f is smooth; it is called *pushforward diffeology* by f and it is denoted by $f_*(\mathcal{D})$. The plots of the pushforward diffeology by f are characterised as follows: $P' \in f_*(\mathcal{D})$ if and only if for any $r \in U$ there exists an open neighbourhood V of r such that $P'|_V$ is constant or there exists a plot $Q : V \to X$ of X such that $P'|_V = f \circ Q$.

Proof. See [I13, 1.43]. □

Definition 1.1.12. Let $f : X \to X'$ be a map between the diffeological spaces $(X, \mathcal{D})$ and $(X', \mathcal{D}')$. The function f is said to be a *subduction* if it is surjective and $f_*(\mathcal{D}) = \mathcal{D}'$.

Proposition 1.1.13 (Universal property of subductions)**.** *Let X, X', X'' be diffeological spaces, $f : X' \to X''$ a map and $\pi : X \to X'$ a subduction. Then f is smooth if and only if $f \circ \pi$ is smooth. Moreover, f is a subduction if and only if $f \circ \pi$ is a subduction.*

Proof. See [I13, 1.51]. □

Definition 1.1.14. Let $(X, \mathcal{D})$ be a diffeological space and let $\mathcal{R}$ be an equivalence relation on X. Let $\pi : X \to X/\mathcal{R}$ be the projection onto the quotient. The *quotient diffeology* on $X/\mathcal{R}$ is defined as $\pi_*(\mathcal{D})$. Therefore a parametrization $P : U \to X/\mathcal{R}$ is a plot of the quotient diffeology if and only if for any $r \in U$ there exists an open neighbourhood V of r such that $P|_V$ is constant or there exists a plot $Q : V \to X$ of X such that $P|_V = \pi \circ Q$. The situation is represented in the following diagram:

$$\begin{tikzcd}
 & & X \arrow[d, "\pi"] \\
V \arrow[r, hook, dashed] \arrow[urr, dashed, "Q"] & U \arrow[r, "P"] & X/\mathcal{R}
\end{tikzcd}$$

Remark 1.1.15. The projection onto the quotient, endowed with the quotient diffeology, is a subduction.

Definition 1.1.16. Let $\{(X_i, \mathcal{D}_i)\}_{i \in I}$ be a collection of diffeological spaces indexed by some I. The *product diffeology* on $\Pi_{i \in I} X_i$ is the coarsest diffeology $\mathcal{D}$ that makes the projections $\pi_i : \Pi_{j \in I} X_j \to X_i$ smooth. The pair $(\Pi_{i \in I} X_i, \mathcal{D})$ is called *diffeological product.*

Remark 1.1.17. Let $f : X \to X'$ be a map between diffeological spaces. Then f can be decomposed as follows:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "s"] & X' \\
X/\sim \arrow[r, "\dot{f}"] & f(X) \arrow[u, hook, "i"']
\end{tikzcd}$$

Where:

- The map f defines an equivalence relation on X: for any $x, y \in X$, $x \sim y$ if and only if $f(x) = f(y)$.
- The map s denotes the projection on the quotient $X/\sim$, which is endowed with the quotient diffeology.
- The map i denotes the inclusion of the image $f(X)$ in X'; the set $f(X)$ is endowed with the subset diffeology.
- The map $\dot{f}$ is a bijection.
- By the universal properties of inductions and subductions applied to the maps i and s, $\dot{f}$ is smooth if and only if f is smooth.

Using the notation of the previous remark it is possible to give the following:

Definition 1.1.18. Let $f : X \to X'$ be a map between diffeological spaces. It is said to be *strict* if $\dot{f}$ is a diffeomorphism.

Remark 1.1.19. An equivalent definition of induction and subduction involve strict maps. As a matter of fact:

- A map $f : X \to Y$ is an induction if and only if it is a strict injection.
- A map $g : X \to Y$ is a subduction if and only if it is a strict surjection.

Let us conclude this section with a proposition from [I85, 1.2.18] and [W12, Proposition 2.10], which allows us to define the *subquotient diffeology.*

Proposition 1.1.20. *Let $s : X \to Y$ be a subduction between diffeological spaces. Let $B \subset Y$ and its preimage $A := s^{-1}(B)$, and endow them with their subset diffeologies. Then $s_{|A} : A \to B$ is also a subduction.*

Proof. For A and B to have subset diffeologies means that $i : A \hookrightarrow X$ and $j : B \hookrightarrow Y$ are inductions, hence in particular smooth. Therefore $s \circ i = j \circ s_{|A}$ is smooth,

$$\begin{tikzcd}
 & & A \arrow[r, hook, "i"] \arrow[d, "s_{|A}"'] & X \arrow[d, "s"] \\
V \arrow[r, hook, dashed] \arrow[urr, dashed, "Q"] & U \arrow[r, "P"] & B \arrow[r, hook, "j"] & Y,
\end{tikzcd} \tag{1.1.21}$$

whence $s_{|A}$ is smooth by the universal property of inductions. Now to say that $s_{|A}$ is a *subduction* in this setting means: given a plot $P : U \to B$ and $u \in U$, there are an open $V \ni u$ and a plot $Q : V \to A$ such that (1.1.21) commutes. To apply this, note that by the same token, s being a subduction and $j \circ P$ a plot of Y give us V and a plot $R : V \to X$ such that $j \circ P_{|V} = s \circ R$. Moreover, since $j \circ P$ takes values in B, R takes values in A; so we have $R = i \circ Q$ for a map Q as indicated, which is smooth by the same universal property. Now $j \circ P_{|V} = j \circ s_{|A} \circ Q$, thus $P_{|V} = s_{|A} \circ Q$, proving the proposition. $\square$

Definition 1.1.22. Let X be a diffeological space, $\mathcal{R}$ an equivalence relation on X and $A \subset X$. By the previous proposition it is equivalent to:

- Endow first A with the subset diffeology in X and then $A/\mathcal{R}$ with the quotient diffeology,
- Endow $X/\mathcal{R}$ with the quotient diffeology and then $A/\mathcal{R}$ with the subset diffeology in $X/\mathcal{R}$.

The diffeology on $A/\mathcal{R}$ that we get in this way is called *subquotient diffeology.*

1.2 Cartan-de Rham calculus on diffeological spaces

Since a diffeology on a set X is defined via parametrizations (plots), it naturally allows the definition of covariant objects such as differential forms. The guiding idea is to transfer, by functoriality, the differential forms defined on domains in $\mathbb{R}^n$ to the diffeological space X [I13, p. 125]. Concretely, a differential form on a diffeological space assigns to each plot a differential form on its domain, subject to a compatibility condition with respect to smooth reparametrizations. This definition generalizes the classical one, and agrees with it when the diffeological space is a smooth manifold. In addition, one can define the bundle of differential forms over a diffeological space and, from it, the cotangent space.

In general, the standard constructions, operations, and results of the Cartan-de Rham calculus on manifolds extend naturally to the diffeological setting.

In this section, we review the key definitions and results concerning diffeological differential forms and cotangent space that will be used in later chapters. Our exposition follows [I13, Chapter 6], to which we refer for a comprehensive treatment.

Definition 1.2.1. Let X be a diffeological space. A *differential k-form* on X is a map $\alpha : P \to \alpha(P)$ that associates to each plot P of X a differential k-form $\alpha(P)$ defined on the domain of P. The map α has to satisfy a smooth compatibility condition, i.e. for any smooth parametrization F in the domain of P:

$$\alpha(P \circ F) = F^*(\alpha(P)). \tag{1.2.2}$$

Remark 1.2.3. In general, we will refer to differential forms both on manifolds and diffeological spaces just as "differential forms", and their nature will be clear depending on the space they are defined on. However, when confusion may arise, we will refer to the first ones as "ordinary differential forms" and to the second ones as "diffeological differential forms".

Definition 1.2.4. Let $\Omega^k(X)$ be the space of diffeological k-forms on a diffeological space X. It can be regarded as a diffeological space when it is endowed with the *functional diffeology*: the latter is composed of parametrizations $\phi : V \to \Omega^k(X)$ such that for any plot $P : U \to X$, the map

$$\begin{aligned} V \times U &\to \Lambda^k(\mathbb{R}^n) \\ (s, r) &\mapsto \phi(s)(P)(r), \end{aligned} \tag{1.2.5}$$

is smooth.

Definition/Proposition 1.2.6. Let X and Y be two diffeological spaces, $f : X \to Y$ a smooth map and α a k-form on Y. Then there exists a diffeological k-form on X, denoted by $f^*(\alpha)$ and defined by

$$f^*(\alpha)(P) = \alpha(f \circ P), \tag{1.2.7}$$

for any P plot of X. The k-form $f^*(\alpha)$ is called *pullback* of α by f.

Proof. See [I13, 6.32]. □

Remark 1.2.8. By [K16, Example 2.14], if we take a diffeological space X, a plot $P : U \to X$ and a differential form α on X, the following coincide as ordinary differential forms on U: $\alpha(P) = P^*\alpha$. Thus we may write $P^*\alpha$ instead of $\alpha(P)$.

The following crucial Theorem ([S85, 2.5c] and [I13, 6.38]) will be applied multiple times in order to deal with the existence of reduced forms.

Theorem 1.2.9 (Souriau's criterion)**.** *Let X and Y be two diffeological spaces. Let $\pi : X \to Y$ be a subduction and let α be a diffeological k-form on X. Then α is the pullback of a k-form β defined on Y if and only if, for any P, Q plots of X such that $\pi \circ P = \pi \circ Q$, then $P^*\alpha = Q^*\alpha$.*

Proof. See [I13, 6.38]. □

The following results are related to the construction of the cotangent bundle to a diffeological space, and the objects that can be defined on it, which play a fundamental role in Section 3.4. We refer to [I13, 6.28, 6.40, 6.45, 6.46, 6.48, 6.49, 6.52, 9.11] for the proofs and a thorough treatment.

Definition 1.2.10. Let X be a diffeological space. A plot $P : U \to X$ is said to be *centered* at $x \in X$ if $0 \in U$ and $P(0) = x$. A p-form α is said to *vanish at x* if and only if for any P centered at x, $P^*\alpha$ vanishes at zero. The set of p-forms on X at x is denoted by

$$\Lambda^p_x(X) := \Omega^p(X)/\{p\text{-forms vanishing at } x\}. \tag{1.2.11}$$

For $p = 1$ the latter is the *cotangent space at x*, which is also denoted by the usual $T^*_x X$.

Definition 1.2.12. The *bundle of p-forms over X* is defined as

$$\Lambda^p(X) := \bigsqcup_{x \in X} \Lambda^p_x(X) = \{(x, a) \,:\, x \in X, a \in \Lambda^p_x(X)\}. \tag{1.2.13}$$

The diffeology of $\Lambda^p(X)$ is the pushforward diffeology of $X \times \Omega^p(X)$ by the projection

$$\begin{aligned} \mathbf{p} : X \times \Omega^p(X) &\to \Lambda^p(X) \\ (x, \alpha) &\mapsto (x, \alpha(x)). \end{aligned} \tag{1.2.14}$$

For $p = 1$ we get the *cotangent space*, which is denoted also by the usual T^*X.

Proposition 1.2.15. *Consider* $\Lambda^p(X)$ *endowed with the pushforward diffeology of the previous definition. A parametrization* $\Pi : U \to \Lambda^p(X)$, $r \mapsto (Q(r), P(r))$ *is a plot of* $\Lambda^p(X)$ *if and only if:*

- *The parametrization* Q *is a plot of* X*.*
- *For any* $r_0 \in U$*, there exist an open neighbourhood* V *of* r_0 *and a plot* $A : V \to \Omega^p(X)$ *of* $\Omega^p(X)$ *such that for any* $r \in V$ *the following holds:* $P(r) = Q^*(A(r))(r)$*.*

Proof. See [I13, 6.46]. □

Definition 1.2.16. Let X be a diffeological space. Define the *tautological p-form of* $X \times \Omega^p(X)$, denoted by Taut, as

$$((Q \times A)^* \mathrm{Taut})(r) = Q^*(A(r))(r), \tag{1.2.17}$$

for every plot $Q \times A : U \to X \times \Omega^p(X)$. Moreover, there exists a p-form on $\Lambda^p(X)$, denoted by Liouv, such that

$$\mathrm{Taut} = \mathbf{p}^* \,\mathrm{Liouv} \quad \text{with} \quad \mathrm{Liouv} \in \Omega^p(\Lambda^p(X)), \tag{1.2.18}$$

where $\mathbf{p}$ denotes the projection $X \times \Omega^p(X) \to \Lambda^p(X)$. This form is called the *Liouville p-form of* $\Lambda^p(X)$, and its existence is proved in [I13, 6.49] by applying Souriau's criterion. For $p = 1$ we get the tautological and the Liouville forms of $X \times \Omega^1(X)$ and T^*X, respectively.

Definition 1.2.19. Let X be a diffeological space and consider the following action of the group of diffeomorphisms of X on $X \times \Omega^p(X)$:

$$\varphi(x, \alpha) = (\varphi(x), \varphi_* \alpha), \tag{1.2.20}$$

where $\varphi \in \mathrm{Diff}(X)$ and $\varphi_* = (\varphi^{-1})^*$. The form Taut is invariant under this action; moreover it descends to an action on $\Lambda^p(X)$, under which the Liouville form is invariant.

We conclude this section with a definition coming from symplectic diffeology. We introduce it here as it will be applied to the cotangent space to a diffeological space.

Definition 1.2.21. Let X be a diffeological space endowed with an exact 2-form $d\alpha$ [I13, 6.73] and a smooth action of a diffeological group G [I13, 7.1, 7.4], preserving $d\alpha$. Then, a moment map can be defined [I13, Chapter 9] and it has the following expression:

$$\begin{aligned} \Phi : X &\to \mathcal{G} \\ x &\mapsto \hat{x}^* \alpha, \end{aligned} \tag{1.2.22}$$

where $\mathcal{G}$ denotes the space of 1-forms on G which are invariant under right translation [I13, 7.12] and $\hat{x} : G \to X$, $\hat{x}(g) = g(x)$.

1.3 Lie groups and actions

In this section we recall some basic definitions and results concerning Lie groups and their actions, referring to e.g. [H15, H12, D00] for a comprehensive treatment. Since Part II of this thesis will make use of the infinite-dimensional Lie group of volume-preserving diffeomorphisms of $\mathbb{R}^3$ [E70, K97], we also highlight some general differences between the finite and infinite-dimensional settings, while postponing some details to Part II.

First, we define Lie groups, Lie algebras, their homomorphisms and the 1-parameter subgroup:

Definition 1.3.1. A *Lie group* G is a manifold endowed with a group structure such that multiplication and inversion are smooth maps.

If H and G are Lie groups, a *Lie group homomorphism* $H \to G$ is a group homomorphism that is also smooth.

Definition 1.3.2. A *Lie algebra* $\mathfrak{g}$ is a vector space endowed with a skew-symmetric bilinear operator $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying the Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0, \forall X,Y,Z \in \mathfrak{g}.$$

A morphism between Lie algebras is a linear map that preserves the Lie bracket.

Let $L_g : h \mapsto gh$ denote the left multiplication by g in G. Then:

Definition 1.3.3. A vector field X on G is said to be *left invariant* if $L_{g*}X(h) = X(gh)$, i.e. $d_hL_gX(h) = X(gh)$, for all g, $h \in G$.

For each $\xi \in T_eG$, the tangent space at the identity of G, we can define a left invariant vector field: $X_\xi(g) := L_{g*}\xi$. Then a Lie bracket can be defined on T_eG as follows:

$$[\xi,\eta] := [X_\xi, X_\eta](e), \tag{1.3.4}$$

where the bracket in the r.h.s is the usual commutator of vector fields.

Definition 1.3.5. The vector space T_eG with this Lie bracket is the *Lie algebra* $\mathfrak{g}$ of the Lie group G.

Remark 1.3.6. The Lie algebra $\mathfrak{g}$ can be equivalently be regarded as the vector space of left invariant vector fields on G, endowed with the commutator of vector fields.

Definition 1.3.7. The 1-*parameter subgroup* in G corresponding to $Z \in \mathfrak{g}$ is the unique Lie group homomorphism $\exp_Z : \mathbb{R} \to G$, denoted as $t \mapsto e^{tZ}$, such that

$$\frac{d}{dt} e^{tZ}\Big|_{t=0} = Z. \tag{1.3.8}$$

This yields the *exponential map* $\exp : \mathfrak{g} \to G$ which maps $Z \mapsto \exp_Z(1) = e^Z$.

Remark 1.3.9. If G is finite-dimensional, the exponential map is a local diffeomorphism between a neighbourhood of $0 \in \mathfrak{g}$ and a neighbourhood of $e \in G$. However, this is not true for infinite-dimensional Lie groups such as the group $\mathrm{Diff}(X)$ of diffeomorphisms on a manifold X (see e.g. [E70, K97]).

Now let us consider Lie group actions on manifolds:

Definition 1.3.10. Let G be a Lie group and X a manifold. A *smooth action* of G on X is a group homomorphism $\rho : G \to \mathrm{Diff}(X)$ such that the map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto \rho(g)(x), \end{aligned} \tag{1.3.11}$$

is smooth. The action of Lie groups on manifolds will be generally denoted by module notation $g(x) := \rho(g)(x)$, $g \in G$, $x \in X$, with the exception of translations and corresponding lifts (see below).

Definition 1.3.12. Let G be a Lie group acting on a manifold X. The *fundamental vector field* corresponding to $Z \in \mathfrak{g}$ is:

$$Z_X(x) := \frac{d}{dt} e^{tZ}\Big|_{t=0}. \tag{1.3.13}$$

The action of G on X naturally lifts to an action of G on the tangent bundle TX: deriving $\rho(g) \in \mathrm{Diff}(X)$, say at $x \in X$, we get

$$\begin{aligned} d_x\rho(g) : T_xX &\to T_{g(x)}X \\ v &\mapsto d_x\rho(g)v. \end{aligned} \tag{1.3.14}$$

We denote this pushforward of tangent vectors by $(g_*)_x(v) := d_x\rho(g)v$; when no confusion may arise, we simply write $g_*(v)$.

The action of G on X also naturally lifts to an action of G on the cotangent bundle T^*X: as $\rho(g^{-1})$ maps $g(x) \mapsto x$, its pullback yields the following

$$\begin{aligned} \rho(g^{-1})^* : T^*_xX &\to T^*_{g(x)}X \\ \alpha_x &\mapsto \rho(g^{-1})^*\alpha_x, \end{aligned} \tag{1.3.15}$$

that is, for any $v \in T_{g(x)}X$,

$$(\rho(g^{-1})^* \alpha_x)(v) = \alpha_x(d_{g(x)}\rho(g^{-1})v). \tag{1.3.16}$$

As before, we denote this pullback of covectors by $g^*\alpha_x := \rho(g)^*\alpha_x$. This naturally generalizes to the pullback of differential forms:

$$(g^*\omega)_x(v_1, ..., v_k) := \omega_{g(x)}(d_x\rho(g)v_1, ..., d_x\rho(g)v_n), \tag{1.3.17}$$

for an n-form ω and $v_1, ..., v_k \in T_xX$.

An important example of Lie group action is the *adjoint action* of the group G on itself:

$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{Aut}(G) \\ g &\mapsto \mathrm{Ad}_g, \end{aligned} \tag{1.3.18}$$

where

$$\begin{aligned} \mathrm{Ad}_g : G &\to G \\ h &\mapsto \mathrm{Ad}_g h := ghg^{-1}. \end{aligned} \tag{1.3.19}$$

The lifted adjoint action (1.3.14) of G on $T_eG = \mathfrak{g}$ yields the *adjoint action of* G *on its Lie algebra* $\mathfrak{g}$:

$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{Aut}(\mathfrak{g}) \\ g &\mapsto \mathrm{Ad}_g, \end{aligned} \tag{1.3.20}$$

where

$$\begin{aligned} \mathrm{Ad}_g : \mathfrak{g} &\to \mathfrak{g} \\ Z &\mapsto \mathrm{Ad}_g Z. \end{aligned} \tag{1.3.21}$$

The adjoint action of G on $\mathfrak{g}$ defines the following smooth map:

$$\begin{aligned} G \times \mathfrak{g} &\to \mathfrak{g} \\ (g, Z) &\mapsto \mathrm{Ad}_g Z. \end{aligned} \tag{1.3.22}$$

Differentiating (1.3.22) with respect to its first argument, we get the *adjoint action of* $\mathfrak{g}$ *on itself*:

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathrm{Der}(\mathfrak{g}) \\ Y &\mapsto \mathrm{ad}_Y, \end{aligned} \tag{1.3.23}$$

where

$$\begin{aligned} \mathrm{ad}_Y : \mathfrak{g} &\to \mathfrak{g} \\ Z &\mapsto \mathrm{ad}_Y Z := [Y, Z], \end{aligned} \tag{1.3.24}$$

and the bracket denotes the Lie bracket.

The lifted coadjoint action (1.3.15) of G on $T^*_eG = \mathfrak{g}^*$ yields the *coadjoint action of* G *on the dual of its Lie algebra* $\mathfrak{g}^*$:

$$\begin{aligned} \mathrm{Ad}^* : G &\to \mathrm{GL}(\mathfrak{g}^*) \\ g &\mapsto \mathrm{Ad}^*_g, \end{aligned} \tag{1.3.25}$$

where

$$\begin{aligned} \mathrm{Ad}^*_g : \mathfrak{g}^* &\to \mathfrak{g}^* \\ \alpha &\mapsto \mathrm{Ad}^*_g\alpha, \end{aligned} \tag{1.3.26}$$

that is $\langle \mathrm{Ad}^*_g\alpha, Z\rangle = \langle \alpha, \mathrm{Ad}_g^{-1} Z\rangle$, for any $Z \in \mathfrak{g}$, and the bracket denotes the duality pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Proceeding as before, we can define the *coadjoint action of* $\mathfrak{g}$ *on the dual of its Lie algebra* $\mathfrak{g}^*$:

$$\begin{aligned} \mathrm{ad}^* : \mathfrak{g} &\to \mathrm{End}(\mathfrak{g}^*) \\ Y &\mapsto \mathrm{ad}^*_Y, \end{aligned} \tag{1.3.27}$$

where

$$\begin{aligned} \mathrm{ad}^*_Y : \mathfrak{g}^* &\to \mathfrak{g}^* \\ \alpha &\mapsto \mathrm{ad}^*_Y\alpha, \end{aligned} \tag{1.3.28}$$

that is $\langle \mathrm{ad}^*_Y\alpha, Z\rangle = \langle \alpha, -\mathrm{ad}_Y Z\rangle$, for any $Z \in \mathfrak{g}$.

We conclude this section by establishing the notational conventions used throughout this Part I for the translation of tangent and cotangent vectors to a Lie group G, following [R22]. For fixed $g, q \in G$, the derivative of the left translation map $q \mapsto gq$ will be denoted:

$$\begin{aligned} T_qG &\to T_{gq}G \\ v &\mapsto gv. \end{aligned} \tag{1.3.29}$$

The contragredient of the same translation will be denoted:

$$\begin{aligned} T^*_qG &\to T^*_{gq}G \\ p &\mapsto gp, \end{aligned} \tag{1.3.30}$$

therefore $\langle gp, v\rangle = \langle p, g^{-1}v\rangle$. Similarly the right translations vg and pg are defined, and $\langle pg, v\rangle = \langle p, vg^{-1}\rangle$. In cases where operator precedence might be ambiguous, we will use a lower dot notation, as in (3.4.9).

1.4 Hamiltonian spaces and reduction

Hamiltonian spaces provide the natural framework for studying symmetries in symplectic geometry. Their structure gives rise to the procedure of symplectic reduction [M74], which, under suitable assumptions, produces new symplectic manifolds by quotienting the level sets of the moment map along the orbits of a Lie group action. In this section we review the fundamental definitions and results concerning Hamiltonian spaces and Hamiltonian reduction (see e.g. [M74, M07, O04] for further details). We also introduce the notions of diffeological reduction and reduced (diffeological) forms, which will be central in later developments.

Definition 1.4.1. A symplectic manifold is a manifold endowed with a closed, non-degenerate 2-form ω, called symplectic form.

Definition 1.4.2. Let G be a Lie group acting on a symplectic manifold (X, ω). The G-action preserves the symplectic form ω if $g^*\omega = \omega$, for all $g \in G$.

Definition 1.4.3. Let (X, ω) be a symplectic manifold and G a Lie group acting on X preserving the symplectic form ω. A *moment map* on X is a map $\Phi : X \to \mathfrak{g}^*$ such that $\iota_{Z_X}\omega = -d\Phi^Z$, where $\Phi^Z(x) := \langle \Phi(x), Z \rangle$ and $Z_X(x)$ denotes the fundamental vector field.

The map Φ is an *equivariant moment map* if it is equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$ and the G-action on X, i.e. $\mathrm{Ad}_g^*\Phi(x) = \Phi(g(x))$. The triple (X, ω, Φ) is said to be a *Hamiltonian G-space.*

Remark 1.4.4. In Part I of this thesis Hamiltonian spaces (and all manifolds and groups) will be finite-dimensional. A different topic, involving infinite-dimensional Hamiltonian spaces, will be investigated in Part II.

Equivariant moment maps have the following property [S70a, M74]:

Proposition 1.4.5. *Let (X, ω, Φ) be a Hamiltonian G-space. Then:*

$$\begin{aligned} &\text{a) } \mathrm{Ker}(d\Phi(x)) = \mathfrak{g}(x)^\omega = \{v \in \mathfrak{X}(X) \,:\, \omega_x(v(x), Z_X(x)) = 0,\ \forall Z \in \mathfrak{g}\} \\ &\text{b) } \mathrm{Im}(d\Phi(x)) = \mathrm{ann}(\mathfrak{g}_x) = \{\mu \in \mathfrak{g}^* \,:\, \langle \mu, Z \rangle = 0,\ \forall Z \in \mathfrak{g}_x\}, \end{aligned} \tag{1.4.6}$$

i.e. the kernel of $d\Phi(x)$ is the symplectic orthogonal complement with respect to ω of the tangent space $\mathfrak{g}(x)$ to the orbit $G(x)$, and $\mathrm{Im}(d\Phi(x))$ *is the annihilator of the infinitesimal stabilizer of x, that is, the Lie algebra of $G_x = \{g \in G \,:\, g(x) = x\}$.*

These properties are closely related to the group actions involved in symplectic reduction (and considered in Chapter 3):

Definition 1.4.7. Let X be a topological space and G a Lie group acting on X (Definition 1.3.10 with diffeomorphisms replaced by homeomorphisms). Then we can define the *Bourbaki map* [B60, III.4.1] $\theta : G \times X \to X \times X$, $(g, x) \mapsto (x, g(x))$.

- The action of G on X is *free* if, for all $x \in X$, the stabilizer $G_x = \{e\}$.
- The action of G on X is *locally free* if, for all $x \in X$, the infinitesimal stabilizer $\mathfrak{g}_x = \{0\}$.
- The action of G on X is *proper* if θ is proper, i.e. preimages of compact sets via θ are compact.

If X is a diffeological space:

- the action of G on X is *strict* if θ is strict (Definition 1.1.18).

Further details and examples on strict actions will be given in Section 3.1.

Now we can state the following [M74]:

Theorem 1.4.8 (Symplectic reduction). *Let (X, ω, Φ) be a Hamiltonian G-space and G act freely and properly on $\Phi^{-1}(0)$. Then $X/\!\!/G := \Phi^{-1}(0)/G$ is a manifold carrying a reduced symplectic form $\omega_{X/\!\!/G}$, i.e. $j^*\omega = \pi^*\omega_{X/\!\!/G}$, where j and π are the natural inclusion and projection in the following diagram:*

$$\begin{tikzcd}
\Phi^{-1}(0) \arrow[r, hook, "j"] \arrow[d, "\pi"'] & X \\
X/\!\!/G. &
\end{tikzcd} \tag{1.4.9}$$

If the freeness assumption is dropped, but $0 \in \mathfrak{g}^*$ is required to be a regular value, $\Phi^{-1}(0)$ is still a manifold and (1.4.6a) implies that the action is locally free. In this case, the quotient is no longer a manifold: it is an orbifold carrying a symplectic orbifold structure [W77].

If the local freeness assumption is also dropped, and the proper action of a Lie group is considered, [S91, B97] proved that the symplectic reduction yields a stratified symplectic space, that is, in particular, a disjoint union of symplectic manifolds (see also [O04, Chapter 8]).

1.4.1 Diffeological symplectic reduction

In this subsection reduced diffeological spaces and 2-forms are introduced combining definitions and results considered so far.

Let G be a Lie group and (X, ω, Φ) a Hamiltonian G-space. By the previous discussion, without any further assumption on the G-action, the *reduced space*

$$X/\!\!/G := \Phi^{-1}(0)/G \tag{1.4.10}$$

need not be a manifold. However it may (and will be) be regarded as a diffeological space endowed with the subquotient diffeology (Definition 1.1.22): first put on $\Phi^{-1}(0)$ the subset diffeology, and then on (1.4.10) the quotient diffeology. Also note that, by Proposition 1.1.20, it is equivalent to endow X/G first the quotient diffeology, and then $X/\!\!/G$ with the subset diffeology. Now differential forms on $X/\!\!/G$ can be defined (Definition 1.2.1), and, even in this more general setting, the question is whether or when it carries a reduced 2-form:

Definition 1.4.11. We say that $X/\!\!/G$ *carries a reduced* 2*-form* if there exists on the latter a (diffeological) 2-form $\omega_{X/\!\!/G}$ such that $j^*\omega = \pi^*\omega_{X/\!\!/G}$, where j and π are the natural inclusion and projection in Diagram 1.4.9. Note that if $\omega_{X/\!\!/G}$ exists, it is unique and closed by [I13, 6.39].

Remark 1.4.12. The existence of the reduced form $\omega_{X/\!\!/G}$ is an open problem in diffeology, solved so far in these contexts:

- If the G-action on $\Phi^{-1}(0)$ is free and proper, by symplectic reduction $\omega_{X/\!\!/G}$ exists as an ordinary 2-form. Thus it is also diffeological by [K16, Example 2.14].
- If the action is only locally free, but still proper, then $X/\!\!/G$ is an orbifold and carries an orbifold 2-form, as we observed previously. As proved in [K16, Appendix A], when orbifolds are regarded as diffeological spaces, orbifold forms define diffeological forms and conversely. So $\omega_{X/\!\!/G}$ exists also in this orbifold setting.

Finally, note that in the diffeological setting, the word "symplectic" can be interpreted in different ways: see [I16, p. 1310]. For this reason the author of the latter reference suggests to call *parasymplectic* any diffeological space endowed with a closed 2-form. Thus we will call "parasymplectic" the reduced space $X/\!\!/G$, when it carries a reduced form.

1.5 Prequantum spaces and prequantum reduction

In Souriau's formulation [S70b], a prequantum space is a contact manifold $(\tilde{X}, \varpi)$ whose Reeb vector field is generated by a free S^1-action. The quotient $\tilde{X}/S^1 =: X$ then inherits a symplectic form $\underline{\omega}$, uniquely determined by $\pi^*\underline{\omega} = d\varpi$, where $\pi : \tilde{X} \to X$ is the projection. In this way, prequantum spaces encode symplectic manifolds together with the additional data required for quantization.

This definition is equivalent to Kostant's description in terms of prequantum line bundle [K70] (see also Section 1.6): the contact form ϖ serves as a connection form on the principal S^1-bundle $\tilde{X} \to X$, whose curvature is the symplectic form $\underline{\omega}$. The associated complex line bundle over $(X, \underline{\omega})$, then carries a natural Hermitian structure, preserved by parallel transport, and a compatible connection whose curvature is $\underline{\omega}$. Thus, passing from Souriau's contact picture to Kostant's line bundle picture is simply a matter of viewpoint.

Just as Hamiltonian spaces admit symplectic reduction, prequantum spaces support an analogous prequantum reduction procedure that respects this additional bundle structure [L01b, R22]. In this section we review the fundamental definitions and results, in parallel with Section 1.4, mainly following [S70b, R22].

Definition 1.5.1. Let $(\tilde{X}, \varpi)$ be a manifold endowed with a contact 1-form, i.e.:

$$\dim(\mathrm{Ker}(d\varpi)) \equiv 1, \qquad \dim(\mathrm{Ker}(\varpi) \cap \mathrm{Ker}(d\varpi)) \equiv 0. \tag{1.5.2}$$

The Reeb vector field ξ on $\tilde{X}$ is defined by:

$$\iota_\xi d\varpi \equiv 0 \qquad \iota_\xi \varpi \equiv 1. \tag{1.5.3}$$

If the Reeb vector field is generated by a free action of S^1 on $\tilde{X}$, then $(\tilde{X}, \varpi)$ is called *prequantum manifold.*

This definition implies that $(\tilde{X}, d\varpi)$ is a presymplectic manifold whose null leaves are the S^1-orbits of the action on $\tilde{X}$. Thus $d\varpi$ descends to a symplectic form $\underline{\omega}$ on the quotient $X = \tilde{X}/S^1$ and we can give the following:

Definition 1.5.4. Let G be a Lie group acting on $\tilde{X}$ preserving ϖ. Then the G-action commutes with the S^1-action and the equivariant *prequantum moment map* can be defined as $\Phi : \tilde{X} \to \mathfrak{g}^*$:

$$\langle \Phi(\tilde{x}), Z \rangle = \varpi(Z_{\tilde{X}}(\tilde{x})), \tag{1.5.5}$$

for all $Z \in \mathfrak{g}$. The prequantum moment map descends to an equivariant moment map $\underline{\Phi} : X \to \mathfrak{g}^*$; thus the Hamiltonian G-space $(X, \underline{\omega}, \underline{\Phi})$ is said to be *prequantized* by the *prequantum G-space* $(\tilde{X}, \varpi, \Phi)$.

In this setting, a prequantum version of the symplectic reduction theorem holds ([L01b, Theorem 2],[R22, 5.5]):

Theorem 1.5.6. *Let* $(\tilde{X}, \varpi, \Phi)$ *as before and assume that* G *acts freely and properly on* $\tilde{X}$*. Then* $\tilde{X}/\!\!/G := \Phi^{-1}(0)/G$ *is a manifold carrying a contact reduced form* $\varpi_{\tilde{X}/\!\!/G}$*, i.e.* $j^*\varpi = \tilde{\pi}^*\varpi_{X/\!\!/G}$*, where* j *and* π *are the natural inclusion and projection in the following:*

$$\begin{tikzcd}
\Phi^{-1}(0) \arrow[r, hook, "j"] \arrow[d, "\pi"'] & \tilde{X} \\
\tilde{X}/\!\!/G. &
\end{tikzcd} \tag{1.5.7}$$

Moreover, $(\tilde{X}/\!\!/G, \varpi_{\tilde{X}/\!\!/G})$ *prequantizes the symplectically reduced space* $X/\!\!/G := \underline{\Phi}^{-1}(0)/G$*.*

If we drop the assumptions on the G-action, we can proceed as in Subsection 1.4.1 and consider the prequantum reduced space:

$$\tilde{X}/\!\!/G := \Phi^{-1}(0)/G, \tag{1.5.8}$$

which can be endowed with the subquotient diffeology. Therefore diffeological forms can be defined and, as before, the question is whether and when (1.5.8) carries a reduced 1-form. More precisely:

Definition 1.5.9. The prequantum reduced space $\tilde{X}/\!\!/G$ carries a *reduced* 1*-form* if it admits a (diffeological) 1-form $\varpi_{\tilde{X}/\!\!/G}$ such that $j^*\varpi = \pi^*\varpi_{\tilde{X}/\!\!/G}$, where j and π are the ones of Diagram 1.5.7. Moreover, if the reduced 1-form exists, it is necessarily unique by [I13, 6.39].

Remark 1.5.10. As in the symplectic context, the existence of the reduced form has been proved so far in these settings:

- When the G-action is free and proper on $\Phi^{-1}(0)$, the reduced space is a manifold carrying a reduced differential 1-form, hence it is also diffeological by [K16, Example 2.14].

- If the identity component G° of G acts in a locally free and proper fashion on $\Phi^{-1}(0)$, by [K16, 5.10] there exists a diffeological reduced 1-form $\varpi_{\tilde{X}/\!\!/G}$ on the reduced space.

1.6 Representation theory and symplectic geometry

The Borel-Weil theorem [S54] establishes a bijective correspondence between irreducible unitary representations of compact, connected Lie groups and integral coadjoint orbits, which are symplectic manifolds [K76]. This result has inspired the development of symplectic constructions that parallel those in representation theory, notably the induced Hamiltonian space [K78, W78] and the space of intertwiners [G82]. Although these symplectic analogues are not known in general to be in bijection with their representation-theoretic counterparts, they satisfy structural properties such as the "induction in stages" [R22] and a symplectic version of Frobenius reciprocity, both necessary conditions for such a bijection to exist. In more detail, "symplectic Frobenius reciprocity" was first established for coadjoint orbits in [G83], and it was later extended to any Hamiltonian space in [R22].

A similar correspondence between representation theory and symplectic geometry also exists for exponential Lie groups [K62, V70]. While in this setting Frobenius reciprocity does not hold [F88], its symplectic counterpart can still capture multiplicity of group representation [V70, C88a, F91].

These ideas have also been extended to the prequantum setting in [R22], in particular to address the representation of solvable Lie groups: for such groups, the correspondence between irreducible representations and integral coadjoint orbits is no longer bijective, in contrast with the Borel-Weil theory or the Kirillov-Bernat theory of exponential groups [K62, F15]. In fact, multiple irreducible representations can correspond to a single integral coadjoint orbit and, when the group is simply connected, the representations are as many as the characters from the fundamental group of the coadjoint orbit to S^1 ([A71]).

In the prequantum framework, analogues of both induction in stages and Frobenius reciprocity were also shown to hold [R22], allowing one to effectively address the representation theory of solvable groups (something the Hamiltonian counterparts could not achieve [R22, §4]).

In this section, we review the theorems and constructions underlying the correspondence between representation theory and symplectic geometry. First, we present a reformulation of the Borel-Weil theorem due to Kostant, which clarifies the motivations that led to the definitions of the induced Hamiltonian space and the space of intertwiners. Then, these definitions are provided, together with the symplectic Frobenius reciprocity in both the formulations of [G83] and [R22]. At this point, the conjecture of [R22] is presented: it claims the Frobenius reciprocity to be a diffeological diffeomorphism between symplectically reduced spaces. In the second part of this section we consider the prequantum analogues of the induced space and the space of intertwiners, followed by a prequantum version of the Frobenius reciprocity and the corresponding conjecture.

1.6.1 The Borel-Weil theorem à la Kostant

The work of Borel-Weil, Kirillov and Kostant ([S54, K62], [G82, §3]) proved a one-to-one correspondence between irreducible unitary representations of compact connected Lie groups and coadjoint orbits. In this subsection we focus on the Borel-Weil theorem in a guise due to Kostant, following [G82, §3]: geometric quantization constructions will be sketched and applied to coadjoint orbits, thereby obtaining the statement of the result.

First, let us give some definitions concerning representations:

Definition 1.6.1. A *representation* of a Lie group G on a vector space V is a group homomorphism $\rho : G \to \mathrm{GL}(V)$.

Definition 1.6.2. A subspace W of V is called *invariant* if $g(w) := \rho(g)(w) \in W$, $\forall g \in G$, $\forall w \in W$. A non-trivial representation is called *irreducible* if it does not have any non-trivial invariant subspaces.

Definition 1.6.3. A representation of a Lie group G on a Hilbert space H is called *unitary* if the representation takes values in the unitary operators on H, i.e. $\rho : G \to \mathrm{U}(H) \subset \mathrm{GL}(H)$.

Now let us introduce coadjoint orbits:

Definition 1.6.4. Let G be a Lie group and $\alpha \in \mathfrak{g}^*$. The coadjoint orbit of G through α is the image of

$$\begin{aligned} G &\to \mathfrak{g}^* \\ g &\mapsto \mathrm{Ad}^*(g)(\alpha), \end{aligned} \tag{1.6.5}$$

where Ad^* was defined in (1.3.25, 1.3.26).

Coadjoint orbits carry a natural symplectic structure given by the Kirillov-Kostant-Souriau 2-form [K76, 15.1, Theorem 1]:

Theorem 1.6.6. *Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and recall that* ad^* *denotes the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$* (1.3.27, 1.3.28). *A coadjoint orbit through $\alpha_0 \in \mathfrak{g}^*$ is a homogeneous symplectic manifold $\mathcal{O} \cong G/G_{\alpha_0}$, where G_{α_0} is the stabilizer of α_0, endowed with the Kirillov-Kostant-Souriau 2-form:*

$$B_\alpha(\mathrm{ad}^*_X\alpha, \mathrm{ad}^*_X\alpha) = B_\alpha(X_\mathcal{O}(x), Y_\mathcal{O}(x)) := \langle \alpha, [X, Y] \rangle, \tag{1.6.7}$$

for all X, $Y \in \mathfrak{g}$, $\alpha \in \mathcal{O}$. The vector fields $X_\mathcal{O}$ and $Y_\mathcal{O}$ denote the fundamental vector fields associated to $X, Y \in \mathfrak{g}$ for the Ad^**-action.*

The natural inclusion of coadjoint orbits in the dual of the Lie algebra yields an equivariant moment map and then coadjoint orbits can be regarded as Hamiltonian spaces.

The coadjoint orbits involved in the Borel-Weil theorem are the *integral orbits* [G82, p. 520]:

Definition 1.6.8. Let $\mathcal{O}$ be the coadjoint orbit through $\alpha \in \mathfrak{g}^*$, G_α the stabilizer group of α and $\mathfrak{g}_\alpha$ its Lie algebra. Define:

$$\begin{aligned} \rho_\alpha : \mathfrak{g}_\alpha &\to i\mathbb{R} \\ Z &\mapsto 2\pi i \langle \alpha, Z \rangle. \end{aligned} \tag{1.6.9}$$

The map ρ_α is an infinitesimal character of G_α, i.e. it vanishes on $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha]$. The element α is called *integral* if there exist a global character $\chi_\alpha : G_\alpha \to S^1$ such that $d\chi_\alpha = \rho_\alpha$. A coadjoint orbit through an integral element of $\mathfrak{g}^*$ is called *integral orbit.*

The fundamental property of these orbits is that they can be endowed with G-invariant prequantum data and polarization. Let us introduce these definitions for a generic symplectic manifold (X, ω), following [K70, G82]:

Definition 1.6.10. A symplectic manifold (X, ω) is said to be *prequantizable* if there exists a complex line bundle L on X endowed with a connection ∇ whose curvature form equals ω and a parallel transport-invariant Hermitian structure $\langle \cdot, \cdot \rangle$; these elements are called *prequantum data.*

The work [K70] proved that a symplectic manifold is prequantizable if and only if $[\omega] \in H^2(X, \mathbb{Z})$, i.e. the cohomology class defined by the symplectic form is integral. Now consider a prequantized (i.e. endowed with prequantum data) Hamiltonian G-space (X, ω, Φ), where G is a connected Lie group; the following defines an action of the Lie algebra $\mathfrak{g}$ of G on the sections of L:

$$s \mapsto \nabla_{Z_X} s + 2\pi i \Phi^Z s, \; Z \in \mathfrak{g}. \tag{1.6.11}$$

Definition 1.6.12. If the action (1.6.11) is induced by a G-action on L, the prequantum data are said to be *G-invariant.*

Now we introduce polarizations [G82, S76, A71]:

Definition 1.6.13. A *polarization* of a symplectic manifold (X, ω) is an integrable Lagrangian subbundle, F, of the complexified tangent bundle $TX \otimes \mathbb{C}$. It is said to be *positive definite* if the fibre F_x is a positive definite Lagrangian subspace of $T_x X \otimes \mathbb{C}$, for all $x \in X$, where the positive definiteness is given by the metric induced by the complexified symplectic form. A section s of the prequantum line bundle L on X is said to be *polarized* if $\nabla_V s = 0$, for all $V \in \overline{F}$.

If X is compact, the space of polarized sections is a finite-dimensional Hilbert space, endowed with the inner product induced by that on L. Moreover, if (X, ω, Φ) is a Hamiltonian G-space endowed with G-invariant pre-quantum data and polarization, a unitary G-action on the Hilbert space of polarized sections is defined [G82, p. 523].

Now we can apply these definitions and results to coadjoint orbits and state Kostant's reformulation of Borel-Weil theorem. If G is a compact, connected Lie group, integral orbits of G can be endowed with G-invariant pre-quantum data [K70, Theorem 5.7.1] and a G-invariant polarization [G82, p. 522], hence the previous steps can be repeated in this setting, leading to a unitary representation of G on the space of polarized sections. Therefore we can state the following [G82, Theorem 3.7]:

Theorem 1.6.14 (Borel-Weil)**.** *Let G be a compact connected Lie group, $\mathcal{O}$ an integral coadjoint orbit of G and $\rho_{\mathcal{O}}$ the representation of G on the space of polarized sections that we have just considered. This representation is irreducible, moreover, the correspondence $\mathcal{O} \mapsto \rho_{\mathcal{O}}$ is a bijection between integral orbits and irreducible unitary representations of G.*

1.6.2 Induced space and space of intertwiners

The Borel-Weil theorem, stated in the previous subsection, establishes a correspondence between irreducible unitary representations of compact connected Lie groups and Hamiltonian spaces. The works [W78, K78, W81, G82] proposed extending this correspondence to other representation-theoretic constructions, including the induced representation and the space of intertwining maps. Although the existence of this extended correspondence remains an open question, results such as symplectic Frobenius reciprocity are essential prerequisites for their potential realization.

First, consider the representation-theoretic induced bundle (for further details see, e.g. [B85, D00]). Let H be a compact Lie group, $P \to M$ be a principal fibre bundle with structure group H and let $\rho : H \to \mathrm{GL}(V)$ be a representation of H on a finite-dimensional vector space V. There exists another vector bundle $E \to M$ with fibre V such that $\mathrm{Hom}_H(P, V) \cong \Gamma(E)$, i.e. the sections of E can be identified with H-equivariant maps from P to V. The vector bundle E is said to be *induced* by the representation ρ.

The symplectic counterpart of this construction was defined in [W78, K78], and the correspondence is motivated by the Borel-Weil theorem. The construction of [W78] is the following: with the same notation as before, let $X := T^*P$ be the cotangent bundle to the principal H-bundle P, $\mathcal{O}$ a coadjoint orbit of H and $\mathcal{O}^-$ the set of the opposites of the elements of $\mathcal{O}$, hence still a coadjoint orbit and a Hamiltonian H-space. The free action of H on P can be lifted to a free action on X, which is Hamiltonian ([M07, 2.2]), thus the *induced symplectic manifold* associated with P and $\mathcal{O}$ is defined as follows:

$$(X \times \mathcal{O}^-)/\!\!/H,$$

and it is actually a symplectic manifold, since the symplectic reduction theorem applies, as we will see in more detail in the general setting of Hamiltonian spaces below (see Remark 1.6.18).

We can also consider the case in which H is a closed subgroup of a Lie group G and $P = G$, thereby obtaining $M = G/H$. In representation theory, these assumptions lead to the definition of the *induced representation* of G on the sections of the associated bundle $G \times_H V \to G/H$. After these substitutions, the induced symplectic manifold writes:

$$(T^*G \times \mathcal{O}^-)/\!\!/H,$$

and it is a Hamiltonian G-space (Remark 1.6.18).

The last definition was extended to Hamiltonian spaces in [K78]; we introduce it following [R22, §1]: let $H \subset G$ be a closed subgroup of a Lie group G and consider a Hamiltonian H-space (Y, ω_Y, Ψ). Let ϖ be the canonical 1-form on T^*G, i.e. $\varpi(\delta p) := \langle p, \delta q\rangle$, where $\pi : T^*G \to G$ is the canonical projection, $\pi(p) = q, \delta p \in T_p(T^*G)$, $\delta q := \pi_*(\delta p) \in T_{\pi(p)}G$. The manifold $L := T^*G \times Y$ can be endowed with the symplectic form $\omega := d\varpi + \omega_Y$ and a $G \times H$-action:

$$(g, h)(p, y) := (gph^{-1}, h(y)),$$

where $h(y)$ is the action of $h \in H$ on $y \in Y$. The equivariant moment map associated to this action is $\phi \times \psi : L \to \mathfrak{g}^* \times \mathfrak{h}^*$, where:

$$\begin{cases} \phi(p, y) &= pq^{-1} \\ \psi(p, y) &= \Psi(y) - q^{-1}p_{|\mathfrak{h}} \end{cases} \tag{1.6.15}$$

using notation (1.3.30) and $p \in T^*_qG$.

Definition 1.6.16. The *induced Hamiltonian space* (or induced space) is the reduction of L at $0 \in \mathfrak{h}^*$:

$$\mathrm{Ind}_H^G Y := L/\!\!/H = \psi^{-1}(0)/H. \tag{1.6.17}$$

Remark 1.6.18. The induced Hamiltonian space is well defined: since the lift of the H-action to T^*G is free and proper, it is free and proper on L as well ([B72, III.1.6]). Thus the symplectic reduction theorem applies and $\mathrm{Ind}_H^G Y$

carries a reduced symplectic form $\omega_{L/\!\!/H}$. Moreover, the G-action on L commutes with the H-action and preserves $\psi^{-1}(0)$, so it descends to a G-action on $\mathrm{Ind}_H^G Y$. Similarly, the moment map ϕ descends to a moment map for the residual G-action $\Phi_{L/\!\!/H} : \mathrm{Ind}_H^G Y \to \mathfrak{g}^*$, thus making $(\mathrm{Ind}_H^G Y, \omega_{L/\!\!/H}, \Phi_{L/\!\!/H})$ a Hamiltonian G-space.

As pointed out in [G05, §2.3] and [S05, Example 6.10], examples of induced G-spaces are the normal forms of Marle-Guillemin-Sternberg [G84a, M85], to which results like (2.1.1) apply.

Now we will focus on the symplectic counterpart of the space of intertwining maps between representations. Following [G82, §6], consider a compact Lie group G and a compact Hamiltonian G-space (X, ω, Φ) endowed with G-invariant prequantum data and polarization. Denote by τ the unitary representation of G on the space of polarized sections on X and let ρ be an irreducible representation of G. By the Borel-Weil theorem there exists a unique integral coadjoint orbit $\mathcal{O}$ of G such that ρ is the representation of G on the space of polarized sections on $\mathcal{O}$. Define V_1 and V_2 as the spaces of polarized sections of $\mathcal{O}$ and X, respectively. Theorem 6.2 of [G82] proves that the space of intertwining operators between V_2 and V_1, that is

$$\mathrm{Hom}_G(V_2, V_1) = \{T : V_2 \to V_1 \ : \ T(\tau s) = \rho T(s)\}, \tag{1.6.19}$$

is isomorphic to the space of polarized sections of

$$(X \times \mathcal{O}^-)/\!\!/G =: \mathrm{Hom}_G(\mathcal{O}, X), \tag{1.6.20}$$

under the assumptions of the symplectic reduction theorem. As the latter is obtained by reduction of the product of two Hamiltonian G-spaces endowed with prequantum data and polarization, the space of polarized sections is well defined. Indeed, the product of prequantized and polarized symplectic manifolds is endowed with prequantum data and polarization and those structures descend to the simplectically reduced manifold by [G82]. The consequences of this result are the following:

- [G82, Corollary 6.3]: If $\mathcal{O}$ is not in the image of Φ, then the representation ρ does not occur in the space of polarized sections on X, i.e. (1.6.20) is empty.

- [G82, Corollary 6.4]: If G acts freely and transitively on $\{x \in X : \Phi(x) \in \mathcal{O}\}$, then ρ occurs in the space of polarized sections on X with multiplicity one, i.e. (1.6.20) consists of a single point.

- [G82, Theorem 6.5, Remark 1]: For large n, the dimension of (1.6.20) is related to the multiplicity with which the irreducible representation corresponding to $\mathcal{O}_n$ occurs in the space of polarized sections on X_n, where the subscript denotes the tensor power.

Thus the symplectic manifold $\mathrm{Hom}_G(\mathcal{O}, X)$ "measures" the multiplicity with which ρ occurs in the space of polarized sections of X.

The symplectic definition of Hom_G was extended to Hamiltonian spaces in [R22], where it is regarded only as a set:

Definition 1.6.21. The *Hamiltonian space of intertwiners* is:

$$\begin{aligned} \mathrm{Hom}_G(X_1, X_2) &:= (X_1^- \times X_2)/\!\!/G \\ &= \Phi^{-1}(0)/G, \end{aligned} \tag{1.6.22}$$

where (X_i, ω_i, Φ_i) are Hamiltonian G-spaces and X_1^- denotes $(X_1, -\omega_1, -\Phi_1)$. The product $X_1^- \times X_2$ is endowed with G-action $g(x_1, x_2) = (g(x_1), g(x_2))$, 2-form $\omega_2 - \omega_1$ and moment map $\Phi(x_1, x_2) = \Phi_2(x_2) - \Phi_1(x_1)$.

When (X_1, ω_1, Φ_1) is trivial, (1.6.22) is just $X_2/\!\!/G$. Moreover, setting $X_1 = \mathcal{O}$, we get again (1.6.20), which coincides with $\Phi^{-1}(\mu)/G_\mu$ if $\mu \in \mathcal{O}$ [G82], that is the Marsden-Weinstein "shifting trick" ([M74]).

Remark 1.6.23. Since the Hamiltonian space of intertwiners (1.6.22) is defined as a reduced space, it carries a symplectic manifold structure only under the assumptions of the symplectic reduction theorem. However, without any further hypothesis, we can repeat the considerations on reduced spaces (1.4.10), and regard (1.6.22) as a diffeological space endowed with the subquotient diffeology (Definition 1.1.22) which may, or may not, carry a reduced diffeological 2-form.

1.6.3 Symplectic Frobenius Reciprocity

In this subsection we formulate the symplectic versions of the Frobenius reciprocity, verified by the induced Hamiltonian space (1.6.17) and by the symplectic space of intertwiners (1.6.22), following [G83] and [R22].

As for Hamiltonian induced spaces and the space of intertwiners, we first consider the formulation of the symplectic Frobenius reciprocity for coadjoint orbits [G83, Theorem 2.2], formulated under the assumptions of the symplectic reduction theorem.

Theorem 1.6.24. *Let H be a closed subgroup of G, and $\mathcal{O}_H$ and $\mathcal{O}_G$ be coadjoint orbits for H and G, respectively. Then the following hold:*

$$\big((T^*G \times \mathcal{O}_H^-)/\!\!/H\big) \times \mathcal{O}_G^-/\!\!/G \;=\; \mathcal{O}_G \times \mathcal{O}_H^-/\!\!/H, \tag{1.6.25}$$

where the equality denotes a symplectomorphism.

Remark 1.6.26. The previous theorem can be rephrased in terms of (1.6.22) and (1.6.17) as follows:

$$\mathrm{Hom}_G(\mathcal{O}_G, \mathrm{Ind}_H^G \mathcal{O}_H) = \mathrm{Hom}_H(\mathrm{Res}_H^G \mathcal{O}_G, \mathcal{O}_H), \tag{1.6.27}$$

where $\mathrm{Res}_H^G \mathcal{O}_G$ denotes the coadjoint orbit $\mathcal{O}_G$ on which the Hamiltonian G-action is restricted to an H-action and the moment map is projected to $\mathfrak{h}^*$.

The theorem was proved as follows: since the left G-action and the right H-action on T^*G commute, Theorem 2.1 of [G83] applies and the left hand side of (1.6.25) can be written as:

$$((T^*G \times \mathcal{O}_G^-)/\!\!/G) \times \mathcal{O}_H^-/\!\!/H.$$

Then, by [G83, Theorem 2.3]:

$$(T^*G \times \mathcal{O}_G^-)/\!\!/G = \mathcal{O}_G. \tag{1.6.28}$$

Therefore the previous space is symplectomorphic to the right hand side of (1.6.25), proving the theorem.

A symplectic Frobenius reciprocity for Hamiltonian spaces was proved in [R22, Theorem 3.4]:

Theorem 1.6.29. *Let H be a closed subgroup of a Lie group G, X be a Hamiltonian G-space and Y a Hamiltonian H-space, then*

$$\mathrm{Hom}_G(X, \mathrm{Ind}_H^G Y) = \mathrm{Hom}_H(\mathrm{Res}_H^G X, Y), \tag{1.6.30}$$

where the equality denotes a bijection between sets.

Proof. See the first part of the proof of Theorem 2.1.1. $\square$

Remark 1.6.31. The Hom-spaces in the theorem need not have a manifold structure, but can be regarded as diffeological spaces endowed with the subquotient diffeology, as specified in Remark 1.6.23. For this reason the authors of [R22] conjectured that the structure preserved by the bijection is the diffeological space one, i.e. the equality denotes a diffeological diffeomorphism. Furthermore, if the left hand or right hand side of (1.6.30) carry a reduced two form, so does the other, and the diffeomorphism preserves those forms. This conjecture will be proved in Theorem 2.1.1.

1.6.4 Induced prequantum space and space of intertwiners

The induced Hamiltonian space and the space of intertwiners, defined in Subsection 1.6.2, have a prequantum analogue, defined in [R22]. In this subsection we introduce these constructions.

Definition 1.6.32. Let H be a closed subgroup of a Lie group G, let $(\tilde{Y}, \varpi_{\tilde{Y}})$ be a prequantum H-space and define $\tilde{L} := T^*G \times \tilde{Y}$. The latter is endowed with a $G \times H$-action $(g,h)(p,\tilde{y}) = (gph^{-1}, h(\tilde{y}))$ and 1-form $\varpi_{T^*G} + \varpi_{\tilde{Y}}$. The S^1-action on $\tilde{L}$ is $z(p,\tilde{y}) := (p, z(\tilde{y}))$, and the prequantum moment map $\phi \times \psi : \tilde{L} \to \mathfrak{g}^* \times \mathfrak{h}^*$ is given by formulae (1.6.15) with $\tilde{y}$ instead of y. The *induced prequantum space* is:

$$\begin{aligned} \operatorname{Ind}_H^G \tilde{Y} &:= (T^*G \times \tilde{Y}) /\!\!/ H \\ &= \psi^{-1}(0)/H. \end{aligned} \tag{1.6.33}$$

Remark 1.6.34. By [L01b, Theorem 2], the induced prequantum space is a manifold carrying a reduced 1-form $\varpi_{\tilde{L}/\!\!/H}$, since, as in the symplectic setting, the H-action on $T^*G \times \tilde{Y}$ is free and proper. Moreover, the actions of G and S^1 descend to (1.6.33), which has a prequantum G-space structure that prequantizes (1.6.17).

Remark 1.6.35. As Marle-Guillemin-Sternberg normal forms are examples of induced spaces in the symplectc setting, local models of Lerman and Willet [L01a, Theorem 4.1] are examples of induced prequantum G-spaces.

Before defining the prequantum space of intertwiners, let us introduce the necessary notions of prequantum dual [S70b, 18.47] and prequantum product [S70b, 18.52]:

Definition 1.6.36. Let $\tilde{X}$ be a prequantum G-space. The *prequantum dual* $\tilde{X}^-$ of $\tilde{X}$ is a prequantum space equal to $\tilde{X}$, but endowed with opposite 1-form, and therefore opposite S^1-action and Reeb vector field. It prequantizes the Hamiltonian space $(X^-, -\omega, -\Phi)$.

Definition 1.6.37. Let $\tilde{X}_1$ and $\tilde{X}_2$ be prequantum G-spaces. The product $\tilde{X}_1 \times \tilde{X}_2$ can be endowed with diagonal G-action, induced $S^1 \times S^1$-action, and 1-form $\varpi_1 + \varpi_2$. Moreover, the orbits of the anti-diagonal subgroup of $S^1 \times S^1$, $\Delta = \{(z^{-1}, z) : z \in S^1\}$, are the characteristic leaves of $\varpi_1 + \varpi_2$. Thus the 1-form descends to the quotient [S70b, 5.21, 18.50]:

$$\tilde{X}_1 \boxtimes \tilde{X}_2 := (\tilde{X}_1 \times \tilde{X}_2)/\Delta, \tag{1.6.38}$$

which is a prequantum G-space called *prequantum product*. The prequantum product prequantizes the symplectic manifold $X_1 \times X_2$.

Remark 1.6.39. In the definition of the prequantum space of intertwiners, not quite $\tilde{X}_1 \boxtimes \tilde{X}_2$, but $\tilde{X}_1^- \boxtimes \tilde{X}_2$ is involved. Combining Definitions 1.6.36 and 1.6.37, the space $\tilde{X}_1^- \boxtimes \tilde{X}_2$ is obtained from $\tilde{X}_1^- \times \tilde{X}_2$ by quotient with respect to the Δ-action, which in this case is $\Delta(\tilde{x}_1, \tilde{x}_2) = (z(\tilde{x}_1), z(\tilde{x}_2))$.

Now we can define the prequantum space of intertwiners [R22, 5.4]:

Definition 1.6.40. Let $\tilde{X}_1$ and $\tilde{X}_2$ be prequantum G-spaces. The *prequantum space of intertwiners* is:

$$\begin{aligned} \mathrm{Hom}_G(\tilde{X}_1, \tilde{X}_2) &:= (\tilde{X}_1^- \boxtimes \tilde{X}_2)/\!\!/G \\ &= \Phi^{-1}(0)/G, \end{aligned} \tag{1.6.41}$$

Remark 1.6.42. As in the definition of the symplectic space of intertwiners, freeness and properness of the action are not assumed and therefore (1.6.41) need not have a manifold structure. However, as before, it may be regarded as a diffeological space endowed with the subquotient diffeology.

1.6.5 Prequantum Frobenius Reciprocity

The prequantum induced space and the prequantum space of intertwiners are involved in a prequantum version of the Frobenius reciprocity theorem, proved in [R22]. This result is applied in [R22, §9] to representations of solvable Lie groups: it explains how an irreducible representation associated to a coadjoint orbit of a solvable Lie group splits when restricted to a subgroup.

The prequantum Frobenius reciprocity theorem is the following [R22, Theorem 8.2]:

Theorem 1.6.43. *Let H be a closed subgroup of a Lie group G, $\tilde{X}$ a prequantum G-space and $\tilde{Y}$ a prequantum H space. Also, denote by $\mathrm{Res}^G_H \tilde{X}$ the prequantum H-space obtained from $\tilde{X}$ by restricting the G-action to an H-action and projecting the moment map to $\mathfrak{h}^*$. Then*

$$\mathrm{Hom}_G(\tilde{X}, \mathrm{Ind}^G_H \tilde{Y}) = \mathrm{Hom}_H(\mathrm{Res}^G_H \tilde{X}, \tilde{Y}), \tag{1.6.44}$$

where the equality denotes a bijection between sets.

Proof. See the first part of the proof of Theorem 2.2.1. □

Remark 1.6.45. As before, the equality denotes a bijection between sets because the Hom-spaces involved in the theorem need not have a manifold structure. However, one could make a conjecture analogue to the symplectic one: the equality in the previous theorem denotes a diffeological diffeomorphism. Moreover, if one side of (1.6.44) carries a reduced 1-form, so does the other and the diffeomorphism relates the 1-forms. This conjecture will be proved in Theorem 2.2.1.

Chapter 2

Diffeological Frobenius reciprocity

In this chapter we present the results of Part I of [B25]: we establish symplectic and prequantum versions of Frobenius reciprocity, thereby proving the conjecture formulated by [R22, 3.5] (Remark 1.6.31) together with its prequantum analogue (Remark 1.6.45).

For background and motivation on the Hamiltonian and prequantum settings, and their relation to representation theory, we refer to Section 1.6 and the references therein.

The main results of this chapter are Theorems 2.1.1 and 2.2.1, which realize symplectic and prequantum Frobenius reciprocity as diffeological diffeomorphisms between the Hom-spaces (1.6.22, 1.6.41), which preserve the reduced forms they may carry. In this way, Frobenius reciprocity acquires a direct geometric interpretation in both the symplectic and prequantum contexts.

The first section of this chapter states and proves the diffeological symplectic Frobenius reciprocity. The second section states and proves the diffeological prequantum Frobenius reciprocity.

2.1 Diffeological Symplectic Frobenius reciprocity

As in Subsection 1.6.3, denote H a closed subgroup of G, and write Res_H^G for the restriction functor from a Hamiltonian G-space to a Hamiltonian H-space, i.e., the G-action is restricted to H and the moment map is composed with the projection $\mathfrak{g}^* \to \mathfrak{h}^*$. The following proves the conjecture [R22, 3.5]:

Theorem 2.1.1. *Let X be a Hamiltonian G-space and Y a Hamiltonian H-space, then we have an isomorphism of reduced spaces*

$$\mathrm{Hom}_G(X, \mathrm{Ind}_H^G Y) = \mathrm{Hom}_H(\mathrm{Res}_H^G X, Y), \tag{2.1.2}$$

i.e. there is a diffeological diffeomorphism t from left to right. Moreover, if one side carries a reduced 2-form, then so does the other, and t relates the 2-forms.

Proof. In the first part of the proof we follow that of [R22, Theorem 3.4], which showed that t is a bijection. Let Φ and Ψ be the moment maps of X and Y, and define

$$M := X^- \times T^*G \times Y, \tag{2.1.3}$$

$$N := X^- \times Y, \tag{2.1.4}$$

which are Hamiltonian spaces. In more detail, M is a Hamiltonian $G \times H$-space with action $(g,h)(x,p,y) = (g(x), gph^{-1}, h(y))$, 2-form $\omega_M = \omega_Y + d\varpi_{T^*G} - \omega_X$ and moment map $\phi_M \times \psi_M : M \to \mathfrak{g}^* \times \mathfrak{h}^*$, and N is an H-space with diagonal H-action, 2-form $\omega_N = \omega_Y - \omega_X$ and moment map $\psi_N : N \to \mathfrak{h}^*$, where

$$\begin{cases} \phi_M(x,p,y) &= pq^{-1} - \Phi(x) \\ \psi_M(x,p,y) &= \Psi(y) - q^{-1}p_{|\mathfrak{h}} \\ \psi_N(x,y) &= \Psi(y) - \Phi(x)_{|\mathfrak{h}}, \end{cases} \tag{2.1.5}$$

and $p \in T_q^*G$. Combining the definitions of the space of intertwiners (1.6.22) and of the induced space (1.6.17) we get that the sides of (2.1.2) are respectively $(M/\!\!/H)/\!\!/G$ and $N/\!\!/H$:

$$\begin{aligned}(M/\!\!/H)/\!\!/G &= (X^- \times (T^*G \times Y)/\!\!/H)/\!\!/G = (X^- \times \mathrm{Ind}_H^G Y)/\!\!/G \\ &= \mathrm{Hom}_G(X, \mathrm{Ind}_H^G Y),\end{aligned} \tag{2.1.6}$$

and

$$N/\!\!/H = (X^- \times Y)/\!\!/H = \mathrm{Hom}_H(\mathrm{Res}_H^G X, Y). \tag{2.1.7}$$

Define a submersion $r : M \to N$ by $r(x,p,y) = (q^{-1}(x), y)$, where q is the base point of $p \in T_q^*G$, and consider the following commutative diagram, where j_1, j_2, j_3 and π_1, π_2, π_3 denote the inclusions and projections involved in the construction of reduced spaces (1.4.9), $\Phi_{M/\!\!/H}$ denotes the moment map of the residual G-action on $M/\!\!/H$ (Remark 1.6.18 and (2.1.6)), j the natural inclusion $(\phi_M \times \psi_M)^{-1}(0) \hookrightarrow \psi_M^{-1}(0)$ and π the restriction of π_1:

$$\begin{tikzcd}
M \arrow[rrr, "r"] & & & N & \\
& \psi_M^{-1}(0) \arrow[ul, hook', "j_1"'] \arrow[dd, "\pi_1"] & & & \\
& & (\phi_M \times \psi_M)^{-1}(0) \arrow[ul, hook', "j"'] \arrow[rr, dashed, "s"] \arrow[dd, "\pi"] & & \psi_N^{-1}(0) \arrow[uul, hook', "j_3"'] \arrow[ddd, "\pi_3"] \\
& M/\!\!/H & & & \\
& & \Phi^{-1}_{M/\!\!/H}(0) \arrow[ul, hook', "j_2"'] \arrow[d, "\pi_2"] & & \\
& & (M/\!\!/H)/\!\!/G \arrow[rr, dashed, "t"] & & N/\!\!/H.
\end{tikzcd} \tag{2.1.8}$$

Also define a right inverse of r, the immersion $r' : N \to M$, by $r'(x, y) = (x, \Phi(x), y)$, where $\mathfrak{g}^*$ is identified with the cotangent space of G at the identity. The maps r and r' send $(\phi_M \times \psi_M)^{-1}(0)$ to $\psi_N^{-1}(0)$ and conversely. In more detail ([R22, 3.8]):

$$\begin{aligned}
\psi_N((r \circ j_1 \circ j)(x, p, y)) &= \psi_N(q^{-1}(x), y) \\
&= \Psi(y) - \Phi(q^{-1}(x))_{|\mathfrak{h}} \\
&= \Psi(y) - q^{-1}\Phi(x)q_{|\mathfrak{h}} \\
&= \Psi(y) - q^{-1}p_{|\mathfrak{h}} \\
&= 0,
\end{aligned} \tag{2.1.9}$$

where the third equality follows by the equivariance of Φ, and the fourth and the fifth by $(x, p, y) \in (\phi_M \times \psi_M)^{-1}(0)$; thus r sends $(\phi_M \times \psi_M)^{-1}(0)$ to $\psi_N^{-1}(0)$. Conversely:

$$\begin{aligned}
\phi_M \times \psi_M((r' \circ j_3)(x, y)) &= \phi_M \times \psi_M(x, \Phi(x), y) \\
&= (0, \Psi(y) - \Phi(x)_{|\mathfrak{h}}) \\
&= 0,
\end{aligned} \tag{2.1.10}$$

where the last equality follows by $(x, y) \in \psi_N^{-1}(0)$, proving that r' sends $\psi_N^{-1}(0)$ to $(\phi_M \times \psi_M)^{-1}(0)$. Therefore r and r' induce a map s as indicated in (2.1.8), and a right inverse s' of it. Next, s sends $G \times H$-orbits to H-orbits and conversely s' sends H-orbits to orbits of the diagonal $\mathrm{diag}(H) \subset G \times H$.

In more detail ([R22, 3.9]):

$$\begin{aligned} s((g,h)(x,p,y)) &= r(g(x), gph^{-1}, h(y)) \\ &= ((gqh^{-1})^{-1}g(x), h(y)) \\ &= (h(q^{-1}(x)), h(y)) \\ &= h(q^{-1}(x), y) \\ &= h(s(x,p,y)); \end{aligned} \tag{2.1.11}$$

conversely:

$$\begin{aligned} s'(h(x,y)) &= r'(h(x), h(y)) \\ &= (h(x), \Phi(h(x)), h(y)) \\ &= (h(x), h\Phi(x)h^{-1}, h(y)) \\ &= (h,h)s'(x,y), \end{aligned} \tag{2.1.12}$$

where the third equality follows by the equivariance of Φ. Therefore s and s' descend to a bijection t, as indicated in (2.1.8), and its inverse t^{-1} (proving Theorem 1.6.29).

Now let us prove that t is smooth. Note first that r is diffeologically smooth, since it is a smooth map between manifolds [I13, 4.3]. Also, by construction of the subquotient diffeologies (1.4.10), j_1, j_2, j_3 are inductions, hence smooth, and π_1, π_2, π_3 are subductions, hence also smooth (see Section 1.1 and [I13, 1.29, 1.36, 1.46, 1.50]). Now endow $(\phi_M \times \psi_M)^{-1}(0)$ with its subset diffeology in $\psi_M^{-1}(0)$: then j is an induction, and π is a subduction by (1.1.20). In particular $r \circ j_1 \circ j$ is smooth, as it is composition of smooth maps [I13, 1.15]. Since this is $j_3 \circ s$, s is smooth by the universal property of inductions. Then $\pi_3 \circ s$ is smooth, and as this is $t \circ \pi_2 \circ \pi$, t is smooth by the universal property of subductions.

To prove that t^{-1} is smooth, note that r' is diffeologically smooth, since it is a smooth map between manifolds. Hence so is $r' \circ j_3$ and, as this is $j_1 \circ j \circ s'$, the universal property of inductions implies that s' is smooth. Then $\pi_2 \circ \pi \circ s'$ is smooth, and as this is $t^{-1} \circ \pi_3$, the universal property of subductions gives smoothness of t^{-1}. Hence t is a diffeomorphism.

Next, assume that both sides of (2.1.2) carry reduced 2-forms and denote them by $\omega_{(M/\!\!/H)/\!\!/G}$ and $\omega_{N/\!\!/H}$. We must prove

$$\omega_{(M/\!\!/H)/\!\!/G} = t^*\omega_{N/\!\!/H}. \tag{2.1.13}$$

To do this, we claim it is enough to show

$$j^*j_1^*\omega_M = j^*j_1^*r^*\omega_N. \tag{2.1.14}$$

Indeed, by commutativity of (2.1.8) and Definition (1.4.11) of a reduced 2-form, which $M/\!\!/H$ always carries, being a Marsden-Weinstein reduction (by (2.1.6) and Remark 1.6.16), we get that (2.1.14) has left-hand side

$$j^*\pi_1^*\omega_{M/\!\!/H} = \pi^* j_2^*\omega_{M/\!\!/H} = \pi^*\pi_2^*\omega_{(M/\!\!/H)/\!\!/G}, \tag{2.1.15}$$

and right-hand side

$$s^* j_3^*\omega_N = s^*\pi_3^*\omega_{N/\!\!/H} = \pi^*\pi_2^* t^*\omega_{N/\!\!/H}. \tag{2.1.16}$$

Since pullback $\pi^*\pi_2^*$ by the subduction $\pi_2 \circ \pi$ is injective [I13, 6.39], we conclude that (2.1.14) implies (2.1.13).

Now (2.1.14) is a genuinely diffeological relation to prove. Indeed, r and j_1 are smooth maps between manifolds, as $\psi_M^{-1}(0)$ is also a manifold, again by (2.1.6) and Remark 1.6.18; however, j can be regarded only as a smooth map between diffeological spaces, since $(\phi_M \times \psi_M)^{-1}(0)$ is not necessarily a manifold. Thus, by definition of diffeological differential form, the identity (2.1.14) means that its two sides coincide after pullback by any plot P of the subset diffeology of $(\phi_M \times \psi_M)^{-1}(0)$, i.e., $(j_1 \circ j \circ P)^*\omega_M = (j_1 \circ j \circ P)^* r^*\omega_N$. Note that $F := j_1 \circ j \circ P$ is an ordinary smooth map $U \to M$, where U is open in some $\mathbb{R}^n$, taking values in $(\phi_M \times \psi_M)^{-1}(0)$, by definition of subset diffeology. Then, proving (2.1.14) is tantamount to proving:

$$F^*\omega_M = F^* r^*\omega_N \tag{2.1.17}$$

for every ordinary smooth map, $F : U \to M$, taking values in $(\phi_M \times \psi_M)^{-1}(0)$. To this end, fix such a map F, write $F(u)$ as $m = (x, p, y)$, denote by q the base point of $p \in T_q^*G$ and regard the variables

$$\mu = pq^{-1}, \qquad \bar\mu = \Phi(x), \qquad \bar x = q^{-1}(x), \qquad n = r(m) = (\bar x, y) \tag{2.1.18}$$

as smooth functions of m and therefore of u. Then derivatives of these functions map each vector $\delta u \in T_uU$ to vectors δm, δx, δp, δy, δq, $\delta\mu$, $\delta\bar\mu$, $\delta\bar x$, δn. Defining also $Z := \delta q.q^{-1} = \Theta(\delta q)$, where Θ is the Maurer-Cartan 1-form, we can write $\delta q = Zq$ and thus

$$\begin{aligned} \delta\bar x &= \frac{\partial[q^{-1}(x)]}{\partial q}(\delta q) + \frac{\partial[q^{-1}(x)]}{\partial x}(\delta x) \\ &= \frac{d}{dt}(\mathrm{e}^{tZ}q)^{-1}(x)\Big|_{t=0} + {q^{-1}}_*(\delta x) \\ &= \frac{d}{dt}\, q^{-1}(\mathrm{e}^{-tZ}(x))\Big|_{t=0} + {q^{-1}}_*(\delta x) \\ &= {q^{-1}}_*(\delta x - Z(x)). \end{aligned} \tag{2.1.19}$$

Now we can compute the following:

$$\begin{align*}
(F^*r^*\omega_N)(\delta u, \delta' u) &= \omega_N(\delta n, \delta' n) \\
&= \omega_Y(\delta y, \delta' y) - \omega_X(\delta \bar{x}, \delta' \bar{x}) \\
&= \omega_Y(\delta y, \delta' y) - \omega_X(\delta x - Z(x), \delta' x - Z'(x)) \\
&= \omega_Y(\delta y, \delta' y) + \omega_X(Z(x), \delta' x) - \omega_X(Z'(x), \delta x) \\
&\qquad\quad - \omega_X(Z(x), Z'(x)) - \omega_X(\delta x, \delta' x) \\
&= \omega_Y(\delta y, \delta' y) + \langle \delta\bar{\mu}, Z' \rangle - \langle \delta'\bar{\mu}, Z \rangle + \langle \bar{\mu}, [Z, Z'] \rangle \tag{2.1.20} \\
&\qquad\quad - \omega_X(\delta x, \delta' x) \\
&= \omega_Y(\delta y, \delta' y) + \langle \delta\mu, Z' \rangle - \langle \delta'\mu, Z \rangle + \langle \mu, [Z, Z'] \rangle \\
&\qquad\quad - \omega_X(\delta x, \delta' x) \\
&= \omega_Y(\delta y, \delta' y) + d\varpi_{T^*G}(\delta p, \delta' p) - \omega_X(\delta x, \delta' x) \\
&= \omega_M(\delta m, \delta' m) \\
&= (F^*\omega_M)(\delta u, \delta' u).
\end{align*}$$

Here the third equality follows by (2.1.19) and G-invariance of ω_X, the fifth by the equivariance of Φ and [S70b, 11.17♯], the sixth by $F(U) \subset \phi_M^{-1}(0)$ and the seventh is the formula for $d\varpi_{T^*G}$ proved in [A78, 4.4.1]. Thus we have proved (2.1.17) and thereby also (2.1.14) and (2.1.13).

Finally, assume that one reduced 2-form exists: $\omega_{(M/\!\!/H)/\!\!/G}$ or $\omega_{N/\!\!/H}$. Then the other can be defined by (2.1.13) and is actually a reduced 2-form. As a matter of fact, a diagram chase in (2.1.8), combined with (2.1.14), which can be applied since its proof involved neither reduced 2-form, shows that $\pi_3^*\omega_{N/\!\!/H} = j_3^*\omega_N$ and respectively $\pi_2^*\omega_{(M/\!\!/H)/\!\!/G} = j_2^*\omega_{M/\!\!/H}$. □

2.2 Diffeological Prequantum Frobenius reciprocity

In Subsections 1.6.4 and 1.6.5, prequantum analogues of the induced Hamiltonian space, the space of intertwiners and symplectic Frobenius reciprocity were introduced. With those in mind, we establish a prequantum version of the diffeological symplectic Frobenius reciprocity, thereby proving the prequantum variant of conjecture [R22, 3.5] (Remark 1.6.45).

Let H be a closed subgroup of G and Res_H^G the restriction functor from prequantum G-spaces to prequantum H-spaces, i.e., the G-action gets restricted to H. Then we can prove the following:

Theorem 2.2.1. *If $\tilde{X}$ is a prequantum G-space and $\tilde{Y}$ a prequantum H-space, then we have an isomorphism of reduced spaces*

$$\operatorname{Hom}_G(\tilde{X}, \operatorname{Ind}_H^G \tilde{Y}) = \operatorname{Hom}_H(\operatorname{Res}_H^G \tilde{X}, \tilde{Y}), \tag{2.2.2}$$

i.e. there is a diffeological diffeomorphism t from left to right. Moreover, if one side carries a reduced 1-form, then so does the other, and t relates the 1-forms.

Proof. As before, the first part of the proof shows that t is a bijection, proving Theorem 1.6.43. First, note that the sides of (2.2.2) can be regarded again as $(M/\!\!/H)/\!\!/G$ and $N/\!\!/H$, but (2.1.3) and (2.1.4) are now replaced by the prequantum $G \times H$-space and H-space

$$M = \tilde{X}^- \boxtimes (T^*G \times \tilde{Y}), \tag{2.2.3}$$

$$N = \tilde{X}^- \boxtimes \tilde{Y}, \tag{2.2.4}$$

respectively. In more detail: define $\check{M}$ and $\check{N}$ by these two formulae with the usual product $\times$ in place of the prequantum one $\boxtimes$, and endow them with the 1-forms $\varpi_{\check{M}} = \varpi_{\tilde{Y}} + \varpi_{T^*G} - \varpi_{\tilde{X}}$ and $\varpi_{\check{N}} = \varpi_{\tilde{Y}} - \varpi_{\tilde{X}}$. Then M and N are the orbit spaces of Δ-actions by Definition 1.6.40. Therefore they come endowed with 1-forms ϖ_M, ϖ_N and group actions, obtained by passing to the quotient from the 1-forms $\varpi_{\check{M}}$, $\varpi_{\check{N}}$ and actions

$$(g, h)(\tilde{x}, p, \tilde{y}) = (g(\tilde{x}), gph^{-1}, h(\tilde{y})), \tag{2.2.5}$$

$$h(\tilde{x}, \tilde{y}) = (h(\tilde{x}), h(\tilde{y})), \tag{2.2.6}$$

respectively.

Being prequantum spaces, M and N are endowed with prequantum moment maps (1.5.5), $\phi_M \times \psi_M : M \to \mathfrak{g}^* \times \mathfrak{h}^*$ and $\psi_N : N \to \mathfrak{h}^*$. Their pullbacks $\phi_{\check{M}} \times \psi_{\check{M}}$ and $\psi_{\check{N}}$ to $\check{M}$ and $\check{N}$, are given by a prequantum version of (2.1.5):

$$\begin{cases} \phi_M(\tilde{x}, p, \tilde{y}) &= pq^{-1} - \Phi(\tilde{x}) \\ \psi_M(\tilde{x}, p, \tilde{y}) &= \Psi(\tilde{y}) - q^{-1}p_{|\mathfrak{h}} \\ \psi_N(\tilde{x}, \tilde{y}) &= \Psi(\tilde{y}) - \Phi(\tilde{x})_{|\mathfrak{h}}, \end{cases} \tag{2.2.7}$$

where Φ and Ψ denote the prequantum moment maps on $\tilde{X}$ and $\tilde{Y}$, respectively. Now define a submersion $\check{r} : \check{M} \to \check{N}$ by $\check{r}(\tilde{x}, p, \tilde{y}) = (q^{-1}(\tilde{x}), \tilde{y})$ where $p \in T_q^*G$. Also define a right inverse immersion of $\check{r}$, $\check{r}' : \check{N} \to \check{M}$, by $\check{r}'(\tilde{x}, \tilde{y}) = (\tilde{x}, \Phi(\tilde{x}), \tilde{y})$, where the dual of the Lie algebra $\mathfrak{g}^*$ is identified with the cotangent space of G at the identity.

These maps send Δ-orbits to Δ-orbits, in more detail:

$$\begin{aligned}\check{r}(\Delta(\tilde{x},p,\tilde{y})) &= \check{r}(z(\tilde{x}),p,z(\tilde{y}))\\ &= (q^{-1}(z(\tilde{x})),z(\tilde{y}))\\ &= (z(q^{-1}(\tilde{x})),z(\tilde{y}))\\ &= \Delta(\check{r}(\tilde{x},p,\tilde{y})),\end{aligned} \tag{2.2.8}$$

where $z \in S^1$ and the expression of the diagonal action is the one described in Remark 1.6.39. Moreover, the third equality follows by commutativity of the actions of S^1 and G. Conversely:

$$\begin{aligned}\check{r}'(\Delta(\tilde{x},\tilde{y})) &= \check{r}'(z(\tilde{x}),z(\tilde{y}))\\ &= (z(\tilde{x}),\Phi(z(\tilde{x})),z(\tilde{y}))\\ &= (z(\tilde{x}),\Phi(\tilde{x}),z(\tilde{y}))\\ &= \Delta(\tilde{x},\Phi(x),\tilde{y})\\ &= \Delta(\check{r}'(\tilde{x},\tilde{y})),\end{aligned} \tag{2.2.9}$$

where z and the expression of Δ are as before and the third equality follows by (1.5.5). Hence $\check{r}$ and $\check{r}'$ induce maps $r : M \to N$ and $r' : N \to M$ which are smooth by [B67, 5.9.6]. At this point, the same diagram (2.1.8) and arguments about it establish that r descends to a bijection $t : (M/\!\!/H)/\!\!/G \to N/\!\!/H$ (proving Theorem 1.6.43), and that t is a diffeological diffeomorphism.

Next, assume that both sides carry reduced 1-forms, denoted by $\varpi_{(M/\!\!/H)/\!\!/G}$ and $\varpi_{N/\!\!/H}$. We must show that $\varpi_{(M/\!\!/H)/\!\!/G} = t^*\varpi_{N/\!\!/H}$. However, as in (2.1.14), it is sufficient to prove

$$j^*j_1^*\varpi_M = j^*j_1^*r^*\varpi_N. \tag{2.2.10}$$

Now we know from Proposition 1.1.20 that the projection $\check{M} \to M$ induces a *subduction* $(\phi_{\check{M}} \times \psi_{\check{M}})^{-1}(0) \to (\phi_M \times \psi_M)^{-1}(0)$. Therefore (2.2.10) will follow if we show that its two sides coincide after pullback by that subduction, i.e., by commutativity of the diagram corresponding to (1.1.21), that

$$\check{\jmath}^*\check{\jmath}_1^*\varpi_{\check{M}} = \check{\jmath}^*\check{\jmath}_1^*\check{r}^*\varpi_{\check{N}}, \tag{2.2.11}$$

where $\check{\jmath}$ and $\check{\jmath}_1$ denote the two inclusions $(\phi_{\check{M}} \times \psi_{\check{M}})^{-1}(0) \hookrightarrow \psi_{\check{M}}^{-1}(0) \hookrightarrow \check{M}$.

As in (2.1.17), note that (2.2.11) will follow if we show

$$F^*\varpi_{\check{M}} = F^*\check{r}^*\varpi_{\check{N}} \tag{2.2.12}$$

for every ordinary smooth map, $F : U \to \check{M}$, taking values in $(\phi_{\check{M}} \times \psi_{\check{M}})^{-1}(0)$. So fix such a map F, write $F(u)$ as $\check{m} = (\tilde{x},p,\tilde{y})$, and regard also q, the base point of $p \in T_q^*G$, and $\check{n} = \check{r}(\check{m}) = (q^{-1}(\tilde{x}), \tilde{y})$ as smooth functions of $\check{m}$ and u. Then derivatives of these functions, and of the right translation by q^{-1}, map each $\delta u \in T_uU$ to vectors $\delta\check{m}$, $\delta\tilde{x}$, δp, $\delta\tilde{y}$, δq, $\delta\check{n}$ and $Z := \delta q.q^{-1} \in \mathfrak{g}$; proceeding as in (2.1.20), we get

$$\begin{aligned}
(F^*\check{r}^*\varpi_{\check{N}})(\delta u) &= \varpi_{\check{N}}(\delta\check{n}) \\
&= \varpi_{\tilde{Y}}(\delta\tilde{y}) - \varpi_{\tilde{X}}(\delta[q^{-1}(\tilde{x})]) \\
&= \varpi_{\tilde{Y}}(\delta\tilde{y}) - \varpi_{\tilde{X}}(\delta\tilde{x} - Z(\tilde{x})) \\
&= \varpi_{\tilde{Y}}(\delta\tilde{y}) + \langle\Phi(\tilde{x}), Z\rangle - \varpi_{\tilde{X}}(\delta\tilde{x}) \qquad (2.2.13) \\
&= \varpi_{\tilde{Y}}(\delta\tilde{y}) + \langle pq^{-1}, Z\rangle - \varpi_{\tilde{X}}(\delta\tilde{x}) \\
&= \varpi_{\tilde{Y}}(\delta\tilde{y}) + \varpi_{T^*G}(\delta p) - \varpi_{\tilde{X}}(\delta\tilde{x}) \\
&= \varpi_{\check{M}}(\delta\check{m}) \\
&= (F^*\varpi_{\check{M}})(\delta u)
\end{aligned}$$

as desired. Here the third equality is obtained by substituting the analogue of (2.1.19), the fourth follows by the definition of prequantum moment map (1.5.5), and the fifth is because $F(U) \subset \phi_{\check{M}}^{-1}(0)$.

Finally, assume that one reduced 1-form exists, $\varpi_{(M/\!\!/H)/\!\!/G}$ or $\varpi_{N/\!\!/H}$. Then we can define the other by $\varpi_{(M/\!\!/H)/\!\!/G} = t^*\varpi_{N/\!\!/H}$, and again chasing in (2.1.8) and using (2.2.10) proves $\pi_3^*\varpi_{N/\!\!/H} = j_3^*\varpi_N$ and respectively $\pi_2^*\varpi_{(M/\!\!/H)/\!\!/G} = j_2^*\varpi_{M/\!\!/H}$. ☐

Chapter 3

Reduced forms

In this chapter, we present the results of Part II of [B25]. Following that reference closely, we establish new sufficient conditions for the existence of reduced forms on reduced spaces.

This problem is related to the classical question of when a differential form pulls back from a quotient space, which has been studied primarily in two settings: when the quotient is the leaf space of a foliation, and when it is the orbit space of a Lie group action. In the first case, differential forms that descend to the leaf space are known as *integral invariants*. They were introduced by Poincaré [P99] and later studied by Cartan [C02, C22], who related them to the characteristic system: $\iota_v \varphi = \iota_v d\varphi = 0$, for all vectors v tangent to the leaves. Integral invariants also play a key role in Souriau's approach to quantization, which involves quotienting along characteristic foliations, as we have seen also in Definition 1.6.37 ([S67, S70b]).

In the second case, differential forms that descend to the orbit space of a Lie group action are called basic forms. A form is basic if it is both G-invariant and horizontal, meaning $g^*\varphi = \varphi$ for all $g \in G$ and $\iota_{Z(\cdot)}\varphi = 0$ for all $Z \in \mathfrak{g}$ ([K53]).

However, integral invariants and basic forms can be pullbacks of differential forms on the quotient only if it is a manifold. This requires the foliation to admit transverse sections [B67, 9.2.9], or the G-action to be free and proper [B72, III.1.5, Prop. 10]. Under these assumptions, the quotients are indeed manifolds, and the pullback defines an isomorphism between the de Rham complex on the quotient and the space of integral invariants or basic forms, respectively. See [S70b, 5.21] for foliations, and [K53, §1] for compact G or [G02, p. 185] in general, for group actions.

Note that when we consider the second case in a Hamiltonian context (1.4.9), i.e. a free and proper Lie group action on $C = \Phi^{-1}(0)$, we recover the assumptions of the symplectic reduction theorem (Theorem 1.4.8). As we have briefly discussed in Section 1.4, this result has been generalized to settings in which the quotient does not have a smooth manifold structure: when the action is locally free and proper it is an orbifold carrying a reduced orbifold form [W77], and when the action is proper it is a Whitney stratified symplectic space [S91, B97], that is, in particular, a disjoint union of symplectic manifolds whose associated Poisson brackets are induced by a global Poisson structure [A91].

In this chapter we regard both $C = \Phi^{-1}(0)$ and the reduced space $X/\!\!/G$ of (1.4.9) as diffeological spaces. Thus, even if the spaces they are defined on are not manifolds, it is possible to ask if the (diffeological) form $j^*\omega$ is the pullback of a (diffeological) form on the quotient, which is unique and global. To address this question we will make use of Souriau's criterion (Theorem 1.2.9), eventually combined with other arguments. This criterion has been extensively applied for aims similar to ours: [H11] generalized the previous result on integral invariants to regular foliations without a transverse section, [W12] generalized the previous result on basic forms to compact, not necessarily free, actions of compact Lie groups; this was further extended in [K16] to proper group actions. Moreover, [W22] extended it to the orbit space of any proper Lie groupoid and [M23] extended [H11] to certain singular foliations. The important difference between the listed results and the ones presented here is that the spaces we are quotienting need not be manifolds.

We are going to prove the existence of reduced forms in both the Hamiltonian and the prequantum setting when the G-action on $\Phi^{-1}(0)$ is either strict (Theorem 3.1.5) or locally free (Theorem 3.2.2), or the G-action on X is proper (Theorem 3.3.16). As a corollary to the last result we find that the reduced diffeological 2-form induces the Sjamaar-Lerman-Bates 2-form on each stratum (Corollary 3.3.28).

In conclusion, we are going to apply the existence result under strict action to induced Hamiltonian spaces (1.6.17) when the subgroup $H \subset G$ is not closed (Corollary 3.4.1), thereby obtaining Frobenius reciprocities under milder assumptions and the Kummer-Marsden-Satzer isomorphism $T^*G/\!\!/H = T^*(G/H)$ when H is dense in G (Theorem 3.4.8).

3.1 Strict G-action

In this section we prove the existence of reduced forms when the G-action on the zero level set of the moment map $C = \Phi^{-1}(0)$ is strict. For applications and examples of this result see Section 3.4.

The definition of strict action was briefly anticipated in Section 1.4; here it is expanded and some examples are given. Whenever a group G acts on a set X, we can consider the resulting Bourbaki map $\theta : (g, x) \mapsto (x, g(x))$ and decompose it as follows (Remark 1.1.17):

$$\begin{tikzcd}
G \times X \arrow[r, "\theta"] \arrow[d, "\mathrm{s}"'] & X \times X \\
(G \times X)/\sim \arrow[r, "\dot{\theta}"] & \theta(G \times X), \arrow[u, "\mathrm{i}"']
\end{tikzcd} \qquad \theta = \mathrm{i} \circ \dot{\theta} \circ \mathrm{s}. \qquad (3.1.1)$$

If G and X are also diffeological spaces, again by Remark 1.1.17 each node is endowed with a natural diffeological space structure, s is a subduction, i is an induction and $\dot{\theta}$ is smooth if and only if θ is smooth. By Definition 1.1.18, the map θ is *strict* if $\dot{\theta}$ is a diffeomorphism, i.e., $\dot{\theta}$ and $\dot{\theta}^{-1}$ are smooth.

Definition 3.1.2. The G-action is said to be *strict* if θ is a strict map.

Examples 3.1.3. Consider a free action. It is strict if and only if it is *principal* in the sense of [I13, 8.11], that is, the corresponding θ is an induction. Indeed, the map θ associated to a free action is an injection, and inductions coincide with strict injections [S85, 1.20] (or Remark 1.1.19). Principal actions are involved in diffeological fibre bundle theory [I13, Chapter 8] and, in particular, in the definition of principal (diffeological) bundle [I13, 8.11]. For instance:

- If G is a Lie group and H an arbitrary subgroup (hence canonically also a Lie group [B72, III.4.5]), then the left and right actions of H on G, $h(g) = hg$ and $h(g) = gh^{-1}$, are principal and hence strict [I13, 8.15].
- More generally, any free smooth action of a Lie group on a manifold is principal and hence strict, by [I85, 3.9.3].

Consider a *transitive* action. It is strict if and only if θ is a subduction. Indeed, the map θ associated to a transitive action is surjective, and subductions coincide with strict surjections [S85, 1.20] (or Remark 1.1.19). For instance:

- Any transitive smooth action of a Lie group on a manifold is strict. In this case, θ is a surjective submersion, hence a subduction [S85, 1.13].
- Examples of non strict actions can be found among actions that are neither free nor transitive: consider for instance the standard action of SO(2) on $\mathbb{R}^2$. By [I13, 1.15] the action is strict if and only if, for

any plot $P, Q : U \to \mathbb{R}^2 \times \mathbb{R}^2$, for all $r \in U$, there exist an open neighbourhood V of r and a plot $R \times S : V \to \mathrm{SO}(2) \times \mathbb{R}^2$ such that $\theta \circ (R \times S) = P \times Q_{|V}$. This implies that $S = P_{|V}$ and

$$Q(u) = R(u)(P(u)), \tag{3.1.4}$$

for all $u \in V$. However, this fails if we consider the 1-plots $P(u) = (0, \mathrm{e}^{-1/u^2})$ and $Q(u) = (0, \mathrm{sign}(u)\mathrm{e}^{-1/u^2})$, both understood to be $(0, 0)$ at $u = 0$. Indeed, by (3.1.4) and by the fact that the first components of both P and Q are zero, follows that $R(u) = \pm \mathrm{id}_{\mathbb{R}^2}$, but substituting in (3.1.4) this would give $Q(u) = \pm P(u)$, which is false independently of the choice of a sign if we take V to be a neighbourhood of 0.

Now return to the setting of symplectic reduced space and prequantum reduced space (1.4.10/1.5.8). The considered G-action is defined on a moment level $C = \Phi^{-1}(0)$, which is not necessarily a manifold, and the corresponding Bourbaki map writes $\theta : G \times C \to C \times C$.

Theorem 3.1.5. *In* (1.4.10/1.5.8), *suppose the G-action on $C = \Phi^{-1}(0)$ is strict. Then $X /\!\!/ G$ carries a reduced* 2*-form, and respectively $\tilde{X} /\!\!/ G$ carries a reduced* 1*-form.*

Proof. Let j and π be the natural inclusion and projection involved in symplectic reduction (1.4.9). The criterion of Theorem 1.2.9 rephrased in this context states that the existence of a reduced 2-form is equivalent to the following: given any two plots of C, $P : U \to C$ and $Q : U \to C$, such that $\pi \circ P = \pi \circ Q$, then

$$P^* j^* \omega = Q^* j^* \omega, \tag{3.1.6}$$

which is an equality of ordinary 2-forms on an open set U in some $\mathbb{R}^n$. To prove (3.1.6), fix P, Q, $u_0 \in U$ and vectors $\delta u_0, \delta' u_0 \in T_{u_0} U$. Note that $\pi \circ P = \pi \circ Q$ implies that $P \times Q$ is a plot of $C \times C$ taking values in $\Gamma := \theta(G \times C)$, since for any $u \in U$, $P(u)$ and $Q(u)$ differ by the action of some $g \in G$. Therefore strictness of θ, as expressed in [I13, 1.54], in this setting is equivalent to the following: for any such $P \times Q$ there exist an open neighborhood V of u_0, and a plot $R \times S : V \to G \times C$, such that $\theta \circ (R \times S) = (P \times Q)_{|V}$. This equality implies that $S = P_{|V}$ and R is a smooth map $V \to G$ such that

$$Q(u) = R(u)(P(u)), \tag{3.1.7}$$

for all $u \in V$. Moreover, by definition of the subset diffeology of C in X, the maps $j \circ P_{|V}$ and $j \circ Q_{|V}$ are smooth from $V \to X$, taking values in C. Thus the following

$$g = R(u), \qquad x = j(P(u)), \qquad y = g(x) = j(Q(u)), \qquad \mu = \Phi(x) \tag{3.1.8}$$

can be regarded as ordinary smooth functions of $u \in V$, taking values respectively in G, X, X, and $\mathfrak{g}^*$. Hence, derivatives of these functions and left translation by g^{-1}, map each $\delta u \in T_u V$ to vectors δg, δx, δy, $\delta \mu$ and an element $Z := g^{-1}\delta g \in \mathfrak{g}$. Computing again as in (2.1.19) and (2.1.20), but this time with $\delta g = gZ$, we get:

$$\begin{aligned} \delta y &= \frac{\partial[g(x)]}{\partial g}(\delta g) + \frac{\partial[g(x)]}{\partial x}(\delta x) \\ &= \frac{d}{dt}(g e^{tZ})(x)\Big|_{t=0} + g_*(\delta x) \\ &= g_*(Z(x) + \delta x) \end{aligned} \tag{3.1.9}$$

and

$$\begin{aligned} (Q^* j^* \omega)(\delta u, \delta' u) &= \omega(\delta y, \delta' y) \\ &= \omega(\delta x + Z(x), \delta' x + Z'(x)) \\ &= \omega(\delta x, \delta' x) + \langle \delta \mu, Z' \rangle - \langle \delta' \mu, Z \rangle + \langle \mu, [Z', Z] \rangle \qquad (3.1.10) \\ &= \omega(\delta x, \delta' x) \\ &= (P^* j^* \omega)(\delta u, \delta' u). \end{aligned}$$

Here the third equality is because Φ is an equivariant moment map, and the fourth because $P(U) \subset \Phi^{-1}(0)$. In particular (3.1.10) holds for our arbitrary pair $\delta u_0, \delta' u_0$ in TU. This proves (3.1.6), and therefore the reduced 2-form exists.

Regarding the prequantum case, to get the reduced 1-form on $\tilde{X}/\!\!/G$ one can proceed as before and only the prequantum version of (3.1.10) need to be proved. Using the notation of prequantum reduction and diffeological prequantum Frobenius reciprocity:

$$\begin{aligned} (Q^* j^* \varpi)(\delta u) &= \varpi(\delta \tilde{y}) \\ &= \varpi(\delta \tilde{x} + Z(\tilde{x})) \\ &= \varpi(\delta \tilde{x}) + \langle \Phi(\tilde{x}), Z \rangle \qquad (3.1.11) \\ &= \varpi(\delta \tilde{x}) \\ &= (P^* j^* \varpi)(\delta u), \qquad \square \end{aligned}$$

where the third equality is because Φ is a prequantum moment map and $\tilde{x} \in \Phi^{-1}(0)$, as this construction is analogous to the previous one.

3.2 Locally free G-action

In this section we prove the existence of reduced forms when the G-action on $\Phi^{-1}(0)$ is locally free. To prove this, we apply [H11, Theorem 3.5], which concerns the descent of differential forms to the leaf space of a foliated manifold, viewed as a diffeological space; such descending forms are called *integral invariants* [S70b, §5.20].

In more detail, when the foliation is regular and sectionable, the leaf space is a manifold, and a form φ is an integral invariant if and only if it satisfies $\iota_v \varphi = \iota_v d\varphi = 0$ for all vectors v tangent to the leaves [S70b, §5.21]. Theorem 3.5 in [H11] generalizes this result to regular foliations that may not be sectionable, by treating the leaf space as a diffeological space.

Remark 3.2.1. Recall that, as we have seen in the introduction to this chapter, when the quotient is the orbit space of a Lie group G action, the differential forms that descend to the quotient are the *basic* forms, which are G-invariant differential form whose kernel contains the tangent space to the orbits. In the following theorem the leaf space is the orbit space, thus basic forms are integral invariants, as one can check applying Cartan's magic formula for the Lie derivative.

Theorem 3.2.2. *In* (1.4.10/1.5.8), *suppose that* G *is connected and the* G*-action on* $C = \Phi^{-1}(0)$ *is locally free. Then* $X/\!\!/G$ *carries a reduced* 2*-form, and respectively* $\tilde{X}/\!\!/G$ *carries a reduced* 1*-form.*

Proof. In (1.4.10), since the G-action is locally free on C, $\mathfrak{g}_x = \{0\}$ for all $x \in C$. Then (1.4.6b) implies that 0 is a regular value of Φ, so C is a manifold and the G-orbits in C have tangent spaces $\mathfrak{g}(x)$, which are of constant dimension equal to the dimension of G. Hence they are the leaves of a *foliation* $\mathcal{F}$ of C ([B67, 9.3.3(iv)], [B06, p. 13]). Furthermore, by the property of the moment map (1.4.6a) and the fact that $\mathrm{Ker}(d\Phi(x)) = T_xC$, follow that $\mathfrak{g}(x) \subset \mathrm{Ker}(\omega_{|C})$ for all $x \in C$. Therefore, being also G-invariant, $\omega_{|C}$ is basic. Hence [H11, Theorem 3.5] applies and shows that $\omega_{|C} = j^*\omega$ is the pullback of a diffeological 2-form on $C/\mathcal{F} = C/G = X/\!\!/G$.

In (1.5.8), we wish to proceed as before, but local freeness does not immediately imply that C is a manifold. To prove this we need to resort to the properties of the moment map $\underline{\Phi}$ to which descends the prequantum one Φ. Let us delve into the details: since the G-action is locally free on C, $\mathfrak{g}_{\tilde{x}} = \{0\}$ for all $\tilde{x} \in C$. Let x denote the circle $S^1(\tilde{x})$, where $\tilde{x}$ is any element in the prequantum space. Then G_x acts on $S^1(\tilde{x})$ via a character $\chi : G_x \to S^1$ with differential $d\chi(e) = i\Phi(\tilde{x})_{|\mathfrak{g}_x}$ ([S88, 4.3d]). Thus we have $\mathfrak{g}_{\tilde{x}} = \mathrm{Ker}(d\chi(e)) \subset \mathfrak{g}_x$, which becomes an equality when $\tilde{x} \in C$, and therefore $\mathfrak{g}_x = \mathfrak{g}_{\tilde{x}} = \{0\}$. Thus, applying (1.4.6b) to $\underline{\Phi}$, we get:

$$\mathrm{Im}(d\Phi(\tilde{x})) = \mathrm{Im}(d\underline{\Phi}(x)) = \mathrm{ann}(\mathfrak{g}_x) = \mathfrak{g}^*. \tag{3.2.3}$$

So again 0 is a regular value of Φ, C is a manifold, and the G-orbits in C are the leaves of a foliation. Moreover, the definition of the prequantum moment map (1.5.5) implies that that $\varpi_{|C}$ is basic, i.e., G-invariant with $\mathfrak{g}(\tilde{x}) \subset \mathrm{Ker}(\varpi_{|C})$ for all $\tilde{x} \in C$. So again [H11, Theorem 3.5] applies and shows that $\varpi_{|C}$ is the pullback of a diffeological 2-form on the quotient. $\square$

3.3 Proper G-action

In this section we prove the existence of reduced forms when the G-action on the Hamiltonian (or prequantum) space is proper. The main tool here is Sjamaar-Lerman-Bates theory of stratified symplectic spaces [S91, B97]. We briefly introduce this topic in Subsection 3.3.1, and present new results in Subsection 3.3.2.

3.3.1 Stratified symplectic spaces

In this subsection we revise the theory of stratified symplectic space, mainly following [S91] and referring to it for a comprehensive treatment. We first introduce the definition of *decomposed space*:

Definition 3.3.1. Let X be a Hausdorff, paracompact topological space and $(I, \leq)$ a partially ordered set of indexes. An *I-decomposition* of X is a locally finite collection of disjoint, locally closed manifolds S_i, one for each $i \in I$, called *pieces*, that fulfill:

i) $X = \bigsqcup_{i \in I} S_i$,

ii) $S_i \cap \overline{S}_j \neq \emptyset \iff S_i \subseteq \overline{S}_j \iff i \leq j$,

where the bar denotes the topological closure into X. Moreover, if $i \leq j$ one writes $S_i \leq S_j$ and, if $S_i \neq S_j$, then $S_i < S_j$. The space X is called *I-decomposed space.*

Definition 3.3.2. Let X be an I-decomposed space. The *depth of a piece* S is:

$$\mathrm{depth}_X S := \sup\{n : \text{exist pieces s.t. } S = S^0 < S^1 < ... < S^n\}. \tag{3.3.3}$$

Then we can also define the *depth of X*:

$$\mathrm{depth}\, X := \sup_{i \in I} \mathrm{depth}_X S_i. \tag{3.3.4}$$

Remark 3.3.5. The depth of a piece is bounded by its codimension. As only finite-dimensional pieces will be considered, the depth will always be finite.

An important non-trivial example of decomposed space is the *cone over a manifold*:

Definition 3.3.6. Let X be a manifold. The *open cone over* X, denoted by $\mathring{C}X$, is obtained by quotienting $X \times [0, \infty)$ with respect to the equivalence relation $(x, 0) \sim (y, 0)$, which gives rise to the vertex.

Remark 3.3.7. A cone over a manifold X can be regarded as a decomposed space with two pieces: $X \times (0, \infty)$ and the vertex. If X is a decomposed space, the cone over X can be defined as before, and its depth is:

$$\operatorname{depth} \mathring{C}X = \operatorname{depth} X + 1. \tag{3.3.8}$$

Now we can define *stratified spaces*:

Definition 3.3.9. A decomposed space X is called *stratified space* if the pieces of X, called *strata*, satisfy the following condition: given a point x in a piece S, there exist an open neighbourhood U of x in X, a compact stratified space L and a homeomorphism

$$\varphi : U \to (U \cap S) \times \mathring{C}L, \tag{3.3.10}$$

that maps strata into strata. The stratified space L is called *link* of x.

The definition of *smooth stratified space* can be obtained from that of stratified space by replacing "homeomorphism" with "diffeomorphism" and "compact stratified space" with "compact smooth stratified space".

Remark 3.3.11. The latter definition of stratified space is that of [S91, G80]; it is possible to give it recursively by finiteness of depth, combined with the fact that L has a smaller depth than S. More general definitions, together with a full account of the theory of stratified spaces can be found in [P01, F20b].

The stratified spaces involved in Sjamaar-Lerman-Bates theory arise from proper Lie group action on manifolds. Consider a group G acting properly on a manifold X, let H be a closed subgroup of G and define the *orbit type* as

$$(H) := \{L \subseteq G \,:\, L = gHg^{-1},\, g \in G\}. \tag{3.3.12}$$

The set of conjugacy classes of closed subgroups of G admits a partial order by defining $(K) \leq (H)$ if and only if H is conjugate to a subgroup of K. The subgroup H induces the *orbit type submanifold*

$$X_{(H)} := \{x \in X \,:\, G_x \in (H)\}, \tag{3.3.13}$$

composed of connected components that may have different dimension (Remark 3.3.15). The collection of the $X_{(H)}$, as H ranges over the closed subgroups of G, partitions X and yields a smooth stratification of X, whose strata are the connected components of the orbit type submanifolds. Likewise, their projections partition the orbit space X/G and yield a (Whitney) smooth stratification thereof. This picture admits a Hamiltonian reformulation: let (X, ω, Φ) be a Hamiltonian G-space, with proper G-action and equivariant moment map Φ. Set $C := \Phi^{-1}(0)$ and define $C_{(H)} := C \cap X_{(H)}$, with quotient $C_{(H)}/G$. The works [S91, B97, O04] (first for G compact, and then for proper G action) showed that the orbit type submanifolds $C_{(H)}$ and their quotients $C_{(H)}/G$ yield a smooth stratification of C and of the reduced space $C/G = X/\!\!/G$, respectively. As before, the strata are given by the connected components of $C_{(H)}$ and their quotients.

Moreover, the stratification of $X/\!\!/G$ is a stratification into symplectic manifolds: the restriction of the symplectic form ω to any orbit type submanifold descends to a symplectic form on the quotient. Explicitly,

$$(j|_{C_{(H)}})^*\omega = (\pi|_{C_{(H)}})^*\omega_{(H)}, \tag{3.3.14}$$

where $\omega_{(H)}$ is a symplectic 2-form on $C_{(H)}/G$, and $j|_{C_{(H)}} : C_{(H)} \to X$ and $\pi|_{C_{(H)}} : C_{(H)} \to C_{(H)}/G$ (same notation used below) denote the inclusion and quotient maps, respectively.

Remark 3.3.15. Following [S91, 1.3], we regard orbit type submanifolds themselves as symplectic manifolds, tacitly allowing for the possibility that a manifold may have connected components of different dimensions. Other references (e.g. [O04, M07]) instead formulate the results in terms of strata, i.e. the connected components of the orbit type submanifolds.

3.3.2 Reduced forms for proper action

Now we state and prove the theorem regarding the existence of reduced forms for proper actions:

Theorem 3.3.16. *In* (1.4.10/1.5.8), *suppose the G-action on X, respectively on $\tilde{X}$, is proper. Then $X/\!\!/G$ carries a reduced* 2*-form, respectively $\tilde{X}/\!\!/G$ carries a reduced* 1*-form.*

Proof (symplectic case). Let j, π, and $\Phi^{-1}(0) = C$ be as in (1.4.9). In order to apply Souriau's criterion we must prove

$$P^*j^*\omega = Q^*j^*\omega \tag{3.3.17}$$

for any two plots $P, Q : U \to C$ with $\pi \circ P = \pi \circ Q$. To this end, consider the commutative diagram

$$\begin{tikzcd}
 & U \arrow[r, "P"] & C \arrow[r, hook, "j"] & X \\
V_t \arrow[r, hook] & U_t \arrow[u, hook] \arrow[r, "P_{|U_t}"] & C_t \arrow[u, hook] \arrow[r, hook] & X_t . \arrow[u, hook]
\end{tikzcd} \tag{3.3.18}$$

Here t denotes an orbit type, that is a conjugacy class of closed subgroups of G, and $X_t := \{x \in X : G_x \in t\}$ is the resulting orbit type piece; C_t is the preimage of X_t under j, and U_t is the preimage of C_t under P, that is: $C_t = C \cap X_t$ and $U_t = P^{-1}(C_t)$. Moreover, V_t is defined as $U_t \cap \operatorname{int}(\operatorname{cl}(U_t))$, where closure and interior are taken in the euclidean topology of U. Note that $\pi \circ P = \pi \circ Q$ implies that U_t is also $Q^{-1}(C_t)$. Indeed, for any fixed $u \in U_t$, there exists $g \in G$ such that $P(u) = g(Q(u))$ and then $G_{P(u)} = gG_{Q(u)}g^{-1}$. Thus the stabilizers of $P(u)$ and $Q(u)$ are in the same conjugacy class for any $u \in U_t$, and therefore the images of the plots lie in the same orbit type piece, giving $P^{-1}(C_t) = Q^{-1}(C_t)$.

By the theory exposed above (see also [B82b, IX.9.4, Theorem 2]), orbit type partitions X into locally finitely many such X_t, all of which are G-invariant embedded submanifolds, and hence locally closed, which are allowed to have connected components of different dimensions (Remark 3.3.15 or [B67, 5.1.8]). By local finiteness of the cover, there exist an open neighbourhood W of $j(P(u_0))$ which intersects only a finite number of X_t. Thus we can shrink U to $P^{-1}(W)$, so that we can assume without loss of generality that only finitely many U_t are nonempty.

Moreover, by [S91, Theorem 2.1] for compact G and [B97] or [O04, Theorem 8.1.1] for proper G-action, the C_t are also G-invariant embedded submanifolds of X, and the quotients C_t/G are symplectic manifolds, carrying 2-forms ω_t characterized by the following:

$$(j_{|C_t})^*\omega = (\pi_{|C_t})^*\omega_t \,, \tag{3.3.19}$$

where $\pi_{|C_t} : C_t \to C_t/G$ (and not on C/G, with a slight abuse of notation). Now if the U_t were open, the theorem would be proved. Indeed, $P_{|U_t}$ and $Q_{|U_t}$ would then be plots of C with values in C_t, i.e., plots of the subset diffeology of C_t in C, or equivalently in X [I13, 1.35]. Therefore they can be regarded as ordinary smooth maps from U_t to the manifold C_t [I13, 4.1, 4.3]. Then, pulling back (3.3.19) by $P_{|U_t}$ we get:

$$P^*_{|U_t}(j_{|C_t})^*\omega = P^*_{|U_t}(\pi_{|C_t})^*\omega_t$$

$$
\begin{aligned}
&= (\pi_{|C_t} \circ P_{|U_t})^* \omega_t \\
&= (\pi_{|C_t} \circ Q_{|U_t})^* \omega_t \\
&= Q_{|U_t}^* (\pi_{|C_t})^* \omega_t \\
&= Q_{|U_t}^* (j_{|C_t})^* \omega, \qquad (3.3.20)
\end{aligned}
$$

where the third equality follows by $\pi \circ P = \pi \circ Q$ and the last one by pulling back (3.3.19) by $Q_{|U_t}$. This implies that

$$(j \circ P)^* \omega \quad \text{and} \quad (j \circ Q)^* \omega \qquad (3.3.21)$$

coincide on every U_t, and thus (3.1.6) holds on the union U of the U_t.

However, the U_t are not necessarily open. We will consider the V_t instead, which have the following properties:

$$\text{(a) each } V_t \text{ is open in } U, \qquad \text{(b) their union is dense in } U. \qquad (3.3.22)$$

To see (b), note that the closures $\mathrm{cl}(U_t)$ cover U and therefore [W12, Lemma 3.11] implies that $\mathrm{int}(\mathrm{cl}(U_t))$ have dense union in U. So it suffices to show that V_t is dense in $\mathrm{int}(\mathrm{cl}(U_t))$, i.e., any nonempty open $O \subset \mathrm{int}(\mathrm{cl}(U_t))$ satisfies $O \cap V_t \neq \emptyset$. This is true because $O \cap V_t$ is just $O \cap U_t$ by definition of V_t, and it is nonempty by openness of O in $\mathrm{cl}(U_t)$ and density of U_t in $\mathrm{cl}(U_t)$.

To see (a), note that U_t is locally closed, being preimage of the locally closed set X_t by the continuous map $j \circ P$ ([B60, I.3.3]); this means that U_t is open in $\mathrm{cl}(U_t)$, which is endowed with its subspace topology σ inside U. It follows that V_t is open in $\mathrm{int}(\mathrm{cl}(U_t))$, endowed with its subspace topology τ inside $\mathrm{cl}(U_t)$. But τ is also the subspace topology directly inside U [B60, I.3.1]; so, being $\mathrm{int}(\mathrm{cl}(U_t))$ open in euclidean topology, τ-open sets are intersections of euclidean-open sets, and therefore are open, thus proving (a).

Now we can repeat the argument that led to (3.3.21) with V_t in place of U_t and get that $(j \circ P)^* \omega$ and $(j \circ Q)^* \omega$ coincide on the open dense union of the V_t, hence everywhere on U by continuity. So (3.1.6) is proved, and so the existence of the 2-form. □

Proof (prequantum case). Let j, π, and $\Phi^{-1}(0) = C$ be as in (1.5.7), and for simplicity, rename $\tilde{X}$ as just X (i.e., drop all tildes in §3). We must prove $P^* j^* \varpi = Q^* j^* \varpi$ for any two plots $P, Q : U \to C$ with $\pi \circ P = \pi \circ Q$. To this end, let $\bigsqcup_t X_t$ and $\bigsqcup_t C_t$ be the orbit type partitions of X and C, so that we again have a diagram (3.3.18). Following the suggestion in [L01a, 2.17], we consider the symplectization of (X, ϖ). It is defined as the Hamiltonian G-space $(\check{X}, \check{\omega}, \check{\Phi})$, where $\check{X} = \mathbb{R} \times X$, it is endowed with a (again proper) G-action $g(s, x) = (s, g(x))$, 2-form $\check{\omega} = d(\mathrm{e}^s \varpi)$, and resulting moment map

$\check{\Phi}(s,x) = \mathrm{e}^s\Phi(x)$. The zero level of the moment map and orbit type pieces are just the product of $\mathbb{R}$ with those of X:

$$\check{C} = \mathbb{R}\times C, \qquad \check{X}_t = \mathbb{R}\times X_t, \qquad \check{C}_t = \mathbb{R}\times C_t. \tag{3.3.23}$$

Rephrasing the theory of stratified symplectic spaces in this context, the $\check{X}_t$ and $\check{C}_t$ are G-invariant embedded submanifolds of $\check{X}$, and the reduced pieces $\check{C}_t/G$ are symplectic manifolds, carrying 2-forms $\check{\omega}_t$ characterized by

$$(\mathrm{id}\times j_{|C_t})^*\check{\omega} = (\mathrm{id}\times \pi_{|C_t})^*\check{\omega}_t\,. \tag{3.3.24}$$

Therefore, X_t, C_t and C_t/G are manifolds, and we claim that $(j_{|C_t})^*\varpi$ descends to C_t/G. To prove this, note that ϖ can be recovered from $\check{\omega}$ as follows:

$$\varpi = I^*\iota_R\check{\omega} \tag{3.3.25}$$

where $I : X \to \check{X}$ denotes the embedding $x \mapsto (0,x)$, and ι_R the interior product with the constant vector field $R(s,x) = (1,0)$. These definitions yield the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "I"] & \check{X} = \mathbb{R}\times X \arrow[r, "R"] & T\check{X} \\
C_t \arrow[u, hook, "j_{|C_t}"] \arrow[r, "I_t"] \arrow[d, "\pi_{|C_t}"'] & \check{C}_t = \mathbb{R}\times C_t \arrow[u, hook, "\mathrm{id}\times j_{|C_t}"'] \arrow[r, "R_t"] \arrow[d, "\mathrm{id}\times\pi_{|C_t}"] & T\check{C}_t \arrow[u, hook, "(\mathrm{id}\times j_{|C_t})_*"'] \arrow[d, "(\mathrm{id}\times\pi_{|C_t})_*"] \\
C_t/G \arrow[r, "J_t"] & \check{C}_t/G = \mathbb{R}\times(C_t/G) \arrow[r, "S_t"] & T(\check{C}_t/G),
\end{tikzcd} \tag{3.3.26}$$

where I_t, J_t denote again the embeddings $(0,\cdot)$, and R_t, S_t the vector fields with constant value $(1,0)$. Note that commutativity of the two squares on the right means that R, R_t, S_t are related, where we call two vector fields $R : A \to TA$ and $S : B \to TB$ related by a smooth map $F : A \to B$ if $F_*(R(a)) = S(F(a))$; this also implies $\iota_R F^* = F^*\iota_S$. Now we can prove the claim:

$$\begin{aligned}
(j_{|C_t})^*\varpi &= (j_{|C_t})^*I^*\iota_R\check{\omega} && \text{by (3.3.25)}\\
&= I_t^*(\mathrm{id}\times j_{|C_t})^*\iota_R\check{\omega} && \text{by commutativity of (3.3.26)}\\
&= I_t^*\iota_{R_t}(\mathrm{id}\times j_{|C_t})^*\check{\omega} && \text{by relatedness of } R, R_t\\
&= I_t^*\iota_{R_t}(\mathrm{id}\times \pi_{|C_t})^*\check{\omega}_t && \text{by (3.3.24)}\\
&= I_t^*(\mathrm{id}\times \pi_{|C_t})^*\iota_{S_t}\check{\omega}_t && \text{by relatedness of } R_t, S_t\\
&= (\pi_{|C_t})^*J_t^*\iota_{S_t}\check{\omega}_t && \text{by commutativity of (3.3.26)}\\
&= (\pi_{|C_t})^*\varpi_t && \text{where we define } \varpi_t = J_t^*\iota_{S_t}\check{\omega}_t\,.
\end{aligned} \tag{3.3.27}$$

This proves our claim that $(j_{|C_t})^*\varpi$ is the pull-back of a unique and ordinary 1-form on the manifold C_t/G. Now we can proceed as before, with (3.3.27) in place of (3.3.19): the same argument of the symplectic proof shows that $P^*j^*\varpi = Q^*j^*\varpi$, proving the Theorem. □

Moreover, the reduced (global and diffeological) form restricts to the Sjamaar-Lerman-Bates 2-form on each reduced piece:

Corollary 3.3.28. *In Theorem 3.3.16, the reduced* 2*-form* $\omega_{X/\!\!/G}$ *on* $X/\!\!/G = C/G$ *restricts to the Sjamaar-Lerman-Bates* 2*-form* ω_t (3.3.19) *on each reduced piece* C_t/G*. Likewise, the reduced* 1*-form* $\varpi_{\tilde{X}/\!\!/G}$ *induces* ϖ_t (3.3.27) *on each reduced piece.*

Proof. Recall that in (3.3.19), $j_{|C_t}$ is regarded as a map $C_t \to X$, while slightly abusively, $\pi_{|C_t}$ as a map $C_t \to C_t/G$. With this notation, we have the commutative diagram:

$$\begin{tikzcd}
C/G & C \arrow[l, "\pi"'] \arrow[r, hook, "j"] & X \\
C_t/G \arrow[u, hook, "i_t"] & C_t \arrow[l, "\pi_{|C_t}"'] \arrow[u, hook, "j_t"] \arrow[ur, "j_{|C_t}"']
\end{tikzcd} \tag{3.3.29}$$

where the inductions i_t and j_t are newly named. We must prove $\omega_t = i_t^*\omega_{X/\!\!/G}$. We have:

$$\begin{aligned}
(\pi_{|C_t})^*\omega_t &= (j_{|C_t})^*\omega && \text{by (3.3.19)} \\
&= j_t^* j^*\omega && \text{by commutativity of (3.3.29)} \\
&= j_t^*\pi^*\omega_{X/\!\!/G} && \text{by definition (1.4.11)} \\
&= (\pi_{|C_t})^* i_t^*\omega_{X/\!\!/G} && \text{by commutativity of (3.3.29).}
\end{aligned} \tag{3.3.30}$$

As $\pi_{|C_t}$ is a subduction (Proposition 1.1.20), $(\pi_{|C_t})^*$ is injective [I13, 6.39], so (3.3.30) implies the desired equality. The proof that $\varpi_t = i_t^*\varpi_{\tilde{X}/\!\!/G}$ is just the same, with (1.4.11) and (3.3.19) replaced by (1.5.9) and (3.3.27). □

3.4 Application: Induction from non-closed subgroups

In this section we apply the existence result under strict action to induced Hamiltonian spaces $\mathrm{Ind}_H^G Y$, dropping the assumption that H is closed in G.

Consider G, a Lie group, and H a non-closed subgroup of G, equipped with its canonical Lie group structure ([B72, III.4.5]). In general, the quotient

G/H is not a manifold; a classic example is the irrational torus $\mathbb{T}_\alpha$, which arises from $G = S^1 \times S^1$ and $H = \mathcal{S}_\alpha$, an irrational winding, or equivalently from $G = \mathbb{R}$ and $H = \mathbb{Z} + \alpha\mathbb{Z}$ ([I13, Exercise 31]).

Nevertheless, the constructions of the induced Hamiltonian space (1.6.17) and the induced prequantum space (1.6.33) remain valid for any Hamiltonian or prequantum H-space Y, even when H is not closed. The resulting space is not a manifold, but the following result hold.

Corollary 3.4.1. *In* (1.6.16/1.6.32), $\mathrm{Ind}_H^G Y$ *carries a reduced* 2*-form, and respectively a* 1*-form, even when* H *is not closed.*

Proof. The cotangent space T^*G is a Lie group and $H \subset G \subset T^*G$ is a subgroup acting on the right on T^*G as follows: $h(p) = ph^{-1}$. By the first item in (3.1.3), this H-action on T^*G is principal, and by (a) of Lemma 3.4.3 below, the H-action $h(p, y) = (ph^{-1}, h(y))$ on $T^*G \times Y$ is also principal, with Y as in (1.6.16/1.6.32). Moreover, by (b) of Lemma 3.4.3 below, the latter H-action restricted to $\psi^{-1}(0) \subset T^*G \times Y$, again with Y as in (1.6.16/1.6.32), is still principal, hence strict. Therefore, the corollary is proved by existence of reduced forms when the action is strict (Theorem 3.1.5). □

Remark 3.4.2. This 2-form, together with the residual G-action and moment map an induced space also carries, make $\mathrm{Ind}_H^G Y$ a parasymplectic Hamiltonian G-space in the sense of [I10, I13, I16]. Indeed, not only it carries a closed 2-form (Remark 1.4.12), but also a moment map.

Now let us prove the lemma used in the proof of the previous corollary.

Lemma 3.4.3. *Let* G *be a diffeological group acting smoothly on diffeological spaces* X_1 *and* X_2*.*

a) *If the* G*-action on* X_1 *is principal, then so is the diagonal* G*-action on* $X = X_1 \times X_2$*.*
b) *If* X_2 *is a* G*-invariant subset of* X_1 *(with subset diffeology) and the* G*-action on* X_1 *is principal or strict, then so is the restricted* G*-action on* X_2*.*

Proof. (a) First of all, G acting freely on X_1 implies that G acts freely on X. Then, to show principality, we need only show that the G-action on X is strict. To prove this, let $P \times Q : U \to X \times X$ be a plot taking values in the graph $\Gamma = \{(x, y) : G(x) = G(y)\}$. Since G acts freely on X, there is a unique $R : U \to G$ such that $Q(u) = R(u)(P(u))$. Note that such R need not be smooth in general (see e.g. [K16, Example 5.2]). Writing $P \times Q = P_1 \times P_2 \times Q_1 \times Q_2$, we see that this is also the unique R such that $Q_1(u) = R(u)(P_1(u))$. Now, since G acts strictly on X_1, this R is smooth in an open neighbourhood V of any $u_0 \in U$ by [I13, 1.54]. So we have (3.1.7), i.e. $Q(u) = R(u)(P(u))$ for any $u \in V$, which, again by [I13, 1.54], is equivalent to strictness of G-action on X.

(b) Let $P_2 \times Q_2 : U \to X_2 \times X_2$ be a plot taking values in the graph $\Gamma_2 = \{(x_2, y_2) : G(x_2) = G(y_2)\}$, and $u_0 \in U$. Composing P_2 and Q_2 with the inclusion $X_2 \hookrightarrow X_1$, we get a plot $P_1 \times Q_1 : U \to X_1 \times X_1$ taking values in $\Gamma_1 = \{(x_1, y_1) : G(x_1) = G(y_1)\}$. Since the G-action on X_1 is strict, there are an open neighbourhood V of u_0, and a plot $R : V \to G$, such that $Q_1(u) = R(u)(P_1(u))$ for all $u \in V$. This implies $Q_2(u) = R(u)(P_2(u))$, i.e. we have (3.1.7) for X_2, proving (b) for strict actions. For principal actions, note that G acting freely on X_1 implies freeness of the action on its subset X_2. $\square$

Corollary 3.4.4. *The reciprocity Theorems 2.1.1 and 2.2.1 remain valid when H is not assumed closed.*

Proof. Dropping the closeness assumption implies that $\operatorname{Ind}_H^G Y$ and $M/\!\!/H$ are not necessarily manifolds anymore. However, they still carry the reduced forms by Corollary 3.4.1, together with residual G-actions and moment maps (3.4.2). Thus the proofs work as before. $\square$

Example: $\mathbf{Ind_H^G\{0\} = T^*(G/H)}$ 3.4.5. If Y is the trivial coadjoint orbit of H, the induced space (1.6.17) is just the reduction of T^*G by the cotangent lift of the right action of H, that is:

$$\operatorname{Ind}_H^G \{0\} = (T^*G)/\!\!/H. \tag{3.4.6}$$

If H is closed in G the Kummer-Marsden-Satzer "cotangent bundle reduction" [M07, Theorem 2.2.2] proves that (3.4.6) is $T^*(G/H)$, and under that identification, its reduced 2-form is just the canonical cotangent bundle 2-form, and thus it also inherits a potential 1-form:

$$\omega_{(T^*G)/\!\!/H} = d\varpi_{T^*(G/H)}. \tag{3.4.7}$$

Now let us consider the non closed case. In the proper group action setting, $T^*G/\!\!/H$ and (3.4.7) were proposed to be taken as the definition of $T^*(G/H)$ and its 2-form in [L93b, 3.3], where G/H is considered as a stratified space. Moreover, it was conjectured in [L93b, 3.7], that this definition does not depend on the way G/H is written as a quotient. Here, in contrast, we have the benefit of the intrinsic notion of the cotangent space T^*X of any diffeological space, which comes endowed with a canonical 2-form denoted $d\,\mathrm{Liouv}$ and a Hamiltonian action of $\mathrm{Diff}(X)$, presented in Section 1.2 (see also [I10, I13]). So the conjecture becomes the following, which we prove when H is dense.

Theorem 3.4.8. *Let G be a Lie group and H any dense subgroup. Then the induced space* (3.4.6) *with its* 2*-form* (3.4.1) *and the cotangent space* $T^*(G/H)$ *of* [I13] *with its* 2*-form* $d\,\mathrm{Liouv}$ *are isomorphic as parasymplectic Hamiltonian G-spaces.*

Proof. For this proof, identify $\mathfrak{g}^*$ with right-invariant 1-forms on G by defining:

$$\mu(\delta q) := \langle \mu, \delta q.q^{-1}\rangle, \tag{3.4.9}$$

where $\mu \in \mathfrak{g}^*, \delta q \in T_qG$. This satisfies the condition $R_g^*\mu = \mu$ for all $g \in G$, where $R_g(q) = qg$, and any right-invariant 1-form can be obtained in this way. Next, recall that H is canonically a Lie group (even if it is not closed) with Lie algebra $\mathfrak{h} = \{Z \in \mathfrak{g} : \mathrm{e}^{tZ} \in H \text{ for all } t \in \mathbb{R}\}$ (see [B72, III.4.5] or [H12, 9.6.13]). A key property in our case, implied by the density assumption, is that G normalizes $\mathfrak{h}$, that is:

$$g\mathfrak{h}g^{-1} = \mathfrak{h} \quad \text{for all } g \in G. \tag{3.4.10}$$

Indeed the normalizer $N_G(\mathfrak{h})$ is always a closed subgroup of G containing H [B72, III.9.4, Prop. 10], so it must be G by our density assumption. Now let $X = G/H$, and Π denote the projection $G \to G/H$, $\Pi(q) = qH$. Now we retrace the constructions regarding the cotangent space in diffeology, introduced in Section 1.2, and apply them in this setting.

Step 1 in Iglesias-Zemmour's definition of T^*X is to form the space $\Omega^1(X)$ of all diffeological 1-forms on X (see Definition 1.2.1). In our case it is finite-dimensional, which is $\dim(\mathfrak{g}/\mathfrak{h})$, as we are going to prove. First we have:

$$\alpha \in \Omega^1(X) \qquad \Rightarrow \qquad \Pi^*\alpha = \mu \quad \text{where } \mu \in \mathrm{ann}(\mathfrak{h}) \subset \mathfrak{g}^* \tag{3.4.11}$$

(notation 1.4.6b, 3.4.9). Indeed, the relation $\Pi \circ R_h = \Pi$ implies $R_h^*\Pi^*\alpha = \Pi^*\alpha$ for all $h \in H$, and since H is dense, the same follows for all $g \in G$: thus $\Pi^*\alpha$ is right-invariant and hence of the form (3.4.9). To see that μ, obtained in (3.4.11) annihilates $\mathfrak{h}$, let $Z \in \mathfrak{h}$ and put $Q(t) = \mathrm{e}^{tZ}$. Then $P := \Pi \circ Q : \mathbb{R} \to G/H$ is the constant plot $t \mapsto eH$, and by [I13, Exercise 96] we have $Q^*\mu = P^*\alpha = 0$. On the other hand, expliciting the left hand side of the previous relation, we find $Q^*\mu = \langle \mu, Z\rangle dt$, thereby concluding that $\langle \mu, Z\rangle = 0$ for all $Z \in \mathfrak{h}$.

Next we claim that, conversely to (3.4.11), every $\mu \in \mathrm{ann}(\mathfrak{h})$ is the pullback of some $\alpha \in \Omega^1(X)$. Indeed, by the same criterion already applied in (3.1.6), this will follow if we show that for any two plots $P : U \to G$ and $Q : U \to G$ with $\Pi \circ P = \Pi \circ Q$, we have $P^*\mu = Q^*\mu$. The relation $\Pi \circ P = \Pi \circ Q$ means that $R(u) := P(u)^{-1}Q(u)$ is smooth, so it defines a plot $R : U \to H$. Now $(g, gh, h) := (P(u), Q(u), R(u))$ are ordinary smooth functions of u, and given $\delta u \in T_u U$ we may compute

$$\begin{aligned}(Q^*\mu)(\delta u) &= \mu(\delta[gh])\\ &= \mu(\delta g.h + g\delta h)\\ &= \langle \mu, [\delta g.h + g\delta h](gh)^{-1}\rangle \qquad \text{by (3.4.9)}\\ &= \langle \mu, \delta g.g^{-1} + g\delta h.h^{-1}g^{-1}\rangle\\ &= \langle \mu, \delta g.g^{-1}\rangle \qquad \text{by (3.4.10, 3.4.11)}\\ &= (P^*\mu)(\delta u).\end{aligned} \tag{3.4.12}$$

This establishes our claim that Π^* is onto $\mathrm{ann}(\mathfrak{h})$. Since Π is a subduction, Π^* is also injective [I13, 6.39], hence we have a linear bijection $\Omega^1(X) \to \mathrm{ann}(\mathfrak{h})$. Writing Π_* for its inverse, we have proved the following:

Proposition 3.4.13. *If H is dense in G, then $\Omega^1(G/H) = \Pi_*(\mathrm{ann}(\mathfrak{h}))$.* □

Proof of Theorem 3.4.8 (continued). In Step 2 of his construction, [I13] defines the cotangent space at $x \in X$ as the quotient

$$T^*_x(X) = \Omega^1(X)/\{\text{1-forms vanishing at } x\}, \tag{3.4.14}$$

where $\alpha \in \Omega^1(X)$ is said to vanish at x if, whenever $P : U \to X$ is a plot with $0 \in U$ and $P(0) = x$, the ordinary 1-form $P^*\alpha \in \Omega^1(U)$ vanishes at 0 (see Definition 1.2.10). In our case (3.4.11), we claim that $\alpha = \Pi_*(\mu)$ vanishes nowhere unless $\mu = 0$. Indeed, given $x = qH \in X$ and $Z \in \mathfrak{g}$, set $Q(t) = \mathrm{e}^{tZ}q$. Then the plot $P = \Pi \circ Q : \mathbb{R} \to X$ satisfies $P(0) = x$ and $P^*\alpha = \langle \mu, Z\rangle dt$, since $P^*\alpha = Q^*\mu = \langle \mu, Z\rangle dt$. The only way this can vanish at $t = 0$ for all $Z \in \mathfrak{g}$ is if $\mu = 0$, so we have our claim. Note that even though this Q is different from the one involved in Step 1, by right invariance of μ, $Q^*\mu$ coincides with the previous one.

Step 3 of [I13] defines T^*X as $\{(x, a) : x \in X, a \in T^*_x(X)\}$ (see Definition 1.2.12). In our setting, (3.4.14) has no denominator, so this is:

$$\begin{aligned}T^*(X) &= X \times \Omega^1(X)\\ &= (G/H) \times \Pi_*(\mathrm{ann}(\mathfrak{h})),\end{aligned} \tag{3.4.15}$$

by Proposition 3.4.13.

Step 4 of [I13] defines the Liouville 1-form on $T^*(X)$; recall that a differential form on a diffeological space is defined by specifying its pullback by every plot of the space. Now the plots of (3.4.15) have the form $P \times A$ for plots $P : U \to X$ and $A : U \to \Omega^1(X)$, so the definition of Liouv is the following:

$$((P \times A)^* \,\mathrm{Liouv})(\delta u) = P^*(A(u))(\delta u), \tag{3.4.16}$$

where $\delta u \in T_u U$. As we have seen in Definition 1.2.16, this defines a tautological 1-form on $X \times \Omega^1(X)$, which then descends as Liouv on the quotient $T^*(X)$. However, as we proved in (3.4.14), here there is no division, so Liouv = Taut.

Step 5 of [I13] (see Definition 1.2.19) defines a Liouv-preserving action of G on (3.4.15) by $g(x, \alpha) = (g(x), g_*(\alpha))$, and here the action is by left translations. Also note that Π and the G-action commute, so $\Pi_* g_* = g_* \Pi_*$. Moreover, since $\Pi^*\alpha = \mu$ is right invariant, we may as well push it forward by $g(\,\cdot\,)g^{-1}$, thereby obtaining:

$$y = (\Pi(q), \Pi_*(\mu)) \quad \Rightarrow \quad g(y) = (\Pi(gq), \Pi_*(g\mu g^{-1})). \tag{3.4.17}$$

Finally, Step 6 of [I13] (see Definition 1.2.21) defines the *moment map* $\Phi : T^*(X) \to \mathfrak{g}^*$ by the following formula, where $y \in T^*(X)$ and $\hat{y} : G \to T^*(X)$ is the map $g \mapsto g(y)$:

$$\Phi(y) = \text{value at } e \text{ of the 1-form } \hat{y}^* \operatorname{Liouv} \in \Omega^1(G). \tag{3.4.18}$$

Now we can compare the cotangent space to G/H (3.4.15–3.4.18) with the reduced space of T^*G (3.4.6, 3.4.1–3.4.2). Indeed, the expression of the moment map (1.6.15) implies that $\operatorname{Ind}_H^G \{0\}$ is the quotient of

$$\begin{aligned} \psi^{-1}(0) &= \{p = \mu q \,:\, q \in G, \mu \in q\operatorname{ann}(\mathfrak{h})q^{-1}\} \\ &= \{p = \mu q \,:\, q \in G, \mu \in \operatorname{ann}(\mathfrak{h})\} \\ &\cong G \times \operatorname{ann}(\mathfrak{h}) \end{aligned} \tag{3.4.19}$$

(notation 1.3.30) by the H-action; here the second equality follows by (3.4.10). Note that under this last identification, the $G \times H$-action on (1.6.17) and the inclusion $j : \psi^{-1}(0) \hookrightarrow T^*G$ become

$$(g, h)(q, \mu) = (gqh^{-1}, g\mu g^{-1}) \qquad \text{and} \qquad j(q, \mu) = \mu q. \tag{3.4.20}$$

In more detail, the identification in (3.4.19) sends $p \mapsto (q, \mu)$, where $p \in T_q^*G$ and $\mu = pq^{-1}$. Thus $(g, h)p = gph^{-1} \in T^*_{gqh^{-1}}G$ and it is mapped to $(gqh^{-1}, g\mu g^{-1})$, while the second equation of (3.4.20) is clear. Now $F(q, \mu) := (\Pi(q), \Pi_*(\mu))$ defines a map from (3.4.19) to (3.4.15) which is G-equivariant with respect to (3.4.20) and (3.4.17), and, by Proposition 3.4.13, descends to a diffeomorphism $\overline{F} : \psi^{-1}(0)/H \to T^*(G/H)$. There remains to see that the latter defines an isomorphism between parasymplectic diffeological spaces, that is:

$$\text{(a)} \quad F^* \operatorname{Liouv} = j^* \varpi_{T^*G} \qquad \text{and} \qquad \text{(b)} \quad \Phi \circ F = \phi \circ j \tag{3.4.21}$$

with ϕ as in (1.6.15). Let us detail why it is enough to prove (a) and (b) in order to prove the isomorphism. The setting and the notation are summarized in the following:

$$\begin{tikzcd}
T^*G & \psi^{-1}(0) \cong G \times \operatorname{ann}(\mathfrak{h}) \arrow[l, hook', "j"'] \arrow[r, "F"] \arrow[d, "\pi"] & T^*(G/H) \\
& \psi^{-1}(0)/H = T^*G/\!\!/H \arrow[ur, "\overline{F}"] &
\end{tikzcd} \tag{3.4.22}$$

Regarding (a), we must prove that it implies that $\overline{F}$ preserves the forms (3.4.1) and Liouv. Indeed, we can write $F = \overline{F} \circ \pi$ and then the left hand side of (3.4.21a) writes $\pi^* \overline{F}^* \operatorname{Liouv}$, while the right hand side writes $\pi^* \varpi_{T^*G/\!\!/H}$, by Corollary 3.4.1. Thus, since the pullback by the subduction π is injective ([I13, 6.39]), if (a) holds, the diffeomorphism preserves the forms. Regarding (b), we must prove that it implies that $\overline{F}$ preserves the moment maps Φ and $\Phi_{T^*G/\!\!/H}$, i.e. the moment map for the residual action of G on $T^*G/\!\!/H$. Indeed, the latter is defined by the following $\Phi_{T^*G/\!\!/H}(\pi(q,\mu)) = \phi \circ j(q,\mu)$, for $(q,\mu) \in \psi^{-1}(0)$; thus (3.4.21b) writes $\Phi \circ \overline{F} \circ \pi = \Phi_{T^*G/\!\!/H} \circ \pi$, so, as before, we get $\Phi \circ \overline{F} = \Phi_{T^*G/\!\!/H}$ as desired.

Now let us prove (3.4.21). The first relation means that $(Q \times M)^* F^* \operatorname{Liouv} = (Q \times M)^* j^* \varpi_{T^*G}$ for all plots $Q \times M : U \to G \times \operatorname{ann}(\mathfrak{h})$. To prove this, note that the left-hand side is Liouv's pullback by $F \circ (Q \times M) = (\Pi \circ Q) \times (\Pi_* \circ M)$, which we can take as the plot $P \times A$ in (3.4.16). Thus we obtain

$$\begin{aligned}
((Q \times M)^* F^* \operatorname{Liouv})(\delta u) &= (\Pi \circ Q)^*((\Pi_* \circ M)(u))(\delta u) && \text{by (3.4.16)} \\
&= Q^*(\Pi^*(\Pi_*(M(u))))(\delta u) \\
&= Q^*(M(u))(\delta u) \\
&= M(u)(\delta[Q(u)]) \\
&= \langle M(u), \delta[Q(u)]Q(u)^{-1} \rangle && \text{by (3.4.9)} \\
&= \langle M(u)Q(u), \delta[Q(u)] \rangle && \text{by (1.3.30)} \\
&= \varpi_{T^*G}(\delta[M(u)Q(u)]) \\
&= \varpi_{T^*G}(\delta[j(Q(u), M(u))]) && \text{by (3.4.20)} \\
&= ((Q \times M)^* j^* \varpi_{T^*G})(\delta u)
\end{aligned} \tag{3.4.23}$$

as desired.

To prove (3.4.21b), note that $(\phi \circ j)(q, \mu) = \phi(\mu q) = \mu$ (1.6.15), whereas if $y = F(q, \mu)$, then G-equivariance gives $\hat{y} = F \circ (q, \mu)\hat{}$ where $(q, \mu)\hat{}(g) = g(q, \mu)$. So (3.4.18) says that $(\Phi \circ F)(q, \mu)$ is the value at e of the (left invariant) 1-form

$$\begin{aligned}
(\hat{y}^* \,\mathrm{Liouv})(\delta g) &= ((q, \mu)\hat{}^{\,*} F^* \,\mathrm{Liouv})(\delta g) \\
&= ((q, \mu)\hat{}^{\,*} j^* \varpi_{T^*G})(\delta g) && \text{by (3.4.21a)} \\
&= \varpi_{T^*G}(\delta[j(g(q, \mu))]) \\
&= \varpi_{T^*G}(\delta[g\mu q]) && \text{by (3.4.20)} \\
&= \langle g\mu q, \delta g.q \rangle \\
&= \langle \mu, g^{-1}\delta g \rangle && \text{by (1.3.30),}
\end{aligned} \tag{3.4.24}$$

and, evaluating at e, $(\Phi \circ F)(q, \mu) = \mu$. So both sides of (3.4.21b) are equal, as claimed. □

Conclusion (Part I)

In this first part, following [B25], we developed a diffeological framework for symplectic and prequantum Frobenius reciprocity and reduction. Motivated by the classical correspondence between representation theory and symplectic geometry, we focused on the geometric analogues of induced representations and spaces of intertwiners. These constructions, expressed through symplectic reduction, naturally assemble into a symplectic version of Frobenius reciprocity.

The first main achievement was the resolution of the conjecture posed in [R22]: we proved that the Frobenius reciprocity bijection is a diffeological diffeomorphism, both in the Hamiltonian and in the prequantum settings. Furthermore, when one of the spaces in question admit a reduced form, so does the other, and this diffeomorphism is shown to preserve them. These results not only uncover the geometry underlying symplectic Frobenius reciprocity but also demonstrate the effectiveness of diffeology in handling objects beyond the smooth category.

We then turned to the problem of existence of reduced forms. We showed that properness, local freeness, or strictness of the group action each suffice to ensure the existence of such forms. As an application, we proved the existence of reduced forms on induced spaces, even when the subgroup $H \subset G$ is not closed, yielding a diffeological version of the Kummer-Marsden-Satzer isomorphism when H is dense.

Several directions for further investigation emerge from this part. The central result used in the proof of Theorem 3.2.2, drawn from [H11], has recently been extended to singular foliations in [M23], suggesting possible generalizations of our assumptions. In Corollary 3.4.1, the density assumption may be dropped by invoking a diffeological version of the symplectic induction in stages [R22, Theorem 2.1], provided such a formulation can be made rigorous.

More generally, the question of characterizing those reduced diffeological spaces that admit reduced forms remains open, and offers a natural continuation of the present work.

Part II

A Clebsch portrait for Schrödinger theory

Introduction to Part II

Vortex structures are ubiquitous in physical systems (see, e.g., [S92] and references therein). In quantum mechanics, they arise as nodal lines: the zero sets of a wave function for a massive spinless particle governed by the Schrödinger equation. Along these lines the probability density vanishes and the phase becomes undefined, leading to subtle analytical and geometric features (see, e.g., [T83, dS16]).

The Madelung transform, which casts the Schrödinger equation into hydrodynamical form, gives the correspondence between vortex and nodal lines. It is also the central topic in [K18, K19], which showed that it yields a moment map and defines a symplectomorphism and an isometry between the (projectivized) space of wave functions and the cotangent bundle to densities.

The present part is motivated by the search for new correspondences of this type. As in [K18, K19], our constructions involve the projective space of wave functions and moment maps, but they also remain valid when restricted to the nodal line. In this way, we extend the geometric interplay between quantum mechanics and fluid dynamics into the singular regime where the wave function vanishes.

Building on the analysis of [S23], where nodal lines were identified as fibres of a Clebsch-type fibration advected by the Schrödinger-Madelung flow, we consider the projective Hilbert space approach of [K18, K19] and symplectic structures arising in hydrodynamics [P89, P92, P00, P98]. We establish moment map relations between these objects and their nodal line restrictions, and synthetize the resulting geometry in a diagrammatic picture.

The results presented here are based on [B24]. Chapter 4 introduces the standard Madelung-Schrödinger picture [M27, B52a, B52b] and the coadjoint orbit portrait of rotational perfect fluids [P92], relating them closely following [S23]. Chapter 5 presents a series of constructions based on [P89, P92, P00] and [K18, K19], which eventually merge together in the main result, Theorem 5.5.

Chapter 4

Tools in symplectic geometry and mechanics

In this chapter we introduce the symplectic-geometric and mechanical tools necessary to deal with the geometry arising in the hydrodynamical formulation of quantum mechanics studied in Part II of the thesis. These provide a connection between wave functions, fluid dynamics, and the geometric structures arising in presence of vorticity. Our exposition draws primarily on [M83, P92, S16, S23, B24].

The first sections explore three interrelated topics that converge in Section 4.4. We begin by introducing a Hamiltonian structure on the infinite-dimensional space of smooth functions from a compact manifold to a symplectic manifold [D99] (Section 4.1). Next, we revise the coadjoint orbit formulation of the dynamics of rotational perfect fluids [M83, P92] (Section 4.2). This is followed by the hydrodynamical reformulation of the Schrödinger equation [M27, B52a, B52b] (Section 4.3).

In Section 4.4, we show how the Hamiltonian structure on the space of wave functions induces a Hamiltonian formulation of the Schrödinger equation. Furthermore, the hydrodynamical perspective enables a reformulation of the latter equation via the coadjoint orbit approach. This yields a Poisson structure for the Schrödinger equation, given by the Kuznetsov-Mikhailov hydrodynamical bracket [K80], and leads to a reinterpretation of the Rasetti-Regge current algebra (Definition 4.2.18) in terms of (co-)moment maps. These maps arise from the action by reparametrization on wave functions of the group of volume-preserving diffeomorphisms of $\mathbb{R}^3$ rapidly approaching the identity at infinity.

The final sections introduce the Clebsch variables and the nodal line, also summarizing some results concerning the latter [S23, B24]. They play a central role in Chapter 5, where the tools developed in this chapter provide a framework to relate various symplectic structures arising from wave functions and their restriction to the nodal line.

4.1 Symplectic geometry on function spaces

In this section we aim at defining a symplectic structure on the (infinite-dimensional) space of functions between a compact manifold and a symplectic one, together with a moment map for the action of the group of the volume preserving diffeomorphisms [D99]. The necessary background on group actions, Hamiltonian spaces, and moment maps is recalled in Sections 1.3 and 1.4. Since the framework of Part II is infinite-dimensional, we also highlight the main differences with the finite-dimensional case. These constructions will be applied to the hydrodynamical formulation of Schrödinger equation in Section 4.4.

Consider a compact manifold S, endowed with a volume form σ, and a symplectic manifold (M, ω), both assumed finite-dimensional. Define $\mathcal{M}$ the infinite-dimensional space of functions $S \to M$ in a fixed homotopy class. The tangent space to $\mathcal{M}$ at $f : S \to M$ can be regarded as the space of sections of the pullback bundle $f^*(TM) := \{(s, X) \in S \times TM \, : \, f(s) = \pi(X)\}$ where $\pi : TM \to M$ [K97, Chapter IX, 42]. Now if v and w are sections of $f^*(TM)$, we can define the following symplectic form on $\mathcal{M}$ [D99]:

$$\Omega_f(v, w) = \int_S \omega_{f(s)}(v, w)\sigma, \tag{4.1.1}$$

where in the integral, with a slight abuse of notation, v and w denote the second component of their image at $s \in S$. In this infinite-dimensional setting, the symplectic form Ω is *weak*, i.e. it does not induce a bijection, but only an injection, between the tangent space to $\mathcal{M}$ and its dual. For further details on symplectic forms on infinite-dimensional spaces see [K97, Chapter X, 48].

Let us endow $\mathcal{M}$ with an action of the Lie group G of volume-preserving diffeomorphisms of S ([E70], [K97, Chapter IX, 43]). It acts on $\mathcal{M}$ by composition on the right: $g(f) := f \circ g^{-1}$ (module notation) and preserves Ω:

$$g^*\Omega_f(v, w) = \int_S \omega_{f \circ g^{-1}(s)}(v \circ g^{-1}, w \circ g^{-1})\sigma = \int \omega_{f(s)}(v, w) g^*\sigma = \Omega_f(v, w), \tag{4.1.2}$$

since G preserves σ.

Under suitable assumptions this action is Hamiltonian [D99]: assume that $f^*([\omega]) = 0$ in the second de Rham cohomology class $H^2(S)$ and that

$H^1(S)$ is also zero. Then, for each $f \in \mathcal{M}$, there exists a 1-form on M, say $a \in \Omega^1(M)$, such that $f^*\omega = da$. Let $\mathfrak{g}$ denote the Lie algebra of G, which is composed of divergence-free vector fields on S, and let $\xi \in \mathfrak{g}$. Then the following pairing is defined:

$$\langle a, \xi \rangle := \int_S a(\xi_S)\sigma, \tag{4.1.3}$$

where ξ_S denotes the fundamental vector field pertaining to ξ for the natural G-action on S. Thus we can define the following:

$$\mu : \mathcal{M} \ni f \mapsto \{\xi \mapsto \langle a, \xi \rangle\} \in \mathfrak{g}^*, \tag{4.1.4}$$

where the dual of the Lie algebra can be regarded as the space of 1-forms modulo closed ones or (via musical isomorphisms) vector fields up to gradients (as in [A21]). Indeed, the 1-form a is defined up to closed 1-forms, which are exact by the assumption $H^1(S) = 0$, and then (4.1.3) is well defined as well as (4.1.4).

Now we show that μ is a moment map (Definition 1.4.3, valid also in infinite dimensions), following [D99]. Let $f \in \mathcal{M}$, v a section of $f^*(TM)$, $\xi \in \mathfrak{g}$ and consider a 1-parameter family of functions $f_t \in \mathcal{M}$ such that

$$f_0 = f \qquad \text{and} \qquad \frac{d}{dt} f_t \bigg|_{t=0} = v. \tag{4.1.5}$$

Then, setting $f_t^*\omega =: da_t$ implies that

$$\frac{d}{dt} da_t \bigg|_{t=0} = \frac{d}{dt} f_t^*\omega \bigg|_{t=0} = f^*(d(\iota_v\omega)), \tag{4.1.6}$$

where the second equality follows by Cartan's magic formula for the Lie derivative. Thus

$$\frac{d}{dt} a_t \bigg|_{t=0} = f^*(\iota_v\omega), \tag{4.1.7}$$

up to close and hence exact 1-forms; then:

$$\begin{aligned} \langle d_f\mu(v), \xi \rangle &= \frac{d}{dt} \langle \mu(f_t), \xi \rangle \bigg|_{t=0} = \left\langle \frac{d}{dt} a_t \bigg|_{t=0}, \xi \right\rangle \\ &= \langle f^*(\iota_v\omega), \xi \rangle = \int_S [f^*(\iota_v\omega)]_s(\xi_S)\sigma = \int_S \omega_{f(s)}(v, f_*(\xi_S))\sigma \\ &= \Omega_f(v, \xi_{\mathcal{M}}). \end{aligned} \tag{4.1.8}$$

By the notation adopted in (4.1.1), $f_*(\xi_S)$ actually denotes the section $s \mapsto (s, f_*(\xi_S))$, which corresponds to the fundamental vector field $\xi_{\mathcal{M}}$ pertaining to ξ for the lifted G-action to $\mathcal{M}$, and then μ is a moment map.

4.2 Preliminaries on coadjoint orbits of rotational perfect fluids

In this section we revise the treatment of coadjoint orbits of rotational perfect fluids, following [P92, §2] and referring to it and the references therein for an extensive treatment. For further details see also [A21, Chapter 1] and [M99, Chapter 14]. This approach and formalism, together with the constructions of the previous section, will be applied to the hydrodynamics of Schrödinger equation in Section 4.4.

We first consider the motion of a perfect fluid in $\mathbb{R}^3$: denote $\mathbf{v}$ its *velocity* field, $\mathbf{w} = \operatorname{curl} \mathbf{v}$ its *vorticity* field and

$$H = \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{v}(x)|^2 \, d^3x \tag{4.2.1}$$

its *Hamiltonian.* In abscence of external forces, the motion of the fluid is governed by *Euler's equation*:

$$\partial_t \mathbf{v} = -(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla p, \tag{4.2.2}$$

where p denotes the pressure. Equation (4.2.2) can be rephrased in terms of vorticity, yielding the following:

$$\partial_t \mathbf{w} = -\operatorname{curl} (\mathbf{w} \times \mathbf{v}). \tag{4.2.3}$$

The latter motivates the coadjoint orbit formulation of the fluid motion. In more detail, consider the Lie group G of volume preserving diffeomorphisms of $\mathbb{R}^3$ rapidly approaching the identity at infinity, a subgroup of the $\mathbb{R}^3$ version of the Lie group introduced in the previous section. Its Lie algebra $\mathfrak{g}$ is composed of divergence-free vector fields on $\mathbb{R}^3$ rapidly vanishing at infinity and, for any $\mathbf{a}, \mathbf{b} \in \mathfrak{g}$, the following identities hold:

$$\begin{aligned} &a) \, [\mathbf{a}, \mathbf{b}] = \operatorname{curl} (\mathbf{a} \times \mathbf{b}) \\ &b) \operatorname{div} (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \operatorname{curl} \mathbf{a} - \mathbf{a} \cdot \operatorname{curl} \mathbf{b}, \end{aligned} \tag{4.2.4}$$

where the square bracket denotes minus the commutator of vector fields.

A pairing on $\mathfrak{g}$ is defined as follows:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \int_{\mathbb{R}^3} \mathbf{a} \cdot \mathbf{b} \, d^3x, \tag{4.2.5}$$

and allows us to identify the dual of the Lie algebra $\mathfrak{g}^*$ with the space of vector valued (tempered) distributions, and thus $\mathfrak{g} \subset \mathfrak{g}^*$. Moreover, it implies, combined with (4.2.4b), the following identity

$$\langle \mathbf{b}, \operatorname{curl} \mathbf{a} \rangle = \langle \mathbf{a}, \operatorname{curl} \mathbf{b} \rangle. \tag{4.2.6}$$

Now let us consider the adjoint and coadjoint action of $\mathfrak{g}$ on itself and on $\mathfrak{g}^*$ (Section 1.3) induced by the right action of G on itself by composition. They have the following expressions:

$$\begin{aligned}&\text{a)}\, ad_{\mathbf{a}}(\mathbf{b}) = [\mathbf{a}, \mathbf{b}]\\ &\text{b)}\, ad^*_{\mathbf{a}}(\mathbf{v}) = -\mathbf{w} \times \mathbf{a} - \nabla(\mathbf{v} \cdot \mathbf{a}),\end{aligned} \tag{4.2.7}$$

where $\mathbf{a}, \mathbf{b} \in \mathfrak{g}$, $\mathbf{v} \in \mathfrak{g}^*$ and the square bracket denotes, as before, minus the commutator of vector fields. See [P92] and e.g. [A21, Chapter 1], [M99, Chapter 14] for further details. Note that the gradient term in (4.2.7b) does not influence the pairing (4.2.5), as the following computation shows:

$$\begin{aligned}\langle ad^*_{\mathbf{a}}\mathbf{v}, \mathbf{b}\rangle &= \langle \mathbf{v}, -ad_{\mathbf{a}}\mathbf{b}\rangle = -\langle \mathbf{v}, [\mathbf{a}, \mathbf{b}]\rangle = -\int_{\mathbb{R}^3} \mathbf{v} \cdot \operatorname{curl}(\mathbf{a} \times \mathbf{b})\, d^3x\\ &= -\int_{\mathbb{R}^3} \mathbf{w} \cdot \mathbf{a} \times \mathbf{b}\, d^3x = \int_{\mathbb{R}^3} (-\mathbf{w} \times \mathbf{a}) \cdot \mathbf{b}\, d^3x\\ &= \langle -\mathbf{w} \times \mathbf{a}, \mathbf{b}\rangle.\end{aligned} \tag{4.2.8}$$

Now we return to (4.2.3); if $\operatorname{div} \mathbf{v} = 0$, by (4.2.4a) it can be written as

$$\partial_t \mathbf{w} = -[\mathbf{w}, \mathbf{v}]. \tag{4.2.9}$$

This implies that the fluid motion evolves on a coadjoint orbit labeled by the vorticity (Remark 4.2.10), which we denote by $\mathcal{O}_{\mathbf{w}}$. Indeed, the solution of equation (4.2.9), given an initial condition for the vorticity, say $\mathbf{w}$, is expressed as $f_t^*\mathbf{w}$. Here, f_t denotes the flow at time t of the divergence-free vector field $\mathbf{v}$ and the pullback $f_t^*\mathbf{w}$ coincides with the coadjoint action of $f_t \in G$ on $\mathbf{w}$, the latter viewed as an element of the dual space $\mathfrak{g}^*$ ([M83, §4]). Then the time evolution of the vorticity takes place on the coadjoint orbit through the initial vorticity.

Remark 4.2.10. Due to the orthogonality (with respect to the pairing (4.2.5)) between gradient fields and divergence-free vector fields, each element of a coadjoint orbit in $\mathfrak{g}^*$ can be identified with an equivalence class $[\mathbf{v}]$, where $\mathbf{v} \sim \mathbf{v}'$ if and only if $\mathbf{v} = \mathbf{v}' + \nabla f$, and f is a smooth function on $\mathbb{R}^3$. This implies that each class admits a representative that is divergence-free, and can be obtained by solving a Poisson equation [P92]. For the same reason, the vector field $ad^*_{\mathbf{a}}(\mathbf{v})$, regarded as an element of $\mathfrak{g}^*$, can be replaced by an equivalent divergence-free representative of the form $-\mathbf{w} \times \mathbf{a} + \nabla f^{\mathbf{a}}$, where the scalar function $f^{\mathbf{a}}$ is determined by solving the Poisson equation

$$\Delta f^{\mathbf{a}} = -\operatorname{div}(\mathbf{w} \times \mathbf{a}). \tag{4.2.11}$$

Alternatively, elements of coadjoint orbits can be regarded as vorticities of vector fields, since any element in $[\mathbf{v}]$ has the same vorticity $\mathbf{w} = \operatorname{curl}\mathbf{v}$. This is the reason why we can label the coadjoint orbit through $[\mathbf{v}]$ with its vorticity $\mathbf{w}$. Switching to differential forms, $[v] \in \mathfrak{g}^*$ denotes the cohomology class represented by the 1-form v, so its elements are $v + df$, with f varying in smooth functions on $\mathbb{R}^3$. As for vector fields, $[v]$ can be represented by dv, which indeed corresponds to the vorticity w via musical isomorphisms [M83, p. 310].

Before proceeding, we clarify the notation used in the previous remark and fix it for the rest of this part:

Remark 4.2.12. The musical isomorphisms, induced by the Euclidean metric on $\mathbb{R}^3$, provide an identification between vector fields and differential forms. In this Part II, we denote vector fields in $\mathbb{R}^3$ using bold font, and the corresponding differential forms via musical isomorphisms using the same letter in normal font. For example, the 1-form corresponding to the velocity field $\mathbf{v}$ will be denoted by v, and the 2-form corresponding to the vorticity field $\mathbf{w}$ will be denoted by w.

As stated in Theorem 1.6.6, coadjoint orbits are naturally endowed with the symplectic Kirillov-Kostant-Souriau strucure (KKS). To express it in this context, recall that (4.2.7b) is also the expression of the fundamental vector field pertaining to $\mathbf{a}$ at $\mathbf{v}$ on $\mathcal{O}_{\mathbf{w}}$; then the KKS form writes:

$$B_{\mathbf{v}}(\operatorname{ad}^*_{\mathbf{a}}\mathbf{v}, \operatorname{ad}^*_{\mathbf{b}}\mathbf{v}) = \langle \mathbf{v}, [\mathbf{a}, \mathbf{b}] \rangle = \langle \mathbf{w}, \mathbf{a} \times \mathbf{b} \rangle. \tag{4.2.13}$$

Now we focus on the relation between the KKS symplectic form and the Kuznetsov-Mikhailov hydrodynamical bracket [K80]. First, let us compute the variational derivative (Fréchet) of a smooth function F on $\mathcal{O}_{\mathbf{w}}$, i.e. a function that depends on $\mathbf{v}$ only through $[\mathbf{v}]$ or $\mathbf{w}$:

$$\begin{aligned} \frac{d}{d\epsilon} F(\mathbf{v} + \epsilon \mathbf{v})\Big|_{\epsilon=0} &= \left\langle \frac{\delta F}{\delta \mathbf{v}}, \delta \mathbf{v} \right\rangle = \left\langle \frac{\delta F}{\delta \mathbf{w}}, \delta \mathbf{w} \right\rangle \\ &= \int_{\mathbb{R}^3} \frac{\delta F}{\delta \mathbf{w}} \cdot \delta \mathbf{w}\, d^3x = \int_{\mathbb{R}^3} \frac{\delta F}{\delta \mathbf{w}} \cdot \operatorname{curl} \delta \mathbf{v}\, d^3x \\ &= \int_{\mathbb{R}^3} \operatorname{curl} \frac{\delta F}{\delta \mathbf{w}} \cdot \delta \mathbf{v}\, d^3x = \left\langle \operatorname{curl} \frac{\delta F}{\delta \mathbf{w}}, \delta \mathbf{v} \right\rangle, \end{aligned} \tag{4.2.14}$$

thus

$$\frac{\delta F}{\delta \mathbf{v}} = \operatorname{curl} \frac{\delta F}{\delta \mathbf{w}}. \tag{4.2.15}$$

This also implies that $\frac{\delta F}{\delta \mathbf{v}} \in \mathfrak{g}$, since div curl $= 0$. Thus if F and G are smooth functions on $\mathcal{O}_{\mathbf{w}}$, by the previous observation $\frac{\delta F}{\delta \mathbf{v}}$ and $\frac{\delta G}{\delta \mathbf{v}} \in \mathfrak{g}$ and then:

$$\begin{aligned} B_{\mathbf{v}}\big(\mathrm{ad}^*_{\frac{\delta F}{\delta \mathbf{v}}}\mathbf{v}, \mathrm{ad}^*_{\frac{\delta G}{\delta \mathbf{v}}}\mathbf{v}\big) &= \left\langle \mathbf{v}, \left[\frac{\delta F}{\delta \mathbf{v}}, \frac{\delta G}{\delta \mathbf{v}}\right]\right\rangle = \left\langle \mathbf{w}, \operatorname{curl}\frac{\delta F}{\delta \mathbf{w}} \times \operatorname{curl}\frac{\delta G}{\delta \mathbf{v}}\right\rangle \\ &= \{F, G\}(\mathbf{v}), \end{aligned} \tag{4.2.16}$$

where the last bracket denotes the Kuznetsov-Mikhailov hydrodynamical bracket [K80]:

$$\{F, G\}(\mathbf{v}) := \left\langle \mathbf{w}, \operatorname{curl}\frac{\delta F}{\delta \mathbf{w}} \times \operatorname{curl}\frac{\delta G}{\delta \mathbf{v}}\right\rangle. \tag{4.2.17}$$

We conclude this section defining the Rasetti-Regge current algebra associated with the G-coadjoint orbit $\mathcal{O}_{\mathbf{w}}$ (see also [R75, P89, P00, P98]):

Definition 4.2.18. The Rasetti-Regge current algebra is composed of functions $\lambda_{\mathbf{b}}$ on $\mathcal{O}_{\mathbf{w}}$, where $\mathbf{b}$ is divergence free and vanishing at infinity, and

$$\lambda_{\mathbf{b}}(\mathbf{v}) = \int_{\mathbb{R}^3} \mathbf{v} \cdot \mathbf{b}\, d^3x = \int_{\mathbb{R}^3} \mathbf{w} \cdot \mathbf{B}\, d^3x, \tag{4.2.19}$$

where $\mathbf{b} = \operatorname{curl}\mathbf{B}$, since $\operatorname{div}\mathbf{b} = 0$. It is an algebra of functions with respect to the bracket (4.2.17), induced by the KKS symplectic form:

$$\{\lambda_{\mathbf{a}}, \lambda_{\mathbf{b}}\}(\mathbf{v}) = B_{\mathbf{v}}(ad^*_{\mathbf{a}}(\mathbf{v}), ad^*_{\mathbf{b}}(\mathbf{v})), \tag{4.2.20}$$

which implies,

$$\{\lambda_{\mathbf{a}}, \lambda_{\mathbf{b}}\}(\mathbf{v}) = \lambda_{[\mathbf{a},\mathbf{b}]}(\mathbf{v}), \tag{4.2.21}$$

for any $\mathbf{a}, \mathbf{b} \in \mathfrak{g}$. Moreover, the current algebra is *Hamiltonian* because

$$d\lambda_{\mathbf{a}}(\mathbf{v}) = (\iota_{ad^*_{\mathbf{a}}(\mathbf{v})}B)_{\mathbf{v}}, \tag{4.2.22}$$

i.e. $\lambda_{\mathbf{a}}$ is a Hamiltonian for the KKS symplectic form and then the Rasetti-Regge currents may also be regarded as co-moment maps (see Remark 4.4.24 and Section 5.4).

Remark 4.2.23. For any $\mathbf{b}$, the function $\lambda_{\mathbf{b}}$ is well defined on the coadjoint orbit $\mathcal{O}_{\mathbf{w}}$. Indeed, it does not vary if $\mathbf{v}$ is replaced by $\mathbf{v}' = \mathbf{v} + \nabla f$, $f \in \mathcal{C}^\infty(\mathbb{R}^3)$, i.e. it only depends on the class $[\mathbf{v}]$, which by Remark 4.2.10 represents an element of the coadjoint orbit.

The fundamental properties of the Rasetti-Regge current algebra are summarized in the following theorem [P92, p. 902] (see also [P89, Theorem 1]):

Theorem 4.2.24. *(i) The Rasetti-Regge current algebra is isomorphic to the Lie algebra* $\mathfrak{g}$.

(ii) The elements of the current algebra satisfy Hamilton equations, i.e. for any $\mathbf{a} \in \mathfrak{g}$,

$$\partial_t \lambda_{\mathbf{a}} = \{\lambda_{\mathbf{a}}, H\}. \tag{4.2.25}$$

(iii) The quantity $Q := \langle \mathbf{w}, \mathbf{v} \rangle$ *is conserved in time.*

4.3 Hydrodynamics of the Schrödinger equation

In this section we revise the hydrodynamics of the Schrödinger equation [M27, B52a, B52b], following [S16, B24]. Other references for this section are [F17, F19, F22, K18, K19, S23].

The quantum wave function ψ for a spinless particle with mass $m > 0$ moving in $\mathbb{R}^3$ depends on $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$, and obeys the *Schrödinger equation*, which, setting $\hbar$ and m equal to 1, writes:

$$\partial_t \psi = -i\hat{H}\psi := -i(-\frac{1}{2}\Delta + V)\psi, \tag{4.3.1}$$

where Δ denotes the Laplace operator and $V = V(x)$ a ("classical") potential. Its polar decomposition, when $\rho > 0$, reads:

$$\psi = \sqrt{\rho}\, e^{iS}, \qquad \rho = |\psi|^2, \tag{4.3.2}$$

and yields the *Madelung transform*

$$\Phi : (\rho, S) \mapsto \psi := \sqrt{\rho} e^{iS}, \tag{4.3.3}$$

which is central in the density manifold approach developed in [K18, K19]. With this change of variables, the Schrödinger equation can be cast into the Madelung-Bohm hydrodynamical form [M27, B52a, B52b]; setting $\mathbf{u} = \nabla S$, it writes

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} & = & -\nabla(V + V_q) \\ \\ \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\, \mathbf{u}) & = & 0 \end{cases} \tag{4.3.4}$$

where $V_q = -\frac{1}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$ is called quantum potential. The first equation is an Euler equation for a *compressible* irrotational fluid, that is $\operatorname{div}\mathbf{u} = \Delta S$ is not zero, in general. The second one is the continuity equation involving the *probability current*:

$$\mathbf{j} := \rho\,\mathbf{u} = \mathrm{Im}(\psi^\dagger\nabla\psi) = \frac{1}{2\,i}(\psi^\dagger\nabla\psi - \psi\nabla\psi^\dagger)\,. \tag{4.3.5}$$

Also note that (4.3.4) can be rephrased in Hamiltonian fashion [B52a, B52b]: setting

$$\mathcal{H} := \langle\psi|\hat{H}\psi\rangle = \int_{\mathbb{R}^3}\psi^\dagger\Big(-\frac{1}{2}\Delta + V(x)\Big)\psi\,d^3x = \int_{\mathbb{R}^3}\Big(\frac{1}{2}\,|\nabla\psi|^2 + V(x)|\psi|^2\Big)\,d^3x, \tag{4.3.6}$$

equations (4.3.4) become:

$$\frac{\partial\rho}{\partial t} = \frac{\delta\mathcal{H}}{\delta S}, \qquad \frac{\partial S}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\rho} \tag{4.3.7}$$

or, in complex terms [S16]:

$$\frac{\partial\psi}{\partial t} = -i\frac{\delta\mathcal{H}}{\delta\psi^\dagger}, \qquad \frac{\partial\psi^\dagger}{\partial t} = i\frac{\delta\mathcal{H}}{\delta\psi}. \tag{4.3.8}$$

4.4 Symplectic-geometric interpretation

In this section we apply the theory and results introduced in Section 4.1 and 4.2 to reformulate geometrically the topics discussed in Section 4.3, following [S16, S23, B24].

Define $\widetilde{\mathcal{M}} := \mathcal{C}^\infty_c(S, M)$, where $S = \mathbb{R}^3$, and $M = \mathbb{R}^2 \cong \mathbb{C}$, whose elements are compactly supported smooth maps $\psi : \mathbb{R}^3 \ni x \mapsto \psi(x) \in \mathbb{C}$. In Section 4.1 the manifold S is assumed compact; here, since $S = \mathbb{R}^3$ is non-compact, we replace compactness by the requirement that maps have compact support. This allows us to extend each map to the one point compactification S^3 and thus equivalently take $S = S^3$ in $\widetilde{\mathcal{M}}$. Moreover, using polar coordinates $\psi(x) = \sqrt{\rho(x)}\,e^{iS(x)}$, the elements of $\widetilde{\mathcal{M}}$ can be regarded as maps $\psi : \mathbb{R}^3 \ni x \mapsto (\rho(x), S(x)) \in \mathbb{R}^2$. However, this substitution can only be performed if ψ does not have nodes, i.e., $\psi(x) \neq 0$; otherwise ρ vanishes, making S indeterminate and giving rise to topological issues (see Section 4.6). We will use both the polar and the ψ formalism, sticking to the latter whenever we need to bypass the limitations imposed by the former.

The symplectic form (4.1.1) on $\widetilde{\mathcal{M}}$ writes (notation [C88b, S16]):

$$\Omega = \int_{\mathbb{R}^3}\delta\rho(x)\wedge\delta S(x)d^3x. \tag{4.4.1}$$

More explicitly, following the notation of [K18, K19], consider the (non-vanishing) function ψ and its polar decomposition $\sqrt{\rho(x)}e^{iS(x)}$; also let $\dot\rho_1$ and $\dot\rho_2$ be two variations at ρ, and $\dot S_1$ and $\dot S_2$ be two variations at S. Then (4.4.1) means:

$$\Omega_{(\rho,S)}\left((\dot\rho_1,\dot S_1),(\dot\rho_2,\dot S_2)\right)=\frac{1}{2}\int_{\mathbb{R}^3}\dot\rho_1\dot S_2-\dot\rho_2\dot S_1\,d^3x. \tag{4.4.2}$$

Moreover, the symplectic form (4.1.1) in complex coordinates writes (notation [C88b, S16]):

$$\Omega=-i\int_{\mathbb{R}^3}\delta\psi^\dagger(x)\wedge\delta\psi(x)d^3x, \tag{4.4.3}$$

since

$$d\mathrm{Re}\,\psi\wedge d\mathrm{Im}\,\psi=\frac{1}{2}d\rho\wedge dS=\frac{i}{2}d\psi\wedge d\psi^\dagger. \tag{4.4.4}$$

More explicitly, again following the notation of [K18, K19], consider $\dot\psi_1$ and $\dot\psi_2$, variations at ψ. Then (4.4.3) writes:

$$\Omega_\psi\left(\dot\psi_1,\dot\psi_2\right)=\mathrm{Im}\int_{\mathbb{R}^3}\overline{\dot\psi}_1\dot\psi_2\,d^3x=-\frac{i}{2}\int_{\mathbb{R}^3}\overline{\dot\psi}_1\dot\psi_2-\dot\psi_1\overline{\dot\psi}_2\,d^3x. \tag{4.4.5}$$

This symplectic structure induces the Hamiltonian form of the hydrodynamical Schrödinger equation given by (4.3.7, 4.3.8). Moreover, it allows us to introduce a Poisson bracket on the smooth functions on $\widetilde{\mathcal{M}}$, which is related to the hydrodynamical bracket (4.2.17). Since we are in an infinite-dimensional setting and the symplectic form is weak, not all functions admit a corresponding Hamiltonian vector field. However, for functions $\mathcal{F}\in\mathcal{C}^\infty(\widetilde{\mathcal{M}})$ that can be expressed as integrated densities, i.e.

$$\mathcal{F}(\rho,S):=\int_{\mathbb{R}^3}f(\rho,S)(x)\,d^3x, \tag{4.4.6}$$

it is possible to find the corresponding Hamiltonian vector field:

$$X_{\mathcal{F}}=\int_{\mathbb{R}^3}\left(\frac{\delta f}{\delta\rho}\frac{\delta}{\delta S}-\frac{\delta f}{\delta S}\frac{\delta}{\delta\rho}\right)d^3x. \tag{4.4.7}$$

Then the Poisson bracket induced by the symplectic form (4.4.1) (or (4.4.3)) on such functions writes [S16]:

$$\{\mathcal{F},\mathcal{G}\}=\int_{\mathbb{R}^3}\left(\frac{\delta f}{\delta\rho}\frac{\delta g}{\delta S}-\frac{\delta f}{\delta S}\frac{\delta g}{\delta\rho}\right)d^3x, \tag{4.4.8}$$

for $\mathcal{F}$ and $\mathcal{G}$ integrated densities.

Thus if we fix a Hamiltonian $\mathcal{H}$ and consider a time-dependent ψ, we get the Hamilton equation [S16]:

$$\partial_t \mathcal{F} = \{\mathcal{F}, \mathcal{H}\}. \tag{4.4.9}$$

This proves that the symplectic form (4.4.1) (or (4.4.3)) induces a Hamiltonian structure for the Schrödinger equation (discussed also in e.g. [B01b, B04]).

We now turn our attention to the Poisson bracket (4.4.8), with the aim of relating it to the hydrodynamical bracket (4.2.17). This connection is established by identifying the quantities arising in the hydrodynamical formulation of the Schrödinger equation with those related to coadjoint orbits of rotational perfect fluids (Section 4.2). As introduced in the previous section, the *probability current* is $\mathbf{j} = \rho \nabla S$. According to equation (4.3.4), it can be interpreted as the velocity field of a (compressible) fluid, whose vorticity is given by

$$\mathbf{w} = \operatorname{curl} \mathbf{j} = \nabla \rho \times \nabla S, \tag{4.4.10}$$

or, equivalently, in differential form notation (Remark 4.2.12),

$$w = dj = d\rho \wedge dS. \tag{4.4.11}$$

This shows that the fluid can be described using the Clebsch variables ρ and S (see [K80, L32, M83] and Section 4.5).

To recover the coadjoint orbits and the Rasetti-Regge current algebra (Section 4.2) in this quantum-hydrodynamical context, we must specify a group action. We consider again the Lie group G, consisting of volume-preserving diffeomorphisms of $\mathbb{R}^3$ that rapidly approach the identity at infinity; its Lie algebra $\mathfrak{g}$ consists of divergence-free vector fields that rapidly vanish at infinity. As noted in general in (4.1.2), G acts (on the right) on $\widetilde{\mathcal{M}}$ preserving the symplectic form (4.4.1). This setup allows us to rephrase Definition 4.2.18 by replacing the velocity field $\mathbf{v}$ with the probability current $\mathbf{j}$ associated to an element $\psi \in \widetilde{\mathcal{M}}$ and setting $\mathbf{w} := \operatorname{curl} \mathbf{j}$:

Definition 4.4.12. The Rasetti-Regge current algebra consists of smooth functions $\lambda_{\mathbf{b}}$ on $\widetilde{\mathcal{M}}$, where $\mathbf{b} \in \mathfrak{g}$. If $\psi \in \widetilde{\mathcal{M}}$ and $\mathbf{j}$ is the corresponding probability current, these functions are defined by

$$\lambda_{\mathbf{b}}(\psi) := \lambda_{\mathbf{b}}(\mathbf{j}) = \int_{\mathbb{R}^3} \mathbf{j} \cdot \mathbf{b} \, d^3x = \int_{\mathbb{R}^3} \rho \nabla S \cdot \mathbf{b} \, d^3x = \int_{\mathbb{R}^3} j(\mathbf{b}) \, d^3x = \int_{\mathbb{R}^3} \mathbf{w} \cdot \mathbf{B} \, d^3x, \tag{4.4.13}$$

where $\mathbf{b} = \operatorname{curl} \mathbf{B}$.

Remark 4.4.14. For any $\mathbf{b}$, the function $\lambda_{\mathbf{b}}(\mathbf{j})$ does not vary if $\mathbf{j}$ is replaced by $\mathbf{j}' = \mathbf{j} + \nabla f$, where f is a smooth function on $\mathbb{R}^3$ (Remark 4.2.23). Thus $\mathbf{j}$ can be chosen divergence free upon solving the Poisson equation $\Delta\phi = -\operatorname{div}\mathbf{j}$. Moreover, as in Section 4.2, the coadjoint orbit through $\mathbf{j}$, regarded as an element of $\mathfrak{g}^*$, is labeled by the corresponding vorticity $\operatorname{curl}\mathbf{j}$. Also note that the equivalence class $[\mathbf{j}] = \{\mathbf{j} + \nabla f \,:\, f \in \mathcal{C}^\infty(\mathbb{R}^3)\}$ defines a functional $\mu \in \mathfrak{g}^*$ which yields a moment map (see Theorem 4.4.15 below).

We are now in a position to state [S16, Theorem 3.1], which highlights the importance of the probability current, establishes some of its fundamental properties and combines the theory and results discussed in Part II so far:

Theorem 4.4.15. *The following hold:*

(i) The G-action on $(\widetilde{\mathcal{M}}, \Omega)$ is Hamiltonian with equivariant moment map:

$$\begin{gathered} \mathfrak{j} : \widetilde{\mathcal{M}} \to \mathfrak{g}^* \\ \{\psi : x \mapsto (\rho(x), S(x))\} \mapsto \mathfrak{j}(\psi) := [\mathbf{j}_\psi]. \end{gathered} \tag{4.4.16}$$

(ii) For any $\psi \in \widetilde{\mathcal{M}}$, the Rasetti-Regge current algebra on $\widetilde{\mathcal{M}}$ is composed of functions:

$$\lambda_{\mathbf{b}}(\psi) = \int_{\mathbb{R}^3} \mathbf{j}_\psi \cdot \mathbf{b}\, d^3x = \int_{\mathbb{R}^3} \mathbf{w} \cdot \mathbf{B}\, d^3x = \langle \mathfrak{j}(\psi), \mathbf{b} \rangle, \tag{4.4.17}$$

where $\mathbf{b} = \operatorname{curl}\mathbf{B}$, and satisfies

$$\{\lambda_{\mathbf{b}}, \lambda_{\mathbf{c}}\} = \lambda_{[\mathbf{b},\mathbf{c}]}, \tag{4.4.18}$$

where the bracket on the left-hand side is (4.4.8) *and the one on the right-hand side is* (4.2.4a).

(iii) The Poisson bracket (4.4.8) *on the currents becomes the hydrodynamical bracket* [K80, M83]*:*

$$\{\lambda_{\mathbf{b}}, \lambda_{\mathbf{c}}\} = \int_{\mathbb{R}^3} \mathbf{w} \cdot \left(\operatorname{curl} \frac{\delta \lambda_{\mathbf{b}}}{\delta \mathbf{w}} \times \operatorname{curl} \frac{\delta \lambda_{\mathbf{c}}}{\delta \mathbf{w}} \right) d^3x. \tag{4.4.19}$$

(iv) The currents evolve in time fulfilling Hamilton equation

$$\partial_t \lambda_{\mathbf{b}} = \{\lambda_{\mathbf{b}}, \mathcal{H}\}. \tag{4.4.20}$$

Proof. The map $\mathfrak{j}$ is a moment map by the construction of Section 4.1, where j substitutes a and $\mathbf{b}$ substitutes ξ. Equation (4.4.18) holds by a direct computation involving (4.4.8) and variational derivatives of $\lambda_{\mathbf{b}}$ with respect to ρ and S ([S16, Lemma 3.1]) and implies the equivariance of the moment map μ (see Remark 4.4.24). Moreover, noting that $\mathbf{b} = \frac{\delta\lambda_{\mathbf{b}}}{\delta\mathbf{j}}$ and substituting (4.2.15) into (4.4.18), gives the desired hydrodynamical bracket (4.4.19). We refer to [S16, Theorem 3.1] for further details and the rest of the proof. $\square$

Remark 4.4.21. Upon representing elements of $\mathfrak{g}^*$ as vorticity fields or 2-forms (Remark 4.2.10, [M83]), the moment map becomes, as in [S23, B24]:

$$\mathfrak{j}(\psi) = \mathbf{w} = \nabla\rho \times \nabla S = -i\nabla\psi^\dagger \times \nabla\psi \tag{4.4.22}$$

or

$$\mathfrak{j}(\psi) = w = dj = d\rho \wedge dS = -id\psi^\dagger \wedge d\psi, \tag{4.4.23}$$

respectively.

Remark 4.4.24. The mapping $\mathfrak{g} \ni \mathbf{b} \mapsto \lambda_{\mathbf{b}} \in \mathcal{C}^\infty(\widetilde{\mathcal{M}})$ yields a *co-moment map*, since by the second item of the previous Theorem

$$\lambda_{\mathbf{b}}(\psi) = \langle \mathfrak{j}(\psi), \mathbf{b} \rangle. \tag{4.4.25}$$

The co-moment map point of view will be applied in Section 5.5 and further explained in Section 5.4. We also remark that even if the functions $\lambda_{\mathbf{b}}$ are defined on $\widetilde{\mathcal{M}}$, they only depend on the probability current $\mathbf{j}$ corresponding to $\psi \in \widetilde{\mathcal{M}}$. More specifically, they depend on the equivalence class $[\mathbf{j}_\psi]$, which can be regarded as an element of a coadjoint orbit of G in $\mathfrak{g}^*$ (Remark 4.2.23), and the previous theorem proves that the Poisson bracket (4.4.8) agrees with hydrodynamical bracket (4.4.19) on the Rasetti-Regge current algebra.

Also note that, as observed in [S23], although the evolution of the probability current can be written in a geometric form (4.4.20), its motion does *not* take place on a coadjoint since the Hamiltonian *does not collectivize* via j in the sense of Guillemin and Sternberg (see e.g. [G84b, M99]).

Other moment map interpretations of j and ρ exist in a density manifold approach to quantum hydrodynamics, see [F17, F19, K18, K19]. There, the Madelung transform yields a moment map for the action of the *semidirect product* [M07, §4.2] $\mathrm{Diff}(\mathbb{R}^3) \ltimes \mathcal{C}^\infty(\mathbb{R}^3)$ on the projective space arising from $L^2(\mathbb{R}^3)$. This holds even if smooth functions are replaced by functions in Sobolev spaces. Moreover, an analogous result was proved in hybrid quantum-classical mechanics in [G20].

4.5 The Clebsch variables

The variables ρ and S, arising from the polar decomposition of the wave function ψ, were referred to as Clebsch variables after (4.4.11). In this section we present their definitions and introduce the Clebsch fibration, following [K80, M83, P92, S23]. This framework enables us to identify the nodal line K (see also Section 4.6 and Chapter 5) as a fibre of the Clebsch fibration.

We begin with the geometric definition of Clebsch variables, following [M83]:

Definition 4.5.1. Clebsch variables on a Poisson manifold P are given by a symplectic manifold M together with a Poisson map $J : M \to P$, that is

$$\{F \circ J, G \circ J\}_M(x) = \{F, G\}_P(J(x)), \tag{4.5.2}$$

for all smooth functions F, G on P. Here, $\{\cdot, \cdot\}_M$ denotes the Poisson bracket on M induced by its symplectic form, and $\{\cdot, \cdot\}_P$ is the Poisson bracket on P.

Now, let $H : P \to \mathbb{R}$ be a Hamiltonian on P, and let X_H be the associated Hamiltonian vector field, defined by

$$\dot{F} = \{F, H\} = X_H(F), \tag{4.5.3}$$

for any smooth function F on P. By Definition 4.5.1, the integral curves of the Hamiltonian vector field on M associated to $H_J := H \circ J$ correspond to integral curves of the Hamiltonian vector field on P pertaining to H. This shows that introducing Clebsch variables allows one to reformulate the dynamics on P in Hamiltonian form [M83, p. 309].

In our setting, the Poisson manifold P is the dual $\mathfrak{g}^*$ of the Lie algebra of G, the Lie group of volume-preserving diffeomorphisms of $\mathbb{R}^3$ rapidly approaching the identity at infinity. Thus the Clebsch variables are provided by the symplectic manifold $\widetilde{\mathcal{M}}$ (see Section 4.1) and the moment map $\mathfrak{j}$ (Theorem 4.4.15), which is G-equivariant and thus Poisson (see, e.g., [M83, p. 307]). Then, since $\mathfrak{j}$ writes

$$\mathfrak{j}(\psi) = \mathfrak{j}(\rho, S) = [\mathbf{j}_\psi] = \rho \nabla S + \nabla f, \tag{4.5.4}$$

by [M83, §5] ρ and S yield *classical* Clebsch variables [K80, P92]. This is coherent with the interpretation of the probability current (4.3.4) as the velocity field of a perfect fluid and of its class or, equivalently, of its vorticity as an element of $\mathfrak{g}^*$ (Remark 4.2.10).

As previously mentioned, the use of Clebsch variables takes the equations of motion in Hamiltonian form. In this setting, the hydrodynamical form of the Schrödinger equation can be recast as (4.3.7) (see also [S16]).

We now introduce the Clebsch fibration, which will be central to our subsequent discussion. Consider the level surfaces defined by fixing the Clebsch variables: $\rho = c_1$ and $S = c_2$, for constants $c_1, c_2 \in \mathbb{R}$. Their intersection yields one-dimensional fibres, which are assumed to be knots in $\mathbb{R}^3$ [K80, S23, B24], or in its one-point compactification S^3. A knot is, briefly, an embedding of S^1 into $\mathbb{R}^3$ up to orientation preserving diffeomorphisms of the circle; see [B93] for the general theory. Since ρ and S are obtained from the polar decomposition of ψ, the fibres can also be regarded as preimages of ψ and, among them there is the nodal line $K := \psi^{-1}(0)$ (at $t = 0$). The latter will be further studied in the following section.

4.6 The nodal line

In this section, we consider the nodal line K, already introduced at the end of the previous section, and review some related results, primarily following [S23, B24]. This object will play a central role in the next chapter, where we analyze the symplectic structures associated with the Schrödinger equation and their restriction to K.

The nodal line is defined as the zero level set of the wave function at time $t = 0$, i.e. $K := \psi^{-1}(0)$. Since ψ is a complex-valued function, its vanishing corresponds to the simultaneous vanishing of its real and imaginary parts. Under suitable regularity assumptions, each of these real-valued functions defines a surface in $\mathbb{R}^3$, and their intersection K is one-dimensional. This is of course coherent with the exposition of the previous section, where K arose as a (1-dimensional) fibre of the Clebsch fibration, and it was assumed to be a knot.

As we already pointed out, the polar decomposition of $\psi \in \widetilde{\mathcal{M}}$ breaks down on the nodal line, being $\rho = 0$ and then S indeterminate. Moreover, definiteness of the wave function requires *quantization* of circulation, see e.g. [T83, S16]:

$$\int_{\mathcal{C}} dS = 2\pi n, \qquad n \in \mathbb{Z} \tag{4.6.1}$$

with $\mathcal{C}$ a closed loop encircling K. This can also be interpreted geometrically as follows [S16]: the *flat* (Maurer-Cartan) $\mathbb{C}^*$-connection

$$\frac{1}{2\pi i}\, d\log\psi = \frac{1}{2\pi i}\left(\frac{1}{2} d\log\rho + i dS\right) \tag{4.6.2}$$

defines an element

$$\left[\frac{1}{2\pi i}d\log\psi\right]=\left[\frac{dS}{2\pi}\right]\in H^1(S^3\setminus K;\mathbb{Z})\cong\mathbb{Z} \tag{4.6.3}$$

and, as such, has trivial holonomy. We refer to [S16] for gauge theoretic aspects of Schrödinger theory.

The nodal line can also rise as a vortex line [L32, M83] if we consider the hydrodynamical reformulation of the Schrödinger equation (4.3.4): in this case $\mathbf{v}=\nabla S$ and then the vorticity is zero everywhere ∇S is defined. Thus the vorticity is concentrated where S is singular, i.e. on K ([B00]). This complicates the analysis of the evolution of K, as "multivalued" fields ([K08, dS16]), or de Rham currents ([dR84, G78]) appear. However, an expression for the velocity of a point on the vortex line has been found in [B00]:

$$\mathbf{u}=\frac{1}{2i}\frac{\mathbf{w}\times\mathbf{w}^\dagger}{|\mathbf{w}\times\mathbf{w}^\dagger|^2}\times(\mathbf{w}\Delta\psi^\dagger+\mathbf{w}^\dagger\Delta\psi), \tag{4.6.4}$$

where $\mathbf{w}=\nabla\psi|_K$. The above difficulties could be overcome by resorting to the probability current $\mathbf{j}$, which shows a more regular behaviour than ∇S (see e.g. [S23, 5.2]), but share the same integral curves, being they proportional outside K. Then, in [S23], the velocity (4.6.4) was proved to be related to the time derivative of the probability current j (here regarded as a 1-form) on the nodal line (at $t=0$):

$$\partial_t j|_K=-\frac{1}{4}[\Delta\psi^\dagger\,d\psi+\Delta\psi\,d\psi^\dagger]. \tag{4.6.5}$$

In more detail, the latter coincides with (4.6.4) up to a $\pi/2$ rotation and scaling. Moreover, the expression for the time derivative of the vorticity $w=dj=-i[d\psi^\dagger\wedge d\psi]$ on K is [S23]:

$$\partial_t w|_K=-\frac{1}{2}[d\psi^\dagger\wedge d(\Delta\psi)-d(\Delta\psi^\dagger)\wedge d\psi]. \tag{4.6.6}$$

Another way to study the evolution of the nodal line pertaining to a wave function is presented in [S23] (see also [B23] for an independent related approach). Consider, at $t=0$, a wave function ψ with nodal line K; then it can be multiplied by a suitable phase term so that it becomes a new wave function ψ_K whose polar coordinates decomposition writes:

$$\psi_K=\sqrt{\rho}e^{iS_K}, \tag{4.6.7}$$

where $S_K(x)$ is the *solid angle function* attached to the node K, i.e. the (oriented) area encircled by the projection of the knot on the unit sphere centered in x (see e.g. [B18]). It is defined up to multiplication by 4π, is harmonic outside K and the following hold:

$$dS_K = B_K, \qquad\qquad dB_K = 2\pi\delta_K, \tag{4.6.8}$$

where the involved forms are actually de Rham currents [dR84, G78] and B_K denotes the magnetic field or the fluid velocity generated by K [F83, 5.2, 5.3], [B20, §2]. The level surfaces of the solid angle function $S_K = c$ provide *Seifert surfaces*, i.e. oriented surfaces with boundary K [F83, 5.3]. In particular, for $c = 0$ one gets the *solid angle framing* [B18, §5].

Now let us take ψ_K as initial wave function and denote $\rho_0\, d^3x$ the corresponding density, which vanishes exactly on K. It evolves via the Schrödinger-Madelung equation (*advection*) [F19], and so does the nodal line. More explicitly the (Eulerian) density writes:

$$\rho(x,t) = \eta_* \rho_0 := \int \rho_0(y)\delta(x - \eta(y,t))d^3y, \tag{4.6.9}$$

where $\eta(\cdot, t) \in \mathrm{Diff}(\mathbb{R}^3)$ and the curves $t \mapsto \eta(y,t)$ are the Bohmian trajectories of the quantum fluid. Therefore the evolution via η_t produces the density at time t, which vanishes exactly on the nodal line at time t, K_t, and corresponds to the Seifert surface Σ_t. Then, given a time-dependent wave function ψ, its phase can be corrected by a time-dependent $U(1)$-action, so that the phase of ψ equals S_{K_t} for any t. Then the corresponding velocity $\mathbf{v} = \nabla S_{K_t}$ is divergence free everywhere it is defined, since S_{K_t} is harmonic outside K_t. In more detail:

$$\tilde{\psi} := g(x,t)\psi(x,t) \equiv \sqrt{\rho(x,t)}\, e^{iS_{K_t}(x)}, \tag{4.6.10}$$

a short calculation then shows that $\tilde{\psi}$ obeys a new "gauged" Schrödinger equation

$$i\partial_t \tilde{\psi} = \tilde{H}\tilde{\psi} \tag{4.6.11}$$

with time-dependent Hamiltonian

$$\tilde{H} = i(\partial_t g)g^{-1} + gHg^{-1}. \tag{4.6.12}$$

One may thus conclude as in [S23] that, so long as the nodal line is concerned, its (compressible) Madelung evolution can be traded for an incompressible one.

Chapter 5

The Clebsch picture

In this chapter, we build on the definitions and results established in the previous chapter to prove the main result of this part, Theorem 5.5. Our approach closely follows the framework developed in [B24].

As shown in Section 4.3, the Madelung transform $\psi(x) \mapsto (\rho(x), S(x))$ casts the Schrödinger equation in hydrodynamical form, allowing quantum-mechanical quantities to be interpreted through fluid dynamics. In this setting nodal lines yield vortex lines, on which the probability density is zero and then the phase, as well as the Madelung transform and the Madelung velocity, are undefined.

We handle these singularities resorting to the Clebsch geometry of the probability current developed in [S23], which yields a collection of closely related symplectic manifolds. They are: the (projectivization of the) wave function manifold, the *Clebsch manifold* [P92] and coadjoint orbits of the Lie group of volume-preserving diffeomorphisms, arising also in fluid dynamics [P92]. We also aim at restricting these manifolds to the nodal line and "localizing" the existing relations among them accordingly, as in [P00].

We develop these connections through a sequence of constructions that form what we call the *Clebsch picture.* Indeed the Clebsch variables ρ and S play a fundamental role in this framework: they yield the probability current and the nodal line can be interpreted as a fibre of the Clebsch fibration; they also mediate between quantum mechanics and fluid dynamics and, in particular, relate wave functions to coadjoint orbits of rotational perfect fluids. Moreover, they admit a reformulation in terms of the $\mathbf{n}$-field representation, which gives rise to the Clebsch manifold [P92].

Section 5.1 introduces the projectivized wave function space equipped with the Fubini-Study Kähler form [K18, K19]. In Section 5.2, this symplectic structure is localized to the nodal line K, connecting it to the symplectic geometry of vortex filaments [M83] or, equivalently, to the space of knots

[B93].

Next, in Section 5.3, we introduce the **n**-field representation, which encodes vorticity via maps $\mathbf{n} : S^3 \to S^2$ and gives rise to a Kähler manifold, as detailed in [P92]. Within this framework, we consider various coadjoint orbits: those associated with the vorticity $\mathbf{w} \in \mathfrak{g}^*$ corresponding to a wave function ψ, with its localized counterpart on the nodal line K, and with the vorticity arising from the **n**-field itself.

These geometric objects are interrelated via localization to the nodal line and through moment maps, as developed in Sections 5.2 and 5.3. Their relations are formalized using a co-moment map framework in Section 5.4. All these constructions converge in Theorem 5.5, which formalizes in diagrammatic form the interplay between quantum hydrodynamics, symplectic geometry, and the nodal line.

5.1 The Fubini-Study Kähler form

In this section, we present the projective version of the wave function manifold $\widetilde{\mathcal{M}}$, introduced in Section 4.4, following the treatment and notation of [K18, K19]. This space arises by considering L^2-normalized wave functions ψ modulo a global phase. The resulting manifold inherits a reduced symplectic structure and a moment map, and we check the well definedness of both. We also briefly review the results concerning the density manifold approach developped in [K18, K19]: these references relate the projective space of wave functions to the density manifold through an isometry and a symplectomorphism given by the Madelung transform. This analysis necessarily excludes the nodal line, as the Madelung transform breaks down on it. However, this still motivates an investigation of the existing relations between the structures arising from wave functions and their localization to the nodal line, which has to be carried out by different means.

The construction we introduce here provides the foundation for relating the projective geometry of wave functions, and its restriction to the nodal line, to other geometric structures that will be analyzed in subsequent sections.

Following [K18, K19], the elements of $\mathcal{M} := \mathbb{P}(\widetilde{\mathcal{M}})$ are written as:

$$[\psi] := \{e^{i\tau}\psi \, : \, \psi \in \widetilde{\mathcal{M}}, \|\psi\|_{L^2} = 1, \tau \in \mathbb{R}\}. \tag{5.1.1}$$

One may, as in the references above, projectivize more generally a Sobolev space, but still using an L^2 product thereon. Here, however, we will always consider smooth functions.

The tangent space at $[\psi]$ to $\mathcal{M}$ is composed of equivalence classes $[\dot{\psi}]$, that is:

$$[\dot{\psi}] = \{ic\psi + \dot{\psi} \,:\, c \in \mathbb{R}\}, \tag{5.1.2}$$

and they are obtained by deriving elements of $[\psi]$. In more detail, let $\tau(t)$ be a curve in $\mathbb{R}$ such that

$$\tau(0) = 0, \qquad \text{and} \qquad \frac{d}{dt}\tau(t)\Big|_{t=0} = c \in \mathbb{R}; \tag{5.1.3}$$

also, let ψ_t be a curve in $\widetilde{\mathcal{M}}$, with

$$\psi(0) = \psi, \qquad \text{and} \qquad \frac{d}{dt}\psi_t\Big|_{t=0} = \dot{\psi}. \tag{5.1.4}$$

Then

$$\frac{d}{dt}e^{i\tau(t)}\psi(t)\Big|_{t=0} = ic\psi + \dot{\psi}, \tag{5.1.5}$$

and if the representative of $[\psi]$ varies, $c \in \mathbb{R}$ varies, thus showing (5.1.2).

Now we can consider the symplectic structure of $\mathcal{M}$, which is equipped with the Fubini-Study Kähler form [K18]:

$$\Omega^{FS}_{\mathcal{M}\,[\psi]}\,([\dot{\psi}]_1, [\dot{\psi}]_2) := \operatorname{Im}\int_{\mathbb{R}^3} \overline{\dot{\psi}_1}\,\dot{\psi}_2\; d^3x, \tag{5.1.6}$$

and it is well defined by Remark 5.1.7 below. This particularly simple expression follows by the fact that we are considering only unitary norm representatives (5.1.1). Without this assumption the integrand of (5.1.6) coincides with the usual expression of the Fubini-Study form on $\mathbb{P}^n(\mathbb{C})$.

Moreover, being S^1-invariant, the G-equivariant moment map $\mathfrak{j}$ (4.4.16) descends to $\mathcal{M}$.

In the following remarks we show that the Fubini-Study form is well defined and can be regarded as a reduced form.

Remark 5.1.7. Let us check that (5.1.6) is well defined: let $ic_1\psi + \dot{\psi}_1$ and $ic_2\psi + \dot{\psi}_2$ be representatives of $[\dot{\psi}]_1$ and $[\dot{\psi}]_2$, respectively. Then

$$\overline{(ic_1\psi + \dot{\psi}_1)}(ic_2\psi + \dot{\psi}_2) = -c_1c_2|\psi|^2 - ic_1\overline{\psi}\dot{\psi}_2 + ic_2\psi\overline{\dot{\psi}}_2 + \overline{\dot{\psi}}_1\dot{\psi}_2. \tag{5.1.8}$$

The integral arising from the first term of (5.1.8) is totally real and thus does not contribute to (5.1.6). Moreover, since we are taking norm 1 ψ's, the following hold:

$$\frac{d}{dt}\|\psi\|^2\Big|_{t=0} = 2\operatorname{Re}\langle\psi, \dot{\psi}_k\rangle = 0, \tag{5.1.9}$$

where $k = 1, 2$ and it depends on the vector we are deriving along. Then, the second (and similarly the third) term of (5.1.8) yields:

$$\begin{aligned}\operatorname{Im}\int_{\mathbb{R}^3} -ic_1\overline{\psi}\dot{\psi}_2 &= \int_{\mathbb{R}^3} -\frac{i}{2}(-ic_1\overline{\psi}\dot{\psi}_2 - ic_1\psi\dot{\overline{\psi}}_2)\,d^3x\\ &= \int_{\mathbb{R}^3} -\frac{c_1}{2}(\overline{\psi}\dot{\psi}_2 + \psi\dot{\overline{\psi}}_2)\,d^3x\\ &= -c_1\operatorname{Re}\langle\psi, \dot{\psi}_2\rangle = 0. \end{aligned}\tag{5.1.10}$$

Remark 5.1.11. The projective space $\mathcal{M}$ can be obtained from $\widetilde{\mathcal{M}}$ by symplectic reduction, just like $\mathbb{P}^n(\mathbb{C})$ is obtained from $\mathbb{C}^{n+1}$. In more detail, S^1 acts naturally on $\widetilde{\mathcal{M}}$ by multiplication, yielding the moment map

$$\varphi : \widetilde{\mathcal{M}} \ni \psi \mapsto -\frac{i}{2}\|\psi\|^2 \in i\mathbb{R}. \tag{5.1.12}$$

Then $\varphi^{-1}(-\frac{i}{2}) = S(\widetilde{\mathcal{M}})$, i.e. the sphere in $\widetilde{\mathcal{M}}$ with respect to the L^2-norm. Equivalently, one can also modify the moment map by adding a constant, thereby getting another moment map whose zero level set is exactly the sphere. Thus $S(\widetilde{\mathcal{M}})/S^1 \cong \mathbb{P}(\widetilde{\mathcal{M}}) = \mathcal{M}$ and if we consider $j : S(\widetilde{\mathcal{M}}) \hookrightarrow \widetilde{\mathcal{M}}$ and $\pi : S(\widetilde{\mathcal{M}}) \to \mathcal{M}$, the pullback via π of the Fubini-Study form (5.1.6) equals the pullback via j of the symplectic form on $\widetilde{\mathcal{M}}$.

Finally, regarding the density manifold approach, by [K18, Theorem 2.4] $\mathcal{M}$ minus the vanishing ψ's, is symplectomorphic to the cotangent space to densities on $\mathbb{R}^3$, that is

$$T^*\mathrm{Dens}(\mathbb{R}^3) = \left\{\rho : \mathbb{R}^3 \to \mathbb{R} \,:\, \rho > 0, \int_{\mathbb{R}^3} \rho\, d^3x = 1\right\} \times \mathcal{C}^\infty(\mathbb{R}^3)/\mathbb{R}, \tag{5.1.13}$$

where $\mathcal{C}^\infty(\mathbb{R}^3)/\mathbb{R} \ni [f] := \{f + c \,:\, f \in \mathcal{C}^\infty(\mathbb{R}^3), c \in \mathbb{R}\}$ and its symplectic form reads [K18]:

$$\Omega^{T^*\mathrm{Dens}}_{(\rho,[S])}((\dot{\rho}_1, [\dot{S}]_1)(\dot{\rho}_2, [\dot{S}]_2)) := \frac{1}{2}\int_{\mathbb{R}^3} \dot{\rho}_1\dot{S}_2 - \dot{\rho}_2\dot{S}_1\, d^3x. \tag{5.1.14}$$

The symplectomorphism is induced by the polar coordinates (Madelung transform) and is:

$$\begin{aligned}\Phi \,:\, T^*\mathrm{Dens}(\mathbb{R}^3) &\to \mathcal{M} \setminus \{[\psi] \in \mathcal{M} \,:\, \exists\, x \in \mathbb{R}^3 \,\text{s.t.}\, \psi(x) = 0\}\\ (\rho, [S]) &\mapsto [\psi].\end{aligned}\tag{5.1.15}$$

5.2 Localizing to K

In this section, we introduce the maps $\mathcal{K}$ and ℓ, which encode the "migration" from a diffuse to a string-like vorticity field concentrated on the nodal line K. This approach provides a framework for comparing localized symplectic structures arising from wave functions.

5.2.1 The map $\mathcal{K}$

In this subsection we introduce the map $\mathcal{K}$: it assigns to each $[\psi] \in \mathcal{M}$ a vorticity field supported on the nodal line of ψ and tangent to it. We show that it is a moment map for $\mathcal{M}$ equipped with a symplectic structure "localized" on K, which is related to the symplectic and complex structure of vortex lines [M83, §10] or, more generally, to that of the space of knots [B93, 3.6].

Consider the map $\mathcal{K}$: it associates to the equivalence class $[\psi] \in \mathcal{M}$, its nodal line K or, equivalently, its Poincaré dual:

$$\mathcal{K} : \mathcal{M} \ni [\psi] \mapsto K = \psi^{-1}(0) \equiv w_K \in \mathcal{O}_{w_K} \subset \mathfrak{g}^*. \tag{5.2.1}$$

Here w_K denotes the singular Poincaré dual of K, see [B82a, G78], i.e. a de Rham current that corresponds to a δ-like vorticity 2-form concentrated on K. This writes, as a vector, $\mathbf{w}_K(x) = \int_K \delta^3(x - y(s))\, dy(s)$ and, since it is still a vorticity whose velocity is given by the Biot-Savart law (4.6.8), $\mathbf{w}_K \in \mathfrak{g}^*$ and we can consider the corresoponding coadjoint orbit through it $\mathcal{O}_{w_K}$.

Let us then localize on the nodal line by pulling back wave functions to K and by replacing $\int_{\mathbb{R}^3}$ by $\int_K$ throughout, in particular in evaluating the Fubini-Study form. The crucial observation is the following: let K be parametrized by its arc-length s and $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ be the standard Frénet trihedron. We assume absence of inflection points in K, which, in any case, yields a measure zero set. Defining

$$\dot{\psi} = \operatorname{Re} \dot{\psi} + i \operatorname{Im} \dot{\psi} \;\equiv\; \dot{\psi}_n + i\, \dot{\psi}_b\,, \tag{5.2.2}$$

gives the following vector field on K:

$$\mathbf{a}_{\dot{\psi}}(s) := \dot{\psi}_n(s)\, \mathbf{n}(s) + \dot{\psi}_b(s)\, \mathbf{b}(s)\,. \tag{5.2.3}$$

Therefore we may set up the identification

$$\dot{\psi}|_K \equiv \mathbf{a}_{\dot{\psi}} \tag{5.2.4}$$

and, given $\dot\psi_1$, $\dot\psi_2$, a computation involving the previous equalities yields the following:

$$\begin{aligned} B_{\mathbf{w}_K}(\mathbf{a}_{\dot\psi_1}, \mathbf{a}_{\dot\psi_1}) &= \int_K \mathbf{t} \cdot (\mathbf{a}_{\dot\psi_1} \times \mathbf{a}_{\dot\psi_2})\, ds \equiv \mathrm{Im} \int_K \overline{\dot\psi_1|_K}\, \dot\psi_2|_K \\ &=: \widetilde{\Omega^{FS}}_{\mathcal{M}}(\dot\psi_1|_K, \dot\psi_2|_K) \end{aligned} \tag{5.2.5}$$

after an appropriate integral replacement in the Fubini-Study form, signalled by the tilde. Here, the Kirillov-Kostant-Souriau form on $\mathcal{O}_{\mathbf{w}_K}$ at $\mathbf{w}_K$ is written (with a slight abuse of notation) as in [M83, §10], in which the tangent space to the coadjoint orbit has been identified with the vector fields normal to K. In further generality, this Kirillov-Kostant-Souriau symplectic form may also be regarded as a symplectic form on the space of knots (evaluated at K) [B93, 3.5, 3.7].

The coadjoint orbit $\mathcal{O}_{\mathbf{w}_K}$ or, equivalently, the space of knots at K, carries a formally integrable complex structure [B93, 3.6], i.e. its Nijenhuis tensor is zero, even though this does not automatically imply integrability due to the infinite-dimensional nature of the setting [L93a]. Since this complex structure is obtained by identifying the normal plane to the curve at each point with a copy of $\mathbb{C}$ [M83, §10], it is compatible with the complex structure of $\mathcal{M}$ by (5.2.2, 5.2.4).

Upon substituting Ω^{FS} with $\widetilde{\Omega^{FS}}$ and considering (5.2.5), the map $\mathcal{K}$ becomes a G-equivariant moment map. Indeed, replacing the image of $\mathfrak{j}$ (4.4.16) with a delta-like vorticity gives rise to $\mathcal{K}$, which is then a moment map for $(\mathcal{M}, \widetilde{\Omega^{FS}})$. The corresponding co-moment map is described in Section 5.4.

Remark 5.2.6. The (infinite-dimensional) "symplectic reduction" yields:

$$\mathcal{K}^{-1}(K)/G_{\mathbf{w}_K} \cong \{K\}, \tag{5.2.7}$$

where $G_{\mathbf{w}_K}$ denotes the stabilizer of $\mathbf{w}_K$ in G. Indeed, $\mathcal{K}^{-1}(K)$ is composed of wave functions having the same zero-set K; then, as the $G_{\mathbf{w}_K}$-action just reparametrizes K, the quotient can be identified with K. Moreover, the functions in (5.2.7) give rise to the same integral cohomology class $[\frac{1}{2\pi i} d \log \psi] \in H^1(S^3 \setminus K; \mathbb{Z})$ (see also the discussion in Section 4.6).

5.2.2 The map ℓ

Let us fix $[\psi] \in \mathcal{M}$, and thus the corresponding nodal line K throughout this subsection. Then we can introduce another localization map ℓ which is defined in view of the analysis carried out in [P00]: it restricts the vorticity $\mathbf{w}$ to a concentrated vorticity $\mathbf{w}_K$ on the string-like domain K.

It is explicitly defined as follows:

$$\ell : \mathcal{O}_{\mathbf{w}} \ni g(\mathbf{w}) \mapsto g(\mathbf{w}_K) \in \mathcal{O}_{\mathbf{w}_K}, \tag{5.2.8}$$

where the action is the coadjoint one of G on $\mathfrak{g}^*$, and it is well defined in view of the inclusion [P92, S16]:

$$G_{\mathbf{w}} \subset G_{\mathbf{w}_K}, \tag{5.2.9}$$

where $G_{\mathbf{w}}$ denotes, as before, the stabilizer group of $\mathbf{w}$ and, similarly, $G_{\mathbf{w}_K}$ that of $\mathbf{w}_K$. The "migration" described by ℓ is also reflected in the corresponding Rasetti-Regge currents $\lambda_{\mathbf{b}}$ (see Section 5.4 for details). Moreover, this map is related to the localization map $\mathcal{K}$ introduced in the previous section as follows:

$$\ell \circ \mathfrak{j} = \mathcal{K}. \tag{5.2.10}$$

Indeed, recalling the definition of the moment map $\mathfrak{j}$ (4.4.16, 4.4.21), which descends to $\mathcal{M}$, one has:

$$\ell \circ \mathfrak{j}([\psi]) = \ell(\mathbf{w}) = \mathbf{w}_K = \mathcal{K}[\psi]. \tag{5.2.11}$$

5.3 The n-field representation

In this section we introduce the **n**-field representation [K80, P92] and some related results; most notably, it gives rise to the *Clebsch manifold* [P92, Theorem 3], a symplectic manifold that takes part in Theorem 5.5.

The **n**-field representation [K80, P92] describes the vorticity in terms of smooth maps $\mathbf{n} : S^3 \to S^2$ via the following *ansatz*:

$$\tilde{\mathbf{w}} = \mathbf{n} \cdot \nabla\mathbf{n} \times \nabla\mathbf{n}, \tag{5.3.1}$$

i.e.

$$(\tilde{\mathbf{w}})_i = \epsilon_{ijk}\mathbf{n} \cdot \partial_j\mathbf{n} \times \partial_k\mathbf{n}, \tag{5.3.2}$$

where ϵ_{ijk} is the Levi-Civita tensor. If we further assume that $\mathbf{n}(x)$ is a regular value for any $x \in S^3$ and that the fibres are diffeomorphic to S^1, the map $\mathbf{n}$ yields a Hopf fibration and the corresponding Hopf number is related to the helicity [K80]. In more detail, let us express $\mathbf{n}$ in spherical coordinates:

$$\mathbf{n} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) \in S^2, \tag{5.3.3}$$

where ϑ and φ are functions of $x \in S^3$ denoting, as usual, colatitude and longitude, the latter counted from the x-axis. Thus, as a 2-form $\tilde{w} = \mathbf{n}^*\sigma_{S^2}$, where $\sigma_{S^2} = \sin\vartheta\, d\vartheta \wedge d\varphi$ is the standard volume form on S^2 [P92]. Moreover, being σ closed, $\tilde{w}$ is also closed and hence exact, as $H^2(S^3) = 0$. Then there exists a 1-form $\tilde{\jmath}$ on S^3, determined up to an exact 1-form, such that $d\tilde{\jmath} = \tilde{w}$, and we can compute the helicity corresponding to $\mathbf{n}$:

$$\mathcal{H}(\mathbf{n}) = \int_{\mathbb{R}^3} \tilde{\jmath} \wedge d\tilde{\jmath}, \tag{5.3.4}$$

representing a multiple of the linking number between two generic fibres of the map $\mathbf{n}$ (see [P92, Appendix], and [B82a] for the general theory).

The $\mathbf{n}$-field representation is closely related to the Clebsch variables [K80, P92]: from the expression of $\mathbf{n} : S^3 \to S^2$ in spherical coordinates (5.3.3) one can obtain Clebsch variables $\alpha := \cos\vartheta$ and $\beta := \varphi$, i.e.

$$\tilde{\mathbf{v}} = \alpha\nabla\beta + \nabla\phi, \tag{5.3.5}$$

and $\operatorname{curl}\tilde{\mathbf{v}} = \tilde{\mathbf{w}}$. Moreover, the linking number between the fibres of the Clebsch fibration rising from $\mathbf{n}$, coincides with the one between the fibres of $\mathbf{n}$ (see also below). Here ϕ coincides with f of Section 4.5 and can be regarded as a fibre parameter [P92].

Now we focus on the geometric structure arising from the set of maps $\mathbf{n}$: it is shown in [P92, Theorem 3] that

$$\widetilde{\mathcal{N}} := \{\mathbf{n} : S^3 \to S^2 \,:\, \mathbf{n}(\infty) = \mathbf{n}_\infty,\, \mathcal{H}(\mathbf{n}) = k \in \mathbb{R}\} \tag{5.3.6}$$

is a symplectic Kähler manifold, called *Clebsch manifold*. Note that the condition on the helicity fixes the linking number for the maps $\mathbf{n} \in \widetilde{\mathcal{N}}$. Its symplectic structure reads, for generic smooth vector fields $\mathbf{b}$ and $\mathbf{c}$ on $\mathbb{R}^3$ [P92, P00]:

$$\begin{aligned}\Omega^{\widetilde{\mathcal{N}}}_{\mathbf{n}}(X_{\mathbf{b}}, X_{\mathbf{c}}) &:= \int_{\mathbb{R}^3} \mathbf{n}(x) \cdot (\mathbf{b}\cdot\nabla\, \mathbf{n}(x) \times \mathbf{c}\cdot\nabla\, \mathbf{n}(x))\, d^3x \\ &= \int_{\mathbb{R}^3} (\sigma_{S^2})_{\mathbf{n}(x)}(\mathbf{b}\cdot\nabla\, \mathbf{n}(x), \mathbf{c}\cdot\nabla\, \mathbf{n}(x))\, d^3x,\end{aligned} \tag{5.3.7}$$

where

$$X_{\mathbf{a}}(\mathbf{n}) := \mathbf{a}\cdot\nabla\mathbf{n},\ \mathbf{a} \in \mathfrak{X}(\mathbb{R}^3) \tag{5.3.8}$$

and such vector fields span the tangent space to $\widetilde{\mathcal{N}}$ [P92, p. 906].

Now let us relate $\mathbf{n} \in \widetilde{\mathcal{N}}$ to $\psi \in \widetilde{\mathcal{M}}$ via stereographic projection (from the south pole):

$$\psi = \tan\frac{\vartheta}{2}\, e^{i\varphi} =: \sqrt{\rho}\, e^{iS}; \tag{5.3.9}$$

indeed ϑ and φ identify a unique $\mathbf{n}$ by (5.3.3). Notice that upon a phase change $\psi \mapsto e^{i\alpha}\,\psi$, the vorticity $\mathbf{w} = -i\nabla\psi^\dagger \times \nabla\psi$ pertaining to ψ does not change and the corresponding $\mathbf{n}$ rotates around the polar axis by an angle α, whence we get a map

$$\mathcal{I} : \mathcal{M} \ni [\psi] \mapsto [\mathbf{n}] \in \mathcal{N} := \widetilde{\mathcal{N}}/S^1. \tag{5.3.10}$$

Let us define a new vorticity, again phase invariant:

$$\hat{w} = \frac{w}{(1+|\psi|^2)^2} = -i\,\frac{d\psi^\dagger \wedge d\psi}{(1+|\psi|^2)^2}. \tag{5.3.11}$$

The latter is pointwise proportional to w and, in particular, coincides with it on K and is tangent to it. Then it can be checked that:

$$\hat{w} = \frac{1}{2}\mathbf{n}^*\sigma_{S^2} = \frac{1}{2}\tilde{w}, \tag{5.3.12}$$

upon substituting (5.3.9) in (5.3.11).

This allows us to recover the moment map for the G-action on $\mathcal{N}$, which is endowed with a symplectic form that still reads as (5.3.7). Indeed, the map $[\psi] \mapsto \hat{w}$ is G-equivariant, and then induces a map between the G-orbits $\mathcal{O}_{[\psi]} \subset \mathcal{M}$ and $\mathcal{O}_{\hat{w}} \subset \mathfrak{g}^*$. Combining this fact with (5.3.12) and (5.3.10), the following is well defined:

$$\begin{aligned} \nu : \mathcal{N} \supset \mathcal{I}(\mathcal{O}_{[\psi]}) &\to \mathcal{O}_{\hat{w}}, \\ [\mathbf{n}] &\mapsto \hat{w} \end{aligned} \tag{5.3.13}$$

and yields a moment map (referred to as μ in [P92]) in view of Remark 4.4.21 and [P92, Theorem 3].

Moreover, proceeding as for $\tilde{w}$, we can show the existence, up to an exact 1-form, of a 1-form $\hat{\jmath}$ such that:

$$d\hat{\jmath} = \hat{w}, \tag{5.3.14}$$

which yields a G-equivariant map

$$\hat{\mathfrak{j}} : \mathcal{M} \ni [\psi] \mapsto [\hat{\jmath}] \in \mathcal{O}_{\hat{w}} \subseteq \mathfrak{g}^*, \tag{5.3.15}$$

analogous to the moment map $\mathfrak{j}$ and pointwise proportional to it. Moreover, even though $\int \hat{\jmath} \wedge \hat{\omega}$ differs from (5.3.4), the linking number of the Clebsch fibration rising from $\hat{w}$ coincides with the one of the fibration rising from w. Indeed, passing from w to $\hat{w}$ the topology does not change, since only a pointwise rescalement is involved.

After this exposition we can naturally define another localization map:

$$\hat{\ell} : \mathcal{O}_{\hat{w}} \to \mathcal{O}_{w_K}, \tag{5.3.16}$$

enjoying the same properties of (5.2.8).

5.4 Rasetti-Regge picture

In this section, we examine the co-moment maps associated with the moment maps introduced earlier in the chapter. These co-moment maps generate Rasetti-Regge-like currents (Definition 4.2.18) and highlight the relation between the geometric structures developed so far. The relations revealed by this approach will be summarized diagrammatically in the following section. Throughout, the co-moment maps are evaluated at $[\psi] \in \mathcal{M}$, with associated vorticity $\mathbf{w}$, and are labeled by the coadjoint orbit to which $\mathcal{O}_{[\psi]}$ is mapped under the corresponding moment map.

Recall that a G-equivariant moment map on a manifold, $\mu : M \to \mathfrak{g}^*$, naturally induces, for any element $\mathbf{b} \in \mathfrak{g}$, a *co-moment map* $\mu_{\mathbf{b}}$, defined as the smooth function on M given by

$$\mu_{\mathbf{b}}(x) := \langle \mu(x), \mathbf{b} \rangle. \tag{5.4.1}$$

Then let us consider the co-moment map induced by $\mathcal{K}$ (5.2.1):

$$\begin{aligned} &\lambda^{\mathcal{K}}_{\mathbf{b}} : \mathcal{M} \to \mathbb{R}, \\ &[\psi] \mapsto \lambda^{\mathcal{K}}_{\mathbf{b}}([\psi]) := \int_{\mathbb{R}^3} \mathbf{w}_K \cdot \mathbf{B}\, d^3x = \int_K \mathbf{B}. \end{aligned} \tag{5.4.2}$$

This expression follows by (4.4.17), where the vorticity is replaced by $\mathbf{w}_K = \mathcal{K}([\psi])$, concentrated on K, and, as before, $\operatorname{curl} \mathbf{B} = \mathbf{b}$.

Moreover, localization via ℓ on an element of the Rasetti-Regge current algebra yields the following:

$$\lambda^{\mathcal{O}_w}_{\mathbf{b}}([\psi]) := \int_{\mathbb{R}^3} \mathbf{w} \cdot \mathbf{B}\, d^3x \;\to\; \int_K \mathbf{B} =: \lambda^{\mathcal{O}_{w_K}}_{\mathbf{b}}([\psi]), \tag{5.4.3}$$

that is, the localization via ℓ substitutes (the vector field corresponding to) $w = d[\mathfrak{j}([\psi])]$ with the delta-like $w_K = d[\ell \circ \mathfrak{j}([\psi])]$. Similarly, for $\hat{\ell}$,

$$\lambda^{\mathcal{O}_{\hat{w}}}_{\mathbf{b}}([\psi]) = \int_{\mathbb{R}^3} \hat{\mathbf{w}} \cdot \mathbf{B}\, d^3x \;\to\; \int_K \mathbf{B} =: \lambda^{\mathcal{O}_{w_K}}_{\mathbf{b}}([\psi]), \tag{5.4.4}$$

where this time the localization via $\hat{\ell}$ substitutes (the vector field corresponding to) $\hat{w} = d[\hat{\mathfrak{j}}([\psi])]$ with the delta-like $w_K = d[\hat{\ell} \circ \hat{\mathfrak{j}}([\psi])]$.

5.5 The main result

We collect the previous findings in a theorem, whose proof immediately follows from the above preparations.

Theorem 5.5.1. (Clebsch portrait). *(i) With the above notation, we have a commutative diagram:*

$$\begin{tikzcd}
 & \mathcal{M} \supset \mathcal{O}_{[\psi]} \arrow[rr, "\mathfrak{j}"] \arrow[dl, "\mathfrak{I}"] \arrow[d, "\hat{\mathfrak{j}}"] \arrow[drr, "\mathcal{K}"] & & \mathcal{O}_w \arrow[d, "\ell"] \\
\mathcal{N} \supset \mathfrak{I}(\mathcal{O}_{[\psi]}) \arrow[r, "\nu"] & \mathcal{O}_{\hat{w}} \arrow[rr, "\hat{\ell}"] & & \mathcal{O}_{w_K}
\end{tikzcd}$$

where all maps other than $\mathfrak{I}$ are G-equivariant moment maps and $\mathcal{K}$, ℓ, $\hat{\ell}$ all "localize" to K.

(ii) The overall diagram is compatible with the dynamics, provided we replace the Schrödinger Hamiltonian H by the time dependent Hamiltonian $\tilde{H}$ (4.6.12)*, thus switching to a perfect fluid flow induced by the gradient of the solid angle function, thereby keeping track of the vortex line evolution only.*

(iii) The complex structures on $\mathcal{M}$, $\mathcal{N}$, $\mathcal{O}_{w_K}$ are compatible, at fixed time.

We conclude this section with two remarks:

Remark 5.5.2. Let us remark *in which sense* the maps involved in the diagram are moment maps. By Theorem 4.4.15, $\mathfrak{j}$ is a moment map on $\widetilde{\mathcal{M}}$ and, as seen in Section 5.1 it descends to $\mathcal{M}$. The map ν is a moment map by [P92, Theorem 3] and $\hat{\mathfrak{j}}$ is pointwise proportional to $\mathfrak{j}$ (Section 5.3), and coincides with it on K. The map $\mathcal{K}$ is a moment map in the sense explained in Section 5.2.1, i.e. for the symplectic structure given by the localized Fubini-Study form $\widetilde{\Omega_{FS}}$. Similarly, the localization maps ℓ and $\hat{\ell}$ are moment maps for the Kirillov-Kostant-Souriau form replaced by a localized version on K.

Remark 5.5.3. The use of two different vorticities w and $\hat{w}$ was motivated by the fact that here the Clebsch variables (ρ and S) are dictated by the quantum-mechanical context and the ensuing $\mathbf{n}$-field representation is ancillary. In the papers [K80, P92, P98], the latter was the primary object whereby vorticities arose.

Conclusion (Part II)

In this part we developed a Clebsch-type picture for the Schrödinger equation, following [B24]. Extending the analysis of [S23], we focused on the evolution of the zero set of a wave function, under the assumption that it forms a knot in $\mathbb{R}^3$. Its Schrödinger dynamics can be reformulated as Euler evolution via the divergence-free probability current $\tilde{\jmath}$, which, unlike the Madelung velocity, is regular on K, though sharing the same trajectories with the latter outside K.

From this perspective, a rich network of symplectic manifolds emerges. The main message is that the standard complex formalism of quantum mechanics, when complemented by the projective geometry of pure states, naturally overcomes the limitations of the polar decomposition of the wave function.

In conclusion, the Clebsch portrait framework may be extended to other quantum-mechanical and hybrid systems. A particularly promising direction would be to establish a direct link with the approaches presented in [B23] and [G20]. The former combines a hydrodynamical formulation of the Gross–Pitaevskii equation with Kleinert's multivalued gauge theory [K08], which employs currents to deal with vortex lines and leads to the proof of a zero-helicity condition for the corresponding system. The latter outlines a hydrodynamical treatment of the Koopman–van Hove equation via the Madelung transform, yielding results parallel to those in [K18, K19]. In both studies, nodal lines emerge naturally, suggesting that an analysis analogous to that of Part II could produce Clebsch portraits in these new contexts.

Bibliography

[A78] Ralph Abraham and Jerrold E. Marsden, *Foundations of mechanics.* Benjamin/Cummings Publishing Co. Inc., Reading, Mass., 2nd edition, 1978.

[A91] Judith M. Arms, Richard H. Cushman, and Mark J. Gotay, A universal reduction procedure for Hamiltonian group actions. In: *The geometry of Hamiltonian systems (Berkeley, 1989)*, Math. Sci. Res. Inst. Publ., vol. 22, pp. 33–51. Springer, New York, 1991.

[A21] Vladimir I. Arnold and Boris A. Khesin, *Topological methods in hydrodynamics*, Applied Mathematical Sciences, vol. 125. Springer Nature Switzerland AG, 2nd edition, 2021.

[A71] Louis Auslander and Bertram Kostant, Polarization and unitary representations of solvable Lie groups. *Invent. Math.* **14** (1971) 255–354.

[B06] Erik P. van den Ban, *Notes on quotients and group actions.* Universiteit Utrecht lecture notes, 2006.

[B25] Gabriele Barbieri, Jordan Watts, and François Ziegler, Remarks on diffeological Frobenius reciprocity. To appear in *Trans. Amer. Math. Soc.*; arXiv:2403.03927.

[B24] Gabriele Barbieri and Mauro Spera, A Clebsch portrait for Schrödinger theory. *Eur. Phys J. Plus* **139** (2024) 1–8.

[B97] Larry M. Bates and Eugene Lerman, Proper group actions and symplectic stratified spaces. *Pacific J. Math.* **181** (1997) 201–229.

[B23] Andrea Belloni and Renzo L. Ricca, On the zero helicity condition for quantum vortex defects. *J. Fluid Mech.* **963** (2023) R2-1–R2-9.

[B04] Alberto Benvegnù, Nicola Sansonetto, and Mauro Spera, Remarks on geometric quantum mechanics. *J. Geom. Phys.* **51** (2004) 229–243.

[B01a] Michael Berry, Knotted zeros in the quantum states of hydrogen. *Found. Phys* **31** (2001), 659–667.

[B00] Iwo Bialynicki-Birula, Zofia Bialynicka-Birula, and Cezary Śliwa, Motion of vortex lines in quantum mechanics. *Phys. Rev. A* **61** (2000) 1–7.

[B18] Jack Binysh and Gareth P. Alexander, Maxwell's theory of solid angle and the construction of knotted fields. *J. Phys. A: Math. Theor.* **51** (2018) 1–20.

[B52a] David Bohm, A suggested interpetation of the quantum theory in terms of "hidden" variables I. *Phys. Rev.* **85** (1952) 166–179.

[B52b] ______, A suggested interpetation of the quantum theory in terms of "hidden" variables II. *Phys. Rev.* **85** (1952) 180–193.

[B20] Maciej Borodzik, Supredee Dangskul, and Andrew Ranicki, Solid angles and Seifert hypersurfaces. *Ann. Glob. Anal. Geom.* **57** (2020) 415–454.

[B82a] Raoul Bott and Loring Tu, *Differential forms in algebraic topology*, Graduate texts in Mathematics, vol. 98. Springer, Berlin, 1982.

[B60] Nicolas Bourbaki, *Topologie générale. Chapitre I: Structures topologiques. Chapitre II: Structures uniformes. Chapitre III: Groupes topologiques. Chapitre IV: Nombres réels.* Hermann, Paris, 1960, 1961.

[B67] ______, *Variétés différentielles et analytiques. Fascicule de résultats. Paragraphes 1 à 7. Paragraphes 8 à 15.* Hermann, Paris, 1967, 1971.

[B72] ______, *Groupes et algèbres de Lie. Chapitre II: Algèbres de Lie libres. Chapitre III: Groupes de Lie.* Hermann, Paris, 1972.

[B82b] ______, *Groupes et algèbres de Lie. Chapitre IX: Groupes de Lie réels compacts.* Masson, Paris, 1982.

[B93] Jean-Luc Brylinski, *Loop spaces, characteristic classes and geometric quantization.* Birkhäuser, Basel, 1993.

[B85] Theodor Bröcker and Tammo tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98. Springer, New York, 1985.

[B01b] D. Brody and L. Hughston, Geometric quantum mechanics. *J. Geom. Phys.* **38** (2001) 19–53.

[C02] Élie Cartan, Sur l'intégration des systèmes différentiels complètement intégrables. *C. R. Acad. Sci. Paris* **134** (1902) 1415–1417, 1564–1566.

[C22] ———, *Leçons sur les invariants intégraux.* Hermann, Paris, 1922.

[C29] ———, Sur les invariants intégraux de certains espaces homogènes clos et les propriétés topologiques de ces espaces. *Ann. Soc. Polon. Math.* **8** (1929) 181–225.

[C50] Henri Cartan, Cohomologie réelle d'un espace fibré principal différentiable. In: notions d'algèbre différentielle, algèbre de Weil d'un groupe de Lie. *Séminaire Henri Cartan*, vol. 2, pp. 19-01–19-10. Secrétariat mathématique, Paris, 1950.

[C77] Kuo Tsai Chen, Iterated path integrals. *Bull. Amer. Math. Soc.* **83** (1977) 831–879.

[C69] Gustave Choquet, *Lectures on analysis. Vol. I.* W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[C88a] Lawrence Corwin and Frederick P. Greenleaf, A canonical approach to multiplicity formulas for induced and restricted representations of nilpotent Lie groups. *Comm. Pure Appl. Math.* **41** (1988) 1051–1088.

[C88b] Čedomir Crnković, Symplectic geometry of the covariant phase space. *Class. Quantum Grav.* **5** (1988) 1557–1575.

[C97] Richard H. Cushman and Larry M. Bates, *Global aspects of classical integrable systems.* Birkhäuser, Basel, 1997.

[dR84] Georges de Rham, *Differentiable manifolds.* A Series of Comprehensive Studies in Mathematics, vol. 266. Springer, Berlin, 1984.

[D99] Simon K. Donaldson, Moment maps and diffeomorphisms. *Asian J. Math.* **3** (1999) 1–16.

[dS16] Francisco E. A. dos Santos, Hydrodynamics of vortices in Bose-Einstein condensates: A defect-gauge field approach. *Phys. Rev. A* **94** (2016) 1–6.

[D00] Johannes J. Duistermaat and Johan A.C. Kolk, *Lie groups.* Universitext, Springer, Berlin, 2000.

[E70] David G. Ebin and Jerrold E. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. of Math.* **92** (1970) 102–163.

[F88] James M. G. Fell and Robert S. Doran, *Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles.* Vol. 2. Pure and Applied Mathematics, vol. 126. Academic Press, San Diego, 1988.

[F83] Roger Fenn, *Techniques of geometric topology.* Cambridge University Press, Cambridge, 1983.

[F20a] Michael S. Foskett, *Geometry of quantum hydrodynamics in theoretical chemistry.* PhD thesis, University of Surrey, 2020; arXiv:2009.13601v1.

[F19] Michael S. Foskett, Darryl D. Holm, and Cesare Tronci, Geometry of nonadiabatic quantum hydrodynamics. *Acta Appl. Math.* **162** (2019) 63–103.

[F22] Michael S. Foskett and Cesare Tronci, Holonomy and vortex structures in quantum hydrodynamics. In *Hamiltonian Systems: Dynamics, Analysis, Applications (Cambridge 2022)*, Math. Sci. Res. Inst. Pub., vol. 72, pp. 173–214. Cambridge University Press, Cambridge, 2024.

[F20b] Greg Friedman, *Singular intersection homology.* New Mathematical Monographs, vol. 33. Cambridge University Press, Cambridge, 2020.

[F82] Alfred Frölicher, Smooth structures. In: *Category theory*, Lecture Notes in Mathematics, vol. 962, pp. 69–81. Springer, Berlin, 1982.

[F91] Hidenori Fujiwara, Sur les restrictions des représentations unitaires des groupes de Lie résolubles exponentiels. *Invent. Math.* **104** (1991) 647–654.

[F15] Hidenori Fujiwara and Jean Ludwig, *Harmonic analysis on exponential solvable Lie groups.* Springer Monographs in Mathematics. Springer, Tokyo, 2015.

[F17] Daniel Fusca, The Madelung transform as a momentum map. *J. Geom. Mech.* **9** (2017) 157–165.

[G20] François Gay-Balmaz and Cesare Tronci, Madelung transform and probability densities in hybrid quantum-classical dynamics. *Nonlinearity* **33** (2020) 5383–5424.

[G80] Mark Goresky and Robert MacPherson, Intersection homology theory. *Topology* **19** (1980) 135–162.

[G78] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry.* J. Wiley & Sons, New York, 1978.

[G02] Victor Guillemin, Viktor Ginzburg, and Yael Karshon, *Moment maps, cobordisms, and Hamiltonian group actions.* Mathematical Surveys and Monographs, vol. 98. Amer. Math. Soc., Providence, RI, 2002.

[G05] Victor Guillemin and Reyer Sjamaar, *Convexity properties of Hamiltonian group actions.* CRM Monograph Series, vol. 26. Amer. Math. Soc., Providence, RI, 2005.

[G82] Victor Guillemin and Shlomo Sternberg, Geometric quantization and multiplicities of group representations. *Invent. Math.* **67** (1982) 515–538.

[G83] ———, The Frobenius reciprocity theorem from a symplectic point of view. *Lecture Notes in Mathematics* **1037** (1983) 242–256.

[G84a] ———, A normal form for the moment map. In: *Differential Geometric Methods in Mathematical Phyisics.* Mathematical Physics Studies, Vol. 6, pp. 161–175. Reidel, Dordrecht, 1984.

[G84b] ———, *Symplectic techniques in physics.* Cambridge University Press, Cambridge, 1984.

[G23] Serap Gürer and Patrick Iglesias-Zemmour, Orbifolds as stratified diffeologies. *Differential Geom. Appl.* **86** (2023) 1–15.

[H15] Brian C. Hall, *Lie groups, Lie algebras, and representations: an elementary introduction.* Graduate Texts in Mathematics, vol. 185. Springer, New York, 2015.

[H11] Gilbert Hector, Enrique Macías-Virgós, and Esperanza Sanmartín-Carbón, De Rham cohomology of diffeological spaces and foliations. *Indag. Math. (N.S.)* **21** (2011) 212–220.

[H12] Joachim Hilgert and Karl-Hermann Neeb, *Structure and geometry of Lie groups.* Springer Monographs in Mathematics. Springer, New York, 2012.

[H09] Darryl D. Holm, Tanya Schmah, and Cristina Stoica, *Geometric mechanics and symmetry: from finite to infinite dimensions.* Oxford University Press, Oxford, 2009.

[I85] Patrick Iglesias, *Fibrations difféologiques et homotopie.* PhD thesis, Université de Provence, 1985.

[I10] Patrick Iglesias-Zemmour, The moment maps in diffeology. *Mem. Amer. Math. Soc.* **207** (2010) 1–72.

[I13] ———, *Diffeology.* Mathematical Surveys and Monographs, vol. 185. Amer. Math. Soc., Providence, RI, 2013.

[I16] ———, Example of singular reduction in symplectic diffeology. *Proc. Amer. Math. Soc.* **144** (2016) 1309–1324.

[K16] Yael Karshon and Jordan Watts, Basic forms and orbit spaces: a diffeological approach. *SIGMA Symmetry Integrability Geom. Methods Appl.* **12** (2016) 1–19.

[K19] Louis H. Kauffman and Samuel J. Lomonaco, Quantum knots and knotted zeros. In: *Quantum Information Science, Sensing, and Computation XI (Baltimore 2019)*, Proc. SPIE, vol. 10984, Spie Digital Library, 7 pp., 2019.

[K78] David A. Kazhdan, Bertram Kostant, and Shlomo Sternberg, Hamiltonian group actions and dynamical systems of Calogero type. *Comm. Pure Appl. Math.* **31** (1978) 481–507.

[K18] Boris Khesin, Gerard Misiołek, and Klas Modin, Geometric hydrodynamics via Madelung transform. *PNAS* **115** (2018) 6165–6170.

[K19] Boris Khesin, Gerard Misiołek, and Klas Modin, Geometry of the Madelung transform. *Arch. Rational Mech. Anal.* **234** (2019) 549–573.

[K62] Aleksandr A. Kirillov, Unitary representations of nilpotent Lie groups. *Uspekhi Mat. Nauk* **17** (1962) 57–110.

[K76] ———, *Elements of the theory of representations.* Grundlehren der mathematischen Wissenschaften, vol. 220. Springer, Berlin, 1976.

[K08] Hagen Kleinert, *Multivalued fields in condensed matter, electromagnetism, and gravitation.* World Scientific, Singapore, 2008.

[K70] Bertram Kostant, Quantization and unitary representations. In: *Lectures in Modern Analysis and Applications III.* Lecture Notes in Mathematics, vol. 170, pp. 87–207. Springer, Berlin-Heidelberg-New York, 1970.

[K53] Jean-Louis Koszul, Sur certains groupes de transformations de Lie. In: *Géométrie différentielle (Strasbourg, 1953).* Colloques Internationaux du Centre National de la Recherche Scientifique, pp. 137–141. CNRS, Paris, 1953. (Reprint: *Selected papers of J.-L. Koszul*, pp. 79–83. World Scientific, 1994.

[K97] Andreas Kriegl and Peter W. Michor, *The convenient setting of global analysis.* Mathematical Survey and Monographs, vol. 53, Amer. Math. Soc., Providence, RI, 1997.

[K80] Evgenii A. Kuznetsov and Alexander V. Mikhailov, On the topological meaning of canonical Clebsch variables. *Phys. Lett. A* **77** (1980) 37–38.

[L32] Horace Lamb, *Hydrodynamics.* Cambridge University Press, Cambridge, 1932.

[L93a] László Lempert, Loop spaces as complex manifolds. *J. Differential Geom.* **38** (1993) 519–543.

[L93b] Eugene Lerman, Richard Montgomery, and Reyer Sjamaar, Examples of singular reduction. In: *Symplectic geometry*, London Math. Soc. Lecture Note Ser., vol. 192, pp. 127–155. Cambridge University Press, Cambridge, 1993.

[L01a] Eugene Lerman and Christopher Willett, The topological structure of contact and symplectic quotients. *Internat. Math. Res. Notices* **2001** (2001) 33–52.

[L19] Yi Lin and Reyer Sjamaar, Convexity properties of presymplectic moment maps. *J. Symplectic Geom.* **17** (2019) 1159–1200.

[L23] Yi Lin and Xiangdong Yang, Basic Kirwan injectivity and its applications. *Q. J. Math.* **74** (2023) 639–657.

[L01b] Frank Loose, Reduction in contact geometry. *J. Lie Theory* **11** (2001) 9–22.

[M27] Erwin Madelung, Quantentheorie in hydrodynamischer form. *Zeit. F. Phys.* **40** (1927) 322–326.

[M85] Charles-Michel Marle, Modèle d'action Hamiltonienne d'un groupe de Lie sur une variété symplectique. *Rend. Sem. Mat. Univers. Politecn. Torino* **43** (1985) 227–251.

[M07] Jerrold E. Marsden, Gerard Misiołek, Juan-Pablo Ortega, Matthew Perlmutter, and Tudor S. Ratiu, *Hamiltonian reduction by stages.* Lecture Notes in Mathematics, vol. 1913 Springer, New York, 2007.

[M99] Jerrold E. Marsden and Tudor S. Ratiu, *Introduction to mechanics and simmetry.* Texts in Applied Mathematics, vol. 17, Springer, New York, 1999.

[M74] Jerrold E. Marsden and Alan Weinstein, Reduction of symplectic manifolds with symmetry. *Rep. Math. Phys.* **5** (1974) 121–130.

[M83] ———, Coadjoint orbits, vortices and Clebsch variables for incompressible fluids. *Physica* **7D** (1983) 305–323.

[M73] Kenneth R. Meyer, Symmetries and integrals in mechanics. In: *Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, pp. 259–272. Academic Press, Inc., New York-London, 1973.

[M23] David Miyamoto, The basic de Rham complex of a singular foliation. *Int. Math. Res. Not. IMRN* **2023** (2023) 6364–6401.

[O04] Juan-Pablo Ortega and Tudor S. Ratiu, *Momentum maps and Hamiltonian reduction.* Progress in Mathematics, vol. 222. Birkhäuser, Boston, 2004.

[P61] Richard S. Palais, On the existence of slices for actions of non-compact Lie groups. *Ann. of Math.* **73** (1961) 295–323.

[P98] Vittorio Penna, Mario Rasetti, and Mauro Spera, Quantum dynamics of 3-D vortices. In: *Secondary Calculus and Cohomological Physics*, Contemp. Math., **219** (1998) 173–193.

[P89] Vittorio Penna and Mauro Spera, A geometric approach to quantum vortices. *J. Math. Phys.* **30** (1989) 2778–2784.

[P92] ———, On coadjoint orbits of rotational perfect fluids. *J. Math. Phys.* **33** (1992) 901–909.

[P00] ———, String limit of vortex current algebra. *Phys.Rev.B* **62** (2000) 14547–14553.

[P01] Markus J. Pflaum, *Analytic and geometric study of stratified spaces.* Lecture Notes in Mathematics, vol. 1768. Springer, Berlin, 2001.

[P99] Henri Poincaré, *Les méthodes nouvelles de la mécanique céleste. Vol. III. Invariants intégraux. Solutions périodiques du deuxième genre. Solutions doublement asymptotiques.* Gauthier-Villars, Paris, 1899.

[R75] Mario Rasetti and Tullio Regge, Vortices in He II, current algebras and quantum knots. *Physica A: Statistical Mechanics and its Applications* **80** (1975) 217–233.

[R22] Tudor S. Ratiu and François Ziegler, Symplectic induction, prequantum induction, and prequantum multiplicities. *Commun. Contemp. Math.* **24** (2022) 13.

[R52] Georges Reeb, Sur certaines propriétés topologiques des trajectoires des systèmes dynamiques. *Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in* 8° **27**, n° 9 (1952) 1–64.

[R11] Renzo L. Ricca and Bernardo Nipoti, Gauss' linking number revisited. *J. Knot Theory Ramif.* **20** (2011) 1325–1343.

[S92] Philip G. Saffman, *Vortex dynamics.* Cambridge monographs on mechanics and applied mathematics. Cambridge University Press, Cambridge, 1992.

[S54] Jean-Pierre Serre, Représentations linéaires et espaces homogènes kähleriens des groupes de Lie compacts (d'après Armand Borel et André Weil). In: *Séminaire Bourbaki, Vol. 2, Exp. No. 100*, pp. 1–8. Secrétariat mathématique, Paris, 1954.

[S72] Roman Sikorski, Differential modules. *Colloq. Math.*, **24** (1971) 45–79.

[S76] David J. Simms and Nicholas M. J. Woodhouse, *Lectures on geometric quantization.* Lecture Notes in Physics, vol. 53. Springer, Berlin, 1976.

[S05] Reyer Sjamaar, A de Rham theorem for symplectic quotients. *Pacific J. Math.* **220** (2005) 153–166.

[S91] Reyer Sjamaar and Eugene Lerman, Stratified symplectic spaces and reduction. *Ann. of Math.* **134** (1991) 375–422.

[S70a] Stephen Smale, Topology and mechanics. I. *Invent. Math.* **10** (1970) 305–331.

[S67] Jean-Marie Souriau, Quantification géométrique. Applications. *Ann. Inst. H. Poincaré Sect. A* **6** (1967) 311–341.

[S70b] ———, *Structure des systèmes dynamiques.* Dunod, Paris, 1970. (Reprint: Éditions Jacques Gabay, Sceaux, 2008. Translation: *Structure of Dynamical Systems. A Symplectic View of Physics.* Progress in Mathematics, vol. 149. Birkhäuser, Boston, 1997.)

[S80] ———, Groupes différentiels. In: *Differential Geometrical Methods in Mathematical Physics*, Lecture Notes in Mathematics, vol. 836, pp. 91–128, Springer, New York, 1980.

[S84] ———, Groupes différentiels et physique mathématique. In: *Feuilletages et quantification géometrique (Lyon, 14–17 juin 1983)*, Collection travaux en cours, pp. 73–119. Hermann, Paris,1984.

[S85] ———, Un algorithme générateur de structures quantiques. In: *Élie Cartan et les mathématiques d'aujourd'hui (Lyon, 25–29 juin 1984)*, Astérisque, vol. Hors Série S131, pp. 341–399. Soc. Math. France, Paris, 1985.

[S86] ———, La structure symplectique de la mécanique décrite par Lagrange en 1811. *Math. Sci. Hum.* **94** (1986) 45–54.

[S88] ———, Quantification géométrique. In: *Physique quantique et géométrie (Paris, 16-20 juin 1986)*, Travaux en Cours, vol. 32, pp. 141–193. Hermann, Paris, 1988.

[S16] Mauro Spera, Moment map and gauge geometric aspects of the Schrödinger and Pauli equations. *Int. J. Geom. Methods Mod. Phys.* **13** (2016) 1–36.

[S23] ———, Hydrodynamics of the probability current in Schrödinger theory. *Geometric Science of Information (Saint-Malo, 2023)*, Lecture Notes in Computer Science, vol. 14072, pp. 272-281. Springer, Cham, 2023.

[T83] Takehiko Takabayasi, Hydrodynamical formalism of quantum mechanics and Aharonov-Bohm effect. *Prog. Theor. Phys.* **69** (1983) 1323–1344.

[V70] Michèle Vergne, Étude de certaines représentations induites d'un groupe de Lie résoluble exponentiel. *Ann. Sci. École Norm. Sup. (4)* **3** (1970) 353–384.

[W16] Charles T. C. Wall, *Differential topology.* Cambridge Stud. Adv. Math., vol. 156. Cambridge University Press, Cambridge, 2016.

[W12] Jordan Watts, *Diffeologies, differential spaces, and symplectic geometry.* PhD thesis, University of Toronto, 2012.

[W22] ______, The orbit space and basic forms of a proper Lie groupoid. In: *Current Trends in Analysis, its Applications and Computation: Proceedings of the 12th ISAAC Congress, Aveiro, Portugal, 2019*, pp. 513–523. Trends Math. Res. Perspect., Birkhäuser, 2022.

[W77] Alan Weinstein, Symplectic v-manifolds, periodic orbits of Hamiltonian systems, and the volume of certain riemannian manifolds. *Comm. Pure and App. Math.* **30** (1977) 265–271.

[W78] ______, A universal phase space for particles in Yang-Mills fields. *Lett. Math. Phys.* **2** (1978) 417–420.

[W81] ______, Symplectic geometry. *Bull. Amer. Math. Soc* **5** (1981) 1–13.

[W05] Robert E. Wyatt, *Quantum dynamics with trajectories.* Springer, New York, 2005.