Essays on Hedonic Product Differentiation

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Abstract

The present work has, as its main goal, that of exposing how the explicit, or *combinable* consumption version of the Lancasterian characteristics model, can be used to study some aspects of product differentiation and imperfect competition. The work is divided into four chapters.

The first chapter provides a concise introduction of the literature on product differentiation and imperfect competition, where we highlight its gap regarding the *combinable consumption model* developed by Lancaster (1966, 1971).

In the second chapter we develop a simple differentiated monopoly market with heterogeneous consumers. We show that, in a model with hedonically differentiated products, where consumers display love for variety, and the market is served by a monopolist exerting its market power by directly shaping the space of marketed varieties, the incentive to increase product differentiation depends on the curvature of the marginal utilities which uniquely define inverse demand functions. The resulting outcome is that the allocative efficiency of the equilibrium may, or may not, be reached. In particular, when marginal utilities are convex, the resulting equilibrium is inefficient. The opposite occurs when marginal utilities are concave.

The third chapter introduces strategic interaction among firms providing differentiated goods. In particular, we will consider an oligpolistic market where firms, given their products' designs, compete à la Cournot. The demand side of the economy is populated by heterogeneous consumers displaying love for variety. Essentially, the model we present can be seen as the partial-equilibrium counterpart, with differentiated goods, of the seminal model proposed Gabszewicz and Vial (1972). Given the complexities

attached to the formulation of a general model, we limit ourselves in presenting a set of examples providing the functioning, the hints and the difficulties embodied in the proposed model.

Finally, in the fourth chapter, we consider a simple duopoly with a representative consumer in where we study, in a two-stage game, the strategic selection of products' designs, assuming that firms differentiate along two possible dimensions (characteristics). A first interesting result of this analysis is the relationship between the characteristics model and the seminal framework introduced by Singh and Vives (1984). By exploiting this new setting, we find that firms' tendency is to select the same product's design. When entry is considered, entrants are shown to also select the same product's design as they enter the market. For, the number of firms at the *free-entry equilibrium* results higher than the *socially optimal* number of firms.

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Contents

Chapter 1

Introduction

How do firms choose the design of their products? This question is not new. It has increasingly been studied in the context of monopolistic competition, as exposed in the seminal work of Chamberlin (1933), and found their apex in the classic works of Dixit and Stiglitz (1977) and Spence (1976). However, the first work dealing explicitly with this question was done by Hotelling (1929). The model developed by Hotelling (1929) became a workhorse for the analysis of product differentiation and firms' incentives to increase or decrease it. The main ingredients of the conceptual framework developed by Hotelling (1929) are: first, the assumption that each consumer buys *only* one unit of its most preferred good; second, that competition among firms is in prices. The conclusion coming from Hotelling's analysis is the famous *principle of minimal product differentiation*. Essentially, this refers to the firms' tendency in locating at the centre of "Main Street"[1]. In terms of product differentiation this translates in the concept that firms, when competing among themselves by setting prices, have an incentive to reduce the degree of variety of the market[2]. Nonetheless, a logical flaw in Hotelling's conclusion was later discovered by D'Aspremont et. al (1979). They showed that actually, when competing in prices, there does not exists a Nash equilibrium for location

[1] In a similar spirit, Salop (1979) introduces the "Circular Market" where, instead of being located on a line, firms locate on a circle. The first main theoretical implication of this modeling strategy is that firms now face neighboring competition.

[2] This tendency, in the original Hotelling (1929) work produces the paradox that firms, in equilibrium, would set a price equal to marginal cost and thus profits will vanish.

pairs that are too close one another when transportation costs are linear. From this, it follows that in order to define an equilibrium for the Hotelling's game, firms shall locate sufficiently far apart. That is, the market should be sufficiently differentiated[3].

A part from the conclusions regarding the differentiation degree of the market, following firms' competition, the Hotelling's tradition as a tool for studying an economy with differentiated goods gave birth to the huge body of literature referring to as *Location Theory*. The *location* paradigm permeated, and still does, the analysis of product differentiation and imperfectly competitive markets. It eventually expanded in what is known as *discrete choice theory*. An exhaustive exposition of this theory in dealing with product differentiation is found in Anderson et. al (1992). Two are the essential features, defining the classic Hotelling's approach to product differentiation. First, each consumer is assumed to seek its *most preferred variety*. Second, firms exert their market power by setting the price of their good. A multitude of works can be cited to give reference of this broad literature[4]. In this broad literature, the explicit specification of firms' location decisions as a two-stage game is not always evident[5]. Strategic behaviour, for example, in the choice of locations was considered, in the standard Hotelling's model, by Neven (1985). His result is particularly simple: in the two-stage game where firms first decides their location and subsequently compete in prices, a subgame perfect Nash equilibrium exists for any location pairs[6]. The key insight, however, coming from Neven (1985), is that the demand faced by a firm selecting its location on "Main Street" is equivalent to the demand faced by a firm selecting an horizontally differentiated product. An exhaustive exposition of the existence problems that may arise in the standard Hotelling's

[3]However, they restored existence in the case of quadratic transportation costs, and showed that in this case firms will totally differentiate their good.

[4]Surveys of this literature are present in Philips and Thisse (1982), Gabszewicz et. al (1986) and Anderson et. al (1992).

[5]In Gabszewicz et. al (1986) this fact is clearly exposed by presenting the standard setting with all the possible combinations of decision variables that can be considered.

[6]This result clearly is in opposition with D'Aspremont et. al (1979). However, in Neven (1985)'s model it is assumed consumers to bear quadratic in-utility transportation costs. D'Aspremont at. al (1979) already restored existence when quadratic transportation cost is assumed.

framework is found in Gabszewicz et. al (1986). From the above succinct discussion it is evident that the Hotelling's paradigm for studying product differentiation is not robust in the sense that, since the existence of a price equilibrium heavily depends on the structure of the transportation costs, the subsequent location decisions, which define the resulting differentiation degree, are consequently affected. Thus, no clear cut results arise regarding the resulting differentiation degree.

There is an alternative way in which firms' location choices can be interpreted. In some settings it is indeed simpler, but also convenient, to see the firms' differentiation choices as the choice of selecting their spot in a multi-dimensional market space. Indeed, when increasing the market space's dimensions, the interpretation of selecting the best spot looses its power. For, standard location models à la Hotelling are usually interpreted, when dealing with multi dimensional decision spaces, as models where firms selects their product's design in the *characteristics space*. The characteristics model was introduced by Lancaster (1966), (1971)[7]. The basic idea of the Lancasterian model is the following. Consumers are not interested in goods *per se*. Indeed, goods are valuable for consumers since they embody different *addresses*, or *characteristics*. For, consumers consumption decisions depend on the specific bundles of characteristics embodied in the set of available products. So, goods are essentially defined as vectors in the characteristics space. Consumers decide their consumption schedules, i.e which goods to acquire, depending on the design of the available products. This interpretation of a differentiated economy is able to deal with two particular consumption patterns. First, if consumers are assumed to have a *most preferred variety*, the characteristics model is said to deal with *non-combinable* consumption. This first specification can be brought back to the standard setting of the standard Hotelling's model. However, in the baseline Hotelling (1929)'s model, a consumer does not have a most-preferred variety *per se* since products are essentially homogeneous, but purchases the closest of the two. On this, Peitz (1997) formally proved the equivalence between the Lancasterian setting and

[7]See also Lancaster (1975) and Lancaster (1979).

the standard Hotelling's setting[8]. However, and this will be the focus of the present work, the Lancasterian framework is able to deal with *combinable* consumption. Essentially, consumers are able to combine the different characteristics acquired after purchasing goods. Thus, consumers are seen to display some sort of *love for variety* is the sense proposed by Dixit and Stiglitz (1977). That is, their consumption schedule is not a single good but instead a *basket* of varieties which will provide consumers a given amount of characteristics. The point here is that, since consumers are interested in acquiring a certain amount of each available characteristics, their consumption decisions, regarding which goods to acquire, depend on the characteristics embodied in the available products.

While the case of *non-combinable* consumption has been widely used as an implicit description of a differentiated market[9], the case of *combinable* consumption has received less attention. The reason for this is to be found in the difficulties attached to the idea of consumption being *combinable*[10]. However, the hypothesis of *combinable* consumption seems reasonable when dealing with mass consumption goods like tea or coffe. On the other hand, the hypothesis of *non-combinable* consumption seems reasonable when dealing with goods such as cars, washing machines or refrigerators. The present work will deal with an economy of mass consumption goods for which the assumption of *combinability* can be embraced.

Models of *combinable* consumption have not received the same attention as their counterparts have. As said, the first reason for this refers to their apparent interpretative difficulty. The second, instead, seems to be referred to the hidden technical difficulties embodied in such models. In fact, as far as we know, there have been two works which explicitly considered the less-studied Lancasterian framework. Namely, Leland (1977) and Dreze and Hagen (1978). These two works, which can be addressed as the two main references of this whole work, show the potentials, but also the dif-

[8]This equivalence result refers to the case in which goods are differentiated just along two dimensions.

[9]Notably, see Irmen and Thisse (1998) where an n-dimensional characteristics space is considered.

[10]Extensive discussion of this point are found in Lancaster (1979) and Friedman (1983).

ficulties around the characteristics model with *combinable* consumption[11]. Leland (1977) exploits the parallelism between the characteristics model and the Arrow-Debreu model with financial markets. This parallelism is indeed fundamental since it connects the theory of product differentiation with the broader theory of incomplete markets[12]. The key intuition is that a financial instrument, having different returns for different possible future states of nature is equivalent, in mathematical terms, as a differentiated product embodying different characteristics. That is, both objects can be described as vectors in a multi-dimensional space. Thus, an economy where the market provides differentiated goods shares the same underlying structure as an economy where a single homogeneous commodity is sold but where consumers' income depends on the future returns deriving from investing in financial instruments. This parallelism is important since all the hints, especially regarding welfare and efficiency, coming from the well established theory of incomplete markets, can equally be applied to the study of an economy providing differentiated goods. This clearly provides a new point of view for the study of product differentiation. Something unusual. Leland (1977) simply focuses on showing this parallelism by means of a simple general equilibrium model where firms are assumed to be perfectly competitive[13]. Moreover, and this is an important aspect of this work, it deals with quality changes *given* a fixed number of products. The result coming from this work is that, if quality changes do not modify the dimensionality of the market space, that is, they do not induce a *drop of rank* then, *any* level of product differentiation results *optimal*. In this setting optimality refers to the fact that in equilibrium the allocation of characteristics is Pareto optimal. More precisely, equilibrium is characterized by the equality, among consumers, of the marginal rates of

[11] In the following discussion we will refer to this model as just *the characteristics model*. When its counterpart is considered we will refer to it as simply *location model*.

[12] For an extensive and very readable exposition of such a beautiful and important theory see Magill and Quinzii (1996).

[13] This aspect is a key simplification. Indeed, product differentiation naturally pairs with an imperfectly competitive market. However, dealing with imperfect competition in a fully general equilibrium model has proven particularly cumbersome. On this topic, there are several streams of literature. Surveys can be found in Bonanno (1990) and Hart (1985).

substitution of characteristics[14].

In the same spirit of Leland (1977), Dreze and Hagen (1978) consider a perfectly competitive market where firms chose the design of their good. That is, firms select the *quality* of their good by selecting the amount of each characteristics contained in the good. However, differently from Leland (1977), Dreze and Hagen (1978) considers the point of view of the *theory of the firm under uncertainty.* That is, they consider the problem of selecting the goods' design as the process undertaken by perfectly competitive firms who must decide on their production plan *given* a specified production set. Their decision must respect some specified *criterion* such as, for example, the maximization of shareholders' value. Thus, the *optimal* level of product differentiation is the one which satisfies all the imposed criteria on the firms. Again, optimality refers to the equilibrium allocation of characteristics.

The parallelism between an economy with differentiated goods and an exchange economy with financial markets has not received much attention in the literature on product differentiation and imperfect competition. However, the financial literature has internalized some hints coming from the analysis of firms and product innovation done in the IO literature, to study the process, and consequences, of *financial innovation.* The fundamental exposition of the theory of financial innovation can be found in Allen and Gale (1994)[15]. But, why is the financial innovation literature related to product differentiation? Again, the connection is made possible given the fact that the two, apparently disjoint topics, share the same underlying mathematical description. From this, it naturally follows that the firm's decision on how to sell to investors its equity, is equivalent, both formally but also logically, to a firm deciding how to design its good(s).

Among the vast financial innovation literature, two recent works are particularly important for understanding the parallelism between product differentiation à la Lancaster and the Arrow-Debreu model with financial markets.

[14]This characterization of Pareto optimality is typical of a *complete* market. See Magill and Quinzii (1996).

[15]This seminal book contains the fundamental articles written by Allen and Gale which provide the full *corpus* of this stream of literature.

Carvajal et.al (2012) showed that in a frictionless market[16], market completeness is endogenously determined by the shape of consumers' marginal utility. Their insight will constitute the building block of the first chapter of the present work, given the similarities between the two models' specifications. More recently, another important contribution is the one of Bejan (2020) which showed that, if the firm is privately owned by one of the consumers, her incentive to increase the set of securities depends on the convexity/concavity of her marginal utility function. In particular, if the firm's owner is sufficiently risk averse, she may have an incentive, not just to increase the set of marketed securities, but even to provide the complete set of assets. By drawing from these two important works on financial innovation, we can characterize, in the first chapter of the present work, an important mechanism that characterizes horizontal product differentiation: even if the marketing of new varieties is essentially costless, a firm selling differentiated produces may, or may not, have an incentive to provide the market with a higher, possibly complete, degree of product differentiation.

The present work has, as its main goal, that of exposing how the explicit version of the Lancasterian characteristics model can be used to study some aspects of product differentiation and imperfect competition. In particular, the work is divided into three chapters. Although each chapter is interesting in its own right, they should not be considered as isolated works. In fact, each of them constitutes a piece of a more general model. The general model we have in mind is a *three*-stage game. In stage one firms select their goods' designs, in stage two they compete in quantities taking as given the goods' designs. Finally, in stage three, differentiated goods, in given designs and quantities, are provided to the market where, via a competitive mechanism, they are allocated among *heterogeneous* consumers. Clearly, there are many difficulties hidden in such a model. Hence, in order to simplify the analysis, we decided to study each stage separately.

Chapter one focuses on the selection of goods' designs in presence of heterogeneous consumers. In order to simplify the analysis, and to highlight

[16]Essentially, this refers to the assumption that firms pay no costs for issuing different financial instruments.

the similarity between the Lancasterian framework of product differentiation and the financial innovation literature, we consider a market dominated by a monopolist. The monopolist's objective is that of maximizing profits. However, the channel through which this is done is new. Namely, as opposed to the standard monopoly setting, our monopolist is not selecting its profit maximizing output, but exerts its market power by deciding how to sell its production output, represented by a certain amount of characteristics. That is, our monopolist decides how to pack the available characteristics into differentiated goods. This behaviour can be seen as a *bundling* decision. Clearly, the monopolist' objective is that of selecting the *bundling strategy* which maximizes profit. We will show how this way of exerting market power directly impacts prices thus making it coherent with the standard idea of a quantity-setting monopolist. Essentially, the structure of the model is that of a two-stage game. In stage one the monopolist selects its profit-maximizing products' designs, i.e how to bundle the different available characteristics. In stage two, a competitive market takes place where, given the available products, heterogeneous consumer formulate their consumption decisions. The general idea of this first chapter is to explicitly present the analogy between a financial firm deciding its market-value maximizing financial structure, and a firm deciding how many products to provide the market in order to maximize its profits. The monopolist's bundling decision impacts consumers' implicit evaluations of the available characteristics. Goods' prices will be functions of these *implicit characteristics prices*[17]. It is immediate to see the monopolist's incentive to provide designs which maximize the values (prices) of the goods, implying *implicit characteristics prices* to be maximal. Since the direct selection of the set of differentiated goods, by impacting the dimensionality of the market space, impacts consumers' evaluation of the available characteristics, we study the incentive for our monopolist to provide the market with the *optimal* number of products, i.e with the set of differentiated goods for which, in equilibrium, marginal rates of substitution of characteristics equates among consumers. We show that allocative efficiency of the equilibrium may, or may not, be reached. In particular, when marginal util-

[17]This terminology was introduced by Leland (1977).

ities are convex, the resulting equilibrium is inefficient. The opposite occurs when marginal utilities are concave.

In chapter two we introduce strategic interaction among firms providing differentiated goods. In particular, we consider an oligpolistic market where firms, given their products' designs, compete à la Cournot. The demand side of the economy is populated by heterogeneous consumers diplaying love for variety. The structure of the model is again that of a two-stage game. In stage one quantity competition takes place. In stage two, the Cournot-Nash equilibrium quantities are provided to the market where, heterogeneous consumers, formulate their consumption decisions coherently with the utility maximization paradigm. Essentially, the model we present can be seen as the partial-equilibrium counterpart, with differentiated goods, of the seminal model proposed Gabszewicz and Vial (1972). Given the complexities attached to the formulation of a general model, we limit ourselves in presenting a set of examples providing the functioning, the hints and the difficulties embodied in the proposed model.

Finally, in chapter three, we consider a simple duopoly with a representative consumer, having a quasi-linear quadratic utility in the spirit of Singh and Vives (1984), in where we study the strategic selection of products' designs, assuming that firms differentiate along two possible dimensions (characteristics). We do this by constructing a simple two stage-game. In stage one firms select their product's design. Subsequently, in stage two, given their designs, they compete in quantities. We thus seek the subgame perfect Nash equilibrium of this game. What we find is that firms' tendency is to select the same design. That is, firms *homogenize* around the same product design. This result is however new since there is no work that deals explicitly with the issue of strategic locations selection when firms compete à la Cournot[18]. One of the novelty of our result is that it arises in a multidimensional product space with a *variety-lover* representative consumer. The

[18]Anderson and Neven (1991) try to address this issue using the standard Hotelling's setting. Nonetheless, they assumed firms to bear the transportation cost. For, firms locate at the center of the market because it is the most *efficient* location from a productive point of view, i.e costs are minimized. Hence, their result has indeed less to do with product differentiation.

representative consumer setting suites itself for a simple analysis of entry and its welfare implications. In order to do this, we retain the assumption that the dimension of the characteristics space remains *fixed*. That is, we do not consider the possibility of *radical innovation*; that is, the situation where a new entrant is able to differentiate its product along a new characteristic that was not available before[19]. What we find is that each entrant will also select the same product's design as the one provided by the already active firms. Thus, the market is seen to pass from a duopoly with an homogeneous good to an oligopoly with the same homogeneous good. Regarding welfare, we find that the number of firms at the *free-entry* equilibrium is higher than the socially optimal number of firms. Although this result is somewhat standard in the literature (see Mankiw and Whinston (1986)), the interesting aspect is that, here, this result arises directly from firms' strategic choice of the product's design. Potentially, this simple setting could be extended so as to embrace the possibility of *innovation*-led entry. Entry produces an expansion of the characteristics space due to the entrant's ability in providing some characteristics that where not available before. In other words, entrant introduces a combination of characteristics that was not available before. To us, this seems a very promising line of research.

In conclusion, the current work wants to stimulate further research on the characteristics model, as originally proposed by Lancaster, given its potentials to enrich the already vast literature on product differentiation and imperfect competition by providing a new, and interesting, point of view. Potential outcomes of such a research program include the possibility of provide a micro-foundation for the process of product innovation, and define a workable alternative to the Dixit and Stiglitz (1977)'s model, a workhorse in macroeconomics, of monopolistic competition. Clearly, the potential impact of such a research program, for the macroeconomic literature, is huge.

[19]It is however important to note that in the characteristics model, *non-radical innovation* refers to the situation in which a potential entrant is able to modify the product space by providing a good that was not produced before *given* the available characteristics. In other words, we will consider the simple situation where, given the number of available characteristics S, what changes in the number of products J. For, a new product is simply defined by a new combination of characteristics in the characteristics space that was not available before.

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Chapter 2

Monopoly Power and Increasing Product Differentiation

2.1 Introduction

In the present work, we will try to answer the following fundamental question: does a market dominated by a single producing firm, the monopolist, provide an efficient level of product differentiation? We will try to tackle this ever-lasting question by developing an alternative framework that slightly differs from the commonly adopted one. In particular, we will complement the base-line setting of the characteristics, or *address* model[1], with intuitions coming from the literature of financial innovation. The parallelism is straightforward once one recognizes that a differentiated good in the characteristics approach is virtually identical, in its fundamental description, to a financial asset.

It is well known that a market dominated by a monopolist is, because of the intrinsic nature of a monopoly, inefficient in providing the optimal quantity of a homogeneous good. This happens because the unmatched market power of the monopolist provides it with the ability to modify its decision variable in order to maximize profit at the expense of consumers' welfare.

[1]We will adopt the baseline model developed by Lancaster (1966), (1975) and (1979).

This unmatched market power erodes consumers' surplus to the advantage of the monopolist. However, when dealing with product differentiation, such a normative conclusion is not so easily reached. The major problem, in our view, is how to define market power in presence of product differentiation. Several attempts have focused on the possibility of optimal provision of quality and have concluded that, again, a monopoly will, generally, provide a non-optimal level of quality[2]. The problem of providing an optimal level of variety has received slightly less attention. On this, by defining market power as the direct possibility for the monopolist to modify the dimensionality of the goods' space, we will provide evidence, by building two examples, that a monopoly may reach an equilibrium outcome in which the efficient level of variety is reached. The determinant element for such an outcome to produce will be shown to be the curvature of the inverse demand functions[3].

In order to find an answer to the initial question, we will consider a model dominated by a single monopolist who strategically chooses the degree of product differentiation of the market which maximizes its profit. Any costs related to increasing product differentiation are neglected so that we are essentially considering a *frictionless market*. The different varieties are offered to competitive consumers who display heterogeneous tastes over characteristics. This heterogeneity comes from the different initial consumers' endowments of characteristics. We show that, depending on the curvature of the resulting inverse demand functions, the monopolist may, or may not, have the incentive to provide the complete set of varieties. From this, it follows that the efficiency of the equilibrium outcome does too depend on the curvature of indirect demands. In particular, with convex inverse demands, the

[2]See for example Sheshinki (1976). Quality is shown to be constantly underprovided by a monopolist. In particular, the extent of this underprovision is shown to depend on goods being substitutes or complements. A similar model is the one of Spence (1975). Two interesting attempts to study quality decisions, using the characteristics approach, are Leland (1977) and Dreze and Hagen (1978). However, for the very nature of their model, the explicit consideration of monopoly has not been taken into consideration.

[3]The role of the curvature of the inverse demand has not been fully recognized in the literature dealing with product differentiation. An important contribution, which however considers an homogeneous good, is provided by Malueg (1994). He shows, in a very simple setting, how the inefficiency, intrinsic in a market dominated by a monopolist, may be reduced depending on the curvature of the inverse demand function.

monopolist will have no incentives to propose a complete set of differentiated goods. Conversely, with concave inverse demands, the monopolist will provide consumers with a complete set of varieties. The main economic insight of the present work is that as far as the monopolist is able, by manipulating the degree of product differentiation, to perfectly discriminate based on the marginal willingness to pay then, the shape of the marginal willingness to pay is what ultimately defines the equilibrium efficiency or inefficiency. Since, as stated above, the approach adopted here is in a certain way non-standard, the related literature we will present below will appear to have a high degree of heterogeneity. However, we see this as a strength.

In order to address the monopolist's incentives to provide the market with an efficient level of variety, i.e. complete product differentiation, we rely on the parallelism between the Lancasterian model and the standard asset model considered in the financial innovation literature. In particular, as Allen and Gale (1994) recognize the importance of acknowledging the hints coming from IO, for the understanding of how financial innovation works, we reverse-engineer this view in order to recognize back the importance of the financial innovation results for further understanding of product innovation in IO. In particular, the recent work by Pellegrino (2019) introduces the characteristics approach in studying the issue of market concentration and market power[4]. Despite its applicability, Pellegrino (2019)'s model makes strong assumptions from a theoretical point of view. In particular, he adopted a representative consumer approach and consider the market structure to be exogenously given. In the present work, we will drop both assumptions

[4]In the past years, an important body of empirical literature has focused on the analysis of market power, markups, and market concentration and their capacity in explaining some important macroeconomic trends. Particularly relevant is the work by De Loecker, Eeckhout, and Unger (2020). They study, in depth, the capacity of increasing market power and market concentration in explaining important macroeconomic secular trends like the decrease in labor share and capital share. They argue that commonly used macroeconomic models do not fully recognize market power, and all its spillovers, as solely able to explain important macro-trends. A step toward the inclusion of market power spillovers into macro models is represented by the analytical framework, developed by Pellegrino (2019). However, in Pellegrino (2019)'s model innovation is not considered: the set of available varieties is assumed to be fixed. Hence, the fact that firms with market power may modify the set of marketed differentiated goods is not taken into account.

by simultaneously introducing a certain degree of heterogeneity among consumers, and by explicitly describing how the market structure could be made endogenous.

In the literature dealing with product differentiation, we find two main attempts that directly recognized the parallelism between the Lancasterian approach to product differentiation and the asset model, Leland (1977) and Dreze and Hagen (1978). These two classic works analyze the optimal provision of quality using a model specification very similar to ours. The central assumption of both the above works is represented by the *spanning condition*. Essentially, this refers to the the possibility of adjusting the basket of goods acquired, subsequently to any change in the composition of the set of available differentiated goods. By assuming *spanning*, they derive conditions ensuring an optimal level of quality. However, they do not directly address the problem of horizontal product differentiation. These two works, although useful from a methodological point of view, by explicitly assuming *spanning*, are unable to address the problem of horizontal differentiation since the latter implies a change in the dimension of the market. Here, instead, we consider a structure able to endogenously generate incomplete product differentiation in an economy dominated by a single producing firm and populated by heterogeneous consumers. The economic mechanism relies on the shape of the consumers' marginal utility function, which changes the evaluation of the single characteristics and hence modifies the monopolist's incentive to provide an efficient level of product differentiation.

Among the vast financial innovation literature, two works are fundamental for understanding the mechanism presented below. Carvajal et.al (2012) showed that in a frictionless market, market completeness is endogenously determined by the shape of consumers' marginal utility. Their insight is indeed the main driver of the present work, given the similarities between the two model specifications. More recently, another important contribution is the one of Bejan (2020) which showed that, if the firm is privately owned by one of the consumers, her incentive to increase the set of securities depends on the convexity/concavity of her marginal utility function. In particular, if the firm's owner is sufficiently risk averse, she may have an incentive, not just to increase the set of marketed securities, but even to provide the complete set of assets. By drawing from these two important works on financial innovation[5], we can characterize an important mechanism that characterizes horizontal product differentiation: even if the marketing of new varieties is essentially costless, the monopolist may not have any incentive to provide the market with a higher, possibly complete, degree of product differentiation.

The paper is organized as follows: Section 2 exposes the baseline model. Section 3 defines and characterizes the concept of equilibrium. Section 4 contains the two main examples of the paper. In section 5 we discuss the evidence coming from the two examples. Finally, section 6 concludes and highlights possible further developments of the model in order to construct an alternative framework where to study product innovation and, more generally, imperfect competition.

2.2 The benchmark model

We will consider a simple production economy populated by $i = 1, \ldots, I$ consumers, and a single production firm, the monopolist, who produces $k = 1, \ldots, K$ differentiated goods over $s = 1, \ldots, S$ characteristics. Each of the K differentiated goods, or varieties, is defined as a vector, $\mathbf{y}$ in the space of available characteristics $C \subseteq \mathbb{R}^S_+$, which will be called characteristics space.

2.2.1 The monopolist

We assume that the economy is dominated by one firm (the monopolist) which, given the exogenous technological level of the economy, can dispose of a vector of fundamental characteristics[6] $\mathbf{y} = (y_1, \ldots, y_S) \in \mathbb{R}^S_+$. We will further assume $\mathbf{y}$ to be *semi-positive*. That is, its has at least one component

[5]The list of important contributions in the financial innovation literature includes the seminal papers of Allen and Gale (1988) and (1991).

[6]We will make the strong assumption that the production plan could be produced without bearing any cost. The model can, however, easily be enlarged by considering the costs connected to the production plan as in Bejan (2020). Moreover, this given vector could be seen as the technological frontier of the firm. That is, given the technological level, the firm can dispose of these characteristics.

which is strictly positive: $\mathbf{y} > 0 \iff \mathbf{y} \in \mathbb{R}^S_+,\ \mathbf{y} \neq \mathbf{0}$. The monopolist's objective will be to select a market structure A, exhausting the available characteristics, so as to maximize its profits. In other words, given the vector of technologically feasible characteristics, the monopolist decision variable is how to bundle these characteristics in order to extract the maximum consumers surplus and hence maximize profits. Since differentiated goods are defined as vectors in the characteristic space C, we can define the *market structure* as the matrix containing all the marketed varieties, i.e the different bundles chosen by the monopolist. We will abstract from any ownership structure consideration[7]. We do this, essentially, to cast the whole model into a more workable setting, resembling the standard partial equilibrium framework commonly adopted in the IO literature. We will further assume the elements of the vector $\mathbf{y}$ to be indivisible. That is, the monopolist cannot split each element of this vector separately. For example given $\mathbf{y} = (y_1, y_2)$, the monopolist cannot split it into two sub-vectors which are just half of $\mathbf{y}$. This ultimately means that the initial vector of fundamental characteristics cannot be modified[8]. However, what can be modified, and indeed is the decision variable of our monopolist, is its bundling.

The monopolist selects a market structure A which exhaust the available characteristics. Without loss of generality, the supply of each differentiated good will be equal to 1. The full set of available differentiated $k = 1, \dots, K$ goods, the *market structure*, is represented by a $S \times K$ matrix $A = (\mathbf{a}_1, \dots, \mathbf{a}_K)$ where $\mathbf{a}_k = (a_{k1}, \dots, a_{kS}) \in \mathbb{R}^S_+$ and $\mathbf{a}_k > 0$ (semi-positive). Now, let $\mathcal{A}$ be the set of all market structures such that $A \cdot \mathbf{1} = \mathbf{y}$. The monopolists is restricted to market structures $A \in \mathcal{A}$. We have

$$A = \begin{bmatrix} a_{11} & \dots & a_{K1} \\ \vdots & \ddots & \vdots \\ a_{1S} & \dots & a_{KS} \end{bmatrix}$$

[7]Also in this case the model could be enlarged to account for ownership. See Bejan (2020).

[8]In the financial innovation literature, this is the parallel of assuming that given a certain equity structure for the j-th firm $\mathbf{y}_j$, it can only be marketed either directly or by issuing K perfectly correlated assets. For this see Allen and Gale (1991).

Note that any vector of fundamental characteristics can be split at most in $K = S$ differentiated goods. The single-producing firm decides on the shape of the market structure. Since the number of possible varieties depends on the initial production plan (technologically feasible vector of characteristics), we impose to the market structure A the following assumptions

Assumption 1. For any $\mathbf{y}$, and for any number of marketed varieties $k = 1, \ldots, K$ it holds $\sum_{k=1}^{K} \mathbf{a}_k = \mathbf{y}$.

Assumption 2. There exists A such that $\mathbf{a}_1 = \ldots = \mathbf{a}_K = \mathbf{0}$.

Assumption 3. For any $\mathbf{y}$ there exists A such that $\mathbf{a}_1 = \mathbf{y}$ and $\mathbf{a}_k = \mathbf{0}$, $\forall k = 2, \ldots, K$.

Assumption 1 is what legitimate a certain matrix to actually represent a valid market structure[9]. Assumption 2 simply represents the possibility of inaction. With assumption 3 we are assuming that the initial vector of fundamental characteristics can directly be marketed, that is $A = \mathbf{y}$. For the sake of simplicity, will assume that the quantity supply of every marketed variety amount to one unit and that these quantities can be produced at no cost.

The firm can choose from various alternative, valid, market structures in the variety set. For example, given S characteristics, the firm can provide consumers with one single variety providing all available characteristics. Another alternative can be to select the *complete* market structure with $K = S$ varieties each of which provides an amount of just one characteristic. The decision of the market structure determines the firm's market value (profit), $\Pi(A) = \sum_{k=1}^{K} p_k = \mathbf{p}(A) \cdot \mathbf{1}$. The objective of the firm, coherently with its monopoly power, is the maximization of its market value. So, in general, the monopolist's goal is the maximization, by choosing a certain market matrix A, of his market value $\Pi(A)$, that is

$$\max_{A} \quad \Pi(A) = \mathbf{p}(A) \cdot \mathbf{1} \tag{2.1}$$

[9]In other words: given any initial production plan, the market matrix A, to be a valid market structure, must exhaust the available amount of each characteristic. Note that this assumption does not impose any restriction on the possible number of varieties in which the firm can split its initial production plan.

From problem 2.1, we see that the monopolist's market value maximization depends on the specific shape of the market structure chosen by the monopolist. That is, since the monopolist perceives its actual position of power then it realizes that a modification of the number of marketed varieties modifies its market value. In other words, our monopolist decides on the market structure taking everyone's decisions as given, it perfectly anticipates the outcome for every possible choice of market structure, and therefore its market value is directly affected by the shape of the market structure[10]. Hence, the monopolist behaves, so to say, strategically, in the sense that the different choices of market structures lead to different equilibrium allocations and hence, since it takes all others' decisions as given, it could perfectly be foreseen all possible equilibria and therefore it can select the market structure that provides the higher market value.

It is standard to consider the monopolist's market power as deriving from its capacity to unilaterally modify its output to increase prices and thus capture a higher share of the consumer's surplus. In the current setting, where we fix output (supply) to be equal to one for any $k = 1, \ldots, K$, the monopolist's market power is expressed in a different, yet particular way. Namely, the monopolist is allowed, given its exogenous available technological frontier, to decide how many different *bundles* to propose the available characteristics to the market. Market power then stems from this unilateral privilege. Hence, what we are considering is a problem of *bundling* that is, how to combine the different attributes which are valuable for the market to maximize profits.

As for the standard case, the monopolist acts unilaterally hence, a starting inefficiency is implicitly present in our model. However, and we will show this later, this intrinsic inefficiency could be eliminated. Therefore, the possibility to decide the *boundling* structure of the available characteristics determines the efficiency properties of the equilibrium.

[10]The monopolist's behavior could be seen as being strongly rational and it encompasses, in a very strong way, the self-fulfilling expectations hypothesis. In other words, we are assuming that the monopolist's expectations about the effect that different market structures produce on equilibrium prices correspond exactly to the actual ones that would verify once a given market structure is effectively chosen.

2.2.2 Consumers

Consumers have preferences defined over the characteristics space $C \subseteq \mathbb{R}^S_+$, that is, utility is derived by consuming specific amounts of the $1, \ldots, S$ available characteristics. These amounts, given the vector of quantities $\mathbf{q}_i = (q_{1i}, \ldots, q_{Ki}) \in \mathbb{R}^K_+$, for every i are defined as follows

$$c_{si} = \sum_{k=1}^{K} a_{ks} q_{ki}, \quad s = 1, \ldots, S \tag{2.2}$$

or, in linear algebra notation[11]

$$\mathbf{c}_i = A\mathbf{q}_i \tag{2.3}$$

where A is the $S \times K$ matrix containing all a_{ks} and q_{ki} is the amount of the k-th variety consumed by the the i-th consumer. Consumers are endowed with an initial amount of some, or all, available characteristics plus some amount of initial resources w_{0i}. The endowment vector for the i-th consumer is

$$\boldsymbol{\omega}_i = (w_{0i}, \mathbf{w}_i) = (w_{0i}, w_{1i}, \ldots, w_{Si}) \tag{2.4}$$

The initial amounts of characteristics detained by each consumer $i = 1 \ldots, I$, act as preference shifters in the sense that they define the initial position of each consumer in the space of characteristics space $C \in \mathbb{R}^S_+$. These endowments should be understood as the amounts of characteristics the consumers derived from prior consumption of goods. That is, we are indeed endowing consumers with a given good providing them with some amounts of characteristics. Moreover, we will assume the aggregate endowment of characteristics to be constant and in particular to be equal to the total amount of characteristics producible by the monopolist; that is, $\sum_{i=1}^{I} \mathbf{w}_i = \bar{\mathbf{w}} = \mathbf{y}$.

We assume that preferences for every i could be represented by a *smooth*

[11] Throughout the paper, we will consider quantity vectors as column vectors whether price vectors will be considered as row vectors.

utility function $U : \mathbb{R}_+^{S+1} \to \mathbb{R}$ of the form[12]

$$\begin{aligned} U(x_{0i}, x_{1i}, \ldots, x_{Si}) &= x_{0i} + V(\mathbf{x}_i) \\ &= x_{0i} + \sum_{s=1}^{S} u(x_{si}), \quad i = 1, \ldots, I \end{aligned} \tag{2.5}$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ are utility indices associated with the total amount of consumption for every of the S available characteristics, $\mathbf{x}_i = (w_{1i} + c_{1i}, \ldots, w_{Si} + c_{Si})$, satisfying standard assumptions of strict concavity, i.e $u' > 0$ and $u'' < 0$, and differentiability. Further, x_{0i} is a homogeneous good which represents the numeraire of the model. In what follows, we will assume the utility index u to be the same for every consumer. An important aspect that we stress is that, in the present setting, the *love for variety* assumption refers to every single characteristic and not to goods. Hence, each consumer $i = 1, \ldots, I$ displays the *love for variety* attitude for each of the available characteristics. From this, what matters to consumers is the amount of characteristics they can acquire, and not the number of differentiated goods directly. A certain degree of consumer heterogeneity is further introduced by assuming that $\mathbf{w}_1 \neq \ldots \neq \mathbf{w}_I$. That is, the endowment vectors of characteristics will differ among consumers.

For each consumer $i = 1, \ldots, I$ the budget, B_i, is defined as

$$B_i = \left\{ \mathbf{q}_i \in \mathbb{R}_+^K \mid x_{0i} + \mathbf{p} \cdot \mathbf{q}_i \leq w_{0i} \right\} \tag{2.6}$$

where $\mathbf{p} = (p_1, \ldots, p_K)$ is the K-dimensional vector of prices and $\mathbf{q} = (q_{1i}, \ldots, q_{Ki})$ is the vector of demanded quantities of each variety. We will assume that the only way in which consumers can acquire the desired amount of characteristics is through the acquisition of differentiated goods (varieties). That is, consumers, decide only the amount of the k-th good to buy. There is no other way in which they are allowed to get characteristics. Moreover,

[12] This particular specification is chosen primarily to maintain tractability while allowing direct connection with the financial innovation literature. The major simplification brought by this particular form is the negligence of any income effects. However, the model could be further expanded by considering income effects as in Carvajal et.al. (2012).

we do impose *non-negativity* constraints on the consumption of differentiated goods, $q_{ki} \geq 0$ for all $i = 1, \ldots, I$ and for all $k = 1, \ldots, K$. That is, consumers are unable to consume a negative amount of any available variety[13].

Each consumer i decides on the quantity of each differentiated good to buy, $\mathbf{q}_i = (q_{1i}, \ldots, q_{Ki})$, to maximize her utility. That is, she solves

$$\begin{aligned} \max_{\mathbf{q}_i \in \mathbb{R}_+^K} \quad & U_i(x_{0i}, \mathbf{x}_i) \\ \text{s.t.} \quad & x_{0i} + \mathbf{p} \cdot \mathbf{q}_i \leq w_{0i}, \\ & \mathbf{c}_i = A\mathbf{q}_i, \\ & \mathbf{q}_i > 0 \end{aligned} \tag{2.7}$$

Before stating the first-order conditions for the above problem, a comment on the constraint $q_{ki} \geq 0$ is in order. This constraint represents the *non-negativity* constraint and, in our current setting, means that consumers cannot short-sell a good. That is, they cannot sell something that they do not possess. Making consumers only endowed with some amount of characteristics and defining goods as the only instruments to acquire more characteristics, naturally prevent the possibility of selling a good without detaining any amount.

Given the specific utility function defined in 2.5, the maximization problem 2.7 can be reduced, by plugging the budget constraints directly into the utility function, as

$$\begin{aligned} \max_{\mathbf{q}_i \in \mathbb{R}^K} \quad & w_{0i} - \mathbf{p} \cdot \mathbf{q}_i + \sum_{s=1}^{S} u(x_{si}) \\ \text{s.t.} \quad & \mathbf{q}_i > 0 \end{aligned} \tag{2.8}$$

with $x_{si} = w_{si} + c_{si} = w_{si} + \sum_{k=1}^{k} a_{ks} q_{ki}$. The optimality conditions for

[13] In the financial innovation literature this amounts to imposing restrictions on short-selling. See Allen and Gale (1991).

the above problem, assuming an internal solution, for every $i = 1, \dots, I$ are

$$-p_k + \sum_{s=1}^{S} u'(x_{si}) a_{ks} = 0, \quad k = 1, \dots, K \tag{2.9}$$

where $u'(x_{si})$, the ratio between the marginal utility of the s-th characteristic and the marginal utility of income[14], defines the *implicit characteristic price* of the s-th characteristic. That is the implicit price that the i-th consumer attaches to the holding of the s-th characteristic. Equation 2.9 can be stated in matrix notation as

$$\mathbf{p} = \nabla V(\mathbf{x}_i) A \tag{2.10}$$

where $\mathbf{p} = (p_1, \dots, p_K)$, $\nabla V(\mathbf{x}_i) = (u'(x_{1i}), ..., u'(x_{Si}))$ and A is the market matrix introduced above. The only constraint coming from the production side of the economy, besides the market structure, is represented by the set of market clearing conditions, which with the normalization imposed for the supplied quantity of each $k = 1, \dots, K$, read

$$\sum_{i=1}^{I} q_{ki} \leq 1, \quad k = 1, \dots, K \tag{2.11}$$

2.3 Market equilibrium

Given our fundamental assumption that the monopolist's only choice variable is the number of varieties through which sells its production plan to consumers, the monopolist is indeed acting strategically by selecting the market structure which maximizes its market value. Hence, since it takes consumers' reactions for every possible (feasible) market structure as given, then, it can select the market structure yielding the higher market value. From this, the equilibrium concept we will adopt consists of two steps. The first step consists in the monopolist's computation, for every possible market

[14]Clearly, by assuming a quasi-linear functional form, the marginal utility of income, i.e the Lagrange multiplier λ_i, is equal to one.

structure A, of the corresponding consumers' reactions and hence of all the attainable profits. The second step consists of the monopolist's selection of the market structure yielding the higher returns in terms of market value in accordance with the problem (1). Finally, the *optimal* market structure, from the monopolist point of view, is proposed to the market and the corresponding market allocation is reached leading the economy to its equilibrium.

In particular, in the first step, for every feasible market structure A, a Walrasian equilibrium allocation is computed, that is, there exist prices $\mathbf{p}^*(A) = (p_1^*, \ldots, p_K^*)$ such that

$$\sum_{i=1}^{I} \mathbf{q}_i(\mathbf{p}^*(A)) = \mathbf{1}$$

Depending on the shape of the market structure A, this Walrasian equilibrium may be different. Now, in the second stage, the monopolist selects, given the anticipated market's reactions, the market structure $\bar{A}$ which maximizes its market value, and hence, a global equilibrium for our economy is reached. Note that given our present setting, namely that the quantity of every possible variety is normalized to be equal to one, given the market clearing condition defined above, the market value of the monopolist is simply $\Pi(A) = \mathbf{p}^*(A) \cdot \mathbf{1}$.

Definition 1. An *hedonic equilibrium* is defined by an array $(\mathbf{q}^*(\bar{A}), \mathbf{p}^*(\bar{A})) \in \mathbb{R}^{K(I+1)}$ and a market structure $\bar{A}$ such that

1. (i) $\forall i\ \mathbf{q}_i^*$ is a solution to program (8);

   (ii) $\sum_i^I q_{ki}^* \leq 1$, for $k = 1, \ldots, K$.

2. $\bar{A}$ maximizes $\mathbf{p}^*(A) \cdot \mathbf{1}$ for all A

Basically, in a hedonic equilibrium, the monopolist selects, knowing all possible reactions, the market structure yielding the higher market value. We will not address the issue of studying the existence and uniqueness of the hedonic equilibrium. Anyway, the efficiency properties of the hedonic equilibrium could be easily analyzed. In particular, the efficiency properties of the hedonic equilibrium deeply depend on the nature, defined by the specific market structure A selected by the monopolist, of the underlying Walrasian equilibrium allocation. In other words, the allocative efficiency of the market equilibrium, in terms of characteristics, depends on the monopolist's decision regarding the different goods' designs, i.e the market structure providing the monopolist the higher market value.

2.3.1 Characterization of the Walrasian equilibrium

It is then important to characterize the allocation of characteristics arising for any possible market structure the monopolist can chose. One major problem in doing this is that we cannot define the span of the market matrix A[15]. The span of A is defined as $\langle A \rangle = \{\mathbf{x} \in \mathbb{R}^S \mid \mathbf{x} = A\mathbf{q}, \mathbf{q} \in \mathbb{R}^K\}$. Note that since we are assuming $\mathbf{a}_k > 0$ and $\mathbf{q}_i > 0$ (semi-positive vectors), for any k and i then, it follows that also $\mathbf{x} > 0$ (semi-positive). Therefore, we are constraining the linear span of A on the positive orthant of $\mathbb{R}^S_+$. We might define with $\langle A \rangle_+$ the *constrained linear span* of A. Intuitively, the constrained linear span is just the linear span of A when the vectors $\mathbf{q}$ are constrained to be semi-positive.

The first step in characterizing the hedonic equilibrium is to prove that given any market structure A chosen by the monopolist, the Walrasian equilibrium (stage 1) allocates characteristics in the same way that an utilitarian planner would while constrained to allocations that are feasible under a given market structure. In order to do so, remember that the total amount of characteristics a consumer acquires is equal to $\mathbf{x}_i = \mathbf{w}_i + A\mathbf{q}_i \in \langle A \rangle_+$. But if this holds for every i then $\sum_{i=1}^{I}(\mathbf{x}_i - \mathbf{w}_i) \in \langle A \rangle_+$. From this, there exist $\mathbf{q} \in \mathbb{R}^K_+$ such that $\sum_{i=1}^{I}(\mathbf{x}_i - \mathbf{w}_i) = A\mathbf{q}$. Since by assumption each differentiated good k in A is in unit supply, and $A \cdot \mathbf{1} = \mathbf{y}$, we have that $\sum_{i=1}^{I}(\mathbf{x}_i - \mathbf{w}_i) = \mathbf{y}$. That is, the total net amount of characteristics acquired by consumers should be equal to the total amount of available characteristics. Hence, transfers (of characteristics) can be implemented such that this

[15]Remember that we are considering semi-positive vectors $\mathbf{z}_i > 0$. Hence, given A we are moving just in the positive orthant of $\mathbb{R}^2$.

condition is satisfied. Given that the initial vector of characteristics $\mathbf{y}$ can be splitted into a set of differentiated goods, defined by A, we can define a correspondence $G : \mathbb{R}_+^{SK} \to \mathbb{R}_+^{SI}$ that for any A defines the set of allocations of characteristics that can be implemented by transfers in $\langle A \rangle_+$. Formally,

$$G(A) = \left\{ \mathbf{x} \in \mathbb{R}_+^{SI} \mid \sum_{i=1}^{I} (\mathbf{x}_i - \mathbf{w}_i) = \mathbf{y}, \quad \text{for} \quad (\mathbf{x}_i - \mathbf{w}_i) \in \langle A \rangle_+ \right\} \qquad (2.12)$$

For any profile $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_I) \in \mathbb{R}_+^{SI}$ define the welfare function, for the utilitarian planner, as $W(x_0, \mathbf{x}) = \sum_{i=1}^{I} U(x_{0i}, \mathbf{x}_i)$. This welfare function aggregates utilities across consumers at an allocation $\mathbf{x}$ of characteristics. We have the following

Lemma 1. *Given a market structure A, let $(\mathbf{q}_1, \ldots, \mathbf{q}_I)$ be an allocation of goods such that $\sum_{i=1}^{I} \mathbf{q}_i = \mathbf{1}$, and $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_I)$ the corresponding allocation of characteristics defined as $\mathbf{x}_i = \mathbf{w}_i + A\mathbf{q}_i$. The allocation $\mathbf{q} \in \mathbb{R}_+^{KI}$ constitute a Walrasian equilibrium under market structure A if, and only if, $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_I)$ maximizes $W(x_0, \mathbf{x})$ with respect to $\mathbf{x} \in G(A)$.*

Proof. To prove necessity, take $(\mathbf{q}, \mathbf{p})$ to be the Walrasian equilibrium corresponding to market structure A. Since $\mathbf{q} \in \mathbb{R}_+^{KI}$, and that by definition $\mathbf{x}_i - \mathbf{w}_i = A\mathbf{q}_i$ we have that the resulting transfers of characteristics lie in $\langle A \rangle_+$. Moreover, $\sum_{i=1}^{I} \mathbf{x}_i = \sum_{i=1}^{I} \mathbf{w}_i + \mathbf{y}$ which implies that $\mathbf{x} = (\mathbf{x}_1, \ldots \mathbf{x}_I) \in G(A)$. Now, because $\mathbf{q}$ is a Walrasian allocation of goods, then it must hold, for every consumer i, that for every other $\mathbf{q}'_i \in \mathbb{R}_+^{K}$ and $\mathbf{x}'_i \in \langle A \rangle_+$ such that $\mathbf{x}'_i - \mathbf{w}_i = A\mathbf{q}'_i$, $u(\mathbf{x}'_i) - \mathbf{p} \cdot \mathbf{q}'_i \le u(\mathbf{x}_i) - \mathbf{p} \cdot \mathbf{q}_i$. Then we have that $\mathbf{p} \cdot \sum_{i=1}^{I} \mathbf{q}'_i = \nabla u(\mathbf{x}) \cdot \sum_{i=1}^{I} A\mathbf{q}'_i = \nabla u(\mathbf{x}) A \cdot \mathbf{1} = \nabla u(\mathbf{x}) \cdot \mathbf{y} = \mathbf{p} \cdot \sum_{i=1}^{I} \mathbf{q}_i$. From this, and by aggregating across consumers the optimality condition above we get that $\sum_{i=1}^{I} u(\mathbf{x}'_i) \le \sum_{i=1}^{I} u(\mathbf{x}_i)$. Hence, $W(\mathbf{x}') = \sum_{i=1}^{I} U(\mathbf{x}'_i) \le \sum_{i=1}^{I} U(\mathbf{x}_i) = W(\mathbf{x})$.

To prove sufficiency, we can appeal to the differentiable setting we are using. For, we may proceed by a first-order conditions argument as follows. Note that the planner's problem could be written as: $\max_{\mathbf{q}_1, \ldots, \mathbf{q}_I} \sum_{i=1}^{I} U(\mathbf{w}_i + A\mathbf{q}_i)$ subject to $\sum_{i=1}^{I} \mathbf{q}_i = \mathbf{1}$. The first order conditions, for every i are (assuming internal solution) equal to $\boldsymbol{\pi}_i A = \boldsymbol{\mu}$. where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_K)$ is the vector of positive Lagrange multipliers associated with the k-th differentiated good. Assume $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_I)$ to be a solution for the system of first order conditions. Then by simply letting $\boldsymbol{\mu} = \mathbf{p}$ it is immediate to see that there is a direct correspondence with the first order conditions of every consumer. Moreover, since the system $\sum_{i=1}^{I} \mathbf{q}_i = \mathbf{1}$ defines the set of market clearing conditions, we conclude that $\mathbf{q} \in \mathbb{R}_+^K$ and the associated price vector $\mathbf{p}$ constitute indeed a Walrasian equilibrium. □

Lemma 1 is useful because it states that whatever market matrix A the monopolist provides, the subsequent allocations of characteristics in the competitive consumption sector corresponds to the one an utilitarian planner would implement while constrained to A. Moreover, remember that the monopolist is able to anticipate, for any possible A, the subsequent Walrasian equilibrium. Hence, it will select the market structure A generating a Walrasian equilibrium yielding the higher possible profit. So, the monopolist is actually unilaterally ranking all the possible Walrasian equilibria, arising each at a different A, and selecting the one having the highest prices.

Before proceeding with the exposition of the suitable definition of optimality, it is necessary to provide some definitions. First, note that the definition of optimality in our framework refers to characteristics, not goods. In particular, given a market matrix A, the corresponding Walrasian equilibrium (if exists) induces a corresponding allocation of characteristics. To start with, note that the simple stetting we constructed provides a very useful result regarding the allocation of characteristics in equilibrium. In particular, given a market matrix A, at the corresponding Walrasian equilibrium $(\mathbf{q}^*, \mathbf{p}^*)$, the resulting allocation of characteristics is such that

$$\sum_i (\mathbf{x}_i^* - \mathbf{w}_i) = \mathbf{y}$$

From this, we can state the following trivial definition regarding an allocation of characteristics.

Definition 2. Given a market structure A, an allocation of characteristics

$(\mathbf{x}_1, \ldots, \mathbf{x}_I) \in \mathbb{R}_+^{SI}$ is *feasible* if

$$\sum_i (\mathbf{x}_i - \mathbf{w}_i) = \mathbf{y}$$

Define the set of all *feasible* allocations of characteristics as

$$F = \left\{ (\mathbf{x}_1, \ldots, \mathbf{x}_I) \in \mathbb{R}_+^{SI} \mid \sum_i (\mathbf{x}_i - \mathbf{w}_i) = \mathbf{y} \right\}$$

Now, we want to find conditions under which the monopolist has an incentive to provide the market with a complete set of differentiated goods. Usually, when trade of goods is allowed, imposing a limit on short-selling hampers the optimality of the equilibrium allocation even in presence of complete markets. It is important to notice that since characteristics are not traded independently of goods, it would be hardly plausible to expect a Pareto optimal allocation of characteristics. Hence, the relevant notion of optimality would be that of *constrained* optimality, which is, in turn, equivalent to that of optimality for the corresponding allocation of goods. The Pareto optimality of the former can be reached only if the monopolist selects a *complete* market structure, that is, a market structure A such that $\text{rank}(A) = S$. If this is the case, since A is invertible, the vector of implicit characteristic prices will be the same for every consumer. As a result, *full* Pareto optimality in the allocation of characteristics is reached. However, apart from this particular case, the allocation of characteristics resulting from a hedonic market equilibrium will generally result in a *constrained* Pareto optimum. We have the following definition

Definition 3. Given A, an allocation of characteristics $(\mathbf{x}_1, \ldots, \mathbf{x}_I) \in \mathbb{R}_+^{SI}$ is constrained Pareto optimal if $(\mathbf{x}_1, \ldots, \mathbf{x}_I) \in F$, and there does not exist a $(\mathbf{x}'_1, \ldots, \mathbf{x}'_I) \in F$ such that $U(\mathbf{x}'_i) \geq U(\mathbf{x}_i)\ \forall i$.

Usually, imposing non-negativity constraints, $\mathbf{q}_i > 0$, will result in a constrained Pareto optimal equilibrium. However, this is so if consumers are allowed to trade the marketed commodities. Since under our current model's specification we ruled out the possibility for consumers to trade goods, imposing non-negativity constraints are seen not to affect the efficiency of the equilibrium outcome. By following a standard argument, we can prove that the allocation of characteristics arising at Walrasian equilibrium is indeed constrained Pareto optimal.

Lemma 2. *If* $(\mathbf{q}, \mathbf{p})$ *is a Walrasian equilibrium for a given* A *then the resulting allocation* $(\mathbf{x}_1^*, \ldots, \mathbf{x}_I^*) \in \mathbb{R}_+^{SI}$ *is constrained Pareto optimal relative to* A*; that is, there exists* $\mathbf{p}^* \in \mathbb{R}_+^K$ *such that the agents' marginal evaluations of characteristics* $\nabla V(\mathbf{x}_i^*) = (u'(x_{1i}^*), \ldots, u'(x_{Si}^*))$, $\forall i$ *satisfy*

$$\nabla V(\boldsymbol{x}_1^*)A = \ldots = \nabla U(\boldsymbol{x}_I^*)A = \boldsymbol{p}^* \tag{2.13}$$

Proof. Given A, since the allocation of characteristics arising at the Walrasian equilibrium corresponds to the one that an utilitarian planner would implement in the same situation, and so $(\mathbf{x}_1^*, \ldots, \mathbf{x}_I^*) \in F$. Then as showed in Lemma 1, there exists prices $\mathbf{p}^* \in \mathbb{R}_+^K$ (strictly positive) such that, for every $i = 1, \ldots, I$, $\mathbf{p}^* = \nabla V(\mathbf{x}_i^*)A$. This is exactly the characterization of a constrained Pareto optimal allocation. □

From all the above we have that the Walrasiam equilibrium arsing given a certain market matrix A always results in a constrained Pareto optimal allocation of characteristics. However, we do not have, until now, characterized the optimality of the hedonic equilibrium. This should not be seen as a major issue since the very definition of an hedonic equilibrium implies the monopolist's unilateral choice of the profit-maximizing market structure A. Hence, an hedonic equilibrium is by definition inefficient. However, what we can investigate are conditions ensuring the market matrix A, constituting an hedonic equilibrium, to provide a set of differentiated goods which enables consumers to better allocate their resources in the acquisition of the desired characteristics.

2.4 Complete product differentiation and efficiency

We have that for an hedonic equilibrium what we can at most hope for is for it to provide consumers with a *complete* set of differentiated goods. By drawing from the theory of incomplete markets we define A to be complete if $rank(A) = K$. We shall now investigate conditions ensuring it. First, we have the following result.

Proposition 1. *Given two market structures, one incomplete A, and one complete A', we have that almost always, in terms of the monopolist's profit: i) if $u''' > 0$, A (weakly) dominates A', and ii) if $u''' < 0$, A' (weakly) dominates A.*

In order to proceed with its proof, we need a little "trick"[16]. Note that if $\mathbf{x} \in V(A)$ is the allocation of characteristics arising at a Walrasian equilibrium then we have that there exist $\mathbf{p}$ such that, in general, for every $i = 1, \ldots, I$, $\mathbf{p} = \nabla UV(\mathbf{x}_i)A = \pi(\mathbf{x}_i)A$. So, no matter if the market matrix A is complete or incomplete, at a Walrasian equilibrium there is a unique price vector. Moreover, given our assumption that $\sum_{k=1}^{K} \mathbf{y}^k = \mathbf{y}$ and that $\sum_{i=1}^{I} \mathbf{w}_i = \bar{\mathbf{w}}$, we can define the following value

$$\mathbf{m}(A) = \frac{1}{I}\sum_{i=1}^{I} \pi_i(\mathbf{x}_i(A)) = \frac{1}{I}\sum_{i=1}^{I} \nabla V(\mathbf{x}_i(A)) \tag{2.14}$$

In words, $\mathbf{m}(A)$ measures the average marginal willingness to pay across consumers in equilibrium. This measure enables us to make a direct comparison between the monopolist's profits arising with different market structures A. Indeed, the monopolist's profit can now be expressed as $\Pi(A) = \mathbf{p} \cdot \mathbf{1} = \mathbf{m}(A)A \cdot \mathbf{1} = \mathbf{m}(A) \cdot \mathbf{y}$.

Proof. Consider first the complete, $S = K$, market structure A'. Given our assumption of constant aggregate endowments, $\sum_{i=1}^{I} \mathbf{w}_i = \bar{\mathbf{w}} > 0$, and the fact that the utility index u is common to all consumers, we have that, in

[16] This way of proceeding was proposed by Carvajal, et. al (2012).

equilibrium $\boldsymbol{\pi}_1 = \ldots = \boldsymbol{\pi}_I$ and hence it follows that $\mathbf{x}_1^* = \ldots = \mathbf{x}_I^* \in \langle A \rangle_+$. Hence, the monopolist's equilibrium profit is given by $\Pi(A') = \sum_{k=1}^K p_k^* = \mathbf{p}^* \cdot \mathbf{1} = \boldsymbol{\pi}^* A \cdot \mathbf{1} = \boldsymbol{\pi}^* \cdot \mathbf{y} = \nabla u((\mathbf{y} + \bar{\mathbf{w}})/I) \cdot \mathbf{y}$. Consider now the incomplete market structure case, A. Because of the incompleteness of the market structure we have that, in equilibrium, $\boldsymbol{\pi}_1 \neq \ldots \neq \boldsymbol{\pi}_I$ and therefore it follows that $\mathbf{x}_1^{**} \neq \ldots \neq \mathbf{x}_I^{**} \in \langle A \rangle_+$. For every characteristic s we have that $\sum_i^I x_{si}^{**} = y_s + \sum_{i=1}^I w_{si}$. Thus, in equilibrium, we have that, for very s, $m_s(A) = 1/I \sum_{i=1}^I u'(x_{si}^{**}) = 1/I \sum_{i=1}^I u'(y_s + \sum_{i=1}^I w_{si})$. Now, if u' is strictly convex, profits should be higher with an incomplete market structure than with a complete one. We have that, for any $s = 1, \ldots, S$, $1/I \sum_{i=1}^I u'(y_s + \sum_{i=1}^I w_{si}) \geq u'(\mathbf{y}/I + 1/I \sum_{i=1}^I w_{si})$. Hence, $\mathbf{m}(A) \geq \mathbf{m}(A')$. Thus, it follows that $\Pi(A) = \mathbf{m}(A) \cdot \mathbf{y} \geq \mathbf{m}(A') \cdot \mathbf{y} = \Pi(A')$. A similar argument follows for ii). □

It is immediate, from the previous proposition, to derive the following

Corollary 1. *Provided $u''' < 0$, the monopolist has an incentive, in terms of profits, to select a complete market structure A.*

The monopolist has an incentive in providing a complete set of differentiated goods provided consumers' marginal utility to be strictly concave. From this, we have the result that, although full efficiency is out of hand given the presence of a monopolist, welfare could however be less hampered by the presence of a monopolist if consumers' marginal utility is concave. Intuitively this can be thought as a situation in which, despite I have to pay exactly my reservation price to acquire what I want, at least I am provided with a way to do so. The monopolist is exacerbating the trade-off between how much I have to pay and how much I actually want to consume.

Hence, the monopolist introduces an intrinsic inefficiency. This inefficiency could be further strengthened by the profit-maximizing objective of the monopolist. In particular, given our definition of hedonic equilibrium, the monopolist acts in a price-discriminatory manner by having the power to select the profit-maximizing market structure. That is, consider two market structures A with $K < S$ and A' with $S = K$. The associated competitive prices, which are perfectly anticipated by the monopolist, are

$\mathbf{p} = (p_1, \dots, p_K)$ and $\mathbf{p}' = (p_1, \dots, p_S)$. Because market value is just a linear function of $\mathbf{p}$, denote it, for either situation, $L(\mathbf{p})$ and $L(\mathbf{p}')$. The monopolist's decision process involves finding $\max\{L(\mathbf{p}), L(\mathbf{p}')\}$. In this way, the monopolist is selecting the market structure, i.e the number of differentiated goods to propose, yielding the maximal marginal willingness to pay. From this, it follows that the intrinsic inefficiency brought by the market being dominated by a single firm can further be worsened given the monopolist's behavior.

2.5 Two simple examples

In this section, we show through two simple examples, that the incentive for a market value maximizing monopolist who perfectly anticipates the Walrasian equilibrium allocation for every particular choice of the market structure, beyond the *status quo* case where the fundamental characteristics vector is directly marketed, depends on the *curvature* of the marginal utility.

In both examples, we first proceed by computing all possible Walrasian equilibrium allocations, perfectly anticipated by the monopolist, each of which arises from a particular market structure A. Then, given the resulting market values, the monopolist selects the market matrix providing the higher one. The hedonic equilibrium is finally reached once the monopolist proposes to the market the selected market structure $\bar{A}$.

Example 1. Consider a population of $i = 1, 2$ consumers. Consider the utility index $u = \ln(x_{si})$ which clearly defines strict love for variety, $u' > 0$ and $u'' < 0$. In particular, for each $i = 1, 2$ we have the following utility structure defined over the $S = 3$ available characteristics

$$U(x_{1i}, x_{2i}, x_{3i}) = \left(w_{0i} - \sum_{k=1}^{K} p_k q_{ki} \right) + \sum_{s=1}^{3} \ln(x_{si}), \quad i = 1, 2$$

and the following constant aggregate endowments distribution

$$\boldsymbol{\omega}_1 = (4, 1, 0, 0), \quad \boldsymbol{\omega}_2 = (4, 0, 1, 1)$$

For simplicity, we assume that the monopolist's technological possibility is defined as $\mathbf{y} = (1, 1, 1)$. Therefore, the first possible market structure the monopolist can offer is just the one containing a single variety reflecting perfectly his production plan, $\mathbf{y} = (1, 1, 1) = \mathbf{1}$, so that the starting market structure is

$$A = \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The optimality conditions for the two consumers are, respectively

$$p_{1,1} = \frac{1}{1 + q_{11}} + \frac{2}{q_{11}}, \quad p_{1,2} = \frac{1}{q_{12}} + \frac{2}{1 + q_{12}} \tag{2.15}$$

By equating them, and using the market clearing condition $q_{11} + q_{12} = 1$, we get that the Walrasian equilibrium allocation associated with the trivial market structure A is $(\mathbf{q}^*(A), \mathbf{p}^*(A)) = \{(q^*_{11} = 0.602, q^*_{12} = 0.398), p^*_1 = 3.945\}$. The associated anticipated market value for the monopolist is $\Pi(\mathbf{p}^*(A)) = 3.945$. The indirect utilities derived from this allocation by the two consumers are $V_1(A) = 0.668$ and $V_2(A) = 1.587$. In this situation in which the monopolist provides directly his production plan, the two consumers' evaluation of the different characteristics differs notably, $\boldsymbol{\pi}^*_1 = (0.624, 1.660, 1.660)$ and $\boldsymbol{\pi}^*_2 = (2.514, 0.715, 0.715)$. From this, we get that, depending on their initial endowments, each consumer evaluates more the lesser held characteristic. Moreover, it is clear that for this market structure, the allocation of characteristics, $\mathbf{x}^* = (\mathbf{x}^*_1, \mathbf{x}^*_2)$, is not efficient.

Now, consider an intermediate situation, in which the firm decides to offer two varieties. In this case, we can parametrize a family of feasible market structures $\mathcal{A}_{3\times2}$ that, given the initial vector of fundamental characteristics $\mathbf{y}$, contains all the 3×2 market matrices A_j, $j = 1, \ldots, J$, for which $\sum_{k=1}^{K} a^k_s = 1$ for $s = 1, \ldots, S$. So, in the present case, the family $\mathcal{A}$ of 3×2 market matrices contains a total of $J = 2^3 = 8$ possible designs[17]. However, two of them

[17] In general, for generic S and $K = S - n$ for $n = 1, \ldots, S - 1$, every family of feasible $S \times (S - n)$ market structures, $\mathcal{A}_n$, contains exactly $(S - n)^S$ possible market structure designs.

correspond to the two cases in which respectively $\mathbf{a}_1 = \mathbf{y}$ and $\mathbf{a}_2 = \mathbf{y}$. We can neglect these two trivial cases since the corresponding reaction will be the one computed above. Consider $\mathbf{a}_1$ and $\mathbf{a}_2$ defining the following market structure in $\mathcal{A}_{3\times 2}$

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The optimality conditions, for the two consumers, now reads

$$p_{1,1} = \frac{1}{1+q_{11}} + \frac{1}{q_{11}}, \quad p_{2,1} = \frac{1}{q_{21}} \tag{2.16}$$

$$p_{1,2} = \frac{1}{q_{12}} + \frac{1}{1+q_{12}}, \quad p_{2,2} = \frac{1}{1+q_{22}} \tag{2.17}$$

By equating them and using the market clearing conditions $\mathbf{q}_1 + \mathbf{q}_2 = \mathbf{1}$, we find that the associated Walrasian equilibrium allocation is $(\mathbf{q}^{**}(A_1), \mathbf{p}^{**}(A_1)) = \{(q_{11}^{**} = 1/2, q_{21}^{**} = 1), (q_{12}^{**} = 1/2, q_{22}^{**} = 0), (p_1^{**} = 2.667, p_2^{**} = 1)\}$ and the relative monopolist's market value is $\Pi(\mathbf{p}^{**}(A_1)) = 3.667$. In this case, the indirect utilities derived by the two consumers are $V_1(A_1) = 1.379$ and $V_2(A_1) = 2.379$. Consumers' marginal evaluations of characteristics are, respectively, $\boldsymbol{\pi}_1^{**} = (0.667, 1, 2)$ and $\boldsymbol{\pi}_2^{**} = (2, 1, 0.667)$.

By looking at the indirect utilities and market values in the two cases it is immediate to notice that passing from A to A_1 on one hand decreases the monopolist's market value and on the other increases consumers' welfare in terms of indirect utility. Stated differently, for a value-maximizing monopolist, enlarging the number of varieties decreases its market value. In other words, the monopolist has no incentive in providing the market with intermediate differentiated goods. Oppositely, utility-maximizing consumers prefer higher variety in terms of the number of available goods. That is, they prefer having a large number of intermediate differentiated goods, each of which provides a given bundle of characteristics, rather than having just one variety providing all available characteristics. Moreover, it is also interesting to notice that although the formal structure of the two market matrices A and A_1 is incomplete, from the consumers' perspective, an incomplete differentiated economy with many available varieties is strictly preferred to another with fewer available varieties. The impact of market incompleteness depends on the degree of product differentiation. Finally, note how consumers' marginal evaluation of characteristics changed now that they have two differentiated goods available. Hence, the allocation of characteristics, $\mathbf{x}^{**} = (\mathbf{x}_1^{**}, \mathbf{x}_2^{**})$, still results to be inefficient.

However, notice that given the number of S characteristics, the number of consumers I, and their endowments, when $K = I$, there always exist $S \times K$ matrices that exactly mimic consumers' endowments. This situation, in the present context, could be seen as a *perfect targeting* situation. In our example, there are two such matrices. The difference between the two is only that the two variety vectors, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are exchanged in their position. Hence, the reaction will be the same for both market structures. Consider, without loss of generality, the following perfect targeting market structure

$$A_{pt} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

With this perfect targeting composition of differentiated goods, it is immediate to see that consumer $i = 1$ will buy only the first good, whether consumer $i = 2$ only the second good. In this situation the Walrasian equilibrium allocation is simply defined as $(\mathbf{q}(A_{pt}), \mathbf{p}(A_{pt})) = \{(q_{11}^{pt} = 1, q_{21}^{pt} = 0, (q_{12}^{pt} = 0, q_{22}^{pt} = 1), p_1^{pt} = 2, p_2^{pt} = 1)\}$, and the corresponding monopolist's anticipated market value is $\Pi(\mathbf{p}(A_{pt})) = 3$. Interestingly enough, and it will be clear why below when we present the complete product differentiation case, although the perfect targeting situation is defined by an incomplete market structure, the efficiency of the allocation of characteristics, $\mathbf{x}^{pt} = (\mathbf{x}_1^{pt}, \mathbf{p}_2^{pt})$ is reached. That is, marginal rates of substitution for characteristics equate across consumers. Hence, this market design, although incomplete, efficiently allocates consumers' tastes. In other words, this Walrasian equilibrium allocation reaches the allocative efficiency[18].

[18]More specifically, what we called perfect targeting situation is nothing more that the particular case of an *effectively complete market*. See Elul (1999).

Consider at last, the extreme case in which the monopolist provides three "Arrow" varieties, $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$, which define the following *complete* market structure

$$A_{cm} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In this last case, the optimality conditions for consumer $i = 1$ are

$$p_{1,1} = \frac{1}{1 + q_{11}}, \quad p_{2,1} = \frac{1}{q_{21}}, \quad p_{3,1} = \frac{1}{q_{31}} \tag{2.18}$$

and, similarly, for consumer $i = 2$

$$p_{1,2} = \frac{1}{q_{12}}, \quad p_{2,2} = \frac{1}{1 + q_{22}}, \quad p_{3,2} = \frac{1}{1 + q_{32}} \tag{2.19}$$

From these, we get that the Walrasian equilibrium allocation is $(\mathbf{q}(A_{cm}), \mathbf{p}(A_{cm})) = \{(q_{11} = 0, q_{21} = 1, q_{31} = 1), (q_{12} = 1, q_{22} = 0, q_{32} = 0), (p_1 = 1, p_2 = 1, p_3 = 1)\}$ and the corresponding monopolist's anticipated market value is $\Pi(\mathbf{p}(A_{cm})) = 3$. Now, unsurprisingly, consumers' marginal evaluation equates across characteristics, $\boldsymbol{\pi}_1 = \boldsymbol{\pi}_2 = (1, 1, 1)$. Hence, the equilibrium allocation of characteristics, $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$, results to be efficient. The indirect utility derived by both consumers is $V_1(A_{cm}) = 2$ and $V_2(A_{cm}) = 3$. These results are not surprising given the current market structure. In fact, A_{cm} defines the situation of complete product differentiation and hence provides consumers with a full set of possible allocations. However, in this last case, we get that the monopolist has no incentive to provide A_{cm} since its anticipated market value will be lower, or at most equal, than the ones attainable from A or any viable $A_j \in \mathcal{A}$. On the contrary, consumers would weakly prefer this last market structure over both A and every $A_j \in \mathcal{A}$. To summarize

$$\forall i \quad V_i(A_{cm}) \geq V_i(\mathcal{A}_{3\times 2}) > V_i(A) \tag{2.20}$$

$$\Pi(\mathbf{p}(A_{cm})) \leq \Pi(\mathbf{p}^{**}(\mathcal{A}_{3\times 2})) < \Pi(\mathbf{p}^{*}(A)) \tag{2.21}$$

From both inequality chains above it is clear that firms and consumers have completely opposite preference orderings over possible market structures. Their behavior is indeed monotonic, although in opposite directions. The monopolist's preferences monotonically decrease as the market structure tends to complete product differentiation[19]. On the other hand, consumers' preferences increase as market structure dimensionality tends towards market completeness.

In the next example, we show that given a utility index, satisfying the lover for variety assumption, but displaying concave marginal utility, the results of Example 1 are reversed. Namely, the monopolist will have the incentive to propose the complete market structure.

Example 2. Consider the following individual utility function, for the generic consumer i of the following form

$$U(\mathbf{x}_i) = x_{0i} + \sum_{s=1}^{S} u(x_{si})$$

with

$$u(x_{si}) = x_{si} - \alpha x_{si}^{1+\gamma}, \quad \forall s \tag{2.22}$$

with $\alpha > 0$ and $\gamma > 1$. We will see that $\alpha > 0$ is too generic, indeed we will specify a feasible interval for this parameter. With $\gamma > 1$ we have that $u' = 1 - (1+\gamma)\alpha x_{si}^{\gamma} > 0$, $u'' = -\gamma(1+\gamma)\alpha x_{si}^{\gamma-1} < 0$, and $u''' = -(\gamma-1)\gamma(1+\gamma)\alpha x_{si}^{\gamma-2} < 0$ for any $\alpha > 0$. For each $i = 1, \ldots, I$ we have the following utility structure defined over the S available characteristics

$$U(\mathbf{x}_i) = \left(w_{0i} - \sum_{s=1}^{S} p_k q_{ki}\right) + \sum_{s=1}^{S} \left(x_{si} - \alpha x_{si}^{1+\gamma}\right)$$

[19]This is in contrast with the financial innovation literature (for a general overview see Allen and Gale (1994)): so far as consumers' marginal valuation of future consumption is heterogeneous, the firm has an incentive in providing the market with more complex securities. Conversely, in our case, the fact that marginal evaluations of characteristic amounts are sufficiently heterogeneous across consumers is precisely what eliminates any incentive for the monopolist to increase the market's variety. Consumers' heterogeneity in the evaluation of characteristics is what increases the monopolist's market value in the presence of non-negativity constraints.

Consider $I = S = 2$, and the following initial endowments $\boldsymbol{\omega}_1 = (w_{01}, 1, 0)$ and $\boldsymbol{\omega}_2 = (w_{02}, 0, 1)$. Assume, for simplicity, that the monopolist's technological possibility frontier, i.e the available characteristics, is given by $\mathbf{y} = (1, 1)$. Now, Consider the incomplete market structure

$$A = \mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

With this market structure, the optimality conditions, assuming internal solution and taking $\gamma = 2$, for the two consumers are

$$p_{1,1} = 2 - 3\alpha[(1 + q_{11})^2 + q_{11}^2] \tag{2.23}$$

$$p_{1,2} = 2 - 3\alpha[(1 + q_{12})^2 + q_{12}^2] \tag{2.24}$$

By equating 2.23 and 2.24, and using the market clearing condition $q_{11} + q_{12} = 1$, we get the Walrasian equilibrium allocation $(\mathbf{q}^*(A), \mathbf{p}^*(A)) = \{(q_{11}^* = 1/2, q_{12}^* = 1/2), p_1^* = 2 - (15/2)\alpha\}$. Clearly, the value of α is crucial for the price to be well defined. Indeed, we must have $\alpha \leq 4/15$ for p_1 to be non-negative. Hence, market value equals $\Pi(p_1^*) = p_1^* = 2 - (15/2)\alpha$ for $\alpha \leq 4/15$. Indirect utilities are

$$V_1 = V_2 = w_{0i} + 1 + \frac{1}{4}\alpha$$

Given the incompleteness of the market, equilibrium allocation of characteristics differs among consumers. In fact easy computations reveal that $\mathbf{x}_1^* = (3/2, 1/2)$ and $\mathbf{x}_2^* = (1/2, 3/2)$. From this, the implicit evaluation of each characteristic differs too, $\boldsymbol{\pi}_1 = (1 - (27/4)\alpha, 1 - (3/4)\alpha)$ and $\boldsymbol{\pi}_2 = (1 - (3/4)\alpha, 1 - (27/4)\alpha)$. Note that $\alpha \leq 4/15$ is indeed not sufficient for ensuring that marginal utilities will be positive in equilibrium. However, it can be seen that this holds for $\alpha \leq 4/27 < 4/15$. As in example 1, with the market structure defined directly by the initial vector of characteristics held by the monopolist, the equilibrium allocation of characteristics is not Pareto optimal.

Now, consider the complete market structure given by

$$A_{cm} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

With this market structure, the optimality conditions for consumer $i = 1$ are

$$p_{1,1} = 1 - 3\alpha(1 + q_{11})^2 \tag{2.25}$$

$$p_{2,1} = 1 - 3\alpha q_{21}^2 \tag{2.26}$$

and for consumer $i = 2$

$$p_{1,2} = 1 - 3\alpha q_{12}^2 \tag{2.27}$$

$$p_{2,2} = 1 - 3\alpha(1 + q_{22})^2 \tag{2.28}$$

Equating 2.25 with 2.26 and 2.27 with 2.28, and using the market clearing condition $\mathbf{q}_1 + \mathbf{q}_2 = \mathbf{1}$, results in the Walrasian equilibrium allocation $(\mathbf{q}^{**}(A_{cm}), \mathbf{p}^{**}(A_{cm})) = \{(q_{11}^{**} = 0, q_{21}^{**} = 1, q_{12}^{**} = 1, q_{22}^{**} = 0), (p_1^{**} = 1-3\alpha, p_2^{**} = 1-3\alpha)\}$. Hence, market value equals $\Pi(A_{cm}) = p_1^{**}+p_2^{**} = 2-6\alpha$ for $\alpha \leq 1/3$. The corresponding indirect utilities are $V_1' = V_2' = w_{0i} + 1 + \alpha$. Product differentiation is complete hence, $\mathbf{x}_1' = \mathbf{x}_2' = (1, 1)$ and $\boldsymbol{\pi}_1' = \boldsymbol{\pi}_2' = (1-3\alpha, 1-3\alpha)$. Clearly, the equilibrium allocation of characteristics is Pareto optimal.

Since $4/27 < 1/3$, for the two solutions to be well defined we must consider $\alpha \in (0, 4/27]$. In this range, it is immediate to see that, surprisingly enough, $\Pi(A) < \Pi(A_{cm})$ and $V_i(A) < V_i(A_{cm})$ for $i = 1, 2$. That is, total surplus will be higher with a complete market structure. The monopolist's objective is aligned with consumers' preferences. This holds since $u''' = -6\alpha < 0$. From this, the hedonic equilibrium corresponds to the array given by $(\mathbf{q}^{**}, \mathbf{p}^{**})$ and the market matrix A_{cm}. As a result, we see that the monopolist has a clear incentive to offer the market matrix providing the higher (complete) degree of product differentiation hence, selecting a differentiation structure leading to the efficient allocation of characteristics. That is, the arising allocative efficiency, in terms of characteristics, of the equilibrium, drove by the monopolist's incentive to increase variety results in higher consumers' indirect utilities. This case is peculiar since, although the monopolist has the incentive to complete the market, making the allocation fully Pareto optimal, market completion absorbs a higher share of consumers' surplus, thus hampering, in general, consumers' welfare. The reason for this is to be found simply in the fact that the monopolist's selects the complete set of products since this is the structure maximizing profit, i.e marginal willingness to pay. Thus, although consumers get the kind of products they want, they have to pay higher prices. Clearly, the quasi-linear utility structure assumed enables us to avoid any complexity related to possible income effects that might cause the allocation of goods, and thus characteristics, in resulting sub-optimal even if the market is complete.

2.6 Discussion

In this section, we will briefly discuss the economic intuition behind the results coming from the two examples above. One element is fundamental for understanding the underlying mechanism, namely, the fact that in our setting love for variety is displayed for each characteristic, not for the quantity of different goods. That is, consumers prefer to have a combination of all the available characteristics instead of a large amount of just some of them. From this, it follows that consumers are not directly evaluating different goods by the quantity they can acquire but, instead, by the relative amount of characteristics each of them provides. That is, their valuation of each variety depends on the implicit evaluation of the characteristics embodied in it.

Consider first the situation of complete product differentiation, $A = I_{S\times K}$. In this case, independently of the monotonicity of u', consumers are buying only the varieties providing them with the missing characteristics. The resulting equilibrium prices however depend on the curvature of u'. In particular, with convex u', willingness to pay, for the missing characteristics, declines as the possibilities to acquire them rise. In equilibrium, the effect of this decline is a consequent low marginal willingness to pay for an additional small increase in any characteristic. The total effect is a generally low level of prices. On the other hand, with u' concave, willingness to pay results higher and slowly declines as missing characteristics are acquired. In other words, consumers display increasing willingness to pay for missing characteristics, and, once provided with the possibility of directly acquiring them, willingness to pay remains high and so does the marginal willingness to pay too[20].

Now, consider the incomplete case and in particular the case of $A = \mathbf{y}$. Conversely from the complete product differentiation case, where the curvature of u' by defining the implicit consumers' evaluation of each characteristic defined equilibrium prices, here the curvature of u' determines two distinct effects. The first is the one generated by a small increase in the held characteristics. The second is the one generated by an equally small increase but in the missing characteristics. The curvature of u' actually determines which of the two prevails. In particular, with u' convex, the second effect prevails: the marginal evaluation of a small increase in the amount of the missing characteristics overcomes the marginal evaluation of an additional small increase in the quantity of held characteristics[21].

Finally, the curvature of u', in this setting, seems also to drive the effective efficiency of the equilibrium. In particular, with convex u' if the monopolist were to be compensated for completing the market, for example via some government intervention, the equilibrium allocation of characteristics would results to be efficient, i.e marginal evaluations are equalized, and indirect utilities result higher than when product differentiation is incomplete. Conversely, with u' concave, an interesting fact arises. Namely, the monopolist's incentive to complete the market, induce higher indirect utilities than with incomplete product differentiation. This means that when marginal utilities

[20]In example 2, with the particular utility index we considered, willingness to pay, defined as $u(x_{si})$ for all $s = 1, \ldots, S$, increases until it reaches the maximum at $\bar{x}_{si} = [1/\alpha(1+\gamma)]^{1/\gamma}$. With $\gamma = 2$ and $\alpha = 0.15$, $\bar{x}_{si} = 1.31$. From this, it is easy to see why, when in equilibrium $x_{si} = \ldots = x_{Si}$=1, willingness to pay and marginal willingness to pay remain high.

[21]To see this consider example 1. In it, $u'(x_{si}) = 1/x_{si}$. A small increase in x_{si}, starting from 0 yields a marginal willingness to pay > 1. Conversely, given an initial amount of characteristics equal to 1, an additional, even if small, increase of it reduces the marginal willingness to pay below 1.

are concave, the monopolist's objective (profit maximization), and the consumers' objective (utility maximization) are interestingly aligned. Even more interestingly, this possibility arises without having assumed any ownership structure. The only, however strong, assumption we made is the monopolist's ability to anticipate market outcomes for any possible market structure. This point seems to deserve further investigation.

2.7 Conclusion

In the present chapter, we showed that a monopolist providing differentiated goods has no incentive in enlarging the number of marketed varieties given an economy populated by heterogeneous variety-lover consumers displaying convex marginal utility. Conversely, in an economy populated, still by variety-lover consumers, however displaying concave marginal utilities, the monopolist has a clear incentive to provide the market with the complete set of varieties. The simple economic intuition is that, when the monopolist can perfectly discriminate then, the curvature of the marginal utility, which defines the marginal willingness to pay, results as the main determinant of the efficiency of the equilibrium outcome. It must be noted that because of the intrinsic nature of a monopoly, *full* Pareto optimality should not be expected at *any* equilibrium outcome even if the monopolist provides the complete set of varieties. However, in our view, the possibility for a monopoly to reach, in equilibrium, the allocative efficiency seems a piece of relevant evidence anyway. This evidence, although arising in a very constrained setting, provides new insight into the ever-lasting discussion concerning the relationship between competition, product differentiation, and consumers' attitude toward the market.

The setting we proposed, in our view, may be suitable for broader generalizations that may be able to shed more light on the mechanisms determining the optimal degree of product differentiation. In particular, many strong assumptions imposed on our hedonic product differentiation model could be relaxed to make the model more general and more able to capture other effects and mechanisms of product differentiation. The usage of such a model in the analysis of innovation incentives for differentiated goods, as already pointed out, is relatively rare in the IO literature. Hence, we think that the present work provides also a new, workable, framework to study important issues arising in the IO theory which, until now, have been studied by means of, yet tractable models, but that are lesser keen to a potential broad generalization, which is indeed provided to our setting by its direct parallelism with the Arrow-Debreu framework.

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Chapter 3

Quantity Competition with Hedonic Product Differentiation: Some Examples

3.1 Introduction

Historically, quantity competition has not received the same attention as has price competition in studying product differentiation. This is so, since, intuitively, price competition better reflects the market power detained by each competing firms when goods are differentiated. Even less attention has been devoted to the study of general equilibrium models with differentiated goods where competition among firms is à la Cournot. Since the seminal work by Dixit and Stiglitz (1977), product differentiation in general equilibrium models has been considered from the point of view of *monopolistic competition*.

In this brief note, we will study, by means of several examples, an oligopoly model where product differentiation is hedonic, i.e goods are defined as *bundles* of characteristics, and consumers are heterogeneous and variety-lovers. We propose an equilibrium concept, the *hedonic* Cournot-Walras equilibrium, which essentially is the subgame perfect Nash equilibrium of a two-stage game where in stage one firms compete in quantities, taking as given the design of their good, and in the second stage, given the available quantities, and goods' designs, consumers formulate their consumption decisions. Thus, the *hedonic* Cournot-Walras equilibrium is the combination of a Cournot-Nash equilibrium and a Walrasian equilibrium.

Essentially, the model we propose is in the spirit of the seminal Gabszewicz and Vial (1972) model. However, its nature is not that of a fully general equilibrium model. The partial equilibrium nature naturally arises in a setting where products are differentiated. Moreover, it seems to us, that this structure may be more suited also from the point of view of having a more workable model in where to study product differentiation. Nonetheless, the model could easily be modified so as to frame it as a full general equilibrium model.

Hedonic product differentiation is considered here as originally proposed by Lancaster (1966), (1975), (1979). Namely, goods are just bundles of different characteristics, and consumers derive their utility by acquiring characteristics buying a particular basket of goods. The advantage of this setting is that it departs from the typical assumption that each consumer has a *most preferred* variety[1]. Departure from this benchmark opens the field to a potential broader generalization of the concept of product differentiation. In particular, this setting shares important features with the Arrow-Debreu model with financial markets. This important similarity[2] provides a powerful framework where consumers' heterogeneity and imperfectly competitive firms, providing differentiated goods, can be simultaneously considered in a coherent setting. The current work has the goal to expose the functioning, and the related difficulties, of this particular framework.

One of the principal issues arising from the seminal model of Gabszewicz and Vial (1972) is that of the definition of a valid *price function*, i.e indirect

[1]This assumption lies at the heart of mostly the whole literature dealing with product differentiation. The reason for this is that it permits to generalize the classic Hotelling (1929) setting and thus permits to study the impact and the implications of product differentiation using the standard *location setting*. For a complete exposition of this theory see Anderson, et. al (1992).

[2]The recognition of this parallelism is not new. It was already recognized by Leland (1977) and Dreze and Hagen (1978). However, since then, not much work has been done to further develop this area.

demands. In order to solve the model they proposed an *ad hoc* normalization for the price function. The implications of this procedure, and its difficulties, have been studied by several authors[3]. However, in this work the issue is not related to which normalization should, if any, be used to define a valid price function. Anyway, the problematic part of the model still refers to the definition of the price function. In particular, we will show, in a simple example with linear demands, that the definition of the price function heavily depends on whether non-negativity constraints are binding, or not. In the case of binding non-negativity constraints, the selection of the price function results rather arbitrary. Moreover, the competitive structure will be altered. Thus, imposing non-negativity constraints, although fully justifiable in the current setting - variety lover consumers hit the market to buy positive quantities of available goods - creates a potential pathological discontinuity which may prevents us from defining a valid price function. In order to avoid this possibility, we introduce a very simple ownership structure whose primary role is to provide consumers with an endowment of goods that they may exchange in the market. This will suppress the need for non-negativity constraints, and also endogenize consumers' income. However, in this modified setting, the outside good will play an important role. In particular, interpreting it as some, sufficiently high, (exogenous) amount of "outside money" detained by consumers, prevent us from dealing with the issue of having binding non-negativity constraints. However, if some consumer disposes only of a limited amount of "outside money", then we have to deal with the difficulties entailed by the *minimum wealth problem*[4].

In what follows we present a series of examples showing the functioning of the model. In particular, for the sake of simplicity, but also because it is typical for product differentiation literature, we consider quasilinear quadratic utilities, generating a linear demand structure[5]. Section 2 presents the model in its various parts. In section 3 we present the concept of hedonic Cournot-Walras equilibrium. Sections 4 and 5 are devoted to the expositions

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[3]For a general, yet precise and clear, exposition of the difficulties connected to *ad hoc* normalizations in a general equilibrium context see Dierker and Dierker (2006).

[4]See Debreu (1959), pp. 62-65.

[5]The fundamental reference is Singh and Vives (1984).

of the examples. Finally, section 6 concludes providing some final remarks and hints for further research.

3.2 The model

We consider an oligopoly where product are hedonically differentiated, i.e defined as vectors in the characteristics space $\mathbb{R}^S_+$, $s = 1, \ldots, S$ being the number of technologically feasible characteristics embodied in different goods. Each firm $j = 1, \ldots, J$ sells only one differentiated good. This good is defined as a vector $\mathbf{a}_j \in \mathcal{A}_j \subseteq \mathbb{R}^S_+$. $\mathcal{A}_j$ is the technologically feasible set from which firms select their specific vector of characteristics that will later be marketed as a differentiated good. Define the $S \times J$ matrix A to be the matrix composed by the vectors $\mathbf{a}_j$. We assume that each vector $\mathbf{a}_j$ is of unit length, $\sum_{s=1}^S a_{js}^2 = 1$ for any $j = 1, \ldots, J$. This *normalization* provides us a way to define the *degree of product differentiation*, or *similarity*, $\gamma = \mathbf{a}_j \cdot \mathbf{a}_k$. This defines the angle between the two vectors. Hence, it is a measure of relative product differentiation. This "circular" normalization is adopted because of its simplifying power. In particular, one could also chose a "linear" normalization such as $\sum_{s=1}^S a_{js} = 1$. However, with the "linear" normalization, the definition of the *similarity* becomes more cumbersome[6]. Given the semi-positiveness[7] of $\mathbf{a}_j$, $\gamma \in [0, 1]$. In particular, $\gamma = 0$ defines independent goods, $\gamma = 1$ perfect substitutes, and for $\gamma \in (0, 1)$ goods results imperfect substitutes. Note how the current structure does not consider the case of complementary goods. Finally, we assume that firms incur in no production costs.

[6] In general, the angle between two vectors is defined implicitly by $\cos\theta_{jk} = \frac{\mathbf{a}_j \cdot \mathbf{a}_k}{||\mathbf{a}_j|| \cdot ||\mathbf{a}_k||}$. Thus, when using the "linear" normalization, an additional term given by the product of the norms must be considered. This increases the complexity of the model without providing any further insight.

[7] A vector $\mathbf{a}_j \in \mathbb{R}^S$ is semi-positive if it is non-negative but not zero. In other words, it has at least one component which is strictly positive.

$$\mathbf{a}_j > 0 \quad \Longleftrightarrow \quad \mathbf{a}_j \in \mathbb{R}^S_+, \quad \mathbf{a}_j \neq \mathbf{0}.$$

3.2.1 Consumers

There are $i = 1, \ldots, I$ active consumers, having preferences defined over the space of characteristics $\mathbb{R}^S_+$. Each consumer is endowed with an initial vector of characteristics, $\mathbf{w}_i$. This initial endowment should be interpreted as deriving from previous consumption of some good. We will assume that $\sum_{i=1}^{I} \mathbf{w}_i = \tilde{\mathbf{w}} > 0$. Each consumer is endowed also with an initial income $w_{0i} >> 0$. Denote the quantities consumed of each good by the i-th consumer by $\mathbf{q}_i = (q_{1i}, q_{2i}, \ldots, q_{Ji}) \in \mathbb{R}^J_+$. Non-negativity constraints are imposed by assuming $\mathbf{q}_i$ to be semi-positive, $\mathbf{q}_i > 0$. The total amount of the s-th characteristic acquired by the i-th consumer is given by $c_{si} = \sum_{j=1}^{J} a_{js} q_{ji}$. Then, from this, we can define the vector of total characteristics for the i-th consumer as $\mathbf{x}_i \in \mathbb{R}^S_+$, $\mathbf{x}_i = \mathbf{w}_i + A\mathbf{q}_i$.

We assume that the utility function $U : \mathbb{R}^S_+ \rightarrow \mathbb{R}_+$ representing consumer's preferences is the same for every consumers. In particular, it is *additively separable* in characteristics and linear in the numeraire good. That is,

$$U(\mathbf{x}_i) = x_{0i} + \sum_{s=1}^{S} u(x_{si}), \quad i = 1, \ldots, I \tag{3.1}$$

where x_{0i} is an homogeneous good being the numeraire of the model. We assume the utility index $u(\cdot)$ to be continuous, concave, and thrice differentiable so that U will be well-behaved. Each consumer $i = 1, \ldots, I$ maximizes (1) subject to

$$B_i = \{\mathbf{q}_i \in \mathbb{R}^J_+ \mid x_{0i} + \sum_{j=1}^{J} p_j q_{ji} \leq w_{0i}\} \tag{3.2}$$

Thus, the problem faced by each consumer $i = 1, \ldots, I$ is that of choosing a vector of quantities $\mathbf{q}_i$ such that 3.1 is maximized. Note that by doing so, each consumer is actually maximizing the amount of characteristics. Thus, we can consider as the actual maximization object the *indirect* utility function derive y plugging the definition of $\mathbf{x}_i$ into 3.1. From this, the problem that the generic consumer i has to solve is the following.

$$\begin{aligned} \max_{q_{1i},\dots,q_{Ji}} \quad & w_{0i} - \sum_{j=1}^{J} p_j q_{ji} + \sum_{s=1}^{S} u\left(w_{si} + \sum_{j=1}^{J} a_{js} q_{ji} \right) \\ \text{s.t.} \quad & q_{ji} \geq 0, \quad j = 1, \dots, J \end{aligned} \tag{3.3}$$

The system of first order conditions (FOC) associated with problem 3.3 is

$$-p_j + \sum_{s=1}^{S} u'(x_{si}) a_{js} \leq 0, \quad q_{ji}\left(-p_j + \sum_{s=1}^{S} u'(x_{si}) a_{js} \right) = 0, \quad j = 1, \dots, J \tag{3.4}$$

Provided $q_{ji} \geq 0$, for every j, the above system directly defines the *inverse demand* schedules for the generic consumer i for every available product j. These represent the key element of the model. In particular, for the generic consumer i, define as $D_i^W : \mathbb{R}_+^J \to \mathbb{R}_+^J$ her Walrasian demand correspondence. Given *interior* solutions to her FOC, for a given price vector, $\mathbf{p}$, and a given market matrix, A, we have that $D_i^W(\mathbf{w}_i, \mathbf{p}) = (q_{1i}(\mathbf{w}_i, \mathbf{p}), \dots, q_{Ji}(\mathbf{w}_i, \mathbf{p}))$. The shape of these demands heavily depends on the shape of the utility index $u(\cdot)$, and on the structure of A.

3.2.2 Firms

Firms jointly provide a set of differentiated goods described by the $S \times J$ market matrix A. Firms' outputs are determined by competition in quantities. In particular, the j-th firm provides $\mathbf{a}_j$ in e certain quantity $y_j \geq 0$, given the output of its rivals. We assume that the only choice the firm is actively making regards the selection of its output y_j. That is, we are assuming that the market matrix A is fixed.

Given a generic market structure A, the j-th firm *strategically* selects y_j so as to maximize its profit. That is, it takes the opponents' outputs $\mathbf{y}_{-j}$ as given and selects y_j which maximizes profit. Firm j's profit function is given by

$$\pi_j = p_j(y_j, \mathbf{y}_{-j}) y_j \tag{3.5}$$

The inverse demand function $p_j(y_j, \mathbf{y}_{-j})$ represents the key object in the current work. In particular, it arises from the Walrasian allocation taking place in the consumption side of the economy *given* a generic output vector $\mathbf{y} = (y_1, \ldots, y_J)$. Given the $i = 1, \ldots, I$ Walrasian demand correspondences $D_i^W(\cdot, \cdot)$, the inverse demand system is a correspondence $P : \mathbb{R}_+^J \to \mathbb{R}_+^J$, such that

$$\sum_i D_i^W(\mathbf{w}_i, P(\mathbf{y})) - \mathbf{y} = \mathbf{0} \tag{3.6}$$

where $P(\mathbf{y}) = (p_1(\mathbf{y}), \ldots, p_J(\mathbf{y}))$. Thus, its existence and behaviour implicitly depend on the specific equilibrium allocation arising in stage two, given a generic output vector $\mathbf{y}$. Nonetheless, it should be clear that a-priori we cannot assume such a price function to exists or, if it does, to be well behaved.

3.3 Equilibrium

The logic of the equilibrium is along the lines of the one developed by Gabszewicz and Vial (1972). In the first stage firms engage in quantity competition à la Cournot by taking the inverse demand system as *given*. In the second stage, given the Cournot-Nash equilibrium outputs arising in stage one, consumers decide how much of each good to consume. From this, a Walrasian equilibrium obtains. We can solve this model by *backward induction*. However, our model is not fully general equilibrium in nature. In particular, we neglect any problem related to the redistribution of profits among consumers. Hence, we also neglect any problem related to the definition of the firm's incentives when stakeholders are heterogeneous. From these, the following model is closer to standard models in industrial organization theory which are typically models of partial equilibrium.

We will assume that for each firm j its production possibilities are in a compact-convex Y_j of $\mathbb{R}_+$. A feasible output for firm j is a point $y_j \in Y_j$. After each firm has chosen an output quantity in its production set, the arising vector of outputs $\mathbf{y} \in Y_1 \times \cdots \times Y_J$ is provided to consumers. At this point, a competitive mechanism takes place. It can be seen as a sort of auction where, given the available quantities $\mathbf{y} \in \mathbb{R}^J_+$, a market-clearing price system $P(\mathbf{y}) \in \mathbb{R}^J_+$ exists, at which each consumer selects her optimal amount of goods $\mathbf{q}_i$. Thus, an allocation is $(\mathbf{q}_1, \ldots, \mathbf{q}_I) \in \mathbb{R}^{JI}_+$. For a set of feasible outputs $\mathbf{y}$, a competitive equilibrium relative to $\mathbf{y}$ is a pair $(P, (\mathbf{q}_1, \ldots, \mathbf{q}_I))$ such that for all i utility is maximized subject to the budget constraint. We define the price function, which is indeed the inverse demand system, as $P : Y_1 \times \cdots \times Y_J \to \mathbb{R}^J_+$. This price function is such that for any output vector $\mathbf{y} = (y_1, \ldots, y_J)$, $(P(\mathbf{y}), (\mathbf{q}_1, \ldots, \mathbf{q}_I))$ is a competitive equilibrium relative to $\mathbf{y}$.

However, the relationship of consumers' endowments and the market matrix A plays a fundamental role. For, it might be the case that for some A, endowments are such that some consumer is forced, facing the output vector $\mathbf{y}$, to sell some quantity of some good. This will violate the non negativity constraint $q_{ji} \geq 0$, bounding to zero the demand of such goods, and hence will result in a corner solution which in turn would complicates the definition of the Walrasian demand correspondence of such a consumer. In the examples below we will show how this affects the behaviour of the model.

The behaviour of firms is the following. Firms engage in quantity competition à la Cournot taking the inverse demand system as given. That this, each firms maximizes $\pi_j(y_j, \mathbf{y}_{-j}) = p_j(y_j, \mathbf{y}_{-j})y_j$. Provide each $p_j(\cdot)$ to be continuous, convex and at least twice differentiable, profit functions are quasi-concave and hence a unique Cournot-Nash equilibrium is expected to exists. We have the following definition

Definition 4. A *hedonic* Cournot-Walras equilibrium is defined as an output vector $\mathbf{y}^C \in \mathbb{R}^J_+$, a price vector $\mathbf{p}^* \in \mathbb{R}^J_+$, and a vector of quantities $\mathbf{q}^* \in \mathbb{R}^{JI}_+$ such that

(i) $\forall i$ $\mathbf{q}_i^*$ maximizes $U(\mathbf{w}_i + A\mathbf{q}_i)$ subject to $x_{0i} + \mathbf{p}^* \cdot \mathbf{q}_i = w_{0i}$ and $\mathbf{q}_i > 0$;

(ii) y_j^C maximizes $\pi_j(\mathbf{y}) = p_j^*(y_j, \mathbf{y}_{-j}^C)y_j$ for all j;

(iii) $p_j^* = p_j^*(\mathbf{y}^C)$ for all j.

The issue of existence, as in the original Gabszewicz and Vial (1972)'s model, faces the problem of defining an *objective demand function* for every firm $j = 1, \ldots, J$, that is, a sufficiently well-behaved price system $P(\mathbf{y})$. However, here the problem does not concern with what *normalization rule* should be used to overcome the issue of having only *relative prices*[8]. Instead, in our setting, it refers to the fact that given consumers' heterogeneity and their love for variety, each of them expresses a demand schedule for each of the available varieties. Hence, it is not clear how to determinate the aggregate demand for every of the $j = 1, \ldots, J$ firms. In what follows, we present a series of examples for which, given the particular form of the utility index $u(\cdot)$, it is possible to easily solve the model.

3.4 The hedonic linear demand system

We will assume firms' direct and inverse demand functions to be derived from consumers' first order conditions. The utility structure we will consider is a workhorse in the industrial organization literature dealing with product differentiation (see Singh and Vives (1984)). Each consumer $i = 1 \ldots, I$ maximizes, taking prices as given,

$$U(x_{0i}, \mathbf{x}_i) = x_{0i} + \sum_{s=1}^{S} \left(x_{si} - \frac{1}{2} x_{si}^2 \right) \tag{3.7}$$

subject to

$$B_i = \{ \mathbf{q}_i \in \mathbb{R}_+^J \mid x_{0i} + \mathbf{p} \cdot \mathbf{q}_i \leq w_{0i} \} \tag{3.8}$$

where x_{0i} is an outside homogeneous good taken as the numeriere, w_{0i} is consumer i's income, and $\mathbf{p}$ is a price vector. The matrix containing the marketed varieties, faced by the consumers, is given in general form by

$$A = \begin{bmatrix} a_{11} & a_{21} & \ldots & a_{J1} \\ a_{12} & a_{22} & \ldots & a_{J2} \\ \vdots & \vdots & \ddots & \\ a_{1S} & a_{2S} & \ldots & a_{JS} \end{bmatrix}$$

[8] This issue arises because of the general equilibrium nature of the original model proposed by Gabszewicz and Vial (1972).

The only assumption imposed on the market matrix A is that of being *non-singular*, i.e rank$(A) \neq 0$. This ensures the matrix invertibility. Consider the simplest case where $S = J = I = 2$. Given a market matrix A, and generic price vector $\mathbf{p}$, the set of first order conditions (FOC), for the generic consumer i, is

$$-p_1 + \alpha_1 - \mathbf{w}_i \cdot \mathbf{a}_1 - q_{1i} - \gamma q_{2i} \leq 0, \quad q_{1i}(-p_1 + \alpha_1 - \mathbf{w}_i \cdot \mathbf{a}_1 - q_{1i} - \gamma q_{2i}) = 0 \tag{3.9}$$

$$-p_2 + \alpha_2 - \mathbf{w}_i \cdot \mathbf{a}_2 - \gamma q_{1i} - q_{2i} \leq 0, \quad q_{2i}(-p_2 + \alpha_2 - \mathbf{w}_i \cdot \mathbf{a}_2 - \gamma q_{1i} - q_{2i}) = 0 \tag{3.10}$$

Where $\alpha_j = a_{j1} + a_{j2}$, and the *cosine similarity* between the two marketed goods is defined as $\gamma = \mathbf{a}_1 \cdot \mathbf{a}_2$. Assume that the non-negativity constraints are not binding for neither of the consumers. Then, the system of optimality conditions of the two consumers, defining their *inverse demand system* could be written, respectively, as

$$\mathbf{p} = (\mathbf{1} - \mathbf{w}_1)A - \Gamma \mathbf{q}_1 \tag{3.11}$$

$$\mathbf{p} = (\mathbf{1} - \mathbf{w}_2)A - \Gamma \mathbf{q}_2 \tag{3.12}$$

where Γ is the *similarity* matrix[9] define by $(A'A)$. From 3.11 and 3.12 we can define the Walrasian demand correspondences for both consumers, $D_1^W(\cdot,\cdot)$ and $D_2^W(\cdot,\cdot)$. In particular, we have that

$$D_1^W(\mathbf{w}_1, \mathbf{p}) = \Gamma^{-1}[(\mathbf{1} - \mathbf{w}_1)A - \mathbf{p}] \tag{3.13}$$

$$D_2^W(\mathbf{w}_2, \mathbf{p}) = \Gamma^{-1}[(\mathbf{1} - \mathbf{w}_2)A - \mathbf{p}] \tag{3.14}$$

Now, given a generic output vector $\mathbf{y}$ and 3.13 and 3.14, we can define the price system $P = (p_1(\mathbf{y}), p_2(\mathbf{y}))$. That is, $P(\cdot)$ is indeed, for given A,

[9]The *similarity* matrix is just the matrix containing the cosine similarities. In particular, in the current setting, we have that

$$\Gamma = \begin{bmatrix} 1 & \gamma \\ \gamma & 1 \end{bmatrix}$$

$(\mathbf{w}_1, \mathbf{w}_2)$ and $\mathbf{y}$, the price system clearing the market. Namely, $P(\cdot)$ is such that

$$\Gamma^{-1}[(\mathbf{1} - \mathbf{w}_1)A - P(\mathbf{y})] + \Gamma^{-1}[(\mathbf{1} - \mathbf{w}_2)A - P(\mathbf{y})] - \mathbf{y} = \mathbf{0} \tag{3.15}$$

Since, $\mathbf{y}$ is given, and the solution of the consumers' optimization problem is interior, hence defined by a linear functions, form 3.15 we have that

$$P(\mathbf{y}) = \mathbf{1}A - \frac{1}{2}(\mathbf{w}_1 + \mathbf{w}_2)A - \frac{1}{2}\Gamma\mathbf{y} \tag{3.16}$$

Provided the non-negativity constraints to not be binding, for any possible $\mathbf{y}$, the price system defining the Walrasian equilibrium in stage two varies according to relation expressed by the above expression. Clearly, 3.16 breaks-up whenever, given $\mathbf{y}$, the non-negativity constraints result binding for at least one of the consumers. From 3.16 we are also able to define firms' choice sets Y_1 and Y_2. For, 3.16 is define in the region of the quantity space where quantities are positive. In the present setting this region is the cone defined by the vectors $((0, \bar{y}_1), (0, \bar{y}_2))$ for which 3.16 is equal to zero. Thus, firms' choice sets are given by, respectively, $Y_1 = [0, \bar{y}_1]$ and $Y_2 = [0, \bar{y}_2]$

System 3.16 defines the inverse demand system faced by firms in stage one. In Cournot quantity competition firms take the indirect demand system as given and maximize profit, given the opponent's strategy, with respect to output. Profit is defined, for any $j = 1, 2$, as $\pi_j(\mathbf{y}) = p_j(\mathbf{y})y_j$. The profit functions for the two firms are, respectively

$$\pi_1(\mathbf{y}) = \left(\alpha_1 - \frac{1}{2}(\mathbf{w}_1 + \mathbf{w}_2) \cdot \mathbf{a}_1\right)y_1 - \frac{1}{2}y_1^2 - \frac{1}{2}\gamma y_2 y_1 \tag{3.17}$$

$$\pi_2(\mathbf{y}) = \left(\alpha_2 - \frac{1}{2}(\mathbf{w}_1 + \mathbf{w}_2) \cdot \mathbf{a}_2\right)y_1 - \frac{1}{2}\gamma y_1 y_2 - \frac{1}{2}y_2^2 \tag{3.18}$$

Firm 1 maximizes 3.17 in Y_1, and firm 2 maximizes 3.18 in Y_2. The system of best replies is given by

$$y_1(y_2) = \alpha_1 - \frac{1}{2}(\mathbf{w}_1 + \mathbf{w}_2) \cdot \mathbf{a}_1 - \frac{\gamma}{2}y_2 \tag{3.19}$$

$$y_2(y_1) = \alpha_2 - \frac{1}{2}(\mathbf{w}_1 + \mathbf{w}_2) \cdot \mathbf{a}_2 - \frac{\gamma}{2} y_1 \tag{3.20}$$

Simple computations yield the Cournot-Nash equilibrium

$$y_1^C = \frac{4\alpha_1 - 2\gamma\alpha_2 - 2(\mathbf{w}_1 + \mathbf{w}_2) \cdot \mathbf{a}_1 + \gamma(\mathbf{w}_1 + \mathbf{w}_2) \cdot \mathbf{a}_2}{(4 - \gamma^2)} \tag{3.21}$$

$$y_2^C = \frac{4\alpha_2 - 2\gamma\alpha_1 - 2(\mathbf{w}_1 + \mathbf{w}_2) \cdot \mathbf{a}_2 + \gamma(\mathbf{w}_1 + \mathbf{w}_2) \cdot \mathbf{a}_1}{(4 - \gamma^2)} \tag{3.22}$$

From the above exposition, we see that when the Walrasian allocation, given $\mathbf{y}$, is interior, a well defined price function for each firm can be derived. Consequently, a unique hedonic Cournot-Walras equilibrium can be defined. However, it may not always be the case of non-binding non-negativity constraints. In those cases, an exact derivation of the price functions results particularly cumbersome. Firms are aware of the constraints imposed to consumers and hence, when selecting their (Cournot-Nash) output, they implicitly complain with these restrictions. When the non-negativity constraints bind consumers' consumption schedules, since firms take as given the price function defined *given* a generic Cournot-Nash output vector $\mathbf{y}^C$, and since this price function define the Walrasian equilibrium in stage two, firms are actually themselves *constraint* by the competitiveness of the market in stage two.

In what follows we present two examples. In the first non-negativity constraints result non-binding for both consumers and thus the identification of an hedonic Cournot-Walras equilibrium results sufficiently simple. In the second example, non-negativity constraints result binding for both consumers making the search for an hedonic Cournot-Walras equilibrium more delicate. This second example also provide a glance of the complexities involved in the model we are developing. The complexities involved result even more striking considering the simple utility function we are using.

3.4.1 The leading examples

Example 3. Consider the following market structure A formed by $\mathbf{a}_1 = (0.71, 0.71)$ and $\mathbf{a}_2 = (0, 1)$. The cosine similarity equals $\gamma = 0.71$. Let endowments be $\mathbf{w}_1 = (1/2, 1/2)$ and $\mathbf{w}_2 = (0.10, 0.20)$. The aggregate endowment vector equals $\tilde{\mathbf{w}} = \sum_i \mathbf{w}_i = (0.60, 0.70)$. Consider the Walrasian allocation in stage two given output vector $\mathbf{y}$. In addition, assume that for this output vector non-negativity constraints result non-binding for both consumers. We can thus proceed as usual. In particular, given any $\mathbf{y}$, there exists a well defined price system $\mathbf{p}^*$ clearing the market. The price system clearing the market, $\mathbf{p}^*$, is derived from the consumers' FOC and the market clearing condition

$$q_{11} + q_{12} = y_1$$

$$q_{21} + q_{22} = y_2$$

After some simple computation we get that the price system clearing the market is given by

$$p_1(\mathbf{y}) = \alpha_1 - \frac{1}{2}\tilde{\mathbf{w}} \cdot \mathbf{a}_1 - \frac{1}{2}y_1 - \frac{1}{2}\gamma y_2$$

$$p_2(\mathbf{y}) = \alpha_2 - \frac{1}{2}\tilde{\mathbf{w}} \cdot \mathbf{a}_2 - \frac{1}{2}\gamma y_1 - \frac{1}{2}y_2$$

Now, in stage one, given the above well defined demand system, firms maximize profits taking the opponent's output as given. From the above we can define the choice set of the two firms. In particular, these choice sets are given, in general, by $Y_j = [0, \bar{y}]$, where $\bar{y} > 0$. In order to simplify the subsequent analysis, we will interpret $\bar{y}_j$ as the output of firm j for which, whenever $\gamma = 0$ or $y_{-j} = 0$, $p_j(\cdot) = 0$. For the present example, we have that $\bar{y}_1 = 1.917$ and $\bar{y}_2 = 1.3$. Thus, $Y_1 = [0, 1.917]$ and $Y_2 = [0, 1.3]$. The two firms solve, respectively

$$\max_{y_1 \in Y_1} \pi_1(\mathbf{y}) = \max_{y_1 \in Y_1} \left\{ \alpha_1 y_1 - \frac{1}{2}\left(\tilde{\mathbf{w}} \cdot \mathbf{a}_1 + y_1 + \gamma y_2 \right) y_1 \right\}$$

$$\max_{y_2 \in Y_2} \pi_2(\mathbf{y}) = \max_{y_2 \in Y_2} \left\{ \alpha_2 y_2 - \frac{1}{2}\left(\tilde{\mathbf{w}} \cdot \mathbf{a}_2 + \gamma y_1 + y_2 \right) y_2 \right\}$$

From standard computations, the resulting Cournot-Nash equilibrium is seen to be given by $y_1^C = 0.89$ and $y_2^C = 0.43$. Now, given the Cournot-Nash output vector $\mathbf{y}^C$, we are now able to characterize the hedonic Cournot-Walras equilibrium. In particular, the price system $\mathbf{p}^*$ can now be explicitly computed given $\mathbf{y}^C$. By definition, this price system clears the market, that is

$$\sum_i D_i^W(\mathbf{w}_i, \mathbf{p}^*) = \mathbf{y}^C \tag{3.23}$$

From 3.23 we are able immediately to derive the market clearing price $\mathbf{p}^*$. This is given by $\mathbf{p}^* = (0.36, 0.12)$. Now, with this at hand, since we assumed the non-negativity constraints not to be binding, the equilibrium allocation $(\mathbf{q}_1^*, \mathbf{q}_2^*)$ can easily be derived from the consumer's system of first order conditions plus the market clearing conditions. That is, the Walrasian allocation is defined as the solution of the following system

$$\begin{aligned}
-p_1^* + 0.71 - q_{11} - 0.71q_{21} &= 0 \\
-p_2^* + 0.5 - 0.71q_{11} - q_{21} &= 0 \\
-p_1^* + 1.21 - q_{12} - 0.71q_{22} &= 0 \\
-p_2^* + 0.80 - 0.71q_{12} - q_{22} &= 0 \\
q_{11} + q_{12} &= 0.89 \\
q_{21} + q_{22} &= 0.43
\end{aligned}$$

After simple computations we get that $\mathbf{q}_1^* = (0.16, 0.27)$ and $\mathbf{q}_2^* = (0.73, 0.16)$. Non-negativity constraints are indeed non-binding in this case! Hence, the hedonic Cournot-Walras equilibrium exists and is well defined and is given by the Cournot-Nash output vector $\mathbf{y}^C$, the price system $\mathbf{p}^*$, and the vector of quantities $\mathbf{q}^* = (\mathbf{q}_1^*, \mathbf{q}_2^*)$. Notice how the fact that the non-negativity constraints do not bind consumers in their consumption decisions ensures the interiority of the solution of the system of fist-order conditions and hence, in return, the possibility to explicitly define the price functions faced by the firms.

Finally, note that despite the interiority of the equilibrium, the allocation of characteristics results not Pareto optimal. To see this, consider the equilibrium values of the consumed characteristics: $\mathbf{x}_1^* = \mathbf{w}_1 + A\mathbf{q}_1^* =$

$(0.6136, 06917)$, $\mathbf{x}_2^* = \mathbf{w}_2 + A\mathbf{q}_2^* = (0.6183, 03136)$. From these, simple computations reveal that

$$\frac{\partial U(x_{01}, \mathbf{x}_1^*)}{\partial x_{11}^*} = 1 - x_{11}^* = 0.3864 \neq 0.3817 = 1 - x_{12}^* = \frac{\partial U(x_{02}, \mathbf{x}_2^*)}{\partial x_{12}^*}$$

and similarly with regard to characteristic $s = 2$. The reason for this inefficiency is to be found in the market power exerted by the two firms.

Example 4. Consider the same market structure A, composed by the two differentiated goods $\mathbf{a}_1 = (0.71, 0.71)$ and $\mathbf{a}_2 = (0, 1)$. Consumers have now the following endowments distribution, $\mathbf{w}_1 = (1, 0)$ and $\mathbf{w}_2 = (0, 1)$ resulting in an aggregate endowment vector equal to $\tilde{\mathbf{w}} = (1, 1)$. We assume that non-negativity constraints are not imposed. This means, in principle, that consumed quantities can be negative.

In stage two, given a generic output vector $\mathbf{y}$, since no non-negativity constraint is imposed, a price system $\mathbf{p}^*$ which clears the market can equally be defined as in the previous example. This price system is given by

$$p_1(\mathbf{y}) = \alpha_1 - \frac{1}{2}\tilde{\mathbf{w}} \cdot \mathbf{a}_1 - \frac{1}{2}y_1 - \frac{1}{2}\gamma y_2 \tag{3.24}$$

$$p_2(\mathbf{y}) = \alpha_2 - \frac{1}{2}\tilde{\mathbf{w}} \cdot \mathbf{a}_2 - \frac{1}{2}\gamma y_1 - \frac{1}{2}y_2 \tag{3.25}$$

From the above, we can readily define, stage one, firms' profit maximization problems. Firms' choice sets are generically given by $Y_j = [0, \bar{y}_j]$, with $\bar{y}_j > 0$. Given the non-symmetric nature of the market matrix A, it may be the case that $\bar{y}_1 \neq \bar{y}_2$. In particular, firm 1's maximal output $\bar{y}_1$, is the one for which, whenever $\gamma = 0$ or $y_2 = 0$, $p_1(\cdot) = 0$. From 3.24 is is immediate to see that $\bar{y}_1 = 1.42$. Similarly, for firm 2 we have that $\bar{y}_2 = 1$. Thus, $Y_1 = [0, 1.42]$, and $Y_2 = [0, 1]$.

Now, in stage one, given the above demand system, firms maximize profits taking the opponent's output as given. That is

$$\max_{y_1 \in Y_1} \pi_1(\mathbf{y}) = \max_{y_1 \in Y_1} \left\{ \alpha_1 y_1 - \frac{1}{2}\left(\tilde{\mathbf{w}} \cdot \mathbf{a}_1 + y_1 + \gamma y_2\right) y_1 \right\}$$

$$\max_{y_2 \in Y_2} \pi_2(\mathbf{y}) = \max_{y_2 \in Y_2} \left\{ \alpha_2 y_2 - \frac{1}{2} \left(\tilde{\mathbf{w}} \cdot \mathbf{a}_2 + \gamma y_1 + y_2 \right) y_2 \right\}$$

The resulting Cournot-Nash equilibrium outputs are $y_1^C = 0.61$ and $y_2^C = 0.29$. We can now compute the price system $\mathbf{p}^*$ which clears the market in stage two. This is done directly from the market clearing conditions. Thus, $\mathbf{p}^*$ is the solution of

$$q_{11}(\mathbf{p}) + q_{12}(\mathbf{p}) = y_1^C$$
$$q_{21}(\mathbf{p}) + q_{22}(\mathbf{p}) = y_2^C$$

Using consumers' FOC, we get that $\mathbf{p}^* = (0.30, 0.14)$. Now, to define the corresponding Walrasian allocation arising in stage two, we solve the following system

$$\begin{aligned} -p_1^* + 0.71 - q_{11} - 0.71 q_{21} &= 0 \\ -p_2^* + 1 - 0.71 q_{11} - q_{21} &= 0 \\ -p_1^* + 0.71 - q_{12} - 0.71 q_{22} &= 0 \\ -p_2^* - 0.71 q_{12} - q_{22} &= 0 \\ q_{11} + q_{12} &= 0.61 \\ q_{12} + q_{22} &= 0.29 \end{aligned}$$

After some simple computations, we have that the solution of the above system is given by $\mathbf{q}_1^* = (-0.41, 1.15)$ for consumer 1, and $\mathbf{q}_2^* = (1.02, -0.86)$ for consumer 2. Note how consumer 1 is selling good 1, and consumer 2 is selling good 2. However, how should we interpret these results? A straightforward, economically meaningful, interpretation seems absent, given the current setting. In particular, how should consumers, given the availability of differentiated goods in *fixed* quantities, be able to generate additional amounts of these goods? No immediate answer comes to mind. Thus, we will leave this as an open question. Having the above allocation at hand, we turn now to the analysis of the more interesting, and economically meaningful, situation where non-negativity constraints are imposed.

Imposing the non-negativity constraints will bind the consumption decisions of both consumers. From the above *unconstrained* consumption schedules, we can directly set $q_{11} = 0$ and $q_{22} = 0$. The problem now is that, by imposing non-negativity constraints, we are now dealing with price correspondences, rather than simple functions. This is so since the system of first order conditions, as we will see, involves several inequalities. We proceed as follows. Knowing that without non-negativity constraints consumer 1 will sell good 1 and consumer 2 good 2, we consider a generic output vector $\mathbf{y} > 0$ and thus impose that $q_{11} = 0$ and $q_{22} = 0$. In this situation, a suitable price system $\mathbf{p}^*$ must solve the following system

$$\begin{aligned} -p_1 + 0.71 - 0.71q_{21} &\leq 0 \\ -p_2 + 1 - q_{21} &= 0 \\ -p_1 + 0.71 - q_{12} &= 0 \\ -p_2 - 0.71q_{12} &\leq 0 \\ q_{12} &= y_1 \\ q_{21} &= y_2 \end{aligned}$$

From the above it is immediate to see that the demand schedules, faced by both firms, now involve an inequality. Consider first firm 1. From the above system we have that

$$p_1 \geq 0.71 - 0.71y_2 \tag{3.26}$$

$$p_1 = 0.71 - y_1 \tag{3.27}$$

and for firm 2 we have

$$p_2 = 1 - y_2 \tag{3.28}$$

$$p_2 \geq -0.71y_1 \tag{3.29}$$

From the above it is immediate to see that any $(y_1, y_2) \in [0, 0.71] \times [0, 1]$ will satisfy the above conditions. Note how the previous Cournot-Nash equilibrium (y_1^C, y_2^C) falls into this set. However, the problem here is how to determine the profit-maximizing output for the two firms. Since any $\mathbf{y} \in [0, 0.71] \times [0, 1]$ will satisfy the inequality conditions above, we can consider the following as suitable profit functions

$$\pi_1 = (0.71 - y_1)y_1 \quad (3.30)$$

$$\pi_2 = (1 - y_2)y_2 \quad (3.31)$$

The two above profit functions, 3.30 and 3.31, actually define a situation in which the two firms behave like monopolists. The profit maximizing outputs are, respectively, $y_1^m = 0.355$ and $y_2^m = 1/2$. Prices are readily derived and reads $p_1^m = 0.355$ and $p_2^m = 1/2$. Note how the binding of the non-negativity constraints eliminates the direct competition among firms. That is, it separates firms' markets from one another. This makes firms behave as *local monopolists*.

At this equilibrium, the values of the consumed characteristic are: $\mathbf{x}_1^* = \mathbf{w}_1 + A\mathbf{q}_1^* = (1, 1/2)$ and $\mathbf{x}_2^* = \mathbf{w} + A\mathbf{q}_2^* = (0.2520, 1)$. Straightforward computations reveal that this equilibrium allocation of characteristics is not efficient. Again, the reason for this is the market power exerted by the two firms.

From the above example it is clear that even if products are imperfect substitutes, that is $\gamma \in (0, 1)$, the fact that non-negativity constraints may bind consumers' consumption decisions might affect the competitive structure of the market. In particular, in example two, the binding non-negativity constraints basically truncated the demand system in such a way that each consumer was forced to buy just one of the two available goods.

An intuitive explanation of the above fact is that when consumption is absent, $\mathbf{q}_i = \mathbf{0}$ for $i = 1, 2$, the particular endowments structure we assumed, *satiate* consumer 1's desire for characteristic 1, and consumer 2's desire for characteristic 2. Indeed, $\partial U(\mathbf{x}_1)/\partial x_{11} = 0$ and $\partial U(\mathbf{x}_2)/\partial x_{21} = 0$. This is what makes the non-negativity constraints bind, and upsetting the competitive structure of the market. Or, stated differently, when non-negativity constraints are binding, example two shows that firms' market areas will be separated although their goods are still imperfect substitutes. Essentially, example 4 describes a situation where each consumer has a *most preferred* variety. This is what activates non-negativity constraints and thus makes firms able to exert some additional degree of market power, even if the goods they provide are imperfect substitutes.

3.4.2 Baseline model with ownership

In the baseline model we considered consumers' endowments as some pre-owned amount of some characteristics which essentially represented utility shifters. Their major consequence was shown to be the possible binding of non-negativity constraints. In this section we instead take a different look on endowments' possible meaning. In particular, let us define the endowment vector $\mathbf{w}_i$ as the vector of goods arising form shares detained by consumer i in active firms $j = 1, \ldots, J$. That is $w_{ji} = \theta_{ji} y_j$, with $\sum_i \theta_{ji} = 1$ for all $j = 1, \ldots, J$. Clearly, different ownership structures $\Theta = \{(\theta_{11}, \ldots, \theta_{J1}), \ldots, (\theta_{I1}, \ldots, \theta_{IJ})\}$, induce different endowments of goods. As before we fix the market matrix A. The functioning of the model will be the following. Given their shares in firms, consumers will thus get an *intermediate endowment* of characteristics arising form their shares structure and the firms' output[10]. So, essentially each consumer is now endowed with a certain amount of some, or all, of the available goods. Define this intermediate endowment, for the i-th consumer as $\boldsymbol{\omega}_i$. In particular, for the i-th consumer we have

$$\boldsymbol{\omega}_i = A\mathbf{w}_i \tag{3.32}$$

The budget faced by the generic consumer i, is given by

$$\begin{aligned} B_i &= \{\mathbf{q}_i \in \mathbb{R}^J_+ \mid x_{0i} + \mathbf{p} \cdot \mathbf{q}_i \leq \mathbf{p} \cdot \mathbf{w}_i\} \\ &= \{\mathbf{q}_i \in \mathbb{R}^J_+ \mid x_{0i} + \sum_j p_j q_{ji} \leq \sum_j p_j \theta_{ji} y_j\} \end{aligned} \tag{3.33}$$

Given that now consumers' income is endogenous, the role of the outside numeriere good x_{0i} must carefully be defined. In particular, we will assume $x_{0i} \in \mathbb{R}_+$, for all $i = 1, \ldots, I$. Intuitively, the numeriere good can be though

[10]The mechanism is the same as the one introduced in Gabszewicz and Vial (1972).

as "outside money" that each consumer holds with full disposability. Moreover, since we implicitly normalize the price of x_0, i.e $p_0 = 1$, the value of each good is actually expressed in terms of *real money*[11]. By considering an initial *non-negative* endowment of the outside good, we avoid any issue related to boundary solutions that might arise.

With this intermediate endowments, exchange takes place among consumers. The utility function for the generic consumer i is now given by

$$U(\mathbf{x}_i) = \mathbf{p} \cdot (\mathbf{w}_i - \mathbf{q}_i) + \sum_s u\left(\sum_j a_{js}(w_{ji} + q_{ji})\right) \tag{3.34}$$

Another set of constraints we introduce is a modified version of the non-negativity constraints. That is $q_{ji} \geq -w_{ji}$, for any i and j. The i-th consumer cannot sell more of good j that the amount she owns, given her share in firm j and firm j's output.

By introducing this simple ownership structure we get rid of the problems involved by *strict* non-negativity constraints, i.e $q_{ji} \geq 0$. Note how *intermediate endowments* provide consumers with some bundle of characteristics, so that, essentially, we are in the situation of the standard model. However, for generic outputs $(y_1, \ldots, y_J)$, these bundles are not jet specified but will depend on the resulting exchange and on firms' profit maximization outputs.

The structure of the model is the following. Given the market matrix A, in stage one, firms compete in quantities. The outcome of this competition is a vector of equilibrium outputs, $\mathbf{y} = (y_1, \ldots, y_J)$. In between stage one and stage two, these output vector is distributed to consumers according to the ownership structure. Intermediate endowments, $(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_I)$ are formed. Finally, in stage two exchange takes place and a competitive allocation is reached. Consumers, nonetheless, are able to modify their bundles of characteristics by exchanging their endowments of goods represented by $\mathbf{w}_i$. The following baseline example exposes the functioning of this modified version of the standard model.

[11]Expressing the value of goods in real money clearly makes sense if p_0 remains exogenously fixed.

Example 5. Consider quadratic-quasilinear utility as defined in 3.7. For simplicity consider $I = 2 = J = S$. The ownership structure is the following, $\Theta = \{(\theta_{11}, \theta_{21}), (\theta_{12}, \theta_{22})\} = \{(1, 0), (0, 1)\}$. That is, consumer 1 owns firm 1, and consumer 2 owns firm 2. The market matrix is the following

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Firms outputs are represented by a vector $\mathbf{y} = (y_1, y_2)$. From this, $\mathbf{w}_1 = (y_1, 0)$, and $\mathbf{w}_2 = (0, y_2)$. *Intermediate endowments* are given, respectively, by

$$(y_1, 0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (y_1, 0) = \boldsymbol{\omega}_1$$

$$(0, y_2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (0, y_2) = \boldsymbol{\omega}_2$$

Given $\mathbf{w}_i, \boldsymbol{\omega}_i$, $i = 1, 2$, we can define the utility for both consumers. We have

$$U(\mathbf{q}_1) = p_1 y_1 - p_1 q_{11} - p_2 q_{21} + \left[y_1 + q_{11} - \frac{1}{2}(y_1 + q_{11})^2 + q_{21} - \frac{1}{2} q_{21}^2 \right] \tag{3.35}$$

$$U(\mathbf{q}_2) = p_2 y_2 - p_1 q_{12} - p_2 q_{22} + \left[q_{12} - \frac{1}{2} q_{12}^2 + y_2 + q_{22} - \frac{1}{2}(y_2 + q_{22})^2 \right] \tag{3.36}$$

Consumer 1 maximizes 3.35 subject to $q_{j1} \geq -y_j$ for $j = 1, 2$, and similarly does consumer 2 with 3.36. Given that consumer 1 is the sole owner of firm 1, and consumer 2 the sole owner of firm, we can define the market clearing conditions as

$$\sum_i q_{ji} = 0, \quad j = 1, 2 \tag{3.37}$$

The system defining the competitive allocation in stage two, provided interior solutions to the FOC of each consumer, is given by

$$\begin{aligned} -p_1 + 1 - y_1 - q_{11} &= 0 \\ -p_2 + 1 - q_{21} &= 0 \\ -p_1 + 1 - q_{12} &= 0 \\ -p_2 + 1 - y_2 - q_{22} &= 0 \\ q_{11} + q_{12} &= 0 \\ q_{12} + q_{22} &= 0 \end{aligned}$$

From the above it is easy to derive the market-clearing price system.

$$p_1 = 1 - \frac{1}{2} y_1 \tag{3.38}$$

$$p_2 = 1 - \frac{1}{2} y_2 \tag{3.39}$$

Together, 3.38 and 3.39 define the inverse demand system faced by firms. We can use 3.38 and 3.39 to define firms' choice set. Given the particular assumptions of the economy's fundamentals we have that the choice set is common for both firms and is given by $Y = [0, 2]$. Firms' profit functions are, respectively, given by

$$\pi_1 = p_1 y_1 = \left(1 - \frac{1}{2} y_1\right) y_1 \tag{3.40}$$

$$\pi_2 = p_2 y_2 = \left(1 - \frac{1}{2} y_2\right) y_2 \tag{3.41}$$

Maximization of the above profit functions yields the equilibrium output vector $\mathbf{y}^* = (y_1^*, y_2^*) = (1, 1)$. Now, these outputs are distributed, according to the specific ownership structure, to consumers. Thus, $\mathbf{w}_1 = (1, 0)$, $\mathbf{w}_2 = (0, 1)$, and *intermediate endowments* are $\boldsymbol{\omega}_1 = (1, 0)$ and $\boldsymbol{\omega}_2 = (0, 1)$. Consumer 1 faces the constraint $q_{11} \geq -1$, while consumer 2 faces $q_{22} \geq -1$. These constraints mean two things. First, given the specific ownership structure assumed, consumer 1 disposes only of good 1, and consumer 2 disposes only of good 2. Second, they cannot sell more that the amount of the good owned. The owned good is the only mean of exchange available to consumers in order to modify the bundle of characteristics owned in such a way that utility is maximized[12].

The equilibrium allocation arising at $\mathbf{y}^* = (1,1)$ is given by $\mathbf{q}_1^* = (-1/2, 1/2)$, $\mathbf{q}_2^* = (1/2, -1/2)$, and $\mathbf{p}^* = (1/2, 1/2)$. That is, consumer 1 sells part of her endowment of good 1 to acquire some amount of good 2. Consumer 2, on the other hand, sells part of her endowment of good 2 to acquire some amount of good 1. Demand and supply, of good 1 and good 2, met at prices $\mathbf{p}^*$. Note how, the exchange of goods is driven by consumers' desire to acquire some amount of the missing characteristic. The resulting equilibrium values of the consumed characteristics are: $\mathbf{x}_1^* = \boldsymbol{\omega}_1 + A\mathbf{q}_1^* = (1/2, 1/2)$ and $\mathbf{x}_2^* = \boldsymbol{\omega}_2 + A\mathbf{q}_2^* = (1/2, 1/2)$. Simple computations reveal that, since

$$\frac{\partial U(x_{01}, \mathbf{x}_1^*)}{\partial x_{s1}} = \frac{\partial U(x_{02}, \mathbf{x}_2^*)}{\partial x_{s2}}, \quad s = 1, 2$$

firms' profit maximization is conducive of a *Pareto optimal* allocation of characteristics. By engaging in the exchange of goods, consumers are able to reach an allocation of characteristics yielding strictly higher utility than if they consumed only their endowments. Consider, for example, consumer 1. We have

$$\begin{aligned} U(\mathbf{q}_1^*) &= \frac{1}{2} + x_{11}^* - \frac{1}{2}(x_{11}^*)^2 + x_{21}^* - \frac{1}{2}(x_{21}^*)^2 \\ &= \frac{5}{4} > \frac{1}{2} = U(\mathbf{w}_1) \end{aligned}$$

The same holds for consumer 2. It is interesting to notice how the particular ownership structure we assumed provided the incentives for exchange to take place. Indeed, the same equilibrium allocation could be reached without any exchange of goods with the ownership structure $\Theta = \{(1/2, 1/2), (1/2, 1/2)\}$. That is, each consumer owns half of each firm. Given the market matrix A, and this ownership structure it is easy to see that no exchange will take place. Simple computations show that the only equilib-

[12]Recall that consumers care only about the amount of characteristics they possess.

rium will be given by $\mathbf{q}_1^* = (0,0)$ and $\mathbf{q}_2^* = (0,0)$. That is, no exchange of goods takes place but consumers just consume the endowments arising given their shares. Finally, note how the above equilibrium could have been reached also without assuming any ownership structure.

The above example is meant to show the functioning of the model when initial endowments arise from a particular ownership structure. In this way, the problem of having non-negativity constraints disappears since now consumers are able to exchange goods among them in order to acquire characteristics. So, ownership is indeed eliminating the problem of binding non-negativity constraints. The above example is nevertheless particular, given the specific market matrix A representing *independent goods*. The next example deals with the case of imperfect substitutes goods.

Example 6. Consider the same quasi-linear quadratic utility function as the one used in the previous examples. The ownership structure is the following $\Theta = \{(1,0),(0,1)\}$, and the market matrix is

$$A = \begin{bmatrix} 1 & 0.71 \\ 0 & 0.71 \end{bmatrix}$$

From the above we thus have consumer 1 owning firm 1, and consumer 2 owning firm 2. Intermediate endowments of goods are, respectively, $\mathbf{w}_1 = (y_1, 0)$ and $\mathbf{w}_2 = (0, y_2)$. From these, endowments of characteristics are $\boldsymbol{\omega}_1 = (y_1, 0)$ and $\boldsymbol{\omega}_2 = (0.71y_2, 0.71y_2)$. Recall that consumer 1 faces the constraint, $q_{11} \geq -y_1$, and consumer 2, $q_{22} \geq -y_2$. Given the non-symmetric structure of the matrix A we have to explicitly recognize the nature of the outside good x_{0i}. Consider for simplicity consumer 1. Denote the Lagrange multiplier by λ. Her FOC are given by

$$1 - y_1 - q_{11} - 0.71q_{21} - \lambda p_1 \leq 0, \quad q_{11}(1 - y_1 - q_{11} - 0.71q_{21} - \lambda p_1) = 0 \tag{3.42}$$

$$1.42 - 0.71(y_1 + q_{11}) - q_{21} - \lambda p_2 \leq 0, \quad q_{21}(1.42 - 0.71(y_1 + q_{11}) - q_{21} - \lambda p_2) = 0 \tag{3.43}$$

$$1 - \lambda \leq 0, \quad x_{01}(1 - \lambda) = 0 \tag{3.44}$$

$$p_1 y_1 - x_{01} - p_1 q_{11} - p_2 q_{21} \leq 0, \quad \lambda(p_1 y_1 - x_{01} - p_1 q_{11} - p_2 q_{21}) = 0 \tag{3.45}$$

In the same way we can define the FOC of consumer 2. It can be shown that if $x_{01} = 0$, thus $\lambda \geq 1$, the model becomes just a modified version of the original example proposed by Gabzsewicz and Vial (1972)[13]. Hence, consider $x_{0i} > 0$ for $i = 1, 2$, so that $\lambda = 1$. Then, given a generic output vector $\mathbf{y} = (y_1, y_2)$, the system defining the competitive equilibrium, assuming interior solutions, is the following.

$$\begin{aligned}
-p_1 + 1 - y_1 - q_{11} - 0.71q_{21} &= 0 \\
-p_2 + 1.42 - 0.71y_1 - 0.71q_{11} - q_{21} &= 0 \\
-p_1 + 1 - 0.71y_2 - q_{12} - 0.71q_{22} &= 0 \\
-p_2 + 1.42 - y_2 - q_{22} - 0.71q_{12} &= 0 \\
q_{11} + q_{12} &= 0 \\
q_{12} + q_{22} &= 0
\end{aligned}$$

From the above, we can derive, for generic $\mathbf{y}$, the corresponding market-clearing price system. Straightforward computations lead to the following

[13] In particular, solving the model implies the imposition of a normalization rule for the price function. Otherwise equilibrium will be defined up to relative prices.

price functions.

$$p_1(\mathbf{y}) = \frac{1}{2}(2 - y_1 - 0.71y_2) \tag{3.46}$$

$$p_2(\mathbf{y}) = \frac{1}{2}(2.84 - 0.71y_1 - y_2) \tag{3.47}$$

With the above, we can readily define, stage one, firms' profit maximization problems. Firms' choice sets are generically given by $Y_j = [0, \bar{y}_j]$, with $\bar{y}_j > 0$. Given the non-symmetric nature of the market matrix A, it may be the case that $\bar{y}_1 \neq \bar{y}_2$. In particular, firm 1's maximal output $\bar{y}_1$, is the one for which, whenever $\gamma = 0$ or $y_2 = 0$, $p_1(\cdot) = 0$. From 3.46 is is immediate to see that $\bar{y}_1 = 2$. Similarly, for firm 2 we have that $\bar{y}_2 = 2.84$. Thus, $Y_1 = [0, 2]$, and $Y_2 = [0, 2.84]$. Firms' profit maximization problems are given, respectively, by

$$\max_{y_1 \in Y_1} \pi_1(\mathbf{y}) = \max_{y_1 \in Y_1} \left\{ \frac{1}{2}(2 - y_1 - 0.71y_2)y_1 \right\} \tag{3.48}$$

$$\max_{y_2 \in Y_2} \pi_2(\mathbf{y}) = \max_{y_2 \in Y_2} \left\{ \frac{1}{2}(2.84 - 0.71y_1 - y_2)y_2 \right\} \tag{3.49}$$

The Cournot-Nash equilibrium, of the quantity game defined by 3.48 and 3.49, is $\mathbf{y}^C = (0.57, 1.22)$. The market-clearing price system is $\mathbf{p}^* = (0.28, 0.61)$. From these, the equilibrium allocation of goods can easily be computed from

$$\begin{aligned} y_1^C + q_{11} + 0.71q_{21} &= 0.71y_2^C + q_{12} + 0.71q_{22} \\ 0.71y_1^C - 0.71q_{11} - q_{21} &= y_2^C + 0.71q_{12} + q_{22} \\ q_{11} + q_{12} &= 0 \\ q_{21} + q_{22} &= 0 \end{aligned}$$

and is given by $\mathbf{q}_1^* = (-0.285, 0.61)$ and $\mathbf{q}_2^* = (0.285, -0.61)$. These, with $\mathbf{y}^*$ and $\mathbf{p}^*$, define an hedonic Cournot-Walras equilibrium. The resulting equilibrium values of characteristics are given, respectively, by $\mathbf{x}_1^* = \boldsymbol{\omega}_1 + A\mathbf{q}_1^* = (0.7451, 0.4331)$ and $\mathbf{x}_2^* = \boldsymbol{\omega}_2 + A\mathbf{q}_2^* = (0.6911, 0.4331)$. It is straightforward to see that this equilibrium allocation of characteristics is not Pareto optimal. However, note that the marginal evaluations for characteristics $s = 2$ in equilibrium coincide.

Note that the constraints on maximum sales are satisfied for both consumers. However, from the above we are able to shed light on the role of "outside money" that we attached to the numeriere good x_{0i}. In particular, assuming $x_{0i} \in \mathbb{R}_+$, turns out to be fundamental especially for consumer 1. In particular, $x_{01} > 0$ is needed in order for consumer 1's budget to be satisfied in equilibrium. To see this, write the equilibrium budget of consumer 1.

$$x_{01} + p_1^C q_{11}^* + p_2^C q_{21}^* = x_{01} + 0.29 \leq 0.16 = p_1^C y_1^C$$

Thus, in order for the above to be satisfied it must be $x_{01} \leq -1.13$. That is, consumer 1 should have a sufficiently high amount of "outside money" in order to be able to engage in the exchange. This is an important point since it shows how, even if consumers are able to exchange goods, making incomes endogenous, giving the model a more general equilibrium flavor, we still have to implicitly assume consumers to detain a sufficiently high mount of the outside good. The alternative to this, will be to define a *price normalization* rule in the spirit of Gabszewicz and Vial (1972). However, assuming that consumers are endowed with some amount of money seems rather more realistic.

Examples 5 and 6 provide also some interesting considerations regarding the possible dynamics arising when firms are able to strategically select their good's design. For, example 5 can be considered as the starting point. Both firms produce a good providing just one of the two characteristics. Then, example 6 can be seen as the result of consumer $i = 2$'s request to firm 2's management board to modify the design of its good. By comparing firm 2's profit and consumer 2's utility arising in example 6, with the corresponding value arising in example 5, we see that this request actually increased both. Denote the values arising in example 5 with $\bar{\cdot}$, and values arising in example 6 with $\hat{\cdot}$. We have

$$\bar{\pi}_2 = \frac{1}{2} < 0.741 = \hat{\pi}_2$$

$$\bar{U}_2 = \frac{3}{2} < 1.83 = \hat{U}_2$$

Thus, passing from $\mathbf{a}_2 = (0,1)$ to $\mathbf{a}_2 = (0.71, 0.71)$, *given* the design of firm 1's good, $\mathbf{a}_1 = (1,0)$, to remain fixed, increased firm 2's profit and hence consumer 2's utility. Computing the corresponding values for firm 1 and consumer 1 reveals that firm 2's design modification reduced firm 1's profit and hence consumer 1's utility. If it were firm 1 to change its good's design, by symmetry, we would have reached the same result. The fundamental element that enables us to derive these interesting insights is again the outside good x_0. Indeed, in examples 5 and 6 we actually implicitly normalized the price of x_0. Because x_0 is taken to be an outside good, normalizing its price does not influence the characterization of the hedonic Cournot-Walras equilibrium. In particular, by doing this we are able to avoid the difficulties highlighted by Dierker and Dierker (2006)[14]. However, interpreting x_{0i} as the amount of outside money detained by the i-th consumer involves the acceptance of an intrinsic possible difficulty. Note that if in example 4 $x_{01} < 1.13$, the equilibrium we found will not be sustainable. Thus, we would have encountered an issue regarding the existence of an hedonic Cournot-Walras equilibrium. However, if we assume x_0 to take any possible *positive* value, which amounts at assuming consumers to have a *sufficiently high* amount of outside money, existence ceases to be an issue.

3.5 Conclusion

In this brief note, we provided several examples showing the functioning of a particular model featuring product differentiation and imperfect competition. Despite maintaining a strong partial equilibrium flavor, the proposed model seems suited for a broader generalization in the spirit of general equilibrium.

The presented model features two unusual elements in the literature of product differentiation and imperfect competition. First, it considers prod-

[14]In particular, in their first example they show that when we normalize the price of one of the traded goods, then may lead to an extreme underprovision of all the other produced, and traded, goods. This will undermine the existence of a Cournot-Walras equilibrium.

uct differentiation as originally modeled by Lancaster in his *combinable consumption* model. Although keen to criticisms, this way of modeling product differentiation is very close to the idea proposed by Dixit and Stiglitz (1977) of *love for variety*. Second, we considered competition among firms to be à la Cournot, i.e in quantities, rather than in prices, as usually considered in the literature on product differentiation. Quantity competition, in our view, seems the natural complement to hedonic product differentiation. For, our model is able, although in a richer setting, to describe the typical mechanism of a differentiated economy in where consumers acquire bundles of goods yielding the optimal amount of *variety*.

However, the present work is not at all conclusive. Rather, it is meant to show the potentials of such a modeling strategy. In particular, the model we proposed can be seen as a counterpart of the typical model of monopolistic competition where product differentiation is studied in presence of heterogeneous consumers. For, it might serve as an alternative which may provide some deeper insights on an economy with differentiated products and imperfectly competitive firms. A step forward would be to study the functioning of the present model for different types of suitable utility structures. Clearly, since no such model is available in the literature, much work has to be done to discover if it is effectively a viable alternative to the commonly adopted models of product differentiation and imperfect competition.

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Chapter 4

Product Selection, Entry and Welfare in a Cournot Duopoly with Hedonic Product Differentiation

4.1 Introduction

The question on whether imperfectly competitive firms have an incentive to concentrate at the same location originated from the seminal work of Hotelling (1929). Hotelling's conclusion was that firms have a tendency toward locating in the centre of the market when competition in prices. Translated in terms of product differentiation this means that firms will tend to produce a single good's design. However, Hotelling's conclusion was later proved wrong by D'Aspremont et. al (1979). They proved, using the same setting originally proposed by Hotelling (1929), that a price equilibrium exists only if firms locate sufficiently far away from each other. Translated in terms of product differentiation, this result means that a well defined Bertrand-Nash equilibrium exists only if firms' products are sufficiently differentiated. Nonetheless, the result of D'Aspremont et. al (1979) do not consider the strategic decision of firms regarding their location. By considering a two-stage game formulation of the original Hotelling location-price setting, Neven (1985) showed that a well defined Bertrand-Nash equilibrium exists for every pair of locations. Neven (1985)'s result arises because the assumption of consumers having in-utility quadratic transportation costs. This, together with explicit consideration of firms' location choices, are sufficient to prove the existence of an equilibrium for every pair of locations. Nonetheless, Neven (1985)'s conclusion applies to a one-dimensional space. The interesting insight, however, coming from Neven (1985) is that the demand faced by a firm selecting its location on "Main Street" is equivalent to the demand faced by a firm selecting an horizontally differentiated product.

The consideration of a multi-dimensional space in which firms locate, in a setting close to the one we propose here, although still considering the Hotelling's model structure with firms competing in prices, is found in Irmen and Thisse (1998). They showed, in a multidimensional characteristics model, that firms have an incentive to homogenize for all but one of the available characteristics. That is, goods will differ with regard to just one of the characteristics. However, their result heavily hinges on the assumption that consumers have a *most preferred variety* and evaluate characteristics differently. By putting together these two aspects it is not difficult to see why firms have an incentive to homogenise their good for all but one characteristic. The sole characteristic which will differentiate firms' products will be the one with the higher *average likelihood*, or, with Irmen and Thisse (1998) wording, the higher average *salience coefficient*. In general, goods' design selection is predominately studied in the typical framework proposed by Hotelling (1929). Given the parallelism between a firm selection its location and a firm selecting the design of its good, all results concerning the strategic selection of locations arises in models were competition is in prices[1].

When turning to imperfectly competitive firms competing in quantities, the question on whether firms have an incentive to concentrate at the same location (homogeneous good) has gathered substantially less attention. The primal reason for this is that when dealing with product differentiation it

[1]The literature on this is too vast to be exhaustively provided here. Friedman (1983) and Gabszewicz and Thisse (1986) provide numerous classic references on this.

is commonly assumed that firms compete in prices. Since firms provide imperfectly substitutable goods each is able to behave in a monopolistic way, thus has the power to set its price without having to worry about getting undercutted by the opponent firms. In this light then the result of D'Aspremont et. al (1979) seems perfectly logical. If firms are able to enjoy some degree of monopoly power over their price by locating sufficiently far from the opponents, then clearly every firm would chose to do it. Things are radically different when firms compete by setting their output.

As far as we know, the only work addressing the question of the optimal location when firms compete in quantities is Anderson and Neven (1991). They show that, in a two-stage game where in stage one firms select their location (variety), and in stage two they subsequently compete in quantities, the tendency of firms is to concentrate towards the centre of the market (homogeneous good). However, their result seems strongly dependant on the assumption of firms bearing the transportation costs (delivered pricing). In this sense, their result simply states that firms locate where production results more efficient. Thus, their result can not be fully considered as being a robust conclusion on what are the firms' location choices when competition is à la Cournot in a typical Hotelling's setting.

The contribution of the present work can be seen as a first attempt to study what are firms' choices concerning thie product's designs when competition is à la Cournot. Indeed, we consider an unusual setting where to study products' designs choices. That is, we assume products to be hedonically differentiated. In this view, goods are valuable to consumers not because of their intrinsic nature of consumption objects, but because they embody different addresses (characteristics) which are what consumers evaluate in making their consumption decisions[2]. To our knowledge, no explicit attempt has been made to address the issue of optimal product differentiation, when competition is in quantities, in an explicit characteristics model à la Lancaster[3]. For, the current work aims at put a step forward in this direction.

[2]The characteristics model was originally proposed by Lancaster (1966) and Lancaster (1975). For an exhaustive exposition of the characteristics model see Lancaster (1979).

[3]There is a vast literature dealing with models of product differentiation which implicitly embody the formal setting proposed by Lancaster. For a complete exposition of such

We also consider entry and its related welfare implications. Entry in a differentiated market where competition is in quantities also has not received much attention in the literature. What has, indeed, received some attention is the consideration of entry in Cournotian markets with homogeneous goods (Mankiw and Whinston (1986)), and in models of Chamberlinian monopolistic competition as formulated by Dixit and Stiglitz (1977). Entry in Cournotian markets with a single homogeneous good is modeled as a two stage game where potential entrants evaluate whether to enter the market, or not, depending on the magnitude of the fixed set-up cost (Vives (1999)). If these costs are sufficiently small, entry will occur. Several effects are observed. First, there occurs the so-called *business stealing effect*, that is, the general decrease in incumbents' output as a new firms enter. However, given the increase in active firms, total output increases. Thus, in general, entry will increase total surplus, although without reaching full efficiency. Indeed, full efficiency is reached only in a *free-entry* equilibrium, i.e when fixed set-up costs are zero. Our conclusions will be similar, but enriched with the consideration of the entrant's product design choice, in addition to the cost consideration. These will generate an interesting dynamics.

In general, we will consider the simple representative consumer formulation of the explicit characteristics model proposed by Pellegrino (2022)[4]. This model, can be viewed as the possible *foundation* of the workhorse model of product differentiation proposed by Singh and Vives (1984). We will consider the simple case in which two firms differentiate along *two* dimensions[5]. Formally, Pellegrino (2022)'s setting is just the original Lancasterian formulation of the characteristics model. However, Pellgrino (2022) considers goods' designs to be exogenously given. What we will do here is endogenize goods' designs in the simplest setting of a duopoly. We do this by building a two-stage game where in the first stage firms select their good's designs and

models see Anderson et al. (1992).

[4]A generalization the characteristics model to a more abstract setting can be found in the network literature. In particular see Ballester et. al (2006). For an application to price competition see Uschchev and Zenou (2016).

[5]The model could be easily generalized to the case of J dimensions. However, to the economic intuition of the results is easily grasped with two differentiation dimensions.

subsequently, given their design, compete in quantities. A tendency towards *aggregation* will arise from this setting. That is, both firms will chose the provide the market with a common good's design, i.e an *homogeneous* good. This amounts, in terms of location, to firms deciding to locate in the same spot, in particular at the centre of the market. So, when firms decide directly on their good's design, i.e explicitly select a feasible vector of characteristics, and given these specifications they subsequently engage in competition à la Cournot, the competitive force draws firms toward the selection of a common product's design. Hence, the market will provide a single *homogeneous* good. This tendency towards *agglomeration* has important consequences for entry. In particular, provided the fixed set-up cost to be sufficiently small, thus inducing entry, we show that potential entrants will also homogenize and thus provide the same homogeneous good already offered by the market. This makes all the standard conclusions about entry in Cournotian markets, with an homogeneous good valid. However, an interesting fact is observed. Namely, we find that the number of active firms at the *free-entry* equilibrium is higher than the *socially optimal* number of firms, even if the market is providing an homogeneous good. Again, the interest for this result steams from the fact that it arises directly from firms' strategic behaviour. This is something that was not recognize in the literature.

The above simple results have, however, important implications given the potential generalization that the characteristics model provides for the study of product differentiation and imperfect competition[6]. Indeed, the characteristics model as originally proposed by Lancaster shares the same formal structure of an Arrow-Debreu model with financial markets[7]. Hence, more broadly, with the theory of incomplete markets[8]. From this, efficiency issues regarding the optimal provision of characteristics can be analysed using the techniques developed in the theory of incomplete markets. In particular, if we depart from the representative consumer specification and introduce some heterogeneity among consumers, then the fact that firms would have

[6]See for example Dreze and Hagen (1978).

[7]This parallelism was realized for example by Leland (1977) and Dreze and Hagen (1978).

[8]For an exhaustive exposition of this theory see Magill and Quinzii (1996).

an incentive to homogenize around a common variety has the consequence of reducing the dimension of the market space. Hence, the optimality in terms of consumers' welfare may be negatively effected by this tendency.

The work is organized as follows. Section 2 provides a description of the model's basic elements. Sections 3 and 4 are devoted to the definition of the Cournot-Nash equilibrium in the present setting. Section 5 contains the main result concerning the strategic selection of products designs. Sections 6 and 7 are devoted, respectively, to the analysis of entry and its welfare implications. Finally, section 8 concludes.

4.2 The model

Consider a market economy where product are hedonically differentiated, i.e defined as vectors in the two-dimensional characteristics space $\mathbb{R}^2_+$, $s = 1, 2$ denoting the two technologically feasible characteristics that can be embodied in different goods. The market is populated by two technologically identical firms which are denoted by $j = 1, 2$. Each firm sells *only one* differentiated good. This good is defined as a vector $\mathbf{a}_j \in \mathcal{A}_j \subseteq \mathbb{R}^S_+$. $\mathcal{A}_j$ is the technologically feasible set from which firms select their specific vector of characteristics that will later be marketed as a differentiated good. Define the 2×2 matrix A to be the matrix composed by the vectors $\mathbf{a}_j$, $j = 1, 2$. We assume that each vector $\mathbf{a}_j$ is of unit length, $\sum_{s=1}^{S} a_{js}^2 = 1$ for any $j = 1, \ldots, J$. This *normalization* provides us a way to define the *degree of product differentiation*, or *similarity*, $\gamma = \mathbf{a}_j \cdot \mathbf{a}_k$. This defines the angle between the two vectors. Hence, it is a measure of relative product differentiation. This "circular" normalization is adopted because of its simplifying power. In particular, one could also chose a "linear" normalization such as $a_{j1} + a_{j2} = 1$. However, with the "linear" normalization, the definition of the *similarity* becomes more cumbersome[9].

[9] In general, the angle between two vectors is defined implicitly by $\cos\theta_{jk} = \frac{\mathbf{a}_j \cdot \mathbf{a}_k}{||\mathbf{a}_j|| \cdot ||\mathbf{a}_k||}$. Thus, when using the "linear" normalization, an additional term given by the product of the norms must be considered. This increases the complexity of the model without providing any further insight.

Given the semi-positiveness[10] of $\mathbf{a}_j$, $\gamma \in [0,1]$. In particular, $\gamma = 0$ defines independent goods, $\gamma = 1$ perfect substitutes, and for $\gamma \in (0,1)$ goods results imperfect substitutes. Note how the current structure does not consider the case of complementary goods. Anyway, it is important to see how in the current setting the so called *degree of substitutability* is directly determined by the products' similarity. Thus, when firms decide on their product's design, they are directly affecting the degree of substitutability. This is an important aspect of this modeling strategy since usually, in the literature on product differentiation[11], the degree of substitutability is taken as being *fully* exogeneous. To see why $\gamma = 0$ and $\gamma = 1$ are two very special configurations note that the two associated market matrices are given, respectively, by

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Looking at the above matrices reveals why the case of $\gamma = 1$ corresponds to the two goods being homogeneous. The characteristics embodied in the two goods are exactly the same. Conversely, for $\gamma = 0$, the two goods are seen to be independent since the bundle of characteristics embodied in them differ. State differently, if we interpret each characteristic as defining a particular market then, $\gamma = 1$ means that for each market firms' market areas coincides whether, $\gamma = 0$, implies that firms' market areas a perfectly disjointed.

In what follows, we will first compute the Cournot-Nash equilibrium associated to a particular pair of locations. After, we will address the problem of choosing the optimal location. The model thus consists of a two-stage game. In stage one, firms select their good's design ("location"), in the second, given the location pair determined from the previous stage, they compete à la Cournot. From these, a Cournot-Nash equilibrium arises. In order to solve the model we assume firms to be *fully rational*, thus we proceed by backward

[10] A vector $\mathbf{a}_j \in \mathbb{R}^S$ is semi-positive if it is non-negative but not zero. In other words, it has at least one component which is strictly positive.

$$\mathbf{a}_j > 0 \iff \mathbf{a}_j \in \mathbb{R}^S_+, \quad \mathbf{a}_j \neq \mathbf{0}.$$

[11] For example Singh and Vives (1984).

induction.

4.3 Demand system

The demand side of the economy is populated by a representative consumer having preferences defined over the space of characteristics $\mathbb{R}^2_+$. In order to acquire characteristics, the representative consumer decides how much of each variety to consume. That is, she has as decision variable the vector of quantities $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2_+$. Given the market matrix A and the vector of quantities $\mathbf{q}$, the total amount of characteristics acquirable by the representative consumer is

$$\mathbf{x} = A\mathbf{q} \tag{4.1}$$

where $x_s = \sum_j a_{js} q_j$ for any $s = 1, 2$. The preferences of the representative consumer are described by the following utility function

$$U(x_0, \mathbf{x}) = x_0 + \sum_s \left(x_s - \frac{1}{2} x_s^2 \right) \tag{4.2}$$

where x_0 is an homogeneous good taken to be the numeriere of the model. The utility function, defined in 4.2, generates the well- known Singh and Vives (1984)'s specification. Thus, our characteristics representation is seen to be the *reasonable* foundation of the quadratic utility setting. In particular, define $\alpha_j = a_{j1} + a_{j2}$ for $j = 1, 2$ to be real positive values accounting for the relative quality of the j-th good. We can write 4.2, expliciting the products' similarity γ, as

$$\begin{aligned} U(x_0, \mathbf{q}) &= x_0 + \alpha_1 q_1 + \alpha_2 q_2 - \frac{1}{2}(q_1^2 + 2\gamma q_1 q_2 + q_2^2) \\ &= x_0 + V(\mathbf{q}) \end{aligned} \tag{4.3}$$

From 4.3 it can be seen that the two extreme cases, $\gamma = 1$ and $\gamma = 0$, give rise to commonly adopted utility functions when goods are homogeneous. However, it must be noted that in the current setting a good is homogeneous when $\gamma = 1$. Nevertheless, homogeneity, from the consumer's point of view refers to the relative amounts of characteristics embodied in the two varieties. That is, from the consumer's prospective, even when $\gamma = 0$, i.e independent goods, since the two varieties embody one unit of either of the two characteristics, the two products seems homogeneous.

1. $\gamma = 0 \longrightarrow$ two homogeneous goods "located" at opposite spots

$$U(x_0, \mathbf{q}) = x_0 + \left(q_1 - \frac{1}{2}q_1^2\right) + \left(q_2 - \frac{1}{2}q_2^2\right)$$

2. $\gamma = 1 \longrightarrow$ one single homogeneous good, $Q = q_1 + q_2$ and $\bar{\alpha}$ is the common good's specification

$$U(x_0, Q) = x_0 + \bar{\alpha}Q - \frac{1}{2}Q^2$$

From the above, it should be evident how the current specification encompasses several standard settings commonly adopted in the product differentiation literature. The key element is represented by the products' similarity γ. This coefficient, which we will treat as exogenous at first but that we will endogenize later, is seen to shape directly the type of competition that will arise in market. This element will be further discussed below.

The representative consumer maximizes 4.2 with respect to the following standard budget constraint

$$B(\mathbf{q}, w_0) = \{(q_1, q_2) \in \mathbb{R}^2_+ \mid x_0 + p_1 q_1 + p_2 q_2 \leq w_0\} \tag{4.4}$$

where $w_0 > 0$ is the consumer's initial wealth (income). With the above budget, the representative consumer solves the following problem.

$$\begin{aligned} \max_{(q_1, q_2) \in \mathbb{R}^2_+} \quad & U(x_0, \mathbf{x}) \\ \text{s.t.} \quad & B(\mathbf{q}, w_0), \\ & \mathbf{x} = A\mathbf{q}, \\ & q_1 \geq 0, q_2 \geq 0 \end{aligned} \tag{4.5}$$

The first order conditions, assuming interior solutions, read

$$-p_1 + \alpha_1 - q_1 - \gamma q_2 = 0$$
$$-p_2 + \alpha_2 - \gamma q_1 - q_2 = 0$$

We can write the above, more concisely, in matrix notation as

$$-\mathbf{p} + \mathbf{1}A - (A'A)\mathbf{q} = \mathbf{0} \quad (4.6)$$

The term $(A'A)$ defines the matrix of products' similarities Γ. Denote $\mathbf{1}A = \boldsymbol{\alpha} = (\alpha_1, \alpha_2)$. We can interpret the α_j's as a measure of *quality* of good j. From 4.6 we can derive the demand system. The inverse demand is readily derived and reads

$$\mathbf{p}(\mathbf{q}) = \boldsymbol{\alpha} - \Gamma\mathbf{q} \quad (4.7)$$

The indirect demand system in 4.7 is well defined in the region of the quantity space for which prices are positive. That is, 4.7 is defined in $\bar{Q} = \{\mathbf{q} \in \mathbb{R}^2_+ \mid 0 \leq q_j < \bar{q}_j\}$. If the matrix A is non-singular then Γ is invertible and we can define $\bar{\mathbf{q}} = \Gamma^{-1}\boldsymbol{\alpha}$. $\bar{q}_j > 0$ holds since for every $\gamma \in [0,1]$ $\gamma < \alpha_1/\alpha_2 < 1/\gamma$ will always be satisfied[12]. Hence, in this range, by taking $\max_j\{\bar{q}_j\} = \bar{q}$, the strategy set for both firm is given by $\bar{Q} = [0, \bar{q}]$.

The magnitude of the products' similarity γ is seen to directly affect the shape of the demand system and hence of the competitiveness of the market. Indeed, the cases of $\gamma = 0$ and $\gamma = 1$ define very peculiar situations. In particular, $\gamma = 0$, i.e *independent goods* in conducive of *local monopoly power*. That is, firms, by focusing on providing a good embodying only one of the available characteristics are *de facto* monopolists in the market of the chosen characteristic. On the other hand, $\gamma = 1$, i.e perfect substitutes, the usual results apply for both price and quantity competition. Instead, whenever $\gamma \in (0,1)$, and goods result imperfect substitutes, the outcome of the quantity competition depends on the intensity of γ.

[12]Note that in the case of A being singular, the model reduces to a standard Cournot duopoly with homogeneous goods. For, we can always assume well-defined strategy sets to be of the form $[0, \bar{q}] \times [0, \bar{q}]$.

4.4 Cournot-Nash equilibrium

In this section we will study the Nash equilibrium arising for *given* goods' designs. That is, for any possible combination of products, we compute the associated Cournot-Nash equilibrium. When competing in quantities, firms take the inverse demand system as given. Consider $\gamma \in (0,1)$. Then, the profit function for firm $j = 1$ is

$$\pi_1(\mathbf{q}) = (\alpha_1 - q_1 - \gamma q_2)q_1 \tag{4.8}$$

and for firm $j = 2$,

$$\pi_2(\mathbf{q}) = (\alpha_2 - q_2 - \gamma q_1)q_2 \tag{4.9}$$

From these, it is easily derived the best reply map $\Phi(\mathbf{q}) : \bar{Q}^2 \to \bar{Q}^2$. The best reply map is given by

$$\phi_1(q_2) = \frac{1}{2}\Big(\alpha_1 - \gamma q_2\Big) \tag{4.10}$$

$$\phi_2(q_1) = \frac{1}{2}\Big(\alpha_2 - \gamma q_1\Big) \tag{4.11}$$

When design specifications are assume to be exogenous, i.e γ is given[13], the concept of Cournot-Nash equilibrium is the standard one. For, since Φ is linear and defined over a compact and convex set, there exists a fixed point, i.e a Cournot-Nash equilibrium, for the quantity game. For $\gamma \in (0,1)$ the equilibrium quantities of the two firms are given by

$$q_1^{DC} = \frac{1}{4-\gamma^2}\Big(2\alpha_1 - \gamma\alpha_2\Big) \tag{4.12}$$

$$q_2^{DC} = \frac{1}{4-\gamma^2}\Big(2\alpha_2 - \gamma\alpha_1\Big) \tag{4.13}$$

The Cournot-Nash equilibrium defined for any $\gamma \in (0,1)$ by 4.12 and 4.13

[13] This can be thought of as firms having already selected their designs. That is, they do not have any power to modify it. Later I will however introduce this possibility.

will be unique *given* γ. For fixed γ there is nothing more to say about the Cournot-Nash equilibrium. Clearly, depending on the magnitude of γ the equilibrium outcomes change.

Consider $\gamma \in [0, 1]$. Since, γ directly affects the shape of the demand system, it then gives rise to different profit functions. In particular, we have that for firm $j = 1$, profit is

$$\Pi_1(q_1, q_2) = \begin{cases} (\alpha_1 - q_2 - \gamma q_2)q_1, & \text{if } \gamma \in (0,1) \\ (1 - q_1)q_1, & \text{if } \gamma = 0 \\ (\bar{\alpha} - q_1 - q_2)q_1, & \text{if } \gamma = 1 \end{cases} \tag{4.14}$$

And symmetrically for firm $j = 2$. The case of $\gamma \in (0, 1)$ was already considered. Hence, the focus is on the two extreme possibilities.

Consider the extreme case of $\gamma = 0$. This represents the situation where competitiveness is the weakest and thus firms enjoy *local monopoly power*. For the j-th firm, equilibrium output and profit are given, respectively, by

$$q_j^{LM} = \frac{1}{2}, \quad j = 1, 2 \tag{4.15}$$

$$\pi_j^{LM} = \frac{1}{4}, \quad j = 1, 2 \tag{4.16}$$

At the other extreme case, $\gamma = 1$, firms are providing the market with a single homogeneous good. The intensity of competition is thus at its maximum. In this case the standard Cournot competition with homogeneous goods takes place. From this, the equilibrium output, q_j^{CC}, and consequent profit for the j-th firm are, respectively

$$q_j^{CC} = \frac{1}{3}\bar{\alpha}, \quad j = 1, 2 \tag{4.17}$$

$$\pi_j^{CC} = \frac{1}{9}\bar{\alpha}^2, \quad j = 1, 2 \tag{4.18}$$

The common good's design is captured by the term $\bar{\alpha}$. Note that given our "circular" normalization, $\bar{\alpha}$ takes values in the closed interval $[1, \sqrt{2}]$, where the two extremes represent, respectively, the case where both firms select a design embodying just one of the two characteristics, or where they select the design embodying the maximal amount of both characteristics. It is straightforward to check that $\pi_j^{LM} > \pi_j^{CC}$, for every $\bar{\alpha} \in [1, \sqrt{2}]$. Thus, ideally, firms would benefit from an agreement in which they decide to establish, each, a local monopoly. However, this might not be the case when firms do not cooperate. For, the question we are interested in is the following. What will be the best, i.e profit maximizing, product's design a firm would chose taking opponent's strategies as given, knowing what will be the outcome of the subsequent Cournot game? This question is relevant since the *non-cooperative* choice of the respective product's designs determines the value of γ which shapes the subsequent Cournot game. Hence, it determines the profitability of firms. Now, at this point one issue arises. This refers to the fact that the parameter γ will be *jointly* determined by the two values α_1 and α_2, which in turns are directly determined by the choice of the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, firms' choices about their respective goods shape the competitive structure of the market.

4.5 Products selection

With the above at hand we can ask ourselves what happens to firms' profitability for different values of γ. Indeed, γ can be seen as a measure of the competition *intensity* with $\gamma \to 0$ representing increasingly monopolistic competition, and $\gamma \to 1$ classic Cournot quantity competition. The former refers to the competitiveness of the market tending towards zero (monopoly). The latter, instead, refers to increasing competitive intensity. However, the parameter γ can also be interpreted as a measure of the *common market area*. That is, the dimension of the shared market space in which firm compete by setting their quantity. Clearly the two interpretations are linked. A larger *common market area* leads to a fiercer competition whether, a smaller *common market area* leads to a weaker competition. Intuition suggests that as competition increases ($\gamma \to 1$) equilibrium outputs will depend on the common quality level of the offered good. The higher the quality, the larger profits will be. On the other hand, as competition weakens ($\gamma \to 0$) equilibrium outputs increase given the increased market power. Firms increasingly concentrate, each, on one of the available markets. So, in the limit each exerts a *local monopoly power*.

From the above, intuition suggests that, if able, firms would want to *collude* in such a way that both can establish a *local monopoly*. This makes perfect sense considering that from 4.16 and 4.18 it is immediate to see that $\pi_j^{LM} > \pi_j^{CC}$ for every $\bar{\alpha} \in [1, \sqrt{2}]$. The question then turns to verify that even if able to establish a *local monopoly*, firms do not have any incentive to deviate from it. That is, if located such that $\gamma = 0$, firms do not have an incentive to *unilaterally* modify their location. In order to address this issue first of all we need to write the profit function for any given $\gamma \in (0, 1)$. Some simple algebra using 4.12 and 4.13, yields the equilibrium profit functions

$$\pi_1(\mathbf{q}) = \left(\frac{2\alpha_1 - \gamma\alpha_2}{4 - \gamma^2} \right)^2 \tag{4.19}$$

$$\pi_2(\mathbf{q}) = \left(\frac{2\alpha_2 - \gamma\alpha_1}{4 - \gamma^2} \right)^2 \tag{4.20}$$

In order to study the location problem we proceed by backward induction, using 4.19 and 4.20 to derive the optimal response in location of firm j, given the location chosen by firm k.

Recall that for every firm, a feasible differentiated good $\mathbf{a}_j$ must be such that $a_{j1}^2 + a_{j2}^2 = 1$. Further, $\gamma = \mathbf{a}_1 \cdot \mathbf{a}_2$, and $\alpha_j = a_{j1} + a_{j2}$ for $j = 1, 2$. By combining these identities we can write γ, α_1 and α_2 in terms of just one of the components of the two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. Without loss of generality we fix the first component of the two vectors, a_{11} and a_{12}, to be the strategic variable of, respectively, firm 1 and firm 2. Thus, we have the following identities

$$\gamma = a_{11}a_{21} + \sqrt{1 - a_{11}^2}\sqrt{1 - a_{21}^2}$$

$$\alpha_1 = a_{11} + \sqrt{1 - a_{11}^2}$$

$$\alpha_2 = a_{21} + \sqrt{1 - a_{21}^2}$$

The interval in which a_{11} and a_{21} are respectively chosen is the unit interval $[0, 1]$. This follows directly from the circular constraint $a_{j1}^2 + a_{j2}^2 = 1$. With the above identities, the profit functions 4.19 and 4.20, can be stated in terms of just the two strategic variables (a_{11}, a_{12}).

$$\pi_1(a_{11}, a_{21}) = \left[\frac{2a_{11} + 2\sqrt{1 - a_{11}^2}}{4 - (a_{11}a_{21} + \sqrt{1 - a_{11}^2}\sqrt{1 - a_{21}^2})} - \frac{(a_{11}a_{21} + \sqrt{1 - a_{11}^2}\sqrt{1 - a_{21}^2})(a_{21} + \sqrt{1 - a_{21}^2})}{4 - (a_{11}a_{21} + \sqrt{1 - a_{11}^2}\sqrt{1 - a_{21}^2})^2}\right]^2 \quad (4.21)$$

$$\pi_2(a_{11}, a_{21}) = \left[\frac{2a_{21} + 2\sqrt{1 - a_{21}^2}}{4 - (a_{11}a_{21} + \sqrt{1 - a_{11}^2}\sqrt{1 - a_{21}^2})} - \frac{(a_{11}a_{21} + \sqrt{1 - a_{11}^2}\sqrt{1 - a_{21}^2})(a_{11} + \sqrt{1 - a_{11}^2})}{4 - (a_{11}a_{21} + \sqrt{1 - a_{11}^2}\sqrt{1 - a_{21}^2})^2}\right]^2 \quad (4.22)$$

Before proceeding with the analysis of design choices, we need to insure, at least, 4.21 and 4.22 to be continuous for every pair $(a_{11}, a_{21}) \in [0, 1] \times [0, 1]$. However, this is not a particularly difficult task since it is easily seen, by just inspecting the two above equations, that, for any pair $(a'_{11}, a'_{21}) \in [0, 1] \times [0, 1]$, it holds

$$\lim_{(a_{11}, a_{21}) \to (a'_{11}, a'_{21})} \pi_j(a_{11}, a_{21}) = \pi_j(a'_{11}, a'_{21}), \quad j = 1, 2$$

Thus, the two profit functions are continuous in the design choices. Unfortunately, ensuring differentiability of profit functions turns out to be a particularly hard task given the complexity associated with computing partial derivatives. Nevertheless, continuity is a sufficient condition for a Nash equilibrium, $(a_{11}^*, a_{21}^*) \in [0, 1] \times [0, 1]$, to exists. As it results evident from the above discussion, proceeding in the usual way, i.e computing best replies, seems unfeasible. In general the system that needs to be solved in order to find the *design equilibrium* (a_{11}^*, a_{21}^*) is

$$\frac{\partial \pi_1(a_{11}, \bar{a}_{21})}{\partial a_{11}} = 0$$

$$\frac{\partial \pi_2(\bar{a}_{11}, a_{21})}{\partial a_{21}} = 0$$

The above system implicitly defines a best reply map $\Psi : [0,1]^2 \to [0,1]^2$. Whether or not Ψ possesses a fixed point (a^*_{11}, a^*_{21}) depends on the behaviour of the profit functions and of the resulting best replies. Given the complexity of the profit functions, I shall not attempt to rigorously prove the existence of such an equilibrium, but I shall nevertheless assume such an equilibrium to exist.

With the issue of existence (partially) avoided we can return to our problem. Namely, intuition suggests that if able firms would increase their profits by collude in such a way that each is able to establish a *local monopoly*. However, in order for this intuitive conclusion to be correct it must be proved that firms have no incentive to *unilaterally* break this coalition, i.e to deviate by changing location *knowing* that the opponent is a *local monopolist*. We have the following result.

Proposition 2. *The subgame perfect Nash equilibrium of the products' design game is given by* $(a^*_{11}, a^*_{21}) = (1/\sqrt{2}, 1/\sqrt{2})$.

Proof. Assume that firm 2 establishes a local monopoly. That is, consider the two cases, $(\bar{a}_{21}, \bar{a}_{22}) = (1, 0)$ and $(\bar{a}'_{21}, \bar{a}'_{22}) = (0, 1)$. Now, we ask whether firm 1, knowing what firm 2 selected, has an incentive to also establish a local monopoly, namely either $(a_{11}, a_{12}) = (0, 1)$ and $(a_{11}, a_{12}) = (1, 0)$, or not. Firm 1's profits are

$$\pi_1(a_{11}, 1) = \left(\frac{a_{11} + 2\sqrt{1 - a_{11}^2}}{4 - a_{11}^2} \right)^2 \tag{4.23}$$

$$\pi_1(a_{11}, 0) = \left(\frac{2a_{11} + \sqrt{1 - a_{11}^2}}{3 + a_{11}^2} \right)^2 \tag{4.24}$$

The maximization of 4.23 and 4.24 yields, respectively, firm 1's best replay in the case of $\bar{\mathbf{a}}_2 = (1, 0)$ and $\bar{\mathbf{a}}'_2 = (0, 1)$. Plotting both 4.23 and 4.24 (Figure 1 below) shows that both functions are *concave* in a_{11}. Thus, a maximum does exists.

In particular, 4.23 attains its maximum at $a^*_{11} \approx 0.67$, and 4.24 attains

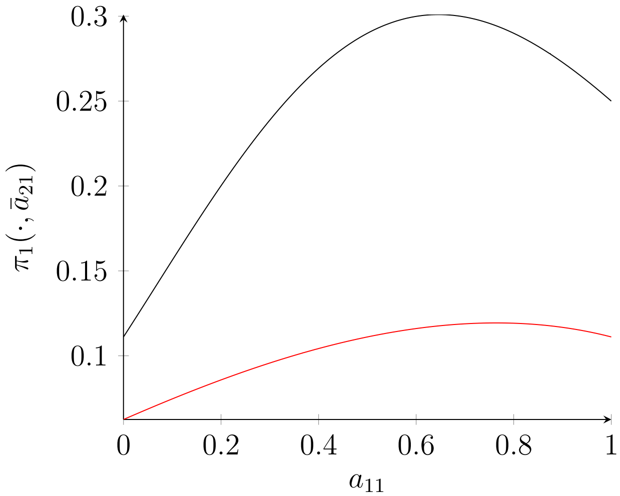

Figure 4.1: Firm 1's profit when $\bar{\mathbf{a}}_2 = (1,0)$ (red curve), and when $\bar{\mathbf{a}}_2 = (0,1)$ (black curve).

its maximum at $a_{11}^* \approx 0.74$. This means that the best reply of firms 1, to firm 2's *local monopoly*, is to contest firm 2's market. Hence, there is no incentive to establish a complementary *local monopoly*. At this point, we can turn to firm 2's side. Knowing that, if locating at, say, $\mathbf{a}_2 = (1,0)$, firm 1 will contest this location by locating at $a_{11}^* = 0.67$, does firm 2 has an incentive to relocate? Plugging a_{11}^* into 4.22 yields

$$\pi_2(a_{11}^*, a_{21}) = \frac{\left[2a_{21} + 2\sqrt{1-a_{21}^2} - \left(0.67a_{21} + 0.742\sqrt{1-a_{21}^2}\right)(0.67+0.742)\right]^2}{\left[4 - \left(0.67a_{21} + 0.742\sqrt{1-a_{21}^2}\right)^2\right]^2} \tag{4.25}$$

The maximum of 4.25 is attained at $a_{21}^* = 1/\sqrt{2} \approx 0.71$. Thus, firm 2 has an incentive to change its location from $\mathbf{a}_2 = (1,0)$ to $\mathbf{a}_2^* = (1/\sqrt{2}, 1/\sqrt{2}) \approx (0.71, 0.71)$ as a best response to firm 1's dispute. A similar argument yields the symmetric result for the case of $\mathbf{a}_1^* = (0.74, 0.672)$, that is the case in which firm initially locates at $\mathbf{a}_2 = (0,1)$. Note that firm 2's best reply to firm 1's action in response to its local monopoly is to locate in the market's "centre". This fact is particularly interesting. It follows that proceeding in the same way considering the best reply of firm 2 to firm 1's *local monopolies* yields the exact same result. From this, if firms act rationally, neither of them has an incentive to establish a local monopoly. This is so since both know that if they choose to establish a local monopoly, the opponent has an incentive to contest it. Hence, neither of them would do it.

With this result at hand we can proceed by asking what happens for every $\gamma \in (0,1)$. Again, the complexity of the profit functions prevents us from proceeding in the usual way. Moreover, given that a firm best reply to the opponent's contestation of its local monopoly is to locate at the "centre" of the market, it can be conjectured that the only *stable* equilibrium, i.e (a_{11}^*, a_{21}^*) such that neither of the firms has an incentive to deviate, is given by both firms locating at the "center" of the market, $(a_{11}^*, a_{21}^*) = (1/\sqrt{2}, 1/\sqrt{2})$. Now, we can consider what would be the best response of either of the firms to the opponent locating in the "centre". That is, we consider the profit of firm 1 given firm 2 locating at the "centre", and symmetrically the profit of firm 2 given firm 1 locating at the "centre". That is

$$\pi_1(a_{11}, 1/\sqrt{2}) = \frac{\left[2a_{11} + 2\sqrt{1-a_{11}^2} - \left(0.71a_{11} + 0.71\sqrt{1-a_{11}^2}\right)(1.42)\right]^2}{\left[4 - \left(0.71a_{11} + 0.71\sqrt{1-a_{11}^2}\right)^2\right]^2} \tag{4.26}$$

$$\pi_1(1/\sqrt{2}, a_{21}) = \frac{\left[2a_{21} + 2\sqrt{1-a_{21}^2} - \left(0.71a_{21} + 0.71\sqrt{1-a_{21}^2}\right)(1.42)\right]^2}{\left[4 - \left(0.71a_{21} + 0.71\sqrt{1-a_{21}^2}\right)^2\right]^2} \tag{4.27}$$

Function 4.26 attains its maximum at $a_{11}^* = 1/\sqrt{2}$, and 4.27 attains its maximum at $a_{21}^* = 1/\sqrt{2}$ as it can be seen in Figure 2 below. In addition, it is interesting to notice that for both firms, as a_{j1} is additionally increased, profit fall to zero. In particular, if any of the firm selects $a_{j1} = 1$ its profit will vanish. Thus, for both firms, the best response to the opponent's locating at the centre of the market is to do the same, i.e to also locate at the centre.

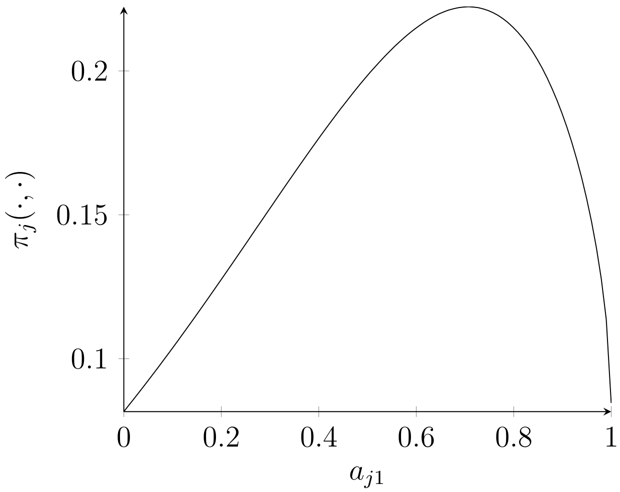

Figure 4.2: Firm j profit as a function of own action a_{j1} when the opponent is located at the "center".

Clearly, from the above argument, once locating at the centre of the market, neither of the firms has an incentive to deviate. Hence, the only Nash equilibrium for the products selection subgame is given by $\mathbf{a}_1^* = (1/\sqrt{2}, 1/\sqrt{2})$ and $\mathbf{a}_2^* = (1/\sqrt{2}, 1/\sqrt{2})$.

□

Proposition 1 establishes a striking result. Namely, the strategic choice goods' designs, yields *aggregation*. That is, the two firms tend to concentrate around the same product specification. The consequence of this is that the market will provide a single homogeneous good. The subsequent quantity competition will thus be the standard one arising when firms provide a single homogeneous good[14]. In particular, we have that $q_j^C = \sqrt{2}/3$ and $\pi_j(\mathbf{q}^C) = 2/9$ for $j = 1, 2$. Note that, although providing the same product reduces firms' profits, the fact that firms are able to design their product in a multidimensional space results in equilibrium profits being higher than the standard Cournot outcome with homogeneous goods.

The intuition for the above result is essentially to be found in the strategic

[14]See for example Mas-Colell, Whinston and Green (1995), Friedman (1983), or Vives (1999).

behaviour of firms. Without any binding agreement allowing firms to split the market, and gain monopoly power on one of the available characteristics, the choice of focusing on the provision of just one characteristics exposes the firm to the risk of being *contested* in its own market with the result of a fall in profit. In order to avoid any such risk, the optimal strategy for a firm is to provide a product embodying an amount of *both* characteristics, yielding the most "qualitative"[15] variety, and for which the opponent cannot do anything if not provide the exact same product. Essentially, this *aggregative* result is driven by the willingness of firms to avoid any potential threats which will cause their profit to fall.

To see this, assume that firms *cooperate* at the product design stage. Cooperation at this stage implies that firms would select the products' design combination yielding the maximum aggregate profits. That is, firms select products' design, (a_{11}^*, a_{21}^*), such that

$$(a_{11}^*, a_{21}^*) = \arg\max\{\pi_1(a_{11}, a_{21}) + \pi_2(a_{11}, a_{21})\} \tag{4.28}$$

Straightforward computations reveals that the above is reached for $(a_{11}^*, a_{21}^*) = (1, 0)$. This clearly implies that, at the *cooperative* equilibrium, $\mathbf{a}_1^* = (1, 0)$ and $\mathbf{a}_2^* = (0, 1)$; that is, firms have an incentive to establish *local monopolies* yielding undoubtedly higher profits compared to the *non-cooperative* equilibrium. Furthermore, in this situation product differentiation is seen to be maximal: firms provide two independent products each of which provides consumers with just one of the available characteristics. This simple argument reveals the importance of cooperation in shaping the outcome of the product selection subgame. When firms choose not to cooperate, the result of Proposition 2 holds[16].

[15]This terminology just refers to the fact that at the equilibrium, $\alpha_j = \sqrt{2}$ for $j = 1, 2$, that is the maximum value attainable by the αs. Given the way we defined α_j, i.e the sum of characteristics embodied in a good, we can see them as a measure of "quality".

[16]It is interesting to connect this last discussion with the model, and results, presented in Chapter 2. In particular, cooperation can equally be seen as a situation where a monopolist choose how to design its products in the characteristics space. It is interesting to see that, despite $u''' = 0$, the monopolist will have an incentive to provide the *complete* market matrix. However, the welfare conclusions will differ from the ones presented in Chapter 2 because of the assumption of a representative consumer.

4.6 Entry

In this section we analyse, by means of a simple example, the case of one firm willing to entering the market. In order to make the analysis as simple as possible, denote the entrant as firm J and assume that there is a *fixed cost* $f > 0$ which accounts for the set-up of both the product design and subsequent output, conditional on entering the market. We assume the following. The entrant firm, J, is aware of the current market situation. That is, it knows that the two duopolistic firms are providing a single homogeneous good. Denote this common good's specification as $\tilde{\mathbf{a}} = (1/\sqrt{2}, 1/\sqrt{2})$. With this information at hand, firm J evaluates whether to enter the market or not, by choosing its good's design. On the other hand, the two duopolistic firms are assumed to be unable to modify their designs in response to firm J's potential entrance. So, the series of events is the following. Firm J, knowing the current market situation evaluates its entrance that corresponds to choosing a certain product's design. Once firm J enters the market, by providing a certain potentially *new* product, quantity competition takes place among firms with the two duopolists' still providing the same homogogeneous good. Thus, the structure is again that of a two stage game where our goal is to find the subgame perfect Nash equilibrium corresponding to firm J's design selection, *given* the opponents' design fixed. This equilibrium, if exists, is interpreted as the product's design that firm J would chose if it enters the market.

Denote firm J's design as $\mathbf{a}_J \in \mathbb{R}^2_+$, with $a^2_{J1} + a^2_{J2} = 1$. The assumption that duopolists' design is fixed and cannot be modified as a reaction to firm J's action simplifies the analysis. However, it is also plausible given that the entrant, J, competes, essentially, against an homogeneous market where the active firms provide the same product's design. Given this, the demand system reads

$$\begin{aligned} p_1(\mathbf{q}) &= \bar{\alpha} - q_1 - q_2 - \gamma_{1J} q_J \\ p_2(\mathbf{q}) &= \bar{\alpha} - q_1 - q_2 - \gamma_{2J} q_J \\ p_J(\mathbf{q}) &= \alpha_J - q_J - \gamma_{J1} q_1 - \gamma_{J2} q_2 \end{aligned}$$

From the above, we can directly write firm J's profit π_J, for any possible product's design. Define $\tilde{\gamma} = \mathbf{a}_J \cdot \tilde{\mathbf{a}}$ to be the similarity between the homogeneous good sold by the two duopolists and firm J's product. Profit of firm J is given by

$$\pi_J(q_J, \mathbf{q}_{-J}) = (\alpha_J - q_j - \tilde{\gamma}(q_1 + q_2))q_J - f \tag{4.29}$$

The two duopolists, $j = 1, 2$, will have the same profit function given that they sell a single homogeneous good. Profit for firm 1 is given by

$$\pi_1(q_1, \mathbf{q}_{-1}) = (\bar{\alpha} - q_1 - q_2)q_1 - \tilde{\gamma}q_J q_1 \tag{4.30}$$

and similarly for firm 2. From the above, it is easily seen that the best reply map, resulting from the simultaneous maximization of the three profit functions, is linear and well behaved. Thus, a Cournot-Nash equilibrium exists for this quantity game. Denote this equilibrium as $\mathbf{q}^e$. We have that

$$q_J^e = \frac{3\alpha_J}{2(3 - \tilde{\gamma}^2)} - \frac{\bar{\alpha}\tilde{\gamma}}{3 - \tilde{\gamma}^2} \tag{4.31}$$

$$q_j^e = \frac{\bar{\alpha}}{3 - \tilde{\gamma}^2} - \frac{\alpha_J\tilde{\gamma}}{2(3 - \tilde{\gamma}^2)}, \quad j = 1, 2 \tag{4.32}$$

Again, the fact that duopolists' design is fixed, simplifies the analysis. Indeed, we have that $\tilde{\gamma} = \mathbf{a}_J \cdot \tilde{\mathbf{a}} = a_{J1}/\sqrt{2} + a_{J2}/\sqrt{2} = (a_{J1} + a_{J2})/\sqrt{2} = \alpha_J/\sqrt{2}$, and $\bar{\alpha} = \sqrt{2}$. Thus, we can write 4.31 and 4.32 as follows

$$q_J^e = \frac{\alpha_J}{6 - \alpha_J^2} \tag{4.33}$$

$$q_j^e = \frac{\bar{\alpha}}{6 - \alpha_J^2}\left(\frac{4 - \alpha_J^2}{2}\right), \quad j = 1, 2 \tag{4.34}$$

By inspecting 4.33 and 4.34, we see that it is more convenient to define α_J as firm J's choice variable regarding good's design, since it is uniquely determined by a_{J1} and a_{J2}. Since we are interested in what will firm J's design if it enters the market, and we assumed duopolists' design to correspond to the subgame perfect Nash equilibrium (a_{11}^*, a_{21}^*), we can just consider firm

J's profit at the Cournot-Nash equilibrium $\mathbf{q}^e$. From it, we will derive the best design for firm J if it has to enter the market. Thus, we search for the Nash equilibrium of the subgame where, given the duopolists' design firm J has to chose its best option in terms of its good's design. At $\mathbf{q}^e(\alpha_J)$, the profit of firm J is given by

$$\pi_J(\mathbf{q}^e(\alpha_J)) = \frac{2\alpha_J^2(5-\alpha_J^2)}{(6-\alpha_J^2)^2} - f \tag{4.35}$$

Now, considering a_{J1} to be the strategic variable of firm J, we can write 4.35 as a function of it. We have

$$\pi_J(a_{J1}) = \frac{2\left(a_{J1} + \sqrt{1-a_{J1}^2}\right)^2 \left[5 - \left(a_{J1} + \sqrt{1-a_{J1}^2}\right)^2\right]}{\left[6 - \left(a_{J1} - \sqrt{1-a_{J1}^2}\right)^2\right]} \tag{4.36}$$

The above function is concave in a_{J1}, thus a maximum exists. The optimal condition for profit maximization, $d\pi_J/da_{J1} = 0$ reads

$$\frac{4(2a_{J1}^2 - 1)\left(14a_{J1}\sqrt{1-a_{J1}^2} - 23\right)}{2a_{J1}^2 + 5\sqrt{1-a_{J1}^2} - 2a_{J1}} = 0$$

For any $a_{J1} \in [0,1]$ the right term of the nominator is always negative hence, the above condition is satisfied whenever $2a_{J1}^2 - 1 = 0$. This holds for $a_{J1}^* = 1/\sqrt{2}$. That is, whenever $f \geq 0$, but sufficiently small so as to incentivize entry, the entrant firm J will provide exactly the same product the market is providing, $\tilde{\mathbf{a}} = (1/\sqrt{2}, 1/\sqrt{2})$.

Clearly, the magnitude of the fixed cost impacts the entry decision of firm J. The final profit of firm J is given by 4.18 decurted of the fixed cost f. We have

$$\pi_J(\mathbf{q}^e) = \frac{1}{16}\bar{\alpha}^2 - f \geq 0 \Longleftrightarrow f \leq 1/8 = \bar{f}$$

So, for positive fixed cost, lower or equal than the threshold $\bar{f}$, firm J will have an incentive to enter the market. Firm J will be the only firm entering the market if $f = \bar{f}$. From this, we can infer that the lower f, below $\bar{f}$, the larger will be the number of potential entrants. Thus, we can push the above reasoning, for the entry decision of a single firm, a bit further in considering the case of the subsequent entry of an unspecified number of $n = 1, \ldots, N$ firms. With an unspecified number of potential entrants, it is hardly plausible that design choices will all converge towards a common design.

In general, assume that $f > 0$, so that entry will effectively occur in the market until profits are exhausted. Now, assume there is a fringe of $n = 1, \ldots, N$ firms who are willing to enter the market, populated by the two duopolists $j = 1, 2$. As for the case of firm J, we assume that all $n = 1, \ldots, N$ potential entrants are aware of the current market situation. For any firm n the decision process is the same as the one exposed from the case of firm J. Denote the effective entrants, i.e the actual number of firms among the $n = 1, \ldots, N$ potential entrants entering the market, at the *free-entry* equilibrium, as N^e. Thus, after the entry process concludes, the total number of active firms will be $2 + N^e$. Note that firms N^e are the ones bearing the fixed cost f. At this point it will be instructive to present how the demand structure is modified when there is an indeterminate number of potential entrants.

$$\begin{aligned}
p_1(\mathbf{q}) &= \bar{\alpha} - q_1 - q_2 - \sum_n \gamma_{1n} q_n \\
p_2(\mathbf{q}) &= \bar{\alpha} - q_1 - q_2 - \sum_n \gamma_{2n} q_n \\
p_n(\mathbf{q}) &= \alpha_n - q_n - \gamma_{n1} q_1 - \gamma_{n2} q_2 - \sum_{n=1, n \neq k} \gamma_{nk} q_k \\
&\vdots \\
p_N(\mathbf{q}) &= \alpha_N - q_N - \gamma_{N1} q_1 - \gamma_{N2} q_2 - \sum_{N \neq n} \gamma_{Nn} q_n
\end{aligned}$$

The similarity matrix Γ, associated to the above linear demand system is easily seen to be

$$\Gamma^e = \begin{bmatrix} 1 & 1 & \gamma_{11} & \cdots & \gamma_{1N} \\ 1 & 1 & \gamma_{21} & \cdots & \gamma_{2N} \\ \vdots & & & & \vdots \\ \gamma_{n1} & \gamma_{n2} & \gamma_{n1} & \cdots & \gamma_{nN} \\ \vdots & & & & \vdots \\ \gamma_{N1} & \gamma_{N2} & \gamma_{N1} & \cdots & 1 \end{bmatrix}$$

From the above similarity matrix it should be immediate to see that even set-up the structure of the two-stage game turn out to be particularly hard when there are multiple potential entrants. Further, as pointed out by Vives (1999), for more than three firms, there may be multiple Cournot-Nash equilibria. Clearly, this potential multiplicity prevents us from setting-up the product selection subgame. Therefore, we will assume that entry will occur sequentially. That is, each firm's entrance is ordered in such a way that entry does not occur in the same exact moment but each subsequent firm enters after the one immediately before it did enter the market. In this way, we may assume that each firm's entry takes place as exposed for the case of just one entrant. Then, each of the new entrant will choose the same product design as the firm who entered before it.

So, each of the N^e subsequent entrants will provide the same homogeneous good determined by $\bar{\alpha}$, we can readily define the final profit for the $2 + N^e$ active firms.

$$\pi_j(\bar{\alpha}) = \frac{\bar{\alpha}^2}{(3 + N^e)^2}, \quad j = 1, 2 \tag{4.37}$$

$$\pi_n(\bar{\alpha}) = \frac{\bar{\alpha}^2}{(3 + N^e)^2} - f, \quad n = 1, \ldots, N^e \tag{4.38}$$

From the above, we are able to derive the total number of active firms that will populate the market of the homogeneous good $\bar{\alpha}$ at the *free-entry* equilibrium. In particular, using 4.38, the *free-entry* equilibrium number of firms satisfies $\pi_n(\bar{\alpha}) = 0$. It is immediate to see that the free-entry equilibrium number of firms is N^e is given by

$$(3 + N^e)^2 = \frac{\bar{\alpha}^2}{f} \tag{4.39}$$

Unless $f = 0$, N^e will be bounded. Thus, and this is a consequence of our assumptions, the two duopolists' profit will still be positive, although small, as N^e is bounded, i.e $f \geq 0$. This claim is immediate to prove by plugging N^e, into π_j, $\pi_j(\bar{\alpha}) = \sqrt{2f} > 0$ for $j = 1, 2$. In particular, the original duopolists, $j = 1, 2$, will detain some residual market power, as their profits remain positive even after entry. This will clearly affect welfare.

Finally, consider the extreme case of $f = 0$. When entrants do not face any set-up cost f, entry is unbounded, that is, $N^e \to \infty$, while profits tend to zero. In other words, when the entry process is *frictionless*, i.e $f = 0$, an indeterminate number of firms will enter the market making profits converge to zero.

The above stylized process is a direct consequence of the implicit product's design subgame that arises for every of the potential entrants $n = 1, \ldots, N$. In addition, and this is an important fact, the above process depends on two, correlated, assumptions. First, that the dimensions through which potential entrants can design their product are only two. Second, that the incumbent firms, the two duopolists, are unable to modify their products in reaction to the entrant's choice. For, the entry problem will be much more complex, although complete, by considering the possibility for the potential entrants $n = 1, \ldots, N$ to enlarge the dimension of the designs space, i.e the number of characteristics. This will definitely accounts for an entry process led by *innovation*. In the next section we consider the welfare implications of our model.

4.7 Welfare

The current setting suits itself to a simple welfare analysis. In particular, we will perform a *second-best* analysis à la Mankiw and Whinston (1986). This provides us a way to define the *optimal* degree of entry. Thus, a comparison between the equilibrium number of firm and the *optimal* one is possible. In particular, given the specific utility function considered, and the representative consumer specification, consumer's surplus and producers' surplus can easily be defined. Producers' surplus is composed by the duopolists' surplus, and, provided $0 \leq f \leq \bar{f}$, the surplus of the entrant firms $1, \ldots, N$. That is

$$PS = \pi_1(\mathbf{q}) + \pi_2(\mathbf{q}) + \sum_{n=1}^{N} \pi_n(\mathbf{q}) = \pi_1(\mathbf{q}) + \pi_2(\mathbf{q}) + \sum_{n=1}^{N} \pi_n(\mathbf{q}) \qquad (4.40)$$

where $q_j = \bar{\alpha}/(3+N)$. Now, from the particular structure of the utility function we considered, U directly defines consumer's surplus as $CS = (w_0 - \mathbf{p} \cdot \mathbf{q}) + V(\mathbf{q})$, where $V(\mathbf{q})$ is the utility derived from acquiring characteristics. From this, it also follows that total surplus is given by $TS = CS + PS = w_0 + V(\mathbf{q}) - Nf$. Formally,

$$TS = w_0 + \alpha_1 q_1 + \alpha_2 q_2 - \frac{1}{2}(q_1^2 + 2\gamma q_1 q_2 + q_2^2) - Nf \qquad (4.41)$$

By using equilibrium quantities $q_j = \bar{\alpha}/(3+N)$ into 4.41 we get

$$TS(N) = w_0 + \frac{2(2+N)}{3+N} - \left(\frac{2+N}{3+N}\right)^2 - Nf \qquad (4.42)$$

From the above, it is easy to derive the *optimal* entry N^*. That is, the number of firms such that $TS(N^*) = \max_N TS(N)$. $dTS(N)/dN = 0$ yields

$$(3+N^*)^3 = \frac{2}{f} \qquad (4.43)$$

It is immediate to see that in our setting $N^* < N^e$. That is, entry will be too high. This result is indeed not new in the product differentiation literature (Mankiw and Whinston (1986)). However, the interesting aspect here is that this tendency of excessive entry arises from the firms' strategic decision of product's design.

4.8 Conclusion

We showed that a duopoly with hedonically differentiated goods, where firms are assumed to compete à la Cournot, yields, when firms are able to modify their product's designs, *aggregation* towards the "centre" of the market. This result is novel in the literature on product differentiation since the strategic choice of a particular good's design was never attempted in a model which specifically considers goods as vectors in the space of characteristics. However, the fact that Cournot competition yields agglomeration when firms are able to strategically choose their locations, is not new. Thus, the result we established here, strengthens the idea that when competition takes place in quantity, firms have an incentive to select a common product (i.e locating in the same spot). Further, the same tendency, was shown to occur when entry in take into account. In particular, potential entrant firms, deciding on their product's design *knowing* the market environment, will enter the market providing the same homogeneous good.

From the consumer's perspective, the above result, is the availability of the only one single variety, even if entry is possible. This outcome implies that consumer enjoys increasing variety in the sense that she will be able to acquire more characteristics. Moreover, as entry occurs, consumer's surplus increases given the general decrease in prices. When locating in the "centre", firms face tougher competition since they provide an homogeneous good, i.e a single variety. However, this single variety is containing the maximum amount of each characteristic. Further, when entry occurs, consumer will additionally gain by both the augmented competition, which lowers prices, and by being able to acquire higher amounts of characteristics. Nonetheless, once we depart from the representative consumer assumption, and assume heterogeneity among consumers in terms of characteristics, this tendency in providing the same variety might very well disappear. This opens the gate to richer welfare implications.

Although not conclusive, the current work has as a general goal that of tease research towards a more deeper study of the characteristics model as originally proposed by Lancaster. In particular, it suggests that the dynamics of product differentiation choices, and its spillovers in terms of market structure, may be richer than thought.

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