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Constant mean curvature hypersurfaces in the Anti-de Sitter space

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Contents

Abstract

Abstract

This thesis is devoted to the study of *constant mean curvature* (CMC) spacelike hypersurfaces, which are a generalization of *minimal surfaces*, in the *Anti-de Sitter space* $\mathbb{H}^{n,1}$, namely the Lorentzian spaceform of negative sectional curvature.

Our first achievement is the complete classification of properly embedded CMC spacelike hypersurfaces, namely we show that every admissible sphere Λ is the boundary of a unique such hypersurface, for any given value $H \in \mathbb{R}$ of the mean curvature. It is known that any admissible sphere Λ is contained in a unique maximal globally hyperbolic Anti-de Sitter manifold, denoted by $\Omega(\Lambda)$, which is called the *invisible domain* of Λ. We also demonstrate that, as H varies in $\mathbb{R}$, these hypersurfaces analytically foliate the invisible domain of Λ: this foliation induces an analytic time-function on $\Omega(\Lambda)$. To conclude the qualitative investigation of CMC hypersurfaces in Anti-de Sitter space, we extend Cheng-Yau Theorem to the Anti-de Sitter space, which establishes the completeness of any entire constant mean curvature hypersurface.

The second main goal of this thesis consists in a quantitative study of properly embedded CMC spacelike hypersurfaces. As main characters of this part, we introduce the notion of $H-$shifted convex hull $\mathcal{CH}_H(\Lambda)$ of a quasi-sphere Λ, and its width $\omega_H(\Lambda)$, namely its timelike diameter.

We bound by $\omega_H(\Lambda)$ the extrinsic curvature of the properly embedded CMC spacelike hypersurface with mean curvature H and the asymptotic boundary Λ, up to a universal constant. As a first application of this result, we produce a *pletora* of CMC hypersurfaces with sectional curvature uniformly negative.

Then, we introduce the notion of *quasi-sphere*, which extends the notion of *quasi-symmetric curve* in higher dimension: we characterize quasi-spheres in term of the width of their $H-$shifted convex hull. Then, we prove that they have nice dynamical properties. Then, we prove that quasi-spheres are a good generalization of the universal Teichmüller space in the context of *higher higher Tiechmüller theory.*

Finally, we focus on the $3-$dimensional case: CMC surfaces are strictly linked to *constant sectional curvature* (CSC) surfaces, which allows us to classify CSC surfaces in $\mathbb{H}^{2,1}$. Moreover, CMC surfaces induce $\theta-$landslide, a special class of diffeomorphisms of the hyperbolic plane $\mathbb{H}^2$: we classify them and prove that their quasiconformal dilatation is bounded by the cross-ratio norm of their extension to $\partial\mathbb{H}^2$, if the latter is small enough.

Sommario

Questa tesi è dedicata allo studio delle ipersuperfici di tipo spazio a *curvatura media costante* (CMC), che sono una generalizzazione delle *superfici minime*, nello *spazio Anti-de Sitter* $\mathbb{H}^{n,1}$, ossia il modello lorentziano di spazio a curvatura sezionale negativa costante.

Il nostro primo risultato è la classificazione completa delle ipersuperfici CMC di tipo spazio propriamente embedded. Mostriamo che ogni sfera ammissibile Λ è il bordo di un'unica tale ipersuperficie, per qualsiasi valore $H \in \mathbb{R}$ della curvatura media. È noto che ogni sfera ammissibile Λ è contenuta in un'unica varietà Anti-de Sitter iperbolica globale massimale, denotata come $\Omega(\Lambda)$, chiamata *dominio invisibile* di Λ. Dimostriamo inoltre che, al variare di H in $\mathbb{R}$, queste ipersuperfici foliano analiticamente il dominio invisibile di Λ: questa foliazione induce una funzione tempo analitica su $\Omega(\Lambda)$. Per concludere l'indagine qualitativa delle ipersuperfici CMC nello spazio Anti-de Sitter, estendiamo il teorema di Cheng-Yau allo spazio Anti-de Sitter, mostrando la completezza di ogni ipersuperficie intera a curvatura media costante.

Il secondo obiettivo di questa tesi consiste in uno studio quantitativo delle ipersuperfici CMC di tipo spazio propriamente embedded. Come principali oggetti di questa parte, introduciamo la nozione di inviluppo convesso $H-$shifted $\mathcal{CH}_H(\Lambda)$ di una quasi-sfera Λ, e il suo spessore $\omega_H(\Lambda)$, ossia il suo diametro temporale.

Limitiamo con $\omega_H(\Lambda)$ la curvatura estrinseca dell'ipersuperficie CMC di tipo spazio propriamente embedded con curvatura media H e bordo asintotico Λ, a meno di una costante universale. Come prima applicazione di questo risultato, produciamo una *pletora* di ipersuperfici CMC con curvatura sezionale uniformemente negativa.

Successivamente, introduciamo la nozione di *quasi-sfera*, che estende la nozione di *curva quasi-simmetrica* in dimensione superiore: caratterizziamo le quasi-sfere in termini dello spessore del loro inviluppo convesso $H-$shifted Λ. Poi, dimostriamo che esse possiedono buone proprietà dinamiche. Dimostriamo inoltre che le quasi-sfere sono una buona generalizzazione dello spazio di Teichmüller universale nel contesto della *teoria di Teichmüller di dimensione e rango superiore.*

Infine, ci concentriamo sul caso tridimensionale: le superfici CMC sono strettamente legate alle superfici a *curvatura sezionale costante* (CSC), il che ci consente di classificare le superfici CSC in $\mathbb{H}^{2,1}$. Inoltre, le superfici CMC inducono $\theta-$landslide, una classe speciale di diffeomorfismi del piano iperbolico $\mathbb{H}^2$: le classifichiamo e dimostriamo che la loro dilatazione quasiconforme è limitata dalla norma del birapporto della loro estensione a $\partial\mathbb{H}^2$, se quest'ultima è sufficientemente piccola.

Résumé

Cette thèse est consacrée à l'étude des hypersurfaces de type espace à *courbure moyenne constante* (CMC), qui sont une généralisation des *surfaces minimales*, dans l'espace *Anti-de Sitter* $\mathbb{H}^{n,1}$, à savoir le modèle lorentzien d'espace de courbure sectionnelle constante négative.

Notre premier résultat est la classification complète des hypersurfaces CMC de type espace proprement plongées. Nous montrons que chaque sphère admissible Λ est le bord d'une unique telle hypersurface, pour toute valeur donnée $H \in \mathbb{R}$ de la courbure moyenne. Il est connu que toute sphère admissible Λ est contenue dans un unique Anti-de Sitter globalement hyperbolique maximale, noté $\Omega(\Lambda)$, que l'on appelle le *domaine invisible* de Λ. Nous démontrons également comme H varie dans $\mathbb{R}$, ces hypersurfaces feuilletent analytiquement le domaine invisible de Λ : cette feuillettage induit une fonction temps analytique sur $\Omega(\Lambda)$. Pour conclure l'investigation qualitative des hypersurfaces CMC dans l'espace Anti-de Sitter, nous étendons le théorème de Cheng-Yau à l'espace Anti-de Sitter, établissant la complétude de toute hypersurface entière à courbure moyenne constante.

Le second objectif de cette thèse consiste en une étude quantitative des hypersurfaces CMC de type espace proprement plongées. Comme principaux objets de cette partie, nous introduisons la notion d'enveloppe convexe $H-$décalée d'une quasi-sphère $\Lambda 0$, noté

$\mathcal{CH}_H(\Lambda)$, et son épaisseur $\omega_H(\Lambda)$, c'est-à-dire son diamètre temporel.

Nous bornons par $\omega_H(\Lambda)$ la courbure extrinsèque de l'hypersurfaces CMC de type espace proprement plongées avec courbure moyenne H et bord asymptotique Λ, quitte à une constante universelle. Comme première application de ce résultat, nous produisons une *pletora* d'hypersurfaces CMC avec courbure sectionnelle uniformément négative.

Ensuite, nous introduisons la notion de *quasi-sphère*, qui étend la notion de *courbe quasi-symétrique* en dimension supérieure : nous caractérisons les quasi-sphères en termes de l'épaisseur de leur enveloppe convexe $H-$décalée. Puis, nous prouvons qu'elles possèdent de bonnes propriétés dynamiques. Ensuite, nous démontrons que les quasi-sphères sont une bonne généralisation de l'espace de Teichmüller universel dans le cadre de la *théorie de Teichmüller de dimension et rang supérieure.*

Enfin, nous nous concentrons sur le cas 3-dimensionnel : les surfaces CMC sont strictement liées aux surfaces à *courbure sectionnelle constante* (CSC), ce qui nous permet de classifier les surfaces CSC dans $\mathbb{H}^{2,1}$. De plus, les surfaces CMC induisent des $\theta-$landslide, une classe spéciale de difféomorphismes du plan hyperbolique $\mathbb{H}^2$: nous les classifions et prouvons que leur dilatation quasiconforme est bornée par la norme du birapport de leur extension à $\partial\mathbb{H}^2$, si cette dernière est suffisamment petite.

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Introduction

Historical background

Constant mean curvature (CMC) hypersurfaces are a classical object in differential geometry. They are the natural generalization of *minimal* surfaces, *i.e.* surfaces whose mean curvature identically vanishes. Minimal surfaces have been widely studied for their wide range applications, with techniques coming from completely different mathematical fields, such as calculus of variations, measure theory, complex analysis and mathematical physics.

In Lorentzian geometry, the study of CMC *spacelike* hypersurfaces is motivated by general relativity (see [DH17; GL22] for recent surveys on the topic). Indeed, using a CMC spacelike hypersurface as initial data for Einstein equations, the associated Cauchy problem greatly simplifies: for this reason, several authors focused on existence, uniqueness and non-existence problems for CMC spacelike hypersurfaces in Lorentzian manifolds (see for example [Cho76; Eck03; Bar21a; Bar21b]). Moreover, foliations by CMC spacelike hypersurfaces define natural time coordinates, useful to understand the global geometry of a Lorentzian manifold (see for example [MT80; Ren96; Ren97]).

The constant sectional curvature cases have been largely studied: for the flat case, *i.e.* the Minkowski space $\mathbb{R}^{n,1}$, see for example [CY76; Tre82; CT90; And+12; BSS19]; for the negatively curved space, *i.e.* the Anti-de Sitter space $\mathbb{H}^{n,1}$, see for example [BBZ07; And+12; Tam19a]. In geometric topology, the study of surfaces in $\mathbb{R}^{2,1}$ and $\mathbb{H}^{2,1}$ has become of great interest since the pioneering work of Mess [Mes07], mostly because of their relation with Teichmüller theory (see [BBZ07; BB09; BS10; Tam19a; Sep19; BS20]). In higher dimension, see [And+12; BM12].

To be more precise, CMC surfaces in $\mathbb{H}^{2,1}$ are linked to a special class of diffeomorphisms of the hyperbolic plane $\mathbb{H}^2$, called $\theta-$landslides, introduced in [BMS13] as smooth version of earthquakes. They generalize another important class of diffeomorphisms: a $(\pi/2)-$lanslide is in fact *minimal Lagrangian* map, namely an area-preserving diffeomorphism whose graph is minimal in $\mathbb{H}^2 \times \mathbb{H}^2$. Minimal Lagrangian maps have been widely studied ([Sch93; Lab92]). The analytical properties of $\theta-$landslides are encoded by the geometry of the corresponding CMC surfaces in the Anti-de Sitter space. In particular, minimal Lagrangian maps correspond to maximal surfaces: this connection have been exploited in several works ([AAW00; KS07; BS10; Tou16; Sep19]).

In higher co-dimension, CMC hypersurfaces generalize to *parallel* mean curvature spacelike $p-$submanifold. Further progress have been made on specific pseudo-Riemannian spaces, notably on the so-called *indefinite space-forms* of signature (p, q): the pseudo-Euclidean space $\mathbb{R}^{p,q}$, the pseudo-hyperbolic space $\mathbb{H}^{p,q}$ and pseudo-spherical space $\mathbb{S}^{p,q}$. See the works [Ish88; KKN91] for estimates on the geometry of parallel mean curvature spacelike $p-$submanifolds in this setting. Recently, maximal spacelike $p-$submanifolds in $\mathbb{H}^{p,q}$ have been studied in relation with higher higher Teichmüller theory (see [DGK18; CTT19; LTW20; LT23; SST23; BK23]).

Generalized Plateau's problem

The Anti-de Sitter space $\mathbb{H}^{n,1}$ is a Lorentzian manifold with constant sectional curvature -1. It is a pseudo-Riemannian symmetric space associated to $O(n,2)$ and it identifies with the space of oriented negative lines of a non-degenerate bilinear form of signature $(n,2)$. As for the hyperbolic space, it admits a conformal boundary $\partial\mathbb{H}^{n,1}$, consisting of oriented degenerate lines of such bilinear form, which is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

For a fixed real number H and a given subset Λ in the asymptotic boundary of $\mathbb{H}^{n,1}$, the asymptotic $H-$Plateau's problem in the Anti-de Sitter space consists in finding the spacelike hypersurfaces with constant mean curvature equal to H, whose asymptotic boundary coincides with Λ. A hypersurface is spacelike if the induced metric is Riemannian, which gives a constraint on the subsets Λ for which the asymptotic $H-$Plateau's problem can be *a priori* solved (Proposition 3.1.5): we call *admissible* a subset $\Lambda \subseteq \partial\mathbb{H}^{n,1}$ satisfying such constraint. For a more intrinsic characterization of admissible boundaries, see Definition 2.1.8. Our first achievement is to solve the asymptotic $H-$Plateau's problem in $\mathbb{H}^{n,1}$: in particular, we prove that the admissibility condition is not only necessary but also sufficient.

Theorem A. *For any $H \in \mathbb{R}$, any admissible boundary in $\partial\mathbb{H}^{n,1}$ bounds a unique spacelike hypersurface with constant mean curvature equal to H.*

Many partial results had been previously obtained. To our knowledge, the first progress has been made by Andersson, Barbot, Béguin and Zeghib [BBZ07; And+12], who solved the problem for Λ invariant by the action of a subgroup Γ of $\mathrm{Isom}(\mathbb{H}^{n,1})$ that acts freely and properly discountinuously on a properly embedded spacelike hypersurface S of $\mathbb{H}^{n,1}$ with S/Γ compact. Equivalently, these subgroups are known in the literature as *convex cocompact*. The result of existence and uniqueness has been extended in the 3-dimensional case $\mathbb{H}^{2,1}$ by Bonsante and Schlenker for the maximal case, *i.e.* $H = 0$ ([BS10]), and by Tamburelli for arbitrary values of H ([Tam19a]), to the class of quasi-symmetric boundary curves. Moreover, [BS10] proved the existence of maximal hypersurfaces in any dimension, for Λ the graph of a strictly $1-$Lipschitz map from $\mathbb{S}^{n-1}$ to $\mathbb{S}^1$. Recentely, Labourie, Toulisse and Wolf solved the asymptotic Plateau problem respectively for spacelike maximal surface in $\mathbb{H}^{2,n}$ (in particular, for the $3-$dimensional Anti-de Sitter space $\mathbb{H}^{2,1}$), for all admissible boundaries ([LTW20]). Seppi, Smith and Toulisse generalised the result to spacelike maximal $p-$submanifold in $\mathbb{H}^{p,q}$ ([SST23]).

It is important to remark that all the above uniqueness results hold in the class of *complete* submanifolds. This is a highly non-trivial requirement in Lorentzian geometry. Indeed, a properly embedded hypersurface in $\mathbb{H}^{n,1}$ might not be complete: if the normal vector degenerates fastly enough, the metric is incomplete. The same phenomenon occurs in $\mathbb{R}^{n,1}$ (see also [BSS22]).

The asymptotic $H-$Plateau problem has been studied in the flat case, *i.e.* in the Minkowski space. In particular, the maximal case has been solved in [CY76], proving that the only properly embedded maximal hypersurfaces in $\mathbb{R}^{n,1}$ are totally geodesic spacelike hypersurfaces, namely affine hyperplane. The case $H \neq 0$ is more delicate: several solutions have been studied ([Tre82; CT90]), until the quite recent classification ([BSS19]). It is worth noticing that, in this setting, the CMC hypersurfaces are proved to be convex ([Tre82]). This does not hold true in the Anti-de Sitter setting: not only for $H = 0$, where a maximal hypersurface is convex if and only if it is totally geodesic, but there are examples of non-convex CMC hypersurfaces also for $H \neq 0$ (see Remark 9.1.1 and Remark 8.1.5).

For $\mathbb{R}^{n,1}$, the uniqueness result holds in the class of properly embedded hypersurfaces, due to the remarkable result [CY76, Corollary of Theorem 1], which states that any properly embedded CMC hypersurface in Minkowski space is complete. We extend this result to the Anti-de Sitter case:

Theorem B. *Any properly embedded spacelike hypersurface with constant mean curvature in $\mathbb{H}^{n,1}$ is complete.*

When $n = 2$ and $H = 0$, the same result have been proved, with different techniques, by [LM19].

In the literature, many results about CMC hypersurfaces in Anti-de Sitter space assume completeness: for example, the aforementioned uniqueness result for maximal hypersurfaces ([BS10; LTW20; SST23]). In Chapter 8, we focus on the estimates on the second fundamental form studied in [Ish88; KKN91].

CMC time function

A *time function* on a Lorentzian manifold (M, g) is a submersion $\tau\colon M \to \mathbb{R}$ which is strictly monotone along timelike path (Definition 3.4.2). Not all Lorentzian manifolds admit time functions: indeed, this is equivalent to being *globally hyperbolic* (Definition 3.2.3). A time functions induces a foliation of M, whose leaves are *Cauchy hypersurfaces*, which allows to study more easily the manifold we are considering.

We study the behaviour of CMC hypersufaces with the same asymptotic boundary Λ, as their mean curvature H varies in $\mathbb{R}$. We prove that they are the level sets of a time function on the *invisible domain* $\Omega(\Lambda)$ (Definition 3.1.1). We recall that $\Omega(\Lambda)$ is the union of entire spacelike hypersurfaces spanning Λ, *i.e* $x \in \Omega(\Lambda)$ if there exists an entire spacelike hypersurface S such that $x \in S$ and $\partial S = \Lambda$. It is known that $\Omega(\Lambda)$ is a globally hyperbolic geodesically convex open subset of $\mathbb{H}^{n,1}$.

Theorem C. *The invisible domain $\Omega(\Lambda)$ of an admissible boundary Λ in $\partial\mathbb{H}^{n,1}$ is real-analytically foliated by complete CMC spacelike hypersurfaces spanning Λ.*

If the boundary Λ is equivariant by the action of a convex cocompact subgroup of $\mathrm{Isom}(\mathbb{H}^{n,1})$, in the quotient we recover the same foliation described in [BBZ07; And+12], since the uniqueness of the CMC hypersurfaces guarantees their invariance by the action of any group preserving the boundary. If $n = 2$ and Λ is a quasi-symmetric boundary, we recover the foliation of [Tam19a], and we improve its regularity.

In the language of [BBZ07; And+12] (see Definition 9.5.1), any maximal globally hyperbolic Cauchy compact Anti-de Sitter manifold admits a unique CMC time function, which is real-analytic. A time function is a *CMC* time function if $\tau^{-1}(H)$ is an $H-$hypersurface, for any $H \in \tau(M)$. Using the same language, Theorem C can be rephrased as follows:

Corollary D. *Any maximal globally hyperbolic Cauchy complete Anti-de Sitter manifold admits a unique CMC-time function, which is real-analytic.*

Cosmological time

A Lorentzian manifold (M, g) is said *time orientable* if the set of timelike vectors, namely the maximal subset of TM where g is negative definite, consits of two connected component. A *time orientation* is the choice of one connected component, and the tangent vectors belonging to it are called *future-directed*. The other component contains *past-directed* timelike vectors. A curve $c\colon I \to M$ is future-directed (resp. past-directed) if $c'(t)$ is future-directed (resp. past-directed) for all $t \in I$. Clearly, this definition depends on the parameterization of c.

Let p be a point in M, the *past* of p, denoted by $I^-(p)$ is the set of endpoints of past-directed curves c starting from p. For $q \in I^-(p)$, the distance $\mathrm{dist}(q,p)$ is the *supremum* of the length of the timelike curves joining p and q. A *cosmological* time function on a Lorentzian manifold (M, g) is defined as

$$\tau(p) := \sup_{q \in I^-(p)} \mathrm{dist}(q, p).$$

Clearly, not all Lorentzian manifolds admit a cosmological time function with a nice behaviour: for example, the cosmological time functions of Anti-de Sitter space and Minkowski space are both the constant function $\tau \equiv +\infty$.

A cosmological time function is called *regular* if takes value in $\mathbb{R}$ and $\tau \to 0$ along any inextensible past-directed causal path (see [AGH98] for details). From a physical point of view, the universe modeled by (M, g) has been in existence for a finite time and the initial singularity coincides with the limit of inextensible past-directed causal curves.

An example of nice cosmological functions are $\tau_{\mathbf{F}}$ and $\tau_{\mathbf{P}}$, introduced in Section 3.4. In Section 10.2, we show that they are a particular case of a more general construction. Indeed, there exists a duality between future-convex and past-convex hypersurfaces (Definition 11.1.6), and we prove that the open convex domain bounded by such pair admits a nice cosmological function (Proposition 10.2.1). Then, we study the geometric properties of the leaves of the induced foliation. A consequence of this investigation is that any maximal globally hyperbolic Cauchy complete Anti-de Sitter manifold admits a Hadamard Cauchy hypersurface (in fact, infinitely many) (Proposition 10.3.3).

In Section 10.4, we specialize to the $3-$dimensional Anti-de Sitter space. The set $\Omega(\Lambda) \setminus \mathcal{CH}(\Lambda)$ consists of two connected component

$$\mathcal{D}_+(\Lambda) = I^-\left(\partial_+\Omega(\Lambda)\right) \cap I^+\left(\partial_-\mathcal{CH}(\Lambda)\right) \qquad \mathcal{D}_-(\Lambda) = I^-\left(\partial_+\mathcal{CH}(\Lambda)\right) \cap I^+\left(\partial_-\Omega(\Lambda)\right).$$

As a consequence of Theorem C and Proposition 10.2.1, we give a complete classification of *constant sectional curvature* (CSC) spacelike surfaces in $\mathbb{H}^{2,1}$.

Theorem E. *Let $\Lambda \subseteq \widetilde{\mathbb{H}}^{2,1}$ be an admissible boundary. For any $K \in (-\infty, 1)$ there exists a unique past-convex (resp. future-convex) achronal surface S^+_K (resp. S^-_K) such that*

- *$\partial S^\pm_K = \Lambda$;*
- *its lightlike part is union of lightlike triangles associated to sawteeth;*
- *its spacelike part is an analytic $K-$surface.*

Moreover, $(S^\pm_K)_{K\in(-\infty,-1)}$ is a real-analitical foliation of $\mathcal{D}_\pm(\Lambda)$.

This result generalizes [BS18, Proposition 9.3], where the uniqueness part was known only for Λ quasi-symmetric (Definition 17.1.1), and improves the regularity of the surfaces.

It is known that $\Omega(\Lambda)\setminus\mathcal{CH}(\Lambda)$ is topologically foliated by CSC spacelike surfaces ([BS18, Theorem 7.8]). We improve the regularity of this foliation, showing that it is real-analytical (Corollary 10.4.3).

$H-$convexity

Theorem A states that properly embedded CMC spacelike hypersurfaces in Anti-de Sitter space are uniquely determined by the value $H \in \mathbb{R}$ of their mean curvature and their admissible asymptotic boundary Λ: we want to quantify this dependence, *i.e.* to extract geometric estimates from objects that only depend on the data (H, Λ).

We introduce the notion of $H-$convexity (Definition 11.1.1), generalizing the usual notion of convexity. The *$H-$shifted convex hull* of Λ is the smallest $H-$convex subset of $\mathbb{H}^{n,1}$ containing Λ, denoted by $\mathcal{CH}_H(\Lambda)$.

By the maximum principle, the CMC hypersurface Σ determined by the data (H, Λ) is contained in $\mathcal{CH}_H(\Lambda)$ (Corollary 11.2.7). The hypersurface Σ is trapped in such portion of space: we prove that the extrinsic curvature of Σ is controlled by the *width* $\omega_H(\Lambda)$ of $\mathcal{CH}_H(\Lambda)$, namely its timelike diameter.

Theorem F. *Let $L \geq K \geq 0$. There exists a universal constant C_L with the following property: let Σ a properly embedded $H-$hypersurface in $\mathbb{H}^{n,1}$ with $H \in [K, L]$, and let B_0 be its traceless shape operator. Then,*

$$\|B_0\|_{C^0(\Sigma)} \leq C_L \sin\left(\omega_K(\partial\Sigma)\right),$$

for ω_K the width of the $K-$shifted convex hull of $\partial\Sigma$.

This result generalizes [Sep19, Theorem 1.A], which focuses on maximal surfaces in $\mathbb{H}^{2,1}$, to any dimension and any values of mean curvature $H \in \mathbb{R}$.

A first application of Theorem F is to produce a class of uniformly negatively curved CMC hypersurfaces. This result is of particular interest in higher dimension: while CMC surfaces in $\mathbb{H}^{2,1}$ are known to be negatively curved, nothing is known so far about the sectional curvature of CMC hypersurfaces in $\mathbb{H}^{n,1}$, for $n > 2$, to the best of our knowledge.

Corollary G. *For any $H \in \mathbb{R}$, there exists a universal constant $K_H > 0$ such that, for any properly embedded $H-$hypersurface Σ, it holds*

$$K_\Sigma \leq -1 - \left(\frac{H}{n}\right)^2 + K_H \sin\left(\omega_H(\partial\Sigma)\right).$$

Extending [Sep19, Proposition 1.C], we prove also the inequality in the opposite direction of Theorem F. Indeed, the width of the $H-$shifted convex hull of Λ cannot explode if the extrinsic curvature of the $H-$hypersurface Σ is small.

Proposition H. *Let Λ be an admissible boundary in $\partial\mathbb{H}^{n,1}$ and $H \in \mathbb{R}$. Let Σ the unique properly embedded spacelike $H-$hypersurface such that $\partial\Sigma = \Lambda$. Then*

$$\omega_H(\Lambda) \leq \arctan\left(\sup_\Sigma \lambda_1\right) - \arctan\left(\inf_\Sigma \lambda_n\right),$$

for $\lambda_1 \geq \cdots \geq \lambda_n$ be the principal curvatures of Σ, decreasingly ordered.

When the traceless shape operator is small enough, which is always the case for $\omega_H(\partial\Sigma)$ small enough, thanks to Theorem F, the inequality of Proposition H can be written in a more expressive way.

Corollary I. *Let Λ be an admissible boundary and $H \in \mathbb{R}$. Let B_0 be the traceless shape operator of the properly embedded spacelike $H-$hypersurface such that $\partial\Sigma = \Lambda$. If $\|B_0\|^2_{C^0(\Sigma)} \leq 1 + (H/n)^2$, the width of $\mathcal{CH}_H(\partial\Sigma)$ satisfies*

$$\tan\left(\omega_H(\Lambda)\right) \leq \frac{2\|B_0\|_{C^0(\Sigma)}}{1 + (H/n)^2 - \|B_0\|^2_{C^0(\Sigma)}}.$$

By Gauss equation, Corollary I translates in terms of sectional curvature, in the 3−dimensional case.

Lemma J. *Let Λ be an admissible boundary in $\partial\mathbb{H}^{2,1}$ and $H \in \mathbb{R}$. Let B_0 be the traceless shape operator of the properly embedded spacelike $H-$hypersurface such that $\partial\Sigma = \Lambda$. Then,*

$$\tan\left(\omega_H(\Lambda)\right) \leq \frac{2\|B_0\|_{C^0(\Sigma)}}{-\sup_\Sigma K_\Sigma}.$$

Universal Teichmüller theory

The *universal Teichmüller space* is the space of *quasi-symmetric* homeomorphism of $\mathbb{S}^1$ (Definition 17.1.1), up to post composition by $\mathbb{P}\mathrm{SL}(2, \mathbb{R})$. The well known Ahlfors-Beuring theorem connects quasi-symmetric homeomorphisms of the circle with quasiconformal map of the disc (Definition 17.1.3):

Theorem ([BA56])**.** *Every quasiconformal map $\Phi\colon \mathbb{D}^2 \to \mathbb{D}^2$ extends to a unique quasi-symmetric map $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$. Conversely, any quasi-symmetric map $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ admits a quasiconformal extension $\Phi\colon \mathbb{D}^2 \to \mathbb{D}^2$.*

The quasiconformal extension is far from being unique, and it is a classical topic in Teichmüller theory to construct a suitable class of quasiconformal extensions to study the universal Teichmüller space. A classical problem consists of the comparison between the cross-ratio norm of the quasi symmetric map ϕ and the *quasiconformal dilatation* (see Equation (17.1)) of the quasiconformal map extension $\Phi\colon \mathbb{H}^2 \to \mathbb{H}^2$ in the preferred class. Classical results in this direction are the estimates contained in [BA56; Leh83; DE86; HM12].

Anti-de Sitter geometry has played an important role in Teichmüller theory since the groudbreaking work of Mess [Mes07]: indeed, spacelike surfaces in $\mathbb{H}^{2,1}$ induce diffeomorphisms of the hyperbolic plane $\mathbb{H}^2$, through the so called *Gauss map* (see Section 15.2). It turns out that *minimal Lagrangian* diffeomorphisms, which have been widely studied (see for example [Sch93; Lab92]), are induced by maximal surfaces: this correspondence has been exploited in [KS07; BS10; Tou16; Sep19] to study such diffeomorphisms using Anti-de Sitter geometry.

Minimal Lagrangian diffeomorphisms are a particular case of $\theta-$landslides (Definition 16.2.1), for $\theta = \pi/2$. $\theta-$landslides have been introduced in [BMS13] as smooth versions of earthquakes: if ϕ is a quasi-symmetric map, then the $\theta-$landslides Φ_θ extending ϕ conjugate the left and the right earthquake extending ϕ, as θ varies in $(0, \pi)$. The diffeomorphism induced by a CMC surface Σ in $\mathbb{H}^{2,1}$ is a $\theta-$landslide, for θ depending on the mean curvature of Σ.

In [Sep19, Theorem 2.A, Theorem 2.B and Corollary 2.D], the quasiconformal dilatation of a quasi-conformal minimal Lagrangian map $\Phi_{\pi/2}$ is bounded by the width of the convex hull of the admissible boundary of the maximal surface corresponding to $\Phi_{\pi/2}$ given by the graph of $\Phi_{\pi/2}|_{\partial\mathbb{H}^2}$. An application of Theorem F is to extend [Sep19, Theorem 2.A] to Φ_θ, for any θ.

Theorem K. *For any $\alpha \in (0, \pi/2)$, there exists universal constants $Q_\alpha, \eta_\alpha > 0$ such that for all $\theta \in [\alpha, \pi - \alpha]$ and ϕ quasi-symmetric map satifing $\|\phi\|_{cr} \leq \eta_\alpha$, then*

$$\ln(K(\Phi_\theta)) \leq Q_\alpha \|\phi\|_{cr},$$

for Φ_θ the unique $\theta-$landslide extending ϕ.

Extensions of circle homeomorphisms

A remarkable result, proved in [BS10, Theorem 1.4] for the minimal Lagrangian case, and in [BS18, Corollary 1.5] in full generality, states that any quasi-symmetric homeomorphism admits a unique $\theta-$landslide extension, for any $\theta \in (0,\pi)$. Moreover, such extension is quasiconformal: hence, the space of quasiconformal $\theta-$landslide maps, for a fixed θ, is a model for the universal Teichmüller space.

We extend this result, removing the quasi-symmetric condition:

Theorem L. *Let $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation preserving homeomorphism. For any $\theta \in (0,\pi)$, there exists a unique $\theta-$landslide $\Phi_\theta\colon \mathbb{H}^2 \to \mathbb{H}^2$ extending ϕ. Moreover, Φ_θ is quasiconformal if and only if ϕ is quasi-symmetric.*

Quasi-spheres

The width ω_H of the $H-$shifted convex hull is bounded as a function on the set of admissible boundaries in $\mathbb{H}^{n,1}$ (Corollary 11.3.2). We characterize the admissible boundaries achieving the maximum of ω_H.

First, we prove a rigidity-like result: if Λ maximizes ω_H for a certain value of H, then it maximizes ω_K, for any $K \in \mathbb{R}$ (Corollary 18.1.5). Hence, the classification reduces to studying the admissible boundaries such that $\omega_0(\Lambda) = \pi/2$.

For $n = 2$, this has already been done in [BS10, Theorem 1.12]: indeed, the space of quasi-symmetric homeomorphisms of the circle (Definition 17.1.1), is identified with the space of admissible boundaries of the 3−dimensional Anti-de Sitter space whose width is strictly less $\pi/2$ (Proposition 15.1.3). Motivated by this result, we call *quasi-sphere* an admissible boundary $\Lambda \subseteq \partial\mathbb{H}^{n,1}$ such that $\omega_0(\Lambda) < \pi/2$.

A *Barbot crown* is defined as the limit set of a Cartan subgroup of the isometry group of the Anti-de Sitter space (Definition 18.2.1). Quasi-symmetric boundaries are the ones whose closure of the orbit by the action of $\mathrm{Isom}_0(\mathbb{H}^{2,1})$ contains no Barbot crown ([BS10, Claim 3.23]).

This dynamical characterization has been used in the recent work [LT23], in the more general context of the pseudo-hyperbolic space $\mathbb{H}^{2,n}$, which is a further generalization of hyperbolic geometry in the pseudo-Riemannian realm, in order to generalize the notion of quasi-symmetric boundary in higher codimension. In fact, a *quasiperiodic loop* is an admissible boundary containing no Barbot crowns in the closure of its orbit by the action of $\mathrm{Isom}(\mathbb{H}^{2,n})$ (see [LT23, Proposition 2.34]).

We show that the dynamical point of view is encoded by the width also in higher dimension.

Theorem M. *Let Λ be an admissible boundary in $\partial\mathbb{H}^{n,1}$. Then Λ is not a quasi-sphere if and only if there exists $\Lambda' \in \overline{G\cdot\Lambda}$ and a totally geodesic copy of $\mathbb{H}^{2,1}$ such that $\Lambda' \cap \partial\mathbb{H}^{2,1}$ is a Barbot crown.*

In both [BS10; LT23], quasi-spheres are characterized also by the fact that they bounds uniformly negatively curved maximal surfaces. We partially generalize this result: indeed, Corollary G shows that quasi-spheres with small width bound uniformly negatively curved maximal hypersurfaces.

Higher higher Teichmüller theory

Let M be a topological manifold of dimension $n \geq 2$ and G a semi-simple Lie group of rank $G \geq 2$. Higher(-dimensional) higher(-rank) Teichmüller theory is the study of connected components of the character variety of $\mathrm{Hom}\,(\pi_1(M), G)$ consisting entirely of discrete and faithful representations (see [Wie18] for a survey on the topic).

The Anti-de Sitter case has been studied in [BM12; Bar15], which proves that the holonomies of maximal globally hyperbolic Cauchy compact $\mathbb{A}\mathrm{d}\mathbb{S}-$manifolds consist of connected components. The general case of the pseudo-hyperbolic space $\mathbb{H}^{p,q}$ has been recently studied in [BK23].

Quasi-periodic loops are also an attempt to define the *universal* higher Teichmüller space for $G = \mathbb{P}\mathrm{O}(2, n)$. The definition of quasi-spheres seems a good attempt to definine the *universal* higher higher Teichmüller space of $\mathbb{P}\mathrm{O}(n, 2)$, as well. Indeed, Theorem M allows to distinguish convex cocompact subgroups of $\mathbb{P}\mathrm{O}(n, 2)$ from $\mathbb{H}^{n,1}-$convex cocompact subgroups of $\mathbb{P}\mathrm{O}(n, 2)$ (Definition 18.3.1), using the width of the $H-$shifted convex core.

Corollary N. *Let Γ be a discrete subset of $\mathrm{Isom}(\mathbb{H}^{n,1})$ acting cocompatly on a closed convex subset of $\mathbb{H}^{n,1}$ with non-empty interior. Then Γ is $\mathbb{H}^{n,1}-$convex cocompact if and only if its limit set is a quasi-sphere.*

Main ingredients

As previously mentioned, completeness is a highly non-trivial condition in the Lorentzian setting. Specifically, in the case of CMC hypersurfaces in Anti-de Sitter space, it has been proved to be equivalent to having bounded second fundamental form. Indeed, any properly embedded spacelike hypersurface with bounded second fundamental form is complete, as proved in [BB09, Proposition 6.3.9], [LTW20, Corollary 3.30] and [SST23, Lemma 3.11] (Lemma 5.3.2). Conversely, the main result of [Ish88; KKN91] provides a universal bound on the second fundamental form of any properly embedded complete CMC hypersurface in $\mathbb{H}^{n,1}$, only depending on the mean curvature H. See Chapter 8 for details.

Bound on the second fundamental form. Through geometric arguments of a global nature, relying on the local estimates contained in [Bar21b; Eck03], we provide uniform bounds on the norm of the second fundamental form and its derivatives (Theorem 5.2.1). More specifically, we specialize these estimates to the Anti-de Sitter space, proving that, for CMC properly embedded hypersurfaces, there exists a bound only depending on the distance between the hypersurface and the boundary of its domain of dependence. Right after, we prove that such distance is uniformly bounded from below, *i.e.* the bound is uniform. Our bounds are not explicit but, *a posteriori*, we retrieve the bounds of [Ish88; KKN91], by applying Theorem B, which is in fact a corollary of Theorem 5.2.1.

Uniqueness. In the context of Anti-de Sitter and, more generally, pseudo-hyperbolic geometry, all the previous results of uniqueness of maximal and CMC submanifolds ([BS10; Tam19a; LTW20; SST23]) are proved in the class of *complete* submanifolds. The proofs rely on an application of the Omori-Yau maximum principle, which indeed requires completeness. Hence, *a priori*, there could exist several properly embedded $H-$hypersurfaces sharing the same boundary, among which only one complete. In light of Theorem B, this is not possible, since there exists no incomplete properly embedded CMC hypersurface.

Here, instead of using Omori-Yau maximum principle, we prove uniqueness by a different method: we prove a maximum principles describing the mutual position of spacelike hypersurfaces, depending on their boundaries and their mean curvature (Theorem O). Roughly speaking, we show that the bigger the mean curvature, the more curved in the past is the hypersurface. More precisely, we prove the following result, stated here in a slightly weaker version:

Theorem O. *Let Σ_1 and Σ_2 be two properly embedded hypersurfaces such that $\partial\Sigma_1$ lies in the past of $\partial\Sigma_2$. If $H_1 \geq H_2$, then Σ_1 lies in the past of Σ_2.*

The proof consists in using isometries and topological arguments to ensure that the maximum of the Lorentzian distance between Σ_1 and Σ_2 is reached in the interior of $\Sigma_1 \times \Sigma_2$. In this case, we can apply a classical maximum principle, which does not require the completeness assumption.

This method has the advantage of, on the one hand, providing barriers (Proposition 4.2.1) which are needed in the proof of the existence of CMC hypersurfaces. On the other hand, the result applies to a more general context, and could be useful in the study of hypersurfaces with prescribed mean curvature in the Anti-de Sitter space.

Existence and compactness. An important ingredient is to show that the limit of a suitable sequence of spacelike hypersurfaces with constant mean curvature is a properly embedded spacelike CMC hypersurface. This result is achieved by the means of *barriers*, *i.e.* hypersurfaces that uniformly bound the geometry of elements of the sequence, whose existence is a consequence of Theorem O.

In particular, for the existence part of Theorem A, we use the leaves of the cosmological time as barriers, similarly to [And+12]. However, since we deal with non-compact hypersurfaces, hence [Ger83, Theorem 5.2] does not apply in our setting. Then, we obtain the entire solution Σ as the limit of compact solutions Σ_k, whose existence is guaranteed by a result of [Eck03].

Incidentally, the compactness result (Proposition 7.0.1), allows us to describe the topology of the moduli space of CMC entire hypersurfaces of $\mathbb{H}^{n,1}$ (Corollary 7.1.1).

CMC foliation. The invisible domain of Λ is topologically foliated by CMC hypersurfaces as a consequence of two results: the compactness result and the strong maximum principle (Proposition 4.2.1), which is a special case of Theorem O.

To promote the regularity of the foliation, we describe the space of spacelike deformations of a single leaf Σ_H as an open subset A of the Banach space of regular functions on Σ_H. Indeed, a function $v\colon \Sigma_H \to \mathbb{R}$ identifies with its normal graph over Σ_H, namely

$$S_v = \{\exp_x (v(x)N(x)), x \in \Sigma_H\}.$$

We prove that the mean curvature, seen as a differential operator on A, is analytically invertible at Σ_H. As a consequence, we deduce that the foliation of CMC hypersurfaces is analytic.

Cosmological functions. The leaves of the cosmological time coincide with the equidistant surfaces. If a surface is convex, these surfaces are regular within a certain range.

For regular surfaces, equidistant surfaces are the level sets of the normal flow: the main results of Section 10.2 follows by the combination of the two points of view.

Constant sectional curvature surfaces. In the 3−dimensional Anti-de Sitter space, CMC surfaces and CSC surface are equidistant (Lemma 10.4.2). Then, Theorem E follows by Theorem A and Corollary 10.4.3 by Theorem C.

Extensions of circle homeomorphisms. An explicit inverse of the Gauss map is built in [BS18, Section 5]: a suitable diffeomorphism of the hyperbolic plane induce a spacelike immersion in the Anti-de Sitter space. Let ϕ be a circle homeomorphism, then $\Lambda := \operatorname{graph}\phi$ is an acausal boundary. The strategy to prove Theorem L is inspired by [BS10; BS18]: the CSC surface S_K^- asymptotic to Λ induces a $\theta-$landslide Φ_θ which extends to ϕ by [BS10, Lemma 3.18] (Lemma 16.1.8). For the uniqueness, consider a $\theta-$landslide Ψ_θ extending ϕ: we recover a CSC surface S by inverting the Gauss map of Ψ_θ: if the boundary of the resulting CSC surface S coincides with Λ, then Theorem E implies $S = S_K^-$, hence $\Psi_\theta = \Phi_\theta$.

However, in [BS10; BS18], the quasi-symmetry of ϕ plays a huge role in the uniqueness part of the proof. Indeed, the quasiconformality of the extension Ψ_θ ensures that the corresponding CSC surface S has bounded principal curvature (or equivalently the norm of traceless shape operator of the corresponding CMC surface is strictly less than 2). This implies that ∂S is a quasisphere: hence, by [BS10, Lemma 3.18], the boundary curve ∂S coincides with the graph of ϕ, that is $\partial S = \Lambda$.

The general case is more delicate: to understand the asymptotic behaviour of the CSC surface induced by a $\theta-$landslide, we study the closure of the image of the Gauss map. We extend the definition of the Gauss map to convex acausal surfaces (Definition 16.1.1), in order to use the machinery developed in Section 10.4. This different approach leads to a generalization of [BS10, Lemma 3.18], in a slightly different flavour.

Corollary P. *Let Λ be an admissible boundary, $K \in (-\infty, -1)$. The diffeomorphism induced by the Gauss map of the CSC surface $S_K^\pm(\Lambda)$ extends to a homeomorphism of $\mathbb{S}^1$ if and only if Λ contains no lightlike segments.*

For technical reason, we need to restrain the statement to CSC surfaces, although we believe that our method could be extended to the wider class of convex acausal surfaces.

Once the asymptotic behaviour of the CSC surface is well understood, the classification of CSC surfaces (Theorem E) provides a classification result for $\theta-$landslides, concluding the proof of Theorem L.

Quasiconformal dilatation. The quasiconformal dilatation of a $\theta-$landslide Φ is bounded from above by an explicit function of the norm of traceless shape operator of the corresponding CMC surface Σ, as enlightened in [Tam19a]. Then, the width of the $H-$shifted convex hull of $\partial\Sigma$ controls the quasiconformal dilatation, as a consequence of Theorem F. Combining Lemma 11.3.3 and [Sep19, Proposition 3.A] (stated here as Lemma 15.1.4), we bound the width with a function of the cross-ratio of the extension of Φ to the boundary of $\mathbb{H}^2$, concluding the proof of Theorem K.

Quasi-spheres The key results of this part are Proposition 18.1.1 and Corollary 18.1.5. The former states that the width of the $H-$shifted convex hull ω_H is a lower semicontinuous as a function over the space of admissible boundaries, while the latter is a rigidity statement: quasi-spheres are characterized as maximizers of ω_H, independently on the choice of H. This allows to study quasi-spheres looking at their convex hull, which leads to the proof of Theorem M.

Nonethless, the information carried by the $H-$shifted convex hull, for $H \neq 0$, is not irrelevant: it can be crucial to distinguish orbits of quasi-spheres (Remark 18.2.3).

Part I.

Preliminaries

Chapter 1.

Anti de-Sitter space

Anti-de Sitter manifolds are the Lorentzian analogous of hyperbolic manifolds, *i.e.* pseudo-Riemannian manifolds with signature $(n, 1)$ and constant sectional curvature -1. In this section, three models of Anti-de Sitter geometry are presented, namely the quadric model $\mathbb{H}^{n,1}$, the universal cover $\widetilde{\mathbb{H}}^{n,1}$ and the Klein (or projective) model $\mathbb{P}\left(\mathbb{H}^{n,1}\right)$.

1.1. Lorentzian geometry

A Lorentzian metric g on a manifold M of dimension $n+1$ is a symmetric non-degenerate $(0,2)-$tensor with signature $(n, 1)$. We distinguish tangent vectors by their causal properties: a vector v is *timelike*, *lightlike* or *spacelike* if $g(v, v)$ is respectively negative, null or positive. Furthermore, a vector is called *causal* if it is not spacelike. A curve $c\colon I \to M$ is called timelike (resp. lightlike, spacelike, causal), if its tangent vector $c'(t)$ is timelike (resp. lightlike, spacelike, causal) for all $t \in I$.

Definition 1.1.1. Two points are *time-related* (resp. *light-related*, *space-related*) if there exists a timelike (resp. lightlike, spacelike) geodesic joining them.

Definition 1.1.2. A C^1-submanifold of M is *spacelike* if the induced metric is Riemannian.

A Lorentzian manifold is *time-orientable* if the set of timelike vectors in TM has two connected components, which will always be our case. A time-orientation is the choice of one of the connected component: vectors are called *future-directed* if they belong to the chosen connected component, *past-directed* otherwise.

Definition 1.1.3. The *cone* of a subset X of M is the set $I(X)$ of points of M that can be joined to X by a timelike curve.

Once a time-orientation is set, one can distinguish the *future* cone $I^+(X)$ and the *past* cone $I^-(X)$, containing respectively the points that can be reached from X along a future-directed or past-directed timelike curve.

1.2. Quadric model

The quadric model for Anti-de Sitter geometry is the equivalent of the hyperboloid model of hyperbolic space, *i.e.*

$$\mathbb{H}^{n,1} := \{x \in \mathbb{R}^{n,2}, \langle x, x\rangle = -1\},$$

$\mathbb{R}^{n,2}$ being the pseudo-Euclidean space of signature $(n, 2)$, namely $\mathbb{R}^{n+2}$ endowed with the bilinear form

$$\langle x, y\rangle := x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1} - x_{n+2}y_{n+2}.$$

As for the hyperbolic space, $\mathbb{H}^{n,1}$ admits a conformal boundary, which consists of the oriented isotropic lines for the bilinear form $\langle \cdot, \cdot \rangle$, and it is conformal to $\partial\mathbb{H}^n \times \mathbb{S}^1$, endowed with the Lorentzian metric $g_{\mathbb{S}^{n-1}} - g_{\mathbb{S}^1}$. One can prove it directly or see it as a consequence of Lemma 1.3.3.

The tangent space $T_x\mathbb{H}^{n,1}$ identifies with $x^\perp = \{y \in \mathbb{R}^{n,2}, \langle x, y \rangle = 0\}$ and the restriction of the scalar product to $T\mathbb{H}^{n,1}$ is a time-orientable Lorentzian metric with constant sectional curvature -1.

One can show that $\mathrm{O}(n, 2)$ is the isometry group of $\mathbb{H}^{n,1}$, and that it is *maximal*: any linear isometry from $T_x\mathbb{H}^{n,1}$ to $T_y\mathbb{H}^{n,1}$ is the tangent map of an isometry of $\mathbb{H}^{n,1}$ sending x to y. In particular, the isometry group acts transitively. Totally geodesic $k-$submanifolds of $\mathbb{H}^{n,1}$ are precisely open subsets of the intersection between $\mathbb{H}^{n,1}$ and $(k+1)-$vector subspaces of $\mathbb{R}^{n,2}$. In the following, we will abusively refer to *maximal* totally geodesic submanifold simply as totally geodesic submanifold, where maximality is to be intended in the sense of inclusion for connected submanifolds.

In particular, geodesics of $\mathbb{H}^{n,1}$ starting from x are of the form

$$\exp_x(tv) = \begin{cases} \cos(t)x + \sin(t)v & \text{if } \langle v, v \rangle = -1; \\ x + tv & \text{if } \langle v, v \rangle = 0; \\ \cosh(t)x + \sinh(t)v & \text{if } \langle v, v \rangle = 1. \end{cases} \tag{1.1}$$

Indeed, any curve $\gamma(t)$ satisfying one of the equations above is contained in $\mathbb{H}^{n,1}$, and an easy computation shows that $\gamma''(t)$, namely the covariant derivative of γ with respect to the flat metric of $\mathbb{R}^{n,2}$, is proportional to $\gamma(t)$, *i.e.* normal to $T_{\gamma(t)}\mathbb{H}^{n,1}$.

Remark 1.2.1. For $x \in \mathbb{H}^{n,1}$, consider $T_x\mathbb{H}^{n,1}$ as the linear subspace $x^\perp \subseteq \mathbb{R}^{n,2}$. By definition, $\mathbb{H}^{n,1} \cap x^\perp$ is the set of unitary timelike vectors in $T_x\mathbb{H}^{n,1}$. By the above discussion, $\mathbb{H}^{n,1} \cap x^\perp$ is a totally geodesic hypersurface, and a direct calculation shows that it is isometric to two disjoint copies of the hyperbolic space $\mathbb{H}^n$. We will denote $P_+(x)$ and $P_-(x)$ these copies of $\mathbb{H}^n$: $P_+(x)$ is the set of *future-directed* unitary timelike vectors and $P_-(x)$ is the set of *past-directed* unitary timelike vectors in $T_x\mathbb{H}^{n,1}$, once a time-orientation is set.

Equation (1.1) shows that timelike geodesics starting at x are periodic curves $\gamma\colon \mathbb{R} \to \mathbb{H}^{n,1}$ such that $\gamma(k\pi) = (-1)^k x$. Moreover, as

$$\exp_x\left(\pm\frac{\pi}{2}v\right) = \pm v,$$

future-directed timelike geodesics intersect orthogonally $P_+(x)$ at $t = \pi/2$ and $P_-(x)$ at $t = -\pi/2$ (see Figure 1.1).

For the purposes of this thesis, the main advantage of this model is that its causal structure is encoded by the scalar product of $\mathbb{R}^{n,2}$.

Lemma 1.2.2. *Let $x, y \in \mathbb{H}^{n,1}$, $x \neq \pm y$. There exists a geodesic joining x and y if and only if $\langle x, y \rangle < 1$. Moreover, x, y are*

1. *time-related* $\iff -1 < \langle x, y \rangle < 1$;
2. *light-related* $\iff \langle x, y \rangle = -1$;
3. *space-related* $\iff \langle x, y \rangle < -1$.

Finally, $\langle x, y \rangle = 1$ (resp. $\langle x, y \rangle > 1$) if and only if $(x, -y)$ and $(-x, y)$ are light-related (resp. space-related).

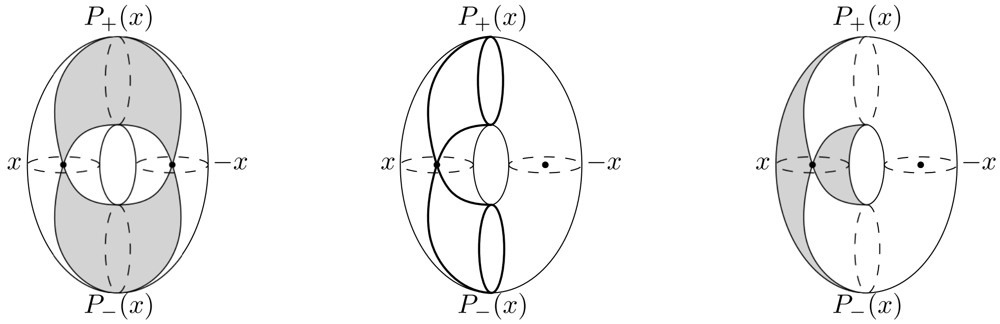

Figure 1.1.: From the left to the right, the set of points time-related, light-related and space-related to x.

The proof follows directly from Equation (1.1). See Figure 1.1 to visualize the set of time-related (resp. light-related, space-related) points to a point x.

Remark 1.2.3. Denote $C(-x)$ the cone of points light-related to $-x$. Lemma 1.2.2 implies the connected component of $\mathbb{H}^{n,1} \setminus C(-x)$ containing x corresponds to the set of points satisfying $\langle x, \cdot \rangle < 1$.

1.3. The universal cover

Let P be a totally geodesic spacelike hypersurface of $\mathbb{H}^{n,1}$, and $p \in P$. Denote N the unitary future-directed normal vector to P at p, and define the map

$$\begin{aligned} \psi_{(p,P)} \colon P \times \mathbb{R} &\longrightarrow \mathbb{H}^{n,1} \\ (x,t) &\longmapsto R_t(x), \end{aligned} \tag{1.2}$$

where R_t is the linear map which is a rotation of angle t restricted to $\mathrm{Span}(p, N)$ and fixes its orthogonal complement $\mathrm{Span}(p, N)^\perp$.

Example 1.3.1. If we choose $p = (0, \dots, 0, 1, 0)$ and $N = (0, 0, \dots, 0, 1)$, namely

$$P = \mathbb{H}^{n,1} \cap N^\perp = \mathbb{H}^n \times \{0\} \subseteq \mathbb{R}^{n,2},$$

then $\psi_{(p,P)}(x,t) = (x_1, \dots, x_n, x_{n+1}\cos(t), x_{n+1}\sin(t))$.

One can easily check that $\psi_{(p,P)}$ is a covering map whose domain is simply connected for any pair (p, P). The pull-back metric can be explicitly computed:

$$\begin{aligned} g_{\widetilde{\mathbb{H}}^{n,1}} := \psi^*_{(p,P)} g_{\mathbb{H}^{n,1}} &= g_P - \langle p, \cdot \rangle^2 dt^2 \\ &= g_{\mathbb{H}^n} - \cosh^2\left(\mathrm{d}_{\mathbb{H}^n}(p, \cdot)\right) dt^2, \end{aligned} \tag{1.3}$$

where g_P is the restriction of $g_{\mathbb{H}^{n,1}}$ to P, which is isometric to $\mathbb{H}^n$, and $\cosh\left(\mathrm{d}_{\mathbb{H}^n}(p, \cdot)\right) = -\langle p, \cdot \rangle$ by Equation (1.1). Hence, the universal cover for the Anti-de Sitter space is $\widetilde{\mathbb{H}}^{n,1} := \mathbb{H}^n \times \mathbb{R}$, endowed with the metric $g_{\widetilde{\mathbb{H}}^{n,1}}$.

Definition 1.3.2. A *splitting* of $\widetilde{\mathbb{H}}^{n,1}$ is the choice of a pair (p, P) identifing $\widetilde{\mathbb{H}}^{n,1}$ with $\mathbb{H}^n \times \mathbb{R}$. We denote $x_0 := (0, \dots, 0, 1) \in \mathbb{H}^n$, namely $\{x_0\} \times \mathbb{R} = \psi^{-1}_{(p,P)}(\gamma)$, for γ the timelike geodesic normal to P at p.

By lifting the isometries of $\mathbb{H}^{n,1}$, it turns out that $\widetilde{\mathbb{H}}^{n,1}$ has maximal isometry group, too. Moreover, since $\psi_{(p,P)}$ restricted to the slices $P \times \{t\}$ is linear, $P \times \{t\}$ is a totally geodesic spacelike hypersurface, for all $t \in \mathbb{R}$. In contrast, $\{x_0\} \times \mathbb{R}$ is the only fiber which is a (timelike) geodesic.

We fix once and for all a time-orientation in the two models for $\mathbb{A}\mathrm{d}\mathbb{S}-$geometry: in the universal cover $\widetilde{\mathbb{H}}^{n,1}$, we choose the one coinciding with the natural orientation of $\mathbb{R}$, and for the quadric model $\mathbb{H}^{n,1}$ the one induced by any covering map ψ as in Equation (1.2).

Lemma 1.3.3. *Let $\mathbb{S}^n_+ \subseteq \mathbb{S}^n$ be an open hemisphere, then $\widetilde{\mathbb{H}}^{n,1}$ is conformal to $\mathbb{S}^n_+ \times \mathbb{R}$ endowed with the metric $g_{\mathbb{S}^n} - dt^2$, hence it has the same causal structure.*

Proof. Let $f \colon \mathbb{H}^n \to \mathbb{D}^n$ be an isometry between $\mathbb{H}^{n,1}$ and the Poincaré disk model, namely

$$f_* g_{\mathbb{H}^n} = \left(\frac{2}{1-r^2}\right)^2 g_{\mathbb{D}^n}.$$

Explicitly, one can take f to be the radial map such that, for $r \geq 0$,

$$r^2 = \frac{x_{n+1}^2 - 1}{(1+x_{n+1})^2}.$$

A direct computation shows that

$$-\langle x_0, x\rangle = x_{n+1} = \frac{1+r^2}{1-r^2}.$$

Using that the spherical metric on a hemisphere has the expression

$$g_{\mathbb{S}^n} = \left(\frac{1-r^2}{1+r^2}\right)^2 g_{\mathbb{D}^n}$$

under the stereographic projection, we get

$$g_{\widetilde{\mathbb{H}}^{n,1}} = g_{\mathbb{H}^n} - \langle x_0, x\rangle^2 dt^2 = \left(\frac{1+r^2}{1-r^2}\right)^2 \left(g_{\mathbb{S}^n} - dt^2\right).$$

□

Remark 1.3.4. In particular, the conformal boundary $\partial\mathbb{H}^{n,1}$ identifies with $\partial\mathbb{H}^n \times \mathbb{R}$, endowed with the Lorentzian conformal structure of the metric $g_{\mathbb{S}^{n-1}} - dt^2$.

1.4. The projective model

The center of $\mathrm{Isom}(\mathbb{H}^{n,1}) = \mathrm{O}(n,2)$ is $\{\pm \mathrm{Id}\}$. It follows that the map $\mathbb{P} \colon \mathbb{R}^{n,1} \to \mathbb{P}(\mathbb{R}^{n,2})$ restricts to a local isometry over $\mathbb{H}^{n,1}$, whose image corresponds to the set of negative lines for $\langle \cdot, \cdot \rangle$.

Definition 1.4.1. The *projective model* (or *Klein model*) is then

$$\mathbb{P}(\mathbb{H}^{n,1}) = \{[x] \in \mathbb{P}(\mathbb{R}^{n,2}), \langle x, x\rangle < 0\},$$

endowed with the pushforward metric $\mathbb{P}_* g_{\mathbb{H}^{n,1}}$.

The map $\mathbb{P} \colon \mathbb{H}^{n,1} \to \mathbb{P}(\mathbb{H}^{n,1})$ is a double covering, and a fundamental domain is the set U_x defined in Section 1.5, for $x \in \mathbb{H}^{n,1}$. The two dual totally geodesic spacelike hypersurfaces $P_\pm(x)$ are mapped to the same totally geodesic spacelike hypersurface $\mathbb{P}(x^\perp)$, and $\mathbb{P}(U_x)$ coincides with the complement of $\mathbb{P}(x^\perp)$ in $\mathbb{P}(\mathbb{H}^{n,1})$.

1.5. Fundamental regions

The length of a piecewise C^1-timelike curve $c\colon (a,b) \to \widetilde{\mathbb{H}}^{n,1}$ is defined as

$$\ell(c) := \int_a^b \sqrt{-g_{\widetilde{\mathbb{H}}^{n,1}}\left(\dot{c}(t), \dot{c}(t)\right)}dt.$$

Definition 1.5.1. Let $p \in \widetilde{\mathbb{H}}^{n,1}$ and $q \in I(p)$. Their Lorentzian distance is

$$\mathrm{dist}(p,q) := \sup\{\ell(c), c \text{ timelike curve joining } p \text{ and } q\},$$

The Lorentzian distance satisfies the reverse triangle inequality

$$\mathrm{dist}(p,q) \geq \mathrm{dist}(p,r) + \mathrm{dist}(r,q), \tag{1.4}$$

provided that the three quantities are well defined and r is chronologically between p and q, that is either $r \in I^+(p) \cap I^-(q)$ or $r \in I^+(q) \cap I^-(p)$ (see Definition 1.1.3).

Definition 1.5.2. For $p \in \widetilde{\mathbb{H}}^{n,1}$, we denote $\mathcal{P}_+(p)$ (resp. $\mathcal{P}_-(p)$) the set at Lorentzian distance $\pi/2$ from p in the future (resp. in the past). We will call it *dual hypersurface* to p in the future (resp. in the past) in $\widetilde{\mathbb{H}}^{n,1}$. We also denote $p_\pm$ the unique point time-related to p contained in $\{\mathrm{dist}(p,\cdot) = \pi\} \cap I^\pm(p)$. To visualize these objects, see Figure 1.2.

Remark 1.5.3. One can prove that the distance between p, q is achieved through a timelike geodesic if $q \in I^-(p_+) \cap I^+(p_-)$ ([BS10, Corollary 2.13, Lemma 2.14]). This condition is not restrictive for our purposes, as we will show in Corollary 3.1.7.

Proposition 1.5.4. *Let $\psi\colon \widetilde{\mathbb{H}}^{n,1} \to \mathbb{H}^{n,1}$ be a splitting. Then,*

$$\langle \psi(p), \psi(q) \rangle = -\cos\left(\mathrm{dist}(p,q)\right),$$

for any pair p, q of time-related points such that $q \in I^-(p_+) \cap I^+(p_-)$.

Proof. Consider two time-related points p, q such that $q \in I^-(p_+) \cap I^+(p_-)$ and let

$$\gamma\colon [0, \mathrm{dist}(p,q)] \to \widetilde{\mathbb{H}}^{n,1}$$

be the timelike geodesic realizing the distance. Since $\psi(\gamma)$ is a timelike-geodesic, $\psi(p)$ and $\psi(q)$ are time-related, too. By Equation (1.1),

$$\psi \circ \gamma(s) = \cos(s)\psi(p) + \sin(s)v,$$

for a unitary timelike tangent vector $v \in T_{\psi(p)}\mathbb{H}^{n,1} = \psi(p)^\perp$. By construction, $\psi(q) = \psi\left(\gamma(\mathrm{dist}(p,q))\right)$, hence

$$\langle \psi(p), \psi(q) \rangle = -\cos\left(\mathrm{dist}(p,q)\right).$$

□

Corollary 1.5.5. *In any splitting, $\psi(\mathcal{P}_\pm(p)) = P_\pm(\psi(p))$ and $\psi(p_\pm) = -\psi(p)$. In particular, $\mathcal{P}_\pm(p)$ is a totally geodesic spacelike hypersurface.*

Proof. The function $\mathrm{dist}(p,\cdot)\colon I(p) \to \mathbb{R}$ is strictly monotone along timelike paths, and $\mathrm{dist}(p, p_\pm) = \pi$, hence

$$\mathcal{P}_+(p) \cup \mathcal{P}_-(p) \subseteq I^-(p_+) \cap I^+(p_-).$$

Hence, we can apply Proposition 1.5.4: it follows that

$$\psi\left(\mathcal{P}_+(p)\cup\mathcal{P}_-(p)\right)=\{\langle\psi(p),\cdot\rangle=0\}=P_+(\psi(p))\cup P_-(\psi(p)).$$

In particular, since ψ preserves the time-orientation, $\psi\left(\mathcal{P}_\pm(p)\right)=P_\pm(\psi(p))$.

By construction, the dual hyperplane to p_+ in the past is $\mathcal{P}_+(p)$. The same argument proves that $\psi(\mathcal{P}_+(p))=P_-(\psi(p_+))$, hence $\psi(p_+)=-p$. The same applies to p_-, observing that its dual hyperplane in the future is $\mathcal{P}_-(p)$. □

Definition 1.5.6. For $p\in\widetilde{\mathbb{H}}^{n,1}$ (resp. $x\in\mathbb{H}^{n,1}$) we define

$$\mathcal{U}_p:=I^-\left(\mathcal{P}_+(p)\right)\cap I^+\left(\mathcal{P}_-(p)\right)$$
$$U_x:=\{y\in\mathbb{H}^{n,1},\ \langle x,y\rangle<0\}$$

Example 1.5.7. To visualise these object, see Figure 1.2: one can check that in a splitting (p,P), $\mathcal{P}_\pm(p)=\mathbb{H}^n\times\{\pm\pi/2\}$, $p_\pm=(p,\pm\pi)$ and $\mathcal{U}_p=\mathbb{H}^n\times(-\pi/2,\pi/2)$.

Remark 1.5.8. The projectivization of $\mathbb{P}\left(P_\pm(x)\right)$ is the spacelike hypersurface orthogonal to x. Hence, $\mathbb{P}(U_x)=\mathbb{P}(\mathbb{H}^{n,1})\setminus\mathbb{P}(x^\perp)$.

These sets are a bridge between the three models: indeed, $\mathcal{U}_p$ isometrically embeds in $\mathbb{H}^{n,1}$ via ψ, and its image is precisely $U_{\psi(p)}$ (Corollary 1.5.9). By Remark 1.5.8, U_x isometrically embeds in $\mathbb{P}\left(\mathbb{H}^{n,1}\right)$ and its image is $\mathbb{P}(U_x)=\mathbb{P}(\mathbb{H}^{n,1})\setminus\mathbb{P}(x^\perp)$. Any properly embedded spacelike hypersurface in $\widetilde{\mathbb{H}}^{n,1}$ is contained in $\mathcal{U}_p$, for a suitable p (Lemma 2.1.7). Hence, the study of properly embedded spacelike hypersurfaces does not depend on the choice of the model of $\mathbb{A}\mathrm{d}\mathbb{S}-$geometry introduced so far (Remark 2.2.2).

Corollary 1.5.9. *Let $p\in\widetilde{\mathbb{H}}^{n,1}$, ψ be a splitting. The restriction of ψ to $\mathcal{U}_p$ is an isometric embedding, whose image is $U_{\psi(p)}$.*

Proof. Without loss of generality, let $p=(x_0,0)$ and $P=\mathbb{H}^n\times\{0\}$. By Corollary 1.5.5, $\mathcal{P}_\pm(p)=\mathbb{H}^n\times\{\pm\pi/2\}$, hence

$$\mathcal{U}_p:=\left\{(x,t)\in\widetilde{\mathbb{H}}^{n,1},-\frac{\pi}{2}<t<\frac{\pi}{2}\right\}.$$

It follows that ψ is injective on $\mathcal{U}_p$, since points in the same fiber as (x,t) are of the form $(x,t+2\pi k)$, $k\in\mathbb{Z}$. Since ψ is a local isometry and it is injective on the open set $\mathcal{U}_p$, the restriction $\psi|_{\mathcal{U}_p}$ is an isometric embedding. To conclude, by Example 1.3.1,

$$\psi(\mathcal{U}_p)=\{x_{n+1}>0\}=\{\langle\psi(p),\cdot\rangle<0\}=U_{\psi(p)}.$$

□

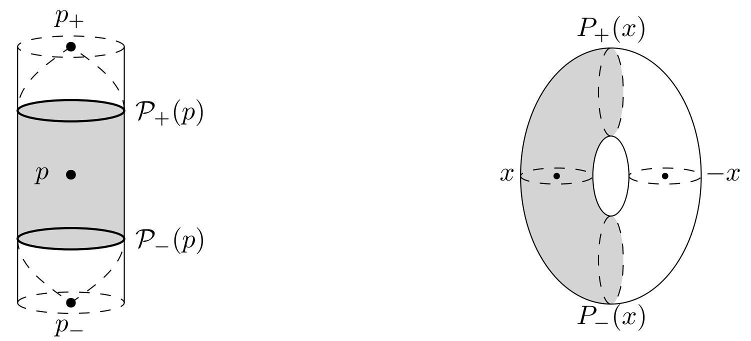

Figure 1.2.: The fundamental regions $\mathcal{U}_p \subseteq \widetilde{\mathbb{H}}^{n,1}$ in a splitting (p, P) (left) and $U_x \subseteq \mathbb{H}^{n,1}$ (right).

Chapter 2.

Graphs in Anti-de Sitter space

All properly embedded spacelike hypersurfaces in $\widetilde{\mathbb{H}}^{n,1}$ are graphs of functions $\mathbb{H}^n \to \mathbb{R}$ (Proposition 2.1.5). Moreover, any properly embedded spacelike hypersurface is contained in a fundamental region (Remark 2.2.2): in other words, properly embedded spacelike hypersurfaces can be studied equivalently in each model of $\mathbb{A}\mathrm{d}\mathbb{S}-$geometry.

In the following, we consider $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$ endowed with the conformal metric $g_{\mathbb{S}^n} - dt^2$.

2.1. Achronal and acausal graphs

The proofs of the following results can be found in [BS20]: even though they are stated for the 3-dimensional case, one can easily check that the arguments do not depend on the dimension.

Definition 2.1.1. A subset X of $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$ is *achronal* (resp. *acausal*) if no pair of points of X can be joined by a timelike (resp. causal) curve of $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$.

Remark 2.1.2. In other words, an achronal subset X is contained in the complement of $I(p)$, for any $p \in X$. Since $\widetilde{\mathbb{H}}^{n,1} \setminus I(p) \subseteq \mathcal{U}_p$, an achronal set X of $\widetilde{\mathbb{H}}^{n,1}$ is contained in the fundamental region $\mathcal{U}_p$, for any $p \in X$.

A splitting (p, P) induces a map

$$\begin{aligned} \pi_{(p,P)} \colon \widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1} = (\mathbb{H}^n \cup \partial\mathbb{H}^n) \times \mathbb{R} &\longrightarrow \mathbb{H}^n \cup \partial\mathbb{H}^n \\ (x,t) &\longmapsto x \end{aligned},$$

which is the projection on the first coordinate.

Definition 2.1.3. An immersed subset $X \subseteq \widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$ is a *graph* if there exists a splitting (p, P) such that the restriction of $\pi_{(p,P)}$ to X is injective. Equivalently, $\pi_{(p,P)}$ restricted to X admits an inverse $u \colon \pi_{(p,P)}(X) \to \mathbb{R}$ and $X = \operatorname{graph} u$. If there exists a splitting (p, P) such that $\pi_{(p,P)}(X) = \mathbb{H}^n$, X is called an *entire* graph.

Lemma 2.1.4. *[BS20, Lemma 4.1.2] For a subset X of $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$, the following statements are equivalent:*

1. *X is achronal (resp. acausal);*
2. *there exists a splitting (p, P) such that X is the graph of a $1-$Lipschitz (resp. strictly $1-$Lipschitz) function;*
3. *for any splitting (p, P), X is the graph of a $1-$Lipschitz (resp. strictly $1-$Lipschitz) function.*

In particular, an entire achronal hypersurface of $\widetilde{\mathbb{H}}^{n,1}$ has a unique $1-$Lipschitz extension to the asymptotic boundary ([McS34, Theorem 1]).

Proposition 2.1.5. *[BS20, Lemma 4.1.3] An achronal hypersurface Σ in $\widetilde{\mathbb{H}}^{n,1}$ is properly embedded if and only if it is an entire graph.*

Definition 2.1.6. For a bounded map $u\colon \mathbb{H}^n \to \mathbb{R}$, we define

$$\operatorname{osc}(u) := \sup_{\mathbb{H}^n} u - \inf_{\mathbb{H}^n} u.$$

Lemma 2.1.7. *[BS20, Lemma 4.1.7] Let Σ be an achronal properly embedded hypersurface in $\widetilde{\mathbb{H}}^{n,1}$, and let $\Sigma = \operatorname{graph} u$ in a splitting. Then $\operatorname{osc}(u) \leq \pi$, with equality if and only if Σ is a totally geodesic degenerate hypersurface.*

Motivated by this result, we give the following definition:

Definition 2.1.8. A set $\Lambda \subseteq \partial\widetilde{\mathbb{H}}^{n,1}$ is an *admissible boundary* if it is the graph of a $1-$Lipschitz map $f\colon \partial\mathbb{H}^n \to \mathbb{R}$ and $\operatorname{osc}(f) < \pi$.

Remark 2.1.9. Even though $\operatorname{osc}(f)$ is not invariant by isometries and depends on the splitting, Lemma 2.1.7 implies that the definition of admissible is intrinsic.

Remark 2.1.10. A direct consequence is the sharpness of Theorem A: indeed, if a boundary is not admissible, it bounds no entire spacelike hypersurface. In particular, no CMC entire spacelike hypersurface.

2.2. Spacelike graphs

As anticipated in Definition 1.1.2, a C^1-embedded hypersurface Σ is spacelike if the induced metric is Riemannian, or equivalently if the normal vector is timelike at every point.

We want to stress that being spacelike is a local property, while being achronal or acausal is a global one. In general, the notions are only partially related. Nonetheless, one can prove that any embedded spacelike hypersurface in $\widetilde{\mathbb{H}}^{n,1}$ is locally the graph of a strictly $1-$Lipschitz function, but this is not true globally.

Proposition 2.2.1. *[BS20, Lemma 4.1.5] An entire spacelike graph Σ in $\widetilde{\mathbb{H}}^{n,1}$ is acausal.*

Proof. Let $\Sigma = \operatorname{graph} u$. By Lemma 2.1.4, it suffices to prove that $u\colon \mathbb{S}^n_+ \to \mathbb{R}$ is strictly $1-$Lipschitz. The hypersurface Σ can be described as the zeros of the function

$$\begin{aligned} \mathbb{H}^n \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (x,t) &\longmapsto u(x) - t \end{aligned}$$

Since being spacelike only depends on the conformal structure, we consider the normal space of Σ with respect to the conformal metric $g_{\mathbb{S}^n_+} - dt^2$. In this setting, $N\Sigma$ is generated by the gradient $\nabla^{\mathbb{S}^n} u - \partial_t$, whose norm is equal to $|\nabla^{\mathbb{S}^n} u|^2 - 1$. It follows that Σ is spacelike if and only if u is strictly $1-$Lipschitz. ☐

Remark 2.2.2. It follows by Proposition 2.2.1 and Remark 2.1.2 that any properly embedded spacelike hypersurface is contained in a fundamental region $\mathcal{U}_p$, for a suitable choice of $p \in \widetilde{\mathbb{H}}^{n,1}$. In other words, there is a natural correspondence between properly embedded spacelike hypersurfaces in the three models of $\mathbb{A}\mathrm{d}\mathbb{S}-$geometry introduced in this thesis.

Chapter 3.

Causal structure

The causal structure of a Lorentzian manifold M is the data of the causal vectors in the tangent space TM. First, we introduce the invisible domain, then the domain of dependence. It turns out that the domain of dependence of a properly embedded achronal hypersurface coincides with the invisible domain of its boundary (Proposition 3.2.5).

Finally, we define the convex hull of an admissible boundary, its past and its future part. In particular, we introduce the time functions (Proposition 3.4.5) that will provide barriers for Chapter 6.

3.1. Invisible domain

The invisible domain makes sense only on *causal* Lorentzian manifold, *i.e.* having no closed causal curve. Since $\mathbb{H}^{n,1}$ is not causal, we define the invisible domain only in the universal cover.

Definition 3.1.1. The *invisible domain* of an achronal subset X of $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$ is the set $\Omega(X) \subseteq \widetilde{\mathbb{H}}^{n,1}$ of points that are connected to X by no causal curve.

Roughly speaking, $\Omega(X)$ is the union of all acausal subsets containing X. In light of Lemma 2.1.4, let $X = \operatorname{graph} u$ in a splitting (p, P): we can equivalently say that $\Omega(X)$ is the union of all acausal graphs of strictly $1-$Lipschitz extensions of u. In particular, one can consider the so called *extremal extension* of u, denoted u^- and u^+. They are extremal in the sense that for any $1-$Lipschitz extension $\tilde{u}$ of u,

$$u^- \leq \tilde{u} \leq u^+.$$

Such extensions exist as they are respectively the supremum and the infimum of a set of bounded $1-$Lipschitz functions. One can prove that their graphs do not depend on the splitting: indeed, for two functions $f, g \colon \mathbb{H}^n \to \mathbb{R}$,

$$f \leq g \iff \operatorname{graph} f \subseteq I^-(\operatorname{graph} g),$$

which does not depend on the splitting.

These ideas are summarized in the following two statements, whose proofs can be found in [BS20, Lemma 4.2.2].

Lemma 3.1.2. *Let X be an achronal subset of $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$, $u^\pm$ its extremal extensions.*

1. $\Omega(X) = I^+(\operatorname{graph} u^-) \cap I^-(\operatorname{graph} u^+)$;
2. $\operatorname{graph} u^\pm$ *is an achronal entire graph.*

Example 3.1.3. The invisible domain of a point p is the set of space-related points to p, namely $\Omega(\{p\}) = \widetilde{\mathbb{H}}^{n,1} \setminus \overline{I(p)}$. The invisible domain of the boundary of the totally geodesic spacelike hypersurface $\mathcal{P}_+(p)$ is $I^-(p_+) \cap I^+(p)$ (see Figure 3.1).

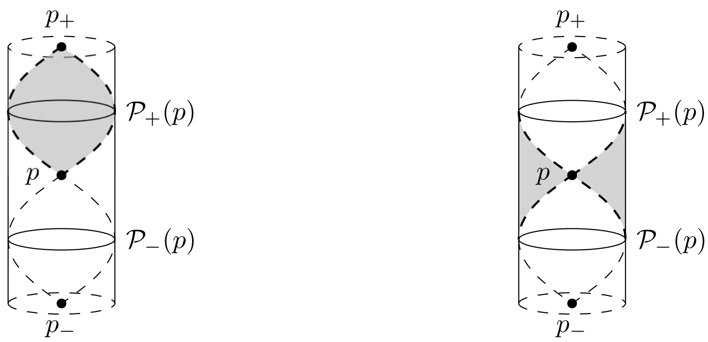

Figure 3.1.: Invisible domain of $\partial\mathcal{P}_+(p)$ (left) and p (right).

Lemma 3.1.4. *For X an achronal closed subset of $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$,*

$$\operatorname{graph} u^+ \cap \operatorname{graph} u^-$$

is the union of X and all lightlike geodesic segments joining points of X.

We are interested in acausal entire graphs, and we can get information through the study of the invisible domain of their asymptotic boundaries: for a graph in $\partial\widetilde{\mathbb{H}}^{n,1}$, admissibility (Definition 2.1.8) is a necessary condition to bound an achronal (or acausal) entire hypersurface (Lemma 2.1.7). In fact, the converse is also true.

Proposition 3.1.5. *For an achronal graph $\Lambda \subseteq \partial\widetilde{\mathbb{H}}^{n,1}$, the following are equivalent:*

1. *Λ is an admissible boundary;*
2. *$\Omega(\Lambda)$ is not empty;*
3. *there exists a properly embedded achronal hypersurface Σ which is not a totally geodesic degenerate hypersurface such that $\partial\Sigma = \Lambda$.*

Proof. We first show (1) $\implies$ (2). If $\Omega(\Lambda)$ is empty, $u^+ = u^-$ by Lemma 3.1.2. In particular, $\operatorname{graph} u^+$ contains a lightlike geodesic connecting two points of Λ (Lemma 3.1.4). By direct computation, this implies that $\operatorname{osc}(u^+) = \pi$, *i.e* Λ is not admissible.

The graph of the extremal extensions u^+ of Λ is an achronal properly embedded hypersurface (Lemma 3.1.2), hence (2) $\implies$ (3): by contradiction, assume that $\operatorname{graph} u^+$ is a totally geodesic degenerate hypersurface, then any point of $\operatorname{graph} u^+$ is connected to Λ by a lightlike geodesic. By Lemma 3.1.4, it follows that $\operatorname{graph} u^+ = \operatorname{graph} u^-$, hence $\Omega(\Lambda) = \emptyset$ (Lemma 3.1.2), contradicting the assumption.

To conclude, the implication (3) $\implies$ (1) is trivial: indeed, the boundary of a properly embedded achronal hypersurface, which is not totally geodesic degenerate, is admissible by Lemma 2.1.7. ☐

Finally, we show that the invisible domain of an admissible boundary isometrically embeds in $\mathbb{H}^{n,1}$, via any projection ψ as in Equation (1.2).

Lemma 3.1.6. *Let Λ_1, Λ_2 two achronal graphs in $\partial\widetilde{\mathbb{H}}^{n,1}$.*

$$\Lambda_1 \subseteq I^-(\Lambda_2) \iff \partial_\pm\Omega(\Lambda_1) \subseteq I^-\left(\partial_\pm\Omega(\Lambda_2)\right).$$

Proof. By Lemma 3.1.2, the boundary of the invisible domain of $\Lambda_i = \operatorname{graph} u_i$ is the graph of its extremal extensions, namely

$$\partial_\pm \Omega(\Lambda_i) = \operatorname{graph} u_i^\pm.$$

In term of graphs, the statement is then equivalent to prove

$$u_1 < u_2 \iff u_1^\pm < u_2^\pm.$$

The implication ($\Leftarrow$) is trivial, since u_i is the restriction of $u_i^\pm$ to the boundary. Conversely, if $u_1 < u_2$, then $v := \max\{u_1^+, u_2^+\}$ is a $1-$Lipschitz function extending u_2. By definition of extremal extension $v \leq u_2^+$, that is $u_1^+ \leq u_2^+$. To obtain the strict inequality, set

$$\varepsilon := \frac{1}{2} \min_{\partial \mathbb{H}^n} (u_2 - u_1) > 0$$

and denote $u_\varepsilon := u_1 + \varepsilon$, which is still a $1-$Lipschitz map, strictly smaller than u_2. The same argument proves that

$$u_1^+ < u_\varepsilon^+ \leq u_2^+,$$

which concludes the proof. □

Corollary 3.1.7. *For an admissible boundary Λ in $\partial \widetilde{\mathbb{H}}^{n,1}$,*

$$\overline{\Omega(\Lambda)} \subseteq I^-(p_+) \cap I^+(p_-), \quad \forall p \in \Omega(\Lambda).$$

Proof. By definition, for any X achronal subset, $X \subseteq \Omega(\Lambda)$ if and only if $\Lambda \subseteq \Omega(X)$.

Take $X = \{p\}$, hence $\Omega(X) = \mathcal{U}_p \setminus \overline{I(p)}$. The frontier of $\mathcal{U}_p$ in $\widetilde{\mathbb{H}}^{n,1}$ is $\partial\mathcal{P}_+(p) \cup \partial\mathcal{P}_-(p)$ (see Example 3.1.3). We apply Lemma 3.1.6 to get

$$\overline{\Omega(\Lambda)} \subseteq I^- \left(\partial_+ \Omega(\partial\mathcal{P}_+(p))\right) \cap I^+ \left(\partial_- \Omega(\partial\mathcal{P}_-(p))\right).$$

As mentioned in Example 3.1.3, since p and $p_\pm$ are the dual points of $\mathcal{P}_\pm(p)$, we have

$$I^- \left(\partial_+ \Omega(\partial\mathcal{P}_+(p))\right) = I^-(p_+)$$
$$I^+ \left(\partial_- \Omega(\partial\mathcal{P}_-(p))\right) = I^+(p_-)$$

which concludes the proof. □

Corollary 3.1.8. *For any admissible boundary Λ in $\partial \widetilde{\mathbb{H}}^{n,1}$, $\overline{\Omega(\Lambda)}$ isometrically embeds in $\mathbb{H}^{n,1}$. Moreover, $\langle \psi(p), \psi(q) \rangle < 1, \quad \forall p, q \in \Omega(\Lambda)$.*

Proof. For $p \in \widetilde{\mathbb{H}}^{n,1}$, $I^-(p_+) \cap I^+(p_-)$ isometrically embeds in the connected component of $\mathbb{H}^{n,1} \setminus C(-\psi(p))$ containing $\psi(p)$ (see Figure 3.2). We recall that this connected component is the set satisfying $\langle \psi(p), \cdot \rangle < 1$ (Remark 1.2.3).

One concludes because $\overline{\Omega(\Lambda)} \subseteq I^-(p_+) \cap I^+(p_-)$, for all $p \in \Omega(\Lambda)$ (Corollary 3.1.7). □

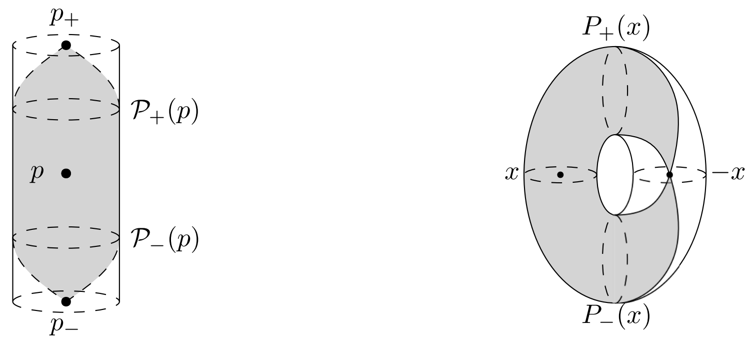

Figure 3.2.: $I^-(p_+) \cap I^+(p_-)$ isometrically embeds in $\{\langle x, \cdot \rangle < 1\}$, for $x = \psi(p)$.

3.2. Domain of dependence

Let X be an acausal subset of a Lorentzian manifold M.

Definition 3.2.1. The *domain of dependence* of X is the set $D(X)$ of points $p \in M$ with the property that any inextensible causal path passing through p intersects X.

Remark 3.2.2. Any inextensible causal path in $\widetilde{\mathbb{H}}^{n,1}$ is properly embedded: indeed, in the same way as in Lemma 2.1.4, one can check that a causal path is the graph of a $1-$Lipschitz map $f\colon (a, b) \subseteq \mathbb{R} \to \mathbb{S}^n_+$. If the path is not properly embedded, without loss of generality we can assume that $a > -\infty$ and $f(t) \to x \in \mathbb{S}^n_+$ for $t \to a^-$. By defining $f(t) = x$ for $t \leq a$, we build a $1-$Lipschitz extension of f, hence graph f was not an inextensible causal path.

Definition 3.2.3. A spacetime (M, g) is called *globally hyperbolic* if there exists an acausal subset X such that $M = D(X)$. In this case, X is called a *Cauchy hypersurface* for M.

Proposition 3.2.4. *Let Σ be an entire acausal graph in $\widetilde{\mathbb{H}}^{n,1}$. A point $p \in \widetilde{\mathbb{H}}^{n,1}$ belongs to $D(\Sigma)$ if and only if $I(p) \cap \Sigma$ is precompact in $\widetilde{\mathbb{H}}^{n,1}$.*

Proof. Without loss of generality, we assume p is in the past of Σ.

First, assume that $I(p) \cap \Sigma$ is precompact in $\widetilde{\mathbb{H}}^{n,1}$. It follows that

$$\overline{I(p)} \cap \Sigma = \overline{I^+(p)} \cap \Sigma$$

is compact in $\widetilde{\mathbb{H}}^{n,1}$, which implies that $\overline{I^+(p)} \cap \overline{I^-(\Sigma)}$ is compact, too: indeed, let $q \in \partial\mathbb{H}^{n,1}$ be an adherence point of $I^+(p) \cap I^-(\Sigma)$, then the future of q in $\partial\mathbb{H}^{n,1}$ is contained in the adherence of $I^+(p)$ and intersects $\partial\Sigma$, contradicting the compactness of $\overline{I^+(p)} \cap \Sigma$. Hence, any future-directed causal curve starting at p and not intersecting Σ, is contained in a compact set. Therefore it is not inextensible (Remark 3.2.2). It follows that any inextensible future-directed causal curve starting at p must intersect Σ, that is $p \in D(\Sigma)$.

Conversely, if the intersection is not compact, there exists a point

$$q \in \overline{I^+(p)} \cap \partial\widetilde{\mathbb{H}}^{n,1} \cap \overline{I^-(\partial\Sigma)} \neq \emptyset.$$

Any inextensible causal line joining p and q does not meet Σ, hence $p \notin D(\Sigma)$. □

In fact, for an entire acausal graph, the domain of dependence only depends on its asymptotic boundary (see [BS10, Corollary 3.8], [BS20, Proposition 4.4.6]):

Proposition 3.2.5. *Let Σ be an entire acausal graph, then $D(\Sigma) = \Omega(\partial\Sigma)$. In particular, two entire acausal graphs in $\widetilde{\mathbb{H}}^{n,1}$ share the same domain of dependence if and only if they share the same boundary.*

3.3. Convex hull

A subset $\mathcal{C}$ of $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$ is *geodesically convex* if any pair of points in $\mathcal{C}$ is joined by at least one geodesic of $\widetilde{\mathbb{H}}^{n,1}$ and any geodesic connecting them lies in $\mathcal{C}$. It follows that the intersection of geodesically convex sets is still geodesically convex.

Definition 3.3.1. For a subset X of $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$ contained in a geodesically convex set $\mathcal{C}$, the *convex hull* of X, denoted $\mathcal{CH}(X)$, is the smallest geodesically convex subset of $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$ containing X.

Remark 3.3.2. In general, a subset X of $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$ might admit no convex neighbourhood: indeed, $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$ is not convex. To check that, take two points $p, q \in \mathbb{H}^{n,1}$ which can be connected by no geodesic, whose existence is due to Lemma 1.2.2: there exists no geodesic connecting any pair of lifting $\tilde{p}$ and $\tilde{q}$ in $\widetilde{\mathbb{H}}^{n,1}$.

Nonetheless, the closure of the invisible domain of an admissible boundary is geodesically convex ([BS10, Proposition 3.9]). It follows that, for X an entire spacelike graph or an admissible boundary, the convex hull is well defined. Moreover, one can prove ([BS10, Lemma 4.7]) that

$$\mathcal{CH}(X) = \bigcap_{p \in X} \overline{\mathcal{U}_p}.$$

In particular, the convex hull is contained in a fundamental region, so it can be projected to $\mathbb{H}^{n,1}$. It turns out that the projection $\psi(\mathcal{CH}(X)) \subseteq \mathbb{H}^{n,1}$ is the intersection of $\mathbb{H}^{n,1}$ and the convex hull of X in $\mathbb{R}^{n,2}$. For a more detailed discussion, we suggest [BS20, Section 4.6]. For our goals, the following characterization is sufficient:

Lemma 3.3.3. *A subset $\mathcal{C}$ of $\mathbb{H}^{n,1} \cup \partial\mathbb{H}^{n,1}$ (resp. $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$) is geodesically convex if and only if $\mathcal{C}$ is connected and $\mathbb{P}(\mathcal{C})$ (resp. $\mathbb{P} \circ \psi(\mathcal{C})$) is convex in $\mathbb{RP}^{n+1}$.*

Motivated by this result, hereafter we will rather call *convex* a geodesically convex subset of $\mathbb{H}^{n,1} \cup \partial\mathbb{H}^{n,1}$ or $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$.

One can prove that $\Omega(\Lambda)$ is convex, for any admissible boundary Λ ([BS20, Proposition 4.6.1]). It follows that $\overline{\Omega(\Lambda)}$ is convex, which implies that $\mathcal{CH}(\Lambda)$ intersects the asymptotic boundary $\partial\widetilde{\mathbb{H}}^{n,1}$ exactly in Λ: indeed, by minimality,

$$\Lambda \subseteq \mathcal{CH}(\Lambda) \cap \partial\widetilde{\mathbb{H}}^{n,1} \subseteq \overline{\Omega(\Lambda)} \cap \partial\widetilde{\mathbb{H}}^{n,1} = \Lambda.$$

Definition 3.3.4. A totally geodesic acausal hypersurface $\mathcal{P} \subseteq \widetilde{\mathbb{H}}^{n,1}$ is a past (resp. future) *support* hypersurface for Λ if $\partial\mathcal{P} \subseteq \overline{I^+(\Lambda)}$ (resp. $\partial P \subseteq \overline{I^-(\Lambda)}$).

Lemma 3.3.5. *Let $\mathcal{P}$ be a support totally geodesic spacelike hypersurface for Λ such that $\mathcal{P} \cap \mathcal{CH}(\Lambda) \neq \emptyset$. Then*

$$\mathcal{CH}(\Lambda) \cap \mathcal{P} = \mathcal{CH}(\Lambda \cap \partial\mathcal{P}).$$

Proof. Since $\mathcal{P}$ is convex and the intersection of convex sets is still convex, by minimality we have

$$\mathcal{CH}(\Lambda \cap \partial\mathcal{P}) \subseteq \mathcal{CH}(\Lambda) \cap \mathcal{P}.$$

The converse inclusion is easily checked in the Klein model. The totally geodesic spacelike hypersurface $P := \psi(\mathcal{P})$ in $\mathbb{H}^{n,1}$ identifies a unique $(n,1)-$hyperplane in $\mathbb{R}^{n,2}$, which we still call P. By Lemma 3.3.3, we can take a affine chart containing $\mathbb{P} \circ \psi(\mathcal{CH}(\Lambda))$: in such affine chart, the projective hyperplane $\mathbb{P}(P)$ is described by the linear equation

$\{\phi = c\}$. Since $\mathcal{P}$ is a support hyperplane for Λ, one can check that $\phi - c$ has a sign restricted to $\mathbb{P} \circ \psi(\Lambda)$. Without loss of generality, we then can assume

$$\mathbb{P} \circ \psi(\Lambda) \subseteq \{\phi \geq c\}.$$

To conclude, fix a point $p \in \mathcal{CH}(\Lambda) \cap \mathcal{P}$. In the affine chart chosen before, describe $\mathbb{P} \circ \psi(p)$ as the convex combination of $q_1, \dots, q_k \in \mathbb{P} \circ \psi(\Lambda)$. Since $\psi(p) \in P$, we have

$$0 = \phi\left(\mathbb{P} \circ \psi(p)\right) = \sum_{i=1}^{k} t_i \phi(q_i) \geq 0,$$

with equality if and only if $\phi(q_i) = 0$ for all $i = 1, \dots, k$. Hence, the q_i's belong to $\mathbb{P}(P)$, namely $p \in \mathcal{CH}(\Lambda \cap \partial\mathcal{P})$, concluding the proof. □

3.4. Past and future part

We introduce two relevant functions on $\Omega(\Lambda)$ and state some of their main properties.

Definition 3.4.1. For $p \in \Omega(\Lambda)$, we denote $\tau_{\mathbf{P}}(p)$ the Lorentzian distance of p from $\partial_-\Omega(\Lambda)$, that is

$$\tau_{\mathbf{P}}(p) := \sup_{q \in \partial_-\Omega(\Lambda) \cap I^-(q)} \operatorname{dist}(p, q).$$

Analogously, $\tau_{\mathbf{F}}$ stands for the Lorentzian distance from $\partial_+\Omega(\Lambda)$.

These functions are *time functions*, namely

Definition 3.4.2. A *time function* on a time-oriented Lorentzian manifold (M, g) is a map $\tau \colon M \to \mathbb{R}$ strictly monotone along timelike paths.

Remark 3.4.3. Usually, in the literature there is a distinction between the cases of strictly increasing and strictly decreasing functions (called *reverse* time functions).

Moreover, $\tau_{\mathbf{P}}$ and $\tau_{\mathbf{F}}$ have further remarkable properties, for which are known in the literature as cosmological times (see for example [AGH98]), when restricted respectively to the past and the future of an admissible boundary.

Definition 3.4.4. For an admissible boundary Λ, we define its *past part* and its *future part* to be

$$\begin{aligned} \mathbf{P}(\Lambda) &:= I^-\left(\partial_+\mathcal{CH}(\Lambda)\right) \cap \Omega(\Lambda); \\ \mathbf{F}(\Lambda) &:= I^+\left(\partial_-\mathcal{CH}(\Lambda)\right) \cap \Omega(\Lambda). \end{aligned} \tag{3.1}$$

To visualize the past and the future of an admissible boundary, see Figure 3.3.

The following result has been proved in [BB09, Proposition 6.19] for the 3-dimensional case. However, the argument does not depend on the dimension, as already remarked in [BS10]. Nonethless, the proof can be found in Section 10.2, where we generalize this result for a special class of convex subsets of the Anti-de Sitter space (Propostion 10.2.1).

Proposition 3.4.5. *Let Λ be an admissible boundary. Then $\tau_{\mathbf{P}}$ is a cosmological time for $\mathbf{P}(\Lambda)$, taking values in $(0, \pi/2)$. Specifically, for every point $p \in \mathbf{P}(\Lambda)$, there exist exactly two points $\rho^{\mathbf{P}}_-(p) \in \partial_-\Omega(\Lambda)$ and $\rho^{\mathbf{P}}_+(p) \in \partial_+\mathcal{CH}(\Lambda)$ such that:*

1. *p belongs to the timelike segment joining $\rho^{\mathbf{P}}_-(p)$ and $\rho^{\mathbf{P}}_+(p)$;*
2. *$\tau_{\mathbf{P}}(p) = \operatorname{dist}(\rho^{\mathbf{P}}_-(p), p)$;*

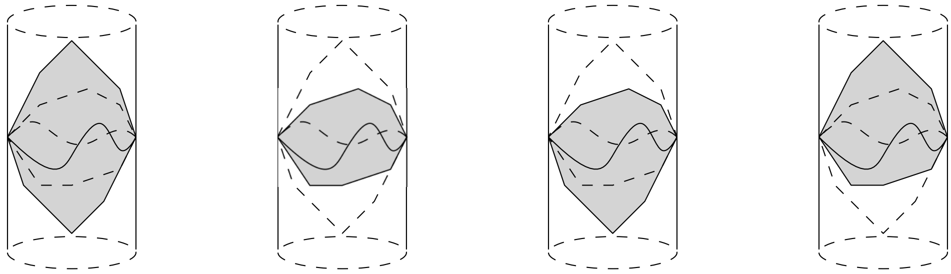

Figure 3.3.: From the left to the right, the invisible domain $\Omega(\Lambda)$, the convex core $\mathcal{CH}(\Lambda)$, the past part $\mathbf{P}(\Lambda)$ and the future part $\mathbf{F}(\Lambda)$ of an admissible boundary Λ.

3. $\mathrm{dist}(\rho_-^{\mathbf{P}}(p), \rho_+^{\mathbf{P}}(p)) = \pi/2$;
4. *$P(\rho_\pm^{\mathbf{P}}(p))$ is a support plane for $\mathbf{P}(\Lambda)$ passing through $\rho_\mp^{\mathbf{P}}(p)$;*
5. *$\tau_{\mathbf{P}}$ is C^1 and $\overline{\nabla}\tau_{\mathbf{P}}(p)$ is the unitary timelike tangent vector such that*

$$\exp_p\left(\tau_{\mathbf{P}}(p)\overline{\nabla}\tau_{\mathbf{P}}(p)\right) = \rho_-^{\mathbf{P}}(p).$$

for $\overline{\nabla}\tau_{\mathbf{P}}$ the gradient of $\tau_{\mathbf{P}}$.

Remark 3.4.6. A symmetric result holds for $\tau_{\mathbf{F}}$ in $\mathbf{F}(\Lambda)$.

Corollary 3.4.7. *Let Λ be an admissible boundary. For any $p \in \Omega(\Lambda)$,*

$$\tau_{\mathbf{F}}(p) + \tau_{\mathbf{P}}(p) \geq \frac{\pi}{2}.$$

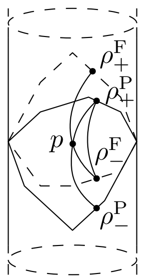

Figure 3.4.: The two longest lines are the geodesic realizing the distance $\pi/2$ between the two pairs $\rho_\pm^{\mathbf{P}}(p)$ and $\rho_\pm^{\mathbf{F}}(p)$. The third one realizes the distance between $\rho_+^{\mathbf{P}}(p)$ and $\rho_-^{\mathbf{F}}(p)$.

Proof. It suffices to check the statement for $p \in \mathcal{CH}(\Lambda)$, that is $\mathbf{P}(\Lambda) \cap \mathbf{F}(\Lambda)$ (see Figure 3.4): indeed, if p is not contained in $\mathbf{P}(\Lambda)$, then $p \in \overline{I^+(\partial_+\mathcal{CH}(\Lambda))}$. Since $\tau_{\mathbf{P}}$ satisfies the reverse triangle inequality (Equation (1.4)), Proposition 3.4.5 ensures that $\tau_{\mathbf{P}}(p) \geq \pi/2$, and $\tau_{\mathbf{F}}$ is non-negative by construction. The same argument applies if $p \notin \mathbf{F}(\Lambda)$.

By Proposition 3.4.5, it follows also that

$$\tau_{\mathbf{P}}(p) = \frac{\pi}{2} - \mathrm{dist}(p, \rho_+^{\mathbf{P}}(p)),$$
$$\tau_{\mathbf{F}}(p) = \frac{\pi}{2} - \mathrm{dist}(p, \rho_-^{\mathbf{F}}(p)),$$

where $\rho_+^{\mathbf{P}}(p)$ and $\rho_-^{\mathbf{F}}(p)$ are respectively the retractions on $\partial_+\mathcal{CH}(\Lambda)$ and $\partial_-\mathcal{CH}(\Lambda))$ induced by $\tau_{\mathbf{P}}$ and $\tau_{\mathbf{P}}$. We deduce that

$$\tau_{\mathbf{P}}(p) + \tau_{\mathbf{F}}(p) = \pi - \operatorname{dist}(p, \rho_+^{\mathbf{P}}(p)) - \operatorname{dist}(p, \rho_-^{\mathbf{F}}(p)).$$

The reverse triangle inequality (Equation (1.4)) concludes the proof:

$$\begin{aligned} &\operatorname{dist}\left(p, \rho_+^{\mathbf{P}}(p)\right) + \operatorname{dist}\left(\rho_-^{\mathbf{F}}(p), p\right) \leq \operatorname{dist}\left(\rho_-^{\mathbf{F}}(p), \rho_+^{\mathbf{P}}(p)\right) \\ &\leq \operatorname{dist}\left(\rho_-^{\mathbf{F}}(p), \partial_+\mathcal{CH}(\Lambda)\right) = \frac{\pi}{2} - \tau_{\mathbf{P}}\left(\rho_-^{\mathbf{F}}(p)\right) \leq \frac{\pi}{2}. \end{aligned}$$

□

The level sets of these time functions will be used as barriers in Chapter 6.

Definition 3.4.8. For $\theta \in [0, \pi/2]$, we denote $W_\theta^{\mathbf{P}}$ the hypersurface at Lorentzian distance θ from $\partial_+\mathcal{CH}(\Lambda)$, that is the level sets

$$\left\{\tau_{\mathbf{P}} = \frac{\pi}{2} - \theta\right\}.$$

In particular, $W_0^{\mathbf{P}} = \partial_+\mathcal{CH}(\Lambda)$, $W_{\pi/2}^{\mathbf{P}} = \partial_-\Omega(\Lambda)$.

Analogously, we denote $W_\theta^{\mathbf{F}}$ the hypersurface at Lorentzian distance θ from $\partial_-\mathcal{CH}(\Lambda)$.

Lemma 3.4.9. *Let Λ be an admissible boundary, then $W_\theta^{\mathbf{P}}$ and $W_\theta^{\mathbf{F}}$ are spacelike Cauchy hypersurfaces for $\Omega(\Lambda)$, for any $\theta \in (0, \pi/2)$.*

Proof. Without loss of generality, we fix $\theta \in (0, \pi/2)$ and prove the statement for $W_\theta^{\mathbf{P}}$, which is the level set of a C^1-submersion. Its normal vector is $\overline{\nabla}\tau_{\mathbf{P}}$, which is timelike by Proposition 3.4.5, hence $W_\theta^{\mathbf{P}}$ is a spacelike hypersurface.

To prove that it is a Cauchy hypersurface, take a point $p \in \Omega(\Lambda)$ and an inextensible future-directed causal path $c\colon (a, b) \to \Omega(\Lambda)$ such that $c(0) = p$: we want to show that c meets $W_\theta^{\mathbf{P}}$.

Being contained in $\Omega(\Lambda)$, c is future-inextensible in $\widetilde{\mathbb{H}}^{n,1}$ if and only if c accumulates at Λ (Remark 3.2.2), which is impossible, since there exists no causal path connecting p to Λ, by definition of invisible domain. Hence, up to reparameterization, we can assume $a, b \in \mathbb{R}$ are finite values, $c(a) \in \partial_-\Omega(\Lambda)$ and $c(b) \in \partial_+\Omega(\Lambda)$.

The boundary $\partial_+\mathcal{CH}(\Lambda)$ disconnects the future and the past boundary of $\Omega(\Lambda)$, hence there exists $t_+ \in (a, b]$ such that $c(t_+) \in \partial_+\mathcal{CH}(\Lambda)$. It follows that

$$\tau_{\mathbf{P}} \circ c\colon (a, t_+) \to (0, \pi/2)$$

is a continuous function from a connected interval whose limit values are 0 and $\pi/2$, hence it is surjective, namely $c(t)$ crosses $W_\theta^{\mathbf{P}}$. □

Corollary 3.4.10. *Λ is an admissible boundary in $\widetilde{\mathbb{H}}^{n,1}$ if and only if there exists a properly embedded spacelike hypersurface Σ such that $\partial\Sigma = \Lambda$.*

Proof. The boundary of a properly embedded spacelike hypersurface is admissibile by Proposition 2.2.1. Conversely, the level sets $W_\theta^{\pm}$ are properly embedded spacelike hypersurfaces with boundary Λ. □

Part II.

Generalized asymptotic Plateau problem

Chapter 4.

Maximum principles for mean curvature

The main object of this article are properly embedded spacelike hypersurfaces with constant mean curvature (hereafter CMC). The second fundamental form of a spacelike C^2-hypersurface Σ is the projection of the ambient Levi-Civita connection $\overline{\nabla}$ on the future-directed normal space $N\Sigma$, which is the symmetric $(0,2)-$tensor on $T\Sigma$ defined by

$$\mathrm{II}(v,w) := \langle \overline{\nabla}_v N, w \rangle,$$

for N the unitary future-directed vector field on Σ.

Definition 4.0.1. The *mean curvature* H of Σ is the trace of the second fundamental form with respect to the induced metric. For an orthonormal frame v_i of $T\Sigma$,

$$H := \sum_{i=1}^{n} \mathrm{II}(v_i, v_i).$$

Remark 4.0.2. The mean curvature is a function $H\colon \Sigma \to \mathbb{R}$, which is invariant under time-orientation preserving isometries of $\widetilde{\mathbb{H}}^{n,1}$. Time-orientation reversing isometries change the sign of the mean curvature.

The first part of this section is devoted to prove a maximum principle (Theorem O), and to show that the uniqueness part of Theorem A follows as a corollary of Theorem O. Finally, we give a stronger version of Theorem O, where one of the two hypersurfaces is a CMC (Proposition 4.2.1), whose corollary (Corollary 4.4.3) will play a key role in the proof of the existence part of Theorem A, provided in Chapter 6.

Remark 4.0.3. Hereafter, a hypersurface Σ will be a C^2-submanifold of co-dimension 1 without boundary. We denote $\partial\Sigma$ its topological frontier in $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$. Moreover, we will consider the causal structure extended to the boundary, *i.e* for $X \subseteq \widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$, $I(X)$ refers to cone of X in $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$.

4.1. Weak maximum principle

Hereafter, for a hypersurface Σ, we will denote ∇ the intrinsic Levi-Civita connection and $\overline{\nabla}$ the exterior Levi-Civita connection, namely the connection of $\widetilde{\mathbb{H}}^{n,1}$ and $\mathbb{H}^{n,1}$. The main result of this section is the following maximum principle:

Theorem O. *Let Σ_1 be a spacelike graph and Σ_2 be an entire spacelike graph with mean curvature respectively H_1, H_2.*

1. *If $H_1(p) \geq H_2(q)$, for every pair $(p,q) \in \Sigma_1 \times \Sigma_2$ of time-related points, and $\partial\Sigma_1 \subseteq \overline{I^-(\Sigma_2)}$, then $\Sigma_1 \subseteq \overline{I^-(\Sigma_2)}$.*

2. *If $H_1(p) \leq H_2(q)$, for every pair $(p,q) \in \Sigma_1 \times \Sigma_2$ of time-related points, and $\partial\Sigma_1 \subseteq \overline{I^+(\Sigma_2)}$, then $\Sigma_1 \subseteq \overline{I^+(\Sigma_2)}$.*

First, let us show how uniqueness follows from Theorem O.

Theorem A (Uniqueness)**.** *Let Σ_1, Σ_2 be two entire spacelike graphs in $\widetilde{\mathbb{H}}^{n,1}$ sharing the same boundary and having the same constant mean curvature. Then $\Sigma_1 = \Sigma_2$.*

Proof. The pair (Σ_1, Σ_2) satisfies the hypotheses of both items of Theorem O, hence

$$\Sigma_1 \subseteq \overline{I^-(\Sigma_2)} \cap \overline{I^+(\Sigma_2)} = \Sigma_2.$$

The other inclusion can be obtained by symmetry or observing that for entire graphs inclusion is equivalent to equality. □

To demonstrate Theorem O, we will apply the maximum principle to the distance between the hypersurfaces, as in [BS10] and [LTW20]. In addition, a topological argument is used to maximize the distance over a compact set, in order to avoid the completeness hypothesis required in the cited approaches.

Proof of Theorem O. We focus on the first part of Theorem O: the second one follows because the map $\phi(x,t) = (x,-t)$ is an isometry of $\widetilde{\mathbb{H}}^{n,1}$ which reverses the time-orientation, hence the sign of the mean curvature.

Step 1: Stronger assumption.

We prove the statement assuming that $\partial\Sigma_1 \subseteq I^-(\Sigma_2)$, instead of $\partial\Sigma_1 \subseteq \overline{I^-(\Sigma_2)}$. The general statement follows directly by a continuity argument: indeed, if $\partial\Sigma_1 \subseteq \overline{I^-(\Sigma_2)}$, it suffices to fix a splitting where $\Sigma_2 = \operatorname{graph} u_2$, to apply the argument to $\Sigma_2(\delta) = \operatorname{graph}(u_2 + \delta)$, $\delta > 0$ and to take the limit as $\delta \to 0$.

The statement reduces to prove that

$$\mathcal{A} := \{(p,q) \in \Sigma_1 \times \Sigma_2 | \, p \in I^+(q)\} = \emptyset.$$

Step 2: $\mathcal{A}$ is precompact in $\Sigma_1 \times \Sigma_2$.

We claim that the projection $\mathcal{A}_i$ of $\mathcal{A}$ over Σ_i is precompact inside Σ_i. If so, $\mathcal{A} \subseteq \mathcal{A}_1 \times \mathcal{A}_2$ is precompact in $\Sigma_1 \times \Sigma_2$.

By definition, $\mathcal{A}_1 = \Sigma_1 \cap I^+(\Sigma_2)$ and by assumption $\partial\Sigma_1 \subseteq I^-(\Sigma_2)$. Hence, $(\Sigma_1 \cup \partial\Sigma_1) \cap I^-(\Sigma_2)$ is an open neighbourhood of $\partial\Sigma_1$ in $\Sigma_1 \cup \partial\Sigma_1$. It follows $\mathcal{A}_1$ is precompact in Σ_1. A symmetric argument does not apply directly: Σ_1 might not be entire, and in that case $I(\Sigma_1) \neq \widetilde{\mathbb{H}}^{n,1} \setminus \Sigma_1$, hence we can not state that $\partial\Sigma_2 \subseteq I^+(\Sigma_1)$. Denote $S_2 := \Sigma_2 \cap I(\Sigma_1)$:

$$\mathcal{A}_2 = \Sigma_2 \cap I^-(\Sigma_1) = S_2 \cap I^-(\Sigma_1),$$

and $\partial S_2 \subseteq I^+(\Sigma_1)$: by the same argument, it follows that $\mathcal{A}_2$ is precompact in S_2, hence in Σ_2, which proves the claim (see Figure 4.1 to visualize the proof).

Step 3: $\mathcal{A}$ is open in $\Sigma_1 \times \Sigma_2$.

The fact that $\mathcal{A}$ is open follows directly from a more general result of Lorentzian geometry. Indeed, the same proof applies to any time-orientable Lorentzian manifold not containing closed causal curve, namely for any time-orientable *causal* Lorentzian manifold. See Figure 4.2 to visualize the proof.

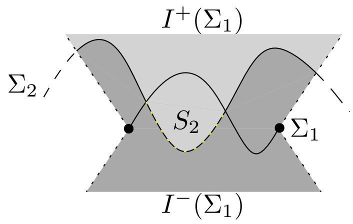

Figure 4.1.: $\mathcal{A}$ is precompact in $\Sigma_1 \times \Sigma_2$.

Fix $(x, y) \in \mathcal{A}$ and pick four points $a, b, c, d \in \widetilde{\mathbb{H}}^{n,1}$ such that

$$a < y < b < c < x < d,$$

where the order is given by the time-orientation, *e.g.* $a < b$ means $a \in I^-(b)$, and the existence of such points a, b, c, d is ensured by the hypothesis $y < x$.

Now, $V'_x := I^+(c) \cap I^-(d)$ is an open neighbourhood of x contained in $I^+(b)$, and $V'_y := I^+(a) \cap I^-(b)$ is an open neighbourhood of y contained in $I^-(c)$. In particular $V_x := V'_x \cap \Sigma_1$ (resp. $V_y := V'_x \cap \Sigma_2$) is an open neighbourhood of x in Σ_1 (resp. of y in Σ_2). By transitivity of the order relation $<$, it follows that

$$V_x \times \{w\} \subseteq \mathcal{A}, \quad \forall w \in V_y,$$
$$\{z\} \times V_y \subseteq \mathcal{A}, \quad \forall z \in V_x,$$

that is $V_x \times V_y \subseteq \mathcal{A}$ is an open neighbourhood of (x, y).

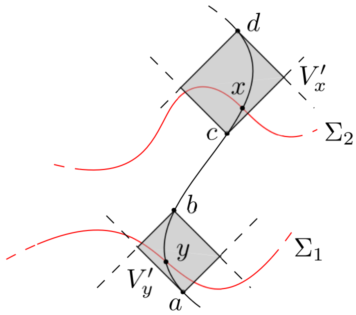

Figure 4.2.: $\mathcal{A}$ is open in $\Sigma_1 \times \Sigma_2$.

Step 4: Project the problem in $\mathbb{H}^{n,1}$.
If $\Sigma_1 \cap \Sigma_2 = \emptyset$, the statement follows directly: indeed, $\Sigma_2 \cup \partial\Sigma_2$ disconnects $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$ by entireness, $\partial\Sigma_1 \subseteq I^-(\Sigma_2)$ by hypothesis and Σ_1 lies in the same connected component as $\partial\Sigma_1$. Otherwise, Σ_1 and Σ_2 are contained in the fundamental region $\mathcal{U}_p$, for $p \in \Sigma_1 \cap \Sigma_2$ (Remark 2.1.2). Since $\mathcal{U}_p$ embeds in $\mathbb{H}^{n,1}$ (Corollary 1.5.9), the distance can be computed through the scalar product (Proposition 1.5.4). In particular, we denote A the image of $\mathcal{A}$ through the embedding $\mathcal{U}_p \times \mathcal{U}_p \hookrightarrow \mathbb{H}^{n,1} \times \mathbb{H}^{n,1}$.

For the rest of the proof, F will denote the scalar product, namely $F(p, q) := \langle p, q\rangle$. The restriction of F to A takes value in $(-1, 1)$, because A is composed by pairs of time-related points in $\mathbb{H}^{n,1}$ (Lemma 1.2.2). By continuity, $F(\overline{A}) \subseteq [-1, 1]$, and $F(\partial A) \subseteq \{\pm 1\}$ since, again by Lemma 1.2.2, $F(p, q) \in (-1, 1)$ if and only if p, q are time-related.

Furthermore, the condition $\partial\Sigma_1 \subseteq I^-(\Sigma_2)$ ensures that $\Sigma_1 \cap \overline{I^+(\Sigma_2)}$ is contained in the domain of dependence of Σ_2: indeed, let $\Sigma_i = \operatorname{graph} u_i$ in a splitting, then

$$\partial\Sigma_1 \subseteq I^-(\Sigma_2) \iff (u_1)|_{\partial\mathbb{H}^n} < (u_2)|_{\partial\mathbb{H}^n}.$$

It follows that $u := \max\{u_1, u_2\}$ is a strictly 1−Lipschitz map such that $u|_{\partial\mathbb{H}^n} = (u_2)|_{\partial\mathbb{H}^n}$, namely graph u is an entire acausal graph sharing the same boundary as Σ_2. By Proposition 3.2.5, graph u is contained in $\Omega(\partial\Sigma_2)$. One concludes by remarking that

$$\Sigma_1 \cap \overline{I^+(\Sigma_2)} = \{(x, u_1(x)), u_1(x) \geq u_2(x)\} \subseteq \operatorname{graph} u.$$

In Step 2, we showed that $\mathcal{A} \subseteq \mathcal{A}_1 \times \mathcal{A}_2$, for $\mathcal{A}_1 = \Sigma_1 \cap I^+(\Sigma_2)$. Hence,

$$\overline{\mathcal{A}} \subseteq \left(\Sigma_1 \cap \overline{I^+(\Sigma_2)}\right) \times \Sigma_2 \subseteq \Omega(\Sigma_2) \times \Omega(\Sigma_2).$$

Then, Corollary 3.1.8 ensures that $F < 1$ on $\overline{A}$, namely $F|_{\partial A} \equiv -1$. It follows that $F(\overline{A})$ is a compact subset of $[-1, 1)$ and, more precisely, $F(\overline{A}) = [-1, \max_{\overline{A}} F]$. In particular, $A = \emptyset$ if and only if $\max_{\overline{A}} F = -1$.

Step 5: Maximum principle.

By contradiction, assume $\max_{\overline{A}} F > -1$, namely F reaches its maximum at $(\bar{p}, \bar{q}) \in A$. We showed in Step 3 that A is open in $\Sigma_1 \times \Sigma_2$: a direct computation leads to

$$d_{(p,q)}F(v, w)\big|_A = \langle p, w\rangle + \langle v, q\rangle. \tag{4.1}$$

At the maximum $(\bar{p}, \bar{q})$, dF vanishes: by Equation (4.1),

$$\bar{p} \in (T_{\bar{q}}\Sigma_2)^\perp = \operatorname{Span}(\bar{q}, N_2(\bar{q}))$$
$$\bar{q} \in (T_{\bar{p}}\Sigma_1)^\perp = \operatorname{Span}(\bar{p}, N_1(\bar{p})).$$

Since both N_1 and N_2 are future-directed and $\bar{p} \in I^+(\bar{q})$, there exists $T > 0$ such that

$$\begin{cases} \bar{p} = \cos(T)\bar{q} + \sin(T)N_2(\bar{q}) \\ \bar{q} = \cos(T)\bar{p} - \sin(T)N_1(\bar{p}) \end{cases} \tag{4.2}$$

Equation (4.2) has two important consequences: first, since $T \neq 0$, at the maximum the tangent spaces are identified. Indeed,

$$T_{\bar{q}}\Sigma_2 = \operatorname{Span}(\bar{q}, N_2(\bar{q}))^\perp = \operatorname{Span}(\bar{p}, N_1(\bar{p}))^\perp = T_{\bar{p}}\Sigma_1.$$

Moreover, we have the following equation:

$$\langle \bar{q}, N_1(\bar{p})\rangle = \sin(T) = -\langle \bar{p}, N_2(\bar{q})\rangle. \tag{4.3}$$

To compute Hess F, we follow [LTW20, Lemma 4.3] and add the proof for completeness. If γ_1 is a geodesic of Σ_1 such that $\gamma_1(0) = p$, $\dot{\gamma}_1(0) = v$,

$$\frac{d^2}{dt^2}\gamma_1(t)|_{t=0} = \mathrm{II}_1(v, v)N_1(p) + \langle v, v\rangle p.$$

The formula can be easily derived by comparing the covariant derivative of γ_1 in Σ, $\mathbb{H}^{n,1}$ and $\mathbb{R}^{n,2}$. The same applies to a geodesic of Σ_2 such that $\gamma_2(0) = q$, $\dot{\gamma}_2(0) = w$, hence

$$\begin{aligned}\operatorname{Hess}_{(p,q)} F\left((v, w), (v, w)\right) &= \frac{d^2}{dt^2}\langle \gamma_1(t), \gamma_2(t)\rangle|_{t=0} \\ &= (\langle v, v\rangle + \langle w, w\rangle)F(p, q) + 2\langle v, w\rangle + \mathrm{II}_1(v, v)\langle N_1(p), q\rangle + \mathrm{II}_2(w, w)\langle N_2(q), p\rangle.\end{aligned}$$

Fix an orthonormal basis $(v_1, \dots, v_n)$ of $T_{\bar{p}}\Sigma_1 = T_{\bar{q}}\Sigma_2$. At the maximum $(\bar{p}, \bar{q})$, the Hessian is semi-negative definite: for every $i = 1, \dots, n$, it holds

$$\begin{aligned} 0 &\geq \mathrm{Hess}_{(\bar{p},\bar{q})} F\left((v_i, v_i), (v_i, v_i)\right) \\ &= 2F(\bar{p}, \bar{q}) + 2 + \mathrm{II}_1(v_i, v_i)\langle N_1(\bar{p}), \bar{q}\rangle + \mathrm{II}_2(v_i, v_i)\langle N_2(\bar{q}), \bar{p}\rangle \\ &= 2F(\bar{p}, \bar{q}) + 2 + (\mathrm{II}_1(v_i, v_i) - \mathrm{II}_2(v_i, v_i)) \sin(T), \end{aligned}$$

where the last equation is due to Equation (4.3). Summing over $i = 1, \dots, n$, one obtains

$$0 \geq 2nF(\bar{p}, \bar{q}) + 2n + \underbrace{(H_1(\bar{p}) - H_2(\bar{q}))}_{\geq 0} \sin(T) \geq 2nF(\bar{p}, \bar{q}) + 2n \geq 0.$$

It follows that $\max_A F = -1$, which is absurd and concludes the proof. □

4.2. Strong maximum principle

If Σ_2 has constant mean curvature, one can promote Theorem O to a strong maximum principle.

Proposition 4.2.1. *Let Σ_1 be a spacelike graph, Σ_2 be an entire CMC spacelike graph with mean curvature respectively H_1, H_2.*

1. *If $H_1 \geq H_2$, and $\partial\Sigma_1 \subseteq \overline{I^-(\Sigma_2)}$, then either $\Sigma_1 \subseteq I^-(\Sigma_2)$ or $\Sigma_1 \subseteq \Sigma_2$.*
2. *If $H_1 \leq H_2$, and $\partial\Sigma_1 \subseteq \overline{I^+(\Sigma_2)}$, then either $\Sigma_1 \subseteq I^+(\Sigma_2)$ or $\Sigma_1 \subseteq \Sigma_2$.*

The proof follows by combining Theorem O with [Esc89, Theorem 1], that is

Theorem 4.2.2. *Let (M, g) be a Lorentzian manifold. Consider two disjoint connected open sets Ω_1, Ω_2 of M with C^2 spacelike boundaries Σ_1, Σ_2, respectively. Assume that $\Omega_1 \subseteq I^-(\Omega_2)$ and that $\Sigma_1 \cap \Sigma_2 \neq \emptyset$. If there exists a constant $c \in \mathbb{R}$ such that*

$$H_1 \geq c \geq H_2,$$

for H_i the mean curvature of Σ_i, then $\Sigma_1 = \Sigma_2$ and $H_1 = H_2 = c$.

Proof of Proposition 4.2.1. As always, we prove only the first part. First, by Theorem O, $\Sigma_1 \subseteq \overline{I^-(\Sigma_2)}$.

If Σ_1 does not intersect Σ_2, we are done. Otherwise, fix a splitting $\widetilde{\mathbb{H}}^{n,1} = \mathbb{H}^n \times \mathbb{R}$ and denote u_i the function whose graph is Σ_i. Let Ω be the domain of u_1, *i.e.* the projection of Σ_1 to $\mathbb{H}^n$, and consider $M := \Omega \times \mathbb{R}$, which is an open subset of $\widetilde{\mathbb{H}}^{n,1}$, hence a Lorentzian manifold.

The condition $\Sigma_1 \subseteq \overline{I^-(\Sigma_2)}$ translates as $u_1 \leq u_2$ on Ω, hence $\Omega_1 := \{(x, t) \in M, t < u_1(x)\}$ and $\Omega_2 := \{(x, t) \in M, t > u_2(x)\}$ satisfy the hypotheses of Theorem 4.2.2: indeed, they are disjoint, their boundaries are C^2, spacelike and intersect by assumption. Moreover, for $c := H_2$, the inequality on the mean curvatures is satisfied: it follows that $\Sigma_1 = \Sigma_2 \cap M$, that is $\Sigma_1 \subseteq \Sigma_2$. □

4.3. Equidistant hypersurfaces

The first example of non-maximal CMC entire graphs we introduce are the ones sharing the boundary with a totally geodesic spacelike hypersurface.

Lemma 4.3.1. *Let $\mathcal{P}$ be a totally geodesic spacelike hypersurface of $\widetilde{\mathbb{H}}^{n,1}$, $\theta \in (0, \pi/2)$. The set*

$$\{p \in \widetilde{\mathbb{H}}^{n,1}, \mathrm{dist}(p, P) = \theta\}$$

is the disjoint union of two entire totally umbilical graphs $\mathcal{P}_\theta^+$ and $\mathcal{P}_\theta^-$, contained respectively in the future and in the past of $\mathcal{P}$, with constant mean curvature $H_\pm(\theta) = \mp n \tan(\theta)$.

Proof. By construction, $\mathcal{P}_\theta^\pm \subseteq I^\pm(\mathcal{P})$, and $\mathcal{P}$ separates $\mathcal{P}_\theta^+$ and $\mathcal{P}_\theta^-$.

To compute the mean curvature, consider the dual past point of $\mathcal{P}$, namely the point $e \in \widetilde{\mathbb{H}}^{n,1}$ such that $\mathcal{P}_+(e) = P$. Denote $P_\theta^\pm := \psi(\mathcal{P}_\theta^\pm)$, by Proposition 1.5.4

$$P_\theta^\pm = \{\cos(\theta)p \pm \sin(\theta)e, p \in P\}\,.$$

It follows that $\mathrm{I}(v, w) = \cos(\theta)\langle v, w\rangle$ and $\mathrm{I\!I}(v, w) = \mp \sin(\theta)\langle v, w\rangle$, hence $B(v) = \mp \tan(\theta)v$, concluding the proof. □

4.4. Barriers

We extend the maximum principle to the leaves of the cosmological time functions $\tau_{\mathbf{P}}$ and $\tau_{\mathbf{F}}$.

Proposition 4.4.1. *Let Σ be an entire spacelike graph, $\theta \in [0, \pi/2)$ and $W_\theta^{\mathbf{P}}, W_\theta^{\mathbf{F}}$ as in Definition 3.4.8, for $\Lambda = \partial\Sigma$.*

1. *If $H \geq n\tan(\theta)$, then either $\Sigma \subseteq I^-\left(W_\theta^{\mathbf{P}}\right)$ or $\Sigma = \mathcal{P}_\theta^-$, for a spacelike hyperplane $\mathcal{P}$;*
2. *if $H \leq -n\tan(\theta)$, then either $\Sigma \subseteq I^+\left(W_\theta^{\mathbf{F}}\right)$ or $\Sigma = \mathcal{P}_\theta^+$, for a spacelike hyperplane $\mathcal{P}$.*

Remark 4.4.2. If $\theta = 0$, namely the mean curvature of Σ is non-negative (resp. non-positive), Proposition 4.4.1 states that Σ is contained in $\mathbf{P}(\partial\Sigma)$ (resp. $\mathbf{F}(\partial\Sigma)$), introduced in Definition 3.4.4. In particular, if Σ is maximal, *i.e.* the mean curvature identically vanishes, we recover a well known fact (see for instance [BS10, Lemma 4.1]), that is that $\Sigma \subseteq \mathcal{CH}(\partial\Sigma)$: indeed, $\mathbf{P}(\partial\Sigma) \cap \mathbf{F}(\partial\Sigma) = \mathcal{CH}(\partial\Sigma)$, by definition (see Figure 3.3).

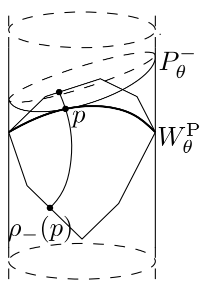

Figure 4.3.: Any point in $W_\theta^{\mathbf{P}}$ is contained in an equidistant hypersurface $\mathcal{P}_\theta^-$, for a suitable support hyperplane $\mathcal{P}$ of $\partial_+\mathcal{CH}(\Lambda)$.

Proof. We focus on the first item. The idea of the proof is to find, for any point $p \in W_\theta^{\mathbf{P}}$, a totally geodesic spacelike hypersurface $\mathcal{P}$ such that $\partial\mathcal{P} \subseteq I^+(\Sigma)$ and $p \in \mathcal{P}_\theta^-$, in order to apply the strong maximum principle (see Figure 4.3).

We recall that by Proposition 3.4.5, we can associates to p the retraction $\rho_-^{\mathbf{P}}(p) \in \partial_-\Omega(\partial\Sigma)$. In particular, the future dual hyperplane $\mathcal{P}_+(\rho_-^{\mathbf{P}}(p))$ is a support hyperplane for $\partial_+\mathcal{CH}(\partial\Sigma)$ containing $\rho_+^{\mathbf{P}}(p)$. By construction, its boundary lies in the future of $\partial\Sigma$ and its distance from p is

$$\operatorname{dist}\left(p, \rho_+^{\mathbf{P}}(p)\right) = \frac{\pi}{2} - \tau_{\mathbf{F}}(p) = \theta,$$

that is $\mathcal{P}_+(\rho_-^{\mathbf{P}}(p))_\theta^-$ contains p.

Applying Proposition 4.2.1 to the two entire spacelike hypersurfaces Σ and $\mathcal{P}_+(\rho_-^{\mathbf{P}}(p))_\theta^-$, we deduce that either Σ does not contain p or Σ coincides with $\mathcal{P}_+(\rho_-^{\mathbf{P}}(p))_\theta^-$. Since p was arbitrary, that concludes the proof. □

With the same argument, one can prove a local version of the proposition.

Corollary 4.4.3. *Let Λ be an admissible boundary, $\theta \in [0, \pi/2)$ and $W_\theta^{\mathbf{P}}, W_\theta^{\mathbf{F}}$ as in Definition 3.4.8. Let Σ be a spacelike graph with mean curvature H.*

1. *If $\partial\Sigma \subseteq I^-\left(W_\theta^{\mathbf{P}}\right)$ and $H \geq n\tan(\theta)$, then $\Sigma \subseteq I^-\left(W_\theta^{\mathbf{P}}\right)$;*
2. *if $\partial\Sigma \subseteq I^+\left(W_\theta^{\mathbf{F}}\right)$ and $H \leq -n\tan(\theta)$, then $\Sigma \subseteq I^+\left(W_\theta^{\mathbf{F}}\right)$.*

Chapter 5.

Estimates and completeness

This section presents estimates on the gradient function, the norm of the second fundamental form and its derivatives. The results rely on more general estimates contained in [Bar21b] and [Eck03].

First, Proposition 5.1.2 furnishes a local estimate for CMC spacelike graphs, crucial in Chapter 6, to prove a compactness result for CMC (not necessarily entire) graphs.

Theorem 5.2.1 provides a global estimate for entire CMC spacelike graphs. This result is probably the most important contained in this chapter: indeed, the uniform bound on the second fundamental form implies completeness. On the other hand, the uniform bounds on the norm $\mathrm{I\!I}$ and $\nabla\mathrm{I\!I}$ play a key role in Chapter 9, to establish a uniform Schauder-type inequality (Proposition 9.3.4).

We denote $|\cdot|$ the norm induced by the Lorentzian metric. In particular, for a spacelike hypersurface Σ,

$$|\mathrm{I\!I}|^2(x) = \sum_{i,j=1}^{n} \mathrm{I\!I}(v_i, v_j)^2,$$

where $v_1, \ldots, v_n$ forms an orthonormal basis of $T_x\Sigma$.

First, some preliminary definitions: a unitary future-directed timelike $C^2(M)-$vector field $T \in \Gamma(TM)$ induces a Riemannian metric on M, given by

$$g_E(v, w) := g(v, w) + 2g(T, v)g(T, w).$$

This reference metric is used to measure the size of tensors: for a tensor field Φ on M, we denote

- $\|\Phi\| := \sup_{x\in M} g_E(\Phi_x, \Phi_x)^{1/2}$;
- $\|\Phi\|_k := \sum_{i=0}^{k} \|\overline{\nabla}^i \Phi\| = \|\Phi\| + \|\overline{\nabla}\Phi\| + \|\overline{\nabla}^2\Phi\| + \cdots + \|\overline{\nabla}^k\Phi\|$.

Here, $\overline{\nabla}$ is the Levi-Civita connection of (M, g) and we abusively denote g_E the extension of the metric to the space of tensors on M.

To observe how fast the metric of a spacelike hypersurface Σ tends to a degenerate one, it is common to fix a suitable reference unitary future-directed timelike vector field T and study the angle between T and the normal vector field $N\Sigma$ via the so called *gradient function*, namely

$$\nu_\Sigma(p) := -g(N_p, T_p), \tag{5.1}$$

where N denotes the unitary future-directed vector field normal to Σ. Hereafter, once a splitting is fixed, we take as T the future-directed unitary vector field spanning ∂_t, *i.e.*

$$T := \frac{1}{\sqrt{-g(\partial_t, \partial_t)}}\partial_t \in \Gamma(T\widetilde{\mathbb{H}}^{n,1}) = \Gamma(T\mathbb{H}^n \times T\mathbb{R}). \tag{5.2}$$

Moreover, for a point $p \in \widetilde{\mathbb{H}}^{n,1}$, we recall that p_+ is the unique point whose dual past hyperplane is $\mathcal{P}_+(p)$ (see Definition 1.5.2).

5.1. Local estimates

The following estimate generalizes [BS10, Lemma 4.13], where the statement is given only for maximal hypersurfaces.

Definition 5.1.1. For $\varepsilon > 0$, define $I_\varepsilon(p) := \{q \in I(p), \operatorname{dist}(p,q) > \varepsilon\}$. As for the cone, we denote to $I_\varepsilon^+(p)$ its future component and $I_\varepsilon^-(p)$ the past one (see Figure 5.1).

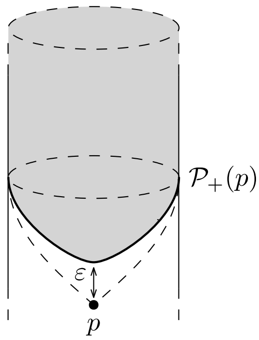

Figure 5.1.: The future $\varepsilon-$cone $I_\varepsilon^+(p)$ at Lorentzian distance ε from p.

Proposition 5.1.2. *For any $p \in \widetilde{\mathbb{H}}^{n,1}$, $\varepsilon > 0$, $L \geq 0$ and $K \subseteq I^-(p_+)$ compact set, there exist constants $C_m = C_m(p, \varepsilon, L, K)$, $m \geq -1$, such that*

$$\sup_{\Sigma \cap I_\varepsilon^+(p) \cap K} \nu_\Sigma \leq C_{-1};$$
$$\sup_{\Sigma \cap I_\varepsilon^+(p) \cap K} |\nabla^m \mathrm{I\!I}|^2 \leq C_m,\ m \geq 0;$$

for any spacelike graph Σ with constant mean curvature $H \in [-L, L]$ such that $p \in D(\Sigma)$.

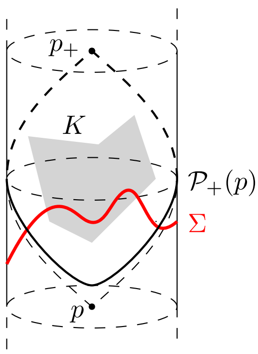

Figure 5.2.: Setting of Proposition 5.1.2.

The proof of the first item relies on [Bar21b, Theorem 3.1]. In the original result, the hypersurfaces are required to satisfy the so called *mean curvature structure condition*, *i.e.* there exists a constant L such that

$$\begin{cases} |H_\Sigma| \leq L\nu_\Sigma \\ |\nabla H_\Sigma| \leq L\left(\nu_\Sigma^2 + \nu_\Sigma |\mathrm{I\!I}|\right) \end{cases} \tag{5.3}$$

where ν_Σ is the gradient function associated to a spacelike hypersurface, defined in Equation (5.1). In our setting, H_Σ is constant and $\nu_\Sigma \geq 1$, so Equation (5.3) is satisfied by any $L \geq |H_\Sigma|$. Hence, the result can be stated as

Theorem 5.1.3 ([Bar21b]). *Let $\tau \in C^2(\widetilde{\mathbb{H}}^{n,1})$ a time function in the region $\{\tau \geq 0\}$. Assume there exist constants $c_0, c_1, c_2, c_3 > 0$ such that on $\{\tau \geq 0\}$ it holds*

$$\langle \overline{\nabla}\tau, \overline{\nabla}\tau \rangle \leq -c_0^{-2}, \qquad \|\tau\|_2 \leq c_1,$$
$$\|T\|_2 \leq c_2, \qquad \|\mathrm{Ric}\| \leq c_3.$$

Then, for every $L, \varepsilon > 0$, there exists a constant $C = C(L, \varepsilon^{-1}, c_0, c_1, c_2, c_3, \tau_{\max})$ such that

$$\sup_{\Sigma \cap \{\tau \geq \varepsilon\}} \nu_\Sigma \leq C,$$

for any spacelike hypersurface Σ in $\widetilde{\mathbb{H}}^{n,1}$, with constant mean curvature $H \in [-L, L]$ such that $\Sigma \cap \{\tau \geq 0\}$ is compact and $\partial\Sigma \cap \{\tau > 0\} = \emptyset$.

The second item follows from [Eck03, Theorem 2.2]. The original setting is the analysis of the mean curvature evolution in Lorentzian manifold satisfying the so called *timelike convergence condition*, namely

$$\mathrm{Ric}(X, X) \geq 0 \quad \text{for any timelike vector field } X \in \Gamma(TM).$$

A direct computation shows that any $\mathbb{A}\mathrm{d}\mathbb{S}-$manifold fulfills this condition.

Given a spacelike immersion $F_0 \colon \Sigma \to M$ in a Lorentzian manifold and a function

$$\mathcal{H} \colon M \times [0, t_0] \to \mathbb{R}$$

such that $X(\mathcal{H}) \geq 0$ for any future-directed timelike vector field X, one considers the solution $F \colon M \times [0, t_0] \to M$ of the prescribed mean curvature flow

$$\begin{cases} \frac{\partial F}{\partial t}(x,t) = \left[(H - \mathcal{H})N\right](x,t) \\ F|_{\Sigma \times \{0\}} = F_0 \end{cases} \tag{MCF$_{\mathcal{H}}$}$$

$H(x,t)$ and $N(x,t)$ being respectively the mean curvature and the future-directed normal vector of $\Sigma_t := F_t(\Sigma)$ at x.

We state [Eck03, Theorem 2.2] in the stationary case, *i.e.* for $\mathcal{H}$ a constant function and $\Sigma_0 = \operatorname{graph} F_0$ a spacelike graph with constant mean curvature $\mathcal{H}$: it follows that $X(\mathcal{H}) = 0$ for any vector field, $F_t = F_0$ and $\Sigma_t = \Sigma$, for all $t \in \mathbb{R}$.

Theorem 5.1.4 ([Eck03]). *Let $\tau \in C^2(\widetilde{\mathbb{H}}^{n,1})$ be a time function in the region $\{\tau \geq 0\}$ and assume there exist constants $c_0, c_1, c_2 > 0$ such that on $\{\tau \geq 0\}$ it holds*

$$\langle \overline{\nabla}\tau, \overline{\nabla}\tau \rangle \leq -c_0^{-2}, \qquad \|\tau\|_2 \leq c_1,$$
$$\|T\|_2 \leq c_2.$$

Then, for every $L, \varepsilon > 0$, there exist constants

$$C_m = C_m(L, n, \varepsilon^{-1}, c_0, c_1, c_2, \|\mathrm{Riem}\|_{m+1}),\ m \in \mathbb{N},$$

such that

$$\sup_{\Sigma \cap \{\tau \geq \varepsilon\}} |\nabla^m \mathrm{I\!I}|^2 \leq C_m.$$

for any spacelike hypersurface Σ in $\widetilde{\mathbb{H}}^{n,1}$, with constant mean curvature $H \in [-L, L]$ such that $\Sigma \cap \{\tau \geq 0\}$ is compact and $\partial\Sigma \cap \{\tau > 0\} = \emptyset$.

Remark 5.1.5. The idea of Proposition 5.1.2 is to apply Theorem 5.1.3 and Theorem 5.1.4 to suitably chosen (τ, T), and use the geometry of $\widetilde{\mathbb{H}}^{n,1}$, in order to obtain bounds that only depend on (p, K).

Proof of Proposition 5.1.2. Fix $p \in \widetilde{\mathbb{H}}^{n,1}$, $\varepsilon > 0$, $L \geq 0$ and $K \subseteq I^-(p_+)$. Let Σ be a spacelike graph such that $p \in D(\Sigma)$. Denote $\tau := \mathrm{dist}(p, \cdot) - \varepsilon/2$. By definition, $\{\tau > 0\} = I_{\varepsilon/2}(p)$ and $\{\tau \geq 0\} = \overline{I_{\varepsilon/2}(p)}$. Both sets are contained in $I(p)$, and $I(p) \cap \Sigma$ is precompact in $\widetilde{\mathbb{H}}^{n,1}$ because $p \in D(\Sigma)$ (Proposition 3.2.4). It follows that the function τ satisfies the conditions of Theorem 5.1.3 and Theorem 5.1.4, namely

$$\Sigma \cap \{\tau \geq 0\} \text{ is compact}, \quad \partial\Sigma \cap \{\tau > 0\} = \emptyset.$$

The region $\{\tau \geq 0\} \cap K$ is contained in $I^+(p) \cap I^-(p_+)$, which we can isometrically embed in $\mathbb{H}^{n,1}$ so that

$$\tau = \mathrm{dist}(p, \cdot) - \frac{\varepsilon}{2} = \arccos(-\langle p, \cdot \rangle) - \frac{\varepsilon}{2} \in C^\infty(I^+(p)).$$

Remark that the conditions on the constants c_0, c_1, c_2 of Theorem 5.1.3 and Theorem 5.1.4 coincide. Hence, it suffices to prove that c_0, c_1, c_2, c_3 only depend on p, ε and K.

Denote $u(x) = \langle p, x \rangle$, then $\overline{\nabla} u(x) = p + \langle p, x \rangle x$. It follows that

$$\overline{\nabla}\tau(x) = -\frac{1}{\sqrt{1 + \langle p, x \rangle^2}} \overline{\nabla} u = -\frac{p + \langle p, x \rangle x}{\sqrt{1 + \langle p, x \rangle^2}}.$$

Over $I^+(p)$, $\tau \geq 0 \iff \langle p, x \rangle^2 \leq \cos^2(\varepsilon/2)$, hence

$$\langle \overline{\nabla}\tau(x), \overline{\nabla}\tau(x) \rangle = -\frac{1 - \langle p, x \rangle^2}{1 + \langle p, x \rangle^2} \leq -\sin^2(\varepsilon/2) =: -c_0(\varepsilon)^{-2}.$$

The compactness of K allows to define

$$c_1 := \max_{K \cap \{\tau \geq 0\}} \|\tau\|_2, \qquad c_2 := \max_{K \cap \{\tau \geq 0\}} \|T\|_2, \qquad c_3 := \max_{K \cap \{\tau \geq 0\}} \|\mathrm{Ric}\|_2.$$

All constants only depend on p, ε, K and meet the requirements of Theorem 5.1.3 and Theorem 5.1.4, which concludes the proof. □

Remark 5.1.6. $\widetilde{\mathbb{H}}^{n,1}$ being a homogeneous space, one could think that c_1, c_2, c_3 in the proof are independent on the choice of p, and so are the constants C_i. This is not the case as T is not invariant by isometries of $\widetilde{\mathbb{H}}^{n,1}$, hence neither the induced Riemannian norm, so the constants C_i do depend on the choice of p.

5.2. Global estimates

We promote the local estimates of Proposition 5.1.2 to global bounds for the second fundamental form and its derivatives of any CMC entire spacelike graph in $\widetilde{\mathbb{H}}^{n,1}$.

Theorem 5.2.1. *Let $L \geq 0$, there exist constants $C_m(L, n)$, $m \in \mathbb{N}$, such that*

$$\sup_\Sigma |\nabla^m \mathrm{I\!I}|^2 \leq C_m(L, n),$$

for any entire spacelike graph Σ in $\widetilde{\mathbb{H}}^{n,1}$ with constant mean curvature $H \in [-L, L]$.

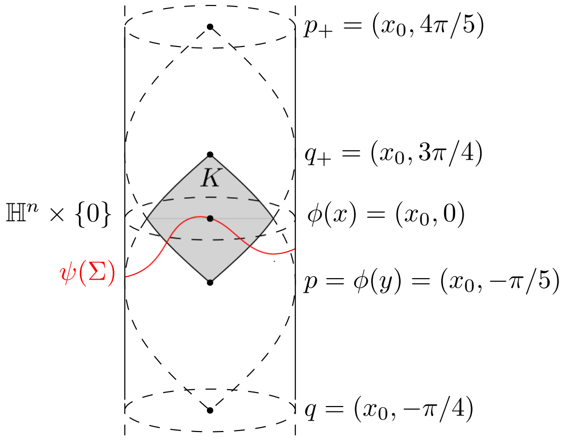

Figure 5.3.: Setting of Proposition 5.2.1.

Proof. In Proposition 5.1.2, we showed that the constants c_0, c_1, c_2 only depend on the choice of p, ε and K: here, we fix suitably p, ε and K and exploit the invariance the second fundamental form by the action of the isometry group, to have a global estimate.

Fix a splitting $\widetilde{\mathbb{H}}^{n,1} = \mathbb{H}^n \times \mathbb{R}$. Take an entire spacelike graph Σ with constant mean curvature H and fix a point $x \in \Sigma$. As remarked in Corollary 3.4.7,

$$\tau_{\mathbf{P}}(x) + \tau_{\mathbf{F}}(x) \geq \frac{\pi}{2}.$$

It follows that there exists a timelike geodesic joining x and $\partial\Omega(\partial\Sigma)$ whose length is at least $\pi/4$. In particular, there exists a point $y \in \Omega(\partial\Sigma)$ such that $\operatorname{dist}(x, y) = \pi/5$. Let $\phi \in \operatorname{Isom}\widetilde{\mathbb{H}}^{n,1}$ such that $\phi(x) = (x_0, 0)$, $\phi(y) = (x_0, -\pi/5) =: p$, whose existence is provided by the transitivity of $\operatorname{Isom}(\widetilde{\mathbb{H}}^{n,1})$ on the bundle of unitary timelike vectors in $T\mathbb{H}^{n,1}$. (Observe that ϕ might not preserve the time-orientation, but in this case $\Sigma' := \phi(\Sigma)$ has constant mean curvature $-H$, which does not affect the result.)

Denote $q := (x_0, -\pi/4)$ and $K := \overline{I^+(p) \cap I^-(q_+)}$, which is a compact set contained in $I^-(p_+)$ because $p \in I^+(q)$.

By construction, $p \in \Omega(\partial\Sigma') = D(\Sigma')$, hence we can apply Proposition 5.1.2, choosing $\varepsilon < \pi/5$, so that $\phi(x) = (x_0, 0) \in I^+_\varepsilon(p)$: at $\phi(x)$,

$$|\nabla^m \mathrm{II}|^2 \leq C_m(L, p, \varepsilon, K, n).$$

The second fundamental form is invariant by the action of $\operatorname{Isom}(\widetilde{\mathbb{H}}^{n,1})$, Σ and x are arbitrary, (p, ε, K) are fixed and that concludes the proof. □

5.3. Completeness

Completeness is a highly non-trivial property in Lorentzian geometry. In fact, a properly embedded spacelike hypersurface can inherits an incomplete metric from the Anti-de Sitter space.

Example 5.3.1. A spacelike hypersurfaces is incomplete if its tangent space rapidly diverges to a degenerate one, *i.e.* the normal vector becomes lightlike fast enough. An easy way to build an example is to consider a smooth strictly $1-$Lipschitz function $F\colon [0, +\infty) \to \mathbb{R}$ such that

$$\int_0^{+\infty} \sqrt{1 - \cosh^2(t) F'(t)^2} dt = A < +\infty.$$

Let $f(x) := F(\|x\|)$; the area of the entire graph $S = \text{graph} f$ is

$$Area(S) = (2\pi)^{n-1}A < +\infty,$$

namely S is an incomplete Riemannian manifold.

A first consequence of Theorem 5.2.1 is

Theorem B (Completeness)**.** *Any CMC entire spacelike graph in $\widetilde{\mathbb{H}}^{n,1}$ is complete.*

Indeed, an entire graph with uniformly bounded second fundamental form in the Anti-de Sitter space is complete. This criterion has been extensively used, see for example [BB09, Proposition 6.3.9], [BS10, Theorem 4.14], [LTW20, Corollary 3.30] or [SST23, Lemma 3.11].

For the sake of completeness, we give here a proof, following [BB09, Proposition 6.3.9].

Lemma 5.3.2. *Let Σ be a properly embedded spacelike hypersurface in $\mathbb{H}^{n,1}$. If the norm of the second fundamental form is uniformly bounded, then Σ is complete.*

Proof. Pick $p_0 \in \Sigma$. Any point $p \in \Sigma$ is space-related to p_0: by Equation 1.1, the length of the (unique) spacelike geodesic of $\mathbb{H}^{n,1}$ joinining p_0 and p is

$$d_{\mathbb{H}^{n,1}}(p_0, p) = \operatorname{arccosh}\left(-\langle p_0, p\rangle\right).$$

Remark that

$$f(p) := -\langle p_0, p\rangle = \cosh\left(d_{\mathbb{H}^{n,1}}(p_0, p)\right).$$

is a proper map over Σ. Indeed, let $(p_k)_{k\in\mathbb{N}}$ be sequence in Σ converging to $p_\infty \in \partial\Sigma$. The geodesics segments γ_k connecting p_0 converges to the geodesic segment γ_∞ connecting p_0 and p_∞. It follows that the sequence $(f(p_k))_{k\in\mathbb{N}}$ is bounded if and only if γ_∞ is a lightlike segment, which is not possible: indeed, Σ is an acausal hypersurface by Proposition 2.2.1, hence it is contained in $\Omega(\partial\Sigma)$ by Lemma 3.1.2. By definition of invisible domain, no causal curve joins p_0 to $\partial\Sigma$, and proving the properness of $f(p)$ over Σ.

We claim there exists $M > 0$ such that

$$\langle \nabla f, \nabla f\rangle \leq M(f^2 - 1).$$

It follows that, for any curve $c(t)$ joining p_0 and p,

$$\begin{aligned} d_{\mathbb{H}^{n,1}}(p_0, p) &= \int \frac{d}{dt}\operatorname{arccosh}\left(f(c(t))\right)dt = \\ &= \int \frac{\langle \nabla f, \dot c(t)\rangle}{\sqrt{f^2-1}}dt \leq \int M^{1/2}|\dot c(t)|dt = M^{1/2}L(c). \end{aligned}$$

By properness of $d_{\mathbb{H}^{n,1}}(p_0, p)$, any curve connecting p_0 and $\partial\Sigma$ has infinite length, concluding the proof.

By Lemma (1.2.2), the point p_0 is a global minimum for f over Σ. Hence, any maximal integral line $c\colon (a, b) \to \Sigma$ of ∇f converges to p_0 as $t \to b$. An explicit computation shows

$$-\nabla f(p) = p_0 + \langle p_0, p\rangle p + \langle p_0, N(p)\rangle N(p), \tag{5.4}$$

It follows that

$$\langle \nabla f, \nabla f\rangle = f^2 - 1 + \langle p_0, N\rangle$$

The function $f \geq 1$, so it suffices to prove that $|\langle p_0, N\rangle| \leq A(f-1)$, for some $A > 0$. Consider $p \in \Sigma$, let $c\colon (a,b) \to \Sigma$ be the maximal integral line of ∇f through p. Define

$$\begin{aligned} F(t) &:= f(c(t)) - f(p_0) = f(c(t)) - 1, \\ G(t) &:= \langle p_0, N(c(t))\rangle. \end{aligned}$$

Since $c(t) \to p_0$ as $t \to b$,

$$\lim_{t\to b} G(t) = \langle N(p_0), p_0\rangle = 0 = \lim_{t\to b} F(t).$$

On the other hand,

$$\begin{aligned} \dot{F}(t) &:= \langle \overline{\nabla} f, \dot{c}\rangle = \langle \nabla f, \nabla f\rangle = \mathrm{I}(\nabla f, \nabla f), \\ \dot{G}(t) &:= \langle p_0, \nabla_{\dot{c}} N, p_0\rangle = \langle \nabla f, \nabla_{\nabla f} N\rangle, \end{aligned}$$

where the last equation one is obtained by combining Equation (5.4) together with

$$\nabla_v N \in T_p\Sigma = \mathrm{Span}(p, N(p))^{\perp},$$

for any $v \in T_p\Sigma$. Denote by $A := \sup_\Sigma \|\mathrm{II}\|^2$, which is finite by hypothesis. Then,

$$|\dot{G}(t)| = |\langle \nabla f, \nabla_{\nabla f} N\rangle| = |\mathrm{II}(\nabla f, \nabla f)| \leq A g(\nabla f, \nabla f) = A\dot{F}(t). \tag{5.5}$$

It follows that $|G(t)| \leq AF(t)$, that is $0 \leq |\langle p_0, N\rangle| \leq A(f-1)$, proving the claim and concluding the proof. □

Remark 5.3.3. In the proof, we implicitly assumed that Σ is C^2: otherwise, the second fundamental form is not defined over Σ, hence the function G could be not differentiable over the integral line c.

However, it is possible to weaken the regularity of Σ to $C^{1,1}$, so that the second fundamental form and $\dot{G}$ are defined almost everywhere, over Σ and c respectively. Hence, the inequality in Equation (5.5) holds almost everywhere, concluding that $|G(t)| \leq AF(t)$.

Since both F and G are continuous, if $|G(t)| \leq AF(t)$ almost everywhere, then $|G| \leq AF$ over c, concluding the proof.

Chapter 6.

Existence

In this section we will prove the first part of Theorem A. The argument generalizes the one used in [BS10].

Theorem A (Existence)**.** *Let Λ be an admissible boundary in $\widetilde{\mathbb{H}}^{n,1}$ and $H \in \mathbb{R}$. There exists a smooth entire spacelike graph Σ with constant mean curvature H and such that $\partial\Sigma = \Lambda$.*

Proof. The proof consists in three steps: in Step 1, we show that the thesis holds true assuming that there exists an acausal Cauchy hypersurface W for $\Omega(\Lambda)$ with the following *barrier* property: for any compact spacelike hypersurface S with constant mean curvature H and boundary contained in W, either $S \subseteq I^+(W)$ or $S \subseteq I^-(W)$.

In Step 2 (resp. Step 3), we provide a barrier hypersurface for the case $H \neq 0$ (resp $H = 0$).

Step 1: Equivalent statement.

The proof consists in building a sequence of compact graphs with given constant mean curvature which smoothly converges to an entire graph which the same constant mean curvature.

Fix a splitting $\widetilde{\mathbb{H}}^{n,1} = \mathbb{H}^n \times \mathbb{R}$ and a barrier hypersurface W. For $r > 0$, denote

$$B_r := B_{\mathbb{H}^n}(0, r), \qquad S_r := W \cap (B_r \times \mathbb{R}).$$

S_r is the intersection between W and the cylinder over B_r. By assumption, W is an entire acausal graph, so S_r is an acausal graph over B_r.

Since S_r is a compact acausal graph, by [Bar21b, Theorem 4.1] there exists a compact smooth spacelike graphs Σ_r with constant mean curvature H, such that $\partial\Sigma_r = \partial S_r$.

By the barrier property of W, there exists an unbounded subset $Y \subseteq \mathbb{R}^+$ such that, for all $r \in Y$, Σ_r is contained in the same connected component of $\widetilde{\mathbb{H}}^{n,1} \setminus W$. Without loss of generality, we assume $\Sigma_r \subseteq I^+(W)$, for all $r \in Y$. Each Σ_r is the graph of a $1-$Lipschitz map u_r defined on an open ball of $\mathbb{S}^n_+$ (Lemma 2.1.4). Moreover, Σ_r is contained in $\Omega(\Lambda)$, then $|u_r|$ is uniformly bounded. By a diagonal process, we can extract a sequence $\Sigma_k = \operatorname{graph} u_k$ converging to an achronal entire graph $\Sigma = \operatorname{graph} u \subseteq \overline{\Omega(\Lambda)}$. In particular, $\partial\Sigma = \Lambda$: indeed,

$$\partial\Sigma \subseteq \overline{\Omega(\Lambda)} \cap \partial\widetilde{\mathbb{H}}^{n,1} = \Lambda,$$

and the other inclusion follows by the fact that $\partial\Sigma$ and Λ are graphs of functions $\partial\mathbb{H}^n \to \mathbb{R}$.

We need to show that Σ is smooth, spacelike and with constant mean curvature H, which are local properties. Hence, it suffices to prove it on $\Sigma \cap (B_R \times \mathbb{R})$, for a fixed $R > 0$. Since $\Sigma_k \subseteq I^+(W)$,

$$K_R := \overline{I^+(W)} \cap \overline{\Omega(\Lambda)} \cap (B_R \times \mathbb{R})$$

is a compact subset of $\overline{\Omega(\Lambda)}$ containing $\Sigma_k \cap (B_R \times \mathbb{R})$, for all $k \in \mathbb{N}$.

We recall that $I_\varepsilon(p)$ is the set $\{\mathrm{dist}(p, \cdot) > \varepsilon\}$ (see Definition 5.1.1). By definition of Cauchy hypersurface,

$$\{I^+_\varepsilon(p), p \in I^-(W) \cap \Omega(\Lambda), \varepsilon > 0\}$$

is an open cover of $\Omega(\Lambda)$: indeed, it trivially covers $\overline{I^-(W)} \cap \Omega(\Lambda)$, while any point in the future of W is connected to W through a timelike past inextensible curve. We can then extract a finite subcover of K_R, namely $I^+_{\varepsilon_i}(p_i)$, $i = 1, \dots, h$, and produce constants

$$C_m(i, R) = C_m(p_i, \varepsilon_i, |H|, K_R),$$

as in Proposition 5.1.2: indeed, by Lemma 3.1.7, $K_R \subseteq I^-(p_+)$, for all $p \in \Omega(\Lambda)$. We define

$$C_m(R) := \max_{i=1,\dots,k} C_m(i, R), \quad m \geq -1.$$

We claim that the sequence p_i is eventually contained in $D(\Sigma_k)$, for all $i = 1, \dots, h$: hence, by Proposition 5.1.2, the gradient function and the second fundamental form of Σ_k, together with all its derivatives, are uniformly bounded on $B_R \times \mathbb{R}$, for k big enough.

To prove the claim, remark that p_i are contained in $\Omega(\Lambda) = D(W)$, hence $I^+(p_i)$ meets W in a precompact set. In particular, there exists $r > 0$ such that $I^+(p_i) \cap W \subseteq B_r \times \mathbb{R}$, for all $i = 1, \dots, h$. In particular, if Σ_k is a graph over B_r, which is the case for k big enough, $I^+(p_i) \cap \Sigma_k$ is precompact, namely $p_i \in D(\Sigma_k)$, for all $i = 1, \dots, h$.

The uniform bound on the gradient function ensures that $\mathrm{graph}(u|_{B_R})$ is spacelike, while the bounds on the derivatives of the second fundamental form imply bounds on all the derivatives of the u_k on B_R, uniformly in k. We have already remarked that $|u_k|$ are uniformly bounded: hence, $\{u_k, k > \bar{k}\}$ is precompact in $C^\infty(B_R)$. Since u_k converges to u over B_R, $u|_{B_R}$ is smooth.

[Bar21a, Equation (2.7)] provides an explicit formula for the mean curvature of a graph $S = \mathrm{graph}\, f$, that is

$$H_S = \frac{1}{\nu_S} \left(\mathrm{div}_S(\varphi \nabla f) + \mathrm{div}_S T \right), \tag{6.1}$$

where $\mathrm{div}_S(X) = \sum_{i=1}^n \langle \overline{\nabla}_{v_i} X, v_i \rangle$, for v_i an orthonormal basis of TS, $X \in \Gamma(T\widetilde{\mathbb{H}}^{n,1})$ and $\varphi := \sqrt{-g(\partial_t, \partial_t)}$, which is known in the literature as *tilt function*. The right hand side of Equation (6.1) is constant for $k \geq \bar{k}$, hence $H_\Sigma = H_{\Sigma_k} = H$ on $B_R \times \mathbb{R}$.

Since the choice of R was arbitrary, Σ is a smooth entire spacelike graph with constant mean curvature H. Hence, we proved the thesis under the further assumption of the existence of a barrier hypersurface W, which concludes the proof of Step 1.

Step 2: $H \neq 0$.

We need to exhibit a barrier hypersurface. For $H > 0$, pick $\theta = \arctan(H/n)$ and choose $W = W^{\mathbf{P}}_\theta$: Corollary 4.4.3 ensures that any compact spacelike hypersurface with constant mean curvature H and whose boundary belongs to W is contained in the future of W. The same argument applies for $H < 0$, choosing $W = W^{\mathbf{F}}_\theta$. Hence, by Step 1, there exists an entire CMC hypersurface Σ_H bounding Λ, for any $H \neq 0$.

Step 3: $H = 0$.

For the maximal case, we choose one entire CMC graph just found, namely $W = \Sigma_H$, for $H \neq 0$. This choice meets the barrier property by Proposition 4.2.1, which concludes the proof. □

Remark 6.0.1. Quite surprisingly, the maximal case is the most delicate to deal with, because, in general, $\partial_+\mathcal{CH}(\Lambda) = W_0^{\mathbf{P}}$ and $\partial_-\mathcal{CH}(\Lambda) = W_0^{\mathbf{F}}$ are not Cauchy hypersurfaces for $\Omega(\Lambda)$. Indeed, as soon as Λ contains transverse lightlike segments, $\partial\mathcal{CH}(\Lambda)$ intersects $\partial\Omega(\Lambda)$. In that case, $\partial_\pm\mathcal{CH}(\Lambda)$ contains lightlike geodesic segments (Lemma 3.1.4), hence it is not an acausal hypersurface. For example, the convex core of all hypersurfaces described in Chapter 8 coincides with the invisible domain of their boundary (Remark 8.1.5).

To our knowledge, a direct way to overcome this problem is still to be found: indeed, [BS20; Tam19a] did not deal with degenerate boundaries, while [LTW20; SST23] solve the Plateau's problem for non-degenerate boundaries and then use a compactness argument to deform them in solutions for degenerate ones.

Chapter 7.

A compactness result

The aim of this short section is to prove the following statement:

Proposition 7.0.1. *Let Σ_k be a sequence of entire spacelike graphs with constant mean curvature H_k, contained in a precompact set of $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$. Up to taking a subsequence, we can assume that Σ_k converges to an entire achronal graph Σ_∞ and H_k converges to $H_\infty \in \mathbb{R} \cup \{\pm\infty\}$.*

Then, exactly one of the following holds:

1. *if $\partial\Sigma_\infty$ is not admissible, then Σ_∞ is a totally geodesic lightlike hypersurface;*
2. *if $\partial\Sigma_\infty$ is admissible and $H_\infty = \pm\infty$, then $\Sigma_\infty = \partial_\mp(\Omega(\partial\Sigma_\infty))$;*
3. *if $\partial\Sigma_\infty$ is admissible and $H_\infty \in \mathbb{R}$, then Σ_∞ is a CMC entire graph with constant mean curvature H_∞.*

Moreover, in the last case, Σ_k converges to Σ_∞ smoothly as a graph, in any splitting.

Proof. In a splitting, Σ_k are the graphs of $1-$Lipschitz functions u_k, which are uniformly bounded: indeed, any precompact set of $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$ is contained in a suitable slice $(\widetilde{\mathbb{H}}^n \cup \partial\mathbb{H}^n) \times [a, b]$. By Ascoli-Arzelà theorem, up to extracting a subsequence, u_k uniformly converges to a $1-$Lipschitz function $u_\infty \colon \mathbb{H}^n \to \mathbb{R}$, *i.e.* Σ_k tends to an entire achronal graph $\Sigma_\infty = \operatorname{graph} u_\infty$. In particular, $\Sigma_\infty \subseteq \overline{\Omega(\partial\Sigma_\infty)}$.

Step 1: $\partial\Sigma_\infty$ not admissible.

The boundary $\partial\Sigma_\infty$ either bounds a totally geodesic lightlike hypersurfaces or is an admissible boundary. In the former case, Σ_∞ has to be a totally geodesic lightlike hypersurface (Lemma 2.1.7), so we proved the first item.

Step 2: $\partial\Sigma_\infty$ admissible, $H_\infty = \pm\infty$.

Without loss of generality, $H_\infty = -\infty$. Denote $a_k := \max_{\partial\mathbb{H}^n} |u_\infty - u_k|$, and replace u_k by $u_k + a_k$, so that

$$u_k|_{\partial\mathbb{H}^n} \geq u_\infty|_{\partial\mathbb{H}^n}, \quad \forall k \in \mathbb{N}.$$

In other words, $\partial\Sigma_k$ is in the future of $\partial\Sigma_\infty$: by Proposition 4.4.1, $\Sigma_k \subseteq I^+(W_\theta^{\mathbf{F}})$, for $\theta < -\arctan(H_k/n)$. It follows that Σ_k is eventually contained in $I^+(W_\theta^{\mathbf{F}})$, for any $\theta \in (0, \pi/2)$.

$$\Sigma_\infty \subseteq \overline{\Omega(\partial\Sigma_\infty)} \cap \bigcap_{\theta \in (0,\pi/2)} I^+\left(W_\theta^{\mathbf{F}}\right) = \partial_+\Omega(\partial\Sigma_\infty),$$

hence $\Sigma_\infty = \partial_+\Omega(\partial\Sigma_\infty)$ by entireness.

Step 3: $\partial\Sigma_\infty$ admissible, $H_\infty \in \mathbb{R}$.
Fix $\varepsilon > 0$, denote $\mathcal{H} := \sup_{k\in\mathbb{N}} |H_k| + \varepsilon$, which is finite as H_k is bounded, and let $\Sigma_{\mathcal{H}} := \operatorname{graph} u_{\mathcal{H}}$ be the unique entire spacelike graph with constant mean curvature $\mathcal{H}$ sharing the same boundary as Σ_∞. As before, consider $a_k := \max_{\partial\mathbb{H}^n} |u - u_k|$ and replace u_k by $u_k + a_k$. By construction, $\partial\Sigma_k$ is in the future of $\partial\Sigma_\infty = \partial\Sigma_{\mathcal{H}}$ and $H_k < \mathcal{H}$. By the maximum principle (Proposition 4.2.1) $\Sigma_k \subseteq I^+(\Sigma_{\mathcal{H}})$, $\forall k \in \mathbb{N}$.

In short, we will use $\Sigma_{\mathcal{H}}$ as a barrier, like in the existence proof: fix a radius $R > 0$ and consider the sequence restricted to the closed cylider $B(0,R) \times \mathbb{R}$.

$$K_R := (B(0,R) \times \mathbb{R}) \cap I^+(\Sigma_{\mathcal{H}}) \cap \Omega(\partial\Sigma_\infty)$$

is a precompact open neighbourhood of $\Sigma_\infty \cap (B(0,R) \times \mathbb{R})$ containing $\Sigma_k \cap (B(0,R) \times \mathbb{R})$. As in the proof of existence, we cover $\overline{K_R}$ with a finite number of cones $I^+_{\varepsilon_i}(p_i)$, for $p_i \in I^-(\Sigma_{\mathcal{H}}) \cap \Omega(\partial\Sigma_\infty)$, and use Proposition 5.1.2 with $L = \mathcal{H}$ to give a uniform bound on the gradient function and on the norm of the derivatives of the second fundamental form, in order to promote the uniform convergence to a smooth one.

To conclude, one can use Equation (6.1) to prove that Σ_∞ has constant mean curvature equal to H_∞. □

7.1. A topological statement

Denote $\mathcal{CMC}$ the space of CMC entire hypersurfaces in $\widetilde{\mathbb{H}}^{n,1}$, equipped with the $C^\infty(\mathbb{H}^n)-$topology, and $\mathcal{B}$ the space of admissible boundaries, which is an open subspace of $\operatorname{Lip}(\mathbb{S}^{n-1})$.

Corollary 7.1.1. *$\mathcal{CMC}$ is homeomorphic to $\mathcal{B} \times \mathbb{R}$.*

Proof. Consider the map

$$\begin{aligned} \mathcal{CMC} &\longrightarrow \mathcal{B} \times \mathbb{R} \\ \Sigma &\longmapsto (\partial\Sigma, H_\Sigma) \end{aligned} .$$

The correspondence is bijective due to Theorem A, and is continuous and proper due to Proposition 7.0.1, hence a homeomorphism. □

Remark 7.1.2. The $C^\infty(\mathbb{H}^n)-$topology is equivalent to the topology induced by $\operatorname{Lip}(\mathbb{S}^n_+)$ on $\mathcal{CMC}$, and the latter compactifies $\mathcal{CMC}$. The boundary $\partial\mathcal{CMC}$ inside $\operatorname{Lip}(\mathbb{S}^n_+)$ is described by Proposition 7.0.1: a diverging sequence $\Sigma_k \subseteq \mathcal{CMC}$ converges either to a totally geodesic degenerate hypersurfaces or to the boundary of an invisible domain.

The boundary of $\mathcal{B}$ in $\operatorname{Lip}(\mathbb{S}^{n-1})$ consists of $1-$Lipschitz maps f such that $\operatorname{osc}(f) = \pi$, namely boundaries of totally geodesic degenerate hypersurfaces, and we compactify $\mathbb{R}$ as $\mathbb{R} \cup \{\pm\infty\}$. It follows that the homeomorphism does not extend to the boundary of $\mathcal{B} \times \mathbb{R}$ in the product $\operatorname{Lip}(\mathbb{S}^{n-1}) \times (\mathbb{R} \cup \{\pm\infty\})$. Indeed, consider the diverging sequence $(\partial\Sigma_k, H_{\Sigma_k})$: if $\partial\Sigma_k$ diverges in $\mathcal{B}$, then Σ_k converges to a totally geodesic degenerate hypersurfaces, independently on the behaviour of H_k. If $\partial\Sigma_k$ converges to $\Lambda \in \mathcal{B}$, Σ_k converges to $\partial_\pm\Omega(\Lambda)$. It follows that

$$\partial\mathcal{CMC} \cong \partial\mathcal{B} \cup (\mathcal{B} \times \{\pm\infty\}) \neq \partial(\mathcal{B} \times \mathbb{R}).$$

Chapter 8.

Explicit bounds

Combining [KKN91, Theorem 1] and Theorem B, we can sharpen the bound $C_0(L,n)$ on the norm of the second fundamental form (Theorem 5.2.1).

Theorem 8.0.1. *Let $L \geq 0$. For any properly embedded hypersurface Σ with constant mean curvature $H \in [-L, L]$ in $\mathbb{H}^{n,1}$, the following holds:*

$$|\mathrm{II}_\Sigma|^2 \leq n\left(1 + \frac{L^2 + L(n-2)\sqrt{L^2 + 4(n-1)}}{2(n-1)}\right). \tag{8.1}$$

Moreover, if the maximum is reached at one point, II_Σ is parallel.

The proof consists in replacing the word *complete* with *properly embedded* in the statement [KKN91, Theorem 1], which is possible because Theorem B makes the two properties equivalent for CMC hypersurfaces.

In the following, we classify properly embedded hypersurfaces with parallel second fundamental form (Proposition 8.2.2) in order to present [KKN91, Theorem 2] from a more geometric point of view (Proposition 8.2.3).

8.1. Cylindrical hypersurfaces

Consider a totally geodesic spacelike submanifold M in $\mathbb{H}^{n,1}$, and let $M_+^\perp$ be its dual in the future, namely

$$M_+^\perp := \bigcap_{x \in M} P_+(x).$$

Each pair $(x,y) \in M \times M_+^\perp$ is connected by the timelike geodesic parameterized by arclength

$$\gamma_{xy}(t) := \cos(t)x + \sin(t)y.$$

Definition 8.1.1. Let M be a totally geodesic spacelike submanifold of dimension $k \in \{0, \ldots, n\}$, and $\theta \in (0, \pi/2)$. We define a cylindrical hypersurface

$$\mathbb{H}(k,\theta) := \{\gamma_{xy}(\theta), x \in M, y \in M_+^\perp\}.$$

Let M' be another totally geodesic spacelike submanifold of dimension k. One can easily check that any time preserving isometry sending M to M' sends $\mathbb{H}(k,\theta)$ to

$$\{\gamma_{xy}(\theta), x \in M', y \in (M')_+^\perp\},$$

namely $\mathbb{H}(k,\theta)$ is well defined, up to isometry. Moreover a time reversing isometry fixing M sends $\mathbb{H}(k,\theta)$ to $\mathbb{H}\left(n-k, \frac{\pi}{2} - \theta\right)$: indeed, it sends M_+ to M_-, and $(M_-)_+ = M$.

Remark 8.1.2. For $k \in \{0, n\}$, we recover the equidistant hypersurfaces described in Section 4.3: in fact, $P_\theta^+ = \mathbb{H}(0,\theta)$ and $P_\theta^- = \mathbb{H}(n,\theta)$.

The following lemmas describes the geometry of the cylindrical hypersurfaces. A direct computation gives:

Lemma 8.1.3. *$\mathbb{H}(k,\theta)$ is a properly embedded spacelike hypersurface isometric to*

$$(\cos\theta)\mathbb{H}^k \times (\sin\theta)\mathbb{H}^{n-k},$$

whose boundary consists of lightlike segments connecting the boundaries of $\partial\mathbb{H}^k$ and $\partial\mathbb{H}^{n-k}$.

Lemma 8.1.4. *$\mathbb{H}(k,\theta)$ has parallel second fundamental form. In particular, it has constant mean curvature $H = k\tan(\theta) - \frac{n-k}{\tan(\theta)}$.*

Proof. To prove that $\mathrm{I\!I}$ is parallel, it suffices to remark that

$$\mathrm{Isom}(\mathbb{H}^k) \times \mathrm{Isom}(\mathbb{H}^{n-k}) \cong \mathrm{O}(k,1) \times \mathrm{O}(n-k,1) \subseteq \mathrm{O}(n,2) \cong \mathrm{Isom}(\mathbb{H}^{n,1}),$$

acts transitively on $\mathbb{H}(k,\theta)$. By a direct computation, analogous as the one contained in the proof of Lemma 4.3.1, it holds

$$B = \left(\tan(\theta)\mathrm{Id}_k, -\frac{1}{\tan(\theta)}\mathrm{Id}_{n-k}\right), \tag{8.2}$$

which concludes the proof. □

Remark 8.1.5. For $k \in \{1, \dots, n-1\}$, one can prove $\mathcal{CH}(\Lambda_k) = \overline{\Omega(\Lambda_k)}$, for $\Lambda_k := \partial\mathbb{H}(k,\cdot)$. It follows that cylindrical hypersurfaces are not convex, for $k \neq 0, n$. By contradiction, assume $\mathbb{H}(k,\theta)$ is convex, for $\theta \in (0,\pi/2)$: hence, either $I^-(\mathbb{H}(k,\theta)) \cap I^+(\partial_-\mathcal{CH}(\Lambda_k))$ or $I^+(\mathbb{H}(k,\theta)) \cap I^-(\partial_+\mathcal{CH}(\Lambda_k))$ is a convex set containing Λ_k and strictly contained in $\mathcal{CH}(\Lambda_k)$, contradicting the minimality of the convex hull.

8.2. Achieving the bound

We exhibit the only CMC entire hypersurfaces achieving the bound of Theorem 8.0.1.

We recall some results of pseudo-Riemannian geometry, which we state for $\mathbb{H}^{n,1}$, but hold in complete generality. Since the second fundamental form $\mathrm{I\!I}$ and the shape operator B are dual with respect to the metric, $\mathrm{I\!I}$ is parallel if and only if B is parallel. Moreover, B is a symmetric $(1,1)-$tensor, hence diagonalizable with respect to an orthonormal basis.

Lemma 8.2.1. *Let Σ be a spacelike hypersurface of $\mathbb{H}^{n,1}$ with parallel shape operator B,*

1. *the eigenvalues of B are constant along Σ;*
2. *the eigenspaces of B are parallel. More precisely, let V_λ be the distribution such that $V_\lambda(x) \subseteq T_x\Sigma$ is the eigenspace of λ at x: then $\nabla_w V_\lambda \subseteq V_\lambda$, for any $w \in T_x\Sigma$;*
3. *the eigenspaces of B are integrable.*

Proof. We recall that B is parallel if and only if

$$\nabla_Y B(X) = B(\nabla_Y X), \qquad \forall X, Y \in \Gamma(T\Sigma). \tag{8.3}$$

To prove (1), consider two point $x, y \in \Sigma$, a curve c in Σ connecting them. Let X be a parallel vector field along c, then $B(X)$ is parallel along c, too: indeed, by Equation (8.3)

$$\nabla_{c'}(B(X)) = B(\nabla_{c'} X) = 0.$$

As a consequence, B commutes with the parallel transport along c, denoted $P_c\colon T_x\Sigma \to T_y\Sigma$, that is

$$P_c \circ B(x) = B(y) \circ P_c\colon T_x\Sigma \to T_y\Sigma.$$

In other words, $B(x)$ and $B(y)$ are conjugate by the linear isometry T_c, hence they have the same characteristic polynomial, which proves that the eigenvalues of B are constant over Σ. Moreover, the dimension of each eigenspace is constant, hence each eigenspace V_λ forms a subbundle of $T\Sigma$.

To prove (2), consider a local eigenvector field $X \in \Gamma(V_\lambda)$, and substitute (1) in Equation (8.3) to obtain

$$B(\nabla_Y X) = \nabla_Y B(X) = Y(\lambda)X + \lambda\nabla_Y X = \lambda\nabla_Y X,$$

that is $\nabla_Y X$ is an eigenvector field relative to λ, *i.e.* $\nabla_Y X \in \Gamma(V_\lambda)$, for any vector field $Y \in \Gamma(T\Sigma)$.

Finally, (3) follows directly by Frobenius Theorem: indeed, for any $X, Y \in \Gamma(V_\lambda)$

$$[X, Y] = \nabla_X Y - \nabla_Y X,$$

which belongs to $\Gamma(V_\lambda)$ by (2), concluding the proof. □

Proposition 8.2.2. *Cylindrical hypersurfaces are the only properly embedded hypersurfaces in $\mathbb{H}^{n,1}$ with parallel second fundamental form.*

Proof. Let Σ be a properly embedded hypersurface of $\mathbb{H}^{n,1}$ with parallel second fundamental form. In particular, it is an entire graph, hence diffeomorphic to $\mathbb{R}^n$. By the fundamental theorem of immersed hypersurfaces, it suffices to prove that Σ has the same induced metric as $\mathbb{H}(k, \theta)$, for a suitable choice of $k = 0, \dots, n$ and $\theta \in (0, \pi/2)$, and that they have the same shape operator.

The eigenspaces V_{λ_i} of B are integrable (Lemma 8.2.1) and parallel, moreover, Σ is a CMC hypersurface, hence complete (Theorem B). By De Rham Decomposition Theorem, Σ is isometric to the product of the integral submanifolds M_i of V_{λ_i} (see [KN63, Chapter IV, Section 6]). Moreover, M_i is a complete totally geodesic submanifold of Σ.

By Gauss equation, the sectional curvature along the tangent 2-plane $\mathrm{Span}\{v_i, v_j\}$, for $v_i \in V_{\lambda_i}$ and $v_j \in V_{\lambda_j}$, is

$$K_\Sigma(\mathrm{Span}\{v_i, v_j\}) = -1 - \lambda_i\lambda_j.$$

It follows that M_i is a simply connected complete manifold with constant sectional curvature $-1 - \lambda_i^2 < 0$ and dimension $k_i = \dim V_{\lambda_i}$, hence it is isometric to $\cos(\theta_i)\mathbb{H}^{k_i}$, for $\theta_i = \arctan(\sqrt{\lambda_i})$ (compare with Lemma 4.3.1). Moreover, since Σ is a product, the sectional curvature vanishes for $i \neq j$, hence B has at most two eigenvalues, and in that case $\lambda_2 = -1/\lambda_1$.

Hence, the metric and the shape operator of Σ coincide with the ones of $\mathbb{H}(k_1, \theta_1)$ (Equation (8.2)), which concludes the proof. □

Proposition 8.2.3. *Let Σ be a properly embedded hypersurface with constant mean curvature H. Assume there exists $x \in \Sigma$ such that*

$$|\mathrm{I\!I}_\Sigma|^2(x) = n\left(1 + \frac{H^2 + |H|(n-2)\sqrt{H^2 + 4(n-1)}}{2(n-1)}\right).$$

- *If* $H = 0$, *then* $\Sigma \cong \mathbb{H}\Big(k, \arctan(\sqrt{(n-k)/k})\Big)$, *for some* $k = 1, \dots, n-1$;
- *otherwise,* $\Sigma \cong \mathbb{H}(1, \theta_H)$,

where $\tan(\theta_H)$ *is the positive solution of* $t^2 - Ht - (n-1) = 0$.

Proof. Denote S the bound. By Theorem 8.0.1, if the bound is achieved at one point, then the second fundamental form is parallel, hence Σ is a cylindrical hypersurface by Proposition 8.2.2. One can then compute explicitely the norm of the second fundamental form of $\mathbb{H}(k, \theta)$, using Equation (8.2).

Denote $S(k, \theta) := |\mathrm{I\!I}_{\mathbb{H}(k,\theta)}|^2$ and S the bound in the statement. For $k = 0, n$, namely $\Sigma = P^{\pm}_{\theta}$, $S(k, \theta) = H^2/n < S$. For $k \notin \{0, n\}$, one obtains

$$S(k, \theta) = n + \frac{nH^2 + |H|(n-2k)\sqrt{H^2 + 4k(n-k)}}{2k(n-k)}.$$

In the maximal case, $S(k, \theta) = S$ for any $k \in \{1, \dots, n-1\}$. For $H \neq 0$, $S(k, \theta) = S$ if and only if at $k = 1, n-1$, concluding the proof. □

Part III.

Time functions on convex domains

Chapter 9.

CMC foliation

We prove that entire CMC hypersurfaces analytically foliate their domain of dependence.

Theorem C. *Let Λ be an admissible boundary. Then $\{\Sigma_H\}_{H\in\mathbb{R}}$ is an analytic foliation of the invisible domain $\Omega(\Lambda)$, where Σ_H is the unique properly embedded hypersurface with constant mean curvature equal to H and boundary Λ.*

For this section, we fix an admissible boundary $\Lambda \subseteq \partial\widetilde{\mathbb{H}}^{n,1}$. The section is organized as follows: first, we show that $\{\Sigma_H\}_{H\in\mathbb{R}}$ is a topological foliation of $\Omega(\Lambda)$. After that, we briefly present the plan to improve the regularity of the foliation. All technical computations are contained in Section 9.3. Finally, we prove Corollary D.

9.1. Continuous foliation

The CMC hypersurfaces $\{\Sigma_H\}_{H\in\mathbb{R}}$ topologically foliate the invisible domain of Λ if any point $p \in \Omega(\Lambda)$ is contained in a unique CMC entire hypersurface Σ_H.

The uniqueness follows from Proposition 4.2.1: indeed, since they have the same boundary, Σ_H does not intersect Σ_K, for $H \neq K$.

To prove that any point $p \in \Omega(\Lambda)$ belongs to a CMC entire hypersurface, denote

$$H^{\pm}(p) := \left\{H \in \mathbb{R},\, p \in I^{\pm}(\Sigma_H)\right\}.$$

In the proof of Proposition 7.0.1, we saw that Σ_H approaches the boundary of $\partial\Omega(\Lambda)$ as H diverges, namely for H big enough, p lies in the future of Σ_H and in the past of Σ_{-H}. In other words $H^{\pm}(p)$ are not empty. By Proposition 4.2.1, if $H \in H^+(p)$, then $[H, +\infty) \subseteq H^+(p)$. Conversely, if $H \in H^-(p)$, then $(-\infty, H] \subseteq H^-(p)$. It follows that

$$\sup H^-(p) = \inf H^+(p) =: H(p).$$

Finally, take a sequence $(H_k^{\pm})_{k\in\mathbb{N}} \subseteq H^{\pm}(p)$ converging to $H(p)$: by Proposition 7.0.1, $\Sigma_{H_k^{\pm}}$ converges to $\Sigma_{H(p)}$. Since $p \in I^{\pm}(\Sigma_{H_k^{\pm}})$ for all k, then

$$p \in \overline{I^+(\Sigma_{H(p)})} \cap \overline{I^-(\Sigma_{H(p)})} = \Sigma_{H(p)}.$$

Remark 9.1.1. The existence of a continuous foliation provides examples of non-convex CMC entire hypersurfaces, in contrast with the flat case [Tre82, Corollary to Proposition 5]. The idea is the following: take an admissible boundary Λ not asymptotic to a totally geodesic hypersurface, so that the maximal hypersurface Σ_0 is contained in the interior of $\mathcal{CH}(\Lambda)$. For H small enough, Σ_H intersects the interior of $\mathcal{CH}(\Lambda)$: if Σ_H was convex, we could build a convex hypersurface strictly contained in the convex core, different from its boundary component, contradicting the minimality of $\mathcal{CH}(\Lambda)$. Furthermore, in Chapter 8, we provide a class of boundaries which bound only non-convex CMC entire hypersurfaces (Remark 8.1.5).

9.2. Regular foliation

In the proof of the existence part of Theorem A, we proved that the leaves Σ_H of the foliation are graph of smooth function on $\mathbb{H}^n$, while the map that associates $H \to \Sigma_H$ is smooth by Proposition 7.0.1.

Following the idea of [BSS19, Section 4], we locally trivialize $\Omega(\Lambda)$ by showing that the mean curvature operator $\mathcal{H}$ is invertible at Σ_H, in the space of deformations of Σ_H in $\widetilde{\mathbb{H}}^{n,1}$. The key result needed for the proof is the uniform bound on the norm of the derivatives of the second fundamental form (Theorem 5.2.1).

By Equation (6.1), if $\Sigma_H = \operatorname{graph} u_H$, for $u_H \in C^\infty(\mathbb{H}^n)$, then u_H satisfies the differential equation $L^H_{\mathbb{H}^n} u = 0$, for

$$L_H^{\mathbb{H}^n} u = \operatorname{div}_S(\varphi \nabla u) + \operatorname{div}_S T - H\nu_S,$$

where $S = \operatorname{graph} u$, which is defined on the class of functions in $C^2(\mathbb{H}^n)$ which are strictly 1−Lipschitz functions with respect to the spherical metric. We recall that $\varphi = \sqrt{-g_{\widetilde{\mathbb{H}}^{n,1}}(\partial_t, \partial_t)}$, $T = \partial_t/\varphi$, and ν_S is the gradient function. The symbol of the differential operator $L_H^{\mathbb{H}^n}$ is a positive multiple of the symbol the Beltrami-Laplace operator on S, hence $L_H^{\mathbb{H}^n}$ is strictly elliptic. We claim that the coefficients are analytic: indeed, the divergence on S can be written as

$$\operatorname{div}_S(X) = \operatorname{div}_{\widetilde{\mathbb{H}}^{n,1}}(X) - \langle x, N \rangle,$$

and an explicit computation gives

$$\nu_S = \sqrt{1 - u^2 + \|\nabla u\|^2}.$$

Hence, $L_H^{\mathbb{H}^n}$ is a rational function of u, its first and second derivatives, φ and T. By Theorem 5.2.1, it follows that the all derivatives of u_H are bounded, hence $u_H \in C^{k,\alpha}(\mathbb{H}^n)$, for any $k \in \mathbb{N}$ and $\alpha \in (0,1)$. By [Hop31], a $C^{2,\alpha}$ solution of a quasi-linear elliptic differential equation $L_H^{\mathbb{H}^n} u = 0$ has the same regularity as the coefficients, namely u_H is analytic. Equivalently, the leaves of the foliation are analytic.

For a fixed H, consider the Banach space $C^{k,\alpha}(\Sigma_H)$ (Definition 9.3.1), for $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Any $v \in C^{k,\alpha}(\Sigma_H)$ induces via the exponential map a deformation of Σ_H in $\widetilde{\mathbb{H}}^{n,1}$, which we denote S_v, defined as the image of the function

$$\begin{aligned} s_v \colon \Sigma_H &\hookrightarrow \widetilde{\mathbb{H}}^{n,1} \\ p &\longmapsto \exp_p\left(v(p)N(p)\right). \end{aligned}$$

To be more explicit, in the quadric model, the map becomes

$$(\psi \circ s_v)(p) = \cos(v(p))\,\psi(p) + \sin(v(p))\,N(\psi(p)). \tag{9.1}$$

In particular, for $v = 0$, $S_v = \Sigma_H$, which is a entire spacelike graph. The uniform bound on $|\mathrm{II}_{\Sigma_H}|$ ensures that there is an open neighbourhood $A^{k,\alpha}$ of 0 inside $C^{k,\alpha}(\Sigma_H)$ such that S_v is a entire spacelike graph and $\partial S_v = \Lambda$, for any $v \in A^{k,\alpha}$ (Lemma 9.3.2). Thus, we define the mean curvature operator $\mathcal{H} \colon A^{k,\alpha} \to C^{k-2,\alpha}(\Sigma_H)$ such that $\mathcal{H}(v)(p)$ is the mean curvature of S_v at the point $s_v(p) = \exp_p(v(p)N(p))$.

Using [Bar21a, Equation (2.7)], $\mathcal{H}$ can be explicitly computed. Denote τ_H the submersion whose level sets are the constant normal graph over Σ_H, namely $\tau_H(x) = t$ if

$x = \exp_p(tN(p))$, for some $p \in \Sigma_H$. One can compute the gradient of τ_H in the quadric model, obtaining that $\overline{\nabla}\tau_H$ is a unitary vector field. Hence, the tilt function is just the constant function $\varphi_H \equiv 1$. If S is a C^2-spacelike hypersurface and it is a normal graph $S = \operatorname{graph} v$ over Σ_H, then

$$\mathcal{H}(v) = \frac{1}{\nu_S^H}\left(\operatorname{div}_S(\nabla v) + \operatorname{div}_S \overline{\nabla}\tau_H\right), \tag{9.2}$$

for $\nu_S^H = -\langle \overline{\nabla}\tau_H, N_S\rangle$. Since Σ_H is analytic, $\mathcal{H}$ is an analytic operator: one can prove it with the same argument as for $L_H^{\mathbb{H}^n}$.

We claim that $\mathcal{H}$ admits an analytic inverse in a neighbourhood of $v = 0$ (Proposition 9.3.4), then we define the path $v_h \subseteq A^{k,\alpha}$ such that $S_{v_h} = \Sigma_h$, which is well defined for h in a neighbourhood of H (Remark 9.3.3). The derivative of v_h with respect to h does not vanish at H (Lemma 9.3.5), that is the map

$$\begin{aligned} S\colon \Sigma_H \times (H - \delta_{k,\alpha}, H + \delta_{k,\alpha}) &\longrightarrow \widetilde{\mathbb{H}}^{n,1} \\ (x, h) &\longmapsto s_{v_h}(x) \end{aligned}$$

is a local C^k-diffeomorphism onto an open neighborhood of Σ_H, *i.e.* a local C^k-trivialization of $\Omega(\Lambda)$.

9.3. Proof of claims

We start introducing the Banach space $C^{k,\alpha}(\Sigma)$.

Definition 9.3.1. Let Σ be a complete Riemannian manifold, $k \in \mathbb{N}$, and $\alpha \in (0, 1)$. We denote $C^{k,\alpha}(\Sigma)$ the completion of $C^\infty(\Sigma)$ with respect to the $(k, \alpha)-$Hölder norm, which is defined as

$$\|v\|_{C^{k,\alpha}(\Sigma)} := \max_{j \le k}\left(\sup_\Sigma |\nabla^j v|\right) + \sup_{d(x,y)<1} \frac{|\nabla^k v(x) - P_{y,x}\nabla^k v(y)|}{\operatorname{dist}(x, y)^\alpha},$$

where $P_{y,x}$ is the parallel transport along the geodesic connecting x and y.

Lemma 9.3.2. *Let Σ_H be a CMC entire hypersurface in $\widetilde{\mathbb{H}}^{n,1}$ with constant mean curvature H. There exists an open neighbourhood $A^{k,\alpha}$ of 0 in $C^{k,\alpha}(\Sigma_H)$ such that S_v is a entire spacelike hypersurface and $\partial S_v = \partial\Sigma$.*

Proof. We claim that if v is sufficiently small in the $C^1(\Sigma)-$norm, then S_v is spacelike. Incidentally, this also proves that s_v is an immersion.

The metric on S_v can be explicitly computed using Equation (9.1): let g_v be the metric on S_v, for $w \in T_p\Sigma$ unitary, it holds

$$(s_v^* g_v)(w, w) = \cos(v)^2 - d_p v(w)^2 + \mathrm{II}(w, w)\sin(2v) + \langle B(w), B(w)\rangle \sin(v)^2.$$

Since $B(w) \in T\Sigma$, $\langle B(w), B(w)\rangle \ge 0$. Moreover, $d_p v(w)^2 \le \|d_p v\|^2$, for $\|\cdot\|$ the operator norm in $\operatorname{Hom}(T_p\Sigma, \mathbb{R})$. Finally, $\mathrm{II}(w, w) \le C_0(|H|, n)$ (Theorem 5.2.1). It follows that

$$(s_v^* g_v)(w, w) \ge \cos(v)^2 - \|d_p v\|^2 - |\sin(2v)|C_0(|H|, n) = 1 + o(|v|) + o\left(\|d_p v\|^2\right),$$

which proves the claim.

To conclude, we claim that, for h close enough to H, Σ_h can be written as a normal graph over Σ_H. In particular, there exist $H_1 < H < H_2$ such that Σ_{H_i} is the normal graph of v_{H_i} over Σ_H. By Proposition 4.2.1, $\Sigma_{H_1} \subseteq I^+(\Sigma)$ and $\Sigma_{H_2} \subseteq I^-(\Sigma)$, that is $v_{H_2} < 0 < v_{H_1}$: it follows that for any $v \in C^{k,\alpha}(\Sigma_H)$ such that $v_{H_2} < v < v_{H_1}$, then $\partial S_v = \partial\Sigma$. Since S_v is spacelike and properly immersed, is properly embedded by [BS20, Lemma 4.5.5].

We prove the claim by contradiction: assume there exist $\varepsilon > 0$ and a sequence $h_k \to H$, such that Σ_{h_k} is not contained in the $\varepsilon-$normal neighborhood of Σ_H, that is there exists a sequence of points $p_k \in \Sigma_H$ such that $\mathrm{dist}(p_k, \Sigma_{h_k}) > \varepsilon$, for any $k \in \mathbb{N}$. For any $k \in \mathbb{N}$, choose an isometry f_k of $\widetilde{\mathbb{H}}^{n,1}$ sending p_k to $(x_0, 0)$ and $N_{\Sigma_H}(p_k)$ to the normal vector to $\mathbb{H}^n \times \{0\}$ at $(x_0, 0)$. Remark that $f_k(\Sigma_H)$ and $f_k(\Sigma_{h_k})$ share the same boundary for any $k \in \mathbb{N}$: by Proposition 7.0.1, up to extracting a subsequence, they converge to the same acausal graph, which can be either an $H-$hypersurface or a totally geodesic degenerate hyperplane. The choice of the normal vector of $f_k(\Sigma_H)$ at $(x_0, 0)$ prevents the latter to happen, hence

$$\mathrm{dist}(p_k, \Sigma_{h_k}) = \mathrm{dist}\left((x_0, 0), f_k(\Sigma_{h_k})\right) \to 0,$$

which contradicts the assumption, proves the claim and concludes the proof. □

Remark 9.3.3. In particular, the leaves of the foliation are contained in $A^{k,\alpha}$, if their mean curvature is sufficiently close to H.

Proposition 9.3.4. *Let Σ be a CMC entire spacelike hypersurface. The operator $\mathcal{H}$ on $C^{k,\alpha}(\Sigma)$ admits an analytic inverse in a neighbourhood of $v = 0$.*

Proof. To prove that $\mathcal{H}$ is invertible at $v = 0$, we first linearize it: denote J the linearization of the mean curvature operator $\mathcal{H}$ at 0. Since $\mathcal{H}$ is analytic, by analytic inverse function theorem its local inverse is analytic, too. By [SST23, Lemma 7.3],

$$J = \Delta - n - |\mathrm{II}|^2, \tag{9.3}$$

for Δ the Laplace-Beltrami operator on Σ. Our goal is to build a bounded inverse J^{-1} at 0 in $C^{k,\alpha}(\Sigma)$.

Step 1: $k = 2$.

The existence of an inverse is equivalent to prove that the differential problem $Ju = f$ has always solution, and that any solution satisfies a Schauder-type inequality

$$\|u\|_{C^{2,\alpha}(\Sigma)} \leq C\|f\|_{C^{0,\alpha}(\Sigma)}, \tag{9.4}$$

for some constant $C > 0$ not depending on f. Indeed, Equation (9.4) then implies that J is injective, hence invertible, and J^{-1} is bounded. First, we build a solution for $Ju = f$, for a fixed $f \in C^{0,\alpha}(\Sigma)$. Since Σ is complete (Theorem B), we can pull-back the problem on $\mathbb{R}^n$ via the exponential map, namely in normal coordinates around a point. By Equation (9.3), J is strictly elliptic: by [GT01, Theorem 6.14] there exists a unique $C^{2,\alpha}(\overline{K_i})$ solution u_i to the Dirichlet problem

$$\begin{cases} Ju = f|_{K_i} \\ u|_{\partial K_i} = 0 \end{cases}$$

for $\{K_i\}_{i\in\mathbb{N}}$ an exhaustion of compact sets of Σ. We claim that there exists a local version of Equation (9.4), namely

$$\|u_i\|_{C^{2,\alpha}(K_i)} \leq C\|f\|_{C^{0,\alpha}(\Sigma)}, \tag{9.5}$$

where C does not depend on i nor f. Hence, we can apply Ascoli-Arzelà Theorem to extract a subsequence converging in $C^{2,\alpha}(\Sigma)$ to a global solution u. In particular, the limit u satisfies Equation (9.4), concluding the proof.

To prove the claim, fix $x \in \Sigma$. In normal coordinates around x, J is a uniformly strictly elliptic operator on bounded sets of $\mathbb{R}^n$, in particular on the ball $B(0,2)$. By [GT01, Theorem 6.2], we obtain

$$\|u\|_{C^{2,\alpha}(B(0,1))} \leq \tilde{C}\left(\|u\|_{C^0(B(0,2))} + \|f\|_{C^{0,\alpha}(B(0,2))}\right), \tag{9.6}$$

where $\tilde{C}$ depends on the uniform bounds of ellipticity of J in $B(0,2)$, hence ultimately $\tilde{C}$ depends on x. Actually, by the uniform bound on the norm of the derivatives of $\mathrm{I\!I}$, the pull-back of J is strictly elliptic uniformly with respect to x, hence one can choose $\tilde{C}$ holding for all $x \in \Sigma$. Moreover, for $v = c$ a constant function,

$$|J(v)| = |c|(n + |\mathrm{I\!I}|^2) \leq |c|\left(n + C_0(|H|, n)\right),$$

for $C_0(|H|, n)$ as in Theorem 5.2.1. It follows that the constant functions

$$u_+ := \frac{\|f\|_{C^{0,\alpha}(B(0,2))}}{n + C_0(|H|, n)} \qquad\qquad u_- := -\frac{\|f\|_{C^{0,\alpha}(B(0,2))}}{n + C_0(|H|, n)}$$

are respectively a supersolution and a subsolution for J. By the strong maximum principle ([GT01, Theorem 3.5]), $u_- < u < u_+$ on $B(0,2)$, namely

$$\|u\|_{C^0(B(0,2))} < \frac{\|f\|_{C^{0,\alpha}(B(0,2))}}{n + C_0(|H|, n)} \leq \frac{\|f\|_{C^{0,\alpha}(\Sigma)}}{n + C_0(|H|, n)}.$$

Substituting in Equation (9.6), one obtains

$$\|u_i\|_{C^{2,\alpha}(B(0,1))} \leq \tilde{C}\left(\frac{1}{n + C_0(|H|, n)} + 1\right)\|f\|_{C^{0,\alpha}(\Sigma)} =: C\|f\|_{C^{0,\alpha}(\Sigma)},$$

which proves Equation (9.5), hence the claim, concluding the proof for $k = 2$.

Step 2: $k > 2$.

It suffices to repeat the argument above for the higher derivatives, remarking that J commutes with the derivatives: let $\beta = (i_1, \dots, i_{|\beta|})$ be a multi index of length $|\beta| \leq k-2$, that is $D^\beta = \partial_{i_1} \dots \partial_{i_{|\beta|}}$. Since $D^\beta u$ is a solution of $Jv = D^\beta f$, the same argument as above implies

$$\|D^\beta u\|_{C^{2,\alpha}(\Sigma)} \leq C\|D^\beta f\|_{C^{0,\alpha}(\Sigma)},$$

hence

$$\|u\|_{C^{k,\alpha}(\Sigma)} \leq \sum_{\beta,\, |\beta| \leq k-2} \|D^\beta u\|_{C^{2,\alpha}(\Sigma)} \leq \sum_{\beta,\, |\beta| \leq k-2} C\|D^\beta f\|_{C^{0,\alpha}(\Sigma)} = C\|f\|_{C^{k-2,\alpha}(\Sigma)},$$

which proves that J is invertible at 0 in $C^{k,\alpha}(\Sigma)$. □

The following lemma allows us to apply the analytic inverse function theorem to the smooth path $v_\bullet \colon (H - \delta_{k,\alpha}, H + \delta_{k,\alpha}) \to C^{k,\alpha}(\Sigma_H)$ such that $S_{v_h} = \Sigma_h$.

Lemma 9.3.5. *The derivative of v_h with respect to h does not vanish at H.*

Proof. By construction, $\mathcal{H}(v_h) = h$. Differentiating both sides, one obtains

$$J\left(\frac{dv_h}{dh}\right) = 1.$$

We recall that, following the proof of Proposition 9.3.4, we can write dv_h/dh as the limit of functions v_i which are solutions of the differential problem

$$\begin{cases} Jv = 1|_{K_i} \\ v|_{\partial K_i} = 0 \end{cases}$$

for K_i an exhaustion of compact set.

The constant function $u = 0$ is a supersolution for the equation, since $J(u) = 0$. As already remarked, J is a strictly elliptic operator on Σ (Equation (9.3)), hence uniformly elliptic over compact set. By the weak maximum principle ([GT01, Theorem 3.1]), the maximum of v_i is reached at the boundary, namely $v_i \le 0$ for any $i \in \mathbb{N}$, hence $dv_h/dh \le 0$. We can then apply the strong maximum principle ([GT01, Theorem 3.5]) to obtain $dv_h/dh < u = 0$, which concludes the proof. $\square$

9.4. Analytic foliation

In the previous section, we have proved that each v_h is an analytic map, and that the path $h \to v_h$ is analytic in $C^{k,\alpha}(\Sigma_H)$. Since the evaluation at a point $p \in \Sigma_H$ is an analytic operator, the map $h \mapsto v_h(p)$ is an analytic map. It follows that $v_\bullet(\cdot)$ is analytic both in the argument $p \in \Sigma$ and $h \in \mathbb{R}$. To prove that the map is jointly analytic, it suffices to pullback the problem using the exponential map, to see $v_\bullet(\cdot)$ as a map $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, and prove that the radius of convergence in both variables are uniformly bounded from below ([Sic69, Theorem (I)]).

Since $h \to v_h$ is an analytic path, the radius of convergence r of $v_\bullet(p)$ at $h = H$ does not depend on p, hence $r/2$ is a uniform lower bound for the radius of convergence of $v_\bullet(p)$ at h, for $h \in (H - r/2, H + r/2)$, $p \in \Sigma_H$. Pick $p \in \Sigma_H$, we claim that there exists $\rho, \varepsilon > 0$ such that v_h has radius of convergence at least ρ at p, for any $h \in (H - \varepsilon, H + \varepsilon)$. As a consequence, for any such h, the radius of convergence of $v_h(\cdot)$ at q is bounded from below by $\rho/2$, for any $q \in B_\Sigma(p, \rho/2)$. It follows that

$$\delta := \min\{r/2, \rho/2, \varepsilon\}$$

is a uniform lower bound for the radii of convergence of both $v_\bullet(q)$ at h for $h \in (H - \varepsilon, H + \varepsilon)$, and $v_h(\cdot)$ at q, for $q \in B_\Sigma(p, \rho/2)$. We conclude that

$$v_\bullet(\cdot)\colon B_\Sigma(p, \rho/2) \times (H - \varepsilon, H + \varepsilon) \to \Omega(\Lambda)$$

is a local analytic trivialization of $\Omega(\Lambda)$. Since p and H are arbitrary, this concludes the proof.

We remark that by Equation (9.2), any v_h solves the analytic non-linear elliptic differential equation $L_h^{\Sigma_H} v = 0$, for

$$L_h^{\Sigma_H} v := \Delta^{S_v} v + \mathrm{div}_{S_v}(\overline{\nabla}\tau_H) - h\nu_{S_v}^H,$$

Without loss of generality, we can assume $p = 0$, so that the claim can be then proved by following the proof of analyticity of [Hop31], which consists in building a complex extension of the solution v in a neighbourhood of 0 and prove that it is analytic on the set

$$(R)_\gamma := \{z = x + iy \in \mathbb{C}^n, \|x\| < R, \|y\| < \gamma(R - \|x\|)\},$$

(see [Hop31, Page 221]), for γ a constant depending continuously on the symbol of the operator (see [Hop31, Equation (6.8)]), which analytically depends on the function u and its derivatives up to the fourth. On the other hand, R has to be bounded from above by many quantities, which continuously depend on the function u and its derivatives up to the sixth (see [Hop31, Equations (6.14), (7.9), (8.6), (8.7)]).

A priori, $R = R(h)$ and $\gamma = \gamma(h)$. However, since the symbol of $L_h^{\Sigma_H}$ does not depend on h, as a function of u and its derivatives up to the second, and v_h is a C^6-foliation, for $h \in (H - \delta_{6,\alpha}, H + \delta_{6,\alpha})$, we can find $\varepsilon < \delta_{6,\alpha}$ such that all forementioned quantities are uniformly bounded: it follows that

$$\bar{R} := \inf_{h \in (H-\varepsilon, H+\varepsilon)} R(h) > 0, \qquad \bar{\gamma} := \inf_{h \in (H-\varepsilon, H+\varepsilon)} \gamma(h) > 0.$$

Hence, the complex extension of v_h is analytic on $(\bar{R})_{\bar{\gamma}}$, for all $h \in (H - \varepsilon, H + \varepsilon)$. Setting $\rho := \min\{\bar{R}, \bar{\gamma}\bar{R}\}/2$, the ball $B_{\mathbb{C}^n}(0, \rho)$ is contained in $(\bar{R})_{\bar{\gamma}}$, hence the radius of convergence of v_h at p is at least ρ, proving the claim and concluding the proof.

9.5. Maximal globally hyperbolic Cauchy complete $\mathbb{A}\mathrm{d}\mathbb{S}-$manifolds

This section is meant to extend [And+12, Theorem 1.5] from maximal globally hyperbolic Cauchy *compact* $\mathbb{A}\mathrm{d}\mathbb{S}-$manifolds to maximal globally hyperbolic Cauchy *complete* $\mathbb{A}\mathrm{d}\mathbb{S}-$manifolds, namely

Corollary D. *Let (M, g) be a maximal globally hyperbolic Cauchy complete Anti-de Sitter manifold. Then (M, g) admits a (unique) globally defined CMC time function $\tau_{cmc} \colon M \to \mathbb{R}$.*

Definition 9.5.1. A globally hyperbolic $\mathbb{A}\mathrm{d}\mathbb{S}-$manifold is called

- *Cauchy compact* if it admits a Cauchy hypersurface which is compact;
- *Cauchy complete* if it admits a Cauchy hypersurface whose induced Riemannian metric is complete;
- *maximal* if every isometric embedding $M \hookrightarrow N$ in another globally hyperbolic $\mathbb{A}\mathrm{d}\mathbb{S}-$manifold is an isometry.

We remark that if a globally hyperbolic manifold is *Cauchy compact*, then any Cauchy hypersurface is compact. On the contrary, it can be easily shown that this is not the case for the Cauchy complete case.

Definition 9.5.2. A time function τ on an time-oriented Lorentzian manifold is called a *CMC* time function if each level set $\tau^{-1}(H)$ is a hypersurface with constant mean curvature H.

Remark 9.5.3. The function $\tau_{cmc} \colon \Omega(\Lambda) \to \mathbb{R}$ which associates to each point $p \in \Omega(\Lambda)$ the unique H such that $p \in \Sigma_H$ is a CMC time function: by definition, $\tau_{cmc}^{-1}(H) = \Sigma_H$, and it is strictly decreasing along future-directed time paths due to the strong maximum principle (Proposition 4.2.1).

The proof reduces to rephrase Theorem C in this setting, using the classification provided by [BB09, Proposition 6.3.1, Corollary 6.3.13]:

Proposition 9.5.4. *Let (M, g) be a maximal globally hyperbolic Cauchy complete* $\mathbb{A}\mathrm{d}\mathbb{S}-$*manifold, then its universal cover* $\widetilde{M}$ *isometrically embeds in* $\widetilde{\mathbb{H}}^{n,1}$ *and its image is the invisible domain of an admissible boundary* Λ*. Moreover,* Λ *is unique, up to isometry of* $\widetilde{\mathbb{H}}^{n,1}$.

Conversely, $\Omega(\Lambda)$ *is a maximal globally hyperbolic Cauchy complete* $\mathbb{A}\mathrm{d}\mathbb{S}-$*manifold, for any admissible boundary* $\Lambda \subseteq \partial_\infty \widetilde{\mathbb{H}}^{n,1}$.

Remark 9.5.5. It follows that any maximal globally hyperbolic Cauchy complete $\mathbb{A}\mathrm{d}\mathbb{S}-$manifold can be written as $\Omega(\Lambda)/\Gamma$, for Γ a subgroup of $\mathrm{Isom}(\widetilde{\mathbb{H}}^{n,1})$. Since M is globally hyperbolic, Γ consists of time-orientation preserving isometries: otherwise, M would not be time-orientable, and in particular not globally hyperbolic.

Proof of Corollary D. Remark 9.5.3 proves the statement for (M, g) simply connected. If $\pi_1(M)$ is not trivial, by Proposition 9.5.4, $(M, g) = \Omega(\Lambda)/\Gamma$, for some Λ admissible boundary and $\Gamma \subseteq \mathrm{Isom}(\widetilde{\mathbb{H}}^{n,1})$. We claim that the CMC time function on $\Omega(\Lambda)$ is invariant over the orbits of Γ.

First, remark that Λ is $\Gamma-$invariant, hence Σ_H is also $\Gamma-$invariant: indeed, for $g \in \Gamma$, $g(\Sigma_H)$ is a CMC entire hypersurface whose boundary is $g(\Lambda) = \Lambda$. Remark 9.5.5 ensures g is time-orientation preserving, hence the mean curvature of $g(\Sigma_H)$ is H. By uniqueness, $g(\Sigma_H)$ coincides with Σ_H. Since the CMC time function on $\Omega(\Lambda)$ associates to a point the mean curvature of the unique CMC entire hypersurface it belongs to, τ_{cmc} is $\Gamma-$invariant, which concludes the proof. □

Chapter 10.

Cosmological time

In this section, we discuss two time functions: the one induced by the normal flow of a spacelike hypersurface, and the one induced by the equidistant hypersurfaces from an embedded achronal hypersurface. It turns out that the two coincides if the hypersurface is regular.

10.1. Normal flow

Let Σ be a C^1-spacelike hypersurface. Denote by N the unit future-directed normal vector field over Σ. The *normal flow* of Σ is the function

$$\begin{aligned} F\colon \Sigma \times \mathbb{R} &\longrightarrow \widetilde{\mathbb{H}}^{n,1} \\ (x,t) &\longmapsto \exp_x(tN(x)). \end{aligned} \tag{10.1}$$

Lemma 10.1.1. *Let Σ be an immersed spacelike C^k-hypersurface in $\widetilde{\mathbb{H}}^{n,1}$, for $k \in \mathbb{N} \cup \{\infty, \omega\}$. Then, the normal flow $F\colon \Sigma \times \mathbb{R} \to \widetilde{\mathbb{H}}^{n,1}$ of Σ is a $C^{k-1}-$map.*

Moreover, if there exists be an interval (a,b) where the leaves of the normal flow Σ_t are non-degenerate, then $(\Sigma_t)_{t\in(a,b)}$ is a C^k-foliation.

Proof. Let $\sigma\colon \Sigma \to \widetilde{\mathbb{H}}^{n,1}$ be a C^k-parameterization of Σ. Then, $N \circ \sigma \in C^{k-1}(\Sigma, T\mathbb{H}^{n,1})$, for N the unit future-directed normal vector to Σ. Since the computation is local, let us consider the problem in $\mathbb{H}^{n,1}$, then

$$F(x,t) = \exp_{\sigma(x)}\left(tN(\sigma(x))\right) = \cos(t)\sigma(x) + \sin(t)N\left(\sigma(x)\right) \in C^{k-1}(\Sigma \times \mathbb{R}, \mathbb{H}^{n,1}).$$

Assume that the Σ_t's are non-degenerate for $t \in (a,b)$. Equivalently, by Frobenius theorem, we get a C^{k-2} distribution $\mathcal{D}$ of tangent $n-$planes. Such distribution is orthogonal to the vector field tangent to the fiber, given by

$$\frac{d}{ds}\exp_{\sigma(x)}\left(sN(\sigma(x))\right)\bigg|_{s=t} = -\sin(t)\sigma(x) + \cos(t)N(\sigma(x)),$$

which is a $C^{k-1}-$vector field since it is analytic as a function of t and C^{k-1} as a function of x. Hence, the distribution $\mathcal{D}$ is C^{k-1}, as well, and the induced foliation is then C^k, concluding the proof. □

The pull-back metric on $\Sigma \times \mathbb{R}$ can be explicitly computed.

Lemma 10.1.2 (Lemma 6.22 in [BS20]). *Let Σ be a C^2-spacelike hypersurface in $\widetilde{\mathbb{H}}^{n,1}$. Denote by $\Sigma_t := F(\Sigma \times \{t\})$ the leaves of the normal flow of Σ. The pull-back to metric on $\Sigma \times \mathbb{R}$ is*

$$(F^* g_{\widetilde{\mathbb{H}}^{n,1}})_{(p,t)}(v,w) = -dt^2 + g\left(\cos(t)v + \sin(t)B(v), \cos(t)w + \sin(t)B(w)\right),$$

for g the metric of Σ and B its shape operator.

We can then compute the shape operator of the leaves of the normal flow, when they are non-degenerate.

Corollary 10.1.3. *Let Σ be a C^2-spacelike properly embedded hypersurface in $\widetilde{\mathbb{H}}^{n,1}$. Assume that the leaf Σ_t is non-degenerate. Then, the principal curvatures of Σ_t are*

$$\lambda_i^t = \tan\left(\arctan(\lambda_i) - t\right),$$

for λ_i the the principal curvatures of Σ.

It follows directly this easy yet crucial lemma.

Lemma 10.1.4. *Let Σ be a C^2-spacelike hypersurface. Denote by $\lambda_1 \geq \cdots \geq \lambda_n$ the principal curvatures of Σ. Define*

$$A_+ = \arctan\left(\inf_\Sigma \lambda_n\right) + \frac{\pi}{2} \qquad A_- = \arctan\left(\sup_\Sigma \lambda_1\right) - \frac{\pi}{2} \tag{10.2}$$

Denote by Σ_t the leaf of the normal flow at time t, g_t the induced metric and $F_t := F|_{\Sigma\times\{t\}}$,. Then, for any $t \in [A_-, A_+]$, we have

$$(F_t)^* g_t \geq \beta(x,t)^2 g, \qquad \text{for } \beta(x,t) = \begin{cases} \cos(t) + \sin(t)\lambda_n(x) & \text{for } t \geq 0 \\ \cos(t) + \sin(t)\lambda_1(x) & \text{for } t \leq 0 \end{cases}.$$

Proof. Let e_i be the unit eigenvector relative to $\lambda_i(x)$. Take a unit tangent vector

$$v = \sum_{i=1}^n a_i e_i \in T_x\Sigma.$$

By Lemma 10.1.2, we have

$$\begin{aligned}(F_t)^* g_t(v,v) &= g\left(\cos(t)v + \sin(t)B(v), \cos(t)v + \sin(t)B(v)\right) \\ &= \sum_{i=1}^n a_i^2 g\left(\cos(t)e_i + \sin(t)B(e_i), \cos(t)e_i + \sin(t)B(e_i)\right) \\ &= \sum_{i=1}^n a_i^2 \left(\cos(t) + \sin(t)\lambda_i(x)\right)^2.\end{aligned}$$

For $t > 0$, denote by $a_+(x) \in (0,\pi)$ the solution of the equation

$$-\frac{1}{\tan(s)} = \lambda_n(x) \iff s = \frac{\pi}{2} + \arctan\left(\lambda_n(x)\right).$$

Since $\cos(t) + \sin(t)\lambda_i(x) \geq 0$ for any $i = 1, \ldots, n$ and $t \in [0, a_+(x)]$, we get

$$\left(\cos(t) + \sin(t)\lambda_i(x)\right)^2 \geq \left(\cos(t) + \sin(t)\lambda_n(x)\right)^2 = \beta(x,t)^2.$$

By definition, $A_+ = \inf_\Sigma a_+(x)$: hence,

$$(F_t)^* g_t(v,v) \geq \beta(x,t)^2 g(v,v), \quad \forall v \in T\Sigma, \forall t \in [0, A_+].$$

The same argument works for $t < 0$, concluding the proof. □

Remark 10.1.5. It is directly checked that $A_+ - A_- \leq \pi$, with equality if and only if Σ is totally umbilical.

Proposition 10.1.6. *Let Σ be a C^2-spacelike hypersurface. Then Σ_t is a spacelike hypersurface, for any $t \in (A_-, A_+)$.*

Moreover, if $\inf_\Sigma \lambda_n$ (resp. $\sup_\Sigma \lambda_1$) is realized, then the leaf Σ_{A_+} (resp. Σ_{A_-}) is degenerate. Otherwise, the leaf Σ_{A_+} (resp. Σ_{A_-}) is a spacelike hypersurface, and there exists $\varepsilon > 0$ such that Σ_t is degenerate for any $t \in (A_+, A_+ + \varepsilon)$ (resp. $t \in (A_- - \varepsilon, A_-)$).

Proof. The proof follows directly by Lemma 10.1.2. Let $x \in \Sigma$ and let $e_n \in T_x\Sigma$ be an eigenvector relative to λ_n such that $g(e_n, e_n) = 1$. Then,

$$\begin{aligned}(F_{A_+})^* g_{A_+}(e_n, e_n) = \beta(x, A_+)^2 g(e_n, e_n) = 0 \iff \beta(x, A_+) = 0 \\ \iff \lambda_n(x) = -\frac{1}{\tan(A_+)} = \inf_\Sigma \lambda_n.\end{aligned}$$

The same argument, using e_1 instead of e_n, verifies the statement for Σ_{A_-}, concluding the proof. □

Corollary 10.1.7. *Let Σ be a C^2-spacelike hypersurface. If Σ is complete, then Σ_t is complete, for any $t \in (A_-, A_+)$.*

Proof. Without loss of generality, assume $t \geq 0$. By Proposition 10.1.6, the leaf Σ_t is a spacelike hypersurface. We claim there exists $b_t > 0$ such that $\beta(x,t) \geq b_t$ over Σ. By Lemma 10.1.2, it follows

$$(F_t)^* g_t \geq \beta(x,t)^2 g \geq b_t g.$$

Hence, the completeness of Σ implies the completeness of Σ_t.

To prove the claim, we recall that $t \in (0, \pi)$, hence $\sin(t) > 0$, and that $A_+ \in [0, \pi]$ is defined by the equation $-1/\tan(A_+) = \inf_\Sigma \lambda_n$. Then,

$$\begin{aligned}\beta(x,t) &= \cos(t) + \sin(t)\lambda_n(x) \geq \cos(t) + \sin(t) \inf_\Sigma \lambda_n \\ &= \sin(t)\left(\frac{1}{\tan(t)} + \inf_\Sigma \lambda_n\right) = \sin(t)\left(\frac{1}{\tan(t)} - \frac{1}{\tan(A_+)}\right) =: b_t.\end{aligned}$$

By hypothesis, $t < A_+$ and $1/\tan(s)$ is a strictly decreasing function over $(0, \pi)$. Hence, b_t is positive, concluding the proof. □

Remark 10.1.8. Clearly, the proof of Corollary 10.1.7 cannot be extended to $\Sigma_{A_\pm}$, even in the case $\Sigma_{A_\pm}$ is a spacelike hypersurface. Indeed, the same argument gives $b_{A_\pm} = 0$.

In fact, our result is sharp: it is not difficult to produce a complete hypersurfaces Σ such that $\Sigma_{A_\pm}$ is smooth but not even properly embedded, hence not complete (see Remark 10.2.5).

10.2. Cosmological time of convex domains

We generalize the construction described in Section 3.4. The notion of duality introduced in Definition 1.5.2 allows to construct the dual hypersurface of a convex hypersurface.

Let S be an achronal, properly embedded, future-convex hypersurface in $\widetilde{\mathbb{H}}^{n,1}$. We define

$$\mathcal{C}(S) := \bigcap_{x \in S} I^-(\mathcal{P}_+(x)) \cap I^+(S).$$

If S is past-convex, then we define

$$\mathcal{C}(S) := \bigcap_{x\in S} I^+\left(\mathcal{P}_-(x)\right) \cap I^-(S).$$

By construction, $\mathcal{C}(S)$ is an open convex subset of $\widetilde{\mathbb{H}}^{n,1}$. If S is future-convex, then its past boundary $\partial_-\mathcal{C}(S)$ coincides with S, while the future boundary $\partial_+\mathcal{C}(S)$ is the achronal, properly embedded, past-convex hypersurface

$$\hat{S} := \{x \in \widetilde{\mathbb{H}}^{n,1}, \mathcal{P}_-(x) \, \text{is a support hyperplane for } S\}.$$

It follows that $\mathcal{C}(S) = \mathcal{C}(\hat{S})$.

One can easily check that, for any admissible boundary Λ, the convex core $\mathcal{CH}(\Lambda)$ is dual to the invisible domain $\Omega(\Lambda)$. More precisely

$$\mathcal{C}\left(\partial_-\Omega(\Lambda)\right) = \mathcal{C}\left(\partial_+\mathcal{CH}(\Lambda)\right) = \mathbf{P}(\Lambda)$$
$$\mathcal{C}\left(\partial_+\Omega(\Lambda)\right) = \mathcal{C}\left(\partial_-\mathcal{CH}(\Lambda)\right) = \mathbf{F}(\Lambda).$$

We generalize Proposition 3.4.5 to these kind of convex sets.

Proposition 10.2.1. *Let S be a past-convex (resp. future-convex) properly embedded achronal hypersurface in $\widetilde{\mathbb{H}}^{n,1}$. Then, $\mathcal{C}(S)$ admits a cosmological time function*

$$\tau(x) := \operatorname{dist}\left(x, \partial_-\mathcal{C}(S)\right)$$

taking values in $(0, \pi/2)$.

For every point $x \in \mathcal{C}(S)$, there exists a unique point $\rho_\pm(x) \in \partial_\pm\mathcal{C}(S)$ realizing the distance between x and the boundary, namely

$$\operatorname{dist}\left(x, \partial_\pm\mathcal{C}(S)\right) = \pm \operatorname{dist}\left(x, \rho_\pm(x)\right).$$

Moreover, $\rho_\pm$ and τ satisfy the following properties:

1. *x lies in the timelike geodesic segment $[\rho_-(x), \rho_+(x)]$;*
2. *$\tau(x)$ is equal to the length of the geodesic segment $[\rho_-(x), x]$;*
3. *the length of $[\rho_-(x), \rho_+(x)]$ is $\pi/2$;*
4. *$\mathcal{P}_\pm\left(\rho_\mp(x)\right)$ is a support plane for $\mathcal{C}(S)$ passing through $\rho_\pm(x)$;*
5. *the map $\rho_\pm$ is continuous;*
6. *the function τ is $C^{1,1}$ and its gradient at x is the unit timelike tangent vector $\nabla\tau(x)$ such that*
$$\exp_x\left(\tau(x)\nabla\tau(x)\right) = \rho_-(x).$$

Proof. We fix once and for all $x \in \mathcal{C}(S)$, and prove all the properties at x. The arbitrary choice of x concludes the proof.

By construction, $\mathcal{P}_-(x) \subseteq I^-\left(\partial_-\mathcal{C}(S)\right)$: it follows that

$$\tau(x) = \operatorname{dist}\left(\partial_-\mathcal{C}(S), x\right) \in (0, \pi/2)$$

and that $I^-(x)$ intersects $\partial_-\mathcal{C}(S)$ in a precompact set. Hence, the distance between x and $\partial_-\mathcal{C}(S)$ is realized, *i.e* there exists a point $y \in \partial_-\mathcal{C}(S)$ such that $\tau(x) = \text{dist}(y, x)$. Moreover, $\tau(x)$ is equal with the length of the timelike geodesic segment γ connecting x and y.

The totally umbilical hypersurface at distance $-\tau(x)$ from x is strictly past-convex, hence it intersects $\partial_-\mathcal{C}(S)$ at most at one point. Then, the function

$$\rho_-(x) := \arg_{\partial_-\mathcal{C}(S)} \max \text{dist}(\cdot, x)$$

is well-defined. In particular, this proves Item (2).

By construction, $\mathcal{P}_+(\rho_-(x))$ is a support hyperplane for $\mathcal{C}(S)$. Let $\mathcal{P}$ be the unique totally geodesic spacelike hyperplane orthogonal to γ at $\rho_-(x)$. By construction, $\mathcal{P}$ is a support plane for $\partial_-\mathcal{C}(S)$. Let $\rho_+(x)$ be the point such that

$$\mathcal{P}_-(\rho_+(x)) = \mathcal{P},$$

namely its future dual point. Then, we have that $[\rho_-(x), \rho_+(x)]$ is a timelike geodesic segment of length $\pi/2$ containing γ, and hence x, proving Items (1),(3),(4).

To prove Item (6), consider the open neighbourhood of x given by

$$U := I^+(\rho_-(x)) \cap I^-(\rho_+(x)),$$

which is contained in $\mathcal{C}(S)$. Consider the two smooth time functions on U defined by

$$\begin{aligned} \tau_-(\cdot) &:= \text{dist}(\rho_-(x), \cdot) \\ \tau_+(\cdot) &:= \text{dist}(\mathcal{P}_-(\rho_+(x)), \cdot) = \frac{\pi}{2} + \text{dist}(\rho_+(x), \cdot). \end{aligned}$$

By construction, $\rho_-(x) \in \partial_-\mathcal{C}(S)$ and $\mathcal{P}_-(\rho_+(x)) \subseteq \overline{I^-(\partial_-\mathcal{C}(S))}$: hence, $\tau_- \leq \tau \leq \tau_+$ over U. An explicit computation shows that

$$\begin{aligned} \tau_+(x) &= \tau_-(x) = \ell([\rho_-(x), x]), \\ \nabla\tau_+(x) &= \nabla\tau_-(x) = v, \end{aligned}$$

for v satisfying $\exp_x(\tau(x)v) = \rho_-(x)$, that is τ is differentiable at x and $\nabla\tau(x) = v$.

Incidentally, we proved that $\rho_\pm$ is constant over $[\rho_-(x), \rho_+(x)]$: indeed, it is straightforword that $\tau_- = \tau_+$ on such geodesic: hence, for any $y \in [\rho_-(x), \rho_+(x)]$

$$\max_{\partial_-\mathcal{C}(S)} \text{dist}(\cdot, y) = \tau(y) = \tau_-(y) = \text{dist}(\rho_-(x), y).$$

We have already showed that the maximizer of $\text{dist}(\cdot, y)$ is unique over $\partial_-\mathcal{C}(S)$ and coincides with $\rho_-(y)$, hence $\rho_-(y) = \rho_-(x)$. Since $\rho_+(y)$ is uniquely determined by $\rho_-(y)$, then $\rho_+(y) = \rho_+(x)$ as well.

Finally, let us prove that ρ_- is continuous at x. Let us fix y in the interior of the geodesic joining x and $\rho_+(x)$: then, $I^-(y)$ is an open neighborhood of x which intersects $\partial_-\mathcal{C}(S)$ in a precompact set of $\widetilde{\mathbb{H}}^{n,1}$.

Consider a sequence $(x_k)_{k\in\mathbb{N}}$ converging to x. Since x_k is eventually contained in $I^-(y)$ and $\overline{I^-(x_k)} \subseteq I^-(y)$, for $x_k \in I^-(y)$, we get that

$$K := \partial_-\mathcal{C}(S) \cap \bigcap_{k\in\mathbb{N}} \overline{I^-(x_k)}$$

is a compact set contained in $\partial_-\mathcal{C}(S) \cap I^-(y)$. Since $\rho_-(x_k) \in I^-(x_k)$, the sequence $(\rho_-(x_k))_{k\in\mathbb{N}}$ eventually belongs to K. By compacteness, we extract a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ such that $\rho_-(x_{k_j})$ converges to $z \in K$. The reverse triangle inequality Equation (1.4) implies that

$$\mathrm{dist}\left(\rho_-(x_{k_j}), y\right) \geq \underbrace{\mathrm{dist}\left(\rho_-(x_{k_j}), x_{k_j}\right)}_{=\tau(x_{k_j})} + \mathrm{dist}\left(x_{k_j}, y\right).$$

Since both τ and $\mathrm{dist}\left(\cdot, y\right)$ are continuous functions over $I^-(y)$, the inequality holds in the limit for $j \to +\infty$, that is

$$\mathrm{dist}\left(z, y\right) \geq \tau(x) + \mathrm{dist}(x, y).$$

Since $x \in [\rho_-(x), y]$ and that $\rho_-(y) = \rho_-(x)$, we conclude

$$\begin{aligned}\mathrm{dist}\left(z, y\right) &\geq \tau(x) + \mathrm{dist}(x, y) = \mathrm{dist}(\rho_-(x), x) + \mathrm{dist}(x, y)\\ &= \mathrm{dist}(\rho_-(x), y) = \mathrm{dist}(\rho_-(y), y) = \tau(y).\end{aligned}$$

We already proved that the uniqueness of the point in $\partial_-\mathcal{C}(S)$ realizing $\tau(y)$, hence

$$z = \rho_-(y) = \rho_-(x).$$

It follows that any converging subsequence $(\rho_-(x_{k_j})_{j\in\mathbb{N}})$ converges to $\rho_-(x)$, namely ρ_- is continuous at x. The proof of the continuity of ρ_+ is analogous, proving Item (5) and concluding the proof. □

Definition 10.2.2. The *acausal part* $\mathrm{Acau}(S)$ of a properly embedded achronal hypersurface S is the set of points $x \in S$ admitting a spacelike support hyperplane.

Corollary 10.2.3. *The image of $\rho_\pm$ coincides with the acausal part of $\partial_\pm\mathcal{C}(S)$.*

Proof. For the sake of definiteness, let us prove it for ρ_-. The proof for ρ_+ is completely analogous.

The image of ρ_- is clearly contained in the acausal part of $\partial_-\mathcal{C}(S)$. Indeed, fix $x \in \mathcal{C}(S)$: by Item 4 of Proposition 10.2.1, $\mathcal{P}_-\left(\rho_+(x)\right)$ is a past spacelike support hyperplane for $\partial_-\mathcal{C}(S)$ at $\rho_-(x)$, that is $\rho_-(x)$ belongs to the acausal part of $\partial_-\mathcal{C}(S)$.

Conversely, fix a point y in the acausal part of $\partial_-\mathcal{C}(S)$: then, there exists a spacelike past support hyperplane $\mathcal{P}$ for $\partial_-\mathcal{C}(S)$ at y. It follows that the future dual point z of $\mathcal{P}$, that is $\mathcal{P}_-\left(z\right) = \mathcal{P}$ belongs to $\partial_+\mathcal{C}(S)$, and the timelike geodesic segment $[y, z]$ is contained in $\mathcal{C}(S)$ and has length $\pi/2$.

Clearly, for any $x \in [y, z]$, the point y satisfies the same properties as $\rho_-(x)$. By uniqueness, $y = \rho_-(x)$, namely y is in the image of ρ_-, concluding the proof. □

In Proposition 10.2.4, we will prove that leaves of the cosmological time of $\mathcal{C}(S)$ are complete. The proof extends [BB09, Proposition 6.3.9], and it is based on the same argument: showing that the second fundamental form of the leaf $S_t := \tau^{-1}(t)$, is uniformly bounded by a constant which only depends on t: in light of Lemma 5.3.2, this conclude the proof.

Proposition 10.2.4. *Let S be a future-convex (resp. past-convex) properly embedded achronal hypersurface in $\widetilde{\mathbb{H}}^{n,1}$. The leaves of the cosmological time $S_t := \tau^{-1}(t)$ are complete $C^{1,1}-$spacelike hypersurfaces, such that $\partial S_t = \partial S$, for any $t \in (0, \pi/2)$.*

Proof. Without loss of generality, assume S to be future-convex. By Proposition 10.2.1, the leaf S_t is a properly embedded spacelike $C^{1,1}-$hypersurface. Since S_t is contained in $\mathcal{C}(S)$, then

$$\partial S_t = \overline{\mathcal{C}(S)} \cap \partial \widetilde{\mathbb{H}}^{n,1} = \partial S.$$

We remark that, *a priori* the second fundamental form is defined only for C^2-hypersurfaces. However, since we proved that the leaves of the cosmological time are $C^{1,1}$ (Proposition 10.2.1) the Levi-Civita connection, hence the second fundamental form, is defined almost everywhere. In light of Remark 5.3.3, we can then apply Lemma 5.3.2. Hence, it suffices to prove that the norm of the second fundamental form is uniformly bounded, where defined. We claim that

$$\mathrm{II}(v,v) \leq \frac{1}{\tan(t)} g(v,v),$$

for g, II respectively the induced metric and the second fundamental form of S_t.

Let us fix $x_0 \in S_t$: as in the proof of Proposition 10.2.1, denote τ_- the distance from $\rho_-(x_0)$, that is

$$\tau_-(x) = \arccos\left(-\langle \rho_-(x_0), x\rangle\right) \in (0,t) \subseteq (0,\pi/2).$$

Since S_t is spacelike, by Lemma 1.2.2 the point x_0 is a global maximum for τ_- restricted to S_t. The cosine is strictly decreasing over $(0,\pi/2)$: hence, we can equivalently say that the Hessian of the function $h(x) := \langle \rho_-(x_0), x\rangle$ is negative semi-definite at x_0.

Let N be future-directed unit vector field normal to S_t, and $v \in T_xS$. By an explicit computation, we get

$$\begin{aligned} \nabla h(x) &= \rho_-(x_0) + \langle \rho_-(x_0), x\rangle x + \langle \rho_-(x_0), N(x)\rangle N(x) \\ \mathrm{Hess}_x\, h(v,v) &= \langle \rho_-(x_0), x\rangle \langle v,v\rangle + \langle \rho_-(x_0), N(x)\rangle \mathrm{II}(v,v) \\ &= \cos\left(\tau_-(x)\right)\langle v,v\rangle - \sin\left(\tau_-(x)\right)\mathrm{II}(v,v). \end{aligned}$$

At x_0, the Hessian is negative semi-definite and $\tau_-(x_0) = \tau(x_0) = t$. Hence, we obtain

$$\mathrm{II}(v,v) \leq \frac{1}{\tan(t)} \langle v,v\rangle,$$

proving the claim and concluding the proof. □

Remark 10.2.5. In light of these results, we can easily produce examples proving the sharpness of Proposition 10.1.6. Indeed, let S be a smooth future-convex achronal non-properly embedded (resp. incomplete) hypersurface. By Lemma 2.1.4, we can write $S = \operatorname{graph} f$, for f a strictly $1-$Lipschitz map defined on an open set of $\mathbb{H}^n$.

Denote by f^+ the extremal extension of f introduced in Section 3.1, and define $S^+ := \operatorname{graph} f^+$, which does not depend on the splitting. One can check that S_+ is a future-convex properly embedded hypersurface whose acausal part coincide with S (compare with Lemma 3.1.2).

By Proposition 10.2.1 and Lemma 5.3.2, the leaves of the cosmological flow of S_t^+ are well-defined complete hypersurfaces, and by Corollary 10.2.3, they coincides with the leaves of the normal flow of S. Since S is smooth, by Lemma 10.1.1, the S_t^+'s are smooth as for $t \in (0,\pi/2)$.

Denote $\Sigma := S_t^+$, for $t \in (0,\pi/2)$. Then Σ is a complete smooth spacelike hypersurface. An explicit computation shows that $A_- = -t$, that is $\Sigma_{A_-} = S$, which is not properly embedded (resp. non-complete), by hypothesis.

10.3. Sectional curvature

Proposition 10.3.1. *Let S be a future-convex (resp. past-convex) properly embedded spacelike hypersurface in $\widetilde{\mathbb{H}}^{n,1}$. If S is C^2, then S_t is negatively curved (in general non uniformly), for all $t \in [0, \pi/2)$.*

Proof. Since S is C^2, its shape operator B is well defined and the equidistant hypersurface S_t coincides with the leaf of the normal flow at time t.

Since S is future-convex, the principal curvatures λ_i of S are non negative. By Corollary 10.1.3, the principal curvatures λ_i^t of S_t satisfy

$$\lambda_i^t = \tan\left(\arctan(\lambda_i) - t\right) = \frac{\lambda_i - t}{1 + \lambda_i t}$$

Let e_i^t be the unit eigenvector relative to λ_i^t. By Gauss equation, the sectional curvature of S_t along the 2−plane spanned by e_i^t, e_j^t is

$$\begin{aligned} -K\left(\mathrm{Span}(e_i^t, e_j^t)\right) &= 1 + \lambda_i^t \lambda_j^t = 1 + \frac{\lambda_i - t}{1 + \lambda_i t} \cdot \frac{\lambda_j - t}{1 + \lambda_j t} \\ &= \frac{1 + (\lambda_i + \lambda_j)t + \lambda_i\lambda_j t^2 + \lambda_i\lambda_j - (\lambda_i + \lambda_j)t + t^2}{1 + (\lambda_i + \lambda_j)t + \lambda_i\lambda_j t^2} \\ &= \frac{(1 + t^2)(1 + \lambda_i\lambda_j)}{1 + (\lambda_i + \lambda_j)t + \lambda_i\lambda_j t^2} > 0 \end{aligned}$$

Now, take two orthonormal vectors

$$u = \sum_{i=1}^n a_i e_i, \qquad v = \sum_{i=1}^n b_i e_i.$$

One can check that $\mathrm{Span}(u, v)$ contains an orthonormal basis which is orthogonal with respect to $\mathrm{I\!I}_t$: we call such basis u, v, again.

$$\begin{aligned} -K\left(\mathrm{Span}(u, v)\right) &= 1 + \mathrm{I\!I}_t(u, u)\mathrm{I\!I}_t(v, v) - \underbrace{\mathrm{I\!I}_t(u, v)^2}_{=0} \\ &= 1 + \sum_{i=1}^n a_i^2 \lambda_i^t \cdot \sum_{j=1}^n b_j^2 \lambda_j^t = 1 + \sum_{i,j=1}^n a_i^2 b_j^2 \lambda_i^t \lambda_j^t. \end{aligned} \tag{10.3}$$

Since u, v are unit vectors,

$$1 = \sum_{i=1}^n a_i^2 \cdot \sum_{j=1}^n b_j^2 = \sum_{i,j=1}^n a_i^2 b_j^2.$$

Substituting in Equation (10.3), we obtain

$$-K\left(\mathrm{Span}(u, v)\right) = \sum_{i,j=1}^n a_i^2 b_j^2 \underbrace{\left(1 + \lambda_i^t \lambda_j^t\right)}_{= -K(\mathrm{Span}(e_i^t, e_j^t)) > 0} > 0.$$

□

Proposition 10.3.3 can be considered a *converse* result. In order to prove it, we need first this technical lemma.

Lemma 10.3.2. *Let Σ be a spacelike hypersurface in $\widetilde{\mathbb{H}}^{n,1}$. Let B be the shape operator of Σ, and denote by $\lambda_1 \geq \cdots \geq \lambda_n$ the eigenvalues of B. Then Σ is non-positively curved at x if and only if $\lambda_1(x)\lambda_n(x) \geq -1$.*

Proof. Assume that Σ is non-positively curved at x. Let e_1, e_n be the eigenvectors relative to $\lambda_1(x), \lambda_n(x)$. By Gauss equation,

$$0 \geq K\left(\mathrm{Span}(e_1, e_n)\right) = -1 - \lambda_1(x)\lambda_n(x),$$

proving that $\lambda_1(x)\lambda_n(x) \geq -1$.

For the converse implication, we distinguish two cases: if $\lambda_1(x)\lambda_n(x) > 0 > -1$, then $B(x)$ is positive (resp. negative) definite, implying that Σ is past (resp. future) convex in a neighborhood U of x. Hence, Σ is strictly negative curved over U: in particular Σ is non-positively curved at x.

Otherwise, $\lambda_1(x)\lambda_n(x) \leq 0$. Hence,

$$\lambda_i(x)\lambda_j(x) \geq \lambda_1(x)\lambda_n(x), \qquad \forall i, j = 1, \dots, n.$$

Consider a tangent $2-$plane W in $T_x\Sigma$ and let

$$u = \sum_{i=1}^n a_i e_i, \quad v = \sum_{i=1}^n b_i e_i$$

be an orthonormal basis of W such that $\mathrm{I\!I}(u, v) = 0$. Then,

$$\begin{aligned} -K(W) =& 1 + \mathrm{I\!I}(u,u)\mathrm{I\!I}(v,v) = 1 + \sum_{i,j=1}^n a_i^2 b_j^2 \lambda_i(x)\lambda_j(x) \\ & \geq 1 + \sum_{i,j=1}^n a_i^2 b_j^2 \lambda_1(x)\lambda_n(x) = 1 + \lambda_1(x)\lambda_n(x) \geq 0, \end{aligned}$$

concluding the proof. □

We are finally ready to prove that $\mathcal{C}(S)$ is foliated by Hadamard manifolds.

Proposition 10.3.3. *Let S be a past-convex (resp. future-convex) properly embedded achronal hypersurface in $\widetilde{\mathbb{H}}^{n,1}$. Let S_t be a C^2-leaf of the cosmological time of $\mathcal{C}(S)$, then S_t is a Hadamard manifold.*

Proof. Denote by $\Sigma := S_t$. By Corollary 10.1.7, S_t is a complete spacelike manifold. Then, we only need to prove that Σ is non-positively curved.

Since Σ is C^2, it is well defined its shape operator B. Denote by $\lambda_1 \geq \cdots \geq \lambda_n$ the eigenvalues of B. By Lemma 10.3.2, it suffices to prove that $\lambda_1\lambda_n \geq -1$ over Σ.

Denote by Σ_s the leaf of the normal flow of Σ at time s. Let $A_\pm$ be the quantities defined in Equation (10.2): by Proposition 10.1.6, (A_-, A_+) is the maximal open interval containing 0 such that Σ_s is a complete spacelike hypersurface.

The leaf Σ_s of the normal flow of Σ coincides with the leaf S_{t+s} of the cosmological time of $\mathcal{C}(S)$. By Corollary 10.1.7, the leaves of cosmological time of $\mathcal{C}(S)$ are spacelike complete hypersurfaces for any

$$s \in \left(-t, \frac{\pi}{2} - t\right).$$

It follows that $A_+ - A_- \geq \pi/2$, then

$$\begin{aligned}\arctan\left(\sup_\Sigma \lambda_1\right) &= A_- + \frac{\pi}{2} \leq A_+ = \arctan\left(\inf_\Sigma \lambda_n\right) + \frac{\pi}{2}\\ \iff \sup_\Sigma \lambda_1 &\leq \frac{-1}{\inf_\Sigma \lambda_n}.\end{aligned}$$

If It follows that

$$\lambda_1(x)\lambda_n(x) \geq \sup_\Sigma \lambda_1 \inf_\Sigma \lambda_n \geq -1, \qquad \forall x \in \Sigma,$$

which concludes the proof, by Lemma 10.3.2. □

10.4. Application to $K-$surface in $\mathbb{H}^{2,1}$

In this section, we briefly study *constant sectional curvature* (CSC) surfaces in the three dimensional Anti-de Sitter space.

Remark 10.4.1. CSC surfaces in $\widetilde{\mathbb{H}}^{2,1}$ with constant sectional curvature $K \in (-\infty, -1)$ are strictly convex.

Indeed, let S be a surface with constant sectional curvature $K \in (-\infty, -1)$, hereafter $K-$surface, and let B be its shape operator. In dimension $(2+1)$, Gauss equation reduces to

$$\det B = -1 - K \in (0, +\infty).$$

It follows that B, which is a 2×2 symmetric matrix, is strictly definite: by Lemma 11.1.12, this is equivalent to strict convexity.

The bridge between the previous two sections is given by the duality which links CSC and CMC surfaces: indeed, it turns out that $K-$surfaces and $H-$ surfaces are equidistant.

Lemma 10.4.2. *Let Σ be a $H-$surface, with $H \in \mathbb{R}$. Then, the equidistant surface $\Sigma_{d(H)_\pm}$ is a $K_\pm(H)-$surface, for*

$$d(H)_\pm = \arctan\left(\frac{H}{2} \pm \sqrt{1 + \frac{H^2}{4}}\right), \qquad K_\pm(H) = -1 - \frac{4}{(H \pm \sqrt{1+H^2})^2}.$$

Conversely, let S be a future-convex (resp. past-convex) $K-$surface, for $K \in (-\infty, -1)$. Then, the equidistance $S_{d(K)}$ (resp. $S_{-d(K)}$) is a $H(K)-$surface (resp. $(-H(K))-$surface), for

$$d(K) = \arctan\left(\frac{1}{\sqrt{-1-K}}\right), \qquad H(K) = \frac{2-K}{\sqrt{-1-K}}. \tag{10.4}$$

Proof. The proof consists in an explicit (local) compution carried using in light of Corollary 10.1.3. □

Combining Theorem C and Lemma 10.4.2, we can classify the $K-$surfaces of $\mathbb{H}^{2,1}$ and improve the regularity of the foliation discovered in [BS18, Theorem 7.8].

Theorem E. *Let $\Lambda \subseteq \widetilde{\mathbb{H}}^{2,1}$ be an admissible boundary. For any $K \in (-\infty, 1)$ there exists a unique past-convex (resp. future-convex) achronal surface S_K^+ (resp. S_K^-) such that*

- *$\partial S_K^\pm = \Lambda$;*
- *its lightlike part is union of lightlike triangles associated to sawteeth;*

- *its spacelike part is an analytic K−surface.*

Proof. We prove the statement for S_K^-. The same argument works for S_K^+.

Let Λ be an admissible boundary. By [BS18, Theorem 7.8], there exists a properly embedded future-convex achronal surface S_K^-

- $\partial S_K^- = \Lambda$;
- its lightlike part is union of lightlike triangles associated to sawteeth;
- its spacelike part is an *smooth* K−surface.

Since S_K^- is an achronal properly embedded future-convex surface, the equidistant surface $(S_K^-)_t$ is a complete spacelike surface for any $t \in (0, \pi/2)$, by Proposition 10.2.4. By Lemma 10.4.2, the surface at distance $d(K)$ is a CMC complete surface. By Theorem C, the surface $(S_K^-)_{d(K)}$ is analytic: it follows that the normal flow is a C^ω−foliation (Lemma 10.1.1), hence S_K^- is analytic where non-degenerate.

The uniqueness part follows by Theorem A: indeed, let S_1, S_2 be two properly embedded achronal future-convex surfaces satisfying the hypothesis of the statement for a fixed $K \in (-\infty, -1)$. Assume S_1, S_2 shares the same asymptotic boundary, denoted by Λ. Since $(S_i)_{d(K)}$ is a complete spacelike $H(K)$−surface asymptotic to Λ, by Theorem A,

$$(S_1)_{d(K)} = (S_2)_{d(K)},$$

hence $S_1 = S_2$, concluding the proof. □

Corollary 10.4.3. *Let Λ be an admissible boundary. Then, $\Omega(\Lambda) \setminus \mathcal{CH}(\Lambda)$ is analytically foliated by $(S_K^\pm)_{K\in(-\infty,-1)}$.*

Proof. Let us consider the future connected component of $\Omega(\Lambda) \setminus \mathcal{CH}(\Lambda)$, that is

$$\mathcal{D}_+(\Lambda) := I^-\left(\partial_+\Omega(\Lambda)\right) \cap I^+\left(\partial_-\mathcal{CH}(\Lambda)\right).$$

By [BS18, Theorem 8.7], the past-convex K−surfaces introduced in Theorem E foliate $\mathcal{D}_+(\Lambda)$, so we only need to promote the regularity of the foliation.

By Theorem C, the H−surfaces bounding Λ analytically foliate $\Omega(\Lambda)$. Denote by

$$\begin{aligned} \sigma\colon \mathbb{H}^n \times \mathbb{R} &\longrightarrow \Omega(\Lambda) \\ (x,t) &\longmapsto \sigma(x,H) \end{aligned}$$

a (local) trivialization of the foliation in H−surfaces, *i.e.* the image of σ restricted to $\mathbb{H}^n \times \{H\}$ is the unique H−surface bounding Λ. By Lemma 10.4.2, σ induces a real-analytic foliation of $\mathcal{D}_+(\Lambda)$ in K−surfaces: indeed, denote $N(x,H)$ the normal vector to $\sigma\left(\mathbb{H}^n \times \{H\}\right)$ at $\sigma(x,H)$, then

$$\begin{aligned} s\colon \mathbb{H}^n \times (-\infty,-1) &\longrightarrow \mathcal{D}_+(\Lambda) \\ (x,K) &\longmapsto s(x,K) := \exp_{\sigma(x,H(K))}\left(d(K)\cdot N(x,H(K))\right). \end{aligned}$$

We already remarked that σ is a real-analytic map, and both $H(K)$, $d(K)$ are real-analytic functions by definition (compare with Equation 10.4), concluding the proof. □

Part IV.

Quantitative estimates

Chapter 11.

$H-$convexity

For an admissible boundary $\Lambda \subseteq \partial\mathbb{H}^{n,1}$, we adapt the notion of convex hull to any value of $H \in \mathbb{R}$: the *$H-$shifted convex hull* $\mathcal{CH}_H(\Lambda)$ (Definition 11.2.1). Then, we study how it controls the geometry of the $H-$hypersurface asymptotic to Λ.

11.1. $H-$convexity

We recalled in Section 3.3 that a subset X in $\mathbb{H}^{n,1}$ is convex if it can be written as intersection of half-spaces. To define *$H-$convexity*, we replace half-spaces with connected components of the complement of totally umbilical spacelike hypersurfaces with mean curvature H. Hereafter, we will denote by $\mathcal{P}_\delta$ the totally umbilical spacelike hypersurfaces with constant mean curvature $H = n\tan(\delta)$. Equivalently, they are the hypersurfaces at oriented distance $-\delta$ from the spacelike totally geodesic hypersurface $\mathcal{P}_0$ with whom they share the same boundary.

Definition 11.1.1. A subset X of $\widetilde{\mathbb{H}}^{n,1}$ is *future-$H-$convex* if it can be written as the intersection of the future of totally umbilical spacelike hypersurfaces of mean curvature H, namely if there exists $(\mathcal{P}^j_{\delta_H})_{j\in J}$, for $\delta_H := \arctan(H/n)$, such that

$$X = \bigcap_{j\in J} I^+(\mathcal{P}^j_{\delta_H}).$$

Conversely, the subset X is *past-$H-$convex* if

$$X = \bigcap_{j\in J} I^-(\mathcal{P}^j_{\delta_H}).$$

Finally, X is *$H-$convex* if $X = X_+ \cap X_-$, for X_+, X_- a future-$H-$convex and a past-$H-$convex subset, respectively.

Remark 11.1.2. One could expect that, for $H = 0$, the actual notion of convexity is recovered. However, this is not the case: $0-$convexity is a weaker notion. For example, one easily check that $\widetilde{\mathbb{H}}^{n,1}$ is $0-$convex but not convex.

Remark 11.1.3. One can easily check that $H-$convexity is preserved by time-preserving isometries, while time-reversing isometries send future-$H-$convex subsets to past-$(-H)-$convex ones. For this reason, most of the following result will be stated and proved for future-$H-$convex subsets, without loss of generality.

The following maximum principle for umbilical hypersurfaces will be useful for discovering properties of $H-$convexity.

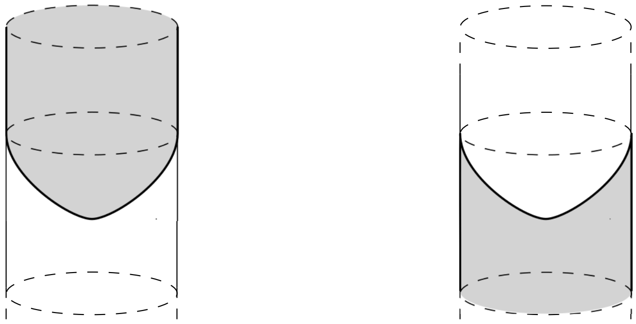

Figure 11.1.: The easiest examples of $H-$convexity, for $H > 0$: to the left, the future of a totally umbilical hypersurface, which is future-$H-$convex, to the right, the past of a totally umbilical hypersurface , which is past-$H-$convex.

Lemma 11.1.4. *Consider two umbilical hypersurfaces $\mathcal{P}_{\delta_H}$ and $\mathcal{P}_{\delta_L}$ in $\widetilde{\mathbb{H}}^{n,1}$. If $H > L$ and they meet tangentially at a point p, then $\mathcal{P}_{\delta_H} \setminus \{p\} \subseteq I^+(\mathcal{P}_{\delta_L})$.*

Proof. For $h \in \mathbb{R}$, denote by $\delta_h = \arctan(h/n)$ and let $f_h \colon \mathbb{H}^n \to \mathbb{R}$ be the function defined by

$$f_h(x) = \arccos\left(\frac{\sin(\delta_h)}{\sqrt{1+\sum_{i=1}^n x_i^2}}\right) - \arccos\left(\sin(\delta_h)\right).$$

We claim that graph f_h is a $h-$umbilical spacelike hypersurface which is tangent to $\mathbb{H}^n \times \{0\}$ at $(x_0, 0)$. In particular, choosing the splitting $(p, T_p\mathcal{P}_{\delta_H}) = (p, T_p\mathcal{P}_{\delta_L})$, we have $\mathcal{P}_{\delta_H} =$ graph f_H and $\mathcal{P}_{\delta_L} =$ graph f_L.

One can easily check that the function $h \mapsto f_h(x)$ is strictly increasing for any $x \in \mathbb{H}^n$, $x \neq x_0$. Hence, $f_H(x) > f_L(x)$ for any $x \neq x_0$, that is

$$P_{\delta_H} \setminus \{p\} \subseteq I^+(P_{\delta_L}),$$

which concludes the proof.

To prove the claim, project the problem on $\mathbb{H}^{n,1}$ through ψ. One can check that $P_{\delta_h} := \psi(\mathcal{P}_{\delta_h})$ is described by the equation

$$\langle p, e\rangle = -\sin(\delta_h). \tag{11.1}$$

for a suitable choice of e. Indeed, in $\mathbb{H}^{n,1}$ the umbilical hypersurface P_{δ_h} is equidistant to a totally geodesic P. Let $e_\pm$ be the point such that $e_\pm^\perp = P$, then $\langle p, e_\pm\rangle = \mp\sin(\delta_h)$, that is $e := e_-$ satisfies Equation (11.1).

Timelike geodesics starting from e intersect P_{δ_h} orthogonally: if $\mathcal{P}_{\delta_h}$ is tangent to $\mathbb{H}^n \times \{0\}$ at $(x_0, 0)$, then $\psi^{-1}(e)$ lies in the fiber $\{x_0\} \times \mathbb{R}$. Denote by $e_h := (x_0, T_h)$ the closest point to $\mathcal{P}_{\delta_h}$ in $\psi^{-1}(e) \cap I^-(\mathcal{P}_{\delta_h})$, namely

$$T_h = \arccos\left(\sin(\delta_h)\right) = \delta_h - \frac{\pi}{2}.$$

Let $q = (x, t) \in \mathcal{P}_{\delta_h}$, for $x = (x_1, \ldots, x_{n+1}) \in \mathbb{H}^n$, then

$$\begin{aligned}\sin(\delta_h) &= -\langle\psi(q), \psi(e_h)\rangle = -\langle\psi(q), e\rangle = x_{n+1}\left(\cos(t)\cos(T_h) + \sin(t)\sin(T_h)\right)\\ &= x_{n+1}\cos(t - T_h) = -\sqrt{1+\sum_{i=1}^n x_i^2}\,\cos(t - T_h).\end{aligned}$$

Hence, in this splitting, the umbilical hypersurface $\mathcal{P}_{\delta_h}$ is the graph of the function

$$f_h(x) = \arccos\left(\frac{\sin(\delta_h)}{\sqrt{1+\sum_{i=1}^n x_i^2}}\right) - T_h,$$

proving the claim and concluding the proof. □

Lemma 11.1.5. *Let X be a non-empty future-$H-$convex (resp. past-$H-$convex) set. Then, there exists a properly embedded achronal hypersurface S such that $\overline{X} = \overline{I^+(S)}$ (resp. $\overline{X} = \overline{I^-(S)}$).*

Proof. Assume that X is future-$H-$convex. In a splitting, the set

$$\overline{X} = \bigcap_{j\in J} \overline{I^+(\mathcal{P}^j_{\delta_H})}$$

is the epigraph of the function

$$f = \sup_{j\in J} f_j,$$

for f_j the strictly $1-$Lipschitz functions describing the $\mathcal{P}^j_{\delta_H}$'s. In particular, the function f is $1-$Lipschitz, *i.e.* $S = \operatorname{graph} f$ is an achronal properly embedded hypersurface by Lemma 2.1.4. □

Definition 11.1.6. With a slight abuse of notation, we will call S a *future-$H-$convex* (resp. *past-$H-$convex*) hypersurface.

Definition 11.1.7. Let S be a properly embedded achronal hypersurface, and $p \in S$. An achronal properly embedded hypersurface $\mathcal{P}$ is a past (resp. future) *support* hypersurface to S at p if it contains p and it is contained in $\overline{I^-(S)}$ (resp. $\overline{I^+(S)}$).

The following characterization descends directly from Lemma 11.1.5.

Corollary 11.1.8. *A properly embedded hypersurface S is future-$H-$convex (resp. past-$H-$convex) if and only if it admits, at any point p, a past (resp. future) support hypersurface which is either an umbilical $H-$hypersurface or a totally geodesic degenerate hypersurface.*

Proof. Let us fix $p \in S$. If S is future-$H-$convex, there exists a collection of umbilical hypersurfaces $(\mathcal{P}^j_{\delta_H})_{j\in J}$, such that $S \subseteq \overline{I^+(\mathcal{P}^j_{\delta_H})}$ for all $j \in J$ and S is the boundary of

$$X = \bigcap_{j\in J} I^+(\mathcal{P}^j_{\delta_H}).$$

Let us extract a sequence $(j_k)_{k\in\mathbb{N}} \subseteq J$ such that

$$\operatorname{dist}(\mathcal{P}^{j_k}_{\delta_H}, p) < 1/k.$$

Thanks to Proposition 7.0.1, the sequence subconverge to a properly embedded achronal hypersurface $\mathcal{P}_\infty$ which is either a totally geodesic degenerate hypersurface or a $H-$hypersurface. In the latter case, it is clear that $\mathcal{P}_\infty$ is a totally umbilical spacelike hypersurface. By construction, the point p belongs to $\mathcal{P}_\infty$ and by continuity $S \subseteq \overline{I^+(\mathcal{P}_\infty)}$, namely $\mathcal{P}_\infty$ is a support hypersuface for S, as requested. Since p was arbitrary, this holds for any point of S.

Conversely, let $\mathcal{P}_p$ be a support totally umbilical spacelike hypersurface or a totally geodesic degenerate hypersurface for S at p. One can easily build a sequence $(P_p^k)_{k\in\mathbb{N}}$ of totally umbilical $H-$hypersurfaces lying in the past of $\mathcal{P}_p$ such that the limit in the Hausdorff topology of $\mathcal{P}_p^k$ is $\mathcal{P}_p$: for the former case, it suffices to take the splitting $(p, \mathcal{P})$, for $\mathcal{P}$ the totally geodesic spacelike hypersurface equidistant to $\mathcal{P}_p$ and define P_p^k to be the totally umbilical $H-$hypersurfaces equidistant to $\mathbb{H}^n \times \{1/k\}$. For the latter one, take totally umbilical spacelike $H-$hypersurface equidistant to the totally geodesic spacelike hypersurface $\mathcal{P}_-(p_k)$, for p_k a sequence of P_p converging to its dual future point. Then, the set

$$X := \bigcap_{p\in S} \bigcap_{k\in N} I^+\left(\mathcal{P}_p^k\right)$$

is future $H-$convex and its boundary coincides with S, concluding the proof.

□

We can deduce that $H-$convexity is a closed property.

Corollary 11.1.9. *Let X_k be a sequence of properly embedded future-$H-$convex subsets of $\mathbb{H}^{n,1}$, converging to X_∞ in the Hausdorff topology. Then, X_∞ is future-$H-$convex.*

Proof. Denote S_k the achronal entire hypersurfaces such that $\overline{X_k} = \overline{I^+(S_k)}$. In a splitting, the S_k's are graphs of $1-$Lipschitz functions f_k, which are uniformly bounded since the sequence X_k converges in the Hausdorff topology. It follows that the sequence f_k converges uniformly to a $1-$Lipschitz function f_∞, *i.e.* the sequence S_k converges to the entire achronal graph $S_\infty = \operatorname{graph} f_\infty$ in the Hausdorff topology, and $\overline{X_\infty} = \overline{I^+(S_\infty)}$.

To prove that S_∞ is future-$H-$convex, let $a_k := \sup |f - f_k|$ and replace f_k by $f_k - a_k$, so that S_k lies in the past of S_∞, for any $k \in \mathbb{N}$. Fix a point $p := f_\infty(x)$ in S_∞, and consider $p_k := f_k(x)$. Since S_k is future-$H-$convex, by Corollary 11.1.8 there exists a past support hypersurface $\mathcal{P}_k$ to S_k at p_k, which is a $H-$umbilical hypersurface or a degenerate totally geodesic hypersurface. In particular, we have

$$S \subseteq \overline{I^+(S_k)} \subseteq \overline{I^+(\mathcal{P}_k)}.$$

We claim that the sequence $\mathcal{P}_k$ converges to a past support hypersurface $\mathcal{P}_\infty$. Indeed, P_∞ contains p and lies in the past of S_∞ by continuity. Moreover, by Proposition 7.0.1, P_∞ is either a $H-$umbilical hypersurface or a degenerate totally geodesic hypersurface. Since the choice of p was arbitrary, by Corollary 11.1.8 we conclude the proof. □

Corollary 11.1.10. *Let S be an achronal properly embedded hypersurface in $\widetilde{\mathbb{H}}^{n,1}$, and $L \in \mathbb{R}$.*

- *If S is future-$H-$convex, for every $H \in (-\infty, L)$, then S is future-$L-$convex;*
- *if S is past-$H-$convex, for every $H \in (L, +\infty)$, then S is past-$L-$convex.*

Proof. The result follows directly from Corollary 11.1.8: fix a point $p \in S$, consider a sequence $H_k \nearrow L$. Since S is future-H_k-convex, it admits at p a past support hypersurface $\mathcal{P}_k$ which is a H_k-totally umbilical spacelike hypersurfaces or a totally geodesic degenerate hypersurface. By Proposition 7.0.1, the sequence $\mathcal{P}_k$ converges to a support hypersurface $\mathcal{P}_\infty$ which is either a $L-$totally umbilical spacelike hypersurfaces or totally geodesic degenerate hypersurface, proving the $L-$convexity of S at p. Since p was arbitrary, this concludes the proof. □

Corollary 11.1.11. *Let S be an achronal properly embedded hypersurface in $\widetilde{\mathbb{H}}^{n,1}$, and $H \geq L$.*

- *If S is future-$H-$convex, then S is future-$L-$convex;*
- *if S is past-$L-$convex, then S is past-$H-$convex.*

In particular, if $H \geq 0$ (resp. $H \leq 0$), then X is future-convex (resp. past-convex).

Proof. Let S be a future-$H-$convex hypersurface. To prove that S future-$L-$convex as well, it suffices to exhibit, at each point $p \in S$, a past support $L-$umbilical hypersurface or a a past support totally geodesic degenerate hypersurface.

Fix $p \in S$: by Corollary 11.1.8, there exists a past support $H-$umbilical hypersurface $\mathcal{P}_{\delta_H}$ or a totally geodesic degenerate hypersurface at p. In the latter case, we are done. Otherwise, by Lemma 11.1.4, the hypersurface $\mathcal{P}_{\delta_L}$ tangent to $\mathcal{P}_{\delta_H}$ at p is a past support hypersurface for S. Since p was arbitrary, this concludes the proof. □

For regular hypersurfaces, the $H-$convexity is strictly linked with the definiteness of the shape operator.

Lemma 11.1.12. *An entire spacelike C^2-hypersurface in $\mathbb{H}^{n,1}$ is future-$H-$convex (resp. past-$H-$convex) if and only if its shape operator B satisfies $B \geq (H/n)\mathrm{Id}$ (resp. $B \leq (H/n)\mathrm{Id}$).*

Proof. Let S be an entire spacelike C^2-hypersurface. Pick a point $p \in S$, an umbilical hypersurface $\mathcal{P}_{\delta_H}$ tangent to S at p. Choose a splitting $\mathbb{H}^{n,1} = \mathbb{H}^n \times \mathbb{R}$ such that $p = (x_0, 0)$ and $T_pS = \mathbb{H}^n \times \{0\}$, and let $f, g \colon \mathbb{H}^n \to \mathbb{R}$ be the $1-$Lipschitz functions realizing the two entire spacelike hypersurface S and $\mathcal{P}_{\delta_H}$ as graphs. By construction, the point x_0 is a critical point for both f and g, and $f(x_0) = g(x_0) = 0$.

First, assume S is future-$H-$convex: it follows that $\mathcal{P}_{\delta_H}$ is a past support umbilical hypersurface for S, namely $f \geq g$. In particular, the function $f - g$ is non-negative and it achieves a global minimum at $p = (x_0, 0)$. Hence, at such a point

$$\mathrm{Hess}(f - g) = B - \frac{H}{n}\mathrm{Id} \geq 0.$$

To prove the converse implication, we assume that B is *strictly* greater than $(H/n)\mathrm{Id}$. The general case follows from Corollary 11.1.10: indeed, if $B \geq (H/n)\mathrm{Id}$, it follows that $B > (L/n)\mathrm{Id}$, hence Σ is future-$L-$convex, for any $L < H$.

Fix a $H-$umbilical hypersurface $\mathcal{P}_{\delta_H}$ tangent to S at a point p: we need to prove that $\mathcal{P}_{\delta_H}$ is a past support hypersurface for Σ at p. In fact, we are proving a strong maximum principle-like statement similar to Lemma 11.1.4, namely that $S \setminus \{p\} \subseteq I^+(\mathcal{P}_{\delta_H})$.

In the splitting (p, T_pS), let $S = \operatorname{graph} f$ and $\mathcal{P}_{\delta_H} = \operatorname{graph} g$. Since $d_{x_0}f = d_{x_0}g$ and

$$\mathrm{Hess}_{x_0}(f - g) = B(x_0) - (H/n)\mathrm{Id} > 0,$$

the point x_0 is a *strict* local minimum for the function $f - g$. Then, there exists an open neighbourhood U of p in S such that $U \setminus \{p\} \subseteq I^+(\mathcal{P}_{\delta_H})$.

By contradiction, assume there exists a point $q \in S \setminus \{p\}$ belonging to $\mathcal{P}_{\delta_H}$. Consider the totally geodesic Lorentzian plane $\mathcal{Q}$ containing the geodesic connecting p and q and the geodesic $\exp_p(\mathbb{R}N_p^S)$. Denote by $\gamma \colon [a, b] \to S$ the curve connecting p and q in $S \cap \mathcal{Q}$, parameterized by arc-length. We claim that γ is contained in the invisible domain of $\partial\mathcal{P}_{\delta_H}$. Indeed, denote by t_1, t_2 the projection of p, q to

$$\mathbb{H}^1 \times \{0\} = (\mathbb{H}^n \times \{0\}) \cap \mathcal{Q}$$

and define the function

$$h(t) = \begin{cases} g(t) & \text{if } t \in [t_1, t_2] \\ f(t) & \text{otherwise.} \end{cases}$$

Since both $p = \gamma(a)$ and $q = \gamma(b)$ lie on the $\mathcal{P}_{\delta_H}$, the function h is continuous. It follows that h is a (strictly) $1-$Lipschitz map which agrees with f at the boundary, hence is contained in the invisible domain of $\partial\mathcal{P}_{\delta_H} \cap \mathcal{Q}$ in $\mathbb{H}^{1,1}$. Denote by e the dual past point of $\mathcal{P}_0$ (see Definition 1.5.2): since e is contained in $\mathcal{Q}$, one can check that $\Omega\left(\partial\mathcal{P}_{\delta_H} \cap Q\right) = \Omega\left(\partial\mathcal{P}_{\delta_H}\right) \cap Q$, hence γ is contained in $\Omega(\partial\mathcal{P}_{\delta_H})$.

It follows that there exists a point $r \in \gamma$ maximising the distance from e. Then, γ is tangent to $\mathcal{P}_{\delta_L}$ at r, for

$$\delta_L = -\operatorname{dist}(e, r) < -\operatorname{dist}\left(e, \gamma(a)\right) = -\operatorname{dist}(e, p) = \delta_H. \tag{11.2}$$

Denote by c the curve $\mathcal{P}_{\delta_L} \cap Q$, parameterized by arc-length. Up to translation, we can assume $\gamma(0) = c(0) = r$. Since both γ and c are plane curves, we have that $\overline{\nabla}_{\gamma'}\gamma'$ and $\overline{\nabla}_{c'}c'$ are proportional at $t = 0$. In particular, they are orthogonal to $\mathcal{P}_{\delta_L}$, since c is a geodesic for $\mathcal{P}_{\delta_L}$. Moreover, since r maximizes the distance from e, γ is contained in $\overline{I^-(\mathcal{P}_{\delta_L})}$. Seeing γ and c as hypersurfaces of $\widetilde{\mathbb{H}}^{1,1}$, the first part of the proof implies that the shape operator of γ is greater or equal than L/n, that is

$$\langle \overline{\nabla}_{\gamma'}\gamma' - \overline{\nabla}_{c'}c', N_r^{\mathcal{P}_{\delta_L}} \rangle \geq 0,$$

for $N^{\mathcal{P}_{\delta_L}}(r)$ the unitary future-directed vector normal to $\mathcal{P}_{\delta_L}$.

It follows that

$$\begin{aligned} \frac{L}{n} &= \langle B_{\mathcal{P}_{\delta_L}}\left(c'(0)\right), c'(0)\rangle = -\langle \overline{\nabla}_{c'}c', N_r^{\mathcal{P}_{\delta_L}}\rangle \\ &\leq -\langle \overline{\nabla}_{\gamma'}\gamma', N_r^{\mathcal{P}_{\delta_L}}\rangle \leq -\langle \overline{\nabla}_{\gamma'}\gamma', N_r^{\Sigma}\rangle = \frac{H}{n}, \end{aligned}$$

contradicting Equation (11.2), since $\delta_H = \arctan(H/n)$. □

11.2. $H-$shifted convex hull

We are finally ready to introduce one of the main objects of this work.

Definition 11.2.1. Let Λ be an admissible boundary in $\partial\widetilde{\mathbb{H}}^{n,1}$ (resp. $\partial\mathbb{H}^{n,1}$). The *$H-$shifted convex hull* of Λ, denoted by $\mathcal{CH}_H(\Lambda)$, is the smallest $H-$convex set of $\widetilde{\mathbb{H}}^{n,1} \cup \partial\widetilde{\mathbb{H}}^{n,1}$ (resp. $\mathbb{H}^{n,1} \cup \partial\mathbb{H}^{n,1}$) containing Λ.

The next proposition shows that the definition is well-posed and explicitly describes $\mathcal{CH}_H(\Lambda)$: indeed, the $H-$shifted convex hull of Λ is the intersection of the future (resp. past) of $H-$umbilical surface whose boundary is in the past (resp.future) of Λ.

Proposition 11.2.2. *Let $\Lambda \subseteq \partial\widetilde{\mathbb{H}}^{n,1}$ be an admissible boundary and denote by $\delta_H = \arctan(H/n)$, then*

$$\mathcal{CH}_H(\Lambda) = \bigcap_{\partial\mathcal{P}_{\delta_H} \subseteq I^-(\Lambda)} I^+\left(\mathcal{P}_{\delta_H}\right) \cap \bigcap_{\partial\mathcal{P}_{\delta_H} \subseteq I^+(\Lambda)} I^-\left(\mathcal{P}_{\delta_H}\right).$$

Proof. Denote by X the set claimed to coincide with $\mathcal{CH}_H(\Lambda)$. By construction, X is $H-$convex and contains Λ, hence $\mathcal{CH}_H(\Lambda) \subseteq X$ by minimality.

Conversely, consider a point $p \notin \mathcal{CH}_H(\Lambda)$. Since $\mathcal{CH}_H(\Lambda)$ contains Λ, it is non-empty, hence it has two boundary components, $\partial_\pm \mathcal{CH}(\Lambda)$, which are properly embedded (see Lemma 11.1.5). Hence, without loss of generality, we can assume that p lies in the past of $\partial_- \mathcal{CH}_H(\Lambda)$. Equivalently, there exists an umbilical hypersurface $\mathcal{P}_{\delta_H}$ such that

$$\begin{cases} \mathcal{CH}_H(\Lambda) \subseteq I^+(\mathcal{P}_{\delta_H}) \\ p \in \overline{I^-(\mathcal{P}_{\delta_H})}. \end{cases}$$

The first condition implies that $\Lambda \subseteq I^+(\partial \mathcal{P}_{\delta_H})$: then, by definition of X, the second condition states that $p \notin X$, that is $X \subseteq \mathcal{CH}_H(\Lambda)$, concluding the proof. □

Lemma 11.2.3. *Let $\Lambda \subseteq \partial \widetilde{\mathbb{H}}^{n,1}$ be an admissible boundary, then $\mathcal{CH}_H(\Lambda)$ is contained in $\overline{\Omega(\Lambda)}$.*

Proof. By Lemma 11.1.5, the boundary components $\partial_\pm \mathcal{CH}_H(\Lambda)$ are properly embedded achronal hypersurfaces. One can easily check that their asymptotic boundary is Λ, hence they are contained in $\overline{\Omega(\Lambda)}$ by Lemma 3.1.7, concluding the proof. □

Remark 11.2.4. Since the restriction of ψ (see Equation (1.2)) to $\overline{\Omega(\Lambda)}$ is a diffeomorphism onto its image for any admissible boundary $\Lambda \subseteq \partial \widetilde{\mathbb{H}}^{n,1}$ (Corollary 3.1.8), we have

$$\mathcal{CH}_H\left(\psi(\Lambda)\right) = \psi\left(\mathcal{CH}_H(\Lambda)\right),$$

namely the $H-$shifted convex hull does not depend on the model.

In light of Lemma 11.2.3, we can characterize $\mathcal{CH}_H(\Lambda)$ in terms of the cosmological time functions $\tau_{\mathbf{P}}, \tau_{\mathbf{F}}$, which have been defined in Equation (3.1).

Corollary 11.2.5. *Let Λ be an admissible boundary in $\widetilde{\mathbb{H}}^{n,1}$. Then*

$$\mathcal{CH}_H(\Lambda) = \left\{ p \in \overline{\Omega(\Lambda)} \mid \tau_{\mathbf{P}}(p) \le \frac{\pi}{2} - \delta_H,\ \tau_{\mathbf{F}}(p) \ge \frac{\pi}{2} + \delta_H \right\},$$

for $\delta_H = \arctan(H/n)$.

Proof. It suffices to prove that

$$\bigcap_{\partial \mathcal{P}_{\delta_H} \subseteq I^+(\Lambda)} I^-\left(\mathcal{P}_{\delta_H}\right) = \left\{ \tau_{\mathbf{P}} \le \frac{\pi}{2} - \delta_H \right\}, \qquad \bigcap_{\partial \mathcal{P}_{\delta_H} \subseteq I^-(\Lambda)} I^+\left(\mathcal{P}_{\delta_H}\right) = \left\{ \tau_{\mathbf{F}} \ge \frac{\pi}{2} + \delta_H \right\}.$$

We claim the first identity holds: then, the second follows by applying a time-orientation reversing isometry, concluding the proof.

To prove the claim, fix a point $p \in \overline{\Omega(\Lambda)} \cap I^+\left(\mathcal{CH}_H(\Lambda)\right)$. Then, there exists a future support hyperplane $\mathcal{P}_0$ for Λ such that $p \in I^+\left(\mathcal{P}_{\delta_H}\right)$. By construction, the distance between p and P_0 is strictly greater then $-\delta_H$, while the distance between $\partial_- \Omega(\Lambda)$ and a future support hyperplane is at least $\pi/2$: indeed $\tau_{\mathbf{P}}^{-1}(\pi/2) = \partial_+ \mathcal{CH}(\Lambda)$ (Proposition 3.4.5). Hence,

$$\tau_{\mathbf{P}}(p) = \operatorname{dist}\left(p, \partial_- \Omega(\Lambda)\right) > \frac{\pi}{2} - \delta_H.$$

Conversely, assume $\tau_{\mathbf{P}}(p) > (\pi/2) - \delta_H$ consider a timelike geodesic γ realizing the distance $\tau_{\mathbf{P}}(p)$ from $\partial_- \Omega(\Lambda)$. Then, γ intersects orthogonally $\partial_- \Omega(\Lambda)$ at the point $\gamma(0)$ and $p = \gamma\left(\tau_{\mathbf{P}}(p)\right)$.

By Proposition 3.4.5, the dual future hyperplane $\mathcal{P}_0 = \gamma(0)_+^\perp$ is a future support hyperplane for $\partial_+ \mathcal{CH}(\Lambda)$ containing $\gamma(\pi/2)$. It follows that $p = \gamma\left(\tau_{\mathbf{P}}(p)\right)$ is at distance

$\tau_{\mathbf{P}}(p) - (\pi/2)$ from $\mathcal{P}_0$, or equivalently p belongs to $\mathcal{P}_{(\pi/2)-\tau_{\mathbf{P}}(p)}$, which is the umbilical hypersurface at oriented distance $\tau_{\mathbf{P}}(p) - (\pi/2)$ from P_0.

Since $\tau_{\mathbf{P}}(p) > (\pi/2) - \delta_H$, the umbilical hypersurface $\mathcal{P}_{(\pi/2)-\tau_{\mathbf{P}}(p)}$ lies in the future of $\mathcal{P}_{\delta_H}$, hence

$$p \notin \bigcap_{\partial\mathcal{P}_{\delta_H} \subseteq I^-(\Lambda)} \overline{I^-(\mathcal{P}_{\delta_H})},$$

proving the claim and concluding the proof. □

Remark 11.2.6. For $H \geq 0$, the future boundary of the $H-$shifted convex hull coincides with a leaf of the past cosmological time. More precisely,

$$\partial_+\mathcal{CH}_H(\Lambda) = \tau_{\mathbf{P}}^{-1}\left(\frac{\pi}{2} - \delta_H\right).$$

Hence, $\partial_+\mathcal{CH}_H(\Lambda)$ is $C^{1,1}$ by Proposition 3.4.5. On the other hand, no regularity is assured for $\partial_-\mathcal{CH}_H(\Lambda)$, except for the fact that it is a $1-$Lipschitz graph, since it is acausal by Lemma 11.1.5.

Conversely, for $H \leq 0$, the past boundary is a leaf of $\tau_{\mathbf{F}}$.

As a consequence of the strong maximum principle for CMC hypersurface (Proposition 4.2.1), we show that $H-$hypersurfaces are contained in the $H-$shifted convex hull of their boundary.

Corollary 11.2.7. *Let $H \in \mathbb{R}$ and Σ_H be a properly embedded spacelike $H-$hypersurface. If Σ_H is totally umbilical, then $\Sigma_H = \mathcal{CH}_H(\partial\Sigma_H)$, otherwise $\Sigma_H \subseteq \mathrm{Int}\,(\mathcal{CH}_H(\partial\Sigma_H))$.*

Proof. As for the convex hull, one can easily check that the interior of $\mathcal{CH}_H(\Lambda)$ is empty if and only if Λ is the boundary of a totally geodesic spacelike hypersurface, or equivalently the boundary of a totally umbilical CMC spacelike hypersurface.

The boundary of $I^\pm(\mathcal{P}_{\delta_H})$ in $\mathbb{H}^{n,1}$ is an umbilical hypersurface with constant mean curvature H. By the strong maximum principle (Proposition 4.2.1), if $\partial\mathcal{P} \subseteq \overline{I^\pm(\Lambda)}$, then $\Sigma \subseteq I^\pm(\mathcal{P}_{\delta_H})$, or $\Sigma = \mathcal{P}_{\delta_H}$ concluding the proof. □

11.3. Width

We need a tool to measure how far an admissible boundary Λ is from being a totally geodesic one. Since there is no metric on the boundary, but only a conformal structure, we will rather measure the timelike diameter of $\mathcal{CH}_H(\Lambda)$.

Definition 11.3.1. Let A be a subset $\subseteq \mathbb{H}^{n,1}$, the *width* of A is the supremum of the length of timelike curves contained in A. Hereafter, $\omega_H(\Lambda)$ will denote the width of the $H-$shifted convex hull $\mathcal{CH}_H(\Lambda)$.

A rough estimate of the width of $\mathcal{CH}_H(\Lambda)$ follows from Corollary 11.2.5.

Corollary 11.3.2. *Let $\Lambda \subseteq \partial\mathbb{H}^{n,1}$ be an admissible boundary, then*

$$\omega_H(\Lambda) \leq \frac{\pi}{2} - |\delta_H|,$$

for $\omega_H(\Lambda)$ the width of $\mathcal{CH}_H(\Lambda)$ and $\delta_H = \arctan(H/n)$.

Proof. Without loss of generality, let us assume $H \geq 0$. In this case, by Corollary 11.2.5 we have

$$\mathcal{CH}_H(\Lambda) \subseteq \tau_{\mathbf{P}}^{-1}\left(\left[0, \frac{\pi}{2} - \delta_H\right]\right),$$

namely $\partial_+\mathcal{CH}_H(\Lambda)$ is the leaf of the cosmological time on the past part of Λ. By definition of the cosmological time, the width of $\tau_{\mathbf{P}}^{-1}(I)$ coincides with the length of the interval I, which concludes the proof. □

Lemma 11.3.3. *Let $\Lambda \subseteq \partial\mathbb{H}^{n,1}$ be an admissible boundary. Let $H, K \in \mathbb{R}$ such that $|H| \geq |K|$ and $HK \geq 0$. Then*

$$\omega_K(\Lambda) + \delta_K - \delta_H \leq \omega_H(\Lambda) \leq \omega_K(\Lambda).$$

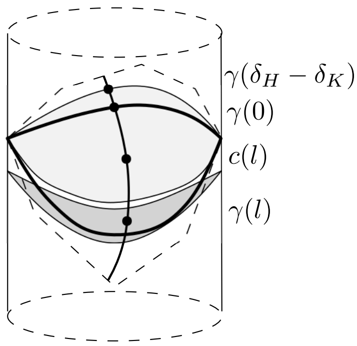

Figure 11.2.: Proof of Lemma 11.3.3: $\mathbf{P}(\Lambda)$ is dashed, $\mathcal{CH}_K(\Lambda)$ in light gray, $\mathcal{CH}_H(\Lambda)$ in heavier line, and $U_{K,H}$ in dark gray.

Proof. Let us fix Λ and admissible boundary and, without loss of generality, assume $H > K \geq 0$. To prove the first inequality, it suffices to remark that

$$\mathcal{CH}_H(\Lambda) \supseteq I^-\left(\partial_+\mathcal{CH}_H(\Lambda)\right) \cap I^+\left(\partial_-\mathcal{CH}_K(\Lambda)\right),$$

which is $\mathcal{CH}_K(\Lambda)$ deprived of

$$\overline{I^-\left(\partial_+\mathcal{CH}_K(\Lambda)\right)} \cap \overline{I^+\left(\partial_-\mathcal{CH}_H(\Lambda)\right)} = \left\{\frac{\pi}{2} - \delta_H \leq \tau_{\mathbf{P}} \leq \frac{\pi}{2} - \delta_K\right\},$$

whose width is $\delta_H - \delta_K$.

To prove the second inequality, consider a timelike curve c of length l contained in $\mathcal{CH}_H(\Lambda)$: we want to construct a curve with the same length in $\mathcal{CH}_K(\Lambda)$.

Assume that c is past-directed. Since it contained in $\mathcal{CH}_H(\Lambda)$, we have that

$$\operatorname{dist}\left(c(l), \partial_+\mathcal{CH}(\Lambda)\right) \geq l.$$

Consider the integral line for $\overline{\nabla}\tau_{\mathbf{P}}$ passing through $c(l)$, that is the timelike geodesic

$$\gamma := \left[\rho_+\left(c(l)\right), \rho_-\left(c(l)\right)\right]$$

(compare with Proposition 3.4.5). If we parameterize γ so that it is past-directed and $\gamma(0) \in \partial_+\mathcal{CH}_H(\Lambda)$. Then, since

$$c(l) = \gamma\left(\operatorname{dist}\left(c(l), \partial_+\mathcal{CH}(\Lambda)\right)\right) \in I^+\left(\gamma(l)\right),$$

we have that $\gamma([0,l])$ is contained in $\mathcal{CH}_H(\Lambda)$.

We claim that the geodesic segment

$$\gamma([\delta_K-\delta_H, l+\delta_K-\delta_H])$$

is contained in $\mathcal{CH}_K(\Lambda)$. In particular, if we choose γ of length $l=\omega_H(\Lambda)-\varepsilon$, we prove that $\omega_K(\Lambda)\geq\omega_H(\Lambda)-\varepsilon$: since ε is arbitrary, we conclude that $\omega_K(\Lambda)\geq\omega_H(\Lambda)$.

To prove the claim, we recall that γ is an integral line for $\overline{\nabla}\tau_{\mathbf{P}}$: hence, it meets orthogonally each level set of $\tau_{\mathbf{P}}$. By construction, the distance between two level sets is precisely the difference of the values they are preimages of, hence

$$\gamma(\delta_K-\delta_H)\in\partial_+\mathcal{CH}_K(\Lambda).$$

To conclude, we need to show that the point $p:=\gamma\left(l+\delta_K-\delta_H\right)$ is contained in $\mathcal{CH}_K(\Lambda)$. Consider an umbilical $K-$hypersurface P_{δ_K} whose boudary lies in the past of Λ. By contradiction, assume that p lies in the past of P_{δ_K}: hence, the geodesic segment

$$\gamma\left([l+\delta_K-\delta_H, l]\right)$$

is contained in the open set

$$U_{K,H}:=I^-\left(P_{\delta_K}\right)\cap I^+\left(P_{\delta_H}\right),$$

for the umbilical $H-$hypersurface P_{δ_H} sharing the same boundary as P_{δ_K}. By openness of $U_{K,H}$, the segment can prolonged, hence $U_{K,H}$ contains a timelike segment whose length is greater then $\delta_H-\delta_K$, which is the width of $U_{K,H}$, leading to a contradiction and concluding the proof. □

The following result descends directly.

Corollary 11.3.4. *Let Λ be an admissible boundary, the function $\omega_\bullet(\Lambda)\colon\mathbb{R}\to[0,\pi/2]$*

1. *is continuous;*
2. *is increasing (resp. decreasing) for $H\leq 0$ (resp. for $H\geq 0$);*
3. *achieves its maximum at $H=0$;*
4. $\lim_{H\to\pm\infty}\omega_\bullet(\Lambda)=0$.

Chapter 12.

Estimates near the Fuchsian locus

In this section, we study CMC hypersurfaces which are close to being totally geodesic. The goal is to show that if the a $H-$hypersurface is closed to be totally umbilical, *i.e.* its traceless shape operator is small, then the $H-$width of its boundary is small. In particular, we prove

Corollary I. *Let Λ be an admissible boundary and $H \in \mathbb{R}$. Let B_0 be the traceless shape operator of the properly embedded spacelike $H-$hypersurface such that $\partial\Sigma = \Lambda$. If $\|B_0\|^2_{C^0(\Sigma)} \leq 1 + (H/n)^2$, the width of $\mathcal{CH}_H(\partial\Sigma)$ satisfies*

$$\tan\left(\omega_H(\Lambda)\right) \leq \frac{2\|B_0\|_{C^0(\Sigma)}}{1 + (H/n)^2 - \|B_0\|^2_{C^0(\Sigma)}}.$$

In fact, we prove the following more general result, which does not require the additional bound on the norm of the traceless shape operator.

Proposition H. *Let Λ be an admissible boundary in $\partial\mathbb{H}^{n,1}$ and $H \in \mathbb{R}$. Let Σ the unique properly embedded spacelike $H-$hypersurface such that $\partial\Sigma = \Lambda$. Then*

$$\omega_H(\Lambda) \leq \arctan\left(\sup_\Sigma \lambda_1\right) - \arctan\left(\inf_\Sigma \lambda_n\right),$$

for $\lambda_1 \geq \cdots \geq \lambda_n$ be the principal curvatures of Σ, decreasingly ordered.

However, the estimate is more interesting near the Fuchsian locus, that is for small values of $\|B_0\|$, and the expression in Corollary I is more direct than the one in Proposition H.

12.1. The width is a lower bound for the extrinsic curvature

The normal evolution allows then to compare the extrinsic curvature of an $H-$hypersurface Σ to the width of the $H-$shifted convex hull of $\partial\Sigma$. In short, by Proposition 11.2.7 Σ is contained in $\mathcal{CH}_H(\partial\Sigma)$, and by Corollary 10.1.3 the principal curvatures are monotone along the normal flow. Hence, we look for the two times T_+, T_- when Σ_t becomes $H-$convex respectively in the past and in the future. In fact, the set $I^-(\Sigma_{T_+}) \cap I^+(\Sigma_{T_-})$ is $H-$convex: by minimality, it contains the $H-$shifted convex hull, dominating its width.

To formalize this idea, we need some preliminary results.

Lemma 12.1.1. *Let Σ be a properly embedded spacelike hypersurface. Denote by $\lambda_1 \geq \cdots \geq \lambda_n$ the principal curvatures of Σ. Define*

$$T_+ := \arctan\left(\sup_\Sigma \lambda_1\right), \qquad T_- := \arctan\left(\inf_\Sigma \lambda_n\right).$$

Let $t \in (A_-, A_+)$ as in Proposition 10.1.6. Then, Σ_t is past-$H-$convex (resp. future-$H-$convex) if and only if $t \geq T_+ - \delta_H$ (resp. $t \leq T_- - \delta_H$).

Proof. Let us prove that Σ_t is past-$H-$convex if and only if $t \geq T_+ + \delta_H$: the other proof is completely analogous.

As remarked in Corollary 10.1.3, the principal curvatures of Σ_t are

$$\lambda_i^t = \tan\left(\arctan(\lambda_i) - t\right).$$

In particular, the function $t \mapsto \lambda_i^t$ is strictly decreasing in t. By Lemma 11.1.12, the leaf Σ_t is past-$H-$convex if and only if $\lambda_i^t \leq H/n$, for all $i = 1, \dots, n$. This is equivalent to require

$$\begin{aligned} \frac{H}{n} \geq \sup_\Sigma \lambda_1^t = \sup_\Sigma \tan\left(\arctan(\lambda_1) - t\right) = \tan\left(\arctan\left(\sup_\Sigma \lambda_1\right) - t\right) \\ \iff \delta_H \geq \arctan\left(\sup_\Sigma \lambda_1\right) - t = T_+ - t, \end{aligned}$$

concluding the proof. □

We can finally bound the width of the $\mathcal{CH}_H(\Lambda)$ using the extrinsic curvature of the unique $H-$hypersurface bounding Λ.

Proposition H. *Let Λ be an admissible boundary in $\widetilde{\mathbb{H}}^{n,1}$ and Σ the unique properly embedded spacelike $H-$hypersurface such that $\partial\Sigma = \Lambda$. Then*

$$\omega_H(\Lambda) \leq \arctan\left(\sup_\Sigma \lambda_1\right) - \arctan\left(\inf_\Sigma \lambda_n\right).$$

Proof. Corollary 11.2.7 implies that Σ is contained in $\mathcal{CH}_H(\Lambda)$. We follow the normal flow of Σ until the leaves become $H-$convex in the future (resp. in the past).

Consider $A_\pm$ as in Equation (10.2) and $T_\pm$ as in Lemma 12.1.1. In particular,

$$A_+ = T_- + \frac{\pi}{2}, \qquad A_- = T_+ - \frac{\pi}{2}.$$

We distinguish three different situation. First, assume that $T_+ - \delta_H \geq A_+$. Then

$$T_+ - \delta_H \geq T_- + \frac{\pi}{2} \iff T_+ - T_- \geq \frac{\pi}{2} + \delta_H \geq \frac{\pi}{2} - |\delta_H|.$$

It follows by Corollary 11.3.2 that $T_+ - T_- \geq \omega_H(\partial\Sigma)$. Second, assume that $T_- - \delta_H \leq A_-$: the same argument proves that

$$T_+ - T_- \geq \frac{\pi}{2} - |\delta_H| \geq \omega_H(\partial\Sigma).$$

The last case is when $A_- < T_- \leq T_+ < A_+$, namely $(T_-, T_+) \subseteq (A_-, A_+)$. Then, by Proposition 10.1.6 and Lemma 12.1.1, the set

$$U := \overline{I^-(\Sigma_{T_+})} \cap \overline{I^+(\Sigma_{T_-})}$$

is a closed $H-$convex set containing Λ and $\omega(U) = T_+ - T_-$. By minimality, U contains $\mathcal{CH}_H(\Lambda)$, hence

$$\arctan\left(\sup_\Sigma \lambda_1\right) - \arctan\left(\inf_\Sigma \lambda_n\right) = T_+ - T_- = \omega(U) \geq \omega_H(\Lambda),$$

concluding the proof. □

Remark 12.1.2. This estimate is meaningful only near the Fuchsian locus. Indeed, for any cylindrical hypersurface $\mathbb{H}(k,\theta)$ with $k \notin \{0,n\}$, we have

$$\lambda_1 \equiv \tan(\theta), \qquad \lambda_n \equiv -\frac{1}{\tan(\theta)},$$

that is

$$\arctan\left(\sup_\Sigma \lambda_1\right) - \arctan\left(\inf_\Sigma \lambda_n\right) = \frac{\pi}{2} \geq \frac{\pi}{2} - \frac{|\delta_H|}{2} \geq \omega_H(\Lambda),$$

hence the estimate is far from sharp far from the Fuchsian locus, for $H \neq 0$.

For $n = 2$, we know that $\omega_0(\Lambda) = \pi/2$ if and only if $T_+ - T_- = \pi/2$ as a consequence of [BS10, Proposition 5.2]. We wonder if the same happens in higher dimension.

Definition 12.1.3. Let Σ be a Riemannian manifold and let B be a tensor of type $(1,1)$ on Σ. We denote the operator norm of $B(x) \in \mathrm{End}(T_x\Sigma, \langle\cdot,\cdot\rangle)$ by

$$\|B(x)\|^2 := \sup_{v \in T_x\Sigma} \frac{\langle B(x)v, B(x)v\rangle}{\langle v, v\rangle}.$$

We recall that $\|B(x)\|^2$ coincides with the maximal eigenvalue of $B(x)^t B(x)$. In particular, $\|B(\cdot)\|\colon \Sigma \to \mathbb{R}$ is a continuous function.

Corollary I. *Let Λ be an admissible boundary in $\partial\widetilde{\mathbb{H}}^{n,1}$. Let B_0 be the traceless shape operator of the properly embedded spacelike $H-$hypersurface such that $\partial\Sigma = \Lambda$. If*

$$\|B_0\|^2_{C^0(\Sigma)} \leq 1 + (H/n)^2,$$

the width of $\mathcal{CH}_H(\partial\Sigma)$ satisfies

$$\tan\left(\omega_H(\Lambda)\right) \leq \frac{2\|B_0\|_{C^0(\Sigma)}}{1 + (H/n)^2 - \|B_0\|^2_{C^0(\Sigma)}}.$$

Proof. By definition of traceless shape operator, we have

$$\|B_0\|_{C^0(\Sigma)} = \max\left\{\|a_1\|_{C^0(\Sigma)}, \|a_n\|_{C^0(\Sigma)}\right\},$$

for $a_i = \lambda_i - (H/n)$ the eigenvalues of B_0. In particular,

$$\sup_\Sigma \lambda_1 = \frac{H}{n} + \|a_1\|_{C^0(\Sigma)}, \qquad \inf_\Sigma \lambda_n = \frac{H}{n} - \|a_n\|_{C^0(\Sigma)}$$

By Proposition H, it follows that

$$\begin{aligned}\omega_H(\Lambda) &\leq \arctan\left(\frac{H}{n} + \|a_1\|_{C^0(\Sigma)}\right) - \arctan\left(\frac{H}{n} - \|a_n\|_{C^0(\Sigma)}\right)\\ &\leq \arctan\left(\|B_0\|_{C^0(\Sigma)} + \frac{H}{n}\right) + \arctan\left(\|B_0\|_{C^0(\Sigma)} - \frac{H}{n}\right) \leq \frac{\pi}{2}\end{aligned}$$

under the hypothesis $\|B_0\|^2_{C^0(\Sigma)} \leq 1 + (H/n)^2$. Since the tangent is strictly increasing over $[0, \pi/2]$, we obtain the result. □

Chapter 13.

The width is an upper bound for the extrinsic curvature

The goal of this section is to prove that the width of the $H-$shifted convex hull is an upper bound of the norm of the traceless shape operator, namely an estimate going in the opposite direction with respect to the one found in Proposition H. To be precise, we will show the following:

Theorem F. *For any $L \geq 0$, there exists a universal constant C_L with the following property: let $K \in [0, L]$ and Σ be a properly embedded $H-$hypersurface in $\mathbb{H}^{n,1}$ with $H \in [K, L]$, and let B_0 be its traceless shape operator. Then,*

$$\|B_0\|_{C^0(\Sigma)} \leq C_L \sin\left(\omega_K(\partial\Sigma)\right),$$

for $\omega_K(\partial\Sigma)$ the width of the $K-$shifted convex hull of $\partial\Sigma$.

Before going into details, we briefly explain the structure of this section. Let us fix an $H-$hypersurface Σ, a point $x \in \Sigma$ and a totally umbilical support $H-$hypersurface $\mathcal{P}_{\delta_H}$ for $\mathcal{CH}_H(\partial\Sigma)$. Let v_H be the sine of the distance between Σ and $\mathcal{P}_{\delta_H}$.

The function v_H is bounded from above from the width $H-$shifted convex hull of Σ, and we prove that the second derivatives of v_H approximate the traceless shape operator of Σ around x (Proposition 13.2.1).

It turns out that v_H solves the elliptic PDE

$$\Delta_\Sigma v_H - n v_H = f_H,$$

for Δ_Σ the Laplace-Beltrami operator on Σ and f_H an explicit function defined in Equation (13.10). Using Schauder-type estimates, we bound the C^2-norm of v_H with its C^0-norm, achieving our goal.

The technical part lies in proving that the procedure does not depend on the choice of the $H-$hypersurface Σ, the point x and the umbilical hypersurface $\mathcal{P}_{\delta_H}$. Under necessary but not restrictive assumptions on $\mathcal{P}_{\delta_H}$ (Definition 13.1.3), we give bounds for the gradient (Proposition 13.2.3) and for the Hessian (Corollary 13.3.2) of v_H over an open ball around x of fixed radius (Corollary 13.2.6), which not depend on the choice of Σ and x.

Finally, we prove that the Laplace-Beltrami operators are uniformly elliptic over the space of $H-$hypersurfaces (Lemma 13.4.2), hence Schauder estimates do not depend on the choice of Σ, concluding the proof of Theorem F.

13.1. Graphs over totally geodesic hypersurfaces

Let us fix a totally geodesic hypersurface $\mathcal{P}$, and denote by e its past dual point, defined in Definition 1.5.2. Let us fix an entire spacelike hypersurface Σ and consider the function

$$\begin{aligned} u \colon \Sigma &\longrightarrow \mathbb{R} \\ x &\longmapsto \langle x, e\rangle \end{aligned}.$$

From a geometric point of view, if $p \in I(e)$, *i.e.* p is time-related to e, then $\langle \psi(p), \psi(e)\rangle$ is the sine of the signed distance between p and $\mathcal{P}$ (compare with Proposition 1.5.4). This function reflects the geometry of Σ, as already exploited in [BS10; Sep19].

Proposition 13.1.1 (Proposition 1.8 in [Sep19])**.**

$$\operatorname{Hess} u - u\mathrm{Id} = \sqrt{1 - u^2 + |\nabla u|^2} B.$$

Proof. We have already remarked that u is the restriction to Σ of the function $U(p) = \langle p, e\rangle$, for e the dual of the totally geodesic spacelike hypersurface P. A direct computation shows

$$\begin{aligned} \overline{\nabla} U &= e + \langle p, e\rangle p \\ \nabla u &= e + \langle p, e\rangle p + \langle \overline{\nabla} U, N\rangle N = e + \langle p, e\rangle p + \langle e, N\rangle N. \end{aligned} \tag{13.1}$$

It follows that

$$\operatorname{Hess} u(w) = \nabla_w \nabla u = \langle e, p\rangle w + \langle e, N\rangle B(w) = u(p) w + \langle e, N\rangle B(w).$$

By Equation (13.1), we have

$$|\nabla u|^2 = -1 + u^2 + \langle e, N\rangle^2,$$

concluding the proof. □

The behaviour of CMC hypersurfaces is strictly linked to the umbilical hypersurfaces with the same mean curvature, which happens to be the equidistant hypersurfaces from a totally geodesic hypersurfaces. Hence, we define a new function encoding the signed distance from an umbilical hypersurface.

Let us fix a spacelike totally geodesic hypersurface $\mathcal{P}$, its past dual point $e = \mathcal{P}^{\perp}_{-}$ and denote by $\mathcal{P}_\delta$ the hypersurface at signed distance $-\delta$ from $\mathcal{P}$. We remark that the mean curvature of $\mathcal{P}_\delta$ is $n\tan(\delta)$. Let us define the function

$$\begin{aligned} v \colon \Sigma &\longrightarrow \mathbb{R} \\ x &\longmapsto \cos(\delta) u + \sin(\delta)\sqrt{1-u^2} \end{aligned}, \tag{13.2}$$

Remark 13.1.2. Restricted to $\Sigma \cap I^+(e)$, the function v equals $\sin(\operatorname{dist}(\cdot, \mathcal{P}_\delta))$: indeed, for any $x \in \Sigma \cap I^+(e)$, we have

$$\operatorname{dist}(x, \mathcal{P}) = \operatorname{dist}(x, \mathcal{P}_\delta) - \delta,$$

since $\mathcal{P}_\delta$ is at constant distance $-\delta$ to $\mathcal{P}$. Then, we conclude

$$\begin{aligned} \sin(\operatorname{dist}(x, \mathcal{P}_\delta)) &= \sin(\operatorname{dist}(x, \mathcal{P}) + \delta) \\ &= \cos(\delta)\sin(\operatorname{dist}(x, \mathcal{P})) + \sin(\delta)\sqrt{1 - \sin(\operatorname{dist}(x, \mathcal{P}))^2} \\ &= \cos(\delta) u(x) + \sin(\delta)\sqrt{1 - u(x)^2} = v(x). \end{aligned}$$

Consider a $H-$hypersurface Σ and a point $x \in \Sigma$. We want to study the geometry of Σ in around x through the function v_H as in Equation (13.2). Since Σ is contained in $\mathcal{CH}_H(\Lambda)$ (Corollary 11.2.7) and the width of the $H-$shifted convex hull is at most $(\pi/2) - |\delta_H|$ (Corollary 11.3.2), we can choose $\mathcal{P}_{\delta_H}$ such that the distance from x is at most $(\pi/4) - (|\delta_H|/2)$. We observe that P_{δ_H} could lie in the past or in the future of Σ.

In order to lighten the notation, we denote

$$U_H(\mathcal{P}) := \left\{ y \in \Sigma, |\operatorname{dist}(y, \mathcal{P}_{\delta_H})| \leq \frac{\pi}{4} - \frac{|\delta_H|}{2} \right\} \tag{13.3}$$

and we introduce the following space.

Definition 13.1.3. Let $\mathcal{P}$ be a totally geodesic spacelike hypersurface, and $L \geq 0$. We denote by $\mathcal{CMC}(L, \mathcal{P})$ the space of properly embedded CMC hypersurfaces Σ such that

1. Σ has mean curvature $H \in [-L, L]$;
2. $\mathcal{P}_{\delta_H}$ is a support hypersurface for $\mathcal{CH}_H(\partial\Sigma)$, for $\delta_H := \arctan(H/n)$;
3. $\Sigma \cap U_H(\mathcal{P}) \neq \emptyset$.

Since the computations in the following have all a local nature, they will be carried over the open neighborhood of $U_H(\mathcal{P})$ given by

$$U_H(\mathcal{P}, \varepsilon) := \left\{ y \in \Sigma, |\operatorname{dist}(y, \mathcal{P}_{\delta_H})| < \frac{\pi}{4} - \frac{|\delta_H|}{2} + \varepsilon \right\}, \tag{13.4}$$

for a suitable choice of ε.

Hereafter, we will assume at least $\varepsilon < \pi/4$, so that $U_H(\mathcal{P}, \varepsilon)$ is contained in $I^+(e)$: as explained in Remark 13.1.2, this ensures that v_H restricted to $U_H(\mathcal{P}, \varepsilon)$ equals to $\sin(\cdot, \mathcal{P}_{\delta_H})$.

13.2. Gradient estimate

Proposition 13.2.1. *Let $\mathcal{P}$ be a totally geodesic spacelike hypersurface. For any spacelike hypersurface Σ, the function v as in Equation (13.2), restricted to $U_L(\mathcal{P}, \varepsilon)$ (see Equation (13.4)) satisfies*

$$\operatorname{Hess} v = v\mathrm{Id} + \sqrt{1 - v^2 + |\nabla v|^2} B - \frac{\tan(\delta)}{\sqrt{1 - v^2} + v\tan(\delta)} \left(\mathrm{Id} + \frac{dv\nabla v}{1 - v^2} \right).$$

Proof. A direct computation shows that

$$\begin{aligned} u &= \cos(\delta)v - \sin(\delta)\sqrt{1 - v^2} \\ \nabla u &= \left(\cos(\delta) + \frac{\sin(\delta)}{\sqrt{1 - v^2}} v \right) \nabla v \\ \operatorname{Hess} u &= \left(\cos(\delta) + \frac{\sin(\delta)}{\sqrt{1 - v^2}} v \right) \operatorname{Hess} v + \frac{\sin(\delta)}{(1 - v^2)^{3/2}} dv\nabla v. \end{aligned} \tag{13.5}$$

We need to compare the last equation with Proposition 13.1.1, which states

$$\operatorname{Hess} u = u\mathrm{Id} + \sqrt{1 - u^2 + |\nabla u|^2} B,$$

for B the shape operator of Σ. One can check that

$$u = \left(\cos(\delta) + \frac{\sin(\delta)}{\sqrt{1-v^2}} v\right) v - \frac{\sin(\delta)}{\sqrt{1-v^2}}$$
$$1 - u^2 = \left(\cos(\delta) + \frac{\sin(\delta)}{\sqrt{1-v^2}} v\right)^2 (1-v^2),$$

hence

$$\operatorname{Hess} u = \left(\cos(\delta) + \frac{\sin(\delta)}{\sqrt{1-v^2}} v\right) \left(v\mathrm{Id} + \sqrt{1-v^2+|\nabla v|^2} B\right) - \frac{\sin(\delta)}{\sqrt{1-v^2}}.$$

We conclude by comparing the above formula with the expression of $\operatorname{Hess} u$ made in Equation (13.5), after the observation that

$$\left(\cos(\delta) + \frac{\sin(\delta)}{\sqrt{1-v^2}} v\right)^{-1} \frac{\sin(\delta)}{\sqrt{1-v^2}} = \frac{\tan(\delta)}{\sqrt{1-v^2} + \tan(\delta) v}.$$

□

We prove here a technical inequality which will be useful througout this section.

Lemma 13.2.2. *Let $\mathcal{P}$ be a totally geodesic spacelike hypersurface, and $L \geq 0$. For any Σ and $H-$hypersurface in $\mathcal{CMC}(L, \mathcal{P})$. Then*

$$\left| \sqrt{1-v_H^2} + v_H \tan(|\delta_H|) \right| \geq \frac{\cos\left(\frac{\pi}{4} + \frac{|\delta_H|}{2} + \varepsilon\right)}{\cos(|\delta_H|)}.$$

over $U_H(\mathcal{P}, \varepsilon)$, for $\varepsilon < (\pi/4) - (|\delta_H|/2)$. In particular, for $\varepsilon < (\pi/8) - (\delta_H/4)$, we have the uniform bound

$$\left| \sqrt{1-v_H^2} + v_H \tan(|\delta_H|) \right| \geq \frac{1}{4}.$$

Proof. To lighten the notation, we assume $H \geq 0$. Recalling that $v_H = \sin(\operatorname{dist}(\cdot, \mathcal{P}_{\delta_H}))$, one can check that

$$\sqrt{1-v_H^2} + v_H \tan(\delta_H) = \frac{\sqrt{1-v_H^2}\cos(\delta_H) + v_H \sin(\delta_H)}{\cos(\delta_H)} = \frac{\cos(\delta_H - \operatorname{dist}(\cdot, \mathcal{P}_{\delta_H}))}{\cos(\delta_H)}.$$

By choice of $\mathcal{P}_{\delta_H}$, we have a lower bound on its distance from Σ:

$$|\delta_H - \operatorname{dist}(\cdot, \mathcal{P}_{\delta_H})| \leq \delta_H + \frac{\pi}{4} - \frac{\delta_H}{2} + \varepsilon = \frac{\pi}{4} + \frac{\delta_H}{2} + \varepsilon < \frac{\pi}{2},$$

for $\varepsilon < (\pi/4) - (\delta_H/2)$. The cosine decreases in absolute value over $[-\pi/2, \pi/2]$. Hence,

$$\left| \sqrt{1-v_H^2} + v_H \tan(\delta_H) \right| \geq \frac{\cos\left(\frac{\pi}{4} + \frac{\delta_H}{2} + \varepsilon\right)}{\cos(\delta_H)}.$$

To conclude, assume $\varepsilon < (\pi/8) - (\delta_H/4)$, so that

$$\frac{\cos\left(\frac{\pi}{4} + \frac{\delta_H}{2} + \varepsilon\right)}{\cos(\delta_H)} > \frac{\cos\left(\frac{\pi}{2} + \frac{1}{4}\left(\delta_H - \frac{\pi}{2}\right)\right)}{\cos(\delta_H)} = -\frac{\sin\left(\frac{1}{4}\left(\delta_H - \frac{\pi}{2}\right)\right)}{\cos(\delta_H)} =: f(\delta_H).$$

Over $[0, \pi/2]$, the function f is decreasing: indeed, the sine is increasing over $[-\pi/8, 0]$ and the cosine is decreasing over $[0, \pi/2]$. Hence, the maximum is achieved at $\pi/2$, namely

$$f(\delta_H) > \lim_{t \to \pi/2} f(t) = \frac{1}{4},$$

concluding the proof. □

Proposition 13.2.3. *For any $L \geq 0$, there exists a universal constant $G_L > 0$ with the following property. Let $\mathcal{P}$ be a totally geodesic spacelike hypersurface, and let $\Sigma \in \mathcal{CMC}(L, \mathcal{P})$ be a $H-$hypersurface. The function v_H as in Equation (13.2) satisfies*

$$|\nabla v_H| \leq G_L |v_H|,$$

over $U_H(\varepsilon)$, for $\varepsilon < (\pi/8) - (\delta_L/4)$.

Proof. We assume $H \geq 0$ to lighten the notation. Let us fix an entire $H-$hypersurface Σ and a point $p \in \Sigma$. Without loss of generality, one can assume $\nabla u(p) \neq 0$ and consider the integral curve γ of the (opposite) normalized gradient flow of v, *i.e.* the solution of

$$\begin{cases} \gamma(0) = p \\ \gamma'(t) = -\frac{\nabla v_H}{|\nabla v_H|} \end{cases}.$$

Denoting $y(t) := |\nabla u(\gamma(t))|$, the structure of the proof is the following: in the first step, we proof that if there exists a constant $A_H > 0$ such that

$$\frac{d}{dt} y(t) \leq A_H \sqrt{1 + y(t)^2}, \tag{13.6}$$

then $\|\nabla v_H\| \leq (A_H^2 + 2A_H)|v_H|$. In the second step, we actually produce the explicit (not necessarely sharp) constant

$$A_H := C_0(H, n) + n\left(1 + 4\tan(\delta_H)\right),$$

for $C_0(H, n)$ as in Theorem 8.0.1, satisfing the inequality in Equation (13.6).

Hence, the universal constant $G_H := A_H^2 + 2A_H$ satisfies the statement for $H-$hypersurfaces. Since the function $H \to G_H$ is increasing for $H \geq 0$, the constant G_L automatically works for any CMC hypersurface having mean curvature in $[0, L]$, concluding the proof.

Step 1: Main argument.

It follows from the claim that

$$-A_H t \leq \int_0^t \frac{y'(s)}{\sqrt{1 + y(s)^2}} ds = \operatorname{arcsinh}(y(t)) - \operatorname{arcsinh}(y(0)).$$

Applying the hyperbolic sine, which is increasing on $[0, +\infty)$, we obtain

$$-y(t) \leq \sinh(A_H t)\sqrt{1 + y(0)^2} - \cosh(A_H t) y(0). \tag{13.7}$$

By construction of γ, we have

$$\begin{aligned} v_H(\gamma(t)) - v_H(p) &= \int_0^t \langle \nabla v_H(\gamma'(s)), \gamma'(s) \rangle ds \\ &= \int_0^t \left\langle \nabla v_H(\gamma'(s)), -\frac{\nabla v_H(\gamma'(s))}{|\nabla v_H(\gamma'(s))|} \right\rangle ds = -\int_0^t y(s) ds. \end{aligned}$$

Integrating Equation (13.7), one obtains

$$v_H(\gamma(t)) - v_H(p) \leq \frac{1}{A_H}\left((\cosh(A_H t) - 1)\sqrt{1+y(0)^2} - \sinh(A_H t)y(0)\right) =: F(t).$$

The function $F(t)$ admits a unique minimum over $[0, +\infty)$: by solving $F'(t) = 0$, one finds that critical point solves

$$\tanh(A_H t_{\min}) = \frac{y(0)}{\sqrt{1+y(0)^2}},$$

hence

$$F(t_{\min}) = \frac{1}{A_H}\left(1 - \sqrt{1+y(0)^2}\right)$$

Now, we recall that $\mathcal{P}_{\delta_H}$ is a support $H-$umbilical hypersurface for Σ. By the strong maximum principle of Proposition 4.2.1, either $\Sigma = \mathcal{P}_{\delta_H}$ or Σ does not intersect $\mathcal{P}_{\delta_H}$. In the former case, the function v_H identically vanishes, hence $\nabla v_H \equiv 0$, concluding the proof. Otherwise, the function v_H never vanishes. Up to a time-inverting isometry, we can assume $v_H > 0$, hence $F(t_{\min}) > -v_H(p)$. It follows, since v_H takes values in $[-1, 1]$, that

$$\begin{aligned}\frac{1}{A_H}\left(1 - \sqrt{1+y(0)^2}\right) \geq -v_H(p) &\iff 1 + y(0)^2 \leq A_H^2 v_H(p)^2 + 2A_H v_H(p) + 1\\ &\iff y(0)^2 \leq A_H^2\left(v_H(p)^2 + 2v_H(p)\right) \leq (A_H^2 + 2A_H)v_H(p).\end{aligned}$$

Recalling that $y(0)^2 = |\nabla u(p)|^2$ and p was arbitrary, we found the seeken universal constant.

Step 2: Producing A_H.

Observe that

$$\begin{aligned}y(t)\left|\frac{d}{dt}y(t)\right| &= \frac{1}{2}\left|\frac{d}{dt}y(t)^2\right| = |\langle \nabla_{\gamma'(t)}\nabla v_H(\gamma(t)), \nabla v_H(\gamma(t))\rangle|\\ &= |\langle \operatorname{Hess} v_H\left(\gamma'(t)\right), \nabla v_H(\gamma(t))\rangle| \leq \|\operatorname{Hess} v_H\|\,|\nabla v_H(\gamma(t))| = \|\operatorname{Hess} v_H\|\, y(t).\end{aligned}$$

Hence, to prove the claim it suffices to bound the norm of the Hessian of v. We recall that, by Proposition 13.2.1, we have

$$\operatorname{Hess} v_H = v_H \mathrm{Id} + \sqrt{1 - v_H^2 + |\nabla v_H|^2}B - \frac{\tan(\delta_H)}{\sqrt{1-v_H^2} + v_H\tan(\delta_H)}\left(\mathrm{Id} + \frac{dv_H \nabla v_H}{1-v_H^2}\right).$$

By Theorem 8.0.1, we have that $\|B\|$ is bounded from above by an explicit universal constant $C_0(H, n)$, Lemma 13.2.2 ensures that the denominator is bounded from below by $1/4$ and $|v_H| \leq 1$ by definition. Hence, an elementary computation shows that

$$\left|\frac{d}{dt}y(t)\right| \leq \|\operatorname{Hess} v_H\| \leq \underbrace{(n + C_0(H,n) + 4n\tan(\delta_H))}_{=:A_H}\sqrt{1+|\nabla v_H|^2}.$$

satisfying the claim stated in Equation (13.6). □

Corollary 13.2.4. *For any $L \geq 0$, there exists an angle α_L as follows: let $\mathcal{P}$ be totally geodesic spacelike hypersurface, and $\Sigma \in \mathcal{CMC}(L, \mathcal{P})$. Let γ be a timelike geodesic orthogonal to $\mathcal{P}_{\delta_H}$ and let $p := \gamma \cap \Sigma$. Then, the angle between γ and the timelike geodesic $\exp_p(tN(p))$ is bounded by α_L.*

Proof. As we have repeatedly done so far, we fix $H \in [0, L]$, claim that

$$\cosh(\alpha_H) := \left(1 + \frac{1}{\cos\left(\frac{\pi}{4} + \frac{\delta_H}{2}\right)}\right)\sqrt{1 + G_H^2},$$

and conclude by the fact that the map $H \to \alpha_H$ is increasing for $H \in [0, L]$.

To have an explicit formula for the exponential map, we work in the quadric model $\mathbb{H}^{n,1}$. We denote by $P := \psi(\mathcal{P})$ and abusively call Σ the image of Σ via ψ. Remark the P_δ's are the level sets of the function $\mathrm{dist}(e, \cdot)$, for e the past dual point of the totally geodesic hypersurface P, namely $P = P_+(e)$. A timelike geodesic orthogonal to P_δ is then of the form $\gamma(t) = \cos(t)e + \sin(t)w$, for $w \in P_+(e) = P$ (compare with Equation (1.1)).

In particular, γ is an integral line for $\overline{\nabla} U$, *i.e.* the extension of $u(p) := \langle p, e\rangle$ to $\mathbb{H}^{n,1}$, hence the angle between γ and Σ coincides with $\cosh\langle\overline{\nabla} U, N\rangle$. By Equation (13.1), we obtain

$$\langle\overline{\nabla} U, N\rangle = \langle e, N\rangle = \sqrt{1 - u^2 + |\nabla u|^2} = \left(\cos(\delta_H) + \frac{\sin(\delta_H)}{\sqrt{1 - v_H^2}}\right)\sqrt{1 - v_H^2 + |\nabla v_H|^2}.$$

By hypothesis, we have

$$\sqrt{1 - v_H^2} = \cos\left(\mathrm{dist}(\cdot, \mathcal{P}_{\delta_H})\right) \geq \cos\left(\frac{\pi}{4} + \frac{\delta_H}{2}\right).$$

By Proposition 13.2.3, we have $|\nabla v_H|^2 \leq G_H^2$, proving the claim and concluding the proof. □

Corollary 13.2.5. *The space $\mathcal{CMC}(L, \mathcal{P})$ is compact, for any $L \in \mathbb{R}$, and for any totally geodesic spacelike hypersurface $\mathcal{P}$.*

Proof. Consider a sequence $(\Sigma_k)_{k\in\mathbb{N}}$ in $\mathcal{CMC}(L, \mathcal{P})$. By Proposition 7.0.1, we can extract a subsequence $(\Sigma_{k_j})_{j\in\mathbb{N}}$ which converges to an entire acausal hypersurface Σ_∞, which is either a $H-$hypersurface or a degenerate hypersurface.

Corollary 13.2.4 prevents the normal vector of Σ_k to degenerate, namely the latter case cannot happen. Since $\mathcal{CMC}(L, \mathcal{P})$ is closed (compare with Definition 13.1.3), Σ_∞ belongs to $\mathcal{CMC}(L, \mathcal{P})$, concluding the proof. □

The last result of this section shows that $U_L(\mathcal{P}, \varepsilon)$ contains all points closed enough to $U_L(\mathcal{P})$, with respect to the induced metric on Σ.

Corollary 13.2.6. *For any $L \geq 0$ and $\varepsilon \in (0, (\pi/8) - (\delta_L/4))$, there exists a universal constant $R_L(\varepsilon) > 0$ with the following property. Let $\mathcal{P}$ a totally geodesic spacelike hypersurface, for any $\Sigma \in \mathcal{CMC}(L, \mathcal{P})$, we have*

$$B_\Sigma\left(U_L(\mathcal{P}), R_L(\varepsilon)\right) \subseteq U_L(\mathcal{P}, \varepsilon),$$

for $U_L(\mathcal{P}), U_L(\mathcal{P}, \varepsilon)$ as in Equation (13.3) and Equation (13.4).

Proof. Let us fix $L \geq 0$ $\varepsilon > 0$ and $\Sigma \in \mathcal{CMC}(L, \mathcal{P})$. For any pair $x, y \in \Sigma$ at distance r, let γ be a length-minimizing geodesic connecting the two, whose existence is guaranteed by Theorem B. Assume γ parameterized by arclength, then

$$|v_H(y) - v_H(x)| = \left|\int_0^r \langle\nabla v_H\left(\gamma(t)\right), \gamma'(t)\rangle dt \nabla\right| \leq \int_0^r \|\nabla v_H\|\|\gamma'\| \leq G_L r$$

for G_L as in Proposition 13.2.3.

If $x \in U_L(\mathcal{P})$, namely $|\operatorname{dist}(x, \mathcal{P}_{\delta_H})| \leq (\pi/2) - (|\delta_H|)/2$, then

$$\begin{aligned}|\operatorname{dist}(y, \mathcal{P}_{\delta_H})| = |\arcsin(v_H(y))| &\leq |\arcsin(v_H(x) + G_L r)| \\ &\leq |\operatorname{dist}(x, \mathcal{P}_{\delta_H})| + \sqrt{2} G_L r \leq \frac{\pi}{4} - \frac{|\delta_H|}{2} + \sqrt{2} G_L r.\end{aligned}$$

It follows that $y \in U_L(\mathcal{P}, \varepsilon)$ for

$$r < \frac{\varepsilon}{\sqrt{2} G_L} =: R_L(\varepsilon),$$

concluding the proof. □

We can finally (locally) bound the C^0−norm of v with the width of the H−shifted convex hull.

Proposition 13.2.7. *For any $L \geq 0$ and $\varepsilon \in (0, (\pi/8) - (\delta_L/4))$, there exists a universal constant E_L with the following property: for any spacelike H−hypersurface Σ with $H \in [-L, L]$ and for any $x \in \Sigma$, there exists a support totally umbilical H−hypersurface for $\mathcal{CH}_H(\partial\Sigma)$ such that*

$$v_H(y) \leq E_L \sin(\omega_H(\partial\Sigma))$$

over $B(x, R_L(\varepsilon))$.

Proof. Let us fix $L \geq 0$ and $\varepsilon \in (0, (\pi/8) - (\delta_L/4))$, a spacelike H−hypersurface Σ with $H \in [-L, L]$ and a point $x \in \Sigma$.

If Σ is totally umbilical, there is nothing to prove: indeed, Σ is a totally umbilical support H−hypersurface for $\mathcal{CH}_H(\partial\Sigma)$, and the corresponding function $v_H = \sin(\operatorname{dist}(\cdot, \Sigma))$ identically vanishes, *i.e.* any choice of E_L satisfies the required inequality.

Otherwise, there exists a totally umbilical support H−hypersurface $\mathcal{P}_{\delta_H}$ for $\mathcal{CH}_H(\partial\Sigma)$ such that

$$v(x) \leq \frac{1}{2} \sin(\omega_H(\partial\Sigma)).$$

In particular, $|v_H(x)| \leq (\pi/2) - (\delta_H/2)$, that is $x \in U_H(\mathcal{P})$.

Let $y \in B_\Sigma(x, R_L(\varepsilon))$, for $R_L(\varepsilon)$ as in Corollary 13.2.6, and let γ be a geodesic of Σ joining x to y, parameterized by arclength so that $\gamma(0) = x$, $\gamma(r) = y$. Define $f(t) := v_H(\gamma(t))$: by Proposition 13.2.3, over $B_\Sigma(x, R_L(\varepsilon))$ we have

$$|f'(t)| = |\langle \nabla v_H(\gamma(t)), \gamma'(t)\rangle| \leq G_L |v_H(\gamma(t))| = G_L |f(t)|.$$

Since $\mathcal{P}_\delta H$ is a totally umbilical support H−hypersurface for $\mathcal{CH}_H(\partial\Sigma)$, by the strong maximum principle it cannot meet Σ, that is f never vanishes over γ. It follows that

$$\left| \ln\left(\frac{v(y)}{v(x)}\right) \right| = \int_0^r \left| \frac{f'(t)}{f(t)} \right| dt \leq G_L r \leq G_L R_L(\varepsilon).$$

Equivalently,

$$|v(y)| \leq e^{G_L R_L(\varepsilon)} |v(x)| \leq e^{G_L R_L(\varepsilon)} \sin(\omega_H(\partial\Sigma)),$$

concluding the proof. □

13.3. Hessian estimate

Let Σ be a properly embedded acausal hypersurface: a splitting naturally induces an embedding of Σ in $\widetilde{\mathbb{H}}^{n,1}$ as follows. Let $f\colon \mathbb{H}^n \to \mathbb{R}$ be the function such that $\Sigma = \operatorname{graph} f$. Then, define

$$\begin{aligned} \sigma\colon \mathbb{H}^n &\longrightarrow \widetilde{\mathbb{H}}^{n,1} \\ x &\longmapsto (x, f(x))\,. \end{aligned}$$

The precomposition by σ induces a bijective correspondence between $C^2(\Sigma)$ and $C^2(\mathbb{H}^n)$.

Lemma 13.3.1. *Let $L \geq 0$, $R > 0$. There exists a universal constant $D_L(R) > 0$ with the following property: let $\mathcal{P}$ be a totally geodesic spacelike hypersurface and $\Sigma \in \mathcal{CMC}(L, \mathcal{P}_0)$. Consider a splitting $(p_0, \mathcal{P})$. Let σ be the embedding of Σ in $\widetilde{\mathbb{H}}^{n,1}$ induced by the splitting. For any function $h \in C^2(\Sigma)$, we have*

$$\| \operatorname{Hess} h\|_{\Sigma\cap(B_{\mathcal{P}}(p_0,R)\times\mathbb{R})} \leq D_L(R)\|h\circ\sigma\|_{C^2(B_{\mathcal{P}}(p_0,R))}.$$

Proof. In the coordinates induced by the splitting, the hessian of a function $h \in C^2(\Sigma)$ writes as

$$\operatorname{Hess} h = \sum_{i,j,k,l=1}^{n} \left(\frac{\partial^2(h\circ\sigma)}{\partial x_k \partial x_l} g^{ki} g^{lj} + \sum_{m=1}^{n} \frac{\partial (h\circ\sigma)}{\partial x_k} g^{kl} g^{jm} \Gamma^i_{lm} \right) \partial_i \otimes \partial_j.$$

Christoffel's symbols depend on the coefficients g_{ij} of the metric expressed, which can be expressed as smooth functions of σ. By compactness (Corollary 13.2.5), we can bound all these quantities by a suitable constant $c_L(R)$ that does not depend on the choice of the hypersurface Σ. It follows that the norm of the hessian is bounded by a polynomial of degree 1 in the derivatives of h up to the second order, which concludes the proof. □

Hereafter, in order to lighten the notation, we will denote by

$$\|h\|_{C^2(B_{\mathcal{P}}(p_0,R))} := \|h\circ\sigma\|_{C^2(B_{\mathcal{P}}(p_0,R))}.$$

Corollary 13.3.2. *For any $L \geq 0$ and $\varepsilon \in (0, (\pi/8) - (\delta_L/4))$, there exists a universal constant $A_L(\varepsilon) > 0$ with the following property: let $\mathcal{P}$ be a totally geodesic spacelike hypersurface and let $\Sigma \in \mathcal{CMC}(L, \mathcal{P})$ be a $H-$hypersurface. For any $p \in U_H(\mathcal{P})$ there exists a point $p_0 \in \mathcal{P}$ such that the function v_H as in Equation (13.2) satisfies*

$$\|B_0\|_{C^0(\Sigma\cap(B_{\mathcal{P}}(p_0,R_L(\varepsilon))\times\mathbb{R}))} \leq A_L(\varepsilon)\|v\|_{C^2(B_{\mathcal{P}}(p_0,R_L(\varepsilon)))},$$

for $R_L(\varepsilon)$ as in Corollary 13.2.6 B_0 the traceless shape operator of Σ.

Proof. Let us fix $L \geq 0$ and $\varepsilon \in (0, (\pi/8) - (\delta_L/4))$. To lighten the notation, assume $H \geq 0$ and denote by $\delta_H = \arctan(H/n)$. We claim that the (non-sharp) constant

$$A_H(\varepsilon) := \sqrt{2}\left(D_H\left(R_L(\varepsilon)\right) + n + 2^{3/4}\tan(\delta_H) G_H^2 + 4n\left(1 + G_H^2 + G_H\tan(\delta_H)\right)\right)$$

satisfies the statement for $H-$hypersurfaces. Since $H \to G_H, \delta_H$ are increasing functions for $H \geq 0$, the function $H \mapsto A_H$ is increasing, too. It follows that if the claim is satisfied for $H-$hypersurfaces, it is automatically satisfied for any CMC hypersurface having mean curvature in $[0, H]$, concluding the proof.

Let us fix a $H-$hypersurface $\Sigma \in \mathcal{CMC}(L, \mathcal{P})$. Since the shape operator of Σ is $B = B_0 + \tan(\delta_H)\mathrm{Id}$, Proposition 13.2.1 gives

$$\sqrt{1 - v_H^2 + |\nabla v_H|^2} B_0 = \mathrm{Hess}\, v_H - v_H \mathrm{Id} + \frac{dv_H \nabla v_H}{(1 - v_H^2)^{3/2}} \tan(\delta_H) \\ + \left(\frac{1}{\sqrt{1 - v_H^2} + v_H \tan(\delta_H)} - \sqrt{1 - v_H^2 + |\nabla v_H|^2} \right) \tan(\delta_H)\mathrm{Id}.$$

Since $p \in U_H(\mathcal{P})$, its distance from $\mathcal{P}_{\delta_H}$ is less than $(\pi/4) - (\delta_H/2)$, hence

$$\sqrt{1 - v_H^2 + |\nabla v_H|^2} \geq \sqrt{1 - v_H^2} \geq \cos\left(\frac{\pi}{4} - \frac{\delta_H}{2}\right) \geq \frac{1}{\sqrt{2}}.$$

It follows that

$$\|B_0\| \leq \sqrt{2}\sqrt{1 - v_H^2 + |\nabla v_H|^2}\|B_0\|.$$

Moreover, by Lemma 13.3.1 and Proposition 13.2.3, we get

$$\left\| \mathrm{Hess}\, v_H - v_H \mathrm{Id} + \frac{dv_H \nabla v_H}{(1 - v_H^2)^{3/2}} \tan(\delta_H) \right\| \leq D_L\left(R_L(\varepsilon)\right) \|v_H\|_{C^2} + n\|v_H\|_{C^0} + 2^{3/4} \tan(\delta_H) G_H^2 \|v_H\|_{C^0}^2 \\ \leq \left(D_L\left(R_L(\varepsilon)\right) + n + 2^{3/4} \tan(\delta_H) G_H^2 \right) \|v_H\|_{C^2}. \tag{13.8}$$

In the last line, we used that $\|v_H\|_{C^0}^2 \leq \|v_H\|_{C^0}$ since $|v_H| \leq 1$ over $U_H(\mathcal{P}, \varepsilon)$, for $\varepsilon < \pi/4$.

To conclude, use Lemma 13.2.2 and Proposition 13.2.3 to get

$$\left\| \frac{1}{\sqrt{1 - v_H^2} + v_H \tan(\delta_H)} - \sqrt{1 - v_H^2 + |\nabla v_H|^2} \right\|_{C^0} \\ \leq 4 \left\| 1 - \sqrt{1 - v_H^2 + |\nabla v_H|^2} + \sqrt{1 - v_H^2 + |\nabla v_H|^2} v_H \tan(\delta_H) \right\|_{C^0} \\ \leq 4 \left\| v_H^2 + |\nabla v_H|^2 + G_H v_H \tan(\delta_H) \right\|_{C^0} \leq 4 \left(1 + G_H^2 + G_H \tan(\delta_H)\right) \|v_H\|_{C^0}, \tag{13.9}$$

proving the claim and concluding the proof. □

Remark 13.3.3. The fact that the C^2 behaviour of v encodes the curvature of Σ should not surprise. The relevance of the statement is due to the universality of the constant $A_L(\varepsilon)$.

13.4. Schauder estimate

At this point, we showed that for a $H-$hypersurface, the norm of the traceless shape operator is a big O of the C^2-norm of the distance from a totally umbilical spacelike hypersurface with the same mean curvature. Through Schauder's methods, we prove that the C^2-norm of v_H is in fact uniformly bounded by its C^0-norm.

The C^0-norm of v_H is related to the width of the $H-$shifted convex hull, since we always choose v_H to be less than $\sin(\omega_H/2)$, while B_0 depends on the C^2-behaviour of v_H.

The plan is to pullback the problem on $\mathbb{H}^n$ by projecting the graph of Σ on a suitable splitting, in order to apply elliptic PDE's theory.

We state a key result for this section, coming from elliptic PDE's theory, in an easier version. The original result can be found in [GT01, Theorem 6.2].

Proposition 13.4.1. *Let $v \in C^2(B(0,R))$ be a bounded solution of the elliptic PDE*

$$Lv = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} v + \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} v - nv = f,$$

for $f \in C^0(B(0,R))$. Assume the tensor $A = (a_{ij})$ is uniformly positive definite and that its coefficients are uniformly bounded, i.e. there exist positive constants $\lambda, \Lambda > 0$ such that

$$A \geq \lambda \mathrm{Id}, \qquad \|a_{ij}\|_{C^0(B(0,R))}, \|b_j\|_{C^0(B(0,R))} \leq \Lambda.$$

Then, there exists a constant $C = C(n, \lambda, \Lambda, R)$ such that

$$\|v\|_{C^2(B(0,R/2))} \leq C \left(\|v\|_{C^0(B(0,R))} + \|f\|_{C^0(B(0,R))} \right).$$

By tracing the expression of the Hessian in Lemma 13.2.1, we obtain that v_H solves the elliptic PDE $\Delta v_H - n v_H = f_H$, for Δ the Laplace-Beltrami operator on Σ and f_H defined by the equation

$$-\frac{f_H}{\tan(\delta_H)} = \frac{|\nabla v_H|^2}{(1 - v_H^2)^{3/2}} - n\sqrt{1 - v_H^2 + |\nabla v_H|^2} + \frac{n}{\sqrt{1 - v_H^2} + v_H \tan(\delta_H)}. \tag{13.10}$$

First, we prove that the Laplace-Beltrami operator on Σ satisfies the conditions of Proposition 13.4.1 and that the constant C does not depend on Σ, for a suitable $R > 0$.

Lemma 13.4.2. *Let $L \geq 0$ and $R > 0$. There exist constants $\lambda_L(R), \Lambda_L(R)$ as follows. Let $(q_0, \mathcal{P})$ be a splitting, and let $\Sigma \in \mathcal{CMC}(L, \mathcal{P})$ be a $H-$hypersurface and let*

$$\Delta_\Sigma := \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}$$

be the Laplace-Beltrami operator of Σ in the coordinates given by the splitting $(q_0, \mathcal{P})$.

Let p_0 be the (unique) point of Σ contained in the fiber $\{x_0\} \times \mathbb{R}$. If p_0 is contained in $U_H(\mathcal{P})$ then, $A \geq \lambda_L(R)\mathrm{Id}$ and

$$\|a_{ij}\|_{C^0(B(x_0,R))}, \|b_j\|_{C^0(B(x_0,R))} \leq \Lambda_L(R).$$

In other words, for any pointed $H-$hypersurface (p_0, Σ), we can find a splitting $(q_0, \mathcal{P})$ such that the Laplace-Beltrami operator of Σ is a uniformly elliptic operato around p_0, and the constants of ellipticity do not depend on the hypersurface nor on the point.

Proof. By Corollary 13.2.5, the space $\mathcal{CMC}(L, \mathcal{P})$ is compact with respect to the Hausdorff topology, which is equivalent to the $C_0^\infty(\mathbb{H}^n)$ on spacelike CMC graph by Proposition 7.0.1. It follows that $\mathcal{CMC}(L, \mathcal{P})$ embedds as a compact subset of $C^\infty\left(\overline{B_{\mathbb{H}^n}(x_0, R)}\right)$.

Pick $\Sigma \in \mathcal{CMC}(L, \mathcal{P})$ and let $\Sigma = \mathrm{graph}\, f$ in the splitting $(q_0, \mathcal{P})$. Then,

$$p_0 = (x_0, f(x_0)) \in U_H(\mathcal{P}) \iff |f(x_0) - \delta_H| \leq \frac{\pi}{4} - \frac{\delta_H}{2}.$$

Hence, the CMC satisfying this technical property are a compact subset of $\mathcal{CMC}(L, \mathcal{P})$ inside $C_0^\infty(\mathbb{H}^n)$. The Laplace-Beltrami operator of Σ can be explicitly written as a smooth function of f, its first and second order derivatives. Hence, the coefficients of Δ_Σ are uniformly bounded over $\overline{B_{\mathbb{H}^n}(x_0, R)}$, and the bound does not depend on Σ, but only on L and R, which concludes the proof. ∎

We are finally ready to prove the main result of this subsection.

Theorem F. *Let $L \geq K \geq 0$. There exists a universal constant C_L with the following property: let Σ a properly embedded $H-$hypersurface in $\mathbb{H}^{n,1}$ with $H \in [K, L]$, and let B_0 be its traceless shape operator. Then,*

$$\|B_0\|_{C^0(\Sigma)} \leq C_L \sin\left(\omega_K(\partial\Sigma)\right).$$

Proof. Let us fix a properly embedded CMC hypersurface Σ, with $H \in [K, L]$, and $p_0 \in \Sigma$. Let us also fix

$$\varepsilon < \frac{\pi}{8} - \frac{\delta_L}{4} \leq \frac{\pi}{8} - \frac{\delta_H}{4},$$

and denote by $R = R_L(\varepsilon)$ as in Corollary 13.2.6.

Take a support $H-$umbilical hypersurface $\mathcal{P}_{\delta_H}$ as in Proposition 13.2.7. In particular,

$$|\operatorname{dist}(p_0, \mathcal{P}_{\delta_H})| \leq \frac{\omega_H(\Lambda)}{2} \leq \frac{\pi}{4} - \frac{|\delta_H|}{4},$$

the last inequality coming from Corollary 11.3.2.

In other words, $p_0 \in U_H(\mathcal{P})$, that is $\Sigma \in \mathcal{CMC}(L, \mathcal{P})$, for $\mathcal{P}$ the totally geodesic spacelike hypersurface equidistant to $\mathcal{P}_{\delta_H}$. Let $q_0 \in \mathcal{P}$ a point realizing the distance $\operatorname{dist}(p_0, \mathcal{P})$. Then, in the splitting $(q_0, \mathcal{P})$, the point belongs to the fiber $\{x_0\} \times \mathbb{R}$.

By tracing the expression of the Hessian find in Lemma 13.2.1, we obtain that v_H solves the elliptic PDE $\Delta_\Sigma v_H - n v_H = f_H$, for Δ_Σ the Laplace-Beltrami operator on Σ and

$$f_H = -\tan(\delta_H)\left(\frac{|\nabla v_H|^2}{(1-v_H^2)^{3/2}} - n\sqrt{1 - v_H^2 + |\nabla v_H|^2} + \frac{n}{\sqrt{1-v_H^2} + v_H \tan(\delta_H)}\right).$$

By Lemma 13.4.2, the Beltrami-Laplacian operator of Σ is an elliptic operator with coefficients bounded by $\lambda_L(R), \Lambda_L(R)$ over $B(x_0, R)$. Hence, by Schauder interior estimates (Proposition 13.4.1), there exists a universal constant

$$C = C(n, \lambda_L(R), \Lambda_L(R), R) =: c_L$$

such that, for any solution $L_\Sigma v = f$, we have

$$\|v\|_{C^2(B(0,R/2))} \leq c_L \left(\|v\|_{C^0(B(0,R))} + \|f\|_{C^0(B(0,R))}\right).$$

Comparing with the proof of Corollary 13.3.2, and more precisely Equation (13.8) and Equation (13.9), we get

$$\|f\|_{C^0(B(0,R))} \leq n A_L(R) \|v\|_{C^0(B(0,R))}.$$

Hence, denoting by $C_L' := c_L A_L(R)\left(1 + n A_L(R)\right)$, we have

$$\|B_0(p_0)\| \leq \|B_0\|_{C^2(B(0,R/2))} \leq C_L' \|v\|_{C^0(B(0,R))} \leq C_L' E_L \sin\left(\omega_H(\partial\Sigma)\right),$$

for $E_L(\varepsilon)$ as in Proposition 13.2.7. Finally, let us denote by $C_L := C_L' E_L(\varepsilon)$: by Lemma 11.3.3, we have

$$\|B_0(p_0)\| \leq C_L \omega_H(\partial\Sigma) \leq C_L \omega_K(\partial\Sigma),$$

which concludes the proof since the choice of p_0 was arbitrary. □

Chapter 14.

Application: sectional curvature

This brief section wants to stress the link between the the width of the $H-$shifted convex hull of an admissible boundary Λ and the sectional curvature of the corresponding $H-$hypersurface.

Let Σ be a spacelike hypersurface in the Anti-de Sitter space. Gauss equation allows to compute the sectional curvature of Σ through its shape operator. Indeed, let $v, w \in T_x\Sigma$ two orthonormal vectors, then

$$\begin{aligned} K_\Sigma\left(\mathrm{Span}(v,w)\right) &= -1 - \mathrm{I\!I}(v,v)\mathrm{I\!I}(w,w) + \mathrm{I\!I}(v,w)^2 \\ &= -1 - \langle B(v), v\rangle\langle B(w), w\rangle + \langle B(v), w\rangle^2. \end{aligned}$$

The orthonormal basis (v, w) can be chosen so that $\langle B(v), w\rangle = 0$, as a consequence of the spectral theorem. It follows that

$$\begin{aligned} -K_\Sigma\left(\mathrm{Span}(v,w)\right) - 1 &= \langle B(v), v\rangle\langle B(w), w\rangle = \left(\langle B_0(v), v\rangle + (H/n)\right)\left(\langle B_0(w), w\rangle + (H/n)\right) \\ &= \langle B_0(v), v\rangle\langle B_0(w), w\rangle + (H/n)^2 + (H/n)\langle B_0(v+w), v+w\rangle \\ &\geq -\|B_0\|(x)^2 + (H/n)^2 - 2(|H|/n)\|B_0\|(x). \end{aligned}$$

In the second line we used that $\langle B(v), w\rangle = 0$ is equivalent to $\langle B_0(v), w\rangle = 0$.

Hence, the sectional curvature of Σ is uniformly bounded by an explicit function of the norm of the traceless operator:

$$\max_{\mathrm{Gr}_2(T_x\Sigma)} K_\Sigma \leq -1 - (H/n)^2 + \|B_0(x)\|^2 + 2(H/n)\|B_0(x)\|. \tag{14.1}$$

To our knowledge, it is still an open question wheter CMC hypersurfaces are Hadamard in higher dimension. However, a direct consequence of Theorem F recovers the existence of many CMC hypersurfaces in Anti-de Sitter space with *uniform* negative sectional curvature.

Corollary G. *For any $H \in \mathbb{R}$, there exists a universal constant $K_H > 0$ such that*

$$\sup_{\mathrm{Gr}_2(T\Sigma)} K_\Sigma < -1 - \left(\frac{H}{n}\right)^2 + K_H \sin\left(\omega_H(\partial\Sigma)\right),$$

for any properly embedded $H-$hypersurface Σ in $\mathbb{H}^{n,1}$.

Proof. The proof consists in comparing Equation (14.1) with the statement of Theorem F.

Indeed, the supremum of the sectional curvature of Σ is bounded by

$$\begin{aligned}\sup_{\mathrm{Gr}_2(T\Sigma)} K_\Sigma &\le -1 - \left(\frac{H}{n}\right)^2 + \frac{2|H|}{n}\|B_0\|_{C^0(\Sigma)} + \|B_0\|^2_{C^0(\Sigma)} \\ &\le -1 - \left(\frac{H}{n}\right)^2 + \left(\frac{2|H|}{n} + \|B_0\|_{C^0(\Sigma)}\right)\|B_0\|_{C^0(\Sigma)}.\end{aligned}$$

By Theorem 8.0.1, the norm of B is bounded by a uniform constant, hence $(2|H|/n) + \|B_0\|_{C^0(\Sigma)}$ is bounded by a constant K'_H, too. By Theorem F, we have

$$\begin{aligned}\sup_{\mathrm{Gr}_2(T\Sigma)} K_\Sigma - 1 - (H/n)^2 + K'_H\|B_0\|_{C^0(\Sigma)} \\ \le -1 - (H/n)^2 + K'_H C_H \sin(\omega_H(\Lambda)).\end{aligned}$$

Then, the constant $K_H := K'_H C_H$ satisfies the statement, concluding the proof. □

Unfortunately, this result cannot be state in terms of neighbourhoods of totally geodesics boundaries with respect to the Hausdorff topology: in the $2-$dimensional case, it is known that if an admissible boundary Λ contains two transverse lightlike rays, then the width of its convex hull is $\pi/2$. Conversely, in [Mor24], a class of admissible boundaries in $\mathbb{H}^{2,1}$ whose associated maximal surface is asymptotically flat is presented: such a class is dense in the space of admissible boundaries with respect to the Hausdorff topology.

For $n = 2$, the sectional curvature of CMC surfaces is non-positive. For the sake of completeness we add an *ad hoc* proof, which need a preliminary result, peculiar of surfaces, which is proved for example in [KS07, Lemma 3.11] or in [ES22, Lemma 3.1]:

Lemma 14.0.1. *Let (Σ, g) be a Riemannian $2-$manifold, and let B_0 be a traceless, $g-$symmetric and $g-$Codazzi $(1,1)-$smooth tensor. Denote by $\chi := \log(-\det B_0)$, then*

$$\frac{1}{4}\Delta^g\chi = K_\Sigma,$$

for $\Delta^g\chi := \mathrm{tr}(\mathrm{Hess}\,\chi)$ the Laplace-Beltrami operator on Σ.

Lemma 14.0.2. *Let Σ be a properly embedded spacelike CMC hypersurface in $\mathbb{H}^{2,1}$, then the sectional curvature of Σ is non-positive.*

Proof. The shape operator B of Σ is automatically a $g-$symmetric and $g-$Codazzi $(1,1)-$tensor. Since Σ has constant mean curvature H, the same yelds for the traceless shape operator $B_0 := B - H\mathrm{Id}$. By Gauss equation,

$$K_\Sigma = -1 - \det B = -1 - (H/2)^2 - \det B_0. \tag{14.2}$$

It follows that

$$\frac{1}{4}\Delta^g\chi = -1 + (H/2)^2 - \det B_0 = e^\chi - 1 - (H/2)^2.$$

By the strong maximum principle, we have $e^\chi \le 1 + (H/2)^2$, that is K_Σ is either strictly less than 0 or identically vanishes. □

Hence, in the $3-$dimensional case, Corollary I can be stated in terms of the sectional curvature of Σ.

Proposition J. *Let Λ be an admissible boundary in $\partial\widetilde{\mathbb{H}}^{2,1}$. Let B_0 be the traceless shape operator of the properly embedded spacelike $H-$hypersurface such that $\partial\Sigma = \Lambda$. Then,*

$$\tan\left(\omega_H(\Lambda)\right) \leq -\frac{2\|B_0\|_{C^0(\Sigma)}}{\sup_\Sigma K_\Sigma}.$$

Proof. In the 3−dimensional case, the eigenvalues of the traceless shape operator are opposite, hence $\|B_0\| = \|a_1\|_{C^0(\Sigma)} = \|a_2\|_{C^0(\Sigma)}$. In particular, by Equation (14.2), we have $\|B_0\|^2 \leq 1 + (H/2)^2$. Hence, by Corollary I, we have

$$\tan\omega_H(\Lambda) \leq \frac{2\|B_0\|_{C^0(\Sigma)}}{1 + (H/2)^2 - \|B_0\|^2} = -\frac{2\|B_0\|_{C^0(\Sigma)}}{\sup_\Sigma K_\Sigma},$$

concluding the proof. □

Part V.

Extensions of circle homeomorphisms

Chapter 15.

The $\mathbb{P}\mathrm{SL}(2,\mathbb{R})-$model

The connection between 3−dimensional Anti-de Sitter geometry and 2−dimensional hyperbolic geometry is due to the fact that $\mathbb{H}^{2,1}$ naturally identifies with the double cover of the Lie group $\mathrm{Isom}_0(\mathbb{H}^2)$. Indeed, $\mathbb{P}\mathrm{SL}(2,\mathbb{R})$ is precisely the space of negative lines of the space of 2×2 matrices $\mathcal{M}(2,\mathbb{R})$ endowed with the quadratic form $q=-\det$, which has signature $(2,2)$. In other words, $\mathrm{SL}(2,\mathbb{R})$ (resp. $\mathbb{P}\mathrm{SL}(2,\mathbb{R})$) is a realization of the quadric (resp. projective) model. It turns out that the Anti-de Sitter metric on these models coincides with a positive multiple of the Killing form.

Remark that $\mathbb{P}\mathrm{SL}(2,\mathbb{R})\times\mathbb{P}\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{P}\mathrm{SL}(2,\mathbb{R})$ by left and right multiplication, *i.e.*

$$(A,B)\cdot X = AXB^{-1}.$$

Binet formula assures that this is an isometric action for the metric induced by the quadratic form $-\det$. A dimensional argument concludes that

$$\mathrm{Isom}_0(\mathbb{P}\mathrm{SL}(2,\mathbb{R})) = \mathbb{P}\mathrm{SL}(2,\mathbb{R})\times\mathbb{P}\mathrm{SL}(2,\mathbb{R}) = \mathrm{Isom}_0(\mathbb{H}^2)\times\mathrm{Isom}_0(\mathbb{H}^2).$$

15.1. The asymptotic boundary

The asymptotic boundary of the projective model is the set of isotropic line for the quadratic form $q=-\det$. Hence, the boundary of $\mathbb{P}\mathrm{SL}(2,\mathbb{R})$ is the projectivization of the cone of rank 1 matrix. A useful description of the boundary follows.

Lemma 15.1.1 (Subsection 3.2 in [BS20]). *The map*

$$\begin{aligned} \partial\mathbb{P}\mathrm{SL}(2,\mathbb{R}) &\longrightarrow \partial\mathbb{H}^2\times\partial\mathbb{H}^2 \\ [X] &\longmapsto (\mathrm{Im}X, \ker X) \end{aligned}$$

is a $\mathbb{P}\mathrm{SL}(2,\mathbb{R})\times\mathbb{P}\mathrm{SL}(2,\mathbb{R})-$*equivariant homeomorphism.*

Remark 15.1.2. The splitting of $\partial\mathbb{P}\mathrm{SL}(2,\mathbb{R})$ as $\partial\mathbb{H}^2\times\partial\mathbb{H}^2$ described in Lemma 15.1.1 does not coincide with the one comping by a splitting (p,P). Indeed, the two circles are lightlike in the former case, while in the latter one is spacelike and the other is spacelike.

Through this identification, we can associate two lightlike geodesic to a point $x\in\partial\mathbb{H}^2$:

$$l_x := \{x_0\}\times\partial\mathbb{H}^2$$
$$r_x := \partial\mathbb{H}^2\times\{x_0\}.$$

As x varies in $\partial\mathbb{H}^2$, the lightlike geodesics l_x, r_x form a double ruling of $\partial\mathbb{PSL}(2,\mathbb{R})$, which gives a more geometric insight of the identification between $\partial\mathbb{PSL}(2,\mathbb{R})$ and $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$. Let us fix a totally geodesic spacelike surface P in $\mathbb{PSL}(2,\mathbb{R})$, which is an isometric copy of the hyperbolic space $\mathbb{H}^2$. We define two projection $\pi_l, \pi_r \colon \partial\mathbb{PSL}(2,\mathbb{R}) \to \partial\mathbb{H}^2$ as

$$\pi_l(x) := l_x \cap \partial P \qquad \pi_r(x) := r_x \cap \partial P. \tag{15.1}$$

Let P be the space of traceless matrix with determinant 1, which is be the orthogonal complement of $\mathrm{Id} \in \mathbb{PSL}(2,\mathbb{R})$, hence a totally geodesic plane. By Cayley-Hamilton theorem, the boundary of P is the projectification of the space of $1-$rank matrix X such that $X^2 = 0$. Equivalently, the image of X coincide with its kernel: hence, we can identify ∂P with $\mathbb{RP}^1$ by sending $[X]$ to $\ker X = \mathrm{Im} X$. It follows that the map described in Equation (15.1) is the same as the map defined in Lemma 15.1.1.

In particular, the graph of a quasi-symmetric homemorphism is an admissible boundary, via this identification. It turns out that this notion can be described purely in terms of Anti-de Sitter geometry, as proved in [BS10, Theorem 1.12]:

Proposition 15.1.3. *A subset Λ in $\mathbb{PSL}(2,\mathbb{R})$ is an acausal boundary if and only if the map*

$$\phi := \pi_r \circ \pi_l^{-1} \colon \pi_l(\Lambda) \subseteq \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$$

is well defined and an orientation preserving homeomorphism.

Moreover, ϕ is quasi-symmetric if and only if $\omega_0(\Lambda) < \pi/2$.

In [Sep19, Proposition 3.A], this result has been improved, from a quantitative point of view:

Lemma 15.1.4. *Let $\phi \colon \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ be a homeomorphism. Then,*

$$\tan(\omega_0(\Lambda)) \leq \sinh\left(\frac{\|\phi\|_{cr}}{2}\right).$$

15.2. The Gauss map

Since the Anti-de Sitter metric on $\mathbb{PSL}(2,\mathbb{R})$ is a multiple of the Killing form, geodesics are $1-$parameter subgroups of $\mathbb{PSL}(2,\mathbb{R})$: one can easily check that $\mathbb{PO}(2)$ is a timelike geodesic. Since $\mathrm{Isom}_0(\mathbb{H}^{2,1})$ acts transitively on the space of (unparameterized) timelike geodesics, such space can be parameterized as follows:

Lemma 15.2.1 (Proposition 3.5.2 in [BS20])**.** *The space of timelike geodesics of the Anti-de Sitter space $\mathbb{PSL}(2,\mathbb{R})$ identifies with $\mathbb{H}^2 \times \mathbb{H}^2$ via the map that associates to a pair $(p,q) \in \mathbb{H}^2 \times \mathbb{H}^2$ the timelike geodesic*

$$L_{p,q} := \{X \in \mathbb{PSL}(2,\mathbb{R}),\ X(q) = p\}.$$

Moreover, this map is $\mathbb{PSL}(2,\mathbb{R}) \times \mathbb{PSL}(2,\mathbb{R})-$equivariant, namely

$$(A,B) \cdot L_{p,q} = L_{A(p),B(q)},$$

for any $A, B \in \mathbb{PSL}(2,\mathbb{R})$.

This fact leads to an intimate connection between embedded spacelike surfaces in $\mathbb{H}^{2,1}$ and immersed surfaces in $\mathbb{H}^2 \times \mathbb{H}^2$, via the so called *Gauss map*. Indeed, let Σ be an embedded spacelike surface in the Anti-de Sitter space $\mathbb{P}\mathrm{SL}(2,\mathbb{R})$: for any point $x \in \Sigma$ passes exactly one timelike geodesic, namely $\exp_x(\mathbb{R}N_x\Sigma)$, which corresponds to a point $\mathbb{H}^2 \times \mathbb{H}^2$ through the map introduced in Lemma 15.2.1.

Let $p_l, p_r \colon \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}^2$ the projection respectively on the first and second coordinate. Denote by $\mathcal{G} = (\mathcal{G}_l, \mathcal{G}_r)$, for $\mathcal{G}_i = \mathcal{G} \circ p_i$.

Lemma 15.2.2 (Lemma 4.2 in[BS18])**.** *Let Σ be convex spacelike surface, then $\mathcal{G}_l, \mathcal{G}_r$ are injective, hence they induce a homeomorphism open subsets of $\mathbb{H}^2$.*

$$\Phi := \mathcal{G}_r \circ \mathcal{G}_l^{-1} \colon \mathcal{G}_l(\Sigma) \to \mathcal{G}_r(\Sigma)$$

This fact has been exploited in order to study homeomorphisms of the hyperbolic space: the pioneer of this tradition is [Mes07], where convex pleated spacelike surfaces are proved to correspond to earthquakes. In [BS10], Gauss map links maximal surfaces in $\mathbb{H}^{2,1}$ and minimal Lagrangian diffeomorphisms of $\mathbb{H}^2$. The latter results is particular case of a bigger picture, studied in [BS18]: surfaces with constant sectional curvature (which are convex by Gauss equation) correspond to the so called $\theta-$landslides (see Definition 16.2.1). Indeed, $(\pi/2)-$landslide are minimal Lagrangian diffeomorphisms and maximal surfaces are equidistant to surfaces with constant sectional curvature $K = -2$ (Lemma 10.4.2), hence they induce the same Gauss map.

Chapter 16.

Application of Anti-de Sitter geometry

16.1. Extension to the boundary

Definition 16.1.1. Let S be a properly embedded achronal convex surface. Let

$$\mathrm{Gr}^0_{(2,0)}(S) := \{(p,P), p \in S, P \text{ spacelike plane of support for } S \text{ at } p\}$$

be the grassmanian of pointed support spacelike plane of S.

We can define the Gauss map to $\mathrm{Gr}^0_{(2,0)}(S)$ by associating (p,P) to the timelike geodesic normal to P at p, and we denote it $\mathcal{G}_S$.

Remark 16.1.2. The image of the projection of $\mathrm{Gr}^0_{(2,0)}(S)$ over S defined by $(p,P) \to p$ is the acausal part of S.

Lemma 16.1.3. *Let S be a properly embedded achronal convex surface. For any leaf S_t of the cosmological time of $\mathcal{C}(S)$, the image of $\mathcal{G}_{S_t}$ coincides with the image of $\mathcal{G}_S$.*

Proof. Assume S to be future convex.

Let $x \in S_t$, which is a $C^{1,1}$ properly embedded spacelike surface, hence it admits a unique tangent plane $T_x S_t$.

The point $(\rho_-(x), \mathcal{P}(\rho_+(x)))$ belongs to $\mathrm{Gr}^0_{(2,0)}(S)$ by Proposition 10.2.1. Moreover, by construction, the timelike geodesic normal to S_t at x is $[\rho_-(x), \rho_+(x)]$, which is normal to $\mathcal{P}(\rho_+(x))$ at $\rho_-(x)$: hence

$$\mathcal{G}_{S_t}(x) = \mathcal{G}_S\big(\rho_-(x), \mathcal{P}(\rho_+(x))\big).$$

Conversely, let $(p,P) \in \mathrm{Gr}^0_{(2,0)}(S)$, then, p belongs to the acausal part of S, which coincides the image of ρ_- by Corollary 10.2.3.

Let q be the dual point to P, which belongs to $\partial_+\mathcal{C}(S)$: the timelike geodesic $[p,q]$ intersects S_t at a point x and has length $\pi/2$, hence $p = \rho_-(x)$, $\rho_+(x)$ and then

$$\mathcal{G}_S(p,P) = \mathcal{G}_S\big(\rho_-(x), \mathcal{P}(\rho_+(x))\big) = \mathcal{G}_{S_t}(x).$$

□

Corollary 16.1.4. *Let S be a properly embedded achronal convex surface. Let $\hat{S}$ be $S_{\pi/2}$, which is a properly embedded achronal convex surface. Then, $\mathcal{G}_S = \mathcal{G}_{\hat{S}}$.*

Proof. It follows trivially by the statement above, since the leaves coincide. □

Definition 16.1.5. The *extremal part* $\mathrm{ext}(\Lambda)$ of an admissible boundary Λ is the complement of the interior of lightlike segments of Λ.

Lemma 16.1.6. *Let Λ be an admissible boundary in $\partial\widetilde{\mathbb{H}}^{2,1}$. Then $p \in \mathrm{ext}(\Lambda)$ if and only if $\overline{I^+(p)} \cap \Lambda = \{p\}$ or $\overline{I^-(p)} \cap \Lambda = \{p\}$.*

Proof. Fix $p \in \Lambda$, and consider $q_\pm \in \overline{I^\pm(p)} \cap \Lambda$.

Since Λ is achronal, $q_\pm$ must lie on the same lightlike geodesic. If both are different from p, then p is in the interior of the lightlike geodesic connecting $q_\pm$, concluding the proof. □

Motivated by this lemma, we give the following definition

Definition 16.1.7. A point p in an admissible boundary Λ in $\widetilde{\mathbb{H}}^{2,1}$ is *future (resp. past) extremal* if $\overline{I^+(p)} \cap \Lambda = \{p\}$ (resp. $\overline{I^-(p)} \cap \Lambda = \{p\}$). We denote the *future (resp. past) extremal part* of Λ by $\mathrm{ext}^+(\Lambda)$ (resp. $\mathrm{ext}^-(\Lambda)$).

This definition can be extended to admissible boundaries in $\partial\mathbb{H}^{2,1}$ by asking that the preimage $\psi^{-1}(p)$ is future (resp. past) extremal in the connected component of $\psi^{-1}(\Lambda)$ p belongs to.

We recall [BS10, Lemma 3.18]. The original statement is a little bit different, but the proof can be extended to the following result.

Lemma 16.1.8. *Let S be a properly embedded convex achronal surface in $\mathbb{H}^{2,1}$. Let p_k be a sequence in S converging to $p_\infty \in \partial S$.*

Let P_k be of support spacelike plane for S at p_k converging to P_∞. If P_∞ is spacelike or the dual plane to p_∞, then $\mathcal{G}_S(p_k)$ converges to $(\pi_l(p_\infty), \pi_r(p_\infty))$.

We recall that the *acausal part* $\mathrm{Acau}(S)$ of a properly embedded achronal hypersurface S is the set of points $x \in S$ admitting a spacelike support hyperplane (see Definition 10.2.2).

Lemma 16.1.9. *Let S be a properly embedded future convex achronal surface, and let $p_\infty \in \overline{\mathrm{Acau}(S)} \cap \mathrm{ext}^+(\partial S)$. Then $(\pi_l(p_\infty), \pi_r(p_\infty))$ is in the closure of $\mathcal{G}_S(S)$.*

Proof. Since $p_\infty \in \overline{\mathrm{Acau}(S)}$, there exists a sequence p_k in $\mathrm{Acau}(S)$ converging to p_∞. By definition of acausal part, each p_k is contained in a spacelike support plane P_k, that is the sequence (p_k, P_k) is contained in $\mathrm{Gr}^0_{(2,0)}(S)$. Up to extracting a subsequence, we can assume that the sequence P_k converges to an acausal plane P_∞, namely that (p_k, P_k) converges to (p_∞, P_∞). If P_∞ is spacelike, we conclude by Lemma 16.1.8.

We then consider the case P_∞ is degenerate. Let q_k be the sequence of future dual points of P_k, namely $P_-(q_k) = P_k$: it follows that $P_\infty = P_-(q_\infty)$, for q_∞ the limit of the q_k's. Remark that $q_\infty \in \partial\hat{S} = \partial S$, and since $q_k \in I^+(p_k)$, we use the future extremality of p_∞ to get

$$q_\infty \in \partial S \cap \overline{I^+(p_\infty)} = \{p_\infty\},$$

which allows us to conclude by applying Lemma 16.1.8. □

Proposition 16.1.10. *Let Λ be an admissible boundary. Then, the frontier of the Gauss map of $S^\pm_K(\Lambda)$ in $\overline{\mathbb{H}^2 \times \mathbb{H}^2}$ contains $\mathrm{ext}(\Lambda)$, for any $K \in (-\infty, -1)$.*

Proof. Let us denote $S := S^-_K$. We will prove that $\mathrm{ext}^+(\Lambda)$ is in the closure of $\mathcal{G}_S(S)$. The same argument proves that $\mathrm{ext}^-(\Lambda)$ is in the closure of $\mathcal{G}_{\hat{S}}(\hat{S})$, which coincide with $\mathcal{G}_S(S)$ by Corollary 16.1.4.

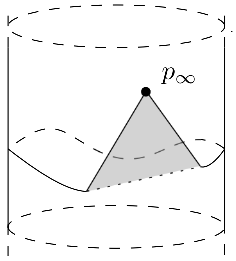

Figure 16.1.: A future-convex sawtooth and its vertex.

Let us fix $p_\infty \in \mathrm{ext}^+(\Lambda)$. We distinguish two cases: if p_∞ is not a vertex of a sawtooth, then it is contained in $\overline{\mathrm{Acau}(S_K^-)}$. Indeed, by Theorem E, the surface S_K^- is real-analytic and spacelike outside the sawteeth. Hence, by Lemma 16.1.9, we have that

$$(\pi_l(p_\infty), \pi_r(p_\infty)) \in \overline{\mathcal{G}_S(S)}.$$

Let p_∞ the vertex of a future-convex sawtooth (Figure 16.1). Using again that S is spacelike outside sawteeth, we can build a sequence (p_k, P_k) in $\mathrm{Gr}^0_{(2,0)}(S)$ such that p_k converges to a point p contained in the geodesic γ defining the sawtooth. We claim that we can build P_k so that it converges to $P_-(p_\infty)$, that is the degenerate past dual plane to the point $p_\infty \in \partial\mathbb{H}^{2,1}$. It follows that the dual sequence (q_k, Q_k) in $\mathrm{Gr}^0_{(2,0)}(\hat{S})$ converges to $(p_\infty, P_+(p))$: since $P_+(p)$ is spacelike, by Lemma 16.1.8, p_∞ belongs to the closure of the Gauss map of the dual surface $\hat{S}$, which coincides $\overline{\mathcal{G}_S(S)}$ by Corollary 16.1.4, concluding the proof.

To prove the claim, let P_∞ be the limit of the sequence P_k. Since P_∞ is a past support plane for Λ, it contains the whole geodesic γ. A spacelike geodesic γ is contained in only two degenerate planes, that is $P_-(x)$ and $P_-(y)$, for x, y the endpoints of the spacelike geodesic dual in the future to γ. By construction, $x = p_\infty$: if Λ is not a Barbot crown, the only degenerate support plane for Λ containing γ is $P_-(p_\infty)$.

Otherwise, P_∞ is spacelike, and it contains the point $p \in \gamma$, that is $(p, P_\infty) \in \mathrm{Gr}_{(2,0)}(S)$. Since $P_-(p_\infty)$ is a support plane for Λ, and the space of acausal planes to Λ is a convex set, we can build a new sequence (p, P_k) in $\mathrm{Gr}_{(2,0)}(S)$ converging to $(p, P_-(p_\infty))$, by tilting the normal vector of P_∞ towards p_∞: for example, denote q_∞ the future dual point to P_∞ and take

$$P_k := P_-\left(\frac{q_\infty + k p_\infty}{|q_\infty + k p_\infty|}\right).$$

This proves the claim and concludes the proof. □

Corollary P. *Let Λ be an admissible boundary. The diffeomorphism induced by the Gauss map of $S_K^\pm(\Lambda)$ extends to a orientation preserving homeomorphism of $\mathbb{S}^1$ if and only if Λ contains no lightlike segments.*

Proof. Assume Λ contains the lightlike segment connecting (a, b) and (a, c) (resp. (b, a) and (c, a)): by Proposition 16.1.10, we have that π_l (resp. π_r) is not injective: it follows that $\pi_r \circ \pi_l^{-1}$ is not well defined (resp. not injective). In both case, it is not homeomorphism.

Conversely, if Λ contains no lightlike segments, then it is an acausal subset of $\partial\mathbb{P}\mathrm{SL}(2, \mathbb{R})$. In particular, every point is extremal, hence the Gauss map extends to the boundary Λ by Proposition 16.1.10. We claim that, if we parameterize $\partial\mathbb{P}\mathrm{SL}(2, \mathbb{R})$ as in Lemma 15.1.1, then Λ is the graph of a orientation preserving homeomorphism $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$, which concludes the proof.

To prove the claim, we will use the construction described in Equation (15.1): fix a totally geodesic spacelike surface P, for a point $x \in \Lambda$ corresponds to the pair

$$(\pi_l(x), \pi_r(x)) = (l_x \cap \partial P, r_x \cap \partial P)\,.$$

for l_x, r_x the two lightlike geodesic containing x. Hence, by construction

$$l_x = l_{\pi_l(x)} \qquad r_x = l_{\pi_r(x)}. \tag{16.1}$$

It follows that the two projection are injective: otherwise, two points of Λ would belong to the same lightlike geodesic, contradicting the acausality of Λ.

For the surjectivity, let $z \in \partial P$: both l_z and r_z are closed causal curves, and $\partial\mathbb{P}\mathrm{SL}(2,\mathbb{R}) \setminus \Lambda$ contains no closed acausal curve. Hence, they intersect Λ at two (not necessarily distinct) points x, y, respectively. By Equation 16.1, $\pi_l(x) = z$ and $\pi_r(y) = z$, which proves the surjectivity of the two projection since the choice z was arbitrary.

Hence, the map $\phi := \pi_r \circ \pi_l^{-1} \colon \mathbb{S}^1 \to \mathbb{S}^1$ is a bijection. The map π_l, π_r are continuous by construction, hence they are continuous restricted to Λ, which is a Lipschitz curve. One can show that ϕ has to be orientation preserving, otherwise it would be a timelike curve ([BS20, Proposition 3.2.3]), concluding the proof. □

16.2. $\theta-$landslide

An important family of diffeomorphisms of the hyperbolic space are the so called $\theta-$landslide, which have been first introduced in [BMS13] to smoothly conjugate the two earthquake extensions associated to a quasi-symmetric homeomorphism.

Definition 16.2.1. Let $\theta \in (0, \pi)$. An orientation preserving diffeomorphism $f\colon \mathbb{H}^2 \to \mathbb{H}^2$ is a $\theta-$*landslide* if there exists a $(1,1)-$tensor $m \in \Gamma\left(\mathrm{End}(T\mathbb{H}^2)\right)$ such that

$$f^* g_{\mathbb{H}^2} = g_{\mathbb{H}^2}\left(\cos(\theta)\mathrm{Id} + \sin(\theta)Jm, \cos(\theta)\mathrm{Id} + \sin(\theta)Jm\right),$$

for J the complex structure of $\mathbb{H}^2$, and

1. $d^\nabla m = 0$;
2. $\det m = 1$;
3. m is positive and self-adjoint for $g_{\mathbb{H}^2}$.

In [BS18, Lemma 4.12], an equivalent characterization for $\theta-$landslides is given.

Lemma 16.2.2. *An orientation preserving diffeomorphism $f\colon \mathbb{H}^2 \to \mathbb{H}^2$ is a $\theta-$landslide if and only if there exists a $(1,1)-$tensor $b \in \Gamma\left(\mathrm{End}(T\mathbb{H}^2)\right)$ such that*

1. $f^* g_{\mathbb{H}^2} = g_{\mathbb{H}^2}(b\,\cdot, b\,\cdot)$;
2. $d^\nabla b = 0$;
3. $\det b = 1$;
4. $\mathrm{tr}\, b = 2\cos(\theta)$;
5. $\mathrm{tr}\, Jb < 0$.

The case $\theta = \pi/2$ is of particular interest. Indeed, in a $(\pi/2)-$landslide is a minimal Lagrangian map.

In this brief section, we classify $\theta-$landslide, proving Theorem L.

Combining the aforementioned results of [BS18] and [Tam19a], we can conclude that $H-$surfaces correspond to $\theta-$landslides. Indeed, the Gauss map of Σ coincides with the Gauss map of Σ_t, for Σ_t the normal evolution of Σ at time t, as soon as Σ_s is not degenerate for any $s \in [0, t]$, by Lemma 10.4.2, the equidistant surface to an $H-$surface Σ and in a 3$-$dimensional spaceforms CMC surfaces are equidistant to CSC surfaces. In particular, the surface at distance

$$d_\pm = \arctan\left(\frac{H}{2} \pm \sqrt{1 + \frac{H^2}{4}}\right)$$

from an $H-$surface has constant sectional curvature

$$K_\pm = -1 - \frac{4}{(H \pm \sqrt{4 + H^2})^2}.$$

Consider the CSC surface in the future of Σ, which is past-convex: by [BS18, Proposition 4.13], the associated diffeomorphism Φ is a $\theta-$landslide with

$$\theta = \theta(H) = 2\arccos\left(\frac{1}{\sqrt{-K}}\right) = 2\arccos\left(\frac{(H + \sqrt{4 + H^2})}{\sqrt{(H + \sqrt{4 + H^2})^2 + 4}}\right). \tag{16.2}$$

In particular, the map $\theta(H) \colon (-\infty, +\infty) \to (0, \pi)$ is a proper diffeomorphism.

Theorem L. *Let $\phi \colon \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation preserving homeomorphism. For any $\theta \in (0, \pi)$, there exists a unique $\theta-$landslide $\Phi_\theta \colon \mathbb{H}^2 \to \mathbb{H}^2$ extending ϕ.*

Proof. Let us fix $\theta \in (0, \pi/2)$ and let $K := -1/\cos^2(\theta/2)$.

By Proposition 15.1.3, the boundary curve $\Lambda := \operatorname{graph}\phi$ is an acausal admissible boundary. By Theorem E, the past-convex $K-$surface $S_K^+(\Lambda)$ bounding Λ is properly embedded, spacelike and convex. The Gauss map of $S_K^+(\Lambda)$ induces a diffeomorphism Φ of $\mathbb{H}^2$ such that $\Phi|_{\partial\mathbb{H}^2} = \phi$ (Lemma 16.1.8). By [BS18, Proposition 4.13], Φ is a $\theta-$landslide. This concludes the existence part of the proof.

The uniqueness part is more delicate. Let Ψ be a $\theta-$landslide extension of ϕ, and let b be the $(1,1)-$tensor as in Definition 16.2.1. By [BS18, Corollary 5.11], the pair (Ψ, b) induces an embedding

$$\sigma_{\Psi,b} \colon \mathbb{H}^2 \to \mathbb{P}\mathrm{SL}(2, \mathbb{R}),$$

whose image is a past-convex $K-$surface, which we denote by S.

To conclude, we need to prove that $S = S_K^+(\Lambda)$. We claim that $S = S_K^+(\Lambda')$, for some admissible boundary Λ'. By Corollary P, then Λ' contains no lightlike segments. Hence, by Proposition 15.1.3, the boundary curve Λ' is the graph of a homeomorphism ψ of the circle. By Lemma 16.1.8, the diffeomorphism induced by the Gauss map of S extends to ψ. By construction, the Gauss of S induces Ψ, which extends to ϕ: it follows that $\psi = \phi$, that is $\Lambda = \Lambda'$. By uniqueness (Theorem E), then $S = S_K^+(\Lambda)$, and $\Psi = \Phi$, concluding the proof.

To prove the claim, let S_- be the past-extremal extension of S (compare with Lemma 3.1.2), which is a properly embedded past-convex acausal surface. The acausal part of S_- coincides with S: by contradiction, let $p \in \operatorname{Acau}(S_-) \setminus S$ and let P the support spacelike plane for S_- at p. Then,

$$(\mathcal{G}_{S_-})_l(p, P) \in \mathbb{H}^2 = \operatorname{Im}(\mathcal{G}_S)_l,$$

namely the Gauss map of S_- is not injective, which is absurd since S_- is convex.

It follows that the leaves of the cosmological time on $\mathcal{C}(S_-)$ are equidistant surfaces to a spacelike $K-$surface. Combining Proposition 10.2.4 Lemma 10.4.2, the leaf at time $t = d(K)$ is a properly embedded $H-$surface Σ. It follows that $S = S_K^+(\Lambda')$, for $\Lambda' = \partial\Sigma$, proving the claim and concluding the proof. $\square$

Chapter 17.

Teichmüller theory

In this chapter, we focus on *quasiconformal* (Definition 17.1.3) $\theta-$landslides. They have been studied in [BS18] from a qualitative point of view. We are interested in a quantitative investigation: the *quasiconformal dilatation* $K(\Phi)$ (Equation (17.1)) of a $\theta-$landslide Φ is bounded by the principal curvatures of the CMC surface whose asymptotic boundary is the graph of $\Phi|_{\partial\mathbb{H}^2}$, as already remarked in [Tam19a]. As a consequence of consequence of Theorem F, we bound $K(\Phi)$ with a multiple of the $H-$width, which is related to the *cross-ratio norm* of $\Phi|_{\partial\mathbb{H}^2}$ thanks to the work [Sep19].

17.1. Universal Teichmüller space

The *cross-ratio* of a quadruple (z_1, z_2, z_3, z_4) of points in $\mathbb{RP}^1$ is

$$cr(z_1, z_2, z_3, z_4) := \frac{z_4 - z_1}{z_2 - z_1}\frac{z_3 - z_2}{z_3 - z_4}.$$

Definition 17.1.1. Let $\phi\colon \mathbb{RP}^1 \to \mathbb{RP}^1$ be an orientation preserving homeomorphism. The *cross-ratio norm* of ϕ is

$$\|\phi\|_{cr} := \sup_{cr(z_1,z_2,z_3,z_4)=-1} \ln|cr\,(\phi(z_1), \phi(z_2), \phi(z_3), \phi(z_4))| \in [0, +\infty].$$

We call ϕ is *quasi-symmetric* $\|\phi\|_{cr} < +\infty$.

Remark 17.1.2. By associating a geodesic to its end points, the space of oriented geodesics of $\mathbb{H}^2$ identifies with $\partial\mathbb{H}^2 \times \mathbb{H}^2 \setminus \Delta$. One can check that $cr(z_1, z_2, z_3, z_4) = -1$ if and only if the geodesics (z_1, z_3) and (z_2, z_4) are orthogonal. The geometric meaning of quasi-symmetric homeomorphism is then that the angle between the geodesics $(\phi(z_1), \phi(z_3))$ and $(\phi(z_2), \phi(z_4))$ is uniformly bounded, for any couple of orthogonal geodesics $(z_1, z_3), (z_2, z_4)$.

The *universal Teichmüller space* is the space of quasi-symmetric homeomorphisms of the circle, *modulo* the action of $\mathbb{PSL}(2, \mathbb{R})$ by postcomposition.

Definition 17.1.3. Let $U \subseteq \mathbb{C}$ be a domain and $f\colon U \to \mathbb{C}$ an orientation preserving homeomorphism onto its image. We say that f is *quasiconformal* if f is absolutely continuous over lines and there exists a constant $k < 1$ such that

$$|\partial_{\bar{z}} f| \leq k|\partial_z f|$$

almost everywhere.

The function $\mu_f := \partial_{\bar{z}} f / \partial_z f$, which is defined almost-everywhere, is called *complex dilatation* of f. From a geometric point of view, f is quasi conformal if and only the distortion of

$$\mathbb{S}^1 = T_z^1 \mathbb{H}^2 \subseteq T_z \mathbb{H}^2$$

through $d_z f$ is uniformely bounded as z varies over $\mathbb{H}^2$. Indeed, one can prove (see for example [Ahl87]) that the ratio between the major and the minor axis of the ellipse $d_z f(\mathbb{S}^1)$ is precisely

$$\frac{|d_z f(\partial_z)| + |d_z f(\partial_{\bar{z}})|}{|d_z f(\partial_z)| + |d_z f(\partial_{\bar{z}})|} = \frac{1 + |\mu_f|}{1 - |\mu_f|} \leq \frac{1 + \|\mu_f\|_{L^\infty(\mathbb{H}^2)}}{1 - \|\mu_f\|_{L^\infty(\mathbb{H}^2)}}$$

The number

$$K(f) := \frac{1 + \|\mu_f\|_{L^\infty(\mathbb{H}^2)}}{1 - \|\mu_f\|_{L^\infty(\mathbb{H}^2)}} \tag{17.1}$$

is the *maximal dilatation* of f and f is said to be $K-$quasiconformal if $K(f) \leq K$.

The relation between quasiconformal maps and quasi-symmetric homeomorphism is classical, thanks to [BA56, Theorem 1], which states:

Theorem 17.1.4 (Ahlfors-Beuring)**.** *Let $f \colon \mathbb{H}^2 \to \mathbb{H}^2$ be a quasiconformal map, then f extends to a unique quasi-symmetric homeomorphism of $\partial \mathbb{H}^2 = \mathbb{RP}^1$.*

Conversely, any quasi-symmetric homeomorphism $\phi \colon \mathbb{RP}^1 \to \mathbb{RP}^1$ admits an extension to the disc which is quasiconformal in the interior.

The quasiconformal extension is far from being unique: it can be useful to choose a specific class of quasiconformal maps to build a bijective correspondence with the universal Teichmüller space.

17.2. Quasiconformal dilatation

In the same spirit of [Sep19], we compare the quasi-conformal dilatation a $\theta-$landslide with the principal curvature of the corresponding $H-$surface. In fact, let Σ be a properly embedded $H-$surface in $\mathbb{H}^{2,1}$, and let $a \in C^0(\Sigma)$ be the non-negative eigenvalue of the traceless shape operator of Σ, that is

$$B = \begin{pmatrix} \frac{H}{2} + a & \\ & \frac{H}{2} - a \end{pmatrix}$$

in a suitable orthonormal frame. Let $x \in \Sigma$, we can apply [Tam19a, Proposition 6.2] to get

$$\mu_{\Phi_\Sigma}(\mathcal{G}_\Sigma(x)) = -a(x) \frac{(H/2) + i}{1 + (H/2)^2}, \tag{17.2}$$

for μ_{Φ_Σ} the complex dilatation of the $\theta-$landslide Φ_Σ associated to Σ.

In light of Theorem F, this allows to estimate the quasiconformal dilatation with the cross-ratio norm of its extension to the boundary.

Theorem K. *For any $\alpha \in (0, \pi/2)$, there exist universal constants $Q_\alpha, \eta_\alpha > 0$ such that*

$$\ln(K(\Phi_\theta)) \leq Q_\alpha \|\phi\|_{cr},$$

for Φ_θ the only $\theta-$landslide extending ϕ, with $\theta \in [\alpha, \pi - \alpha]$ and $\|\phi\|_{cr} \leq \eta_\alpha$.

Proof. Let us fix $\theta \in [\alpha, \pi - \alpha]$, and a quasi-symmetric homeomorphism ϕ. Let $\Phi_\theta \colon \mathbb{H}^2 \to \mathbb{H}^2$ be the unique $\theta-$landslide extending ϕ and denote by Σ the corresponding $H-$surface, *i.e.* Σ is the unique $H-$surface bounded by $\Lambda = \operatorname{graph} \phi$. By Equation (17.2), the maximal dilatation of Φ_θ at $z = \mathcal{G}_\Sigma(x)$ is

$$|\mu_{\Phi_\theta}(\mathcal{G}_\Sigma(x))|^2 = \frac{a(x)^2}{1 + (H/2)^2}.$$

By substituting in Equation (17.1), the maximal dilatation of Φ_θ is

$$K(\Phi_\theta) = \frac{1 + |\mu_{\Phi_\theta}|^2_{L^\infty(\mathbb{H}^2)}}{1 - |\mu_{\Phi_\theta}|^2_{L^\infty(\mathbb{H}^2)}} = \frac{1 + (H/2)^2 + \|B_0\|^2_{C^0(\Sigma)}}{1 + (H/2)^2 - \|B_0\|^2_{C^0(\Sigma)}}. \tag{17.3}$$

The function $\theta = \theta(H)$ introduced in Equation (16.2) is proper, hence there exists $L \geq 0$ such that

$$\theta^{-1}([\alpha, \pi - \alpha]) \subseteq [-L, L].$$

In particular, combining Theorem F, Lemma 11.3.3 and Lemma 15.1.4 we obtain

$$\begin{aligned} \|B_0\|^2_{C^0(\Sigma)} &\leq C_L \sin(\omega_H(\partial\Sigma)) \leq C_L \sin(\omega_0(\partial\Sigma)) \\ &\leq C_L \tan(\omega_0(\partial\Sigma)) \leq C_L \sinh\left(\frac{\|\phi\|_{cr}}{2}\right). \end{aligned}$$

The last formula in Equation (17.3) is increasing in $\|B_0\|^2_{C^0(\Sigma)}$, hence

$$\begin{aligned} K(\Phi_\theta) &\leq \frac{1 + (H/2)^2 + C_L \sinh\left(\frac{\|\phi\|_{cr}}{2}\right)}{1 + (H/2)^2 - C_L \sinh\left(\frac{\|\phi\|_{cr}}{2}\right)} \\ &\leq \frac{1 + C_L \sinh\left(\frac{\|\phi\|_{cr}}{2}\right)}{1 - C_L \sinh\left(\frac{\|\phi\|_{cr}}{2}\right)}, \end{aligned}$$

since the function is decreasing in H^2. The inequality is valid by setting $\eta_\alpha < 2\operatorname{arcsinh}(C_L^{-1})$, which ultimately depends on α since L depends on it.

Finally, we remark that

$$\frac{d}{dx} \ln\left(\frac{1 + C_L \sinh(x)}{1 - C_L \sinh(x)}\right) = \frac{2C_L \cosh(x)}{1 - C_L^2 \sinh^2(x)},$$

hence there is a constant $Q_\alpha > 0$ depending on C_L, η_α, hence ultimately only on α, such that

$$\ln\left(\frac{1 + C_L \sinh\left(\frac{\|\phi\|_{cr}}{2}\right)}{1 - C_L \sinh\left(\frac{\|\phi\|_{cr}}{2}\right)}\right) \leq Q_\alpha \|\phi\|_{cr},$$

which concludes the proof. □

Part VI.

Quasi-spheres

Chapter 18.

Quasi-spheres

The notion of quasi-simmetric boundary comes from hyperbolic geometry. However, it can be characterized in terms of $\mathbb{A}\mathrm{d}\mathbb{S}-$geometry, as remarked in [BS10]. We introduce *quasi-spheres*, generalizing this notion in higher dimension. Then, we give an equivalent dynamical definition, and apply it in the context of *higher higher Teichmüller* theory.

18.1. Rigidity

Denote by $\mathcal{B}$ the space of admissible boundaries, endowed with the Hausdorff topology.

Proposition 18.1.1. *Let $H \in \mathbb{R}$, then $\omega_H(\cdot)\colon \mathcal{B} \to \mathbb{R}$ is lower semicontinuous.*

Proof. Consider a converging sequence $\Lambda_k \to \Lambda$ in $\partial\widetilde{\mathbb{H}}^{n,1}$. We need to prove that

$$\omega_H(\Lambda) \leq \liminf_{k\to+\infty} \omega_H(\Lambda_k).$$

Extract a subsequence $(k_m)_{m\in\mathbb{N}}$ such that $\omega_H(\Lambda_{k_m})$ converges to

$$W := \liminf_{m\to+\infty} \omega_H(\Lambda_{k_m}).$$

Observe that $X_{k_m} := \mathcal{CH}_H(\Lambda_{k_m})$ is a sequence of $H-$convex sets which is bounded for the Hausdorff topology: up to extracting another subsequence, we can assume X_{k_m} converges in the Hausdorff topology to a subset X, whose width is exactly W. Indeed, the width is continuous with respect to the Hausdorff topology on closed subsets of $\mathbb{H}^{n,1}$.

Since the sequence $X_{k_m} \cap \partial\widetilde{\mathbb{H}}^{n,1} = \Lambda_{k_m}$ converges to Λ, the asymptotic boundary of X coincides with Λ. Morevover, X is $H-$convex by Corollary 11.1.9, hence it contains $\mathcal{CH}_H(\Lambda)$ by minimality. It follows that $\omega_H(\Lambda) \leq W$, which concludes the proof. □

Remark 18.1.2. Although the width is continuous with respect to the Hausdorff topology on subsets of $\widetilde{\mathbb{H}}^{n,1}$, the function ω_H is not continuous with respect to the Hausdorff topology on admissible boundaries. The reason is due to the fact that the limit of $H-$shifted convex hulls is not in general a $H-$shifted convex hull.

To produce examples, consider the case $n = 2$. A *lightlike polygon* is an admissible boundary consisting of lightlike segment (these boundaries have been studied for in [Tam19b] and, in a more general context, in [Mor24]). In other words, a lightlike polygon is the graph of a piecewise isometry $\mathbb{S}^1 \to \mathbb{R}$.

It is clear that lightlike polygons form a dense subset of the set of admissible boundaries. On the other hand, combining Corollary 18.1.5 and Corollary 18.3.3, for any $H \in \mathbb{R}$, the function $\omega_H(\cdot)$ is identically $(\pi/2) - |\delta_H|$ over lightlike poligons.

Clearly, $G := \mathrm{Isom}(\widetilde{\mathbb{H}}^{n,1})$ acts on $\mathcal{B}$. As a consequence of Proposition 18.1.1, we can extend the definition of width to $\mathcal{B}/G$. Moreover, for an admissible boundary Λ, we define

$$\omega_H(\overline{G\cdot\Lambda}) := \sup_{\Lambda'\in\overline{G\cdot\Lambda}},$$

that is we extend the definition of width to the closure of the orbit of Λ.

Corollary 18.1.3. *Let Λ be an admissible boundary. For any $H\in\mathbb{R}$,*

$$\omega_H(\overline{G\cdot\Lambda}) = \omega_H(\Lambda).$$

Proof. Let $\Lambda'\in\overline{G\cdot\Lambda}$. Then there exists a sequence $(g_k)_{k\in\mathbb{N}}$ of isometries in G such that

$$\Lambda' = \lim_{k\to+\infty} g_k(\Lambda).$$

By Proposition 18.1.1, we have

$$\omega_H(\Lambda') \le \lim_{k\to+\infty}\omega_H(g_k(\Lambda)) = \lim_{k\to+\infty}\omega_H(\Lambda) = \omega_H(\Lambda),$$

since the width is invariant by isometries. □

Corollary 18.1.4. *Let Λ be an admissible boundary, $H\in\mathbb{R}$. There exists an admissible boundary $\Lambda'\in\overline{G\cdot\Lambda}$ such that*

$$\omega_H(\Lambda') = \omega_H(\Lambda)$$

and the width $\omega_H(\Lambda')$ is realized by a timelike geodesic γ.

Proof. Build a sequence of timelike curves $(\gamma_k)_{k\in\mathbb{N}}$ contained in $\mathcal{CH}_H(\Lambda)$ such that

$$\ell(\gamma_k) > \omega_H(\Lambda) - \frac{1}{k}.$$

Without loss of generality, we can assume γ_k to be a timelike geodesic.

Let g_k be an isometry such that

$$g_k(\gamma_k(0)) = \gamma_1(0), \qquad d_{\gamma_k(0)}g_k\left(\gamma_k'(0)\right) = \gamma_1'(0).$$

Denote by $\Lambda' := \lim_{k\to+\infty} g_k(\Lambda)$. By construction, the limit curve

$$\gamma := \lim_{k\to+\infty} g_k(\gamma_k)$$

is a timelike geodesic of length $\omega_H(\Lambda)$ contained in $\mathcal{CH}_H(\Lambda')$. By Corollary 18.1.3, we have

$$\ell(\gamma) \le \omega_H(\Lambda') \le \omega_H(\Lambda) = \ell(\gamma).$$

It follows that the two $H-$shifted convex hulls have the same width and that γ realizes the width of $\mathcal{CH}_H(\Lambda')$, concluding the proof. □

We now can prove that the estimate for $\omega_H(\Lambda)$ found in Corollary 11.3.2 is rigid.

Corollary 18.1.5. *Let Λ an admissible boundary. If there exists H such that $\omega_H(\Lambda) = (\pi/2) - |\delta_H|$, then*

$$\omega_L(\Lambda) = \frac{\pi}{2} - |\delta_L|, \qquad \forall L\in\mathbb{R}.$$

Proof. Without loss of generality, assume $H \geq 0$. First, take $K \in [0, H]$: comparing Corollary 11.3.2 and Lemma 11.3.3, we have

$$\frac{\pi}{2} - \delta_K \geq \omega_K(\Lambda) \geq \omega_H(\Lambda) + \delta_H - \delta_K = \left(\frac{\pi}{2} - \delta_H\right) + \delta_H - \delta_K = \frac{\pi}{2} - \delta_K.$$

In particular, we proved that $\omega_0(\Lambda) = \pi/2$.

Consider $\Lambda' \in \overline{G \cdot \Lambda}$ as in Corollary 18.1.4, and let γ a timelike geodesic realizing the width of $\mathcal{CH}_0(\Lambda')$, *i.e.*

$$\ell(\gamma) = \omega_0(\Lambda') = \omega_0(\Lambda) = \frac{\pi}{2}.$$

The curve γ is contained in the convex hull of Λ, hence in both $\mathbf{P}(\Lambda)$ and $\mathbf{F}(\Lambda)$, and it has length $\pi/2$. By Proposition 3.4.5, its endopoints lies in the intersection of the boundary of the convex hull and the invisible domain of Λ More presciesly, we can parameterize γ by arclength so that

$$\gamma(0) \in \partial_- \mathcal{CH}(\Lambda') \cap \partial_- \Omega(\Lambda'), \qquad \gamma\left(\frac{\pi}{2}\right) \in \partial_+ \mathcal{CH}(\Lambda') \cap \partial_+ \Omega(\Lambda').$$

Moreover, again by Proposition 3.4.5, γ is an integral line for both $\nabla \tau_{\mathbf{P}}$ and $\nabla \tau_{\mathbf{F}}$: it follows that

$$\begin{cases} \tau_{\mathbf{P}}(\gamma(t)) = t, \\ \tau_{\mathbf{F}}(\gamma(t)) = \frac{\pi}{2} - t. \end{cases}$$

For $K > 0$, we have $\partial_- \mathcal{CH}_K(\Lambda') \subseteq \overline{I^- \partial_- \mathcal{CH}(\Lambda')}$, namely $\gamma(0) \in \partial_- \mathcal{CH}_K(\Lambda')$. Since

$$\partial_+ \mathcal{CH}_K(\Lambda') = \tau_{\mathbf{P}}^{-1}\left(\frac{\pi}{2} - \delta_K\right),$$

the geodesic segment $\gamma([0, (\pi/2) - \delta_K])$ is a timelike curve of length $(\pi/2) - \delta_K$ contained in $\mathcal{CH}_K(\Lambda')$: it follows that $\omega_K(\Lambda') = (\pi/2) - \delta_K$. Since $\Lambda' \in \overline{G \cdot \Lambda}$, we conclude by Corollary 18.1.3 that

$$\omega_K(\Lambda) \geq \omega_K(\Lambda') = \frac{\pi}{2} - \delta_K.$$

For $K < 0$, the same argument applies, by replacing the past boundaries with the future boundaries and viceversa. ☐

Motivated by this result, we give the following definition.

Definition 18.1.6. An admissible boundary $\Lambda \subseteq \partial \mathbb{H}^{n,1}$ is a *quasi-sphere* if $\omega_0(\Lambda) < \pi/2$.

18.2. Dynamical characterization

A *Barbot crown* is the boundary of the unique (up to isometry) cylindrical surface in $\mathbb{H}^{2,1}$.

Definition 18.2.1. A *Barbot crown* is the (only) admissible boundary in the 3−dimensional Anti-de Sitter space $\mathbb{H}^{2,1}$ containing four points a_i, b_i, $i = 1, 2$, such that

$$\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle < 0 \qquad \langle a_i, b_j \rangle = 0, \qquad \forall i, j = 1, 2.$$

From a Lie-group perspective, Barbot crowns are characterized by being the limit sets of the orbit of Cartan subgroups of $\mathbb{P}0(2, 2)$.

The proof of Theorem M follows almost directly from the following lemma:

Lemma 18.2.2. *Let Λ be an admissible boundary such that $\omega_0(\Lambda)$ is realized. Then Λ is not a quasi-sphere if and only if there exists a $(2, 2)-$space $W \subseteq \mathbb{R}^{n,2}$ such that $\Lambda \cap W$ is a Barbot crown.*

Proof. If Λ contains a Barbot crown, by convexity $\mathcal{CH}_0(\Lambda)$ contains the complete spacelike geodesic γ_a (resp. γ_b) joining a_1 to a_2 (resp. b_1 to b_2). The two geodesics are dual, hence each pair of points $(p_a, p_b) \in \gamma_a \times \gamma_b$ is connected by a timelike geodesic γ of length $\pi/2$. By convexity, γ is contained in $\mathcal{CH}_0(\Lambda)$.

Conversely, assume that the width $\omega_0(\Lambda)$ is realized by a timelike geodesic γ of length $\pi/2$. Parameterize γ by arclength and so that it is future-directed: then, γ meets the spacelike totally geodesic hypersurface $\mathcal{P}_0 := \mathcal{P}_-(\gamma(\pi/2))$ orthogonally at $\gamma(0)$. In particular, $\mathcal{P}_0$ is a past support hyperplane for $\mathcal{CH}_0(\Lambda)$: indeed, $\gamma(0) \in \mathcal{P}_0 \cap \mathcal{CH}_0(\Lambda)$, and by Proposition 3.4.5, $\mathcal{P}_0$ is a past support hyperplane for $\Omega(\Lambda)$ at $\gamma(0)$, hence

$$\partial_-\mathcal{CH}_0(\Lambda) \subseteq \overline{I^+(\partial_-\Omega(\Lambda))} \subseteq \overline{I^+(\mathcal{P}_0)}.$$

By Lemma 3.3.5, for a support hyperplane $\mathcal{P}_0$ intersecting the convex hull, it holds

$$\mathcal{CH}_0(\Lambda) \cap \mathcal{P}_0 = \mathcal{CH}_0(\Lambda \cap \mathcal{P}_0).$$

It follows that $\Lambda \cap \partial P_0$ contains at least two points a_1, a_2, since $\gamma(0)$ is contained in $\mathcal{CH}_0(\Lambda) \cap P_0$ by construction.

By convexity, the whole spacelike geodesic γ_a joining a_1 and a_2 is contained in the $\mathcal{CH}_0(\Lambda)$, and in particular in its past boundary. Take a point $p \in \gamma_a$: the hyperplane $\mathcal{P}_1 := \mathcal{P}_+(p)$ is a future support hyperplane for $\mathcal{CH}_0(\Lambda)$: indeed, by Proposition 3.4.5, $\mathcal{P}_1$ is a support future hyperplane for $\Omega(\Lambda)$, and it meets $\partial\mathcal{CH}_0(\Lambda)$ at $\gamma(\pi/2)$, since $p \in \mathcal{P}_0 = \mathcal{P}_-(\gamma(\pi/2))$ by construction.

Repeating the argument above, also $\Lambda \cap \partial\mathcal{P}_1$ contains at least two points b_1, b_2 and the geodesic γ_b joining them lies on the future boundary of $\mathcal{CH}_0(\Lambda)$. Let $q \in \gamma_b \subseteq \mathcal{P}_1$, then p, q are connected by a timelike geodesic of length $\pi/2$ contained in $\mathcal{CH}_0(\Lambda)$, which is orthogonal to both $\mathcal{P}_0$ and $\mathcal{P}_1$.

To conclude the proof, we need to prove that $\langle \psi(a_i), \psi(b_j)\rangle = 0$. In the splitting $(p, \mathcal{P}_0)$, we have that

$$a_i = \left((-1)^i x, 0\right), \qquad (p, \mathcal{P}_0) = \psi\left((x_0, 0), \mathbb{H}^n \times \{0\}\right),$$
$$b_i = \left((-1)^i y, \pi/2\right), \qquad (q, \mathcal{P}_1) = \left((x_0, \pi/2), \mathbb{H}^n \times \{\pi/2\}\right).$$

Compare with Equation (1.3) to get

$$\langle \psi(x,t), \psi(y,s)\rangle = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}\cos(t-s).$$

In our case, $t = 0$ and $s = \pi/2$, then $\cos(t-s) = 0$. Moreover, Λ is the graph of a 1−Lipschitz map f, hence

$$d_{\mathbb{S}^n}(x,y) \geq |f(x) - f(y)| = \left|\frac{\pi}{2} - 0\right| = \frac{\pi}{2}.$$

Since the cosine is a decreasing function over $[0, \pi/2]$, we get

$$|\langle \psi(a_i), \psi(b_j)\rangle| = |x_1y_1 + \cdots + x_ny_n| = \cos\left(d_{\mathbb{S}^n}(x,y)\right) \leq \cos(\pi/2) = 0,$$

concluding the proof □

We are now ready to prove the main result of this part.

Theorem M. *Let Λ be an admissible boundary. Then Λ is not a quasi-sphere if and only if there exists $\Lambda' \in \overline{G \cdot \Lambda}$ and a totally geodesic copy of $\mathbb{H}^{2,1}$ such that $\Lambda' \cap \partial\mathbb{H}^{2,1}$ is a Barbot crown.*

Proof. By Corollary 18.1.5, there exists $\Lambda' \in \overline{G\cdot\Lambda}$ such that $\omega_0(\Lambda') = \omega_0(\Lambda)$ and the width is achieved. By Lemma 18.2.2, the width is $\pi/2$, *i.e.* Λ is not a quasi-sphere, if and only if Λ' contains a Barbot crown, concluding the proof. $\square$

Remark 18.2.3. Denote by Ξ the Barbot crown. For $n = 2$, Theorem M implies that, we have

$$\bigcap_{\omega_0(\Lambda)=\pi/2} \overline{G\cdot\Lambda} = \overline{G\cdot\Xi}.$$

There is no generalization of this result in higher dimension: on the contrary, for $n > 2$, we claim that

$$\bigcap_{\omega_0(\Lambda)=\pi/2} \overline{G\cdot\Lambda} = \emptyset.$$

Indeed, take $k = 1, \dots, n-1$ and consider a copy of $\mathrm{O}(k,1)\oplus\mathrm{O}(n-k,1)$ inside $\mathrm{Isom}(\mathbb{H}^{n,1})$. Each group identifies, up to isometry, a unique admissible boundary Λ_k such that $\omega_0(\Lambda_k) = \pi/2$.

The CMC hypersurfaces asymptotic to Λ_k are precisely the cylindrical hypersurfaces $\mathbb{H}(k,\theta)$, which have been introduced in Chapter 8. In Proposition 8.2.3, we gave an explicit formula to compute the (convstant) norm of the second fundamental form of maximal hypersurface bounding Λ_k: for $H \in \mathbb{R}$ and $\theta_{k,H}$ the positive root of $kt^2 - Ht - (n-k)$, we have

$$S(k,\theta_{k,H}) = n + \frac{H^2 + |H|(n-2k)\sqrt{H^2+4k(n-k)}}{2k(n-k)}.$$

In particular, if we fix $H \neq 0$, we proved $S(k,\theta_{k,H}) \neq S(j,\theta_{k,H})$ for $j \notin \{k, n-k\}$. Since $S(k,\theta_{k,H})$ is invariant by the action of G, we conclude that

$$\overline{G\cdot\Lambda_k} \cap \overline{G\cdot\Lambda_j} = \emptyset$$

for $j \neq k, n-k$, proving the claim.

18.3. Application to Higher Higher Teichmüller theory

We can apply this result within the framework of higher higher Teichmüller theory.

Definition 18.3.1. A discrete subgroup Γ of $G = \mathrm{O}(n,2)$ is *convex cocompact* if it acts properly discontinuously and cocompactly on some properly convex closed subset $\mathcal{C}$ of $\mathbb{H}^{n,1}$. It is $\mathbb{H}^{n,1}$*−convex cocompact* if moreover $\partial\mathcal{C}$ contains no non-trivial lightlike segments.

To prove that admissible boundaries containing non-trivial lightlike segment are not quasi-spheres, we need the following result of 3−dimensional Anti-de Sitter geometry. The result can be found in [BS10, Claim 3.23]. For the sake of completeness, we add the proof here.

Lemma 18.3.2. *Let Λ be and admissible boundary in the boundary of the 3−dimensional Anti-de Sitter space $\partial\mathbb{H}^{2,1}$. If Λ contains a non-trivial lightlike segment, then $\overline{G\cdot\Lambda}$ contains a Barbot crown.*

Proof. Let $a_1, b_1 \in \Lambda$ be the endpoints of the non-trivial lightlike segment contained in Λ, namely $a_1 \oplus b_1$ is an isotropic plane. Equivalently, $\langle a_1, b_1\rangle = 0$. Up to isometry, we can assume that

$$\begin{cases} a_1 = [1:0:0:1] \\ b_1 = [0:1:1:0]. \end{cases}$$

Consider the 1−parameter subgroup of isometries

$$g_t = \begin{pmatrix} \cosh(t) & 0 & 0 & -\sinh(t) \\ 0 & \cosh(t) & -\sinh(t) & 0 \\ 0 & -\sinh(t) & \cosh(t) & 0 \\ -\sinh(t) & 0 & 0 & \cosh(t) \end{pmatrix} \in \mathrm{O}(2,2) = \mathrm{Isom}(\mathbb{H}^{2,1}).$$

Let us give the following parameterization of the boundary:

$$\begin{aligned} [0,2\pi]\times[0,2\pi] &\longrightarrow \partial\mathbb{H}^{2,1} = \mathbb{S}^1\times\mathbb{S}^1 \\ (\alpha,\beta) &\longmapsto [\cos(\alpha):\sin(\alpha):\sin(\beta):cos(\beta)]. \end{aligned}$$

To study the action of g_t over the boundary, it suffices to consider its projective class:

$$\begin{aligned} \frac{1}{\cosh(t)} g_t \cdot \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \\ \sin(\beta) \\ \cos(\beta) \end{bmatrix} &= \begin{pmatrix} 1 & 0 & 0 & -\tanh(t) \\ 0 & 1 & -\tanh(t) & 0 \\ 0 & -\tanh(t) & 1 & 0 \\ -\tanh(t) & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \\ \sin(\beta) \\ \cos(\beta) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha) - \tanh(t)\cos(\beta) \\ \sin(\alpha) - \tanh(t)\sin(\beta) \\ \sin(\beta) - \tanh(t)\sin(\alpha) \\ \cos(\beta) - \tanh(t)\cos(\alpha) \end{bmatrix}. \end{aligned}$$

Denote by $a_2 := [-1:0:0:1]$, $b_2 := [0:-1:1:0]$: one directly check that the isotropic plane $a_1 \oplus b_1$ (resp. $a_2 \oplus b_2$) consists of repulsive (resp. attracting) fixed points for the action of g_t, for $t > 0$. We claim that

$$\Lambda' := \lim_{k\to+\infty} g_k(\Lambda)$$

is a Barbot crown of vertex a_1, b_1, a_2, b_2, which concludes the proof.

To prove the claim, let $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the graph of Λ in the splitting $(x_0, P = \{x_4 = 0\})$: since the lightlike segment between a_1 and b_1 is contained in Λ, we have $\phi(\alpha) = \alpha$ over $[0, \pi/2]$, and $(\alpha, \phi(\alpha))$ is pointwise fixed by g_t. Let us study the pointwise limit of $g_k g_k(\alpha, \phi(\alpha))$, for $\alpha \in (\pi/2, 2\pi)$, we have

$$\begin{aligned} g_\infty(\alpha) := \lim_{k\to+\infty} g_k(\alpha,\phi(\alpha)) &= \begin{bmatrix} \cos(\alpha) - \cos(\phi(\alpha)) \\ \sin(\alpha) - \sin(\phi(\alpha)) \\ \sin(\phi(\alpha)) - \sin(\alpha) \\ \cos(\phi(\alpha)) - \cos(\alpha) \end{bmatrix} \\ &= \begin{bmatrix} -2\sin\left(\frac{\alpha+\phi(\alpha)}{2}\right)\sin\left(\frac{\alpha-\phi(\alpha)}{2}\right) \\ 2\cos\left(\frac{\alpha+\phi(\alpha)}{2}\right)\sin\left(\frac{\alpha-\phi(\alpha)}{2}\right) \\ -2\cos\left(\frac{\alpha+\phi(\alpha)}{2}\right)\sin\left(\frac{\alpha-\phi(\alpha)}{2}\right) \\ 2\sin\left(\frac{\alpha+\phi(\alpha)}{2}\right)\sin\left(\frac{\alpha-\phi(\alpha)}{2}\right) \end{bmatrix} = \begin{bmatrix} -\sin\left(\frac{\alpha+\phi(\alpha)}{2}\right) \\ \cos\left(\frac{\alpha+\phi(\alpha)}{2}\right) \\ -\cos\left(\frac{\alpha+\phi(\alpha)}{2}\right) \\ \sin\left(\frac{\alpha+\phi(\alpha)}{2}\right) \end{bmatrix}. \end{aligned}$$

It follows that

$$\lim_{\alpha\to 2\pi^+ = 0^-} g_\infty(\alpha) = b_2 \qquad \lim_{\alpha\to\pi/2^-} g_\infty(\alpha) = a_2,$$

concluding that a_2, b_2 belong to Λ'. Incidentally, this proves that Λ' cannot be the boundary of a totally geodesic degenerate surface: hence it is an admissible boundary containing a_1, b_1, a_2, b_2. The only such boundary is a Barbot crown, proving the claim and concluding the proof. □

Corollary 18.3.3. *Let Λ be an admissible boundary in $\partial\mathbb{H}^{n,1}$ containing a non-trivial lightlike segment. Then Λ is not a quasi-sphere.*

Proof. Let $a_1, b_1 \in \Lambda$ be the endpoints of the non-trivial lightlike segment contained in Λ, namely $a_1 \oplus b_1$ is an isotropic plane. Choose a $(2,2)-$subspace W of $\mathbb{R}^{n,2}$ containing $a_1 \oplus b_1$. Up to isometry, we can assume that $\mathbb{R}^{n,2} = W \times \mathbb{R}^{n-2}$

Then, $\Lambda \cap W$ is an admissible boundary in a totally geodesic copy $\mathbb{H}^{2,1}$ containing a non-trivial lightlike segment. By Lemma 18.3.2, there exists a sequence $g_k \in \mathrm{Isom}(\mathbb{H}^{2,1})$ such that

$$\lim_{k\to+\infty} g_k(\Lambda \cap W)$$

is a Barbot crown. Define $h_k := g_k \oplus \mathrm{Id}_{n-2} \in \mathrm{Isom}(\mathbb{H}^{2,1})$, and let

$$\Lambda' := \lim_{k\to+\infty} g_k(\Lambda \cap W).$$

By construction, $\Lambda' \cap W$ is a Barbot crown: incidentally, this proves that Λ' cannot bound a totally degenerate hyperplane, hence it is an admissible boundary. Combining Lemma 18.1.3 and Lemma 18.2.2, we have

$$\omega_0(\Lambda) \geq \omega_0(\Lambda') = \frac{\pi}{2},$$

which concludes the proof. □

This result allows us to characterize $\mathbb{H}^{n,1}-$convex cocompact subgroups of $\mathrm{Isom}(\mathbb{H}^{n,1})$ in terms of the width of their convex core.

Corollary N. *Let Γ be a convex cocompact subgroup of $\mathrm{Isom}(\mathbb{H}^{n,1})$ whose limit set Λ_Γ is an admissible boundary. Then Γ is $\mathbb{H}^{n,1}-$convex cocompact if and only if its limit set is a quasi-sphere.*

Proof. Since Γ acts cocompactly on $\mathcal{CH}_0(\Lambda_\Gamma)$, the width of the convex hull of Λ_Γ is realized.

Let $\omega_0(\Lambda_\Gamma) = \pi/2$: by Lemma 18.2.2, Λ_Γ contains a Barbot crown, hence in particular a non-trivial lightlike segment, *i.e.* Γ is not $\mathbb{H}^{n,1}-$convex cocompact.

Conversely, let $\omega_0(\Lambda_\Gamma) < \pi/2$: by Corollary 18.3.3, Λ_Γ contains no non-trivial lightlike segment, hence Γ is a $\mathbb{H}^{n,1}-$convex cocompact subgroup of $\mathrm{Isom}(\mathbb{H}^{n,1})$, concluding the proof. □

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