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Invariants and zeta functions associated to totally disconnected locally compact groups

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Abstract

The study of invariants and asymptotic properties relative to groups plays a prominent role in several classification problems. This thesis focuses on some cohomological and geometric invariants and certain growth series associated to totally disconnected locally compact (for short, t.d.l.c.) groups, with a specific attention on those groups acting properly and cocompactly on trees or buildings.

We first deal with the *number of ends* of a compactly generated t.d.l.c. group. Generalising a result of E. Specker, we express this invariant in terms of the dimension of low-degree cohomology groups of the relevant t.d.l.c. group.

Then we focus on the *rational discrete cohomological dimension* of a t.d.l.c. group. If the group acts properly and cocompactly on a building, we prove that this invariant coincides with the (more accessible) rational cohomological dimension of the Coxeter group describing the type of the building.

Another cohomological invariant we deal with is the *Euler–Poincaré characteristic* of a unimodular t.d.l.c. group. We establish a result on the sign of this invariant and then use it to generalise the famous Stallings–Swan theorem to a class of unimodular t.d.l.c. groups.

The second part of the thesis focuses on two types of Dirichlet series arising from counting problems pertaining (directly and indirectly) to a t.d.l.c. group. The first type arises from counting double cosets in a group by their size. Within the cases of groups acting on locally finite trees or buildings, we provide explicit formulae of the relevant zeta functions in terms of local data of the action. Following a result of I. Castellano, G. Chinello and T. Weigel, we establish a surprising connection between the evaluation at -1 and the Euler–Poincaré characteristic of the group. The second series we considered arise from counting submodules of a free finitely generated module which are invariant under the action of a unipotent matrix group. If the coefficient ring in the ring of integers of a non-Archimedean local field, we highlight a sufficient condition to ensure the validity of T. Rossmann's conjecture about the behaviour of the function in a neighbourhood of 0. Moreover, in the case of finite fields as coefficient rings, we provide explicit combinatorial formulae for the relevant series.

Sommario

Lo studio di invarianti e proprietà asintotiche relative ad un gruppo ha un ruolo rilevante in molto problemi di classificazione. Questa tesi si concentra sullo studio di invarianti coomologici e geometrici e di alcune serie di crescita associati a gruppi totalmente sconnessi e localmente compatti (in breve, gruppi t.s.l.c.). Si dedice una particolare attenzione a gruppi t.s.l.c. che agiscono propriamente e cocompattamente su alberi o edifici.

Inizialmente ci concentriamo sul *numero di fini* di un gruppo t.s.l.c. compattamente generato. Generalizzando un risultato di E. Specker, esprimiamo tale invariante in termini delle dimensioni dei gruppi di coomologia di basso grado del gruppo.

In seguito, ci dedichiamo allo studio della *dimensione coomologica razionale discreta* di un gruppo t.s.l.c.. Nel caso in cui il gruppo agisca propriamente e cocompattamente su un edificio, dimostriamo che tale invariante coincide con la (più accessibile) dimensione coomologica razionale del gruppo di Coxeter che descrive il tipo dell'edificio.

Un altro invariante coomologico che consideriamo è la *caratteristica di Eulero–Poincaré* di un gruppo t.s.l.c. unimodulare. Forniamo un risultato sul segno di tale invariante, ed inseguito sfruttiamo questa informazione per generalizzare il famoso teorema di Stallings–Swan ad una classe di gruppi t.s.l.c. unimodulari.

La seconda parte della tesi si concentra su due serie di Dirichlet derivanti da problemi di conteggio associati (direttamente o indirettamente) a un gruppo. La prima classe di serie derivano dal conteggio dei doppi laterali di un gruppo in base al loro "volume". All'interno della famiglia di gruppi che agiscono su alberi o edifici, forniamo formule esplicite di tali serie in termini di dati locali dell'azione. Inoltre, generalizzando un risultato di I. Castellano, G. Chinello e T. Weigel, stabiliamo una sorprendente connessione tra il valore di tali funzioni in -1 e la caratteristica di Eulero–Poincaré del gruppo. La seconda famiglia di serie che consideriamo derivano dal conteggio di sottomoduli di un modulo libero e finitamente generato invarianti rispetto all'azione di un gruppo matriciale unipotente. Se l'anello dei coefficienti è l'anello degli interi di un campo locale non-archimedeo, delineamo una condizione sufficiente affinchè una congettura di T. Rossmann riguardante il comportamento della funzione in un intorno di 0 sia soddisfatta. In seguito, spostiamo l'attenzione sul caso in cui l'anello dei coefficienti sia un campo finito e produciamo formule esplicite delle serie in questione.

Zusammenfassung

Die Untersuchung der Invarianten von Gruppen auf ihre asymptotischen Eigenschaften spielt eine wichtige Rolle für verschiedene Klassifikationsprobleme. Die vorliegende Arbeit behandelt die Theorie kohomologischer und geometrischer Invarianten und bestimmter Wachstumsreihen von total unzusammenhangenden, lokal kompakten (kurz: t.u.l.k.) Gruppen, mit besonderem Augenmerk auf solche, die treu und kokompakt auf Bäumen oder Gebäuden wirken.

Wir befassen uns zunächst mit der *Anzahl der Enden* einer kompakt erzeugten t.u.l.k. Gruppe. Indem wir ein Ergebnis von E. Specker verallgemeinern, drücken wir diese Invariante durch die Dimension der Kohomologiegruppen niedrigen Grades der betreffenden t.u.l.k. Gruppe aus.

Im folgenden gehen wir zur Untersuchung des *rationalen diskreten kohomologischen Maßes* einer t.u.l.k. Gruppe über. Unter der Annahme, dass die Gruppe treu und kokompakt auf ein Gebäude wirkt, beweisen wir, dass diese Invariante mit der (viel zugänglicheren) rationalen kohomologischen Dimension der Coxeter-Gruppe übereinstimmt, die den Typ des Gebäudes beschreibt.

Eine weitere kohomologische Invariante mit der wir uns beschäftigen, ist die *Euler–Poincaré-Charakteristik* einer unimodularen t.d.l.c. Gruppe. Wir stellen einen Satz über das Vorzeichen dieser Invariante auf und verwenden ihn, um den berühmten Satz von Stallings und Swan auf eine Klasse von unimodularen t.d.l.c. Gruppen zu verallgemeinern.

Der zweite Teil der Arbeit behandelt zwei Types von Dirichlet-Reihen, welche sich aus Zählproblemen ergeben, die (direkt und indirekt) t.u.l.k. Gruppen betreffen. Die erste Reihe resultiert aus der Zählung von Doppelnebenklassen in einer Gruppe gestaffelt nach ihrer Größe. Im Fall von Gruppen, die auf lokal endlichen Bäumen oder Gebäuden wirken, liefern wir explizite Formeln für die relevanten Zeta-Funktionen via des lokalen Datums der Wirkung. Darüber hinaus stellen wir in Analogie zu einem Ergebnis von I. Castellano, G. Chinello und T. Weigel eine überraschende Verbindung zwischen der Funktionswert an der Stelle -1 und der Euler-Poincaré-Charakteristik der Gruppe her. Die zweite von uns betrachtete Reihe resultiert aus der Zählung von Untermodulen eines gegebenen freien, endlich erzeugten Moduls, welche unter der Wirkung einer unipotenten Matrixgruppe invariant sind. Wenn der Koeffizientenring im Ganzheitsring eines nicht-archimedischen lokalen Körpers enthalten ist, stellen wir eine hinreichende Bedingung auf, um die Gültigkeit einer Vermutung von T. Rossmanns über das Verhalten der Funktion in

einer Umgebung der 0 sicherzustellen. Schlussendlich richten wir unsere Aufmerksamkeit auf endliche Körper als Koeffizientenringe, um explizite kombinatorische Formeln für die relevanten Reihen zu liefern.

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Contents

Introduction

Broadly speaking, an invariant is a property of a class of mathematical objects (e.g., groups, topological spaces, etc.) that remains unchanged under suitable transformations (e.g., isomorphisms, homeomorphisms, etc.). The notion of invariants pervades so many areas of mathematics and beyond that we have been used to deal with it since our first studies. For instance, the area of a triangle in an Euclidean plane is invariant under isometries, the degree of a polynomial is invariant under linear change of variables and so on [Wik]. Or, citing an example from physics, the local volume of an incompressible fluid is invariant under motion [Tre]. One of the main motivations for which invariants are so widely spread is the key role they may play in classification problems. In this thesis, we focus on certain invariants for groups, mostly arising from cohomology and geometric group theory, or asymptotic algebra.

The subject matter: t.d.l.c. groups

Almost all the groups we consider in this thesis have a common feature: they are endowed with a *totally disconnected locally compact* (for short, t.d.l.c.) *topology* that is compatible with their algebraic structure. Recall that a *t.d.l.c. space* is a topological space in which every point is contained in an open neighbourhood with compact closure, and the connected components of the space are singletons. A *t.d.l.c. group* is a group endowed with a t.d.l.c. topology with respect to which the multiplication and inversion maps are continuous. Discrete groups form a (fundamental but somehow trivial) class of examples of t.d.l.c. groups and, with a slight abuse of notation, they can be identified with groups in the algebraic sense. In these terms, the class of t.d.l.c. groups generalises the one of abstract groups.

On the other hand, t.d.l.c. groups constitute one of the two building blocks of a key class of topological (Hausdorff) groups: the one of *locally compact* (= l.c.) *groups*. Indeed, every l.c. group G modulo the connected component of the unit is a t.d.l.c. group. In particular, every l.c. group arises as a group extension of a connected l.c. group and a t.d.l.c. group. Moreover, the class of l.c. groups has two main subclasses: connected l.c. groups and t.d.l.c. groups. In certain cases, this subdivision even turns into a dichotomy. For instance, every locally compact field (for short *local field*) is either connected or totally disconnected. In detail, connected local fields are either $\mathbb{R}$ or $\mathbb{C}$, and t.d.l.c. fields are either a finite-degree extension of the field $\mathbb{Q}_p$ of p-adic numbers, for some prime p, or the field $\mathbb{F}_q(\!(t)\!)$ of the formal Laurent series over some finite field $\mathbb{F}_q$. Returning to the general l.c. case, one may ideally split the study of several structural problems for l.c. groups (e.g., the classification of topologically simple l.c. groups) into connected and totally disconnected cases. For connected l.c. groups, the solution of Hilbert's fifth problem sheds light on their structure: every connected l.c. group can be written as an inverse limit of connected Lie groups (it is a consequence of Gleason–Yamabe's theorem, cf. [Cas23, Theorem 10.8]). On the other hand, the structure of t.d.l.c. groups is still less clear, and this has motivated a relevant part of the contemporary research on l.c. groups. A notable theorem regarding the structure of t.d.l.c. groups is due to D. van Dantzig [Van36]: t.d.l.c. groups are exactly all those Hausdorff groups that admit a fundamental local basis at 1 given by compact open subgroups. We may informally rephrase this by saying that t.d.l.c. groups are all those Hausdorff groups that are *locally profinite*. Profinite groups are exactly all those t.d.l.c. groups that are compact, and they constitute another building block (in addition to discrete groups) of the class of t.d.l.c. groups.

T.d.l.c. groups naturally arise in at least two contexts, which are both represented in this thesis. The first context pertains to non-Archimedean local fields and algebraic groups over them. As we partially mentioned before, local fields that are not $\mathbb{R}$ or $\mathbb{C}$ are t.d.l.c. and they are called *non-Archimedean local fields*. Given a non-Archimedean local field K, all matrix groups (e.g., $\mathrm{GL}_n(K)$, $\mathrm{SL}_n(K)$, ...) are t.d.l.c. with respect to the induced topology from the product topology of $K^{n\times n} \simeq \mathrm{Mat}_n(K)$. More generally, if $\mathcal{G}$ is an algebraic affine group scheme over K, the set $\mathcal{G}(K)$ of all K-points of $\mathcal{G}$ determines a t.d.l.c. group with the induced topology from $\mathrm{GL}_n(K)$, for some $n \geq 1$.

The second context involves permutation group theory and geometric group theory. The symmetric group $\mathrm{Sym}(X)$ of a discrete set X carries a natural group topology, the *permutation topology*, which is the roughest topology for which all stabilisers of finite subsets of X are open. The permutation topology on $\mathrm{Sym}(X)$ is totally disconnected. Hence, taking closed subgroups of $\mathrm{Sym}(X)$, one produces totally disconnected groups, which in several cases turn to be also locally compact. For instance, if Σ is either a locally finite tree or a locally finite building, then the group of automorphisms $\mathrm{Aut}(\Sigma)$ of Σ is a closed l.c. subgroup of the symmetric group of the set of vertices or the set of chambers of Σ, respectively. In particular, $\mathrm{Aut}(\Sigma)$ is a t.d.l.c. group. It is not surprising that t.d.l.c. groups arise somehow "naturally" in the context of geometric group theory.

In this thesis, we will study invariants of t.d.l.c. groups in the broader sense. That is, we will first consider "classical" invariants for these groups, i.e., suitable real numbers arising – in our case – while studying the geometry and cohomology of the groups. However, we will also regard as invariants all growth series arising from counting suitable subobjects associated to a given t.d.l.c. group.

Geometric and cohomological invariants for t.d.l.c. groups

A common finiteness condition on a group G one may ask is its *finite generation*. For each finite generating set S of G, one can construct a locally finite connected graph called *Cayley graph associated to* (G, S). A notable feature of Cayley graphs of a given finitely generated group is that they are pairwise quasi-isometric. Informally speaking, this means that they pairwise share the same large-scale properties. There is a wealth of properties that are defined on locally finite connected graphs and that are invariant under quasi-isometries. One of them is the *number of ends* of a graph (in the sense of R. Halin [Hal94]) which, informally speaking, is the number of branches of the graph at infinity. The reader is referred to Section 2.2 for a formal definition. Having a family of pairwise quasi-isometric graphs attached to a group allows us to transfer properties (which are invariant under quasi-isometries) from the graphs to the group. For instance, given a finitely generated group G, its number of ends $e(G)$ is defined as the number of ends of an arbitrary Cayley graph of G.

The fact that the Cayley graphs are quasi-isometric is strictly connected to the fact that they are locally finite and thus the fact that the generating sets they come from are finite. From the viewpoint of topological groups, finitely generated groups are countable and thus admit only one possible locally compact (Hausdorff) group topology: the discrete one. For arbitrary topological groups, the concept that replaces "finite generation" is *compact generation*. For compactly generated t.d.l.c. groups, H. Abels [Abe74] introduced a generalisation of Cayley graphs that we now call *Cayley–Abels graph*. A Cayley–Abels graph for a compactly generated t.d.l.c. group G is a locally finite connected graph on which G acts vertex-transitively and with compact open stabilisers. If G is discrete, every Cayley graph of G is a Cayley–Abels graph. Since every two Cayley–Abels graphs of a given group are quasi-isometric, one can again transfer quasi-isometric invariants from Cayley–Abels graphs to the relevant compactly generated t.d.l.c. group G. In particular, the number of ends $e(G)$ of G is defined as the number of ends of an arbitrary Cayley–Abels graph of G.

Number of ends and accessibility of compactly generated t.d.l.c. groups

By Hopf's theorem [Hop94], the number of ends of a compactly generated t.d.l.c. group might be only 0, 1, 2 or ∞. E. Specker [Spe49] provided an explicit formula of the number of ends of a finitely generated group G as follows:

$$e(G) = 1 - \dim_{\mathbb{Q}} \mathrm{H}^0(G, \mathbb{Q}[G]) + \dim_{\mathbb{Q}} \mathrm{H}^1(G, \mathbb{Q}[G]),$$

where $\mathrm{H}^k(G, \mathbb{Q}[G])$ is the k-th cohomology group of G with coefficient in the rational group algebra $\mathbb{Q}[G]$. In this thesis, we generalise Specker's result to compactly generated t.d.l.c. groups. Dealing with t.d.l.c. groups, we replaced the standard cohomology theory for abstract groups with one of its closest generalisations, the *discrete cohomology theory* *for t.d.l.c. groups* (with rational coefficients). This theory, developed by I. Castellano and T. Weigel in [CW16], is the only cohomology theory for t.d.l.c. groups appearing in this thesis. In this setting, the k-th cohomology group $\mathrm{H}^k(G, \mathbb{Q}[G])$ is replaced by the discrete k-th cohomology group $\mathrm{dH}^k(G, \mathrm{Bi}(G))$ of a t.d.l.c. group G with coefficient in the so-called standard rational discrete bimodule $\mathrm{Bi}(G)$. The latter bi-module is a suitable generalisation of the rational group algebra $\mathbb{Q}[G]$ to t.d.l.c. groups, and it is (non-canonically) isomorphic to $\mathbb{Q}[G]$ if G is discrete (cf. [CW16]). Thus, we obtain the following.

Theorem A (Theorem 2.2.25)**.** *Let G be a compactly generated t.d.l.c. group. Then*

$$e(G) = 1 - \dim_{\mathbb{Q}} \mathrm{dH}^0(G, \mathrm{Bi}(G)) + \dim_{\mathbb{Q}} \mathrm{dH}^1(G, \mathrm{Bi}(G)).$$

The number of ends provides a first (rough) classification of compactly generated t.d.l.c. groups. Among these groups, the 0-ended ones are precisely the compact t.d.l.c. groups (i.e., the profinite groups).

A famous result due to J. Stallings, generalised to t.d.l.c. groups by I. Castellano [Cas20], states that a compactly generated t.d.l.c. group G has at least 2 ends if, and only if, G *splits non-trivially over a compact open subgroup*, i.e., there is a compact open subgroup $U \leq G$ such that either G is the free amalgamated product of two open subgroups X and Y with respect to the common proper subgroup U, or G is the HNN-extension of an open subgroup $X \gneq U$ with respect to U. In turn, the factors of these two decompositions are shown to be compactly generated t.d.l.c. groups. Provided they have at least 2 ends, they split non-trivially over a compact open subgroup and so on. One can encode the successive decompositions of the factors of G by expressing G itself as the fundamental group of a finite graph of t.d.l.c. groups with profinite edge-groups and compactly generated vertex-groups. Roughly speaking, we keep track of the splitting of each vertex-group by extending the current graph of t.d.l.c. groups which G is the fundamental group by adding a new edge-pair to the correspondence vertex. The new graph of t.d.l.c. groups still has G as a fundamental group.

A compactly generated t.d.l.c. group G is said to be *accessible* if it is topologically isomorphic to the fundamental group of a finite graph of t.d.l.c. groups with profinite edge-groups and whose vertex-groups are compactly generated t.d.l.c. groups with at most one end. In general, compactly generated t.d.l.c. groups might not be accessible (cf. M. Dunwoody's class of examples [Dun93]). On the other side, compactly presented t.d.l.c. groups are accessible by a result of Y. Cornulier [Cor18]. The challenge is to find the right finiteness conditions to add to compact generation to obtain accessible t.d.l.c. groups. For discrete groups, an example is given by the *Stallings–Swan theorem*, which in a formulation due to M. Dunwoody [Dun79] asserts the following: the accessibility of a finitely generated group G is guaranteed if its rational cohomological dimension $\mathrm{cd}_{\mathbb{Q}}(G)$ is at most 1. This theorem had a long story and was initially proved by J. Stallings for finitely presented groups. In the t.d.l.c. case, I. Castellano [Cas20] proved that a compactly presented t.d.l.c. group G with $\mathrm{cd}_{\mathbb{Q}}(G) \leq 1$ is accessible. Whether this statement is true or not replacing compact presentation with compact generation remains an open problem for the moment. In this thesis, we provide an intermediate result towards a general Stallings–Swan theorem within the class of unimodular compactly generated t.d.l.c. groups. Following I. Castellano's approach [Cas20], we replace the standard rational cohomological dimension of an abstract group with the *rational discrete cohomological dimension* $\mathrm{cd}_{\mathbb{Q}}(G)$ of a t.d.l.c. group G. The result is the following.

Theorem B (Theorem 2.5.15)**.** *Let G be a compactly generated t.d.l.c. group G. Assume that $\mathrm{cd}_{\mathbb{Q}}(G) \leq 1$ and that, for some (and hence every) Haar measure on G, the following quantity is finite:*

$$\|G\|_\mu := \sup\{\mu(U) \mid U \leq G \text{ compact open subgroup}\}.$$

Then G is accessible.

Unlike Dunwoody's theorem, our argument requires the additional condition that $\|G\|_\mu$ is finite. This extra assumption was motivated by the fact that Dunwoody's proof relies on some peculiar properties of discrete groups (e.g., having a minimal compact open subgroup) which break down for non-discrete t.d.l.c. groups. Our method is inspired by a result due to P. Linnel [Lin83], which states the following: a finitely generated group G satisfying $\|G\|_\mu < \infty$, where μ is the counting measure on G, is accessible. The strategy to achieve Theorem B relies on another cohomological invariant of the relevant group G, its *Euler–Poincaré characteristic*. Using this invariant, we produce a finite upper bound on the number of successive decompositions G might have. The Euler–Poincaré characteristic we consider is the one defined by I. Castellano, G. Chinello and T. Weigel [CCW24] for every unimodular t.d.l.c. group G of type FP. In this thesis, we refer to it as a real number $\chi(G, \mu)$ defined once fixed an arbitrary Haar measure μ on G. One of the key tools used to prove Theorem B is the following result.

Theorem C (Theorem 2.4.41)**.** *Let G be a compactly generated unimodular t.d.l.c. group. If $\mathrm{cd}_{\mathbb{Q}}(G) = 1$, then $\chi(G, \mu) \leq 0$ for every Haar measure μ on G.*

Rational discrete cohomological dimension and Euler–Poincaré characteristic of t.d.l.c. groups

We now comment on the two cohomological invariants mentioned in the previous lines: the *rational discrete cohomological dimension* and the *Euler–Poincaré characteristic* of a t.d.l.c. group G. Already for discrete groups, determining their exact values is in general a non-trivial problem. Nevertheless, there are some exceptions: e.g., profinite groups or groups having a favourable action on a contractible locally finite simplicial complex. Profinite groups are exactly all t.d.l.c. groups having vanishing rational discrete cohomological dimension (cf. [CW16, Proposition 3.7(a)]). Moreover, every profinite group G satisfies

$$\chi(G, \mu) = \mu(G)^{-1}.$$

In this thesis, we focus on t.d.l.c. groups having a (simplicial) action either on locally finite trees or on the Davis' realisations of locally finite buildings. Given a t.d.l.c. group G, we say that a simplicial G-action on a simplicial complex Σ is *proper* (resp. *cocompact*) if the G-action on each skeleton of Σ has compact open stabilisers (resp. finitely many orbits).

The case of a t.d.l.c. group G acting properly and cocompactly on a locally finite tree T was already largely understood by I. Castellano and T. Weigel in [CW16]. First, we have $\mathrm{cd}_{\mathbb{Q}}(G) \leq 1$. To be precise, by Theorem B, one may deduce that compactly generated t.d.l.c. groups G have $\mathrm{cd}_{\mathbb{Q}}(G) \leq 1$ and $\|G\|_\mu < \infty$ if, and only if, they are unimodular and admit a proper cocompact action on a locally finite tree (cf. Corollary 2.5.16). Second, for a t.d.l.c. group G acting properly and cocompactly on a tree, there is an explicit criterion – due to H. Bass and R. Kulkarni [BK90] – to check whether G is unimodular (cf. Proposition 3.5.2). If G is unimodular, then $\chi(G, \mu)$ is an explicit alternating sum of the Euler–Poincaré characteristic of selected vertex- and edge-stabilisers (cf. Proposition 2.4.37). In this thesis, under additional properties on the action, we rephrase this description of $\chi(G, \mu)$ in terms of local data of the action (cf. Proposition 3.5.4).

For a t.d.l.c. group G acting properly and cocompactly on (the Davis' realisation of) a locally finite building Δ, we deduce the following theorem:

Theorem D (Theorem 2.3.20)**.** *Let G be a t.d.l.c. group acting properly and cocompactly on a locally finite building of type (W, S). Then*

$$\mathrm{cd}_{\mathbb{Q}}(G) = \mathrm{cd}_{\mathbb{Q}}(W).$$

Theorem D characterises $\mathrm{cd}_{\mathbb{Q}}(G)$ in terms of a local datum of the building on which G acts: the type (W, S), which is a Coxeter group. The advantage of this reduction is that there is a wealth of results and methods (like the one of M. Bestvina [Bes93]) to compute the rational cohomological dimension of a Coxeter group. In addition to spherical and affine Coxeter groups – whose cohomological dimension was already well known – in this thesis we provide a general formula for the rational cohomological dimension of a Coxeter group of hyperbolic type (cf. Propositions 2.3.16 and 2.3.17). Moreover, provided the G-action on Δ is *Weyl-transitive* and the building has uniform thickness $q < \infty$, one can conveniently describe the Euler–Poincaré characteristic of G in terms of local data of the building. The explicit formula, obtained by I. Castellano, G. Chinello and T. Weigel [CCW24] by rephrasing a result of J. Dymara [Dym06], is summarised by the following identity:

$$\chi(G, \mu) = W(q)^{-1}\mu(B), \tag{0.0.1}$$

where $W(t)$ is the Poincaré series of (W, S) and B is a chamber stabiliser. The formula in (0.0.1) applies if, for instance, G is the set of K-points of a semisimple algebraic group over a non-Archimedean local field K, or if G is the geometric completion of a Kac–Moody group over a finite field. It is quite straightforward to generalise (0.0.1) to all t.d.l.c. groups acting properly and Weyl-transitively on locally finite buildings of arbitrary thickness (cf. Section 3.4.1 and Theorem 3.5.1). With this, we can include to the list of examples above all t.d.l.c. groups acting properly and Weyl-transitively on locally finite right-angled buildings, for instance.

The identity (0.0.1) motivated I. Castellano, G. Chinello and T. Weigel [CCW24] to study a new growth series attached to the relevant pair (G, B), namely

$$\zeta_{G,B}(s) := \sum_{BgB \in B\backslash G/B} |BgB/B|^{-s},$$

which in the particular cases they consider coincides with $W(q^{-s})$. By (0.0.1), the meromorphic continuation of $\zeta_{G,B}(s)$ at $s = -1$ recovers the Euler–Poincaré characteristic of G. The series $\zeta_{G,B}(s)$, as a function in the complex variable s, is called the *double-coset zeta function of* (G, B) and can be actually defined for all pairs (G, B) of t.d.l.c. groups where $B \leq G$ is a compact open subgroup for which the series $\sum_{BgB \in B\backslash G/B} |BgB/B|^{-s}$ converges at some $s \in \mathbb{C}$. One of the main questions raised in [CCW24] asks under which conditions the pair (G, B) satisfies the so-called *Euler–Poincaré identity*, i.e.,

$$\chi(G, \mu) = \zeta_{G,B}(-1)^{-1}\mu(B), \qquad (0.0.2)$$

for every Haar measure μ on G. The study of this open question is one of the main goals of the thesis, as outlined below.

Zeta functions associated to t.d.l.c. groups

Zeta functions initially arose in number theory, although nowadays they are also an established tool in studying groups, rings and algebras (see M. du Sautoy's survey [Sau03] for motivation). By a *zeta function* we mean the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ associated to a sequence $(a_n)_{n\geq 1}$ of complex numbers having *polynomial growth*, i.e., there is $\alpha \in \mathbb{R}_{\geq 0}$ such that $a_n = O(n^{\alpha})$ for every $n \gg 1$. The generating sequence often arises from a counting problem in a given structure: for instance, given a finitely generated group G, one may set a_n as the number of subgroups of G of index n. In group theory, the seminal work of F. Grunewald, D. Segal, G. Smith [GSS88] initiated the study of numerous zeta functions associated to finitely generated nilpotent groups or profinite groups. The double-coset zeta functions have been one of the first instances of zeta functions attached to possibly non-discrete and non-profinite t.d.l.c. groups.

In this thesis, we will deal with two kinds of zeta functions: the *double-coset zeta functions associated to a group and two prescribed subgroups* and the *submodule zeta functions associated to a unipotent matrix group*. In both cases, the relevant generating sequence arises from counting subobjects related to the given group. In the first case, the problem is to count double-cosets by their size; in the second one, given a unital commutative ring R, one counts finite-index submodules of R^n invariant under the action of a prescribed unipotent matrix group over R by their additive index in R^n. We will give a more detailed overview in the next lines.

The zeta functions arising from counting subobjects in a given structure (e.g., counting double-cosets of a group) are potential sources of invariants for the structure itself. Indeed, the analytic properties of zeta functions (i.e., convergence properties of the defining series, the meromorphic continuation of the associated function, functional equations, specific evaluations, ...) are a primer for defining invariants or properties of the relevant structure.

Double-coset zeta functions

In this thesis, we deal with a slight generalisation of the double-coset zeta functions introduced in [CCW24]. Namely, we consider a group G and subgroups $H, K \leq G$ satisfying $|HgK/K| < \infty$ for all $g \in G$. The case of G being a t.d.l.c. group and $H = K \leq G$ a compact open subgroup is a special instance. A triple (G, H, K) is said to have the *double-coset property* if, for every $n \geq 1$,

$$a_n(G, H, K) := |\{HgK \in H\backslash G/K \,:\, |HgK/K| = n\}| < \infty. \tag{0.0.3}$$

If $\big(a_n(G, H, K)\big)_{n\geq 1}$ grows polynomially in n, we say that (G, H, K) has *polynomial double-coset growth*. In this case, the following Dirichlet series

$$\zeta_{G,H,K}(s) := \sum_{n=1}^{\infty} a_n(G, H, K) n^{-s} = \sum_{HgK \in H\backslash G/K} |HgK/K|^{-s} \tag{0.0.4}$$

defines the so-called *double-coset zeta function of* (G, H, K). If G is a t.d.l.c. group and $H = K$ is compact open in G, then $\zeta_{G,K,K}(s)$ is exactly the double-coset zeta function of (G, K) defined in [CCW24].

As mentioned before, the introduction of double-coset zeta functions was motivated by their surprising behaviour at $s = -1$ they showed in the case of t.d.l.c. groups acting Weyl-transitively on uniform locally finite buildings (cf. (0.0.2)). In [CCW24], the authors also provided an explicit meromorphic continuation of double-coset zeta functions for those groups with respect to a spherical parabolic subgroup. Their results plainly extend to our more general double-coset zeta functions even by dropping the hypothesis of uniformity on the building (cf. Section 3.4.1).

The main contribution of this thesis to the theory of double-coset zeta functions pertains to the case of groups acting on trees. One may argue that trees are specific instances of buildings, so this setting would seem only a particular case of the previous one. However, our goal is to focus on groups acting on buildings with an easy combinatorics (i.e., trees) and then weaken the transitivity assumption (i.e., the Weyl-transitivity) we assumed before.

We highlight two specific classes of group actions on trees for which checking the double-coset property and computing the double-coset zeta functions reduce to a more accessible problem involving local data of the action. These two classes are the one of (P)-closed group actions on trees and the one of weakly locally ∞-transitive actions on trees. The first one is well-established in literature (cf. [Tit70; BEW15; RS22] for instance). The second one is introduced by the author as a generalisation of the well-known notion of locally ∞-transitive group action on a tree (cf. [BM00; Rad17]). In both cases, we deduce the following characterisation.

Theorem E (Theorems 3.3.21 and 3.3.22)**.** *Let G be a group acting cocompactly on a locally finite tree T. Assume that the action (G, T) is weakly locally ∞-transitive or (P)-closed. Then the following are equivalent, for all $t_1, t_2 \in T = VT \sqcup ET$:*

(i) (G, G_{t_1}, G_{t_2}) *has the double-coset property;*

(ii) (G, G_{t_1}, G_{t_2}) *has polynomial double-coset growth;*

(iii) there is $k \geq 1$ such that, for every geodesic $\mathfrak{p}$ in T of length $l \geq 1$, the pointwise stabiliser of $\mathfrak{p}$ does not fix any geodesic in T of length $l + k$ extending $\mathfrak{p}$.

Before, G_{t_i} denotes the pointwise stabiliser of t_i. Moreover, VT and ET denote the vertex-set and the edge-set of T, respectively.

Condition (iii) can be further characterised in terms of local data of the action, as discussed in Section 3.3.2. If a group action on a tree (G, T) falls into one of the two classes above, we also provide an explicit formula of double-coset zeta functions $\zeta_{G,G_{t_1},G_{t_2}}(s)$. The key has been to translate, in both cases, the problem of counting (G_{t_1}, G_{t_2})-double cosets by their size to the one of counting certain weighted path in a graph-like local datum of the action by a suitable weight. To underline that there is an analogous pattern for two different cases, in the following theorem we label $\zeta_{G,G_{t_1},G_{t_2}}(s)$ with a superscript $\bullet \in \{(w), (p)\}$ to distinguish whether (G, T) is weakly locally ∞-transitive ($\bullet = (w)$) or (P)-closed ($\bullet = (p)$).

Theorem F (Theorems 3.4.20 and 3.4.29)**.** *Let (G, T) be a group action on a tree that is weakly locally ∞-transitive or (P)-closed. Let $t_1, t_2 \in T$ be such that (G, G_{t_1}, G_{t_2}) has polynomial double-coset growth. Then*

$$\zeta^{\bullet}_{G,G_{t_1},G_{t_2}}(s) = \frac{\det(I^{\bullet} - \mathcal{E}^{\bullet}(s) + \mathcal{U}^{\bullet}_{t_1,t_2}(s))}{\det(I - \mathcal{E}^{\bullet}(s))} + \epsilon^{\bullet}_{t_1}(t_2),$$

for explicitly defined square matrices $\mathcal{E}^{\bullet}(s)$ and $\mathcal{U}^{\bullet}_{t_1,t_2}(s)$ whose entries are entire functions in $s \in \mathbb{C}$, and for a determined integer $\epsilon^{\bullet}_{t_1}(t_2)$. Here, $I^{\bullet}$ denotes the identity matrix of the same dimension as $\mathcal{E}^{\bullet}(s)$. In particular, $\zeta^{\bullet}_{G,G_{t_1},G_{t_2}}(s)$ extends to a meromorphic function on $\mathbb{C}$.

In Theorem F, the matrix $\mathcal{E}^{\bullet}(s)$ can be interpreted as a weighted adjacency matrix of the graph-like local structure in which we count the paths. The matrix $\mathcal{U}^{\bullet}_{t_1,t_2}(s)$ and the integer $\epsilon^{\bullet}_{t_1}(t_2)$ can be regarded as "perturbation data" given by the choice of t_1 and t_2.

Having an explicit meromorphic continuation allows us to evaluate the relevant zeta functions at $s = -1$ and to prove the following.

Theorem G (Theorem 3.5.12 and Corollary 3.5.13)**.** *Let G be a unimodular t.d.l.c. group acting properly and cocompactly on a tree T. Assume that the quotient graph of the action does not have cycles of length ≥ 2, and that the action is weakly locally ∞-transitive or (P)-closed. Then, for every $t \in T$ such that (G, G_t, G_t) has polynomial double-coset growth,*

$$\chi(G, \mu) = \zeta_{G,G_t,G_t}(-1)^{-1}\mu(G_t)^{-1},$$

for every Haar measure μ.

The proof of Theorem G is far from being just a direct evaluation at $s = -1$, but makes use of simplifications and splitting formulae which we discuss in Section 3.4.3.

The reduction of the problem to counting paths in a graph-like structure also suggests the introduction of a new zeta function which solely pertains to a graph and an edge weight. We introduce it in Definition 3.4.14. Remarkably, the evaluation at $s = -1$ of this zeta function has an explicit connection with the value at 1 of another growth series attached to an edge-weighted graph: its *weighted Ihara zeta function* (as introduced by A. Deitmar in [Dei19]). The detailed connection is described in Section 3.5.3.

Submodule zeta functions of unipotent matrix groups

The second types of zeta functions we focus on are the one associated to a unipotent matrix group U over a ring R, and they arise from counting U-submodules of R^n by their index. More precisely, we consider a commutative unital ring R which has *polynomial submodule growth*, i.e., every finitely generated R-module has finitely many submodules of index m, for every $m \geq 1$. Examples are finite commutative unital rings, rings of integers of number fields or of non-Archimedean local fields. Given a ring R with polynomial submodule growth and an integer $n \geq 1$, one of the most natural counting problems one may consider is the following: count, for every $m \geq 1$, the number $a_m(R^n)$ of submodules of R^n of additive index m. The Dirichlet series generated by $(a_m(R^n))_{m\geq 1}$ is called the *submodule zeta function of M*.

In this thesis, we focus on a variation of this problem. We set R and n as above and consider a subgroup of $U \leq \mathrm{GL}_n(R)$ acting on R^n by right-multiplication. Typically, U is a subgroup of the group $\mathrm{Uni}_n(R)$ of all uni-upper-triangular matrices over R. For every $m \geq 1$, we count the number $a_m(U \curvearrowright R^n)$ of all U-submodules of R^n (i.e., R-submodule of R^n which are invariant under the action of each element of U) of additive index m. Note that $a_m(R^n) = a_m(\{1\} \curvearrowright R^n)$ for every $m \geq 1$. The Dirichlet series generated by $(a_m(U \curvearrowright R^n))_{m\geq 1}$, i.e.,

$$\zeta_{U \curvearrowright R^n}(s) := \sum_{m=1}^{\infty} a_m(U \curvearrowright R^n) m^{-s}, \tag{0.0.5}$$

is called the *submodule zeta function of U acting on R^n*. These kinds of submodule zeta functions were initially defined by T. Rossmann [Ros15] for associative subalgebras $\mathcal{E}$ of

$\mathrm{Mat}_n(R)$ acting on R^n, as a generalisation of the ideal zeta functions of Lie rings. We will give a more detailed introduction to them in Section 4.2. Here we only outline how they connect with the zeta function defined in (0.0.5). Namely, every subgroup U of $\mathrm{Uni}_n(R)$ comes with a (nilpotent associative) algebra $\mathcal{E}_U := U - I_n = \{u - I_n \mid u \in U\}$, which is obtained by subtracting the identity matrix I_n to each element of U. The submodule zeta functions of U acting on R^n and of $\mathcal{E}_U$ acting on R^n (both by right-multiplication) equal (cf. Remark 4.2.8(v)). The subfamily of the submodule zeta functions we focus on already produces a wealth of challenges and open problems. Moreover, following the literature so far, we focus only on the cases of R being either the ring of integers of a non-Archimedean local field of characteristic zero (following for instance [Ros15; Ros17; CSV24; Van23; Vol19]), or a finite field (following for instance [Lee22]). Note that in both cases we are dealing with a compact t.d.l.c. ring.

We first focus on the case of a ring of integers R of a non-Archimedean local field K of characteristic zero. In this setting, in view of the submodule zeta functions in (0.0.5), one can benefit from techniques from p-adic integration [SG00] or the fact that finite-index submodules of R^n – identified up to homotheties – describe the set of vertices of the Bruhat–Tits building of $\mathrm{SL}_n(K)$ [Vol19]. For instance, using each of these two approaches, one proves that

$$\zeta_{\{1\}\curvearrowright R^n}(s) = \prod_{i=1}^{n} \frac{1}{1 - q^{-s+i-1}},$$

where q is the residue field size of K. More generally, with these techniques, one may prove that $\zeta_{U\curvearrowright R^n}(s)$ meromorphically extends to $\mathbb{C}$ as a rational function in q^{-s}. A more detailed exposition is given in Sections 4.2.1 and 4.2.3.

Beyond the case $U = \{1\}$, the problem of giving explicit formulae of $\zeta_{U\curvearrowright R}(s)$ revealed to be extremely challenging already for small n (e.g., $n = 5$, cf. [Ros15]). Although the formulae available at the moment are not many, they seem to share common features. One of them pertains to the behaviour at $s = 0$ and led T. Rossmann to formulate the following conjecture (here stated only for unipotent groups):

Conjecture 0.0.1 ([Ros15, §8.3])**.** Let R be the ring of integers of a non-Archimedean local field of characteristic zero, and let $n \geq 1$. Then, for every subgroup $U \leq \mathrm{Uni}_n(R)$, we have

$$\left. \frac{\zeta_{U\curvearrowright R^n}(s)}{\zeta_{\{1\}\curvearrowright R^n}(s)} \right|_{s=0} = 1.$$

What makes the proof of Conjecture 0.0.1 more difficult is that one can not directly evaluate $\zeta_{U\curvearrowright R^n}(s)$ at $s = 0$, as the defining series diverges. Thus, one has to rely on the meromorphic continuation of $\zeta_{U\curvearrowright R^n}(s)$ whose computation, with the nowadays techniques, seems an extremely challenging problem to tackle. In this thesis, we provide a sufficient condition ensuring the validity of Conjecture 0.0.1 if the associated nilpotent algebra $\mathcal{E}_U$ of U is generated by elementary matrices (cf. Theorem H below). Our strategy is to go around the problem of finding an explicit formula for $\zeta_{U \curvearrowright R^n}(s)$. We introduce a multivariate version, say $\zeta_{U \curvearrowright R^n}(\mathbf{s})$ where $\mathbf{s} = (s_1, \dots, s_n)$, of the zeta function $\zeta_{U \curvearrowright R^n}(s)$, which allows more flexibility in the evaluation at $s = 0$. The multivariate zeta function $\zeta_{U \curvearrowright R^n}(\mathbf{s})$ is a rational function in $q^{-s_1}, \dots, q^{-s_n}$, and recovers $\zeta_{U \curvearrowright R^n}(s)$ as follows:

$$\zeta_{U \curvearrowright R^n}(s-1, s-2, \dots, s-n) = \zeta_{U \curvearrowright R^n}(s).$$

In particular, provided $\mathbf{s}_0 = (-1, -2, \dots, -n)$, we have

$$\Big((1 - q^{-s_1 - 1})\zeta_{U \curvearrowright R^n}(\mathbf{s})\Big)\Big|_{\mathbf{s}=\mathbf{s}_0} = \Big((1 - q^{-s})\zeta_{U \curvearrowright R^n}(s)\Big)\Big|_{s=0}.$$

Our main result is the following.

Theorem H (Theorem 4.3.22)**.** *Let $n \geq 1$ and let $U \leq \mathrm{Uni}_n(R)$ be a group whose associated algebra $\mathcal{E}_U$ is generated by elementary matrices. Assume that*

$$\lim_{\mathbf{s} \to \mathbf{s}_0} \Big((1 - q^{-s_1 - 1})\zeta_{U \curvearrowright R^n}(\mathbf{s})\Big) \text{ exists in } \mathbb{C} \cup \{\infty\}. \tag{0.0.6}$$

Then,

$$\left.\frac{\zeta_{U \curvearrowright R^n}(\mathbf{s})}{\zeta_{\{0\} \curvearrowright R^n}(\mathbf{s})}\right|_{\mathbf{s}=\mathbf{s}_0} = 1. \tag{0.0.7}$$

The last part of the thesis deals with submodule zeta functions of a subgroup U of $\mathrm{Uni}_n(\mathbb{F}_q)$, where $\mathbb{F}_q$ is the field of finite size q. In this case, taking advantage of the divisibility of $\mathbb{F}_q$, we give an explicit formula for $\zeta_{U \curvearrowright \mathbb{F}_q^n}(s) = \zeta_{\mathcal{E}_U \curvearrowright \mathbb{F}_q^n}(s)$. More generally, given any subalgebra $\mathcal{E}_I$ of $\mathrm{Mat}_n(\mathbb{F}_q)$ generated by a set $\{E_{ij}\}_{(i,j) \in I}$ of elementary matrices, we obtain an explicit formula for $\zeta_{\mathcal{E}_I \curvearrowright \mathbb{F}_q^n}(s)$ in terms of combinatorial properties of I (cf. Theorem I below). Our description involves the set $\mathrm{Clos}(I)$ of all *closed subsets* of I, i.e., subsets $R \subseteq I$ which are either empty or satisfy the following: for every $(i,j) \in I$ such that $(j,k) \in R$ for some k, we have $(i,j) \in R$. In detail, we prove the following.

Theorem I (Theorem 4.4.16)**.** *Under the hypotheses above, we have*

$$\zeta_{\mathcal{E}_I \curvearrowright \mathbb{F}_q^n}(s) = \sum_{\substack{\mathcal{R} \subseteq \mathrm{Clos}(I): \\ \mathcal{R} \neq \emptyset \text{ and } C_{\mathcal{R}} \subseteq R_{\mathcal{R}}}} (-1)^{|\mathcal{R}|+1} q^{-s(n - \gamma_{\mathcal{R}} - \rho_{\mathcal{R}})} \cdot \zeta_{\{0\} \curvearrowright \mathbb{F}_q^{\rho_{\mathcal{R}}}}(s).$$

where the sets $C_{\mathcal{R}}$ and $R_{\mathcal{R}}$ are explicitly defined subsets of $\{1, \dots, n\}$ satisfying $\gamma_{\mathcal{R}} := |C_{\mathcal{R}}|$ and $\rho_{\mathcal{R}} := |R_{\mathcal{R}}| - \gamma_{\mathcal{R}}$.

Theorem I has very little overlap with S. Lee's work [Lee22]. Nevertheless, it continues showing a feature emerging in the formulae computed in [Lee22]: the multi-polynomial dependence of the relevant submodule zeta functions with respect to q and q^{-s}. According to [Lee22], this uniformity dependence would be a "symptom" of an analogous behaviour for submodule zeta functions of algebras over rings of integers of non-Archimedean local fields. Establishing a uniformity result for the latter class of zeta functions is a wide open problem at the moment.

Structure of the thesis

The thesis consists of four chapters.

Chapter 1 introduces the general background required for the subsequent chapter. Most of the chapter is an elaboration of already known literature, except for some definitions in Section 1.3.5 (as specified therein) and Section 1.7.

Chapter 2 deals with three invariants of t.d.l.c. groups arising from cohomology and geometric group theory: the number of ends (cf. Section 2.2), the rational discrete cohomological dimension (cf. Section 2.3), and the Euler–Poincaré characteristic (cf. Section 2.4). The last section of the chapter, namely Section 2.5, presents a generalisation of the Stallings–Swan theorem to t.d.l.c. groups. The proof of the theorem exemplifies how using invariants of a group can be helpful in deducing structural information on the group itself.

Chapter 3 focuses on the double-coset zeta functions. We first introduce the setup of the counting problem from which these growth series arise (cf. Sections 3.2 and 3.3). Then we introduce the relevant zeta functions (cf. Section 3.4), providing – in two prescribed classes of groups – explicit formulae (cf. Sections 3.4.1 and 3.4.2) and a description of the value at $s = -1$ in terms of the Euler–Poincaré characteristic of the group (cf. Section 3.5). The two main classes of groups we focus on are: groups having a Bruhat decomposition (or, equivalently, groups acting Weyl-transitively on buildings) and groups with an action on a tree which is weakly locally ∞-transitive or (P)-closed. In those cases, we describe the relevant zeta functions in terms of accessible local data of the action. This description allows us to find a connection with another zeta function arising from graphs: the weighted Ihara zeta function (cf. Section 3.5.3).

Finally, Chapter 4 is concerned with another growth series: the submodule zeta functions of algebras of endomorphisms. In the introduction, we have sketched how they generalise the concept of submodule zeta functions of unipotent matrix groups (which, however, will be basically the only case we consider in the chapter). In Section 4.2 we introduce the main concepts and briefly contextualise the relevant zeta functions in the world of zeta functions associated to groups (cf. Section 4.2.2). In Section 4.3 we focus on the case in which the ring of coefficients is the ring of integers of a non-Archimedean local field. Therein we delineate a sufficient condition to ensure the validity of a conjecture of T. Rossmann (cf. Section 4.3.2), supporting this condition with examples (cf. Section 4.3.1). In Section 4.4, we focus on the case in which the ring of coefficients is a finite field $\mathbb{F}_q$. Under this assumption, we provide an explicit formula in the case of algebras generated by elementary matrices, deducing also a uniformity result (cf. Section 4.4.5).

At the beginning of each chapter, we give a more detailed overview of the structure, providing references to the main results.

Associated preprints

The thesis is partially based on three preprints: two of them written in collaboration with I. Castellano and T. Weigel [CMW24b; CMW24a], and one by the author [Mar24]. They cover the original contributions of the first three chapters of the thesis. More precisely, regarding Chapter 1, Sections 1.3.3 and 1.3.5 appear in the author's paper [Mar24], while Section 1.7 appears in [CMW24a]. Regarding Chapter 2, Sections 2.2 and 2.3 collect results from [CMW24a], while Sections 2.4 and 2.5 follow [CMW24b]. Moreover, in Chapter 3, all the results concerning groups acting on trees are taken after [Mar24].

Conventions

Here below, we list some general notation and conventions that we follow throughout the thesis.

About sets

- For $n \in \mathbb{Z}_{\geq 1}$, denote by $[n]$ the set $\{1, \dots, n\}$. Moreover, set

$$\begin{aligned} [n]^2 &= [n] \times [n]; \\ [n]^2_{\neq} &= \{(i,j) \in [n]^2 \mid i \neq j\}; \\ [n]^2_{\leq} &= \{(i,j) \in [n]^2 \mid i \leq j\}; \\ [n]^2_{<} &= [n]^2_{\neq} \cap [n]^2_{\leq}. \end{aligned}$$

- The disjoint union operator is denoted by $\sqcup$.
- The inclusion operator is denoted by $\subseteq$, and the strict inclusion operator is denoted either as $\subset$ or as $\subsetneq$.
- Given a set X, for every subset $A \subseteq X$ denote by $\mathbb{1}_A \colon X \to \{0,1\}$ the indicator function of A.
- Whenever there is no ambiguity, we may denote the 1-point set $\{x\}$ by the element x itself. For instance, we may write $\mathbb{1}_x$ instead of $\mathbb{1}_{\{x\}}$.
- Given a set X, denote by $\mathcal{P}(X)$ the power set of X. Moreover, for every $n \in \mathbb{Z}_{\geq 0}$, denote by $\mathcal{P}_n(X)$ the set of all subsets of X of size n.

About (topological) groups

- For the coproduct operator in the category of groups, we use the symbols $*$ and $\amalg$ interchangeably.
- Every topological group is by default assumed to be Hausdorff.
- "l.c." means "locally compact", and "t.d.l.c." means "totally disconnected locally compact".

- given a l.c. group G and a compact open subgroup $K \leq G$, we denote by μ_K the left-Haar measure on G such that $\mu_K(K) = 1$.

About group actions on sets

Let X be a non-empty set.

- Denote by $\mathrm{Sym}(X)$ the *symmetric group* of X, i.e., the group of all self-bijections of X. By default, we regard $\mathrm{Sym}(X)$ as a topological group with respect to the *permutation topology*, i.e., the roughest group topology on $\mathrm{Sym}(X)$ for which the stabiliser of each finite subset of X is open. If X is finite, this topology is exactly the discrete group topology, as $\{\mathrm{id}_X\}$ (and so every singleton) is an open subset of $\mathrm{Sym}(X)$.

- Given a topological group G, a *G-action* on X is a continuous group homomorphism $\sigma\colon G \to \mathrm{Sym}(X)$. We usually do not specify σ while giving a group action and, for all $g \in G$ and $x \in X$, we usually write $g \cdot x$ instead of $\sigma(g)(x)$. For every $x \in X$, we denote by G_x the stabiliser of x in G. More generally, for all non-empty subsets $S \subseteq X$, we denote by G_S the pointwise stabiliser of S in G.

- Let G and H be topological groups with actions $\sigma_G\colon G \to \mathrm{Sym}(X)$ and $\sigma_H\colon H \to \mathrm{Sym}(Y)$ on the sets X and Y, respectively. The actions σ_G and σ_H are said to be *permutational isomorphic* if there are an isomorphism of topological groups $\varphi\colon \sigma_G(G) \to \sigma_H(H)$ and a bijection $f\colon X \to Y$ satisfying $f(g \cdot x) = \varphi(g) \cdot f(x)$, for all $x \in X$ and $g \in \sigma_G(G)$.

About matrix algebras

Given a commutative ring R and an integer $n \geq 1$, we always regard $\mathrm{Mat}_n(R)$ as an associative non-unital R-algebra with respect to the entry-wise addition and the row-by-column multiplication. Thus, a *subalgebra* of $\mathrm{Mat}_n(R)$ is an additive subgroup of $\mathrm{Mat}_n(R)$ that is closed under the row-by-column product of its elements. Even if R is a unital ring, in our setting a subalgebra of $\mathrm{Mat}_n(R)$ might not have a unit.

About graphs

By default, every graph is intended to be in the sense of J-P. Serre (cf. Section 1.2.1). While drawing Serre-graphs, for simplicity we usually write the vertices and edges belonging to a certain edge orientation.

About sequences

A sequence of complex numbers $(a_n)_{n\geq 1}$ is said to *grow polynomially* if there is $\alpha \in \mathbb{R}_{\geq 0}$ such that $a_n = O(n^\alpha)$ for every $n \geq 1$.

Chapter 1

Preliminaries

1.1 Structure of the chapter

In this chapter, we introduce the main concepts studied in the thesis and put the basis for the next chapters.

In Section 1.2, we introduce three notions of *graphs*, i.e., Serre graphs, undirected and directed graphs, and how these definitions interplay with each other. If we do not specify any adjective, a "graph" is meant to be a Serre graph.

Section 1.3 deals with *group actions on graphs*. After some general definitions (cf. Sections 1.3.1 and 1.3.2), we entirely focus on group actions on trees, which are one of the main sources of examples occurring in the thesis. We introduce two classes of actions on trees that will play a relevant role in computing the double-coset zeta functions of groups acting on trees (cf. Chapter 3). The first class consists of (P)-closed actions on trees (cf. Section 1.3.3): it is a well-known class in the literature, being a notable source of examples of simple groups acting on trees (cf. [Tit70, Théorème 4.5], [BEW15, Theorem 7.3] or [RS22, Theorem 1.8]). The second class has been introduced by the author in [Mar24] as a generalisation of the well-known class of locally ∞-transitive actions on trees (cf. Section 1.3.5). In Section 1.3.4, we recall the concept of a local action diagram, which is a local data set attached to the action that has been introduced by C. Reid and S. Smith [RS22] and which will be widely used in Chapter 3. In Section 1.3.6 we recall the basics of Bass–Serre theory for t.d.l.c. groups which recur above all in Chapter 2.

In Section 1.4 we introduce background notions about *Coxeter groups*. After recalling some well-established concepts, we deal with the more recent notions of ∞-decompositions and visual graph of groups decompositions (cf. Section 1.4.2). The latter concepts play a role in constructing trees from buildings (cf. Section 1.7) and can be used to characterise certain invariants or properties of Coxeter groups.

In Sections 1.5 and 1.6 we introduce the basic concepts regarding buildings and groups acting on them. The sections mainly consist of well-known definitions and facts. The only exception is Section 1.7, introduced in [CMW24a] by the author and I. Castellano and T. Weigel. This section gives a method to construct trees from buildings whose type admits a visual graph of groups decomposition and to translate group actions on these buildings to group actions on the induced tree. This construction, however, might be regarded as a compendium to the general theory of buildings, and it does not play a crucial role in the next chapters of the thesis.

Section 2.3 gives the bases of the rational discrete cohomology theory for t.d.l.c. groups. It is the only cohomology theory used in this thesis (mainly in Chapter 2). Finally, Section 1.9 introduces basic concepts regarding the Dirichlet series and gives the general background for Chapters 3 and 4.

1.2 Graphs

In this thesis, we will mainly deal with *Serre graphs* (or graphs in the sense of J-P. Serre, cf. Section 1.2.1). This is why we develop more terminology for them. By default, a graph is a Serre graph. However, in certain situations, it suffices to consider another notion of graph, namely the one of *undirected graph* (cf. Section 1.2.2). In Section 1.2.3, we recall how to associate an undirected graph with a Serre graph and vice-versa.

1.2.1 Serre graphs

A *graph* (in the sense of J-P. Serre [Ser80]) consists of a set $\Gamma = V\Gamma \sqcup E\Gamma$ partitioned into two subsets $V\Gamma$ and $E\Gamma$ (called the *set of vertices* and the *set of edges* of Γ, respectively), together with two maps $o, t\colon E\Gamma \to V\Gamma$ (called *origin* and *terminus* maps, respectively) and an involution $\bar{\cdot}\colon E\Gamma \to E\Gamma$ (called *edge inversion*) satisfying $\bar{e} \neq e$ and $o(\bar{e}) = t(e)$, for every $e \in E\Gamma$. For a graph Γ, we introduce the following notation.

Notation 1.2.1. Given $u \in \Gamma$, we use the associated capital letter U to denote the set $\{u\}$ if $u \in V\Gamma$, and the set $\{u, \bar{u}\}$ if $u \in E\Gamma$.

An *orientation* in a graph Γ is a set $E\Gamma^+ \subseteq E\Gamma$ satisfying $|\{e, \bar{e}\} \cap E\Gamma^+| = 1$, for every $e \in E\Gamma$, and $E\Gamma = E\Gamma^+ \cup \{\bar{e} \mid e \in E\Gamma^+\}$. A *subgraph* of a graph Γ is a subset Λ such that $o(E\Gamma \cap \Lambda), t(E\Gamma \cap \Lambda) \subseteq V\Gamma \cap \Lambda$ and $\overline{e} \in E\Gamma \cap \Lambda$ for every $e \in E\Gamma \cap \Lambda$. The subset Λ inherits a graph structure from Γ. A subgraph Λ of Γ is *proper* if $\Lambda \neq \Gamma$. A vertex v in Γ is said to be *terminal* if $|o^{-1}(v)| = 1$. An edge e in Γ with $o(e) = t(e)$ is called 1-*loop*. A n-*bouquet of loops (based at c)* is a graph with one vertex c and edge-set $\{a_i, \bar{a}_i \mid 1 \leq i \leq n\}$, where each a_i is a 1-loop starting at c. A 1-*segment* is a graph Γ with two distinct vertices and an edge-couple $\{e, \bar{e}\}$ connecting them.

Given two graphs Γ and Λ, a *graph morphism* is a map $\varphi\colon \Gamma \to \Lambda$ satisfying $\varphi(V\Gamma) \subseteq V\Lambda$, $\varphi(E\Gamma) \subseteq E\Lambda$, $\varphi(o(e)) = o(\varphi(e))$ and $\varphi(\bar{e}) = \overline{\varphi(e)}$ for every $e \in E\Gamma$. A *graph monomorphism* (resp. *epimorphism*, *isomorphism*) is a graph morphism which is injective (resp. surjective, bijective). Given a graph Γ, let $\mathrm{Aut}(\Gamma)$ be the group of all automorphisms (= self-isomorphisms) of Γ. We always regard $\mathrm{Aut}(\Gamma)$ as a topological group with the subspace topology induced by $\mathrm{Sym}(\Gamma)$.

Paths

Let Γ be a graph. A *path* in Γ is a sequence of vertices and edges $\mathfrak{p} = (v_0, e_1, v_1, \dots, e_n, v_n)$, $n \geq 0$, with $o(e_i) = v_{i-1}$ and $t(e_i) = v_i$ for every $1 \leq i \leq n$. We say that $\mathfrak{p}$ starts at v_0 (or at e_1) and ends at v_n (or at e_n), has *reverse path* is $\overline{\mathfrak{p}} = (v_n, \bar{e}_n, v_{n-1}, \dots, \bar{e}_1, v_0)$, and length n (written $\ell(\mathfrak{p}) = n$). If $n \geq 1$, we may without ambiguity specify only the sequence of edges of $\mathfrak{p}$. If $n = 0$, the 1-term sequence $\mathcal{O}_{v_0} = (v_0)$ is called the *trivial path at* v_0. Denote by $\mathcal{P}_\Gamma$ the set of all paths in Γ. Given non-empty subsets $X, Y \subseteq \Gamma$, let $\mathcal{P}_\Gamma(X \to Y)$ be the set of all paths in Γ starting at some $x \in X$ and ending at some $y \in Y$. The *product* of two paths $\mathfrak{p} = (e_1, \dots, e_m)$ and $\mathfrak{q} = (f_1, \dots, f_n)$ is defined only if $t(e_m) = o(f_1)$ and it is the path $\mathfrak{p} \cdot \mathfrak{q} = (e_1, \dots, e_m, f_1, \dots, f_n)$. If $\mathfrak{p}$ is a path starting and ending at the same vertex, denote by $\mathfrak{p}^d$ the d-th power of $\mathfrak{p}$ with respect to the product defined before. A path $\mathfrak{p}$ is *reduced* if either $\ell(\mathfrak{p}) = 0$ or $\mathfrak{p} = (e_1, \dots, e_n)$ and $e_{i+1} \neq \bar{e}_i$ for every $1 \leq i \leq n-1$. For $n \geq 1$, an *n-cycle* is a reduced path $\mathfrak{p} = (e_1, \dots, e_n)$ with $o(e_1) = t(e_n)$ and $t(e_i) \neq t(e_j)$ for all $1 \leq i, j \leq n$ with $i \neq j$.

Finally, a *ray* in a graph is a sequence of edges $(e_i)_{i \in \mathbb{Z}_{\geq 1}}$ such that $o(e_i) \neq o(e_j)$ and $t(e_i) = o(e_{i+1})$, for all $i, j \in \mathbb{Z}_{\geq 1}$ with $i \neq j$.

Properties on graphs

A graph Γ is:

- *non-empty* if $V\Gamma \neq \emptyset$;
- *locally finite* if $|o^{-1}(c)| < \infty$ for all $c \in V\Gamma$;
- *combinatorial* if the map $e \in E\Gamma \mapsto (o(e), t(e)) \in V\Gamma \times V\Gamma$ is injective or, in other words, if every edge is uniquely determined by its origin and its terminus vertices.
- *connected* if for all $v, w \in V\Gamma$ there is a path from v to w. A subgraph of Γ is a *connected component* if it is a maximal connected subgraph of Γ. A graph is the disjoint union of all its connected components.
- a *tree* if it is connected and has no n-cycles, for every $n \geq 1$.

Given a connected graph Γ, the set of vertices $V\Gamma$ is endowed with a metric $\mathrm{d}\colon V\Gamma \times V\Gamma \to \mathbb{Z}_{\geq 0}$, called *geodesic distance* and defined on all $v, w \in V\Gamma$ as follows:

$$\mathrm{d}(v, w) = \min\{\ell(\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{P}_\Gamma(v \to w)\}. \tag{1.2.1}$$

If T is a tree and $e \in ET$, then the graph $T \setminus \{e, \bar{e}\}$ has two connected components, $T_e^+ \ni t(e)$ and $T_e^- \ni o(e)$. Set $T_{\geq e} := T_e^+ \sqcup \{e\}$ and $T_{\geq \bar{e}} := T_e^- \sqcup \{\bar{e}\}$.

A tree T is *uniquely geodesic*, i.e., for all $v, w \in VT$ there is a unique reduced path $[v, w]$ from v to w, which we call *geodesic from v to w*. Recall that $[v, w]$ is the path of minimal length in T from v to w. Moreover, given $e, f \in ET$, there is a geodesic $(e_1, \ldots, e_n)$ in T with $e_1 = e$ and $e_n = f$ if, and only if, $f \in T_{\geq e}$. In general, for $t_1, t_2 \in T$ we denote by $[t_1, t_2]$ the geodesic from t_1 to t_2 in T (if it exists). Moreover, for non-empty subsets $X, Y \subseteq T$, denote by $\mathrm{Geod}_T(X \to Y)$ the set of all geodesics in T from some $x \in X$ to some $y \in Y$.

Remark 1.2.2. Let Γ be a connected graph without n-cycles for every $n \geq 2$. Then Γ has a unique maximal subtree: the subgraph Λ obtained from Γ by removing all its 1-loops. In particular, for all $v, w \in V\Gamma$ the geodesic $[v, w] = (v = v_0, e_1, v_1, \ldots, e_n, v_n = w)$ in Λ is the path of minimal length in Γ from v to w. Thus, if $v \neq w$ then $v_i \neq v_j$ for all $0 \leq i, j \leq n$ with $i \neq j$.

1.2.2 (Un)directed graphs

An *undirected graph* (V, E) consists of a non-empty set of vertices V together with a set of edges $E \subseteq \mathcal{P}_2(V)$. A *subgraph* (U, F) of (V, E) is an undirected graph such that $U \subseteq V$ and $F \subseteq E$. An undirected graph (V, E) is said to be *complete* if $E = \mathcal{P}_2(V)$. A complete subgraph (U, F) of an undirected graph (V, E) satisfying $|U| = n$ is said to be an *n-clique*. A subgraph (U, F) of an undirected graph (V, E) is said to be *induced* if $F = \mathcal{P}_2(U) \cap E$. For instance, any n-clique is an induced subgraph. A *path* in (V, E) is a finite sequence of vertices $(v_0, \ldots, v_n)$ such that $\{v_i, v_{i+1}\} \in E$ for every $0 \leq i \leq n - 1$.

In Section 4.2.5 (and only there), we consider a third notion of graphs, i.e., *directed graphs*. A directed graph $(V, \vec{E})$ consists of a set V (called the set of vertices) and a set $\vec{E} \subseteq V \times V$ (called the set of edges). Every $(v, w) \in \vec{E}$ has origin v and terminus w. However, unlike Serre graphs, given $(v, w) \in \vec{E}$ we might not have $(w, v) \in \vec{E}$.

1.2.3 From undirected graphs to Serre graphs and vice-versa

Every undirected graph $\Gamma = (V, E)$ defines a combinatorial Serre graph $\vec{\Gamma}$ by taking $V\vec{\Gamma} := V$ and

$$E\vec{\Gamma} := \{(v, w), (w, v) \mid \{v, w\} \in E\},$$

by setting the origin and terminus maps as the projections on the first and the second coordinate, respectively, and by setting edge-inversion map $\bar{\cdot} \colon \vec{E} \to \vec{E}$ as the map interchanging the first and second coordinates.

Conversely, every combinatorial Serre graph $\Gamma = V\Gamma \sqcup E\Gamma$ defines an undirected graph $\ddot{\Gamma} = (V, E)$ by setting $V := V\Gamma$ and

$$E := \big\{\{o(e), t(e)\} \mid e \in E\Gamma\big\}.$$

1.3 Groups acting on graphs

1.3.1 Background notions

Let Γ be a graph, and let G be a topological group. A *G-action* (G,Γ) *on* Γ is a continuous group homomorphism $G \to \mathrm{Aut}(\Gamma)$. The action (G,Γ) is said to be *without edge-inversions* if $g \cdot e \neq \bar{e}$, for all $g \in G$ and $e \in E\Gamma$. Moreover, (G,Γ) is said to be *without global fixed points* if, for all $v \in V\Gamma$, there is $g \in G$ such that $g \cdot v \neq v$.

Given $\gamma \in \Gamma$, denote by G_γ the stabiliser of γ in G. More generally, for every subset $X \subseteq \Gamma$, let G_X denote the pointwise stabiliser of X. If $\mathfrak{p} = (e_i)_{1 \leq i \leq n}$ is a path, $G_{\mathfrak{p}}$ denotes the pointwise stabiliser of the set $\{e_1, \ldots, e_n\}$. The action (G,Γ) is said to be *proper* (resp. *cocompact*) if G has compact open vertex-stabilisers (resp. the action has finitely orbits on both $V\Gamma$ and $E\Gamma$).

An action (G,Γ) is *edge-transitive* if $E\Gamma = G\cdot e \sqcup G\cdot \bar{e}$ for some (and hence every) $e \in E\Gamma$. Moreover, if Γ is a tree, (G,Γ) is *locally ∞-transitive* if, for every $v \in V\Gamma$ and $d \geq 0$, the stabiliser G_v acts transitively on $\{\mathfrak{p} \in \mathrm{Geod}_\Gamma(v \to \Gamma) \mid \ell(\mathfrak{p}) = d\}$ (cf. [BM00, §0.2]). One checks that locally ∞-transitive actions are edge-transitive. Examples of groups admitting a locally ∞-transitive action on a tree are the k-points of algebraic k-groups of relative rank 1, where k is a non-Archimedean local field (cf. [Ser80, pp. 91 and 95]), and the Burger–Mozes universal groups $U(F)$ associated to 2-transitive groups $F \leq \mathrm{Sym}(\{1,\ldots,d\})$ acting on a d-regular tree (cf. the lines before [BM00, §3.1]).

1.3.2 The quotient graph and its standard edge weight

Let $(G,\tilde{\Gamma})$ be a group action on a graph without edge inversions. The *quotient graph* $\Gamma = G\backslash\tilde{\Gamma}$ of the action is the graph with $V\Gamma := G\backslash V\tilde{\Gamma}$, $E\Gamma := G\backslash E\tilde{\Gamma}$ and, given $G\cdot e \in E\Gamma$, its origin is $G \cdot o(e)$, its terminus is $G \cdot t(e)$ and its inverse edge is $G \cdot \bar{e}$. One checks that these definitions are independent of the choice of e in $G \cdot e$. Note that if $\tilde{\Gamma}$ is connected, so is Γ.

The assignment $\pi\colon \gamma \in \tilde{\Gamma} \longmapsto G \cdot \gamma \in \Gamma$ yields a graph epimorphism which is called *quotient map* of (G,T). The map π entrywise extends to a map (denoted with the same symbol) from the set of all paths in $\tilde{\Gamma}$ to the set of all paths in Γ.

The *standard edge weight* on Γ is the map $\omega\colon E\Gamma \to \mathbb{Z}_{\geq 1} \cup \{\infty\}$ defined on every $a \in E\Gamma$ by choosing $v \in V\tilde{\Gamma}$ with $\pi(v) = o(a)$ and setting

$$\omega(a) := |\{e \in E\tilde{\Gamma} \,:\, o(e) = v \text{ and } \pi(e) = a\}|. \tag{1.3.1}$$

In other words, $\omega(a)$ counts how many edges in $\tilde{\Gamma}$ starting at v lift a via π. It is straightforward to check that the assignment in (1.3.1) does not depend on the choice of the vertex v.

Starting from a connected graph Γ and a function $\omega\colon E\Gamma \to \mathbb{Z}_{\geq 1}$, the following example shows how to construct a group action on a tree having quotient graph Γ and standard edge weight ω.

Example 1.3.1. Let Γ be a connected graph with a function $\omega\colon E\Gamma \to \mathbb{Z}_{\geq 1}$ (called *edge weight*). Following [Car18, §3.5], the pair (Γ, ω) admits an essentially unique universal cover (T, π), which consists of a tree T and a graph epimorphism $\pi\colon T \to \Gamma$ with the following property: for all $a \in E\Gamma$ and $v \in VT$ with $\pi(v) = o(a)$, the number of edges $e \in ET$ with origin v and $\pi(e) = a$ is exactly $\omega(a)$. The *group of deck transformations* of (Γ, ω) is

$$\mathrm{Aut}_\pi(T) := \{\varphi \in \mathrm{Aut}(T) \mid \pi \circ \varphi = \pi\}.$$

The group $\mathrm{Aut}_\pi(T)$ is t.d.l.c. with respect to the subspace topology induced by $\mathrm{Aut}(T)$. Since $\bar{a} \neq a$ for every $a \in E\Gamma$, note that $\mathrm{Aut}_\pi(T)$ acts on T without edge inversions.

1.3.3 (P)-closed actions on trees

The study of the group $\mathrm{Aut}_\pi(T)$ as in Example 1.3.1 initiated the study of (P)-closed group actions on trees (cf. [BEW15]), a class which stems from the more general class of group actions on trees with the Tits' independence property (cf. [Tit70, §4.2]).

Definition 1.3.2. A group action on a tree (G, T) is *(P)-closed* if $G \leq \mathrm{Aut}(T)$ is closed and, for every $e \in ET$,

$$G_e = G_{T_{\geq \bar{e}}} \cdot G_{T_{\geq e}}. \tag{1.3.2}$$

Recall from Section 1.2.1 that $T_{\geq e} = \{e\} \cup T_e^+$ and $T_{\geq \bar{e}} = \{\bar{e}\} \cup T_e^-$, where T_e^+ and T_e^- are the connected components of $T \setminus \{e, \bar{e}\}$ containing $t(e)$ and $o(e)$, respectively.

In (1.3.2), note that the inclusion $\supseteq$ is automatic and the product is an inner direct product.

Proposition 1.3.3. *Let (G, T) be a (P)-closed action on a tree and $(e_1, \ldots, e_n)$ be a geodesic in T of length $n \geq 2$. Then, for every $k < n$, we have*

$$G_{(e_1,\ldots,e_k)} \cdot (e_{k+1}, \ldots, e_n) = G_{e_k} \cdot (e_{k+1}, \ldots, e_n). \tag{1.3.3}$$

Proof. The inclusion $\subseteq$ is clear. Moreover, since $e_1, \ldots, e_{k-1} \in ET_{\geq \bar{e}_k}$ and $e_{k+1}, \ldots, e_n \in ET_{\geq e_k}$, we have

$$\begin{aligned} G_{(e_1,\ldots,e_k)} \cdot (e_{k+1}, \ldots, e_n) &\supseteq G_{T_{\geq \bar{e}_k}} \cdot (e_{k+1}, \ldots, e_n) \\ = G_{T_{\geq \bar{e}_k}} \cdot G_{T_{\geq e_k}} \cdot (e_{k+1}, \ldots, e_n) &= G_{e_k} \cdot (e_{k+1}, \ldots, e_n). \end{aligned}$$

□

C. Reid and S. Smith [RS22] provide a parametrisation of (P)-closed group actions on trees in terms of *local action diagrams*, as recalled in Section 1.3.4.

1.3.4 Local action diagrams and their associated universal groups

Definition 1.3.4 ([RS22, Definition 3.1])**.** A *local action diagram* is a triple

$$\Delta = (\Gamma, (X_a)_{a\in E\Gamma}, (G(c))_{c\in V\Gamma})$$

consisting of the following data:

(i) a connected graph Γ;

(ii) a family of non-empty pairwise disjoint sets $(X_a)_{a\in E\Gamma}$. For every $c \in V\Gamma$, set $X_c := \bigsqcup_{a\in o^{-1}(c)} X_a$ and $X := \bigsqcup_{a\in E\Gamma} X_a$;

(iii) for every $c \in V\Gamma$, a closed subgroup $G(c)$ of $\mathrm{Sym}(X_c)$ whose orbits are given by $G(c)\backslash X_c = \{X_a\}_{a\in o^{-1}(c)}$.

Notation 1.3.5. If there is no ambiguity, we write $\Delta = (\Gamma, (X_a), (G(c)))$ in place of $\Delta = (\Gamma, (X_a)_{a\in E\Gamma}, (G(c))_{c\in V\Gamma})$. Moreover, given $u \in \Gamma$, set $X_U := X_u$ if $u \in V\Gamma$ and $X_U := X_u \sqcup X_{\bar{u}}$ if $u \in E\Gamma$.

Local action diagrams can be constructed from a group action on a tree (G, T) as follows.

Definition 1.3.6 ([RS22, Definition 3.6])**.** Let π be the quotient map on (G, T) and choose a set of representatives V^* of the G-orbits on VT. The *local action diagram associated to* (G, T) *and* V^* is defined as follows:

(i) $\Gamma = G\backslash T$ is the quotient graph of (G, T);

(ii) for every $a \in E\Gamma$, let $v^* \in V^*$ be such that $\pi(v^*) = o(a)$ and define

$$X_a := \{e \in ET \mid o(e) = v^* \text{ and } \pi(e) = a\};$$

(iii) for every $c \in V\Gamma$ with representative $c^* \in V^*$, let $G(c)$ be the closure in $\mathrm{Sym}(X_c)$ of the permutation group induced by G_{c^*} acting on $X_c = \bigsqcup_{a\in o^{-1}(c)} X_a$.

Note that the standard edge weight on the quotient graph Γ is given by $\omega(a) = |X_a|$, for every $a \in E\Gamma$. Up to isomorphism of local action diagrams (cf. [RS22, Definition 3.2]), every group action on a tree (G, T) has a unique associated local action diagram (cf. [RS22, Lemma 3.7]). Thus, we refer to *the* local action diagram associated to (G, T). Moreover, one of the key results in [RS22] (recalled in Theorem 1.3.17) shows that every local action diagram arises as the local action diagram associated to a group action on a tree.

Example 1.3.7.

(i) Let Γ be a connected graph with an edge weight $\omega\colon E\Gamma \to \mathbb{Z}_{\geq 1}$. Let T and $G = \mathrm{Aut}_\pi(T)$ as in Example 1.3.1. Then the local action diagram $\Delta = (\Gamma, (X_a), (G(c)))$ associated to (G, T) is given by taking, for every $a \in E\Gamma$, a set X_a of cardinality $\omega(a)$, and by setting

$$G(c) := \{\sigma \in \mathrm{Sym}(X_c) \mid \forall\, a \in o^{-1}(c),\, \sigma(X_a) = X_a\},$$

for every $c \in V\Gamma$. By design, $G(c)\backslash X_c = \{X_a\}_{a \in o^{-1}(c)}$. More precisely, for every $a \in o^{-1}(c)$, the $G(c)$-action on X_a is permutational isomorphic to the action of $\mathrm{Sym}(X_a)$ on the same set.

Note that $G(c)$ is closed in $\mathrm{Sym}(X_c)$. In fact, if $\sigma \in \mathrm{Sym}(X_c)$ satisfies $\sigma(x) \notin X_a$ for some $x \in X_a$, then $\sigma \cdot \mathrm{Sym}(X_c)_x$ is an open neighbourhood of σ in $\mathrm{Sym}(X_c)$ which is contained in $\mathrm{Sym}(X_c) \setminus G(c)$.

(ii) Given a prime p, denote by $\mathbb{Q}_p$ the field of p-adic numbers, by $\mathbb{Z}_p$ the ring of p-adic integers, and by $\mathbb{F}_p \simeq \mathbb{Z}_p/p\mathbb{Z}_p$ the field of size p. Here below, we report and rephrase some results from [Ser80, §II.1.4]. The action of $G = \mathrm{SL}_2(\mathbb{Q}_p)$ on its Bruhat–Tits tree T – which is a $(p+1)$-regular tree – is without edge inversions and has a 1-segment as quotient graph Γ. Set $E\Gamma = \{a, \bar{a}\}$. Moreover, there is $e \in ET$ with $G \cdot e = a$ satisfying $G_{o(e)} = \mathrm{SL}_2(\mathbb{Z}_p) \simeq G_{t(e)}$. Both the $\mathrm{SL}_2(\mathbb{Z}_p)$-action on $\{f \in ET \mid o(f) = o(e)\}$ and the $G_{t(e)}$-action on $\{f \in ET \mid o(f) = o(\bar{e})\}$ are permutational isomorphic to the faithful action of $\mathrm{PSL}_2(\mathbb{F}_p)$ on the projective line $\mathbb{P}^1(\mathbb{F}_p)$. Hence, the local action diagram $\Delta = (\Gamma, (X_b)_{b \in E\Gamma}, (G(c))_{c \in V\Gamma})$ associated to (G, T) is given by setting $X_a = X_{\bar{a}} = \mathbb{P}^1(\mathbb{F}_p)$ and $G(o(a)) = G(t(a)) = \mathrm{PSL}_2(\mathbb{F}_p)$.

We now expand the vocabulary of local action diagrams, introducing tools of key importance in Chapter 3.

Definition 1.3.8. Let $\Delta = (\Gamma, (X_a), (G(c)))$ be a local action diagram. For $n \geq 1$, an *n-path* in Δ is a sequence $\xi = (x_1, \dots, x_n)$ obtained by starting with a path $\mathfrak{p}_\xi = (a_1, \dots, a_n)$ in Γ and selecting, for each $1 \leq i \leq n$, an element $x_i \in X_{a_i}$. One says that ξ starts at x_1, ends with x_n, has *length* n (written $\ell(\xi) = n$), and $\mathfrak{p}_\xi$ is called the *underlying path* of ξ in Γ. The *0-path* O_c in Δ at $c \in V\Gamma$ is the empty sequence of elements in X with the trivial path at c as underlying path in Γ. The path O_c has length zero. A *path* in Δ is an n-path for some $n \geq 0$. Given paths $\xi = (x_1, \dots, x_m)$ and $\eta = (y_1, \dots, y_n)$ in Δ with $\mathfrak{p}_\xi = (a_1, \dots, a_m)$ and $\mathfrak{p}_\eta = (b_1, \dots, b_n)$, the product $\xi \cdot \eta$ is defined only if $t(a_m) = o(b_1)$ and it is the path $(x_1, \dots, x_m, y_1, \dots, y_n)$ in Δ with underlying path $\mathfrak{p}_\xi \cdot \mathfrak{p}_\eta$. Put also $\xi \cdot O_{t(a_m)} = \xi$ and $O_{o(b_1)} \cdot \eta = \eta$. Given a path ξ and a non-empty set of paths $\mathcal{E}$ in Δ such that the product $\xi \cdot \eta$ is defined for every $\eta \in \mathcal{E}$, set $\xi \cdot \mathcal{E} = \{\xi \cdot \eta \mid \eta \in \mathcal{E}\}$. If $\xi = (x)$ has length 1, we write $x \cdot \mathcal{E}$ in place of $(x) \cdot \mathcal{E}$.

A map $\iota\colon X \to X$ is said to be an *inversion* in Δ if $\iota(X_a) \subseteq X_{\bar{a}}$ for every $a \in E\Gamma$. A path ξ in Δ is *reduced* in (Δ, ι) if either $\ell(\xi) = 0$ or $\xi = (x_1, \dots, x_n)$, for some $n \geq 1$, and $x_{i+1} \neq \iota(x_i)$ for every $1 \leq i \leq n-1$. Note that, even if ξ is reduced, the underlying path $\mathfrak{p}_\xi$ needs not to be reduced. Denote by $\mathcal{P}_{(\Delta,\iota)}$ the set of all reduced paths in (Δ, ι). For non-empty subsets $X_1, X_2 \subseteq X$, let also $\mathcal{P}_{(\Delta,\iota)}(X_1 \to X_2)$ be the collection of all reduced paths in (Δ, ι) starting at some $x_1 \in X_1$ and ending at some $x_2 \in X_2$.

An inversion in a local action diagram is *not* required to be an involution. In fact, the sizes of X_a and $X_{\bar{a}}$ might differ. The term "inversion" here refers to the edge inversion on the labels in the partition $X = \bigsqcup_{a \in E\Gamma} X_a$.

Definition 1.3.9 ([RS22, Definition 3.4])**.** Let $\Delta = (\Gamma, (X_a), (G(c)))$ be a local action diagram. A Δ-*tree* $(T, \pi, \mathcal{L})$ consists of a tree T, a graph epimorphism $\pi \colon T \to \Gamma$ and a map $\mathcal{L} \colon ET \to X$ which restricts, for all $v \in VT$ and $a \in E\Gamma$ with $o(a) = \pi(v)$, to a bijection

$$\mathcal{L}_{v,a} \colon \{e \in ET \mid o(e) = v \text{ and } \pi(e) = a\} \longrightarrow X_a.$$

In particular, for every $v \in VT$, the map $\mathcal{L}$ restricts to a bijection

$$\mathcal{L}_v \colon o^{-1}(v) \longrightarrow X_{\pi(v)}. \tag{1.3.4}$$

Note that the definition of a Δ-tree is independent of $(G(c))_{c \in V\Gamma}$.

According to [Car18, §3.5], the pair (T, π) is the universal cover of the edge-weighted graph (Γ, ω), where $\omega(a) = |X_a|$ for every $a \in E\Gamma$ (cf. Example 1.3.1). Therefore, for every two Δ-trees $(T, \pi, \mathcal{L})$ and $(T', \pi', \mathcal{L}')$ there is a graph isomorphism $\varphi \colon T \to T'$ such that $\pi = \pi' \circ \varphi$ (cf. [RS22, Lemma 3.5]). Moreover, for every Δ-tree we may define $\mathrm{Aut}_\pi(T)$ as in Example 1.3.1.

The standard Δ-tree associated to ι and c_0

Let $\Delta = (\Gamma, (X_a), (G(c)))$ be a local action diagram. Following the proof of [RS22, Lemma 3.5], we recall the construction of an explicit family of Δ-trees which plays an important role in Chapter 3.

Set an inversion $\iota \colon X \to X$ in Δ and $c_0 \in V\Gamma$, and define a graph $T = T(\Delta, \iota, c_0)$ as follows: The set of vertices of T is

$$VT := \mathcal{P}_{(\Delta,\iota)}(X_{c_0} \to X). \tag{1.3.5}$$

The vertex $v_0 = O_{c_0}$ is called the *root of* T. The edges of T are the pairs (v, w) and (w, v) of reduced paths in Δ of the form $v = (x_1, \dots, x_n)$ and $w = (x_1, \dots, x_n, x_{n+1})$, for some $n \geq 0$. Every edge (v, w) of T as before is said to be a *positive edge*. Denote by ET^+ the set of all positive edges of T. The origin, terminus and inversion maps of T are given by $o(v, w) = v$, $t(v, w) = w$ and $\overline{(v, w)} = (w, v)$, for every $(v, w) \in ET$. For every $(v, w) \in ET^+$ with $w = (x_1, \dots, x_{n+1})$, set $\mathcal{L}(v, w) = x_{n+1}$ and $\mathcal{L}(w, v) = \iota(x_{n+1})$.

Remark 1.3.10. Every $e \in ET(\Delta, \iota, c_0)^+$ satisfies $\mathcal{L}(\bar{e}) = \iota(\mathcal{L}(e))$. This is generally not true if $e \in ET(\Delta, \iota, c_0) \setminus ET(\Delta, \iota, c_0)^+$, as a given $x \in X$ might differ from $\iota(\iota(x))$.

More generally, for every path $\mathfrak{p} = (e_1, \dots, e_n)$ in $T = T(\Delta, \iota, c_0)$ we define

$$\mathcal{L}(\mathfrak{p}) := (\mathcal{L}(e_1), \dots, \mathcal{L}(e_n)). \tag{1.3.6}$$

Remark 1.3.11. By Remark 1.3.10, a path $\mathfrak{p} = (e_1, \dots, e_n)$ in T with $e_1, \dots, e_n \in ET^+$ is reduced if, and only if, $\mathcal{L}(\mathfrak{p})$ is reduced in (Δ, ι). Moreover, by (1.3.5), for every $v = (x_1, \dots, x_n) \in VT$ there is a unique reduced path from v_0 to v in T, namely $\mathcal{O}_{c_0}$ if $v = v_0$, and $(v_0, e_1, v_1, \dots, e_n, v_n = v)$ with $v_i = (x_1, \dots, x_i)$ for every $1 \leq i \leq n$ otherwise. In particular, T is connected.

More precisely, T is a tree. In fact, if T admits a n-cycle γ for some $n \geq 1$, there is a vertex v of γ such that the reduced path $\mathfrak{p}$ from v_0 to v does not share edges with γ. Up to an edge-relabelling, we may assume that $\gamma = (e_1, \dots, e_n)$ and $o(e_1) = v$. Then $\mathfrak{p}$ and $\mathfrak{p} \cdot \gamma$ are two distinct reduced paths in T from v_0 to v, which is impossible.

Being T a tree, the map $\mathcal{L}$ in (1.3.6) restricts to a bijection

$$\mathcal{L}^{v_0} \colon \mathrm{Geod}_T(v_0 \to T) \longrightarrow \mathcal{P}_{(\Delta,\iota)}(X_{c_0} \to X). \tag{1.3.7}$$

Remark 1.3.12. By (1.3.6), every edge lying on a geodesic from v_0 in T belongs to ET^+. Indeed, if $(e_1, \dots, e_n) \in \mathrm{Geod}_T(v_0 \to T)$ and $\mathcal{L}(e_1, \dots, e_n) = (x_1, \dots, x_n)$, then $e_i = \big((x_1, \dots, x_{i-1}), (x_1, \dots, x_i)\big)$ for every $1 \leq i \leq n$.

Moreover, if $e_1 \in ET^+$, then for every $e_2 \in o^{-1}(t(e_1)) \setminus \{\bar{e}_1\}$ the path $[v_0, e_1] \cdot e_2$ is a geodesic from v_0 and thus $e_2 \in ET^+$.

We define a graph epimorphism $\pi \colon T \to \Gamma$ by putting $\pi(O_{c_0}) = c_0$ and, provided $v = (x_1, \dots, x_n) \in VT$ $(n \geq 1)$ has underlying path $(a_1, \dots, a_n)$ in Γ, by $\pi(v) = t(a_n)$. The triple $(T = T(\Delta, \iota, c_0), \pi, \mathcal{L})$ is a Δ-tree that we call the *standard Δ-tree associated to ι and c_0*.

Since the map in (1.3.7) is a bijection, we deduce the following.

Lemma 1.3.13. *Consider $e \in ET^+$ with $\mathcal{L}(e) = x$. The map $\mathcal{L}$ in (1.3.6) restricts to the following bijection:*

$$\mathcal{L}^{e} \colon \mathrm{Geod}_T(e \to T) \longrightarrow \mathcal{P}_{(\Delta,\iota)}(x \to X). \tag{1.3.8}$$

If in particular $o(e) = v_0$, the map $\mathcal{L}$ in (1.3.6) restricts also to the following bijection:

$$\mathcal{L}^{\bar{e}} \colon \mathrm{Geod}_T(\bar{e} \to T) \longrightarrow \iota(x) \cdot \mathcal{P}_{(\Delta,\iota)}(X_{c_0} \setminus \{x\} \to X). \tag{1.3.9}$$

Proof. Write $e = (v, w)$, where $w = (x_1, \dots, x_m, x) \in \mathcal{P}_{(\Delta,\iota)}(X_{c_0} \to X)$. Note that $\mathcal{L}([v_0, v]) = (x_1, \dots, x_m)$. The bijection $\mathcal{L}^{v_0}$ in (1.3.7) restricts to a bijective map

$$[v_0, v] \cdot \mathrm{Geod}_T(e \to T) \longrightarrow (x_1, \dots, x_m) \cdot \mathcal{P}_{(\Delta,\iota)}(x \to X).$$

This implies that $\mathcal{L}^e$ is a bijection. Moreover, we observe that

$$\mathrm{Geod}_T(\bar{e} \to T) = \bigsqcup_{f \in o^{-1}(v),\, f \neq e} \bar{e} \cdot \mathrm{Geod}_T(f \to T) \tag{1.3.10}$$

and

$$\iota(x) \cdot \mathcal{P}_{(\Delta,\iota)}(X_{\pi(v)} \setminus \{x\} \to X) = \bigsqcup_{y \in X_{\pi(v)},\, y \neq x} \iota(x) \cdot \mathcal{P}_{(\Delta,\iota)}(y \to X). \tag{1.3.11}$$

If $v = v_0$, then $o^{-1}(v) \subseteq ET^+$. From (1.3.10), (1.3.11) and the first part of the statement we conclude that $\mathcal{L}^{\bar{e}}$ is bijective. □

Remark 1.3.14. Although it is not necessary for the discussion, for every $e = (v, w) \in ET^+$ one may restrict the map $\mathcal{L}$ in (1.3.6) to a bijection $\mathcal{L}^{\bar{e}}$ from $\mathrm{Geod}_T(\bar{e} \to T)$ to a suitable set of paths in (Δ, ι). One proceeds inductively on $\ell([v_0, v]) = l \geq 0$. The case $l = 0$ is done by Lemma 1.3.13. If $l \geq 1$, one assumes the claim true for $l - 1$ and observes that $o^{-1}(v) \setminus \{e\}$ has exactly one edge that does not belong to ET^+: it is the edge whose reverse f_0 is the last edge of $[v_0, v]$. By Lemma 1.3.13, for all $f \in o^{-1}(v) \setminus \{e, \bar{f}_0\}$, the map $\mathcal{L}^f$ as in (1.3.8) is bijective. Moreover, $\ell([v_0, o(f_0)]) = \ell([v_0, v]) - 1$ and by induction one has that $\mathcal{L}^{\bar{f}_0}$ is bijective. Using the decomposition in (1.3.10), one determines the image of $\mathrm{Geod}_T(\bar{e} \to T)$ via $\mathcal{L}$ and thus the bijection $\mathcal{L}^{\bar{e}}$.

The universal group $U(\Delta, \mathbb{T})$

Let Δ be a local action diagram with a Δ-tree $\mathbb{T} = (T, \pi, \mathcal{L})$. For all $g \in \mathrm{Aut}_\pi(T)$ and $v \in VT$, there is an induced permutation $\sigma(g, v)\colon X_{\pi(v)} \to X_{\pi(v)}$ given by

$$\sigma(g, v)(x) := \Big(\mathcal{L}_{g \cdot v} \circ g \circ (\mathcal{L}_v)^{-1}\Big)(x),$$

where $\mathcal{L}_v$ is the map as in (1.3.4).

Definition 1.3.15 ([RS22, Definition 3.8])**.** The *universal group* associated to Δ and $\mathbb{T}$ is

$$U(\Delta, \mathbb{T}) := \{g \in \mathrm{Aut}_\pi(T) \mid \forall\, v \in VT,\ \sigma(g, v) \in G(\pi(v))\}.$$

In other words, $U(\Delta, \mathbb{T})$ collects all the elements of $\mathrm{Aut}_\pi(T)$ acting on $o^{-1}(v)$ as a permutation in $G(\pi(v))$, for every $v \in VT$. If $\mathbb{T}$ is the standard Δ-tree associated to ι and c_0 (cf. Section 1.3.4), we write $U(\Delta, \iota, c_0)$ instead of $U(\Delta, \mathbb{T})$. If T is locally finite, the group $U(\Delta, \mathbb{T})$ is a t.d.l.c. group with the subspace topology from $\mathrm{Aut}(T)$. Indeed, $U(\Delta, \mathbb{T})$ is a closed subgroup of the t.d.l.c. group $\mathrm{Aut}(T)$ (cf. [RS22, §6]). If additionally Γ is finite, then $U(\Delta, \mathbb{T})$ is a compactly generated t.d.l.c. group (cf. [RS22, Proposition 6.5]).

The group $U(\Delta, \mathbb{T})$ simultaneously generalises the three following notable examples.

Example 1.3.16. Let $\Delta = (\Gamma, (X_a), (G(c)))$ be a local action diagram and $\mathbb{T} = (T, \pi, \mathcal{L})$ a Δ-tree.

(i) For every $c \in V\Gamma$, let

$$G(c) := \{\sigma \in \mathrm{Sym}(X_c) \mid \forall\, a \in o^{-1}(c),\, \sigma(X_a) = X_a\}.$$

Then $U(\Delta, \mathbb{T}) = \mathrm{Aut}_\pi(T)$.

(ii) Let Γ be a 1-segment with $E\Gamma = \{a, \bar{a}\}$, $c = o(a)$ and $d = t(a)$. Assume that $G(c)$ and $G(d)$ act transitively on X_c and X_d, respectively.

(iia) Set $F := G(c)$ and assume that $G(d) = C_2$, $|X_c| = k \geq 2$ and $|X_d| = 2$. Adapting [RS23, Example 11] to actions on trees without edge inversion, the action $(U(\Delta, \mathbb{T}), T)$ is permutational isomorphic to the action of the Burger–Mozes universal group $U(F)$ on the barycentric subdivision T_k' of the k-regular tree T_k (cf. [BM00, §3.2]). Here we consider T_k' and not T_k because $U(F)$ acts vertex-transitively (and thus with edge inversions) on T_k.

(iib) Set $F_1 := G(c)$ and $F_2 := G(d)$. Following [RS23, Example 12], the $U(\Delta, \mathbb{T})$-action on T is permutational isomorphic to $(U(F_1, F_2), T)$, where $U(F_1, F_2)$ is the group introduced by S. Smith in [Smi17].

The following theorem collects some key results of the work of C. Reid and S. Smith [RS22]. It motivates why, throughout the paper, we focus on actions of type $(U(\Delta, \iota, c_0), T(\Delta, \iota, c_0))$ while considering (P)-closed actions on trees with associated local action diagram Δ.

Theorem 1.3.17.

(i) ([RS22, Lemma 3.5, Theorem 3.12]) *Let Δ be a local action diagram. For all Δ-trees $\mathbb{T} = (T, \pi, \mathcal{L})$ and $\mathbb{T}' = (T', \pi', \mathcal{L}')$ and $v \in VT$, $v' \in VT'$ with $\pi(v) = \pi'(v')$, there is a graph isomorphism $\phi\colon T \to T'$ such that $\phi(v) = v'$, $\pi = \pi' \circ \phi$ and $\phi U(\Delta, \mathbb{T})\phi^{-1} = U(\Delta, \mathbb{T}')$.*

(ii) ([RS22, Theorem 3.9]) *Let Δ be a local action diagram and $\mathbb{T} = (T, \pi, \mathcal{L})$ be a Δ-tree. Then the local action diagram associated to $(U(\Delta, \mathbb{T}), T)$ is isomorphic to Δ (in the sense of [RS22, Definition 3.2]).*

(iii) ([RS22, Theorem 3.10]) *Let (G, T) be a (P)-closed action on a tree with associated local action diagram Δ. Then (G, T) is permutational isomorphic to the action $(U(\Delta, \iota, c_0), T(\Delta, \iota, c_0))$, for every inversion ι in Δ and every $c_0 \in V\Gamma$.*

1.3.5 Weakly locally ∞-transitive actions on trees

In this section, every group action on a tree (G, T) is assumed to be without edge-inversion. Moreover, the tree T is assumed to have no leaves and to satisfy $ET \neq \emptyset$.

Definition 1.3.18. Let (G,T) be a group action on a tree with quotient map $\pi\colon T \to \Gamma = G\backslash T$. Then (G,T) is said to be *weakly locally ∞-transitive at* $v \in VT$ if, for every path $\mathfrak{p}$ in Γ starting at $\pi(v)$, the stabiliser G_v acts transitively on the set

$$\{\tilde{\mathfrak{p}} \in \mathrm{Geod}_T(v \to T) \mid \pi(\tilde{\mathfrak{p}}) = \mathfrak{p}\}. \tag{1.3.12}$$

Moreover, (G,T) is said to be *weakly locally ∞-transitive* if it is weakly locally ∞-transitive at every $v \in VT$.

Remark 1.3.19.

(i) The action (G,T) is weakly locally ∞-transitive at v if, and only if, it is weakly locally ∞-transitive at every $u \in G \cdot v$.

(ii) Let (G,T) be weakly locally ∞-transitive at v, and consider $\tilde{\mathfrak{p}} = (e_1,\dots,e_n) \in \mathrm{Geod}_T(v \to T)$ with $\pi(\tilde{\mathfrak{p}}) = \mathfrak{p}$. Then, for every $i < n$ the group $G_{(e_1,\dots,e_i)}$ acts transitively on

$$\{\widetilde{\mathfrak{q}} = (f_1,\dots,f_n) \in \mathrm{Geod}_T(v \to T) \mid \pi(\widetilde{\mathfrak{q}}) = \mathfrak{p} \text{ and } \forall\, j \le i, f_j = e_j\}.$$

As the name suggests, weakly locally ∞-transitive actions generalise locally ∞-transitive ones as follows.

Lemma 1.3.20. *Let (G,T) be a group action on a tree with quotient graph Γ. Then (G,T) is locally ∞-transitive if, and only if, it is weakly locally ∞-transitive and Γ is a 1-segment.*

Proof. Let $\pi\colon T \to \Gamma = G\backslash T$ be the quotient map of (G,T). Given $v \in VT$ with $\pi(v) = c$ and for every $d \ge 0$, we observe that

$$\{\tilde{\mathfrak{p}} \in \mathrm{Geod}_T(v \to T) \mid \ell(\tilde{\mathfrak{p}}) = d\} = \bigsqcup_{\substack{\mathfrak{p} \in \mathcal{P}_\Gamma(c \to \Gamma),\\ \ell(\mathfrak{p}) = d}} \{\tilde{\mathfrak{p}} \in \mathrm{Geod}_T(v \to T) \mid \pi(\tilde{\mathfrak{p}}) = \mathfrak{p}\}. \tag{1.3.13}$$

Note that Γ is a 1-segment if, and only if, $|\{\mathfrak{p} \in \mathcal{P}_\Gamma(c \to \Gamma) : \ell(\mathfrak{p}) = d\}| = 1$ for all $d \ge 0$ and $c \in V\Gamma$. If (G,T) is weakly locally ∞-transitive and Γ is a 1-segment, then G_v acts transitively on $\{\tilde{\mathfrak{p}} \in \mathrm{Geod}_T(v \to T) \mid \ell(\tilde{\mathfrak{p}}) = d\}$, for all $v \in VT$ and $d \ge 0$. Conversely, if (G,T) is locally ∞-transitive, then (G,T) is edge-transitive. More precisely, since G_v acts transitively on $o^{-1}(v)$ for every $v \in VT$, we deduce that Γ is a 1-segment. Indeed, if Γ is a 1-loop, for every $v \in VT$ the group G_v has two orbits on $o^{-1}(v)$. Moreover, for all $v \in VT$ and $d \ge 0$, there is a unique path $\mathfrak{p}$ in Γ starting at $\pi(v)$ of length d. By (1.3.13), we conclude that (G,T) is weakly locally ∞-transitive. □

The next lemma provides a local characterisation of weakly locally ∞-transitive (P)-closed actions on trees. An analogous result has already been proved for Burger–Mozes universal groups [BM00, Lemma 3.1.1 and the lines before it].

Proposition 1.3.21. *Let (G, T) be a group action on a tree with associated local action diagram $\Delta = (\Gamma, (X_a), (G(c)))$, and let $v \in VT$ with $\pi(v) = c$. If (G, T) is weakly locally ∞-transitive at v, then*

($\diamondsuit$) *for all $a, b \in o^{-1}(c)$ and $x \in X_a$, the group $G(c)_x$ acts transitively on $X_b \setminus \{x\}$.*

Conversely, if (G, T) is (P)-closed, condition ($\diamondsuit$) implies that (G, T) is weakly locally ∞-transitive at v.

In ($\diamondsuit$), note that $X_b \setminus \{x\} = X_b$ unless $a = b$. Moreover, if $a = b$, condition ($\diamondsuit$) is equivalent to say that $G(c)$ acts 2-transitively on X_a.

Proof. To prove the first part of the assertion, let $a, b \in o^{-1}(c)$. By essential uniqueness of the local action diagram associated to (G, T) (cf. Section 1.3.4), we may assume that $X_a, X_b \subseteq o^{-1}(v)$. Since (G, T) is weakly locally ∞-transitive at v, for every $e \in X_a$ the stabiliser G_e acts transitively on

$$\{\mathfrak{p} \in \mathrm{Geod}_T(\bar{e} \to T) \mid \pi(\mathfrak{p}) = (\bar{a}, b)\} = \{(\bar{e}, f) \mid f \in X_b \setminus \{e\}\}$$

and thus on $X_b \setminus \{e\}$.

Let now (G, T) be (P)-closed and suppose that ($\diamondsuit$) holds. Without loss of generality, set $G = U(\Delta, \iota, c)$ and $T = T(\Delta, \iota, c)$. Consider two geodesics in T, say $[v, w_1] = (e_1, \ldots, e_n)$ and $[v, w_2] = (f_1, \ldots, f_n)$ for some $n \geq 1$ and $w_1, w_2 \in VT$, with the same image in Γ. For every $1 \leq i \leq n$, since $\pi(e_i) = \pi(f_i)$, there exists $g_i \in G$ such that $f_i = g_i \cdot e_i$. Note that $o(g_i \cdot e_{i+1}) = g_i \cdot t(e_i) = t(f_i) = o(f_{i+1})$ and $\pi(g_i \cdot e_{i+1}) = \pi(e_{i+1}) = \pi(f_{i+1})$. Hence both $\mathcal{L}(g_i \cdot e_{i+1})$ and $\mathcal{L}(f_{i+1})$ belong to $X_{\pi(f_{i+1})}$. By ($\diamondsuit$) applied to the vertex $c_i = o(\pi(f_{i+1}))$, for every $i < n$ there is $h_i \in G_{\bar{f}_i}$ such that $f_{i+1} = h_i g_i \cdot e_{i+1}$. For every $i < n$, set $k_i := h_i g_i$ and observe that

$$(k_i \cdot e_i, k_i \cdot e_{i+1}) = (f_i, f_{i+1}). \tag{1.3.14}$$

In particular, $k_1 \in G_v$ and $k_i \cdot e_{i+1} = f_{i+1} = k_{i+1} \cdot e_{i+1}$ for every $i < n - 1$. For every $i < n - 1$ write $u_{i+1} := k_i^{-1} k_{i+1} \in G_{e_{i+1}}$, and let $u_{i+1}^- \in G_{T_{\geq \bar{e}_{i+1}}}$ and $u_{i+1}^+ \in G_{T_{\geq e_{i+1}}}$ be such that $u_{i+1} = u_{i+1}^- u_{i+1}^+$ (recall that (G, T) is (P)-closed). Set $\tilde{k}_1 := k_1$ and, for every $2 \leq i \leq n-1$, define inductively $\tilde{k}_i := \tilde{k}_{i-1} u_i^-$. For each $2 \leq i \leq n-1$, note that $k_i = \tilde{k}_i u_i^+$ and, since u_i^+ fixes both e_i and e_{i+1}, from (1.3.14) we have

$$(\tilde{k}_i \cdot e_i, \tilde{k}_i \cdot e_{i+1}) = (k_i \cdot e_i, k_i \cdot e_{i+1}) = (f_i, f_{i+1}). \tag{1.3.15}$$

Moreover, we claim that $\tilde{k}_i$ fixes v for every $1 \leq i \leq n - 1$. If $i = 1$, this is clear. For $i \geq 2$, assuming inductively that $\tilde{k}_{i-1}$ fixes v, the fact that u_i^- fixes $T_{\geq \bar{e}_i}$ pointwise implies that $\tilde{k}_i = \tilde{k}_{i-1} u_i^-$ fixes v. In particular, $\tilde{k}_{n-1} \in G_v$ and (1.3.15) yields

$$\tilde{k}_{n-1} \cdot w_1 = \tilde{k}_{n-1} \cdot t(e_n) = t(\tilde{k}_{n-1} \cdot e_n) = t(f_n) = w_2.$$

Being T a tree, we conclude that $[v, w_2] = [v, \tilde{k}_{n-1} \cdot w_1] = \tilde{k}_n \cdot [v, w_1]$. ∎

Proposition 1.3.21 gives a recipe for constructing (P)-closed actions that are weakly locally ∞-transitive. We collect some explicit examples below.

Example 1.3.22. (i) Let (T, π) be the universal cover of a connected edge-weighted graph (Γ, ω) as in Example 1.3.1. According to Example 1.3.16(i), one checks that $(\diamondsuit)$ is satisfied for every $c \in V\Gamma$. Hence, by Proposition 1.3.21, $(\mathrm{Aut}_\pi(T), T)$ is weakly locally ∞-transitive. This gives an alternative proof of [CW20, Theorem 3.1 and the comment thereafter].

(ii) Let $U(F)$ be the Burger–Mozes universal group associated to a transitive group $F \leq \mathrm{Sym}(\{1, \dots, k\})$, $k \geq 2$, acting on the barycentric subdivision T'_k of the k-regular tree. According to Example 1.3.16(iia), condition $(\diamondsuit)$ of Proposition 1.3.21 is satisfied for every $c \in V\Gamma$ if, and only if, F is 2-transitive. Moreover, $U(F)\backslash T'_k$ is a 1-segment. Hence, $(U(F), T'_k)$ is weakly locally ∞-transitive if, and only if, it is locally ∞-transitive (cf. Lemma 1.3.20). By Proposition 1.3.21, we deduce that $(U(F), T'_k)$ is locally ∞-transitive if, and only if, F is 2-transitive. This gives an alternative proof of what was observed in the first lines of [BM00, §3].

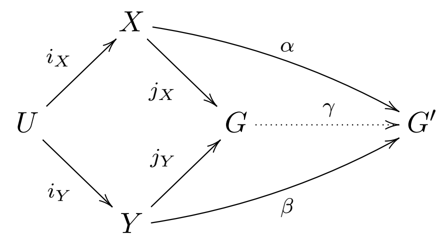
(iii) Let $U(F_1, F_2)$ be the Smith's group associated to two transitive groups $F_1 \leq \mathrm{Sym}(\{1, \dots, k_1\})$ and $F_2 \leq \mathrm{Sym}(\{1, \dots, k_2\})$, $k_1, k_2 \geq 2$. According to Example 1.3.16(iib), condition $(\diamondsuit)$ is satisfied for every $c \in V\Gamma$ if, and only if, both F_1 and F_2 are 2-transitive. Moreover, the $U(F_1, F_2)$-action on the (k_1, k_2)-biregular tree T_{k_1,k_2} has quotient graph a 1-segment. Hence, as in (ii), $(U(F_1, F_2), T_{k_1,k_2})$ is weakly locally ∞-transitive if, and only if, it is locally ∞-transitive. By Proposition 1.3.21 we conclude that $(U(F_1, F_2), T_{k_1,k_2})$ is locally ∞-transitive if, and only if, both F_1 and F_2 are 2-transitive.

1.3.6 Graphs of groups

In geometric group theory, two basic operations on groups recur on several occasions: *amalgamated free products* and *HNN-extensions*. We now recall them.

Amalgamated free products

Fact 1.3.23. *Let X, Y and U be groups with two monomorphisms $i_X \colon U \hookrightarrow X$ and $i_Y \colon U \hookrightarrow Y$. Then there is a group G, uniquely determined up to canonical isomorphisms, together with two monomorphisms $j_X \colon X \hookrightarrow G$ and $j_Y \colon Y \hookrightarrow G$ satisfying the following universal property:*



for every group G' and for all homomorphisms $\alpha\colon X \to G'$ and $\beta\colon Y \to G'$ satisfying $\alpha(i_X(u)) = \beta(i_Y(u))$ for all $u \in U$, there is a unique homomorphism $\gamma\colon G \to G'$ such that $\gamma \circ j_X = \alpha$ and $\gamma \circ j_Y = \beta$.

Definition 1.3.24. The group G in Fact 1.3.23 is called the *amalgamated free product of X and Y with respect to i_X and i_Y* and it is denoted by $X *_{U,i_X,i_Y} Y$.

If i_X and i_Y are clear from the context, we say that G is the *amalgamated free product of X and Y with respect to U* and we denote it by $X *_U Y$.

Proposition 1.3.25 ([Bog08, Chapter 3, §11], [CH16, Proposition 8.B.9])**.**

(i) Up to isomorphisms one has

$$X *_{U,i_X,i_Y} Y = (X * Y)/\langle\langle i_X(u)i_Y(u)^{-1},\, u \in U\rangle\rangle,$$

where $\langle\langle i_X(u)i_Y(u)^{-1},\, u \in U\rangle\rangle$ is the normal closure of $\{i_X(u)i_Y(u)^{-1},\, u \in U\}$.

(ii) Let X and Y be groups admitting the presentations $\langle S_X \mid R_X\rangle$ and $\langle S_Y \mid R_Y\rangle$, respectively. Then, for all monomorphisms $i_X\colon U \hookrightarrow X$ and $i_Y\colon U \hookrightarrow Y$, the group $G = X *_{U,i_X,i_Y} Y$ admits the following presentation:

$$\langle S_X \sqcup S_Y \mid R_X \sqcup R_Y;\ \forall\, u \in U,\ i_X(u) = i_Y(u)\rangle.$$

Moreover, the monomorphisms $j_X\colon X \hookrightarrow G$ and $j_Y\colon Y \hookrightarrow G$ in Fact 1.3.23 can be regarded as inclusion maps.

(iii) Let X, Y and U be topological groups together with two continuous open monomorphisms $i_X\colon U \hookrightarrow X$ and $i_Y\colon U \hookrightarrow Y$. Then $X *_{U,i_X,i_Y} Y$ has a unique group topology which makes the maps j_X and j_Y in Fact 1.3.23 continuous open monomorphisms. In particular, if X and Y are (compactly generated) t.d.l.c. groups, then $X *_{U,i_X,i_Y} Y$ is a (compactly generated) t.d.l.c. group.

HNN-extensions

Fact 1.3.26. *Let X be a group together with a subgroup $U \leq X$ and a monomorphism $\varphi\colon U \hookrightarrow X$. Then there is a group G, uniquely determined up to canonical isomorphisms, a monomorphism $\iota_X\colon X \hookrightarrow G$ and an element $t \in G$ such that the following universal property holds:*
for every group G' with an element $t' \in G'$, and for every homomorphism $\alpha\colon X \to G'$ such that $\alpha(\varphi(u)) = t'\alpha(u)(t')^{-1}$ for all $u \in U$, there is a unique homomorphism $\gamma\colon G \to G'$ such that $\gamma \circ \iota_X = \alpha$ and $\gamma(t) = t'$.

Definition 1.3.27. The group G in Fact 1.3.26 is called the *HNN-extension of X with respect to U and φ and with stable letter t*, and it is denote by $X*^t_{U,\varphi}$.

If φ is clear from the context, we denote $X*^t_{U,\varphi}$ simply by $X*^t_U$.

Proposition 1.3.28 ([Bog08, Chapter 3, §14], [CH16, Proposition 8.B.10])**.**

(i) *Up to isomorphisms one has*

$$X*^t_{U,\varphi} = (X * \langle t\rangle)/\langle\langle \varphi(u)tu^{-1}t^{-1},\, u \in U\rangle\rangle,$$

where $X * \langle t\rangle$ is the free product of X and the infinite cyclic group $\langle t\rangle$, and $\langle\langle \varphi(u)tu^{-1}t^{-1},\, u \in U\rangle\rangle$ is the normal closure of the set $\{\varphi(u)tu^{-1}t^{-1},\, u \in U\}$.

(ii) *Let X be a group with presentation $\langle S_X \mid R_X\rangle$. Then, for every subgroup $U \leq X$ and every monomorphism $\varphi\colon U \hookrightarrow Y$, the group $G = X*^t_{U,\varphi}$ has the following presentation:*

$$\langle S_X \mid R_X;\ \forall\, u \in U,\, tut^{-1} = \varphi(u)\rangle.$$

Moreover, the monomorphism $\iota_X\colon X \hookrightarrow G$ in Fact 1.3.26 can be regarded as the inclusion map.

(iii) *Let X be a topological group, $U \leq X$ be an open subgroup and $\varphi\colon U \hookrightarrow Y$ be a continuous open monomorphism. Then $G = X*^t_{U,\varphi}$ has a unique group topology which makes the inclusion map $\iota_X\colon X \hookrightarrow G$ a continuous open monomorphism. In particular, if X is a (compactly generated) t.d.l.c. group, then $X*^t_{U,\varphi}$ is a (compactly generated) t.d.l.c. group.*

Graphs of t.d.l.c. groups and their t.d.l.c. fundamental group

Adapting [Ser80, Definition 8, p. 37] to the t.d.l.c. context, one obtains the notion of graph of t.d.l.c. groups as follows.

Definition 1.3.29. A *graph of t.d.l.c. groups* $(\mathcal{G}, \Lambda)$ consists of:

(G1) a connected graph Λ;

(G2) a family of t.d.l.c. groups $\{\mathcal{G}_\lambda\}_{\lambda\in\Lambda}$ satisfying $\mathcal{G}_e = \mathcal{G}_{\bar{e}}$, for every $e \in E\Lambda$;

(G3) for every $e \in E\Lambda$, there is a continuous open monomorphism $\eta_e : \mathcal{G}_e \to \mathcal{G}_{t(e)}$.

The graph of groups $(\mathcal{G}, \Lambda)$ is called *proper* if, whenever η_e is surjective, then $o(e) = t(e)$.

The initial definition of J-P. Serre for abstract groups is recovered by dropping all the topological assumptions appearing in Definition 1.3.29.

Definition 1.3.30. Let $(\mathcal{G}, \Lambda)$ be a graph of t.d.l.c. groups and T be a maximal subtree of Λ. The (abstract) *fundamental group* $\pi_1(\mathcal{G}, \Lambda, T)$ *of* $(\mathcal{G}, \Lambda, T)$ is defined as follows:

$$\pi_1(\mathcal{G}, \Lambda, T) := \left(\coprod_{v \in V\Lambda} \mathcal{G}_v * \langle \{t_e\}_{e\in E\Lambda} \mid \forall\, e \in E\Lambda,\, t_{\bar{e}} = t_e^{-1} \rangle \right) \Big/ \langle\!\langle \mathcal{R} \rangle\!\rangle \tag{1.3.16}$$

where

$$\mathcal{R} = \left\{ t_e \eta_e(g) t_e^{-1} \eta_{\bar{e}}(g)^{-1} \,\middle|\, e \in E\Lambda,\, g \in \mathcal{G}_e \right\} \sqcup \left\{ t_f \,\middle|\, f \in ET \right\}$$

Proposition 1.3.31 ([Bou98, §III.2, Proposition 1])**.** *Let $(\mathcal{G}, \Lambda)$ be a graph of t.d.l.c. groups and T be a maximal subtree of Λ. Then the group $\pi_1(\mathcal{G}, \Lambda, T)$ can be equipped with a group topology such that*

(F1) $\pi_1(\mathcal{G}, \Lambda, T)$ is a t.d.l.c. group; and

(F2) the natural inclusions $\mathcal{G}_v \hookrightarrow \pi_1(\mathcal{G}, \Lambda)$ and $\mathcal{G}_e \hookrightarrow \pi_1(\mathcal{G}, \Lambda)$, for $v \in V\Lambda$ and $e \in E\Lambda$, are continuous open monomorphisms;

From now on, the group $\pi_1(\mathcal{G}, \Lambda, T)$ is regarded as a t.d.l.c. group.

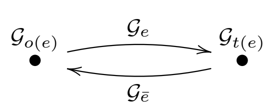
The construction of $\pi_1(\mathcal{G}, \Lambda, T)$ depends on the choice of a maximal subtree $T \subseteq \Lambda$. However, given arbitrary maximal trees T_1 and T_2 of Λ, one proves that the canonical inclusions $\mathcal{G}_v \hookrightarrow \pi_1(\mathcal{G}, \Lambda, T_2)$, $\langle t_e \rangle \hookrightarrow \pi_1(\mathcal{G}, \Lambda, T_2)$, for $v \in V\Lambda$ and $e \in E\Lambda$, induce an isomorphism of topological groups $\pi_1(\mathcal{G}, \Lambda, T_1) \to \pi_1(\mathcal{G}, \Lambda, T_2)$ (cf. [Bog08, Chapter 3, Corollary 16.7]). This motivates the following.

Notation 1.3.32. We write $\pi_1(\mathcal{G}, \Lambda)$ unless specifying the maximal subtree is relevant.

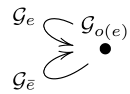
The fundamental group of a graph of (t.d.l.c.) groups is a generalisation of the following three notable examples.

Example 1.3.33.

(i) Let Λ be a 1-segment graph with $E\Lambda = \{e, \bar{e}\}$. By Proposition 1.3.25, for every graph of t.d.l.c. groups $(\mathcal{G}, \Lambda)$, the t.d.l.c. group $\pi_1(\mathcal{G}, \Lambda)$ is topologically isomorphic to $\mathcal{G}_{o(e)} *_{\mathcal{G}_e, \eta_{\bar{e}}, \eta_e} \mathcal{G}_{t(e)}$.

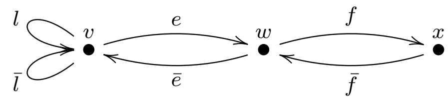


(ii) Let Λ be a 1-bouquet of loops with $E\Lambda = \{e, \bar{e}\}$. By Proposition 1.3.28, for every graph of t.d.l.c. groups $(\mathcal{G}, \Lambda)$, the t.d.l.c. group $\pi_1(\mathcal{G}, \Lambda)$ is topologically isomorphic to $\mathcal{G}_{o(e)} *^{t_e}_{\eta_e(\mathcal{G}_e), \eta_{\bar{e}}}$.



(iii) Let $(\mathcal{G}, \Lambda)$ be a graph of t.d.l.c. groups with $\mathcal{G}_v = \{1\}$ for every $v \in V\Lambda$. Then $\pi_1(\mathcal{G}, \Lambda)$ is isomorphic to the fundamental group of the connected graph Λ (cf. [Bog08, Chapter 2, §4]).

Remark 1.3.34. It is well known that a connected graph is obtained by successively adding edges with different endpoints and 1-loops. For instance, let Λ be the following graph.



(1.3.17)

Then Λ is obtained, for instance, as follows: we start with the vertex v, we add the 1-loop couple $\{l, \bar{l}\}$ at v, then we hang the 1-segment $\{e, \bar{e}\}$ so that $v = o(e)$, and finally hang the 1-segment $\{f, \bar{f}\}$ so that $w = t(e) = o(f)$. Note that the procedure is not unique: we could have started by first adding the 1-segment $\{e, e\}$ to v, or also have started from w or x.

According to [Bog08, Chapter 3, Examples 16.4], an analogous phenomenon happens for (t.d.l.c.) fundamental groups of graphs of (t.d.l.c.) groups. Namely, let $(\mathcal{G}, \Lambda)$ be an arbitrary graph of t.d.l.c. groups. Then $\pi_1(\mathcal{G}, \Lambda)$ is obtained by a successive application of amalgamated free products and HNN-extensions of t.d.l.c. groups. For example, consider a graph of t.d.l.c. groups $(\mathcal{G}, \Lambda)$, where Λ is the graph in (1.3.17). Then $\pi_1(\mathcal{G}, \Lambda)$ is topologically isomorphic to each of the following t.d.l.c. groups:
$\big((\mathcal{G}_v *_{\mathcal{G}_e} \mathcal{G}_w) *_{\mathcal{G}_f} \mathcal{G}_x\big) *^{t_l}_{\mathcal{G}_l}$, $\big(\mathcal{G}_v *_{\mathcal{G}_e} (\mathcal{G}_w *_{\mathcal{G}_f} \mathcal{G}_x)\big) *^{t_l}_{\mathcal{G}_l}$, $\big((\mathcal{G}_v *^{t_l}_{\mathcal{G}_l}) *_{\mathcal{G}_e} \mathcal{G}_w\big) *_{\mathcal{G}_f} *\mathcal{G}_x$, etc ... .

Two of the most elementary transformations one applies on a graph are the contraction of a subgraph to a single vertex or the expansion of a vertex with a new graph. We recall them in the following definition.

Definition 1.3.35 ([Coh73, p. 30])**.** Let Λ be a connected graph and $\{\Lambda_i\}_{i \in I}$ be a collection of mutually disjoint non-empty connected subgraphs of Λ. For every $i \in I$, choose a vertex v_i of Λ_i. The *graph obtained from* Λ *by contracting* $\{\Lambda_i\}_{i \in I}$ *to* $\{v_i\}_{i \in I}$ is the connected graph $\tilde{\Lambda}$ defined as follows:

$$V\tilde{\Lambda} := \left(V\Lambda \setminus \left(\bigsqcup_{i \in I} V\Lambda_i\right)\right) \sqcup \{v_i \mid i \in I\} \quad \text{and} \quad E\tilde{\Lambda} := E\Lambda \setminus \left(\bigsqcup_{i \in I} E\Lambda_i\right).$$

For every $e \in E\tilde{\Lambda}$ with $o(e), t(e) \notin \bigsqcup_{i \in I} V\Lambda_i$, the origin and terminus of e in $\tilde{\Lambda}$ are $o(e)$ and $t(e)$, respectively. Moreover, for every $e \in E\tilde{\Lambda}$ with $o(e) \notin \bigsqcup_{i \in I} V\Lambda_i$ and $t(e) \in V\Lambda_i$

for some $i \in I$, the origin and terminus of e in $\tilde{\Lambda}$ are $o(e)$ and v_i and the origin and terminus of $\bar{e}$ are v_i and $o(e)$, respectively. Finally, the edge inversion on $\tilde{\Lambda}$ is induced by the edge inversion on Λ. Given $\tilde{\Lambda}$, the graph Λ is said to be *obtained from* Λ *by expanding* $\{v_i\}_{i\in I}$ *with* $\{\Lambda_i\}_{i\in I}$.

Generalising the approach of D. Cohen [Coh73] to t.d.l.c. groups, after a preliminary notion of "conjugate graph of t.d.l.c. groups", we now present how the transformations of contraction and expansion of graphs carry over the realm of graphs of t.d.l.c. groups.

Definition 1.3.36 ([Coh73, pp. 27-28])**.** Two graphs of t.d.l.c. groups $(\mathcal{G}, \Lambda)$ and $(\mathcal{H}, \Lambda)$ are said to be *conjugated* if:

(cj1) for every $\lambda \in \Lambda$ we have $\mathcal{G}_\lambda = \mathcal{H}_\lambda$;

(cj2) for every $e \in E\Lambda$, the monomorphism $\eta'_e \colon \mathcal{H}_e \hookrightarrow \mathcal{H}_{t(e)}$ in $(\mathcal{H}, \Lambda)$ is the monomorphism $\eta_e \colon \mathcal{G}_e \hookrightarrow \mathcal{G}_{t(e)}$ followed by the conjugation map by an element of $\mathcal{G}_{t(e)}$.

Lemma 1.3.37 ([Coh73, Lemma 1])**.** *Let* $(\mathcal{G}, \Lambda)$ *and* $(\mathcal{H}, \Lambda)$ *be conjugated graphs of t.d.l.c. groups. Then there is an isomorphisms of t.d.l.c. groups* $\theta \colon \pi_1(\mathcal{G}, \Lambda) \to \pi_1(\mathcal{H}, \Lambda)$ *with the property that, for each* $v \in V\Lambda$*, there is* $c_v \in \pi_1(\mathcal{H}, \Lambda)$ *such that* $\theta(g) = c_v g c_v^{-1}$*, for all* $g \in \mathcal{G}_v$.

Definition 1.3.38 ([Coh73, p. 30])**.** Let $(\mathcal{G}, \Lambda)$ be a graph of t.d.l.c. groups with a maximal subtree T. Consider a collection $\{\Lambda_i\}_{i\in I}$ of pairwise disjoint connected non-empty subgraphs of Λ and, for every $i \in I$, choose a vertex $v_i \in V\Lambda_i$. The graph of t.d.l.c. groups $(\mathcal{H}, \tilde{\Gamma})$ *obtained from* $(\mathcal{G}, \Lambda, T)$ *by contracting* $\{\Lambda_i\}_{i\in I}$ *with* $\{v_i\}_{i\in I}$ is the following graph of t.d.l.c. groups:

(ct1) $\tilde{\Lambda}$ is the graph obtained from Λ by contracting $\{\Lambda_i\}_{i\in I}$ with $\{v_i\}_{i\in I}$;

(ct2) for every $e \in E\tilde{\Lambda}$, set $\mathcal{H}_e := \mathcal{G}_e$;

(ct3) for every $v \in V\tilde{\Lambda} \setminus \{v_i\}_{i\in I}$, set $\mathcal{H}_v := \mathcal{G}_v$. Moreover, for every $i \in I$ set $\mathcal{H}_{v_i} := \pi_1(\mathcal{G}|_{\Lambda_i}, \Lambda_i, T \cap \Lambda_i)$;

(ct4) for every $e \in E\tilde{\Lambda}$ such that $t(e) \notin \{v_i\}_{i\in I}$, the monomorphism $\mathcal{H}_e \hookrightarrow \mathcal{H}_{t(e)}$ is given by the monomorphism $\mathcal{G}_e \hookrightarrow \mathcal{G}_{t(e)}$ in $(\mathcal{G}, \Lambda)$. Moreover, for every $e \in E\tilde{\Lambda}$ with $t(e) = v_i$ for some $i \in I$, the monomorphism $\mathcal{H}_e \hookrightarrow \mathcal{H}_{t(e)}$ is given by the continuous open monomorphism $\mathcal{G}_e \hookrightarrow \pi_1(\mathcal{G}|_{\Lambda_i}, \Lambda_i, T \cap \Lambda_i)$.

Given $(\mathcal{H}, \tilde{\Lambda})$ and a maximal subtree T of Λ, the graph of t.d.l.c. groups $(\mathcal{G}, \Lambda)$ is said to be *obtained from* $(\mathcal{H}, \tilde{\Lambda})$ *by expanding* $\{v_i\}_{i\in I}$ *with* $\{\Lambda_i\}_{i\in I}$.

Whenever there is no necessity to underline which maximal tree T involved, we simply refer to the *graph of t.d.l.c. groups* $(\mathcal{H}, \tilde{\Lambda})$ *obtained from* $(\mathcal{G}, \Lambda)$ *by contracting* $\{\Lambda_i\}_{i\in I}$ *to* $\{v_i\}_{i\in I}$.

Lemma 1.3.39 (Expansion of graphs of groups - [Coh73, Lemma 2])**.** *Let $(\mathcal{G},\Lambda)$ be a graph of t.d.l.c. groups with compact edge-groups, and let $G := \pi_1(\mathcal{G},\Lambda)$. Let $\mathcal{V} \subseteq V\Lambda$ and, for every $v \in \mathcal{V}$, assume that there is a graph of t.d.l.c. groups $(\mathcal{H}^{(v)},\Gamma^{(v)})$ with compact edge-groups such that $\mathcal{G}_v = \pi_1(\mathcal{H}^{(v)},\Gamma^{(v)})$. Then, for every $e \in E\Lambda$ with $t(e) \in \mathcal{V}$, there exists $x = x_e \in V\Gamma^{(t(e))}$ such that the image of $\mathcal{G}_e$ in $\mathcal{G}_{t(e)}$ lies in a $\mathcal{G}_{t(e)}$-conjugate of the vertex-group $(\mathcal{H}^{(t(e))})_x$. Moreover, there exists a graph of t.d.l.c. groups $(\mathcal{H},\Gamma)$ obtained from a conjugate of $(\mathcal{G},\Lambda)$ by expanding $\mathcal{V}$ with $\{\Gamma^{(v)}\}_{v\in\mathcal{V}}$, such that $G \simeq \pi_1(\mathcal{H},\Gamma)$.*

Proof. The first part of the statement is due to the Bruhat–Tits fixed-point theorem. To prove the second part, one proceeds as in the proof of [Coh73, Lemma 2]. □

Finally, we state the main theorem of Bass–Serre theory translated for t.d.l.c. groups. Essentially, it creates a bijective correspondence between graphs of (t.d.l.c.) groups and t.d.l.c. groups acting on trees.

Theorem 1.3.40.

(i) Let $(\mathcal{G},\Lambda)$ be a graph of t.d.l.c. groups and Λ_0 be a maximal subtree of Λ. Then there is a tree T on which the t.d.l.c. group $G := \pi_1(\mathcal{G},\Lambda,\Lambda_0)$ acts and such that $G\backslash T = \Lambda$.

(ii) Let G be a t.d.l.c. group acting on a tree T with quotient graph Λ. Then G is topologically isomorphic to $\pi_1(\mathcal{G},\Lambda)$, for every graph of t.d.l.c. groups $(\mathcal{G},\Lambda)$ associated to the action (G,T) (as defined in [Ser80, §I.5.4, p. 54]).

The tree T in Theorem 1.3.40(i) is called the *universal tree of* $(\mathcal{G},\Lambda)$. It has an explicit description in terms of $(\mathcal{G},\Lambda)$ which can be based, for example, on [Ser80, §I.5.3, p. 51].

1.4 Coxeter groups

1.4.1 Background knowledge on Coxeter groups

A *Coxeter matrix* $[m_{st}]_{s,t\in S}$ is a symmetric matrix with diagonal entries equal to 1 and whose non-diagonal entries are contained in $\mathbb{Z}_{\geq 2}\cup\{\infty\}$. One can associate to M the *Coxeter group* (W,S), which is the group generated by S subject to the relations $(st)^{m_{st}} = 1$, whenever $m_{st} \neq \infty$. It turns out that $m_{s,t}$ coincides with the order of st in W. Here, with abuse of notation, we name the pair (W,S) as a "Coxeter group". To be precise, the pair (W,S) is called a "Coxeter system" and the group W is the "Coxeter group".

A *special subgroup* W_J of (W,S) is a subgroup generated by a subset J of S. Notice that (W_J,J) is a Coxeter group for every $J \subseteq S$ (cf. [Hum90, Theorem 5.5(a)]).

Classically, to each Coxeter group (W,S) one associates the so-called *Coxeter diagram* $\Gamma(W,S)$, which is the edge-labelled undirected graph with set of vertices S, set of edges

$$E(\Gamma(W,S)) = \big\{\{s,t\} \mid s,t \in S,\, m_{st} \neq 2\big\},$$

and edge-labelling $m\colon E(\Gamma(W,S)) \to \mathbb{Z}_{\geq 3} \cup \{\infty\}$, $m(\{s,t\}) := m_{st}$. One usually omits the label of all edges $\{s,t\}$ in $\Gamma(W,S)$ satisfying $m_{st} = 3$. Sometimes, it is convenient to consider another graph $\Gamma_\infty(W,S)$ associated to (W,S), the so-called *presentation diagram* of (W,S) (cf. [MT09]). It is the edge-labelled undirected graph with vertex set S and edge set

$$E(\Gamma_\infty(W,S)) = \big\{\{s,t\} \mid s,t \in S,\, m_{st} \neq \infty\big\},$$

and every edge $\{s,t\}$ is labelled with m_{st}.

Each of the two edge-labelled undirected graphs, $\Gamma(W,S)$ and $\Gamma_\infty(W,S)$, determines uniquely the Coxeter matrix $[m_{st}]_{s,t\in S}$ and thus the Coxeter group (W,S).

Let (W,S) be a Coxeter group with $|S| = n$. Given $V = \mathrm{span}_{\mathbb{R}}\{\alpha_s \mid s \in S\}$, let $B\colon V \times V \to \mathbb{R}$ be the symmetric bilinear form on V given by

$$B(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{m_{st}}\right), \quad \forall\, s,t \in S.$$

We recall that a Coxeter group (W,S) is said to be:

- *irreducible*, if the Coxeter diagram $\Gamma(W,S)$ is a connected graph (cf. [Hum90, §6.1]);
- *crystallographic*, if $m_{st} \in \{2,3,4,6,\infty\}$ for all $s,t \in S$ with $s \neq t$, and every closed path in $\Gamma(W,S)$ has an even number of edges labelled by 4 and an even number of edges labelled by 6 (cf. [Hum90, §6.6]);
- *spherical*, if W is finite or, equivalently, the bilinear form B of (W,S) is positive definite (cf. [Hum90, §6.4]). Irreducible spherical Coxeter groups are classified by their Coxeter diagram (cf. [Hum90, Figure 1, p. 32]);
- *affine*, if the bilinear form B of (W,S) is positive semidefinite but not positive definite (cf. [Hum90, §6.5]). Affine Coxeter groups are classified by their Coxeter diagram (cf. [Hum90, Figure 2, p. 34]);
- *of positive type*, if the bilinear form B of (W,S) is positive semidefinite (cf. [Hum90, p. 31]);
- *of hyperbolic type*, if (W,S) is irreducible, the bilinear form B of (W,S) is non-degenerate and not positive definite and, for every $s \in S$, the special subgroup $(W_{S\setminus\{s\}}, S \setminus \{s\})$ is of positive type (cf. [Hum90, §6.8]);
- *right-angled*, if $m_{st} \in \{2,\infty\}$ for all $s,t \in S$ with $s \neq t$ (cf. [Dav08, Chapter 1]).

Given a Coxeter group (W,S), we say that a subset $J \subseteq S$ is *spherical* if its associated special subgroup (W_J, J) is a spherical Coxeter group.

Let (W, S) be a Coxeter group. Following [Dav08, §17.1], we now recall the definition of *generalised Poincaré series* of a subset X of W. Let $\mathbf{t} = (t_s)_{s\in S}$ be a vector of pairwise commuting variables. For every $X \subseteq W$, define the formal series

$$X(\mathbf{t}) := \sum_{w\in X} \mathbf{t}_w, \tag{1.4.1}$$

where $\mathbf{t}_w := t_{s_1} \cdots t_{s_n}$ for *any* reduced expression $w = s_1 \cdots s_n$ of w in (W, S). Note that $\mathbf{t}_w$ is well-defined: indeed, for every two reduced expressions $w = s_1 \cdots s_n = s'_1 \cdots s'_m$ in (W, S), we have $\{s_1, \ldots, s_n\} = \{s'_1, \ldots, s'_m\}$ (cf. [BB06, Corollary 1.4.8(ii)]). If $X = W$ and if $t_s = t$ for every $s \in S$, then (1.4.1) recovers the classical univariate Poincaré series of (W, S).

If (W, S) is a spherical irreducible Coxeter group, then the associated univariate Poincaré series $W(t)$ can be described by the following formula (cf. [BB06, Theorem 7.1.5]):

$$W(t) = \prod_{i=1}^{|S|} \frac{1 - t^{e_i+1}}{1 - t}, \tag{1.4.2}$$

where $e_1, \ldots, e_{|S|}$ are the exponents of (W, S) (as listed in [BB06, Appendix A1]).

If (W, S) is an affine Coxeter group with $|S| = n + 1$, then *Bott's formula* (cf. [BB06, Theorem 7.1.10]) yields

$$W(t) = \prod_{i=1}^{n} \frac{(1 - t^{e_i+1})}{(1 - t)(1 - t^{e_i})}, \tag{1.4.3}$$

where $e_1, \ldots, e_n$ are the exponents of the spherical Coxeter group associated to (W, S).

Remark 1.4.1. For every $I \subseteq S$, the series $W_I(\mathbf{t})$ depends only on the variables $\mathbf{t}_I := (t_i)_{i\in I}$. Indeed, for every $w \in W_I$, every reduced expression $w = s_1 \cdots s_n$ in (W, S) satisfies $s_1, \ldots, s_n \in I$ (cf. [Hum90, Theorem 5.5(b)]). Hence,

$$W_I(\mathbf{t}) = W_I(\mathbf{t}_I).$$

Lemma 1.4.2 ([Dav08, Lemma 17.1.13])**.** *Let (W, S) be a reducible Coxeter group, say $(W, S) = (W_1 \times W_2, S_1 \sqcup S_2)$ for some $S_1, S_2 \subseteq S$. Then,*

$$W(\mathbf{t}) = W_{S_1}(\mathbf{t}) \cdot W_{S_2}(\mathbf{t}).$$

1.4.2 ∞-Decompositions and visual graph of groups decompositions

Definition 1.4.3. Let (W, S) be a Coxeter group. Consider subsets $S_+, S_- \subseteq S$ of S with $S = S_+ \cup S_-$, and let $S_0 := S_+ \cap S_-$. Then $S = S_+ \cup S_-$ is said to be an ∞*-decomposition* for (W, S) if

$$m_{st} = \infty, \quad \forall\, s \in S_+ \setminus S_0,\, t \in S_- \setminus S_0.$$

The concept of ∞-decomposition was initially introduced by F. Haglund and F. Paulin [HP03, §4.1] under the French name *scindement*.

Fact 1.4.4 ([HP03, Remarques]). *For a Coxeter group* (W, S)*, the following holds:*

(i) (W, S) *admits an* ∞*-decomposition if, and only if, there are* $s, t \in S$ *with* $m_{st} = \infty$*;*

(ii) Let $S_+, S_- \subseteq S$ *be subsets with* $S = S_+ \cup S_-$*, and set* $S_0 = S_+ \cap S_-$*. Then* $S = S_+ \cup S_-$ *is an* ∞*-decomposition if, and only if,* $W \simeq W_{S_+} *_{W_{S_0}} W_{S_-}$*.*

We say that an ∞-decomposition $S = S_+ \cup S_-$ of (W, S) is:

- *non-trivial*, if both $S_+ \setminus S_0$ and $S_- \setminus S_0$ are non-empty;
- *spherical*, if W_{S_0} is finite.

Spherical ∞-decompositions can be characterised as follows.

Theorem 1.4.5 ([Dav08, Theorem 8.7.2], [MT09, Corollary 16]). *For a Coxeter group* (W, S) *the following are equivalent:*

(i) the group W *has at least* 2 *ends;*

(ii) (W, S) *has a non-trivial spherical* ∞*-decomposition.*

Having Fact 1.4.4(ii) in mind, we now present a generalisation of the concept of ∞-decompositions due to M. Mihalik and S. Tschantz [MT09].

Definition 1.4.6. A *graph of special subgroups of* (W, S) a graph of groups $(\mathcal{G}, \Lambda)$ assigning to each $\lambda \in V\Lambda \sqcup E\Lambda$ the special subgroup $\mathcal{G}_\lambda = W_{S_\lambda}$ of (W, S) and so that, for every $e \in E\Lambda$, the monomorphism $\mathcal{G}_e \hookrightarrow \mathcal{G}_{t(e)}$ is the inclusion map. This graph of groups $(\mathcal{G}, \Lambda)$ is said to be a *visual graph of groups decomposition* of (W, S) if the group homomorphism $\tau\colon \pi_1(\mathcal{G}, \Lambda) \to W$ induced by the inclusion maps $W_{S_\lambda} \hookrightarrow W$, $\lambda \in V\Lambda \sqcup E\Lambda$, is an isomorphism.

Remark 1.4.7. Let Λ be a 1-segment with $E\Lambda = \{e, \bar{e}\}$. Then, every visual graph of groups decomposition $(\mathcal{G}, \Lambda)$ based on Λ determines an ∞-decomposition $S = S_{o(e)} \cup S_{t(e)}$ of (W, S). Conversely, every ∞-decomposition $S = S_+ \cup S_-$ of (W, S) produces a visual graph of groups decomposition by taking the following graph of special subgroups $(\mathcal{G}, \Lambda)$ of (W, S):

$$\mathcal{G}_{o(e)} := W_{S_+}, \quad \mathcal{G}_{t(e)} := W_{S_-} \quad \text{and} \quad \mathcal{G}_e = \mathcal{G}_{\bar{e}} := W_{S_0}.$$

A visual graph of groups decomposition $(\mathcal{G}, \Lambda)$ of (W, S) is said to be *completely spherical* if all vertex-groups of $(\mathcal{G}, \Lambda)$ are finite. The counterpart of Theorem 1.4.5 for the completely spherical visual graph of groups decompositions is the following.

Theorem 1.4.8 ([MT09, Theorem 34]). *Let (W,S) be a Coxeter group. Then the following are equivalent:*

(i) *W is virtually free;*

(iii) *(W,S) admits a completely spherical visual graph of groups ∞-decomposition;*

(iv) *every clique $\Lambda \subseteq \Gamma_\infty(W,S)$ is spherical and $\Gamma_\infty(W,S)$ is chordal.*

1.5 Buildings

1.5.1 Background knowledge on buildings

Here below, we mainly follow [Dav08, §18.1].

Definition 1.5.1. A *chamber system over a set S* is a set $\mathcal{C}$ together with a family of equivalence relations $\{\sim_s\}_{s\in S}$ on $\mathcal{C}$ labelled by S. The elements of $\mathcal{C}$ are called *chambers*. A *building* $\Delta = (\mathcal{C}, \{\sim_s\}_{s\in S}, \delta)$ *of type* (W,S) consists of a chamber system $(\mathcal{C}, \{\sim_s\}_{s\in S})$ together with a function $\delta : \mathcal{C} \times \mathcal{C} \longrightarrow W$ (called *Weyl distance*) satisfying the following axioms:

(B1) for all $c \in \mathcal{C}$ and $s \in S$, there exists $d \in \mathcal{C}$ with $d \neq c$ such that $c \sim_s d$;

(B2) for every $w \in W$ and every reduced expression $w = s_1 \cdot \ldots \cdot s_n$ of w in (W,S), one has

$$\delta(c,d) = w$$

if, and only if, there exists a tuple of chambers $(c_0 = c, c_1, \ldots, c_n = d)$, with $n \geq 0$, such that $c_{i-1} \sim_{s_i} c_i$ and $c_{i-1} \neq c_i$, for every $1 \leq i \leq n$.

Remark 1.5.2. By (B2), note that $\delta(c,c) = 1$ for every $c \in \mathcal{C}$. More precisely,

$$\delta(c,d) = 1 \Longleftrightarrow c = d. \tag{1.5.1}$$

Moreover, given $s \in S$, (B2) implies that

$$\delta(c,d) = s \iff c \sim_s d \text{ and } c \neq d.$$

For this reason, while giving a building, we may specify only the set of chambers and the Weyl-distance.

Definition 1.5.3. The *chamber graph* Γ_Δ associated to a building $\Delta = (\mathcal{C}, \delta)$ (cf. [Kra22, Definition 4.1]) is the graph defined by setting $V\Gamma_\Delta := \mathcal{C}$ and

$$E\Gamma_\Delta := \{(c,d) \in \mathcal{C} \times \mathcal{C} \mid \delta(c,d) \in S\}.$$

The origin and terminus maps of Γ_Δ are the projections on the first and second coordinates, respectively. The edge-inversion map of Γ_Δ is the interchange of the first and second coordinates.

Since $\delta(d,c) = \delta(c,d)^{-1}$ for all $c, d \in \mathcal{C}$ and the elements of S are involutions in W, note that the edge-inversion map is well-defined.

Fact 1.5.4 ([Dav08, Example 18.1.6], [Bro89, p. 87])**.** *Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S). Then the following are equivalent:*

(a) (W, S) is of type $\tilde{A}_1$, i.e., $S = \{s, t\}$ and $W = \langle s, t \mid s^2 = t^2 = 1 \rangle \simeq D_\infty$;

(b) Γ_Δ is a tree without leaves (in the sense of J-P. Serre).

Convention 1.5.5. After Fact 1.5.4, with a slight abuse of notation, we use the terms "buildings of type $\tilde{A}_1$" and "trees without leaves" (in the sense of J-P. Serre) interchangeably.

Definition 1.5.6. A *gallery γ (from a chamber c to a chamber d)* in a building $\Delta = (\mathcal{C}, \delta)$ is a sequence of chambers $(c_0 = c, c_1, \ldots, c_n = d)$ that satisfy $\delta(c_{i-1}, c_i) = s_i \in S$, for every $1 \leq i \leq n$. The sequence $(s_1, \ldots, s_n)$ is called *type* of γ.

Note that (B2) rephrases as follows for the relevant $w = s_1 \cdot \ldots \cdot s_n$: one has $\delta(c,d) = w$ if, and only if, there is a gallery in Δ from c to d of type $(s_1, \ldots, s_n)$.

Definition 1.5.7 ([AB08, Definition 5.26])**.** For all $J \subseteq S$ and $c \in \mathcal{C}$, the *J-residue of Δ centred at c* is defined as

$$\mathrm{Res}_J(c) := \{d \in \mathcal{C} \mid \delta(c,d) \in W_J\}.$$

In other words, $\mathrm{Res}_J(c)$ is the collection of all $d \in \mathcal{C}$ for which there is a gallery from c to d in Δ whose type $(s_1, \ldots, s_n)$ either is the empty sequence or satisfies $s_i \in J$, for every $1 \leq i \leq n$.

In particular,

$$\mathrm{Res}_J(c) = \mathrm{Res}_J(d) \iff \delta(c,d) \in W_J. \tag{1.5.2}$$

A J-residue is called *spherical* if J is a spherical subset of S (cf. Section 1.4.1). Denote by $\mathcal{R}$ the set of all residues of Δ, by $\mathcal{R}_{\mathrm{sph}}$ the set of all spherical ones and, given $J \subseteq S$, let $\mathcal{R}_J$ be the set of all J-residues of Δ.

Definition 1.5.8. A building Δ is said to be *thin* (resp. *thick*, *locally finite*) if all $\{s\}$-residues have two (resp. at least three, finitely many) elements, for every $s \in S$.

Definition 1.5.9. Given two buildings $\Delta = (\mathcal{C}, \delta)$ and $\Delta' = (\mathcal{C}', \delta')$ of type (W, S), a map $\varphi\colon \mathcal{C} \to \mathcal{C}'$ is a *W-isometry* if, for all $c_1, c_2 \in \mathcal{C}$,

$$\delta'(\varphi(c_1), \varphi(c_2)) = \delta(c_1, c_2). \tag{1.5.3}$$

By (1.5.1), a W-isometry is necessarily injective. In particular, Δ and Δ' are said to be *isomorphic* if there is a surjective W-isometry $\varphi\colon \mathcal{C} \to \mathcal{C}'$.

Example 1.5.10 ([Dav08, Example 18.1.3])**.** The *abstract Coxeter complex of type* (W, S) is the building $\Sigma(W, S)$ with set of chambers W and with Weyl distance δ_W defined as follows:

$$\delta_W(w_1, w_2) = w_1^{-1} w_2, \quad \forall\, w_1, w_2 \in W.$$

By [AB08, Proposition 5.65], every thin building of type (W, S) is isomorphic to $\Sigma(W, S)$.

Example 1.5.11 ([Dav08, Example 18.1.11])**.** Let $\mathbb{F}$ be a field and $n \in \mathbb{Z}_{\geq 2}$. Set $V = \mathbb{F}^n$ and denote by $\mathcal{C}$ the set of all complete flags of subspaces of V, namely

$$V_1 \subset \ldots \subset V_{n-1} \subset V_n = V,$$

where $\dim_{\mathbb{F}} V_k = k$ for every $k \geq 1$. Given $i \in [n]$, we say that two complete flags $(V_k)_{k\in[n]}$ and $(W_k)_{k\in[n]}$ are i*-adjacent* if $V_i \neq W_i$ and $V_k = W_k$ for all $k \in [n] \setminus \{i\}$. For every $i \in [n]$, "being i-adjacent" yields an equivalence relation denoted by $\sim_i$. Hence $\Delta_n := (\mathcal{C}, \{\sim_i\}_{i\in[n]})$ is a chamber system. More precisely, it is a building of type A_n with Weyl-distance δ defined below. For all chambers $c = (V_k)_{k\in[n]}$ and $d = (W_k)_{k\in[n]}$, put $V_0 = W_0 = \{0\}$ and define a map $\sigma_{(c,d)} \colon [n] \to [n]$ by setting:

$$\sigma_{(c,d)}(i) := \min\{j \mid W_i \subset W_{i-1} + V_j\}, \quad \forall\, i \in [n]. \tag{1.5.4}$$

For all $c, d \in \mathcal{C}$, $\sigma_{(c,d)}$ is a permutation in S_n and therefore one may define

$$\delta(c, d) = \sigma_{c,d}.$$

Following [Dav08, p. 333], for every building $\Delta = (\mathcal{C}, \delta)$ of type (W, S) there is a W-isometry $W \to \mathcal{C}$ mapping 1_W to a prescribed chamber $c \in \mathcal{C}$. An *apartment* of Δ is any W-isometric image of W in $\mathcal{C}$.

Fact 1.5.12 ([Dav08, p. 333], [Kra22, §4])**.** *Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S).*

(i) For every W-isometry $\alpha \colon W \to \mathcal{C}$, the chamber system $(\alpha(W), \delta|_{\alpha(W)})$ is a thin building of type (W, S).

(ii) There exists a collection of apartments $\mathfrak{A}$, called atlas *of Δ, with the following property: every two chambers c and d of Δ are both contained in the set of chambers of some $\Sigma \in \mathfrak{A}$. In particular, one has that*

$$\mathcal{C} = \bigcup_{\Sigma\in\mathfrak{A}} \mathcal{C}(\Sigma),$$

where $\mathcal{C}(\Sigma)$ is the set of chambers of the apartment Σ.

1.5.2 The Davis' complex of a building

A possible way to associate a simplicial complex to a building $\Delta = (C, \delta)$ of type (W, S) is the following.

Definition 1.5.13 ([Kra22, §5])**.** The *Davis' complex* Δ_{Dav} *of* Δ is the simplicial complex whose k-simplices, for $k \geq 0$, are all chains of spherical residues

$$\mathrm{Res}_{J_0}(c) \subsetneq \cdots \subsetneq \mathrm{Res}_{J_k}(c), \tag{1.5.5}$$

where $c \in \mathcal{C}$ and $J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_m$.

The simplicial complex Δ_{Dav} is locally finite if, and only if, Δ is locally finite (cf. [Kra22, p. 22]). By [Dav08, Corollary 18.3.6], the geometric realisation $|\Delta_{\mathrm{Dav}}|$ of Δ_{Dav} is contractible (with respect to the weak topology) and is usually called the *Davis' realisation of* Δ. Moreover, the augmented cellular chain complex associated to $|\Delta_{\mathrm{Dav}}|$ is exact (cf. [CW16, §A.3]).

1.6 Groups acting on buildings

1.6.1 Background notions

Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S).

Definition 1.6.1. The *group of type-preserving automorphisms* of Δ is defined as

$$\mathrm{Aut}_0(\Delta) := \{g \mid g : \mathcal{C} \to \mathcal{C} \text{ surjective } W\text{-isometry of } \Delta\}. \tag{1.6.1}$$

The group $\mathrm{Aut}_0(\Delta)$ is a totally disconnected group with the permutation topology induced by $\mathrm{Sym}(\mathcal{C})$. If Δ is locally finite, then for every finite subset F of $\mathcal{C}$ the group $\mathrm{Aut}_0(\Delta)_F$ is profinite, and therefore $\mathrm{Aut}_0(\Delta)$ is t.d.l.c. (cf. [Kra22, Theorem 5.1]).

Definition 1.6.2. A topological group G *acts on* Δ if there is a continuous group homomorphism $\varphi\colon G \to \mathrm{Aut}_0(\Delta)$. A G-action on a building Δ is said to be *transitive* (or *chamber-transitive*) if the associated G-action on $\mathcal{C}$ is transitive. Moreover, a G-action on a building Δ is said to be *cocompact* (resp. *proper*) if the associated G-action on $\mathcal{C}$ has finitely many orbits (resp. has compact open stabilisers).

Note that, if G has a transitive action of a building $\Delta = (\mathcal{C}, \delta)$ of type (W, S), then for all $c \in \mathcal{C}$, $g \in G$ and subset $J \subseteq S$, one has

$$\mathrm{Res}_J(g \cdot c) = g \cdot \mathrm{Res}_J(c). \tag{1.6.2}$$

1.6.2 Chamber-transitivity and local thickness

Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S) admitting a transitive group action. For every $s \in S$ and for an arbitrarily chosen $c \in \mathcal{C}$, define

$$q_s := |\{d \in \mathcal{C} \,:\, \delta(c,d) = s\}| = |\mathrm{Res}_{\{s\}}(c)|, \tag{1.6.3}$$

Note that the assignment in (1.6.3) does not depend on the choice of c in $\mathcal{C}$ (cf. 1.6.2). The vector (of cardinal numbers) $\mathbf{q} = (q_s)_{s\in S}$ is called the *thickness vector of* Δ. In particular, $\mathbf{q}$ is said to be *uniform* if $q_s = q_t$ for all $s, t \in S$. With an abuse of notation, we say that "Δ has uniform thickness q" (where $q = q_s$ for every $s \in S$).

Note that

$$\begin{array}{ccc} \Delta \text{ is locally finite} & \iff & q_s < \infty, \quad \forall\, s \in S; \\ \Delta \text{ is thin} & \iff & q_s = 1, \quad \forall\, s \in S; \\ \Delta \text{ is thick} & \iff & q_s \geq 2, \quad \forall\, s \in S. \end{array}$$

1.6.3 Weyl-transitive actions and Bruhat decompositions

Definition 1.6.3 ([AB08, §6.1.3])**.** a A topological group G acts *Weyl-transitively* on Δ if, for every $w \in W$, the G-action is transitive on the set

$$\{(c,d) \in \mathcal{C} \times \mathcal{C} \mid \delta(c,d) = w\}, \tag{1.6.4}$$

Note that a Weyl-transitive action on Δ is transitive.

For buildings of type $\tilde{A}_1$ (i.e., for trees without leaves, see Fact 1.5.4), we have the following characterisation.

Lemma 1.6.4 ([Ser80, §II.1.7, Exercises])**.** *A group action on a tree (G, T) is Weyl-transitive if, and only if, it is locally ∞-transitive (cf. Section 1.3.1).*

In group-theoretical terms, Weyl-transitive actions can be characterised as follows.

Definition 1.6.5 ([AB08, Definition 6.31])**.** Let G be a group and B be a subgroup of G. A *Bruhat decomposition of type* (W, S) (written $G = BWB$) is a bijection $C\colon W \to B\backslash G/B$ with the property, for all $s \in S$ and $w \in W$, that

$$C(sw) \subseteq C(s) \cdot C(w) \subseteq C(sw) \cup C(w),$$

and such that $C(sw) = C(s) \cdot C(w)$ whenever $\ell(s \cdot w) = \ell(w) + 1$. We call (W, S) the *Weyl group* (or the *type*) of the Bruhat decomposition $G = BWB$.

Note that C induces an injective function $W \hookrightarrow G$ mapping each $w \in W$ to a representative of the B-double coset $C(w)$. The easiest example of a Bruhat decomposition of type (W, S) is given by taking $G = W$ and $B = \{1\}$.

Notation 1.6.6. In this thesis, given a Bruhat decomposition $G = BWB$, we regard W as a subset of G. In particular, we write $C(w) = BwB$ for every $w \in W$. Thus G decomposes as a disjoint union of B-double cosets as follows:

$$G = \bigsqcup_{w \in W} BwB.$$

Let $G = BWB$ be a Bruhat decomposition of type (W, S). By [AB08, Proposition 6.34], the coset space G/B is the set of chambers of a building $\Delta(G, B)$ of type (W, S) on which G acts Weyl-transitively by left translation.

Definition 1.6.7. We call $\Delta(G, B)$ the *building associated to the Bruhat decomposition* $G = BWB$. In particular, the subgroup B is the stabiliser of the chamber $c_0 = 1B$, usually called the *fundamental chamber* of $\Delta(G, B)$. A Bruhat decomposition $G = BWB$ is called *thin* (resp. *thick, locally finite*) if $\Delta(G, B)$ is a thin (resp. thick, locally finite) building.

Conversely, let G be a group acting Weyl-transitively on a building of type (W, S) and denote by B the stabiliser of a chamber. Again by [AB08, Proposition 6.34], G admits a Bruhat decomposition $G = BWB$ of type (W, S) and Δ is isomorphic to $\Delta(G, B)$.

Essentially, there is a bijective correspondence between Weyl-transitive actions and Bruhat decompositions. ("Essentially" because a Weyl-transitive group action on a building induces a Bruhat decomposition for each choice of a chamber). In this thesis, we will use these concepts interchangeably.

A notable property of every Bruhat decomposition of type (W, S), say $G = BWB$, is the following: for every $I \subseteq S$, the following subset

$$P_I := \bigsqcup_{w \in W_I} BwB \tag{1.6.5}$$

is a subgroup of G.

Definition 1.6.8. The group P_I in (1.6.5) is called the *standard I-parabolic subgroup* of G (with respect to the Bruhat decomposition $G = BWB$).

If $G = W$ and $B = \{1\}$, note that $P_I = W_I$ is the special subgroup of W generated by I.

Proposition 1.6.9. *Let $G = BWB$ be a thick Bruhat decomposition of type (W, S), and let $I \subseteq S$.*

(i) The group P_I admits a Bruhat decomposition $P_I = BW_IB$ of type (W_I, I). Moreover, $\Delta(P_I, B)$ is isomorphic to the residue $\mathrm{Res}_I(c_0)$ of $\Delta(G, B)$, where c_0 is the fundamental chamber of $\Delta(G, B)$.

(ii) ([AB08, Theorem 6.43(2)]) *Every $g \in G$ satisfying $B \subseteq gP_Ig^{-1}$ belongs to P_I.*

(iii) ([Dav08, Corollary 18.1.18]) *Assume that the thickness vector* $\mathbf{q} = (q_s)_{s\in S}$ *associated to the Bruhat decomposition* $G = BWB$ *has finite entries. Then, for every spherical subset* $I \subseteq S$,

$$|P_I : B| = |\mathrm{Res}_I(c_0)| = W_I(\mathbf{q}),$$

where $\mathrm{Res}_I(c_0)$ *is the* I*-residue of* $\Delta(G, B)$ *(cf. Definition 1.6.7) centred at the fundamental chamber* $c_0 = 1B$*, and* $W_I(\mathbf{q})$ *is the generalised Poincaré series of* $W_I \subseteq W$ *evaluated at* $\mathbf{q}$ *(cf. 1.4.1).*

Remark 1.6.10. For completeness, we mention that there is another notable transitivity condition that one can assume on a group action on a building, that is *strongly transitivity.* We will never really need this concept in this thesis. The reader may refer for instance to [AB08, §6.1.1] for details.

We just mention that all strongly transitive actions are always Weyl-transitive, although the converse does not hold in general (cf. [AB08, §6.10]). However, strongly transitive actions are Weyl-transitive if either Δ is a spherical building (cf. [AB08, Proposition 6.15]) or Δ is an Euclidean building and the relevant group acts with compact stabilisers on its set of chambers (cf. [Kra22, Theorem 5.16]).

According to [AB08, Theorem 6.56], there is a bijective correspondence between strongly transitive actions on buildings and BN-pairs. The latter ones are particular Bruhat decompositions introduced, for instance, in [AB08, §6.2.6]. Every Bruhat decomposition mentioned in the next Example 1.6.11 is a BN-pair.

In the following, we record (almost all) the main examples of t.d.l.c. groups having a Bruhat decomposition known so far.

Example 1.6.11.

(i) Let (W, S) be a Coxeter group. Then the pair $(W, \{1\})$ clearly admits a Bruhat decomposition of type (W, S). In addition, $\Delta(W, \{1\})$ is isomorphic to the standard Coxeter complex $\Sigma(W, S)$ (cf. Section 1.5.1).

(ii) Let $\mathbb{F}$ be a field. For $n \geq 1$, let $G = \mathrm{GL}_{n+1}(\mathbb{F})$ and denote by B the subgroup of G of all upper triangular matrices. Then (G, B) admits a Bruhat decomposition of type A_n. The Weyl group W can be identified with the subgroup of G of all permutation matrices, i.e., matrices in $\mathrm{GL}_{n+1}(\mathbb{F})$ obtained from the identity I_{n+1} by permuting its rows and then its columns according to a prescribed element in the symmetric group S_{n+1}. Moreover, $\Delta(G, B)$ is isomorphic to the building in Example 1.5.11, and the Davis' realisation of $\Delta(G, B)$ is isomorphic to the flag complex of the vector space $\mathbb{F}^{n+1}$. The reader is referred to [Bro89, §V.5] for details.

(iii) Let $\mathcal{G}$ be a semisimple algebraic group defined over a non-Archimedean local field K, and let $G = \mathcal{G}(K)$ be the set of K-points of $\mathcal{G}$. For instance, one may think of $\mathcal{G} = \mathrm{SL}_n$ for some $n \geq 1$. Then G has a subgroup I, called the *standard Iwahori* *subgroup*, such that (G, I) has a Bruhat decomposition of affine type. It is worth remarking that $\Delta(G, I)$ is isomorphic to so-called the Bruhat–Tits building of G. One may refer to [AB08, §C.11] for details.

For example, let $G = \mathrm{SL}_n(K)$ $(n \geq 2)$ and denote by R the ring of integers of K with a prescribed uniformiser π. Then I is the group of all matrices in $\mathrm{SL}_n(R)$ that are congruent to an upper triangular matrix modulo π. In this case, $(\mathrm{SL}_n(K), I)$ has a Bruhat decomposition of type $\tilde{A}_{n-1}$. See [AB08, §6.9.2] for details.

(iv) Here we follow [BRW05, §1.2]. Let G be a group with a locally finite twin root datum of (crystallographic) type (W, S). Then G admits a Weyl-transitive (actually, strongly transitive) action on the positive part Δ_+ of a twin building of type (W, S). The building Δ_+ is locally finite and of type (W, S). Denote by $\bar{G}$ the closure of the image of G in $\mathrm{Aut}_0(\Delta_+)$. Then $\bar{G}$ acts Weyl-transitively (actually, strongly transitively) on Δ_+. Provided $\bar{B}$ is the stabiliser of a chamber of Δ_+, the pair $(\bar{G}, \bar{B})$ admits a Bruhat decomposition of type (W, S). The topological group $\bar{G}$ is called *geometric completion of* G (cf. [Mar18, p. 187]). For example, if G is an abstract Kac–Moody group defined over a finite field, then $\bar{G}$ gives the so-called *complete Kac–Moody group* [RR06].

(v) Let Δ be a right-angled building, i.e., the type (W, S) of Δ is a right-angled Coxeter group. Then $G = \mathrm{Aut}_0(\Delta)$ acts Weyl-transitively (actually, strongly transitively) on Δ (cf. [Cap14, Proposition 6.1]). Equivalently, if B denotes the stabiliser in G of a chamber of Δ, then (G, B) admits a Bruhat decomposition of type (W, S) and $\Delta(G, B)$ is isomorphic to Δ.

(vi) Let $d \geq 2$ and, given a subgroup $F \leq \mathrm{Sym}(d)$, let $U(F) \leq \mathrm{Aut}(T_d)$ be the Burger–Mozes universal group associated to the group F. Then $U(F)$ acts Weyl-transitively on T_d if, and only if, U is 2-transitive (cf. [BM00, §3.1] and Lemma 1.6.4).

1.7 From buildings to trees

Generalising an idea due to F. Haglund and F. Paulin [HP03], we present a new method for constructing trees from buildings whose type (W, S) admits a visual graph of groups decomposition (cf. Section 1.4.2). Our discussion culminates with Theorem 1.7.10. As a consequence, we describe how to induce transitive actions on a building whose type admits a visual graph of groups decomposition $W = \pi_1(\mathcal{G}, \Lambda)$ to actions on the tree constructed from the relevant building (cf. Corollary 1.7.12).

Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S). For $T \subseteq S$ and $\mathcal{M} \subseteq \mathcal{C}$, set

$$\mathcal{R}_T(\mathcal{M}) := \{\mathrm{Res}_T(c) \mid c \in \mathcal{M}\}, \tag{1.7.1}$$

(cf. Section 1.5.1). If necessary, one writes $\mathrm{Res}_T^{\Delta}(c)$ and $\mathcal{R}_T^{\Delta}(\mathcal{M})$ in place of $\mathrm{Res}_T(c)$ and $\mathcal{R}_T(\mathcal{M})$, respectively.

Definition 1.7.1. Let (W,S) be a Coxeter group and $\Delta=(\mathcal{C},\delta)$ be a building of type (W,S) and $(\mathcal{G},\Lambda)$ be a graph of special subgroups of (W,S). For every non-empty subset $\mathcal{M}\subseteq\mathcal{C}$ define a graph $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{M})=(V\mathcal{M},E\mathcal{M})$ in the sense of J-P. Serre as follows:

$$V\mathcal{M}:=\bigsqcup_{v\in V\Lambda}\mathcal{R}_{S_v}(\mathcal{M})\quad\text{and}\quad E\mathcal{M}:=\bigsqcup_{e\in E\Lambda}\mathcal{R}_{S_e}(\mathcal{M}).$$

For every $e\in E\Lambda$, although $S_e=S_{\bar{e}}$, here $\mathcal{R}_{S_e}(\mathcal{M})$ and $\mathcal{R}_{S_{\bar{e}}}(\mathcal{M})$ are regarded as disjoint copies of the same set. The edge-reversing, the origin and the terminus maps are defined, respectively, as follows:

$$\overline{\mathrm{Res}_{S_e}(c)}:=\mathrm{Res}_{S_{\bar{e}}}(c),\quad o(\mathrm{Res}_{S_e}(c)):=\mathrm{Res}_{S_{o(e)}}(c),\quad t(\mathrm{Res}_{S_e}(c)):=\mathrm{Res}_{S_{t(e)}}(c),\qquad(1.7.2)$$

for all $e\in E\Lambda$ and $c\in\mathcal{M}$.

Remark 1.7.2. Since W is generated by involutions, if $(\mathcal{G},\Lambda)$ is a visual graph of groups decomposition, then Λ is necessarily a tree.

Remark 1.7.3.

(i) The assignments in (1.7.2) are well defined. For the inversion map, it is clear. Moreover, for $e\in E\Lambda$ and $v\in\{o(e),t(e)\}$, observe that $\mathrm{Res}_{S_e}(c)=\mathrm{Res}_{S_e}(d)$ implies $\delta(c,d)\in W_{S_e}\subseteq W_{S_v}$ and then, by (1.5.2), $\mathrm{Res}_{S_v}(c)=\mathrm{Res}_{S_v}(d)$.

(ii) If Λ is a combinatorial graph, so is $\Gamma=\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{M})$. In other terms, if the map $e\in E\Lambda\longmapsto(o(e),t(e))\in V\Lambda\times V\Lambda$ is injective, so is the map $\mathrm{Res}_{S_e}(c)\in E\Gamma\longmapsto(\mathrm{Res}_{S_{o(e)}}(c),\mathrm{Res}_{S_{t(e)}}(c))\in V\Gamma\times V\Gamma$. Indeed, $\mathrm{Res}_{S_{o(e)}}(c)\cap\mathrm{Res}_{S_{t(e)}}(c)=\mathrm{Res}_{S_e}(c)$ for all $e\in E\Lambda$ and $c\in\mathcal{M}$.

Example 1.7.4. Given a Coxeter group (W,S), let $S_+,S_-\subseteq S$ such that $S=S_+\cup S_-$, and put $S_0=S_+\cap S_-$. Let Λ be a 1-segment (in the sense of J-P. Serre) with edge-pair $\{e,\bar{e}\}$ and define a graph of groups $(\mathcal{G},\Lambda)$ by putting $\mathcal{G}_{o(e)}=W_{S_+}$, $\mathcal{G}_{t(e)}=W_{S_+}$ and $\mathcal{G}_e=\mathcal{G}_{\bar{e}}=W_{S_0}$. Let $\Sigma=\Sigma(W,S)=(W,\delta_W)$ be the abstract Coxeter complex of type (W,S) (cf. Example 1.5.10). Recall that for every $\times\in\{+,-,0\}$, the residue $\mathrm{Res}^{\Sigma}_{S_\times}(w)$ can be identified with the left coset $wW_{S_\times}$.

(i) The graph $\Gamma^{\Sigma}_{(\mathcal{G},\Lambda)}(W)$ is isomorphic to the graph $X=X(W;W_{S_+},W_{S_-})$ defined in [Ser80, §I.4, proof of Theorem 7]. By [Ser80, §I.4, Theorems 6 and 7],

$$\Gamma^{\Sigma}_{(\mathcal{G},\Lambda)}(W)\text{ is a tree}\iff W\simeq W_{S_+}*_{W_{S_0}}W_{S_-}.$$

(ii) More generally, let $\Delta=(\mathcal{C},\delta)$ be a building of type (W,S). Then $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{C})$ coincides with the graph introduced by F. Haglund and F. Paulin in [HP03, §4.2]. By [HP03, p. 146, Remarques] and [HP03, Lemme 4.3] (or Corollary 1.7.11 below),

$$\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{C})\text{ is a tree}\iff W\simeq W_{S_+}*_{W_{S_0}}W_{S_-}.$$

From now on, the graph $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{C})$ constructed here will be denoted by $\Gamma^{\Delta}_{\pm}(\mathcal{C})$.

Lemma 1.7.5. *Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S), let $\mathcal{M} \subseteq \mathcal{C}$ be non-empty and $(\mathcal{G}, \Lambda)$ be a graph of special subgroups of (W, S). Denote by $\Sigma(W, S)$ the standard Coxeter complex of type (W, S).*

(i) Every W-isometry $\alpha\colon W \to \mathcal{C}$ induces a graph isomorphism

$$\varphi_\alpha : \Gamma^{\Sigma(W,S)}_{(\mathcal{G},\Lambda)}(W) \longrightarrow \Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\alpha(W))$$

mapping wW_{S_x} to $\mathrm{Res}^{\Delta}_{S_x}(\alpha(w))$, for all $x \in V\Lambda \sqcup E\Lambda$ and $w \in W$.

(ii) For $c \in \mathcal{C}$, let $\mathcal{A}_c$ be the set of all apartments in an atlas of Δ containing c as a chamber (cf. Fact 1.5.12(ii)). Then

$$\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{C}) = \bigcup_{\Sigma \in \mathcal{A}_c} \Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{C}(\Sigma)).$$

Proof. Part (ii) is straightforward from $\mathcal{C} = \bigcup_{\Sigma \in \mathcal{A}_c} \mathcal{C}(\Sigma)$ (cf. Fact 1.5.12(ii)). Thus, it suffices to prove (i). For every $x \in V\Lambda \sqcup E\Lambda$, the assignment

$$\varphi_{\alpha,x} : wW_{S_x} \longmapsto \mathrm{Res}^{\Delta}_{S_x}(\alpha(w)), \quad \forall\, w \in W$$

yields a well-defined bijection from $\mathcal{R}^{\Sigma(W,S)}_{S_x}(W)$ to $\mathcal{R}^{\Delta}_{S_x}(\alpha(W))$. Indeed, $\varphi_{\alpha,x}$ is well-defined and injective by the following:

$$\begin{aligned} w_1 W_{S_x} = w_2 W_{S_x} \iff & \delta_W(w_1, w_2) = \delta(\alpha(w_1), \alpha(w_2)) \in W_{S_x} \\ \iff & \mathrm{Res}^{\Delta}_{S_x}(\alpha(w_1)) = \mathrm{Res}^{\Delta}_{S_x}(\alpha(w_2)), \end{aligned}$$

for all $w_1, w_2 \in W$ (cf. (1.5.2)). Finally, the surjectivity of $\varphi_{\alpha,x}$ is part of the definition. □

Lemma 1.7.6. *Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S), $\mathcal{M} \subseteq \mathcal{C}$ be non-empty, and $(\mathcal{G}, \Lambda)$ be a graph of special subgroups of (W, S). Assume that $S = \bigcup_{v \in V\Lambda} S_v$. Let $\gamma = (c_0, c_1, \ldots, c_n)$ be a gallery in Δ with $c_i \in \mathcal{M}$ for every $0 \leq i \leq n$. Then, for all $v, w \in V\Lambda$, γ induces a path $\mathfrak{p}_{\gamma, v \to w}$ in $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{M})$ from $\mathrm{Res}_{S_v}(c_0)$ to $\mathrm{Res}_{S_w}(c_n)$.*

Proof. It suffices to show the statement for $E\Lambda \neq \emptyset$ and for a gallery $\gamma = (c, d)$ of length 2. That is, given $c, d \in \mathcal{M}$ with $\delta(c, d) = s \in S$ and $v, w \in V\Lambda$, there is a path in $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{M})$ from $\mathrm{Res}_{S_v}(c)$ to $\mathrm{Res}_{S_w}(d)$. By hypothesis, there is $z \in V\Lambda$ such that $s \in S_z$, and $\mathrm{Res}_{S_z}(c) = \mathrm{Res}_{S_z}(d)$ (cf. (1.5.2)). If $v = w = z$, the path $\mathfrak{p}_{\gamma, v \to w}$ is the trivial path. In all the remaining cases, let $(e_1, \ldots, e_n)$ be a path in Λ from v to w that runs through z, i.e., $o(e_m) = z$ for some $1 \leq m \leq n$. If $m = 1$, then $v = z$ and $\mathrm{Res}_{S_v}(c) = \mathrm{Res}_{S_z}(c)$ is the origin of $\mathrm{Res}_{S_{e_1}}(d)$ in $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{M})$ (cf. (1.7.2)). Thus the sequence

$$(\mathrm{Res}_{S_{e_1}}(d), \ldots, \mathrm{Res}_{S_{e_n}}(d))$$

gives a path in $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{M})$ from $\mathrm{Res}_{S_v}(c)$ to $\mathrm{Res}_{S_w}(d)$. Similarly, if $m \geq 2$ one has $t(e_{m-1}) = z = o(e_m)$ and then

$$\mathrm{Res}_{S_{t(e_{m-1})}}(c) = \mathrm{Res}_{S_z}(c) = \mathrm{Res}_{S_z}(d) = \mathrm{Res}_{S_{o(e_m)}}(d).$$

Hence the sequence

$$(\mathrm{Res}_{S_{e_1}}(c), \dots, \mathrm{Res}_{S_{e_{m-1}}}(c), \mathrm{Res}_{S_{e_m}}(d), \dots, \mathrm{Res}_{S_{e_n}}(d))$$

defines a path in $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{M})$ from $\mathrm{Res}_{S_v}(c)$ to $\mathrm{Res}_{S_w}(d)$. □

Definition 1.7.7 ([AB08, Definition 5.43])**.** Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S). A non-empty subset $\mathcal{M} \subseteq \mathcal{C}$ is *gallery connected* if, for all $c, d \in \mathcal{M}$, there exists a gallery $(c_0 = c, c_1, \dots, c_k = d)$ in Δ with $c_i \in \mathcal{M}$ for all $0 \leq i \leq k$.

For example, the set $\mathcal{C}$ is gallery connected (cf. [AB08, Example 5.44(a)]).

Lemma 1.7.6 implies the following.

Corollary 1.7.8. *Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S) and $(\mathcal{G}, \Lambda)$ be a graph of special subgroups of (W, S). Assume that $S = \bigcup_{v \in V\Lambda} S_v$. Then, for every gallery connected subset $\mathcal{M} \subseteq \mathcal{C}$, the graph $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{M})$ is connected. In particular, $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{C})$ is connected.*

Moreover, consider a collection $\{\Sigma_i = (\mathcal{C}(\Sigma_i), \delta|_{\Sigma_i})\}_{i \in I}$ of apartments of Δ satisfying $\bigcap_{i \in I} \mathcal{C}(\Sigma_i) \neq \emptyset$. Then

$$\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}\left(\bigcap_{i \in I} \mathcal{C}(\Sigma_i)\right) = \bigcap_{i \in I} \Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{C}(\Sigma_i)),$$

$$\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}\left(\bigcup_{i \in I} \mathcal{C}(\Sigma_i)\right) = \bigcup_{i \in I} \Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{C}(\Sigma_i)),$$

and $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}\left(\bigcap_{i \in I} \mathcal{C}(\Sigma_i)\right)$ and $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}\left(\bigcup_{i \in I} \mathcal{C}(\Sigma_i)\right)$ are both connected.

Proof. For the first part of the statement, apply Lemma 1.7.6. The second part of the statement follows from the first part and the fact that, since $\bigcap_{i \in I} \mathcal{C}(\Sigma_i) \neq \emptyset$, the sets $\bigcap_{i \in I} \mathcal{C}(\Sigma_i)$ and $\bigcup_{i \in I} \mathcal{C}(\Sigma_i)$ are gallery connected (cf. [AB08, Example 5.44(c)]). □

Proposition 1.7.9. *Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S) and $(\mathcal{G}, \Lambda)$ be a graph of special subgroups of (W, S). Assume that $S = \bigcup_{v \in V\Lambda} S_v$. Then $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{C})$ is a tree if, and only if, $\Gamma^{\Sigma(W,S)}_{(\mathcal{G},\Lambda)}(W)$ is a tree.*

Proof. Let first $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{C})$ be a tree and $\alpha\colon W \to \mathcal{C}$ be a W-isometry. Then $\alpha(W)$ defines an apartment of Δ. By Corollary 1.7.8 one has that $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\alpha(W))$ is connected and thus a tree. By Lemma 1.7.5(i), also $\Gamma^{\Sigma(W,S)}_{(\mathcal{G},\Lambda)}(W)$ is a tree.

Suppose conversely that $\Gamma^{\Sigma(W,S)}_{(\mathcal{G},\Lambda)}(W)$ is a tree. For $c \in \mathcal{C}$, let $\mathcal{A}_c$ be the collection of all apartments in some atlas of Δ having c as a chamber. By Lemma 1.7.5, for every $\Sigma \in \mathcal{A}_c$ the graph $\Gamma_\Sigma := \Gamma_{(\mathcal{G},\Lambda)}(\mathcal{C}(\Sigma))$ is a subtree of $\Gamma := \Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{C})$, and $\Gamma = \bigcup_{\Sigma\in\mathcal{A}_c} \Gamma_\Sigma$. To conclude that Γ is a tree, it is sufficient to show that, for every finite non-empty subset $\mathcal{S} \subseteq \mathcal{A}_c$, the graph $\Gamma_{\mathcal{S}} := \bigcup_{\Sigma\in\mathcal{S}} \Gamma_\Sigma$ is a subtree of Γ. In fact, every closed path in Γ without backtrackings must be contained in the union of finitely many Γ_Σ's. We proceed by induction on the cardinality of $\mathcal{S}$. For $|\mathcal{S}| = 1$ there is nothing to prove. Let now $\mathcal{S} \subseteq \mathcal{A}_c$ with $|\mathcal{S}| \geq 2$ and suppose that the claim holds for all subsets of $\mathcal{A}_c$ of cardinality $|\mathcal{S}| - 1$. Let $\Sigma \in \mathcal{S}$ and $\mathcal{S}' := \mathcal{S} \setminus \{\Sigma\}$. By induction, both Γ_Σ and $\Gamma_{\mathcal{S}'}$ are trees. Moreover, by Corollary 1.7.8,

$$\Gamma_\Sigma \cap \Gamma_{\mathcal{S}'} = \bigcup_{\Sigma'\in\mathcal{S}} (\Gamma_\Sigma \cap \Gamma_{\Sigma'})$$

is a non-empty connected subgraph of Γ_Σ and thus a tree. By van Kampen's theorem, $\Gamma_{\mathcal{S}} = \Gamma_\Sigma \cup \Gamma_{\mathcal{S}'}$ is a tree. □

The following theorem generalises a result due to F. Haglund and F. Paulin (cf. Corollary 1.7.11) to arbitrary visual graph of groups decompositions.

Theorem 1.7.10. *Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S) and $(\mathcal{G}, \Lambda)$ be a graph of special subgroups of (W, S). Then the following conditions are equivalent:*

(i) $(\mathcal{G}, \Lambda)$ is a visual graph of groups decomposition of (W, S);

(ii) $S = \bigcup_{v\in V\Lambda} S_v$ and $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{C})$ is a tree.

Proof. Note that if $(\mathcal{G}, \Lambda)$ is a visual graph of groups decomposition of (W, S), then W is generated by the union of the vertex groups and therefore $S = \bigcup_{v\in V\Lambda} S_v$ (cf. [BB06, Corollary 1.4.8(iii)]). The statement is a consequence of Proposition 1.7.9 and the main theorem of Bass–Serre theory (cf. [Ser80, §I.5.4, Theorem 13]), because (i) is equivalent to the fact that $\Gamma^{\Sigma(W,S)}_{(\mathcal{G},\Lambda)}(W)$ is a tree. Indeed, as $\mathcal{R}^{\Sigma(W,S)}_T(W) = W/W_T$ for $T \subseteq S$, the graph $\Gamma^{\Sigma(W,S)}_{(\mathcal{G},\Lambda)}(W)$ is the universal Bass–Serre graph of $(\mathcal{G}, \Lambda)$ (cf. [Ser80, p. 51]). □

Corollary 1.7.11 ([HP03, Lemme 4.3, Remarques at p. 146])**.** *Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S) and $S_+, S_- \subseteq S$ such that $S = S_+ \cup S_-$. Put $S_0 = S_+ \cap S_-$ and let $\Gamma^{\Delta}_{\pm}(\mathcal{C})$ be as in Example 1.7.4(ii). Then the following conditions are equivalent:*

(i) $S = S_+ \cup S_-$ is an ∞-decomposition of (W, S);

(ii) $W \simeq W_{S_+} *_{W_{S_0}} W_{S_-}$;

(iii) $\Gamma^{\Delta}_{\pm}(\mathcal{C})$ *is a tree.*

Proof. The statement follows by combining Theorem 1.7.10 and the fact that $S = S_+ \cup S_-$ is an ∞-decomposition of (W, S) if, and only if, $W \simeq W_{S_+} *_{W_{S_0}} W_{S_-}$ (cf. Fact 1.4.4). □

Let $\Delta = (\mathcal{G}, \delta)$ be a building of type (W, S) and $(\mathcal{G}, \Lambda)$ be a graph of special subgroups of (W, S). Then every group G acting on Δ has an induced action on $\Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{M})$, for every (non-empty) G-invariant subset $\mathcal{M} \subseteq \mathcal{C}$.

Corollary 1.7.12. *Let G be a σ-compact t.d.l.c. group acting chamber-transitively on a building $\Delta = (\mathcal{C}, \delta)$ of type (W, S), and let $c \in \mathcal{C}$. Suppose that (W, S) admits a visual graph of groups decomposition $(\mathcal{G}, \Lambda)$. Then G is topologically isomorphic to the fundamental group of the tree of t.d.l.c. groups $(\widetilde{\mathcal{G}}, \Lambda)$, where $\widetilde{\mathcal{G}}_\lambda$ is the setwise stabiliser of $\mathrm{Res}_{S_\lambda}(c)$ for every $\lambda \in V\Lambda \sqcup E\Lambda$, and the continuous open monomorphisms $\widetilde{\mathcal{G}}_e \hookrightarrow \widetilde{\mathcal{G}}_{t(e)}$, $e \in E\Lambda$, are given by inclusion. Moreover, the universal Bass–Serre tree of $\pi_1(\widetilde{\mathcal{G}}, \Lambda)$ is isomorphic to $\Gamma^{\Delta}_{(\widetilde{\mathcal{G}},\Lambda)}(\mathcal{C})$.*

Proof. By Theorem 1.7.10, $\Gamma = \Gamma^{\Delta}_{(\mathcal{G},\Lambda)}(\mathcal{C})$ is a tree. Since G acts chamber-transitively on Δ, one has that $g \cdot \mathrm{Res}_T(c) = \mathrm{Res}_T(g \cdot c)$ for all $g \in G$ and $T \subseteq S$. In particular, for every $\lambda \in V\Lambda \sqcup E\Lambda$, the group G acts transitively and with open stabilisers on $\mathcal{R}_{S_\lambda}(\mathcal{C})$. Therefore, G acts on Γ with open stabilisers and $G\backslash\Gamma = \Lambda$. Let $(\widetilde{\mathcal{G}}, \Lambda)$ be the graph of t.d.l.c. groups defined in the statement. By the main theorem of Bass–Serre theory, the homomorphism $\varphi\colon \pi_1(\widetilde{\mathcal{G}}, \Lambda) \to G$ induced by inclusion maps $\widetilde{\mathcal{G}}_\lambda \hookrightarrow G$, $\lambda \in V\Lambda \sqcup E\Lambda$, is an isomorphism of abstract groups. Moreover, the universal Bass–Serre tree of $\pi_1(\widetilde{\mathcal{G}}, \Lambda)$ is isomorphic to $\Gamma^{\Delta}_{(\widetilde{\mathcal{G}},\Lambda)}(\mathcal{C})$. It remains to prove that φ is open and has an open inverse. Recall that the group topology of $\pi_1(\widetilde{\mathcal{G}}, \Lambda)$ has a neighbourhood basis at 1 formed by the open subgroups of the vertex-groups of $(\widetilde{\mathcal{G}}, \Lambda)$. In particular, $\pi_1(\widetilde{\mathcal{G}}, \Lambda)$ is a t.d.l.c. group and φ is open. Moreover, by the open mapping theorem (cf. [Str06, Theorem 6.19]), the homomorphism φ^{-1} is open as well. □

1.8 Rational discrete cohomology of t.d.l.c. groups

This section summarises some background knowledge regarding the *rational discrete cohomology theory for t.d.l.c. groups* introduced by I. Castellano and T. Weigel in [CW16]. The overview we give is not complete, but sufficient for the aim of this thesis. As a general observation, we recall that the choice of $\mathbb{Q}$ as a field of coefficients can be replaced almost every time with no changes with any field of characteristic zero.

We also use this occasion to recall some notions of the ordinary cohomology theory of abstract groups. Indeed, for discrete groups, the two above-mentioned theories are equivalent.

1.8.1 Discrete left $\mathbb{Q}[G]$-modules

Let G be a t.d.l.c. group.

Definition 1.8.1 ([CW16, §2.2])**.** A *discrete left* $\mathbb{Q}[G]$*-module* M is an abstract left $\mathbb{Q}[G]$-module equipped with the discrete topology such that the G-action $\cdot\colon G\times M\to M$ is a continuous map. Here, $G\times M$ is endowed with the product topology.

If G is discrete, with abuse of notation, we regard a discrete left $\mathbb{Q}[G]$-module just as an abstract left $\mathbb{Q}[G]$-module.

Fact 1.8.2. *Let G be t.d.l.c. group. A left $\mathbb{Q}[G]$-module M is discrete if, and only if, for every $m\in M$ the group* $\mathrm{stab}_G(m)=\{g\in G\mid g\cdot m=m\}$ *is open in G.*

Let ${}_{\mathbb{Q}[G]}\mathbf{mod}$ denote the category of abstract left $\mathbb{Q}[G]$-modules. The category ${}_{\mathbb{Q}[G]}\mathbf{dis}$ is the full subcategory of ${}_{\mathbb{Q}[G]}\mathbf{mod}$ whose objects are discrete left $\mathbb{Q}[G]$-modules. By [CW16, Fact 2.2 and Proposition 3.2], ${}_{\mathbb{Q}[G]}\mathbf{dis}$ is an abelian category with enough injectives and enough projectives. In particular, for every $\mathcal{O}\in\mathcal{CO}(G)$, the *transitive left* $\mathbb{Q}[G]$*-permutation module* $\mathbb{Q}[G/\mathcal{O}]$ is projective in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ (cf. [CW16, Proposition 3.2]). By Fact 1.8.2, every $M\in\mathrm{ob}({}_{\mathbb{Q}[G]}\mathbf{dis})$ admits a collection of compact open subgroups $\{\mathcal{O}_i\}_{i\in I}$ of G together with an epimorphism

$$\pi_M\colon \coprod_{i\in I}\mathbb{Q}[G/\mathcal{O}_i]\twoheadrightarrow M. \tag{1.8.1}$$

In particular, M is *projective* if and only if it is a direct summand of $\coprod_{i\in I}\mathbb{Q}[G/\mathcal{O}_i]$ (cf. [CW16, Corollary 3.3]). Moreover, M is said to be *finitely generated* in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ if the set I in (1.8.1) can be chosen finite. One easily checks that this is equivalent to the existence of finitely many elements $m_1,\dots,m_n$ in M satisfying $M=\sum_{i=1}^{n}\mathbb{Q}[G]\cdot m_i$.

1.8.2 The n-th rational discrete cohomology functor

According to [CW16, §3], for $n\geq 0$ the n^{th} *rational discrete cohomology functor* of G, denoted by

$$\mathrm{dH}^n(G,\_)\colon {}_{\mathbb{Q}[G]}\mathbf{dis}\longrightarrow {}_{\mathbb{Q}}\mathbf{vect}, \tag{1.8.2}$$

is defined as the n^{th} right-derived functor of the fixed-point functor $(\_)^G$ from ${}_{\mathbb{Q}[G]}\mathbf{dis}$ to ${}_{\mathbb{Q}}\mathbf{vect}$.

1.8.3 The rational discrete standard $\mathbb{Q}[G]$-bimodule

Unless G is discrete, the group algebra $\mathbb{Q}[G]$ is not an object of ${}_{\mathbb{Q}[G]}\mathbf{dis}$ (cf. Fact 1.8.2). A suitable substitute of the group algebra, as introduced in [CW16, §4.2], is the *rational discrete standard* $\mathbb{Q}[G]$*-bimodule* $\mathrm{Bi}(G)$. It is defined as the direct limit of $\{\mathbb{Q}[G/\mathcal{O}]\mid\mathcal{O}\in\mathcal{CO}(G)\}$, where $\mathcal{CO}(G)$ is ordered by reverse inclusion (cf. [CW16, Equation (4.14)] for the transition maps). Alternatively, one may consider the convolution algebra $C_c(G,\mathbb{Q})$ of all continuous functions with compact support from G to $\mathbb{Q}$, where $\mathbb{Q}$ has the discrete topology. According to [CW16, §4.7], $C_c(G,\mathbb{Q})$ has a structure of left discrete $\mathbb{Q}[G]$-bimodule with respect to the following G-action:

$$(g_1 \cdot f \cdot g_2)(x) := f(g_1^{-1} x g_2^{-1}), \quad \forall\, x \in G, \tag{1.8.3}$$

for all $g_1, g_2 \in G$ and $f \in C_c(G,\mathbb{Q})$. As observed in [CW16, p. 128], the discrete left $\mathbb{Q}[G]$-modules $\mathrm{Bi}(G)$ and $C_c(G,\mathbb{Q})$ are (non-canonically) isomorphic as discrete left $\mathbb{Q}[G]$-modules. More precisely, for every compact open subgroup $U \leq G$, there is an isomorphism of discrete $\mathbb{Q}[G]$-bimodule $\psi^{(U)} \colon \mathrm{Bi}(G) \to C_c(G,\mathbb{Q})$. The isomorphism becomes an isomorphism of discrete $\mathbb{Q}[G]$-bimodules if the t.l.d.c. group G is unimodular. For instance G is discrete, both $\mathrm{Bi}(G)$ and $C_c(G,\mathbb{Q})$ are isomorphic to $\mathbb{Q}[G]$ as (discrete) left-$\mathbb{Q}[G]$ bimodules. The reader is referred to [CW16, Equation (4.64), Proposition 4.11 and Remark 4.13] for details. Depending on the context, we will prefer either $\mathrm{Bi}(G)$ or $C_c(G,\mathbb{Q})$.

1.8.4 Rational discrete cohomological dimension

Definition 1.8.3 ([CW16, p. 115])**.** The *rational discrete cohomological dimension* of a t.d.l.c. group G is defined as follows:

$$\mathrm{cd}_{\mathbb{Q}}(G) := \sup\{n \in \mathbb{Z}_{\geq 0} \mid \mathrm{dH}^n(G,M) \neq 0 \text{ for some } M \in \mathrm{ob}({}_{\mathbb{Q}[G]}\mathbf{dis})\}.$$

Remark 1.8.4. If G is a discrete group, the category ${}_{\mathbb{Q}[G]}\mathbf{dis}$ is isomorphic to ${}_{\mathbb{Q}[G]}\mathbf{mod}$. Therefore, $\mathrm{cd}_{\mathbb{Q}}(G)$ is the ordinary rational cohomological dimension of G (cf. [Bro12, Chapter VIII, §2]).

We now collect some basic facts that will be frequently used throughout the chapter.

Proposition 1.8.5. *Let G be a t.d.l.c. group.*

(i) ([CW16, Proposition 3.7(a)]) *We have $\mathrm{cd}_{\mathbb{Q}}(G) = 0$ if, and only if, G is compact;*

(ii) ([CW16, Proposition 3.7(c)]) *For every closed subgroup $H \leq G$, we have*

$$\mathrm{cd}_{\mathbb{Q}}(H) \leq \mathrm{cd}_{\mathbb{Q}}(G).$$

(iii) ([CW16, Proposition 4.7]) *If G is of type* FP *(cf. Section 1.8.5), then*

$$\mathrm{cd}_{\mathbb{Q}}(G) = \max\{n \in \mathbb{Z}_{\geq 0} \mid \mathrm{dH}^n(G,\mathrm{Bi}(G)) \neq 0\} < \infty,$$

where $\mathrm{Bi}(G)$ *is the rational discrete standard bimodule of G (cf. Section 1.8.3).*

(iv) ([CW16, Theorem 6.7, Equation (6.15)]) *Let Σ be a contractible locally finite simplicial complex of finite dimension. Assume that G acts simplicially on Σ with compact open simplex stabilisers and has finitely many orbits on each skeleton of Σ. Then*

$$\mathrm{cd}_{\mathbb{Q}}(G) = \max\{n \in \mathbb{Z}_{\geq 0} \mid \mathrm{H}^n_c(|\Sigma|, \mathbb{Q}) \neq 0\}.$$

Here $\mathrm{H}^n_c(|\Sigma|, \_)$ denotes the cohomology functor with compact support of the geometric realisation $|\Sigma|$ of Σ (cf. [CW16, Appendix A.4]).

1.8.5 The finiteness property FP_n

According to [CW16, §3.6 and §4.5], a discrete left $\mathbb{Q}[G]$-module M is of *type* FP_n ($n \in \mathbb{Z}_0$) if it admits a partial projective resolution in ${}_{\mathbb{Q}[G]}\mathbf{dis}$

$$P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \tag{1.8.4}$$

with $P_0, \ldots, P_n$ finitely generated. The discrete left $\mathbb{Q}[G]$-module M is of *type* FP_∞ if it is of type FP_n for all $n \geq 0$, and M is of *type* FP if there exists a projective resolution in ${}_{\mathbb{Q}[G]}\mathbf{dis}$

$$0 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \tag{1.8.5}$$

which has finite length $n \geq 0$ and in which every P_i are finitely generated.

The t.d.l.c. group G is of *type* FP_n, $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ if $\mathbb{Q}$ is of type FP_n as trivial discrete left $\mathbb{Q}[G]$-module. As for abstract groups, the properties FP_n's on t.d.l.c. groups are a cohomological generalisation of the concept of compact generation. Indeed, a t.d.l.c. group is compactly generated if, and only if, it is of type FP_1 (cf. [CW16, Proposition 5.3]). According to [CW16, §4.5], G is of *type* FP if, and only if, it is of type FP_∞ and $\mathrm{cd}_{\mathbb{Q}}(G) < \infty$ or, equivalently, $\mathbb{Q}$ is of type FP as a trivial discrete left $\mathbb{Q}[G]$-module.

1.9 Dirichlet series

In this section, we outline the basic concepts of the Dirichlet series. For further details, one may refer to [HR15; Ser73] for example.

Definition 1.9.1. Let $\mathbf{a} = (a_n)_{n \geq 1}$ be a sequence in $\mathbb{R}_{\geq 0}$. The *(ordinary) Dirichlet series associated to* $\mathbf{a}$ is the formal series

$$D_{\mathbf{a}}(s) := \sum_{n=1}^{\infty} a_n n^{-s}. \tag{1.9.1}$$

Regarded as a series in the complex variable s, if $D_{\mathbf{a}}(s)$ converges at some $s_0 \in \mathbb{C}$, then it absolutely converges at every $s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) > \mathrm{Re}(s_0)$ (cf. [Ser73, §II.2.2, Corollary 1]). By [HR15, §I.1, Theorem 3], there is $\alpha \in \mathbb{R} \cup \{\pm\infty\}$ such that $D_{\mathbf{a}}(s)$ converges whenever $\mathrm{Re}(s) > \alpha$, and it does not converge whenever $\mathrm{Re}(s) < \alpha$. The extended real number $\alpha = \alpha(D_{\mathbf{a}}(s))$ is called the *abscissa of convergence* of $D_{\mathbf{a}}(s)$. In particular,

$$\begin{aligned} &\alpha(D_{\mathbf{a}}(s)) < +\infty \iff (a_n)_{n\geq 1} \text{ grows polynomially in } n; \\ &\alpha(D_{\mathbf{a}}(s)) = -\infty \iff \text{there is } n_0 \geq 1 \text{ such that } a_n = 0 \text{ for all } n \geq n_0. \end{aligned} \tag{1.9.2}$$

Chapter 2

Some geometric and cohomological invariants for t.d.l.c. groups

2.1 Structure of the chapter

This chapter deals with three notable invariants associated to t.d.l.c. groups: their number of ends (cf. Section 2.2), their rational discrete cohomological dimension (cf. Section 2.3), and their Euler–Poincaré characteristic (cf. Section 2.4). At the beginning of each of the above-mentioned sections, we briefly summarise their structure and highlight the main results. In Section 2.5, we provide a version of the Stallings–Swan theorem for a suitable subclass of compactly generated t.d.l.c. groups (cf. Theorem 2.5.15). The proof of this result based on information on the sign of the Euler–Poincaré characteristic that we provide in the main theorem of Section 2.4.3 (cf. Theorem 2.4.41).

2.1.1 Notation of the chapter

- Given a t.d.l.c. group G, let

  $$\mathcal{CO}(G) := \{\mathcal{O} \leq G \mid \mathcal{O} \text{ compact open subgroup}\}$$

  be the set of compact open subgroups of G. By van Dantzig's theorem [Van36], $\mathcal{CO}(G)$ is non-empty.

- If not specifically mentioned, an isomorphism between topological groups is meant to be an *isomorphism of topological groups*, i.e., a continuous isomorphism with continuous inverse.

2.2 Number of ends of compactly generated t.d.l.c. groups

This section deals with the *number of ends* of a compactly generated t.d.l.c. group. This invariant is inherited from graph theory, cf. Definition 2.2.11. The number of ends of a compactly generated t.d.l.c. group is defined as the number of ends of any of its Cayley–Abels graphs. The latter ones generalise Cayley graphs to compactly generated t.d.l.c. groups. We recall their definition in Section 2.2.1.

Despite their "geometric" definition, thanks to a result of E. Specker [Spe49], the number of ends of a finitely generated t.d.l.c. group can be explicitly described in terms of the dimension of low-degree cohomology spaces of the group itself. In Theorem 2.2.25 (and Corollary 2.2.26) this result is generalised to every compactly generated t.d.l.c. group. In order to achieve it, we characterise the relevant cohomology spaces in terms of almost invariant functions, which we briefly recall in Section 2.2.3.

Throughout Section 2.2.2, we also cite the generalisation of the well-known Stallings' decomposition theorem to compactly generated t.d.l.c. groups (cf. Theorems 2.2.16 and 2.2.18). A repeated application to this theorem leads to the concept of *accessibility* of the relevant group, around which Section 2.5 (especially Section 2.5.1) is centred.

Finally, Section 2.2.5 focuses on the case of compactly generated t.d.l.c. groups acting properly and cocompactly on locally finite buildings. In this setting, we use our cohomological characterisation of the number of ends to connect the number of ends of the relevant group with the number of ends of the Coxeter group describing the type of the building (cf. Proposition 2.2.31).

2.2.1 Cayley–Abels graphs and number of ends

Cayley graphs for finitely generated groups

Cayley graphs have become an important tool in the study of finitely generated groups. We briefly recall them to make the comparison with the generalisation exposed in Section 2.2.1 more evident.

Definition 2.2.1 ([Bog08, Chapter 2, Definition 1.15])**.** Consider a group G with a finite generating set $S \not\ni 1$ which is *symmetric* (i.e., $s^{-1} \in S$ for every $s \in S$). The *Cayley graph* $\Gamma(G,S)$ *associated to* (G,S) is the graph defined by

$$V\Gamma(G,S) = G \qquad \text{and} \qquad E\Gamma(G,S) = \{(g, gs) \mid g \in G,\, s \in S\}.$$

The origin, terminus and edge-inversion maps on $\Gamma(G,S)$ are defined, respectively, as:

$$o(g,gs) = g, \quad t(g,gs) = gs \quad \text{and} \quad \overline{(g,gs)} = (gs,g) = (gs, gss^{-1}), \quad \forall\, g \in G,\, s \in S.$$

One checks that $\Gamma(G,S)$ is a locally finite connected combinatorial connected graph, and the G-action on $V\Gamma(G,S)$ by left-translation is free (i.e., with trivial stabilisers) and transitive. Moreover, given two finite symmetric generating sets S_1 and S_2 of G, the graphs $\Gamma(G,S_1)$ and $\Gamma(G,S_2)$ are *quasi-isometric* as metric spaces with respect to the geodesic distance (cf. [BH11, Example 8.17(3)]). The notion of quasi-isometry, recalled below, was first introduced by M. Gromov [Gro84] in the more general setting of metric spaces.

Definition 2.2.2. Two metric spaces (X_1, d_1) and (X_2, d_2) are said to be *quasi-isometric* if there is a map $\varphi\colon X_1 \to X_2$ and real constants $a \geq 1$ and $b \geq 0$ such that the following two conditions are satisfied:

(i) for all $x, y \in X_1$,

$$a^{-1} \cdot \big(d_1(x, y) - b\big) \leq d_2(\varphi(x), \varphi(y)) \leq a \cdot \big(d_1(x, y) + b\big);$$

(ii) for all $y \in X_2$,

$$d_2(y, \varphi(X_1)) := \inf_{x \in X_1} d_2(y, \varphi(x)) \leq b.$$

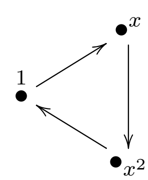
A map φ satisfying (i) and (ii) is called *quasi isometry.*

Informally speaking, the condition (i) requires that φ is an "isometry up to a bounded error", and the condition (ii) entails that φ is "almost surjective". It is reasonable to think that quasi-isometric spaces have the same large-scale behaviour, although they locally look non-isometric. One checks that "being quasi-isometric" is an equivalence relation on metric spaces.

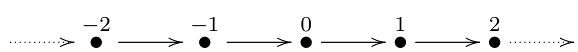
Returning to Cayley graphs, we list some classical examples.

Example 2.2.3 ([Bog08, pp. 49-50])**.** Here below, although every graph are Serre graphs, for simplicity we draw only their vertex set and an orientation on edges.

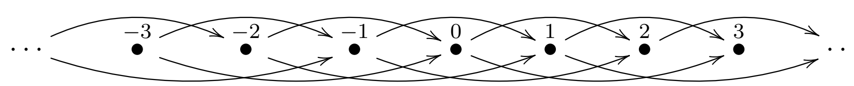
(i) For $n \geq 1$, let $G = C_n = \langle x \mid x^n = 1 \rangle$ be the cyclic group of order n and let $S = \{x, x^{-1}\}$. Then $\Gamma(G, S)$ is an n-cycle graph. Here below, we picture $\Gamma(C_3, \{x, x^{-1}\})$.



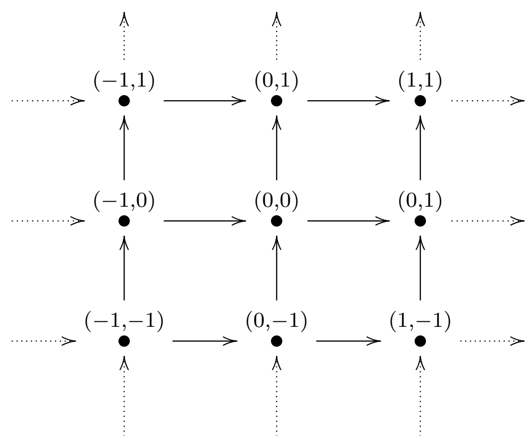
(ii) Consider the additive group $G = \mathbb{Z}$ and let $S = \{\pm 1\}$. One checks that $\Gamma(G, S)$ is a bi-infinite line, as pictured below.



Moreover, setting $S' = \{\pm 2, \pm 3\}$, the graph $\Gamma(G, S')$ is pictured as follows.



(iii) Consider the additive group $G = \mathbb{Z}^2$, and let $S = \{(\pm 1, 0), (0, \pm 1)\}$. Then $\Gamma(G, S)$ is the "infinite 2-dimensional square grid" (lying in $\mathbb{R}^2$) sketched below.



More generally, for $n \geq 2$ and denoting by e_i the i-th vector of the canonical $\mathbb{Z}$-basis of $\mathbb{Z}^n$, the Cayley graph $\Gamma(\mathbb{Z}^n, \{\pm e_i \mid i \in [n]\})$ is the "infinite grid made up of n-cubes" lying in $\mathbb{R}^n$.

(iv) Let $G = F_2 = \langle a, b \mid \emptyset \rangle$ be the free group of rank 2 and set $S = \{a^{\pm 1}, b^{\pm 1}\}$. Then $\Gamma(G, S)$ is a 4-regular tree, i.e., an infinite tree in which each vertex has 4 outgoing edges.

(v) Let (W, S) be a Coxeter system. Then the Cayley graph $\Gamma(W, S)$ is the chamber graph associated to the standard Coxeter complex $\Sigma(W, S)$ (cf. Section 1.5.1 and Example 1.5.10).

In Example 2.2.3(ii), one might build a naïve intuition of the concept of quasi-isometry: zooming out the pictures of $\Gamma(G, S)$ and $\Gamma(G, S')$, they both roughly look like bi-infinite lines.

Let $\mathcal{C}$ be the family of connected combinatorial locally finite graphs. There are several invariants (e.g., the number of ends) defined on graphs in $\mathcal{C}$ that are *quasi-isometric invariants*, i.e., every two elements of $\mathcal{C}$ that are quasi-isometric have the same invariant. Similarly, there are several properties $\mathcal{P}$ (e.g., accessibility, hyperbolicity, ...) defined on the graphs in $\mathcal{C}$ that are *quasi-isometric properties*, i.e., for all $\Gamma_1, \Gamma_2 \in \mathcal{C}$, the graph Γ_1 has property $\mathcal{P}$ if, and only if, the graph Γ_2 has property $\mathcal{P}$.

Since all Cayley graphs of a finitely generated group G are pairwise quasi-isometric, one may transfer quasi-isometric properties or invariants from graphs to groups.

Remark 2.2.4. Strictly speaking, the definition of $\Gamma(G, S)$ carries over *verbatim* even by dropping the finiteness assumption on S. With this more general definition, $\Gamma(G, S)$ is a connected combinatorial graph and it is locally finite if, and only if, S is finite. However, given two arbitrary symmetric generating sets S_1 and S_2, one sees that $\Gamma(G, S_1)$ and $\Gamma(G, S_2)$ are *not* quasi-isometric in general.

For instance, given $G = (\mathbb{Z}, +)$, set $S_1 = \{\pm 1\}$ and $S_2 = G \setminus \{0\}$. We claim that $\Gamma_1 := \Gamma(G, S_1)$ and $\Gamma_2 := \Gamma(G, S_2)$ are not quasi-isometric. Indeed, let $\mathrm{d}_1 \colon V\Gamma_1 \times V\Gamma_1 \to \mathbb{Z}_{\geq 0}$ and $\mathrm{d}_2 \colon V\Gamma_2 \times V\Gamma_2 \to \mathbb{Z}_{\geq 0}$ be the geodesic distances on Γ_1 and Γ_2, respectively (cf. (1.2.1)). Recalling Example 2.2.3(ii), one checks that $\mathrm{d}_1(x, y) = |x - y|$ and $\mathrm{d}_2(x, y) = 1 - \mathbb{1}_x(y)$, for all $x, y \in \mathbb{Z}$. If there is a quasi-isometry $\varphi \colon V\Gamma_1 \to V\Gamma_2$, then there exist real constants $a \geq 1$ and $b \geq 0$ such that

$$a^{-1}\big(\mathrm{d}_1(x, y) - b\big) \leq \mathrm{d}_2(x, y) \leq 1, \quad \forall\, x, y \in \mathbb{Z}. \tag{2.2.1}$$

This would imply that the metric space $(V\Gamma_1, \mathrm{d}_1)$ is bounded, which is impossible.

Cayley–Abels graphs for compactly generated t.d.l.c. groups

Cayley graphs can be defined for arbitrary symmetric generating sets of a given group, but might be no more pairwise quasi-isometric (cf. Remark 2.2.4). Finitely generated groups are countable, thus the only locally compact group topology they admit is the discrete one (cf. [Cas23, Proposition 10.6]). The problem of having a good replacement of locally finite Cayley graphs arises while dealing, for instance, with non-discrete t.d.l.c. groups (they can not be finitely generated, as we observed above).

However, for a compactly generated t.d.l.c. group G, one may count on the following property: for every compact open subgroup $K \leq G$, there is a finite symmetric set $S \subseteq G$ with $K \cap S = \emptyset$ and such that $K \cup S$ generates G as an abstract group. Indeed, given an arbitrary compact generating set X for G and a compact open subgroup $K \leq G$, there is a finite subset S of $X \setminus K$ such that $X \subseteq \bigsqcup_{s \in S} sK \sqcup K$. Up to adding inverses of its elements, we may assume that S is also symmetric.

Definition 2.2.5. The pair (K, S) as above is called a *compact generating system* of G.

Definition 2.2.6 ([KM08, §2, Construction 2])**.** Let G be a compactly generated t.d.l.c. group with a compact generating system (K, S). The *Cayley–Abels graph* $\Gamma(G, K, S)$ *associated to* (K, S) is a connected combinatorial graph defined as follows:

$$V\Gamma(G, K, S) := G/K \quad \text{and} \quad E\Gamma(G, K, S) := \{(gK, gsK) \mid g \in G, s \in S\}.$$

The origin, terminus and inversion maps are given by

$$o(gK, gsK) = gK, \quad t(gK, gsK) = gsK, \quad \text{and} \quad \overline{(gK, gsK)} = (gsK, gK), \quad \forall\, g \in G, s \in S.$$

Since S is symmetric, the inversion map in Definition 2.2.6 is well-defined. Namely, for all $g \in G$ and $s \in S$ one has $(gsK, gK) = (gsK, gss^{-1}K)$.

More generally, one has the following.

Definition 2.2.7 ([KM08, §2, Definition 1])**.** Let G be a compactly generated t.d.l.c. group. A *Cayley–Abels graph of* G is a connected locally finite combinatorial graph on which G acts vertex-transitively and with compact open vertex stabilisers.

Proposition 2.2.8.

(i) ([KM08, §2, Theorem 1]) *A t.d.l.c. group admits a Cayley–Abels graph if, and only if, it is compactly generated.*

(ii) ([KM08, §2, Theorem 2]) *Every two Cayley–Abels graphs of a compactly generated t.d.l.c. group are quasi-isometric.*

Example 2.2.9. Let $\Delta = (\mathcal{C}, \delta)$ be a locally finite building of type (W, S), and denote by Γ_Δ its associated chamber graph (cf. Section 1.5.1). Then Γ_Δ is a Cayley–Abels graph of every group G acting (chamber-)transitively and with compact open (chamber) stabilisers on Δ.

Further information on Cayley–Abels graphs can be deduced from the following theorem due to B. Krön and R. Möller.

Theorem 2.2.10 ([KM08, Theorem 16])**.** *Let G be a compactly generated t.d.l.c. group.*

(i) Some (and hence every) Cayley–Abels graph of G is quasi-isometric to a tree if, and only if, G acts properly and cocompactly on a locally finite tree T without edge-inversion (cf. Section 1.3.1). In particular, T is quasi-isometric to every Cayley–Abels graph of G.

(ii) Let G be unimodular. Then some (and hence every) Cayley–Abels graph of G is quasi-isometric to a tree if, and only if, G admits a finitely generated free cocompact lattice.

The advantage of Theorem 2.2.10 is that we bypass the problem of finding vertex-transitive actions on a locally finite connected graph. We will see an application of Theorem 2.2.10 in Example 2.2.15(iv).

2.2.2 Number of ends: a geometric viewpoint

We recall a possible notion of ends of a graph due to R. Halin.

Definition 2.2.11 ([Hal94])**.** Let Γ be a locally finite connected combinatorial graph. Given a subgraph Γ' of Γ, let $\Gamma - \Gamma'$ be the subgraph of Γ with vertex set $V(\Gamma - \Gamma') := V\Gamma \setminus V\Gamma'$ and edge set $E(\Gamma - \Gamma') := \{e \in E\Gamma \mid o(e), t(e) \notin E\Gamma'\}$.

The graph Γ is said to have *at least* n *ends* if there is a finite subgraph Γ' such that $\Gamma - \Gamma'$ has at least n infinite connected components. The graph Γ has n *ends* (written $e(\Gamma) = n$) if it has at least n ends but not at least $n+1$ ends. Moreover, Γ has *infinitely many ends* (written $e(\Gamma) = \infty$) if, for all $n \in \mathbb{Z}_{\geq 1}$, Γ has at least n ends.

Remark 2.2.12. Let Γ be a locally finite connected combinatorial graph. Then $e(\Gamma) = 0$ if, and only if, Γ is finite.

Proposition 2.2.13 ([Krö01, Proposition 1])**.** *Let Γ_1 and Γ_2 be locally finite connected combinatorial graphs. If Γ_1 and Γ_2 are quasi-isometric, then they have the same number of ends.*

Let G be a compactly generated t.d.l.c. group. The *number of ends* $e(G)$ *of* G is defined to be the number of ends of a(ny) Cayley–Abels graph of G. Note that this number is well-defined since the Cayley–Abels graphs of G are pairwise quasi-isometric. Moreover, we have the following.

Theorem 2.2.14 (Hopf's theorem – [Hop94], [KM08, Theorem 5])**.** *Let G be a compactly generated t.d.l.c. group. Then $e(G) \in \{0, 1, 2, \infty\}$.*

Example 2.2.15.

(i) Let G be a compactly generated t.d.l.c. group. Then $e(G) = 0$ if, and only if, G is compact. Indeed, both the two conditions before are equivalent to require that G admits a finite Cayley–Abels graph. Then Remark 2.2.12 applies.

(ii) From Example 2.2.3 one checks that $e(\mathbb{Z}) = 2$ and, for every $n \geq 2$, that $e(\mathbb{Z}^n) = 1$.

(iii) Let F_2 be the free group of rank 2. From Example 2.2.3 one checks that $e(F_2) = \infty$.

(iv) The following is a generalisation of item (iii). Let G be a t.d.l.c. group acting properly and cocompactly on a locally finite tree T without edge-inversions (cf. Section 1.3.1). By Theorem 2.2.10, the group G has the same number of ends as T. For instance, if T is the d-regular tree for some $d \geq 2$, then T has 2 ends if $d = 2$, and it has infinitely many ends if $d \geq 3$. In particular, for p prime, the group $\mathrm{SL}_2(\mathbb{Q}_p)$ has infinitely many ends as it acts properly, edge-transitively and without edge-inversions on a $(p + 1)$-regular tree (its Bruhat–Tits tree).

In Example 2.2.15(i) we observe that 0-ended compactly generated t.d.l.c. groups are exactly profinite groups. Another piece of structural information can be deduced from the so-called *Stallings' decomposition theorem*. It was initially proved by J. Stallings [Sta72] for discrete groups, and then extended by H. Abels (cf. [Abe74, Struktursatz 5.7, Korollar 5.8]) to the topological case. Here below we state the t.d.l.c. version of Abels' theorem.

Theorem 2.2.16. *For a compactly generated t.d.l.c. group G, the following are equivalent:*

(i) $e(G) \geq 2$*;*

(ii) there is a compact open subgroup $U \leq G$ such that G is topologically isomorphic either to an amalgamated free product $X *_U Y$ of two open subgroups $X, Y \leq G$ satisfying $U \lneqq X, Y$, or to an HNN-extension $X*_U^t$ with respect to an open subgroup $X \leq G$ satisfying $U \lneqq X$.

If a compactly generated t.d.l.c. group G satisfies condition (ii) for a compact open subgroup U, then G is said to *split non-trivially over* U, and the groups X (and Y, if it occurs) are called *factors of (the splitting of)* G. It might be useful to recall the following.

Proposition 2.2.17 ([Cas20, Proposition 4.1])**.** *Let G be a compactly generated t.d.l.c. group which splits non-trivially over a compact open subgroup. Then the factors of G are compactly generated t.d.l.c. groups.*

Combining Theorem 2.2.16 and Proposition 2.2.17, given a compactly generated t.d.l.c. group G which splits non-trivially over a compact open subgroup, one deduces that the factors of G either have at most 1 end or split non-trivially over a compact open subgroup. We will see in Section 2.5.1 that an iterated decomposition of the factors of G leads to another property on compactly generated t.d.l.c. groups that is invariant under quasi-isometries: the property of *being accessible*.

Stallings' decomposition theorem gives a structural characterisation of compactly generated t.d.l.c. groups with more than one end. This characterisation can be furthermore rephrased in cohomological terms. This is the content of the following result due I. Castellano, which extends a result of W. Dicks and M. Dunwoody (cf. [DD89, Theorem IV 6.10]) from discrete to t.d.l.c. groups.

Theorem 2.2.18 ([Cas20, Theorem A*])**.** *Let G be a compactly generated t.d.l.c. group. Then $e(G) \geq 2$ if, and only if,* $\mathrm{dH}^1(G, \mathrm{Bi}(G)) \neq 0$.

The key ingredient to prove Theorem 2.2.18 is to observe that $\mathrm{dH}^1(G, \mathrm{Bi}(G))$ is the direct limit of spaces $\mathrm{dH}^1(G, \mathbb{Q}[G/K])$, for $K \leq G$ compact open subgroup, and then describe each $\mathrm{dH}^1(G, \mathbb{Q}[G/K])$ in terms of the so-called space of (G, K)-*invariant functions*. We recall the definition of the latter space in the next Section 2.2.3. In Section 2.2.4 we will use it to give an explicit formula of $e(G)$ in terms of the dimensions of the $\mathbb{Q}$-vector spaces $\mathrm{dH}^0(G, \mathrm{Bi}(G))$ and $\mathrm{dH}^1(G, \mathrm{Bi}(G))$.

2.2.3 Almost (G, K)-invariance

In the following, G denotes a compactly generated t.d.l.c. group.

Almost (G, K)-invariant sets

Definition 2.2.19 ([Cas20; Dun79])**.** Let K be a compact open subgroup of G. A set $B \subseteq G/K$ is said to be *almost* (G, K)*-invariant* if, for all $g \in G$, $gB =_a B$ (i.e., the symmetric difference of gB and B is finite) and, for all $k \in K$, $kB = B$.

Note that a finite set $B \subseteq G/K$ is almost (G, K)-invariant if, and only if, $B = B_1 \sqcup \ldots \sqcup B_n$ where each B_i denotes the set of all left K-cosets that are necessary to cover a single K-double-coset lying in the preimage of B in G.

Example 2.2.20. Let G be a compactly generated t.d.l.c. group, (K, S) be a compact generating system (cf. Definition 2.2.5) and $\Gamma = \Gamma(G, K, S)$ the associated Cayley–Abels graph. Given a finite connected subgraph Γ' of Γ, denote by $\Gamma - \Gamma'$ the subgraph spanned by all the vertices of Γ that are not in Γ'. Every connected component C of $\Gamma - \Gamma'$ determines the almost (G, K)-invariant set as follows:

$$B_C := \{gK \in G/K \mid g^{-1}K \in VC\}. \tag{2.2.2}$$

Clearly, $kB_C = B_C$ for every $k \in K$. Therefore, since G is algebraically generated by $K \cup S$, it suffices to prove that $sB_C =_a B_C$ for every $s \in S$. Given $s \in S$, assume that $gK \in B_C$ and $sgK \notin B_C$. Then $g^{-1}s^{-1}K$ belongs to the boundary of C, i.e., $g^{-1}s^{-1}K$ is not a vertex of C but there is an edge in Γ connecting it to a vertex of C (which in this case is $g^{-1}K$). As Γ' is finite, the boundary of C is finite. Since $S^{-1} = S$, we conclude that $sB_C =_a B_C$.

Almost (G, K)-invariant functions

Let (K, S) be a compact generating system of G. In this section we recall the relation between the rational discrete first-degree cohomology of G and the almost (G, K)-invariant sets. The $\mathbb{Q}$-vector space $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[G/K], \mathbb{Q})$ admits the left G-action given by

$$g \cdot \alpha(x) = \alpha(g^{-1}x), \quad \alpha \in \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[G/K], \mathbb{Q}), g \in G, x \in G/K.$$

One writes $\alpha =_a \beta$ if $\alpha(x) = \beta(x)$ for all but finitely many elements $x \in G/K$.

Definition 2.2.21. An element $\alpha \in \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[G/K], \mathbb{Q})$ is said to be *almost* (G, K)-*invariant* if, for all $g \in G$, $g \cdot \alpha =_a \alpha$ and, for all $k \in K$, $k \cdot \alpha = \alpha$.

Example 2.2.22. The characteristic function $\mathbb{1}_B$ of an almost (G, K)-invariant set B is an almost (G, K)-invariant function.

By [Cas20, Proposition 3.15], for every compact open subgroup K of a t.d.l.c. group G we have

$$\mathrm{dH}^1(G, \mathbb{Q}[G/K]) \cong \frac{\mathrm{AInv}_K(G, \mathbb{Q})}{C(G/K) + \mathbb{Q}[G/K]^K}, \tag{2.2.3}$$

where $\mathrm{AInv}_K(G, \mathbb{Q})$ is the $\mathbb{Q}$-vector space of all almost (G, K)-invariant functions, $C(G/K)$ denotes the set of all linear functions from $\mathbb{Q}[G/K]$ to $\mathbb{Q}$ which are constant on G/K, and $\mathbb{Q}[G/K]^K$ is the largest K-invariant submodule of $\mathbb{Q}[G/K]$. Here $\mathbb{Q}[G/K]^K$ is regarded as the space of all almost zero functions of $\mathrm{AInv}_K(G, \mathbb{Q})$. By [Cas20, Lemma 4.4], the $\mathbb{Q}$-vector space $\mathrm{AInv}_K(G, \mathbb{Q})$ is generated by the set

$$\{\mathbb{1}_B \mid B \subset G/K \text{ almost } (G, K)\text{-invariant set}\}.$$

2.2.4 Number of ends: a cohomological viewpoint

Theorem 2.2.23. *Let G be a non-compact compactly generated t.d.l.c. group and K be a compact open subgroup. If $e(G) < \infty$, then the $\mathbb{Q}$-vector space $\mathrm{dH}^1(G, \mathbb{Q}[G/K])$ has dimension $e(G) - 1$. Moreover, $e(G) = \infty$ if and only if $\mathrm{dH}^1(G, \mathbb{Q}[G/K])$ is infinite-dimensional.*

Proof. The argument is based on the proof of [Spe49, Satz IV]. Let (K, S) be a compact generating system of G. It suffices to prove the following statements, for all $n, m \geq 1$: if the associated Cayley–Abels graph $\Gamma(G, K, S)$ has at least n ends, then the dimension of $\mathrm{dH}^1(G, \mathbb{Q}[G/K])$ is at least $n-1$; if the dimension of $\mathrm{dH}^1(G, \mathbb{Q}[G/K])$ is at least m, then $\Gamma(G, K, S)$ has at least $m+1$ ends.

Assume that $\Gamma(G, K, S)$ has at least n ends. Then there exists a finite subgraph Γ' such that $\Gamma(G, K, S) - \Gamma'$ has at least n infinite connected components $C_1, \ldots, C_n$. Let $B_{C_1}, \ldots, B_{C_n}$ be the infinite almost (G, K)-invariant subsets determined by $C_1, \ldots, C_n$, respectively (cf. Example 2.2.20). We now observe that $\mathbb{1}_{B_{C_1}}, \ldots, \mathbb{1}_{B_{C_{n-1}}}$ are linearly independent modulo $C(G/K) + \mathbb{Q}[G/K]^K$. Indeed, any $\mathbb{Q}$-linear combination $t_1 \mathbb{1}_{B_{C_1}} + \cdots + t_{n-1} \mathbb{1}_{B_{C_{n-1}}}$ vanishes (at least) on the infinite set B_{C_n}. Since $B_{C_i} \cap B_{C_j} = \emptyset$ for all $i \neq j$, the only $\mathbb{Q}$-linear combination $t_1 \mathbb{1}_{B_{C_1}} + \cdots + t_{n-1} \mathbb{1}_{B_{C_{n-1}}}$ that is constant almost everywhere is the one with $t_1 = \ldots = t_{n-1} = 0$. By (2.2.3), we conclude that the dimension of $\mathrm{dH}^1(G, \mathbb{Q}[G/K])$ is at least $n-1$.

For every $B \subseteq G/K$ let $C_B = \{gK \in G/K \mid g^{-1}K \in B\}$. One checks that B is finite if, and only if, C_B is finite. Assume that the dimension of $\mathrm{dH}^1(G, \mathbb{Q}[G/K])$ is at least m. By (2.2.3) and [Cas20, Lemma 4.4], there exist m almost (G, K)-invariant subsets $B_1, \ldots, B_m$ of G/K whose characteristic functions $\mathbb{1}_{B_1}, \ldots, \mathbb{1}_{B_m}$ are linearly independent modulo $C(G/K) + \mathbb{Q}[G/K]^K$. Denote by δC_{B_i} the set of vertices in $\Gamma(G, K, S)$ which are not in C_{B_i} but are adjacent to a vertex of C_{B_i}. As B_i is infinite and almost (G, K)-invariant, C_{B_i} is infinite and δC_{B_i} is finite (for the finiteness of δC_{B_i} one can argue as in Example 2.2.20). Let Γ' be the subgraph of $\Gamma(G, K, S)$ spanned by $\delta C_{B_1} \cup \ldots \cup \delta C_{B_m}$, and denote by $D_1, \ldots, D_p$ the sets of vertices of the p infinite connected components of $\Gamma(G, K, S) - \Gamma'$, respectively. Observe that

$$G/K = V\Gamma(G, K, S) = F \sqcup \bigsqcup_{j=1}^{p} D_j,$$

where F is the (finite) set collecting all the elements of $\delta C_{B_1} \cup \ldots \cup \delta C_{B_m}$ and all the vertices of any finite connected component of $\Gamma(G, K, S) - \Gamma'$. In particular, for every $1 \leq i \leq m$ we have

$$C_{B_i} = (C_{B_i} \cap F) \sqcup \bigsqcup_{j=1}^{p} (C_{B_i} \cap D_j). \tag{2.2.4}$$

We claim that, for all $1 \leq i \leq m$ and $1 \leq j \leq p$, either $C_{B_i} \supseteq D_j$ or $C_{B_i} \cap D_j = \emptyset$. Suppose indeed that there is $gK \in D_j \setminus C_{B_i}$. Since $D_j \cap \delta C_{B_i} = \emptyset$, every adjacent vertex of any $hK \in D_j \setminus C_{B_i}$ does not belong to C_{B_i}. Hence, every path in $\Gamma(G, K, S)$ with vertices in D_j and starting at gK has all its vertices in $D_j \setminus C_{B_i}$. Since D_j spans a connected subgraph of $\Gamma(G, K, S)$, every vertex in D_j can be connected to gK by such a path. Then $D_j = D_j \setminus C_{B_i}$, which yields $D_j \cap C_{B_i} = \emptyset$. Therefore, by (2.2.4), for every $1 \leq i \leq m$ there is $\{j_1, \ldots, j_{r_i}\} \subseteq \{1, \ldots, p\}$ such that

$$\mathbb{1}_{C_{B_i}} = \mathbb{1}_{C_{B_i} \cap F} + \sum_{k=1}^{r_i} \mathbb{1}_{D_{j_k}}. \tag{2.2.5}$$

Denote by $\mathcal{AC}(G/K, \mathbb{Q})$ the space of all functions $f : G/K \longrightarrow \mathbb{Q}$ which are *constant almost everywhere*, i.e., f is constant on all but finitely many elements of G/K. We regard $\mathcal{AC}(G/K, \mathbb{Q})$ as a subspace of the space $\mathcal{F}(G/K, \mathbb{Q})$ of all maps from G/K to $\mathbb{Q}$. Since $\mathbb{1}_{C_{B_i} \cap F} \in \mathcal{AC}(G/K, \mathbb{Q})$ for every $1 \leq i \leq m$, (2.2.5) implies that the $\mathbb{Q}$-subspace $\mathcal{S}_1$ in $\mathcal{F}(G/K, \mathbb{Q})/\mathcal{AC}(G/K, \mathbb{Q})$ generated by $\{\mathbb{1}_{C_{B_i}} + \mathcal{AC}(G/K, \mathbb{Q})\}_{i=1}^m$ is contained in the $\mathbb{Q}$-subspace $\mathcal{S}_2$ generated by $\{\mathbb{1}_{D_j} + \mathcal{AC}(G/K, \mathbb{Q})\}_{j=1}^p$. Note that $\dim_{\mathbb{Q}} \mathcal{S}_1 = m$. Indeed, $\mathbb{1}_{B_i}, \ldots, \mathbb{1}_{B_m}$ are linearly independent modulo $C(G/K) + \mathbb{Q}[G/K]^K$ (and so modulo $\mathcal{AC}(G/K, \mathbb{Q})$) and, for all $t_1, \ldots, t_m \in \mathbb{Q}$ and $x \in G$, we have

$$\Big(t_1 \mathbb{1}_{C_{B_1}} + \ldots + t_m \mathbb{1}_{C_{B_m}}\Big)(xK) = \Big(t_1 \mathbb{1}_{B_1} + \ldots + t_m \mathbb{1}_{B_m}\Big)(x^{-1}K).$$

Moreover, arguing as in the first part of the proof, $\mathbb{1}_{D_1}, \ldots, \mathbb{1}_{D_{p-1}}$ are linearly independent modulo $\mathcal{AC}(G/K, \mathbb{Q})$. However, $\mathbb{1}_{D_1}, \ldots, \mathbb{1}_{D_p}$ are linearly dependent modulo $\mathcal{AC}(G/K, \mathbb{Q})$: observe for example that $\sum_{j=1}^p \mathbb{1}_{D_j}$ is the constant function 1 almost everywhere. Thus $p - 1 = \dim_{\mathbb{Q}} \mathcal{S}_2 \geq \dim_{\mathbb{Q}} \mathcal{S}_1 = m$. Since every D_j spans an infinite connected component of $\Gamma(G, K, S) - \operatorname{span}(F)$, we conclude that the number of ends of $\Gamma(G, K, S)$ is at least $p \geq m + 1$. □

By Theorem 2.2.23 and recalling that $\mathrm{dH}^1(G, \_) \equiv 0$ whenever G is profinite, we deduce the following.

Corollary 2.2.24. *Given a compactly generated t.d.l.c. group G, the dimension of the $\mathbb{Q}$-vector space $\mathrm{dH}^1(G, \mathbb{Q}[G/K])$ is the same for all compact open subgroups K of G.*

Let $\mathcal{CO}(G)$ be the poset of all compact open subgroups of G ordered by reverse inclusion $\prec$. Recall that the rational discrete standard bimodule of G is defined in [CW16, Section 4.2] as

$$\mathrm{Bi}(G) := \varinjlim_{K \in \mathcal{CO}(G)} \Big(\mathbb{Q}[G/K], \eta_{K,H}\Big),$$

where, whenever $K \prec H$, $\eta_{K,H} : \mathbb{Q}[G/K] \to \mathbb{Q}[G/H]$ is the $\mathbb{Q}[G]$-module homomorphism given by $\eta_{K,H}(K) = |K : H|^{-1} \sum_{r \in \mathcal{R}_{K,H}} rH$. Here $\mathcal{R}_{K,H}$ denotes a set of representatives of the left cosets of H in K that contains 1.

Theorem 2.2.25. *Let G be a non-compact compactly generated t.d.l.c. group. The $\mathbb{Q}$-vector space $\mathrm{dH}^1(G, \mathrm{Bi}(G))$ has dimension $e(G)-1$ if G has finitely many ends. Moreover, G has infinitely many ends if and only if $\mathrm{dH}^1(G, \mathrm{Bi}(G))$ is infinite-dimensional.*

Proof. For all compact open subgroups $H \subseteq K$ of G, $\eta_{K,H} \colon \mathbb{Q}[G/K] \to \mathbb{Q}[G/H]$ induces the injective map $\mathrm{dH}^1(\eta_{H,K}) \colon \mathrm{dH}^1(G, \mathbb{Q}[G/K]) \to \mathrm{dH}^1(G, \mathbb{Q}[G/H])$ (cf. the end of the proof of [CW16, Proposition 4.7]). By [Cas20, Remark 3.4], we deduce

$$\mathrm{dH}^1(G, \mathrm{Bi}(G)) \cong \varinjlim_{K \in \mathcal{CO}(G)} \big(\mathrm{dH}^1(G, \mathbb{Q}[G/K]), \mathrm{dH}^1(\eta_{H,K})\big),$$

where the set of all compact open subgroups $\mathcal{CO}(G)$ is ordered by reverse inclusion.

The claim then follows by Corollary 2.2.24 and the subsequent two facts:

Fact 1. If $\dim_{\mathbb{Q}} \mathrm{dH}^1(G, \mathbb{Q}[G/\mathcal{O}]) = \infty$ for some $\mathcal{O} \in \mathcal{CO}(G)$, then the space

$$\varinjlim_{K \in \mathcal{CO}(\mathcal{G})} \big(\mathrm{dH}^1(G, \mathbb{Q}[G/K]), \mathrm{dH}^1(\eta_{H,K})\big)$$

has infinite dimension (cf. [RZ00, Proposition 1.2.4(c)]).

Fact 2. If $\dim_{\mathbb{Q}} \mathrm{dH}^1(G, \mathbb{Q}[G/\mathcal{O}]) = d < \infty$ for some $\mathcal{O} \in \mathcal{CO}(G)$, then the space

$$\varinjlim_{K \in \mathcal{CO}(G)} \big(\mathrm{dH}^1(G, \mathbb{Q}[G/K]), \mathrm{dH}^1(\eta_{H,K})\big)$$

has dimension d.

In order to prove Fact 2, it is sufficient to observe that, by Corollary 2.2.24, $\mathrm{dH}^1(\eta_{H,K})$ is an isomorphism for all H and K. Finally, [RZ00, Proposition 1.2.4(c)-(d)] yields the claim. □

Corollary 2.2.26. *Let G be a compactly generated t.d.l.c. group. Then,*

$$e(G) = 1 - \dim_{\mathbb{Q}} \mathrm{dH}^0(G, \mathrm{Bi}(G)) + \dim_{\mathbb{Q}} \mathrm{dH}^1(G, \mathrm{Bi}(G)).$$

Proof. The statement follows from the fact that $e(G) = 0$ if, and only if, G is compact (cf. Example 2.2.15(a)), from Theorem 2.2.25 and [CW16, Proposition 4.3(b)]. □

2.2.5 Number of ends and groups acting on buildings

In the following, we apply results from Section 2.2.4 to t.d.l.c. groups acting properly and cocompactly on locally finite buildings.

Given a locally finite building Δ, denote by $\mathrm{H}_c^\bullet(|\Delta_{\mathrm{Dav}}|, \mathbb{Q}) = \mathrm{H}_c^\bullet(|\Delta_{\mathrm{Dav}}|, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ the cohomology with compact support of the geometric realisation of Δ_{Dav} with rational coefficients (cf. [CW16, Appendix A.4]). Furthermore, let $C_c(G, \mathbb{Q})$ be the space of all locally constant functions from G to $\mathbb{Q}$ with compact support (cf. Section 1.8.3), and denote by $\mathrm{cd}_{\mathbb{Q}}(W)$ the ordinary rational cohomological dimension of W.

Lemma 2.2.27. *Let G be a t.d.l.c. group acting properly and cocompactly on a locally finite building Δ of type (W, S). Then, for every $k \geq 0$, we have a canonical isomorphism of $\mathbb{Q}$-vector spaces*

$$\mathrm{dH}^k(G, C_c(G, \mathbb{Q})) \simeq \mathrm{H}_c^k(|\Delta_{\mathrm{Dav}}|, \mathbb{Q}). \tag{2.2.6}$$

Proof. The simplicial complex Δ_{Dav} is finite-dimensional, locally finite and has contractible geometric realisation (cf. Section 1.5.2). By hypothesis, G acts on Δ_{Dav} with compact open vertex-stabilisers (cf. [Kra22, Lemma 5.13]). By (1.6.2), G has finitely many orbits on each skeleton of Δ_{Dav}. Now the argument used to prove [CW16, Eq. (6.13)] can be transferred verbatim. □

Remark 2.2.28. There exist analogous isomorphisms to (2.2.6) when replacing Δ_{Dav} by any locally finite contractible simplicial complex X of type F_∞ on which G acts properly (cf. [CW16, Theorem 6.7 and Equation (6.13)] for details).

Combining Lemma 2.2.27 and Corollary 2.2.26, we deduce what follows.

Corollary 2.2.29. *Let Δ be a locally finite building and G be a t.d.l.c. group acting properly and cocompactly on Δ. Then*

$$e(|\Delta_{\mathrm{Dav}}|) = e(G) = 1 - \dim_{\mathbb{Q}} \mathrm{H}_c^0(|\Delta_{\mathrm{Dav}}|, \mathbb{Q}) + \dim_{\mathbb{Q}} \mathrm{H}_c^1(|\Delta_{\mathrm{Dav}}|, \mathbb{Q}), \tag{2.2.7}$$

where $e(|\Delta_{\mathrm{Dav}}|)$ is the number of ends of $|\Delta_{\mathrm{Dav}}|$ as defined in [Dav08, Appendix G.3]. In particular, if G is non-compact then $e(G) = 1 + \dim_{\mathbb{Q}} \mathrm{H}_c^1(|\Delta_{\mathrm{Dav}}|, \mathbb{Q})$.

Proof. By Lemma 2.2.27 (together with the isomorphism in [CW16, Equation (4.64)]) and Corollary 2.2.26, we have

$$e(G) = 1 - \dim_{\mathbb{Q}} \mathrm{H}_c^0(|\Delta_{\mathrm{Dav}}|, \mathbb{Q}) + \dim_{\mathbb{Q}} \mathrm{H}_c^1(|\Delta_{\mathrm{Dav}}|, \mathbb{Q})$$

and, if G is non-compact, $e(G) = 1 + \dim_{\mathbb{Q}} \mathrm{H}_c^1(|\Delta_{\mathrm{Dav}}|, \mathbb{Q})$. It suffices now to observe that G acts geometrically on the proper geodesic space $|\Delta_{\mathrm{Dav}}|$ and then $e(G) = e(|\Delta_{\mathrm{Dav}}|)$ (cf. [CH16, Theorem 4.C.5]). □

Remark 2.2.30. Let G be a t.d.l.c. group acting properly and cocompactly on a locally finite building Δ of type (W, S). By Theorem 2.3.20 (which will be proved in the next pages), we have

$$\begin{aligned} e(G) = 0 \iff & \ G \text{ compact} \iff \mathrm{cd}_{\mathbb{Q}}(G) = 0 \\ \iff & \ \mathrm{cd}_{\mathbb{Q}}(W) = 0 \iff W \text{ finite} \iff e(W) = 0. \end{aligned}$$

The following proposition provides an alternative proof of [CMR22, Proposition 5.15], and is based on a result of [Dav+08].

Proposition 2.2.31. *Let G be a t.d.l.c. group acting properly and cocompactly on a locally finite building Δ of type (W,S). Then $e(W) \le e(G)$ and*

$$\begin{aligned} &e(W)=0 \iff W \text{ is finite} \iff G \text{ is compact} \iff e(G)=0;\\ &e(W)=1 \iff e(G)=1. \end{aligned} \tag{2.2.8}$$

Proof. For every $k \ge 0$, we claim that

$$\mathrm{H}_c^k(|\Delta_{\mathrm{Dav}}|,\mathbb{Q})=0 \iff \mathrm{H}_c^k(|\Sigma(W,S)_{\mathrm{Dav}}|,\mathbb{Q})=0. \tag{2.2.9}$$

Indeed, from the main theorem of [Dav+08] we have

$$\mathrm{H}_c^k(|\Delta_{\mathrm{Dav}}|,\mathbb{Q}) \simeq \bigoplus_{\substack{T\subseteq S,\\ T \text{ spherical}}} \mathrm{H}^k(K,K^{S\setminus T})\otimes \widehat{A}^T(\Delta) \tag{2.2.10}$$

and

$$\mathrm{H}_c^k(|\Sigma(W,S)_{\mathrm{Dav}}|,\mathbb{Q}) \simeq \bigoplus_{\substack{T\subseteq S,\\ T \text{ spherical}}} \mathrm{H}^k(K,K^{S\setminus T})\otimes \widehat{A}^T(\Sigma(W,S)). \tag{2.2.11}$$

The Davis chamber K depends only on the Coxeter group (W,S) and not on the building. By [Dav+08, Remark at p. 570 and Definition 7.4], the abelian groups $\widehat{A}^T(\Delta)$ and $\widehat{A}^T(\Sigma(W,S))$ are non-vanishing for every spherical subset $T\subseteq S$. Therefore, $\mathrm{H}_c^k(|\Delta_{\mathrm{Dav}}|,\mathbb{Q})=0$ if, and only if, $\mathrm{H}^k(K,K^{S-T})=0$ for every $T\subseteq S$ spherical (cf. (2.2.10)). By (2.2.11), an analogous equivalence holds for $\mathrm{H}_c^k(|\Sigma(W,S)_{\mathrm{Dav}}|,\mathbb{Q})$.

We first prove that

$$e(G)=0 \iff G \text{ is compact} \iff W \text{ is finite} \iff e(W)=0. \tag{2.2.12}$$

By [CW16, Proposition 6.6(c)], both G and W are of type FP in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ and ${}_{\mathbb{Q}[W]}\mathbf{mod}$, respectively. Therefore, by Proposition 1.8.5(i)-(iv), we have

$$\begin{array}{ccccc} G \text{ is compact} & \iff & \mathrm{cd}_{\mathbb{Q}}(G)=0 & \iff & \mathrm{H}_c^k(|\Delta_{\mathrm{Dav}}|,\mathbb{Q})=0,\ \forall\, k\ge 1;\\ W \text{ is finite} & \iff & \mathrm{cd}_{\mathbb{Q}}(W)=0 & \iff & \mathrm{H}_c^k(|\Sigma(W,S)_{\mathrm{Dav}}|,\mathbb{Q})=0,\ \forall\, k\ge 1, \end{array}$$

and the claim in (2.2.12) follows.

From now onwards, assume that G is non-compact. By (2.2.12), W is infinite. Moreover, by Corollary 2.2.29 and Remark 2.2.30,

$$e(G)=1+\dim_{\mathbb{Q}}\mathrm{H}_c^1(|\Delta_{\mathrm{Dav}}|,\mathbb{Q}) \quad \text{and} \quad e(W)=1+\dim_{\mathbb{Q}}\mathrm{H}_c^1(|\Sigma(W,S)_{\mathrm{Dav}}|,\mathbb{Q}). \tag{2.2.13}$$

According to [Dav08, p. 338], there is a continuous map $\rho : |\Delta_{\mathrm{Dav}}| \longrightarrow |\Sigma(W,S)_{\mathrm{Dav}}|$ which is a retraction (i.e., it admits a right-inverse continuous map). By functoriality, ρ induces the surjective linear maps

$$\rho_k^*\colon \mathrm{H}_c^k(|\Delta_{\mathrm{Dav}}|,\mathbb{Q}) \longrightarrow \mathrm{H}_c^k(|\Sigma(W,S)_{\mathrm{Dav}}|,\mathbb{Q}), \quad k\ge 0. \tag{2.2.14}$$

By Corollary 2.2.29 and (2.2.14), we deduce that

$$e(W) \leq 1 + \dim_{\mathbb{Q}} \mathrm{H}^1_c(|\Sigma(W,S)_{\mathrm{Dav}}|, \mathbb{Q}) \leq 1 + \dim_{\mathbb{Q}} \mathrm{H}^1_c(|\Delta_{\mathrm{Dav}}|, \mathbb{Q}) = e(G).$$

Finally, since $\mathrm{H}^1_c(|\Delta_{\mathrm{Dav}}|, \mathbb{Q}) = 0 \iff \mathrm{H}^1_c(|\Sigma(W,S)_{\mathrm{Dav}}|, \mathbb{Q}) = 0$, from (2.2.13) we also conclude that

$$e(G) = 1 \iff e(W) = 1. \qquad \square$$

Remark 2.2.32. In Proposition 2.2.31 the equality between $e(W)$ and $e(G)$ does not hold in general. For example, consider $G = \mathrm{SL}_2(\mathbb{Q}_p)$ acting on its Bruhat–Tits building Δ. It is well known that the G-action on Δ is proper and cocompact, and that Δ is a $(p+1)$-regular tree. By Example 2.2.15(iv) one has $e(\mathrm{SL}_2(\mathbb{Q}_p)) = \infty$. On the other hand, the Weyl group associated to Δ is the infinite dihedral group, which is two-ended.

2.3 The rational discrete cohomological dimension of t.d.l.c. groups

This section focuses on the *rational discrete cohomological dimension* of t.d.l.c. groups acting properly and cocompactly on locally finite buildings. We have recalled the general definition of this invariant in Section 1.8.4. The case in which the building is a tree is a straightforward consequence of what was already observed in [CW16], and it is understood even by dropping the assumptions of properness and cocompactness on the action (cf. Section 2.3.1). For arbitrary buildings, we prove that the relevant cohomological dimension coincides with the (standard) rational cohomological dimension of the Coxeter group that describes the type of the building (cf. Theorem 2.3.20). This description allows to conveniently transfer methods for computing the cohomological dimension of a Coxeter group to the more general (but, in principle, implicit) setting mentioned above. In Section 2.3.2 we collect some standard results on the rational discrete cohomological dimension of a Coxeter group, providing new explicit formulae for Coxeter groups of hyperbolic type (cf. Propositions 2.3.16 and 2.3.17).

2.3.1 The case of groups acting on trees

Proposition 2.3.1. *Let G a t.d.l.c. group acting on a tree T without edge-inversions. Let $\mathcal{V}$ and $\mathcal{E}^+$ be arbitrary sets of representatives for the G-orbits on VT and on a given orientation in T. Then, for every $M \in \mathrm{ob}({}_{\mathbb{Q}[G]}\mathbf{dis})$, there is a long exact sequence*

$$\cdots \longrightarrow \mathrm{dH}^n(G, M) \longrightarrow \prod_{v \in \mathcal{V}} \mathrm{dH}^n(G_v, M) \longrightarrow \prod_{e \in \mathcal{E}^+} \mathrm{dH}^n(G_e, M) \longrightarrow \cdots$$

In particular,

$$\sup_{v \in \mathcal{V}} \{\mathrm{cd}_{\mathbb{Q}}(G_v)\} \leq \mathrm{cd}_{\mathbb{Q}}(G) \leq \sup \{\mathrm{cd}_{\mathbb{Q}}(G_v), 1 + \mathrm{cd}_{\mathbb{Q}}(G_e) \mid v \in \mathcal{V},\, e \in \mathcal{E}^+\}. \qquad (2.3.1)$$

Proof. The first part of the statement is [CW16, Proposition 5.4(a)]. From this, one easily deduces the second inequality of (2.3.1). Finally, the first inequality in (2.3.1) is due to Proposition 1.8.5(ii). □

Corollary 2.3.2. *Let G be a t.d.l.c. group acting properly and cocompactly on a (locally finite) tree T without edge-inversions. Then $\mathrm{cd}_{\mathbb{Q}}(G) \leq 1$ and $\mathrm{cd}_{\mathbb{Q}}(G) = 0$ if, and only if, G is compact.*

Note that the locally finiteness of T follows from the fact that the G-action on T is proper and cocompact. Indeed, if ω is the standard edge weight on $G\backslash T$ (cf. Section 1.3.2), then $\omega(a) < \infty$ for every $a \in G\backslash ET$. Hence, for every $v \in VT$ one has

$$|o^{-1}(v)| = \sum_{a \in G\backslash ET:\, o(a) = G\cdot v} \omega(a) < \infty.$$

Proof. By Proposition 1.8.5(i), for every $t \in T$ we have $\mathrm{cd}_{\mathbb{Q}}(G_t) = 0$. Hence, Proposition 2.3.1 implies $\mathrm{cd}_{\mathbb{Q}}(G) \leq 1$. Moreover, again by Proposition 1.8.5(i), the condition $\mathrm{cd}_{\mathbb{Q}}(G) = 0$ is equivalent to have G compact. □

2.3.2 The case of Coxeter groups

Coxeter groups constitute a notable family of discrete groups of type FP over $\mathbb{Q}$. By Proposition 1.8.5(iii) (or [Bro12, Chapter VIII, Proposition 6.7]), note that

$$\mathrm{cd}_{\mathbb{Q}}(W) = \max\{n \geq 0 \mid \mathrm{H}^n(W, \mathbb{Q}[W]) \neq 0\}. \tag{2.3.2}$$

Another possible characterisation of $\mathrm{cd}_{\mathbb{Q}}(W)$ is the following.

Lemma 2.3.3. *For every locally finite building Δ of type (W, S), we have*

$$\mathrm{cd}_{\mathbb{Q}}(W) = \max\{n \geq 0 \mid \mathrm{H}^n_c(|\Delta_{\mathrm{Dav}}|, \mathbb{Q}) \neq 0\},$$

where Δ_{Dav} is the Davis' complex associated to Δ (cf. Section 1.5.2).

Proof. The argument used to prove the equality (6.26) in [CW16] transfers verbatim. □

In Section 2.3.3, we discuss how the formula in Lemma 2.3.3 can be extended to a wider class of groups acting on locally finite buildings (cf. Theorem 2.3.20).

Alternatively to the concept of cohomological dimension, one may consider the *virtual cohomological dimension* of a group. In the case of a Coxeter group W, the latter invariant provides a convenient upper-bound for $\mathrm{cd}_{\mathbb{Q}}(W)$ (cf. Fact 2.3.11 below).

In order to define the virtual cohomological dimension of a Coxeter group W, it is worth recall that W is *virtually torsion-free*, i.e., it has a torsion-free subgroup of finite index. This is due to the fact that W admits a faithful linear representation (cf. [Dav08, Corollary D.1.4]). For virtually torsion-free groups, the following well-known result of J-P. Serre holds.

Theorem 2.3.4 ([Bro12, Chapter VIII, Theorem 3.1]). *Let G be a discrete group and let R be either $\mathbb{Z}$ or a field. Then, for all torsion-free subgroups $H_1, H_2 \leq G$ of finite index, we have* $\mathrm{cd}_R(H_1) = \mathrm{cd}_R(H_2)$.

Thanks to Theorem 2.3.4, the following invariant is well-defined.

Definition 2.3.5. Let G be a virtually torsion-free discrete group, and let R be either $\mathbb{Z}$ or a field. The *virtual R-cohomological dimension* of G is defined as

$$\mathrm{vcd}_R(G) := \mathrm{cd}_R(H),$$

where $H \leq G$ is any torsion-free subgroup of finite index.

If $R = \mathbb{Z}$, the subscript "R" in $\mathrm{vcd}_R(\_)$ is usually omitted.

In 1993, M. Bestvina [Bes93] provided an effective method to compute the virtual cohomological dimension of finitely generated Coxeter groups. Namely, for R being either $\mathbb{Z}$ or a field, Bestvina's construction – recalled below – produces an acyclic complex $B_R(W,S)$ of dimension $\mathrm{vcd}_R(W)$ on which W acts proper discontinuously and cocompactly. In the following, given a Coxeter group (W,S), we denote by $\mathcal{S}$ the collection of all spherical subsets of (W,S).

Definition 2.3.6 (*Panel complex*, cf. [Bes93, p. 20]). A *panel complex* of (W,S) is a pair $(K,\mathcal{P})$ consisting of a compact polyhedron K and a collection of acyclic polyhedra of K, say $\mathcal{P} = \{P_J \subseteq K \mid J \in \mathcal{S}\}$, with $P_\emptyset = K$ and such that, for every $J_1, J_2 \in \mathcal{S}$,

$$P_{J_1} \cap P_{J_2} = \begin{cases} P_{J_1 \cup J_2}, & \text{if } J_1 \cup J_2 \in \mathcal{S}; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Fact 2.3.7 ([Bes93, p. 20]). *Let $(K,\mathcal{P})$ be a panel complex of (W,S). Define the following equivalence relation on $W \times K$:*

$$(w_1,k_1) \sim (w_2,k_2) \iff \begin{cases} k_1 = k_2; \\ w_1 w_2^{-1} \in \langle \{s \in S \mid k_1 \in P_{\{s\}}\} \rangle \leq W. \end{cases} \tag{2.3.3}$$

Then W acts proper discontinuously and cocompactly on $X := (W \times K)/_{\sim}$ as a reflection group.

As a particular instance of a panel complex, one defines the following.

Definition 2.3.8 (*Bestvina's panel complex*, cf. [Bes93, p. 20]). Let (W,S) be a Coxeter group and order the elements $\mathcal{S}$ by inclusion. Let also R be either $\mathbb{Z}$ or a field. We inductively define the panel complex $(B_R(W,S),\mathcal{P})$ as follows:

(a) for every $J \in \mathcal{S}$ maximal, let P_J be a point;

(b) let $J \in \mathcal{S}$ and assume to have defined P_I for every $I \in \mathcal{S}$ with $J \subsetneq I$. Let P_J be any R-acyclic polyhedron of the least possible dimension among the R-acyclic polyhedra containing $\bigcup_{I \in \mathcal{S},\, J \subsetneq I} P_I$. Recall that an *$R$-acyclic polyhedron* is a polyhedron whose geometric realisation has vanishing homology groups with coefficient in R.

Finally, define $B_R(W, S) = P_\emptyset$.

Theorem 2.3.9 ([Bes93]). *Let (W, S) be a Coxeter group, and R be either $\mathbb{Z}$ or a field. Then $\mathrm{vcd}_R(W)$ equals the dimension of the polyhedron $B(W, S)$.*

Remark 2.3.10. Let (W, S) be a Coxeter group, and $|\Sigma(W, S)_{\mathrm{Dav}}|$ be the geometric realisation of the Davis' complex of the standard Coxeter complex of (W, S) (cf. Example 1.5.10). As observed in [Bes93, p. 20], the space $|\Sigma(W, S)_{\mathrm{Dav}}|$ arises from the panel complex $(K', \mathcal{P}')$ constructed inductively as follows:

(a') for every $J \in \mathcal{S}$ maximal (with respect to the inclusion), let P'_J to be a point;

(b') let $J \in \mathcal{S}$ and assume to have defined P'_I for every $I \in \mathcal{S}$ with $J \subsetneq I$. Then define P'_J as the cone of $\bigcup_{I \in \mathcal{S},\, J \subsetneq I} P'_I$.

Finally, define $K' := P'_\emptyset$ as the cone of $\bigcup_{I \in \mathcal{S}} P'_I$. One proves that $|\Sigma(W, S)_{\mathrm{Dav}}| = (W \times K')/_\sim$, where $\sim$ is as in (2.3.3), and $\mathrm{vcd}_R(W) \leq \dim \Sigma(W, S)_{\mathrm{Dav}}$ (cf. [Dav08, §7] and [Bes93, p. 20]).

What makes Bestvina's complex the optimal choice of computing the virtual cohomological dimension is the following. The constructions of $(B_R(W, S), \mathcal{P})$ and $(K', \mathcal{P}')$ are analogous except for one detail: each P'_I is the cone of $\bigcup_{J \in \mathcal{S},\, J \subsetneq I} P_J$, while P_I is any R-acyclic polyhedron of the least possible dimension containing $\bigcup_{J \in \mathcal{S},\, J \subsetneq I} P_J$ (cf. Definition 2.3.8). In particular, the dimension of $B_R(W, S)$ is not greater than the dimension of K' (i.e., of $\Sigma(W, S)$).

The difference is evident if, for instance, (W, S) is spherical with $S \neq \emptyset$. In this case, $B_R(W, S)$ has dimension $0 = \mathrm{vcd}_R(W)$ and $|\Sigma(W, S)_{\mathrm{Dav}}|$ has dimension $|S| \geq 1$.

Fact 2.3.11. *Let (W, S) be a Coxeter group. Then $\mathrm{cd}_{\mathbb{Q}}(W) = \mathrm{vcd}_{\mathbb{Q}}(W) \leq \mathrm{vcd}(W)$.*

Remark 2.3.12. Let (W, S) be a Coxeter group.

(i) We have $\mathrm{vcd}(W) = 0$ if, and only if, $\mathrm{cd}_{\mathbb{Q}}(W) = 0$. Moreover, by Proposition 1.8.5 $\mathrm{cd}_{\mathbb{Q}}(W) = 0$ if and only if W is finite.

(ii) By Stallings–Swan's theorem [Swa69] and by (i), we have

$$\mathrm{cd}_{\mathbb{Q}}(W) = 1 \iff \mathrm{vcd}(W) = 1 \iff W \text{ is infinite and virtually free.}$$

Recall from Theorem 1.4.8 that the virtually freeness of W can be explicitly rephrased in terms of the presentation diagram of (W, S).

(iii) By (i), (ii) and Fact 2.3.11, if $\mathrm{vcd}(W) \leq 2$ then $\mathrm{cd}_{\mathbb{Q}}(W) = \mathrm{vcd}(W)$. The latter equality might not hold anymore if $\mathrm{vcd}(W) = 3$ (cf. [Dav08, Example 8.5.8]).

(iv) Let (W, S) be affine. Then

$$\mathrm{cd}_{\mathbb{Q}}(W) = \mathrm{vcd}(W) = |S| - 1.$$

Indeed, W has a finite-index subgroup which is isomorphic to $\mathbb{Z}^{|S|-1}$. Therefore, $\mathrm{vcd}(W) = \mathrm{cd}_{\mathbb{Z}}(\mathbb{Z}^{|S|-1}) = |S| - 1$. Moreover, by Proposition 1.8.5(ii) and Fact 2.3.11 we have $|S| - 1 = \mathrm{cd}_{\mathbb{Q}}(\mathbb{Z}^{|S|-1}) \leq \mathrm{cd}_{\mathbb{Q}}(W) \leq \mathrm{vcd}(W)$, which yields the claim.

Remark 2.3.12 gives explicit formulae for $\mathrm{cd}_{\mathbb{Q}}(W)$ and $\mathrm{vcd}(W)$ if (W, S) is either spherical or affine. The next section provides explicit computations of these invariants if (W, S) belongs to another relevant class of Coxeter groups.

(Virtual) cohomological dimension of Coxeter groups of hyperbolic type

In this section we compute the rational and the virtual integral cohomological dimensions of a Coxeter groups of hyperbolic type. We have recalled the definition of Coxeter groups of hyperbolic type in Section 1.4.1. We now add further facts that are going to be useful for the next discussion.

Remark 2.3.13. Let (W, S) be a Coxeter group of hyperbolic type with $|S| = n$. Then the following holds.

(i) One has $3 \leq n \leq 10$ (cf. [Hum90, Exercise 6.8]). In particular, there are finitely many Coxeter groups of hyperbolic type with $n \geq 4$ and they are listed by their Coxeter diagram in [Hum90, p. 141].

(ii) For a Coxeter group of hyperbolic type (W, S), the image of the geometric representation $\sigma\colon W \to \mathrm{GL}_n(\mathbb{R})$ is contained in $O(n-1, 1)$ (cf. Section 1.4.1). In particular, $\mathrm{im}(\sigma)$ is a lattice in $O(n-1, 1)$ (cf. [Hum90, Remark 6.8], [Bou81, pp. 131-135] and [Kos67]).

(iii) If (W, S) is additionally a crystallographic Coxeter group, then $\mathrm{im}(\sigma)$ stabilises some lattice $\Lambda \subset V$ (cf. [Hum90, §6.6]). Thus, if $\mathcal{B} = \{\lambda_i \mid 1 \leq i \leq |S|\}$ is a free generating system of Λ, then there exists $e \in \mathbb{Z}_{\geq 1}$ such that $B(\lambda_i, \lambda_j) \in \frac{1}{e}\mathbb{Z}$. Defining

$$\mathbf{G}(\mathbb{Z}) := \{\alpha \in \mathrm{GL}_{\mathbb{Z}}(\Lambda) \mid B(\alpha(v_i), \alpha(v_j)) = B(v_i, v_j)\}$$

yields an affine algebraic group scheme $\mathbf{G}$ defined over $\mathbb{Z}$, and

$$W \subseteq \mathbf{G}(\mathbb{Z}) \subset \mathbf{G}(\mathbb{R}) = O(n-1, 1).$$

Since W is a lattice in $O(n-1, 1)$, so is $\mathbf{G}(\mathbb{Z})$ and $|\mathbf{G}(\mathbb{Z})\colon W| < \infty$.

As a consequence of Remark 2.3.13(iii), we deduce the following.

Fact 2.3.14. *Let (W,S) be a hyperbolic crystallographic Coxeter group. Then W is an arithmetic lattice in $O(n-1,1)$, where $n=|S|$.*

Definition 2.3.15 ([Hum90, §6.9])**.** A Coxeter group of hyperbolic type (W,S) is said to be *of compact hyperbolic type* if one of the two equivalent conditions holds:

(i) the group W is cocompact in $O(n-1,1)$;

(ii) for every $s\in S$, the special subgroup $W_{S\setminus\{s\}}$ is finite.

If (W,S) is of hyperbolic type but not of compact hyperbolic type, it is said to be of *non-compact hyperbolic type.*

Proposition 2.3.16. *Let (W,S) be a Coxeter group of compact hyperbolic type. Then*

$$\mathrm{cd}_{\mathbb{Q}}(W)=\mathrm{vcd}(W)=|S|-1.$$

Proof. By Definition 2.3.15(ii), every proper special subgroup of (W,S) is spherical. Hence, the poset of all spherical special subgroups of (W,S) is isomorphic to the poset of all spherical special subgroups of an affine Coxeter group (W',S') with $|S'|=|S|$, where the two posets are ordered by inclusion. Therefore, $B_R(W,S)$ and $B_R(W',S')$ (for $R\in\{\mathbb{Z},\mathbb{Q}\}$) have the same dimension. By Theorem 2.3.9, we conclude that

$$\mathrm{cd}_{\mathbb{Q}}(W)=\mathrm{cd}_{\mathbb{Q}}(W')=|S|-1=\mathrm{vcd}(W')=\mathrm{vcd}(W).$$ ☐

Proposition 2.3.17. *Let (W,S) be a Coxeter group of non-compact hyperbolic type. Then*

$$\mathrm{cd}_{\mathbb{Q}}(W)=\mathrm{vcd}(W)=|S|-2.$$

Proof. Let $|S|=n$. By Definition 2.3.15, every Coxeter group (W,S) of non-compact hyperbolic type contains an affine special subgroup (W',S') with $|S'|=n-1$. Hence,

$$\mathrm{vcd}(W)\geq \mathrm{cd}_{\mathbb{Q}}(W)\geq \mathrm{cd}_{\mathbb{Q}}(W')=n-2. \tag{2.3.4}$$

It remains to show that $\mathrm{vcd}(W)\leq n-2$. We first deal with the crystallographic case. Let X denote the symmetric space associated to $O(n-1,1)$. Then,

$$\dim(X)=\binom{n}{2}-\binom{n-1}{2}=n-1. \tag{2.3.5}$$

Since $W\backslash X$ is non-compact, we have $\partial X\neq\emptyset$ (cf. [BS73, Theorem 9.3]). Equivalently, the $\mathbb{Q}$-rank of $\mathbf{G}$ (cf. Remark 2.3.13(iii)) satisfies $\ell(\mathbf{G})\geq 1$, which by [BS73, Corollary 11.4.3] yields

$$\mathrm{vcd}(W)=\dim(X)-\ell(\mathbf{G})\leq n-2.$$

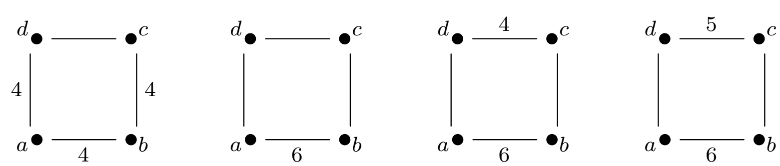
In the remaining non-crystallographic cases, applying Fact 2.3.18 below, we deduce that the Bestvina's complex $B_{\mathbb{Z}}(W,S)$ has at most $n-1$ vertices and thus it can be embedded in $\mathbb{R}^{n-2}$. By Theorem 2.3.9, we conclude that $\mathrm{vcd}(W)\leq n-2$. ☐

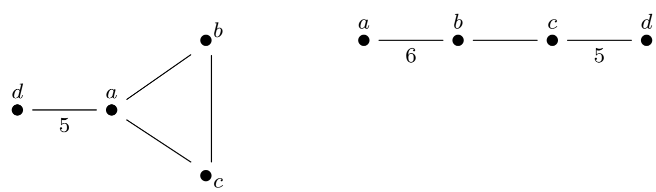
Fact 2.3.18. *Let (W, S) be a Coxeter group of non-compact hyperbolic type. Suppose that either $|S| = 3$ or (W, S) is non-crystallographic with $|S| \geq 4$. Then (W, S) admits exactly $|S| - 1$ maximal spherical subsets.*

In the following proof, by an *r-subset* of S we mean a subset with r elements. Moreover, we make use of the classifications Coxeter groups of spherical, affine and hyperbolic types by their Coxeter diagram (cf. [Hum90, pp. 32, 34, 142–144]). If (W, S) is of hyperbolic type, every r-subset of S with $r \leq |S| - 2$ is spherical (cf. [Hum90, Proposition 6.8], recalling that every proper special subgroup of a Coxeter group of positive type is spherical). Hence the maximal spherical subsets of S have cardinality either $|S| - 1$ or $|S| - 2$.

Proof. Let first $|S| = 3$, say $S = \{a, b, c\}$. Since (W, S) is non-compact hyperbolic, at least one edge of $\Gamma(W, S)$ is labelled by ∞, say $m_{ab} = \infty$. Moreover, being (W, S) not crystallographic, either $m_{ac} < \infty$ or $m_{bc} < \infty$. If both m_{ac} and m_{bc} are finite, then the maximal spherical subsets of S are $\{a, c\}$ and $\{b, c\}$. If $m_{bc} = \infty$ and $m_{ac} < \infty$, then the maximal spherical subsets are $\{a, c\}$ and $\{b\}$. An analogous conclusion holds if $m_{ac} = \infty$ and $m_{bc} < \infty$.

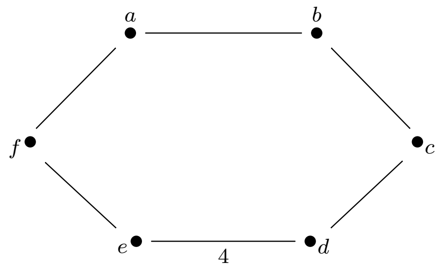
Assume now that $|S| \geq 4$. By [Hum90, §6.9, Figure 3], there are only seven cases: six with $|S| = 4$ and one with $|S| = 6$. If $S = \{a, b, c, d\}$, then (W, S) has one of following Coxeter diagrams:


(2.3.6)


(2.3.7)

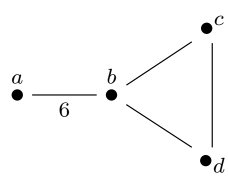
Let first (W, S) be associated to one of the Coxeter diagrams in (2.3.6). Among the 3-subsets of S, only $\{a, c, d\}$ and $\{b, c, d\}$ are spherical. Note also that $\{a, b\}$ is the only 2-subset of S which is not contained in both $\{a, c, d\}$ and $\{b, c, d\}$. Thus, $\{a, c, d\}$ and $\{b, c, d\}$ and $\{a, b\}$ are the maximal spherical subsets of S. Assume now that (W, S) is associated to one of the Coxeter diagrams in (2.3.7). Among the 3-subsets of S, the following ones are spherical: $\{a, b, d\}$, $\{a, c, d\}$ and $\{b, c, d\}$. Hence, they are the maximal spherical subsets of S.

Finally, let $S = \{a, b, c, d, e, f\}$. Then (W, S) has the following Coxeter diagram:



For $x \in S$, the set $S \setminus \{x\}$ is spherical if and only if $x \notin \{a, b\}$. In particular, the 4-subset $S \setminus \{x, y\}$ is not contained in a spherical 5-subset of S if, and only if, $\{x, y\} = \{a, b\}$. We conclude that the maximal spherical subsets of S are $S \setminus \{x\}$, for some $x \notin \{a, b\}$, and $S \setminus \{a, b\}$. □

Remark 2.3.19. Fact 2.3.18 might *not* be true for arbitrary crystallographic Coxeter groups of non-compact hyperbolic type. For example, let (W, S) be the Coxeter group defined by the following Coxeter diagram:



Among the 3-subsets of $S = \{a, b, c, d\}$, only $\{a, c, d\}$ is spherical. Moreover, there are three spherical 2-subsets of S, namely $\{a, b\}$, $\{b, c\}$ and $\{b, d\}$, that are not contained in $\{a, c, d\}$. As a conclusion, (W, S) admits exactly $4 = |S|$ maximal spherical subsets and so does not satisfy the conclusion of Fact 2.3.18.

2.3.3 The case of groups acting on buildings

Given a locally finite building Δ, denote by $\mathrm{H}_c^\bullet(|\Delta_{\mathrm{Dav}}|, \mathbb{Q})$ the cohomology with compact support of the geometric realisation of Δ_{Dav} with rational coefficients (cf. [CW16, Appendix A.4]). Furthermore, let $C_c(G, \mathbb{Q})$ be the space of all locally constant functions from G to $\mathbb{Q}$ with compact support, and let $\mathrm{cd}_{\mathbb{Q}}(W)$ be the ordinary rational cohomological dimension of W (cf. Section 2.3.2).

Theorem 2.3.20. *Let G be a t.d.l.c. group acting properly and cocompactly on a locally finite building Δ of type (W, S). Then*

$$\mathrm{cd}_{\mathbb{Q}}(G) = \mathrm{cd}_{\mathbb{Q}}(W). \tag{2.3.8}$$

In particular G is compact if, and only if, W is spherical.

Proof. Arguing as in the proof of Lemma 2.2.27, the group G acts on Δ_{Dav} with compact open stabilisers and finitely many orbits on each skeleton. In particular, $\mathrm{cd}_{\mathbb{Q}}(G) \leq \dim(\Delta_{\mathrm{Dav}})$ and G is of type FP_∞ in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ (cf. [CW16, §3.6 and §6.5]). By [CW16, Proposition 4.7, Equation (4.73)], we obtain

$$\mathrm{cd}_{\mathbb{Q}}(G) = \max\{n \geq 0 \mid \mathrm{dH}^n(G, C_c(G, \mathbb{Q})) \neq 0\}$$

and therefore, by Lemma 2.2.27,

$$\mathrm{cd}_{\mathbb{Q}}(G) = \max\{n \geq 0 \mid \mathrm{H}^n_c(|\Delta_{\mathrm{Dav}}|, \mathbb{Q}) \neq 0\}.$$

By Lemma 2.3.3, we deduce that $\mathrm{cd}_{\mathbb{Q}}(G) = \mathrm{cd}_{\mathbb{Q}}(W)$. The last assertion of the statement is now a consequence of Proposition 1.8.5(a). □

Corollary 2.3.21. *Let G be a unimodular t.d.l.c. group acting properly and cocompactly on a locally finite building Δ of type (W, S). If W is infinite and virtually free, then G is non-compact and has a finitely generated free subgroup that is cocompact and discrete.*

Proof. By [Dun79, Corollary 1.2], we have $\mathrm{cd}_{\mathbb{Q}}(W) = 1$. Hence, by Theorem 2.3.20, $\mathrm{cd}_{\mathbb{Q}}(G) = 1$ as well. Moreover G acts on Δ_{Dav} with compact open stabilisers and finitely many orbits and so G is compactly presented (cf. [CC20, Theorem 4.9]). Thus G is the fundamental group of a finite graph of profinite groups (cf. [Cas20, Theorem B]). As G is unimodular, Theorem 2.2.10(ii) yields the claim. □

In the following examples, we collect consequences of Theorem 2.3.20.

Example 2.3.22. Let $\mathcal{G}$ be a semisimple algebraic group defined over a non-Archimedean local field K, and let $G = \mathcal{G}(K)$. We have mentioned in Example 1.6.11(iii) that the Bruhat–Tits building Δ of G is a locally finite building of affine type (W, S) and G acts properly and Weyl-transitively (and hence cocompactly) on it. By Remark 2.3.13(iv) and Theorem 2.3.20, we deduce that

$$\mathrm{cd}_{\mathbb{Q}}(G) = |S| - 1.$$

For instance, if $\mathcal{G} = \mathrm{SL}_n$ for some $n \geq 2$, then $\mathrm{cd}_{\mathbb{Q}}(\mathrm{SL}_n(K)) = n - 1$. These conclusions agree with what was already observed in [CW16, §6.11].

Example 2.3.23. Let $\bar{G}$ be the geometric completion of a Kac–Moody group G over a finite field associated to a finite root datum of type (W, S). According to Example 1.6.11(iv) and Theorem 2.3.20, we have

$$\mathrm{cd}_{\mathbb{Q}}(\bar{G}) = \mathrm{cd}_{\mathbb{Q}}(W).$$

These conclusions agree with what was already observed in [CW16, §6.12]. However, after Propositions 2.3.16 and 2.3.17, we can give a more explicit description of $\mathrm{cd}_{\mathbb{Q}}(\bar{G})$ if (W, S) is a Coxeter group of hyperbolic crystallographic type. Namely, in the latter case, we have

$$\mathrm{cd}_{\mathbb{Q}}(\bar{G}) = \begin{cases} |S| - 1, & \text{if } (W,S) \text{ is compact hyperbolic;} \\ |S| - 2, & \text{if } (W,S) \text{ is non-compact hyperbolic.} \end{cases}$$

Since there is a group $\bar{G}$ as above for every crystallographic Coxeter group (W, S), the previous observations refer to non-empty families of cases.

2.4 The Euler–Poincaré characteristic of t.d.l.c. groups

This section deals with the *Euler–Poincaré characteristic* $\tilde{\chi}_G$ of a t.d.l.c. group G, which is a cohomological invariant defined whenever G is unimodular and of type FP (cf. Definition 2.4.33). Roughly speaking, $\tilde{\chi}_G$ is the "dimension" of the trivial discrete left $\mathbb{Q}[G]$-module $\mathbb{Q}$. While dealing with modules over non-Noetherian rings, one has to be careful with the notion of "dimension". In this case, by "dimension" of the relevant module we mean its *Hattori–Stallings rank*. This quantity is first defined for finitely generated projective discrete left $\mathbb{Q}[G]$-modules (which are the counterpart in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ of finite-dimensional vector spaces over a field) as the trace of the identity map (cf. Definition 2.4.24). Via finite projective resolutions of finite type, we extend the notion of Hattori–Stallings rank to all discrete left $\mathbb{Q}[G]$-modules of type FP (cf. Definition 2.4.26).

In Section 2.4.1 we briefly recall the definition of trace required for the Hattori–Stallings rank, and put the bases for the main theorem of the section: Theorem 2.4.41. In this result we determine the sign of the Euler–Poincaré characteristic of a compactly generated t.d.l.c. group G with $\mathrm{cd}_{\mathbb{Q}}(G) = 1$. This information plays a key role in the proof of a Stallings–Swan like theorem (cf. Theorem 2.5.15) in the next section. The proof of Theorem 2.4.41 is achieved by a convenient estimate of the Hattori–Stallings rank of a specific finitely generated and projective module: the *augmentation ideal* N_U^G *with respect to a compact open subgroup* $U \leq G$ (cf. Section 2.4.2). Our strategy, which takes inspiration from a work of P. Linnel [Lin83], is to approximate the operator whose trace is the Hattori–Stallings rank of N_U^G via projections in a suitable von Neumann algebra. The result containing the final estimate is Proposition 2.4.32, while the introduction of the relevant von Neumann algebra is given by Definition 2.4.9.

2.4.1 A trace map

The spaces $C_c(G, \mathbb{K})^{\mathcal{O}}$, $\mathcal{H}_{\mathbb{K}}(G, \mathcal{O})$ and $W(G, \mathcal{O})$

Definition 2.4.1. Let G be a t.d.l.c. group. Denote by $C_c(G, \mathbb{K})$ the space of all continuous functions with compact support from G to a discrete field $\mathbb{K}$. One checks that

$$C_c(G, \mathbb{K}) = \mathrm{span}_{\mathbb{K}}\{\mathbb{1}_{g\mathcal{O}} \mid g \in G,\ \mathcal{O} \in \mathcal{CO}(G)\}.$$

Moreover, given $\mathcal{O} \in \mathcal{CO}(G)$, define the two following $\mathbb{K}$-subspaces of $C_c(G, \mathbb{K})$:

$$C_c(G, \mathbb{K})^{\mathcal{O}} := \mathrm{span}_{\mathbb{K}}(\{\mathbb{1}_{g\mathcal{O}} \mid g \in G\}) \text{ and } \mathcal{H}_{\mathbb{K}}(G, \mathcal{O}) := \mathrm{span}_{\mathbb{K}}(\{\mathbb{1}_{\mathcal{O}g\mathcal{O}} \mid g \in G\}).$$

Remark 2.4.2. Let G be a t.d.l.c. group and set $\mathcal{O} \in \mathcal{CO}(G)$.

(i) The space $C_c(G,\mathbb{K})^{\mathcal{O}}$ has a structure of a discrete left $\mathbb{K}[G]$-module given by

$$h \cdot \mathbb{1}_{g\mathcal{O}} := \mathbb{1}_{hg\mathcal{O}}, \quad \forall\, h, g \in G.$$

In particular, the map

$$C_c(G,\mathbb{K})^{\mathcal{O}} \longrightarrow \mathbb{K}[G/\mathcal{O}], \quad \mathbb{1}_{g\mathcal{O}} \longmapsto g\mathcal{O}$$

is an isomorphism in ${}_{\mathbb{K}[G]}\mathbf{dis}$ (cf. [CCW24, Fact 3.2(a)]).

(ii) Let $\mu_{\mathcal{O}}$ be the left-invariant Haar measure on G satisfying $\mu_{\mathcal{O}}(\mathcal{O}) = 1$. Then the $\mathbb{K}$-vector space $\mathcal{H}_{\mathbb{K}}(G,\mathcal{O})$ has a structure of an associative unital algebra given by the following convolution product:

$$(f_1 *_{\mu_{\mathcal{O}}} f_2)(x) := \int_G f_1(y) f_2(y^{-1}x)\mathrm{d}\mu_{\mathcal{O}}(y), \quad \forall\, x \in G$$

for all $f_1, f_2 \in \mathcal{H}_{\mathbb{K}}(G,\mathcal{O})$. The unit of $\mathcal{H}_{\mathbb{K}}(G,\mathcal{O})$ is $\mathbb{1}_{\mathcal{O}}$. For details, see [CCW24, Section 3.5].

The algebra $\mathcal{H}_{\mathbb{K}}(G,\mathcal{O})$ in Remark 2.4.2(ii) is usually called the *Hecke* $\mathbb{K}$*-algebra associated to the Hecke pair* $(G,\mathcal{O})$.

We now focus on the case $\mathbb{K} = \mathbb{C}$ (regarded as a discrete field). We will see in the next Definition 2.4.5 that $\mathcal{H}_{\mathbb{C}}(G,\mathcal{O})$ admits a structure of an algebra of operators defined on the Hilbert space $L^2(G,\mathcal{O})$ defined below.

Definition 2.4.3. Let G be t.d.l.c. group. Consider the space

$$L^2(G,\mathbb{C}) := \left\{ f\colon G \to \mathbb{C} \text{ measurable} \;\middle|\; \int_G |f(w)|^2 \mathrm{d}\mu_{\mathcal{O}}(w) < \infty \right\},$$

equipped with the inner product

$$\langle f_1, f_2 \rangle = \int_G f_1(w)\overline{f_2(w)}\mathrm{d}\mu_{\mathcal{O}}(w),$$

and let $\|\_\|_2\colon L^2(G,\mathbb{C})^{\mathcal{O}} \to \mathbb{R}_0^+$ be the Hilbert norm associated to $\langle \_, \_ \rangle$. For all $g, h \in G$ and $f \in L^2(G,\mathcal{O})$, the assignment

$$(g \cdot f \cdot h)(x) := f(g^{-1}xh^{-1}), \quad \forall\, x \in G$$

simultaneously defines left and right G-actions on $L^2(G,\mathbb{C})$.

For every $\mathcal{O} \in \mathcal{CO}(G)$, set

$$L^2(G,\mathbb{C})^{\mathcal{O}} := \{ f \in L^2(G,\mathbb{C}) \mid f \cdot \omega = f, \forall\, \omega \in \mathcal{O} \}.$$

Denote by $\mathcal{B}(G,\mathcal{O})$ the algebra of bounded linear operators $T\colon L^2(G,\mathbb{C})^{\mathcal{O}} \to L^2(G,\mathbb{C})^{\mathcal{O}}$ that commute with the left G-action, i.e., such that

$$T(g\cdot f) = g\cdot T(f), \quad \forall\, g\in G,\ f\in L^2(G,\mathbb{C})^{\mathcal{O}}.$$

The space $\mathcal{B}(G,\mathcal{O})$ comes with the standard operator norm $\|\_\|\colon \mathcal{B}(G,\mathcal{O}) \to \mathbb{R}_0^+$, i.e., for every $T\in\mathcal{B}(G,\mathcal{O})$ we set

$$\|T\| := \max\left\{ \|T(f)\|_2 \,\middle|\, f\in L^2(G,\mathbb{C})^2 \text{ s.th. } \|f\|_2 = 1 \right\}.$$

Moreover, $\mathcal{B}(G,\mathcal{O})$ admits an adjoint map $(\_)^*\colon \mathcal{B}(G,\mathcal{O})^{\mathrm{op}} \to \mathcal{B}(G,O)$ defined in the usual way: i.e., given $T\in\mathcal{B}(G,\mathcal{O})$, let $T^*\colon L^2(G,C)^{\mathcal{O}} \to L^2(G,\mathbb{C})^{\mathcal{O}}$ be the (unique) linear bounded operator satisfying

$$\langle T(f_1), f_2\rangle = \langle f_1, T^*(f_2)\rangle, \tag{2.4.1}$$

for all $f_1, f_2\in L^2(G,\mathbb{C})^{\mathcal{O}}$.

Remark 2.4.4. Let G be a t.d.l.c. group and set $\mathcal{O}\in\mathcal{CO}(G)$.

(i) Given a set $\mathcal{R}\subseteq G$ of representatives for $G/\mathcal{O}$, the set $\{\mathbb{1}_{g\mathcal{O}} \mid g\in\mathcal{R}\}$ is an orthonormal (Hilbert) basis of the Hilbert space $L^2(G,\mathbb{C})^{\mathcal{O}}$. In other words,

$$L^2(G,\mathbb{C})^{\mathcal{O}} = \left\{ \sum_{g\in\mathcal{R}} \lambda_g \mathbb{1}_{g\mathcal{O}} \,\middle|\, \lambda_g\in\mathbb{C},\ \sum_{g\in\mathcal{R}} |\lambda_g|^2 < \infty \right\},$$

(cf. [CCW24, Fact 3.3]). In particular, $L^2(G,\mathbb{C})^{\mathcal{O}}$ is the closure of $C_c(G,\mathbb{C})^{\mathcal{O}}$ in $L^2(G,\mathbb{C})$ with respect to the topology induced by $\|\_\|_2$.

(ii) The algebra $\mathcal{B}(G,\mathcal{O})$ is a C^*-algebra. Recall that a *C^*-algebra* is a Banach algebra A with an involution $(\_)^*$ (i.e., a conjugate-linear self-map of A such that $x^{**}=x$ and $(xy)^* = y^*x^*$ for all $x,y\in A$) which satisfies $\|x^*x\| = \|x\|^2$ for all $x\in A$.

Definition 2.4.5 ([CCW24, Proposition 3.7])**.** Let G be a t.d.l.c. group and set $\mathcal{O}\in\mathcal{CO}(G)$. Denote by $\mu_{\mathcal{O}}$ the left Haar measure on G satisfying $\mu_{\mathcal{O}}(\mathcal{O})=1$, and by $\mathcal{H}_{\mathbb{C}}(G,\mathcal{O})^{\mathrm{op}}$ the *opposite algebra of* $\mathcal{H}_{\mathbb{C}}(G,\mathcal{O})$, i.e., the algebra whose underlying space is the same as $\mathcal{H}_{\mathbb{C}}(G,\mathcal{O})$ and whose product is

$$f_1 *^{\mathrm{op}}_{\mu_{\mathcal{O}}} f_2 := f_2 *_{\mu_{\mathcal{O}}} f_1,$$

for all $f_1,f_2\in\mathcal{H}_{\mathbb{C}}(G,\mathcal{O})$. Then the *left regular representation* of $\mathcal{H}_{\mathbb{C}}(G,\mathcal{O})$ is defined by

$$\phi\colon \mathcal{H}(G,\mathcal{O})^{op}_{\mathbb{C}} \to \mathcal{B}(G,\mathcal{O}), \quad (\phi(f))(\mathbb{1}_{g\mathcal{O}}) = \mathbb{1}_{g\mathcal{O}} *_{\mu_{\mathcal{O}}} f, \quad \forall g\in G. \tag{2.4.2}$$

Remark 2.4.6.

(i) As shown in [CCW24, Proposition 3.7], the map ϕ in (2.4.2) is an injective homomorphism of algebras. Moreover, ϕ is continuous when $\mathcal{H}(G,\mathcal{O})_{\mathbb{C}}$ is endowed with the topology induced by $\|\_\|_1$, where

$$\|f\|_1 := \int_G |f(w)| \mathrm{d}\mu_{\mathcal{O}}(w), \quad \forall\, f \in \mathcal{H}(G,O)_{\mathbb{C}}.$$

(ii) The uniform closure $\overline{\mathcal{H}}(G,\mathcal{O})$ of $\phi(\mathcal{H}(G,\mathcal{O})_{\mathbb{C}}^{op})$ in $\mathcal{B}(G,\mathcal{O})$, endowed with the standard operator norm, is a C^*-subalgebra of $\mathcal{B}(G,\mathcal{O})$ (cf. Remark 2.4.4(ii)). Such an operator algebra is called the *C^*-Hecke algebra associated to the Hecke pair* $(G,\mathcal{O})$. We refer for instance to [CCW24, §3.7] for further details. Instead of looking at $\overline{\mathcal{H}}(G,\mathcal{O})$, in the section we focus on the closure of $\phi(\mathcal{H}(G,\mathcal{O})_{\mathbb{C}}^{op})$ in $\mathcal{B}(G,\mathcal{O})$ with respect to the weak operator topology, which is coarser than the uniform topology. The reader is referred to the next Definition 2.4.9 for details.

W^*-algebras and traces on them

Definition 2.4.7. A *W^*-algebra* (or *von Neumann algebra*) is a weakly closed self-adjoint algebra of operators on a complex Hilbert space.

Definition 2.4.8. Let A be an algebra of operators which contains the identity operator and which is closed under taking adjoints. For every subset $S \subseteq A$, let $S' := \{a \in A : as = sa, \forall\, s \in S\}$ be the *commutant of* S. Then $S'' := (S')'$ is called the *bicommutant of* S. By [Arv12, Theorem 1.2.1], the weak closure of A is equal to the bicommutant A'' of A and is called the *W^*-algebra generated by* A.

Definition 2.4.9. Let G be a t.d.l.c. group. Given $\mathcal{O} \in \mathcal{CO}(G)$, denote by $\phi\colon \mathcal{H}_{\mathbb{C}}(G,\mathcal{O})^{\mathrm{op}} \to \mathcal{B}(G,\mathcal{O})$ the left regular representation of $\mathcal{H}_{\mathbb{C}}(G,\mathcal{O})$ (cf. Definition 2.4.5). Define $W(G,\mathcal{O})$ as the closure of $\phi(\mathcal{H}(G,\mathcal{O})_{\mathbb{C}}^{op})$ in $\mathcal{B}(G,\mathcal{O})$ with respect to the weak operator topology.

Let A be a W^*-algebra acting on the complex Hilbert space H. Denote by $\mathrm{Mat}_n(A)$ the set of all n-dimensional matrices with entries in A. The algebra $\mathrm{Mat}_n(A)$ acts on the n-fold direct sum $H^n = H \oplus \cdots \oplus H$ through the usual matrix action on the column vectors. Therefore, $\mathrm{Mat}_n(A)$ can be regarded as a self-adjoint subalgebra of $\mathcal{B}(H^n)$ where, for every matrix $M = [m_{jk}] \in \mathrm{Mat}_n(A)$, the matrix $M^* = [n_{jk}]$ is defined by $n_{jk} := m_{kj}^*$.

Lemma 2.4.10. *If A is a W^*-algebra, then* $\mathrm{Mat}_n(A)$ *is a W^*-algebra for every* $n \geq 1$.

Proof. It suffices to check that the double commutant $\mathrm{Mat}_n(A)''$ coincides with $\mathrm{Mat}_n(A)$. Since the commutant $\mathrm{Mat}_n(A)'$ is $A' \cdot I_n$ (where I_n is the identity matrix and A' is the commutant of A acting on H), it is easily verified that

$$\mathrm{Mat}_n(A)'' = M_n(A'') = \mathrm{Mat}_n(A). \qquad \square$$

Corollary 2.4.11. *Let G be a t.d.l.c. group and $\mathcal{O} \in \mathcal{CO}(G)$. If $W(G, \mathcal{O})$ is as in Definition 2.4.9, then $\mathrm{Mat}_n(W(G, \mathcal{O}))$ is a W^*-algebra for every $n \geq 1$.*

Definition 2.4.12. A (finite) *trace* on a W^*-algebra A is a $\mathbb{C}$-linear function $\tau\colon A \to \mathbb{C}$ satisfying $\tau(ab) = \tau(ba)$ for all $a, b \in A$. A trace τ is *positive* if $\tau(a^*a) \geq 0$ for every $a \in A$. Moreover, a trace is said to be *faithful* if for every $a \in A$ we have $\tau(a^*a) = 0$ if, and only if, $a^*a = 0$.

Remark 2.4.13. If A is a W^*-algebra with trace τ, then $\mathrm{Mat}_n(A)$ is a W^*-algebra with trace

$$(\tau \otimes \mathrm{id}_n)(M) := \tau(m_{11}) + \ldots + \tau(m_{nn}), \tag{2.4.3}$$

for every matrix $M = [m_{ij}] \in \mathrm{Mat}_n(A)$.

Remark 2.4.14. Despite the C^*-case, no intrinsic axioms are known for W^*-algebras. I. Kaplanski [Kap51] proposed an algebraic generalisation of W^*-algebras based on the assumption of least upper bounds in the poset of projections (= self-adjoint idempotents) of the operator algebra: the so called *AW^*-algebras*. Every W^*-algebra is an AW^*-algebra.

The main motivation for considering $(A)W^*$-algebras (and, in particular, for introducing $W(G, \mathcal{O})$) is the following.

Proposition 2.4.15 ([Ber72, §7, Corollary, p. 43])**.** *Let A be an AW^*-algebra and consider $x \in A$ with $x \neq 0$. For every $\varepsilon > 0$, there exists y in the bi-commutator of $\{x^*x\}$ such that $(x^*x)y^2 = p$, for some non-zero projection p, and $\|x - xp\| < \varepsilon$.*

A trace map on $W(G, \mathcal{O})$

Let G be a unimodular t.d.l.c. group. In [CCW24, §3.7], for every $\mathcal{O} \in \mathcal{CO}(G)$ the authors defined a $\mathbb{C}$-valued trace map on the C^*-algebra $\overline{\mathcal{H}}(G, \mathcal{O})_{\mathbb{C}}$. In the next proposition, we extend this map to the weak closure of $\mathcal{H}(G, \mathcal{O})_{\mathbb{C}}$.

Proposition 2.4.16. *The $\mathbb{C}$-linear map*

$$\tau_0\colon \phi(\mathcal{H}(G, \mathcal{O})_{\mathbb{C}}^{op}) \to \mathbb{C}, \quad \tau_0(\phi(f)) = f(1),$$

can be extended to the $\mathbb{C}$-linear map $\tau\colon W(G, \mathcal{O}) \longrightarrow \mathbb{C}$ as

$$\tau(F) = \langle F(\mathbb{1}_{\mathcal{O}}), \mathbb{1}_{\mathcal{O}} \rangle \quad \text{for } F \in W(G, \mathcal{O}).$$

The map τ is a positive and faithful trace satisfying $\tau(F^*) = \overline{\tau(F)}$ for all $F \in W(G, O)$.

Proof. Clearly, the map τ is $\mathbb{C}$-linear. Let $\mathcal{R} \subseteq G$ denote a set of representatives for $\mathcal{O} \backslash G / \mathcal{O}$, and consider $f = \sum_{r \in \mathcal{R}} f(r) \mathbb{1}_{\mathcal{O} r \mathcal{O}} \in \mathcal{H}(G, \mathcal{O})_{\mathbb{C}}$ (recall that f vanishes for all but finitely many r). Then

$$\begin{aligned}\tau(\phi(f)) = \langle \mathbb{1}_{\mathcal{O}} *_{\mu_{\mathcal{O}}} f, \mathbb{1}_{\mathcal{O}} \rangle &= \sum_{r\in\mathcal{R}} f(r)\langle \mathbb{1}_{\mathcal{O}} *_{\mu_{\mathcal{O}}} \mathbb{1}_{\mathcal{O}r\mathcal{O}}, \mathbb{1}_{\mathcal{O}}\rangle = \sum_{r\in\mathcal{R}} f(r)\langle \mathbb{1}_{\mathcal{O}r\mathcal{O}}, \mathbb{1}_{\mathcal{O}}\rangle \\ &= \sum_{r\in\mathcal{R}} f(r)\int_G \mathbb{1}_{\mathcal{O}r\mathcal{O}}(w)\mathbb{1}_{\mathcal{O}}(w)\mathrm{d}\mu_{\mathcal{O}}(w) \\ &= f(1)\int_G (\mathbb{1}_{\mathcal{O}}(w))^2 \mathrm{d}\mu_{\mathcal{O}}(w) = f(1)\mu_{\mathcal{O}}(\mathcal{O}) = f(1).\end{aligned} \tag{2.4.4}$$

Therefore, $\tau(\phi(f)) = \tau_0(\phi(f))$ for every $f \in \mathcal{H}(G,\mathcal{O})_{\mathbb{C}}$. In general, the multiplication map in $W(G,\mathcal{O})$ is not continuous with respect to the weak topology as a function of two variables. However, it is continuous as a function of one variable when the other is kept fixed. Then, for every fixed $F \in W(G,\mathcal{O})$ and every sequence $(G_n)_{n\geq 1}$ in $\phi(\mathcal{H}(G,\mathcal{O})_{\mathbb{C}}^{op})$ that weakly converges to $G \in W(G,\mathcal{O})$, the sequence $\tau(FG_n) - \tau(G_nF)$ converges to $\tau(FG) - \tau(GF)$. Hence, to prove that τ satisfies the trace property on $W(G,\mathcal{O})$, it suffices to verify that $\tau(F_1F_2) = \tau(F_2F_1)$ for arbitrary elements F_1, F_2 of a basis of $\phi(\mathcal{H}(G,\mathcal{O})_{\mathbb{C}}^{op})$. More precisely, it suffices to show that $\langle \mathbb{1}_{\mathcal{O}} *_{\mu_{\mathcal{O}}} \mathbb{1}_{\mathcal{O}g\mathcal{O}}, \mathbb{1}_{\mathcal{O}} *_{\mu_{\mathcal{O}}} (\mathbb{1}_{\mathcal{O}h\mathcal{O}})^*\rangle = \langle \mathbb{1}_{\mathcal{O}} *_{\mu_{\mathcal{O}}} \mathbb{1}_{\mathcal{O}h\mathcal{O}}, \mathbb{1}_{\mathcal{O}} *_{\mu_{\mathcal{O}}} (\mathbb{1}_{\mathcal{O}g\mathcal{O}})^*\rangle$ for all $g, h \in G$. Now the end of the proof of [CCW24, Theorem 3.10] can be transferred verbatim.

For an arbitrary $F \in W(G,\mathcal{O})$, we have

$$\tau(F^*F) = \langle\, F^*F(\mathbb{1}_{\mathcal{O}}), \mathbb{1}_{\mathcal{O}}\,\rangle = \langle\, F(\mathbb{1}_{\mathcal{O}}), F(\mathbb{1}_{\mathcal{O}})\,\rangle \geq 0. \tag{2.4.5}$$

Moreover, if $\tau(F^*F)$ vanishes, then $F(\mathbb{1}_{\mathcal{O}}) = 0$ (cf. (2.4.5)) and, since F commutes with the left G-action, $F(\mathbb{1}_{g\mathcal{O}}) = 0$ for every $g \in G$. By linearity, F vanishes over $C_c(G,\mathbb{C})^{\mathcal{O}}$ and then, by [CCW24, Fact 3.3], $F = 0$. Hence, τ is a faithful trace on $W(G,\mathcal{O})$.

Finally, for every $F \in W(G,\mathcal{O})$ we have

$$\tau(F^*) = \langle\, F^*(\mathbb{1}_{\mathcal{O}}), \mathbb{1}_{\mathcal{O}}\,\rangle = \langle\, \mathbb{1}_{\mathcal{O}}, F(\mathbb{1}_{\mathcal{O}})\,\rangle = \overline{\langle\, F(\mathbb{1}_{\mathcal{O}}), \mathbb{1}_{\mathcal{O}}\,\rangle} = \overline{\tau(F)}. \qquad \square$$

Fact 2.4.17 ([CCW24, Proposition 3.7(b)])**.** *Let G be a t.d.l.c. group with a compact open subgroup $\mathcal{O}$. Let $\phi_*\colon \mathcal{H}(G,\mathcal{O})_{\mathbb{Q}}^{op} \to \mathrm{End}_G(C_c(G,\mathbb{Q})^{\mathcal{O}})$ denote the isomorphism of* [CCW24, Proposition 3.7(b)]*. Then:*

(a) there is an isomorphism of $\mathbb{Q}$-algebras

$$\Phi_*\colon M_n\Big(\mathcal{H}(G,\mathcal{O})_{\mathbb{Q}}^{op}\Big) \to \mathrm{End}_G\Big((C_c(G,\mathbb{Q})^{\mathcal{O}})^n\Big)$$

defined, for every $f = [f_{ij}] \in M_n(\mathcal{H}(G,\mathcal{O})_{\mathbb{Q}}^{op})$, as

$$\Phi_*(f)\Big(\,[0,\ldots,\overset{j\text{-}th}{\mathbb{1}_{g\mathcal{O}}},\ldots,0]^t\,\Big) := \Big[\phi_*(f_{1j})(\mathbb{1}_{g\mathcal{O}}),\ldots,\phi_*(f_{nj})(\mathbb{1}_{g\mathcal{O}})\Big]^t,\ \forall j\in[n];$$

(b) the inverse $\Psi_*\colon \mathrm{End}_G\Big((C_c(G,\mathbb{Q})^{\mathcal{O}})^n\Big) \to M_n\Big(\mathcal{H}(G,\mathcal{O})^{op}_{\mathbb{Q}}\Big)$ *of* Φ_* *is*

$$(\Psi_*(\alpha))_{ij} := (\phi_*)^{-1}(\alpha_{ij}), \quad \forall i,j \in [n].$$

Here, for all $\alpha \in \mathrm{End}_G\Big((C_c(G,\mathbb{Q})^{\mathcal{O}})^n\Big)$ *and* $i,j \in [n]$*, the map* $\alpha_{ij} \in \mathrm{End}_G(C_c(G,\mathbb{Q})^{\mathcal{O}})$ *is defined as*

$$\alpha_{ij}(\mathbb{1}_{g\mathcal{O}}) := pr_i\Big(\alpha\Big(\Big[0,...,\overset{j\text{-}th}{\mathbb{1}_{g\mathcal{O}}},...,0\Big]\Big)\Big), \qquad \forall g \in G,$$

where $pr_i : (C_c(G,\mathbb{Q})^{\mathcal{O}})^n \longrightarrow C_c(G,\mathbb{Q})^{\mathcal{O}}$ *projects on the* i*-th component.*

The following fact is analogous to [CCW24, Fact 2.2].

Corollary 2.4.18. *The map*

$$\bar{\tau}\colon M_n(W(G,\mathcal{O})) \to \mathbb{C}, \quad [F_{ij}] \mapsto \sum_{i=1}^{n} \tau(F_{ii}),$$

is a positive and faithful trace over the W^**-algebra* $M_n(W(G,\mathcal{O}))$ *satisfying* $\bar{\tau}(F^*) = \overline{\bar{\tau}(F)}$*, for every* $F \in M_n(W(G,\mathcal{O}))$*.*

Lemma 2.4.19. *For every* $F \in M_n(W(G,\mathcal{O}))$*, we have*

$$|\bar{\tau}(F)| \leq n \cdot \|F\|,$$

where $\|F\| = \max\{\|F(v)\|_2 : v \in (L^2(G,\mathbb{C})^{\mathcal{O}})^n, \|v\|_2 = 1\}$ *and* $\|v\|_2 := \sqrt{\sum_{i=1}^n \|v_i\|_2^2}$*, for every* $v = [v_1,...,v_n]^t \in (L^2(G,\mathbb{C})^{\mathcal{O}})^n$*.*

Proof. By Corollary 2.4.18 and the Cauchy–Schwarz inequality in $L^2(G,\mathbb{C})^{\mathcal{O}}$, we observe that

$$|\bar{\tau}(F)| \leq \sum_{j=1}^{n} |\tau(F_{jj})| = \sum_{j=1}^{n} |\langle F_{jj}(\mathbb{1}_{\mathcal{O}}), \mathbb{1}_{\mathcal{O}} \rangle| \leq \sum_{j=1}^{n} \|F_{jj}(\mathbb{1}_{\mathcal{O}})\|_2.$$

For $j \in [n]$, let $\mathbb{1}_{\mathcal{O}j}$ be the vector in the n-fold of $L^2(G,\mathbb{C})^{\mathcal{O}}$ having the i^{th} component equals to $\mathbb{1}_{\mathcal{O}} \cdot \delta_{ij}$ for every $i \in [n]$. Since $\|\mathbb{1}_{\mathcal{O}j}\|_2 = 1$, we have

$$\|F\| \geq \|F(\mathbb{1}_{\mathcal{O}j})\|_2 = \sqrt{\sum_{i=1}^{n} \|F_{ij}(\mathbb{1}_{\mathcal{O}})\|_2^2} \geq \|F_{jj}(\mathbb{1}_{\mathcal{O}})\|_2.$$

As a conclusion, $|\bar{\tau}(F)| \leq \sum_{j=1}^{n} \|F_{jj}(\mathbb{1}_{\mathcal{O}})\|_2 \leq n \cdot \|F\|$. □

Definition 2.4.20. A *projection* in a W^*-algebra is a self-adjoint idempotent.

Theorem 2.4.21. *For every idempotent $e \in M_n(W(G, \mathcal{O}))$, there exists a projection $p \in M_n(W(G, \mathcal{O}))$ such that $eM_n(W(G, \mathcal{O})) = pM_n(W(G, \mathcal{O}))$ and $\bar{\tau}(e) = \bar{\tau}(p)$.*

Proof. The proof strategy is the same to prove [Kap68, Theorem 26]. Namely, for $p = ee^*(1 + (e - e^*)(e^* - e))^{-1}$ we have $p = ep$ and $e = pe$. As $\bar{\tau}$ is a trace, we conclude that $\bar{\tau}(p) = \bar{\tau}(ep) = \bar{\tau}(pe) = \bar{\tau}(e)$. □

Corollary 2.4.22. *For every idempotent $e \in M_n(W(G, \mathcal{O}))$ we have $\bar{\tau}(e) \in [0, n]$. Moreover, $\bar{\tau}(e) = 0$ if and only if $e = 0$.*

Proof. By Theorem 2.4.21, for every idempotent $e \in M_n(W(G, \mathcal{O}))$ there exists a projection $p \in M_n(W(G, \mathcal{O}))$ such that $\bar{\tau}(e) = \bar{\tau}(p) = \bar{\tau}(p^*p)$. Then,

$$\begin{aligned} \bar{\tau}(e) = & \sum_{j\in[n]} \tau((p^*p)_{jj}) & = \sum_{i,j\in[n]} \tau((p^*)_{ji}p_{ij}) \\ = & \sum_{i,j\in[n]} \tau(p^*_{ij}p_{ij}) & = \sum_{i,j\in[n]} \langle\, p_{ij}(\mathbb{1}_{\mathcal{O}}), p_{ij}(\mathbb{1}_{\mathcal{O}})\,\rangle. \end{aligned}$$

Hence $\bar{\tau}(e) \geq 0$ and $\tau(e) = 0$ if, and only if, $p_{ij} = 0$ for every $i, j \in [n]$. As a consequence, $\bar{\tau}(e) = 0$ if and only if $0 = pM_n(W(G, \mathcal{O})) = eM_n(W(G, \mathcal{O}))$, which is in turn equivalent to the condition that $e = 0$.

To prove that $\bar{\tau}(e) \leq n$, it suffices to observe that $I_n - e$ is an idempotent of $M_n(W(G, \mathcal{O}))$ and so $0 \leq \bar{\tau}(I_n - e) = n - \bar{\tau}(e)$. □

Remark 2.4.23. We collect some properties that will be useful later on.

(a) From the proof of Corollary 2.4.22 it follows that

$$\bar{\tau}(p) = \sum_{i,j\in[n]} \|p_{ij}(\mathbb{1}_{\mathcal{O}})\|_2^2,$$

for every projection $p \in M_n(W(G, \mathcal{O}))$.

(b) Let $p, q \in M_n(W(G, \mathcal{O}))$ be projections. By [Kap68, Proposition 1] we have

$$pM_n(W(G, \mathcal{O})) \subseteq qM_n(W(G, \mathcal{O})) \Longleftrightarrow p = qp.$$

Consequently, if $pM_n(W(G, \mathcal{O})) \subseteq qM_n(W(G, \mathcal{O}))$ then $\bar{\tau}(p) \leq \bar{\tau}(q)$. Indeed, since $p = qp$, we obtain that

$$pq = p^*q^* = (qp)^* = p^* = p = qp$$

and thus $q - p$ is a projection. By Theorem 2.4.21, we conclude that

$$\bar{\tau}(q) - \bar{\tau}(p) = \bar{\tau}(q - p) \geq 0.$$

2.4.2 Hattori–Stallings rank in ${}_{\mathbb{Q}[G]}\mathbf{dis}$

Definition and general properties

Let G be a unimodular t.d.l.c. group and $\mathcal{O} \in \mathcal{CO}(G)$. Denote by $\mathbf{h}(G)$ the $\mathbb{Q}$-vector space $\mathbb{Q} \cdot \mu_{\mathcal{O}}$. Note that $\mathbf{h}(G)$ contains all rational multiples of the Haar measures on G which are equal to 1 on some compact open subgroup of G. Indeed, let $\mathcal{O}_1, \mathcal{O}_2 \leq G$ be compact open subgroups and μ is any Haar measure on G. Note that

$$\mu(\mathcal{O}_1) = |\mathcal{O}_1 : \mathcal{O}_1 \cap \mathcal{O}_2| \cdot \mu(\mathcal{O}_1 \cap \mathcal{O}_2) = \frac{|\mathcal{O}_1 : \mathcal{O}_1 \cap \mathcal{O}_2|}{|\mathcal{O}_2 : \mathcal{O}_1 \cap \mathcal{O}_2|} \cdot \mu(\mathcal{O}_2).$$

Hence, by the essential uniqueness of the Haar measure,

$$\mu_{\mathcal{O}_1} = \frac{1}{\mu(\mathcal{O}_1)}\mu = \frac{|\mathcal{O}_2 : \mathcal{O}_1 \cap \mathcal{O}_2|}{|\mathcal{O}_1 : \mathcal{O}_1 \cap \mathcal{O}_2| \cdot \mu(\mathcal{O}_2)}\mu = \frac{|\mathcal{O}_2 : \mathcal{O}_1 \cap \mathcal{O}_2|}{|\mathcal{O}_1 : \mathcal{O}_1 \cap \mathcal{O}_2|}\mu_{\mathcal{O}_2} \in \mathbb{Q}^{\times} \cdot \mu_{\mathcal{O}_2}. \tag{2.4.6}$$

Definition 2.4.24. Let $P \in \mathrm{ob}({}_{\mathbb{Q}[G]}\mathbf{dis})$ be finitely generated and projective. The *Hattori–Stallings rank* $\tilde{\rho}(P)$ of P is defined as

$$\tilde{\rho}(P) := \rho_P(\mathrm{id}_P) \in \mathbf{h}(G),$$

where ρ_P is the map from $\mathrm{End}_G(P)$ to $\mathbf{h}(G)$ defined in [CCW24, §4.3].

In this thesis, we never need the explicit definition of ρ_P. It suffices to recall two facts: First, for every compact open subgroup $\mathcal{O} \leq G$, the Hattori–Stallings rank of $\mathbb{Q}[G/\mathcal{O}]$ is equal to $\mu_{\mathcal{O}} \in \mathbf{h}(G)$. The second fact is the following.

Fact 2.4.25 (cf. the proof of [CCW24, Theorem 4.4])**.** *Let G be a unimodular t.d.l.c. group and $P \in \mathrm{ob}({}_{\mathbb{Q}[G]}\mathbf{dis})$ be finitely generated and projective. Then there exist $n \in \mathbb{Z}_{\geq 1}$, $\mathcal{O} \in \mathcal{CO}(G)$ and an idempotent $e \in M_n(\mathcal{H}(G,\mathcal{O})_{\mathbb{Q}})$ such that $P \simeq e \cdot \mathbb{Q}[G/\mathcal{O}]^n$ and*

$$\tilde{\rho}(P) = \bar{\tau}(e) \cdot \mu_{\mathcal{O}}. \tag{2.4.7}$$

We now extend the definition of the Hattori–Stallings rank to every $M \in \mathrm{ob}({}_{\mathbb{Q}[G]}\mathbf{dis})$ of type FP as follows.

Definition 2.4.26. Let $P_0, ..., P_n \in \mathrm{ob}({}_{\mathbb{Q}[G]}\mathbf{dis})$ be finitely generated projective such that

$$0 \longrightarrow P_n \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is a projective resolution of M in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ (cf. Section 1.8.5). A projective resolution as above is called a *finite projective resolution of finite type* of M in ${}_{\mathbb{Q}[G]}\mathbf{dis}$. Define the *Hattori–Stallings rank of M* as

$$\tilde{\rho}(M) := \sum_{i=0}^{n}(-1)^i\tilde{\rho}(P_i). \tag{2.4.8}$$

By the additivity of $\tilde{\rho}$ (cf. [CCW24, Equation (5.1)]) and [Bro12, Chapter VII, Lemma 4.4], the definition in (2.4.8) is independent of the choice of the projective resolution of M. The value $\tilde{\rho}(M)$ is sometimes called the *Lefschetz number* of the module M.

From the Horseshoe Lemma (cf. [Bro12, Chapter VII, Lemma 4.4]) we conclude the following property.

Proposition 2.4.27. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence in ${}_{\mathbb{Q}[G]}\mathbf{dis}$, and assume that A and C are of type* FP*. Then so is B, and $\tilde{\rho}(B) = \tilde{\rho}(A) + \tilde{\rho}(C)$.*

The augmentation module N_U^G

Let G be a t.d.l.c. group. For every $U \in \mathcal{CO}(G)$, the map

$$\varepsilon_U \colon C_c(G, \mathbb{Q})^U \to \mathbb{Q}, \quad f \mapsto \int_G f(\omega) \mathrm{d}\mu_U(\omega) \tag{2.4.9}$$

is a surjective morphism in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ mapping every $\mathbb{1}_{gU}$ to 1. The kernel of ε_U, denoted by N_U^G, is called the *discrete left augmentation* $\mathbb{Q}[G]$*-module relative to* U. Note that $C_c(G, \mathbb{Q})^U$ is isomorphic to $\mathbb{Q}[G/U]$ in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ via the map $\mathbb{1}_{gU} \mapsto gU$, $g \in G$.

Proposition 2.4.28. *Let G be a t.d.l.c. group and U be a compact open subgroup.*

(a) If N_U^G is finitely generated, then there are $n \in \mathbb{Z}_{\geq 0}$ and $\{s_1, \ldots, s_n\} \subseteq G \setminus U$ such that the set $\{\mathbb{1}_U - \mathbb{1}_{s_iU} \mid i \in [n]\}$ generates N_U^G as a discrete left $\mathbb{Q}[G]$-module. In particular, provided $\mathcal{O} = U \cap s_1 U s_1^{-1} \cap \ldots \cap s_n U s_n^{-1}$, there is an epimorphism in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ from the n-fold of $C_c(G, \mathbb{Q})^{\mathcal{O}}$ to N_U^G as follows:

$$\begin{array}{rccc} p_{\mathcal{O},U} \colon & (C_c(G,\mathbb{Q})^{\mathcal{O}})^n & \longrightarrow & N_U^G \\ & [0, \ldots, \overset{j\text{-}th}{\mathbb{1}_{\mathcal{O}}}, \ldots, 0]^t & \longmapsto & \mathbb{1}_U - \mathbb{1}_{s_jU}. \end{array} \tag{2.4.10}$$

(b) N_U^G is finitely generated if, and only if, G is compactly generated.

(c) N_U^G is projective if, and only if, $\mathrm{cd}_{\mathbb{Q}}(G) \leq 1$.

Proof. For (a) and (b) see [CW16, Proposition 5.3], for (c) see [CW16, Lemma 3.6]. □

Consequently, if G is compactly generated and $\mathcal{O} \leq U$ are compact open subgroups of G, we define the endomorphism

$$\pi: \ (C_c(G,\mathbb{Q})^{\mathcal{O}})^n \xrightarrow{p_{\mathcal{O},U}} N_U^G \leq C_c(G,\mathbb{Q})^U \xrightarrow{\eta_{U,\mathcal{O}}} C_c(G,\mathbb{Q})^{\mathcal{O}} \xrightarrow{\ \xi\ } (C_c(G,\mathbb{Q})^{\mathcal{O}})^n, \tag{2.4.11}$$

where $\eta_{U,\mathcal{O}} \colon C_c(G, \mathbb{Q})^U \to C_c(G, \mathbb{Q})^{\mathcal{O}}$ is the injective homomorphism given by

$$\eta_{U,\mathcal{O}}(\mathbb{1}_{gU}) = \frac{1}{|U : \mathcal{O}|} \sum_{r \in \mathrm{Rep}(U/\mathcal{O})} \mathbb{1}_{gr\mathcal{O}}$$

(cf. [CW16, §4.2]), and ξ is the embedding into the first component, i.e.,

$$\xi(\mathbb{1}_{g\mathcal{O}}) = [\mathbb{1}_{g\mathcal{O}}, 0, \ldots, 0]^t, \quad \forall\, g \in G.$$

Fact 2.4.29. *Let G be a compactly generated t.d.l.c. group. According to Proposition 2.4.28, the following properties hold:*

(i) the image of π is isomorphic to N_U^G in ${}_{\mathbb{Q}[G]}\mathbf{dis}$;

(ii) if $\mathrm{cd}_{\mathbb{Q}}(G) \leq 1$, there exists a morphism in ${}_{\mathbb{Q}[G]}\mathbf{dis}$

$$\iota\colon \mathrm{im}(\pi) \to (C_c(G, \mathbb{Q})^{\mathcal{O}})^n$$

such that $\pi\iota = \mathrm{id}_{\mathrm{im}(\pi)}$;

(iii) the element $\alpha := \iota\pi$ satisfies $\alpha^2 = \alpha$ and $\pi\alpha = \pi$.

Proof. The only part which does not follow directly by construction is the existence of the morphism ι. It is instead a consequence of the fact that, since $\mathrm{im}(\pi)$ is projective in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ (cf. Proposition 2.4.28(c)), every epimorphism onto $\mathrm{im}(\pi)$ admits a right inverse. □

Remark 2.4.30. By Fact 2.4.25, Fact 2.4.29 and Corollary 2.4.22, we have

$$\tilde{\rho}(N_U^G) = \bar{\tau}(\alpha) \cdot \mu_{\mathcal{O}} \geq 0.$$

From now on the maps π and α will be identified with their images in the operator algebra $M_n(\phi(\mathcal{H}(G, \mathcal{O})^{op}_{\mathbb{Q}})) \subseteq M_n(W(G, \mathcal{O}))$ (cf. Fact 2.4.17(b) and (2.4.2)). In particular, $\pi = [\pi_{ij}]$ is the matrix with entries in $\phi(\mathcal{H}(G, \mathcal{O})^{op}_{\mathbb{Q}})$ defined by

$$\pi_{ij}(\mathbb{1}_{g\mathcal{O}}) := \begin{cases} 0, & \text{if } i > 1; \\ \dfrac{1}{|U : \mathcal{O}|} \displaystyle\sum_{r \in \mathrm{Rep}(U/\mathcal{O})} (\mathbb{1}_{gr\mathcal{O}} - \mathbb{1}_{gs_j r\mathcal{O}}), & \text{if } i = 1. \end{cases} \tag{2.4.12}$$

Recall that $\{\mathbb{1}_{g\mathcal{O}} \mid g \in \mathrm{Rep}(G/\mathcal{O})\}$ is an orthonormal basis of the Hilbert space $L^2(G, \mathbb{C})^{\mathcal{O}}$ (cf. Remark (2.4.4)). For simplicity, let $\pi_j := \pi_{1j}$ for every $j \in [n]$.

The following fact is relevant for the proof of Proposition 2.4.32. In particular, in item (c) we introduce the matrix $e \in M_n(W(G, \mathcal{O}))$ such that

$$M_n(W(G, \mathcal{O}))e\pi = M_n(W(G, \mathcal{O}))\pi.$$

We will see that the trace $\bar{\tau}(e\pi)$ produces a lower bound for the rank $\tilde{\rho}(N_U^G)$ which is independent of the number n of generators of the discrete left $\mathbb{Q}[G]$-module N_U^G (cf. Proposition 2.4.32).

Fact 2.4.31. *According to (2.4.11), the following properties hold:*

(a) $\pi^* M_n(W(G,\mathcal{O})) \subseteq \alpha^* M_n(W(G,\mathcal{O}))$.

(b) *There exists a projection* $p \in M_n(W(G,\mathcal{O}))$ *such that*

$$\alpha^* M_n(W(G,\mathcal{O})) = p M_n(W(G,\mathcal{O})) \quad \text{and} \quad \bar{\tau}(p) = \bar{\tau}(\alpha^*) = \bar{\tau}(\alpha).$$

(c) *If* e *denotes the matrix such that* $e_{ij} = 1_{W(G,\mathcal{O})}$ *for all* $i, j \in [n]$*, then we have* $n = |U : \mathcal{O}| \cdot \bar{\tau}(e\pi)$.

Proof. By Fact 2.4.29(iii), $M_n(W(G,\mathcal{O}))\pi = M_n(W(G,\mathcal{O}))\pi\alpha \subseteq M_n(W(G,\mathcal{O}))\alpha$ and (a) holds. Fact 2.4.29(iii) also implies that α^* is an idempotent. Then Theorem 2.4.21 applies and, since $\bar{\tau}(\alpha) \in \mathbb{Q}$, (b) is proved. Finally, (c) follows from $\bar{\tau}(e\pi) = \sum_{i=1}^n \bar{\tau}(\pi_i)$ and $\bar{\tau}(\pi_i) = \langle \pi_i(\mathbb{1}_{\mathcal{O}}), \mathbb{1}_{\mathcal{O}} \rangle = 1/|U : \mathcal{O}|$. □

Proposition 2.4.32. *Let* G *be a compactly generated unimodular t.d.l.c. group with* $\mathrm{cd}_{\mathbb{Q}}(G) \leq 1$*. Then, for every compact open subgroup* $U \lneq G$*, we have*

$$\tilde{\rho}(N_U^G) \geq \frac{1}{2}\mu_U. \tag{2.4.13}$$

Proof. We use here the notation of Fact 2.4.29, Remark 2.4.30 and Fact 2.4.31. In particular, $\alpha \in M_n(W(G,\mathcal{O}))$ is the idempotent such that $\tilde{\rho}(N_U^G) = \bar{\tau}(\alpha)\mu_{\mathcal{O}}$ and $p \in M_n(W(G,\mathcal{O}))$ is the projection such that $\alpha^* M_n(W(G,\mathcal{O})) = p M_n(W(G,\mathcal{O}))$ and $\bar{\tau}(p) = \bar{\tau}(\alpha^*) = \bar{\tau}(\alpha)$. Moreover, by Proposition 2.4.28, N_U^G is $\mathbb{Q}[G]$-generated by the elements $\mathbb{1}_{s_i U} - \mathbb{1}_U \in C_c(G,\mathbb{Q})^U$, for $i \in [n]$.

Denote by $e \in M_n(W(G,\mathcal{O}))$ the matrix such that $e_{ij} = 1_{W(G,\mathcal{O})}$ for all $i, j \in [n]$, and let $\varepsilon > 0$. By Proposition 2.4.15, there exists a non-zero projection $f = f_\varepsilon \in \pi^* e^* M_n(W(G,\mathcal{O})) \subseteq \pi^* M_n(W(G,\mathcal{O})) \subseteq p M_n(W(G,\mathcal{O}))$ such that $\|e\pi - e\pi f\| < \varepsilon$. By Fact 2.4.31 and since $|\bar{\tau}(e\pi - e\pi f)| \leq n \cdot \|e\pi - e\pi f\| < n \cdot \varepsilon$ (cf. Lemma 2.4.19), we have

$$\frac{n}{|U : \mathcal{O}|} = \bar{\tau}(e\pi) = |\bar{\tau}(e\pi)| \leq |\bar{\tau}(e\pi - e\pi f)| + |\bar{\tau}(e\pi f)| \leq n \cdot \varepsilon + |\bar{\tau}(e\pi f)|. \tag{2.4.14}$$

Notice that all the rows of the matrix $e\pi f$ are equal to the vector

$$\left[\sum_{i \in [n]} \pi_i f_{i1}, \ldots, \sum_{i \in [n]} \pi_i f_{in} \right].$$

Hence the properties of $\bar{\tau}$ imply that

$$|\bar{\tau}(e\pi f)| = \left| \sum_{i,j \in [n]} \tau(\pi_i f_{ij}) \right| = \left| \sum_{i,j \in [n]} \tau(f_{ij} \pi_i) \right| = \left| \sum_{i,j \in [n]} \langle f_{ij} \pi_i(\mathbb{1}_{\mathcal{O}}), \mathbb{1}_{\mathcal{O}} \rangle \right|$$

and so the Cauchy–Schwarz inequality yields

$$|\bar{\tau}(e\pi f)| \leq \sum_{i,j\in[n]} |\langle \pi_i(\mathbb{1}_{\mathcal{O}}), f_{ij}^*(\mathbb{1}_{\mathcal{O}})\rangle| \leq \sum_{i,j\in[n]} \|\pi_i(\mathbb{1}_{\mathcal{O}})\|_2 \cdot \|f_{ij}^*(\mathbb{1}_{\mathcal{O}})\|_2. \tag{2.4.15}$$

By (2.4.12), given a set $\mathrm{Rep}(U/\mathcal{O}) \subseteq U$ of representatives of $U/\mathcal{O}$, for each $i \in [n]$ we have

$$\|\pi_i(\mathbb{1}_{\mathcal{O}})\|_2^2 = \frac{1}{|U:\mathcal{O}|^2}\left\langle \sum_{r\in\mathrm{Rep}(U/\mathcal{O})} (\mathbb{1}_{r\mathcal{O}} - \mathbb{1}_{s_ir\mathcal{O}}), \sum_{r'\in\mathrm{Rep}(U/\mathcal{O})} (\mathbb{1}_{r'\mathcal{O}} - \mathbb{1}_{s_ir'\mathcal{O}})\right\rangle = \frac{2}{|U:\mathcal{O}|},$$

having $\mathbb{1}_{r\mathcal{O}} \neq \mathbb{1}_{s_ir'\mathcal{O}}$ and $\mathbb{1}_{r'\mathcal{O}} \neq \mathbb{1}_{s_ir\mathcal{O}}$ for all $r, r' \in \mathrm{Rep}(U/\mathcal{O})$ (because $U \cap s_iU = \emptyset$). Then, the inequalities (2.4.14) and (2.4.15) imply that

$$\frac{n}{|U:\mathcal{O}|} \leq n\cdot\varepsilon + \sqrt{\frac{2}{|U:\mathcal{O}|}}\left(\sum_{i,j\in[n]} \|f_{ij}^*(\mathbb{1}_{\mathcal{O}})\|_2\right). \tag{2.4.16}$$

Using the explicit equivalence between the ℓ^1-norm and the ℓ^2-norm in $\mathbb{R}^{n^2}$ of the vector $[\|f_{ij}^*(\mathbb{1}_{\mathcal{O}})\|_2]_{i,j\in[n]}$, we obtain that

$$\sum_{i,j\in[n]} \|f_{ij}^*(\mathbb{1}_{\mathcal{O}})\|_2 \leq n \cdot \sqrt{\sum_{i,j\in[n]} \|f_{ij}^*(\mathbb{1}_{\mathcal{O}})\|_2^2}.$$

Therefore, (2.4.16) and Remark 2.4.23(a) yield

$$\frac{n}{|U:\mathcal{O}|} \leq n\cdot\left(\varepsilon + \sqrt{\frac{2}{|U:\mathcal{O}|}\sum_{i,j\in[n]} \|f_{ij}^*(\mathbb{1}_{\mathcal{O}})\|_2^2}\right) = n\cdot\left(\varepsilon + \sqrt{\frac{2}{|U:\mathcal{O}|}\bar{\tau}(f)}\right), \tag{2.4.17}$$

for every $\varepsilon > 0$. Since $\bar{\tau}(\alpha) = \bar{\tau}(p) \geq \bar{\tau}(f)$ for every $\varepsilon > 0$ (cf. Fact 2.4.31(b) and Remark 2.4.23(b)), it follows from (2.4.17) that

$$\bar{\tau}(\alpha) \geq \frac{1}{2|U:\mathcal{O}|}.$$

By Remark 2.4.30, we conclude that

$$\tilde{\rho}(N_U^G) = \bar{\tau}(\alpha)\mu_{\mathcal{O}} \geq \frac{1}{2|U:\mathcal{O}|}\mu_{\mathcal{O}} = \frac{1}{2}\mu_U. \qquad \square$$

2.4.3 Euler–Poincaré characteristic of a t.d.l.c. group

Definition and general properties

Definition 2.4.33 ([CCW24, Equation (5.2)])**.** The *Euler–Poincaré characteristic* $\tilde{\chi}_G$ of a unimodular t.d.l.c. group G of type FP is the Hattori–Stallings rank of the trivial discrete left $\mathbb{Q}[G]$-module $\mathbb{Q}$. In other words,

$$\tilde{\chi}_G = \sum_{i=0}^{n} (-1)^n \tilde{\rho}(P_k),$$

where $0 \to P_n \to \ldots \to P_0 \twoheadrightarrow \mathbb{Q} \to 0$ is an arbitrary finite projective resolution of finite type of the trivial module $\mathbb{Q}$ in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ (cf. Definition 2.4.26). Moreover, for every Haar measure μ on G, let $\chi(G, \mu)$ be the real number satisfying

$$\tilde{\chi}_G = \chi(G, \mu) \cdot \mu.$$

Remark 2.4.34. If μ and μ' be Haar measures on G. Then $\mu' = c \cdot \mu$ for some $c \in \mathbb{R}_{>0}$, and

$$\chi(G, \mu) = c \cdot \chi(G, \mu'). \tag{2.4.18}$$

Example 2.4.35.

(i) If G is a profinite group then $\tilde{\chi}_G = 1 \cdot \mu_G$ (cf. [CCW24, Remark 1.3]).

(ii) Let G be an abstract group of type FP in ${}_{\mathbb{Q}[G]}\mathbf{mod}$. Then G, with the discrete group topology, is of type FP in ${}_{\mathbb{Q}[G]}\mathbf{dis}$. Moreover,

$$\tilde{\chi}_G = \chi(G) \cdot \mu_{\{1\}},$$

where $\chi(G)$ is the Euler–Poincaré characteristic introduced by I. Chiswell in [Chi76, p. 4], and $\mu_{\{1\}}$ is the counting measure on G (cf. [CCW24, Remark 1.2]).

Proposition 2.4.36. *Let G be a unimodular t.d.l.c. group of type* FP*.*

(a) If $G = X *_U Y$ for some open subgroups X, Y, U of G of type FP*, then*

$$\tilde{\chi}_G = \tilde{\chi}_X + \tilde{\chi}_Y - \tilde{\chi}_U.$$

(b) If $G = X *_U^t$ for some open subgroups X, U of G of type FP*, then*

$$\tilde{\chi}_G = \tilde{\chi}_X - \tilde{\chi}_U.$$

Proof. The argument in the proof of [CCW24, Theorem 5.2] applies. □

Proposition 2.4.36 implies the following generalisation of [CCW24, Corollary C].

Proposition 2.4.37. *Let G be a unimodular t.d.l.c. group which is isomorphic to $\pi_1(\mathcal{G}, \Lambda)$, for some finite graph $(\mathcal{G}, \Lambda)$ of t.d.l.c. groups of type* FP. *Then G is of type* FP *and, for a given orientation $\mathcal{E}^+$ in Λ, we have*

$$\tilde{\chi}_G = \sum_{v \in V\Lambda} \tilde{\chi}_{\mathcal{G}_v} - \sum_{e \in \mathcal{E}^+} \tilde{\chi}_{\mathcal{G}_e}.$$

Remark 2.4.38. From the main theorem of Bass–Serre theory (cf. Theorem 1.3.40), the statement of Proposition 2.4.37 can be rephrased as follows: Let G be a unimodular t.d.l.c. group acting properly and cocompactly on a tree T without edge-inversion. As observed after Corollary 2.3.2, these hypotheses imply that T is locally finite. Consider a set of representatives $\mathcal{V}$ of VT modulo G and a set of representatives $\mathcal{E}^+ \subseteq ET$ of a given orientation in $G \backslash T$. Then, for every Haar measure μ on G we have

$$\chi(G, \mu) = \sum_{v \in \mathcal{V}} \frac{1}{\mu(G_v)} - \sum_{e \in \mathcal{E}^+} \frac{1}{\mu(G_e)}. \tag{2.4.19}$$

In this case, the quantity $\chi(G, \mu)$ coincides with the Euler–Poincaré characteristic of G with respect to μ as defined by H. Petersen, R. Sauer and A. Thom in [PST18, Definition 4.8]. The formula (2.4.19) will be made even more explicit in terms of local data of the action in Proposition 3.5.4.

In [CCW24, Theorem 5.6], a generalisation of Proposition 2.4.37 is given for every unimodular t.d.l.c. group acting properly and cocompactly on a contractible simplicial complex Σ of finite dimension. Here we only report only a specific case, i.e., we consider Σ as the Davis' realisation Δ_{Dav} of a locally finite building Δ. We additionally assume that the relevant t.d.l.c. group acts properly and Weyl-transitively on Δ. Recall that this kind of groups is unimodular (cf. [BRW05, Corollary 5]) and of type FP (cf. [CW16, Proposition 6.6(c)]). Therefore, its Euler–Poincaré characteristic in the sense of Definition 2.4.33 is defined.

Proposition 2.4.39 ([CCW24, Corollary 5.12, Equations (5.12) and (5.13)])**.** *Let G be a t.d.l.c. group acting properly and Weyl-transitively on a locally finite building Δ of type (W, S). Denote by $\mathbf{q} = (q_s)_{s \in S}$ the thickness vector of Δ, and by B the stabiliser in G of a chamber c_0 of Δ. Then*

$$\tilde{\chi}_G = \sum_{\substack{T \subseteq S, \\ T \text{ spherical}}} \big(1 - \chi(L_T)\big) \mu_{P_T} = \sum_{\substack{T \subseteq S, \\ T \text{ spherical}}} \frac{1 - \chi(L_T)}{W_T(\mathbf{q})} \mu_B,$$

where P_T denotes the setwise stabiliser of $\mathrm{Res}_T(c_0)$, for every $T \subseteq S$ spherical,

$$\chi(L_T) = (-1)^{|T|+1} \cdot \sum_{\substack{T \subsetneq U \subseteq S, \\ U \text{ spherical}}} (-1)^{|U|}$$

(cf. [Dav08, Equation (17.10)]), and $W_T(\mathbf{q})$ *is the generalised Poincaré series of* $W_T \subseteq W$ *evaluated at* $\mathbf{q}$ *(cf. Equation* (1.4.1) *and Section 1.6.2).*

Proof. One proceeds as in the proof of [CCW24, Corollary 5.12], recalling the following: by Weyl-transitivity, for every spherical subset $T \subseteq S$ we have $P_T = BW_TB$ (cf. (1.6.5)) and

$$\mathrm{Res}_T(c_0) = \{bw \cdot c_0 \mid b \in B,\ w \in W_T\}.$$

Therefore, P_T acts transitively on $\mathrm{Res}_T(c_0)$ and

$$\mu_B(P_T) = |P_T : B| = |\mathrm{Res}_T(c_0)| = W_T(\mathbf{q}),$$

see [Dav08, Corollary 18.1.18] for the third equality. □

2.4.4 On the sign of the Euler–Poincaré characteristic

The main result of the section is Theorem 2.4.41. Before stating it, from Proposition 2.4.32 we deduce the following preliminary result.

Lemma 2.4.40. *Let* G *be a compactly generated unimodular t.d.l.c. group satisfying* $\mathrm{cd}_{\mathbb{Q}}(G) \leq 1$. *Then* $\tilde{\chi}_G \leq \frac{1}{2}\mu_U$ *for every* $U \in \mathcal{CO}(G)$ *with* $U \neq G$. *In particular,*

$$\tilde{\chi}_G = \frac{1}{2}\mu_U \iff \tilde{\rho}(N_U^G) = \frac{1}{2}\mu_U. \tag{2.4.20}$$

Proof. By Propositions 2.4.27 and 2.4.28, for every $U \in \mathcal{CO}(G)$ we have

$$\tilde{\chi}_G = \tilde{\rho}(\mathbb{Q}) = \tilde{\rho}(C_c(G,\mathbb{Q})^U) - \tilde{\rho}(N_U^G) = \mu_U - \tilde{\rho}(N_U^G).$$

The claim now follows from Proposition 2.4.32. □

Theorem 2.4.41. *Let* G *be a compactly generated unimodular t.d.l.c. group having* $\mathrm{cd}_{\mathbb{Q}}(G) = 1$. *Then* $\tilde{\chi}_G \leq 0$. *Moreover* $\tilde{\chi}_G = 0$ *if, and only if,* $G = X *_U Y$ *for some compact open subgroups* $X, Y, U \leq G$ *satisfying* $|X : U| = |Y : U| = 2$.

Proof. First note that $\tilde{\chi}_{\mathcal{O}} = \tilde{\rho}(\mathbb{Q}[G/\mathcal{O}]) = \mu_{\mathcal{O}}$ for every $\mathcal{O} \in \mathcal{CO}(G)$ (cf. Section 2.4.2). Moreover, since G is of type FP, one has $\mathrm{dH}^1(G, \mathrm{Bi}(G)) \neq 0$ (cf. Proposition 1.8.5(iii)). Hence, by Theorems 2.2.16 and 2.2.18, G splits non-trivially over a compact open subgroup $U \in \mathcal{CO}(G)$. Suppose first that $G = X *_U Y$, for some compact open subgroup U and compactly generated open subgroups X, Y of G different to U (cf. Proposition 2.2.17). By Proposition 2.4.36(a) and Lemma 2.4.40, we have

$$\tilde{\chi}_G = \tilde{\chi}_X + \tilde{\chi}_Y - \tilde{\chi}_U \leq \frac{1}{2}\mu_U + \frac{1}{2}\mu_U - \mu_U \leq 0. \tag{2.4.21}$$

Similarly, let $G = X*_U^t$ for some compact open subgroup $U \leq G$ and compactly generated open subgroup $X \leq G$ different to U. From Proposition 2.4.36(b) and Lemma 2.4.40 we deduce that

$$\tilde{\chi}_G = \tilde{\chi}_X - \tilde{\chi}_U \leq \frac{1}{2}\mu_U - \mu_U < 0. \tag{2.4.22}$$

If $\tilde{\chi}_G = 0$, then (2.4.21) and (2.4.22) imply that $G = X *_U Y$ for some compact open subgroup $U \leq G$ and compactly generated open subgroups $X, Y \leq G$ different to U. Moreover, (2.4.21) yields to

$$z := \tilde{\chi}_X - \frac{1}{2}\mu_U = \frac{1}{2}\mu_U - \tilde{\chi}_Y. \tag{2.4.23}$$

By Lemma 2.4.40 applied to X and Y, respectively, we deduce that z is simultaneously non-positive and non-negative and therefore equal to 0. Since $\mathrm{cd}_{\mathbb{Q}}(X), \mathrm{cd}_{\mathbb{Q}}(Y) \leq \mathrm{cd}_{\mathbb{Q}}(G) = 1$ (cf. Proposition 1.8.5(ii)), the first part of the theorem implies that $\mathrm{cd}_{\mathbb{Q}}(X) = \mathrm{cd}_{\mathbb{Q}}(Y) = 0$ and then X and Y are compact (cf. Proposition 1.8.5(i)). Therefore,

$$\mu_X = \tilde{\chi}_X = \frac{1}{2}\mu_U = \tilde{\chi}_Y = \mu_X.$$

Since $\mu_{\mathcal{O}_1} = |\mathcal{O}_1 : \mathcal{O}_2|^{-1}\mu_{\mathcal{O}_2}$ for all compact open subgroups $\mathcal{O}_2 \leq \mathcal{O}_1$ of G, this yields the "only if" part of the claim. The "if" part is a direct consequence of Proposition 2.4.36(a). □

Remark 2.4.42. By Theorem 2.4.41, the equivalence in (2.4.20) can be refined, for every $U \in \mathcal{CO}(G)$ with $U \neq G$, as follows:

$$\tilde{\chi}_G = \frac{1}{2}\mu_U \iff \tilde{\rho}(N_U^G) = \frac{1}{2}\mu_U \iff |G : U| = 2.$$

2.5 Application: a Stallings–Swan theorem for unimodular t.d.l.c. groups

2.5.1 Accessibility of compactly generated t.d.l.c. groups

Here below, we provide more details of what was already mentioned after the statement of Theorem 2.2.16.

Definition 2.5.1. A t.d.l.c. group G is said to *split over a compact open subgroup* U if one of the following two conditions occurs:

(α) $G \simeq \pi_1(\mathcal{G}, \Lambda)$, where Λ is a 1-segment with edge set $\{e, \bar{e}\}$ and $\mathcal{G}_e = U$. In other words, G is isomorphic to the t.d.l.c. free amalgamated product $\mathcal{G}_{o(e)} *_{\mathcal{G}_e, \eta_{\bar{e}}, \eta_e} \mathcal{G}_{t(e)}$ (cf. Definition 1.3.24).

(β) $G \simeq \pi_1(\mathcal{G}, \Lambda)$, where Λ is a 1-bouquet of loops with edge set $\{e, \bar{e}\}$ and $\mathcal{G}_e = U$. In other words, G is isomorphic to the t.d.l.c. HNN-extension $\mathcal{G}_{o(e)} *_{\mathcal{G}_e, \eta_e}^{t_e}$ (cf. Definition 1.3.27).

In particular, G splits *non-trivially over* U if $U = \mathcal{G}_e$ is embeds as a proper subgroup in each vertex-group.

By construction, the vertex-groups of $(\mathcal{G}, \Lambda)$ are open subgroups of G. Moreover, by Proposition 2.2.17, if G is compactly generated and splits as above, then the vertex-groups of $(\mathcal{G}, \Lambda)$ are compactly generated.

The notion of accessibility carries over to the t.d.l.c. context as follows.

Definition 2.5.2 ([KM08, Definition 8], [Cas20])**.** A compactly generated t.d.l.c. group G is said to be *accessible* if there is a finite proper graph of t.d.l.c. groups $(\mathcal{G}, \Lambda)$ with compact edge-groups and whose vertex-groups have at most 1 end.

According to Theorem 2.2.18, the concept of accessibility can be reformulated in the following terms.

Definition 2.5.3. A compactly generated t.d.l.c. group G is *accessible* if

($A1$) G is isomorphic to the fundamental group of a finite graph of t.d.l.c. groups $(\mathcal{G}, \Lambda)$ with compact edge-groups; and

($A2$) $\mathrm{dH}^1(\mathcal{G}_v, \mathrm{Bi}(\mathcal{G}_v)) = 0$ for every $v \in V\Lambda$.

Proposition 2.5.4. *Let G be a compactly generated t.d.l.c. group with $\mathrm{cd}_{\mathbb{Q}}(G) = 1$. Then G splits non-trivially over a compact open subgroup. Moreover, given a finite graph of t.d.l.c. groups $(\mathcal{G}, \Lambda)$ with compact edge-groups satisfying $G \simeq \pi_1(\mathcal{G}, \Lambda)$, for every $v \in V\Lambda$ the group $\mathcal{G}_v$ either is compact (if $\mathrm{cd}_{\mathbb{Q}}(\mathcal{G}_v) = 0$) or it is compactly generated and splits non-trivially over a compact open subgroup (if $\mathrm{cd}_{\mathbb{Q}}(\mathcal{G}_v) = 1$). In particular G is accessible if, and only if, G is isomorphic to the fundamental group of a finite proper graph of profinite groups.*

Proof. Since G is of type FP, by Proposition 1.8.5(iv) one has $\mathrm{dH}^1(G, \mathrm{Bi}(G)) \neq 0$. By Theorems 2.2.16 and 2.2.18, G splits non-trivially over a compact open subgroup. Let $(\mathcal{G}, \Lambda)$ be a graph of t.d.l.c. groups with compact edge-groups satisfying $G \simeq \pi_1(\mathcal{G}, \Lambda)$. Recall that for every $v \in V\Lambda$ the group $\mathcal{G}_v$ can be regarded as an open subgroup of $G \simeq \pi_1(\mathcal{G}, \Lambda)$. Thus $\mathcal{G}_v$ is compactly generated and $\mathrm{cd}_{\mathbb{Q}}(\mathcal{G}_v) \leq 1$ (cf. Proposition 2.2.17 and Proposition 1.8.5(ii)). Therefore, $\mathcal{G}_v$ either is compact (if $\mathrm{cd}_{\mathbb{Q}}(\mathcal{G}_v) = 0$) or splits non-trivially over a compact open subgroup (if $\mathrm{cd}_{\mathbb{Q}}(\mathcal{G}_v) = 1$). Hence, the first and second statements of the claim hold. The last part of the assertion follows from Definition 2.5.3 and the fact that a t.d.l.c. group H of type FP satisfying $\mathrm{cd}_{\mathbb{Q}}(H) \leq 1$ and $\mathrm{dH}^1(H, \mathrm{Bi}(H)) = 0$ is necessarily compact (cf. Proposition 1.8.5(i)-(iv)). □

The following fact has been proved in [CW16, Proposition 5.6]. It can be directly deduced from (2.3.1) and Proposition 1.8.5(a).

Proposition 2.5.5. *Let G be the t.d.l.c. fundamental group of a graph of profinite groups. Then* $\mathrm{cd}_{\mathbb{Q}}(G) \le 1$.

As a consequence of Lemma 1.3.39, we obtain the following result.

Proposition 2.5.6. *Suppose that a t.d.l.c. group G is isomorphic to $\pi_1(\mathcal{G}, \Lambda)$, where $(\mathcal{G}, \Lambda)$ is a finite graph of t.d.l.c. groups with compact edge-groups and whose vertex-groups are accessible compactly generated t.d.l.c. groups. Then G is accessible.*

The following proposition focuses on a situation arising in the next Proposition 2.5.9, and will play a key role in the proof of Corollary 2.5.10. Contrary to the previous results, this situation can happen only for non-discrete t.d.l.c. groups. Indeed, it requires an infinite strictly decreasing sequence of compact open subgroups of the relevant group.

Proposition 2.5.7. *Let G be a compactly generated unimodular t.d.l.c. group of type* FP. *Let $\big((\mathcal{G}_n, \Lambda_n)\big)_{n\ge 1}$ be an infinite sequence of finite graphs of t.d.l.c. groups with compact edge-groups, and with $G \simeq \pi_1(\mathcal{G}_n, \Lambda_n)$ for every $n \ge 1$. Assume that, for every $n \ge 2$, there are a bijection $\varphi_n\colon V\Lambda_n \to V\Lambda_1$ and $e_n \in E\Lambda_n$ with $v_n := o(e_n)$ such that:*

- *(i) $(\mathcal{G}_n)_{v_n}$ is inaccessible, $\varphi_n(v_n) = \varphi_{n+1}(v_{n+1})$, and $(\mathcal{G}_{n+1})_{e_{n+1}}$ is G-conjugated to a proper subgroup of $(\mathcal{G}_n)_{e_n}$;*
- *(ii) for every $w \in V\Lambda_n \setminus \{v_n\}$, the group $(\mathcal{G}_n)_w$ is G-conjugated to $(\mathcal{G}_1)_{\varphi_n(w)}$.*

Then $\mathrm{cd}_{\mathbb{Q}}(G) \ne 1$.

Proof. Assume that $\mathrm{cd}_{\mathbb{Q}}(G) = 1$ and let $n \ge 2$. By Proposition 2.4.36, given an orientation $\mathcal{E}_n^+$ in Λ_n with $e_n \in \mathcal{E}_n^+$, we have

$$\tilde{\chi}_G = \underbrace{\sum_{\substack{w\in V\Lambda_n,\\ w\neq v_n}} \tilde{\chi}_{(\mathcal{G}_n)_w} - \sum_{\substack{f\in\mathcal{E}_n^+,\\ f\neq e_n}} \mu_{(\mathcal{G}_n)_f}}_{=:\eta_n(G)} + \tilde{\chi}_{(\mathcal{G}_n)_{v_n}} - \mu_{(\mathcal{G}_n)_{e_n}}. \tag{2.5.1}$$

By (ii), we have

$$\tilde{\chi}_{(\mathcal{G}_n)_w} = \tilde{\chi}_{(\mathcal{G}_1)_{\varphi_n(w)}}, \quad \forall\, w \in V\Lambda_n \setminus \{v_n\}. \tag{2.5.2}$$

We now find a quantity $\eta(G)$, independent of n, such that

$$\eta_n(G) - \mu_{(\mathcal{G}_n)_{e_n}} \le \eta(G) - \mu_{(\mathcal{G}_n)_{e_n}}, \quad \forall\, n \ge 2. \tag{2.5.3}$$

In detail, let $v := \varphi_2(v_2)$ and recall that, by (i), we have $v = \varphi_n(v_n)$ for every $n \ge 2$. Then,

$$\eta_n(G) - \mu_{(\mathcal{G}_n)_{e_n}} = \sum_{\substack{w\in V\Lambda_n,\\ w\neq v_n}} \tilde{\chi}_{(\mathcal{G}_n)_w} - \underbrace{\sum_{\substack{f\in\mathcal{E}_n^+,\\ f\notin\{e_n,\bar{e}_n\}}} \mu_{(\mathcal{G}_n)_f}}_{\le 0} - \mu_{(\mathcal{G}_n)_{e_n}} \overset{(2.5.2)}{\le} \underbrace{\sum_{\substack{w\in V\Lambda_1,\\ w\neq v}} \tilde{\chi}_{(\mathcal{G}_1)_w}}_{=:\eta(G)} - \mu_{(\mathcal{G}_n)_{e_n}}.$$

By (i), there is a sequence $(\mathcal{O}_n)_{n\geq 1}$ of compact open subgroups of G satisfying $\mathcal{O}_1 \geq \mathcal{O}_2 > \ldots > \mathcal{O}_n > \ldots$ and such that, for all $n \geq 1$, the group $\mathcal{O}_n$ is G-conjugated to $(\mathcal{G}_n)_{e_n}$. Hence, for every $n \geq 2$,

$$\mu_{(\mathcal{G}_n)_{e_n}} = \mu_{\mathcal{O}_n} = |\mathcal{O}_1 : \mathcal{O}_n| \cdot \mu_{\mathcal{O}_1} = \left(\prod_{i=1}^{n-1} |\mathcal{O}_i : \mathcal{O}_{i+1}|\right)\mu_{\mathcal{O}_1} \geq 2^{n-1}\mu_{\mathcal{O}_1}. \tag{2.5.4}$$

Moreover, Theorem 2.4.41 yields $\tilde{\chi}_{(\mathcal{G}_n)_{v_n}} \leq 0$. This fact, together with (2.5.1), (2.5.3) and (2.5.4), implies that

$$\tilde{\chi}_G \leq \eta_n(G) - \mu_{(\mathcal{G}_n)_{e_n}} = \eta_n(G) - \mu_{\mathcal{O}_n} \leq \eta(G) - 2^{n-1}\mu_{\mathcal{O}_1},$$

for every $n \geq 2$. This contradicts the fact that $\tilde{\chi}_G = r \cdot \mu_{\mathcal{O}_1}$ for some fixed $r \in \mathbb{Q}$. □

The following lemma is preliminary to the proof of the next Proposition 2.5.9.

Lemma 2.5.8. *Let G be a compactly generated inaccessible t.d.l.c. group. Consider a finite proper graph of t.d.l.c. groups $(\mathcal{G}, \Lambda)$ with compact edge-groups such that $G \simeq \pi_1(\mathcal{G}, \Lambda)$, and let $v \in V\Lambda$ such that $\mathcal{G}_v$ is inaccessible. Then there are two finite graphs of t.d.l.c. groups $(\mathcal{H}, \Gamma)$ and $(\mathcal{G}', \Lambda')$ with compact edge-groups such that: $(\mathcal{G}', \Lambda')$ is proper, $|E\Gamma| = |E\Lambda| + 2$, $|E\Lambda'| \in \{|E\Lambda|, |E\Lambda| + 2\}$, and $G \simeq \pi_1(\mathcal{H}, \Gamma) \simeq \pi_1(\mathcal{G}', \Lambda')$.*

Moreover, the pair $((\mathcal{H}, \Gamma), (\mathcal{G}', \Lambda'))$ satisfies exactly one of the following conditions:

(A) *$(\mathcal{H}, \Gamma)$ is proper, $(\mathcal{G}', \Lambda') = (\mathcal{H}, \Gamma)$ and there is an injective map $\iota\colon V\Lambda \hookrightarrow V\Lambda'$ such that $\mathcal{G}'_{\iota(v)}$ is G-conjugated to an inaccessible proper subgroup of $\mathcal{G}_v$;*

(B) *$(\mathcal{H}, \Gamma)$ is not proper. Moreover, there are bijections*

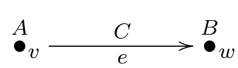
$$\varphi\colon V\Lambda' \to V\Lambda \quad \text{and} \quad \psi\colon E\Lambda' \to E\Lambda,$$

with ψ commuting with the edge-inversion maps, such that:

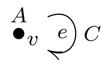
(B1) *$\mathcal{G}'_{\varphi^{-1}(v)}$ is an inaccessible subgroup of $\mathcal{G}_v$; and,*

(B2) *for every $w \in V\Lambda' \setminus \{\varphi^{-1}(v)\}$, the group $\mathcal{G}'_w$ is G-conjugated to $\mathcal{G}_{\varphi(w)}$ and, for every $f' \in E\Lambda'$, the group $\mathcal{G}_{f'}$ is G-conjugated to a subgroup of $\mathcal{G}_{\psi(f')}$. In particular, there is $e \in E\Lambda'$ with $o(e) = \varphi^{-1}(v)$ such that $\mathcal{G}'_e$ is G-conjugated to a proper subgroup of $\mathcal{G}_{\psi(e)}$.*

Proof. By Proposition 2.5.4, $\mathcal{G}_v$ splits non-trivially over a compact open subgroup C, say either $\mathcal{G}_v = A *_C B$ or $\mathcal{G}_v = A*^t_C$. Without loss of generality, assume that A is inaccessible. By Lemma 1.3.39, there is a finite graph of t.d.l.c. groups $(\mathcal{H}, \Gamma)$ obtained from a conjugate of $(\mathcal{G}, \Lambda)$ by expanding v with either a 1-segment (if $\mathcal{G}_v = A *_C B$)

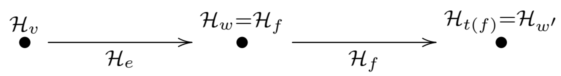


or a 1-loop (if $\mathcal{G}_v = A*_C^t$).

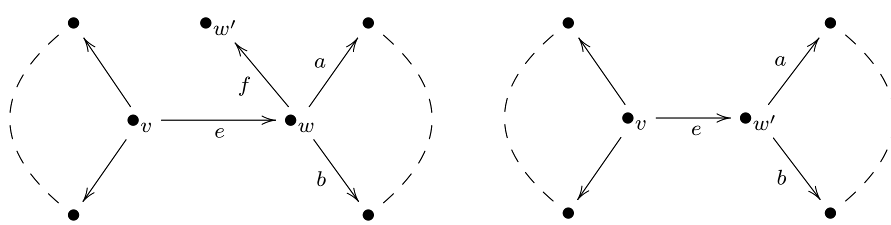


If $\mathcal{G}_v = A *_C B$, we may assume that $V\Gamma = V\Lambda \sqcup \{w\}$, $E\Gamma = E\Lambda \sqcup \{e, \overline{e}\}$, $o(e) = v$, $t(e) = w$, $\mathcal{H}_v = A$, $\mathcal{H}_w = B$, and $\mathcal{H}_e = C$. Similarly, if $\mathcal{G}_v = A*_C^t$, we may assume that $V\Gamma = V\Lambda$, $E\Gamma = E\Lambda \sqcup \{e, \overline{e}\}$, $o(e) = t(e) = v$, $\mathcal{H}_v = A$ and $\mathcal{H}_e = C$. In both cases, we have $|E\Gamma| = |E\Lambda| + 2$ and the vertex-group $\mathcal{H}_v = A$ is an inaccessible subgroup of $\mathcal{G}_v$.

If $(\mathcal{H}, \Gamma)$ is proper, setting $(\mathcal{G}', \Lambda') := (\mathcal{H}, \Gamma)$ one easily checks that (A) holds. Assume that $(\mathcal{H}, \Gamma)$ is not proper. Since $(\mathcal{G}, \Lambda)$ is proper and $\mathcal{G}_v$ is non-compact, necessarily $\mathcal{G}_v = A *_C B$ and there is a path (e, f) in Γ with $w = o(f) \neq t(f) = w'$ such that $\mathcal{H}_w = \mathcal{H}_{o(f)} = \mathcal{H}_f \subsetneq \mathcal{H}_{t(f)} = \mathcal{H}_{w'}$.



Let $(\mathcal{G}', \Lambda')$ be the finite graph of t.d.l.c. groups obtained from $(\mathcal{H}, \Gamma)$ by contracting the edge f, i.e., $V\Lambda' := V\Gamma \setminus \{w\} = V\Lambda$, $E\Lambda' := E\Gamma \setminus \{f, \overline{f}\}$, and the origin and the terminus of e in Λ' are the vertices v and w', respectively (cf. Definition 1.3.38). All edges of Λ with origin w that survive in Λ' have now origin in w' after the relabelling.



Moreover, up to G-conjugation, $\mathcal{G}'_x = \mathcal{H}_x$ for all $x \in V\Lambda' \sqcup E\Lambda'$. Since $\mathcal{H}_{w'} \simeq \mathcal{H}_w *_{\mathcal{H}_f} \mathcal{H}_{w'}$, we have $G \simeq \pi_1(\mathcal{G}', \Lambda')$ (cf. Lemma 1.3.39). In addition, $(\mathcal{G}', \Lambda')$ is proper. Indeed, since $(\mathcal{G}, \Lambda)$ is proper, the only possibility for $(\mathcal{G}', \Lambda')$ not being proper is that there exists $a \in E\Lambda' \setminus \{\overline{e}\}$ with $o(a) = w'$ such that $\mathcal{G}'_{w'} = \mathcal{G}'_a$. Necessarily a was an edge in Λ with origin w, and we have that $\mathcal{H}_a \subsetneq \mathcal{H}_w$. The inclusion is strict because $(\mathcal{H}, \Gamma)$ is proper. Therefore, $\mathcal{G}'_a = \mathcal{H}_a \subsetneq \mathcal{H}_w = \mathcal{H}_f \subsetneq \mathcal{H}_{t(f)} = \mathcal{G}'_{w'}$. Define $\varphi := \mathrm{id}_{V\Lambda}$ and

$$\psi \colon E\Lambda' \to E\Lambda, \quad \psi(f') := \begin{cases} f', & \text{if } f' \in E\Lambda' \setminus \{e, \overline{e}\}; \\ f, & \text{if } f' = e; \\ \bar{f}, & \text{if } f' = \bar{e}. \end{cases}$$

Up to G-conjugation, note that $\mathcal{G}'_e = \mathcal{H}_e = C \lneq B = \mathcal{H}_w = \mathcal{H}_f = \mathcal{G}'_f$. It is now easy to check that the pair $((\mathcal{H}, \Gamma), (\mathcal{G}', \Lambda'))$ satisfies condition (B) of the claim. $\square$

Proposition 2.5.9. *Let G be a compactly generated inaccessible t.d.l.c. group. Then there exists an infinite sequence $\big((\mathcal{G}_n, \Lambda_n)\big)_{n\geq 1}$ of finite proper graphs of t.d.l.c. groups having compact edge-groups satisfying $G \simeq \pi_1(\mathcal{G}_n, \Lambda_n)$ for every $n \geq 1$. Moreover, exactly one of the following conditions holds:*

(a) $|E\Lambda_n| = 2n$ *for every* $n \geq 1$*;*

(b) for every $n \geq 2$, there are bijections $\varphi_n \colon V\Lambda_n \to V\Lambda_1$ and $\psi_n \colon E\Lambda_n \to E\Lambda_1$, with ψ_n commuting with the edge-inversion maps, and there is $e_n \in E\Lambda_n$ with $v_n := o(e_n)$ such that:

(i) $(\mathcal{G}_n)_{v_n}$ is inaccessible;

(ii) for every $w \in V\Lambda_n \setminus \{v_n\}$, the group $(\mathcal{G}_n)_w$ is G-conjugated to $(\mathcal{G}_1)_{\varphi_n(w)}$ and, for every $f \in E\Lambda_n$, the group $(\mathcal{G}_n)_f$ is G-conjugated to a subgroup of $(\mathcal{G}_1)_{\psi_n(f)}$.

Moreover, for every $n \geq 2$, we have $\varphi_n(v_n) = \varphi_{n+1}(v_{n+1})$ and $(\mathcal{G}_{n+1})_{e_{n+1}}$ is G-conjugated to a proper subgroup of $(\mathcal{G}_n)_{e_n}$.

Proof. By Proposition 2.5.4, there is a finite proper graph of t.d.l.c. groups $(\mathcal{G}_1, \Lambda_1)$ with compact edge-groups such that $|E\Lambda_1| = 2$ and $G \simeq \pi_1(\mathcal{G}_1, \Lambda_1)$. Repeatedly applying Lemma 2.5.8, one constructs two infinite sequences of finite graphs of t.d.l.c. groups $\big((\mathcal{G}_n, \Lambda_n)\big)_{n\geq 1}$ and $\big((\mathcal{H}_n, \Gamma_n)\big)_{n\geq 1}$ with compact edge-groups satisfying, for every $n \geq 1$, what follows: $(\mathcal{G}_n, \Lambda_n)$ is proper, $|E\Gamma_{n+1}| = |E\Lambda_n| + 2$, $|E\Lambda_{n+1}| \in \{|E\Lambda_n|, |E\Lambda_n| + 2\}$, $G \simeq \pi_1(\mathcal{G}_n, \Lambda_n) \simeq \pi_1(\mathcal{H}_n, \Gamma_n)$, and the pair $((\mathcal{H}_n, \Gamma_n), (\mathcal{G}_{n+1}, \Lambda_{n+1}))$ meets either condition (A) or condition (B) of Lemma 2.5.8. If there are infinitely many $n \geq 1$ for which condition (A) is satisfied, there is a subsequence of $\big((\mathcal{G}_n, \Lambda_n)\big)_{n\geq 1}$ satisfying (a).

Assume now that there is $n_0 \geq 1$ such that condition (B) holds for every $n \geq n_0$. Let $v \in V\Lambda_{n_0}$ such that $(\mathcal{G}_{n_0})_v$ is inaccessible (cf. Proposition 2.5.6). By Lemma 2.5.8, for every $n \geq n_0$ there exist bijections $\tilde{\varphi}_{n+1} \colon V\Lambda_{n+1} \to V\Lambda_n$ and $\tilde{\psi}_{n+1} \colon E\Lambda_{n+1} \to E\Lambda_n$, with $\tilde{\psi}_{n+1}$ commuting with the edge-inversion maps, and there is $v_{n+1} \in V\Lambda_{n+1}$ such that $\tilde{\varphi}_{n_0+1}(v_{n_0+1}) = v$ and $\tilde{\varphi}_{n+1}(v_{n+1}) = v_n$ for every $n \geq n_0 + 1$. Moreover, for every $n \geq n_0$, the following holds:

(b1) $(\mathcal{G}_{n+1})_{v_{n+1}}$ is inaccessible;

(b2) for every $w \in V\Lambda_{n+1} \setminus \{v_{n+1}\}$, the group $(\mathcal{G}_{n+1})_w$ is G-conjugated to $(\mathcal{G}_n)_{\tilde{\varphi}_{n+1}(w)}$ and, for every $f \in E\Lambda_{n+1}$, the group $(\mathcal{G}_{n+1})_f$ is G-conjugated to a subgroup of $(\mathcal{G}_n)_{\tilde{\psi}_{n+1}(f)}$. In particular, there is $e \in E\Lambda_{n+1}$ with $o(e) = v_{n+1}$ such that $(\mathcal{G}_{n+1})_e$ is G-conjugated to a proper subgroup of $(\mathcal{G}_n)_{\tilde{\psi}_{n+1}(e)}$.

Replacing $\big((\mathcal{G}_n, \Lambda_n)\big)_{n\geq n_0}$ with one of its subsequences if necessary, we may assume that for every $n \geq n_0$ there is $e_n \in E\Lambda_n$ with $o(e_n) = v_n$ such that $(\mathcal{G}_{n+1})_{e_{n+1}}$ is G-conjugated to a proper subgroup of $(\mathcal{G}_n)_{e_n}$. This is indeed a consequence of (b2) and the fact that Λ_{n_0}

has finitely many edges. Defining $\varphi_n := \tilde{\varphi}_n \circ \ldots \circ \tilde{\varphi}_{n_0+1}$ and $\psi_n := \tilde{\psi}_n \circ \ldots \circ \tilde{\psi}_{n_0+1}$ for every $n > n_0$, we conclude that $\big((\mathcal{G}_n, \Lambda_n)\big)_{n \geq n_0}$ satisfies condition (b) of the claim. □

Combining Propositions 2.5.7 and 2.5.9, we deduce an important tool for the proof of the next Theorem 2.5.15.

Corollary 2.5.10. *Let G be a compactly generated unimodular t.d.l.c. group satisfying $\mathrm{cd}_{\mathbb{Q}}(G) = 1$. If G is inaccessible, then there is an infinite sequence $\big((\mathcal{G}_n, \Lambda_n)\big)_{n \geq 1}$ of finite proper graphs of t.d.l.c. groups having compact edge-groups, with $|E\Lambda_n| = 2n$ and $G \simeq \pi_1(\mathcal{G}_n, \Lambda_n)$ for every $n \geq 1$.*

2.5.2 $\mathcal{CO}$-bounded t.d.l.c. groups

In view of Theorem 2.5.15, we introduce the following property on t.d.l.c. groups.

Definition 2.5.11. A t.d.l.c. group G is said to be *$\mathcal{CO}$-bounded* if there is a left-invariant Haar measure μ on G such that

$$\|G\|_\mu := \sup\{\mu(U) : U \in \mathcal{CO}(G)\} < \infty. \tag{2.5.5}$$

Being $\mathcal{CO}$-bounded does not depend on the choice of the Haar measure. Indeed, if λ is another left-invariant Haar measure on G, there is $c > 0$ such that $\lambda = c \cdot \mu$.

For instance, a discrete group is $\mathcal{CO}$-bounded if (and only if) there is a uniform bound on the order of its finite subgroups.

Proposition 2.5.12. *Every $\mathcal{CO}$-bounded t.d.l.c. group is unimodular.*

Proof. Let $\Delta\colon G \to \mathbb{Q}_{>0}$ be the modular function of G and, for a given $U \in \mathcal{CO}(G)$, let μ_U be the left-invariant Haar measure on G such that $\mu_U(U) = 1$. Observe that $\Delta(g) = \mu_U(g^{-1}Ug) \leq \|G\|_{\mu_U}$ for every $g \in G$. Since $\mathbb{Q}_{>0}$ has no non-trivial bounded subgroups, we conclude that $\mathrm{im}(\Delta) = \{1\}$. □

Proposition 2.5.13. *Let G be a $\mathcal{CO}$-bounded t.d.l.c. group. Then every strictly increasing sequence of compact open subgroups of G is finite.*

Proof. It suffices to observe what follows. Every strictly increasing sequence of compact open subgroups $U_1 < U_2 < \cdots < U_n < \cdots$ of G yields a bounded (and hence finite) strictly increasing sequence of positive integers $1 < \mu_{U_1}(U_2) < \cdots < \mu_{U_1}(U_n) < \cdots$. Indeed, for every $i \geq 2$, the set $U_i \setminus U_{i-1}$ is a non-empty open subset in G and, therefore, $0 < \mu_{U_1}(U_i \setminus U_{i-1}) = \mu_{U_1}(U_i) - \mu_{U_1}(U_{i-1})$. □

Example 2.5.14. Let G be a t.d.l.c. group.

(a) If G is unimodular and admits an $\underline{\mathrm{E}}_{\mathcal{CO}}(G)$-space with finitely many orbits on the 0-skeleton, then G is $\mathcal{CO}$-bounded. Indeed, every $\mathcal{O} \in \mathcal{CO}(G)$ stabilises a point of X and hence it is subconjugated to a vertex-stabiliser (cf. [CW16, §6.6]). By unimodularity and since there are only finitely many conjugacy classes of vertex stabilisers, we conclude that G is $\mathcal{CO}$-bounded.

For instance, if G is unimodular and $G \simeq \pi_1(\mathcal{G}, \Lambda)$ for some graph $(\mathcal{G}, \Lambda)$ of profinite groups with $|V\Lambda| < \infty$, then G is $\mathcal{CO}$-bounded. Indeed, the universal tree of $(\mathcal{G}, \Lambda)$ is a 1-dimensional model for $\underline{\mathrm{E}}_{\mathcal{CO}}(G)$ (cf. [Lüc05, Theorem 4.9]) and G has finitely many orbits on the 0-skeleton. Hence, if G is a t.d.l.c. group having $\mathrm{cd}_{\mathbb{Q}}(G) \leq 1$ and such that G is either finitely generated and discrete or unimodular and compactly presented, then G is $\mathcal{CO}$-bounded.

(b) One can introduce the Bredon cohomology for G with respect to the family of all compact open subgroups (cf. [MV03, p. 14ff]). A natural generalisation of (a) is given by unimodular t.d.l.c. groups which are of *type Bredon* FP_0, i.e., there is a finite subset $\mathcal{A}$ of $\mathcal{CO}(G)$ such that every $\mathcal{O} \in \mathcal{CO}(G)$ is subconjugate to some element of $\mathcal{A}$ (cf. [KMN09, §3]).

2.5.3 A Stallings–Swan theorem for unimodular t.d.l.c. groups

We now state one of the main results of the chapter. Its proof exemplifies how an invariant of a group (in this case, its Euler–Poincaré characteristic) might be used to deduce structural properties of the group itself.

Theorem 2.5.15. *Let G be a compactly generated t.d.l.c. group satisfying $\mathrm{cd}_{\mathbb{Q}}(G) = 1$. If G is $\mathcal{CO}$-bounded, then G is accessible.*

Proof. Suppose that G is inaccessible. Let $\big((\mathcal{G}_n, \Lambda_n)\big)_{n\geq 1}$ be the sequence produced by Corollary 2.5.10. For every $n \geq 1$ fix an orientation $\mathcal{E}_n^+ \subseteq E\Lambda_n$, a maximal subtree $\mathcal{T}_n$ of Λ_n and $v_n \in V\Lambda_n$ such that $(\mathcal{G}_n)_{v_n}$ is non-compact (such a vertex exists because G is inaccessible, see Proposition 2.5.6). By [Ser80, §I.2, Proposition 12], there exists a bijection $\mathcal{E}_n^+ \cap E\mathcal{T}_n \longrightarrow V\mathcal{T}_n \setminus \{v_n\} = V\Lambda_n \setminus \{v_n\}$ (we will also prove the existence of this bijection in Remark 3.5.5(i)). Replacing $e \in \mathcal{E}_n^+ \cap E\mathcal{T}_n$ by $\bar{e}$ if necessary, we may assume that the bijection is given by the origin map $e \mapsto o(e)$. By Proposition 2.4.37, for every $n \geq 1$ we have

$$
\begin{aligned}
\tilde{\chi}_G &= \sum_{v\in V\Lambda_n} \tilde{\chi}_{(\mathcal{G}_n)_v} - \sum_{e\in\mathcal{E}_n^+} \mu_{(\mathcal{G}_n)_e} \\
&= \tilde{\chi}_{(\mathcal{G}_n)_{v_n}} + \sum_{e\in\mathcal{E}_n^+\cap E\mathcal{T}_n} (\tilde{\chi}_{(\mathcal{G}_n)_{o(e)}} - \mu_{(\mathcal{G}_n)_e}) - \sum_{e\in\mathcal{E}_n^+\setminus E\mathcal{T}_n} \mu_{(\mathcal{G}_n)_e} \qquad (2.5.6) \\
&\leq \sum_{e\in\mathcal{E}_n^+\cap E\mathcal{T}_n} (\tilde{\chi}_{(\mathcal{G}_n)_{o(e)}} - \mu_{(\mathcal{G}_n)_e}) - \sum_{e\in\mathcal{E}_n^+\setminus E\mathcal{T}_n} \mu_{(\mathcal{G}_n)_e},
\end{aligned}
$$

where the last inequality holds since $\tilde{\chi}_{(\mathcal{G}_n)_{v_n}} \leq 0$ (cf. Theorem 2.4.41). Let μ be an arbitrary left-invariant Haar measure on G. By Lemma 2.4.40, for every $e \in \mathcal{E}_n^+ \cap E\mathcal{T}_n$ we have

$$\tilde{\chi}_{(\mathcal{G}_n)_{o(e)}} - \mu_{(\mathcal{G}_n)_e} \leq -\frac{1}{2}\mu_{(\mathcal{G}_n)_e} = -\frac{1}{2\mu((\mathcal{G}_n)_e)}\mu \leq -\frac{1}{2\|G\|_\mu}\mu. \tag{2.5.7}$$

Moreover, for every $e \in \mathcal{E}_n^+ \setminus E\mathcal{T}_n$ we observe that

$$\mu_{(\mathcal{G}_n)_e} = \frac{1}{\mu((\mathcal{G}_n)_e)}\mu \geq \frac{1}{\|G\|_\mu}\mu > \frac{1}{2\|G\|_\mu}\mu. \tag{2.5.8}$$

Combining (2.5.6), (2.5.7) and (2.5.8), we conclude that

$$\begin{aligned}\tilde{\chi}_G &\leq \left(-\frac{|\mathcal{E}_n^+ \cap E\mathcal{T}_n|}{2\|G\|_\mu} - \frac{|\mathcal{E}_n^+ \setminus E\mathcal{T}_n|}{2\|G\|_\mu} \right) \cdot \mu \\ &= -\frac{|\mathcal{E}_n^+|}{2\|G\|_\mu}\mu = -\frac{|E\Lambda_n|}{4\|G\|_\mu}\mu = -\frac{n}{2\|G\|_\mu}\mu, \quad \text{for every } n \geq 1,\end{aligned}$$

which is impossible. Therefore, G is accessible. □

Theorem 2.5.15 yields the following characterisation.

Corollary 2.5.16. *For a compactly generated t.d.l.c. group G, the following are equivalent:*

(a) G is $\mathcal{CO}$-bounded and $\mathrm{cd}_{\mathbb{Q}}(G) \leq 1$;

(b) G is unimodular and $G \simeq \pi_1(\mathcal{G}, \Lambda)$, for some finite graph $(\mathcal{G}, \Lambda)$ of profinite groups;

(c) G is unimodular and some (and hence every) Cayley–Abels graph of G is quasi-isometric to a tree;

(d) G has a finitely generated free cocompact lattice.

Proof. The implication (a)⇒(b) is given by Proposition 2.5.12, Theorem 2.5.15 and then Proposition 2.5.4. The converse is Proposition 2.5.5 and Example 2.5.14(a). Finally, Theorem 2.2.10(ii) yields (b)⇔(c)⇔(d). □

Chapter 3

Double-coset zeta functions

3.1 Structure of the chapter

This chapter deals with *double-coset zeta functions* for groups with respect to two chosen subgroups. We will give a particular emphasis on groups acting Weyl transitively on locally finite buildings (*case (a)*), and to groups acting on locally finite trees (*case (b)*) with a weakly locally-∞ transitive or a (P)-closed action (*case (b1)* and *case (b2)*, respectively).

In Section 3.2, we introduce the main tools needed to define the relevant zeta functions. After some general notions, we specialise the discussion to the cases (a) and (b). In both cases, we conveniently enumerate the relevant double cosets and describe their size in terms of local data of the action (cf. Fact 3.2.3 and Proposition 3.2.4 for case (a); cf. Propositions 3.2.7 and 3.2.12 for case (b1); cf. Propositions 3.2.13 and 3.2.19 for case (b2)).

In Section 3.3 we introduce the concepts of *double-coset property* and *polynomial double-coset growth*, which are specific finiteness conditions on the sequence that generates the relevant zeta functions (cf. Definition 3.3.1). After a general independence result (cf. Theorem 3.3.6), we specialise the study to the cases (a) and (b), providing characterisations of the two mentioned properties in terms of local data of the actions. For case (a), one may refer to Proposition 3.3.10; for case (b1) to Theorem 3.3.21, and for case (b2) to Theorem 3.3.22. In case (b), the characterisation we provide involves a property on the action that we introduce for the purpose in Definition 3.3.13.

In Section 3.3.3, we discuss other structural implications on the group that can be deduced by studying the double-coset property and the polynomial double-coset growth. We contextualise our results in a more general landscape, recalling famous theorems regarding other sequences associated to groups.

Section 3.4 introduces the main object of the chapter: the *double-coset zeta functions* (cf. Definition 3.4.1). As in the previous sections, we soon specialise the discussion to the cases (a) and (b), providing explicit formulae and a meromorphic continuation for the relevant zeta functions in terms of local data of the action. Specifically, for case (a) see Proposition 3.4.7 and Theorem 3.4.11; for case (b1) see Theorem 3.4.20 and for case (b2) see Theorem 3.4.29. For case (a) the techniques adopted here are just a variation of those introduced in [CCW24]. For case (b), instead, we use a fairly new strategy: namely, we reduce the initial problem to the more accessible enumeration of weighted paths in a suitable "graph-like structure" (which changes from case (b1) to case (b2)). The reader who only wants to build up an idea of the strategy (without many technicalities) can refer to the results related to case (b1) (cf. Section 3.4.2). To study the problem in case (b1), we introduce an auxiliary zeta function counting weighted paths in a graph (cf. Definition 3.4.14). For this latter zeta function, we prove some splitting formulae that are crucial for studying the behaviour at $s = -1$ of the relevant double-coset zeta functions. This zeta function also satisfies a non-trivial connection with another generating function associated to a weighted graph, the so-called *Ihara zeta function of a weighed graph*. For details, see Section 3.5.3.

Finally, in Section 3.5 we prove that the *Euler–Poincaré identity* holds within the three classes of groups studied in this chapter. In case (a), this is an adaptation of a result appearing in [CCW24] (cf. Theorem 3.5.1). The reader may refer to the proof of this theorem to get a brief idea of the strategy used for cases (b1) and (b2). For these two cases, we first characterise the Euler–Poincaré characteristic of the relevant group in terms of local data of the action (cf. Proposition 3.5.4). Later, we reduced the problem to case (b1) (cf. Lemma 3.5.16) and use the splitting formulae provided in Section 3.4.3 to deduce the claimed identity (cf. Theorem 3.5.12 and Corollary 3.5.13).

3.1.1 Conventions of the chapter

In this chapter,

- every group action on a tree (G, T) is assumed without edge-inversions, and the tree T satisfies $ET \neq \emptyset$ and has no leaves;
- given a group G and subgroups $H, K \leq G$, we denote by $\mathrm{Rep}(H\backslash G/K)$ an arbitrarily chosen set of representatives of the (H, K)-double-cosets in G.

3.2 Double-cosets and their size

Let G be a group and $H, K \leq G$ be subgroups. Recall that the (H, K)-*double cosets* are subsets of G of the form

$$HgK = \{hgk \mid h \in H,\ k \in K\},$$

for $g \in G$. Denote by $H\backslash G/K$ the set of all (H, K)-double cosets. Then

$$G = \bigsqcup_{g \in \mathrm{Rep}(H\backslash G/K)} HgK. \tag{3.2.1}$$

For each double-coset $HgK \in H\backslash G/K$, we assign a *size* as follows:

$$|HgK/K| := |\{xK \in G/K \mid xK \subseteq HgK\}|. \tag{3.2.2}$$

In (3.2.2), we implicitly identify the K-coset xK with its respective subset of G, namely $\{xk \mid k \in K\}$. In other words, $|HgK/K|$ is the cardinality of the H-orbit of gK in G/K. By the orbit-stabiliser theorem, we observe that

$$|HgK/K| = |H : H \cap gKg^{-1}|, \quad \forall\, g \in G. \tag{3.2.3}$$

A *commensurable triple* is a triple of groups (G, H, K) in which H, K are subgroups of G that satisfy $|HgK/K| < \infty$ for every $g \in G$.

Example 3.2.1.

(i) Let G be a group and $H, K \leq G$ be subgroups of G with K being *normal* in G. By (3.2.3), (G, H, K) is a commensurable triple if, and only if, $|H : H \cap K| < \infty$.

(ii) Let G be a topological group and $H, K \leq G$ be open subgroups such that H is *compact* in G. If, for example, G is a discrete group, then H and K are subgroups of G with H finite. Then (G, H, K) is a commensurable triple. Indeed, for every $g \in G$, the subgroup $H \cap gKg^{-1}$ is open in H. Since H is compact, we have $|H : H \cap gKg^{-1}| < \infty$.

Remark 3.2.2 (*From double-cosets to cosets*)**.** Let (G, H, K) be a commensurable triple. For every $n \geq 1$, consider $a_n(G, H, K)$ as in (3.3.1) and define

$$b_n(G, H, K) := |\{gK \in G/K \,:\, |HgK/K| = n\}|. \tag{3.2.4}$$

We claim that $b_n(G, H, K) < \infty$ if, and only if, $a_n(G, H, K) < \infty$. Moreover, if $b_n(G, H, K) < \infty$ then

$$b_n(G, H, K) = n \cdot a_n(G, H, K). \tag{3.2.5}$$

To see this, consider the map $\varphi\colon gK \in G/K \longmapsto HgK \in H\backslash G/K$. Hence, for every $g \in G$, we have $\varphi^{-1}(HgK) = \{hgK \in G/K \mid h \in H\}$ and $|\varphi^{-1}(HgK)| = |H : H \cap gKg^{-1}| = |HgK/K|$.

3.2.1 The case of groups having a Bruhat decomposition

In the following, we collect some well-known facts about cosets and double-cosets of groups having a Bruhat decomposition. Recall that the latter class coincides with the one of groups acting Weyl-transitively on buildings (cf. Section 1.6.3). Our background references are [AB08] and [APV17]

We begin with the "base case" of a Coxeter group (W, S). Recall that each left-coset wW_J, for $w \in W$ and $J \subseteq S$, contains a unique minimal length representative w_1. More generally, given $I, J \subseteq S$ and $w \in W$, the double-coset $W_I w W_J$ contains a unique minimal length representative, which is called a (I, J)-*reduced* element in (W, S) (cf. [Dav08, Lemma 4.3.1]). Denote by $R(I, J)$ the collection of all (I, J)-reduced elements in W. Clearly,

$$W = \bigsqcup_{w \in R(I,J)} W_I w W_J. \tag{3.2.6}$$

The case of a Coxeter group acting on its standard Coxeter complex is an instance of a group acting Weyl-transitively on a building (cf. Example 1.6.11(i)). As we recall in the following fact, this case plays a key role in enumerating double-cosets for arbitrary groups having a Bruhat decomposition.

Fact 3.2.3 ([AB08, Exercise 6.38])**.** *Let G be a group with a Bruhat decomposition $G = BWB$ of type (W, S). Then, for all $I, J \subseteq S$, the following map is a bijection:*

$$\varphi_{I,J} \colon R(I,J) \longrightarrow P_I \backslash G / P_J, \quad \varphi_{I,J}(w) = P_I w P_J.$$

Under the notation of Fact 3.2.3, if I and J are both spherical subsets of S, then each size $|P_I w P_J / P_J|$ is finite and it admits an explicit formula in terms of (W, S) and the thickness vector of Δ. This formula relies on the generalised Poincaré series of a subset X of (W, S) (cf. Section 1.4.1).

Proposition 3.2.4 ([APV17, Theorem 2.1, Proposition 5.3])**.** *Let G be a group with a Bruhat decomposition $G = BWB$ of type (W, S) and with finite thickness vector $\mathbf{q} = (q_s)_{s \in S}$. Then, for all spherical subsets $I, J \subseteq S$ and $w \in R(I, J)$, we have*

$$|P_I w P_J / P_J| = \frac{W_I(\mathbf{q})}{W_{I \cap wJw^{-1}}(\mathbf{q})} \mathbf{q}_w, \tag{3.2.7}$$

where $\mathbf{q}_w = q_{s_1} \cdots q_{s_n}$ for some reduced expression $w = s_1 \cdots s_n$ in (W, S).

According to [APV17, §1.5], the integer $|P_I w P_J / P_J|$ can be interpreted as the size of a suitable sphere in the building $\Delta(G, B)$ associated to the Bruhat decomposition $G = BWB$.

Recall that there is an explicit formula for the univariate Poincaré series of a spherical (irreducible) Coxeter group (cf. (1.4.2)). Hence, in principle, the ratio appearing in (3.2.7) can be expressed in terms of $(q_s)_{s \in S}$, at least if $q_s = q$ for all $s \in S$. The difficulty, in general, is to determine the type of the Coxeter group $(W_{I \cap wJw^{-1}}, I \cap wJw^{-1})$.

3.2.2 The case of groups acting on trees

Since a tree is uniquely geodesic and by the orbit-stabiliser theorem, we observe what follows.

Fact 3.2.5. *Let (G, T) be a group action on a tree and let $v \in VT$, $e \in ET$. For every $t \in VT$, there are G-equivariant bijections $\varphi_{v,t} \colon G/G_t \to \mathrm{Geod}_T(v \to G \cdot t)$ and $\varphi_{e,t} \colon G/G_t \to \mathrm{Geod}_T(\{e, \bar{e}\} \to G \cdot t)$ defined as follows:*

$$\varphi_{v,t}(gG_t) := [v, g \cdot t] \quad \textit{and} \quad \varphi_{e,t}(gG_t) := \begin{cases} [e, g \cdot t], & \textit{if } g \cdot t \in T_{\geq e}; \\ [\bar{e}, g \cdot t], & \textit{if } g \cdot t \in T_{\geq \bar{e}}. \end{cases}$$

Similarly, for every $t \in ET$ there are G-equivariant bijections $\varphi_{v,t}\colon G/G_t \to \mathrm{Geod}_T(v \to G\cdot\{t,\bar{t}\})$ and $\varphi_{e,t}\colon G/G_t \to \mathrm{Geod}_T(\{e,\bar{e}\} \to G\cdot\{t,\bar{t}\})$ defined as follows:

$$\varphi_{v,t}(gG_t) := \begin{cases} [v, g\cdot t], & \text{if } [v,g\cdot t] \text{ exists in } T; \\ [v, g\cdot \bar{t}], & \text{if } [v,g\cdot \bar{t}] \text{ exists in } T; \end{cases} \qquad \text{and}$$

$$\varphi_{e,t}(gG_t) := \begin{cases} [e, g\cdot t], & \text{if } g\cdot t \in T_{\geq e}; \\ [e, g\cdot \bar{t}], & \text{if } g\cdot \bar{t} \in T_{\geq e}; \\ [\bar{e}, g\cdot t], & \text{if } g\cdot t \in T_{\geq \bar{e}}; \\ [\bar{e}, g\cdot \bar{t}], & \text{if } g\cdot \bar{t} \in T_{\geq \bar{e}}. \end{cases}$$

Lemma 3.2.6. *Let (G,T) be a group action on a tree. According to Fact 3.2.5, for all $t_1, t_2 \in T$ we have*

$$|G_{t_1} g G_{t_2}/G_{t_2}| = |G_{t_1} \cdot \varphi_{t_1,t_2}(gG_{t_2})|.$$

Proof. By (3.2.3), for every $g \in G$ we observe that

$$|G_{t_1} g G_{t_2}/G_{t_2}| = |G_{t_1} : G_{t_1} \cap g G_{t_2} g^{-1}|.$$

Since $gG_{t_2}g^{-1} = G_{g\cdot t_2}$ and T is uniquely geodesic, the group $G_{t_1} \cap gG_{t_2}g^{-1}$ is the pointwise stabiliser of the geodesic $\varphi_{t_1,t_2}(gG_{t_2})$. Then the orbit-stabiliser theorem yields the claim. □

The case of groups acting weakly locally ∞-transitively on trees

Let (G,T) be a weakly locally ∞-transitive group action on a tree. Denote by $\pi\colon T \to \Gamma = G\backslash T$ the quotient map of (G,T), and let ω be the standard edge weight on Γ. Given $u_1, u_2 \in \Gamma$ and $t_1 \in \pi^{-1}(u_1)$, let $\mathcal{P}^{\mathrm{lift}}_{\Gamma,t_1}(u_1 \to u_2)$ be the set of all paths $\mathfrak{p} \in \mathcal{P}_\Gamma(u_1 \to u_2)$ that can be lifted to a geodesic in T from t_1 via π.

Proposition 3.2.7. *In the hypotheses previously presented, let $v \in VT$ with $\pi(v) = c$ and $e \in ET$ with $\pi(e) = a$. According to Fact 3.2.5, we have the following bijections for every $t \in VT$ with $\pi(t) = u$:*

$$\begin{aligned} \Psi_{v,t}\colon G_v\backslash G/G_t &\longrightarrow \mathcal{P}^{\mathrm{lift}}_{\Gamma,v}(c \to U), \quad \Psi_{v,t}(G_v g G_t) = \pi(\varphi_{v,t}(gG_t)); \\ \Psi_{e,t}\colon G_e\backslash G/G_t &\longrightarrow \mathcal{P}^{\mathrm{lift}}_{\Gamma,e}(A \to U), \quad \Psi_{e,t}(G_e g G_t) = \pi(\varphi_{e,t}(gG_t)). \end{aligned}$$

Here the sets A and U are as in Notation 1.2.1.

For $t_1, t_2 \in T$, $g \in G$ and $h \in G_{t_1}$, note that

$$\pi(\Psi_{t_1,t_2}(hgG_{t_2})) = \pi(h \cdot \Psi_{t_1,t_2}(gG_{t_2})) = \pi(\Psi_{t_1,t_2}(gG_{t_2})).$$

Hence the map Ψ_{t_1,t_2} in Proposition 3.2.7 is well-defined for all $t_1, t_2 \in T$.

Proof. We only prove that $\Psi_{v,t}$ is bijective for $t \in VT$, as for the remaining cases one may argue analogously. Let $\pi_{v,t} \colon \mathrm{Geod}_T(v \to G \cdot t) \longrightarrow \mathcal{P}^{\mathrm{lift}}_{\Gamma,v}(c \to u)$ be the map defined as

$$\pi_{v,t}([v, g \cdot t]) := \pi([v, g \cdot t]).$$

Clearly, $\pi_{v,t}$ is G-equivariant and surjective. Moreover, since the G-action on T is weakly locally ∞-transitive, for every $g \in G$ the $\pi_{v,t}$-fibre of $\pi([v, g \cdot t])$ is $G_v \cdot [v, g \cdot t]$. Thus $\pi_{v,t}$ induces a 1–to–1 map

$$\tilde{\Psi}_{v,t} \colon G_v \backslash \mathrm{Geod}_T(v \to G \cdot t) \longrightarrow \mathcal{P}^{\mathrm{lift}}_{\Gamma,v}(c \to u).$$

Composing the bijection $G_v \backslash G / G_t \to G_v \backslash \mathrm{Geod}_T(v \to G \cdot t)$ induced by $\varphi_{v,t}$ with $\tilde{\Psi}_{v,t}$, we obtain $\Psi_{v,t}$. □

In view of Proposition 3.2.12, we introduce a weight on the paths of $G \backslash T$ which extends the standard edge weight ω on $G \backslash T$ and such that $|G_{t_1} g G_{t_2} / G_{t_2}|$ coincides with the weight of $\Psi_{t_1,t_2}(G_{t_1} g G_{t_2})$, for all $t_1, t_2 \in T$. Such a weight can be defined even in the following more general framework.

Definition 3.2.8. Let Γ be a graph with a function $\omega \colon E\Gamma \to \mathbb{Z}_{\geq 1}$ (called *edge weight*). Define two functions $N_{\mathrm{edg}} = N^{\omega}_{\mathrm{edg}}, N_{\mathrm{vert}} = N^{\omega}_{\mathrm{vert}} \colon \mathcal{P}_\Gamma \to \mathbb{Z}_{\geq 0}$ as follows. For every path $\mathfrak{p}$ in Γ, let

$$N_{\mathrm{edg}}(\mathfrak{p}) := \begin{cases} \ell(\mathfrak{p}), & \text{if } \ell(\mathfrak{p}) \leq 1 \\ \displaystyle\prod_{i=1}^{n-1} \Big(\omega(a_{i+1}) - \mathbb{1}_{\{\bar{a}_i\}}(a_{i+1})\Big), & \text{if } \mathfrak{p} = (a_1, \dots, a_n),\ n \geq 2 \end{cases} \tag{3.2.8}$$

and

$$N_{\mathrm{vert}}(\mathfrak{p}) := \begin{cases} 1, & \text{if } \ell(\mathfrak{p}) = 0 \\ \omega(a_1) \cdot N_{\mathrm{edg}}(\mathfrak{p}), & \text{if } \mathfrak{p} = (a_1, \dots, a_n),\ n \geq 1. \end{cases} \tag{3.2.9}$$

Notation 3.2.9. For $(a_1, \dots, a_n) \in \mathcal{P}_\Gamma$, we write $N_{\mathrm{edg}}(a_1, \dots, a_n)$ and $N_{\mathrm{vert}}(a_1, \dots, a_n)$ in place of $N_{\mathrm{edg}}((a_1, \dots, a_n))$ and $N_{\mathrm{vert}}((a_1, \dots, a_n))$, respectively.

Remark 3.2.10. Let Γ be the quotient graph of a group action on a tree (G, T), and denote by $\omega \colon E\Gamma \to \mathbb{Z}_{\geq 1}$ its standard edge weight. For every path $\mathfrak{p} = (a_1, \dots, a_n)$ in Γ of positive length and for every $e \in ET$ with $\pi(e) = a_1$, one checks that

$$N^{\omega}_{\mathrm{edg}}(\mathfrak{p}) = |\{\widetilde{\mathfrak{p}} \in \mathrm{Geod}_T(e \to T) \, : \, \pi(\widetilde{\mathfrak{p}}) = \mathfrak{p}\}|.$$

Similarly, for every path $\mathfrak{p}$ in Γ starting at $c \in V\Gamma$ and for every $v \in VT$ with $\pi(v) = c$, we have

$$N^{\omega}_{\mathrm{vert}}(\mathfrak{p}) = |\{\widetilde{\mathfrak{p}} \in \mathrm{Geod}_T(v \to T) \, : \, \pi(\widetilde{\mathfrak{p}}) = \mathfrak{p}\}|.$$

These observations applies if in particular Γ is the underlying graph of a local action diagram $\Delta = (\Gamma, (X_a), (G(c)))$, and if $T = T(\Delta, \iota, c_0)$ is a standard Δ-tree (cf. Section 1.3.4). In this case, given a path $\mathfrak{p} = (a_1, \dots, a_n)$ in Γ of positive length and for every $e \in ET^+$ with $\pi(e) = a_1$, we have

$$N^{\omega}_{\mathrm{edg}}(\mathfrak{p}) = |\{\xi \in \mathcal{P}_{(\Delta,\iota)}(\mathcal{L}(e) \to X) \,:\, \mathfrak{p}_\xi = \mathfrak{p}\}|, \tag{3.2.10}$$

(cf. Definition 1.3.8 and Lemma 1.3.13). Thus a reduced path ξ in (Δ, ι) of positive length can be lifted to a geodesic in T if, and only if, $N^{\omega}_{\mathrm{edg}}(\mathfrak{p}_\xi) \geq 1$.

Remark 3.2.11. Under the hypotheses of Proposition 3.2.7, suppose that $\omega(a) \geq 2$ for every $a \in E\Gamma$. Then every path $\mathfrak{p}$ can be lifted to a geodesic in T via π (cf. Remark 3.2.10). In particular, the bijections in Proposition 3.2.7 are onto $\mathcal{P}_\Gamma(c \to U)$ and $\mathcal{P}_\Gamma(A \to U)$, respectively.

Proposition 3.2.12. *Let (G, T) be a weakly locally ∞-transitive group action on a tree with quotient graph Γ. Assume that the standard edge weight ω on Γ takes finite values. Then, for all $g \in G$ and $t_1, t_2 \in T$, we have*

$$|G_{t_1} g G_{t_2} / G_{t_2}| = |G_{t_1} : G_{\varphi_{t_1,t_2}(gG_{t_2})}| = \begin{cases} N^{\omega}_{\mathrm{vert}}(\Psi_{t_1,t_2}(G_{t_1} g G_{t_2})), & \text{if } t_1 \in VT; \\ N^{\omega}_{\mathrm{edg}}(\Psi_{t_1,t_2}(G_{t_1} g G_{t_2})), & \text{if } t_1 \in ET. \end{cases}$$

Here the map Ψ_{t_1,t_2} is as in Proposition 3.2.7.

Proof. We may assume that $\varphi_{t_1,t_2}(gG_{t_2}) = [t_1, g \cdot t_2]$. Indeed, in the other cases the argument is analogous. Let $\mathfrak{p} := \Psi_{t_1,t_2}(G_{t_1} g G_{t_2}) = \pi([t_1, g \cdot t_2])$, where π denotes the quotient map of (G, T) (extended entrywise to all paths). By Lemma 3.2.6 and by weak local ∞-transitivity, we deduce that

$$|G_{t_1} g G_{t_2} / G_{t_2}| = |G_{t_1} \cdot [t_1, g \cdot t_2]| = |\{\widetilde{\mathfrak{p}} \in \mathrm{Geod}_T(t_1 \to T) \,:\, \pi(\widetilde{\mathfrak{p}}) = \mathfrak{p}\}|.$$

Now Remark 3.2.10 applies. □

The case of (P)-closed group actions on trees

Let Δ be a local action diagram and $(T = T(\Delta, \iota, c_0), \pi, \mathcal{L})$ be the standard Δ-tree associated to an inversion map ι in Δ and a chosen $c_0 \in V\Gamma$ (cf. Section 1.3.4). Let $G \leq U(\Delta, \iota, c_0)$ be a subgroup acting on T with local action diagram Δ. For the definition of $U(\Delta, \iota, c_0)$, see Section 1.3.4.

The following proposition rephrases Fact 3.2.5 in the language of local action diagrams.

Proposition 3.2.13. *Let $G \leq U(\Delta, \iota, c_0)$ and $T = T(\Delta, \iota, c_0)$ as before, and denote by v_0 the root of T. Consider also $e \in o^{-1}(v_0)$ with $\mathcal{L}(e) = x$, and $t \in T$ with $\pi(t) = u$. Let $\varphi_{u,t}$*

and $\varphi_{e,t}$ be the maps introduced in Fact 3.2.5, and denote by $\mathcal{L}$ the map defined in (1.3.6). Then the following two maps are bijective:

$$\begin{array}{rcl} \mathcal{L}\circ\varphi_{v_0,t}\colon G/G_t & \longrightarrow & \mathcal{P}_{(\Delta,\iota)}(X_{c_0}\to X_U); \\ \mathcal{L}\circ\varphi_{e,t}\colon G/G_t & \longrightarrow & \mathcal{P}_{(\Delta,\iota)}(x\to X_U)\sqcup\iota(x)\cdot\mathcal{P}_{(\Delta,\iota)}(X_{c_0}\setminus\{x\}\to X_U). \end{array}$$

For X_U see Notation 1.3.5.

Proof. It is a direct consequence of Fact 3.2.5 and Lemma 1.3.13. □

Proceeding similarly to the weakly locally ∞-transitive case, we define a weight on paths in Δ as follows.

Definition 3.2.14. Let $\Delta = (\Gamma, (X_a), (G(c)))$ be a local action diagram and recall that $X = \bigsqcup_{a\in E\Gamma} X_a$. The *standard weight* on Δ is the function $\mathcal{W}\colon X\times X \longrightarrow \mathbb{Z}_{\geq 0}\cup\{\infty\}$ defined, for all $x\in X_a$ and $y\in X_b$ with $a,b\in E\Gamma$, as follows:

$$\mathcal{W}(x,y) := \begin{cases} |G(t(a))_{\iota(x)}\cdot y|, & \text{if } t(a)=o(b); \\ 0, & \text{otherwise.} \end{cases}$$

Define also $\mathcal{W}_{\mathrm{rev}}\colon X\times X\longrightarrow\mathbb{Z}_{\geq 0}\cup\{\infty\}$ as follows, for all $x\in X_a$, $y\in X_b$ with $a,b\in E\Gamma$:

$$\mathcal{W}_{\mathrm{rev}}(x,y) := \begin{cases} |G(o(a))_{x}\cdot y|, & \text{if } o(a)=o(b); \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, for every sequence $\xi = (x_1,\dots,x_n)$ of elements of X of length $n\geq 0$, define

$$\mathcal{W}(\xi) := \begin{cases} 1, & \text{if } n\leq 1; \\ \displaystyle\prod_{i=1}^{n-1}\mathcal{W}(x_i,x_{i+1}), & \text{if } n\geq 2. \end{cases} \tag{3.2.11}$$

Remark 3.2.15. For $x,y\in X$ note that $\mathcal{W}(x,y)\neq 0$ if, and only if, (x,y) is a path in Δ. More generally, given a sequence ξ of elements of X we have $\mathcal{W}(\xi)\neq 0$ if, and only if, ξ is a path in Δ.

Notation 3.2.16. For a sequence $\xi = (x_1,\dots,x_n)$, we write $\mathcal{W}(x_1,\dots,x_n)$ in place of $\mathcal{W}((x_1,\dots,x_n))$.

Remark 3.2.17. Let $\Delta = (\Gamma, (X_a), (G(c)))$ be a local action diagram, set $T = T(\Delta,\iota,c_0)$ and $G = U(\Delta,\iota,c_0)$. Denote by ω and $\mathcal{W}$ be the standard edge weights on Γ and Δ, respectively. Hence, for every $e\in ET$ we have

$$|G_{o(e)}\cdot e| = \omega(\pi(e)).$$

Moreover, let $e, f \in ET$ with $t(e) = v = o(f)$. If $e \in ET^+$, then $\mathcal{L}(\bar{e}) = \iota(\mathcal{L}(e))$ (cf. Remark 1.3.10) and

$$|G_e \cdot f| = |G_{\bar{e}} \cdot f| = |G(\pi(v))_{\iota(\mathcal{L}(e))} \cdot \mathcal{L}(f)| = \mathcal{W}(\mathcal{L}(e), \mathcal{L}(f)).$$

Moreover, if $e \in ET \setminus ET^+$ we have

$$|G_e \cdot f| = |G_{\bar{e}} \cdot f| = |G(\pi(v))_{\mathcal{L}(\bar{e})} \cdot f| = \mathcal{W}_{\text{rev}}(\mathcal{L}(\bar{e}), \mathcal{L}(f)).$$

Example 3.2.18. Let $\Delta = (\Gamma, (X_a), (G(c)))$ be a local action diagram and consider $a, b \in E\Gamma$ with $t(a) = c = o(b)$. Assume that, for every $x \in X_a$, the group $G(c)_{\iota(x)}$ acts transitively on $X_b \setminus \{\iota(x)\}$. Then, for all $x \in X_a$ and $y \in X_b \setminus \{\iota(x)\}$, we have

$$\mathcal{W}(x, y) = |G(c)_{\iota(x)} \cdot y| = |X_b \setminus \{\iota(x)\}| = w(b) - \mathbb{1}_{\{\bar{a}\}}(b).$$

The transitivity condition before leads back to Proposition 1.3.21. For explicit examples satisfying it, see Example 1.3.22.

Proposition 3.2.19. *Let $G = U(\Delta, \iota, c_0)$. Consider a geodesic $\mathfrak{p} = (e_1, \ldots, e_n)$ in $T = T(\Delta, \iota, c_0)$ with $n \geq 1$, $e_1, \ldots, e_n \in ET^+$ and $\mathcal{L}(\mathfrak{p}) = (x_1, \ldots, x_n)$. Then,*

$$|G_{o(e_1)} : G_{\mathfrak{p}}| = \omega(\pi(e_1)) \cdot \mathcal{W}(\xi) \quad \textit{and} \quad |G_{e_1} : G_{\mathfrak{p}}| = \mathcal{W}(\xi).$$

Proof. By the orbit stabiliser theorem,

$$\begin{aligned} |G_{o(e_1)} : G_{\mathfrak{p}}| &= |G_{o(e_1)} : G_{e_1}| \cdot \prod_{i=1}^{n-1} |G_{(e_1,\ldots,e_i)} : G_{(e_1,\ldots,e_{i+1})}| \\ &= |G_{o(e_1)} \cdot e_1| \cdot \prod_{i=1}^{n-1} |G_{(e_1,\ldots,e_i)} \cdot e_{i+1}|. \end{aligned} \tag{3.2.12}$$

Similarly,

$$|G_{e_1} : G_{\mathfrak{p}}| = \prod_{i=1}^{n-1} |G_{(e_1,\ldots,e_i)} \cdot e_{i+1}|. \tag{3.2.13}$$

By Proposition 1.3.3, for every $1 \leq i \leq n-1$ we have

$$G_{(e_1,\ldots,e_i)} \cdot e_{i+1} = G_{e_i} \cdot e_{i+1}. \tag{3.2.14}$$

Combining Remark 3.2.17, (3.2.12), (3.2.13) and (3.2.14), we conclude the claim. □

Corollary 3.2.20. *Let $G = U(\Delta, \iota, c_0)$ and denote by v_0 be the root of $T = T(\Delta, \iota, c_0)$. Let $e \in ET \setminus ET^+$ with $t(e) = v_0$, and consider $t \in T$ such that $[v_0, t]$ is defined. Set $\mathfrak{p}_t := e \cdot [v_0, t] = (e_1 = e, e_2, \ldots, e_n)$, $\mathcal{L}(\bar{e}) = x_1$ and $\mathcal{L}(\mathfrak{p}_t) = (\iota(x_1), x_2, \ldots, x_n)$. Then*

$$|G_e : G_{\mathfrak{p}_t}| = \begin{cases} 1, & \textit{if } n = 1; \\ \mathcal{W}_{\text{rev}}(x_1, x_2) \cdot \mathcal{W}(x_2, \ldots, x_n), & \textit{otherwise.} \end{cases}$$

Proof. First, note that

$$|G_e : G_{\mathfrak{p}_t}| = |G_e : G_{(e_1,e_2)}| \cdot |G_{(e_1,e_2)} : G_{\mathfrak{p}_t}|.$$

By Remark 3.2.17, we deduce that

$$|G_e : G_{(e_1,e_2)}| = |G_{e_1} \cdot e_2| = \mathcal{W}_{\mathrm{rev}}(x_1, x_2).$$

From Remark 1.3.12 observe that $e_2, \dots, e_n \in ET^+$. Hence, Proposition 1.3.3 and Remark 3.2.17 yield

$$\begin{aligned} |G_{(e_1,e_2)} : G_{\mathfrak{p}_t}| &= \prod_{i=2}^{n-1} |G_{(e_1,\dots,e_i)} : G_{(e_1,\dots,e_{i+1})}| = \prod_{i=2}^{n-1} |G_{e_i} \cdot e_{i+1}| \\ &= \prod_{i=2}^{n-1} \mathcal{W}(x_i, x_{i+1}) = \mathcal{W}(x_2, \dots, x_n). \end{aligned}$$

□

Corollary 3.2.21. *Let $G = U(\Delta, \iota, c_0)$, $T = T(\Delta, \iota, c_0)$ and assume that $|X_a| \geq 2$ for every $a \in E\Gamma$. Then, for each geodesic $\mathfrak{p} = (e_1, \dots, e_n)$ in T of length $n \geq 1$, there is $g \in G_{o(e_1)}$ such that $g \cdot e_i \in ET^+$ for every $i \leq n$ and*

$$|G_{e_1} : G_{\mathfrak{p}}| = \mathcal{W}(\mathcal{L}(g \cdot \mathfrak{p})).$$

Proof. If $e_1 \in ET^+$, by Remark 1.3.12 we may take $g = 1$ and Proposition 3.2.19 applies. Assume now that $e_1 \in ET \setminus ET^+$ and let $(f_1, \dots, f_r = \bar{e}_1)$ be the geodesic from the root v_0 of T to $\bar{e}_1$. Set $\mathcal{L}(f_i) = x_i$ for every $1 \leq i \leq r$. In particular, $\mathcal{L}(\bar{e}_1) = x_r$ and then $\mathcal{L}(e_1) = \iota(x_r)$ (cf. Remark 1.3.10). Since $|X_{\pi(e_1)}| \geq 2$, there is $y \in X_{\pi(e_1)} \setminus \{\iota(x_r)\}$ such that $(x_1, \dots, x_r, y)$ is a reduced path in (Δ, ι). Then there is $f \in ET^+$ such that $o(f) = o(e_1)$ and $t(f)$ corresponds to $(x_1, \dots, x_r, y)$ (cf. Section 1.3.4). Since $\mathcal{L}(f) = y$ and $\mathcal{L}(e_1)$ belong to $X_{\pi(e_1)}$, there is $g \in G_{o(e_1)}$ such that $f = g \cdot e_1$. By Remark 1.3.12, every edge of $g \cdot \mathfrak{p} = (g \cdot e_1, \dots, g \cdot e_n)$ belongs to ET^+. Moreover,

$$|G_{e_1} \cdot \mathfrak{p}| = |gG_{e_1} \cdot \mathfrak{p}| = |G_{g \cdot e_1} g \cdot \mathfrak{p}|$$

and Proposition 3.2.19 applies. □

3.3 Double-coset property and polynomial double-coset growth

In this section, we outline [CCW24, §6] slightly generalising the framework.

Definition 3.3.1. A commensurable triple (G, H, K) is said to have the *double-coset property* if, for every $n \geq 1$,

$$a_n(G, H, K) := |\{HgK \in H\backslash G/K \,:\, |HgK/K| = n\}| < \infty. \tag{3.3.1}$$

Moreover, a triple (G, H, K) with the double-coset property is said to have *polynomial double-coset growth* if the sequence $(a_n(G, H, K))_{n \geq 1}$ grows polynomially.

The concept of double-coset property was already introduced in [CCW24, p. 5] for triples (G, K, K), where G is a t.d.l.c. group and K is a compact open subgroup.

Remark 3.3.2. Let G be a unimodular locally compact group and let $K \leq G$ be a compact open subgroup. By Example 3.2.1(ii), (G, K, K) is a commensurable triple. Moreover, given $KgK \in K\backslash G/K$, we observe that

$$|KgK/K| = 1 \iff KgK = gK \iff \forall\, k_1 \in K, \exists\, k_2 \in K \text{ s.th. } k_1 g = g k_2 \iff g^{-1}Kg \subseteq K,$$

Let μ be a Haar measure on G. If $g^{-1}Kg \subseteq K$, then $\mu(K \backslash g^{-1}Kg) = \mu(K) - \mu(g^{-1}Kg) = 0$ and, since $K\backslash g^{-1}Kg$ is open in G, we deduce that $K\backslash g^{-1}Kg = \emptyset$. Indeed, if $K\backslash g^{-1}Kg = \emptyset$ is non-empty then, by van Dantzig's theorem [Van36], $K \setminus g^{-1}Kg$ contains a translated of some compact open subgroup of G, which has positive measure. Hence,

$$g^{-1}Kg \subseteq K \iff g^{-1}Kg = K \iff g \in N_G(K),$$

where $N_G(K)$ denotes the normaliser of K in G. We conclude that

$$a_1(G, K, K) = |K\backslash N_G(K)/K| = |N_G(K) : K|.$$

Example 3.3.3. Let (G, H, K) be a commensurable triple and assume that K is normal in G. Then $|HgK/K| = |H : H \cap K|$ for every $g \in G$ (cf. (3.2.3)). Hence (G, H, K) has the double-coset property if, and only if, $H\backslash G/K$ is finite. In this case, having the double-coset property and polynomial double-coset growth are equivalent conditions.

In particular, we obtain the following sub-examples.

(i) Assume furthermore that G is a topological group and that $H, K \leq G$ are compact open in G (cf. Example 3.2.1(ii)). Then (G, H, K) has the double-coset property if, and only if, G is compact. Indeed, the decomposition in (H, K)-double cosets (cf. 3.2.1) is a partition of G in non-empty compact open subsets.

(ii) Let in particular $K = \{1\}$. Then (G, H, K) has the double-coset property if, and only if, H has finite index in G.

Lemma 3.3.4. *Let G be a topological group, G_0 be the connected component of G containing 1, and consider open subgroups $H, K \leq G$. Denote by $\bar{\cdot}: G \longrightarrow G/G_0$ the canonical projection. Then $\overline{H}, \overline{K} \leq \overline{G}$ are open subgroups of $\overline{G}$ satisfying*

$$|\overline{H} : \overline{H} \cap g\overline{K}g^{-1}| = |H : H \cap gKg^{-1}|, \quad \forall\, g \in G. \tag{3.3.2}$$

Moreover,

$$a_n(\overline{G}, \overline{H}, \overline{K}) = a_n(G, H, K), \quad \forall\, n \geq 1. \tag{3.3.3}$$

Proof. Since $H \cap K$ is open, we have $G_0 \subseteq H \cap K$ (cf. [Bou98, §III.2.2, Proposition 6]). Moreover,

$$\overline{H \cap K} = \overline{H} \cap \overline{K}.$$

Indeed, we clearly have $\overline{H \cap K} \subseteq \overline{H} \cap \overline{K}$. Moreover, every element in $\overline{H} \cap \overline{K}$ has the form hG_0 for some $h \in H$ satisfying $k^{-1}h \in G_0$ for some $k \in K$. Since $G_0 \subseteq K$, the latter two conditions imply $h \in H \cap K$ and $\overline{H} \cap \overline{K} \subseteq \overline{H \cap K}$.

It remains to prove (3.3.2). Given $g \in G$, set $m = |H : H \cap gKg^{-1}|$ and let $h_1, \ldots, h_m \in H$ be such that $H = \bigsqcup_{i=1}^m h_i(H \cap gKg^{-1})$. Then, $\overline{H} = \bigcup_{i=1}^m \overline{h_i}(\overline{H \cap gKg^{-1}})$. The latter is a disjoint union: indeed, if $\overline{h_i}(\overline{H \cap gKg^{-1}}) = \overline{h_j}(\overline{H \cap gKg^{-1}})$, then $\overline{h_j^{-1}h_i} \in \overline{H \cap gKg^{-1}}$ and there are $k \in H \cap gKg^{-1}$ and $g_0 \in G_0 \subseteq H \cap gKg^{-1}$ such that $h_j^{-1}h_i = kg_0 \in H \cap gKg^{-1}$. Therefore, $|\overline{H} : \overline{H} \cap \overline{gKg^{-1}}| = m$.

To conclude, we observe that $\bar{\cdot}$ induces a bijection $HgK \longmapsto \overline{H}\overline{g}\overline{K}$, $g \in G$, from $H\backslash G/K$ to $\overline{H}\backslash\overline{G}/\overline{K}$. This map is clearly surjective. For the injectivity, let $g_1, g_2 \in G$ satisfy $\overline{H}\overline{g_1}\overline{K} = \overline{H}\overline{g_2}\overline{K}$. Then $\overline{g_2} \in \overline{H}\overline{g_1}\overline{K} = \overline{Hg_1K}$, i.e., there are $h \in H$, $k \in K$ such that $g_2G_0 = hg_1kG_0$. Hence, there is $g_0 \in G_0$ such that $g_2 = hg_1kg_0$. As a conclusion, $g_2 \in Hg_1K$ and $Hg_1K = Hg_2K$. □

Remark 3.3.5. Let G be a locally compact group. By Lemma 3.3.4, while studying the sequence $(a_n(G, H, K))_{n\geq 1}$, where (G, H, K) is a commensurable triple with $H, K \leq G$ open subgroups, there is no loss of generality to assume that G also totally disconnected.

Provided H, K vary in a suitable family of subgroups of a given group, one can prove the following independence result.

Theorem 3.3.6. *Let G be a group. Consider a family $\mathcal{U}$ of subgroups of G that is closed under finite intersections of its elements, and such that (G, H, K) is commensurable for all $H, K \in \mathcal{U}$. If (G, H, K) has the double-coset property (resp. polynomial double-coset growth) for some $H, K \in \mathcal{U}$, then (G, H, K) has the double-coset property (resp. polynomial double-coset growth) for all $H, K \in \mathcal{U}$.*

If G is a t.d.l.c. group, the family $\mathcal{U}$ of all compact open subgroups satisfies the hypotheses of Theorem 3.3.6. In this case, a version of Theorem 3.3.6 for the double-coset property is provided by [CCW24, Proposition 6.2].

Other examples of families $\mathcal{U}$ that satisfy the hypotheses of Theorem 3.3.6 are:

- the collection of all spherical standard parabolic subgroups of a group having a locally finite Bruhat decomposition or equivalently, a group acting Weyl-transitively on a locally finite building (cf. Proposition 3.3.10);
- the collection of all vertex or edge stabilisers of a group having an action on a locally finite tree which is either weak locally ∞-transitive or (P)-closed (cf. Theorems 3.3.21 and 3.3.22).

For simplicity, we divide the proof of Theorem 3.3.6 into steps, which are included in the statements of the subsequent lemmas. Recall that we are considering the $a_n(G,H,K)$'s as elements in $\mathbb{Z}_{\geq 0} \cup \{+\infty\}$. The latter set is endowed with the usual addition and multiplication, and with a total order $\leq$ extending the usual one in $\mathbb{Z}_{\geq 0}$. For every commensurable triple (G,H,K) and for all $n \geq 1$, define

$$D_n(G,H,K) := \{HgK \in H\backslash G/K \ : \ |HgK/K| = n\} \tag{3.3.4}$$

and note that $a_n(G,H,K) = |D_n(G,H,K)|$.

Lemma 3.3.7. *Let (G,H,K) be a commensurable triple, and consider a subgroup $L \leq G$ satisfying $L \geq H$ and $|L : H| = m < \infty$. Then (G,L,K) is a commensurable triple. Moreover, for every $n \geq 1$,*

$$a_n(G,L,K) \leq \sum_{k=1}^{n} a_k(G,H,K) \quad \textit{and} \quad a_n(G,H,K) \leq m \cdot \sum_{k|mn} a_k(G,L,K).$$

In particular, (G,H,K) has the double-coset property (resp. polynomial double-coset growth) if, and only if, (G,L,K) has the double-coset property (resp. polynomial double-coset growth).

Proof. For every $g \in G$, note first that

$$|L : L\cap gKg^{-1}| \cdot |L\cap gKg^{-1} : H\cap gKg^{-1}| = |L : H\cap gKg^{-1}| = \overbrace{|L : H|}^{=m} \cdot |H : H\cap gKg^{-1}|. \tag{3.3.5}$$

Therefore, (G,L,K) is a commensurable triple. Consider the map

$$\pi_{H,L} \colon H\backslash G/K \longrightarrow L\backslash G/K, \quad \pi_{H,L}(HgK) = LgK \quad \forall\, g \in G.$$

By (3.3.5), the integer $|L : L\cap gKg^{-1}|$ divides $m \cdot |H : H\cap gKg^{-1}|$. Hence,

$$\pi_{H,L}(D_n(G,H,K)) \subseteq \bigsqcup_{k|mn} D_k(G,L,K), \quad \forall\, n \geq 1. \tag{3.3.6}$$

Moreover, for every $g \in G$, a well-known property of the index implies that

$$|L\cap gKg^{-1} : H\cap gKg^{-1}| \leq |L : H| = m. \tag{3.3.7}$$

Hence, by (3.3.5),

$$|H : H\cap gKg^{-1}| \leq |L : L\cap gKg^{-1}|, \quad \forall\, g \in G \tag{3.3.8}$$

and for every $n \geq 1$ we deduce what follows:

$$\begin{aligned}\pi_{H,L}^{-1}(D_n(G,L,K)) &= \bigsqcup_{LgK \in D_n(G,L,K)} \left\{ HlgK \,\middle|\, l \in \mathrm{Rep}\big(H \backslash L/(L \cap gKg^{-1})\big) \right\} \\ &\subseteq \bigsqcup_{k=1}^{n} D_k(G,H,K).\end{aligned} \tag{3.3.9}$$

The first equality in (3.3.9) implies that

$$|\pi_{H,L}^{-1}(D_n(G,L,K))| \leq m \cdot |D_n(G,L,K)|. \tag{3.3.10}$$

Moreover, the second inclusion in (3.3.9) yields

$$a_n(G,L,K) = |D_n(G,L,K)| \leq |\pi_{H,L}^{-1}(D_n(G,L,K))| \leq \sum_{k=1}^{n} a_k(G,H,K). \tag{3.3.11}$$

We also notice that

$$\begin{aligned}D_n(G,H,K) \subseteq \pi_{H,L}^{-1}(\pi_{H,L}(D_n(G,H,K))) &\overset{(3.3.6)}{\subseteq} \pi_{H,L}^{-1}\left(\bigsqcup_{k \mid mn} D_k(G,L,K) \right) \\ &= \bigsqcup_{k \mid mn} \pi_{H,L}^{-1}(D_k(G,L,K)).\end{aligned} \tag{3.3.12}$$

Therefore, for every $n \geq 1$ we conclude that

$$a_n(G,H,K) = |D_n(G,H,K)| \overset{(3.3.12)}{\leq} \sum_{k \mid mn} |\pi_{H,L}^{-1}(D_k(G,L,K))| \overset{(3.3.10)}{\leq} m \cdot \sum_{k \mid mn} a_k(G,L,K).$$

□

Lemma 3.3.8. *Let (G,H,K) be a commensurable triple, and consider a subgroup $L \leq G$ satisfying $L \geq K$ and $|L : K| = m < \infty$. Then (G,H,L) is a commensurable triple. Moreover, for every $n \geq 1$,*

$$a_n(G,H,L) \leq \sum_{k=1}^{mn} a_k(G,H,K) \quad \text{and} \quad a_n(G,H,K) \leq m \cdot \sum_{k \mid n} a_k(G,H,L).$$

In particular, (G,H,L) has the double-coset property (resp. polynomial double-coset growth) if, and only if, (G,H,K) has the double-coset property (resp. polynomial double-coset growth).

Proof. The fact that (G, H, L) is commensurable follows from the fact that $|H : H \cap gLg^{-1}|$ divides $|H : H \cap gKg^{-1}|$, for every $g \in G$. Consider the map

$$\eta_{K,L} \colon H\backslash G/K \longrightarrow H\backslash G/L, \quad \eta_{K,L}(HgK) = HgL \quad \forall\, g \in G.$$

Since $|H : H \cap gLg^{-1}|$ divides $|H : H \cap gKg^{-1}|$, we have

$$\eta_{K,L}(D_n(G, H, K)) \subseteq \bigsqcup_{k|n} D_k(G, H, L). \tag{3.3.13}$$

Moreover,

$$\begin{aligned}\eta_{K,L}^{-1}(D_n(G, H, L)) &= \bigsqcup_{HgL \in D_n(G,H,L)} \{HglK \mid l \in \mathrm{Rep}(g^{-1}Hg\backslash L/K)\} \\ &\subseteq \bigsqcup_{k=1}^{mn} D_k(G, H, K).\end{aligned} \tag{3.3.14}$$

In (3.3.14), the second inclusion follows from the general fact that

$$\begin{aligned}|H : H \cap xKx^{-1}| &= |H : H \cap xLx^{-1}| \cdot |H \cap xLx^{-1} : H \cap xKx^{-1}| \\ &\leq |H : H \cap xLx^{-1}| \cdot \overbrace{|xLx^{-1} : xKx^{-1}|}^{=m}, \quad \forall\, x \in G.\end{aligned} \tag{3.3.15}$$

From (3.3.14), we deduce two consequences. First,

$$a_n(G, H, L) = |D_n(G, H, L)| \leq |\eta_{K,L}^{-1}(D_n(G, H, L))| \leq \sum_{k=1}^{mn} a_k(G, H, K). \tag{3.3.16}$$

Second, since $\mathrm{Rep}\big((g^{-1}Hg)\backslash L/K\big)$ has size $\leq m$, for every $n \geq 1$ we have

$$|\eta_{L,K}^{-1}(D_n(G, H, L))| \leq m \cdot a_n(G, H, L) \tag{3.3.17}$$

and thus

$$\begin{aligned}a_n(G, H, K) = |D_n(G, H, K)| &\leq |\eta_{L,K}^{-1}(\eta_{L,K}(D_n(G, H, K)))| \\ &\overset{(3.3.13)}{\leq} \left|\eta_{L,K}^{-1}\left(\bigsqcup_{k|n} D_k(G, H, L)\right)\right| = \sum_{k|n} |\eta_{L,K}^{-1}(D_k(G, H, L))| \\ &\overset{(3.3.17)}{\leq} m \cdot \sum_{k|n} a_k(G, H, L).\end{aligned} \tag{3.3.18}$$

$\square$

We conclude with the proof of Theorem 3.3.6.

Proof of Theorem 3.3.6. Let $H, K, H', K' \in \mathcal{U}$. By hypothesis, both $H \cap H'$ and $K \cap K'$ belong to $\mathcal{U}$. By Lemmas 3.3.7 and 3.3.8, the triple (G, H, K) has the double-coset property (resp. polynomial double-coset growth) if, and only if, so has $(G, H \cap H', K)$ or, equivalently, if so has $(G, H \cap H', K \cap K')$. With a similar argument, one proves that (G, H', K') has the double-coset property (resp. polynomial double-coset growth) if, and only if, so has $(G, H \cap H', K \cap K')$. □

After Theorem 3.3.6, we may introduce the following.

Definition 3.3.9. Let G be a group. A family $\mathcal{U}$ of subgroups of G is said to be *admissible* if it is non-empty, it is closed under finite intersections of its elements, and (G, H, K) is a commensurable triple for all $H, K \in \mathcal{U}$. Given an admissible family $\mathcal{U}$ of subgroups, the group G is said to have the *double-coset property with respect to* $\mathcal{U}$ (resp. *polynomial double-coset growth with respect to* $\mathcal{U}$) if (G, H, K) has the double-coset property (resp. polynomial double-coset growth) for some (and hence all) $H, K \in \mathcal{U}$.

3.3.1 The case of groups having a Bruhat decomposition

The following section continues Section 3.2.1. It provides local characterisations of the double-coset property and the polynomial double-coset growth within the class of groups having a Bruhat decomposition.

Proposition 3.3.10. *Let G be a group having a Bruhat decomposition $G = BWB$ of type (W, S) and finite thickness vector $\mathbf{q} = (q_s)_{s \in S}$. Then the family $\mathcal{P} = \mathcal{P}(G)$ of spherical standard parabolic subgroups of G is admissible in the sense of Definition 3.3.9. Moreover, the following assertions hold:*

(a) G has the double-coset property with respect to $\mathcal{P}$ if, and only if, $|\{w \in W : \mathbf{q}_w = n\}|$ is finite for every $n \geq 1$;

(b) G has polynomial double-coset growth with respect to $\mathcal{P}$ if, and only if, $|\{w \in W : \mathbf{q}_w = n\}|$ grows polynomially in n.

Proof. We first prove that $\mathcal{P}$ is admissible. For all $I, J \subseteq S$, note that $P_I \cap P_J = P_{I \cap J}$ (because $W_I \cap W_J = W_{I \cap J}$, see [Hum90, Theorem 5.5(c)]). Moreover, if I and J are spherical, then $|P_I g P_J / P_J| < \infty$ for every $g \in G$ (cf. Proposition 3.2.4). Hence, by Theorem 3.3.6, it suffices to prove that (G, B, B) has the double-coset property (resp. polynomial double-coset growth) if, and only if, $|\{w \in W : \mathbf{q}_w = n\}|$ is finite (resp. grows polynomially in n), for every $n \geq 1$.

By Fact 3.2.3, the (B, B)-double cosets are bijection with the elements of $R(\emptyset, \emptyset) = W$ via $w \in W \longmapsto BwB \in B\backslash G/B$. Moreover, by Proposition 3.2.4 we have $|BwB/B| = \mathbf{q}_w$

for every $w \in W$. Therefore,

$$a_n(G,B,B) = |\{w \in W \,:\, \mathbf{q}_w = n\}|, \quad \forall\, n \geq 1, \tag{3.3.19}$$

and the statement follows. □

Corollary 3.3.11. *Let G be a group having a thick locally finite Bruhat decomposition $G = BWB$ of type (W,S). Then G has polynomial double-coset growth with respect to the family of all spherical standard parabolic subgroups of G.*

Proof. Let $\mathbf{q} = (q_s)_{s\in S}$ be the thickness vector of $G = BWB$. By hypothesis, $2 \leq q_s < \infty$ for every $s \in S$. In particular, $2^{\ell(w)} \leq \mathbf{q}_w < \infty$ for every $w \in W$, provided ℓ is the word length of (W,S). If $|S| = 1$, one has $|\{w \in W \,:\, \mathbf{q}_w = n\}| = O(1)$. If $|S| \geq 1$, for every $n \geq 1$ we have

$$\begin{aligned} |\{w \in W \,:\, \mathbf{q}_w = n\}| &\leq |\{w \in W \,:\, \ell(w) \leq \lfloor \log_2 n \rfloor\}|. \\ &\leq \sum_{l=0}^{\lfloor \log_2 n \rfloor} |S|^l = O(|S|^{\log_2(n)}) = O(n^{\log_2 |S|}). \end{aligned} \tag{3.3.20}$$

Now Proposition 3.3.10 applies. □

Corollary 3.3.12. *Let G be a group with a Bruhat decomposition $G = BWB$ of type (W,S) with uniform thickness $1 \leq q < \infty$. Let $\mathcal{P}$ be the family of spherical standard parabolic subgroups of G. Then the following are equivalent:*

(i) G has the double-coset property with respect to $\mathcal{P}$;

(ii) G has polynomial double-coset growth with respect to $\mathcal{P}$;

(iii) either W is finite or $q \geq 2$.

Proof. Let ℓ be the word length function of (W,S). Note that

$$X_n := |\{w \in W \,:\, \ell(w) = n\}| \leq |S|^n < \infty, \quad \forall\, n \geq 0.$$

Hence,

$$|\{w \in W \mid q^{\ell(w)} = n\}| = \begin{cases} |W| \cdot \mathbb{1}_{\{1\}}(n), & \text{if } q = 1; \\ X_{\log_q n} \cdot \mathbb{1}_{\mathbb{Z}_{\geq 0}}(\log_q n), & \text{if } q \geq 2. \end{cases}$$

Moreover, if $q \geq 2$ and $\log_q n \in \mathbb{Z}_{\geq 1}$, we have $X_{\log_q n} = O(|S|^{\log_q n}) = O(n^{\log_q |S|})$. Thus, Proposition 3.3.10 yields (iii)⇒(ii) and (i)⇒(iii), and the claim follows. □

3.3.2 The case of groups acting on trees

Here below, we introduce a family of properties (labelled with positive integers) on group actions on trees (cf. Definition 3.3.13). In Proposition 3.3.20, we exploit that one of these properties is satisfied to deduce that the group has polynomial double-coset growth with respect to vertex or edge stabilisers. The latter result can be refined to a characterisation in case that the action is weakly locally ∞-transitive (cf. Theorem 3.3.21) or (P)-closed (cf. Theorem 3.3.22).

The property $(*_k)$

Definition 3.3.13. Let $k \geq 1$. A group action on a tree (G,T) has *property* $(*_k)$ if, for every geodesic $(e_1, \dots, e_{l+k})$ in T with $l \geq 1$, we have

$$|G_{(e_1,\dots,e_l)} \cdot (e_{l+1}, \dots, e_{l+k})| \geq 2. \tag{3.3.21}$$

Remark 3.3.14. Note that property $(*_k)$ implies property $(*_{k+1})$, for every $k \geq 1$. Indeed, for every geodesic $(e_1, \dots, e_{l+k+1})$ in T with $l \geq 1$, we have

$$|G_{(e_1,\dots,e_l)} \cdot (e_{l+1}, \dots, e_{l+k+1})| \geq |G_{(e_1,\dots,e_{l+1})} \cdot (e_{l+2}, \dots, e_{l+k+1})| \geq 2.$$

Remark 3.3.15. Let T be a tree and $G \leq \mathrm{Aut}(T)$ be a subgroup with the subspace topology induced by $\mathrm{Aut}(T)$. If (G,T) has property $(*_k)$ for some $k \geq 1$, then G is non-discrete.

In detail, assume that (G,T) has property $(*_k)$. If we prove that all vertex-stabilisers in G are infinite, then [BEW15, Lemma 2.1] yields the claim. Let $v \in VT$. Since T has no leaves (cf. Section 3.1.1), there is a ray $(e_i)_{i \in \mathbb{Z}_{\geq 1}}$ in T with $o(e_1) = v$. For every $h \in \mathbb{Z}_{\geq 1}$, set $\mathfrak{p}_h = (e_i)_{1 \leq i \leq hk}$ and note that $|G_{\mathfrak{p}_h} \cdot (e_{hk+1}, \dots, e_{hk+k})| = |G_{\mathfrak{p}_h} : G_{\mathfrak{p}_{h+1}}| \geq 2$. Therefore,

$$|G_v : G_{\mathfrak{p}_h}| = |G_v : G_{\mathfrak{p}_1}| \cdot \prod_{i=1}^{h-1} |G_{\mathfrak{p}_i} : G_{\mathfrak{p}_{i+1}}| \geq 2^{h-1}, \quad \forall\, h \geq 1$$

and G_v is infinite.

Proposition 3.3.16. *Let (G,T) be a weakly locally ∞-transitive action on a locally finite tree. Denote by ω the standard edge weight on $\Gamma = G\backslash T$, consider $N_{\mathrm{edg}} = N^{\omega}_{\mathrm{edg}}$ as in Definition 3.2.8, and let $k \geq 1$. Then (G,T) has property $(*_k)$ if, and only if, $N_{\mathrm{edg}}(\rho) \geq 2$ for every path ρ in Γ of length $k+1$ which can be lifted to a geodesic in T.*

Proof. Let $k \geq 1$ and $\mathfrak{p} = (e_1, \dots, e_{l+k})$ be a geodesic in T with $l \geq 1$. Denote by $\pi \colon T \to \Gamma$ the quotient map of (G,T) and set $\pi(e_i) = a_i$ for every $1 \leq i \leq l+k$. By Remark 1.3.19(ii), $G_{(e_1,\dots,e_l)}$ acts transitively on

$$\{\mathfrak{q} = (f_1, \dots, f_{l+k}) \in \mathrm{Geod}_T(e_1 \to T) \,:\, \pi(\mathfrak{q}) = \pi(\mathfrak{p}) \text{ and } \forall\, i \leq l,\ f_i = e_i\}.$$

Hence, by Remark 3.2.10,

$$\begin{aligned}
&|G_{(e_1,\dots,e_l)} \cdot (e_{l+1},\dots,e_{l+k})| = |G_{(e_1,\dots,e_l)} \cdot \mathfrak{p}| = \\
&= \Big|\{\mathfrak{q} = (f_1,\dots,f_{l+k}) \in \mathrm{Geod}_T(e_1 \to T) \,:\, \pi(\mathfrak{q}) = \pi(\mathfrak{p}) \text{ and } \forall\, i \le l,\ f_i = e_i\}\Big| \\
&= \prod_{i=l}^{l+k-1} N_{\mathrm{edg}}(a_i, a_{i+1}) = N_{\mathrm{edg}}(a_l,\dots,a_{l+k}).
\end{aligned}$$

This yields the "if" part of the statement. For the "only if" part, let $\rho = (a_1,\dots,a_{k+1})$ be an arbitrary path in Γ which can be lifted to a geodesic $\mathfrak{p} = (e_1,\dots,e_{k+1})$ in T. Remark 3.2.10 now yields $|G_{e_1} \cdot \mathfrak{p}| = N_{\mathrm{edg}}(\rho) \ge 2$. □

Corollary 3.3.17. *Let (G,T) be a weakly locally ∞-transitive group action on a tree with quotient graph Γ and standard edge weight ω. Assume that $\omega(E\Gamma) \subseteq \mathbb{Z}_{\ge 2}$. Then the following are equivalent:*

(i) (G,T) has property $(*_k)$ for some $k \ge 2$;

(ii) (G,T) has property $(*_2)$;

(iii) $\omega(a) \ge 3$ or $\omega(\bar{a}) \ge 3$ for every $a \in E\Gamma$.

Moreover (G,T) has property $(*_1)$ if, and only if, $\omega(a) \ge 3$ for every $a \in E\Gamma$.

By Remark 3.2.10, the hypothesis that $\omega(E\Gamma) \subseteq \mathbb{Z}_{\ge 2}$ guarantees that all paths in Γ can be lifted to a geodesic.

Proof. Given $a, b \in E\Gamma$ with $t(a) = o(b)$, note that

$$N_{\mathrm{edg}}(a,b) = 1 \iff b = \bar{a} \text{ and } \omega(b) = 2. \tag{3.3.22}$$

The statement now follows from (3.3.22) and by Proposition 3.3.16. □

Proposition 3.3.18. *Let (G,T) be a (P)-closed action on a tree with associated local action diagram Δ, and let ι be an inversion on Δ. Assume that the standard weight $\mathcal{W}$ on Δ (cf. Definition 3.2.14) takes values in $\mathbb{Z}_{\ge 1}$, and that $|X_a| \ge 2$ for every $a \in E\Gamma$. Let also $k \ge 1$. Then (G,T) has property $(*_k)$ if, and only if, every reduced path ξ in (Δ,ι) of length $k+1$ has $\mathcal{W}(\xi) \ge 2$.*

Proof. Let $k \ge 1$ and consider a geodesic $\mathfrak{p} = (e_1,\dots,e_{l+k})$ in T with $l \ge 1$. By Proposition 1.3.3 and Corollary 3.2.21,

$$\begin{aligned}
|G_{(e_1,\dots,e_l)} \cdot (e_{l+1},\dots,e_{l+k})| &= |G_{e_l} \cdot (e_{l+1},\dots,e_{l+k})| \\
&= \mathcal{W}(\mathcal{L}(g \cdot (e_l,\dots,e_{l+k}))),
\end{aligned}$$

for some $g \in G_{o(e_l)}$. Since $\mathcal{L}(g \cdot (e_l, \ldots, e_{l+k}))$ is a reduced path in (Δ, ι) (cf. Lemma 1.3.13), this yields the "if" part of the statement. For the "only if" part, let $\xi = (x_1, \ldots, x_{k+1}) \in \mathcal{P}_{(\Delta,\iota)}$ and $e \in ET^+$ such that $\mathcal{L}(e) = x_1$. By Lemma 1.3.13, there is a geodesic $\mathfrak{p} = (e_1, \ldots, e_{k+1})$ in T with $e_1 = e$ such that $\mathcal{L}(\mathfrak{p}) = \xi$. By Proposition 3.2.19 we conclude that

$$|G_{e_1} \cdot (e_2, \ldots, e_k)| = \mathcal{W}(\xi) \geq 2. \qquad \square$$

Convergence properties

The main goal of what follows is to study the double-coset property and the polynomial double-coset growth of triples (G, G_{t_1}, G_{t_2}), where G is a group acting on a tree T and $t_1, t_2 \in T$.

Lemma 3.3.19. *Let (G, T) be a group action on a tree.*

(i) Assume that $C := \sup_{e,f \in ET : t(e)=o(f)} |G_e \cdot f|$ is finite. Then, for every geodesic $\mathfrak{p} = (e_1, \ldots, e_l)$ in T of length $l \geq 1$, we have $|G_{e_1} : G_{\mathfrak{p}}| \leq C^{l-1}$.

(ii) Suppose that (G, T) have property $(*_k)$ for some $k \geq 1$. Then, for every geodesic $\mathfrak{p} = (e_1, \ldots, e_l)$ in T of length $l \geq 1$, we have $|G_{e_1} : G_{\mathfrak{p}}| \geq 2^{\frac{l-k}{k}}$.

Proof. Let $\mathfrak{p} = (e_1, \ldots, e_l)$ be a geodesic in T of length $l \geq 1$. Arguing as for (3.2.13) we have

$$|G_{e_1} : G_{\mathfrak{p}}| = \prod_{i=1}^{l-1} |G_{(e_1,\ldots,e_i)} \cdot e_{i+1}|. \tag{3.3.23}$$

Since $|G_{(e_1,\ldots,e_i)} \cdot e_{i+1}| \leq |G_{e_i} \cdot e_{i+1}|$ for every $1 \leq i \leq l-1$, we obtain (i).

To prove (ii), we may assume that $l \geq k+1$. We claim that

$$|G_{e_1} : G_{\mathfrak{p}}|^k \overset{(3.3.23)}{=} \prod_{i=1}^{l-1} |G_{(e_1,\ldots,e_i)} \cdot e_{i+1}|^k \geq \prod_{i=1}^{l-k} \prod_{j=i}^{k+i-1} |G_{(e_1,\ldots,e_j)} \cdot e_{j+1}|. \tag{3.3.24}$$

To prove the latter inequality in (3.3.24), set $A_j = |G_{(e_1,\ldots,e_j)} \cdot e_{j+1}|$ for every $j \in [l-1]$. Then the product on the right-hand side of (3.3.24) becomes

$$\prod_{i=1}^{l-k} \prod_{j=i}^{k+i-1} A_j = \prod_{j=1}^{l-1} \prod_{i=\max\{1,j-k+1\}}^{\min\{j,l-k\}} A_j = \prod_{j=1}^{l-1} A_j^{\alpha_j},$$

where, for every $j \in [l-1]$,

$$\alpha_j = |\{i \in \mathbb{Z}_{\geq 0} : \max\{1, j-k+1\} \leq i \leq \min\{j, l-k\}\}|.$$

It remains to show that $\alpha_j \leq k$ for every $j \in [l-1]$. If $j \leq k-1$, then $\alpha_j = |\{i : 1 \leq i \leq \min\{j, l-k\}\}| \leq j < k$. If $k \leq j \leq l-k$, then $\alpha_j = |\{i : j-k+1 \leq i \leq j\}| = k$. Finally, if $j \geq k$ and $j \geq l-k$ then $\alpha_j = |\{i \,:\, j-k+1 \leq j \leq l-k\}| = l-j \leq k$. Hence (3.3.24) holds. By (3.3.24), the orbit-stabiliser theorem and the fact that (G,T) has property $(*_k)$, we conclude that

$$\begin{aligned}|G_{e_1} : G_{\mathfrak{p}}|^k &\geq \prod_{i=1}^{l-k} \prod_{j=i}^{k+i-1} |G_{(e_1,\ldots,e_j)} : G_{(e_1,\ldots,e_{j+1})}| = \\ &= \prod_{i=1}^{l-k} |G_{(e_1,\ldots,e_i)} \cdot (e_{i+1},\ldots,e_{i+k})| \geq 2^{l-k}.\end{aligned}$$ □

Let (G,T) be a group action on a tree and consider $t_1,t_2 \in T$ such that $|G_{t_1} g G_{t_2}/G_{t_2}| < \infty$ for every $g \in G$. For $i \in \{1,2\}$, set $T_i = \{t_i\}$ if $t_i \in VT$ and $T_i = \{t_i, \bar{t}_i\}$ if $t_i \in ET$. By Fact 3.2.5 and Lemma 3.2.6, we have

$$b_n(G,G_{t_1},G_{t_2}) = \left|\left\{\mathfrak{p} \in \mathrm{Geod}_T(T_1 \to G \cdot T_2) \,:\, |G_{t_1} : G_{\mathfrak{p}}| = n\right\}\right|, \quad \forall\, n \geq 1. \tag{3.3.25}$$

Proposition 3.3.20. *Let (G,T) be a group action on a locally finite tree with finite quotient graph, and set $M := \sup_{v \in VT} |o^{-1}(v)|$. If (G,T) has property $(*_k)$ for some $k \geq 1$, then $3 \leq M < \infty$ and, for all $t_1,t_2 \in T$ and $n \geq 1$,*

$$a_n(G,G_{t_1},G_{t_2}) = O(n^{k \cdot \log(M-1)-1}).$$

In particular, for all $t_1,t_2 \in T$, the triple (G,G_{t_1},G_{t_2}) has polynomial double-coset growth.

Proof. Let $t_1,t_2 \in T$ and consider T_1 and T_2 as defined before (3.3.25). We first prove that $3 \leq M \leq \infty$. Indeed, $|o^{-1}(v)| = |o^{-1}(g \cdot v)|$ for all $v \in VT$ and $g \in G$. Since T is locally finite and $G \backslash VT$ is finite, we have $M < \infty$. Moreover, $M \geq 2$ because T is assumed to have no leaves (cf. Section 3.1.1). Since (G,T) has property $(*_k)$, then T cannot be a bi-infinite line and then $M \geq 3$.

Given $l \geq 0$ and $t \in T$, note that the number of geodesics $\mathfrak{p}$ from t in T with $\ell(\mathfrak{p}) = l$ is at most 1 if $l = 0$, and it is at most $M(M-1)^{l-1}$ otherwise. Hence, by (3.3.25) and Lemma 3.3.19(ii), the following holds for every $n \geq 1$:

$$\begin{aligned} b_n(G,G_{t_1},G_{t_2}) &= |\{\mathfrak{p} \in \mathrm{Geod}_T(T_1 \to T) \,:\, |G_{t_1} : G_{\mathfrak{p}}| = n\}| \\ &\leq |\{\mathfrak{p} \in \mathrm{Geod}_T(T_1 \to T) \,:\, \ell(\mathfrak{p}) \leq \lfloor k \cdot \log_2 n \rfloor + k\}| \\ &= \sum_{t \in T_1} |\{\mathfrak{p} \in \mathrm{Geod}_T(t \to T) \,:\, \ell(\mathfrak{p}) \leq \lfloor k \cdot \log_2 n \rfloor + k\}| \\ &\leq \sum_{t \in T_1} \left(1 + \sum_{l=1}^{\lfloor k \cdot \log_2 n \rfloor + k} M(M-1)^{l-1}\right) \\ &= |T_1| \cdot \frac{(M-1)^{\lfloor k \cdot \log_2 n \rfloor + k} - 2}{M-2}. \end{aligned}$$

Hence,

$$b_n(G, G_{t_1}, G_{t_2}) = O((M-1)^{k \cdot \log n}) = O(n^{k \cdot \log(M-1)}).$$

The latter claim of the statement now follows from (3.2.5). □

Theorem 3.3.21. *Let (G,T) be a weakly locally ∞-transitive group action on a tree with finite quotient graph Γ. Assume that the standard edge weight ω on Γ takes values in $\mathbb{Z}_{\geq 2}$. Then the following are equivalent for all $t_1, t_2 \in T$:*

(i) (G, G_{t_1}, G_{t_2}) *has the double-coset property;*

(ii) (G, G_{t_1}, G_{t_2}) *has polynomial double-coset growth;*

(iii) (G,T) *has property* $(*_k)$ *for some* $k \geq 1$*;*

(iv) $\omega(a) \geq 3$ *or* $\omega(\bar{a}) \geq 3$ *for every* $a \in E\Gamma$*.*

Proof. Clearly, (ii)⇒(i). Moreover, Proposition 3.3.20 and Corollary 3.3.17 imply (iii)⇒(ii) and (iii)⇔(iv), respectively. It remains to prove (i)⇒(iv).

Assume that there is $a \in E\Gamma$ such that $\omega(a) = \omega(\bar{a}) = 2$, and let $\mathfrak{p} = (a, \bar{a})$. Then $N_{\text{edg}}(\mathfrak{p}^d) = (\omega(\bar{a}) - 1)^d (\omega(a) - 1)^{d-1} = 1$ for every $d \geq 1$. Consider two paths $\mathfrak{q}_1 = (a_1, \ldots, a_h)$ and $\mathfrak{q}_2 = (b_1, \ldots, b_k)$ of positive length in Γ from $\pi(t_1)$ to $o(a)$ and from $o(a)$ to $\pi(t_2)$, respectively. Then, for every $d \geq 1$ we have

$$\begin{aligned} N_{\text{edg}}(\mathfrak{q}_1 \cdot \mathfrak{p}^d \cdot \mathfrak{q}_2) &= N_{\text{edg}}(\mathfrak{q}_1) N_{\text{edg}}(a_h, a) N_{\text{edg}}(\mathfrak{p}^d) N_{\text{edg}}(\bar{a}, b_1) N_{\text{edg}}(\mathfrak{q}_2) \\ &= N_{\text{edg}}(\mathfrak{q}_1) N_{\text{edg}}(a_h, a) N_{\text{edg}}(\bar{a}, b_1) N_{\text{edg}}(\mathfrak{q}_2) =: N \geq 1. \end{aligned}$$

Since ω takes values in $\mathbb{Z}_{\geq 2}$, for every $d \geq 1$ there is $\tilde{\mathfrak{q}}_d \in \text{Geod}_T(T_1 \to G \cdot T_2)$ satisfying $\pi(\tilde{\mathfrak{q}}_d) = \mathfrak{q}_1 \cdot \mathfrak{p}^d \cdot \mathfrak{q}_2$ (cf. Remark 3.2.11). By Proposition 3.2.12, for every $d \geq 1$ we have

$$|G_{t_1} : G_{\tilde{\mathfrak{q}}_d}| = \begin{cases} N_{\text{edg}}(\mathfrak{q}_1 \cdot \mathfrak{p}^d \cdot \mathfrak{q}_2) = N, & \text{if } t_1 \in ET; \\ N_{\text{vert}}(\mathfrak{q}_1 \cdot \mathfrak{p}^d \cdot \mathfrak{q}_2) = \omega(a_1) \cdot N =: N', & \text{if } t_1 \in VT. \end{cases}$$

Since $\tilde{\mathfrak{q}}_d \neq \tilde{\mathfrak{q}}_{d'}$ for all $d \neq d'$, by (3.3.25) we conclude that $b_N(G, G_{t_1}, G_{t_2}) = \infty$ if $t_1 \in ET$, and $b_{N'}(G, G_{t_1}, G_{t_2}) = \infty$ if $t_1 \in VT$. □

Theorem 3.3.22. *Let (G,T) be a (P)-closed group action on a locally finite tree. Assume that the quotient graph is finite and its standard edge weight takes values in $\mathbb{Z}_{\geq 2}$. Then the following are equivalent for all $t_1, t_2 \in T$:*

(i) (G, G_{t_1}, G_{t_2}) *has the double-coset property;*

(ii) (G, G_{t_1}, G_{t_2}) *has polynomial double-coset growth;*

(iii) (G,T) *has property* $(*_k)$ *for some* $k \geq 1$*.*

Proof. The implication (ii)⇒(i) is immediate, and (iii)⇒(ii) follows from Proposition 3.3.20. It remains to prove (i)⇒(iii). Assume that (G, T) does not have property $(*_k)$ for every $k \geq 1$. Let Δ be the local action diagram associated to (G, T) and consider an inversion ι in Δ. Without loss of generality, $G = U(\Delta, \iota, c_0)$ and $T = T(\Delta, \iota, c_0)$ for some $c_0 \in V\Gamma$ (cf. Theorem 1.3.17(iii)). By Proposition 3.3.18, there is a reduced path $(x_1, \dots, x_k)$ in (Δ, ι) of length $k \geq |X|^2 + 2$ such that $\mathcal{W}(x_1, \dots, x_k) = 1$, i.e., $\mathcal{W}(x_i, x_{i+1}) = 1$ for every $1 \leq i \leq k-1$. Since $k \geq |X|^2 + 2$, we have $(x_i, x_{i+1}) = (x_{i+l}, x_{i+1+l})$ for some $i, l \geq 1$. Set $\eta := (x_j)_{i \leq j \leq i+l-1}$. For every $d \geq 1$, the power η^d is a reduced path in (Δ, ι) satisfying

$$\mathcal{W}(\eta^d) = \mathcal{W}(\eta)^d \cdot \mathcal{W}(x_{i+l-1}, x_i)^{d-1} = \mathcal{W}(\eta)^d \cdot \mathcal{W}(x_{i+l-1}, x_{i+l})^{d-1} = 1.$$

Choose arbitrary reduced paths of positive length in (Δ, ι), say $\xi_1 = (y_1, \dots, y_h)$ and $\xi_2 = (z_1, \dots, z_r)$, such that $\xi_1 \cdot x_1 \in \mathcal{P}_{(\Delta,\iota)}(X_{\pi(t_1)} \to x_1)$ and $x_{i+l-1} \cdot \xi_2 \in \mathcal{P}_{(\Delta,\iota)}(x_{i+l-1} \to X_{\pi(t_2)})$. Such reduced paths exist because the standard edge weight of $G \backslash T$ takes values in $\mathbb{Z}_{\geq 2}$ (cf. Remark 3.2.10). For every $d \geq 1$, the path $\xi_1 \cdot \eta^d \cdot \xi_2$ is reduced in (Δ, ι) and

$$\begin{aligned} \mathcal{W}(\xi_1 \cdot \eta^d \cdot \xi_2) &= \mathcal{W}(\xi_1)\mathcal{W}(y_h, x_i)\mathcal{W}(\eta^d)\mathcal{W}(x_{i+l-1}, z_1)\mathcal{W}(\xi_2) \\ &= \mathcal{W}(\xi_1)\mathcal{W}(y_h, x_i)\mathcal{W}(x_{i+l-1}, z_1)\mathcal{W}(\xi_2) =: N. \end{aligned}$$

By Lemma 1.3.13, for every $d \geq 1$ there is $\tilde{\mathfrak{q}}_d \in \mathrm{Geod}_T(T_1 \to G \cdot T_2)$ such that $\mathcal{L}(\tilde{\mathfrak{q}}_d) = \xi_1 \cdot \eta^d \cdot \xi_2$. By Corollary 3.2.21, we may assume that all edges of $\tilde{\mathfrak{q}}_d$ are in ET^+ for every $n \geq 1$. Note that each $\tilde{\mathfrak{q}}_d$ has the same first edge, say e_1. By Proposition 3.2.19, for every $d \geq 1$ we have

$$|G_{t_1} : G_{\tilde{\mathfrak{q}}_d}| = \begin{cases} \mathcal{W}(\mathcal{L}(\tilde{\mathfrak{q}}_d)) = N, & \text{if } t_1 \in ET; \\ \omega(\pi(e_1)) \cdot \mathcal{W}(\mathcal{L}(\tilde{\mathfrak{q}}_d)) =: N', & \text{if } t_1 \in VT. \end{cases}$$

Since $\tilde{\mathfrak{q}}_d \neq \tilde{\mathfrak{q}}_{d'}$ for all $d \neq d'$, from (3.3.25) we conclude that $b_N(G, G_{t_1}, G_{t_2}) = \infty$ if $t_1 \in ET$, and $b_{N'}(G, G_{t_1}, G_{t_2}) = \infty$ if $t_1 \in VT$. □

3.3.3 Structural implications of the double-coset property

From a group-theoretic perspective, one may wonder whether a sequence counting subobjects in a group G (e.g., elements of a prescribed length, subgroups, double-cosets etc.) carries some structural pieces of information on the group itself. To be more precise, we state some notable theorems in the literature that exemplify this connection.

The first one is due to M. Gromov and pertains to the word length growth of a group.

Theorem 3.3.23 ([Gro81, Main Theorem])**.** *Let G be a group with a finite generating set S, and denote by ℓ the word length of (G, S). For every $n \geq 1$, let $a_n(G, S)$ be the number of elements $g \in G$ with $\ell(g) = n$. Then the following are equivalent:*

(i) $(a_n(G, S))_{n \geq 1}$ grows polynomially in n;

(ii) G is virtually nilpotent.

The second result, due to M. du Sautoy and F. Grunewald, pertains the subgroup growth of a group.

Theorem 3.3.24 ([SG00, Theorem 1.1])**.** *Let G be a finitely generated residually finite group. For every $n \geq 1$, let $a_n^{\leq}(G)$ be the number of subgroups of index n in G. Then the following are equivalent:*

(i) $(a_n^{\leq}(G))_{n\geq 1}$ grows polynomially in n;

(ii) G has a subgroup of finite index which is soluble and of finite rank.

In the same spirit, we list and establish some structural properties of the group that can be deduced by studying the sequence $(a_n(G, H, K))_{n\geq 1}$ introduced in Definition 3.3.1.

We have already stated characterisations of the double-coset property and polynomial double-coset growth in specific families of groups in Proposition 3.3.10, Theorem 3.3.21 and Theorem 3.3.22. We now add some further properties.

The first property is an easy consequence of Example 3.3.3(ii).

Fact 3.3.25 ([CCW24, Fact 1.4])**.** *Let G be a discrete group. Then G has the double-coset property with respect to the family of finite groups if, and only if, G is finite.*

The second property is an alternative formulation of a result due to R. Möller [Möl10, Corollary 2.14].

Proposition 3.3.26 ([CCW24, Proposition 6.3])**.** *Let G be a t.d.l.c. group and $K \leq G$ be a compact open subgroup. Then G has a compact open normal subgroup if, and only if, there is $m_K \in \mathbb{Z}_{\geq 1}$ such that $a_n(G, K, K) = 0$ for every $n \geq m_K$. In particular, if G is non-compact then G does not have the double-coset property with respect to the family of all its compact open subgroups.*

The third property gives a sufficient condition on a t.d.l.c. group of being N-compact. A t.d.l.c. group G is said to be *N-compact* if every compact open subgroup of G has compact normaliser (cf. [CW16, §3.5]). We establish the following.

Proposition 3.3.27. *Let G be a t.d.l.c. group having a thick locally finite Bruhat decomposition $G = BWB$ of type (W, S) with respect to a compact open subgroup $B \leq G$. Then G is N-compact.*

Proof. By Corollary 3.3.11, (G, B, B) has the double-coset property. Note that the family $\mathcal{CO}(G)$ of all compact open subgroups of G is admissible (cf. Example 3.2.1(ii)) and B belongs to $\mathcal{CO}(G)$. Hence Theorem 3.3.6 implies G has the double-coset property with respect to $\mathcal{CO}(G)$. Let $K \leq G$ be a compact open subgroup. Since G is unimodular (cf. [BRW05, Corollary 5]), by Remark 3.3.2 we deduce that K has finite index in its normaliser $N_G(K)$ in G. Since K is compact, we conclude that also $N_G(K)$ is compact. □

3.4 Double-coset zeta functions

Generalising the definition given in [CCW24], we introduce the following generating function of Dirichlet type. For some general preliminaries on Dirichlet series, one may refer to Section 1.9.

Definition 3.4.1. Let (G, H, K) be a commensurable triple with polynomial double-coset growth (cf. Definition 3.3.1). The *double-coset zeta function associated to* (G, H, K) is the Dirichlet series (in the complex variable s) generated by $(a_n(G, H, K))_{n\geq 1}$, i.e.,

$$\zeta_{G,H,K}(s) := \sum_{n=1}^{\infty} a_n(G, H, K)n^{-s} = \sum_{HgK\in H\backslash G/K} |HgK/K|^{-s}. \tag{3.4.1}$$

Moreover, denote by $\alpha(G, H, K) \in \mathbb{R}_{\geq 0} \sqcup \{-\infty\}$ the *abscissa of convergence* of $\zeta_{G,H,K}(s)$.

Question 3.4.2. Let G be a group with an admissible family of subgroups $\mathcal{U}$. In Theorem 3.3.6 we have proved that, for all $H, K, H', K' \in \mathcal{U}$, the triple (G, H, K) has polynomial double-coset growth if, and only if, so has (G, H', K'). Is is also true that $\alpha(G, H, K) = \alpha(G, H', K')$ for all $H, K, H', K' \in \mathcal{U}$?

Remark 3.4.3. Let (G, H, K) be a commensurable triple with polynomial double-coset growth. By Remark 3.2.2, we have

$$\zeta_{G,H,K}(s) = \sum_{n=1}^{\infty} b_n(G, H, K) \cdot n^{-s-1} = \sum_{gK\in G/K} |HgK/K|^{-s-1}. \tag{3.4.2}$$

3.4.1 The case of groups having a Bruhat decomposition

The following part is a follow-up of Section 3.2.1 and generalises [CCW24, Section 6]. In more detail, we first the Dirichlet series defining $\zeta_{G,P_I,P_J}(s)$ in terms of local data of the Bruhat decomposition $G = BWB$ (cf. Proposition 3.4.5). This allows us to deduce a rationality result for the relevant zeta functions (cf. Proposition 3.4.7) and, thanks to a result of M. Davis, an explicit formula (cf. Theorem 3.4.11).

Notation 3.4.4. Let X be a finite non-empty set. For every $s \in \mathbb{C}$ and for every tuple of integers $\mathbf{t} = (t_x)_{x\in X}$, let $\mathbf{t}^s = ((t_x)^s)_{x\in S}$ be the pointwise s-power of $\mathbf{t}$.

General properties of $\zeta_{G,P_I,P_J}(s)$

We obtain the following generalisation of [CCW24, Proposition 6.4].

Proposition 3.4.5. *Let G be a group with a Bruhat decomposition $G = BWB$ of type (W, S) and with finite thickness vector $\mathbf{q} = (q_x)_{x\in S}$. Assume that G has polynomial double-coset* *growth with respect to the family of spherical standard parabolic subgroups. Then, for all spherical subsets $I, J \subseteq S$ and for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \mathrm{abs}(G, P_I, P_J)$, we have*

$$\zeta_{G,P_I,P_J}(s) = \sum_{w \in R(I,J)} \left(\frac{W_I(\mathbf{q})}{W_{I \cap wJw^{-1}}(\mathbf{q})} \right)^{-s} \mathbf{q}_w^{-s}.$$

Proof. It is a direct consequence of Fact 3.2.3 and Proposition 3.2.4. □

In particular, $\zeta_{G,B,B}(s)$ recovers the generalised Poincaré series $W(\mathbf{t})$ of (W, S) as follows.

Corollary 3.4.6. *Under the hypotheses of Proposition 3.4.5, we have*

$$\zeta_{G,B,B}(s) = W(\mathbf{q}^{-s}).$$

Assume the setting of Proposition 3.4.5, and let $I, J \subseteq S$ be spherical subsets. For every $Q \subseteq I$, define

$$P_{I,J,Q} := \{w \in R(I,J) \mid I \cap wJw^{-1} = Q\}. \tag{3.4.3}$$

The sets $P_{J,J,Q}$ have already been defined in the proof of [CCW24, Proposition 6.9] (with the symbol $p_{Q,J}$). An immediate consequence of Proposition 3.4.5 is that

$$\zeta_{G,P_I,P_J}(s) = \sum_{Q \subseteq I} \left(\frac{W_I(\mathbf{q})}{W_Q(\mathbf{q})} \right)^{-s} \cdot P_{I,J,Q}(\mathbf{q}^{-s}). \tag{3.4.4}$$

Proposition 3.4.7. *In the hypotheses of Proposition 3.4.5, for all spherical subsets $I, J \subseteq S$, $\zeta_{G,P_I,P_J}(s)$ is a rational function in $\mathbf{q}^{-s}$ and in $\{\big(W_I(\mathbf{q})/W_Q(\mathbf{q})\big)^{-s}\}_{Q \subseteq I}$ with integral coefficients.*

Proof. The proof of [CCW24, Proposition 6.9] is transferred to our context with minor adaptations. For the reader's convenience, we sketch the argument below. It suffices to prove that, for a given $Q \subseteq I$, the series $P_{I,J,Q}(\mathbf{q}^{-s})$ produces a rational function in $\mathbf{q}^{-s}$. By [Edg09, Section 5], it suffices to prove that $P_{I,J,Q}$ belongs to the Boolean subalgebra $\mathcal{B}$ of the power set of W generated by the following sets, for all $x, y \in W$ and $m \in \mathbb{Z}$:

$$W_{x,y,m} = \{w \in W \mid \ell(ywx) = \ell(y) + \ell(w) + \ell(x) - 2m\},$$

where ℓ denotes the length function on (W, S). To this end, for every $w \in R(I,J)$, note that

$$I \cap wJw^{-1} = Q \iff Q \subseteq wJw^{-1} \text{ and } (I \setminus Q) \cap wJw^{-1} = \emptyset.$$

Hence,

$$P_{I,J,Q} = \{w \in R(I,J) \mid w^{-1}Qw \subseteq J\} \cap \{w \in R(I,J) \mid w^{-1}(I \setminus Q)w \cap J = \emptyset\}.$$

The statement follows by observing that each of the following sets belongs to $\mathcal{B}$:

$$\begin{aligned}
&R(I,J) = \bigcap_{j\in J} W_{j,1,0} \cap \bigcap_{i\in I} W_{1,i,0};\\
&\{w \in R(I,J) \mid w^{-1}iw = j\} = W_{j,i,1} \cap R(I,J), \quad \forall\, i \in I,\, j \in J;\\
&\{w \in R(I,J) \mid w^{-1}iw \in J\} = \bigsqcup_{j\in J} \{w \in R(I,J) \mid w^{-1}iw = j\}, \quad \forall\, i \in I;\\
&\{w \in R(I,J) \mid w^{-1}iw \notin J\} = R(I,J) \setminus \{w \in R(I,J) \mid w^{-1}iw \in J\}, \quad \forall\, i \in I;\\
&\{w \in R(I,J) \mid w^{-1}Qw \subseteq J\} = \bigcap_{i\in Q} \{w \in R(I,J) \mid w^{-1}iw \in J\};\\
&\{w \in R(I,J) \mid w^{-1}(I\setminus Q)w \cap J = \emptyset\} = \bigcap_{i\in I\setminus Q} \{w \in R(I,J) \mid w^{-1}iw \notin J\}.
\end{aligned} \tag{3.4.5}$$

Once shown that the equalities in the first two lines in (3.4.5) hold, the subsequent ones follow immediately. For the first one, recall from [Dav08, Lemma 4.3.1] the following: given $w \in R(I,J)$, every element $v \in W_I w W_J$ can be written in a unique way as $v = xwy$, where $x \in W_I$ and $y \in W_I$, and $\ell(ywx) = \ell(y) + \ell(w) + \ell(x)$. We claim that, for $w \in W$, one has

$$\ell(iw) = \ell(i) + \ell(w) \text{ and } \ell(wj) = \ell(w) + \ell(j),\ \forall\, i \in I, j \in J \iff w \in R(I,J).$$

The implication ($\Leftarrow$) is clear from the lemma recalled before. For the converse, let $w \in W$ satisfy $\ell(wj) = \ell(w)+\ell(j) = \ell(w)+1$ and $\ell(iw) = \ell(i)+\ell(w) = 1+\ell(w)$ for all $i \in I, j \in J$. Suppose by contradiction that $w \notin R(I,J)$, i.e., that there are $x \in W_I$, $w' \in R(I,J)$ and $y \in W_J$ such that $w' \neq w$, $w = xw'y$ and $\ell(w) = \ell(x) + \ell(w') + \ell(y)$. Since $x \in W_I$, there is $i \in I$ such that $\ell(ix) = \ell(x) - 1$. Applying again [Dav08, Lemma 4.3.1], we have

$$\ell(iw) = \ell((ix)w'y) = \ell(ix) + \ell(w') + \ell(y) = \ell(x) + \ell(w') + \ell(y) - 1 = \ell(w) - 1,$$

impossible. We conclude that $w \in R(I,J)$ and the first equality in (3.4.5) is proved. For the second one, observe from [Dav08, Lemma 4.3.1] that, for $w \in R(I,J)$, there is a unique element in $W_I w W_J$ of minimal length: w itself. Hence, given $w \in R(I,J)$ and for all $i \in I$, $j \in J$, we deduce that

$$\ell(iwj) = \ell(w) \iff iwj = w \iff w^{-1}iw = j.$$

This implies that also the second equality of (3.4.5) holds. □

Splitting formulae

Lemma 3.4.8. *Assume the hypotheses of Proposition 3.4.5, and suppose that* $(W,S) = (W_1 \times W_2, S_1 \sqcup S_2)$ *for some* $S_1, S_2 \subseteq S$. *Then,*

$$\zeta_{G,B,B}(s) = \zeta_{P_{S_1},B,B}(s) \cdot \zeta_{P_{S_2},B,B}(s).$$

Lemma 3.4.9. *Let (W,S) be a Coxeter group and $I,J \subseteq S$ be spherical subsets. Then,*

$$W(\mathbf{t}) = W_J(\mathbf{t}) \cdot \sum_{Q\subseteq I} \frac{W_I(\mathbf{t})}{W_Q(\mathbf{t})} P_{I,J,Q}(\mathbf{t}).$$

Proof. Given $w \in R(I,J)$, let $M_I(J,w)$ be the set of all minimal length representatives of $W_I/W_{I\cap wJw^{-1}}$. By [Dav08, Lemma 17.1.2] and Remark 1.4.1, note that

$$W_I(\mathbf{t}) = \big(M_I(J,w)\big)(\mathbf{t}) \cdot W_{I\cap wJw^{-1}}(\mathbf{t}). \tag{3.4.6}$$

By [APV17, Lemma 1.1], for every $v \in W$ there are uniquely determined $w \in R(I,J)$, $x \in M_I(J,w)$ and $y \in W_J$ such that $v = xwy$ and $\ell(v) = \ell(x) + \ell(w) + \ell(y)$ (here ℓ denotes the word length of (W,S)). Hence, if $x = s_1 \cdots s_l$, $w = t_1 \cdots t_m$ and $y = u_1 \cdots u_n$ are reduced expressions in (W,S), the fact that $\ell(v) = l+m+n$ implies that $v = s_1 \cdots s_l t_1 \cdots t_m u_1 \cdots u_n$ is a reduced expression in (W,S). In particular,

$$\mathbf{t}_v = \mathbf{t}_x \cdot \mathbf{t}_w \cdot \mathbf{t}_y.$$

Therefore,

$$\begin{aligned} W(\mathbf{t}) &= \sum_{w\in R(I,J)} \left(\sum_{x\in M_I(J,w),\, y\in W_J} \mathbf{t}_x \mathbf{t}_y \right) \mathbf{t}_w = \sum_{w\in R(I,J)} M_I(J,w)(\mathbf{t}) \cdot W_J(\mathbf{t})\mathbf{t}_w = \\ &\overset{(3.4.6)}{=} W_J(\mathbf{t}) \cdot \sum_{w\in R(I,J)} \frac{W_I(\mathbf{t})}{W_{I\cap wJw^{-1}}(\mathbf{t})} \mathbf{t}_w = W_J(\mathbf{t}) \cdot \sum_{Q\subseteq I} \left(\frac{W_I(\mathbf{t})}{W_Q(\mathbf{t})} \cdot \sum_{\substack{w\in R(I,J):\\ I\cap wJw^{-1}=Q}} \mathbf{t}_w \right), \end{aligned} \tag{3.4.7}$$

and the statement follows. ☐

Corollary 3.4.10. *Assume the hypotheses of Proposition 3.4.5 and let $I,J \subseteq S$ be spherical subsets. Then,*

$$\zeta_{G,B,B}(-1) = \zeta_{G,P_I,P_J}(-1) \cdot \zeta_{P_J,B,B}(-1). \tag{3.4.8}$$

Proof. By Corollary 3.4.6, (3.4.4) and Lemma 3.4.9, we have

$$\zeta_{G,B,B}(-1) = W_J(\mathbf{q}) \cdot \sum_{Q\subseteq I} \frac{W_I(\mathbf{q})}{W_Q(\mathbf{q})} P_{I,J,Q}(\mathbf{q}) = \zeta_{P_J,B,B}(-1) \cdot \zeta_{G,P_I,P_J}(-1).$$ ☐

Thanks to Corollary 3.4.6, we rephrase well-known results due to M. Davis as follows.

Theorem 3.4.11 ([Dav08, Theorems 17.1.9 and 17.1.10])**.** *In the hypotheses of Proposition 3.4.5, we have*

$$\zeta_{G,B,B}(s)^{-1} = \sum_{\substack{T\subseteq S\\ T\text{ spherical}}} (-1)^{|T|} \zeta_{P_T,B,B}(-s)^{-1} = \sum_{\substack{T\subseteq S,\\ T\text{ spherical}}} (1-\chi(L_T))\zeta_{P_T,B,B}(s)^{-1},$$

where

$$\chi(L_T) = (-1)^{|T|+1} \cdot \sum_{\substack{T \subsetneq U \subseteq S, \\ U \text{ spherical}}} (-1)^{|U|}.$$

According to [Dav08, Equation (17.10)], the quantity $\chi(L_T)$ is the Euler–Poincaré characteristic of the link L_T of the simplex corresponding to T in the Davis' complex associated to the standard Coxeter complex $\Sigma(W, S)$.

In the case of uniform thickness, there are explicit formulae for $\zeta_{P_T, P_\emptyset, P_\emptyset}(-s)^{-1} = W_T(\mathbf{q}^{-s})$, for $T \subseteq S$ spherical (cf. (1.4.2) and [BB06, Appendix A.1]). In this case, Theorem 3.4.11 provides an explicit formula for $\zeta_{G,P_I,P_J}(s)$.

Example 3.4.12. Let G be a group having a Bruhat decomposition $G = BWB$ of affine type (W, S) and uniform thickness $q > 1$. Set $|S| = n \geq 2$. Then, by Bott's formula (cf. (1.4.3)), we have that

$$\zeta_{G,B,B}(s) = \prod_{k=1}^{n-1} \frac{1 - q^{-s(e_k+1)}}{(1 - q^{-s})(1 - q^{-s \cdot e_k})},$$

where $e_1, \ldots, e_n$ are the exponents of the spherical Coxeter group associated to (W, S) (cf. [BB06, Appendix A.1]).

For instance, let K be a non-Archimedean local field with ring of integers O_K and set $n \geq 2$. Consider $G = \mathrm{SL}_n(K)$ and let B be the standard Iwahori subgroup of G (that is, the subgroup of $\mathrm{SL}_n(O_K)$ whose entries strictly below the diagonal belong to the maximal ideal of O_K). Then G admits a Bruhat decomposition of type $\tilde{A}_{n-1}$ with respect to B. Since the exponents of a Coxeter group of type A_{n-1} are $1, 2, \ldots, n-1$, we deduce that

$$\zeta_{\mathrm{SL}_n(K),B,B}(s) = \prod_{k=1}^{n-1} \frac{1 - q^{s(k+1)}}{(1 - q^{-s})(1 - q^{-sk})}.$$

3.4.2 The case of groups acting on trees

The case of groups acting weakly locally ∞-transitively on trees

Setting [WLIT]. Let (G, T) be a weakly locally ∞-transitive group action on a locally finite tree with quotient map $\pi \colon T \to \Gamma = G\backslash T$. Assume that Γ is finite, and that its standard edge weight ω takes values in $\mathbb{Z}_{\geq 2}$ and satisfies $\omega(a) \geq 3$ or $\omega(\bar{a}) \geq 3$ for every $a \in E\Gamma$. Let also $N_{\text{edg}} = N^{\omega}_{\text{edg}}$ and $N_{\text{vert}} = N^{\omega}_{\text{vert}}$ be as in Definition 3.2.8.

Setting [WLIT] guarantees that the series defining $\zeta_{G,G_{t_1},G_{t_2}}(s)$ converges at some $s \in \mathbb{C}$, for all $t_1, t_2 \in T$ (cf. Theorem 3.3.21).

Proposition 3.4.13. *Suppose Setting [WLIT], and let $t \in T$ with $\pi(t) = u$. Then, for every $v \in VT$ with $\pi(v) = c$, we have*

$$\zeta_{G,G_v,G_t}(s) = \sum_{\mathfrak{p} \in \mathcal{P}_\Gamma(c \to U)} N_{\mathrm{vert}}(\mathfrak{p})^{-s}.$$

Moreover, for every $e \in ET$ with $\pi(e) = a$, we have

$$\zeta_{G,G_e,G_t}(s) = \varepsilon_a(u) + \sum_{\substack{\mathfrak{p} \in \mathcal{P}_\Gamma(A \to U),\\ \ell(\mathfrak{p}) \geq 2}} N_{\mathrm{edg}}(\mathfrak{p})^{-s},$$

where $\varepsilon_a(u) = \mathbb{1}_{\{o(a),t(a)\}}(u)$ if $u \in V\Gamma$ and $\varepsilon_a(u) = \mathbb{1}_A(u)$ if $u \in E\Gamma$.

Proof. It is a direct consequence of Proposition 3.2.7 (recalling Remark 3.2.11) and Proposition 3.2.12. □

Proposition 3.4.13 suggests the following generalisation.

Definition 3.4.14. Let Γ be a non-empty graph with an edge weight $\omega\colon E\Gamma \to \mathbb{Z}_{\geq 2}$, and let $c \in V\Gamma$, $a \in E\Gamma$ and $u \in \Gamma$. Define the following formal Dirichlet series:

$$\mathcal{Z}_{\Gamma,c\to u}(s) := \sum_{\mathfrak{p} \in \mathcal{P}_\Gamma(c \to U)} N_{\mathrm{vert}}(\mathfrak{p})^{-s},$$

$$\mathcal{Z}_{\Gamma,a\to u}(s) := \varepsilon_a(u) + \sum_{\substack{\mathfrak{p} \in \mathcal{P}_\Gamma(A \to u),\\ \ell(\mathfrak{p}) \geq 2}} N_{\mathrm{edg}}(\mathfrak{p})^{-s},$$

where $\varepsilon_a(u) = \mathbb{1}_{\{o(a),t(a)\}}(u)$ if $u \in V\Gamma$ and $\varepsilon_a(u) = \mathbb{1}_A(u)$ if $u \in E\Gamma$.

Remark 3.4.15. In Definition 3.4.14, we may assume that Γ is connected. Indeed, given $u_1, u_2 \in \Gamma$, if there is a connected component Λ of Γ containing both u_1 ad u_2, then $\mathcal{Z}_{\Gamma,u_1\to u_2}(s) = \mathcal{Z}_{\Lambda,u_1\to u_2}(s)$. If such a connected component does not exist, the function $\mathcal{Z}_{\Gamma,u_1\to u_2}(s)$ is identically zero.

Remark 3.4.16. By Proposition 3.4.13, for all $t_1, t_2 \in T$ with $\pi(t_1) = u_1$ and $\pi(t_2) = u_2$ we have

$$\zeta_{G,G_{t_1},G_{t_2}}(s) = \mathcal{Z}_{\Gamma,u_1\to u_2}(s).$$

In view of an explicit formula for $\mathcal{Z}_{\Gamma,u_1\to u_2}(s)$, we introduce the following linear operator.

Definition 3.4.17. Let Γ be a non-empty graph with an edge weight $\omega\colon E\Gamma \to \mathbb{Z}_{\geq 2}$. Let $\mathbb{C}[\![E\Gamma]\!]$ be the complex vector space of all formal sums $\sum_{a\in E\Gamma} \gamma_a a$, where $\gamma_a \in \mathbb{C}$ for every $a \in E\Gamma$. For each $u \in \Gamma$, define $e_u \in \mathbb{C}[\![E\Gamma]\!]$ as follows:

$$e_u := \begin{cases} u, & \text{if } u \in E\Gamma; \\ \sum_{a \in t^{-1}(u)} a, & \text{if } u \in V\Gamma. \end{cases} \tag{3.4.9}$$

For every $s \in \mathbb{C}$, the *Bass operator* $\mathcal{E}(s) = \mathcal{E}^{(\Gamma,\omega)}(s) \colon \mathbb{C}[\![E\Gamma]\!] \to \mathbb{C}[\![E\Gamma]\!]$ *of* Γ *at* $s \in \mathbb{C}$ is the linear extension of the following assignment:

$$\mathcal{E}(s)(a) := \sum_{b \in E\Gamma} \mathcal{E}(s)(a,b)b, \quad \forall\, a \in E\Gamma \tag{3.4.10}$$

where, for all $a, b \in E\Gamma$,

$$\mathcal{E}(s)(a,b) := \begin{cases} N_{\text{edg}}(a,b)^{-s}, & \text{if } t(a) = o(b); \\ 0, & \text{otherwise.} \end{cases} \tag{3.4.11}$$

Notation 3.4.18.

(i) We will usually write $\mathcal{E}(s)$ instead of $\mathcal{E}^{(\Gamma,\omega)}(s)$. If we want to specify Γ (but ω is clear from the context), we write $\mathcal{E}^{\Gamma}(s)$ instead of $\mathcal{E}(s)$ or $\mathcal{E}^{(\Gamma,\omega)}(s)$, and e_u^{Γ} instead of e_u, for all $u \in \Gamma$.

(ii) We implicitly set a total order on $E\Gamma$. Thus, provided $|E\Gamma| < \infty$, we can regard $\mathcal{E}(s)$ as a $|E\Gamma|$-dimensional matrix $[\mathcal{E}(s)(a,b)]_{a,b\in E\Gamma}$ with complex entries, and the e_u's in (3.4.9) as row vectors in $\mathbb{C}^{|E\Gamma|}$. For all $a, b \in E\Gamma$, note that $e_a \mathcal{E}(s) e_b^t = \mathcal{E}(s)(a,b)$.

The term "Bass operator" is taken after [Dei19, Definition 3.10]. The reader is referred to Section 3.5.3 for further connections with [Dei19].

Remark 3.4.19. Let $\mathcal{E}(s)$ be as in Definition 3.4.17. For every $n \geq 1$, let $\mathcal{E}(s)^n$ be the n-th power of $\mathcal{E}(s)$, and $\mathcal{E}(s)^0$ be the identity operator on $\mathbb{C}[\![E\Gamma]\!]$. Then, for all $n \geq 0$ and $a, b \in E\Gamma$, we observe that

$$\mathcal{E}(s)^n(a,b) = \sum_{\substack{\mathfrak{p} \in \mathcal{P}_{\Gamma}(a \to b) \\ \ell(\mathfrak{p}) = n+1}} N_{\text{edg}}(\mathfrak{p})^{-s}.$$

If $n \leq 1$, it is clear. For every $n \geq 2$, it suffices to observe that

$$\mathcal{E}(s)^n(a,b) = \sum_{a_2,\dots,\, a_n \in E\Gamma} \mathcal{E}(s)(a,a_2) \cdot \ldots \cdot \mathcal{E}(s)(a_n,b).$$

As we did in Setting [WLIT], we fix a setting which ensures that the series $\mathcal{Z}_{\Gamma,u_1\to u_2}(s)$ as in Definition 3.4.14 converges at some $s \in \mathbb{C}$, for all $u_1, u_2 \in \Gamma$.

Setting $[\Gamma]$**.** Let Γ be a finite connected non-empty graph with an edge weight $\omega \colon E\Gamma \to \mathbb{Z}_{\geq 2}$ satisfying $\omega(a) \geq 3$ or $\omega(\bar{a}) \geq 3$, for every $a \in E\Gamma$.
Convention: Every subgraph $\Gamma' \subseteq \Gamma$ is endowed "by default" with the restricted edge weight from ω.

Theorem 3.4.20. *Let (Γ,ω) be an edge-weighted graph satisfying Setting [Γ]. Then, for all $u,w\in\Gamma$,*

$$\mathcal{Z}_{\Gamma,u\to w}(s)=\frac{\det(I-\mathcal{E}(s)+\mathcal{U}_{u,w}(s))}{\det(I-\mathcal{E}(s))}+\epsilon_u(w), \tag{3.4.12}$$

where I be the identity matrix in $\mathrm{Mat}_{|E\Gamma|}(\mathbb{C})$ and

$$\mathcal{U}_{u,w}(s):=\begin{cases}\sum_{a\in o^{-1}(u)}\omega(a)^{-s}e_w^t\cdot e_a, & \text{if } u,w\in V\Gamma;\\ \sum_{a\in o^{-1}(u)}\omega(a)^{-s}(e_w+e_{\bar{w}})^t\cdot e_a\mathcal{E}(s), & \text{if } u\in V\Gamma,\ w\in E\Gamma;\\ e_w^t\cdot(e_u+e_{\bar{u}})\mathcal{E}(s), & \text{if } u\in E\Gamma,\ w\in V\Gamma;\\ (e_w+e_{\bar{w}})^t\cdot(e_u+e_{\bar{u}})\mathcal{E}(s), & \text{if } u,w\in E\Gamma;\end{cases}$$

$$\epsilon_u(w):=\begin{cases}\mathbb{1}_{\{u\}}(w)-1, & \text{if } u,w\in V\Gamma;\\ \mathbb{1}_{o^{-1}(u)}(w)\cdot\omega(w)-1, & \text{if } u\in V\Gamma,\ w\in E\Gamma;\\ \mathbb{1}_{\{o(u),t(u)\}}(w)-1, & \text{if } u\in E\Gamma,\ w\in V\Gamma;\\ \mathbb{1}_{\{u,\bar{u}\}}(w)-1, & \text{if } u,w\in E\Gamma.\end{cases}$$

In particular, for all $u,w\in\Gamma$ the function $\mathcal{Z}_{\Gamma,u\to w}(s)$ is a meromorphic function over $\mathbb{C}$.

The proof of Theorem 3.4.20 makes use of the following fact.

Fact 3.4.21 (Matrix Determinant Lemma, cf. [Har97])**.** *Consider $A\in\mathrm{GL}_n(\mathbb{C})$ with adjugate matrix $\mathrm{adj}(A)$, and let $u,v\in\mathbb{C}^n$ be row vectors. Then,*

$$\frac{\det(A+u^t\cdot v)}{\det(A)}=1+vA^{-1}u^t.$$

Proof of Theorem 3.4.20. Let $s\in\mathbb{C}$ be such that $\sum_{n=0}^{\infty}\mathcal{E}(s)$ converges. Recall that

$$\sum_{n=0}^{\infty}\mathcal{E}(s)=(I-\mathcal{E}(s))^{-1}.$$

By Remark 3.4.19, if $u\in V\Gamma$ then

$$\mathcal{Z}_{\Gamma,u\to w}(s)=$$
$$=\begin{cases}\mathbb{1}_{\{u\}}(w)+\displaystyle\sum_{n=0}^{\infty}\sum_{a\in o^{-1}(u)}\omega(a)^{-s}e_a\mathcal{E}(s)^n e_w^t, & \text{if } w\in V\Gamma;\\ \mathbb{1}_{o^{-1}(u)}(w)\cdot\omega(w)^{-s}+\displaystyle\sum_{n=1}^{\infty}\sum_{a\in o^{-1}(u)}\omega(a)^{-s}e_a\mathcal{E}(s)^n(e_w+e_{\bar{w}})^t, & \text{if } w\in E\Gamma.\end{cases}$$

Similarly, if $u\in E\Gamma$ then

$$\mathcal{Z}_{\Gamma,u\to w}(s)=\begin{cases}\mathbb{1}_{\{o(u),t(u)\}}(w)+\displaystyle\sum_{n=1}^{\infty}(e_u+e_{\bar{u}})\mathcal{E}(s)^n e_w^t, & \text{if } w\in V\Gamma;\\ \mathbb{1}_{\{u,\bar{u}\}}(w)+\displaystyle\sum_{n=1}^{\infty}(e_u+e_{\bar{u}})\mathcal{E}(s)^n(e_w^t+e_{\bar{w}}^t), & \text{if } w\in E\Gamma.\end{cases}$$

We now focus on the case in which $u, w \in V\Gamma$, as the other cases are analogous. Namely, if $u, w \in V\Gamma$ then

$$\begin{aligned}\mathcal{Z}_{\Gamma,u\to w}(s) &= \mathbb{1}_{\{u\}}(w) + \sum_{a\in o^{-1}(u)} \omega(a)^{-s} e_a \left(\sum_{n=0}^{\infty} \mathcal{E}(s)^n\right) e_w^t \\ &= \mathbb{1}_{\{u\}}(w) + \left(\sum_{a\in o^{-1}(u)} \omega(a)^{-s} e_a\right)(I - \mathcal{E}(s))^{-1} e_w^t\end{aligned}$$

and Fact 3.4.21 applies. □

In view of Section 3.5.2, we provide some explicit formulae for $\mathcal{Z}_{\Gamma,u\to u}(s)$ in case that Γ has one edge-pair.

Example 3.4.22. Let Γ be a 1-segment with $E\Gamma = \{a, \bar{a}\}$, $c = o(a)$ and $d = t(a)$. Set $\omega(a) := \alpha + 1$ and $\omega(\bar{a}) := \beta + 1$, where $\alpha, \beta \in \mathbb{Z}_{\geq 1}$ with $\alpha \geq 2$ or $\beta \geq 2$. Set the order $\leq$ on $E\Gamma$ such that $a < \bar{a}$, and identify $\mathbb{C}[\![E\Gamma]\!]$ with $\mathbb{C}^2$, $e_a = e_d$ with $(1,0)$ and $e_{\bar{a}} = e_c = (0,1)$. Then, for every $s \in \mathbb{C}$,

$$\begin{aligned}\mathcal{E}(s) &= \begin{bmatrix}\mathcal{E}(s)(a,a) & \mathcal{E}(s)(a,\bar{a})\\ \mathcal{E}(s)(\bar{a},a) & \mathcal{E}(s)(\bar{a},\bar{a})\end{bmatrix} = \begin{bmatrix}0 & \beta^{-s}\\ \alpha^{-s} & 0\end{bmatrix};\\ \mathcal{U}_{c,c}(s) &= (\alpha+1)^{-s} e_c^t \cdot e_a = \begin{bmatrix}0 & 0\\ (\alpha+1)^{-s} & 0\end{bmatrix};\\ \mathcal{U}_{a,a}(s) &= (e_a + e_{\bar{a}})^t \cdot (e_a + e_{\bar{a}}) \cdot \mathcal{E}(s) = \begin{bmatrix}\alpha^{-s} & \beta^{-s}\\ \alpha^{-s} & \beta^{-s}\end{bmatrix}.\end{aligned}$$

Let I be the identity matrix in $\mathrm{Mat}_2(\mathbb{C})$. By Theorem 3.4.20,

$$\begin{aligned}\mathcal{Z}_{\Gamma,c\to c}(s) &= \frac{1 + ((\alpha+1)^{-s} - \alpha^{-s}) \cdot \beta^{-s}}{1 - \alpha^{-s}\beta^{-s}};\\ \mathcal{Z}_{\Gamma,a\to a}(s) &= \frac{(1+\alpha^{-s})(1+\beta^{-s})}{1-\alpha^{-s}\beta^{-s}}.\end{aligned} \tag{3.4.13}$$

In particular, let (G, T) be a locally ∞-transitive action on a locally finite tree with quotient graph Γ and standard edge weight ω as before. By Remark 3.4.16 and (3.4.13), we have explicit formulae for $\zeta_{G,G_v,G_v}(s)$ and $\zeta_{G,G_e,G_e}(s)$ for all $v \in VT$ with $G \cdot v = c$ and $e \in ET$ with $G \cdot e = a$. For instance, one may take $G = \mathrm{SL}_2(\mathbb{Q}_p)$ and T the Bruhat–Tits tree of G (cf. Example 1.3.7(ii)). In this case $\alpha = \beta = p$. Let $v \in VT$ be the vertex with $G_v = \mathrm{SL}_2(\mathbb{Z}_p)$, and $e \in ET$ be the edge whose pointwise stabiliser is the standard Iwahori subgroup. Then,

$$\zeta_{G,G_v,G_v}(s) = \frac{1 + ((p+1)^{-s} - p^{-s}) \cdot p^{-s}}{1 - p^{-2s}} \quad \text{and} \quad \zeta_{G,G_e,G_e}(s) = \frac{1+p^{-s}}{1-p^{-s}}.$$

Other examples can be obtained from Example 1.3.22. Note that the formulae before agree with [CCW24, Example 1.7] in case that G is the group of automorphisms of a bi-coloured tree T.

Example 3.4.23. Let Γ be a 1-bouquet of loops with $E\Gamma = \{a, \bar{a}\}$ and $c = o(a) = t(a)$. Set $\omega(a) := \alpha + 1$ and $\omega(\bar{a}) := \beta + 1$, for some $\alpha, \beta \in \mathbb{Z}_{\geq 1}$ with $\alpha \geq 2$ or $\beta \geq 2$. Consider the order $\leq$ on $E\Gamma$ such that $a < \bar{a}$, and identify $\mathbb{C}[\![E\Gamma]\!]$ with $\mathbb{C}^2$, e_a with the vector $(1, 0)$, $e_{\bar{a}}$ with $(0, 1)$ and $e_c = e_a + e_{\bar{a}}$ with $(1, 1)$. Then, for every $s \in \mathbb{C}$,

$$\mathcal{E}(s) = \begin{bmatrix} \mathcal{E}(s)(a,a) & \mathcal{E}(s)(a,\bar{a}) \\ \mathcal{E}(s)(\bar{a},a) & \mathcal{E}(s)(\bar{a},\bar{a}) \end{bmatrix} = \begin{bmatrix} (\alpha+1)^{-s} & \beta^{-s} \\ \alpha^{-s} & (\beta+1)^{-s} \end{bmatrix};$$

$$\mathcal{U}_{c,c}(s) = (\alpha+1)^{-s} e_c^t \cdot e_a + (\beta+1)^{-s} e_c^t \cdot e_{\bar{a}} = \begin{bmatrix} (\alpha+1)^{-s} & (\beta+1)^{-s} \\ (\alpha+1)^{-s} & (\beta+1)^{-s} \end{bmatrix};$$

$$\mathcal{U}_{a,a}(s) = (e_a + e_{\bar{a}})^t \cdot (e_a + e_{\bar{a}})\mathcal{E}(s) = \begin{bmatrix} (\alpha+1)^{-s} + \alpha^{-s} & (\beta+1)^{-s} + \beta^{-s} \\ (\alpha+1)^{-s} + \alpha^{-s} & (\beta+1)^{-s} + \beta^{-s} \end{bmatrix}.$$

Let I be the identity matrix in $\mathrm{Mat}_2(\mathbb{C})$. By Theorem 3.4.20, we have

$$\begin{aligned} \mathcal{Z}_{\Gamma,c\to c}(s) &= \frac{1 - \Big((\alpha+1)^{-s} - \alpha^{-s}\Big) \cdot \Big((\beta+1)^{-s} - \beta^{-s}\Big)}{\Big(1 - (\alpha+1)^{-s}\Big) \cdot \Big(1 - (\beta+1)^{-s}\Big) - \alpha^{-s}\beta^{-s}}; \\ \mathcal{Z}_{\Gamma,a\to a}(s) &= \frac{(\alpha^{-s}+1)(\beta^{-s}+1) - (\alpha+1)^{-s}(\beta+1)^{-s}}{\Big(1 - (\alpha+1)^{-s}\Big) \cdot \Big(1 - (\beta+1)^{-s}\Big) - \alpha^{-s}\beta^{-s}}. \end{aligned} \tag{3.4.14}$$

If $\alpha = \beta$, after basic algebraic manipulations, the formulae in (3.4.14) become

$$\mathcal{Z}_{\Gamma,c\to c}(s) = \frac{1 - \alpha^{-s} + (\alpha+1)^{-s}}{1 - \alpha^{-s} - (\alpha+1)^{-s}} \quad \text{and} \quad \mathcal{Z}_{\Gamma,a\to a}(s) = \frac{1 + \alpha^{-s} + (\alpha+1)^{-s}}{1 - \alpha^{-s} - (\alpha+1)^{-s}}. \tag{3.4.15}$$

Consider a weakly locally ∞-transitive group action on a locally finite tree (G, T) with quotient graph Γ and standard edge weight ω. For explicit examples, see Example 1.3.22(i). By Remark 3.4.16, the computations in (3.4.14) provide explicit formulae for $\zeta_{G,G_v,G_v}(s)$ and $\zeta_{G,G_e,G_e}(s)$ whenever $v \in VT$ and $e \in ET$ satisfy $G \cdot v = c$ and $G \cdot e = a$ (or $G \cdot e = \bar{a}$), respectively.

The case of (P)-closed group actions on trees

Setting [(P)-cl]. Let $\Delta = (\Gamma, (X_a), (G(c)))$ be a local action diagram based on a non-empty finite connected graph Γ. Choose an inversion ι in Δ and $c_0 \in V\Gamma$. Denote by $(T = T(\Delta, \iota, c_0), \pi, \mathcal{L})$ the standard Δ-tree associated to ι and c_0, let v_0 be the root of T (cf. Section 1.3.4), and set $G = U(\Delta, \iota, c_0)$. Assume that the standard edge weight ω on Γ and the standard weight $\mathcal{W}$ on Δ take values in $\mathbb{Z}_{\geq 2}$ and $\mathbb{Z}_{\geq 0}$, respectively. Finally, assume that (G, T) has property $(*_k)$ for some $k \geq 1$.

Setting [(P)-cl] guarantees that the series defining $\zeta_{G,G_{t_1},G_{t_2}}(s)$ converges at some $s \in \mathbb{C}$, for all $t_1, t_2 \in T$ (cf. Theorem 3.3.22). Thanks to the following remark, from now on we can focus only on the case where $t_1 \in \{v_0\} \cup o^{-1}(v_0)$ while studying $\zeta_{G,G_{t_1},G_{t_2}}(s)$.

Remark 3.4.24. Let Δ be a local action diagram and $\mathbb{T} = (T, \pi, \mathcal{L})$ be Δ-tree. Consider an inversion ι in Δ and denote by $(T(\Delta, \iota, c_0), \pi_0, \mathcal{L}_0)$ the standard Δ-tree associated to ι and some $c_0 \in V\Gamma$. By Theorem 1.3.17(i), there is a graph isomorphism $\phi\colon T \to T(\Delta, \iota, c_0)$ such that $\pi = \pi_0 \circ \phi$ and $U(\Delta, \iota, c_0) = \phi U(\Delta, \mathbb{T})\phi^{-1}$. Set $G = U(\Delta, \mathbb{T})$ and $H = U(\Delta, \iota, c_0)$. Then, for all $t \in T$, the following map is bijective:

$$G/G_t \longrightarrow H/H_{\phi(t)}, \quad gG_t \longmapsto \phi g \phi^{-1} H_{\phi(t)}.$$

Moreover, for all $t_1, t_2 \in T$ and $g \in G$, provided $h := \phi g \phi^{-1}$ we have

$$|G_{t_1} : G_{t_1} \cap gG_{t_2}g^{-1}| = |H_{\phi(t_1)} : H_{\phi(t_1)} \cap hH_{\phi(t_2)}h^{-1}|$$

and then

$$\zeta_{G,G_{t_1},G_{t_2}}(s) = \zeta_{H,H_{\phi(t_1)},H_{\phi(t_2)}}(s),$$

when the series before are defined. In particular, by Theorem 1.3.17(i), given $v \in VT$ with $\pi(v) =: c_0$ one may take ϕ so that $\phi(v)$ is the root v_0 of $T(\Delta, \iota, c_0)$. Then, for all $t \in T$,

$$\zeta_{G,G_v,G_t}(s) = \zeta_{H,H_{v_0},H_{\phi(t)}}(s).$$

Moreover, for all $e \in o^{-1}(v)$, we have $\phi(e) \in o^{-1}(v_0)$ and then, for all $t \in T$,

$$\zeta_{G,G_e,G_t}(s) = \zeta_{H,H_{\phi(e)},H_{\phi(t)}}(s).$$

The analogue of Proposition 3.4.13 for (P)-closed actions is the following.

Proposition 3.4.25. *Let (G, T) be as in Setting [(P)-cl], and let $t \in T$ with $\pi(t) = u$. Then*

$$\zeta_{G,G_{v_0},G_t}(s) = \mathbb{1}_{\{c_0\}}(u) + \sum_{\substack{a \in o^{-1}(c_0),\\ \xi \in \mathcal{P}_{(\Delta,\iota)}(X_a \to X_U)}} \omega(a)^{-s-1} \mathcal{W}(\xi)^{-s-1}. \tag{3.4.16}$$

Moreover, for every $e \in ET$ with $\pi(e) = a$ and $\mathcal{L}(e) = x \in X_{c_0}$, we have

$$\zeta_{G,G_e,G_t}(s) = \eta_a(u) + \sum_{\substack{\xi \in \mathcal{P}_{(\Delta,\iota)}(x \to X_U),\\ \ell(\xi) \geq 2}} \mathcal{W}(\xi)^{-s-1} + \sum_{\substack{y \in X_{c_0} \setminus \{x\},\\ \xi \in \mathcal{P}_{(\Delta,\iota)}(y \to X_U),\, \ell(\xi) \geq 1}} \mathcal{W}_{\text{rev}}(x, y)^{-s-1} \mathcal{W}(\xi)^{-s-1}, \tag{3.4.17}$$

where $\eta_a(u) = \mathbb{1}_{\{o(a), t(a)\}}(u)$ if $u \in V\Gamma$ and $\eta_a(u) = \mathbb{1}_A(u)$ if $u \in E\Gamma$.

Proof. The statement follows from (3.4.2), Proposition 3.2.13, Proposition 3.2.19 and Corollary 3.2.20. □

As in Section 3.4.2, we introduce a linear operator to express the series defining $\zeta_{G,G_{t_1},G_{t_2}}(s)$ within Setting [(P)-cl].

Definition 3.4.26. Let $\Delta = (\Gamma, (X_a), (G(c)))$ be a local action diagram together with a function $\mathcal{W}\colon X \times X \to \mathbb{Z}_{\geq 0}$ (recall that $X = \bigsqcup_{a \in E\Gamma} X_a$). Let $\mathbb{C}[\![X]\!]$ be the complex vector space of all formal sums $\sum_{x \in X} \gamma_x x$, where $\gamma_x \in \mathbb{C}$ for every $x \in X$. For every non-empty set $S \subseteq X$, set

$$f_S := \sum_{x \in S} x \in \mathbb{C}[\![X]\!].$$

Given $s \in \mathbb{C}$ and for all $x \in X_a, y \in X_b$ with $a, b \in E\Gamma$, define

$$\mathcal{F}(s)(x,y) := \begin{cases} \mathcal{W}(x,y)^{-s}, & \text{if } t(a) = o(b) \text{ and } y \neq \iota(x); \\ 0, & \text{otherwise.} \end{cases}$$

The *Bass operator* $\mathcal{F}(s)\colon \mathbb{C}[\![X]\!] \longrightarrow \mathbb{C}[\![X]\!]$ *of* $(\Delta, \mathcal{W})$ *at* $s \in \mathbb{C}$ is defined by linearly extending the following assignment, for all $x \in X$:

$$\mathcal{F}(x) := \sum_{y \in X} \mathcal{F}(s)(x,y) y.$$

Notation 3.4.27. Technically, $\mathcal{F}(s)$ depends on Δ and $\mathcal{W}$. In our case, since Δ and $\mathcal{W}$ will be always clear from the context (in particular, $\mathcal{W}$ will be always the standard weight on Δ), we avoid underlying this dependence.

In what follows, we implicitly fix a total order on X. In this way, we can regard $\mathcal{F}(s)$ and the f_x's as a $|X|$-dimensional matrix $[\mathcal{F}(s)(x,y)]_{x,y \in X}$ and as $|X|$-dimensional row vectors with complex entries, respectively. For all $x, y \in X$, note that $f_x \mathcal{F}(s) f_y^t = \mathcal{F}(s)(x,y)$.

Continuing the analogy with the weakly locally ∞-transitive case (cf. Remark 3.4.19), we observe the following.

Remark 3.4.28. Let $\mathcal{F}(s)$ be as in Definition 3.4.26. For every $n \geq 1$, let $\mathcal{F}(s)^n$ be the n-th power of $\mathcal{F}(s)$, and $\mathcal{F}(s)^0$ be the identity operator on $\mathbb{C}[\![X]\!]$. For $n \geq 0$ and for all $x, y \in X$, we claim that

$$f_x \cdot \mathcal{F}(s)^n \cdot f_y^t = \mathcal{F}(s)^n(x,y) = \sum_{\xi \in \mathcal{P}_{(\Delta,\iota)}(x \to y)\colon \ell(\xi) = n+1} \mathcal{W}(\xi)^{-s}, \tag{3.4.18}$$

where $\mathcal{W}(\xi) = 1$ if $\ell(\xi) = 1$, and $\mathcal{W}(\xi) = \prod_{i=1}^{l-1} \mathcal{W}(x_i, x_{i+1})$ if $\xi = (x_1, \ldots, x_l)$ for some $l \geq 2$. Indeed, (3.4.18) is immediate if $n \leq 1$. For $n \geq 2$, one argues as in Remark 3.4.19.

Theorem 3.4.29. *Let $G = U(\Delta, \iota, c_0)$ and $T = T(\Delta, \iota, c_0)$ be as in Setting* [(P)-cl]. *Let $t \in T$ with $\pi(t) = u$ and $e \in o^{-1}(v_0)$ with $\mathcal{L}(e) = x$. Then, for every $r \in \{v_0, e\}$, we have*

$$\zeta_{G,G_r,G_t}(s) = \frac{\det(I - \mathcal{F}(s+1) + \mathcal{Y}_{\pi(r),u}(s+1))}{\det(I - \mathcal{F}(s+1))} + \kappa_{\pi(r)}(u),$$

where I is the identity matrix in $\mathrm{Mat}_{|X|}(\mathbb{C})$,

$$\mathcal{Y}_{\pi(r),u}(s) = \begin{cases} \sum_{a\in o^{-1}(c_0)} \omega(a)^{-s} f^t_{X_U} f_{X_a}, & \text{if } r = v_0; \\ f^t_{X_U}\Big(f_x \mathcal{F}(s) + \sum_{y\in X_{c_0}\setminus\{x\}} \mathcal{W}_{\mathrm{rev}}(x,y)^{-s} f_y\Big), & \text{if } r = e; \end{cases}$$

and

$$\kappa_{\pi(r)}(u) = \begin{cases} \mathbb{1}_{\{c_0\}}(u) - 1, & \text{if } r = v_0; \\ \mathbb{1}_{X_A}(x) - 1, & \text{if } r = e. \end{cases}$$

Proof. One proceeds analogously as in the proof of Theorem 3.4.20. Let $s \in \mathbb{C}$ such that $\sum_{n=0}^{\infty} \mathcal{F}(s+1)$ converges. By Proposition 3.4.25 and Remark 3.4.28, we deduce what follows:

$$\zeta_{G,G_v,G_t}(s) = \mathbb{1}_{\{c_0\}}(u) + \sum_{a\in o^{-1}(c_0)} \omega(a)^{-s-1} f_{X_a}\left(\sum_{n=0}^{\infty}\mathcal{F}(s+1)^n\right) f^t_{X_U}.$$

$$\begin{aligned}\zeta_{G,G_e,G_t}(s) &= \eta_a(u) + f_x\left(\sum_{n=1}^{\infty}\mathcal{F}(s+1)^n\right) f^t_{X_U} + \\ &\qquad + \sum_{y\in X_{c_0}\setminus\{x\}} \mathcal{W}_{\mathrm{rev}}(x,y)^{-s-1} f_y\left(\sum_{n=0}^{\infty}\mathcal{F}(s+1)^n\right) f^t_{X_U} \\ &= \mathbb{1}_{X_U}(x) + \Big(f_x\mathcal{F}(s+1) + \sum_{y\in X_{c_0}\setminus\{x\}} \mathcal{W}_{\mathrm{rev}}(x,y)^{-s-1} f_y\Big)\left(\sum_{n=0}^{\infty}\mathcal{F}(s+1)^n\right) f^t_{X_U}.\end{aligned}$$

Since $\sum_{n=0}^{\infty} \mathcal{F}(s+1)^n = (I - \mathcal{F}(s+1))^{-1}$, Fact 3.4.21 yields the claim. □

3.4.3 The reciprocal of $\mathcal{Z}_{\Gamma,u\to u}(s)$

In view of Section 3.5.2, we present some formulae involving the reciprocal of the function $\mathcal{Z}_{\Gamma,u\to u}(s)$, for $u \in \Gamma$, introduced in Definition 3.4.14. Recall that this function is a generalisation of $\zeta_{G,G_t,G_t}(s)$, where (G,T) is a weakly locally ∞-transitive group action on a locally finite tree and $t \in T$ (cf. Remark 3.4.16).

Definition 3.4.30. Let Γ be a non-empty graph with an edge weight $\omega\colon E\Gamma \to \mathbb{Z}_{\geq 1}$. Consider $\mathcal{E}(s)$, for $s \in \mathbb{C}$, and $\{e_u\}_{u\in\Gamma}$ as in Definition 3.4.17. For all $c \in V\Gamma$ and $a \in E\Gamma$, define

$$\mathcal{G}_c(s) := \mathcal{E}(s) - \mathcal{U}_{c,c}(s) \quad \text{and} \quad \mathcal{G}_a(s) := \mathcal{E}(s) - \mathcal{U}_{a,a}(s),$$

where $\mathcal{U}_{c,c}(s) = \sum_{a\in o^{-1}(c)} \omega(a)^{-s} e^t_c e_a$ and $\mathcal{U}_{a,a}(s) = (e_a + e_{\bar{a}})^t \cdot (e_a + e_{\bar{a}})\mathcal{E}(s)$ (cf. Theorem 3.4.20).

Notation 3.4.31. If necessary, we write $\mathcal{G}^{\Gamma}_{\bullet}(s)$, $\mathcal{E}^{\Gamma}(s)$, $\mathcal{U}^{\Gamma}_{\bullet,\bullet}(s)$, $e^{\Gamma}_{\bullet}$ and I^{Γ} instead of $\mathcal{G}_{\bullet}(s)$, $\mathcal{E}(s)$, $\mathcal{U}_{\bullet,\bullet}(s)$, $e_{\bullet}$ and the identity matrix in $\mathrm{Mat}_{|E\Gamma|}(\mathbb{C})$, respectively.

Lemma 3.4.32. *Let (Γ,ω) satisfy Setting [Γ], and denote by I the identity matrix on $\mathrm{Mat}_{|E\Gamma|}(\mathbb{C})$. Then, for all $c \in V\Gamma$ and $s \in \mathbb{C}$ such that $I - \mathcal{G}_c(s)$ is invertible, we have*

$$\mathcal{Z}_{\Gamma,c\to c}(s)^{-1} = 1 - \sum_{a\in o^{-1}(c)} \omega(a)^{-s} e_a (I - \mathcal{G}_c(s))^{-1} e_c^t.$$

Moreover, for all $a \in E\Gamma$ and $s \in \mathbb{C}$ such that $I - \mathcal{G}_a(s)$ is invertible, we have

$$\mathcal{Z}_{\Gamma,a\to a}(s)^{-1} = 1 - (e_a + e_{\bar{a}})\mathcal{E}(s)(I - \mathcal{G}_a(s))^{-1}(e_a + e_{\bar{a}})^t.$$

Proof. By Theorem 3.4.20, we deduce that

$$\mathcal{Z}_{\Gamma,c\to c}(s)^{-1} = \frac{\det(I - \mathcal{G}_c(s) - \mathcal{U}_{c,c}(s))}{\det(I - \mathcal{G}_c(s))};$$
$$\mathcal{Z}_{\Gamma,a\to a}(s)^{-1} = \frac{\det(I - \mathcal{G}_a(s) - \mathcal{U}_{a,a}(s))}{\det(I - \mathcal{G}_a(s))}.$$

The statements now follow from Fact 3.4.21. □

Proposition 3.4.33. *Let (Γ,ω) satisfy Setting [Γ]. Consider two subgraphs Γ_1 and Γ_2 of Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \{c\}$, for some $c \in V\Gamma$. Then,*

$$\mathcal{Z}_{\Gamma,c\to c}(s)^{-1} = \mathcal{Z}_{\Gamma_1,c\to c}(s)^{-1} + \mathcal{Z}_{\Gamma_2,c\to c}(s)^{-1} - 1.$$

Proof. Let $s \in \mathbb{C}$ such that $I^{\Gamma} - \mathcal{G}^{\Gamma}_c(s)$ is invertible, and set $\Gamma_3 = \Gamma_1 \cap \Gamma_2$. By Definition 3.4.30, for all $a, b \in E\Gamma$ we have

$$\begin{aligned}(I^{\Gamma} - \mathcal{G}^{\Gamma}_c(s))(a,b) &= e^{\Gamma}_a \cdot \left(I^{\Gamma} - \mathcal{G}^{\Gamma}_c(s)\right) \cdot (e^{\Gamma}_b)^t \\ &= (I^{\Gamma} - \mathcal{E}^{\Gamma}(s))(a,b) + \sum_{a'\in o^{-1}(c)} \omega(a')^{-s}(e^{\Gamma}_a(e^{\Gamma}_c)^t)\cdot(e^{\Gamma}_{a'}(e^{\Gamma}_b)^t) \\ &= (I^{\Gamma} - \mathcal{E}^{\Gamma}(s))(a,b) + \mathbb{1}_{t^{-1}(c)}(a)\mathbb{1}_{o^{-1}(c)}(b)\omega(b)^{-s}.\end{aligned} \tag{3.4.19}$$

Similarly, for $1 \leq i \leq 3$ and for all $a, b \in E\Gamma_i$ we have

$$(I^{\Gamma_i} - \mathcal{G}^{\Gamma_i}_c(s))(a,b) = (I^{\Gamma_i} - \mathcal{E}^{\Gamma_i}(s))(a,b) + \mathbb{1}_{t^{-1}(c)\cap E\Gamma_i}(a)\mathbb{1}_{o^{-1}(c)\cap E\Gamma_i}(b)\omega(b)^{-s}. \tag{3.4.20}$$

Combining (3.4.19) and (3.4.20), for every $1 \leq i \leq 3$ we deduce that

$$(I^{\Gamma} - \mathcal{G}^{\Gamma}_c(s))(a,b) = (I^{\Gamma_i} - \mathcal{G}^{\Gamma_i}_c)(a,b), \quad \forall\, a, b \in E\Gamma_i \tag{3.4.21}$$

and Lemma 3.4.32 implies that

$$\mathcal{Z}_{\Gamma_i,c\to c}(s)^{-1} = 1 - \sum_{\substack{a\in o^{-1}(c)\cap E\Gamma_i,\\ b\in t^{-1}(c)\cap E\Gamma_i}} \omega(a)^{-s}(I_{\Gamma} - \mathcal{G}^{\Gamma}_c(s))^{-1}(a,b). \tag{3.4.22}$$

Moreover, for all $a, b \in E\Gamma$ with $t(a) = c = o(b)$ we have $(I^\Gamma - \mathcal{G}_c^\Gamma(s))(a, b) = \mathbb{1}_{\{a\}}(b) - (\omega(b) - \mathbb{1}_{\{\bar{a}\}}(b))^{-s} + \omega(b)^{-s}$ and then

$$(I^\Gamma - \mathcal{G}_c^\Gamma(s))(a, b) = 0, \quad \forall\, b \in o^{-1}(c) \setminus \{a, \bar{a}\}. \tag{3.4.23}$$

We claim that

$$(I^\Gamma - \mathcal{G}_c^\Gamma(s))(a, b) = 0, \quad \forall\, (a, b) \in (E\Gamma_1 \times E\Gamma_2) \cup (E\Gamma_2 \times E\Gamma_1).$$

Indeed, recall that $E\Gamma_1 \cap E\Gamma_2 = \emptyset$ and $V\Gamma_1 \cap V\Gamma_2 = \{c\}$. Hence, for all $a \in E\Gamma_1$ and $b \in E\Gamma_2$, we have $b \notin \{a, \bar{a}\}$ and either $t(a) \neq o(b)$ or $t(a) = c = o(b)$. Now (3.4.19) and (3.4.23) apply. A similar argument holds for all $a \in E\Gamma_2$ and $b \in E\Gamma_1$.

Therefore, once fixed a total order $\leq$ on $E\Gamma$ so that $a < b$ for all $a \in E\Gamma_1$ and $b \in E\Gamma_2$, we have the following decomposition in diagonal blocks:

$$I^\Gamma - \mathcal{G}_c^\Gamma(s) = \begin{bmatrix} I^{\Gamma_1} - \mathcal{G}_c^{\Gamma_1}(s) & 0 \\ 0 & I^{\Gamma_2} - \mathcal{G}_c^{\Gamma_2}(s) \end{bmatrix}. \tag{3.4.24}$$

Since

$$o^{-1}(c) = (o^{-1}(c) \cap E\Gamma_1) \sqcup (o^{-1}(c) \cap E\Gamma_2),$$

by Lemma 3.4.32, (3.4.24) and then (3.4.22), we conclude that

$$\begin{aligned} \mathcal{Z}_{\Gamma,c\to c}(s)^{-1} &= 1 - \sum_{a,b \in o^{-1}(c)} \omega(a)^{-s} (I^\Gamma - \mathcal{G}_c^\Gamma(s))^{-1}(a, \bar{b}) \\ &= 1 - \sum_{a,b \in o^{-1}(c) \cap E\Gamma_1} \omega(a)^{-s} (I^{\Gamma_1} - \mathcal{G}_c^{\Gamma_1}(s))^{-1}(a, \bar{b}) + \\ &\quad - \sum_{a,b \in o^{-1}(c) \cap E\Gamma_2} \omega(a)^{-s} (I^{\Gamma_2} - \mathcal{G}_c^{\Gamma_2}(s))^{-1}(a, \bar{b}) \\ &= \mathcal{Z}_{\Gamma_1,c\to c}(s)^{-1} + \mathcal{Z}_{\Gamma_2,c\to c}(s)^{-1} - 1. \quad \square \end{aligned}$$

Corollary 3.4.34. *Let* (Γ, ω) *satisfy Setting* [Γ]. *Assume that there are subgraphs* Λ_1 *and* Λ_2 *of* Γ *such that* $\Gamma = \Lambda_1 \cup \Lambda_2$ *and* $\Lambda_1 \cap \Lambda_2 = \{c\}$, *for some vertex* $c \in V\Gamma$. *Then, for all subgraphs* Γ_1 *and* Γ_2 *of* Γ *satisfying* $\Gamma_i \supseteq \Lambda_i$ *for every* $i \in \{1, 2\}$, *we have*

$$\mathcal{Z}_{\Gamma,c\to c}(s)^{-1} = \mathcal{Z}_{\Gamma_1,c\to c}(s)^{-1} + \mathcal{Z}_{\Gamma_2,c\to c}(s)^{-1} - \mathcal{Z}_{\Gamma_1\cap\Gamma_2,c\to c}(s)^{-1}.$$

Proof. Let Γ_1 and Γ_2 be as in the statement, and set $\Gamma_3 := \Gamma_1 \cap \Gamma_2$. Note that $\Lambda_1 \cap \Gamma_3 = \Lambda_1 \cap \Gamma_2$ and $\Lambda_2 \cap \Gamma_3 = \Lambda_2 \cap \Gamma_1$. Therefore,

$$\begin{array}{lll} \Gamma = \Lambda_1 \cup \Lambda_2 & \text{and} & \Lambda_1 \cap \Lambda_2 = \{c\}; \\ \Gamma_1 = \Lambda_1 \cup (\Lambda_2 \cap \Gamma_1) & \text{and} & \Lambda_1 \cap (\Lambda_2 \cap \Gamma_1) = \{c\}; \\ \Gamma_2 = (\Lambda_1 \cap \Gamma_2) \cup \Lambda_2 & \text{and} & (\Lambda_1 \cap \Gamma_2) \cap \Lambda_2 = \{c\}; \\ \Gamma_3 = (\Lambda_1 \cap \Gamma_2) \cup (\Lambda_2 \cap \Gamma_1) & \text{and} & (\Lambda_1 \cap \Gamma_2) \cap (\Lambda_2 \cap \Gamma_1) = \{c\}. \end{array} \tag{3.4.25}$$

Applying Proposition 3.4.33 to each decomposition in (3.4.25) yields the claim. $\square$

The hypotheses of Proposition 3.4.33 are satisfied for every $c \in V\Gamma$ if, for instance, Γ is a connected graph without n-cycles, for every $n \geq 2$.

Lemma 3.4.35. *Let Γ be a connected graph without n-cycles, for every $n \geq 2$. For all $c \in V\Gamma$, there are connected subgraphs Γ_1 and Γ_2 of Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \{c\}$. If in particular $|o^{-1}(c)| \geq 2$, one may take Γ_1 and Γ_2 to be proper subgraphs of Γ.*

Proof. Let $\{\Xi_i\}_{i \in I}$ be the collection of all the connected components of the graph $\Gamma \setminus (o^{-1}(c) \cup \overline{o^{-1}(c)} \cup \{c\})$. Recall that $\Gamma \setminus (o^{-1}(c) \cup \overline{o^{-1}(c)} \cup \{c\}) = \bigsqcup_{i \in I} \Xi_i$. For every $i \in I$, there is exactly one edge $a_i \in o^{-1}(c)$ such that $t(a_i) \in \Xi_i$. In fact, assume that there are $a, b \in o^{-1}(c)$ with $a \neq b$ such that $x = t(a), y = t(b) \in V\Xi_i$. Then the reduced path $[x, y]$ as in Remark 1.2.2 is contained in Ξ_i. Hence, $a \cdot [x, y] \cdot \bar{b}$ is a cycle of length ≥ 2 in Γ, impossible.

Consider subsets $\mathcal{E}_1, \mathcal{E}_2$ of $o^{-1}(c)$ with the following properties: $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$, $o^{-1}(c) = \mathcal{E}_1 \cup \mathcal{E}_2$ and, for every $k \in \{1, 2\}$, every 1-loop a in $\mathcal{E}_k$ satisfies $\bar{a} \in \mathcal{E}_k$. Provided $\overline{\mathcal{E}}_k = \{\bar{a} \mid a \in \mathcal{E}_k\}$ for every $k \in \{1, 2\}$, note that $(\mathcal{E}_1 \cup \overline{\mathcal{E}}_1) \cap (\mathcal{E}_2 \cup \overline{\mathcal{E}}_2) = \emptyset$. Moreover, if $|o^{-1}(c)| \geq 2$, take $\mathcal{E}_1$ and $\mathcal{E}_2$ so that $\mathcal{E}_1 \neq \emptyset$ and $\mathcal{E}_2 \neq \emptyset$. For $k \in \{1, 2\}$, set also $I_k := \{i \in I \mid a_i \in \mathcal{E}_k\}$. Note that $I_1 \cap I_2 = \emptyset$ and $I = I_1 \cup I_2$. For $k \in \{1, 2\}$, define the following subgraph of Γ:

$$\Gamma_k := \{c\} \cup \mathcal{E}_k \cup \overline{\mathcal{E}}_k \cup \bigsqcup_{i \in I_k} \Xi_i.$$

One checks that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \{c\}$. If in particular $|o^{-1}(c)| \geq 2$, then $\Gamma_1 \setminus \Gamma_2 \supseteq \mathcal{E}_1 \neq \emptyset$ and $\Gamma_2 \setminus \Gamma_1 \supseteq \mathcal{E}_2 \neq \emptyset$. Therefore, both Γ_1 and Γ_2 are proper subgraphs of Γ. □

Proposition 3.4.36. *Let (Γ, ω) satisfy Setting [Γ]. Consider subgraphs Γ_1 and Γ_2 of Γ satisfying $\Gamma = \Gamma_1 \cup \Gamma_2$ and such that $\Gamma_3 := \Gamma_1 \cap \Gamma_2$ is a 1-segment with edge set $\{a, \bar{a}\}$. Then,*

$$\mathcal{Z}_{\Gamma, a \to a}(s)^{-1} = \mathcal{Z}_{\Gamma_1, a \to a}(s)^{-1} + \mathcal{Z}_{\Gamma_2, a \to a}(s)^{-1} - \mathcal{Z}_{\Gamma_3, a \to a}(s)^{-1}.$$

Proof. Fix $s \in \mathbb{C}$ such that $I^\Gamma - \mathcal{G}_a^\Gamma(s)$ is invertible, and set $c = o(a)$, $d = t(a)$. By Lemma 3.4.32, we have

$$\begin{aligned} &\mathcal{Z}_{\Gamma, a \to a}(s)^{-1} = \\ &= 1 - \left(\sum_{\substack{b \in E\Gamma, \\ o(b) = d}} \mathcal{E}^\Gamma(s)(a, b) e_b^\Gamma + \sum_{\substack{b \in E\Gamma, \\ o(b) = c}} \mathcal{E}^\Gamma(s)(\bar{a}, b) e_b^\Gamma \right) (I^\Gamma - \mathcal{G}_a^\Gamma(s))^{-1} (e_a^\Gamma + e_{\bar{a}}^\Gamma)^t. \end{aligned} \tag{3.4.26}$$

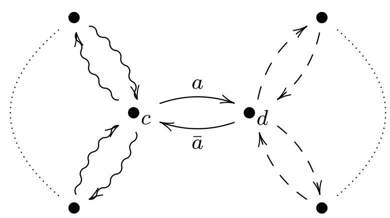
By Definition 3.4.30, for all $b_1, b_2 \in E\Gamma$ we observe that

$$\begin{aligned} &(I^\Gamma - \mathcal{G}_a^\Gamma(s))(b_1, b_2) = \\ = \;& \mathbb{1}_{\{b_1\}}(b_2) - \mathcal{E}^\Gamma(s)(b_1, b_2) + e_{b_1}^\Gamma (e_a^\Gamma + e_{\bar{a}}^\Gamma)^t (e_a^\Gamma + e_{\bar{a}}^\Gamma) \mathcal{E}^\Gamma(s) (e_{b_2}^\Gamma)^t \\ = \;& \mathbb{1}_{\{b_1\}}(b_2) - \mathcal{E}^\Gamma(s)(b_1, b_2) + \mathbb{1}_{\{a, \bar{a}\}}(b_1) \cdot \Big(\mathcal{E}^\Gamma(s)(a, b_2) + \mathcal{E}^\Gamma(s)(\bar{a}, b_2) \Big). \end{aligned} \tag{3.4.27}$$

Hence, for every $i \in \{1,2\}$,

$$\begin{aligned} (I^{\Gamma_i} - \mathcal{G}_a^{\Gamma_i}(s))(b_1,b_2) &= (I^{\Gamma} - \mathcal{G}_a^{\Gamma}(s))(b_1,b_2), \quad \forall\, b_1,b_2 \in E\Gamma_i; \\ (I^{\Gamma_i} - \mathcal{G}_a^{\Gamma_i}(s))(b_1,b_2) &= (I^{\Gamma_3} - \mathcal{G}_a^{\Gamma_3}(s))(b_1,b_2), \quad \forall\, b_1,b_2 \in \{a,\bar{a}\}. \end{aligned} \tag{3.4.28}$$

Let Λ_1 (resp. Λ_2) be the graph obtained from Γ_1 (resp. Γ_2) by removing $a, \bar{a}$ and d (resp. $a, \bar{a}$ and c). The following picture sketches an example of Λ_1 (with wavy edges) and Λ_2 (with dashed edges).



Note that $c \in V\Lambda_1$, $d \in V\Lambda_2$ and $V\Lambda_1 \cap V\Lambda_2 = \emptyset$. In particular, no edges of Λ_1 end in a vertex of Λ_2. We claim that

$$\begin{aligned} (I^{\Gamma} - \mathcal{G}_a^{\Gamma}(s))(b_1,b_2) &= 0, \quad \forall\,(b_1,b_2) \in (E\Lambda_1 \sqcup \{a\}) \times (E\Lambda_2 \sqcup \{\bar{a}\}); \\ (I^{\Gamma} - \mathcal{G}_a^{\Gamma}(s))(b_1,b_2) &= 0, \quad \forall\,(b_1,b_2) \in (E\Lambda_2 \sqcup \{\bar{a}\}) \times (E\Lambda_1 \sqcup \{a\}). \end{aligned} \tag{3.4.29}$$

Indeed, let $b_1 \in E\Lambda_1 \sqcup \{a\}$ and $b_2 \in E\Lambda_2 \sqcup \{\bar{a}\}$. If $b_1 \in E\Lambda_1$, then $V\Lambda_1 \ni t(b_1) \neq o(b_2) \in V\Lambda_2$ and (3.4.27) implies $I^{\Gamma} - \mathcal{G}_a^{\Gamma}(s) = -\mathcal{E}^{\Gamma}(b_1,b_2) = 0$. If $b_1 = a$, then $c = t(\bar{a}) \neq o(b_2) \in E\Lambda_2$ and (3.4.27) implies that $I^{\Gamma} - \mathcal{G}_a^{\Gamma}(s) = \mathcal{E}^{\Gamma}(s)(\bar{a},b_2) = 0$. The second line of (3.4.29) can be proved analogously.

Fix a total order $\leq$ on $E\Gamma = E\Lambda_1 \sqcup \{a,\bar{a}\} \sqcup E\Lambda_2$ so that $b_1 < a < \bar{a} < b_2$ for all $b_1 \in E\Lambda_1$ and $b_2 \in E\Lambda_2$. Set also

$$\begin{aligned} A &:= [(I^{\Gamma_1} - \mathcal{G}_a^{\Gamma_1}(s))(b_1,b_2)]_{b_1,b_2 \in E\Lambda_1 \sqcup \{a\}}; \\ B &:= [(I^{\Gamma_2} - \mathcal{G}_a^{\Gamma_2}(s))(b_1,b_2)]_{b_1,b_2 \in \{\bar{a}\} \sqcup E\Lambda_2}; \\ \alpha &:= \omega(a) - 1 \quad \text{and} \quad \beta := \omega(\bar{a}) - 1. \end{aligned}$$

From (3.4.28) we observe that $A(a,a) = 1 + \alpha^{-s}$ and $B(\bar{a},\bar{a}) = 1 + \beta^{-s}$. Moreover, by (3.4.28) and (3.4.29), we have the following decompositions in diagonal blocks:

$$\begin{aligned} I^{\Gamma} - \mathcal{G}_a^{\Gamma}(s) &= \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad \text{and} \quad (I^{\Gamma} - \mathcal{G}_a^{\Gamma}(s))^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix}; \\ I^{\Gamma_1} - \mathcal{G}_a^{\Gamma_1}(s) &= \begin{bmatrix} A & 0 \\ 0 & B(\bar{a},\bar{a}) \end{bmatrix} \quad \text{and} \quad (I^{\Gamma_1} - \mathcal{G}_a^{\Gamma_1}(s))^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & B(\bar{a},\bar{a})^{-1} \end{bmatrix}; \\ I^{\Gamma_2} - \mathcal{G}_a^{\Gamma_2}(s) &= \begin{bmatrix} A(a,a) & 0 \\ 0 & B \end{bmatrix} \quad \text{and} \quad (I^{\Gamma_2} - \mathcal{G}_a^{\Gamma_2}(s))^{-1} = \begin{bmatrix} A(a,a)^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix}. \end{aligned} \tag{3.4.30}$$

Note that

$$o^{-1}(c) = \{a\} \sqcup (o^{-1}(c) \cap E\Lambda_1) \text{ and } o^{-1}(d) = \{\bar{a}\} \sqcup (o^{-1}(d) \cap E\Lambda_2). \tag{3.4.31}$$

Therefore, by (3.4.30) and (3.4.31), we rewrite (3.4.26) as follows:

$$\begin{aligned} \mathcal{Z}_{\Gamma,a\to a}(s)^{-1} = 1 - \beta^{-s}B^{-1}(\bar{a},\bar{a}) - \sum_{b\in o^{-1}(d)\cap E\Lambda_2} \mathcal{E}^{\Gamma_2}(s)(a,b)\cdot B^{-1}(b,\bar{a}) + \\ - a^{-s}A^{-1}(a,a) - \sum_{b\in o^{-1}(c)\cap E\Lambda_1} \mathcal{E}^{\Gamma_1}(s)(\bar{a},b)\cdot A^{-1}(b,a). \end{aligned} \tag{3.4.32}$$

A formula analogous to (3.4.26) holds for Γ_1, namely

$$\begin{gathered} \mathcal{Z}_{\Gamma_1,a\to a}(s)^{-1} = \\ 1 - \left(\mathcal{E}^{\Gamma_1}(s)(a,\bar{a})e_{\bar{a}}^{\Gamma_1} + \mathcal{E}^{\Gamma_1}(s)(\bar{a},a)e_{a}^{\Gamma_1} + \sum_{b\in o^{-1}(c)\cap E\Lambda_1} \mathcal{E}^{\Gamma_1}(s)(\bar{a},b)e_{b}^{\Gamma_1} \right)\cdot \\ \cdot\left(I^{\Gamma_1} - \mathcal{G}_a^{\Gamma_1}(s) \right)^{-1} \cdot (e_a^{\Gamma_1} + e_{\bar{a}}^{\Gamma_1})^t \end{gathered}$$

which, by (3.4.30), yields

$$\mathcal{Z}_{\Gamma_1,a\to a}(s)^{-1} = 1 - \beta^{-s}B(\bar{a},\bar{a})^{-1} - \alpha^{-s}A^{-1}(a,a) - \sum_{b\in o^{-1}(c)\cap E\Lambda_1} \mathcal{E}^{\Gamma_1}(s)(\bar{a},b)\cdot A^{-1}(b,a). \tag{3.4.33}$$

Similarly, we deduce that

$$\mathcal{Z}_{\Gamma_2,a\to a}(s)^{-1} = 1 - \alpha^{-s}A(a,a)^{-1} - \beta^{-s}B^{-1}(\bar{a},\bar{a}) - \sum_{b\in o^{-1}(d)\cap E\Lambda_2} \mathcal{E}^{\Gamma_2}(s)(a,b)\cdot B^{-1}(b,\bar{a}). \tag{3.4.34}$$

By Example 3.4.22, one also checks that

$$\mathcal{Z}_{\Gamma_3,a\to a}(s)^{-1} = \frac{1-\alpha^{-s}\beta^{-s}}{(1+\alpha^{-s})(1+\beta^{-s})} = 1 - \alpha^{-s}A(a,a)^{-1} - \beta^{-s}B(\bar{a},\bar{a})^{-1}. \tag{3.4.35}$$

Combining (3.4.32), (3.4.33), (3.4.34) and (3.4.35), we conclude the claim. □

Note that the strategy to prove Proposition 3.4.36 strictly depends on the fact that a has distinct endpoints (cf. (3.4.29)).

Corollary 3.4.37. *Let (Γ,ω) satisfy Setting [Γ]. Assume that there are subgraphs Λ_1 and Λ_2 of Γ satisfying $\Gamma = \Lambda_1 \cup \Lambda_2$ and such that $\Lambda_1 \cap \Lambda_2$ is a 1-segment graph with edge set $\{a,\bar{a}\}$. Then, for all subgraphs Γ_1 and Γ_2 of Γ such that $\Gamma_i \supseteq \Lambda_i$ for every $i \in \{1,2\}$, we have*

$$\mathcal{Z}_{\Gamma,a\to a}(s)^{-1} = \mathcal{Z}_{\Gamma_1,a\to a}(s)^{-1} + \mathcal{Z}_{\Gamma_2,a\to a}(s)^{-1} - \mathcal{Z}_{\Gamma_1\cap\Gamma_2,a\to a}(s)^{-1}.$$

Proof. Let Γ_1 and Γ_2 as in the statement. For simplicity, set $\Gamma_3 = \Gamma_1 \cap \Gamma_2$ and $\Gamma_a = \Lambda_1 \cap \Lambda_2$. Since $\Lambda_1 \subseteq \Gamma_1$ and $\Lambda_2 \subseteq \Gamma_2$, we have $\Lambda_1 \cap \Gamma_2 = \Lambda_1 \cap \Gamma_3$ and $\Lambda_2 \cap \Gamma_1 = \Lambda_2 \cap \Gamma_3$. Therefore, the following decompositions hold:

$$\begin{aligned} &\Gamma = \Lambda_1 \cup \Lambda_2 && \text{and} \quad \Lambda_1 \cap \Lambda_2 = \Gamma_a; \\ &\Gamma_1 = \Lambda_1 \cup (\Lambda_2 \cap \Gamma_1) && \text{and} \quad \Lambda_1 \cap (\Lambda_2 \cap \Gamma_1) = \Gamma_a; \\ &\Gamma_2 = (\Lambda_1 \cap \Gamma_2) \cup \Lambda_2 && \text{and} \quad (\Lambda_1 \cap \Gamma_2) \cap \Lambda_2 = \Gamma_a; \\ &\Gamma_3 = (\Lambda_1 \cap \Gamma_2) \cup (\Lambda_2 \cap \Gamma_1) && \text{and} \quad (\Lambda_1 \cap \Gamma_2) \cap (\Lambda_2 \cap \Gamma_1) = \Gamma_a. \end{aligned} \tag{3.4.36}$$

Applying Proposition 3.4.36 to each decomposition in (3.4.36) yields the claim. □

For completeness, in analogy to Lemma 3.4.35 we observe the following.

Remark 3.4.38. Let Γ be a connected graph without n-cycles for every $n \geq 2$, and let $a \in E\Gamma$. By Lemma 3.4.35, there are connected subgraphs Λ_1 and Λ_2 of Γ such that $\Gamma = \Lambda_1 \cup \Lambda_2$ and $\Lambda_1 \cap \Lambda_2 = \{o(a)\}$. Denote by Γ_1 and Γ_2 the smallest subgraphs of Γ containing $\Lambda_1 \cup \{a, \bar{a}\}$ and $\Lambda_2 \cup \{a, \bar{a}\}$, respectively. Then $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2$ is the subgraph of Γ with edge set $\{a, \bar{a}\}$.

In view of the next proofs, it might be useful to recall the following well-known fact [Tab21]. Given a 2×2 block-matrix $M = [M_{ij}]_{1 \leq i,j \leq 2} \in \mathrm{Mat}_n(\mathbb{C})$ with M_{22} invertible, one has

$$\det(M) = \det(M_{22}) \cdot \det(M_{11} - M_{12} M_{22}^{-1} M_{21}). \tag{3.4.37}$$

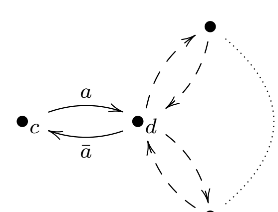
Proposition 3.4.39. *Let (Γ, ω) satisfy Setting [Γ]. Consider $a \in E\Gamma$ with $o(a) =: c \neq d := t(a)$, and assume that c is a terminal vertex in Γ. Put $\omega(a) = \alpha + 1$, $\omega(\bar{a}) = \beta + 1$, and denote by Λ the graph obtained from Γ by removing a, $\bar{a}$ and c. Then,*

$$\mathcal{Z}_{\Gamma, c \to c}(s)^{-1} = \frac{(1 + \alpha^{-s}\xi(\beta, s)) \cdot \mathcal{Z}_{\Lambda, d \to d}(s)^{-1} - \alpha^{-s}(\beta+1)^{-s}}{(1 - \xi(\alpha, s)\xi(\beta, s)) \cdot \mathcal{Z}_{\Lambda, d \to d}(s)^{-1} + \xi(\alpha, s)(\beta+1)^{-s}};$$

$$\mathcal{Z}_{\Gamma, a \to a}(s)^{-1} = \frac{(1 + \alpha^{-s}\xi(\beta, s)) \cdot \mathcal{Z}_{\Lambda, d \to d}(s)^{-1} - \alpha^{-s}(\beta+1)^{-s}}{(1 + \alpha^{-s})\Big((1 - \xi(\beta, s)) \cdot \mathcal{Z}_{\Lambda, d \to d}(s)^{-1} + (\beta+1)^{-s}\Big)},$$

where $\xi(\alpha, s) = (\alpha+1)^{-s} - \alpha^{-s}$ and $\xi(\beta, s) = (\beta+1)^{-s} - \beta^{-s}$.

The picture below sketches a possible setting for Proposition 3.4.39. The edges of Λ are dashed.



Proof. Let $s \in \mathbb{C}$ such that $\mathcal{Z}_{\Gamma,a\to a}(s) \neq 0$, and consider a total order $\leq$ on $E\Gamma$ such that $a < \bar{a} < b$ for all $b \in E\Lambda$. Then $\mathcal{E}^\Gamma(s)$ admits the following block decomposition:

$$\mathcal{E}^\Gamma(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{3.4.38}$$

where

$$A = [\mathcal{E}^\Gamma(s)(b_1,b_2)]_{b_1,b_2\in\{a,\bar{a}\}}; \qquad B = [\mathcal{E}^\Gamma(s)(b_1,b_2)]_{b_1\in\{a,\bar{a}\},\, b_2\in E\Lambda};$$
$$C = [\mathcal{E}^\Gamma(s)(b_1,b_2)]_{b_1\in E\Lambda,\, b_2\in\{a,\bar{a}\}}; \qquad D = [\mathcal{E}^\Gamma(s)(b_1,b_2)]_{b_1,b_2\in E\Lambda} = \mathcal{E}^\Lambda(s).$$

For $i \in \{1,2\}$, let B_i and C^i denote the i-th row of B and the i-th column of C, respectively. One checks that

$$B_1 = \sum_{b\in o^{-1}(d)\cap E\Lambda} \omega(b)^{-s} e_b^\Lambda; \quad B_2 = \underline{0}; \quad C^1 = \underline{0}^t \quad \text{and} \quad C^2 = (\beta+1)^{-s}(e_d^\Lambda)^t, \tag{3.4.39}$$

where $\underline{0}$ denotes the row zero vector in $\mathbb{C}^{|E\Lambda|}$. Moreover, denote by E_{21} and $\underline{1} \in \mathrm{Mat}_2(\mathbb{C})$ the elementary matrix associated to $(2,1)$ and the matrix with all the entries equal to 1, respectively. Hence,

$$\mathcal{U}_{cc}^\Gamma(s) = (\alpha+1)^{-1}(e_{\bar{a}}^\Gamma)^t(e_a^\Gamma) = \begin{bmatrix} (\alpha+1)^{-s}E_{21} & 0_{2\times|E\Lambda|} \\ 0_{|E\Lambda|\times 2} & 0_{|E\Lambda|\times|E\Lambda|} \end{bmatrix};$$
$$\mathcal{U}_{aa}^\Gamma(s) = (e_a^\Gamma + e_{\bar{a}}^\Gamma)^t(e_a^\Gamma + e_{\bar{a}}^\Gamma)\mathcal{E}^\Gamma(s) = \begin{bmatrix} \underline{1}\cdot A & \underline{1}\cdot B \\ 0_{|E\Lambda|\times 2} & 0_{|E\Lambda|\times|E\Lambda|} \end{bmatrix}.$$

Denoting by I_2 the identity matrix in $\mathrm{Mat}_2(\mathbb{C})$, the following holds:

$$\begin{aligned} I^\Gamma - \mathcal{E}^\Gamma(s) &= \begin{bmatrix} I_2 - A & -B \\ -C & I_\Lambda - D \end{bmatrix}; \\ I^\Gamma - \mathcal{E}^\Gamma(s) + \mathcal{U}_{c,c}^\Gamma(s) &= \begin{bmatrix} I_2 - A + (\alpha+1)^{-s}E_{21} & -B \\ -C & I_\Lambda - D \end{bmatrix}; \\ I^\Gamma - \mathcal{E}^\Gamma(s) + \mathcal{U}_{a,a}^\Gamma(s) &= \begin{bmatrix} I_2 + (\underline{1} - I_2)A & (\underline{1} - I_2)B \\ -C & I_\Lambda - D \end{bmatrix}; \\ I_2 - A = \begin{bmatrix} 1 & -\beta^{-s} \\ -\alpha^{-s} & 1 \end{bmatrix}; &\qquad I_2 + (\underline{1} - I_2)A = \begin{bmatrix} 1+\alpha^{-s} & 0 \\ 0 & 1+\beta^{-s} \end{bmatrix}. \end{aligned} \tag{3.4.40}$$

By Theorem 3.4.20, (3.4.37) and (3.4.40), we deduce that

$$
\begin{aligned}
\mathcal{Z}_{\Gamma,c\to c}(s)^{-1} &= \frac{\det(I^\Gamma - \mathcal{E}^\Gamma(s))}{\det(I - \mathcal{E}^\Gamma(s) + \mathcal{U}^\Gamma_{c,c}(s))} \\
&= \frac{\det\left(I_2 - A - \overbrace{B(I_\Lambda - D)^{-1}C}^{=:X}\right)}{\det\left(I_2 - A + (\alpha+1)^{-s}E_{21} - \underbrace{B(I_\Lambda - D)^{-1}C}_{=X}\right)} \\
&= \frac{(1-X(a,a))(1-X(\bar a,\bar a)) - (X(a,\bar a)+\beta^{-s})(X(\bar a,a)+\alpha^{-s})}{(1-X(a,a))(1-X(\bar a,\bar a)) + (\xi(\alpha,s) - X(\bar a,a))(\beta^{-s}+X(a,\bar a))}.
\end{aligned}
\tag{3.4.41}
$$

Similarly,

$$
\begin{aligned}
\mathcal{Z}_{\Gamma,a\to a}(s)^{-1} &= \frac{\det(I^\Gamma - D)}{\det(I - D + \mathcal{U}^\Gamma_{a,a}(s))} \\
&= \frac{\det\left(I_2 - A - \overbrace{B(I_\Lambda - D)^{-1}C}^{=X}\right)}{\det\left(I_2 + (\underline{1} - I_2)A + \underbrace{(\underline{1} - I_2)B(I_\Lambda - D)^{-1}C}_{=:Y}\right)} \\
&= \frac{(1-X(a,a))(1-X(\bar a,\bar a)) - (X(a,\bar a)+\beta^{-s})(X(\bar a,a)+\alpha^{-s})}{(1+\alpha^{-s}+Y(a,a))(1+\beta^{-s}+Y(\bar a,\bar a)) - Y(a,\bar a)Y(\bar a,a)}.
\end{aligned}
\tag{3.4.42}
$$

It remains to study the entries of X and Y. Observe that $(\underline{1} - I_2)B$ is the matrix obtained from B by interchanging its two rows. Since $B_2 = \underline{0}$ and $C^1 = \underline{0}^t$, we deduce the following:

$$
\begin{aligned}
X(a,a) = Y(\bar a,a) &= B_1(I_\Lambda - \mathcal{E}^\Lambda(s))^{-1}C^1 = 0; \\
X(\bar a,a) = Y(a,a) &= B_2(I_\Lambda - \mathcal{E}^\Lambda(s))^{-1}C^1 = 0; \\
X(\bar a,\bar a) = Y(a,\bar a) &= B_2(I_\Lambda - \mathcal{E}^\Lambda(s))^{-1}C^2 = 0.
\end{aligned}
$$

Moreover, by Theorem 3.4.20, Fact 3.4.21 and since $\mathcal{E}^\Lambda(s) = D$,

$$
\begin{aligned}
X(a,\bar a) = Y(\bar a,\bar a) &= B_1(I_\Lambda - \mathcal{E}^\Lambda(s))^{-1}C^2 \\
&= (\beta+1)^{-s}\cdot \sum_{b\in o^{-1}(d)\cap E\Lambda} \omega(b)^{-s} e^\Lambda_b (I_\Lambda - \mathcal{E}^\Lambda(s))^{-1}(e^\Lambda_d)^t \\
&= (\beta+1)^{-s}(\mathcal{Z}_{\Lambda,d\to d}(s) - 1). \qquad \square
\end{aligned}
$$

The claim now follows by substitution and elementary algebraic manipulations.

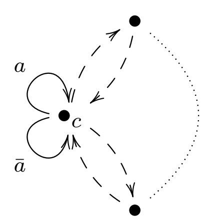
Proposition 3.4.40. *Let (Γ,ω) satisfy Setting [Γ], and consider $a \in E\Gamma$ with $o(a) = t(a) = c$. Set $\omega(a) = \alpha+1$, $\omega(\bar{a}) = \beta+1$, and consider the subgraph of Γ given by $\Lambda := \Gamma \setminus \{a, \bar{a}\}$. Then*

$$\mathcal{Z}_{\Gamma,a\to a}(s)^{-1} = \frac{\xi_1(\alpha,\beta)\cdot \mathcal{Z}_{\Lambda,c\to c}(s)^{-1} - \eta(\alpha,\beta)}{\xi_2(\alpha,\beta)\cdot \mathcal{Z}_{\Lambda,c\to c}(s)^{-1} + \eta(\alpha,\beta)},$$

where

$$\begin{aligned}
\xi_1(\alpha,\beta) &= 1 - \Big(\alpha^{-s} - (\alpha+1)^{-s}\Big)\Big(\beta^{-s} - (\beta+1)^{-s}\Big);\\
\xi_2(\alpha,\beta) &= \Big(1+\alpha^{-s} - (\alpha+1)^{-s}\Big)\Big(1+\beta^{-s} - (\beta+1)^{-s}\Big);\\
\eta(\alpha,\beta) &= (\alpha^{-s}+1)(\beta+1)^{-s} + (\alpha+1)^{-s}(\beta^{-s}+1) - 2(\alpha+1)^{-s}(\beta+1)^{-s}.
\end{aligned}$$

The picture below sketches the setting of Proposition 3.4.40. The edges of Λ are dashed.



Remark 3.4.41. In Proposition 3.4.40, if $\alpha = \beta$, after elementary manipulations the given formula becomes

$$\mathcal{Z}_{\Gamma,a\to a}(s)^{-1} = \frac{(1-\alpha^{-s}+(\alpha+1)^{-s})\cdot \mathcal{Z}_{\Lambda,c\to c}(s)^{-1} - 2(\alpha+1)^{-s}}{(1+\alpha^{-s}-(\alpha+1)^{-s})\cdot \mathcal{Z}_{\Lambda,c\to c}(s)^{-1} + 2(\alpha+1)^{-s}}.$$

Proof of Proposition 3.4.40. The strategy of the proof is analogous to the one of Proposition 3.4.39. Thus, we keep the same notation and proof structure, and we only specify what needs to be changed. First, instead of (3.4.39), the rows B_1 and B_2 of B and the columns C^1 and C^2 of C are the following:

$$\begin{aligned}
B_1 = B_2 &= \sum_{b\in o^{-1}(c)\cap E\Lambda} \omega(b)^{-s} e_b^{\Lambda};\\
C^1 &= \sum_{b\in t^{-1}(c)\cap E\Lambda} (\alpha+1)^{-s}(e_b^{\Lambda})^t = (\alpha+1)^{-s}(e_c^{\Lambda})^t;\\
C^2 &= \sum_{b\in t^{-1}(c)\cap E\Lambda} (\beta+1)^{-s}(e_b^{\Lambda})^t = (\beta+1)^{-s}(e_c^{\Lambda})^t.
\end{aligned} \tag{3.4.43}$$

Moreover, in (3.4.40) the only matrices that change are the following:

$$\begin{aligned}
I_2 - A &= \begin{bmatrix} 1-(\alpha+1)^{-s} & -\beta^{-s} \\ -\alpha^{-s} & 1-(\beta+1)^{-s} \end{bmatrix};\\
I_2 + (\underline{1} - I_2)A &= \begin{bmatrix} 1+\alpha^{-s} & (\beta+1)^{-s} \\ (\alpha+1)^{-s} & 1+\beta^{-s} \end{bmatrix}.
\end{aligned} \tag{3.4.44}$$

Analogously to (3.4.42), we deduce that

$$\mathcal{Z}_{\Gamma,a\to a}(s)^{-1} = \frac{\det\left(I_2 - A - \overbrace{B(I_\Lambda - D)^{-1}C}^{=X}\right)}{\det\left(I_2 + (\underline{1} - I_2)A + \underbrace{(\underline{1} - I_2)B(I_\Lambda - D)^{-1}C}_{=Y}\right)}. \tag{3.4.45}$$

In this case $X = Y$, because $B_1 = B_2$ and then $(\underline{1} - I_2)B = B$. Moreover, recalling Theorem 3.4.20 and Fact 3.4.21, we have

$$\begin{aligned} X(a,a) = X(\bar{a},a) &= B_1(I_\Lambda - D)^{-1}C^1 \\ &= (\alpha+1)^{-s} \cdot \sum_{b\in o^{-1}(c)\cap E\Lambda} \omega(b)^{-s} e_b^\Lambda (I_\Lambda - \mathcal{E}^\Lambda(s))^{-1}(e_c^\Lambda)^t \\ &= (\alpha+1)^{-s}(\mathcal{Z}_{\Lambda,c\to c}(s) - 1). \end{aligned} \tag{3.4.46}$$

Similarly,

$$X(a,\bar{a}) = X(\bar{a},\bar{a}) = B_1(I_\Lambda - D)^{-1}C^2 = (\beta+1)^{-s}(\mathcal{Z}_{\Lambda,c\to c}(s) - 1). \tag{3.4.47}$$

The statement now follows by (3.4.46), (3.4.47) and elementary algebraic manipulations. □

3.5 About the behaviour at $s = -1$

In [CCW24], the main motivation for introducing double-coset zeta functions pertained to their behaviour at $s = -1$, see [CCW24, Theorem E]. In the following result, we state the latter theorem in a slightly more general setting (i.e., we focus on groups having Bruhat decomposition of possibly non-uniform thickness).

Theorem 3.5.1. *Let G be a t.d.l.c. group having a Bruhat decomposition $G = BWB$ with respect to a compact open subgroup $B \leq G$. Assume that (G, P_I, P_I) has polynomial double-coset growth for some (and hence every) spherical subset $I \subseteq S$. Then, for every spherical subset $I \subseteq S$ we have*

$$\chi(G, \mu_{P_I}) = \zeta_{G,P_I,P_I}(-1)^{-1}.$$

Note that the assumption that B is compact open in G implies that the Bruhat decomposition $G = BWB$ is locally finite (i.e., $|BsB/B|$ is finite for every $s \in S$). Moreover, by [BRW05, Corollary 5], G is unimodular.

Proof. By Proposition 1.6.9(iii), for every spherical subset $I \subseteq S$ we have $|P_I : B| = W_I(\mathbf{q})$. Then $\mu_{P_I} = W_I(\mathbf{q})\mu_B$ and, by Remark 2.4.34,

$$\chi(G, \mu_B) = W_I(\mathbf{q}) \cdot \chi(G, \mu_{P_I}). \tag{3.5.1}$$

By Corollary 3.4.10, it suffices to prove the statement for $I = \emptyset$, i.e., for $P_I = B$.

By Proposition 2.4.39, we have

$$\chi(G, \mu_B) = \sum_{\substack{T \subseteq S \\ T \text{ spherical}}} \frac{1 - \chi(L_T)}{W_T(\mathbf{q})}, \tag{3.5.2}$$

where $\chi(L_T) = (-1)^{|T|+1} \cdot \sum_{T \subsetneq U \subseteq S,\, U \text{ spherical}} (-1)^{|U|}$. Moreover, for every spherical subset $T \subseteq S$, proceeding as in (3.5.1) and then by Corollary 3.4.6, we have that

$$\chi(P_T, \mu_B) = W_T(\mathbf{q}) \cdot \overbrace{\chi(P_T, \mu_{P_T})}^{=\mu_{P_T}(P_T)^{-1}=1} = \zeta_{P_T,B,B}(-1)^{-1}. \tag{3.5.3}$$

Combining (3.5.2), (3.5.3) and Theorem 3.4.11, we conclude that

$$\chi(G, \mu_B) = \zeta_{G,B,B}(-1)^{-1}. \qquad \square$$

The purpose of the following pages is to provide a result analogous to Theorem 3.5.1 in the case of unimodular t.d.l.c. groups having a weak locally ∞-transitive or (P)-closed proper and cocompact action on a locally finite tree. The proof of Theorem 3.5.1 delineates the pattern that we are also going to follow in this case. Namely,

(1) in analogy to (3.5.2), we first give a formula of the Euler–Poincaré characteristic for the relevant groups acting on trees in terms of local data of the action (cf. Proposition 3.5.4);

(2) after that, we prove some splitting formulae which allow us to reduce the problem to easier cases (cf. Lemmas 3.5.10 and 3.5.11);

(3) finally, in Section 3.5.2 we prove that a result analogous to Theorem 3.5.1 holds for the above-mentioned t.d.l.c. groups acting on trees (cf. Theorem 3.5.12 and Corollary 3.5.13).

3.5.1 Euler–Poincaré characteristic and groups acting on trees

In the present note, we focus on the case of t.d.l.c. groups acting properly and cocompactly on a tree. For these groups, their unimodularity and their Euler–Poincaré characteristic can be characterised in terms of local data of the action as shown in Proposition 3.5.2 and Proposition 3.5.4, respectively.

Proposition 3.5.2 ([BK90, Propositions 1.2 and 3.6], [Car18, §3.6])**.** *Let G be a t.d.l.c. group acting properly on a tree T. Let Γ be the quotient graph of (G, T), and denote by ω its standard edge weight. Then G is unimodular if, and only if, for every closed path $(a_1, \ldots, a_n)$ in Γ we have*

$$\prod_{i=1}^{n} \omega(a_i) = \prod_{i=1}^{n} \omega(\bar{a}_i). \tag{3.5.4}$$

Remark 3.5.3. Let $\mathfrak{p} = (a_1, \dots, a_m)$ and $\mathfrak{q} = (b_1, \dots, b_n)$ be reduced paths in Γ with $o(a_1) = o(b_1)$ and $t(a_m) = t(b_n)$. Hence $N_{\mathrm{vert}}(\mathfrak{p}) = \prod_{i=1}^{m} \omega(a_i)$. Moreover, $N_{\mathrm{edg}}(\mathfrak{p}) = 1$ if $m = 1$ and $N_{\mathrm{edg}}(\mathfrak{p}) = \prod_{i=2}^{m} \omega(a_i)$ if $m \geq 2$. Similar observations hold for $\overline{\mathfrak{p}}$, $\mathfrak{q}$ and $\overline{\mathfrak{q}}$. By Proposition 3.5.2, we deduce that

$$N_{\mathrm{vert}}(\mathfrak{p}) N_{\mathrm{vert}}(\overline{\mathfrak{q}}) = N_{\mathrm{vert}}(\mathfrak{q}) N_{\mathrm{vert}}(\overline{\mathfrak{p}}).$$

If in particular $a_m = b_n$, we also have

$$N_{\mathrm{vert}}(\mathfrak{p}) N_{\mathrm{edg}}(\overline{\mathfrak{q}}) = N_{\mathrm{vert}}(\mathfrak{q}) N_{\mathrm{edg}}(\overline{\mathfrak{p}}).$$

Moreover, if $a_1 = b_1$ and $a_m = b_n$ then

$$N_{\mathrm{edg}}(\mathfrak{p}) N_{\mathrm{edg}}(\overline{\mathfrak{q}}) = N_{\mathrm{edg}}(\mathfrak{q}) N_{\mathrm{edg}}(\overline{\mathfrak{p}}).$$

Proposition 3.5.4. *Let G be a unimodular t.d.l.c. group acting on a tree T with compact open vertex stabilisers, finite quotient graph Γ, and such that (G, T) is weakly locally ∞-transitive or (P)-closed. Let ω be the standard edge weight on Γ, and let $N_{\mathrm{vert}} = N_{\mathrm{vert}}^{\omega}$, $N_{\mathrm{edg}} = N_{\mathrm{edg}}^{\omega}$ be as in Definition 3.2.8. Let $c \in V\Gamma$ and $\Lambda \subseteq \Gamma$ be a maximal subtree, and consider an orientation $E\Lambda^+$ in Λ such that the restricted origin map $o\colon E\Lambda^+ \to V\Lambda \setminus \{c\}$ is a bijection. Let also $E\Gamma^+$ be an arbitrary orientation in Γ such that $E\Gamma^+ \cap \Lambda = E\Lambda^+$.*

Then, for every $v \in VT$ with $\pi(v) = c$, we have

$$\chi(G, \mu_{G_v}) = 1 + \sum_{a \in E\Gamma^+ \cap E\Lambda} (1 - \omega(a)) \frac{N_{\mathrm{vert}}(\mathfrak{p}_{c,o(a)})}{N_{\mathrm{vert}}(\overline{\mathfrak{p}_{c,o(a)}})} - \sum_{b \in E\Gamma^+ \setminus E\Lambda} \frac{N_{\mathrm{vert}}(\mathfrak{q}_{c,b})}{N_{\mathrm{edg}}(\overline{\mathfrak{q}_{c,b}})}, \tag{3.5.5}$$

for arbitrary reduced paths $\mathfrak{p}_{c,o(a)} \in \mathcal{P}^{\mathrm{lift}}_{\Gamma,v}(c \to o(a))$ and $\mathfrak{q}_{c,b} \in \mathcal{P}^{\mathrm{lift}}_{\Gamma,v}(c \to b)$, for all $a \in E\Gamma^+ \cap E\Lambda$ and $b \in E\Gamma^+ \setminus E\Lambda$.

Remark 3.5.5. In the following, we comment on the choices made in the statement of Proposition 3.5.4.

(i) Let Γ be a finite tree. For every $c \in V\Gamma$, there is an orientation $E\Gamma^+$ for which the origin map restricts to a bijection $o\colon E\Gamma^+ \to V\Gamma \setminus \{c\}$.

Indeed, let $\mathcal{E}^+$ be an arbitrary orientation in Γ and set

$$E\Gamma^+ := \{a \mid a \in \mathcal{E}^+,\, o(a) \neq c\} \sqcup \{\bar{a} \mid a \in \mathcal{E}^+,\, o(a) = c\}.$$

Then $E\Gamma^+$ is an orientation. Moreover, since $o(a) \neq t(a)$ for every $a \in E\Gamma$, the origin map in Γ restricts to a map $o\colon E\Gamma^+ \to V\Gamma \setminus \{c\}$. By [Ser80, §I.2, Proposition 12] we have $|E\Gamma^+| = |V\Gamma| - 1$ and then $o\colon E\Gamma^+ \to V\Gamma \setminus \{c\}$ is bijective.

(ii) By Remark 3.5.3, the right-hand side of (3.5.5) does not depend on the choice of specific reduced paths $\mathfrak{p}_{c,o(a)}$ and $\mathfrak{q}_{c,b}$ from c to $o(a)$ and from c to b, respectively.

(iii) A formula analogous to the one in (3.5.5) holds for $\chi(G, \mu_{G_e})$, $e \in ET$. Indeed, since $\mu_{G_e} = |G_v : G_e| \cdot \mu_{G_v} = \omega(\pi(e)) \cdot \mu_{G_v}$, by (2.4.18) we have

$$\chi(G, \mu_{G_e}) = \omega(\pi(e))^{-1}\chi(G, \mu_{G_{o(e)}}).$$

Then Proposition 3.5.4 applies.

Proof of Proposition 3.5.4. Let $\pi\colon T \to \Gamma$ be the quotient map and consider a set of representatives $\mathcal{E}^+ \subseteq ET$ for $E\Gamma^+$. Up to replacing elements of $E\Gamma^+ \setminus E\Lambda$ with their reverse, we may assume that for every $e \in \mathcal{E}^+$ with $\pi(e) \notin E\Lambda$ the geodesic from v to e is defined in T. Since $o\colon E\Lambda^+ \to V\Gamma \setminus \{c\}$ is bijective, note that $\mathcal{V} := \{v\} \sqcup \{o(e) \mid e \in \mathcal{E}^+ \text{ and } \pi(e) \in E\Lambda\}$ is a set of representatives for $V\Gamma$. Moreover, $\mu_{G_v}(G_v) = 1$ and, for every $e \in \mathcal{E}^+$,

$$\mu_{G_v}(G_{o(e)}) = |G_{o(e)} : G_e| \cdot \mu_{G_v}(G_e) = \omega(\pi(e)) \cdot \mu_{G_v}(G_e).$$

By Remark 2.4.38, we have

$$\begin{aligned}\chi(G, \mu_{G_v}) &= 1 + \sum_{\substack{e\in\mathcal{E}^+:\\ \pi(e)\in E\Lambda^+}} \left(\frac{1}{\mu_{G_v}(G_{o(e)})} - \frac{1}{\mu_{G_v}(G_e)}\right) - \sum_{\substack{e\in\mathcal{E}^+:\\ \pi(e)\in E\Gamma^+\setminus\Lambda}} \frac{1}{\mu_{G_v}(G_e)}\\ &= 1 + \sum_{\substack{e\in\mathcal{E}^+:\\ \pi(e)\in E\Lambda^+}} (1-\omega(\pi(e)))\frac{1}{\mu_{G_v}(G_{o(e)})} - \sum_{\substack{e\in\mathcal{E}^+:\\ \pi(e)\in E\Gamma^+\setminus\Lambda}} \frac{1}{\mu_{G_v}(G_e)}.\end{aligned}$$

Let $e \in \mathcal{E}^+$ and set $\pi(e) = a$. For $t \in \{o(e), e\}$, consider the geodesic $[v, t] = (e_1, \dots, e_n)$ in T lifting $\mathfrak{p}_{c,o(a)}$ if $t = o(e)$, and lifting $\mathfrak{q}_{c,a}$ if $t = e$. Then,

$$\begin{aligned}\frac{1}{\mu_{G_v}(G_t)} &= \frac{\overbrace{|G_v : G_v \cap G_t| \cdot \mu_{G_v}(G_v \cap G_t)}^{=\mu_{G_v}(G_v)=1}}{\underbrace{|G_t : G_v \cap G_t| \cdot \mu_{G_v}(G_v \cap G_t)}_{=\mu_{G_v}(G_t)}} = \frac{|G_v : G_{[v,t]}|}{|G_t : G_{\overline{[v,t]}}|}\\ &= \frac{|G_v \cdot e_1|}{|G_t \cdot \bar{e}_n|}\prod_{k=1}^{n-1}\frac{|G_{(e_1,\dots,e_k)} \cdot e_{k+1}|}{|G_{(\bar{e}_n,\dots,\bar{e}_{k+1})} \cdot \bar{e}_k|}.\end{aligned} \tag{3.5.6}$$

For the latter equality in (3.5.6), see (3.2.12) and (3.2.13). Note that $|G_v \cdot e_1| = \omega(\pi(e_1))$. Moreover, $|G_t \cdot \bar{e}_n| = 1$ if $t = e$ (because $e_n = t$) and $|G_t \cdot \bar{e}_n| = \omega(\pi(\bar{e}_n))$ if $t = o(e)$ (because $o(\bar{e}_n) = t$). For $1 \le k \le n-1$ we claim that

$$|G_{(e_1,\dots,e_k)} \cdot e_{k+1}| = |G_{e_k} \cdot e_{k+1}| \quad \text{and} \quad |G_{(\bar{e}_n,\dots,\bar{e}_{k+1})} \cdot \bar{e}_k| = |G_{\bar{e}_{k+1}} \cdot \bar{e}_k|. \tag{3.5.7}$$

If (G, T) is (P)-closed, (3.5.7) follows from (3.2.14). If (G, T) is weakly locally ∞-transitive, Remark 1.3.19(ii) and Proposition 3.2.12 yield

$$|G_{(e_1,\dots,e_k)} \cdot e_{k+1}| = N_{\mathrm{edg}}(\pi(e_k), \pi(e_{k+1})) = |G_{e_k} \cdot e_{k+1}|.$$

A similar argument holds for $|G_{(\bar{e}_n,\ldots,\bar{e}_{k+1})}\cdot\bar{e}_k|$.

For $1\le k\le n-1$, we now prove that

$$\frac{|G_{e_k}\cdot e_{k+1}|}{|G_{\bar{e}_{k+1}}\cdot\bar{e}_k|}=\frac{\omega(\pi(e_{k+1}))}{\omega(\pi(\bar{e}_k))}. \tag{3.5.8}$$

To see this, set $v_k=t(e_k)$ and $H=G_{e_k}\cap G_{e_{k+1}}$. Since $G_{e_k}=G_{\bar{e}_k}$ and $G_{e_{k+1}}=G_{\bar{e}_{k+1}}$, we have

$$|G_{v_k}:H|=|G_{v_k}:G_{e_{k+1}}|\cdot|G_{\bar{e}_{k+1}}:G_{(\bar{e}_{k+1},\bar{e}_k)}|=\omega(\pi(e_{k+1}))\cdot|G_{\bar{e}_{k+1}}\cdot\bar{e}_k|$$

and, at the same time,

$$|G_{v_k}:H|=|G_{v_k}:G_{\bar{e}_k}|\cdot|G_{e_k}:G_{(e_k,e_{k+1})}|=\omega(\pi(\bar{e}_k))\cdot|G_{e_k}\cdot e_{k+1}|.$$

Combining (3.5.6), (3.5.7) and (3.5.8), we deduce that

$$\frac{1}{\mu_{G_v}(G_t)}=\frac{\omega(\pi(e_1))}{|G_t\cdot\bar{e}_n|}\prod_{i=1}^{n-1}\frac{\omega(\pi(e_{k+1}))}{\omega(\pi(\bar{e}_k))}, \tag{3.5.9}$$

where $|G_t\cdot\bar{e}_n|$ equals 1 if $t=e$, and it equals $\omega(\pi(\bar{e}_n))$ if $t=o(e)$. By design, $\pi([v,t])=:(a_1,\ldots,a_n)$ is a reduced path in Γ. Then

$$N_{\mathrm{vert}}(\pi([v,t]))=\prod_{i=1}^{n}\omega(a_i)\ \text{ and }\ N_{\mathrm{edg}}(\pi([v,t]))=\begin{cases}1, & \text{if } n=1;\\ \prod_{i=2}^{n}\omega(a_i), & \text{if } n\ge 2.\end{cases} \tag{3.5.10}$$

By (3.5.9) and (3.5.10), we conclude that

$$\frac{1}{\mu_{G_v}(G_{o(e)})}=\frac{N_{\mathrm{vert}}(\mathfrak{p}_{c,o(a)})}{N_{\mathrm{vert}}(\overline{\mathfrak{p}_{c,o(a)}})},\quad\forall\, e\in\mathcal{E}^+\text{ with }\pi(e)\in E\Lambda;$$
$$\frac{1}{\mu_{G_v}(G_e)}=\frac{N_{\mathrm{vert}}(\mathfrak{q}_{c,a})}{N_{\mathrm{edg}}(\overline{\mathfrak{q}_{c,a}})},\quad\forall\, e\in\mathcal{E}^+\text{ with }\pi(e)\notin E\Lambda.$$ ☐

Remark 3.5.6. Let (G_1,T_1) and (G_2,T_2) be group actions on trees that satisfy the hypotheses of Proposition 3.5.4. Let (Γ_1,ω_1) and (Γ_2,ω_2) be the quotient graphs of (G_1,T_1) and (G_2,T_2) endowed with their standard edge weights, respectively. Assume there is a graph isomorphism $\varphi\colon\Gamma_1\to\Gamma_2$ such that $\omega_2(\varphi(a))=\omega_1(a)$ for every $a\in E\Gamma_1$. Let $v_1\in VT_1$ and $v_2\in VT_2$ be vertices satisfying $G_1\cdot v_1=c_1$ and $G_2\cdot v_2=\varphi(c_1)$. By Proposition 3.5.4,

$$\chi(G_1,\mu_{(G_1)_{v_1}})=\chi(G_2,\mu_{(G_2)_{v_2}}),$$

where $\mu_{(G_i)_{v_i}}$ is the Haar measure of G_i normalised with respect to $(G_i)_{v_i}$.

A notable consequence of Proposition 3.5.4 is that the value $\chi(G,\mu_{G_v})$ depends only on (Γ,ω). This suggests the following definition.

Definition 3.5.7. Let Γ be a finite connected non-empty graph with an edge weight $\omega\colon E\Gamma \to \mathbb{Z}_{\geq 1}$. Let (T, π) be the universal cover of (Γ, ω), and set $G = \mathrm{Aut}_\pi(T)$ (cf. Example 1.3.1). The pair (Γ, ω) is said to be *unimodular* if $\mathrm{Aut}_\pi(T)$ is unimodular.

Let (Γ, ω) be unimodular. For all $c \in V\Gamma$ and $a \in E\Gamma$ and given arbitrary $v \in VT$ and $e \in ET$ satisfying $\pi(v) = c$ and $\pi(e) = a$, define

$$\chi(\Gamma, c) := \chi(G, \mu_{G_v}) \quad \text{and} \quad \chi(\Gamma, a) := \chi(G, \mu_{G_e}). \tag{3.5.11}$$

Since G is unimodular, the assignments in (3.5.11) do not depend on the choice of $v \in \pi^{-1}(c)$ and of $e \in \pi^{-1}(a)$, respectively.

Remark 3.5.8. Let G be a unimodular t.d.l.c. group acting on a tree T with compact open vertex-stabilisers and finite quotient graph Γ. Denote by ω the standard edge weight, and assume that (G, T) is weakly locally ∞-transitive or (P)-closed. For every $t \in T$ with $G \cdot t = u$, from Proposition 3.5.4 we have

$$\chi(\Gamma, u) = \chi(G, \mu_{G_t}).$$

Example 3.5.9.

(i) Let Γ be a 1-segment with $E\Gamma = \{a, \bar{a}\}$. Since Γ is a tree, (Γ, ω) is unimodular for every $\omega\colon E\Gamma \to \mathbb{Z}_{\geq 1}$ (cf. Proposition 3.5.2). Moreover, Γ is its only maximal subtree. Consider an edge weight $\omega\colon E\Gamma \to \mathbb{Z}_{\geq 1}$. Set $c = o(a)$, $E\Gamma^+ = \{\bar{a}\}$ and let $\mathfrak{p}_{c,o(\bar{a})}$ be the 1-edge path a. Then Proposition 3.5.4 implies that

$$\chi(\Gamma, c) = 1 + (1 - \omega(\bar{a}))\frac{\omega(a)}{\omega(\bar{a})}.$$

With a similar strategy, one computes $\chi(\Gamma, t(a))$.

(ii) Let Γ be a n-bouquet of loops based on the vertex c. Note that the 1-point subgraph is the only maximal subtree of Γ. By Proposition 3.5.2, for an edge weight $\omega\colon E\Gamma \to \mathbb{Z}_{\geq 1}$ the pair (Γ, ω) is unimodular if, and only if, $\omega(a) = \omega(\bar{a})$ for every $a \in E\Gamma$. Provided (Γ, ω) is unimodular and $E\Gamma = \{a_i, \bar{a}_i \mid 1 \leq i \leq n\}$, from Proposition 3.5.4 we deduce that

$$\chi(\Gamma, c) = 1 - \sum_{i=1}^{n} \omega(a_i).$$

Lemma 3.5.10. *Let Γ be a finite connected non-empty graph, and let $\omega\colon E\Gamma \to \mathbb{Z}_{\geq 2}$ be such that (Γ, ω) is unimodular. Then, for every $a \in E\Gamma$,*

$$\chi(\Gamma, o(a)) = \omega(a) \cdot \chi(\Gamma, a). \tag{3.5.12}$$

Moreover, for all $c, d \in V\Gamma$,

$$\chi(\Gamma, c) = \frac{N_{\mathrm{vert}}(\mathfrak{p})}{N_{\mathrm{vert}}(\bar{\mathfrak{p}})}\chi(\Gamma, d), \tag{3.5.13}$$

where $\mathfrak{p}$ is any reduced path in Γ from c to d. Similarly, for all $a, b \in E\Gamma$ for which there is a reduced path in Γ from a to b, we have

$$\chi(\Gamma, a) = \frac{N_{\mathrm{edg}}(\mathfrak{q})}{N_{\mathrm{edg}}(\overline{\mathfrak{q}})} \chi(\Gamma, b), \tag{3.5.14}$$

where $\mathfrak{q}$ is any reduced path from a to b in Γ.

In Lemma 3.5.10, since Γ is connected, replacing a with $\bar{a}$ or b with $\bar{b}$ if necessary, we can always find a reduced path from a to b in Γ.

Proof. First, (3.5.12) follows from Remark 3.5.5. By Remark 3.5.3, the ratios in (3.5.13) and (3.5.14) do not depend on the choices of $\mathfrak{p}$ and $\mathfrak{q}$, respectively. Moreover, if we prove (3.5.13) and (3.5.14) for $\ell(\mathfrak{p}) = 1$ and $\ell(\mathfrak{q}) = 2$, the general statements follow iteratively. It remains to observe what follows. First, for every $a \in E\Gamma$ we have

$$\chi(\Gamma, o(a)) = \omega(a) \cdot \chi(\Gamma, a) = \frac{\omega(a)}{\omega(\bar{a})} \chi(\Gamma, t(a)). \tag{3.5.15}$$

Moreover, let (a, b) is a length-2 reduced path in Γ and set $t(a) = c = o(b)$. Then (3.5.12) and (3.5.15) imply that

$$\chi(\Gamma, a) = \chi(\Gamma, \bar{a}) = \frac{1}{\omega(\bar{a})} \chi(\Gamma, c) = \frac{\omega(b)}{\omega(\bar{a})} \chi(\Gamma, b). \qquad \square$$

Lemma 3.5.11. *Let Γ be a finite connected non-empty graph with an edge weight $\omega \colon E\Gamma \to \mathbb{Z}_{\geq 1}$ such that (Γ, ω) is unimodular. Suppose that there are connected subgraphs Γ_1 and Γ_2 of Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \{c\}$, for some $c \in V\Gamma$. Then $(\Gamma_i, \omega|_{E\Gamma_i})$ is unimodular for every $i \in \{1, 2\}$, and*

$$\chi(\Gamma, c) = \chi(\Gamma_1, c) + \chi(\Gamma_2, c) - 1. \tag{3.5.16}$$

Proof. Let Λ be a maximal subtree of Γ. We claim that $\Lambda_i := \Gamma_i \cap \Lambda$ is a maximal subtree of Γ_i, for every $i \in \{1, 2\}$. Clearly, both Λ_1 and Λ_2 are subtrees of Γ_1 and Γ_2, respectively. We prove the maximality for $i = 1$, as for $i = 2$ one may proceed analogously. For every subtree $\Xi_1 \subseteq \Gamma_1$ with $\Xi_1 \supseteq \Lambda_1$, we have $\Xi_1 \cap \Lambda_2 = \{c\}$ and thus $\Xi_1 \cup \Lambda_2$ is a subtree of Γ containing Λ. Hence $\Lambda = \Xi_1 \cup \Lambda_2$ and

$$\Lambda_1 = \Gamma_1 \cap \Lambda = (\Gamma_1 \cap \Xi_1) \cup (\Gamma_1 \cap \Lambda_2) = \Xi_1 \cup \{c\} = \Xi_1.$$

Consider an orientation $E\Gamma^+$ in $E\Gamma$ such that the origin map in Γ restricts to a bijection $o \colon E\Gamma^+ \cap E\Lambda \to V\Gamma \setminus \{c\}$ (cf. Remark 3.5.5). For every $i \in \{1, 2\}$, the set $E\Gamma_i^+ := E\Gamma^+ \cap E\Gamma_i$ is an orientation in $E\Gamma_i$ and the origin map in Γ_i restricts to a bijection $o_i \colon E\Gamma_i^+ \cap E\Lambda_i \to V\Gamma_i \setminus \{c\}$. By Proposition 3.5.4, we conclude that

$$\begin{aligned} \chi(\Gamma, c) &= 1 + \sum_{i=1}^{2} \left(\sum_{a \in E\Gamma_i^+ \cap E\Lambda_i} (1 - \omega(a)) \frac{N_{\mathrm{vert}}(\mathfrak{p}_{c,o(a)})}{N_{\mathrm{vert}}(\overline{\mathfrak{p}_{c,o(a)}})} - \sum_{a \in E\Gamma_i^+ \setminus E\Lambda_i} \frac{N_{\mathrm{vert}}(\mathfrak{q}_{c,a})}{N_{\mathrm{edg}}(\overline{\mathfrak{q}_{c,a}})} \right) \\ &= \chi(\Gamma_1, c) + \chi(\Gamma_2, c) - 1. \qquad \square \end{aligned}$$

3.5.2 The evaluation at $s = -1$ and the Euler–Poincaré characteristic

The main goal of the next pages is to prove the following theorem.

Theorem 3.5.12. *Let (Γ, ω) satisfy Setting [Γ]. If (Γ, ω) is unimodular, then*

$$\chi(\Gamma, u) = \mathcal{Z}_{\Gamma, u\to u}(-1)^{-1}, \quad \forall\, u \in \Gamma.$$

By Theorem 3.5.12 and Lemma 3.5.16, we deduce the following.

Corollary 3.5.13. *Let G be a unimodular t.d.l.c. group acting on a locally finite tree T with compact open vertex stabilisers. Assume that the quotient graph is finite and does not have cycles of length ≥ 2. Suppose also that (G, T) is weakly locally ∞-transitive or (P)-closed. Then, for every $t \in T$ such that (G, G_t, G_t) has polynomial double-coset growth, we have*

$$\widetilde{\chi}_G = \zeta_{G,G_t,G_t}(-1)^{-1}\mu_{G_t}.$$

In view of Theorem 3.5.12, we first formulate a version of Lemma 3.5.10 for $\mathcal{Z}_{\Gamma,u\to u}(-1)^{-1}$.

Lemma 3.5.14. *Let (Γ, ω) be a unimodular edge-weighted graph satisfying Setting [Γ] and such that Γ has no cycles of length ≥ 2. Then, for every $a \in E\Gamma$,*

$$\mathcal{Z}_{\Gamma,o(a)\to o(a)}(-1)^{-1} = \omega(a) \cdot \mathcal{Z}_{\Gamma,a\to a}(-1)^{-1}. \tag{3.5.17}$$

Moreover, for all $c, d \in V\Gamma$ and all $a, b \in E\Gamma$ such there is a reduced path from a to b in Γ, we have

$$\begin{aligned} \mathcal{Z}_{\Gamma,c\to c}(-1)^{-1} &= \frac{N_{\mathrm{vert}}(\mathfrak{p})}{N_{\mathrm{vert}}(\overline{\mathfrak{p}})}\mathcal{Z}_{\Gamma,d\to d}(-1)^{-1}, \\ \mathcal{Z}_{\Gamma,a\to a}(-1)^{-1} &= \frac{N_{\mathrm{edg}}(\mathfrak{q})}{N_{\mathrm{edg}}(\overline{\mathfrak{q}})}\mathcal{Z}_{\Gamma,b\to b}(-1)^{-1}, \end{aligned} \tag{3.5.18}$$

where $\mathfrak{p}$ and $\mathfrak{q}$ are arbitrary reduced paths in Γ from c to d and from a to b, respectively.

Proof. Once proved (3.5.17) (which is analogous to (3.5.12)), arguing as in the proof of Lemma 3.5.10 one can deduce (3.5.18). We first prove (3.5.17) for every 1-loop a. Namely, let $a \in E\Gamma$ with $o(a) = t(a) = c$. Since (Γ, ω) is unimodular, note that $\omega(a) = \omega(\bar{a})$. Let Λ be the graph obtained from Γ removing a and $\bar{a}$, and let L_a be the subgraph of Γ with $VL_a = \{c\}$ and $EL_a = \{a, \bar{a}\}$. By Remark 3.4.41, we have

$$\mathcal{Z}_{\Gamma,a\to a}(-1)^{-1} = \omega(a)^{-1} \cdot \Big(\mathcal{Z}_{\Lambda,c\to c}(-1)^{-1} - \omega(a)\Big). \tag{3.5.19}$$

Moreover, Proposition 3.4.33 and (3.4.15) yield

$$\mathcal{Z}_{\Gamma,c\to c}(-1)^{-1} = \mathcal{Z}_{\Lambda,c\to c}(-1)^{-1} - \omega(a). \tag{3.5.20}$$

Combining (3.5.19) and (3.5.20), we deduce (3.5.17).

For all edges a with $o(a) \neq t(a)$, the relation in (3.5.17) is proved by induction on $|E\Gamma|/2 =: k(\Gamma) \geq 1$. If $k(\Gamma) = 1$, then Γ is a 1-segment and (3.5.17) follows from (3.4.13). Let $k(\Gamma) \geq 2$ and assume that the claim holds for every graph Γ' with $k(\Gamma') < k(\Gamma)$. Let $a \in E\Gamma$ be such that $o(a) =: c \neq d := t(a)$. If $o^{-1}(c) = \{a\}$, then Proposition 3.4.39 directly implies the claim. In case that $o^{-1}(d) = \{\bar{a}\}$, let Λ be the graph obtained from Γ by removing a and $\bar{a}$. Then Proposition 3.4.39 yields

$$\mathcal{Z}_{\Gamma,a\to a}(-1)^{-1} = \mathcal{Z}_{\Gamma,\bar{a}\to\bar{a}}(-1)^{-1} = \frac{\mathcal{Z}_{\Lambda,c\to c}(-1)^{-1}}{\omega(a)} - \frac{\omega(\bar{a}) - 1}{\omega(\bar{a})}. \tag{3.5.21}$$

On the other hand, let Γ_a denote the 1-segment subgraph of Γ with $E\Gamma_a = \{a, \bar{a}\}$. By Proposition 3.4.33 and Example 3.4.22,

$$\begin{aligned}\mathcal{Z}_{\Gamma,c\to c}(-1)^{-1} &= \mathcal{Z}_{\Lambda,c\to c}(-1)^{-1} + \mathcal{Z}_{\Gamma_a,c\to c}(-1)^{-1} - 1 \\ &= \mathcal{Z}_{\Lambda,c\to c}(-1)^{-1} + \omega(a)\frac{\omega(\bar{a}) - 1}{\omega(\bar{a})}.\end{aligned} \tag{3.5.22}$$

Hence (3.5.17) follows from (3.5.21) and (3.5.22). Finally, assume that both $|o^{-1}(c)| \geq 2$ and $|o^{-1}(d)| \geq 2$. Denote by Ξ_1 and Ξ_2 be the connected components of $\Gamma \setminus \{a, \bar{a}\}$ containing c and d, respectively. There are exactly two connected components because Γ has no cycles of length ≥ 2. Since $|o^{-1}(c)| \geq 2$ and $|o^{-1}(d)| \geq 2$, both $E\Xi_1$ and $E\Xi_2$ are non-empty. Moreover, $E\Gamma = E\Xi_1 \sqcup \{a, \bar{a}\} \sqcup E\Xi_2$. For $i \in \{1, 2\}$, let Γ_i be the smallest subgraph of Γ containing $\Lambda_i \cup \{a, \bar{a}\}$, and note that $k(\Gamma_i) < k(\Gamma)$. Let also Γ_a be the 1-segment subgraph with edge set $\{a, \bar{a}\}$, and observe that $\Gamma_1 \cap \Gamma_2 = \Gamma_a$. Moreover, if $\Lambda_1 = \Xi_1$ and $\Lambda_2 = \Gamma_2$, we have $\Gamma = \Lambda_1 \cup \Lambda_2$, $\Lambda_1 \cap \Lambda_2 = \{c\}$ and $\Lambda_i \subseteq \Gamma_i$ for every $i \in \{1, 2\}$. Hence, Corollary 3.4.34 and Corollary 3.4.37 imply

$$\begin{aligned}\mathcal{Z}_{\Gamma,c\to c}(-1)^{-1} &= \mathcal{Z}_{\Gamma_1,c\to c}(-1)^{-1} + \mathcal{Z}_{\Gamma_2,c\to c}(-1)^{-1} - \mathcal{Z}_{\Gamma_a,c\to c}(-1)^{-1}; \\ \mathcal{Z}_{\Gamma,a\to a}(-1)^{-1} &= \mathcal{Z}_{\Gamma_1,a\to a}(-1)^{-1} + \mathcal{Z}_{\Gamma_2,a\to a}(-1)^{-1} - \mathcal{Z}_{\Gamma_a,a\to a}(-1)^{-1}.\end{aligned}$$

The induction hypothesis now yields (3.5.17). □

By Lemma 3.5.10 and Lemma 3.5.14, we deduce the following.

Corollary 3.5.15. *Let (Γ, ω) be a unimodular edge-weighted graph satisfying Setting [Γ] and such that Γ has no cycles of length ≥ 2. If $\chi(\Gamma, u) = \mathcal{Z}_{\Gamma,u\to u}(-1)^{-1}$ for some $u \in \Gamma$, then $\chi(\Gamma, u) = \mathcal{Z}_{\Gamma,u\to u}(-1)^{-1}$ for every $u \in \Gamma$.*

Proof. Let $u \in \Gamma$. Since Γ is connected, for all $c \in V\Gamma$ and $a \in E\Gamma$ there are reduced paths $\mathfrak{p} \in \mathcal{P}_\Gamma(U \to c)$ and $\mathfrak{q} \in \mathcal{P}_\Gamma(U \to A)$ (cf. Notation 1.2.1). Moreover, note that $\mathcal{Z}_{\Gamma,b\to b}(s) = \mathcal{Z}_{\Gamma,\bar{b}\to\bar{b}}(s)$ and $\chi(\Gamma, b) = \chi(\Gamma, \bar{b})$ for every $b \in E\Gamma$. Hence, we may assume that $\mathfrak{p} \in \mathcal{P}_\Gamma(u \to c)$ and $\mathfrak{q} \in \mathcal{P}_\Gamma(u \to a)$. The statement follows from Lemma 3.5.10 and Lemma 3.5.14. □

Proof of Theorem 3.5.12. By Corollary 3.5.15, it suffices to prove that

$$\mathcal{Z}_{\Gamma,c\to c}(-1)^{-1} = \chi(\Gamma, c) \tag{3.5.23}$$

for some vertex $c \in V\Gamma$. We prove it by induction on $|E\Gamma|/2 =: k(\Gamma) \geq 1$. Let first $k(\Gamma) = 1$, i.e., $E\Gamma = \{a, \bar{a}\}$. If $o(a) \neq t(a)$, Example 3.4.22 and Example 3.5.9(i) yield

$$\mathcal{Z}_{\Gamma,c\to c}(-1)^{-1} = 1 + (1 - \omega(\bar{a}))\frac{\omega(a)}{\omega(\bar{a})} = \chi(\Gamma, c).$$

If $o(a) = t(a)$, from (3.4.15) and Example 3.5.9(ii) we deduce that

$$\mathcal{Z}_{\Gamma,c\to c}(-1)^{-1} = 1 - \omega(a) = \chi(\Gamma, c).$$

Let now $k(\Gamma) \geq 2$ and assume that the statement holds for all graphs Γ' with $k(\Gamma') < k(\Gamma)$. Without loss of generality, we may take $c \in V\Gamma$ such that $|o^{-1}(c)| \geq 2$. Note that this vertex exists because $k(\Gamma) \geq 2$. By Lemma 3.4.35, there are proper connected subgraphs Λ_1 and Λ_2 of Γ such that $\Gamma = \Lambda_1 \cup \Lambda_2$ and $\Lambda_1 \cap \Lambda_2 = \{c\}$. Then Proposition 3.4.33 and Lemma 3.5.11 yield the claim. □

In view of the proof of Corollary 3.5.13, we observe what follows.

Lemma 3.5.16. *Let $\Delta = (\Gamma, (X_a), (G(c)))$ be a local action diagram. Let $G = U(\Delta, \iota, c_0)$ and $T = T(\Delta, \iota, c_0)$ be as in Setting [(P)-cl]. Let $\tilde{\Delta} = (\Gamma, (X_a), (\tilde{G}(c)))$ be a local action diagram such that $\tilde{G} := U(\tilde{\Delta}, \iota, c_0)$ acts weakly locally ∞-transitively on T. Then, for all $t_1, t_2 \in T$ we have*

$$\zeta_{G,G_{t_1},G_{t_2}}(-1) = \zeta_{\tilde{G},\tilde{G}_{t_1},\tilde{G}_{t_2}}(-1).$$

By Example 1.3.22(i), note that $\tilde{\Delta}$ as in Lemma 3.5.16 exists.

Proof. Both (G, T) and $(\tilde{G}, T)$ are (P)-closed actions on trees satisfying Setting [(P)-cl]. The statement now follows by applying Theorem 3.4.29 to both $\zeta_{G,G_{t_1},G_{t_2}}(s)$ and $\zeta_{\tilde{G},\tilde{G}_{t_1},\tilde{G}_{t_2}}(s)$. In detail, in both cases the matrices $\mathcal{F}(0)$ and $\mathcal{Y}_{\pi(t_1),\pi(t_2)}(0)$ involved in the statement of Theorem 3.4.29, as well as the integer $\kappa_{\pi(t_1)}(\pi(t_2))$, depend only on $X = \bigsqcup_{a\in E\Gamma} X_a$, on Γ and its standard edge weight ω, and on the inversion map ι. The latter quantities do not vary by passing from Δ to $\tilde{\Delta}$, and the statement follows. □

Proof of Corollary 3.5.13. By Remark 3.4.24 and Lemma 3.5.16, it suffices to prove the statement for (G, T) being weakly locally ∞-transitive. In this case, the claim follows from Remark 3.4.16, Remark 3.5.8 and Theorem 3.5.12. □

3.5.3 The behaviour at $s = -1$ and the Ihara zeta function of a weighted graph

In [Dei19, §3], a generalisation of the classical Ihara zeta function has been defined for every finite graph Γ with a transition weight. Although it is not necessary here, we mention that the finiteness hypothesis on Γ can be relaxed. According to [Dei19, Definition 3.3], a *transition weight* on a finite graph Γ is a map $W\colon E\Gamma \times E\Gamma \longrightarrow \mathbb{R}_{\geq 0}$ such that, whenever $W(a,b) \neq 0$, then $t(a) = o(b)$.

According to the definition of graph in [Dei19] (cf. [Dei19, Definition 3.1]), every edge is supposed to be uniquely determined by its endpoints. However, for the results involved below, this hypothesis has no influence and thus we do not assume it.

The *Ihara zeta function* $Z_{(\Gamma,W)}(x)$ of (Γ, W) has been defined in [Dei19, Definition 3.8] as a suitable infinite product of meromorphic functions on $\mathbb{C}$ converging for all $x \in \mathbb{C}$ with $|x| \ll 1$. Here we only need the following characterisation of the reciprocal of $Z_{(\Gamma,W)}(x)$, cf. [Dei19, Theorem 3.11]. Namely,

$$Z_{(\Gamma,W)}(x)^{-1} = \det(I - xT), \tag{3.5.24}$$

where I is the identity matrix of dimension $|E\Gamma|$ and $T = [T(a,b)]_{a,b\in E\Gamma} \in \mathrm{Mat}_n(\mathbb{R})$ is defined as $T(a,b) = W(a,b)$ for all $a, b \in E\Gamma$ (assuming to have set a total order on $E\Gamma$). The matrix T is called the *Bass operator* of (Γ, W) (cf. [Dei19, Definition 3.10]). Note that (3.5.24) gives a meromorphic continuation of $Z_{(\Gamma,W)}(x)$ to $\mathbb{C}$.

Example 3.5.17. Let Γ be a finite graph with an edge weight $\omega\colon E\Gamma \to \mathbb{Z}_{\geq 2}$. Let $N_{\mathrm{edg}} = N^{\omega}_{\mathrm{edg}}$ and $\mathcal{E}(s)$ be as in Definition 3.2.8 and Definition 3.4.17, respectively. Consider the map $W = W_{(\Gamma,\omega)}\colon E\Gamma \times E\Gamma \to \mathbb{Z}_{\geq 0}$ defined by

$$W(a,b) := \mathcal{E}(-1)(a,b) = \begin{cases} N_{\mathrm{edg}}(a,b), & \text{if } t(a) = o(b); \\ 0, & \text{otherwise.} \end{cases}$$

Then W yields a transition weight on Γ. Note that assuming that $\omega(a) \geq 2$ is necessary to have $W(\bar{a}, a) \neq 0$, for all $a \in E\Gamma$. In particular, by (3.5.24) we have

$$Z_{(\Gamma,W)}(x)^{-1} = \det(I - x\mathcal{E}(-1)).$$

Theorem 3.5.18. *Let (Γ, ω) be an edge-weighted graph satisfying Setting [Γ]. Let Γ_1, Γ_2 be subgraphs of Γ satisfying $\Gamma = \Gamma_1 \cup \Gamma_2$ and such that $\Gamma_1 \cap \Gamma_2$ is a 1-segment with edge set $\{a, \bar{a}\}$. Assume also that $t(a)$ and $o(a)$ are terminal vertices in Γ_1 and Γ_2, respectively. Let W, W_1 and W_2 be the transition weights defined in Example 3.5.17 on Γ, Γ_1 and Γ_2, respectively. Then,*

$$\frac{\mathcal{Z}_{\Gamma,a\to a}(-1)}{\mathcal{Z}_{\Gamma_1,a\to a}(-1)\cdot \mathcal{Z}_{\Gamma_2,a\to a}(-1)} = \frac{1}{\omega(a)\omega(\bar{a})}\cdot \frac{Z_{(\Gamma,W)}(1)}{Z_{(\Gamma_1,W_1)}(1)\cdot Z_{(\Gamma_2,W_2)}(1)}.$$

Proof. Denote by $\mathcal{E}(-1)$, $\mathcal{E}_1(-1)$, $\mathcal{E}_2(-1)$ the Bass operators of Γ, Γ_1 and Γ_2 at -1. Let also I, I_1 and I_2 denote the identity matrices with complex entries of dimension $|E\Gamma|$, $|E\Gamma_1|$ and $|E\Gamma_2|$, respectively. By Theorem 3.4.20 and Example 3.5.17,

$$\mathcal{Z}_{\Gamma,a\to a}(-1) = Z_{(\Gamma,W)}(1)\cdot \det(I-M). \tag{3.5.25}$$

where

$$M = [M(h,k)]_{h,k\in E\Gamma} := \mathcal{E}(-1) - \mathcal{U}_{a,a}(-1) = (I-(e_a+e_{\bar a})^t(e_a+e_{\bar a}))\mathcal{E}(-1).$$

For all $h,k\in E\Gamma$, observe that

$$\begin{aligned} M(h,k) &= e_h\Big(I-(e_a+e_{\bar a})^t(e_a+e_{\bar a})\Big)\mathcal{E}(-1)e_k^t \\ &= e_h\mathcal{E}(-1)e_k^t - e_h(e_a^t+e_{\bar a}^t)\Big((e_a+e_{\bar a})\mathcal{E}(-1)e_k^t\Big) \\ &= \mathcal{E}(-1)(h,k) - \mathbb{1}_{\{a,\bar a\}}(h)\Big(\mathcal{E}(-1)(a,k)+\mathcal{E}(-1)(\bar a,k)\Big). \end{aligned} \tag{3.5.26}$$

Similarly, for every $i\in\{1,2\}$ we have

$$\mathcal{Z}_{\Gamma_i,a\to a}(-1) = Z_{(\Gamma_i,W_i)}(1)\cdot\det(I_i-M_i), \tag{3.5.27}$$

where $M_i = [M_i(h,k)]_{h,k\in E\Gamma_i}$ is the $|E\Gamma_i|$-dimensional given by

$$M_i(h,k) = \mathcal{E}_i(-1)(h,k) - \mathbb{1}_{\{a,\bar a\}}(h)\Big(\mathcal{E}_i(-1)(a,k)+\mathcal{E}_i(-1)(\bar a,k)\Big). \tag{3.5.28}$$

By (3.5.26) and (3.5.28), for every $i\in\{1,2\}$ we deduce that

$$M(h,k) = M_i(h,k), \quad \forall\, h,k\in E\Gamma_i. \tag{3.5.29}$$

Let $\tilde M_1 := [M(h,k)]_{h,k\in E\Gamma_1\setminus\{\bar a\}}$ and $\tilde M_2 := [M(h,k)]_{h,k\in E\Gamma_2\setminus\{a\}}$. Set also $\tilde I_1$ and $\tilde I_2$ be the identity matrices in $\mathrm{Mat}_{|E\Gamma_1|-1}(\mathbb{C})$ and $\mathrm{Mat}_{|E\Gamma_2|-1}(\mathbb{C})$, respectively. We claim that M, M_1 and M_2 have the following decompositions in diagonal blocks:

$$M = \begin{bmatrix}\tilde M_1 & 0\\ 0 & \tilde M_2\end{bmatrix};\ M_1 = \begin{bmatrix}\tilde M_1 & 0\\ 0 & M(\bar a,\bar a)\end{bmatrix};\ M_2 = \begin{bmatrix}M(a,a) & 0\\ 0 & \tilde M_2\end{bmatrix}. \tag{3.5.30}$$

Before proving (3.5.30), we use it to conclude the argument. From (3.5.30) we deduce that

$$\begin{aligned} \det(I-M) &= \det(\tilde I_1-\tilde M_1)\cdot\det(\tilde I_2-\tilde M_2) \\ &= \frac{\det(I_1-M_1)\cdot\det(I_2-M_2)}{(1-M(\bar a,\bar a)(1-M(a,a)))}. \end{aligned} \tag{3.5.31}$$

Moreover, (3.5.26) yields $M(a,a) = -\mathcal{E}(-1)(\bar a,a) = 1-\omega(a)$ and $M(\bar a,\bar a) = -\mathcal{E}(-1)(a,\bar a) = 1-\omega(\bar a)$. Combining (3.5.25), (3.5.27) and (3.5.31) we conclude the statement.

It remains to prove (3.5.30). By (3.5.28), it suffices to show that $M(h,k) = 0$ if either $(h,k) \in (E\Gamma_1 \setminus \{\bar{a}\}) \times (E\Gamma_2 \setminus \{a\})$ or $(h,k) \in (E\Gamma_2 \setminus \{a\}) \times (E\Gamma_1 \setminus \{\bar{a}\})$. Recall that the only edge of Γ_1 (resp. Γ_2) ending at $t(a)$ (resp. $o(a)$) is a (resp. $\bar{a}$). Hence, if $h \in E\Gamma_1 \setminus \{a, \bar{a}\}$ then $t(h) \in V\Gamma_1 \setminus \{t(a)\}$, and every $k \in E\Gamma_2 \setminus \{a\}$ satisfies $o(k) \in V\Gamma_2 \setminus \{o(a)\}$. Since $V\Gamma_1 \setminus \{t(a)\}$ and $V\Gamma_2 \setminus \{o(a)\}$ are disjoint, for such h and k we have $t(h) \neq o(k)$ and (3.5.26) implies $M(h,k) = \mathcal{E}(-1)(h,k) = 0$. Similarly, if $h \in E\Gamma_2 \setminus \{a, \bar{a}\}$ and $k \in E\Gamma_1 \setminus \{\bar{a}\}$, then we have $M(h,k) = \mathcal{E}(-1)(h,k) = 0$. Moreover, $M(a,k) = -\mathcal{E}(-1)(\bar{a},k) = 0$ for every $k \in E\Gamma_2 \setminus \{a\}$ as $t(\bar{a}) \neq o(k)$. Similarly, $M(\bar{a},k) = -\mathcal{E}(-1)(a,k) = 0$ for every $k \in E\Gamma_1 \setminus \{\bar{a}\}$. □

Chapter 4

Submodule zeta functions of algebras of endomorphisms

4.1 Structure of the chapter

In Section 4.2 we introduce some general definitions and results about the submodule zeta functions of algebras of endomorphisms. In particular, Section 4.2.3 (especially Theorem 4.2.16) provides the preliminary notions and motivations in view of Section 4.3. Moreover, in Section 4.2.6 the reader can find some general results about p-adic integration that might be helpful for the computations in Section 4.3.1.

In Section 4.3, we focus on the case in which the ring of coefficients is the ring of integers of a non-Archimedean local field. We introduce a multivariate version of the relevant submodule zeta function (cf. Definition 4.3.1), which is used to prove one of the main results of the chapter: Theorem 4.3.22. For a quick reading, the reader might read only Definition 4.3.1 and go directly to Section 4.3.2. In between, explicit computations support the fact that the hypothesis in Theorem 4.3.22 seems to occur often.

Section 4.4 deals with the case in which the coefficient ring is a finite field. In view of Section 4.4, it might be convenient to read Section 4.2.4 for motivation. The main result of Section 4.4 is Theorem 4.4.16. To prove it, we need some terminology regarding the flag complex of $\mathbb{F}_q^n$ (cf. Section 4.4.1) and some combinatorial tools (cf. Section 4.4.3). Section 4.4.2 provides explicit examples that do not make use of Theorem 4.4.16. We conclude the chapter with some observations and questions about the formula obtained in Theorem 4.4.16 (cf. Corollary 4.4.24 and Question 4.4.25).

4.1.1 Notation of the chapter

Additionally to the general conventions of the thesis, we set the following notation and conventions for the present chapter.

Matrices

Let A be a unital commutative ring.

- For $m, n \in \mathbb{Z}_{\geq 1}$, denote by $0_{m\times n}$ the zero matrix in $\mathrm{Mat}_{m\times n}(A)$ and by I_n the identity matrix in $\mathrm{Mat}_n(A)$.

- Denote by $\mathrm{Tr}_n(A)$ and $\mathrm{Up}_n(A)$ the associative subalgebras of $\mathrm{Mat}_n(A)$ of all upper-triangular and all strictly upper-triangular matrices over A, respectively. Throughout the chapter, we often identify $\mathrm{Tr}_n(A)$ with $A^{n(n+1)/2}$ as an additive abelian group. Set also

$$\mathrm{Tr}_n^*(A) := \left\{x = [x_{ab}]_{1\leq a\leq b\leq n} \in \mathrm{Tr}_n(A) \,\middle|\, \forall\,(a,b) \in [n]^2_{\leq},\, x_{ab} \neq 0\right\}.$$

- Given $x = [x_{ij}]_{1\leq i,j\leq n} \in \mathrm{Mat}_n(A)$ and $(p,q),(r,s) \in [n]^2_{\leq}$, define the submatrix

$$x^{[p,q]\times[r,s]} = [x_{ij}]_{p\leq i\leq q,\, r\leq j\leq s} \in \mathrm{Mat}_{(q-p+1)\times(s-r+1)}(A).$$

For instance, for $n = 3$,

$$x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix};\ x^{[1,2]\times[1,3]} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix};\ x^{[1,2]\times[2,3]} = \begin{bmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{bmatrix}.$$

Informally speaking, $x^{[p,q]\times[r,s]}$ is the matrix obtained from x by first selecting its k-rows, where $p \leq k \leq q$, and then selecting the h-th columns, where $r \leq h \leq s$, of the obtained matrix.

- For $(p,q) \in [n]^2_{\leq}$, let $x^{\vee(p,q)} \in \mathrm{Mat}_{n-1}(A)$ be the matrix obtained from x by deleting its p-th row and its q-th column. For example, for $n = 3$,

$$x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \quad \text{and} \quad x^{\vee(2,3)} = \begin{bmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{bmatrix}.$$

Non-Archimedean local fields

Let K denote a non-Archimedean local field with absolute value $|\_|$ and ring of integers R, i.e., $R = \{x \in K \,:\, |x| \leq 1\}$.

- For a non-empty subset $A \subseteq K$, define

$$\|A\| := \sup_{a\in A} |a| \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

- For $N \geq 1$, denote by μ_N the Haar measure on the additive abelian group K^N normalised with respect to R^N.

Remark 4.1.1. Provided $N = n(n+1)/2$ and identifying $\mathrm{Tr}_n(K)$ with K^N, we have

$$\mu_N\Big(\mathrm{Tr}_n(R) \setminus \mathrm{Tr}_n^*(R)\Big) = 0.$$

In particular, for every integrable function $f\colon A \to \mathbb{C}$ over a Borel-measurable subset A of $\mathrm{Tr}_n(R)$, one observes that

$$\int_A f(x)\mathrm{d}\mu_N(x) = \int_{A\cap \mathrm{Tr}_n^*(R)} f(x)\mathrm{d}\mu_N(x).$$

4.2 Submodule growth and submodule zeta functions

4.2.1 Background concepts

Definition 4.2.1 ([Seg97])**.** A ring R is said to have *polynomial submodule growth* if, for every finitely generated R-module M, the number of finite-index submodules of M with additive index m is the m-th term of a sequence growing polynomially in m.

Example 4.2.2. Every finite ring R has polynomial submodule growth. Indeed, every finitely generated R-module is a quotient of R^n, for some $n \geq 1$, and therefore it is finite.

Example 4.2.3. Every field K has polynomial submodule growth. If K is finite, the statement follows from Example 4.2.2. Let now K be infinite. Note that every finitely generated K-module is isomorphic to K^n, for some $n \geq 0$. Moreover, given $n \geq 0$ and for every subspace $N \leq K^n$, the quotient space K^n/N is isomorphic to $K^{n-\dim_K(N)}$. Thus, the only finite index subspace of K^n is the space itself and K has polynomial submodule growth.

To produce more interesting examples, one may use the following characterisation due to D. Segal. It is based on the concept of *Krull dimension* $\mathrm{kr}(R)$ of a ring R, i.e.,

$$\mathrm{kr}(R) := \sup\big\{n \in \mathbb{Z}_{\geq 0} \,\big|\, \text{there exists a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n \text{ of primes ideals of } R\big\}.$$

Theorem 4.2.4 ([Seg97, Theorem 1])**.** *Let R be a ring which is finitely generated over $\mathbb{Z}$ or is semi-local (i.e., it has finitely many maximal ideals) with finite residue fields. Then R has polynomial submodule growth if, and only if,* $\mathrm{kr}(R) \leq 2$*.*

Example 4.2.5. Applying Theorem 4.2.4, we obtain the following.

(i) Let R be a principal ideal domain which is not a field. Since every non-zero prime ideal is principal and generated by a prime of R, we have $\mathrm{kr}(R) = 1$. If in particular R is finitely generated over $\mathbb{Z}$ (e.g., $R = \mathbb{Z}$) or is semilocal with finite residue fields (e.g., R is the ring of integers of a non-Archimedean local field), then R has polynomial submodule growth.

(ii) Let R be a Dedekind domain which is not a field. Then every non-zero proper ideal of R factors uniquely (up to reordering of the factors) into prime ideals and then $\mathrm{kr}(R) = 1$. If R is finitely generated over $\mathbb{Z}$ (e.g., R is the ring of integers of a number field) or is semilocal with finite residue fields, then R has polynomial submodule growth.

Definition 4.2.6. Let R be a ring with polynomial submodule growth, and let M be a finitely generated R-module. Given $m \geq 1$, let $a_m(M)$ be the number of R-submodules of M of additive index m. The Dirichlet series associated to $\big(a_m(M)\big)_{m\geq 1}$, i.e.,

$$\zeta_M(s) := \sum_{m=1}^{\infty} a_m(M) m^{-s} = \sum_{N \leq_R M,\, |M:N|<\infty} |M:N|^{-s},$$

is called the *submodule zeta function of M*.

Definition 4.2.6 has the following generalisation, which will be the main object of study in the present chapter.

Definition 4.2.7. Let R be a ring with polynomial submodule growth, M be a finitely generated R-module, and consider a subalgebra $\mathcal{E} \leq \mathrm{End}_R(M)$ of R-linear endomorphisms of M. A R-submodule $N \leq_R M$ is *$\mathcal{E}$-invariant* (written $N \leq_{\mathcal{E}} M$) if $\xi(N) \subseteq N$ for every $\xi \in \mathcal{E}$. For $m \geq 1$, define

$$a_m(\mathcal{E} \curvearrowright M) := |\{N \leq_R M \,:\, N \leq_{\mathcal{E}} M \text{ and } |M:N| = m\}|, \tag{4.2.1}$$

where $|M:N|$ denotes the additive index of N in M. The Dirichlet series associated to the sequence $\big(a_m(\mathcal{E} \curvearrowright M)\big)_{m\geq 1}$, i.e.,

$$\zeta_{\mathcal{E}\curvearrowright M}(s) := \sum_{m=1}^{\infty} a_m(\mathcal{E} \curvearrowright M) m^{-s} = \sum_{N \leq_{\mathcal{E}} M,\, |M:N|<\infty} |M:N|^{-s}, \tag{4.2.2}$$

defines a function called the *submodule zeta function of $\mathcal{E}$ acting on M*. Denote by $\mathrm{abs}(\mathcal{E} \curvearrowright M) \in \mathbb{R}_{\geq 0} \cup \{-\infty\}$ the abscissa of convergence of $\zeta_{\mathcal{E}\curvearrowright M}(s)$, which is finite by the assumption on R (cf. Section 1.9).

Remark 4.2.8. Let R, M and $\mathcal{E}$ be as in Definition 4.2.7.

(i) (**Recovering submodule zeta function of M**). Let $\mathcal{E} = \{0\}$. Then every submodule of M is $\mathcal{E}$-invariant and $\zeta_{\mathcal{E}\curvearrowright M}(s)$ is the submodule zeta function of M (cf. Definition 4.2.6).

(ii) (**Invariance under homotheties**). Given a R-submodule $N \leq M$, we have

$$N \leq_{\mathcal{E}} M \Longrightarrow rN \leq_{\mathcal{E}} M, \quad \forall\, r \in R.$$

Moreover,

$$N \leq_{\mathcal{E}} M \Longleftrightarrow rN \leq_{\mathcal{E}} M, \quad \forall\, r \in R^{\times}.$$

(iii) (**Reduction to generators**). Let $\mathcal{E} = \langle S \rangle$ be the R-subalgebra of $\mathrm{End}_R(M)$ generated by a non-empty set S. Then, for every R-submodule $N \leq M$, one checks that

$$N \leq_{\mathcal{E}} M \Longleftrightarrow \xi(N) \subseteq N, \quad \forall \xi \in S.$$

(iv) (**Invariance under conjugation**). Let $\eta \in \mathrm{End}_R(M)$ be invertible. Then, for every R-submodule $N \leq M$, one checks that

$$N \leq_{\mathcal{E}} M \Longleftrightarrow \eta(N) \leq_{\eta \mathcal{E} \eta^{-1}} M. \tag{4.2.3}$$

Moreover, for $m \geq 1$ we have

$$a_m(\mathcal{E} \curvearrowright M) = a_m(\eta \mathcal{E} \eta^{-1} \curvearrowright M) \tag{4.2.4}$$

This is a consequence of (4.2.3) and the fact that η induces a self-bijection in the set of all R-submodules $N \leq M$ of index m. By (4.2.4) we deduce that

$$\zeta_{\mathcal{E} \curvearrowright M}(s) = \zeta_{\eta \mathcal{E} \eta^{-1} \curvearrowright M}(s).$$

(v) (**Adding the identity**) Let $\eta \in \mathrm{End}_R(M)$ and denote by id_M the identity map on M. Then, for every R-submodule $N \leq_R M$, we have

$$\eta(N) \subseteq N \Longleftrightarrow (\mathrm{id}_M + \eta)(N) \subseteq N.$$

Hence, if $\mathcal{E} \leq \mathrm{End}_M(R)$ is a subalgebra then

$$\zeta_{\mathcal{E} \curvearrowright R^n}(s) = \zeta_{(\mathrm{id}_M + \mathcal{E}) \curvearrowright R^n}(s).$$

Here we are implicitly extending Definition 4.2.7 to every subset of $\mathrm{End}_R(M)$.

Convention 4.2.9. From now on, we focus on the case $M = R^n$ for some $n \geq 1$. The elements of R^n are regarded as row vectors and $\{e_i \mid 1 \leq i \leq n\}$ denotes the canonical R-basis of R^n. The relevant algebra $\mathcal{E}$ is regarded as a subalgebra of $\mathrm{Mat}_n(R)$ acting on R^n by right row-by-column multiplication.

Remark 4.2.10. For every $A \in \mathrm{Mat}_n(R)$, the R-module $R^n A$ is generated by the rows $e_1 A, \ldots, e_n A$ of A. Hence, for all $A, B \in \mathrm{Mat}_n(R)$ we have

$$R^n A \subseteq R^n B \Longleftrightarrow e_1 A, \ldots, e_n A \in R^n B. \tag{4.2.5}$$

4.2.2 A particular case: ideal zeta functions of R-Lie lattices

Let R be a unital commutative ring. Prominent examples of submodule zeta functions of algebras of endomorphisms acting on R^n are the *ideal zeta functions of R-Lie lattices*. Following for instance [Vol19, p. 2], an *R-Lie lattice* is a free finitely generated R-module L of finite rank n with an antisymmetric bi-additive form $[\_,\_]$ satisfying the Jacobi identity. Assume that, for every $m \geq 1$, the number $a^{\triangleleft}_m(L)$ of ideals of L of additive index m is finite and grows polynomially in m. This latter condition is met if, for instance, R has polynomial submodule growth (cf. Definition 4.2.1 and the examples thereafter). If L is as before, then the *ideal zeta function* of L is defined as the following Dirichlet series in the complex variable s:

$$\zeta^{\triangleleft}_L(s) := \sum_{m=1}^{\infty} a^{\triangleleft}_m(L) m^{-s}. \tag{4.2.6}$$

Remark 4.2.11. The ideal zeta function of the ring of integers O_K of a number field K is usually called the *Dedekind zeta function of K* and it is denoted by $\zeta_K(s)$ (cf. [Neu13, §5]). For instance, $\zeta_{\mathbb{Q}}(s)$ is the Riemann zeta function.

We now motivate how the ideal zeta functions fit into the framework of submodule zeta functions of algebras of endomorphisms. Given an R-Lie lattice L, consider its *adjoint representation*, i.e.,

$$\mathrm{ad}\colon L \longrightarrow \mathrm{End}_R(L), \qquad \mathrm{ad}_x(y) = [x, y],\ \forall\, x, y \in L. \tag{4.2.7}$$

Then we have the following.

Fact 4.2.12 ([Vol19, p. 2])**.** *Let R be a unital commutative ring of polynomial submodule growth. Let L be an R-Lie ring and $\mathcal{E}(L)$ be the subalgebra of $\mathrm{End}_R(L)$ generated by $\mathrm{ad}(L)$. Then*

$$\zeta_{\mathcal{E}(L)\curvearrowright L}(s) = \zeta^{\triangleleft}_L(s).$$

Concretely, one may represent the elements $\mathcal{E}(L)$ as in Fact 4.2.12 as matrices in the following way. We fix an ordered R-basis $(x_1, \ldots, x_n)$ on L. Note that, for all $i, j \in [n]$, there are uniquely determined coefficients $\{\gamma^{ij}_k\}_{k\in[n]} \subseteq R$ such that

$$[x_i, x_j] = \sum_{k\in[n]} \gamma^{ij}_k \cdot x_k. \tag{4.2.8}$$

Hence, for every $i \in [n]$, we may regard ad_{x_i} as the matrix $C_i = [C_i(j,k)]_{j,k\in[n]}$ in $\mathrm{Mat}_n(R)$ given by:

$$C_i(j,k) := \gamma^{ij}_k, \quad \forall\, k \in [n]. \tag{4.2.9}$$

Example 4.2.13. Let R be a unital commutative ring of polynomial submodule growth. Denote by L the *Heisenberg R-ring*, i.e., the R-Lie ring given by the following presentation:

$$L = \langle x_1, x_2, x_3 \mid [x_1, x_2] = x_3,\ [x_1, x_3] = 0 = [x_2, x_3] \rangle.$$

Consider the ordered R-basis (x_1, x_2, x_3) of L. Then, according to (4.2.8) and (4.2.9), we have

$$C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \quad C_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad C_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Remark 4.2.14. The ideal zeta functions are a cornerstone in the theory of zeta functions associated to groups and rings. According to Fact 4.2.12, they constitute a notable subclass of zeta functions of algebras of endomorphisms. Moreover, they are related to the so-called *normal subgroup zeta functions* of $\mathcal{T}$-groups (i.e., finitely generated residually finite torsion-free groups). Namely, given a $\mathcal{T}$-group G, its normal subgroup zeta function is defined as

$$\zeta_G^{\triangleleft}(s) := \sum_{m=1}^{\infty} a_m^{\triangleleft}(G) m^{-s} = \sum_{H \trianglelefteq G,\, |G:H| < \infty} |G : H|^{-s}, \tag{4.2.10}$$

where $a_m^{\triangleleft}(G)$ is the number of normal subgroups of G of index m (cf. [GSS88]). The Mal'cev correspondence assigns the group G a $\mathbb{Z}$-Lie ring $\mathcal{L}(G)$. By [GSS88, §4], for all but finitely many primes p, the normal subgroup zeta function of the pro-p completion $\widehat{G}^p$ of G satisfies

$$\zeta_{\widehat{G}^p}^{\triangleleft}(s) = \zeta_{\mathcal{L}(G) \otimes_{\mathbb{Z}} \mathbb{Z}_p}^{\triangleleft}(s), \tag{4.2.11}$$

where $\mathbb{Z}_p$ denotes the ring of p-adic integers. In turn, by [GSS88, Proposition 4], the following Euler product decomposition holds:

$$\zeta_G^{\triangleleft}(s) = \prod_{p \text{ prime}} \zeta_{\widehat{G}^p}^{\triangleleft}(s). \tag{4.2.12}$$

Thus, for almost all prime numbers the ideal zeta functions occur as local factors of the normal subgroup zeta function of a $\mathcal{T}$-group in the Euler product decomposition in (4.2.12).

So far, ideal zeta functions have been the most studied class of submodule zeta functions of algebras of endomorphisms. If R is the ring of integers of a non-Archimedean local field, the reader may refer, for instance, to the articles of A. Carnevale, M. Schein and C. Voll [CSV24], C. Voll [Vol20], and L. Woodward [Woo08]. If R is a finite field, one may refer to the work of S. Lee [Lee22].

4.2.3 General results and problems

So far, most of the literature on submodule zeta functions of algebras of endomorphisms has been focused on the case in which the coefficient ring is either the ring of integers $R = O_K$ of a number field K, or the completion $R_\mathfrak{p}$ of R at some non-zero prime ideal $\mathfrak{p}$. Recall that the quotient field $K_\mathfrak{p}$ of $R_\mathfrak{p}$ is a non-Archimedean local field with absolute value $|\_|_\mathfrak{p}$, and $R_\mathfrak{p}$ is the ring of integers of $K_\mathfrak{p}$, i.e.,

$$R_\mathfrak{p} = \{x \in K_\mathfrak{p} \,:\, |x|_\mathfrak{p} \leq 1\}. \tag{4.2.13}$$

The following theorem recalls a notable interplay between submodule zeta functions of R and of its completions $R_\mathfrak{p}$'s. It generalises the well-known Euler product decomposition of the Riemann zeta function:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$

Theorem 4.2.15 (cf. [SG00, Theorem 1.2], [Ros15, Lemma 2.3])**.** *Let R be the ring of integers of a number field K, and denote by $\mathcal{P}(R)$ the set of all non-zero prime ideals of R. Consider a subalgebra $\mathcal{E} \leq \mathrm{Mat}_n(R)$. For every $\mathfrak{p} \in \mathcal{P}(R)$, set $\mathcal{E}_\mathfrak{p} := \mathcal{E} \otimes_R R_\mathfrak{p}$ and assume that $\mathcal{E}_\mathfrak{p}$ acts on $R_\mathfrak{p}^n$ by right row-by-column multiplication. Then*

$$\zeta_{\mathcal{E} \curvearrowright R^n}(s) = \prod_{\mathfrak{p} \in \mathcal{P}(R)} \zeta_{\mathcal{E}_\mathfrak{p} \curvearrowright R_\mathfrak{p}^n}(s). \tag{4.2.14}$$

The decomposition in (4.2.14) is called the *Euler product decomposition* of $\zeta_{\mathcal{E} \curvearrowright R^n}(s)$. For every $\mathfrak{p} \in \mathcal{P}(R)$, the zeta function $\zeta_{\mathcal{E} \otimes_R R_\mathfrak{p} \curvearrowright R_\mathfrak{p}^n}(s)$ is said to be the $\mathfrak{p}$-*local factor* of $\zeta_{\mathcal{E} \curvearrowright R^n}(s)$. Theorem 4.2.15 focuses on what we call the *local submodule zeta functions*, i.e., submodule zeta functions of algebras of endomorphisms over the ring of integers of a non-Archimedean local field.

Considering local submodule zeta functions might be convenient for multiple reasons. First, every ring of integers R of a non-Archimedean local field is a principal ideal domain, so every matrix with entries in this ring admits a Smith normal form. This fact has been repeatedly applied, for instance, in several works by C. Voll [Vol10; Vol19; Vol20] (to cite some references). Moreover, at least if the quotient field K of R has characteristic zero, one can reformulate the series defining the relevant zeta functions via a p-adic integral, as recalled in the following result.

Theorem 4.2.16 ([Ros15, Theorem 2.6], [SG00, Theorem 1.2])**.** *Consider a non-Archimedean local field K of characteristic zero with absolute value $|\_|$, ring of integers R and residue field cardinality q. Consider a subalgebra $\mathcal{E} \leq \mathrm{Mat}_n(R)$, for some $n \geq 1$. Then, for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \mathrm{abs}(\mathcal{E} \curvearrowright R^n)$,*

$$\zeta_{\mathcal{E} \curvearrowright R^n}(s) = \frac{1}{(1-q^{-1})^n} \int_{\{x \in \mathrm{Tr}_n(R) \,:\, R^n x \leq_\mathcal{E} R^n\}} \prod_{i \in [n]} |x_{ii}|^{s-i} \mathrm{d}\mu_N(x),$$

where $N = n(n+1)/2$. Moreover, there is a rational function $W_{\mathcal{E},R}(t) \in \mathbb{Z}(t)$ such that

$$\zeta_{\mathcal{E}\curvearrowright R^n}(s) = W_{\mathcal{E},R}(q^{-s}).$$

Remark 4.2.17. The rationality mentioned in Theorem 4.2.16 implies that the sequence $\big(a_m(\mathcal{E} \curvearrowright R^n)\big)_{m\geq 1}$ satisfies a linear recurrence with integral coefficients, i.e., there are integers $k \geq 1$ and $\lambda_1, \dots, \lambda_k$ such that

$$a_m(\mathcal{E} \curvearrowright R^n) = \sum_{i=1}^{k} \lambda_i \cdot a_{m-i}(\mathcal{E} \curvearrowright R^n),$$

for every $m \geq k+1$. The integers k and $\lambda_1, \dots, \lambda_k$ are explicitly related to the denominator of the rational function $W_{\mathcal{E},R}(t)$ reduced in the lowest terms (cf. [Sta11, Theorem 4.1.1]).

Remark 4.2.18 (*Why do we assume that K has characteristic zero?*)**.** Initially, local subobject zeta functions (e.g., subalgebra zeta functions, submodule zeta functions, etc.) were introduced for the p-adic field $\mathbb{Q}_p$, where p is an arbitrary prime (cf. [GSS88; SG00; Vol10] to cite some examples). Several results are carried over by replacing $\mathbb{Q}_p$ with an arbitrary non-Archimedean local field of characteristic zero (via methods from algebraic geometry such as *Hironaka resolution of singularities*). However, some arguments – like the one for proving the rationality of $\zeta_{\mathcal{E}\curvearrowright R^n}(s)$ (cf. [SG00, Theorem 1.2]) – break down for non-Archimedean local fields of positive characteristic. As observed in [Vol19], there is a way to transfer the results from the “characteristic zero case” to the “positive characteristic case”. The strategy is to use the so-called *transfer principle*, as introduced by R. Cluckers and F. Loeser (cf. [CL10, Theorem 9.2.4]). Let O_K be the ring of integers of a number field K and $\mathcal{E} \leq \mathrm{End}_{O_K}(O_K^n)$ be an algebra of endomorphisms. In our case the transfer principle yields the following: for all but finitely many non-zero prime ideals $\mathfrak{p} \trianglelefteq O_K$, provided R is the completion of R at $\mathfrak{p}$ and $P \triangleleft R$ is the maximal ideal of R, we have

$$\zeta_{\left(\mathcal{E}\otimes_{O_K} R\right)\curvearrowright R^n}(s) = \zeta_{\left(\mathcal{E}\otimes_{O_K}(R/P)[\![T]\!]\right)\curvearrowright R^n}(s).$$

In this thesis, for simplicity, we focus only on the “characteristic zero case”.

Unfortunately, the argument in [SG00] to prove the existence of a rational function $W_{\mathcal{E},R}(t)$ is not constructive. It turns out that finding the explicit form of $W_{\mathcal{E},R}(t)$ is an extremely challenging problem. In the last decade, T. Rossmann [Ros16a; Ros15] provided explicit calculations for some $\zeta_{\mathcal{E}\curvearrowright R^n}(s)$ where $\mathcal{E}$ is the algebra of strictly upper-triangular n-dimensional matrices $\mathrm{Up}_n(R)$ over R (or slight variations of this algebra) and $n \leq 5$. These formulae have been obtained via `Zeta`, a programme by T. Rossmann [Ros16b], and through which other explicit formulae can be produced for small n. T. Rossmann [Ros17] also proved a general formula for arbitrary n in the case where $\mathcal{E}$ consists of only one matrix. Other general formulae for certain nilpotent algebras generated by elementary matrices have been obtained by M. Vantomme in her PhD thesis [Van23]. Explicit formulae for arbitrary n have also been provided for a particular instance of the submodule zeta functions of algebras of endomorphisms, namely the ideal zeta function of certain R-Lie lattices (cf. Section 4.2.2). One may refer, for instance, to the articles of A. Carnevale, M. Schein and C. Voll [CSV24], C. Voll [Vol20] and L. Woodward [Woo08].

So far, the known formulae for $\zeta_{\mathcal{E} \curvearrowright R^n}(s)$ are not many and turn out to be already very complicated for $n = 5$. However, they seem to exhibit certain common features. A recurrent phenomenon refers to the behaviour at $s = 0$ and led T. Rossmann to formulate the following conjecture.

Conjecture 4.2.19 (*Semisimplicity conjecture - nilpotent case*, [Ros15, §8.3])**.** Let R be the ring of integers of a non-Archimedean local field of characteristic zero, and let $n \geq 1$. Then, for every nilpotent subalgebra $\mathcal{E}$ of $\mathrm{End}_R(R^n)$, we have

$$\left.\frac{\zeta_{\mathcal{E} \curvearrowright R^n}(s)}{\zeta_{\{0\} \curvearrowright R^n}(s)}\right|_{s=0} = 1.$$

The series defining each $\zeta_{\mathcal{E} \curvearrowright R^n}(s)$ never converges at $s = 0$. This means that the ratio in Conjecture 4.2.19 cannot be verified by a direct evaluation at $s = 0$, and this substantially increases the difficulty of the problem.

The formulation of Conjecture 4.2.19 has been subsequently extended by T. Rossmann (cf. [Ros17, Conjecture E]) to every associative unital subalgebra $\mathcal{E}$ of $\mathrm{Mat}_n(R)$, replacing $\{0\}$ with the quotient of $\mathcal{E}$ modulo its nil-radical. The latter quotient is – as the name of the conjecture suggests – a semisimple algebra.

In this thesis, we focus only on the case of nilpotent algebras generated by elementary matrices. In this setting, we find a sufficient condition for Conjecture 4.2.19 to hold, cf. Theorem 4.3.22.

4.2.4 Geometric descriptions of the submodule zeta functions

Theorem 4.2.16 expresses the submodule zeta function $\zeta_{\mathcal{E} \curvearrowright R^n}(s)$, where R is the ring of integers of a non-Archimedean local field of characteristic zero, as a p-adic integral over $\mathrm{Tr}_n(R)$. In [Vol19], C. Voll rewrote this zeta function as a sum over the vertices of the Bruhat–Tits building of $\mathrm{SL}_n(K)$. This characterisation turned out to be useful, for instance, to prove functional equations for $\zeta_{\mathcal{E} \curvearrowright R^n}(s)$, where $\mathcal{E}$ varies within a suitable class of nilpotent algebras (cf. [Vol19, Theorem 1.2]).

In what follows, we recall the above-mentioned characterisation (cf. Section 4.2.4). Then we provide a similar description replacing R with a finite field (cf. Section 4.2.4).

Local submodule zeta functions and the Bruhat–Tits building of $\mathrm{SL}_n(K)$

Let K be a non-Archimedean local field (of characteristic zero) with ring of integers R. Fix a uniformiser $\pi \in R$ (i.e., an element of R with p-adic norm equal to 1) and set $n \geq 1$.

We first recall a description of the set of vertices of the Bruhat–Tits building of $\mathrm{SL}_n(K)$ which is necessary to state the main result of the section.

Recall that an *R-lattice* of K^n is the R-linear span of a K-basis of K^n. Two R-lattices L and L' of K^n are said to be *equivalent* if there is $k \in K^\times$ such that $L' = kL$. This equivalence relation induces a partition on the set of all R-lattices into homothety classes, each of them written as $[L] = \{\pi^k L : k \in \mathbb{Z}\}$ for some lattice L of K^n. Every homothety class $[L]$ admits a unique element L_0 that is maximal among the elements of $[L]$ contained in R^n. Let

$$\mathcal{F}_n^{(0)} := \{[L] \mid L \text{ is a } R\text{-lattice of } K^n\}. \tag{4.2.15}$$

According to [Gar12, §19.1], one constructs a building $X_n = X_n(K)$ of type $\tilde{A}_{n-1}$ from the set of all maximal simplices of the flag complex $\mathcal{F}_n$ of all R-lattices of K^n (analogously as we did in Example 1.5.11). It turns out that $\mathrm{SL}_n(K)$ acts strongly transitively on X_n and $\mathrm{SL}_n(R)$ is the stabiliser of a vertex of $\mathcal{F}_n$, namely $[R^n]$. The building X_n is called the *Bruhat–Tits building* of $\mathrm{SL}_n(K)$.

Proposition 4.2.20 ([Vol19, §2.1.1])**.** *With the notation mentioned before, we have*

$$\zeta_{\mathcal{E}\curvearrowright R^n}(s) = \frac{1}{1-q^{-ns}} \cdot \sum_{[L_0]\in\mathcal{F}_n^{(0)}\,:\,L_0\leq_{\mathcal{E}} R^n} |R^n : L_0|^{-s}.$$

Proof. An R-lattice L of K^n satisfies $\eta(L) \subseteq L$ for every $\eta \in \mathcal{E}$ if, and only if, every lattice $L' \in [L]$ satisfies $\eta(L') \subseteq L'$ for every $\eta \in \mathcal{E}$. Therefore,

$$\begin{aligned}
\zeta_{\mathcal{E}\curvearrowright R^n}(s) &= \sum_{L\leq_{\mathcal{E}} R^n} |R^n : L|^{-s} = \sum_{[L_0]\in\mathcal{F}_n^{(0)}\,:\,L_0\leq_{\mathcal{E}} R^n} \sum_{k=0}^{\infty} |R^n : \pi^k L_0|^{-s} \\
&= \sum_{[L_0]\in\mathcal{F}_n^{(0)}\,:\,L_0\leq_{\mathcal{E}} R^n} \left(\sum_{k=0}^{\infty} q^{-nks}\right) \cdot |R^n : L_0|^{-s} \\
&= \frac{1}{1-q^{-ns}} \cdot \sum_{[L_0]\in\mathcal{F}_n^{(0)}\,:\,L_0\leq_{\mathcal{E}} R^n} |R^n : L_0|^{-s}.
\end{aligned}$$

□

Submodule zeta functions over finite fields and spherical buildings

In the following, we provide a result analogous to Proposition 4.2.20 if $R = \mathbb{F}_q$ is a finite field of cardinality q. What replaces the Bruhat–Tits building of $\mathrm{SL}_n(K)$ is now the building Δ_n associated to the flag complex of non-zero subspaces of $\mathbb{F}_q^n$ (cf. Example 1.5.11). For simplicity, we also denote by Δ_n the flag complex of $\mathbb{F}_q^n$. Note that the set of vertices of the latter simplicial complex is

$$\Delta_n^{(0)} = \{L \mid L \text{ non-zero subspace of } \mathbb{F}_q^n\}. \tag{4.2.16}$$

For every subspace L of $\mathbb{F}_q^n$, observe that

$$|\mathbb{F}_q^n : L| = q^{n - \dim_{\mathbb{F}_q} L}. \tag{4.2.17}$$

Hence, it is straightforward to check the following.

Proposition 4.2.21. *For every subalgebra $\mathcal{E} \leq \mathrm{Mat}_n(\mathbb{F}_q)$ we have*

$$\zeta_{\mathcal{E} \curvearrowright \mathbb{F}_q^n}(s) = q^{-ns}\left(1 + \sum_{L \in \Delta_n^{(0)} \,:\, L \leq_{\mathcal{E}} \mathbb{F}_q^n} q^{s \cdot \dim_{\mathbb{F}_q} L}\right).$$

In Section 4.4, we use the description provided by Proposition 4.2.21 to give an explicit formula for $\zeta_{\mathcal{E} \curvearrowright \mathbb{F}_q^n}(s)$, where $\mathcal{E}$ is a subalgebra of $\mathrm{Mat}_n(\mathbb{F}_q)$ generated by any set of elementary matrices.

4.2.5 Algebras generated by elementary matrices and incidence algebras

Let R be a unital commutative ring and $n \in \mathbb{Z}_{\geq 1}$. In the next sections, we will mainly focus on submodule zeta functions of specific subalgebras of $\mathrm{Mat}_n(R)$: the ones generated by a set of elementary matrices. In the following lines, we explain that this family of algebras coincides with the one of the so-called *incidence algebras*. In view of this, we first introduce some terminology.

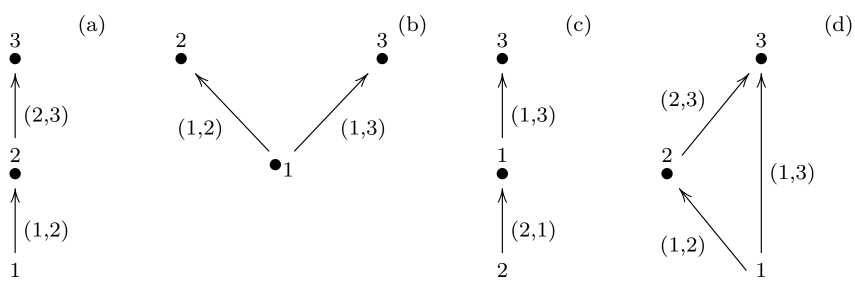
We call a *poset with a natural labelling* any poset $(P, \leq_P)$ whose underlying set P is a subset of $[n]$, and such that $i \leq_P j$ implies that $i \leq j$ (cf. [Sta11, §3.15.1]). Up to isomorphisms of posets, it is proved that every poset $([n], \leq_P)$ is isomorphic to a natural poset over $[n]$.

Every poset $\mathbb{P} = (P, \leq_P)$ has an associated *Hasse diagram*, i.e., a directed graph $\Gamma_{\mathbb{P}}$ with set of vertices P and set of edges

$$\vec{E} = \vec{E}(\mathbb{P}) := \left\{(i,j) \,\middle|\, i <_P j \text{ and } \big(i \leq_P k \leq_P j \Rightarrow k \in \{i,j\}\big)\right\}. \tag{4.2.18}$$

For every pair (i, j) satisfying the conditions in (4.2.18), one says that j *covers* i in $\mathbb{P}$.

Example 4.2.22. Consider the following directed graphs.



(4.2.19)

All graphs in (4.2.19) produce a partial order $\leq_P$ on $P = [3]$ as follows: $i \leq_P j$ if, and only if, there is a path in the relevant graph (possibly consisting of one vertex) from i to j.

In particular, the graphs in (a), (b) and (c) turn to be the Hasse diagrams of posets on P defined as $1 <_P 2 <_P 3$ in case (a), $1 <_P 2$ and $1 <_P 3$ in case (b), and $2 <_P 1 <_P 3$ in case (c), respectively. Note that the cases (a) and (b) produce structures of posets with natural labellings on P.

The graph in case (d) induces a poset structure on $P = [3]$ by setting $1 <_P 2 <_P 3$ (the same poset structure produced in case (a)). However, this graph can not be the Hasse diagram of the above-mentioned poset: indeed, it has $(1, 3)$ as an edge but 3 does not cover 1 in $(P, \leq_P)$.

Following [Sta11, §3.3.6], given a poset $\mathbb{P} = (P, \leq_P)$ with $P \subseteq [n]$ and a unital commutative ring R, the *incidence (R, n)-algebra associated to* $\mathbb{P}$ is

$$\mathcal{E}_{\mathbb{P}} := \left\{x \in \mathrm{Mat}_n(R) \,\middle|\, x_{ij} = 0,\, \forall\, (i, j) \in [n] \text{ with } i \not\leq_P j \text{ or } i = j\right\}.$$

One proves that $\mathcal{E}_{\mathbb{P}}$ is an associative subalgebra of $\mathrm{Mat}_n(R)$, and it is generated by $\{E_{ij}\}_{(i,j)\in\vec{E}}$, where $\vec{E} = \vec{E}(\mathbb{P})$ is the edge set of the Hasse diagram associated to $\mathbb{P}$. Moreover, if $\mathbb{P}$ is a poset with a natural labelling, then $\mathcal{E}_{\mathbb{P}}$ is contained in $\mathrm{Up}_n(R)$ and thus is a nilpotent algebra.

We return to the case of an arbitrary set $I \subseteq [n]^2$. Consider the directed graph $\Lambda_I = (P, I)$ with edge set I and with vertex set given by

$$P = P_I := \{i \in [n] \mid \exists\, j \in [n] \text{ s.th. } (i, j) \in I \text{ or } (j, i) \in I\}.$$

Consider the binary relation $\leq_I$ on P defined as follows:

$$i \leq_I j \iff \text{there is a path in } \Lambda_I \text{ with vertex sequence } (i_0 = i, i_1, \ldots, i_l = j),\ l \geq 0. \tag{4.2.20}$$

If $I \subseteq [n]^2_<$, one verifies that $\leq_I$ is a partial ordering on P. Note that we might not have the anti-symmetry of $\leq_P$ if we drop the hypothesis that I is contained in $[n]^2_<$. Then $\Gamma_I = (P, I)$ is the Hasse diagram of $(P, \leq_I)$ and one deduces the following.

Lemma 4.2.23. *Let $I \subseteq [n]^2_<$ and $\Gamma_I = (P, \vec{E}_I)$ as before. Then the subalgebra of $\mathrm{Mat}_n(R)$ generated by $\{E_{ij}\}_{(i,j)\in I}$ is the incidence (R, n)-algebra associated to the poset $(P, \leq_I)$.*

Remark 4.2.24. Let R be a unital commutative ring and L be the R-Lie ring given by the following presentation:

$$L = \left\langle x_1, \ldots, x_n \,\middle|\, \forall\, (i, j) \in [n]^2_\leq,\ [x_i, x_j] = \sum_{k\in[n]} C_i(j, k) x_k \right\rangle,$$

where $\{C_i(j, k)\}_{(i,j)\in[n]^2_\leq,\, k\in[n]} \subseteq R$ is a given set. Following (4.2.9), the endomorphism $\mathrm{ad}_{x_i} \colon L \to L$ is represented by the matrix $C_i = [C_i(j, k)]_{j,k\in[n]}$ in $\mathrm{Mat}_n(R)$. Then the algebra $\mathcal{E}(L)$ appearing in Fact 4.2.12 is the R-subalgebra of $\mathrm{Mat}_n(R)$ generated by the matrices $\{C_1, \ldots, C_n\}$.

Let $i \in [n]$. Then the matrix C_i equals the elementary matrix E_{ab}, with $(a, b) \in [n]^2$, in $\mathrm{Mat}_n(R)$ if, and only if,

$$C_i(j,k) = \begin{cases} 1, & \text{if } (j,k) = (a,b); \\ 0, & \text{otherwise.} \end{cases}$$

For instance, if L is the Heisenberg R-Lie ring, then the matrices $\{C_1, C_2, C_3\}$ as in Example 4.2.13 are elementary matrices and $\mathcal{E}(L)$ is the incidence $(R, 3)$-algebra associated to the poset $([3], \preceq)$, where $1 \prec 3$ and $2 \prec 3$.

4.2.6 Preliminaries on p-adic integration

In the following, we collect some well-known facts about p-adic integration that will be repeatedly used in the next Section 4.3. Let K be a non-Archimedean local field with absolute value $|\_|$ and ring of integers R. Let $\pi \in R$ be a uniformiser and set $q := |R : \pi R|$.

Fact 4.2.25.

(i) For every $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq 0$ and every $\alpha \in \mathbb{Z}$, we have

$$\int_{\{x \in K \,:\, |x| \leq q^{-\alpha}\}} |x|^s \mathrm{d}\mu_1(x) = \frac{q^{-\alpha(s+1)}(1 - q^{-1})}{1 - q^{-s-1}}.$$

(ii) Let $n \geq 2$ and consider $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n$ with $\mathrm{Re}(s_i) > 0$ for every $i \in [n]$. Then

$$I_n(\mathbf{s}) := \int_{\{x \in R^n \,:\, |x_i| \leq |x_{i+1}|, \forall\, i \in [n-1]\}} \prod_{i \in [n]} |x_i|^{s_i} \mathrm{d}\mu_n(x) = \prod_{j \in [n]} \frac{1 - q^{-1}}{1 - q^{-\sum_{i \in [j]} s_i - j}}.$$

*(iii) (**Polynomial change of variables**) Let $A, B \subseteq K^n$ be Borel–measurable sets, and consider a bijection $f = (f_1, \ldots, f_n) \colon A \to B$ such that $f_i(x)$ is polynomial in the entries $x = (x_1, \ldots, x_n)$ for all $i \in [n]$. Then, for a polynomial function $g \colon B \to K$ and for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, we have*

$$\int_B |g(y)|^s \mathrm{d}\mu_n(y) = \int_A |(g \circ f)(x)|^s \cdot |\det(\mathrm{Jac} f(x))| \mathrm{d}\mu_n(x),$$

where $\mathrm{Jac} f(x) = \left[\frac{\partial f_i}{\partial x_j}\right]_{1 \leq i,j \leq n}$ is the Jacobian matrix of f.

Proof. (i) Without loss of generality, we can integrate over $\{x \in K^\times : |x| \leq q^{-\alpha}\}$. In fact, we discard only a measure zero set (namely $\{0_K\}$) from the initial domain of integration. We now observe that

$$\{x \in K^\times \,:\, |x| \leq q^{-\alpha}\} = \bigsqcup_{k \in \mathbb{Z}_{\geq 0}} \{x \in K \,:\, |x| = q^{-\alpha - k}\} = \bigsqcup_{k \in \mathbb{Z}_{\geq 0}} \pi^{\alpha + k} R^\times.$$

Therefore,

$$\int_{\{x\in K\,:\,|x|\le q^{-\alpha}\}} |x|^s \mathrm{d}\mu_1(x) = \sum_{k=0}^{\infty} \overbrace{\mu_1(\pi^{\alpha+k}R^\times)}^{=(1-q^{-1})q^{-\alpha-k}} \cdot q^{-(\alpha+k)s}$$

$$= (1-q^{-1})q^{-\alpha(s+1)} \sum_{k=0}^{\infty} q^{-(s+1)k}$$

and the statement follows.

(ii) Notice that

$$\{x\in R^n\setminus\{0\} : |x_i|\le|x_{i+1}|,\,\forall\, i\in[n-1]\} = \bigsqcup_{(k_1,\dots,k_n)\in\mathbb{Z}^n_{\ge 0}} \{x\in R^n \,:\, |x_i| = q^{-\sum_{j=i}^n k_j},\,\forall\, i\in[n]\}$$

$$= \bigsqcup_{(k_1,\dots,k_n)\in\mathbb{Z}^n_{\ge 0}} \underbrace{\prod_{i\in[n]} \pi^{\sum_{j=i}^n k_j} R^\times}_{=:A_{k_1,\dots,k_n}}.$$

Moreover, for all $(k_1,\dots,k_n)\in\mathbb{Z}^n_{\ge 0}$ we have

$$\mu_n(A_{k_1,\dots,k_n}) = \prod_{i\in[n]} \mu_1\Big(\pi^{\sum_{j=i}^n k_j} R^\times\Big) = (1-q^{-1})^n q^{-\sum_{i=1}^n \sum_{j=i}^n k_j} = (1-q^{-1})^n q^{-\sum_{j=1}^n jk_j}$$

and, for every $x\in A_{k_1,\dots,k_n}$, we observe that

$$\prod_{i\in[n]} |x_i|^{s_i} = q^{-\sum_{i=1}^n (\sum_{j=i}^n k_j)s_i} = q^{-\sum_{j=1}^n (\sum_{i=1}^j s_i)k_j}.$$

Hence,

$$I_n(\mathbf{s}) = \sum_{(k_1,\dots,k_n)\in\mathbb{Z}^n_{\ge 0}} \mu_n(A_{k_1,\dots,k_n})\cdot q^{-\sum_{j=1}^n (\sum_{i=1}^j s_i)k_j}$$

$$= (1-q^{-1})^n \sum_{(k_1,\dots,k_n)\in\mathbb{Z}^n_{\ge 0}} q^{-\sum_{j=1}^n (\sum_{i=1}^j s_i + j)k_j}$$

$$= (1-q^{-1})^n \prod_{j\in[n]} \left(\sum_{k_j=0}^{\infty} q^{-\left(\sum_{i=1}^j s_i + j\right)k_j} \right),$$

which yields the claim.

(iii) It is a particular case of [Igu00, Proposition 7.4.1]. □

4.3 A multivariate local submodule zeta function

We introduce a multivariate generalisation of the integral appearing in Theorem 4.2.16. This tool is key for the proof of one of the main results of the chapter, i.e., Theorem 4.3.22.

For this section, K denotes a non-Archimedean local field of characteristic zero with absolute value $|\_|$, ring of integers R and residue field cardinality q. We fix a uniformiser $\pi \in R$, an integer $n \geq 1$, and set $N = n(n+1)/2$. Moreover, we fix an arbitrary subalgebra $\mathcal{E} \leq \mathrm{Mat}_n(R)$. Recall that we are following Convention 4.2.9.

Definition 4.3.1. For every $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n$ with $\mathrm{Re}(s_i) \geq 0$ for each $i \in [n]$, define

$$\zeta_{\mathcal{E} \curvearrowright R^n}(\mathbf{s}) := \frac{1}{(1-q^{-1})^n} \int_{\{x \in \mathrm{Tr}_n(R)\,:\, R^n x \leq_{\mathcal{E}} R^n\}} \prod_{i \in [n]} |x_{ii}|^{s_i} \mathrm{d}\mu_N(x).$$

The function $\zeta_{\mathcal{E} \curvearrowright R^n}(\mathbf{s})$ is called the *multivariate submodule zeta function of $\mathcal{E}$ acting on R^n* (by right multiplication).

Whenever $\mathrm{Re}(s_i) \geq 0$ for every $i \in [n]$, Fact 4.2.25(i) yields

$$|\zeta_{\mathcal{E} \curvearrowright R^n}(\mathbf{s})| \leq \frac{1}{(1-q^{-1})^n} \int_{\mathrm{Tr}_n(R)} \prod_{i \in [n]} |x_{ii}|^{\mathrm{Re}(s_i)} \mathrm{d}\mu_N(x) = \prod_{i \in [n]} \frac{1}{1 - q^{-\mathrm{Re}(s_i)-1}} < \infty. \quad (4.3.1)$$

Remark 4.3.2. In the notation of Definition 4.3.1, the following holds.

(i) (**Rationality**). By [SG00, Theorem 1.2], there is a rational function $W_{\mathcal{E},R}(t_1, \ldots, t_n) \in \mathbb{Z}(t_1, \ldots, t_n)$ such that

$$\zeta_{\mathcal{E} \curvearrowright R^n}(\mathbf{s}) = W_{\mathcal{E},R}(q^{-s_1}, \ldots, q^{-s_n}).$$

Being a ratio of holomorphic functions in $\mathbb{C}^n$, one can extend $\zeta_{\mathcal{E} \curvearrowright R^n}(\mathbf{s})$ to every $\mathbf{s} \in \mathbb{C}$ for which the evaluation of $W_{\mathcal{E},R}(q^{-s_1}, \ldots, q^{-s_n})$ exists and is finite.

(ii) (**Recovering the univariate case**). For every $s \in \mathbb{C}$ we have

$$\zeta_{\mathcal{E} \curvearrowright R^n}(s) = \zeta_{\mathcal{E} \curvearrowright R^n}(s-1, \ldots, s-i, \ldots, s-n). \quad (4.3.2)$$

In fact, the two functions of (4.3.2) coincide for $\mathrm{Re}(s) \gg 0$ (cf. Theorem 4.2.16 and Definition 4.3.1). Moreover, they are both ratios of holomorphic functions in the complex variable s. Hence, according to the identity principle, (4.3.2) holds for every $s \in \mathbb{C}$.

In contrast to the univariate case, the multivariate zeta function of $\mathcal{E}$ acting on R^n allows more flexibility in certain evaluations (cf. Section 4.3.2). This has been the main motivation for introducing it. On the other hand, unlike the univariate case, $\zeta_{\mathcal{E} \curvearrowright R^n}(\mathbf{s})$ depends on the choice of a basis of R^n. In other words, in contrast to Remark 4.2.8(iv), one might have $\zeta_{\mathcal{E} \curvearrowright R^n}(\mathbf{s}) \neq \zeta_{\eta \mathcal{E} \eta^{-1} \curvearrowright R^n}(\mathbf{s})$ for some $\eta \in \mathrm{GL}_n(R)$ (cf. Remark 4.3.9).

4.3.1 Some explicit formulae of $\zeta_{\mathcal{E}\curvearrowright R^n}(\mathbf{s})$

In what follows, we collect some explicit formulae of $\zeta_{\mathcal{E}\curvearrowright R^n}(\mathbf{s})$.

Example 4.3.3. We have

$$\zeta_{\{0\}\curvearrowright R^n}(\mathbf{s}) = \prod_{i\in[n]} \frac{1}{1-q^{-s_i-1}}. \tag{4.3.3}$$

Indeed, if $\mathrm{Re}(s_i) \geq 0$ for every $i \in [n]$, by basic manipulations and Fact 4.2.25(i) we observe that

$$\begin{aligned}
\zeta_{\{0\}\curvearrowright R^n}(\mathbf{s}) &= \frac{1}{(1-q^{-1})^n} \int_{R^n} \prod_{i\in[n]} |x_i|^{s_i} \mathrm{d}\mu_n(x) \\
&= \frac{1}{(1-q^{-1})^n} \prod_{i\in[n]} \left(\int_{\{x\in R\,:\,|x|\leq 1\}} |x|^{s_i} \mathrm{d}\mu_1(x) \right) \\
&= \prod_{i\in[n]} \frac{1}{1-q^{-s_i-1}}.
\end{aligned}$$

By Remark 4.3.2(i), the equality extends to all $\mathbf{s} \in \mathbb{C}^n$ for which the evaluation of the right-hand side of (4.3.3) is defined and finite.

In addition to Example 4.3.3, another favourable (and less straightforward) case is when $\mathcal{E}$ is generated by the nilpotent matrix $A(\lambda)$ introduced in [Ros17]. We now briefly recall the definition of $A(\lambda)$.

Set $n \geq 1$ and let $\lambda = (\lambda_1, \ldots, \lambda_r) \in (\mathbb{Z}_{\geq 1})^r$ be a *decreasing partition of* n, i.e., $\lambda_1 \geq \ldots \geq \lambda_r$ and $\sum_{i\in[n]} \lambda_i = n$. Following [Ros17, §4.1], define $A(\lambda) \in \mathrm{Mat}_n(R)$ by $A(\lambda) = 0$ if $r \leq 1$ and, if $r \geq 2$, by the following block decomposition:

$$A(\lambda) = \begin{bmatrix}
0_{\lambda_1\times\lambda_1} & A(\lambda_2) & 0_{\lambda_1\times\lambda_3} & \ldots & 0_{\lambda_1\times\lambda_r} \\
0_{\lambda_2\times\lambda_1} & 0_{\lambda_2\times\lambda_2} & A(\lambda_3) & \ldots & 0_{\lambda_2\times\lambda_r} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{\lambda_{r-1}\times\lambda_1} & 0_{\lambda_{r-1}\times\lambda_2} & 0_{\lambda_{r-1}\times\lambda_3} & \ldots & A(\lambda_r) \\
0_{\lambda_r\times\lambda_1} & 0_{\lambda_r\times\lambda_2} & 0_{\lambda_r\times\lambda_3} & \ldots & 0_{\lambda_r\times\lambda_r}
\end{bmatrix}, \tag{4.3.4}$$

where, for every $2 \leq i \leq r$,

$$A(\lambda_i) = \begin{bmatrix} I_{\lambda_i} \\ 0_{(\lambda_{i-1}-\lambda_i)\times\lambda_i} \end{bmatrix} \in \mathrm{Mat}_{\lambda_{i-1}\times\lambda_i}(R).$$

Following [Ros17, §1.5], for every $i \in \mathbb{Z}_{\leq r}$ we also set

$$\sigma_i(\lambda) := \begin{cases} \sum_{k\in[i]} \lambda_k, & \text{if } i > 0; \\ 0, & \text{if } i \leq 0 \end{cases} \tag{4.3.5}$$

and, for every $j \in [\lambda_i]$, we put

$$f(i,j) := \sigma_{i-1}(\lambda) + j. \tag{4.3.6}$$

Adapting [Ros17, Proposition 4.13] and [Ros17, Theorem 4.1], we prove the following.

Proposition 4.3.4. *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a decreasing partition of an integer $n \geq 1$. Then, for every $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$,*

$$\zeta_{\langle A(\lambda)\rangle \curvearrowright R^n}(\mathbf{s}) = \prod_{i=1}^{r} \prod_{j=1}^{\lambda_i} \frac{1}{1 - q^{-\sum_{a=1}^{i}(s_{f(a,j)} + f(a,j)) + f(i,j) - 1}}.$$

Proof. Transferring the proof of [Ros17, Proposition 4.13] verbatim, we obtain that

$$\zeta_{\langle A(\lambda)\rangle \curvearrowright R^n}(\mathbf{s}) = \frac{1}{(1-q^{-1})^n} \int_{V_\lambda(R)} \prod_{i \in [n]} |x_i|^{s_i} \mathrm{d}\mu_N(x),$$

where $N = n(n+1)/2$ and $V_\lambda(R)$ is the set of all vectors $z = (z_a)_{1 \leq a \leq N} \in R^N$ satisfying the following divisibility conditions, provided the $y_{i,j,k}$'s denote unspecified variables among $\{z_i\}_{n+1 \leq i \leq N}$ with the condition that $(i,j,k) \neq (i',j',k')$ implies $y_{i,j,k} \neq y_{i',j',k'}$:

(0) $z_{aa} \neq 0$ for every $a \in [n]$;

(1) $z_{\sigma_{i-1}(\lambda)+j}$ divides $z_{\sigma_{i-2}(\lambda)+j}, y_{i,j,1}, \dots, y_{i,j,j-1}$, for $2 \leq i \leq r$ and $1 \leq j \leq \lambda_i$;

(2) z_j divides $y_{i,j,n+1}, \dots, y_{i,j,n+\lambda_{i-2}}$, for every $3 \leq i \leq r$ and $\sigma_{i-1}(\lambda) < j \leq n$.

It was possible to proceed as in the proof of [Ros17, Proposition 4.13] because the argument therein involves only the domain of integration (and not the integrand) of the p-adic integral characterising $\zeta_{\langle A(\lambda)\rangle \curvearrowright R^n}(s)$.

Proceeding as in the proof of [Ros17, Theorem 4.1], we deduce that

$$\zeta_{\langle A(\lambda)\rangle \curvearrowright R^n}(\mathbf{s}) = \frac{1}{(1-q^{-1})^n} \int_{U_\lambda(R)} \prod_{i \in [n]} F_\lambda(z) \mathrm{d}\mu_N(z), \tag{4.3.7}$$

where

$$U_\lambda(R) = \left\{ z \in R^n \,:\, z_{f(i,j)} \mid z_{f(i-1,j)} \text{ for all } 2 \leq i \leq r \text{ and } 1 \leq j \leq \lambda_i \right\}$$

and

$$\begin{aligned} F_\lambda(z) &= \left(\prod_{i=1}^{r} \prod_{j=1}^{\lambda_i} |z_{f(i,j)}|^{s_{f(i,j)}} \right) \cdot \left(\prod_{i=2}^{r} \prod_{j=1}^{\lambda_i} |z_{f(i,j)}|^{j-1} \right) \cdot \left(\prod_{a=3}^{r} \prod_{i=a}^{r} \prod_{j=1}^{\lambda_i} |z_{f(i,j)}|^{\lambda_{a-2}} \right) \\ &= \left(\prod_{j=1}^{\lambda_1} |z_{f(1,j)}|^{s_{f(1,j)}} \right) \cdot \left(\prod_{i=2}^{r} \prod_{j=1}^{\lambda_i} |z_{f(i,j)}|^{s_{f(i,j)} + j - 1 + \sigma_{i-2}(\lambda)} \right). \end{aligned}$$

Whenever $\mathrm{Re}(s), \mathrm{Re}(t) \geq 0$, according to [Ros17, Equation (4.9)] we have

$$\int_{\{(x,y)\in R^2 \,:\, x|y\}} |x|^s|y|^t \mathrm{d}\mu_2(x,y) = \int_{R^2} |x|^{s+t+1}|y|^t \mathrm{d}\mu_2(x,y). \tag{4.3.8}$$

For all $2 \leq i \leq n$ and $1 \leq j \leq \lambda_i$, note that

$$\begin{aligned} s_{f(i,j)} + j - 1 + \sigma_{i-2}(\lambda) + s_{f(1,j)} + 1 + \sum_{a=2}^{i-1} \Big(\big(s_{f(a,j)} + j - 1 + \sigma_{a-2}(\lambda)\big) + 1 \Big) \\ = \sum_{a=1}^{i} s_{f(a,j)} + j(i-1) + \sum_{a=1}^{i-2} \sigma_a(\lambda). \end{aligned} \tag{4.3.9}$$

Applying (4.3.8) in (4.3.7) (recalling (4.3.9)) and then Fact 4.2.25(i), we deduce the following:

$$\begin{aligned} \zeta_{\langle A(\lambda)\rangle \curvearrowright R^n}(\mathbf{s}) &= \frac{1}{(1-q^{-1})^n} \int_{R^n} F_\lambda(z) \mathrm{d}\mu_N(z) \\ &= \frac{1}{(1-q^{-1})^n} \int_{R^n} \prod_{i=1}^{r} \prod_{j=1}^{\lambda_i} |z_{f(i,j)}|^{\sum_{a=1}^{i} s_{f(a,j)} + \sum_{a=1}^{i-2} \sigma_a(\lambda) + j(i-1)} \mathrm{d}\mu(z) \\ &= \frac{1}{(1-q^{-1})^n} \prod_{i=1}^{r} \prod_{j=1}^{\lambda_i} \frac{1-q^{-1}}{1-q^{-\sum_{a=1}^{i} s_{f(a,j)} - \sum_{a=1}^{i-2} \sigma_a(\lambda) - j(i-1) - 1}}. \end{aligned}$$

The statement now follows by observing, for every $1 \leq i \leq r$, that

$$\sum_{a=1}^{i-2} \sigma_a(\lambda) + j(i-1) = \sum_{a=1}^{i-2} (\sigma_a(\lambda) + j) + j = \sum_{a=2}^{i-1} f(a,j) + j = \sum_{a=1}^{i-1} f(a,j). \qquad \square$$

Another case in which we provide a partially explicit formula for $\zeta_{\mathcal{E} \curvearrowright R^n}(\mathbf{s})$ is when $\mathcal{E}$ is generated by an elementary strictly upper-triangular matrix, as stated below.

Proposition 4.3.5. *Let $1 \leq a < b \leq n$. Then*

$$\begin{aligned} &\zeta_{\langle E_{ab}\rangle \curvearrowright R^n}(s_1, \ldots, s_n) = \\ &\quad = (1 - q^{-s_a - a}) \cdot \zeta_{\{0\} \curvearrowright R^{b-1}}(s_1, \ldots, s_{b-1}) \cdot \zeta_{\langle E_{12}\rangle \curvearrowright R^{n-b+2}}(s_a + a - 1, s_b, \ldots, s_n). \end{aligned}$$

Proof. Set $N = n(n+1)/2$ and recall that

$$\zeta_{\langle E_{ab}\rangle \curvearrowright R^n}(s_1, \ldots, s_n) = \frac{1}{(1-q^{-1})^n} \int_A \prod_{i\in[n]} |x_{ii}|^{s_i} \mathrm{d}\mu_N(x),$$

where $A := \{x \in \mathrm{Tr}_n^*(R) \mid R^n x \cdot E_{ab} \subseteq R^n x\}$. Observe that

$$A = \bigsqcup_{S \subseteq [a],\, S \neq \emptyset} \underbrace{\left\{ x \in A \,\middle|\, x_{ra} \in x_{sa} R^\times \text{ and } x_{ka} \in x_{sa}\pi R,\ \forall\, r, s \in S,\ \forall\, k \in [a] \setminus S \right\}}_{=: A_S}.$$

Therefore,

$$\zeta_{\langle E_{ab}\rangle \curvearrowright R^n}(s_1, \ldots, s_n) = \sum_{S \subseteq [a],\, S \neq \emptyset} \underbrace{\frac{1}{(1 - q^{-s})^n} \int_{A_S} \prod_{i \in [n]} |x_{ii}|^{s_i} \mathrm{d}\mu_N(x)}_{=: I_S}. \tag{4.3.10}$$

From now on, we fix a non-empty subset $S \subseteq [a]$ and we focus on I_S. Define

$$s := \begin{cases} \min S, & \text{if } a \notin S; \\ a, & \text{otherwise.} \end{cases}$$

Note that $s < a$ unless $a \in S$. Moreover, for $x \in \mathrm{Tr}_n^*(R)$, consider the submatrix

$$x' := \begin{bmatrix} x_{sa} & x_{b-1,b} & \cdots & x_{b-1,n} \\ 0 & & & \\ \vdots & & x^{[b,n]\times[b,n]} & \\ 0 & & & \end{bmatrix} \in \mathrm{Tr}_{n-b+2}^*(R). \tag{4.3.11}$$

By Remark 4.2.10, given $x \in A_S$ we have

$$\begin{aligned} R^n x \cdot E_{ab} \subseteq R^n x &\iff x_{ka} \cdot (1, 0, \ldots, 0) \in R^{n-b+1} x^{[b,n]\times[b,n]},\ \forall\, k \in [a] \\ &\iff x_{sa} \cdot (1, 0, \ldots, 0) \in R^{n-b+1} x^{[b,n]\times[b,n]} \\ &\iff R^{n-b+2} x' \cdot E_{12} \subseteq R^{n-b+2} x'. \end{aligned}$$

Therefore, given $x \in \mathrm{Tr}_n^*(R)$ we have

$$x \in A_S \iff \begin{cases} R^{n-b+2} x' \cdot E_{12} \subseteq R^{n-b+2} x'; \\ x_{ra} \in x_{sa} R^\times, \quad \forall\, r \in S \setminus \{s\}; \\ x_{ka} \in x_{sa}\pi R, \quad \forall\, k \in [a] \setminus S. \end{cases}$$

For every $x \in A_S$, the entries x_{ij} with either ($1 \leq i < j \leq b-1$ and $j \neq a$) or ($1 \leq i \leq b-2$ and $b \leq j \leq n$), are only required to belong to $R \setminus \{0\}$ and do not appear in the integrand of I_S. Hence, by Fact 4.2.25(i) and Fubini's theorem,

$$\begin{aligned} I_S &:= \frac{1}{(1 - q^{-1})^n} \left(\int_{\{\xi = (x_{ii})_{i \in [b-1],\, i \neq a} \in R^{b-2}\}} \prod_{i \in [b-1],\, i \neq a} |x_{ii}|^{s_i} \mathrm{d}\mu_{b-2}(\xi) \right) \cdot J_S \\ &= \left(\frac{1}{(1 - q^{-1})^{n-b+2}} J_S \right) \cdot \prod_{i \in [b-1],\, i \neq a} \frac{1}{1 - q^{-s_i - 1}}, \end{aligned} \tag{4.3.12}$$

where, provided $N_b = (n-b)(n-b+1)/2$,

$$J_S = \int_{\{x' \in \mathrm{Tr}^*_{n-b+2}(R)\,:\, R^{n-b+2}x' \cdot E_{12} \subseteq R^{n-b+2}x'\}} \prod_{i=b}^{n} |x_{ii}|^{s_i} \cdot K_S(x_{sa}) \mathrm{d}\mu_{N_b}(x'), \tag{4.3.13}$$

and, for every $x_{sa} \in R \setminus \{0\}$,

$$K_S(x_{sa}) := \int_{\{y=(x_{ka})_{k\in[a],\,k\neq s} \in R^{a-1}\,:\, x_{ra} \in x_{sa}R^\times \text{ and } x_{ka} \in x_{sa}\pi R, \forall\, r \in S\setminus\{s\}, \forall\, k \in [a]\setminus S\}} |x_{aa}|^{s_a} \mathrm{d}\mu_{a-1}(y). \tag{4.3.14}$$

We first focus on $K_S(x_{sa})$. If $s = a$ (i.e., $a \in S$), then

$$\begin{aligned} K_S(x_{sa}) &= |x_{aa}|^{s_a} \cdot \prod_{r\in S,\, r\neq a} \mu_1(x_{aa}R^\times) \cdot \prod_{k\in[a]\setminus S} \mu_1(x_{aa}\pi R) \\ &= |x_{aa}|^{s_a} \cdot \prod_{r\in S,\, r\neq a} \Big(|x_{aa}|(1-q^{-1})\Big) \cdot \prod_{k\in[a]\setminus S} \Big(|x_{aa}|q^{-1}\Big) \\ &= \frac{(q-1)^{|S|}q^{-a}}{1-q^{-1}} |x_{aa}|^{s_a+a-1}. \end{aligned} \tag{4.3.15}$$

If $s \neq a$ (i.e., $a \notin S$), from Fact 4.2.25(i) we deduce that

$$\begin{aligned} K_S(x_{sa}) &= \prod_{r\in S\setminus\{s\}} \mu_1(x_{sa}R^\times) \cdot \prod_{k\in[a-1]\setminus S} \mu_1(x_{sa}\pi R) \cdot \int_{\{x_{aa}\in R\,:\, |x_{aa}| \leq |\pi \cdot x_{sa}|\}} |x_{aa}|^{s_a} \mathrm{d}\mu_1(x_{aa}) \\ &= \prod_{r\in S\setminus\{s\}} \Big(|x_{sa}|(1-q^{-1})\Big) \cdot \prod_{k\in[a-1]\setminus S} \Big(|x_{sa}|q^{-1}\Big) \cdot \frac{(1-q^{-1})q^{-s_a-1} \cdot |x_{sa}|^{s_a+1}}{1-q^{-s_a-1}} \\ &= \frac{(q-1)^{|S|}q^{-s_a-a}}{1-q^{-s_a-1}} |x_{sa}|^{s_a+a-1}. \end{aligned} \tag{4.3.16}$$

Hence, if $a \in S$ then (4.3.15) implies that

$$\begin{aligned} J_S(x) &= \frac{(q-1)^{|S|}q^{-a}}{1-q^{-1}} \int_{\{x\in \mathrm{Tr}^*_{n-b+2}(R)\,:\, R^{n-b+2}x' \cdot E_{12} \subseteq R^{n-b+2}x'\}} |x_{aa}|^{s_a+a-1} \prod_{i=b}^{n} |x_{ii}|^{s_i} \mathrm{d}\mu_{N_b}(x') \\ &= (q-1)^{|S|}q^{-a}(1-q^{-1})^{n-b+1} \cdot \zeta_{\langle E_{12}\rangle \curvearrowright R^{n-b+2}}(s_a+a-1, s_b, \ldots, s_n). \end{aligned} \tag{4.3.17}$$

Similarly, if $a \notin S$ then (4.3.16) yields

$$\begin{aligned} J_S(x) &= \frac{(q-1)^{|S|}q^{-s_a-a}}{1-q^{-s_a-1}} \int_{\{x\in \mathrm{Tr}^*_{n-b+2}(R)\,:\, R^{n-b+2}x' \cdot E_{12} \subseteq R^{n-b+2}x'\}} |x_{sa}|^{s_a+a-1} \prod_{i=b}^{n} |x_{ii}|^{s_i} \mathrm{d}\mu_{N_b}(x') \\ &= \frac{(q-1)^{|S|}q^{-s_a-a}(1-q^{-1})^{n-b+2}}{1-q^{-s_a-1}} \zeta_{\langle E_{12}\rangle \curvearrowright R^{n-b+2}}(s_a+a-1, s_b, \ldots, s_n). \end{aligned} \tag{4.3.18}$$

Combining (4.3.17) and (4.3.18), we have

$$\begin{aligned}\frac{1}{(1-q^{-1})^{n-b+2}} \sum_{S\subseteq[a],\, S\neq\emptyset} J_S &= \zeta_{\langle E_{12}\rangle \curvearrowright R^{n-b+2}}(s_a+a-1, s_b, \ldots, s_n)\cdot \\ &\cdot\left(\frac{q^{-a}}{1-q^{-1}} \cdot \sum_{S\subseteq[a],\, a\in S} (q-1)^{|S|} + \frac{q^{-s_a-a}}{1-q^{-s_a-1}} \cdot \sum_{S\subseteq[a-1],\, S\neq\emptyset} (q-1)^{|S|}\right).\end{aligned} \tag{4.3.19}$$

Observe that

$$\sum_{T\subseteq[a-1]} (q-1)^{|T|} = \sum_{k=0}^{a-1} \binom{a-1}{k} (q-1)^k = q^{a-1}.$$

Hence,

$$\sum_{S\subseteq[a],\, a\in S} (q-1)^{|S|} = \sum_{T\subseteq[a-1]} (q-1)^{|T|+1} = (q-1)q^{a-1} = (1-q^{-1})q^a \tag{4.3.20}$$

and

$$\sum_{S\subseteq[a-1],\, S\neq\emptyset} (q-1)^{|S|} = q^{a-1}-1. \tag{4.3.21}$$

Combining (4.3.19), (4.3.20) and (4.3.21), we obtain that

$$\frac{1}{(1-q^{-1})^{n-b+2}} \sum_{S\subseteq[a],\, S\neq\emptyset} J_S = \frac{1-q^{-s_a-a}}{1-q^{-s_a-1}} \cdot \zeta_{\langle E_{12}\rangle \curvearrowright R^{n-b+2}}(s_a+a-1, s_b, \ldots, s_n). \tag{4.3.22}$$

The statement now follows from (4.3.22), from the fact that

$$\zeta_{\langle E_{ab}\rangle \curvearrowright R^n}(s_1, \ldots, s_n) = \prod_{\substack{i\in[b-1],\\ b\neq a}} \frac{1}{1-q^{-s_1-1}} \cdot \left(\frac{1}{(1-q)^{n-b+2}} \sum_{\substack{S\subseteq[a],\\ S\neq\emptyset}} J_S\right),$$

(cf. (4.3.10) and (4.3.12)) and from Example 4.3.3. □

Proposition 4.3.5 allows to focus on $\zeta_{\langle E_{12}\rangle \curvearrowright R^n}(\mathbf{s})$ while studying the multivariate submodule zeta function of an elementary matrix. In the following, we collect some explicit computations.

Example 4.3.6. We have

$$\zeta_{\langle E_{12}\rangle \curvearrowright R^2}(\mathbf{s}) = \frac{1}{(1-q^{-s_1-1})(1-q^{-s_1-s_2-2})} = \zeta_{\{0\} \curvearrowright R^2}(s_1, s_1+s_2+1). \tag{4.3.23}$$

Indeed, for $x \in \mathrm{Tr}_2^*(R)$, by Remark 4.2.10 we observe that

$$R^2 x \cdot E_{12} = R^2 \begin{bmatrix} 0 & x_{11} \\ 0 & 0 \end{bmatrix} \subseteq R^2 x \iff x_{11} \in x_{22}R \iff |x_{11}| \leq |x_{22}|.$$

Hence, whenever $\mathrm{Re}(s_1), \mathrm{Re}(s_2) \geq 0$, we have

$$\zeta_{\langle E_{12}\rangle \curvearrowright R^2}(s_1,s_2) = \frac{1}{(1-q^{-1})^2}\int_{\{x\in \mathrm{Tr}_2^*(R)\,:\,|x_{11}|\leq |x_{22}|\}} |x_{11}|^{s_1}|x_{22}|^{s_2}\mathrm{d}\mu_3(x)$$

and Fact 4.2.25(ii) applies.

Example 4.3.7. We have

$$\begin{aligned}\zeta_{\langle E_{12}\rangle \curvearrowright R^3}(\mathbf{s}) &= \frac{1-q^{-s_1-s_3-3}}{(1-q^{-s_1-1})(1-q^{-s_1-s_2-2})(1-q^{-s_3-2})(1-q^{-s_1-s_3-2})} \\ &= \frac{1-q^{-s_1-s_3-3}}{1-q^{-s_3-2}}\cdot \zeta_{\{0\}\curvearrowright R^3}(s_1, s_1+s_2+1, s_1+s_3+1).\end{aligned} \tag{4.3.24}$$

In fact, following Remark 4.2.10, for every $x \in \mathrm{Tr}_3^*(R)$ we deduce that

$$\begin{aligned}R^3x\cdot E_{12}\subseteq R^3x &\iff [x_{11},0]\in R^2\cdot \begin{bmatrix} x_{22} & x_{23}\\ 0 & x_{33}\end{bmatrix} \\ &\iff x_{11}\in x_{22}R \text{ and } x_{11}x_{22}^{-1}x_{23}\in x_{33}R \\ &\iff |x_{11}x_{22}^{-1}|\leq 1 \text{ and } |x_{11}x_{22}^{-1}x_{23}x_{33}^{-1}|\leq 1.\end{aligned}$$

Hence,

$$\{x\in \mathrm{Tr}_3^*(R) \mid R^3x\cdot E_{12}\subseteq R^3x\} = A\sqcup B, \tag{4.3.25}$$

where

$$\begin{aligned} A &:= \left\{x\in \mathrm{Tr}_3^*(R)\,\middle|\, |x_{11}x_{22}^{-1}|\leq 1 \text{ and } |x_{23}x_{33}^{-1}|\leq 1\right\};\\ B &:= \left\{x\in \mathrm{Tr}_3^*(R)\,\middle|\, |x_{11}x_{22}^{-1}|\leq 1, |x_{33}x_{23}^{-1}|< 1 \text{ and } |(x_{11}x_{22}^{-1})(x_{33}x_{23}^{-1})^{-1}|\leq 1\right\}\\ &= \left\{x\in \mathrm{Tr}_3^*(R)\,\middle|\, |x_{11}x_{22}^{-1}|< 1, |x_{33}x_{23}^{-1}|< 1 \text{ and } |(x_{11}x_{22}^{-1})(x_{33}x_{23}^{-1})^{-1}|\leq 1\right\}.\end{aligned}$$

In particular, by (4.3.25) we have

$$\zeta_{\langle E_{12}\rangle \curvearrowright R^3}(\mathbf{s}) = \underbrace{\frac{1}{(1-q^{-1})^3}\int_A \prod_{i\in[3]}|x_{ii}|^{s_i}}_{=:I_A} + \underbrace{\frac{1}{(1-q^{-1})^3}\int_B \prod_{i\in[3]}|x_{ii}|^{s_i}}_{=:I_B}.$$

Fact 4.2.25(ii) implies that

$$I_A = \frac{1}{(1-q^{-1})^3}I_2(s_1,s_2)\cdot I_2(0,s_3) = \frac{1}{(1-q^{-s_1-1})(1-q^{-s_1-s_2-2})(1-q^{-s_3-2})}. \tag{4.3.26}$$

Moreover, up to renaming $x_{11}, x_{22}, x_{33}, x_{23}$ with x_1, x_2, x_3, x_4 respectively,

$$I_B = \int_{\{x \in (R\setminus\{0\})^4 \,:\, |x_1 x_2^{-1}|<1,\, |x_3 x_4^{-1}|<1 \text{ and } |(x_1 x_2^{-1})(x_3 x_4^{-1})^{-1}|\leq 1\}} \prod_{i\in[3]} |x_i|^{s_i} \mathrm{d}\mu_4(x). \tag{4.3.27}$$

Consider the change of variables $\varphi\colon (R\setminus\{0\})^4 \to (R\setminus\{0\})^4$, $\varphi(x) = (y_1, \dots, y_4)$, given by

$$y_1 = \pi^{-1} x_1 x_2^{-1}, \quad y_2 = x_2, \quad y_3 = \pi^{-1} x_3 x_4^{-1} \quad \text{and} \quad y_4 = x_4.$$

Note that $(x_1, \dots, x_4)$ belongs to the domain of the integral in (4.3.27) if, and only if, $(y_1, \dots, y_4)$ belongs to $(R\setminus\{0\})^4$ and satisfies $|y_1| \leq |y_3|$. Moreover, it is easy to check that the Jacobian matrix $(\mathrm{Jac}\varphi)_x = [\partial y_i/\partial x_j]_{i,j\in[4]}$ has determinant $\pi^{-2} x_2^{-1} x_4^{-1}$. By Fact 4.2.25, we deduce that

$$\begin{aligned} I_B &= \frac{q^{-s_1-s_3-2}}{(1-q^{-1})^3} \int_{\{y\in R^4 \,:\, |y_1|\leq|y_3|\}} |y_1|^{s_1}|y_2|^{s_1+s_2+1}|y_3|^{s_3}|y_4|^{s_3+1} \mathrm{d}\mu_4(y) \\ &= \frac{q^{-s_1-s_3-2}}{(1-q^{-s_1-s_2-2})(1-q^{-s_3-2})(1-q^{-1})} \int_{\{y=(y_1,y_3)\in R^2 \,:\, |y_1|\leq|y_3|\}} |y_1|^{s_1}|y_3|^{s_3} \mathrm{d}\mu_2(y) \\ &= \frac{(1-q^{-1})q^{-s_1-s_3-2}}{(1-q^{-s_1-1})(1-q^{-s_1-s_2-2})(1-q^{-s_3-2})(1-q^{-s_1-s_3-2})}. \end{aligned} \tag{4.3.28}$$

The formula in (4.3.24) now follows.

From Proposition 4.3.5 we obtain the following.

Corollary 4.3.8. *For every $1 \leq a < n$, for every $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$ we have*

$$\zeta_{\langle E_{an}\rangle \curvearrowright R^n}(\mathbf{s}) = \zeta_{\{0\}\curvearrowright R^n}(s_1, \dots, s_{n-1}, s_a + s_n + a). \tag{4.3.29}$$

Moreover, for all $1 \leq a < n-1$ and $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$, we have

$$\zeta_{\langle E_{a,n-1}\rangle \curvearrowright R^n}(\mathbf{s}) = \frac{1-q^{-s_a-s_n-a-2}}{1-q^{-s_n-2}} \cdot \zeta_{\{0\}\curvearrowright R^n}(\mathbf{s}). \tag{4.3.30}$$

Proof. It is a direct consequence of Example 4.3.6, Example 4.3.7, Proposition 4.3.5 and immediate simplifications. □

Remark 4.3.9. By Corollary 4.3.8, one sees that in general the multivariate submodule zeta function $\zeta_{\mathcal{E}\curvearrowright R^n}(\mathbf{s})$ is not invariant under conjugation of the set $\mathcal{E}$. Namely, set $n \geq 3$ and $1 < a < n-1$. Denote by $P \in \mathrm{GL}_n(R)$ the permutation matrix obtained from I_n by exchanging its last two columns and then its last two rows. Then, since $a < n-1$, we have

$$E_{an} = P \cdot E_{a,n-1} \cdot P.$$

However, Corollary 4.3.8 shows that

$$\zeta_{\langle E_{an}\rangle \curvearrowright R^n}(\mathbf{s}) \neq \zeta_{\langle E_{a,n-1}\rangle \curvearrowright R^n}(\mathbf{s}).$$

In contrast, as observed in Remark 4.2.10(iv), in the univariate case we have

$$\zeta_{\langle E_{an}\rangle \curvearrowright R^n}(s) \neq \zeta_{\langle E_{a,n-1}\rangle \curvearrowright R^n}(s).$$

Note that the latter equality can be recovered also by substituting s_i with $s-i$, for every $i \in [n]$, in the formulae in Corollary 4.3.8.

To conclude, we discuss how to reduce the computation of $\zeta_{\mathcal{E}\curvearrowright R^n}(\mathbf{s})$ if $\mathcal{E}$ contains the elementary matrix E_{in}, for some $i < n$. In view of this, given $k \in K$ and $i \in [n]$, denote by $D_i(k) \in \mathrm{Mat}_n(K)$ the diagonal matrix with i-th entry equal to k, and with all the other diagonal entries equal to 1. Clearly, for $k \neq 0$, the matrix $D_i(k)$ is invertible and $D_i(k)^{-1} = D_i(k^{-1})$.

Proposition 4.3.10. *Let $1 \leq i < n$ and consider $r \geq 1$ matrices $A_1, \ldots, A_r$ in $\mathrm{Mat}_n(R)$ such that, for all $j \in [r]$,*

$$D_i(k) \cdot A_j \cdot D_i(k)^{-1} = A_j, \quad \forall\, k \in K^{\times}. \tag{4.3.31}$$

Then

$$\zeta_{\langle A_1,\ldots,A_r,E_{in}\rangle \curvearrowright R^n}(s_1,\ldots,s_n) = \zeta_{\langle A_1,\ldots,A_r\rangle \curvearrowright R^n}(s_1,\ldots,s_{n-1},s_i+s_n+i).$$

Proof. Given $x = [x_{ab}]_{1\leq a\leq b\leq n} \in \mathrm{Tr}_n^*(R)$, observe that xE_{in} is a matrix whose first $n-1$ columns are zero and whose n-th column is the i-th column of x. Hence, by Remark 4.2.10,

$$\begin{aligned} R^n x E_{in} \subseteq R^n x &\iff x_{ki} \in R \cdot x_{nn},\ \forall\, k \in [i] \\ &\iff x \cdot D_i(x_{nn}^{-1}) \in \mathrm{Tr}_n(R). \end{aligned} \tag{4.3.32}$$

Consider the function $f\colon \mathrm{Tr}_n^*(R) \longrightarrow \mathrm{Tr}_n^*(R)$ defined by the assignment $f(x) = x \cdot D_i(x_{nn}^{-1})$, for every $x \in \mathrm{Tr}_n^*(R)$. Namely, for all $1 \leq a \leq b \leq b$, the (a,b)-th entry $f_{ab}(x)$ of $f(x)$ is

$$f_{ab}(x) = \begin{cases} x_{ab}x_{nn}^{-1}, & \text{if } b = i; \\ x_{ab}, & \text{otherwise}, \end{cases} \tag{4.3.33}$$

We now use f to change the variables in the integral defining $\zeta_{\langle A_1,\ldots,A_r,E_{in}\rangle \curvearrowright R^n}(\mathbf{s})$, which is

$$\zeta_{\langle A_1,\ldots,A_r,E_{in}\rangle \curvearrowright R^n}(s_1,\ldots,s_n) = \frac{1}{(1-q^{-1})^n} \int_{\mathcal{V}} \prod_{l\in[n]} |x_{ll}|^{s_l} \mathrm{d}\mu_N(x), \tag{4.3.34}$$

where

$$\mathcal{V} = \{x \in \mathrm{Tr}_n^*(R) \mid R^n x \cdot B \subseteq R^n x \text{ for every } B \in \{A_1,\ldots,A_r,E_{in}\}\}.$$

Denote by $\mathrm{Jac}f(x) = \left[\frac{\partial f_{ab}}{\partial x_{cd}}\right]_{(a,b),(c,d)\in[n]^2_{\leq}}$ the Jacobian matrix of f. Then,

$$\frac{\partial f_{ab}}{\partial x_{cd}} = \begin{cases} 0, & \text{if } (a,b) \neq (c,d); \\ x_{nn}^{-1}, & \text{if } (a,b) = (c,d) \text{ and } b = i; \\ 1, & \text{otherwise.} \end{cases}$$

In particular, $\mathrm{Jac}f(x)$ is a diagonal matrix and therefore

$$|\det \mathrm{Jac}f(x)| = \prod_{1\leq a\leq b\leq n} \left|\frac{\partial f_{ab}}{\partial x_{ab}}\right| = \prod_{l=1}^{i} |x_{nn}^{-1}| = |x_{nn}|^{-i}.$$

Moreover,

$$f(\mathcal{V}) = \{x \in \mathrm{Tr}_n(R) \mid R^n x A_k \subseteq R^n x, \forall\, k \in [r]\}.$$

Changing the variables with f in (4.3.34) (cf. Fact 4.2.25(iii)), we conclude that

$$\begin{aligned} \zeta_{\langle A_1,\dots,A_r,E_{in}\rangle \curvearrowright R^n}(s_1,\dots,s_n) &= \frac{1}{(1-q^{-1})^n} \int_{f(\mathcal{V})} \prod_{l\in[n-1]} |x_{ll}|^{s_l} \cdot |x_{nn}|^{s_n - s_i - i} \mathrm{d}\mu_N(x) \\ &= \zeta_{\langle A_1,\dots,A_r\rangle \curvearrowright R^n}(s_1,\dots,s_{n-1}, s_i + s_n + i). \end{aligned}$$

□

Corollary 4.3.11. *Given integers $i < n \geq 2$ and $1 \leq a < b \leq n$ such that $a \neq i \neq b$. Then,*

$$\zeta_{\langle E_{ab},E_{in}\rangle \curvearrowright R^n}(s_1,\dots,s_n) = \zeta_{\langle E_{ab}\rangle \curvearrowright R^n}(s_1,\dots,s_{n-1}, s_i + s_n + i). \tag{4.3.35}$$

Proof. For every $k \in K^\times$, observe that

$$D_i(k) \cdot E_{ab} \cdot D_i(k)^{-1} = k^{\mathbb{1}_{\{i\}}(a)} \cdot E_{ab} \cdot k^{-\mathbb{1}_{\{i\}}(b)} = E_{ab},$$

where the 1-th power of an element of $K^\times$ is the element itself, and its 0-th power is the unit of $\mathbb{K}$. The claim now follows from Proposition 4.3.10. □

Remark 4.3.12. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a decreasing partition of $n \geq 2$, and let $A(\lambda)$ be the matrix defined in (4.3.4). Given $1 \leq i < n$ and an arbitrary $k \in K^\times$, we claim that

$$D_i(k) \cdot A(\lambda) \cdot D_i(k)^{-1} = A(\lambda) \Longleftrightarrow \lambda_2 \leq i \leq \lambda_1. \tag{4.3.36}$$

In fact, $D_i(k) \cdot A(\lambda)$ is the matrix obtained from $A(\lambda)$ by multiplying the i-th row by k. Observe also that, whenever $\lambda_1 + \ldots + \lambda_k + \lambda_{k+2} \leq i \leq \lambda_1 + \ldots + \lambda_k + \lambda_{k+1}$ for some $k \leq r-2$, then the i-th row of $A(\lambda)$ is zero. In all the other cases, the i-th row of $A(\lambda)$ has exactly one non-vanishing entry. Hence, $D_i(k) \cdot A(\lambda) = A(\lambda)$ if $\lambda_1 + \ldots + \lambda_k + \lambda_{k+2} \leq i \leq \lambda_1 + \ldots + \lambda_k + \lambda_{k+1}$ for some $k \leq r-2$, and $D_i(k) \cdot A(\lambda) = k \cdot A(\lambda)$ otherwise.

Similarly, $A(\lambda)\cdot D_i(k)^{-1} = A(\lambda)\cdot D_i(k^{-1})$ is the matrix obtained from $A(\lambda)$ by multiplying the i-th column by k^{-1}. By definition, the first λ_1 columns of $A(\lambda)$ are zero, and all the remaining columns have exactly one non-zero entry. Hence, $A(\lambda)\cdot D_i(k^{-1}) = A(\lambda)$ if $i \leq \lambda_1$ and $A(\lambda)\cdot D_i(k^{-1}) = k^{-1}A(\lambda)$ otherwise. This yields (4.3.36).

In view of Theorem 4.3.22, we conclude with the following.

Remark 4.3.13. Proposition 4.3.4 and Corollary 4.3.8 provide instances of multivariate submodule zeta functions $\zeta_{\mathcal{E}\curvearrowright R^n}(\mathbf{s})$ of subalgebras $\mathcal{E} \leq \mathrm{Up}_n(R)$ such that

$$\lim_{\mathbf{s}\to\mathbf{s}_0}\Big((1-q^{-s_1-1})\zeta_{\mathcal{E}\curvearrowright R^n}(\mathbf{s})\Big) \tag{4.3.37}$$

exists (and it is finite). The existence of this limit will be one of the key hypotheses of Theorem 4.3.22. Thanks to Propositions 4.3.5 and 4.3.10, in other cases one may simplify the problem of checking the existence of the limit in (4.3.37) by looking at algebras with a lower R-rank than the algebra we start with. However, it remains an open problem to check whether the limit in 4.3.37 exists for arbitrary subalgebras of $\mathrm{Up}_n(R)$.

4.3.2 The case of algebras generated by strictly upper triangular weighted elementary matrices

In this section, K denotes a non-Archimedean local field of characteristic zero with absolute value $|\_|$, ring of integers R and residue field size q. We also set a uniformiser $\pi \in R$, an integer $n \geq 1$ and $N = n(n+1)/2$.

The goal is to study the multivariate submodule zeta function of the subalgebra $\mathcal{E}_{I,r} \leq \mathrm{Mat}_n(R)$ generated by

$$\{\pi^{r_{ij}}E_{ij}\}_{(i,j)\in I} \subseteq \mathrm{Mat}_n(R), \tag{4.3.38}$$

for some $I \subseteq [n]^2$ and $r = (r_{ij})_{(i,j)\in I} \in \mathbb{Z}^I_{\geq 0}$. Given $\mathbf{s} = (s_1,\dots,s_n) \in \mathbb{C}^n$ with $\mathrm{Re}(s_i) \geq 0$ for every $i \in [n]$, note that

$$\zeta_{\mathcal{E}_{I,r}\curvearrowright R^n}(\mathbf{s}) = \frac{1}{(1-q^{-1})^n}\int_{\mathcal{V}_{I,r}}\prod_{i\in[n]}|x_{ii}|^{s_i}\mathrm{d}\mu_N(x), \tag{4.3.39}$$

where

$$\mathcal{V}_{I,r} := \Big\{x \in \mathrm{Tr}_n^*(R)\,\Big|\, R^n x\cdot \pi^{r_{ij}}E_{ij} \subseteq R^n x,\ \forall\,(i,j)\in I\Big\}. \tag{4.3.40}$$

In Section 4.3.2, we characterise the domain $\mathcal{V}_{I,r}$ as a set of the form

$$\{x \in \mathrm{Tr}_n^*(R) \mid \forall\, i \in [n],\ \|\mathfrak{g}_i(x)\| \leq 1\}, \tag{4.3.41}$$

where every $\mathfrak{g}_i(x)$ is a determined family of Laurent polynomials in $x = [x_{ab}]_{1\leq a\leq b\leq n}$. In Section 4.3.2, we assume that $I \subseteq [n]^2_<$ and exploit the description of $\mathcal{V}_{I,r}$ in (4.3.41) to prove that Conjecture 4.2.19 holds whenever a determined analytic hypothesis on $\zeta_{\mathcal{E}_{I,r}\curvearrowright R^n}(\mathbf{s})$ is satisfied.

A characterisation of the set $\mathcal{V}_{I,r}$

We introduce a function Ψ which maps every invertible upper triangular matrix over K in its inverse. This allows us to give a compact description of each invariance condition defining the set $\mathcal{V}_{I,r}$ (cf. Lemma 4.3.17). After this, the characterisation of $\mathcal{V}_{I,r}$ as in (4.3.41) will be a straightforward consequence (cf. Corollary 4.3.18 and Remark 4.3.21).

Definition 4.3.14 (*The map* Ψ)**.** For $x \in \mathrm{Mat}_n(K)$, the *adjugate matrix* $\mathrm{adj}(x)$ of x is the matrix whose (a,b)-th entry equals $(-1)^{a+b}\det(x^{\vee(b,a)})$ (see Section 4.1.1 for $x^{\vee(b,a)}$). Let $\Psi\colon \mathrm{Tr}_n^{\times}(K) \to \mathrm{Tr}_n^{\times}(K)$ be the function mapping x in its inverse, and observe that

$$\Psi(x) = \left(\prod_{l\in[n]} x_{ll}^{-1}\right)\mathrm{adj}(x), \quad \forall\, x \in \mathrm{Tr}_n^{\times}(K). \tag{4.3.42}$$

For $(a,b) \in [n]^2_{\leq}$, let $\Psi_{ab}(x)$ denote the (a,b)-th entry of $\Psi(x)$ and note that

$$\Psi_{ab}(x) := (-1)^{a+b}\left(\prod_{l\in[n]} x_{ll}^{-1}\right)\det(x^{\vee(b,a)}). \tag{4.3.43}$$

Convention 4.3.15. Technically, the map Ψ depends on n. However, we avoid specifying it as we will always evaluate Ψ at a matrix whose dimension is clear from the context.

Lemma 4.3.16. *Let $x \in \mathrm{Tr}_n^{\times}(K)$. For $(a,b) \in [n]^2_{\leq}$ we have*

$$\Psi_{ab}(x) = (-1)^{a+b}\left(\prod_{l=a}^{b} x_{ll}^{-1}\right)p_{ab}(x),$$

where $p_{ab}(x) \equiv 1$ if $a=b$, and $p_{ab}(x) = \det\left(x^{[a,b-1]\times[a+1,b]}\right)$ if $a<b$. In particular,

$$\Psi_{ab}(x) = \Psi_{1,b-a+1}(x^{[a,b]\times[a,b]}). \tag{4.3.44}$$

Proof. By (4.3.43) we have

$$\Psi_{ab}(x) = (-1)^{a+b}\left(\prod_{l\in[n]} x_{ll}^{-1}\right)\det(x^{\vee(b,a)}).$$

The matrix $x^{\vee(b,a)}$ has the following decomposition as a block upper-triangular matrix (possibly with blocks of dimension zero):

$$x^{\vee(b,a)} = \begin{bmatrix} x^{[1,a-1]\times[1,a-1]} & x^{[1,a-1]\times[a+1,b]} & x^{[1,a-1]\times[b+1,n]} \\ 0_{(b-a)\times(a-1)} & x^{[a,b-1]\times[a+1,b]} & x^{[a,b-1]\times[b+1,n]} \\ 0_{(n-b)\times(a-1)} & 0_{(n-b)\times(b-a)} & x^{[b+1,n]\times[b+1,n]} \end{bmatrix}.$$

Hence,

$$\begin{aligned}
\det(x^{\vee(b,a)}) &= \det(x^{[1,a-1]\times[1,a-1]}) \cdot \det(x^{[a,b-1]\times[a+1,b]}) \cdot \det(x^{[b+1,n]\times[b+1,n]}) \\
&= \left(\prod_{l=1}^{a-1} x_{ll}\right) \cdot p_{ab}(x) \cdot \left(\prod_{l=b+1}^{n} x_{ll}\right) \\
&= \left(\prod_{l\in[n]} x_{ll}\right) \cdot \left(\prod_{l=a}^{b} x_{ll}^{-1}\right) \cdot p_{ab}(x). \qquad \square
\end{aligned}$$

Lemma 4.3.17. *Let $1 \le i, j \le n$, $r \in \mathbb{Z}$ and $x \in \mathrm{Tr}_n^*(R)$. Then $R^n x \cdot \pi^r E_{ij} \subseteq R^n x$ if, and only if, $|\pi^r x_{ki} \Psi_{jm}(x)| \le 1$ for all $1 \le k \le i$ and $j \le m \le n$.*

Proof. By Remark 4.2.10, we observe that

$$R^n x \cdot \pi^r E_{ij} \subseteq R^n x \iff \pi^r x_{ki} e_j^{(n)} \in R^n x, \quad \forall\, k \in [i], \tag{4.3.45}$$

where $e_j^{(n)}$ denotes the j-th canonical row vector of R^n. Set now $1 \le k \le i$ and denote by $e_1^{(n-j+1)}$ the first canonical row vector of R^{n-j+1}. Since the first $j-1$ entries of each $\pi^r x_{ki} e_j^{(n)}$ are vanishing, we deduce that

$$\begin{aligned}
\pi^r x_{ki} e_j^{(n)} \in R^n x &\iff \pi^r x_{ki} e_1^{(n-j+1)} \in R^{n-j+1} x^{[j,n]\times[j,n]} \\
&\iff \pi^r x_{ki} \cdot e_1^{(n-j+1)} \Psi(x^{[j,n]\times[j,n]}) \in R^{n-j+1} \\
&\iff \pi^r x_{ki} \Psi_{jm}(x) \in R, \quad \forall\, j \le m \le n.
\end{aligned} \tag{4.3.46}$$

In the last equivalence, we have applied (4.3.44). $\square$

Corollary 4.3.18. *Let $\mathcal{V}_{I,r}$ be the set defined in (4.3.40). For an arbitrary $x \in \mathrm{Tr}_n^*(R)$ we have $x \in \mathcal{V}_{I,r}$ if, and only if, for all $1 \le k \le i$ and $j \le m \le n$ with $(i,j) \in I$ we have*

$$|\pi^{r_{ij}} x_{ki} \Psi_{jm}(x)| \le 1.$$

Definition 4.3.19. Let $\{\pi^{r_{ij}} E_{ij}\}_{(i,j)\in I}$, for some $I \subseteq [n]^2$ and $r = (r_{ij})_{(i,j)\in I} \subseteq \mathbb{Z}_{\ge 0}^I$. For every $i \in [n]$ and $x \in \mathrm{Tr}_n^\times(K)$, define

$$\mathfrak{f}_i(x) = \mathfrak{f}_i^{I,r}(x) := \{\pi^{r_{ij}} \Psi_{jm}(x) \mid (i,j) \in I,\ j \le m \le n\} \cup \{1\}.$$

Note that $\mathfrak{f}_i(x) = \{1\}$ if there is no $j \in [n]$ such that $(i,j) \in I$.

Remark 4.3.20. For $i \in [n]$, let

$$j_i := \min\big\{\{j \in [n] \mid (i,j) \in I\} \cup \{n\}\big\}.$$

Recalling (4.3.44), each element of $\mathfrak{f}_i(x)$ depends only on the entries of $x^{[j_i,n]\times[j_i,n]}$. If in particular $j_i > i$, this implies that each member of $\mathfrak{f}_i(x)$ never depends on the variables x_{ab}, for all $a \le b \le n$ such that $a \le i$. This observation will play an important role in the proof of Theorem 4.3.22.

Remark 4.3.21. For every $i \in [n]$, since $1 \in \mathfrak{f}_i(x)$, note that

$$\|\mathfrak{f}_i(x)\| \geq 1.$$

Moreover, the set $\mathcal{V}_{I,r}$ defined in (4.3.40) becomes

$$\mathcal{V}_{I,r} = \left\{x \in \mathrm{Tr}_n^*(R) \,\middle|\, |x_{ki}| \leq \|\mathfrak{f}_i(x)\|^{-1},\ \forall 1 \leq k \leq i \leq n\right\}. \tag{4.3.47}$$

Setting $\mathfrak{g}_i(x) = \mathfrak{g}_i^{I,r}(x) := \{x_{ki} \cdot f(x) \mid k \in [i],\ f(x) \in \mathfrak{f}_i(x)\}$ for every $i \in [n]$, one obtains that

$$\mathcal{V}_{I,r} = \left\{x \in \mathrm{Tr}_n^*(R) \,\middle|\, \|\mathfrak{g}_i(x)\| \leq 1,\ \forall\, i \in [n]\right\}. \tag{4.3.48}$$

About the behaviour of $\zeta_{\mathcal{E}_{I,r} \curvearrowright R^n}$ at $\mathbf{s} = (-1, -2, \ldots, -n)$

Provided

$$\mathbf{s}_0 := (-1, -2, \ldots, -n) \in \mathbb{C}^n, \tag{4.3.49}$$

the goal of the next pages is to prove the following theorem.

Theorem 4.3.22. *Let $\mathcal{E}_{I,r}$ be the subalgebra of $\mathrm{Mat}_n(R)$ generated by $\{\pi^{r_{ij}} E_{ij}\}_{(i,j)\in I} \subseteq \mathrm{Mat}_n(K)$, for some $I \subseteq [n]^2_<$ and $r = (r_{ij})_{(i,j)\in I} \subseteq \mathbb{Z}^I_{\geq 0}$. Assume that*

$$\lim_{\mathbf{s}\to\mathbf{s}_0} \left((1 - q^{-s_1-1})\zeta_{\mathcal{E}_{I,r} \curvearrowright R^n}(\mathbf{s})\right) \text{ exists in } \mathbb{C} \cup \{\infty\}. \tag{4.3.50}$$

Then

$$\left((1 - q^{-s_1-1})\zeta_{\mathcal{E}_{I,r} \curvearrowright R^n}(\mathbf{s})\right)\Big|_{\mathbf{s}=\mathbf{s}_0} = \prod_{i=1}^{n-1} \frac{1}{1-q^i} = \left((1 - q^{-s_1-1})\zeta_{\{0\} \curvearrowright R^n}(\mathbf{s})\right)\Big|_{\mathbf{s}=\mathbf{s}_0}. \tag{4.3.51}$$

In particular,

$$\left.\frac{\zeta_{\mathcal{E}_{I,r} \curvearrowright R^n}(\mathbf{s})}{\zeta_{\{0\} \curvearrowright R^n}(\mathbf{s})}\right|_{\mathbf{s}=\mathbf{s}_0} = 1. \tag{4.3.52}$$

Recalling Remark 4.3.2(iii), having (4.3.52) implies that the univariate zeta function $\zeta_{\mathcal{E}_{I,r} \curvearrowright R^n}(s)$ satisfies Conjecture 4.2.19.

To prove Theorem 4.3.22, we introduce auxiliary functions (cf. Definition 4.3.24) which are based on the definition of the $\mathfrak{f}_i(x)$'s (cf. Definition 4.3.19) and the characterisation of $\mathcal{V}_{I,r}$ given in Remark 4.3.21.

Notation 4.3.23. Given $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n$ and $1 \leq j \leq n$, denote by $\mathbf{s}^{(j)}$ the vector obtained from $\mathbf{s}$ by deleting the first $j-1$ entries, i.e.,

$$\mathbf{s}^{(j)} := (s_j, \ldots, s_n) \in \mathbb{C}^{n-j+1}.$$

Definition 4.3.24. For $1 \leq j \leq n$, define

$$\mathcal{V}_{I,r}^{(j)} := \Big\{ x = [x_{ab}]_{j \leq a \leq b \leq n} \in \mathrm{Tr}_{n-j+1}^{\times}(K) \,\Big|\, |x_{ki}| \leq \|\mathfrak{f}_i(x)\|^{-1}, \forall\, j \leq k \leq i \leq n \Big\}. \quad (4.3.53)$$

Moreover, for all $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n$ with $\mathrm{Re}(s_i) \geq 0$ for every $i \geq 1$, and all $\mathbf{t} = (t_0, \ldots, t_n) \in \mathbb{C}^{n+1}$ with $\mathrm{Re}(t_i) \leq 0$ for every $i \geq 0$, define

$$\mathcal{Z}_{\mathcal{E}_{I,r}}^{(j)}(\mathbf{s}^{(j)}, \mathbf{t}^{(j-1)}) := \frac{1}{(1-q^{-1})^{n-j+1}} \int_{\mathcal{V}_{I,r}^{(j)}} \prod_{i=j}^{n} |x_{ii}|^{s_i} \cdot \prod_{i=j-1}^{n} \|\mathfrak{f}_i(x)\|^{t_i} \mathrm{d}\mu_{N_j}(x). \quad (4.3.54)$$

where $N_j = (n-j+1)(n-j+2)/2$.

Remark 4.3.25.

(i) The integral in (4.3.54) converges for the prescribed $\mathbf{s}$ and $\mathbf{t}$. The argument is analogous to (4.3.1).

(ii) By Definition 4.3.1, we observe that

$$\zeta_{\mathcal{E} \curvearrowright R^n}(\mathbf{s}) = \mathcal{Z}_{\mathcal{E}}^{(1)}(\mathbf{s}, \mathbf{t})\Big|_{\mathbf{t}=(0,\ldots,0)}.$$

(iii) Arguing as in Remark 4.3.2(i), for every $1 \leq j \leq n$ there is a rational function $W_{\mathcal{E}}^{(j)}(X_j, \ldots, X_n, Y_{j-1}, \ldots, Y_n)$ with integral coefficients such that

$$\mathcal{Z}_{\mathcal{E}_{I,r}}^{(j)}(\mathbf{s}^{(j)}, \mathbf{t}^{(j-1)}) = W_{\mathcal{E}}^{(j)}(q^{-s_j}, \ldots, q^{-s_n}, q^{-t_{j-1}}, \ldots, q^{-t_n}),$$

whenever $\mathrm{Re}(s_i) \geq 0$ for every $i \geq 1$ and $\mathrm{Re}(t_i) \leq 0$ for every $i \geq 0$. By the identity principle, this allows us to continue $\mathcal{Z}_{\mathcal{E}_{I,r}}^{(j)}(\mathbf{s}^{(j)}, \mathbf{t}^{(j-1)})$ to a function defined in every $(\mathbf{s}^{(j)}, \mathbf{t}^{(j-1)}) \in \mathbb{C}^{n-j+1} \times \mathbb{C}^{n-j+2}$ for which the evaluation

$$W_{\mathcal{E}}^{(j)}(q^{-s_j}, \ldots, q^{-s_n}, q^{-t_{j-1}}, \ldots, q^{-t_n})$$

is defined and is finite.

Lemma 4.3.26. *Assume that $I \subseteq [n]^2_<$. Then, for every $1 \leq j \leq n-1$, we have*

$$\mathcal{Z}_{\mathcal{E}_{I,r}}^{(j)}(\mathbf{s}^{(j)}, \mathbf{t}^{(j-1)})\Big|_{t_{j-1}=0} = \frac{1}{1-q^{-s_j-1}} \cdot \mathcal{Z}_{\mathcal{E}_{I,r}}^{(j+1)}\Big(\mathbf{s}^{(j+1)}, \mathbf{t}^{(j)} - (s_j+1, 1, \ldots, 1)\Big).$$

Proof. Since the two functions involved in the statement are both a ratio of holomorphic functions, it suffices to prove the claim for every $(\mathbf{s}^{(j)}, \mathbf{t}^{(t_j)}) \in \mathbb{C}^{n-j+1} \times \mathbb{C}^{n-j+2}$ satisfying

$\mathrm{Re}(s_i) \geq 0$ for every $i \geq 1$ and $\mathrm{Re}(t_i) \leq 0$ for every $i \geq 1$. Given $x = [x_{ab}]_{j \leq a \leq b \leq n} \in \mathrm{Tr}_{n-j+1}(K)$, consider the submatrix $x' = [x_{ab}]_{j+1 \leq a \leq b \leq n} \in \mathrm{Tr}_{n-j}(K)$ and observe that

$$x \in \mathcal{V}_{I,r}^{(j)} \iff x' \in \mathcal{V}_{\mathcal{E}}^{(j+1)} \text{ and } |x_{jl}| \leq \|\mathfrak{f}_l(x)\|^{-1}, \forall\, j \leq l \leq n. \tag{4.3.55}$$

Moreover, for $j \leq l \leq n$, every function in $\mathfrak{f}_l(x)$ depends only on the entries of x' (cf. Remark 4.3.20). Thus we may write $\mathfrak{f}_l(x')$ in place of $\mathfrak{f}_l(x)$.

Set $N_k = (n-k+1)(n-k+2)/2$ for all $k \in [n]$. Hence we have the following:

$$\begin{aligned}
\mathcal{Z}_{\mathcal{E}_{I,r}}^{(j)}(\mathbf{s}^{(j)}, \mathbf{t}^{(j-1)})\Big|_{t_{j-1}=0} &= \frac{1}{(1-q^{-1})^{n-j+1}} \int_{\mathcal{V}_{I,r}^{(j)}} \prod_{i=j}^{n} (|x_{ii}|^{s_i} \cdot \|\mathfrak{f}_i(x)\|^{t_i}) \mathrm{d}\mu_{N_j}(x) \\
\overset{(4.3.55)}{=} & \frac{1}{(1-q^{-1})^{n-j+1}} \int_{x'=[x_{ab}]_{j+1 \leq a \leq b \leq n} \in \mathcal{V}_{\mathcal{E}}^{(j+1)}} \prod_{i=j+1}^{n} |x_{ii}|^{s_i} \cdot \prod_{i=j}^{n} \|\mathfrak{f}_i(x')\|^{t_i} \cdot \\
& \cdot \left(\int_{|x_{jj}| \leq \|\mathfrak{f}_j(x')\|^{-1}} |x_{jj}|^{s_j} \mathrm{d}\mu_1(x_{jj}) \right) \cdot \prod_{l=j+1}^{n} \left(\int_{|x_{jl}| \leq \|\mathfrak{f}_l(x')\|^{-1}} 1 \mathrm{d}\mu_1(x_{jl}) \right) \mathrm{d}\mu_{N_{j+1}}(x') \\
= & \frac{1}{(1-q^{-1})^{n-j}} \int_{x' \in \mathcal{V}_{\mathcal{E}}^{(j+1)}} \prod_{i=j+1}^{n} (|x_{ii}|^{s_i} \|\mathfrak{f}_i(x')\|^{t_i - 1}) \cdot \frac{\|\mathfrak{f}_j(x')\|^{t_j - s_j - 1}}{1 - q^{-s_j - 1}} \mathrm{d}\mu_{N_{j+1}}(x').
\end{aligned}$$

The third equality before is due to Fact 4.2.25(i). □

Proof of Theorem 4.3.22. Let $\mathbf{s} \in \mathbb{C}^n$ with $\mathrm{Re}(s_i) \geq 0$ for every $i \in [n]$, and recall that $\mathbf{s}_0^{(2)} = (-2, \ldots, -n)$ (cf. (4.3.49) and Notation 4.3.23). By Remark 4.3.25(ii) and Lemma 4.3.26, we observe that

$$\begin{aligned}
\left((1-q^{-s_1-1})\zeta_{\mathcal{E}_{I,r} \curvearrowright R^n}(\mathbf{s})\right)\Big|_{s_1=-1} &= \left((1-q^{-s_1-1}) \cdot \mathcal{Z}_{\mathcal{E}_{I,r}}(\mathbf{s}, \mathbf{t})|_{\mathbf{t}=(0,\ldots,0)}\right)\Big|_{s_1=-1} \\
&= \mathcal{Z}_{\mathcal{E}_{I,r}}^{(2)}(\mathbf{s}^{(2)}, \mathbf{t}^{(1)})|_{\mathbf{t}^{(1)}=(0,-1,\ldots,-1)}.
\end{aligned}$$

We now give a general observation that will be of key importance for what follows. Given $m \geq 2$, consider a function $f\colon \mathbb{C}^m \to \mathbb{C}$ in the complex variables $\mathbf{u} = (u_1, \ldots, u_m)$ and let $\mathbf{u}_0 = ((u_0)_1, \ldots, (u_0)_m) \in \mathbb{C}^m$. Recall that if the limit of $f(\mathbf{u})$ for $\mathbf{u} \to \mathbf{u}_0$ exists, then for every $i < m$ one has

$$\lim_{\mathbf{u} \to \mathbf{u}_0} f(\mathbf{u}) = \lim_{\mathbf{u}^{(i+1)} \to \mathbf{u}_0^{(i+1)}} f(\mathbf{u})|_{u_1=(u_0)_1, \ldots, u_i=(u_0)_i}. \tag{4.3.56}$$

Repeatedly applying this latter observation and Lemma 4.3.26, we obtain the following:

$$
\begin{aligned}
\lim_{\mathbf{s}\to\mathbf{s}_0}(1-q^{-s_1-1})\zeta_{\mathcal{E}_{I,r}\curvearrowright R^n}(\mathbf{s}) &= \lim_{\mathbf{s}^{(2)}\to\mathbf{s}_0^{(2)}}(1-q^{-s_1-1})\zeta_{\mathcal{E}_{I,r}\curvearrowright R^n}(\mathbf{s})\Big|_{s_1=-1}\\
&= \lim_{\mathbf{s}^{(2)}\to\mathbf{s}_0^{(2)}} \mathcal{Z}^{(2)}_{\mathcal{E}_{I,r}}(\mathbf{s}^{(2)},\mathbf{t}^{(1)})|_{\mathbf{t}^{(1)}=(0,-1,\dots,-1)}\\
&= \lim_{\mathbf{s}^{(2)}\to\mathbf{s}_0^{(2)}} \frac{\mathcal{Z}^{(3)}_{\mathcal{E}_{I,r}}(\mathbf{s}^{(3)},\mathbf{t}^{(2)})|_{\mathbf{t}^{(2)}=(-s_2-2,-2,\dots,-2)}}{1-q^{-s_2-1}}.\\
&= \frac{1}{1-q}\cdot \lim_{\mathbf{s}^{(3)}\to\mathbf{s}_0^{(3)}} \mathcal{Z}^{(3)}_{\mathcal{E}_{I,r}}(\mathbf{s}^{(3)},\mathbf{t}^{(2)})|_{\mathbf{t}^{(2)}=(0,-2,\dots,-2)}\\
&= \frac{1}{1-q}\cdot \lim_{\mathbf{s}^{(3)}\to\mathbf{s}_0^{(3)}} \frac{\mathcal{Z}^{(4)}_{\mathcal{E}_{I,r}}(\mathbf{s}^{(4)},\mathbf{t}^{(3)})|_{\mathbf{t}^{(3)}=(-s_3-3,-3,\dots,-3)}}{1-q^{-s_3-1}}\\
&= \frac{1}{(1-q)(1-q^2)}\cdot \lim_{\mathbf{s}^{(4)}\to\mathbf{s}_0^{(4)}} \mathcal{Z}^{(4)}_{\mathcal{E}_{I,r}}(\mathbf{s}^{(4)},\mathbf{t}^{(3)})|_{\mathbf{t}^{(3)}=(0,-3,\dots,-3)}\\
&= \dots\\
&= \prod_{l=1}^{n-2}\frac{1}{1-q^l}\cdot \lim_{\mathbf{s}^{(n)}\to\mathbf{s}_0^{(n)}} \mathcal{Z}^{(n)}_{\mathcal{E}}(\mathbf{s}^{(n)},\mathbf{t}^{(n-1)})|_{\mathbf{t}^{(n-1)}=(0,-n+1)}.
\end{aligned}
$$

Since $I\subseteq[n]^2_<$, then $n\notin\mathrm{row}(I)$ and $\mathfrak{f}_n(x)=\{1\}$ for every $x\in K^\times$ (cf. Definition 4.3.19). Then Fact 4.2.25(i) yields

$$
\mathcal{Z}^{(n)}_{\mathcal{E}}(\mathbf{s}^{(n)},\mathbf{t}^{(n-1)})\Big|_{\mathbf{t}^{n-1}=(0,-n+1)} = \frac{1}{1-q^{-1}}\int_{\{x_{nn}\in K\,:\,|x_{nn}|\le 1\}}|x_{nn}|^{s_n}\mathrm{d}\mu(x_{nn}) = \frac{1}{1-q^{-s_n-1}}
$$

and the statement follows. □

4.4 Submodule zeta functions over finite fields

Let $\mathbb{F}_q$ be the finite field of cardinality q and set an integer $n\ge 1$. The purpose of this section is to give an explicit formula of $\zeta_{\mathcal{E}\curvearrowright\mathbb{F}_q^n}(s)$, where $\mathcal{E}\subseteq\mathrm{Mat}_n(\mathbb{F}_q)$ is an algebra generated by an arbitrary collection of elementary matrices. Recall that we are following Convention 4.2.9.

A relevant part of our argument is based on the setting introduced in Section 4.2.4, which we now briefly expand.

4.4.1 Background on the spherical building associated to $\mathrm{GL}_n(\mathbb{F}_q)$

For $n\ge 1$, set $G=\mathrm{GL}_n(\mathbb{F}_q)$ and denote by B is the subgroup of G of all upper-triangular matrices (the so-called *Borel subgroup of* $\mathrm{GL}_n(\mathbb{F}_q)$). The flag complex of the vector space $\mathbb{F}_q^n$

defines a building Δ_n of type A_n on which G act strongly transitively and B is the stabiliser of a chamber (cf. [AB08, §6.5] or Examples 1.5.11 and 1.6.11(ii)). The Weyl group (W, S) associated to Δ_n is of type A_n, and we may take $S = \{1, \dots, n\}$.

For simplicity, in the following we still denote the flag complex associated to $\mathbb{F}_q^n$ by Δ_n. In particular, recall that the set of vertices of Δ_n is

$$\Delta_n^{(0)} = \{L \mid L \text{ non-zero subspace of } \mathbb{F}_q^n\}.$$

Two vertices L and M are said to be *joinable* if $L \subseteq M$ or $M \subseteq L$. Recall that the chambers of Δ_n are exactly the maximal flags of non-zero subspaces of $\mathbb{F}_q^n$. Hence L and M are joinable if, and only if, L and M are vertices of a common chamber of Δ_n.

An *apartment* of Δ_n is associated to a $\mathbb{F}_q$-basis $\lambda = \{\lambda_1, \dots, \lambda_n\}$ of $\mathbb{F}_q^n$ and it is the subcomplex $\Sigma(\lambda)$ of Δ_n spanning the following set of vertices:

$$\left\{\bigoplus_{i\in I} \mathbb{F}_q \lambda_i \,\middle|\, I \subseteq [n],\, I \neq \emptyset\right\}.$$

Note that there is not a bijective correspondence between $\mathbb{F}_q$-bases and apartments. Indeed, given $\mathbb{F}_q$-bases λ and μ, we have $\Sigma(\lambda) = \Sigma(\mu)$ if, and only if, there is $g \in \mathrm{GL}_n(\mathbb{F}_q)$ inducing a bijection (by left-multiplication) from the set λ to the set μ. Denote the apartment of Δ_n associated to the canonical basis $\{e_1, \dots, e_n\}$ by $\Sigma_{n,0}$, and its vertex set by $\Sigma_{n,0}^{(0)}$.

Notation 4.4.1. For every $i \in [n]$, set

$$S_i := \mathbb{F}_q e_i.$$

More generally, for every non-empty set $I \subseteq [n]$, define

$$S_I := \bigoplus_{i\in I} S_i.$$

With this notation, observe that

$$\Sigma_{n,0}^{(0)} = \{S_I \mid \emptyset \neq I \subseteq [n]\}.$$

For $(i, j) \in [n]^2$ with $i \neq j$, the (i, j)-*root* is the subcomplex of $\Sigma_{n,0}$ spanning the following set of vertices:

$$\alpha_{ij} := \{S_I \mid \emptyset \neq I \subseteq [n] \text{ s.th. either } i \notin I \text{ or } i, j \in I\}. \tag{4.4.1}$$

The *boundary* and the *interior of the* (i, j)-*th root* are the subcomplexes of $\Sigma_{n,0}$ spanning the following sets of vertices, respectively:

$$\begin{aligned} \partial\alpha_{ij} &= \{S_I \mid \emptyset \neq I \subseteq [n] \text{ s.th. either } i, j \notin I \text{ or } i, j \in I\}; \\ \alpha_{ij}^{\circ} &= \{S_I \mid \emptyset \neq I \subseteq [n] \text{ s.th. } i \notin I \text{ and } j \in I\}. \end{aligned} \tag{4.4.2}$$

Clearly, $\alpha_{ij} = \partial\alpha_{ij} \sqcup \alpha_{ij}^\circ$, $\alpha_{ij} \cap \alpha_{ji} = \partial\alpha_{ij} = \partial\alpha_{ji}$ and $\Sigma_{n,0}^{(0)} = \alpha_{ij} \cup \alpha_{ij}$.

One checks that α_{ij} is the set of vertices L in $\Sigma_{n,0}^{(0)}$ which are E_{ij}-invariant, i.e., $L \cdot E_{ij} \subseteq L$. Indeed, given an non-empty set $I \subseteq [n]$, by Remark 4.2.10 we observe that

$$S_I \cdot E_{ij} \subseteq S_I \iff e_k E_{ij} \in S_I, \quad \forall\, k \in I. \tag{4.4.3}$$

Note that $e_k E_{ij} = e_k(i) \cdot e_j$, where $e_k(i)$ is the i-th entry of e_k. We conclude that

$$S_I \cdot E_{ij} = S_I \iff \text{if } i \in I, \text{ then } j \in I \iff S_I \in \alpha_{ij}.$$

With a similar argument, for every $i \in [n]$ the set of E_{ii}-invariant vertices of $\Sigma_{n,0}$ is $\Sigma_{n,0}^{(0)}$.

More generally, for every $(i,j) \in [n]^2$ define

$$\mathcal{F}_{ij} := \{L \in \Delta_n^{(0)} \mid L \cdot E_{ij} \subseteq L\}. \tag{4.4.4}$$

Remark 4.4.2. Let $(i,j) \in [n]^2$ with $i \neq j$ and set

$$U_{ij} := I_n + E_{ij} \in \mathrm{GL}_n(\mathbb{F}_q).$$

By Remark 4.2.8(v), for every $L \in \Delta_n^{(0)}$ we have

$$L \cdot E_{ij} \subseteq L \iff L \cdot U_{ij} = L. \tag{4.4.5}$$

In particular, $\mathcal{F}_{ij}$ is the set of all fixed points of U_{ij} in $\Delta_n^{(0)}$.

The following characterisation is of key importance for what follows.

Lemma 4.4.3 (cf. [AB08, Lemma 7.41])**.** *Let $(i,j) \in [n]^2$. Then $\mathcal{F}_{ij}$ is the collection of all vertices of $\Delta_n^{(0)}$ that are joinable to some vertex of α_{ij}°, i.e.,*

$$\mathcal{F}_{ij} = \{L \in \Delta_n^{(0)} \mid S_j \subseteq L \textit{ or } L \subseteq S_{[n]\setminus\{i\}}\}. \tag{4.4.6}$$

Proof. Our strategy is based on the proof of [AB08, Lemma 7.41]. However, since we have changed some conventions (e.g., we are considering $\mathrm{GL}_n(\mathbb{F}_q)$ acting on Δ_n on the right), we rewrite the argument below. First, we prove that the set of vertices in $\Delta_n^{(0)}$ that are joinable to some vertex of α_{ij}° is

$$\{L \in \Delta_n^{(0)} \mid S_j \subseteq L \text{ or } L \subseteq S_{[n]\setminus\{i\}}\}. \tag{4.4.7}$$

In fact, every vertex of α_{ij}° contains S_j and is contained in $S_{[n]\setminus\{i\}}$. It remains to prove that $\mathcal{F}_{ij}$ equals the set in (4.4.7). Let $L \in \Delta_n^{(0)}$, that is $L = \bigoplus_{k\in[l]} \mathbb{F}_q\lambda_l$ for some $\mathbb{F}_q$-basis $\{\lambda_1, \dots, \lambda_n\}$ of $\mathbb{F}_q^n$ and $l \in [n]$. Arguing as in (4.4.3) we have

$$L \in \mathcal{F}_{ij} \iff \lambda_k \cdot E_{ij} = \lambda_k(i) \cdot e_j \in L, \quad \forall\, k \in [l]. \tag{4.4.8}$$

Here $\lambda_k(i)$ denotes the i-th entry of the vector λ_k. Since $\mathbb{F}_q$ is a field, if $\lambda_k(i) \neq 0$ then

$$\lambda_k(i) \cdot e_j \in L \iff e_j \in L.$$

By (4.4.8) we conclude that

$$L \in \mathcal{F}_{ij} \iff e_j \in L \text{ or } \Big(\lambda_k(i) = 0, \forall\, k \in [l]\Big) \iff S_j \subseteq L \text{ or } L \subseteq S_{[n]\setminus\{i\}}. \qquad \square$$

4.4.2 Some explicit examples

In what follows, we provide some explicit formulae for the submodule zeta function of a set of elementary matrices in $\mathrm{Mat}_n(\mathbb{F}_q)$. Each of them can be derived from the definition or Proposition 4.2.21.

Recall that the *Gaussian binomial* (in the variable Y) is a polynomial defined, for all integers $0 \leq k \leq n$, as

$$\binom{n}{k}_Y := \frac{[n]_Y!}{[k]_Y![n-k]_Y!}, \tag{4.4.9}$$

where $[i]_Y!$ is the *Y-factorial of $i \in \mathbb{Z}_{\geq 0}$*, i.e., $[0]_Y! := 1$ and $[i]_Y! := [1]_Y \cdot [2]_Y \cdot \ldots \cdot [i]_Y$ for every $i \geq 1$, provided

$$[i]_Y = \frac{Y^i - 1}{Y - 1}, \quad \forall\, i \in \mathbb{Z}_{\geq 1}.$$

Remark 4.4.4. Taking the limit for $Y \to 1$ in (4.4.9), we recover the binomial coefficient $\binom{n}{k}$. Moreover, if q is a prime power, it is well known that

$$\binom{n}{k}_q = \text{number of } k\text{-dimensional subspaces of } \mathbb{F}_q^n. \tag{4.4.10}$$

Remark 4.4.5. Note that

$$\binom{n}{k}_Y = \prod_{r=1}^{k} \frac{Y^{n-r+1} - 1}{Y^r - 1}. \tag{4.4.11}$$

Although it appears as a rational function, the Gaussian binomial is actually an integral polynomial in Y. Namely, from (4.4.9) we observe that

$$\begin{aligned} \binom{n}{k}_Y &= \frac{\prod_{r=1}^{k}(Y^{n-(k-r)} - 1)}{\prod_{r=1}^{k}(Y^r - 1)} = \prod_{r=1}^{k} \frac{(Y^r)^{n-k} - 1}{Y^r - 1} \\ &= \prod_{r=1}^{k} \left(\sum_{l=0}^{n-k-1} Y^{rl} \right) = \sum_{0 \leq l_1, \ldots, l_k \leq n-k-1} Y^{\sum_{i \in [k]} i \cdot l_i}. \end{aligned} \tag{4.4.12}$$

Example 4.4.6. By (4.4.10) and since $\binom{n}{k}_Y = \binom{n}{n-k}_Y$ for every $k \leq n$, we deduce that

$$\zeta_{\{0\} \curvearrowright \mathbb{F}_q^n}(s) = \sum_{k=0}^{n} \binom{n}{k}_q \cdot q^{-s(n-k)} = \sum_{k=0}^{n} \binom{n}{k}_q q^{-sk}, \quad \forall\, s \in \mathbb{C}. \tag{4.4.13}$$

For instance,

$$\begin{aligned} &\zeta_{\{0\} \curvearrowright \mathbb{F}_q}(s) = 1 + q^{-s}; \\ &\zeta_{\{0\} \curvearrowright \mathbb{F}_q^2}(s) = 1 + (q+1)q^{-s} + q^{-2s}; \\ &\zeta_{\{0\} \curvearrowright \mathbb{F}_q^3}(s) = 1 + (q^2+q+1)q^{-s} + (q^2+q+1)q^{-2s} + q^{-3s}; \end{aligned}$$

From (4.4.13) one checks that

$$\zeta_{\{0\}\curvearrowright\mathbb{F}_q^n}(s) = q^{-ns} \cdot \zeta_{\{0\}\curvearrowright\mathbb{F}_q^n}(-s). \tag{4.4.14}$$

Example 4.4.7. Given $n \geq 1$, denote by $\mathrm{Up}_n(\mathbb{F}_q)$ the subalgebra of $\mathrm{Mat}_n(\mathbb{F}_q)$ of all strictly upper-triangular matrices. Let also $U_n(\mathbb{F}_q) \leq \mathrm{GL}_n(\mathbb{F}_q)$ be the associated unipotent group, i.e.,

$$U_n(\mathbb{F}_q) = I_n + \mathrm{Up}_n(\mathbb{F}_q) = \{I_n + E \mid E \in \mathrm{Up}_n(\mathbb{F}_q)\}.$$

Recall that the Borel subgroup B of $G = \mathrm{GL}_n(\mathbb{F}_q)$ consists of all invertible n-dimensional upper-triangular matrices over $\mathbb{F}_q$. Hence B is the inner semidirect product of U_n and the subgroup T of all diagonal matrices in G, namely:

$$B = T \ltimes U_n(\mathbb{F}_q). \tag{4.4.15}$$

Recall that B is the pointwise stabiliser of the fundamental chamber c_0 of Δ_n, i.e., the chamber with set of vertices $\{S_{[k]} \mid 1 \leq k \leq n\}$. Moreover, denoting by $W \leq G$ the subgroup of G of all permutation matrices, we have

$$T = \bigcap_{w \in W} w^{-1} B w. \tag{4.4.16}$$

Indeed, T is the pointwise stabiliser of the set of chambers of the fundamental apartment of Δ_n, i.e., $\{c_0 \cdot w \mid w \in W\}$ (cf. [BRW05, §V.1B, §V.5]).

By Remark 4.4.2,

$$\begin{aligned} \mathcal{F}_{\mathrm{Up}_n(\mathbb{F}_q)} &:= \{L \in \Delta_n^{(0)} \mid L \cdot E \subseteq L, \forall\, E \in \mathrm{Up}_n(\mathbb{F}_q)\} \\ &= \{L \in \Delta_n^{(0)} \mid L \cdot U = L, \forall\, U \in U_n(\mathbb{F}_q)\}. \end{aligned} \tag{4.4.17}$$

We claim that $\mathcal{F}_{\mathrm{Up}_n(\mathbb{F}_q)}$ consists of the vertices of the fundamental chamber c_0 of Δ_n, i.e.,

$$\mathcal{F}_{\mathrm{Up}_n(\mathbb{F}_q)} = \{S_{[k]} \mid 1 \leq k \leq n\}. \tag{4.4.18}$$

Since $U_n(\mathbb{F}_q) \leq B$ and B is the pointwise stabiliser of c_0, we have

$$\mathcal{F}_{\mathrm{Up}_n(\mathbb{F}_q)} \supseteq \{S_{[k]} \mid 1 \leq k \leq n\}.$$

To prove that the reverse inclusion also holds, it suffices to show that $U_n(\mathbb{F}_q)$ is not contained in the stabiliser of any vertex in Δ_n which is different from $S_{[k]}$, for some $1 \leq k \leq n$. Recall that Δ_n is the thick building associated to the Bruhat decomposition (G, B) of type A_n. By [AB08, Corollary 6.44], for every $k \in [n]$ there is a $\mathrm{GL}_n(\mathbb{F}_q)$-invariant bijection from the set of vertices of Δ_n of type k to the coset space $P_{[n]\setminus\{k\}} \backslash \mathrm{GL}_n(\mathbb{F}_q)$, where $P_{[n]\setminus\{k\}} = BW_{[n]\setminus\{k\}}B$ (cf. Section 4.4.1). This bijection maps the vertex $S_{[k]}$ to the coset $P_{[n]\setminus\{k\}}1$. If $U_n(\mathbb{F}_q)$ is contained in the stabiliser of a vertex which is not in c_0, there are $k \in [n]$ and $g \in \mathrm{GL}_n(\mathbb{F}_q)$ with $g \notin P_{[n]\setminus\{k\}}$ satisfying

$$U_n(\mathbb{F}_q) \subseteq g^{-1} P_{[n]\setminus\{k\}} g. \tag{4.4.19}$$

We may write $g = b_1 w b_2$, for some $b_1, b_2 \in B$ and $w \in W \setminus W_{[n]\setminus\{k\}}$. Then $g^{-1} P_{[n]\setminus\{k\}} g = b_2^{-1} w^{-1} P_I w b_2$ and, since $U_n(\mathbb{F}_q)$ is normal in B, (4.4.19) implies that

$$U_n(\mathbb{F}_q) \subseteq w^{-1} P_{[n]\setminus\{k\}} w. \tag{4.4.20}$$

Combining (4.4.15), (4.4.16) and (4.4.20), we deduce that there is $w \in W \setminus P_{[n]\setminus\{k\}}$ such that

$$B = T \cdot U_n(\mathbb{F}_q) \subseteq w^{-1} P_{[n]\setminus\{k\}} w,$$

which contradicts Proposition 1.6.9(ii). As a conclusion, (4.4.18) holds and

$$\zeta_{\mathrm{Up}_n(\mathbb{F}_q) \curvearrowright \mathbb{F}_q^n}(s) = \sum_{k=0}^{n} q^{-sk}. \tag{4.4.21}$$

If $n \geq 2$, recall that $\mathrm{Up}_n(\mathbb{F}_q)$ is generated – as an associative subalgebra of $\mathrm{Mat}_n(\mathbb{F}_q)$ – by the set of elementary matrices $\{E_{i,i+1} \mid 1 \leq i \leq n-1\}$. Thus, by Remark 4.2.8(iii),

$$\mathcal{F}_{\mathrm{Up}_n(\mathbb{F}_q)} = \bigcap_{i \in [n-1]} \mathcal{F}_{i,i+1}.$$

We will deduce the same conclusion as in (4.4.21) by characterising the intersection $\bigcap_{1 \leq i \leq n-1} \mathcal{F}_{i,i+1}$ (cf. Example 4.4.15) or by applying the subsequent Theorem 4.4.16 (cf. Example 4.4.22).

Example 4.4.8. Let E_{ij} be an elementary matrix in $\mathrm{Mat}_3(\mathbb{F}_q)$ with $1 \leq i, j \leq 3$ and $i \neq j$. Then

$$\zeta_{\{E_{ij}\} \curvearrowright \mathbb{F}_q^3}(s) = 1 + (q+1)q^{-s} + (q+1)q^{-2s} + q^{-3s}. \tag{4.4.22}$$

For (i,j) with $i \neq j$, let $P_{ij} \in \mathrm{GL}_3(\mathbb{F}_q)$ be the matrix obtained from the identity matrix I_3 by exchanging the first and the i-th rows and then the second and the j-th columns. Then $P_{ij} = P_{ij}^{-1}$ and $P_{ij} E_{12} P_{ij} = E_{ij}$. Hence, by Remark 4.2.8(iv), we may focus on the case where $(i,j) = (1,2)$.

Let c be the chamber of Δ_3 containing both S_2 and $S_{\{2,3\}}$ as vertices, i.e.,

$$c = \{S_2, S_{\{2,3\}}, \mathbb{F}_q^3\}.$$

Let $L \in \Delta_3^{(0)}$. By Lemma 4.4.3, $L \in \mathcal{F}_{12}$ if and only if there is a chamber d in Δ_3 that contains L and at least one between S_2 and $S_{\{2,3\}}$. Each chamber of Δ_3 has 3 vertices and always contains $\mathbb{F}_q^3$. Therefore, $L \in \mathcal{F}_{12}$ if and only if L satisfies exactly one of the following conditions:

(a) L is a vertex of c, i.e., $L \in \{S_2, S_{\{2,3\}}, \mathbb{F}_q^3\}$;

(b) L is in a chamber d with vertices $\{L, S_{\{2,3\}}, \mathbb{F}_q^3\}$. There are exactly q vertices L satisfying this, and each of them has dimension 1;

(c) L is in a chamber d with vertices$\{S_2, L, \mathbb{F}_q^3\}$. There are exactly q vertices L satisfying this, and each of them has dimension 2.

By Proposition 4.2.21 and Lemma 4.4.3, one deduces (4.4.22).

Example 4.4.9. Let E_{12} and E_{13} be the elementary matrices in $\mathrm{Mat}_3(\mathbb{F}_q)$ associated to $(1,2)$ and $(1,3)$, respectively. Then

$$\zeta_{\{E_{12},E_{13}\}\curvearrowright\mathbb{F}_q^3}(s) = 1 + (q+1)q^{-s} + q^{-2s} + q^{-3s}. \tag{4.4.23}$$

Indeed, by Proposition 4.2.21,

$$\zeta_{\{E_{12},E_{13}\}\curvearrowright\mathbb{F}_q^3}(s) = q^{-3s} + \sum_{L\in\mathcal{F}_{12}\cap\mathcal{F}_{13}} q^{-s(3-\dim_{\mathbb{F}_q} L)}. \tag{4.4.24}$$

By Lemma 4.4.3,

$$\mathcal{F}_{12} \cap \mathcal{F}_{13} = \{L \in \Delta_3^{(0)} \mid S_{\{2,3\}} \subseteq L \text{ or } L \subseteq S_{\{2,3\}}\}.$$

In other words, $L \in \mathcal{F}_{12} \cap \mathcal{F}_{13}$ if and only if L is a vertex of a chamber in $\Delta_3^{(0)}$ having $S_{\{2,3\}}$ (and, of course $\mathbb{F}_q^3$) as vertice. Therefore,

$$\mathcal{F}_{12} \cap \mathcal{F}_{13} = \{L \in \Delta_3^{(0)} \mid \{L, S_{\{2,3\}}, \mathbb{F}_q^3\} \text{ is a chamber of } \Delta_3^{(0)}\} \cup \{S_{\{2,3\}}, \mathbb{F}_q^3\}.$$

In particular, $\mathcal{F}_{12} \cap \mathcal{F}_{13}$ consists of q dimensional subspaces of $\mathbb{F}_q^3$ of dimension 1, one subspace of dimension 2, and $\mathbb{F}^3$ itself. The formula in (4.4.23) now follows from (4.4.24).

4.4.3 Characterising the $\mathcal{E}_I$-invariance for subspaces of $\mathbb{F}_q^n$

Let I be a subset of $[n]^2$, and denote by $\mathcal{E}_I$ the associative subalgebra of $\mathrm{Mat}_n(\mathbb{F}_q)$ generated by the set of elementary matrices $\{E_{ij}\}_{(i,j)\in I}$. Recalling (4.4.4), for every $(i,j) \in I$ we set

$$\mathcal{F}_I := \bigcap_{(i,j)\in I} \mathcal{F}_{ij}. \tag{4.4.25}$$

In other words, $\mathcal{F}_I$ is the set of all vertices L of $\Delta_n^{(0)}$ such that $L \cdot E_{ij} \subseteq L$ for all $(i,j) \in I$. Note that

$$\{L \mid L \leq_{\mathcal{E}_I} \mathbb{F}_q^n\} = \mathcal{F}_I \sqcup \{\{0\}\}.$$

We now provide an alternative characterisation of the set $\mathcal{F}_I$, which turns out to be convenient in the subsequent Section 4.4.4. Before this, given $I \subseteq [n]^2$, we define

$$\begin{aligned} \mathrm{row}(I) &:= \{i \in [n] \mid \exists\, j \in [n] \text{ s.th. } (i,j) \in I\}; \\ \mathrm{col}(I) &:= \{j \in [n] \mid \exists\, i \in [n] \text{ s.th. } (i,j) \in I\}. \end{aligned} \tag{4.4.26}$$

Lemma 4.4.10. *For every $L \in \Delta_n^{(0)}$, the following are equivalent:*

(i) $L \in \mathcal{F}_I$*;*

(ii) there are subsets R and J of I satisfying $I = R \cup J$, and such that

$$S_{\mathrm{col}(J)} \subseteq L \subseteq S_{\mathrm{row}(R)^c}.$$

If (ii) holds, then necessarily $\mathrm{col}(J) \subseteq \mathrm{row}(R)^c$ (i.e., $\mathrm{row}(R) \cap \mathrm{col}(J) = \emptyset$).

Proof. Assume that (i) holds. By Lemma 4.4.3, for every $(i, j) \in I$ one has $S_j \subseteq L$ or $L \subseteq S_{[n]\setminus\{i\}}$. Then (ii) follows by setting

$$R := \{(i, j) \in I \mid L \subseteq S_{[n]\setminus\{i\}}\} \quad \text{and} \quad J := \{(i, j) \in I \mid S_j \subseteq L\}.$$

Conversely, if there are subsets $R, J \subseteq I$ as in (ii) for L, then $L \in \mathcal{F}_{ij}$ for every $(i, j) \in I = R \cup J$ (cf. Lemma 4.4.3) and (i) holds. $\square$

Lemma 4.4.10 focuses on pairs (R, J) of subsets of I satisfying $I = R \cup J$ and $\mathrm{row}(R) \cap \mathrm{col}(J) = \emptyset$. We now provide a convenient characterisation of those pairs of subsets (cf. Lemma 4.4.11), after introducing the following definition. We say that a subset $R \subseteq I$ is *closed* if it is empty or, for every $(i, j) \in I$ with $j \in \mathrm{row}(R)$, then $(i, j) \in R$. Denote by $\mathrm{Clos}(I)$ the set of all closed subsets of I.

Lemma 4.4.11. *Let $I \subseteq [n]^2$. Then the following are equivalent for every pair (R, J) of subsets of I:*

(i) $I = R \cup J$ *and* $\mathrm{row}(R) \cap \mathrm{col}(J) = \emptyset$*;*

(ii) R *is closed,* $J \supseteq I \setminus R$ *and* $R \cap J \subseteq \{(i, j) \in I \mid j \notin \mathrm{row}(R)\}$.

Proof. Assume first that $I = R \cup J$ and $\mathrm{row}(R) \cap \mathrm{col}(J) = \emptyset$. Then $J \supseteq I \setminus R$ and $R \cap J \subseteq \{(i, j) \in I \mid j \notin \mathrm{row}(R)\}$. Moreover, every $(i, j) \in I$ with $j \in \mathrm{row}(R)$ satisfies $(i, j) \in I \setminus J \subseteq R$, which means that R is closed. This proves the implication (i)⇒(ii).

Assume conversely that (ii) holds. Since $J \supseteq I \setminus R$, we have $I = J \cup R$. Moreover, the fact that R is closed implies that

$$\mathrm{row}(R) \cap \mathrm{col}(J) = \{j \in \mathrm{row}(R) \mid \exists\, i \in [n] \text{ s.th. } (i, j) \in R \cap J\}. \tag{4.4.27}$$

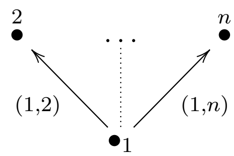
Having $R \cap J \subseteq \{(i, j) \in I \mid j \notin \mathrm{row}(R)\}$, then (4.4.27) yields $\mathrm{row}(R) \cap \mathrm{col}(J) = \emptyset$. $\square$

Example 4.4.12. For $n \geq 2$, let $I = \{(i, i + 1) \mid 1 \leq i \leq n\}$. We can visualise I as the edge set of the following graph.

The only possible closed subsets of I are those of the form $\{(i, i+1) \mid 1 \leq i \leq k\}$, for some $k \leq n-1$. Hence the only possible pairs (R, J) of subsets of I satisfying condition (ii) of Lemma 4.4.11 are the following:

- $(R, J) = (\emptyset, I)$ or $(R, J) = (I, \emptyset)$;
- $R = \{(i, i+1) \mid 1 \leq i \leq k\}$ for some $1 \leq k < n-1$, and either $J = I \setminus R$ or $J = (I \setminus R) \cup \{(k, k+1)\}$. In fact, $\{(i, i+1) \in I \mid i+1 \notin \mathrm{row}(R)\} = \{(i, i+1) \in I \mid i+1 > k\}$.

Example 4.4.13. For $n \geq 3$, let $I = \{(1, i) \mid 2 \leq i \leq n\}$. We can visualise I as the edge set of the following graph.



Note that every pair (R, J) of subsets of I with $I = R \cup J$ satisfies $\mathrm{row}(R) \cap \mathrm{col}(J) = \emptyset$.

For every pair (R, J) of subsets of I satisfying $I = R \cup J$ and $\mathrm{row}(R) \cap \mathrm{col}(J) = \emptyset$, define

$$\mathcal{F}^{(R,J)} := \left\{ L \in \Delta_n^{(0)} \,\middle|\, S_{\mathrm{col}(J)} \subseteq L \subseteq S_{\mathrm{row}(R)^c} \right\}. \tag{4.4.28}$$

Lemma 4.4.14. *Let $I \subseteq [n]^2$ be a set.*

(i) Consider pairs (R_1, J_1) and (R_2, J_2) of subsets of I satisfying $I = R_1 \cup J_1 = R_2 \cup J_2$ and $\mathrm{row}(R_1) \cap \mathrm{col}(J_1) = \mathrm{row}(R_2) \cap \mathrm{col}(J_2) = \emptyset$. If $R_1 \supseteq R_2$ and $J_1 \supseteq J_2$, then

$$\mathcal{F}^{(R_1,J_1)} \subseteq \mathcal{F}^{(R_2,J_2)}.$$

(ii) The following holds:

$$\mathcal{F}_I = \bigcup_{R \in \mathrm{Clos}(I)} \mathcal{F}^{(R, I \setminus R)}.$$

Proof. Part (i) is immediate from the fact that

$$S_{\mathrm{col}(J_1)} \supseteq S_{\mathrm{col}(J_2)} \quad \text{and} \quad S_{\mathrm{row}(R_1)^c} \subseteq S_{\mathrm{row}(R_2)^c}.$$

To prove part (ii), by Lemmas 4.4.10 and 4.4.11 we first observe that

$$\mathcal{F}_I = \bigcup_{R \in \mathrm{Clos}(I)} \bigcup \left\{ \mathcal{F}^{(R,J)} \,\middle|\, I \setminus R \subseteq J \subseteq I \text{ s.th. } R \cap J \subseteq \{(i,j) \in I \mid j \notin \mathrm{row}(R)\} \right\}.$$

Finally, by part (i), for every closed subset $R \subseteq I$ we have

$$\bigcup \left\{ \mathcal{F}^{(R,J)} \mid I \setminus R \subseteq J \subseteq I : \, R \cap J \subseteq \{(i,j) \in I \mid j \notin \mathrm{row}(R)\} \right\} = \mathcal{F}^{(R, I \setminus R)}. \qquad \square$$

4.4.4 The submodule zeta function of a matrix algebra over $\mathbb{F}_q$ generated by elementary matrices

Let $I \subseteq [n]^2$ and denote by $\mathcal{E}_I$ the subalgebra of $\mathrm{Mat}_n(\mathbb{F}_q)$ generated by the set of elementary matrices $\{E_{ij}\}_{(i,j)\in I}$. Moreover, for every non-empty subset $\mathcal{R} \subseteq \mathrm{Clos}(I)$, define

$$\begin{aligned} C_{\mathcal{R}} &:= \bigcup_{R\in\mathcal{R}} \mathrm{col}(I \setminus R); \qquad & \gamma_{\mathcal{R}} &:= |C_{\mathcal{R}}|; \\ R_{\mathcal{R}} &:= \bigcap_{R\in\mathcal{R}} \mathrm{row}(R)^c; \qquad & \rho_{\mathcal{R}} &:= |R_{\mathcal{R}}| - \gamma_{\mathcal{R}}. \end{aligned} \tag{4.4.29}$$

Example 4.4.15. Let $n \geq 2$ and consider $I = \{(i, i+1) \mid 1 \leq i \leq n-1\}$. By Example 4.4.12, let

$$\mathrm{Clos}(I) = \Big\{ R_k := \{(i,i+1) \mid i \in [k]\} \,\Big|\, 0 \leq k \leq n-1 \Big\}.$$

Consider a subset $\mathcal{R} = \{R_{i_1}, \ldots, R_{i_l}\}$ of $\mathrm{Clos}(I)$, for some $0 \leq i_1 < i_2 < \ldots < i_l$ and $l \geq 1$. Observe that $R_{i_1} = \bigcap_{j\in[l]} R_{i_j}$ and $R_{i_l} = \bigcup_{j\in[l]} R_{i_j}$. Hence,

$$\begin{aligned} \mathcal{C}_{\mathcal{R}} &= \bigcup_{j\in[l]} \mathrm{col}(I \setminus R_{i_j}) = \mathrm{col}(I \setminus R_{i_1}) = \{k \mid i_1 + 2 \leq k \leq n\}; \\ R_{\mathcal{R}} &= \bigcap_{j\in[l]} \mathrm{row}(R_{i_j})^c = \mathrm{row}(R_{i_l})^c = \{k \mid i_l + 1 \leq k \leq n\}. \end{aligned}$$

If $l \geq 2$ and $C_{\mathcal{R}} \subseteq R_{\mathcal{R}}$, we deduce that $i_l = i_1 + 1$ (as $i_l + 2 > i_1 + 2 \geq i_l + 1$) and $l = 2$. Therefore,

$$C_{\mathcal{R}} \subseteq R_{\mathcal{R}} \iff \text{either } l = 1 \text{ or } \big(l = 2 \text{ and } i_l = i_1 + 1\big).$$

Moreover,

- if $\mathcal{R} = \{R_k\}$ for some $0 \leq k \leq n-1$, then

$$\gamma_{\mathcal{R}} = |\mathrm{col}(I \setminus R_k)| = n - k - 1 \quad \text{and} \quad \rho_{\mathcal{R}} = |\mathrm{row}(R_k)^c| - \gamma_{\mathcal{R}} = 1;$$

- if $\mathcal{R} = \{R_k, R_{k+1}\}$ for some $0 \leq k \leq n-2$, then

$$\gamma_{\mathcal{R}} = |\mathrm{col}(I \setminus R_k)| = n - k - 1 \quad \text{and} \quad \rho_{\mathcal{R}} = |\mathrm{row}(R_{k+1})^c| - \gamma_{\mathcal{R}} = 0.$$

The goal of the next pages is to prove the following theorem.

Theorem 4.4.16. *For every $s \in \mathbb{C}$ we have*

$$\zeta_{\mathcal{E}_I \curvearrowright \mathbb{F}_q^n}(s) = \sum_{\substack{\mathcal{R}\subseteq \mathrm{Clos}(I): \\ \mathcal{R}\neq\emptyset \text{ and } C_{\mathcal{R}}\subseteq R_{\mathcal{R}}}} (-1)^{|\mathcal{R}|+1} q^{-s(n-\gamma_{\mathcal{R}}-\rho_{\mathcal{R}})} \cdot \zeta_{\{0\}\curvearrowright \mathbb{F}_q^{\rho_{\mathcal{R}}}}(s).$$

Remark 4.4.17. For every non-empty subset $\mathcal{R} \subseteq \mathrm{Clos}(I)$, note that

$$\bigcap_{R\in\mathcal{R}} \mathrm{row}(R)^c = [n] \setminus \left(\bigcup_{R\in\mathcal{R}} \mathrm{row}(R) \right).$$

In particular,

$$n - \gamma_{\mathcal{R}} - \rho_{\mathcal{R}} = \left| \bigcup_{R\in\mathcal{R}} \mathrm{row}(R) \right|. \tag{4.4.30}$$

In view of the proof of Theorem 4.4.16, it is convenient to set the following auxiliary function. Namely, for every $X \subseteq \Delta_n^{(0)}$, define

$$\mathcal{Z}_X(s) := \sum_{L\in X} q^{-s\cdot \dim_{\mathbb{F}_q} L}, \quad \forall\, s \in \mathbb{C}. \tag{4.4.31}$$

If $X = \emptyset$, we conventionally set $\mathcal{Z}_X(s) \equiv 0$.

Remark 4.4.18. Let $\mathcal{F}_I$ be the set introduced in (4.4.25). By Proposition 4.2.21, we observe that

$$\zeta_{\mathcal{E}_I \curvearrowright \mathbb{F}_q^n}(s) = q^{-ns}\Big(1 + \mathcal{Z}_{\mathcal{F}_I}(-s)\Big), \quad \forall\, s \in \mathbb{C}.$$

Lemma 4.4.19. *Given a set $I \subseteq [n]^2$, we have*

$$\mathcal{Z}_{\mathcal{F}_I}(s) = \sum_{\mathcal{R}\subseteq \mathrm{Clos}(I),\, \mathcal{R}\neq\emptyset} (-1)^{|\mathcal{R}|+1} \cdot \mathcal{Z}_{\bigcap_{R\in\mathcal{R}} \mathcal{F}_I^{(R,I\setminus R)}}(s), \quad \forall\, s \in \mathbb{C}. \tag{4.4.32}$$

Proof. By Lemma 4.4.14(ii) and the inclusion-exclusion principle on characteristic functions, we observe that

$$\mathbb{1}_{\mathcal{F}_I} = \sum_{\mathcal{R}\subseteq \mathrm{Clos}(I),\, \mathcal{R}\neq\emptyset} (-1)^{|\mathcal{R}|+1} \mathbb{1}_{\bigcap_{R\in\mathcal{R}} \mathcal{F}_I^{(R,I\setminus R)}}.$$

Hence,

$$\begin{aligned} \mathcal{Z}_{\mathcal{F}_I}(s) &= \sum_{L\in\Delta_n^{(0)}} \mathbb{1}_{\mathcal{F}_I}(L) \cdot q^{-s\cdot \dim_{\mathbb{F}_q} L} \\ &= \sum_{\mathcal{R}\subseteq \mathrm{Clos}(I),\, \mathcal{R}\neq\emptyset} (-1)^{|\mathcal{R}|+1} \left(\sum_{L\in\Delta_n^{(0)}} \mathbb{1}_{\bigcap_{R\in\mathcal{R}} \mathcal{F}_I^{(R,I\setminus R)}}(L) \cdot q^{-s\cdot \dim_{\mathbb{F}_q} L} \right) \end{aligned}$$

and the statement follows. □

Lemma 4.4.19 allows us to focus only on $\mathcal{Z}_X(s)$, where $X = \bigcap_{R\in\mathcal{R}} \mathcal{F}_I^{(R,I\setminus R)}$ for a non-empty subset $\mathcal{R} \subseteq \mathrm{Clos}(I)$. For such a set $\mathcal{R}$, from (4.4.28) it is immediate to observe that

$$\bigcap_{R\in\mathcal{R}} \mathcal{F}_I^{(R,I\setminus R)} = \left\{ L \in \Delta_n^{(0)} \,\middle|\, S_{\bigcup_{R\in\mathcal{R}} \mathrm{col}(I\setminus R)} \subseteq L \subseteq S_{\bigcap_{R\in\mathcal{R}} \mathrm{row}(R)^c} \right\}. \tag{4.4.33}$$

Hence, for all functions $\mathcal{Z}_X(s)$ as above, the following more general result applies.

Proposition 4.4.20. *Let $A, B \subseteq [n]$ with $A \subseteq B$, and set*

$$X_{A,B}^n := \{L \in \Delta_n^{(0)} \mid S_A \subseteq L \subseteq S_B\}.$$

Then

$$\mathcal{Z}_{X_{A,B}^n}(s) = q^{-s|A|} \cdot \sum_{k=0}^{|B|-|A|} \binom{|B|-|A|}{k}_q \cdot q^{-sk} - \mathbb{1}_{\{0\}}(|A|).$$

Proof. First, if $A = \emptyset$ then

$$\mathcal{Z}_{X_{A,B}^n}(s) = \sum_{L \leq_{\mathbb{F}_q} S_B,\, L\neq\{0\}} q^{-s\cdot \dim_{\mathbb{F}_q} L} = \sum_{L \leq_{\mathbb{F}_q} \mathbb{F}_q^{|B|},\, L\neq\{0\}} q^{-s\cdot \dim_{\mathbb{F}_q} L} = \sum_{k=0}^{|B|} \binom{|B|}{k}_q \cdot q^{-sk} - 1,$$

see Example 4.4.6.

Assume now that $A \neq \emptyset$. For every $L \in X_{A,B}^n$, denote by L' the complement of S_A in L, i.e., the unique $\mathbb{F}_q$-subspace of L such that $L = L' \oplus S_A$. Thus, we have a bijection

$$\varphi\colon X_{A,B}^n \longrightarrow \{M \mid M \leq_{\mathbb{F}_q} S_B/S_A\}, \quad \varphi(L) := L',$$

and clearly $\dim_{\mathbb{F}_q} L = \dim_{\mathbb{F}_q} \varphi(L) + |A|$ for every $L \in X_{A,B}^n$. Therefore, by Example 4.4.6,

$$\begin{aligned}\mathcal{Z}_{X_{A,B}^n}(s) &= q^{-s|A|} \cdot \sum_{M \leq_{\mathbb{F}_q} S_B/S_A} q^{-s\cdot\dim_{\mathbb{F}_q} M} = q^{-s|A|} \cdot \sum_{M \leq_{\mathbb{F}_q} \mathbb{F}_q^{|B|-|A|}} q^{-s\cdot\dim_{\mathbb{F}_q} M} \\ &= q^{-s|A|} \cdot \sum_{k=0}^{|B|-|A|} \binom{|B|-|A|}{k}_q \cdot q^{-sk}. \end{aligned}$$

□

We conclude with a proof of the main theorem of the section.

Proof of Theorem 4.4.16. By Remark 4.4.18 and Lemma 4.4.19, we obtain that

$$\zeta_{\mathcal{E}_I \curvearrowright \mathbb{F}_q^n}(s) = q^{-ns}\Big(1 + \mathcal{Z}_{\mathcal{F}_I}(-s)\Big) = q^{-ns}\left(1 + \sum_{\substack{\mathcal{R}\subseteq \mathrm{Clos}(I),\\ \mathcal{R}\neq\emptyset}} (-1)^{|\mathcal{R}|+1} \mathcal{Z}_{\bigcap_{R\in\mathcal{R}} \mathcal{F}_I^{(R,I\setminus R)}}(-s)\right). \tag{4.4.34}$$

From (4.4.33), observe that

$$\bigcap_{R\in\mathcal{R}} \mathcal{F}_I^{(R,I\setminus R)} = \{L \in \Delta_n^{(0)} \mid S_{C_\mathcal{R}} \subseteq L \subseteq S_{R_\mathcal{R}}\}, \tag{4.4.35}$$

where $C_\mathcal{R}$ and $R_\mathcal{R}$ are as in (4.4.29). Combining (4.4.35) and Proposition 4.4.20, we have

$$\mathcal{Z}_{\bigcap_{R\in\mathcal{R}} \mathcal{F}_I^{(R,I\setminus R)}}(-s) = \begin{cases} q^{s\gamma_\mathcal{R}} \sum_{k=0}^{\rho_\mathcal{R}} \binom{\rho_\mathcal{R}}{k}_q \cdot q^{sk} - \mathbb{1}_{\{0\}}(\gamma_\mathcal{R}), & \text{if } C_\mathcal{R} \subseteq R_\mathcal{R}; \\ 0, & \text{otherwise.} \end{cases} \tag{4.4.36}$$

By (4.4.34) and (4.4.36), we deduce that

$$\zeta_{\mathcal{E}_I \curvearrowright \mathbb{F}_q^n}(s) = q^{-ns} + \sum_{\substack{\mathcal{R}\subseteq \mathrm{Clos}(I): \\ \mathcal{R}\neq\emptyset \text{ and } C_\mathcal{R}\subseteq R_\mathcal{R}}} (-1)^{|\mathcal{R}|+1} \left(\sum_{k=0}^{\rho_\mathcal{R}} \binom{\rho_\mathcal{R}}{k}_q \cdot q^{-s(n-\gamma_\mathcal{R}-k)} - q^{-ns} \cdot \mathbb{1}_{\{0\}}(\gamma_\mathcal{R}) \right). \tag{4.4.37}$$

Note that the only non-empty subset $\mathcal{R}$ of $\mathrm{Clos}(I)$ which satisfies $\gamma_\mathcal{R} = 0$ is $\mathcal{R} = \{I\}$. Therefore, by (4.4.37),

$$\begin{aligned} \zeta_{\mathcal{E}_I \curvearrowright \mathbb{F}_q^n}(s) &= q^{-ns} + \left(\sum_{\substack{\mathcal{R}\subseteq \mathrm{Clos}(I): \\ \mathcal{R}\neq\emptyset \text{ and } C_\mathcal{R}\subseteq R_\mathcal{R}}} (-1)^{|\mathcal{R}|+1} \cdot \sum_{k=0}^{\rho_\mathcal{R}} \binom{\rho_\mathcal{R}}{k}_q \cdot q^{-s(n-\gamma_\mathcal{R}-k)} \right) - q^{-ns} \\ &= \sum_{\substack{\mathcal{R}\subseteq \mathrm{Clos}(I): \\ \mathcal{R}\neq\emptyset \text{ and } C_\mathcal{R}\subseteq R_\mathcal{R}}} (-1)^{|\mathcal{R}|+1} q^{-s(n-\gamma_\mathcal{R}-\rho_\mathcal{R})} \cdot \sum_{k=0}^{\rho_\mathcal{R}} \binom{\rho_\mathcal{R}}{k}_q \cdot q^{-s(\rho_\mathcal{R}-k)} \end{aligned}$$

Recalling Example 4.4.6, one concludes the argument. □

Applying Theorem 4.4.16, we obtain the following.

Corollary 4.4.21. *Let $n \geq 2$ and $1 \leq i, j \leq n$. Then,*

$$\zeta_{\langle E_{ij}\rangle \curvearrowright \mathbb{F}_q^n}(s) = (1 + q^{-s}) \cdot \zeta_{\{0\}\curvearrowright \mathbb{F}_q^{n-1}}(s) - q^{-s} \cdot \zeta_{\{0\}\curvearrowright \mathbb{F}_q^{n-2}}(s).$$

Proof. If $I = \{(i, j)\}$, then $\mathrm{Clos}(I) = \{\emptyset, I\}$. One checks that:

- if $\mathcal{R} = \{\emptyset\}$, then $\gamma_\mathcal{R} = |\mathrm{col}(I)| = 1$ and $\rho_\mathcal{R} = |\mathrm{row}(\emptyset)^c| - \gamma_\mathcal{R} = n - 1$;
- if $\mathcal{R} = \{I\}$, then $\gamma_\mathcal{R} = |\mathrm{col}(\emptyset)| = 0$ and $\rho_\mathcal{R} = |\mathrm{row}(I)^c| - \gamma_\mathcal{R} = n - 1$;
- if $\mathcal{R} = \{\emptyset, I\}$, then $\gamma_\mathcal{R} = |\mathrm{col}(I)| = 1$ and $\rho_\mathcal{R} = |\mathrm{row}(I)^c| - \gamma_\mathcal{R} = n - 2$.

Now Theorem 4.4.16 applies. □

If $n=3$, Corollary 4.4.21 and Example 4.4.6 recover the formula in (4.4.22).

Example 4.4.22. Let $n \geq 2$ and consider $I = \{(i, i+1) \mid 1 \leq i \leq n-1\}$. As observed in Example 4.4.7, the subalgebra $\mathcal{E}_I$ of $\mathrm{Mat}_n(\mathbb{F}_q)$ generated by $\{E_{i,i+1}\}_{1 \leq i \leq n-1}$ is precisely the algebra of all strictly upper-triangular matrices. By Theorem 4.4.16 and Example 4.4.15, we have

$$\begin{aligned}
\zeta_{\mathcal{E}_I \curvearrowright \mathbb{F}_q^n}(s) &= \overbrace{\sum_{k=0}^{n-1} q^{-s(n-\gamma_{\{R_k\}}-\rho_{\{R_k\}})} \cdot \zeta_{\{0\} \curvearrowright \mathbb{F}_q^{\rho_{\{R_k\}}}}(s)}^{(a)} \\
&\quad - \overbrace{\sum_{k=0}^{n-2} q^{-s(n-\gamma_{\{R_k,R_{k+1}\}}-\rho_{\{R_k,R_{k+1}\}})} \cdot \zeta_{\{0\} \curvearrowright \mathbb{F}_q^{\rho_{\{R_k,R_{k+1}\}}}}(s)}^{(b)} \\
&= \overbrace{\sum_{k=0}^{n-1} q^{-sk}(1+q^{-s})}^{(a)} - \overbrace{\sum_{k=0}^{n-2} q^{-s(k+1)}}^{(b)} = \sum_{k=0}^{n} q^{-sk}.
\end{aligned}$$

Remark 4.4.23. By Fact 4.2.12, the ideal zeta function of a $\mathbb{F}_q$-Lie ring L is equal to the submodule zeta function of a determined $\mathbb{F}_q$-algebra $\mathcal{E}(L) \leq \mathrm{Mat}_n(\mathbb{F}_q)$ acting on $\mathbb{F}_q^n$ by right row-by-column multiplication. In [Lee22], S. Lee gave explicit formulae for the ideal zeta function of a $\mathbb{F}_q$-Lie ring L given by a suitable presentation. Using (4.2.9), from the presentation of L one obtains matrices $C_1, \dots, C_n$ as in (4.2.8) that generate $\mathcal{E}(L)$ as a $\mathbb{F}_q$-algebra. Among the examples appearing in [Lee22], one checks that the only $\mathbb{F}_q$-Lie ring L whose matrices $C_1, \dots, C_n$ as above are elementary matrices is the Heisenberg $\mathbb{F}_q$-Lie ring (cf. Example 4.2.13). However, this does not exclude *a priori* that $\mathcal{E}(L)$ can be generated by a set of elementary matrices.

4.4.5 Uniformity

From Example 4.4.6 and Remark 4.4.5, for every $n \geq 0$ we have

$$\zeta_{\{0\} \curvearrowright \mathbb{F}_q^n}(s) = W_{\emptyset,n}(q, q^{-s}),$$

where $W_{\emptyset,n}(x,y) \in \mathbb{Z}[x,y]$ is given by

$$W_{\emptyset,n}(x,y) = \sum_{k=0}^{n} \binom{n}{k}_x \cdot y^{n-k}, \quad \text{where } \binom{n}{k}_x := \prod_{r=1}^{k} \left(\sum_{l=0}^{n-r-1} x^{lr} \right). \tag{4.4.38}$$

As a consequence of Theorem 4.4.16, we obtain the following generalisation of (4.4.38).

Corollary 4.4.24. *Let $I \subseteq [n]^2$, and denote by $\mathcal{E}_I$ the subalgebra of $\mathrm{Mat}_n(\mathbb{F}_q)$ generated by the set of elementary matrices $\{E_{ij}\}_{(i,j)\in I}$. Then $\zeta_{\mathcal{E}_I \curvearrowright \mathbb{F}_q^n}(s)$ is uniform in q and q^{-s}, i.e., there is $W_{I,n}(x,y) \in \mathbb{Z}[x,y]$, namely*

$$W_{I,n}(x,y) = \sum_{\substack{\mathcal{R}\subseteq \mathrm{Clos}(I): \\ \mathcal{R}\neq\emptyset \text{ and } C_{\mathcal{R}}\subseteq R_{\mathcal{R}}}} (-1)^{|\mathcal{R}|+1} y^{n-\gamma_{\mathcal{R}}-\rho_{\mathcal{R}}} \cdot W_{\emptyset,\rho_{\mathcal{R}}}(x,y), \tag{4.4.39}$$

such that

$$\zeta_{\mathcal{E}_I \curvearrowright \mathbb{F}_q^n}(s) = W_{I,n}(q, q^{-s})$$

for every prime power q.

4.4.6 Towards a functional equation

Since

$$\binom{n}{k}_x = \binom{n}{n-k}_x, \quad \forall\, k \leq n, \tag{4.4.40}$$

the formal substitution $y \mapsto y^{-1}$ in $W_{\emptyset,n}(x,y)$ yields

$$W_{\emptyset,n}(x,y) = y^n \cdot W_{\emptyset,n}(x, y^{-1}), \tag{4.4.41}$$

see (4.4.41).

Question 4.4.25. For which $I \subseteq [n]^2$ is $W_{I,n}(x,y) = y^n \cdot W_{I,n}(x,y^{-1})$ satisfied?

Bibliography

[AB08] P. Abramenko and K. S. Brown. *Buildings*. Vol. 248. Graduate Texts in Mathematics. Theory and applications. Springer, New York, 2008, pp. xxii+747.

[Abe74] H. Abels. “Specker-Kompactifizierungen von lokal kompakten topologischen Gruppen”. In: *Math. Z.* 135 (1974), pp. 325–361.

[APV17] P. Abramenko, J. Parkinson, and H. Van Maldeghem. “Distance regularity in buildings and structure constants in Hecke algebras”. In: *J. Algebra* 481 (2017), pp. 158–187.

[Arv12] W. Arveson. *An invitation to C^*-algebras*. Vol. 39. Springer Science & Business Media, 2012.

[BB06] A. Bjorner and F. Brenti. *Combinatorics of Coxeter Groups*. Graduate Texts in Mathematics. Springer Berlin Heidelberg, 2006.

[Ber72] S. K. Berberian. *Baer $*$-Rings*. Vol. 195. Springer, 1972.

[Bes93] M. Bestvina. “The Virtual Cohomological Dimension of Coxeter Groups”. In: *Geometric Group Theory*. Ed. by G. A. Niblo and M. A.Editors Roller. London Mathematical Society Lecture Note Series. Cambridge University Press, 1993, pp. 19–23.

[BEW15] C. Banks, M. Elder, and G. A. Willis. “Simple groups of automorphisms of trees determined by their actions on finite subtrees”. In: *J. Group Theory* 18.2 (2015), pp. 235–261.

[BH11] M. R. Bridson and A. Häfliger. *Metric Spaces of Non-Positive Curvature*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2011.

[BK90] H. Bass and R. Kulkarni. “Uniform tree lattices”. In: *J. Amer. Math. Soc.* 3.4 (1990), pp. 843–902.

[BM00] M. Burger and S. Mozes. “Groups acting on trees: From local to global structure”. In: *Publ. Math. Inst. Hautes Études Sci.* 92 (2000), pp. 113–150.

[Bog08] O. V. Bogopolskij. *Introduction to Group Theory*. eng. EMS textbooks in mathematics. European Mathematical Society, 2008. ISBN: 9783037190418.

[Bou81] N. Bourbaki. “Groupes et algebres de Lie, Ch. 4-6”. In: *MR* 39 (1981), p. 1590.

[Bou98] N. Bourbaki. *General Topology: Chapters 1-4*. Elem. Math. Springer, 1998.

[Bro12] K. S. Brown. *Cohomology of Groups*. Graduate Texts in Mathematics. Springer New York, 2012.

[Bro89] K. S. Brown. *Buildings*. Springer Science+Business Media New York, 1989.

[BRW05] U. Baumgartner, B. Rémy, and G. A. Willis. “Flat rank of automorphism groups of buildings”. In: *Transform. Groups* 12 (2005), pp. 413–436.

[BS73] A. Borel and J-P. Serre. “Corners and arithmetic groups”. In: *Comment. Math. Helv* 48.1 (1973), pp. 436–491.

[Cap14] P.-E. Caprace. “Automorphism groups of right-angled buildings: simplicity and local splittings”. eng. In: *Fund. Math.* 224.1 (2014), pp. 17–51.

[Car18] M. Carette. “Commability of groups quasi-isometric to trees”. In: *Ann. Inst. Fourier* 68.4 (2018), pp. 1365–1398.

[Cas20] I. Castellano. “Rational discrete first degree cohomology for totally disconnected locally compact groups”. In: 168.2 (2020), pp. 361–377.

[Cas23] I. Castellano. “An Introduction to Totally Disconnected Locally Compact Groups and Their Finiteness Conditions”. In: *Modern Trends in Algebra and Representation Theory*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2023, pp. 335–369.

[CC20] I. Castellano and G. Corob Cook. “Finiteness properties of totally disconnected locally compact groups”. In: *J. Algebra* 543 (2020), pp. 54–97.

[CCW24] I. Castellano, G. Chinello, and Th. Weigel. *The Hattori–Stallings rank, the Euler–Poincaré characteristic and zeta functions of totally disconnected locally compact groups*. ArXiv preprint. 2024. arXiv: 2405.08105 [math.GR].

[CH16] Y. Cornulier and P. de la Harpe. *Metric geometry of locally compact groups*. Vol. 25. EMS Tracts in Mathematics. Winner of the 2016 EMS Monograph Award. European Mathematical Society (EMS), Zürich, 2016, pp. viii+235.

[Chi76] I. M. Chiswell. “Euler Characteristics of Groups”. In: *Math. Z.* 147 (1976), pp. 1–12.

[CL10] R. Cluckers and F. Loeser. “Constructible exponential functions, motivic Fourier transform and transfer principle”. In: *Ann. of Math.* 171 (2010), pp. 1011–1065.

[CMR22] P.-E. Caprace, T. Marquis, and C. D. Reid. “Locally normal subgroups and ends of locally compact Kac–Moody groups”. In: *Münster J. Math.* 15.2 (2022), pp. 473–498.

[CMW24a] I. Castellano, B. Marchionna, and T. Weigel. *Weyl-invariants of totally disconnected locally compact groups acting cocompactly on buildings*. 2024. arXiv: 2408.15716 [math.GR]. URL: https://arxiv.org/abs/2408.15716.

[CMW24b] I. Castellano, B. Marchionna, and Th. Weigel. *Unimodular totally disconnected locally compact groups of rational discrete cohomological dimension one*. 2024. arXiv: 2201.10847 [math.GR]. URL: https://arxiv.org/abs/2201.10847.

[Coh73] D. E. Cohen. "Group with free subgroups of finite index". In: *Conference on group theory*. Springer. 1973, pp. 26–44.

[Cor18] Y. Cornulier. "On the quasi-isometric classification of locally compact groups". In: *New Directions in Locally Compact Groups*. Ed. by P.-E. Caprace and N. Monod. London Mathematical Society Lecture Note Series. Cambridge University Press, 2018, pp. 275–342.

[CSV24] A. Carnevale, M. M. Schein, and C. Voll. "Generalized Igusa functions and ideal growth in nilpotent Lie rings". In: *Algebra and Number Theory* 18.2 (2024), pp. 537–582.

[CW16] I. Castellano and Th. Weigel. "Rational discrete cohomology for totally disconnected locally compact groups". In: *J. Algebra* 453 (2016), pp. 101–159.

[CW20] M. Carter and G. A. Willis. "Decomposition theorems for automorphism groups of trees". In: *Bull. Aust. Math. Soc.* 103 (2020), pp. 104–112.

[Dav+08] M. W. Davis et al. "Compactly supported cohomology of buildings". In: *Comment. Math. Helv.* 85 (2008).

[Dav08] M. W. Davis. *The geometry and topology of Coxeter groups*. Vol. 32. London Mathematical Society Monographs Series. Princeton University Press, Princeton, NJ, 2008, pp. xvi+584.

[DD89] W. Dicks and M. J. Dunwoody. *Groups acting on graphs*. Vol. 17. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1989, pp. xvi+283. ISBN: 0-521-23033-0.

[Dei19] A. Deitmar. "Weighted prime geodesic theorems". In: *Eur. J. Math.* 9 (2019).

[Dun79] M. J. Dunwoody. "Accessibility and Groups of Cohomological Dimension One". In: *Proc. Lond. Math. Soc.* (1979), pp. 193–215.

[Dun93] M. Dunwoody. "An Inaccessible Group". In: *Geometric Group Theory*. Ed. by G. Niblo and M. Roller. London Mathematical Society Lecture Note Series. Cambridge University Press, 1993, pp. 75–78.

[Dym06] J. Dymara. "Thin buildings". In: *Geom. Top.* 10 (2006).

[Edg09] T. Edgar. "Dominance and regularity in Coxeter groups". PhD thesis. University of Notre Dame, 2009.

[Gar12] P. B. Garrett. *Buildings and Classical Groups*. Springer Netherlands, 2012.

[Gro81] M. Gromov. “Groups of polynomial growth and expanding maps (with an appendix by Jacques Tits)”. In: *Publ. Math. Inst. Hautes Études Sci.* 53 (1981), pp. 53–78.

[Gro84] M. Gromov. “Infinite Groups as Geometric Objects”. In: *Proceedings of the International Congress of Mathematics* 1 (1984), pp. 385–392.

[GSS88] F. Grunewald, D. Segal, and G. C. Smith. “Subgroups of finite index in nilpotent groups”. In: *Invent. Math.* 93 (1988), pp. 185–223.

[Hal94] R. Halin. “Über undendliche Wege in Graphen”. In: *Math. Ann.* 157 (1994), pp. 125–137.

[Har97] D. A. Harville. *Matrix Algebra From a Statistician's Perspective*. New York: Springer-Verlag, 1997.

[Hop94] H. Hopf. “Enden offener Räume und endliche diskontinuierlich Gruppen”. In: *Comment. Math. Helv.* 16 (1994), pp. 81–100.

[HP03] F. Haglund and F. Paulin. “Constructions arborescentes d'immeubles”. In: *Math. Ann.* 325 (Jan. 2003), pp. 137–164.

[HR15] G. H. Hardy and M. Riesz. *The General Theory of Dirichlet's Series*. Cambridge Tracts in Mathematics. Cambridge University Press, 2015.

[Hum90] J. E. Humphreys. *Reflection groups and Coxeter groups*. Vol. 29. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1990, pp. xii+204.

[Igu00] J. Igusa. *An Introduction to the Theory of Local Zeta Functions*. AMS/IP studies in advanced mathematics. American Mathematical Society, 2000.

[Kap51] I. Kaplansky. “Projections in Banach algebras”. In: *Ann. of Math.* (1951), pp. 235–249.

[Kap68] I. Kaplansky. *Rings of operators*. W. A. Benjamin, Inc., New York-Amsterdam, 1968, pp. viii+151.

[KM08] B. Krön and R. G. Möller. “Analogues of Cayley graphs for topological groups”. In: *Math. Z.* 258.3 (2008), pp. 637–675.

[KMN09] P. H. Kropholler, C. Martínez-Pérez, and B. E. A. Nucinkis. “Cohomological finiteness conditions for elementary amenable groups”. In: *Walter de Gruyter GmbH and Co. KG* 673 (2009), pp. 49–62.

[Kos67] J. L. Koszul. “Lectures on hyperbolic Coxeter groups”. In: *(No Title)* (1967).

[Kra22] L. Kramer. “Some remarks on proper actions, proper metric spaces, and buildings”. In: *Adv. Geom.* 22.4 (2022), pp. 541–559.

[Krö01] B. Krön. “End compactifications in non-locally-finite graphs”. In: *Math. Proc. Camb. Phil. Soc.* 131 (2001), pp. 427–443.

[Lee22] Seungjai Lee. *Zeta functions of $\mathbb{F}_p$-Lie algebras and finite p-groups*. 2022. arXiv: 2010.02268 [math.RA]. URL: https://arxiv.org/abs/2010.02268.

[Lin83] P. A. Linnell. “On accessibility of groups”. In: *Journal of Pure and Applied Algebra* 30.1 (1983), pp. 39–46.

[Lüc05] W. Lück. *Survey on classifying spaces for families of subgroups*. 2005.

[Mar18] T. Marquis. *An Introduction to Kac-Moody Groups Over Fields*. EMS Textbooks in Mathematics. European Mathematical Society Publishing House, 2018.

[Mar24] B. Marchionna. *Double-coset zeta functions for groups acting on trees*. 2024. arXiv: 2409.01860 [math.GR]. URL: https://arxiv.org/abs/2409.01860.

[Möl10] R. Möller. “Graphs, permutations and topological groups”. 2010. URL: https://arxiv.org/abs/1008.3062.

[MT09] M. Mihalik and S. Tschantz. “Visual decompositions of Coxeter groups”. In: *Groups Geom. Dyn.* 3.1 (2009), pp. 173–198.

[MV03] G. Mislin and A. Valette. *Proper Group Actions and the Baum-Connes Conjecture*. Springer Basel AG, 2003. ISBN: 978-3-0348-8089-3.

[Neu13] J. Neukirch. *Algebraic Number Theory*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2013. ISBN: 9783662039830.

[PST18] H. D. Petersen, R. Sauer, and A. Thom. “L^2-Betti numbers of totally disconnected groups and their approximation by Betti numbers of lattices”. In: *J. Topol.* 11.1 (2018), pp. 257–282.

[Rad17] N. Radu. “A classification theorem for boundary 2-transitive automorphism groups of trees”. In: *Invent. Math.* 209 (July 2017).

[Ros15] T. Rossmann. “Computing topological zeta functions of groups, algebras, and modules, I”. In: *Proc. London Math. Soc.* 3 (2015), pp. 1099–1134.

[Ros16a] T. Rossmann. “Computing local zeta functions of groups, algebras, and modules”. 2016. arXiv: 1602.00919 [math.GR]. URL: https://arxiv.org/abs/1602.00919.

[Ros16b] T. Rossmann. “Zeta, a SageMath package for computing zeta functions (version 0.3.2)”. In: (2016). URL: https://maths.nuigalway.ie/~rossmann/Zeta/.

[Ros17] T. Rossmann. “Enumerating submodules invariant under an endomorphism”. In: *Math. Ann.* 368 (June 2017).

[RR06] B. Rémy and M. Ronan. “Topological groups of Kac-Moody type, right-angled twinnings and their lattices”. In: *Comment. Math. Helv.* 81.1 (2006), pp. 191–219.

[RS22] C. D. Reid and S. M. Smith. *Groups acting on trees with Tits' independence property (P)*. ArXiv preprint. 2022. arXiv: 2002.11766 [math.GR].

[RS23] C. D. Reid and S. M. Smith. *An introduction to the local-to-global behaviour of groups acting on trees and the theory of local action diagrams*. ArXiv preprint. 2023. arXiv: 2309.05065 [math.GR].

[RZ00] L. Ribes and P. Zalesskii. *Profinite Groups*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2000, p. 16.

[Sau03] M. du Sautoy. “Zeta functions of groups: the quest for order versus the flight from ennui”. In: *Groups St Andrews 2001 in Oxford*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2003, pp. 150–189.

[Seg97] D. Segal. “On the growth of ideals and submodules”. In: *J. London Math. Soc.* 56.2 (1997), pp. 245–263.

[Ser73] J-P. Serre. *A Course in Arithmetic*. Graduate texts in mathematics. Springer, 1973.

[Ser80] J-P. Serre. *Trees*. Translated from the French by John Stillwell. Springer-Verlag, Berlin-New York, 1980, pp. ix+142. ISBN: 3-540-10103-9.

[SG00] M. du Sautoy and F. Grunewald. “Analytic properties of zeta functions and subgroup growth”. eng. In: *Ann. of Math.* 152.3 (2000), pp. 793–833.

[Smi17] S. M. Smith. “A product for permutation groups and topological groups”. In: *Duke Math. J.* 166 (2017), pp. 2965–2999.

[Spe49] E. Specker. “Die erste Cohomologiegruppe von Überlagerungen und Homotopie-Eigenschaften dreidimensionaler Mannigfaltigkeiten”. In: *Comment. Math. Helv.* 23 (1949), pp. 303–333.

[Sta11] R. P. Stanley. *Enumerative Combinatorics, Volume I*. 2nd ed. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2011.

[Sta72] J. Stallings. *Group Theory and Three-dimensional Manifolds*. (James K. Whittemore lectures in mathematics). Yale University Press, 1972.

[Str06] M. Stroppel. *Locally Compact Groups*. EMS series of lectures in mathematics. European Mathematical Society, 2006.

[Swa69] R. G. Swan. “Groups of cohomological dimension one”. In: *J. Algebra* 12 (1969), pp. 585–610.

[Tab21] M. Taboga. “Determinant of a block-matrix”. In: *Lectures on matrix algebra* (2021).

[Tit70] J. Tits. "Sur le groupe des automorphismes d'un arbre". In: *Essays on Topology and Related Topics: Memoires dédiés à Georges de Rham*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1970, pp. 188–211.

[Tre] Enciclopedia Italiana Treccani. *Invariante*. URL: https://www.treccani.it/enciclopedia/invariante_(Enciclopedia-Italiana)/.

[Van23] M. Vantomme. "Combinatorial Submodule and Subalgebra Zeta Functions". PhD thesis. Bielefeld University, 2023.

[Van36] D. Van Dantzig. "Zur topologischen Algebra. III. Brouwersche und Cantorsche Gruppen". In: *Compos. Math.* 3 (1936), pp. 408–426.

[Vol10] C. Voll. "Functional equations for zeta functions of groups and rings". In: *Ann. of Math.* 172 (2010).

[Vol19] C. Voll. "Local functional equations for submodule zeta functions associated to nilpotent algebras of endomorphisms". In: *Int. Math. Res. Not. IMRN* 2019.7 (2019), pp. 2137–2176.

[Vol20] C. Voll. "Ideal zeta functions associated to a family of class-2-nilpotent Lie rings". In: *Quart. J. Math.* 71.3 (June 2020), pp. 959–980.

[Wik] Wikipedia. *Invariant*. URL: https://en.wikipedia.org/wiki/Invariant_(mathematics).

[Woo08] L. Woodward. "Zeta functions of Lie rings of upper-triangular matrices". In: *J. Lond. Math. Soc.* 77.2 (2008), pp. 69–82.