

## Tight entropic uncertainty relations for systems with dimension three to five

Alberto Riccardi, Chiara Macchiavello, and Lorenzo Maccone

*Dipartimento Fisica and INFN Sez. Pavia, University of Pavia, via Bassi 6, I-27100 Pavia, Italy*

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We consider two (natural) families of observables  $O_k$  for systems with dimension  $d = 3, 4, 5$ : the spin observables  $S_x, S_y$ , and  $S_z$ , and the observables that have mutually unbiased bases as eigenstates. We derive tight entropic uncertainty relations for these families, in the form  $\sum_k H(O_k) \geq \alpha_d$ , where  $H(O_k)$  is the Shannon entropy of the measurement outcomes of  $O_k$  and  $\alpha_d$  is a constant. We show that most of our bounds are stronger than previously known ones. We also give the form of the states that attain these inequalities.

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### I. INTRODUCTION

Entropic uncertainty relations [1–10] nicely express the concept of quantum uncertainty since their lower bound is typically state independent, in contrast to the Heisenberg-Robertson ones [11,12]. The most used one is the Maassen-Uffink relation [3],

$$H(A) + H(B) \geq -2 \log_2 c = q_{MU}, \quad (1)$$

where  $H(A)$  and  $H(B)$  are the Shannon entropies of the measurement outcomes of two observables  $A$  and  $B$ , and  $c = \max_{j,k} |\langle a_j | b_k \rangle|$  is the maximum overlap between their eigenstates. It is a state-independent bound, meaningful even if the observables share some common eigenstates. The bound (1) is tight if  $A$  and  $B$  have mutually unbiased bases (MUBs) as eigenstates. Stronger bounds for arbitrary observables, which involve the second largest term in  $|\langle a_j | b_k \rangle|$ , have been found recently in [13] and [14]. If one considers more than two observables, tight bounds were proven only in a few cases, most of them in dimension  $d = 2$ . For a complete set of MUBs, the strongest bounds were derived by Ivanovic in [15] for odd  $d$ , and by Sanchez in [16] for even  $d$ . Moreover, some bounds for an incomplete set of MUBs are in [17].

In this paper, we derive tight entropic uncertainty relations for more than two observables for systems of dimensions  $d = 3, 4, 5$ , both for spin observables and for arbitrary numbers of MUBs. On one hand, for spin observables, we find

$$H(S_x) + H(S_y) + H(S_z) \geq \gamma_s, \quad (2)$$

with  $\gamma_s = 2, 3 - \frac{3}{4} \log_2 3 \simeq 3.62, 3.12$  for spin  $s = 1, \frac{3}{2}, 2$ , respectively. These inequalities are all stronger than previously known results. The case of  $s = 1$  has been derived analytically, while the rest numerically. For half-integer spins, the inequality is saturated by any of the eigenstates of the three spin observables, while for integer spins, the inequality is saturated only by null projection states. Moreover, we find

$$H(S_j) + H(S_k) \geq \xi_s \quad (3)$$

for all  $j, k = x, y, z$  ( $j \neq k$ ) with  $\xi_s = 1, 1.71, 1.56$  for spin  $s = 1, \frac{3}{2}, 2$ . The case  $s = 1$  coincides with (1), but the other cases are equal or stronger than previous results [7,8]. On the other hand, for observables  $\{A_j\}$  with MUBs as eigenstates [the eigenvalues are irrelevant for entropic uncertainty relations (EURs)], we find, for dimension  $d = 3$  (where up to four

MUBs exist),

$$H(A_1) + H(A_2) + H(A_3) \geq 3, \quad (4)$$

$$H(A_1) + H(A_2) + H(A_3) + H(A_4) \geq 4; \quad (5)$$

for dimension  $d = 4$  (where up to five MUBs exist),

$$H(A_1) + H(A_2) + H(A_3) \geq 3, \quad (6)$$

$$H(A_1) + H(A_2) + H(A_3) + H(A_4) \geq 5, \quad (7)$$

$$H(A_1) + H(A_2) + H(A_3) + H(A_4) + H(A_5) \geq 7; \quad (8)$$

and, finally, for dimension  $d = 5$ ,

$$H(A_1) + H(A_2) + H(A_3) \geq 2 \log_2 5, \quad (9)$$

$$H(A_1) + H(A_2) + H(A_3) + H(A_4) \geq 6.34, \quad (10)$$

$$\sum_{j=1}^5 H(A_j) \geq 8.33, \quad (11)$$

$$\sum_{j=1}^6 H(A_j) \geq 10.25. \quad (12)$$

In addition to the above bounds, we also provide the form of the states that saturate them and we compare them to previous results in the literature.

The paper is organized as follows. In Sec. II we consider spin observables. The case  $s = 1$  is developed analytically from a recent parametrization of the state [18], while the other cases are solved numerically. In Sec. III we consider the observables with MUBs as eigenstates: after a brief review of the previous results, we derive tight entropic uncertainty relations through numerical methods. In all cases, we detail the classes of states that saturate the obtained relations. In the Appendix, we give the details of the numerical procedures that we employed.

### II. ENTROPIC UNCERTAINTY RELATIONS FOR SPIN OBSERVABLES

We start by considering the entropic uncertainty relations (EURs) relative to the spin observables  $S_x, S_y$ , and  $S_z$  for systems of different dimensions.

### A. Spin 1

The state of a three-dimensional system can be written in terms of  $S_x$ ,  $S_y$ , and  $S_z$  as [18]

$$\rho = \sum_{j=x,y,z} \left[ \omega_j (\mathbb{I} - S_j^2) + \frac{a_j S_j + q_j Q_j}{2} \right], \quad (13)$$

where  $Q_j$  is the anticommutator of  $S_k$  and  $S_l$ , with  $j \neq k, l$ , i.e.,  $Q_j = \{S_k, S_l\}$ , and

$$\omega_j = 1 - \langle S_j^2 \rangle, \quad a_j = \langle S_j \rangle, \quad q_j = \langle Q_j \rangle, \quad (14)$$

with  $0 \leq \omega_j \leq 1$  and  $|a_j| \leq 1$ . In matrix form, (13) is

$$\rho = \begin{pmatrix} \omega_x & \frac{-ia_z - q_z}{2} & \frac{ia_y - q_y}{2} \\ \frac{ia_z - q_z}{2} & \omega_y & \frac{-ia_x - q_x}{2} \\ \frac{-ia_y - q_y}{2} & \frac{ia_x - q_x}{2} & \omega_z \end{pmatrix}. \quad (15)$$

The condition  $\text{Tr}[\rho] = 1$  implies

$$\omega_x + \omega_y + \omega_z = 1. \quad (16)$$

Since  $\rho$  [19] is positive semidefinite, all principal minors of the right-hand-side of (15) are non-negative, which implies the three inequalities  $4\omega_k \omega_l \geq a_j^2$ , for  $k, l = x, y, z$  and  $j \neq k, j \neq l$ . These inequalities can be expressed also as

$$-2\sqrt{\omega_k \omega_l} \leq a_j \leq 2\sqrt{\omega_k \omega_l}. \quad (17)$$

In the representation where  $S_j^2$  are diagonal, the spin components are

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad (18)$$

$$S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (19)$$

The eigenstates of  $S_x$  are then given by

$$|S_x = 0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |S_x = \pm 1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mp i \\ 1 \end{pmatrix}, \quad (20)$$

and similar relations for the other observables. The probabilities of  $S_j$  are then given by

$$p_{m=0} = \omega_j, \quad p_{m=\pm 1} = \frac{1}{2}(1 - \omega_j \mp a_j), \quad (21)$$

whence one can calculate the Shannon entropies of  $S_j$  as

$$\begin{aligned} H(S_j) &= -\frac{1}{2}(1 - \omega_j + a_j) \log_2 \left[ \frac{1}{2}(1 - \omega_j + a_j) \right] \\ &\quad -\frac{1}{2}(1 - \omega_j - a_j) \log_2 \left[ \frac{1}{2}(1 - \omega_j - a_j) \right] \\ &\quad -\omega_j \log_2 \omega_j. \end{aligned} \quad (22)$$

For two observables, we find the optimal EUR (1),

$$H(S_i) + H(S_j) \geq 1; \quad (23)$$

indeed,  $c = \frac{1}{\sqrt{2}}$  and, moreover, the above inequality is tight when calculated on the null projection state of any of the two observables. For three observables, we obtain an EUR by

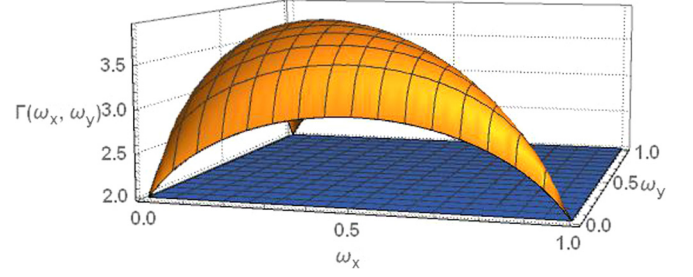


FIG. 1. Plot of the function  $\Gamma(\omega_x, \omega_y)$ .

finding an upper bound to  $-\sum_j H(S_j)$ . To this aim, we can use the conditions (17), employing the monotonicity of the logarithm as

$$\begin{aligned} &\frac{1}{2}(1 - \omega_j \pm a_j) \log_2 \left[ \frac{1}{2}(1 - \omega_j \pm a_j) \right] \\ &\leq \frac{1}{2}(1 - \omega_j + 2\sqrt{\omega_k \omega_l}) \log_2 \left[ \frac{1}{2}(1 - \omega_j + 2\sqrt{\omega_k \omega_l}) \right]. \end{aligned} \quad (24)$$

Then we have

$$\begin{aligned} -\sum_j H(S_j) &\leq \sum_j \omega_j \log_2 \omega_j + (1 - \omega_j + 2\sqrt{\omega_k \omega_l}) \\ &\quad \times \log_2 \left[ \frac{1 - \omega_j + 2\sqrt{\omega_k \omega_l}}{2} \right]. \end{aligned} \quad (25)$$

The right-hand side is a function  $-\Gamma(\omega_x, \omega_y)$ , which depends only on  $\omega_x$  and  $\omega_y$  since the  $\omega_j$ 's are constrained by (16). Inverting the inequality, we find the EUR,

$$\sum_j H(S_j) \geq \Gamma(\omega_x, \omega_y) \geq 2. \quad (26)$$

The lower bound  $\Gamma$  is plotted in Fig. 1. Its minimum value  $\Gamma = 2$  is found for  $\omega_j = 1$  and  $\omega_k = \omega_l = 0$ . These conditions imply that  $a_j = 0$  for all  $j$  through (17). Thus,  $\sum_j H(S_j) = 2$  is attained on null projection states.

This result shows a different behavior of the EUR for spin observables in the case of integer spin with respect to the half-integer case. A simple example of the latter is the qubit case: it was shown in [16] that for qubits, we have  $\sum H(S_j) \geq 2$ , but the minimum is achieved by any of the eigenstates of one of the  $S_j$  in contrast to the qutrit case obtained here. This difference in behavior between integer and half-integer spins is true also for larger spin numbers (see below).

A straightforward generalization of (23) is obtained by repeating that inequality for pairs of observables, obtaining  $\sum H(S_j) \geq \frac{3}{2}$ . It is weaker than our bound (26).

### B. Spin $\frac{3}{2}$

For a four-dimensional system, we are unaware of a representation of the density matrix in terms of the spin observables and we cannot reproduce the derivation given for  $s = 1$ . We thus develop a simple computational method that gives tight EUR for small system dimensions  $d$ .

An arbitrary pure state  $|\psi\rangle$  of a  $d$ -dimensional system depends on  $2d - 2$  real parameters. It is sufficient to consider pure states because of the concavity of the Shannon entropy: mixed states have greater entropy. The probability on  $|\psi\rangle$

of the measurement outcomes is  $p(a_k) = |\langle a_k | \psi \rangle|^2$  for an arbitrary observable  $A = \sum_k a_k |a_k\rangle\langle a_k|$ , whence the entropy is  $H(A) = -\sum_k p(a_k) \log_2 [p(a_k)]$ . Considering  $n$  observables  $A_1, A_2, \dots, A_n$ , we can calculate the quantity  $\sum_{j=1}^n H(A_j)$ , which can be seen as a function of the  $2d - 2$  parameters representing the state. This function can then be numerically minimized over this parameter space. In addition to finding the minimum, we then also find the states that saturate the bounds, which are then tight. In the Appendix we give more details on the computational procedure; here we present only the results.

For the case of two spin observables, we find

$$H(S_j) + H(S_k) \geq 1.71, \quad (27)$$

with  $j, k = x, y, z$  and  $j \neq k$ .

To compare this result with the previous results of [13] and [14], we can express these as [20]

$$H(A) + H(B) \geq \max(q_{CP}, q_{RPZ}), \quad (28)$$

with

$$q_{CP} = 2 \left[ -\log_2 c + \frac{1}{2} (1 - \sqrt{c}) \log_2 \frac{c}{c_2} \right], \quad (29)$$

$$q_{RPZ} = 2 \left\{ -\log_2 c - \log_2 \left[ b^2 + \frac{c_2}{c} (1 - b^2) \right] \right\}, \quad (30)$$

where  $b = \frac{1+\sqrt{c}}{2}$ ,  $c = \max_{j,k} |\langle a_j | b_k \rangle|$  is the maximum overlap among eigenstates of  $A$  and  $B$ , and  $c_2$  is the second maximum overlap. Both  $q_{CP}$  and  $q_{RPZ}$  are greater than  $q_{MU}$  of (1). Our result (27) is an even stronger bound than both  $q_{CP}$  and  $q_{RPZ}$ . Indeed, for  $s = 1$ , we have  $c = \frac{1}{2}\sqrt{\frac{3}{2}}$  and  $c_2 = \frac{1}{2\sqrt{2}}$ , so  $q_{CP} = 1.59$  and  $q_{RPZ} = 1.68$ .

The bound (27) is not saturated by one of the eigenstates of  $S_j$ ; indeed, for any eigenstate, we have  $H(S_j) + H(S_k) \geq 1.81$ . Instead, it is saturated by the state

$$|\psi\rangle = \sin(15^\circ)|0\rangle + \cos(15^\circ)|2\rangle, \quad (31)$$

and by similar superpositions weighted by the angle  $\alpha = 15^\circ$ . The bound (27) is in agreement with the numerical bound found in [14], but here we also find the state that achieves the minimum.

For the case of three spin observables, we find

$$H(S_x) + H(S_y) + H(S_z) \geq 3 - \frac{3}{4} \log_2 3 = 3.62. \quad (32)$$

If we employ (27) to obtain a bound for three observables (by applying it to each pair of observables), we find

$$H(S_x) + H(S_y) + H(S_z) \geq \frac{3}{2}(1.71) = 2.56, \quad (33)$$

which is weaker than (32). The same argument applied to  $q_{RPZ}$  of (30) leads to  $H(S_x) + H(S_y) + H(S_z) \geq \frac{3}{2}(1.68) = 2.52$ : also in this case, our result (32) is stronger than previous ones.

The lower bound (32) is achieved by the eigenstates of any of three observables  $S_j$ . This generalizes the result found by Sanchez in [16]; indeed, in this case the MUBs also represent the spin components. As mentioned above, the EUR for half-integer and integer spin values are attained for different classes of states.

### C. Spin 2

A spin-2 system has dimension  $d = 5$ . Using the same algorithm detailed in the previous section, we find

$$H(S_j) + H(S_k) \geq 1.56, \quad (34)$$

$$H(S_x) + H(S_y) + H(S_z) \geq 3.12. \quad (35)$$

Both of the above inequalities are saturated by the eigenstates corresponding to the eigenvalue 0 of any of the three observables  $S_j$ , the null projection state (as in the case  $s = 1$ ). For example, the above inequalities are saturated by the state

$$|S_x = 0\rangle = \frac{1}{2}\sqrt{\frac{3}{2}}|0\rangle - \frac{1}{2}|2\rangle + \frac{1}{2}\sqrt{\frac{3}{2}}|4\rangle. \quad (36)$$

The comparison of (34) with the previously known bounds  $q_{MU}$ ,  $q_{CP}$ , and  $q_{RPZ}$  shows that, again, our result is stronger. In fact, in this case, we have  $c = \frac{1}{2}\sqrt{\frac{3}{2}}$  and  $c_2 = \frac{1}{2}$ . Therefore,  $q_{MU} = 1.41$ ,  $q_{CP} = 1.48$ , and  $q_{RPZ} = 1.53$ , which are weaker than (34). Instead, the numerical bound found in [14] agrees with ours, but we also provide the states that saturate it. If we consider the application of (35) to three spin observables, we would obtain  $H(S_x) + H(S_y) + H(S_z) \geq \frac{3}{2}(1.56) = 2.34$ , which is weaker than (35); the three-observable bound is again stronger than the ones obtained by joining two-observable bounds.

## III. ENTROPIC UNCERTAINTY RELATIONS FOR ARBITRARY NUMBERS OF MUBS

We now consider the EURs relative to observables that have mutually unbiased bases (MUBs) as eigenstates (the eigenvalues are irrelevant for the EURs). To obtain the EURs, we use the same procedure detailed in Sec. II B. However, we must also calculate the MUBs for each dimension. In a  $d$ -dimensional Hilbert space, there exist  $d + 1$  MUBs if  $d$  is a power of a prime, otherwise only three bases are known to exist [21]. The properties of the MUBs strongly depend on the dimension, e.g., for a qubit, MUBs are also the eigenbases of the spin observables, but this is not true for  $d > 2$ . The problem of finding MUBs can be translated into finding Hadamard matrices: the columns of such matrices are the states of the MUBs. This problem was solved in [22] for dimensions  $d = 2, 3, 4, 5$ . Here we use that result to study EURs: for each dimension  $d = 3, 4, 5$ , we consider up to  $d + 1$  observables  $A_1, A_2, \dots, A_{d+1}$  that have MUBs as eigenstates.

We now briefly review previous results for EURs with MUB observables. For any number  $L$  of these observables, we can construct an EUR with a trivial generalization of Maassen and Uffink's relation (1) by applying (1) to pairs of bases, obtaining

$$\sum_{i=1}^L H(A_i) \geq \frac{L}{2} \log_2 d. \quad (37)$$

However, this inequality is almost never tight. A better bound was given in [15] for  $L = d + 1$ ,

$$\sum_{i=1}^L H(A_i) \geq (d + 1)[\log_2(d + 1) - 1] = q_L, \quad (38)$$

which is also not always tight. For even dimension  $d$ , a stronger bound was given in [16],

$$\sum_{i=1}^L H(A_i) \geq \left( \frac{d}{2} \log_2 \frac{d}{2} + \frac{d+1}{2} \log_2 \frac{d+1}{2} \right) = q_S, \quad (39)$$

which is tight only in dimension two. For  $L < d + 1$  in [23], it has been shown that if the Hilbert space dimension is a square, that is,  $d = r^2$ , then for  $L < r + 1$  the inequality (37) is tight, namely,

$$\sum_{i=1}^L H(A_i) \geq \frac{L}{2} \log_2 d = q_{BW}. \quad (40)$$

A further bound for  $L < d + 1$  was given in [17],

$$\sum_{i=1}^L H(A_i) \geq -L \log_2 \left( \frac{d+L-1}{dL} \right) = q_A. \quad (41)$$

For more details on the above bounds, we refer to [20]. We now present our results which are tight for all dimensions and all numbers  $L$  of MUBs.

### A. Dimension three

In dimension  $d = 3$ , four MUBs exist ( $A_1, A_2, A_3$ , and  $A_4$ ), whose states are respectively given by the columns of the Hadamard matrices,

$$\begin{aligned} M_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & M_2 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \\ M_3 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & \omega & 1 \\ 1 & \omega & \omega^2 \end{pmatrix}, & M_4 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ 1 & \omega^2 & \omega \end{pmatrix}, \end{aligned} \quad (42)$$

with  $\omega = \exp(\frac{2\pi i}{3})$ . If the system is prepared in an eigenstate of any of the MUBs, the entropy of that observable is null while the other entropies are maximal: e.g., if  $H(A_1) = 0$ , then we have  $H(A_1) + H(A_2) + H(A_3) = 2 \log_2 3$ . In contrast to the qubit case  $d = 2$ , this is *not* the state that gives the strongest EUR for  $d = 3$ . Indeed, the state  $\frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$  has entropies for all MUBs equal to 1:  $H(A_i) = 1$ . Therefore,

$$H(A_1) + H(A_2) + H(A_3) \geq 3, \quad (43)$$

$$H(A_1) + H(A_2) + H(A_3) + H(A_4) \geq 4. \quad (44)$$

We have numerically shown that the above inequalities are the optimal ones. In addition to the above state, they are also saturated by the following states:

$$\frac{e^{i\varphi}|0\rangle + |1\rangle}{\sqrt{2}}, \quad \frac{e^{i\varphi}|0\rangle + |2\rangle}{\sqrt{2}}, \quad \frac{e^{i\varphi}|1\rangle + |2\rangle}{\sqrt{2}}, \quad (45)$$

where  $\varphi = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ . Our bound (43) is stronger than (37), which in this case gives  $H(A_1) + H(A_2) + H(A_3) = \frac{3}{2} \log_2 3 = 2.38$ . For  $L = 3$ , the bound (41) gives  $q_A = 2.54$ , which is also weaker than (43). For a complete set of MUBs  $L = 4$ , the bound (38) gives  $q_I = 4$  and is then equal to our

relation (44). However, here we have proven that (44) is a tight relation for  $d = 3$ , and we have provided that the states achieve the minimum.

### B. Dimension four

In dimension  $d = 4$ , five MUBs exist, whose states are given by the columns of the Hadamard matrices,

$$\begin{aligned} M_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ M_2 &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \\ M_3 &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -i & i & i & -i \\ i & -i & i & -i \end{pmatrix}, \\ M_4 &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ i & -i & i & -i \\ -1 & -1 & 1 & 1 \\ i & -i & -i & i \end{pmatrix}, \\ M_5 &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ i & -i & i & -i \\ i & i & -i & i \\ -1 & -1 & 1 & 1 \end{pmatrix}. \end{aligned} \quad (46)$$

Since  $d = 4 = r^2$  is a square for  $r = 2$ , then for  $L < r + 1 = 3$ , the inequality (37) is tight [23]. This is then the best bound up to  $L = 2$ . However, for  $d = 4$ , we can consider up to  $L = 5$ . For example, for  $L = 3$ , we find that the optimal bound is

$$H(A_1) + H(A_2) + H(A_3) \geq 3. \quad (47)$$

It is achieved by the four states

$$(|0\rangle \pm |1\rangle)\sqrt{2}, \quad (|2\rangle \pm |3\rangle)\sqrt{2}. \quad (48)$$

By symmetry, similar relations hold by permuting the MUBs observables, but involving the superposition of different eigenstates. For example,  $H(A_1) + H(A_2) + H(A_4) \geq 3$  has lower bound achieved by  $(|0\rangle \pm |2\rangle)\sqrt{2}$  and  $(|1\rangle \pm |3\rangle)\sqrt{2}$ . Note that in  $d = 4$ , it is possible to consider different triplets of MUBs, which are unitarily inequivalent from the one considered here. Indeed, it was shown in [22,24] that there exists a three-parameter family of inequivalent triplets. However, (47) does not change with different choices of MUBs.

In the case of  $L = 4$  observables, we find

$$H(A_1) + H(A_2) + H(A_3) + H(A_4) \geq 5 \quad (49)$$

as the optimal bound, which is saturated by the states

$$\begin{aligned} &(|0\rangle \pm |1\rangle)/\sqrt{2}, \quad (|0\rangle \pm |2\rangle)/\sqrt{2}, \quad (|0\rangle \pm i|3\rangle)/\sqrt{2}, \\ &(|1\rangle \pm i|2\rangle)/\sqrt{2}, \quad (|1\rangle \pm |3\rangle)/\sqrt{2}, \quad (|2\rangle \pm |3\rangle)/\sqrt{2}. \end{aligned} \quad (50)$$

Compare our bound (49) to (40) and (41): for  $L = 4$ , we find  $q_{BW} = 4$  and  $q_A = 4.77$ . Therefore, our inequality is stronger than both.

In the case of  $L = 5 = d + 1$  observables (the complete set of MUBs), we find

$$H(A_1) + H(A_2) + H(A_3) + H(A_4) + H(A_5) \geq 7, \quad (51)$$

which is saturated by states of the following form:

$$|\psi_{jk}\rangle = \frac{1}{\sqrt{2}}(|j\rangle \pm (i)^t|k\rangle), \quad (52)$$

with  $t = 0, 1$  and  $j$  and  $k$  are the eigenstates of  $A_1$ . For  $d = 4$ , the inequality (39) gives  $q_S = 2 + \frac{5}{2} \log_2 5 = 5.30$ , so that (39) is weaker than our bound (51) in this case.

### C. Dimension five

In dimension  $d = 5$ , six MUBs exist, whose states are given by the columns of the Hadamard matrices,

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (53)$$

$$M_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega & \omega^3 \\ 1 & \omega^3 & \omega & \omega^4 & \omega^2 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{pmatrix}, \quad (54)$$

$$M_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \omega & \omega^2 & \omega^3 & \omega^4 & 1 \\ \omega^4 & \omega & \omega^3 & 1 & \omega^2 \\ \omega^4 & \omega^2 & 1 & \omega^3 & \omega \\ \omega & 1 & \omega^4 & \omega^3 & \omega^2 \end{pmatrix}, \quad (55)$$

$$M_4 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \omega^3 & \omega^4 & 1 & \omega & \omega^2 \\ \omega^2 & \omega^4 & \omega & \omega^3 & 1 \\ \omega^2 & 1 & \omega^3 & \omega & \omega^4 \\ \omega^3 & \omega^2 & \omega^3 & 1 & \omega^4 \end{pmatrix}, \quad (56)$$

$$M_5 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \omega^2 & \omega^3 & \omega^4 & 1 & \omega \\ \omega^3 & 1 & \omega^2 & \omega^4 & \omega \\ \omega^3 & \omega & \omega^2 & \omega^2 & 1 \\ \omega^2 & \omega & 1 & \omega^4 & \omega^3 \end{pmatrix}, \quad (57)$$

$$M_6 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \omega^4 & 1 & \omega & \omega^2 & \omega^3 \\ \omega & \omega^3 & 1 & \omega^2 & \omega^4 \\ \omega & \omega^4 & \omega^2 & 1 & \omega^3 \\ \omega^4 & \omega^3 & \omega^2 & \omega & 1 \end{pmatrix}, \quad (58)$$

with  $\omega = \exp(\frac{2\pi i}{5})$ . For three MUB observables, we find that the optimal bound is

$$H(A_1) + H(A_2) + H(A_3) \geq 2 \log_2 5, \quad (59)$$

which is saturated by any eigenstate of any of the three MUBs, as in the qubit  $d = 2$  case (also there the EUR for three complementary observables is saturated by the eigenstates of the observables). The bound (59) is the only known entropic uncertainty relation, apart from the qubit case, with more than two observables that has this property. In this respect, it is somewhat similar to Maassen and Uffink's (1): they are both achieved by the eigenstates of one of the observables (so that the entropies of the others are maximum). For  $L = 3$  in (41), we have  $q_A = 3.30$  while  $2 \log_2 5 = 4.64$ . Our bound is also stronger than these in this case.

For four MUBs, we find that the optimal bound is

$$H(A_1) + H(A_2) + H(A_3) + H(A_4) \geq 6.34, \quad (60)$$

and the minimum is achieved by states that are superposition of four basis states, e.g.,

$$|\psi\rangle = 0.19e^{i\frac{5}{3}\pi}|0\rangle + 0.19|1\rangle + 0.68e^{i\frac{9}{5}\pi}|3\rangle + 0.68|4\rangle. \quad (61)$$

In this case, we have  $q_A = 5.28$ , which is again weaker than our bound (60). For five MUBs, we find

$$H(A_1) + H(A_2) + H(A_3) + H(A_4) + H(A_5) \geq 8.33. \quad (62)$$

And, finally, for the complete set of six MUBs, we find

$$\sum_{i=1}^6 H(A_i) \geq 10.25. \quad (63)$$

The two above inequalities are again minimized by states that can be expressed by the superposition of four basis states, having the same form of (61). For  $L = 5$ , we can compare (62) to (41) which gives a weaker bound  $q_A = 7.34$ , while for the complete set of MUBs we can compare (63) to (38), which gives a weaker bound  $q_I = 9.51$ .

## IV. CONCLUSIONS

In this paper, we have found several tight entropic uncertainty relations for two classes of observables: the spin observables  $S_x, S_y, S_z$  and the observables  $\{A_j\}$  with MUB eigenstates.

For the case of spin observables, for  $s = 1$  we found a tight relation (26) for the complete set of spin observables; its minimum value is achieved by null projection states of any of the three observables. The same types of states also saturate the inequality (35) for the case of  $s = 2$ . Instead, in the case  $s = \frac{3}{2}$ , the inequality (32) is minimized by eigenstates of any of the three spin observables. For both  $s = \frac{3}{2}$  and  $s = 2$ , we have also found tight inequalities for two spin observables,



which are in agreement with the optimal bound found in [14], and we have given the states that minimize them. In the case of  $s = 2$ , they are the null projections states.

For the case of MUB observables, we have derived several tight inequalities for dimensions  $d = 3, 4, 5$ . For  $d = 3$  the results (44) equals the previous bound (38) but here we also found the class of states that saturates it. In contrast, for  $d = 4, 5$ , the bounds (51) and (63) represent stronger EUR than known ones. Inequalities have been also found for incomplete sets of MUBs in every dimension: in each case, the bounds are tight and we have derived the states that achieve the minimum. We note the peculiar behavior of (59), which is achieved by any eigenstate of one of the three MUBs, resembling the behavior of qubit systems.

### APPENDIX : NUMERICAL METHODS

Here we detail the numerical methods used to derive most of our entropic uncertainty relations. We have used the software package MATHEMATICA. For the sake of illustration, we consider the case of  $s = \frac{3}{2}$ . The most general pure state of a quantum system for  $d = 4$  is

$$|\psi\rangle = e^{i\chi_0} \sin a_0 \sin a_1 \cos a_2 |0\rangle + e^{i\chi_1} \sin a_0 \sin a_1 \sin a_2 |1\rangle + e^{i\chi_2} \sin a_0 \cos a_1 |2\rangle + \cos a_0 |1\rangle, \quad (\text{A1})$$

where  $a_i \in [0, \frac{\pi}{2}]$  and  $\chi_0 \in [0, 2\pi]$ . To compute the probability distributions of  $S_x, S_y$ , and  $S_z$  over the state  $|\psi\rangle$ , we work in the representation of eigenstates of  $S_z$ . In this representation, the spin matrices are

$$S_x = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad (\text{A2})$$

$$S_y = \frac{1}{2i} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}, \quad (\text{A3})$$

$$S_z = \frac{1}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \quad (\text{A4})$$

The probability distribution of  $S_z$  is

$$p(S_z = +2) = \sin^2 a_0 \sin^2 a_1 \cos^2 a_2, \quad (\text{A5})$$

$$p(S_z = +1) = \sin^2 a_0 \sin^2 a_1 \sin^2 a_2, \quad (\text{A6})$$

$$p(S_z = -1) = \sin^2 a_0 \cos^2 a_1, \quad (\text{A7})$$

$$p(S_z = -2) = \cos^2 a_0. \quad (\text{A8})$$

Then the entropy is  $H(S_z) = -\sum_l p_l(S_z = l) \log_2 p_l(S_z = l)$ , which depends only on the three parameters  $a_j$ .

To calculate the entropy for  $S_x$ , consider its eigenstates

$$|S_x = \pm 2\rangle = \frac{1}{2\sqrt{2}}(|0\rangle \pm \sqrt{3}|1\rangle - \sqrt{3}|2\rangle \pm |3\rangle),$$

$$|S_x = \pm 1\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|0\rangle \pm |1\rangle - |2\rangle \mp \sqrt{3}|3\rangle). \quad (\text{A9})$$

We can compute the probability distribution of  $S_x$  over  $|\psi\rangle$  with

$$p(S_x = \pm l) = |\langle \psi | S_x = \pm l \rangle|^2. \quad (\text{A10})$$

This expression depends on all six parameters of  $|\psi\rangle$ , and we can use it to calculate the entropy  $H(S_x) = -\sum_l p_l(S_x = l) \log_2 p_l(S_x = l)$ .

An analogous procedure can be used for  $S_y$ , whose eigenstates are

$$|S_y = \pm 2\rangle = \frac{1}{2\sqrt{2}}(|0\rangle \pm i\sqrt{3}|1\rangle - \sqrt{3}|2\rangle \mp i|3\rangle), \quad (\text{A11})$$

$$|S_y = \pm 1\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|0\rangle \pm i|1\rangle + |2\rangle \pm i\sqrt{3}|3\rangle), \quad (\text{A12})$$

whence we can calculate the probabilities and the entropy.

To obtain the optimal EUR, we need to minimize the sum of two or three of the above entropies. Due to their nonlinear dependence on the parameters, it is highly nontrivial to find the minimum analytically. We have therefore resorted to numerical methods: MATHEMATICA permits the minimization of a function  $f(x_1, \dots, x_n)$  that depends on  $n$  parameters with the routine

$$\text{NMinimize}[\{f(x_1, \dots, x_n), \gamma(x_1, \dots, x_n)\}, \{x_1, \dots, x_n\}], \quad (\text{A13})$$

where  $\gamma$  represents possible constraints. This routine returns both the minimum value of the function and also the parameter values that attain it, which in our case identify the states that minimize the EUR. For example, if we define

$$f(a_0, a_1, a_2, \chi_0, \chi_1, \chi_2) = H(S_x) + H(S_z), \quad (\text{A14})$$

the instruction

$$\text{NMinimize}[f(a_i, \chi_i), \{a_0, a_1, a_2, \chi_0, \chi_1, \chi_2\}] \quad (\text{A15})$$

returns

$$\left[ 1.71, \left\{ a_0 \rightarrow \frac{\pi}{12}, a_1 \rightarrow \frac{\pi}{4}, a_2 \rightarrow \frac{\pi}{4}, \chi_1 \rightarrow \pi \right\} \right], \quad (\text{A16})$$

which implies (34). The other relations that we derived can be similarly obtained.

For example, for the case of spin 2, we can repeat the above procedure. Again, we can choose the representation of the eigenbasis of  $S_z$ , which gives

$$S_x = \frac{1}{2} \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}, \quad (\text{A17})$$

$$S_y = \frac{1}{2i} \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & -\sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & -\sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix}, \quad (\text{A18})$$

$$S_z = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{A19})$$

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