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Frobenius type structures and manifolds from stability conditions on quiver categories

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Introduction

This thesis is concerned with constructing Frobenius type structures as well as semisimple Frobenius manifolds using the data of stability conditions and invariants enumerating semistable objects in categories.

Frobenius manifolds were introduced by Dubrovin in order to give a geometric interpretation of a special system of partial differential equations appearing in 2-dimensional topological field theory. The theory of Frobenius manifolds has since developed as an independent geometric tool.

Roughly speaking a Frobenius structure on a manifold M is the datum of a Frobenius algebra structure with associative multiplication $\circ$ and quadratic form g on the fibres of the tangent bundle of M. The structure coefficients of $\circ$ must be given everywhere locally by the third derivatives $\partial_a \partial_b \partial_c \Phi$ of a single function Φ (raising indices with g), known as the potential. A Frobenius manifold may be described equivalently in terms of a family of meromorphic connections on a vector bundle on the complex projective line $\mathbb{P}^1$, having constant (generalised) monodromy.

There is a natural notion of Frobenius type structure on a general holomorphic vector bundle $K \to M$, introduced by Hertling in his study of geometric structures on unfolding spaces of singularities. Such a structure is a collection of holomorphic objects $(\nabla^r, C, \mathcal{U}, \mathcal{V}, g)$, with values in the bundle K, where ∇^r is a flat connection, g is a non-degenerate bilinear pairing, C is a Higgs field, and $\mathcal{U}, \mathcal{V}$ are endomorphisms on the fibres of K, satisfying a set of partial differential equations. It can be equivalently described in terms of a flat meromorphic connection on the pull-back of K to $\mathbb{P}^1 \times M$ or as a family of meromorphic connections on $\mathbb{P}^1$ with constant generalised monodromy.

It turns out that often (e.g. for many unfolding spaces) Frobenius and Frobenius type structures are in fact part of more refined "Cecotti-Vafa" (CV) structures, a notion which was also formalised by Hertling in terms of C^∞ objects $(D, C, \widetilde{C}, \kappa, h, \mathcal{U}, \mathcal{Q})$ with values in K, where D is a connection, $C, \widetilde{C}$ are (anti)-Higgs fields, κ is an involution, h a hermitian form and $\mathcal{U}, \mathcal{Q}$ are endomorphism, satisfying a set of partial differential equations.

There exists a special theory of (formal) Frobenius manifolds called Quantum Cohomology which is related to classical enumerative problems in algebraic geometry. Let V be a projective algebraic manifold and write $H = H^*(V, \mathbb{C})$. Let Φ_q be a suitable generating function for the number of rational curves on V of fixed degree (the genus zero Gromov-Witten invariants). Φ_q is a formal power series in an auxiliary variable q and coordinates on H. Kontsevich and Manin [34] proved that Φ_q is the potential for a formal Frobenius structure on $H \otimes \mathbb{C}[\![q]\!]$.

Following ideas of Bridgeland-Toledano Laredo [10] and Joyce [31], in this thesis we define and study Frobenius type structures appearing in a different kind of enumerative geometry. Our approach is based on the theory of stability conditions on a triangulated category developed by Bridgeland [8] as well as the notion of holomorphic generating functions for invariants enumerating semistable objects in abelian and triangulated categories introduced by Joyce. We concentrate on the special but instructive setup of categories described by quivers (oriented graphs) endowed with a potential [9].

Here is a brief outline of the thesis and its main contributions.

Chapter 1 reviews the theory of (semisimple) Frobenius manifolds and its connection with the theory of isomonodromic deformations. In particular the Stokes matrix $\mathcal{S}$ and maximal analytic continuation of a semisimple Frobenius manifold to the configuration space C_n are introduced. We recall how analytic continuation can be understood in terms of an action of the braid group on the Stokes matrices. Finally the notions of Frobenius type and Cecotti-Vafa (CV) structures are recalled.

Chapter 2 reviews the theory of quivers with potential (Q, W) and their mutations. We recall the canonical construction of a 3-Calabi-Yau triangulated category $\mathcal{D}(Q, W)$ with a finite heart $\mathcal{A}(Q, W) \subset \mathcal{D}(Q, W)$. Mutations (Q', W') have equivalent $\mathcal{D}(Q', W')$, while the corresponding hearts are related by a pair of simple tilts. The general theory of Bridgeland stability conditions is briefly recalled, and applied to the special case of stability conditions $(\mathcal{A}, Z)$ (pairs of a heart and a central charge) supported on finite hearts $\mathcal{A} \subset \mathcal{D}(Q, W)$. The invariants $\mathrm{DT}(\alpha, Z)$ enumerating Z-semistable objects in $\mathcal{D}(Q, W)$ with Grothendieck class α are introduced and combined in formal, infinite sums $f^{\alpha}(Z)$ known as Joyce holomorphic generating functions.

Chapter 3 uses the previous theory to construct Frobenius type and CV-structures on the space of stability conditions $\mathrm{Stab}(\mathcal{A})$ supported on a finite heart $\mathcal{A} = \mathcal{A}(Q, W)$: this is the content of Theorems 3.26 and 3.27. These are formal families taking values in the trivial bundle K with fibre $\mathbb{C}[\mathcal{K}(\mathcal{A})]$ ($\mathcal{K}(\mathcal{A})$ denoting the Grothendieck group, i.e. the free abelian group spanned by the isomorphism classes of simple objects $[S_1], \dots, [S_n]$). More precisely these Frobenius type and CV-structures take values in $\mathbb{C}[\mathcal{K}(\mathcal{A})][\![\mathbf{s}]\!]$ where $\mathbf{s} = (s_1, \dots, s_n)$ are formal parameters (one for each class of a simple object). Thus the coefficients of these structures are formal power series in $\mathbf{s}$, and it is natural to ask if they have positive radius of convergence. Theorem 3.30 (proved in Appendix A) gives such a convergence result for the CV-structures for all sufficiently large central charges Z, in a precise sense. We explain that the same argument fails for the Frobenius type structures because of a scale-invariance property.

In Chapter 4 we study a natural operation of pullback on the infinite-dimensional families of Frobenius type structures of Chapter 3, from K to the tangent bundle $T\,\mathrm{Stab}(\mathcal{A})$. The pullback families are also parametrised by $\mathbf{s}$ of course. This operation depends on the choice of a section $\zeta \in \mathcal{O}(K)$. We give necessary and sufficient conditions on the section ζ so that the pullback to $T\,\mathrm{Stab}(\mathcal{A})$ is a family of structures which is tangent to (or osculates to higher order) a family of genuine semisimple Frobenius structures: this is the content of Theorems 4.26 and 4.27. The guiding examples of the quivers A_2 and A_3 are discussed in detail. In the case of A_n we show that the choice of a suitable section ζ produces a family which contains (at the special value $\mathbf{s} = (1, \dots, 1)$) a branch of the Saito-Dubrovin manifold for the unfolding space of A_n.

In Chapter 5 we study the effect of quiver mutations on the constructions of Chapter 4. We focus on the set of mutations of the basic A_n quiver. We show that for all mutations of the basic quiver one can construct an admissible section ζ. The corresponding (families of) semisimple Frobenius structures can be analytically continued to the whole configuration space C_n, and can be specialised at $\mathbf{s} = (1, \dots, 1)$. Our main result 5.12 says that at least for $n \leq 5$ two such structures are always branches of the same semisimple Frobenius manifold structure on C_n, that is they are always related by analytic continuation. In

particular this allows to construct other branches of the Saito-Dubrovin manifold from stability conditions. We write down explicit relations between the corresponding Stokes matrices in the braid group. In simple cases (e.g. for a single mutation) there is a natural bijective correspondence between mutations and braids, but the general situation seems much more complicated (Cotti, Dubrovin and Guzzetti have informed us that a similar difficulty arises in their ongoing study of the quantum cohomology of Grassmannians). Finally we discuss the possibility of extending the result to all values of n, and we give some further examples of quivers and mutations to which our methods apply.

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Contents

Chapter 1

Introduction to Frobenius manifolds and some generalizations

A Frobenius manifold is a manifold with a Frobenius algebra structure on the fibers of its tangent bundle varying smoothly. This concept was introduced by Dubrovin in [16] to formulate in geometrical terms the WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations (1.1.2). Due to its generalizations, it is also related to Hodge theory. In this Chapter Frobenius and Frobenius-like structures are described, together with some results in the deformation theory of connections.

The first section presents Frobenius structures; it is based on Manin's book [37, Chapter 1]. Other good references for an introduction to Frobenius manifolds are [38] by Sabbah and [27] by Hitchin. Sections 1.6 and 1.7 present instead the concepts of Frobenius type and CV- structure respectively. The first is a generalization of a Frobenius structure from the tangent bundle to an auxiliary general vector bundle, the latter is a more complicated structure. The datum of a special class of Frobenius (the semisimple ones) or Frobenius type structure can be encoded in flat or isomonodromic families of connections. Sections 1.2 gives an introduction to the theory of isomonodromic deformations. In the next chapters, we will be most interested in connections over $\mathbb{P}^1$ with a double pole at 0. They are considered in Section 1.3. The description of a semisimple Frobenius manifold in terms of isomonodromic connections over $\mathbb{P}^1$ is given in Section 1.4.

1.1 Frobenius manifolds

1.1.1 Frobenius algebra and integrability property

Let $(A, \cdot, +)$ be a finite-dimensional commutative and associative algebra with identity $\mathbf{1}$ over a field k. We call it a *Frobenius algebra* if there exists a linear form $\theta \in A^*$ such that contraction of the multiplication with θ is a non-degenerate symmetric bilinear form $(a, b) := \theta(a \cdot b)$ and $(a \cdot b, c) = (a, b \cdot c)$.

A similar structure can be defined on a vector space V of dimension n. It is the datum of three objects: a function $\theta \in V^*$, a symmetric pairing $g \in S^2V^*$ and a three-symmetric tensor $A \in S^3V^*$. The operation of multiplication is defined by the conditions $g(u, v) = \theta(u \cdot v)$ and $A(u, v, w) = \theta(u \cdot v, w)$. Requiring that $\cdot$ is associative imposes strong algebraic constraints on θ, g, A.

A Frobenius structure on a manifold M is a structure of Frobenius algebra on each tangent space satisfying some axioms. The definition is reproduced in the next Section. Frobenius structures may be defined over C^∞, real, analytic, super- manifolds. In this thesis, however, only complex manifolds and their holomorphic tangent bundles will be considered.

1.1.2 Complex Frobenius manifolds

Let M be a complex manifold and g a non-degenerate symmetric bilinear $\mathbb{C}-$valued form on its tangent bundle T_M. Even if it is not a Riemannian metric, there exists a natural notion of Levi-Civita connection ∇^g of g. We denote by $\mathcal{O}_M$ the sheaf of holomorphic functions, and by $\mathcal{T}_M$ its tangent sheaf.

Definition 1.1 (Frobenius manifold)**.** Let M be a complex manifold, with a non-degenerate symmetric bilinear $\mathbb{C}-$valued pairing g on the fibers of its holomorphic tangent bundle T_M. Assume that there exists a subsheaf $\mathcal{T}_M^f \subset \mathcal{T}_M$ of ∇^g-flat (thus commuting) vector fields such that $\mathcal{T}_M = \mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{T}_M^f$. M is called a pre-Frobenius manifold if on fibers of T_M it is defined an $\mathcal{O}_M$-bilinear symmetric commutative and associative multiplication $\circ$, i.e. a Frobenius algebra structure. It is called a *Frobenius manifold* if moreover the structure of Frobenius algebra is compatible with the pairing, in the sense that there exists everywhere a local function Φ, called *potential*, such that, for all vector fields $X, Y, Z \in T_M^f$

$$XYZ\Phi = g(X \circ Y, Z) = g(X, Y \circ Z). \tag{1.1.1}$$

The unit vector field for the multiplication is unique and it is usually denoted by e.

The two equalities in (1.1.1) are called respectively the *potentiality* and the *compatibility properties.*

Abusing notation, the pairing g is often referred to as a *metric*. The subsheaf $\mathcal{T}_M^f$ define an affine structure: the structure group induced by the reduction of $\mathcal{T}_M$ to $\mathcal{T}_M^f$ consists of linear affine transformations.

Notice that the tensor $A(X, Y, Z) := g(X \circ Y, Z)$ is totally symmetric. On the other hand, any totally symmetric three tensor $A : S^3(\mathcal{T}_M) \to \mathcal{O}_M$ determines a bilinear multiplication (and hence a pre-Frobenius structure), setting

$$X \circ Y = T \quad \overset{\text{DEF}}{\Longleftrightarrow} \quad A(X, Y, Z) = g(T, Z) \ \forall \ Z.$$

Symmetry of A gives the compatibility of g and $\circ$.

Denote by x_a a basis of local coordinates such that $\partial_a := \partial/\partial x_a$ is a local basis of flat vector fields. In terms of A

- the multiplication is explicitly given by

$$\partial_a \circ \partial_b = \sum_c A_{ab}^c \partial_c$$

  where $A_{abc} := A(\partial_a, \partial_b, \partial_c)$, $A_{ab}^c := \sum_e A_{abe} g^{ec}$, $(g^{ab}) := (g_{ab})^{-1}$;

- the potential satisfies $(XYZ)\Phi = A(X, Y, Z)$ for any flat X, Y, Z: in particular

$$\Phi_{abc} := \partial_a \partial_b \partial_c \Phi = A_{abc}.$$

  It is unique up to a polynomial in flat coordinates of degree less or equal than two;

- the *associativity* of $\circ$ is equivalent to the following (generically) highly non-linear system of PDE

$$\sum_{ef} \Phi_{abe} g^{ef} \Phi_{fcd} = \sum_{ef} \Phi_{bce} g^{ef} \Phi_{fad} \quad \forall \text{ indices } a, b, c, d. \tag{1.1.2}$$

Equations (1.1.2) are deduced from

$$(\partial_a \circ \partial_b) \circ \partial_c = \left(\sum_e A^e_{ab} \partial_e \right) \circ \partial_c = \sum_{ef} A^e_{ab} A^f_{ec} \partial_f$$
$$\partial_a \circ (\partial_b \circ \partial_c) = \partial_a \circ \left(\sum_e A^e_{bc} \partial_e \right) = \sum_{ef} A^e_{bc} A^f_{ae} \partial_f$$

expressing A_{abc} trough a potential. They are called *Associativity Equations* or WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations. In low dimension it is possible to solve the WDVV equations explicitly. In the next paragraph, they will be interpreted as the flatness condition for a one-parameter family of connections.

A Frobenius structure often comes equipped with an Euler field, that induces a grading of the tangent bundle.

Definition 1.2. An *Euler field* E is a non-zero vector field such that

$$\mathrm{Lie}_E(g) = Dg, \tag{1.1.3}$$
$$\mathrm{Lie}_E(\circ) = d(\circ) \tag{1.1.4}$$

for constants D, $d \in \mathbb{C}$. In other words

$$E(g(X,Y)) - g([E,X],Y) - g(X,[E,Y]) = Dg(X,Y)$$
$$[E, X \circ Y] - [E,X] \circ Y - X \circ [E,Y] = dX \circ Y$$

for any vector fields X, Y.

D is called the *conformal dimension.*

Clearly, any scalar multiple of an Euler field is also an Euler field. So, provided that $d \neq 0$, we can normalize E by requiring that $d = 1$.

Proposition 1.3 ([37, Chapt. 1, Sec. 2])**.** *a) In flat coordinates* $E = \sum_a E^a(x) \partial_a$*, where* $E^a(x)$ *are polynomials of degree less or equal than 1. b) A conformal vector field E (satisfying* (1.1.3)*) is Euler if and only if*

$$E\Phi = (D + d)\Phi + \textit{ quadratic terms in flat coordinates.}$$

c) The Euler field induces a grading of the tangent sheaf

$$\mathcal{T}^\bullet_M := \bigoplus_{r \in \mathbb{C}} \mathcal{T}_M(r), \quad \mathcal{T}_M(r) := \{ X \in \mathcal{T}_M | [E,X] = (r-d)X \} \, .$$

We call *spectrum* of E the set of its eigenvalues together with d and D. In particular if the identity e is flat, then $[e, E] = de$ and we will put $\partial_0 := \frac{\partial}{\partial x_0} := e$. The spectrum is reordered $(d_0 = d, d_1, \ldots, d_{n-1}, D)$. If e is not flat, the subscript 0 is not used and the spectrum is $(d, d_1, \ldots, d_n, D)$.

When e is flat and $g(e,e)=0$ the basis of flat vector fields can be normalized in such a way that the metric in flat coordinates is represented by the matrix

$$\begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix} \tag{1.1.5}$$

If $g(e,e)\neq 0$, then there exists a basis of flat vector fields such that

$$g=\begin{pmatrix} g_{00} & 0 & 0 & 0 & 0 \\ 0 & & & & 1 \\ \vdots & & & \iddots & \\ 0 & & 1 & & \\ 0 & 1 & & & \end{pmatrix}.$$

1.1.3 Structure connections

Denote by ∇^g or ∇_0 the Levi-Civita connection associated to the metric g of a Frobenius manifold (M,g,A). It can be deformed to a family of flat connections ∇_λ depending on a complex parameter $\lambda\neq 0$ defined as

$$\begin{aligned} &\nabla_\lambda : \mathcal{T}_M \to \Omega^1_M \otimes \mathcal{T}_M \\ &\nabla_{\lambda,X}Y = \nabla^g_X Y + \lambda X\circ Y \end{aligned} \tag{1.1.6}$$

Definition 1.4. The pencil of connections ∇_λ (1.1.6) is called the *structure connection* of (M,g,A).

Remark. In flat coordinates $\nabla_{0,\partial_a}(\partial_b)=0$ and $\nabla_{\lambda,\partial_a}(\partial_b)=\lambda\partial_a\circ\partial_b=\lambda\sum_e A_{ab}{}^e\partial_e=\lambda\partial_b\circ\partial_a=\nabla_{\lambda,\partial_b}(\partial_a)$.

Theorem 1.5 ([37, Chapt. 1, Theo. 1.5])**.** *Let (M,g,A) be a pre-Frobenius manifold. Then the curvature of ∇_λ vanishes identically in λ if and only if M is Frobenius, that is to associativity and potentiality conditions hold.*

Sketch of the proof. ∇_0 being the Levi-Civita of g, it is flat. Therefore the curvature of ∇_λ is $\nabla^2_\lambda=R_2\lambda^2+R_1\lambda$, for some coefficients R_2 and R_1. Clearly $R_{2,XY}(Z)=X\circ(Y\circ Z)-Y\circ(X\circ Z)=0$ if and only if the commutative multiplication is associative. One can also show that $R_1=0$ if and only if (M,g,A) is potential.In fact, in flat coordinates $R_{1,\partial_a\partial_b}(\partial_c)=\partial_a(\sum_e A_{bc}{}^e\partial_e)-\partial_b(\sum_e A_{ac}{}^e\partial_e)=0$ if and only if

$$\forall\ e \quad \partial_a A_{bce}=\partial_b A_{ace}. \tag{1.1.7}$$

If M is potential (1.1.7) is easily verified replacing A_{abc} with $\partial_a\partial_b\partial_c\Phi$ since the flat vector fields commute. On the other side, if one assumes (1.1.7), then for all c,e, the form $\sum_b \mathrm{d}\,x_b A_{bce}$ is closed and hence locally exact because of the Poincaré Lemma. Thus there exist local functions $B_{ce}=B_{ec}$ such that $A_{bce}=\partial_b B_{ce}=\partial_c B_{be}=A_{cbe}$ as A is symmetric. By the same argument, for all e, $\sum_c \mathrm{d}\,x_c B_{ce}$ is closed and locally $B_{ce}=\partial_c C_e$ and finally $C_e=\partial_e\Phi$. It follows that there exists a function Φ such that $\partial_b\partial_c\partial_e\Phi=A_{bce}$. □

If, in addition, M is endowed with an Euler field E with $d = 1$, one can define the *extended structure connection* $\overline{\nabla}$ on an auxiliary bundle. Consider the product of the variety M with the projective line with coordinate λ, $\widehat{M} := M \times \mathbb{P}^1_\lambda$. Call $p = p_1$ and p_2 the natural projection maps to M and $\mathbb{P}^1$, and consider the pull-back of the tangent bundle $p^*T_M \to \widehat{M}$. Abusing notation, let X denote also a vector field along the M-direction in $\mathcal{T}_{\widehat{M}}$. A system of flat vector fields X is completed to a basis for $\mathcal{T}_{\widehat{M}}$ by a new generator for the $\mathbb{P}^1$-direction. On the other hand, a basis of sections for p^*T_M consists of $\widehat{X} := p^*X$ for $X \in \mathcal{T}^f_M$.

Definition 1.6. The *extended structure connection* for M is the meromorphic connection $\overline{\nabla}$ on $p^*T_M \to \widehat{M} := M \times \mathbb{P}^1_\lambda$ defined by the following formulae for arbitrary $Y \in T_M$

$$\begin{aligned} \overline{\nabla}_X(p^*Y) &= p^*(\nabla_{0,X}(Y)) - \lambda p^*(X \circ Y), \\ \overline{\nabla}_{\partial/\partial\lambda}(p^*Y) &= p^*(\nabla_{0,E}(Y)) + p^*(E \circ Y) - \frac{1}{\lambda}p^*[E, Y]. \end{aligned} \tag{1.1.8}$$

It is an irregular connection with pole divisors at $M_0 := \{0\} \times M$ and $M_\infty := \{\infty\} \times M$ (see 1.2 for a brief introduction to meromorphic connections) and it gives an equivalent description of a Frobenius structure.

Theorem 1.7 ([37, Chapt. 1, Theo. 2.5.5])**.** *The extended structure connection $\overline{\nabla}$ of a pre-Frobenius manifold M is flat away from its poles if and only if M is Frobenius and E is an Euler field with constant $d = 1$.*

Proof. The proof follows via direct computations from Theorem 1.5. □

Extend the immersion $\{\lambda\} \to \mathbb{P}^1$ to $i_\lambda : \{\lambda\} \times M \to \mathbb{P}^1 \times M$. For any λ (including $\lambda = 0, \infty$) there is an isomorphism of bundles

$$i^*_\lambda(p^*T_M) \simeq (p^*T_M)_{|M} \simeq T_M$$

With this identification, the restriction of $\overline{\nabla}$ to M coincides with the first structure connection.

It is convenient to change the local coordinate on $\mathbb{P}^1$ by the automorphism $\lambda \mapsto \frac{1}{\lambda}$, in order to swap the positions of the poles. We call the new coordinate z and we write the extended structure connection as

$$\begin{aligned} \overline{\nabla}_X(Y) &= \nabla_{0,X}(Y) + \frac{1}{z}X \circ Y \\ \overline{\nabla}_{\partial/\partial z}(Y) &= -\left(\nabla_{0,E}(Y) + \frac{1}{z^2} \circ Y - \frac{1}{z}[E, Y]\right) \end{aligned} \tag{1.1.9}$$

From now on, we will refer equivalently either to (1.1.9) or (1.1.8) as extended structure connection.

Their restriction to $\mathbb{P}^1$ will play a special role in the case of semisimple structures defined below, which can be classified depending on the monodromy of a differential operator. This is briefly described in Section 1.4.

Theorem 1.7 above gives a characterization of a Frobenius structure on M that leads to change perspective and to describe it as a collection of objects that can be combined in a flat connection on the pull-back of the tangent bundle over $\mathbb{P}^1 \times M$. This description is named after K. Saito in [38]. I want to emphasize that the equivalence of these presentations is very natural: flatness of such meromorphic connection is exactly the same of solving the WDVV equations. The specificity of the tangent bundle is marked by the symmetry imposed on the one-form Θ appearing in Definition 1.8. At the same time, this point of view is most important for generalizations.

Definition 1.8. A *Saito structure endowed with a metric* on M consists of

a) a symmetric non-degenerate $\mathcal{O}_M$-bilinear form g on T_M, with its Levi-Civita connection ∇.

b) a 1-form Θ with values in $\mathrm{End}(T_M)$, symmetric if considered as a bilinear map $\mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{T}_M \to \mathcal{T}_M$

c) two global vector fields e and $\mathcal{E}$ of $\mathcal{T}_M$,

subject to the following conditions

1. e is flat ($\nabla e = 0$) and $\Theta(e) = -\mathrm{Id}$;
2. the meromorphic connection $\widetilde{\nabla}$ on the vector bundle p^*T_M over $\mathbb{A}^1_z \times M$

$$\widetilde{\nabla} = \pi^*\nabla + \frac{\pi^*\Theta}{z} - \left(\frac{\Theta(\mathcal{E})}{z} + \nabla\mathcal{E}\right)\frac{\mathrm{d}\,z}{z}$$

is flat.

Denote by * the adjoint for g, we also require

3. $\Theta^* = \Theta$, that is $\forall X, Y, Z \in \mathcal{T}_M$ one has $g(\Theta(X)(Y), Z) = g(X, \Theta(Y)(Z))$
4. there exists a rational number q and an integer ω such that $\big(\nabla(\mathcal{E}) - q\,\mathrm{Id}\,\big)^* + \big(\nabla(\mathcal{E}) - q\,\mathrm{Id}\,\big) = -\omega\,\mathrm{Id}$.

The next Proposition and Theorem show that a Saito structure is equivalent to a Frobenius structure with Euler field and rational conformal dimension. In particular the meromorphic connection $\widetilde{\nabla}$ coincides with the extended structure connection of the corresponding Frobenius structure.

Proposition 1.9 ([38])**.** *Given a Saito structure $(g, \Theta, e, \mathcal{E})$ on a complex manifold M, it is possible to define a Frobenius structure with an Euler field $E = \mathcal{E}$.*

Sketch of the proof. One define an $\mathcal{O}_M$-bilinear product $\circ$ on the tangent bundle T_M by the formula $X \circ Y = -\Theta(X)(Y)$. The symmetry of the Higgs field Θ implies that the product is commutative. From flatness of $\widetilde{\nabla}$: $\Theta \wedge \Theta = 0$, $\nabla\Theta = 0$ and $\nabla(-\Theta(\mathcal{E})) = 0$. $\Theta \wedge \Theta = 0$ is equivalent to associativity of $\circ$: chose local coordinates x_i and the frame $\partial_i = \partial/\partial x_i$, then $\partial_i \circ (\partial_j \circ \partial_k) = \Theta(\partial_i)(\Theta(\partial_j)(\partial_k))$ and $(\partial_i \circ \partial_j) \circ \partial_k = \Theta(\partial_k)(\Theta(\partial_i)(\partial_j))$ and their difference is $\big((\Theta \wedge \Theta)(\partial_i, \partial_j)\big)(\partial_k) = 0$.

Finally one shows that $\mathcal{E}$ is an Euler field. □

Theorem 1.10 ([38])**.** *There is a bijective correspondence between Saito structures with metric and Frobenius structures endowed with flat identity and Euler field.*

Proof. One implication is proved in Proposition 1.9 above. To define a Saito structure on a Frobenius manifold, define a 1-form $\Theta : \mathcal{T}_M \to \mathcal{T}_M \otimes_{\mathcal{O}_M} \Omega^1_M$ by $\Theta(X)(Y) = -X \circ Y$ for any pair of holomorphic vector fields X and Y. Symmetry of Θ follows from commutativity of the multiplication on T_M. Put $\nabla = \nabla_0$, $\mathcal{E} = E$, and e equal to the identity. The extended structure connection is flat and can be rewritten as

$$\begin{aligned}
\overline{\nabla}_X(p^*Y) &= p^*(\nabla_{0,X}(Y) + \frac{1}{z}\Theta(X)(Y)) \\
\overline{\nabla}_{\partial/\partial z}(p^*Y) &= p^*\left(\frac{1}{z}\nabla_{0,E}(Y) + \frac{1}{z^2}E \circ Y - \frac{1}{z}[E, Y]\right) \\
&= -p^*\left(\frac{1}{z}\nabla_{0,Y}(E) + \frac{1}{z^2}E \circ Y\right) = -p^*\left(\frac{\Theta(\mathcal{E})}{z} + \nabla\mathcal{E}\right)\frac{\mathrm{d}\, z}{z}
\end{aligned}$$

It is enough to restrict it to the affine chart $\mathbb{A}^1 \times M$ centerd at 0.

Property 3. is the compatibility of the multiplication. Let D be the conformal dimension. To prove property 4., we evaluate the expression $\nabla(\mathcal{E}) - \frac{D}{2}\,\mathrm{Id}$ on a flat vector field X. By (1.1.3), the endomorphism $(\nabla(\mathcal{E}) - \frac{D}{2}\,\mathrm{Id})(X) = [X, E] - \frac{D}{2}X$ is skew-symmetric. Then for any integer ω, one has that $(\nabla(\mathcal{E}) - \frac{D+\omega}{2}\,\mathrm{Id})^* + (\nabla(\mathcal{E}) - \frac{D+\omega}{2}\,\mathrm{Id}) = -\omega\,\mathrm{Id}$. ☐

1.1.4 Semisimple Frobenius manifolds

A Frobenius structure is *semisimple* if there is a local isomorphism of sheaves

$$(\mathcal{T}_M, \circ) \simeq (\mathcal{O}^n_M, \text{componentwise multiplication}).$$

A handier definition is the following.

Definition 1.11. A Frobenius manifold M is called semisimple if there exists a local basis of vector fields $\{e_1, \ldots, e_n\}$ such that $e_i \circ e_j = \delta_{ij} e_j$. Such a basis of idempotents is well-defined up to renumbering.

It imples that the structure group of the tangent bundle T_M can be reduced to S_n. In general this reduction is not compatible with that induced by $GL(n)$, for which $\mathcal{T}_M = \mathcal{O}_M \otimes \mathcal{T}^f_M$. This is equivalent to saying that e_i, $i = 1, \ldots, n$, are in general not flat.

We set $e_i := \frac{\partial}{\partial u_i}$, $i = 1, \ldots, n$, for a suitable system of local coordinates u_i over M, called *canonical coordinates.*

In presence of a semisimple structure, the potentiality condition and flatness of g can be encoded in a function η, called the *metric potential.*

Theorem 1.12 ([37, Chapt. 1, Theo. 3.3])**.** *Let M be a complex manifold. Suppose we have*

a) a reduction of the structure group of T_M to S_n, specified by a choice of local bases $\{e_1, \ldots, e_n\}$ and dual bases $\{\nu_1, \ldots, \nu_n\}$,

b) a flat symmetric bilinear diagonal pairing $g = \sum \eta_i \nu_i^{\otimes 2}$,

c) a diagonal totally symmetric cubic tensor $A = \sum_i \eta_i \nu_i^{\otimes 3}$.

Then, M is a complex semisimple Frobenius manifold if and only if

1) $[e_i, e_j] = 0$, or equivalently $e_i = \frac{\partial}{\partial u_i}$ for a local coordinate system $u_1, \ldots, u_n$ defined up to renumbering or constant shifts, *2) $\eta_i = e_i\eta$ for a local function η defined up to addition of a constant.*

In the presence of a flat identity e, the metric potential is described by a formula

$$\eta = \sum_a x_a g(\partial_a, e) + \ const.$$

It is worth noticing that the associativity equations (1.1.2) of a semisimple Frobenius structure, are automatically satisfied when expressed in canonical coordinates. Moreover, canonical coordinates can be always renormalized in such a way that

$$E = d\sum_i u_i e_i.$$

We introduce the operator $\mathcal{U}$ and the $\mathcal{O}_M$-linear skew-symmetric operator $\mathcal{V} : T_M \to T_M$ defined by

$$\begin{aligned} \mathcal{U}(X) &:= E \circ X \\ \mathcal{V}(X) &:= \nabla_{0,X}(E) - \frac{D}{2}X \end{aligned} \tag{1.1.10}$$

In canonical coordinates, $\mathcal{U}$ acts diagonally: $\mathcal{U}(e_i) = u_i e_i$. Moreover M is semisimple if and only if $\mathcal{U}$ has distinct eigenvalues [24]. The endomorphism $\mathcal{V}$ is represented in the basis $\{e_i\}$ by a matrix with entries

$$\mathcal{V}_{ij} = \sum_{j\neq i}(u_j - u_i)\frac{(e_i e_j \eta)}{2\sqrt{(e_i\eta)(e_j\eta)}}.$$

Canonical coordinates define a local diffeomorphism sending a point m to the eigenvalues of $\mathcal{U}$ evaluated at that point up to permutation

$$\begin{aligned} &M \simeq \frac{\mathbb{C}^n \setminus diagonals}{S^n} \\ &m \mapsto [u_1(m), \dots, u_n(m)]_{S^n} = [eig(\mathcal{U})]_{S^n}. \end{aligned}$$

Equations (1.1.9) can be rewritten as

$$\begin{aligned} \overline{\nabla}_X(Y) &= \nabla^g_X(Y) + \frac{1}{z}X \circ Y \\ \overline{\nabla}_{\partial/\partial\lambda}(Y) &= -\left[\frac{1}{z^2}\mathcal{U} + \frac{1}{z}(\mathcal{V} + \frac{D}{2}\,\mathrm{Id})\right](Y) \end{aligned}$$

and the extended structure connection takes the form

$$\overline{\nabla} = \mathrm{d}_{\mathbb{P}^1} + \nabla_z + \frac{1}{z^2}\mathcal{U}\,\mathrm{d}\,z + \frac{1}{z}(\mathcal{V} + \frac{D}{2}\,\mathrm{Id})\,\mathrm{d}\,z$$

1.2 Isomonodromy of meromorphic connections

In this section we briefly recall some basic theory about the generalized monodromy of meromorphic connections over the projective line and their deformation theory. The goal is also to fix the notation.

Let Z be a complex manifold and F a vector bundle on it. We denote by Ω^1_Z the sheaf of holomorphic one-forms on Z. Meromorphic one-forms with pole divisor D are denoted $\Omega^1_Z(D)$. The sheaf of one-forms valued in a vector space V is $\Omega^1(V)$. If F is a holomorphic vector bundle, we indicate with $\mathcal{O}(F)$ the sheaf of its holomorphic sections: it is a locally free $\mathcal{O}_Z$-module.

Let D be a divisor in Z. A *flat meromorphic connection with pole divisor* D on a vector bundle F over Z is a connection $\nabla : \mathcal{O}(F) \to \mathcal{O}(F) \otimes_{\mathcal{O}_Z} \Omega^1_Z(D)$, which is flat away from the support $\mathrm{supp}(D)$ of D.

Say $(z_0, \dots, z_N)$ are local coordinates on Z such that locally $\mathrm{supp}(D) = \{z_0 = 0\}$. The order or *rank* of singularity along the divisor D is $r+1$ if locally the connection matrix A of $\nabla = \mathrm{d} - A \, \mathrm{d}\, \mathbf{z}$ can be written as

$$A = A^0(z_0, \dots, z_N)\frac{\mathrm{d}\, z_0}{z_0^{r+1}} + \sum_{i=1}^{N} A^i(z_0, \dots, z_N)\frac{\mathrm{d}\, z_i}{z_0^r}, \quad A^0 \, \mathrm{d}\, z_0, A^i \, \mathrm{d}\, z_i \in \Omega^1_Z.$$

When $Z = \mathbb{CP}^1_z$, the pole order at ∞ is $r-1$ if the local expression of A around ∞ is $A = -z^{r-1} A^\infty \, \mathrm{d}\, z$. r is called the Poincaré rank. The rank of a polar divisor does not depend on the chosen frame. We have a *logarithmic pole* or *regular singularity* when $r = 0$. If $r \geq 1$ the singularity is said *irregular*. Accordingly, meromorphic connections and, in general, linear systems of ODE, are said regular or irregular as well.

The *principal part* of a connection on a pole of any rank is $A^0(0, z_1, \dots, z_N)$. If $r \geq 1$, it depends on the choice of local coordinates, and it is multiplied by invertible local functions on D when coordinates change. Hence, its spectrum is well defined globally on D only for logarithmic singularities, but the possible semiplicity of the spectrum makes sense for any r.

In the following, we will be interested in the monodromy and the fundamental solutions of a (irregular) meromorphic connections on a holomorphically trivial vector bundle over $\mathbb{P}^1 := \mathbb{CP}^1$.

When singularities are at most regular, the monodromy group of $\mathbb{P}^1$ with poles removed describes analytic continuations of local fundamental solutions. For any point z^* not in the singular locus, there exists a neighborhood $\mathcal{U}_{z^*}$ of z^* and a *fundamental solution* of ∇, that is a holomorphic solution Y in $\mathcal{U}_{z^*}$ which solves the system of differential equations $\mathrm{d}\, Y(z) = A(z)Y(z)$. A fundamental solution can be analytically continued along paths not containing any pole. Two analytic continuations differ by the action of the monodromy group. The monodromy representation is a map

$$\pi_1^{z^*} := \pi_1\Big(\mathbb{P}^1(\mathbb{C}) \setminus \{poles\}, z^*\Big) \to GL_n(\mathbb{C})$$

It depends by conjugation on the choice of the base point z^* and of a basis of $\pi_1^{z^*}$. Given two analytic continuations $\widetilde{Y}$ and $\widetilde{Y}'$, defined along γ and $\gamma' = \gamma \cdot \delta$ respectively, then $\widetilde{Y}' = \widetilde{Y} M_\delta$, where M_δ is the monodromy matrix of the class of δ in $\pi_1^{z^*}$.

When the system $\mathrm{d}\, Y(z) = A(z)Y(z)\, \mathrm{d}\, z$ has irregular singularities, the situation is much more involved and the monodromy is described in terms of *generalized monodromy data*, consisting of Stokes rays and Stokes multipliers.

Let a be a pole of Poicaré rank $r \geq 1$. It is a standard result that there exist sectors of the disc Δ_a punctured in a, where fundamental solutions with prescribed asymptotic behavior as $z \to a$ are well-defined. Expressing meromorphic functions as power series centered in a, we can express A around the pole a as $A(z) = \sum_{k=0}^r A_k(z)\frac{\mathrm{d}\, z}{(z-a)^{r+1-k}}$. Assume that the principal part, which coincides with A_0, is diagonal with n distinct ordered eigenvalues $u_i = e^{i\pi\phi(u_i)}$, counterclock-wise ordered according to their phase $\phi(u_1) < \dots < \phi(u_n)$. The *Stokes rays* are defined as the rays $\ell_{ij} := \mathbb{R}_{>0}(u_i - u_j)$. Distinct Stokes rays can be labelled counterclock-wise ℓ_1, $\ell_2, \ldots$ such that $\ell_h = \mathbb{R}_{>0} \exp\{i\pi\phi_h\}$ and $\phi_h < \phi_{h+1}$ with respect to the positive real ray. Then

Theorem 1.13 ([1, Theorem A]; [40]). *There are $2r$ open regions $\mathcal{R}_j$ given by*

$$\mathcal{R}_j = \left\{ z \in \Delta_a | \frac{(j-1)\pi}{r} < \arg z < \frac{j\pi}{r} + \epsilon \right\}, \quad j = 1, \ldots 2r, \epsilon > 0 \textit{ small},$$

and unique fundamental solutions $Y_j(z)$ in $\mathcal{R}_j$ with a prescribed asymptotic expansion $\hat{Y}$ as $z \to a$ in $\mathcal{R}_j$. This is given by $\hat{Y} = z^{A_r} \exp\{T(z)\}$, where

$$T(z) = -\left(\frac{A_0}{r(z-a)^r} + \frac{A_1}{(r-1)(z-a)^{r-1}} + \cdots + \frac{A_{r-1}}{(z-a)} \right).$$

The definition domain of Y_j can be enlarged to $\overline{\mathcal{R}}_j$ up to the nearest Stokes rays, not included. The solutions are still unique. Moreover, there exist matrices $\mathcal{S}_j$ such that in the overlapping $\overline{\mathcal{R}}_j \cap \overline{\mathcal{R}}_{j+1}$, $j = 1, \ldots, 2r-1$, we have $Y_{j+1}(z) = Y_j(z)\mathcal{S}_j$, and in the last intersection $\overline{\mathcal{R}}_{2r} \cap \overline{\mathcal{R}}_1$ $Y_1(z) = Y_{2r}(z)\mathcal{S}_{2r}$.

The matrices $\mathcal{S}_j$ are called *Stokes multipliers* at the point a.

Remark. If A_0 is diagonalizable, but not in the form as above, then $\hat{Y} = W(z) z^{A_r} \exp T(z)$, where W is such that $A_0' = W^{-1}(z) A_0(z) W(z)$ is diagonal with n distinct ordered eigenvalues, and the Stokes multipliers change by conjugation.

This finishes the description of the local generalized monodromy around irregular singularities. There is also a notion of global monodromy which generalizes the representation of π_1. We don't go into details; a reference for its description is [24, Section 1.6.1].

1.2.1 Isomonodromic deformations

The aim of the deformation theory of connections is to describe a family of differential equations with coefficients $A(z) = A(z, x)$ depending on parameters $x = (x_1, \ldots, x_N) \in X$, that share the same (generalized) monodromy data irrespective to x.

Definition 1.14. A family of meromorphic connections $\{\nabla(x) = \mathrm{d}_{\mathbb{P}^1} - A(x)(z)\}_{x \in X}$ over $\mathbb{P}^1(\mathbb{C})$ parameterized by a space X of deformation parameters is said *isomonodromic* if for any x^0 there exists a neighborhood $\mathcal{U}_0 \subset X$ of x^0 such that all the matrix-valued one-forms $A(x)$, $x \in \mathcal{U}_0$, share the same Stokes multipliers defined in common sectors, and the global monodromy representation is constant.

We present here the key result by Jimbo, Miwa and Ueno [29] (1980) relating isomonodromic families of connections over $\mathbb{P}^1$ and flat meromorphic connections over families of $\mathbb{P}^1$'s. It essentially says that a family $\mathrm{d} - A(x)$ of connection on $\mathbb{P}^1$ parametrized by $x \in X$ is isomonodromic if and only if its solutions Y also satisfy $\mathrm{d}Y = \Omega(z, x) Y$, for a one-form Ω on X valued in $GL_n(\mathbb{C})$, uniquely determined by $A(z, x)$. This is a flatness condition for a connection $\overline{\nabla} = \mathrm{d}_{\mathbb{P}^1 \times X} - (A(z, x) + \Omega(z, x))$ over $\mathbb{P}^1 \times X$.

Theorem 1.15 ([29, Theo. 3.1]). *Let X be a simply connected space of deformation parameters, $D = \sum_i (r_i + 1)\{a_i\} \times X$ a divisor, for a finite set of points $a_i \in \mathbb{P}^1$. Suppose we have a flat meromorphic connection $\overline{\nabla}$ on the trivial bundle F of rank n over $\mathbb{P}^1 \times X$, with poles and their ranks specified by D. Then the restricted bundles $F_{|\mathbb{P}^1 \times \{x\}}$ are isomorphic* *for any $x \in X$ and the set* $\left\{\nabla(x) := \overline{\nabla}_{|\mathbb{P}^1\times\{x\}}\right\}_x$ *is a family of isomonodromic meromorphic connections over* $\mathbb{P}^1$.

Conversely, given a family of isomonodromic meromorphic connections

$$\nabla(x) = \mathrm{d}_{\mathbb{P}^1} - A(x)(z),$$

varying on a space of parameters X*, then for any* $x^0 \in X$ *the pull back of* $\nabla(x^0)$ *over* $\mathbb{P}^1 \times X$ *is the restriction to* $\mathbb{P}^1 \times \{x^0\}$ *of a flat meromorphic connection* $\overline{\nabla} = \mathrm{d}_{\mathbb{P}^1\times X} - \widetilde{A}$*, whose one-forms matrix* $\widetilde{A}$ *depends only on* $A(z,x)$.

$\overline{\nabla}$ is sometimes referred to as "full connection". $\widetilde{A}(z,x)$ has the form $A(z,x)+\Omega(z,x)$, for A regarded as a two variable one-form, and

$$\Omega = (\mathrm{d}_X Y(z,x))Y(z,x)^{-1}$$

where Y is a fundamental solution of $\mathrm{d} - A(z,x)$ on $\mathbb{P}^1$ regarded as a matrix valued holomorphic function of the parameter x. Flatness of $\overline{\nabla}$ translates into two equations

$$\begin{cases} \mathrm{d}_X \Omega = -\Omega \wedge \Omega \\ \mathrm{d}_X A = -\mathrm{d}_{\mathbb{P}^1} \Omega - [A, \Omega] \end{cases}$$

called the *deformation equations*. A detailed proof of the result is in [6, Chapter 6].

1.3 Monodromy of $\mathrm{d} - \left(\frac{U}{z^2} + \frac{V}{z}\right)\mathrm{d}\, z$

Let us describe the monodromy of the differential operator $\mathrm{d} - \left(\frac{U}{z^2} + \frac{V}{z}\right)\mathrm{d}\, z$ in more detail. It has a order 2 pole at 0 and a logarithmic pole at ∞. The pole divisor is $2\cdot 0 + 1\cdot\infty$. In this special case the global monodromy representation is completely determined by the local monodromy around the double pole, thus we study only the irregular singularity. Assume $U \in GL_n(\mathbb{C})$ is a constant diagonal matrix with distinct eigenvalues,

$$U = \begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{pmatrix}$$

The Stokes rays are $\ell_{ij} = \mathbb{R}_{>0}(u_i - u_j)$, $i \neq j$. We say that a ray $l \subset \mathbb{C}$ is admissible if it is not a Stokes ray. For any admissible ray l, we denote by $\mathbb{H}_l$ the corresponding upper-half plane

$$\mathbb{H}_l = \{z = vw | v \in l, Re(w) > 0\} \subset \mathbb{C}^*.$$

According to Theorem 1.13, for any admissible ray l emanating from the origin, there exists a unique solution Y_l in $\mathbb{H}_l$ such that $Y_l \cdot \exp\{\frac{1}{z}U\} \to \mathrm{Id}$ as $z \to 0$ in $\mathbb{H}_l$, [11, Theorem 2.5]. It can be continued in some sectorial neighborhood of $\mathbb{H}_l$ clock-wise and counter-clock-wise to $Y_{l,+}$ and $Y_{l,-}$ respectively. On the other hand we have a similar solution Y_{-l} in $\mathbb{H}_{-l}$. In the intersections of the neighborhoods with $\mathbb{H}_{-l}$, the *Stokes multipliers* are defined by

$$Y_{l,\pm}(z) = Y_{-l}(z) \cdot \mathcal{S}_\pm.$$

We introduce an ordered product. Say $\phi_1 < \cdots < \phi_m$ are angles in the complex plane and S_{ϕ_i} are matrices labelled by ϕ_i, then the ordered product $\overset{\curvearrowright}{\prod}$ is defined as $\overset{\curvearrowright}{\prod} S_{\phi_i} := S_{\phi_m}\cdots S_{\phi_1}$.

Proposition 1.16 ([11, Section 2]; [15, Appendix F])**.** *$\mathcal{S}_+$ remains constant under a perturbation of l so long as l or $-l$ does not intersect any Stokes rays. There exist invertible matrices $\mathcal{S}_{\ell_{ij}}$ such that, if ℓ_{ij} crosses clock-wise l then*

$$\mathcal{S}_+ \mapsto \mathcal{S}_{\ell_{ij}}\mathcal{S}_+\mathcal{S}^{-1}_{-\ell_{ij}} \quad \textit{and} \quad Y_l \mapsto Y_l \mathcal{S}^{-1}_{\ell_{ij}}.$$

A similar result holds for $\mathcal{S}_-$. $\mathcal{S}_{\ell_{ij}}$ are called Stokes factors *of the operator ∇.*

Moreover, the Stokes factors determine the Stokes multipliers for any ray l

$$\mathcal{S}_+ = \prod_{\ell_{ij}\subset i\mathbb{H}_l}^{\curvearrowright} \mathcal{S}_{\ell_{ij}}, \quad \mathcal{S}_- = \prod_{\ell_{ji}\subset i\mathbb{H}_{-l}}^{\curvearrowright} \mathcal{S}^{-1}_{\ell_{ji}}.$$

In fact, a stronger result holds: the Stokes multipliers of a single admissible ray l determine all the Stokes factors, [10, Lemma 2.10]. Therefore, the generalised monodromy data around the irregular singularity may be equivalently defined as the set of Stokes rays and Stokes factors.

A special characterization holds when V is skew-symmetric.

Proposition 1.17 ([15, Prop. 3.10])**.** *If V is skew-symmetric, the Stokes multipliers $\mathcal{S}_\pm$ are such that $\mathcal{S}_+ = (\mathcal{S}_-)^t =: \mathcal{S}$ and the Stokes factors satisfy $\mathcal{S}_{\ell_{ji}} = \mathcal{S}^{-t}_{\ell_{ij}}$.*

$\mathcal{S} = (s_{ij})_{ij}$ decomposes as

$$\mathcal{S} = \mathrm{Id} + \sum_{\ell_{ij}\subset i\mathbb{H}_l} s_{ij}E_{ij},$$

where E_{ij} are the matrices whose entries (hk) are $\delta_{hi}\delta_{kj}$.

A natural choice for l is the positive real ray. In this case $\mathcal{S}$ is given by the clockwise ordered product

$$\mathcal{S} = \prod_{\ell\subset\mathbb{H}}^{\curvearrowright} \mathcal{S}_\ell.$$

of Stokes matrices over Stokes rays contained in the open upper-half plane $\mathbb{H} := \{z \in \mathbb{C} \mid 0 < \arg z < \pi\}$. The generalized monodromy is therefore encoded in $(\mathcal{S}, \{\ell_{ij}\})$ and one may call $\mathcal{S}$ the *Stokes matrix.*

Notice that the Stokes factors of two gauge-equivalent connections are the same up to conjugation by the gauge matrix. The same thus holds also for Stokes multipliers. Whenever it is possible, we will define the Stokes matrix of a connection with poles $2\cdot 0 + 1\cdot\infty$ and diagonal principal part on 0, as the Stokes multiplier $\mathcal{S}_+$ referred to $\mathbb{H}$ of the gauge-equivalent connection which has diagonal principal part with distinct eigenvalues and skew-symmetric residue.

Since by assumptions the global monodromy is determined by the Stokes data around the origin, isomonodromic families of connections of the form $\nabla(x) = \mathrm{d} - \left(\frac{U}{z^2} + \frac{V}{z}\right)\mathrm{d}\,z$ are defined as follows.

Definition 1.18 ([10, Def. 2.11])**.** The family of connections $\nabla(x)$ parametrized by a space of deformation parameters X is isomonodromic if, for any $x^0 \in X$ there exists an open neighborhood $\mathcal{U}_0 \subset X$ of x^0 and a ray r such that $\pm r$ is admissible for all $\nabla(x)$, $x \in \mathcal{U}_0$, and the Stokes multipliers $\mathcal{S}_\pm(x)$ of $\nabla(x)$ relative to r are constant on $\mathcal{U}_0$.

Let $\Sigma \subset \mathbb{C}^*$ be a convex open sector whose boundary rays are admissible for all $x \in \mathcal{U}_0$.

Proposition 1.19 ([10, Section 2.11]). *The isomonodromy may also be defined as the constancy of the clock-wise ordered product*

$$\prod_{\ell\subset\Sigma}^{\curvearrowright} \mathcal{S}_\ell(x)$$

as x varies in $\mathcal{U}_0$.

The deformation equations consist in the system of differential equations

$$\mathrm{d}\, V_{ij} = -\sum_k V_{ik}V_{kj}\big(\mathrm{d}\log(u_i-u_k) - \mathrm{d}\log(u_k-u_j)\big).$$

1.3.1 Bridgeland and Toledano-Laredo formulae for the Stokes map

The monodromy of the connection $\nabla = \mathrm{d} - \left(\frac{U}{z^2}+\frac{V}{z}\right)\mathrm{d}\, z$ can be expressed in terms of multilogarithms. The explicit formula was proved by Bridgeland and Toledano-Laredo in [11] for a connection with structure group $GL_n(\mathbb{C})$ or an arbitrary complex affine algebraic group G. We consider here the the first case (Theorem 1.22).

Let $G = GL_n(\mathbb{C})$, P be the holomorphically trivial, principal G-bundle on $\mathbb{P}^1_z$ and ∇ a connection of the form

$$\nabla = \mathrm{d} - \Big(\frac{U}{z^2}+\frac{V}{z}\Big)\,\mathrm{d}\, z, \tag{1.3.1}$$

where $U, V \in \mathfrak{gl}_n(\mathbb{C})$, U is diagonalizable with distinct eigenvalues $u_1,\dots,u_n$, and V is skew-symmetric.

Introduce the following iterated integrals.

Definition 1.20. Let $\omega_1,\dots,\omega_k$ be one-forms defined on a domain $\mathcal{U}_0\subset\mathbb{C}$, and $\delta : [0,1]\to\mathcal{U}_0$ a path in $\mathcal{U}_0$. Let

$$\Delta = \{(t_1,\dots,t_n)\in[0,1]^n : 0\le t_i\le\cdots\le t_n\le 1\}\subset[0,1]^n$$

be the unit simplex. Define

$$\int_\delta \omega_1\circ\cdots\circ\omega_k := \int_\Delta f_1(t_1)\cdots f_k(t_k)\,\mathrm{d}\, t_1\cdots\mathrm{d}\, t_k$$

where $\delta^*\omega_i = f_i(t)\,\mathrm{d}\, t$.

Definition 1.21. Set $M_1(z_1) = 2\pi i$ and, for $k\ge 2$, define the function $M_k : (\mathbb{C}^*)^k\to\mathbb{C}$ by

$$M_k(z_1,\dots,z_k) = (-1)^{k-1}2\pi i\int_{\mathcal{C}}\frac{\mathrm{d}\, t}{t-s_1}\circ\cdots\circ\frac{\mathrm{d}\, t}{t-s_{k-1}} \tag{1.3.2}$$

where $s_i = z_1+\cdots+z_i$, $1\le i\le k$ and the path of integration $\mathcal{C}$ is the line segment $]0,s_k[$, perturbed if necessary to avoid any point $s_i\in[0,s_k]$ by small counterclockwise arcs.

We choose the branch of the complex logarithm with branch-cut along $[0,\infty[$. Therefore, for instance

$$M_2(z_1,z_2) = -2\pi i\Big(\log\frac{z_2}{z_1}-\pi i\Big).$$

In general, the multilogarithms M_n are hard to compute already for $n=3$.

Introduce the set of roots $\Phi^U = \{u_i-u_j, i\ne j\}\subset\mathbb{C}$, that is the set of eigenvalues of $\operatorname{ad} U$. If $\gamma = u_i-u_j$, V_γ stands for the entry V_{ij}. Call E_{ij} the elementary matrices with only one non-zero entry, in position (i,j).

Theorem 1.22 ([11, Section 9])**.** *The Stokes factor $\mathcal{S}_\ell$ of* (1.3.1) *corresponding to a Stokes ray ℓ is given by*

$$\mathcal{S}_\ell = \mathrm{Id} + \sum_{\substack{\gamma=u_i-u_j\\ \gamma\in\ell}} \sum_{k\geq 1} \sum_{\substack{\gamma_1,\dots,\gamma_k\in\Phi^U\\ \gamma_1+\dots+\gamma_k=\gamma}} M_k(\gamma_1,\dots,\gamma_k) V_{\gamma_1}\cdots V_{\gamma_k} E_{ij}. \tag{1.3.3}$$

The result extends to the case when U has repeated eigenvalues or G is an arbitrary complex algebraic group, [11]. Moreover, it is possible to invert the formula 1.3.3

Proposition 1.23 ([11, Theorem 4.8])**.** *Provided the convergence of the resulting formula,* (1.3.3) *can be inverted, and one can reconstruct canonically V in terms of the components of the Stokes factors and the multilogarithms.*

1.4 Stokes data of a semisimple Frobenius manifold

One of the main feature of semisimple Frobenius manifolds is that the local moduli space of semisimple Frobenius structures can be identify with a space of Stokes matrices. This gives a characterization of Frobenius structures on manifolds of given dimension. The result is due to Dubrovin and is proven in [15, Section 3].

Consider the differential operator

$$\nabla = \mathrm{d} - \Big(\frac{U}{z^2} + \frac{V}{z}\Big)\,\mathrm{d}\,z \tag{1.4.1}$$

with diagonal U and skew-symmetric V. On the space of upper triangular matrices, consider the equivalence given by

$$S \mapsto P^{-1}\mathcal{I}S\mathcal{I}P \tag{1.4.2}$$

where $P \in \Sigma^n$ is a permutation matrix, $\mathcal{I}$ an arbitrary diagonal matrix with ± 1 diagonal entries.

Let M be a Frobenius manifold with flat coordinates x_a and potential Φ. Define the multiplication-preserving transformation S_k, $k = 1,\dots,n$, $x_a \mapsto \hat{x}_a$

$$\hat{x}_a = \partial_a\partial_k\Phi(x_1,\dots,x_n). \tag{1.4.3}$$

The transformations S_k preserve the spectrum of the structure up to permutation. Moreover $\frac{\hat{\partial}^2\hat{\Phi}}{\hat{\partial}_a\hat{\partial}_b} = \frac{\partial^2\Phi}{\partial_a\partial_b}$ and $\hat{\eta}_{ab} = \eta_{ab}$.

Theorem 1.24 ([15, Theo. 3.2])**.** *There exists a local one-to-one correspondence between*

$$\{\textit{semisimple Frobenius structures modulo transformations } (1.4.3)\}$$
$$\textit{and}$$
$$\{\textit{Stokes matrices } \mathcal{S} \textit{ of connections } (1.4.1) \textit{ up to equivalence } (1.4.2)\}.$$

Without going into the detail of the construction of the correspondence, let us try to describe the picture.

Let $\{\nabla(x)\}_{x\in X}$ a family of connections of type (1.4.1) with prescribed generalized monodromy. On the space of deformation parameters X there exists a natural Frobenius structure. This structure is determined uniquely up to the symmetry (1.4.2). Conversely canonical coordinates of a Frobenius structure on a manifold M define a local diffeomorphism between the locus of semisimple points $\mathcal{M}$ in M and the space of parameters

$(\mathbb{C}^n \setminus \{diagonals\})/\Sigma^n$ via the map

$$\mathcal{M} \simeq \frac{\mathbb{C}^n \setminus \{diagonals\}}{\Sigma^n}$$
$$m \mapsto [u_1(m), \ldots, u_n(m)]_{\Sigma^n} = [eig(\mathcal{U})]_{\Sigma^n}.$$

sending a point m to the $n-$tuple, up to permutations, of eigenvalues of the endomorphism $\mathcal{U}$ (Eq. (1.1.10)) of the operator $\mathrm{d} - \left(\frac{\mathcal{U}}{z^2} + \frac{1}{z}(\mathcal{V} + \frac{D}{2}\,\mathrm{Id})\right)\mathrm{d}\,z$ on the tangent bundle, evaluated at that point m. Therefore, the space of deformation parameters X is (locally) identified with an open set in a Frobenius manifold M. In particular the rank of the bundle on which ∇ lives equals the dimension of the manifold M.

At the same time, Theorem 1.15 by Jimbo, Miwa and Ueno suggests that the extended structure connections of the structure on M define a special family of connections over $\mathbb{P}^1$ parametrized by M. Dubrovin proved that the WDVV equations are indeed equivalent to the isomonodromic deformation equations of the operator (1.4.1).

Definition 1.25. The Stokes matrix $\mathcal{S}$ of the operator (1.4.1) (considered modulo transformations (1.4.2)) is called the *Stokes matrix of the Frobenius manifold.*

Having assumed that the endomorphism V is skew-symmetric, the moduli space of semisimple Frobenius manifolds is identified with the corresponding space of upper triangular Stokes matrices of the connection (1.4.1). The local moduli space of semisimple Frobenius manifold has dimension $n(n-1)/2$.

1.5 Analytic continuation of Frobenius structures

We will now define and briefly describe the analytic continuation of a Frobenius manifold. The correspondence of Theorem 1.24 is local in the sense that it refers to germs of semisimple Frobenius structures defined in a small neighborhood $\mathcal{U}_0$ of $u^{(0)} = (u_1^{(0)}, \ldots, u_n^{(0)}) \in \mathbb{C}^n$ such that $u_i^{(0)} \neq u_j^{(0)}$ for $i \neq j$. The diagonals $\{u_i = u_j\}_{ij}$ are a critical locus for the space of isomonodromy deformation parameters, which thus presents monodromy itself. The canonical coordinates and the potential of the Frobenius structure are functions of $u \in \mathcal{U}_0$ and can be analytically continued to meromorphic functions on the universal covering of $\mathbb{C}^n \setminus \{diagonals\}$, [17, Section 4]. This defines analytic continuations of a Frobenius manifold structure. Unfortunately, their properties are not clear from this description, and comparing the analytic continuation at a point $u^{(0)}$ along a nontrivial path with the original semisimple Frobenius structure is a hard problem.

At the same time, canonical coordinates are well defined up to reordering. Therefore, the analytic continuation of a Frobenius structure with given Stokes matrix is described by the action of the fundamental group

$$\pi_1\big((\mathbb{C}^n \setminus \{diagonals\})/\Sigma^n, u^{(0)}\big) = \mathcal{B}_n$$

on the Stokes matrix at point $u^{(0)}$. $\mathcal{B}_n$ is called the Braid group. Its standard generators are $n-1$ elements $\beta_{1,2}, \beta_{2,3}, \ldots, \beta_{n-1,n}$ with relations

$$\beta_{i,i+1}\beta_{j,j+1} = \beta_{j,j+1}\beta_{i,i+1} \text{ for } i+1 \neq j,\ j+1 \neq i$$
$$\beta_{i,i+1}\beta_{i+1,i+2}\beta_{i,i+1} = \beta_{i+1,i+2}\beta_{i,i+1}\beta_{i+1,i+2}$$

$\beta_{i,i+1}$ corresponds to a loop moving u_i counter-clockwise around u_{i+1} then interchanging u_i, u_{i+1}.

Theorem 1.26 ([17, Theorem 4.6]). *The analytic continuation of a semisimple Frobenius manifold is described by the following action:*

$$\mathcal{S} \mapsto \beta_{i,i+1}(\mathcal{S}) := B_i(\mathcal{S})\mathcal{S}B_i^t(\mathcal{S}) \tag{1.5.1}$$

where the matrix $B_i(\mathcal{S})$ has entries

$$\begin{gathered}(B_i(S))_{kk} = 1,\ \ k = 1, \dots, n,\ \ k \neq i, i+1 \\ (B_i(S))_{i,i+1} = (B_i(S))_{i+1,i} = 1, \quad (B_i(S))_{i+1,i+1} = -S_{i,i+1}, \\ (B_i(S))_{hk} = 0 \ \textit{elsewhere.}\end{gathered}$$

The representation of the inverse $\beta^{-1}_{i,i+1}$ (corresponding to moving u_i and u_{i+1} clockwise) is the matrix $B_i^-(\mathcal{S})$ with entries

$$\begin{gathered}(B_i^-(S))_{kk} = 1,\ \ k = 1, \dots, n, k \neq i, i+1 \\ (B_i^-(S))_{i,i+1} = (B_i^-(S))_{i+1,i} = 1 \quad (B_i^-(S))_{i,i} = -S_{i,i+1} \\ (B_i^-(S))_{hk} = 0 \text{ elsewhere.}\end{gathered}$$

It follows that two analytic continuations of a Frobenius structure are parametrized by Stokes matrices $\mathcal{S}$ and $\mathcal{S}'$ possibly related by a the action of a sequence of permutations P, matrices $\mathcal{I}$ as in (1.4.2), and braids $\beta_{i,i+1}$ or $\beta^{-1}_{i,i+1}$.

1.6 Frobenius type structures

The part of a Frobenius manifold structure which makes sense on an abstract bundle is called a Frobenius type structure. As far as I know, it is a concept due to Hertling.

Definition 1.27 ([26, Def. 5.6]). A *Frobenius type structure* on a holomorphic vector bundle $K \to M$ is a collection of holomorphic objects $(\nabla^r, C, \mathcal{U}, \mathcal{V}, g)$, with values in the bundle K, where

a) ∇^r is a flat connection,

b) C is a Higgs field, that is a 1-form with values in endomorphisms of K, with $C \wedge C = 0$,

c) $\mathcal{U}, \mathcal{V}$ are endomorphisms,

d) g is a symmetric $\mathbb{C}-$valued nondegenerate bilinear form on K (sometimes referred to as "metric"),

satisfying the conditions

$$\begin{gathered}\nabla^r(C) = 0, \quad [C, \mathcal{U}] = 0, \quad \nabla^r(\mathcal{V}) = 0, \\ \nabla^r(\mathcal{U}) - [C, \mathcal{V}] + C = 0\end{gathered} \tag{1.6.1}$$

plus the conditions on the "metric" g

$$\begin{aligned}\nabla^r(g) &= 0, \\ g(C_X a, b) &= g(a, C_X b), \\ g(\mathcal{U}a, b) &= g(a, \mathcal{U}b), \\ g(\mathcal{V}a, b) &= -g(a, \mathcal{V}b).\end{aligned} \tag{1.6.2}$$

There exists an equivalent definition in terms of an extended connection on a holomorphic vector bundle $H \to \mathbb{C} \times M$.

Theorem 1.28 ([26, Theorem 5.7])**.** *Fix $\omega \in Z$ and the "metric" on $K \to M$. A Frobenius type structure on a holomorphic vector bundle $K \to M$ is equivalent to a triple (H, ∇, ω) where*

- *H is the space of a holomorphic vector bundle over $\mathbb{P}^1 \times M$.*
- *∇ is a flat meromorphic connection on H;*
- *the restriction of ∇ to the affine chart $\mathbb{C} \times M$ has a pole of Poincaré rank 1 in $\{0\} \times M$ and is flat in $(\mathbb{C} \setminus \{0\}) \times M$, the restriction to the affine chart $(\mathbb{P}^1 \setminus \{0\}) \times M$ has a logarithmic pole at $\{\infty\} \times M$ and is flat and holomorphic out of the singular locus.*

Starting from $(\nabla^r, C, \mathcal{U}, \mathcal{V})$, the triple (H, ∇, ω) is the datum of

- an integer ω,
- $H := p^*K$, $p : \mathbb{P}^1 \times M \to M$,
- $\nabla = \nabla^r + \frac{1}{z}C + \left(\frac{1}{z}\mathcal{U} - \mathcal{V} + \frac{\omega}{2}\,\mathrm{Id}\right)\frac{\mathrm{d}\,z}{z}$ for $\nabla^r, C, \mathcal{U}, \mathcal{V}$ canonically extended via pull-back to H.

Hertling proved also that a Frobenius manifold structure on M gives rise to a Frobenius type structure on $T_M \to M$.

Lemma 1.29 ([26, Lemma 5.11])**.** *Let M be a Frobenius manifold with flat identity e and Euler field E of conformal dimension d. Denote by g_M the metric and by ∇^{g_M} its Levi-Civita connection. Define*

- *a Higgs field C^M by $C^M_X(Y) = -X \circ Y$,*
- *endomorphisms $\mathcal{U}^M$ and $\mathcal{V}^M$ by*

$$\begin{aligned} \mathcal{U}^M &= E \circ (\bullet), \\ \mathcal{V}^M &= \nabla^{g_M}_{\bullet} E - \frac{2-d}{2} I. \end{aligned}$$

Then $(\nabla^{g_M}, C^M, \mathcal{U}^M, \mathcal{V}^M, g_M)$ is a Frobenius type structure on the fibers of T_M.

It is clear then that a Frobenius type structure is indeed a generalization of a Frobenius manifold on a generic auxiliary vector bundle and that a Saito structure is equivalent to a Frobenius type structure on the tangent bundle.

A natural question is whether a Frobenius type structure on a generic $K \to M$ may induce a genuine Frobenius structure over M. It turns out that the answer is positive provided that K admits a special global section $\zeta \in \mathcal{O}(K)$.

Let $K \to M$ be a holomorphic vector bundle on a complex manifold M, with Higgs field C. A holomorphic section $\zeta \in \mathcal{O}(K)$ can be contracted with the Higgs field to give a map

$$v := -C_{\bullet}(\zeta) \colon T_M \to K \tag{1.6.3}$$

i. e. the derivative of the section ζ along the Higgs field.

Theorem 1.30 ([26, Theorem 5.12]). *Suppose that $K \to M$ has a Frobenius type structure $(\nabla^r, C, \mathcal{U}, \mathcal{V}, g)$ and that ζ is a global section of K such that*

- *it is a flat section with respect to the flat connection of the Frobenius type structure, $\nabla^r(\zeta) = 0$,*
- *it is homogeneous with respect to the endomorphism $\mathcal{V}$, i.e. we have $\mathcal{V}(\zeta) = \frac{d}{2}\zeta$ for some $d \in \mathbb{C}$,*
- *the map* (1.6.3) *is an isomorphism.*

Then the pullback of $(\nabla^r, C, \mathcal{U}, \mathcal{V}, g)$ along the map (1.6.3) *gives a Frobenius manifold structure on M with unit field given by the pullback of the section ζ and with conformal dimension $2 - d$.*

This structure actually corresponds to the Frobenius type structure (according to Lemma 1.29) on the tangent bundle

$$(\nabla^{r,M}, C^M, \mathcal{U}^M, \mathcal{V}^M, g_M),$$

given by

$$\begin{aligned}
\nabla^{r,M} &= v^{-1}\nabla^r v, \quad C^M = v^{-1}Cv, \\
\mathcal{U}^M &= v^{-1}\mathcal{U}v, \quad \mathcal{V}^M = v^{-1}\mathcal{V}v, \\
g_M &= g(v(-), v(-)).
\end{aligned}$$

The proof is well-illustrated in [26]. Here only the main ingredients are recalled. The multiplication is uniquely characterized by the property

$$C_X C_Y = -C_{X \circ Y} \tag{1.6.4}$$

and it is given by

$$X \circ Y = v^{-1}(C_X v(Y)) = -C_X^M Y.$$

The flat identity e must satisfies $-C_e = \mathrm{Id}$ and it is in fact given by

$$e = v^{-1}(\zeta). \tag{1.6.5}$$

$E := \mathcal{U}(e)$ is an Euler field and satisfies $\mathrm{Lie}_E(g) = (2-d)g$, where the metric g coincide with g^M. The Levi-Civita connection ∇_0 of g is the pullback of ∇^r. Denote by $\overline{\nabla}$ the structure connection of the Frobenius manifold on $\mathbb{P}^1 \times M$. It equals

$$\begin{aligned}
\overline{\nabla} &= v^*\nabla \\
&= \nabla_0 + \frac{1}{z}C + \left(\frac{1}{z}\mathcal{U} - v^{-1}\mathcal{V}v + \frac{\omega}{2}\,\mathrm{Id}\right)\frac{\mathrm{d}\,z}{z}.
\end{aligned}$$

1.7 CV structures

The last structure I want to briefly present was introduced by Hertling and is called *CV*-structure after Cecotti and Vafa (with reference to [13] and [12]). The CV-geometry is defined on a generic vector bundle K. However, it shares some common feature with Frobenius geometry. In particular, a CV-structure embodies (due to an anti-involution κ)

the so-called reality property of a Frobenius structure. A Frobenius manifold is said to be *real* if it admits an antiholomorphic automorphism $\tau : M \to M$, or equivalently if the solutions of the WDVV equations are real. Real Frobenius manifold are discussed in [15].

First the preliminary notion of $DC\widetilde{C}$-structure, which is also due to Hertling, is introduced.

Definition 1.31. A $(DC\widetilde{C})$*-structure* on a C^∞ complex vector bundle $K \to M$ is the collection of C^∞ objects $(D, C, \widetilde{C})$ with values in K where

- D is a connection,
- C is a $(1,0)$-form with values in endomorphisms of K,
- $\widetilde{C}$ is a $(0,1)$-form with values in endomorphisms of K;

satisfying the conditions

$$\begin{aligned} (D'' + C)^2 = 0, \quad (D' + \widetilde{C})^2 = 0, \\ D'(C) = 0, \quad D''(\widetilde{C}) = 0, \\ D'D'' + D''D' = -(C\widetilde{C} + \widetilde{C}C) \end{aligned} \tag{1.7.1}$$

where D' and D'' are the $(1,0)$ and $(0,1)$ parts of D respectively.

A CV-structure is a $DC\widetilde{C}$-structure together with a metric and an anti-involution. Two endomorphisms of the bundle are also defined.

Definition 1.32. A *CV-structure* on a C^∞ complex bundle $K \to M$ is a collection of C^∞ objects $(D, C, \widetilde{C}, \kappa, h, \mathcal{U}, \mathcal{Q})$ with values in K where

- $(D, C, \widetilde{C})$ is a $(DC\widetilde{C})$-structure,
- κ is an antilinear involution with $D(\kappa) = 0$ which intertwines C and $\widetilde{C}$, $\kappa C \kappa = \widetilde{C}$,
- h is a hermitian (not necessarily positive) metric, which satisfies $D(h) = 0$, $h(C_X a, b) = h(a, \widetilde{C}_{\bar{X}} b)$ for $(1,0)$ fields X and which is real-valued on the real subbundle $K_{\mathbb{R}} \subset K$ defined by κ,
- $\mathcal{U}$ and $\mathcal{Q}$ are endomorphisms,

satisfying the conditions

$$\begin{aligned} &= 0, \\ D'(\mathcal{U}) - [C, \mathcal{Q}] + C &= 0, \\ D''(\mathcal{U}) &= 0, \\ D'(\mathcal{Q}) + [C, \kappa\mathcal{U}\kappa] &= 0, \\ \mathcal{Q} + \kappa\mathcal{Q}\kappa &= 0, \\ h(\mathcal{U}a, b) &= h(a, \kappa\mathcal{U}\kappa b), \\ h(\mathcal{Q}a, b) &= h(a, \mathcal{Q}b). \end{aligned} \tag{1.7.2}$$

Similarly to Theorem 1.28, also the CV-geometry of K can be described in terms of a flat irregular connections on the lifted bundle $p^*K \to \mathbb{P}^1 \times M$. With this language, it is the datum of $(\hat{H}, \nabla, H_{\mathbb{R}}, g)$, where $\hat{H}$ is $\hat{H} = p^*K$, ∇ is the flat meromorhic connection $\hat{\mathcal{H}} \to \hat{\mathcal{H}} \otimes_{\mathcal{O}_{\mathbb{P}^1 \times M}} \Omega^1_{\mathbb{P}^1 \times M}(\{0, \infty\} \times M)$

$$\nabla = D + \frac{1}{z}C + z\tilde{C} + \left(\frac{1}{z}\mathcal{U} - \mathcal{Q} + \frac{\omega}{2}\operatorname{Id} - z\kappa\mathcal{U}\kappa\right)\frac{\mathrm{d}\, z}{z},$$

for $D, C, \tilde{C}, \mathcal{U}, \mathcal{Q}, \kappa$ canonically extended, and $H_{\mathbb{R}}$ is a real ∇-flat subbundle of $\hat{H}_{|\mathbb{C}^* \times M}$ is flat on $\hat{H}_{|\mathbb{C} \times M}$.

Chapter 2

Quivers, stability conditions and generating functions

This Chapter collects some wide background material about

- CY triangulated categories $\mathcal{C}$ associated to a quiver with potential,
- the space of stability conditions $\mathrm{Stab}(\mathcal{C})$, and
- Joyce generating functions for invariant counting semistable objects.

For this reason it might appear fragmentary. Some notions on triangulated categories are summed up in Section 2.1. One of the goals is also to fix the notation used henceforth. Given a category $\mathcal{C}$, there is a concept of stability due to Bridgeland [8]. Stability conditions are an interesting generalization of slope stability for coherent sheaves, but they are also interesting in their own right. The set of stability conditions on $\mathcal{C}$ is a complex manifold $\mathrm{Stab}(\mathcal{C})$ called the *space of stability conditions*. It is briefly described in Section 2.3. In particular, if $\mathcal{C}$ admits a so-called bounded t-structure with a finite length heart, an open subset of $\mathrm{Stab}(\mathcal{C})$ can be described as a union of cells, one for each bounded $t-$structure of finite length. The way these cells glue together along their boundaries is controlled by an operation on $\mathcal{C}$ called *tilt*.

Quivers with potential (Q, W) are oriented graphs together with the choice of a distinguished collection of cycles. Mutations are transformations of quivers. For any (Q, W) there is a special triangulated category $\mathcal{C} = \mathcal{D}(Q, W)$. The $\mathcal{D}(Q, W)$ provide a big class of examples for which the space of stability conditions admits a nice description. Mutations (at the quiver level) incarnate simple tilts (at the category level), thus cells of $\mathrm{Stab}(\mathcal{C})$ are in natural correspondence with the mutation classes of (Q, W). Quivers and their associated categories are presented in Section 2.2.

Section 2.4 is devoted to generating functions. It is the starting point for Chapters 3 and 4. For any choice of a stability condition σ, an object in $Obj(\mathcal{C})$ is either σ-semistable or unstable. Under some assumptions, we can attach to any stability condition σ a system of invariants virtually enumerating σ-semistable objects. A basic question is whether these invariants -which jump discontinuously- can be combined in functions which are at least continuous. This problem was first studied in depth by Joyce in [31]. He introduced a family of holomorphic generating functions, with interesting geometric implications. The PDE these functions satisfy allows to regard $\mathrm{Stab}(\mathcal{C})$ as the deformation space of a family of isomonodromic irregular connections over $\mathbb{P}^1$.

2.1 Triangulated categories

The concept of a tringulated category, introduced by Verdier [39], is modeled on the one of bounded derived category $D^b(\mathcal{A})$ of an abelian category. The definition is reproduced for completeness. For the interested reader, a good reference for an introductory exposition about triangulated and derived categories is [28]. In this section we focus on the notions of hearts of a t-structure and the operation of tilting. The next few pages also set once for all the notation.

We start by giving the axioms of a triangulated structure and by setting the notation.

Definition 2.1. A *triangulated category* is an additive $\mathbb{C}$-linear category $\mathcal{C}$ together with an additive autoequivalence $[1] : \mathcal{C} \to \mathcal{C}$ (the *shift functor*) and a collection of *distinguished triangles*

$$X \to Y \to Z \to X[1]$$

satisfying the following axioms

TR0 any triangle isomorphic to a distinguished triangle is again a distinguished triangle;

TR1 the identity morphism $X \xrightarrow{id} X$ can be completed to a distinguished triangle $X \xrightarrow{id} X \to 0 \to X[1]$;

TR2 if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is distinguished, then so is its *rotation* $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{f[1]} Y[1]$, and vice-versa;

TR3 any morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ can be completed to a distinguished triangle whose third object Z we call a *cone of* f;

TR4 given distinguished triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]$, then each commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd}$$

can be completed to a (possibly not unique) morphism of triangles with $h : Z \to Z'$;

TR5 [4, octahedral axiom] given a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$, there exist cones X', T' and Y' of f, $f \circ g$ and g respectively, forming a distinguished triangle $X' \to T' \to Y' \to X'[1]$, such that the four distinguished triangles fit into a octahedtron-shaped commuting diagram.

Distinguished triangles are called also "exact" triangles. In some sense, they are the replacement of short exact sequences in abelian categories, in the sense that they are analogue of extensions. A distinguished triangle $X \to Y \to Z \to X[1]$ is also represented by

$$\begin{tikzcd} X \arrow[rr] & & Y \arrow[dl] \\ & Z \arrow[ul, dashed] & \end{tikzcd}$$

With the classical notation

$$\mathrm{Hom}^m_{\mathcal{C}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y[m]).$$

We say that the triangulated category is of *finite type* if, for all objects $X, Y \in \mathcal{C}$,

$$\dim \oplus_{m\in\mathbb{Z}} \mathrm{Hom}^m_{\mathcal{C}}(X,Y) < \infty.$$

Definition 2.2. The Grothendieck group $\mathcal{K}(\mathcal{C})$ of a triangulated category is the free abelian group generated by isomorphism classes $[X]$ of objects X of $\mathcal{C}$ modulo the relation

$$[Y] = [X] + [Z]$$

whenever $X \to Y \to Z \to X[1]$ is a distinguished triangle.

The shift functor acts as an involution on $\mathcal{K}(\mathcal{C})$: $[X[1]] = -[X]$.

Results of Chapter 3 and 4 hold for finite dimensional $\mathcal{K}(\mathcal{C}) \simeq \mathbb{Z}^r$. In fact, later we will restrict to triangulated categories which admit a finite bounded $t-$structure. In this case $\mathrm{rk}\,\mathcal{K}(\mathcal{C}) < \infty$. Moreover, under the finite type condition, the Grothendieck group carries a bilinear form $\chi(\cdot,\cdot) : \mathcal{K}(\mathcal{C}) \times \mathcal{K}(\mathcal{C}) \to \mathbb{Z}$, the *Euler pairing*, defined by the formula

$$\chi(X,Y) := \sum_{n\in\mathbb{Z}} (-1)^n \dim \mathrm{Hom}^n_{\mathcal{C}}(X,Y).$$

If necessary, $\mathcal{K}(\mathcal{C})$ is replaced by the *numerical Grothendieck group* $\mathcal{K}_N(\mathcal{C}) = \mathcal{K}(\mathcal{C})/\mathcal{K}(\mathcal{C})^{\perp}$, for $\mathcal{K}(\mathcal{C})^{\perp} := \{[E] \in \mathcal{K}(\mathcal{C}) | \chi([E],[F]) = 0 \ \forall [F] \in \mathcal{K}(\mathcal{C})\}$. Restricted to $\mathcal{K}_N(\mathcal{C})$ the Euler form is non-degenerate.

In this thesis, we are interested in Calabi-Yau categories of dimension three.

Definition 2.3. A triangulated category $\mathcal{C}$ is said three-Calabi-Yau if there are functorial isomorphisms

$$\mathrm{Hom}^m_{\mathcal{C}}(A,B) \simeq \mathrm{Hom}^{3-m}_{\mathcal{C}}(B,A)^* \text{ for all } A, B \in \mathcal{C}.$$

For a 3CY category, the Euler form is skew-symmetric

$$\chi(A,B) = \dim \mathrm{Hom}(A,B) - \dim \mathrm{Ext}^1(A,B) - \dim \mathrm{Hom}(B,A) + \dim \mathrm{Ext}^1(B,A).$$

2.1.1 $t-$structures and tilting

Bounded t-structures provide a way to see different abelian categories embedded in a given triangulated category. They were introduced in [4] in 1982.

Definition 2.4. A *t-structure* on a triangulated category $\mathcal{C}$ is the datum of a full additive subcategory $\mathcal{T} \subset \mathcal{C}$, stable under shift ($\mathcal{T}[1] \subset \mathcal{T}$), such that, setting

$$\mathcal{T}^{\perp} := \{V \in \mathcal{C} : \mathrm{Hom}_{\mathcal{C}}(T, V) = 0 \ \forall \ V \in \mathcal{T}\},$$

for every object $E \in \mathcal{C}$ there exists a triangle

$$T_1 \to E \to T_2 \to T_1[1]$$

with $T_1 \in \mathcal{T}$ and $T_2 \in \mathcal{T}^{\perp}$.
It is said to be *bounded* if $\mathcal{C} = \bigcup_{m\in\mathbb{Z}} \mathcal{T}[-m] \cap \mathcal{T}^{\perp}[m]$.

The *heart* of a t-structure $\mathcal{T} \subset \mathcal{C}$ is the full subcategory $\mathcal{A} = \mathcal{T} \cap \mathcal{T}^{\perp}[1] \subset \mathcal{C}$. It is an abelian category, [4].

A bounded t-structure $\mathcal{T} \subset \mathcal{C}$ is determined by its heart $\mathcal{A} \subset \mathcal{C}$. Moreover, the following Lemma holds.

Lemma 2.5 ([8, Lemma 3.2]). *Let $\mathcal{A} \subset \mathcal{C}$ be a full additive subcategory of a triangulated category $\mathcal{C}$. Then $\mathcal{A}$ is the heart of a bounded t-structure if and only if the following two conditions hold:*

1. *if $k_1 > k_2$ are integers and A, B are objects of $\mathcal{A}$, then* $\mathrm{Hom}_{\mathcal{C}}(A[k_1], B[k_2]) = 0$,
2. *for every $E \in \mathcal{C}$, $E \neq 0$, there exists a finite sequence of integers $k_1 > k_2 > \cdots > k_m$ and a collection of triangles*

$$\begin{tikzcd}[column sep=small] 0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{m-1} \arrow[rr] & & E_m = E \arrow[dl] \\ & A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_m \arrow[ul, dashed] & \end{tikzcd}$$

with $A_j \in \mathcal{A}[k_j]$ for all j.

The class $[E]$ in the Grothendieck group $\mathcal{K}(\mathcal{C})$ decomposes as

$$[E] = \sum_j [A_j].$$

It follows that we can map the class $[E] \in \mathcal{K}(\mathcal{C})$ of an element $E \in \mathcal{C}$ to the sum

$$\sum_j (-1)^{k_j} [A_j[-k_j]] \in \mathcal{K}(\mathcal{A}).$$

Proposition 2.6. *If $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{C}$, then $\mathcal{K}(\mathcal{C}) \simeq \mathcal{K}(\mathcal{A})$.*

For $\mathcal{C} = D^b(\mathcal{A})$, $\mathcal{A}$ is a heart of a t-structure. It is called the *standard heart*. The *standard t-structure* $\mathcal{T}$ of $D^b(\mathcal{A})$ consists of those complexes concentrated in degree less or equal than 0. This realizes an embedding of $\mathcal{A}$ in $D^b(\mathcal{A})$. Suppose we have an equivalence of derived categories $D^b(\mathcal{A}) \simeq D^b(\mathcal{B})$. Then $\mathcal{A}$ is embedded in $D^b(\mathcal{A})$, but generically it is not mapped to $\mathcal{B}$ by the restricted isomorphism of categories. However, its image $\mathcal{A}'$ is the heart of a $t-$structure on $D^b(\mathcal{B})$.

Definition 2.7. A heart is called *of finite length* if any object $A \in \mathcal{A}$ has a filtration $0 \subset A_1 \subset \cdots \subset A$ such that all A_i/A_{i-1} are simple. It is called *finite* if, moreover, has a finite number of simple objects.

Whenever we will be concerned in a triangulated category, in Chapters 3 and 4, we will assume it admits a bounded t-structure with a finite heart. This ensure that its Grothendieck group has finite rank.

Given a bounded t-structure $(\mathcal{T}, \mathcal{T}^{\perp})$ or its heart $\mathcal{A}$, one can construct many non-trivial t-structures with a procedure called *tilting*. We say that a pair of hearts $(\mathcal{A}_1, \mathcal{A}_2)$ in $\mathcal{C}$ is a *tilting pair* if $\mathcal{A}_2 \subset \langle \mathcal{A}_1, \mathcal{A}_1[-1] \rangle$ and $\mathcal{A}_1 \subset \langle \mathcal{A}_2[1], \mathcal{A}_2 \rangle$. The brackets $\langle \mathcal{A}, \mathcal{B} \rangle$ denote the extension-closure of $\mathcal{A}$ and $\mathcal{B}$, that is the smallest full subcategory of $\mathcal{C}$ containing both $\mathcal{A}$ and $\mathcal{B}$ and closed with respect to the extension: $X, Y \in \langle \mathcal{A}, \mathcal{B} \rangle$ and $X \to Z \to Y \to X[1]$ implies $Z \in \langle \mathcal{A}, \mathcal{B} \rangle$.

The special construction we will be most interested in is the so-called (left and right) *simple tilt*. Given a finite-length heart $\mathcal{A} \subset \mathcal{C}$ and a simple object $S \in \mathcal{A}$, denote by $\langle S \rangle$ the full subcategory consisting of direct sums of simple factors isomorphic to S. Define then the full subcategories

$$S^{\perp} := \{E \in \mathcal{A} | \operatorname{Hom}_{\mathcal{A}}(S, E) = 0\},$$
$$^{\perp}S := \{E \in \mathcal{A} | \operatorname{Hom}_{\mathcal{A}}(E, S) = 0\}.$$

$\mu_S^-(\mathcal{A}) := \langle S[1], ^{\perp} S \rangle$ and $\mu_S^+(\mathcal{A}) := \langle S^{\perp}, S[-1] \rangle$ are hearts of bounded t-structures in $\mathcal{C}$, [9]. They are called respectively *left and right simple tilts of the heart $\mathcal{A}$ at the simple S.*

More generally, we can produce a tilted heart from any torsion pair $(\mathcal{T}, \mathcal{F})$ in a given heart.

Definition 2.8. A torsion pair in an abelian category $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full additive subcategories of $\mathcal{A}$ with $\mathcal{F} \subseteq \mathcal{T}^{\perp}$, such that for all $E \in \mathcal{A}$ there exists a short exact sequence $0 \to \mathcal{T}(E) \to E \to \mathcal{F}(E) \to 0$, $\mathcal{T}(E) \in \mathcal{T}$, $\mathcal{F}(E) \in \mathcal{F}$.

A torsion pair inherits its name from the exemplifying case of the abelian category $\mathcal{A} = Coh(X)$ of coherent sheaves on a smooth projective curve X. The subcategories of torsion sheaves $\mathcal{T}$ and torsion-free sheaves $\mathcal{F}$ are torsion pair.

Although the subcategory $\mathcal{T} \subset \mathcal{A}$ above must be not confused with the subcategory $\mathcal{T} \subset \mathcal{C}$ of Definition 2.4, we can notice the similarity between the two definitions due to the decomposition of an object into parts lying in a subcategory and its orthogonal.

Proposition 2.9 ([25]). *Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, the following define hearts of bounded t-structures in $\mathcal{C}$:*

$$\mathcal{A}^{\#} = \langle \mathcal{F}[1], \mathcal{T} \rangle,$$
$$A^{\flat} = \langle \mathcal{F}, \mathcal{T}[-1] \rangle.$$

Vice-versa, given a tilting pair $(\mathcal{A}_1, \mathcal{A}_2)$ in $\mathcal{C}$, then the subcategories $\mathcal{T} = \mathcal{A}_1 \cap \mathcal{A}_2[1]$ and $\mathcal{F} = \mathcal{A}_1 \cap \mathcal{A}_2$ form a torsion pair $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}_1$.

2.2 Quivers with potential

In this section we introduced quivers with potential and their mutations. The goal is to associate a 3CY triangulated category to a quiver with potential and relate mutations of quivers on quivers to tilts of categories. I learned the material contained in this section from [33] and [9].

Quiver and Ext −quivers

A quiver is an oriented graph. In general infinitely many vertices and arrows between two vertices are possible. Here we restrict to finite quivers. Formally a finite *quiver* Q is a quadruple $Q = (Q_0, Q_1, s, t)$ consisting of two finite sets Q_0 (*vertices*) and Q_1(*arrows*), together with two maps s (*source* function - giving the starting vertex) and t (*target* function - giving the ending vertex) $s, t : Q_1 \to Q_0$.

An (oriented) *path* a of length l, $l < \infty$, in Q is a finite sequence of consecutive oriented arrows $a = (a_1, \ldots, a_l)$, with $t(a_p) = s(a_{p+1})$. We can think of a path as a word in $a_i \in Q_1$.

It is convenient to consider also the lazy path e_i of length 0 at each vertex. We can identify e_i with the corresponding vertex i. The *path algebra* (or quiver algebra) $\mathbb{C}Q$ of Q over $\mathbb{C}$ is the $\mathbb{C}$-vector space spanned by the oriented paths in Q, with a multiplication defined by the concatenation if the starting point of the second coincides with the tail of the first, vanishing otherwise,

$$ab := \begin{cases} (a_1 \cdots a_n b_1 \cdots b_n) & \text{if } t(a_n) = s(b_1) \\ 0 & \text{otherwise} \end{cases}.$$

The path algebra is a graded algebra $\mathbb{C}Q = \bigoplus_{d=0}^{\infty} \mathbb{C}Q^d$, the grading being induced by the length of a path. A path basis is the union over $d \in \mathbb{N}$ of length-d-path bases. For $d = 0$ a basis consists on the set of vertices Q_0, for $d = 1$ it is the set of arrows Q_1. Inductively, a basis for $\mathbb{C}Q^d$ is given by products $a_1 \cdots a_d$, $a_i \in Q_1$, $t(a_i) = s(a_{i+1})$ for $1 \leq i < d$. The following definitions, however, are independent of the choice of a basis.

A representation of Q is a module over the path algebra $\mathbb{C}Q$. One can shows that this is the same as the datum of vector spaces associated to vertices together with a map for any arrows. For any quiver Q, the abelian category of finite-dimensional representations of Q is $\mathcal{A} = Q$-Rep with objects $\underline{V} = (V_h)_{h \in Q_0}, V_{h'} \xrightarrow{f_a} V_{h''}$ for any arrow $a = (h' \to h'') \in Q_1$.

Proposition 2.10 ([5])**.** *When Q has no oriented cycles*

$$\begin{aligned} \chi_{Q-Rep}([A],[B]) &= \dim \operatorname{Hom}(A,B) - dim \operatorname{Ext}^1(A,B) \\ &= \sum_{i \in Q_0} \dim A_i \dim B_i - \sum_{a \in Q_1} \dim A_{s(a)} \dim B_{t(a)} \end{aligned}$$

for any classes $[A],[B] \in \mathcal{K}(Q - Rep)$. Moreover, the simple modules are exactly the representations S_i concentrated at each vertex i, with dimension vector $(0, \ldots, \overset{i}{1}, 0, \ldots, 0)$.

Simple tilts have nice combinatorical expressions, obtained immediately by the commutative diagrams representing morphisms between finite dimensional representations. For any vertex $i \in Q_0$, say $\mathcal{T} := \langle S_i \rangle = \{\oplus_n S_i^n\}$ the subcategory of representations supported at a fixed vertex i. It is easy to verify that $\mathcal{F} := \langle S_i \rangle^{\perp}$ coincides with $\mathcal{F} = \left\{ \underline{V} : \bigcap_{a \in Q_1 : s(a) = i} \ker f_a = 0 \right\}$. On the other hand, if one choses $\mathcal{F} := \langle S_i \rangle$, then $\mathcal{T} :=^{\perp} \langle S_i \rangle = \left\{ \underline{V} : \oplus_{a \in Q_1 : t(a) = i} \operatorname{im} f_a = V_i \right\}$.

On the other hand, for any finite abelian category or finite heart $\mathcal{A} \subset \mathcal{C}$ of a triangulated category there is a finite quiver.

Definition 2.11. The quiver $Q(\mathcal{A})$, whose vertices are indexed by the isomorphism classes of simple objects $S_i \in \mathcal{A}$ and whose adjacency matrix η has entries $\eta_{ij} = \dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{A}}(S_i, S_j)$ is called the Ext-*quiver of* $\mathcal{A}$.

Any acyclic quiver Q can be recovered as the Ext-quiver of its abelian category of representations.

Potential and Ginzburg algebra

The *complete path algebra* $\widehat{\mathbb{C}Q}$ is the completion of $\mathbb{C}Q$ with respect to the ideal $\mathfrak{m}$ generated by the arrows of Q. It is the set of formal sums $\alpha = \sum_p \alpha_p$ indexed by oriented paths in Q, and with multiplication defined by $\sum_p \alpha_p * \sum_q \beta_q = \sum_{pq} \alpha_p \beta_q$.

A quiver can be equipped with relations. A *potential* on Q is an element of the closure in $\widehat{\mathbb{C}Q}$ of the vector space generated by all nontrivial cyclic paths of Q, up to cyclic equivalence. Two linear combinations of cycles are *cyclically equivalent* if their difference is in the closure of the vector space generated by all differences $a_1 \dots a_s - a_2 \dots a_s a_1$, where $a_1 \dots a_s$ is a cycle.

Definition 2.12. A potential is said *reduced* if it does not contain cycles of length less than three.

Definition 2.13 ([14, Section 4])**.** A *quiver with potential* (Q, W) is a finite quiver Q without loops together with a potential W.

Definition 2.14. Two quivers with potential (Q, W) and (Q', W') are *right-equivalent* if Q and Q' have the same set of vertices and there exists an algebra isomorphism $\varphi : \widehat{\mathbb{C}Q} \to \widehat{\mathbb{C}Q'}$ whose restriction on the vertices is the identity map and $\varphi(W)$ and W' are cyclically equivalent. Such an isomorphism φ is called a *right-equivalence*.

For an arrow a, the *cyclic derivative* $\partial_a W$ is defined as the sum of words in $\mathbb{C}Q$ obtained by deleting a from all the cycles which compose W, where a has been cycled to the beginning of each word. $\langle \partial_a W : a \in Q_1 \rangle$ is a left-and-right ideal in $\widehat{\mathbb{C}Q}$. The Jacobi algebra $J(Q, W)$ of a quiver with potential is the quotient of $\widehat{\mathbb{C}Q}$ by the closure of the ideal generated by the derivatives $\{\partial_a W\}_{a \in Q_1}$ of the potential.

Note that if Q is acyclic then the only potential on Q is the zero potential and $J(Q, 0) \simeq \mathbb{C}Q$.

The good candidate for the construction of a 3CY triangulated category from a quiver is a refinement of the Jacobi algebra, introduced by Ginzburg in [21].

We define a graded quiver as a quiver equipped with a grading function $\deg : Q_1 \to \mathbb{Z}$. Given a quiver with potential (Q, W) construct a graded quiver Q^{gr} with the same vertices $Q_0^{gr} = Q_0$ and $Q_1^{gr} \supset Q_1$ [33]. Q_1^{gr} is define by reverting all the arrows (for all $a = (i \to j) \in Q_1$ add $a^* = (j \to i)$) and adding a loop $t_i = (i \to i)$ for every vertex i and assigning degree 0 to a, -1 to a^*, $a \in Q_1$, and -2 to t_i. As for the path algebra of a quiver Q, we can consider the complete graded path algebra $\widehat{\mathbb{C}Q^{gr}}$ in the category of graded algebras with respect to the ideal generated by the arrows in Q_1^{gr}. It is called *Ginzburg dg algebra* and denoted by $\Gamma(Q, W)$. On Q^{gr} is defined a differential d as the unique linear endomorphism homogeneous of degree 1 satisfying

1. $\mathrm{d}\, a = 0$ for every $a \in Q_1$,

2. $\mathrm{d}\, a^* = \partial_a W$ for every $a^*, a \in Q_1$,

3. $\mathrm{d}\, t_i = \sum_{a:s(a)=i} [a, a^*]$ for every vertex i,

4. $\mathrm{d}(uv) = (\mathrm{d}\, u)v + (-1)^p u(\mathrm{d}\, v)$, for all u homogeneous of degree p and all v (Leibniz rule).

The Jacobi algebra $J(Q, W)$ of a quiver with potential is the 0-th cohomology of the complete Ginzburg algebra, i. e.

$$J(Q, W) = H^0(\Gamma(Q, W)).$$

Mutations

Quiver mutations are elementary operations on a quiver that create new quivers with the same set of vertices. We will see later that it is the incarnation of the operation of tilting, once that the heart of a triangulated category has been associated to a quiver with potential. Assume (Q, W) is a quiver with reduced potential without loops and two-cycles. Then, for any vertex i we can define a new quiver with potential $\mu_i(Q, W) = (Q', W')$ with the same vertices set. The set of arrows Q'_1 is constructed as follows:

1. for any pair of arrows $a, b \in Q_1$ with $t(a) = i = s(b)$, add a new arrow $c : s(a) \to t(b)$,

2. replace any arrow a with source or target i with the reverse a^*.

W' is the reduced potential cyclically equivalent to $W'_1 + W'_2$, where W'_1 is obtained from W replacing every composition ab with c, and $W'_2 = \sum_{a,b} cb^*a^*$, a, b, c as above. The operation μ_i is called the *mutation at the vertex* i.

Generically, a quiver with potential is *non-degenerate*, that is $\mu_i(Q, W)$ has no loops or 2-cycles, so the process of mutation can be iterated. $\mu_i^2(Q, W)$ is right-equivalent to (Q, W) and μ_i acts as an involution on the right-equivalence classes of reduced quivers with potential without loops and 2-cycles, [14].

2.2.1 $3CY$-categories associated to (Q, W)

Let $\mathcal{D}$ be a 3CY-category with a bounded t-structure and a finite heart $\mathcal{A} \subset \mathcal{D}$. Associated to the heart $\mathcal{A}$ there is a quiver $Q(\mathcal{A})$ with skew-symmetric adiacency matrix η, $\eta_{ij} = \chi_{\mathcal{A}}(S_i, S_j)$ and without 2-cycles. Moreover, if all the simples $S_1, \ldots, S_n$ in $\mathcal{A}$ are spherical (that is $\mathrm{Hom}^r_{\mathcal{D}}(S_i, S_i)$ has dimension 1 for $r = 0, 3$, and 0 otherwise), then $Q(\mathcal{A})$ has no loops. Arguments in [33, sec. 5 and 7] show that the process can be reversed, provided that a quiver Q is equipped with a potential W.

Let be given (Q, W) and consider the derived category of the complete Ginzburg algebra $D(\Gamma(Q, W))$. The subcategory $\mathcal{D}(Q, W)$ consisting of objects with finite-dimensional cohomology is $3CY$. Moreover $J(Q, W) = H^0(\Gamma(Q, W))$ and $\mathcal{A} = \mathrm{Rep}(J(Q, W))$ is the heart of a canonical bounded t-structure on $\mathcal{D}(Q, W)$. So we have the following result.

Theorem 2.15 ([33]; [9, sec. 7.4])**.** *For any quiver with reduced potential* (Q, W) *there exists a* $3CY$ *triangulated category* $\mathcal{D}(Q, W)$ *of finite type, with a bounded* t*-structure whose heart* $\mathcal{A} = \mathcal{A}(Q, W) \subset \mathcal{D}(Q, W)$ *is of finite length and has associated* Ext$-$*quiver* $Q(\mathcal{A})$*.*

We want to emphasize that $\mathcal{D}(Q, W)$ is not the bounded derived category of $\mathcal{A}(Q, W) = \mathrm{Rep}(J(Q, W))$, [33].

We can now relate tiltings in the triangulated Calabi-Yau-3 category $\mathcal{D}(Q, W)$ with mutations of the corresponding quiver (Q, W). This is again a result by Keller and Yang [33].

Theorem 2.16 ([33, Theo. 3.2]; [9, Theo. 7.3])**.** *Let* (Q, W) *be a quiver with no loops or 2-cycles and with reduced potential. Denote by* $\mu_k(Q, W)$ *its mutation at the vertex* k*. There are two equivalences of triangulated categories*

$$\Phi_k^{\pm} : \mathcal{D}(\mu_k(Q, W)) \to \mathcal{D}(Q, W)$$

which induce tilts in the simple object S_k *in the sense that*

$$\Phi_k^{\pm}(\mathcal{A}(\mu_k(Q, W))) = \mu_{S_k}^{\pm}(\mathcal{A}(Q, W))$$

and which moreover induce the natural bijection on simple objects.

This statement implies that the operation of mutation on quivers with potential (Q, W) is the incarnation of a simple tilt on $\mathcal{D}(Q, W)$ and that any tilted heart $\mu(\mathcal{A})$ can be thought as $\mathcal{A}(\mu(Q, W))$, where μ denotes respectively a sequence of quiver mutations or a sequence of simple tilts.

The bijection between vertices of (Q, W) and $\mu_k(Q, W)$ is equivalent to the existence of the isomrphism between the Grothendieck groups $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A}_k^{\pm}) := \mu_k^{\pm}\mathcal{A}$ induced by (2.6). The bijection

$$\mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{A}_k^{+})$$

on classes of simple objects of $\mathcal{A}$ and $\mathcal{A}_k^{+}$ is given by the following "*cluster mutation rule*":

$$[S_j] \mapsto \begin{cases} -[S_j] & \text{if } j = k \\ [S_j] + [\chi([S_k], [S_j])]_+ [S_k] & \text{otherwise} \end{cases}$$

where $[x]_+$ denotes the positive part of x, i. e. the maximum of 0 and x. The involution $\mathcal{A} = (\mathcal{A}_k^{+})_k^{-}$ inverts the relation. By quiver with potential, here we mean a reduced quiver in its right-equivalent class. The adjacency matrix η of Q depends on χ and changes accordingly:

$$\eta_{ji} \mapsto \begin{cases} -\eta_{ji} & \text{if } i \in \{j, k\} \\ \eta_{ji} + |\eta_{jk}|\eta_{ki} & \text{if } i \neq j, k \text{ and } \eta_{jk}\eta_{ki} > 0 \\ \eta_{ji} & \text{otherwise} \end{cases}$$

Definition 2.17. We say that a heart $\mathcal{A}$ of $\mathcal{D}(Q, W)$ is reachable if it can be obtained from the standard heart $\mathcal{A}(Q, W)$ by a finite sequence of simple tilts.

2.3 The space of stability conditions

In this section we review the definition of Bridgeland stability conditions on abelian and triangulated categories. For our pourpose, the most interesting feature is that the set of stability conditions $Stab(\mathcal{C})$ of a triangulated category $\mathcal{C}$, introduced in [8], is a complex manifold. It is covered by the closures of domains $U(\mathcal{A})$ of stability conditions supported on given hearts $\mathcal{A}$ of bounded t-structures in $\mathcal{C}$. In the case when the hearts $\mathcal{A}$ are of finite length, the connected components of $Stab(\mathcal{C})$ have a so-called wall and chamber structures, with chambers the $U(\mathcal{A})$ and walls related to simple tilts.

Definition 2.18. A stability function on an abelian category $\mathcal{A}$ is a group homomorphism

$$Z : \mathcal{K}(\mathcal{A}) \to \mathbb{C}$$

such that for any non-zero object $E \in \mathcal{A}$, $Z([E]) \in \bar{\mathbb{H}} := \left\{ re^{i\pi\theta} | r \in \mathbb{R}_{>0}, 0 < \theta \leq 1 \right\}$.

Definition 2.19. An object in $\mathcal{A}$ is called *semistable* if the phase $\phi(F) := \frac{1}{\pi} \arg Z([F])$ of any non-zero proper subobject of E satisfies $\phi(F) \leq \phi(E)$. It is called *stable* if the inequality holds strictly.

Stability conditions on a triangulated category can be described via stability functions supported on hearts $\mathcal{A} \subset \mathcal{C}$, satisfying the properties described below.

Definition 2.20 (Bridgeland stability condition)**.** A *stability condition* on a triangulated category $\mathcal{C}$ is a pair $\sigma = (Z, \mathcal{A})$, where

- $\mathcal{A} \subset \mathcal{C}$ is the heart of a bounded $t-$structure
- $Z : \mathcal{K}(\mathcal{A}) \to \mathbb{C}$ is a stability function on $\mathcal{A}$ which
    1. has the Harder-Narashima property,
    2. satisfies the support property.

The Harder-Narasimhan property concerns the existence of a filtration (HN filtration)

$$0 = E_0 \subset E_1 \cdots \subset E_m = E$$

such that the quotients E_i/E_{i-1} are in $\mathcal{A}$, have decreasing phase ϕ_i and are semistable, for any object E. As $\mathcal{A}$ is the heart of a bounded $t-$structure, we can identify $\mathcal{K}(\mathcal{C}) \simeq \mathcal{K}(\mathcal{A})$. The stability function Z induces a *central charge* $Z : \mathcal{K}(\mathcal{C}) \to \mathbb{C}$ and the HN filtration of an object $X \in \mathcal{C}$ is indeed a filtration of its cohomology objects.

If $\mathcal{K}(\mathcal{C})$ is finite-dimensional, we say that $\sigma = (Z, \mathcal{A})$ satisfies the *support property* if there exists a norm $||\cdot||$ on $\mathcal{K}(\mathcal{C})$ such that for any $X \in \mathcal{C}$, $0 \neq |Z([X])| \geq ||[X]||$. Otherwise, we require the existence of a finite-dimensional lattice Γ together with a map $\gamma : \mathcal{K}(\mathcal{C}) \to \Gamma$, such that $Z : \mathcal{K}(\mathcal{C}) \to \mathbb{C}$ factors via Γ. A typical choice for Γ might be the numerical Grothendieck group $\mathcal{K}_N(\mathcal{C})$. Fixing an arbitrary norm $||\cdot||$ on $\Gamma_{\mathbb{R}} = \Gamma \otimes \mathbb{R}$, the definition lifts naturally.

The support property allows one to continuously deform a stability condition. The set $Stab(\mathcal{C})$ of all stability conditions on $\mathcal{C}$ has a Hausdorff topology [8, Section 8] induced by the generalized metric

$$d(\sigma_1, \sigma_2) := \sup_{0 \neq E \in \mathcal{C}} \left\{ |\phi^-_{\sigma_2}(E) - \phi^-_{\sigma_1}(E)|, |\phi^+_{\sigma_2}(E) - \phi^+_{\sigma_1}(E)|, ||Z_2 - Z_1|| \right\} \in [0, +\infty]$$

for $\sigma_i = (Z_i, \mathcal{A}_i)$, $\phi^-(E) := \phi(E_1/E_0)$, $\phi^+(E) := \phi(E_m/E_{m-1})$.

Moreover, if $\mathcal{K}(\mathcal{C})$ has finite rank, any connected component $\Sigma \subset Stab(\mathcal{C})$ is a finite-dimensional complex manifold, locally homeomorphic to $\mathrm{Hom}(\mathcal{K}(\mathcal{C}), \mathbb{C})$. In the rest of the thesis we will consider triangulated categories with finite rank Grothendieck group.

Theorem 2.21 ([8, Theo. 1.2]; [3, Theo. A.5])**.** *Let $\mathcal{C}$ be a triangulated category with $\mathcal{K}(\mathcal{C}) \simeq \mathbb{Z}^r$. Restricted to any connected component $\Sigma \subset Stab(\mathcal{C})$, the forgetful map $Z : Stab(\mathcal{C}) \to \mathrm{Hom}(\mathcal{K}(\mathcal{C}), \mathbb{C})$ sending $\sigma = (Z, \mathcal{A})$ to its central charge Z, is a local isomorphism onto its image.*

In this thesis, by *stability condition* we always mean a stability condition in the sense of Bridgeland (Def. 2.20).

2.3.1 Wall and chamber structure

We say a stability condition is *supported* on a heart $\mathcal{A}$ if it is of the form $\sigma = (Z, \mathcal{A})$ and write $U(\mathcal{A})$ for the set of stability condition supported on $\mathcal{A}$.

Suppose that $\mathcal{A}$ is of finite length with n isomorphism classes of simple objects, then $U(\mathcal{A}) \simeq \bar{\mathbb{H}}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n | 0 < \arg z_i \leq \pi\}$. If $\sigma = (Z, \mathcal{A})$ is on the boundary of $U(\mathcal{A})$, there is some i such that $Z([S_i])$ lies on the real axis. Under this hypothesis, the following holds

Lemma 2.22 ([7, Lemma 5.5]). *The codimension one boundary components of a region $U(\mathcal{A})$ are parametrized by simple objects S_k together with a sign. Each of them consists of stability conditions $\sigma = (Z, \mathcal{A})$ for which $Z(S_k) \in \mathbb{R}_{<0}$ or $Z(S_k) \in \mathbb{R}_{>0}$, for a unique k. In the first case a neighborhood of σ is contained in $\overline{U(\mathcal{A})} \cup \overline{U(\mu_k^+ \mathcal{A})}$, else in $\overline{U(\mathcal{A})} \cup \overline{U(\mu_k^- \mathcal{A})}$, provided that $\mu^{\pm}(\mathcal{A})$ is again of finite type.*

If all the tilted hearts are again of finite type, the process can be iterated and subsets of the space of stability conditions gain a so-called *wall and chamber* structure. If moreover $\mathcal{A}$ has only a finite number of indecomposables, then the whole connected component of $Stab(\mathcal{C})$ containing $U(\mathcal{A})$ is covered by chambers $U(\mathcal{A}')$ isomorphic to $Stab(\mathcal{A}') \simeq \overline{\mathbb{H}}$ (Woolf, [41]), where $\mathcal{A}'$ is reachable from $\mathcal{A}$ and two regions $U(\mathcal{A}')$ and $U(\mathcal{A}'')$ are glued together along boundaries when $\mathcal{A}'$, $\mathcal{A}''$ are related by tilts at simple objects. For this reason, when $\mathcal{C} = \mathcal{D}(Q, W)$, mutation classes of Q give a combinatorical description of part of its space of stability conditions.
Codimension-one boundaries of sets $U(\mathcal{A})$ are called *walls of the second kind*.

A different type of *walls* exists in any region $U(\mathcal{A}) \subset Stab(\mathcal{C})$, for a fixed heart $\mathcal{A}$. Let $\alpha, \beta \in \mathcal{K}(\mathcal{A})$ be two classes not proportional to each other. The set $W_\alpha(\beta)$ of central charges Z such that $Z(\beta)/Z(\alpha) \in \mathbb{R}_{>0}$ is a codimension one submanifold of $Stab(\mathcal{A}) \simeq U(\mathcal{A})$ and it is called a *wall of marginal stability*. Let $\alpha = \sum_{i=1}^n \alpha_i [S_i]$, $\alpha_i \in \mathbb{N}$. Then there are only finitely many classes $\beta \in \mathcal{K}(\mathcal{A})$ such that $\beta = \sum_{i=1}^n \beta_i [S_i]$ and $\beta_i \leq \alpha_i$. We write $\beta < \alpha$ if the inequality holds strictly for at least one index i. One may consider the complement $\mathcal{C}_\alpha \subset U(\mathcal{A})$ of

$$\bigcup_{\beta < \alpha} \overline{W_\alpha(\beta)}.$$

Lemma 2.23. *In any connected component $C \subseteq \mathcal{C}_\alpha$, an object E of class α is (semi)stable with respect to a central charge $Z \in C$ if and only of it is (semi)stable with respect to all the central charges $Z' \in C$.*

2.4 Joyce holomorphic generating functions

In this section we introduce the Joyce generating functions for invariants counting semistable objects and we review their geometric meaning as studied by Bridgeland and Toledano-Laredo.

It is conjectured [30], and proven in some cases [32, Chapter 7], that given a suitable 3CY category $\mathcal{C}$ and a stability condition $Z \in \mathrm{Stab}(\mathcal{C})$, one can define enumerative invariants "counting" Z-semistable objects in $\mathcal{C}$ of a given class $\alpha \in \mathcal{K}(\mathcal{C})$. These invariants should be extension of the Donaldson-Thomas invariants DT for the derived category $D^b(X)$ of coherent sheaves on a Calabi-Yau three-fold X. In his groundbreaking work [31] Joyce studied how to combine these Donaldson-Thomas type invariants (which we will refer to again as DT) into continuous and holomorphic functions f^α of the form

$$f^\alpha = \sum_{n \geq 1} \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} J_n(Z(\alpha_1), \ldots, Z(\alpha_n)) \prod_{i=1}^n \mathrm{DT}(\alpha_i, Z) c(\alpha_1, \ldots, \alpha_n) x_\alpha.$$

The f^α are defined over the space of stability conditions $\mathrm{Stab}(\mathcal{C})$ and take values in a graded Lie algebra generated by elements x_α, $\alpha \in \mathcal{K}(\mathcal{C})$.

Following Joyce, we consider the DT invariants just as $\mathbb{Q}$-valued locally constant functions on the space $\mathrm{Stab}(\mathcal{C})$, which jump discontinuously when Z crosses a wall of marginal stability and obey some specific transformation laws, [31, Section 2]. Their discontinuity is balanced by universal functions $J_n : (\mathbb{C}^*)^n \to \mathbb{C}$, depending only on their "wall-crossing" behavior. In particular, the function J_n are independent on the underlying category. We refer to these holomorphic functions as the *Joyce generating functions*.

In the same article, Joyce also pointed out a number of basic convergence problems in the theory, and showed how the f^α lead formally to very interesting new geometric structures.

One of the ideas underlying [31] was to describe a picture with some analogies with the Gromov-Witten (GW) theory. In GW theory, a formal power series Φ (the Gromov-Witten potential) encoding invariants counting rational curves on a manifold M satisfies a special PDE, the WDVV equation. This PDE can be interpreted as the flatness of a 1-parameter family of connections and makes Φ into the potential of a (formal) Frobenius structure (called the Quantum Cohomology $H^*_q(M)$). This point of view was later explored by Bridgeland and Toledano-Laredo [10], which reinterpreted the Joyce's formulae as defining an irregular connection on $\mathbb{P}^1$.

Assume that $\mathcal{C}$ admits a well-defined numerical Donaldson-Thomas type theory with invariants $\mathrm{DT}(\alpha, Z)$, virtually enumerating Z-semistable objects with class $\alpha \in \mathcal{K}(\mathcal{C})$, $Z \in \mathrm{Stab}(\mathcal{C})$ and satisfying the assumptions of [31, Section 2.2]. We write Z for a stability function on an abelian category or a point $\sigma = (\mathcal{A}, Z) \in \mathrm{Stab}(\mathcal{C})$ in the triangulated case.

We define the Kontsevich-Soibelman algebra of a category $\mathcal{C}$ with Grothendieck group $\mathcal{K}(\mathcal{C})$.

Definition 2.24. The Kontsevich-Soibelman Poisson algebra $\mathfrak{g}_{\mathcal{K}(\mathcal{C})}$ is the associative group algebra $\mathbb{C}[\mathcal{K}(\mathcal{C})]$ generated by formal elements x_α, $\alpha \in \mathcal{K}(\mathcal{C})$, endowed with Lie bracket induced by the Euler form $\langle -, - \rangle$ on $\mathcal{C}$

$$[x_\alpha, x_\beta] = (-1)^{\langle \alpha, \beta \rangle} \langle \alpha, \beta \rangle x_{\alpha+\beta}.$$

It is a graded algebra of infinite sums $\mathbb{C}[\mathcal{K}(\mathcal{C})] = \prod_{\alpha \in \mathcal{K}(\mathcal{C})} \mathbb{C}x_\alpha$ with commutative multiplication

$$\begin{aligned} \mathbb{C}x_\alpha \times \mathbb{C}x_\beta &\to \mathbb{C}x_{\alpha+\beta} \\ x_\alpha \cdot x_\beta &= (-1)^{\langle \alpha, \beta \rangle} x_{\alpha+\beta} \end{aligned}$$

and with a derivation: a central charge $Z \in \mathrm{Hom}(\mathcal{K}(\mathcal{C}), \mathbb{C})$ defines an endomorphism of $\mathbb{C}[\mathcal{K}(\mathcal{C})]$ satisfying the Leibniz rule by $Z(x_\alpha) := Z(\alpha) x_\alpha$.

We introduce the following coefficients

$$c(\alpha_1, \cdots, \alpha_n) = \sum_T \frac{1}{2^{n-1}} \prod_{\{i \to j\} \subset T} (-1)^{\langle \alpha_i, \alpha_j \rangle} \langle \alpha_i, \alpha_j \rangle, \tag{2.4.1}$$

given by a sum over connected trees T with vertices labelled by $\{1, \ldots, n\}$, endowed with an orientation respecting the total order (edge $i \to j$ implies $i < j$).

Theorem 2.25 ([31]). *a) There exist universal functions $J_n : (\mathbb{C}^*)^n \to \mathbb{C}$ continuous and holomorphic out of the locus where $z_i / z_{i+1} \in \mathbb{R}_{>0}$ for some $1 \leq i < n$, such that*

$$f^\alpha(Z) := \sum_{n \geq 1} \sum_{\substack{\alpha_1 + \cdots + \alpha_n = \alpha, \\ Z(\alpha_i) \neq 0}} J_n(Z(\alpha_1), \ldots, Z(\alpha_n)) \prod_{i=1}^n \mathrm{DT}(\alpha_i, Z) x_\alpha \tag{2.4.2}$$

is a formally continuous and holomorphic function taking values in the completion $\widehat{\mathbb{C}[\mathcal{K}_{>0}(\mathcal{C})]}$ *of* $\mathbb{C}[\mathcal{K}_{>0}(\mathcal{C})]$ *with respect to the ideal generated by* $x_{[S_1]}, \dots, x_{[S_n]}$ *when* $\mathcal{C}$ *is finite abelian, or* $\prod_{\alpha\in\mathcal{K}(\mathcal{C})\setminus\{0\}} \mathbb{C}x_\alpha$ *otherwise.*

b) The functions J_n*,* $n \geq 1$*, satisfy the differential equation*

$$\mathrm{d}\, J_n(z_1, \dots, z_n) = \sum_{n=1}^{n-1} J_i(z_1, \dots, z_i) J_{n-i}(z_{i+1}, \dots, z_n)\, \mathrm{d}\log\Big(\frac{z_{i+1} + \cdots + z_n}{z_1 + \cdots + z_i}\Big) \tag{2.4.3}$$

and are unique provided that they satisfy the conditions $J_1(z_1) \equiv (2\pi i)^{-1}$*, and*

$$J_n(\lambda z_1, \dots, \lambda z_n) = J(z_1, \dots, z_n),$$
$$J_n(z_1, \dots, z_n) = 0 \quad \text{if } z_1 + \cdots + z_n = 0,$$

for $n \geq 2$*,* $\lambda \in \mathbb{C}^*$*.*

It is important to note that (2.4.2) are infinite sums and are treated as "convergent in as strong sense as necessary" in [31]. Convergence issues are considered in [31, Section 5]. Only in the case when $\mathcal{C}$ is a finite abelian category, these functions are known to converge. In general they are infinite sums and their convergence may depend on the summation order. For instance, this happens for triangulated 3CY categories. Indeed, the DT invariants are symmetric with respect to the shift functor, in the sense that $\mathrm{DT}(\alpha, Z) = \mathrm{DT}(-\alpha, Z)$, where the involution $\alpha \mapsto -\alpha$ in $\mathcal{K}(\mathcal{C})$ is induced by the shift functor $[1] : \mathcal{C} \to \mathcal{C}$. This implies that there are infinitely many decompositions $\alpha_1 + \cdots + \alpha_n$ with $\prod_{i=1}^n \mathrm{DT}(\alpha_i, Z) \neq 0$ and the sum (2.4.2) contains in general infinitely many terms: the convergence problem is ill-posed.

In Section 3.2.2 we reformulate the problem transforming the infinite sum f^α into well-defined formal power series $f^\alpha_{\mathbf{s}}$ in an auxiliary vector of variables $\mathbf{s}$ of the form

$$f^\alpha_{\mathbf{s}}(Z) = \sum_{n\geq 1} \sum_{\substack{\alpha_1+\cdots+\alpha_n=\alpha,\\ Z(\alpha_i)\neq 0}} J_n(Z(\alpha_1), \dots, Z(\alpha_n)) \prod_i \mathbf{s}^{\alpha_i}\, \mathrm{DT}(\alpha_i, Z) c(\alpha_1, \dots, \alpha_n) x_\alpha.$$

Again, we don't claim convergence results for $f^\alpha_{\mathbf{s}}$ and an abstract setting is required to be fully rigorous. However the coefficients are always well-defined since there are only finitely many decompositions modulo "powers of $\mathbf{s}$".

In the rest of the Section we deal with Joyce generating functions restricting to the finite abelian case or ignoring convergence problems.

We define f as the formal infinite sum $f(Z) = \sum_{\alpha\in\mathcal{K}(\mathcal{C})\setminus\{0\}} f^\alpha$ of elements in $\widehat{\mathbb{C}[\mathcal{K}_{>0}(\mathcal{C})]}$ or $\prod_{\alpha\in\mathcal{K}(\mathcal{C})\setminus\{0\}} \mathbb{C}x_\alpha$. We may write $f^\alpha(Z)$, $\alpha \in \mathcal{K}(\mathcal{C})$ as

$$f^\alpha(Z) = \hat{f}^\alpha(Z) x_\alpha,$$

where the holomorphic functions $\hat{f}^\alpha(Z)$ are given by

$$\hat{f}^\alpha(Z) = \sum_{n\geq 1} \sum_{\substack{\alpha_1+\cdots+\alpha_n=\alpha,\\ Z(\alpha_i)\neq 0}} c(\alpha_1, \dots, \alpha_n) J_n(Z(\alpha_1), \dots, Z(\alpha_n)) \prod_{i=1}^n \mathrm{DT}(\alpha_i, Z). \tag{2.4.4}$$

Equation (2.4.3) is equivalent to the following important PDEs for f^α

$$\mathrm{d}\, f^\alpha = -\sum_{\beta+\gamma=\alpha} [f^\beta, f^\gamma]\, \mathrm{d}\log Z(\beta), \tag{2.4.5}$$

and for $\hat{f}^\alpha$

$$\mathrm{d}\,\hat{f}^\alpha = -\sum_{\beta+\gamma=\alpha} (-1)^{\langle\beta,\gamma\rangle}\langle\beta,\gamma\rangle \hat{f}^\beta \hat{f}^\gamma \,\mathrm{d}\log Z(\beta). \tag{2.4.6}$$

Joyce also proved that, using the generating functions f^α and (2.4.5), it is possible to define a flat connection on an infinite dimensional bundle on the space $\mathrm{Stab}(\mathcal{C})$ with fibres $\mathbb{C}[\widehat{\mathcal{K}_{>0}(\mathcal{C})}]$ or $\prod_{\alpha\in\mathcal{K}(\mathcal{C})\setminus\{0\}} \mathbb{C}x_\alpha$.

Theorem 2.26. *The connection* $\nabla^J = \mathrm{d} + \Gamma(Z)$*, where* d *is the differential on* $\mathrm{Stab}(\mathcal{C})$ *and*

$$\Gamma(Z) := \sum_{\alpha\in\mathcal{K}(\mathcal{C}), Z(\alpha)\neq 0} f^\alpha(Z)\frac{\mathrm{d}\,Z(\alpha)}{Z(\alpha)},$$

has identically vanishing curvature $\mathrm{d}\,\Gamma + \Gamma\wedge\Gamma$*.*

Equations (2.4.5) play an important rôle in the work by Bridgeland and Toledano-Laredo [10]. They constructed an isomonodromic family of connections $\nabla^{BTL}(Z)$ over $\mathbb{P}^1$ parametrized by the space $\mathrm{Stab}(\mathcal{A})$ of stability functions of an abelian category $\mathcal{A}$, whose isomonodromic deformation equations are exactly (2.4.5).

Let $\mathcal{A}$ be a finite abelian category and $Z \in \mathrm{Stab}(\mathcal{A})$. We define the principal bundle P on $\mathbb{P}^1_z$, with fibers the automorphism group $\mathrm{Aut}(\hat{\mathfrak{g}}_{\mathcal{K}(\mathcal{A})})$ of the completion of the Kontsevich-Soibelman Lie algebra $\mathfrak{g}_{\mathcal{K}(\mathcal{A})} = \mathbb{C}[\mathcal{K}(\mathcal{A})]$ with respect to the ideal $(x_{[S_1]},\dots,x_{[S_n]})$. Let Φ^Z be the root system associated to the endomorphism Z.

Theorem 2.27 (Theo. 6.5 (ii),(iii)[10])**.** *There exists a unique connection* $\nabla^{BTL}(Z)$ *on* P *of the form*

$$\nabla^{BTL}(Z) = \mathrm{d} - \Big(\frac{Z}{z^2} + \frac{\mathrm{ad}\, f}{z}\Big)\,\mathrm{d}\,z \tag{2.4.7}$$

where the components of $f = \sum_{\alpha\in\mathcal{K}_{>0}(\mathcal{A})} f^\alpha$ *are the (positive) Joyce generating functions* (2.4.2)*, with Stokes factors the exponential in the algebra of derivations* $D(\hat{\mathfrak{g}}_{\mathcal{K}(\mathcal{C})})$

$$\mathcal{S}_\ell = \exp_{D(\hat{\mathfrak{g}}_{\mathcal{K}(\mathcal{C})})}\Big\{\sum_{\alpha: Z(\alpha)\in\ell} \mathrm{DT}(\alpha, Z)[x_\alpha, -]\Big\}.$$

As Z *varies in* $\mathrm{Stab}(\mathcal{A})$*, the family of connections* $\nabla^{BTL}(Z)$ *varies isomonodromically. The isomonodromy property is equivalent to the system of PDE*

$$\mathrm{d}\, f^\alpha = -\sum_{\substack{\beta+\gamma=\alpha,\\ \beta,\gamma\in\mathcal{K}_{>0}(\mathcal{A})}} [f^\beta, f^\gamma]\,\mathrm{d}\log Z(\beta).$$

Remark. Theorem 2.27 was originally stated for the Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ of constructible functions. I refer to [10, Section 4] for the theory of Hall algebras, and to [18, Section 5] for the correspondence with the formulation above.

Extending the result of Jimbo, Miwa and Ueno to the case of interest, the isomonodromy of $\nabla^{BTL}(Z)$ as Z vary in a subset $\mathcal{U}_0 \subset \mathrm{Stab}(\mathcal{A})$ may also be expressed in term of flatness of the connection

$$\overline{\nabla} = \mathrm{d} - \Big(\frac{Z}{z^2} + \frac{\mathrm{ad}\, f(Z)}{z}\Big) + \sum_{\alpha\in\mathcal{K}_{>0}(\mathcal{A})} f^\alpha(Z)\frac{\mathrm{d}\,Z(\alpha)}{Z(\alpha)} - \frac{1}{z}\,\mathrm{d}\,Z.$$

on $\mathbb{P}^1\times\mathcal{U}_0$.

Bridgeland and Toledano-Laredo also proved an inversion formula giving the coefficients J_n in terms of the multilogarithms M_m (Definition 1.21).

Theorem 2.28 ([11, Section 11]). *The functions J_n appearing in Joyce generating functions can be expressed explicitly as*

$$\begin{cases} J_1(z_1) = (2\pi i)^{-1} \\ (2\pi i)^n J_n(z_1, \dots, z_n) = \sum_T \left(\frac{-1}{2\pi i}\right)^{|V(T)|} L_T(z_1, \dots, z_n) \end{cases} \tag{2.4.8}$$

where T and L_T are defined below.

The sum in (2.4.8) is over rooted plane tree T with n leaves decorated with $z_1, \dots, z_n$, vertices $v \in V(T)$ with valency $val(v) \geq 3$, and outgoing arrows a from w decorated with $s_a = \sum_{i \in I(v)} s_i$ depending on the incoming arrows in $I(v)$.
$L_T(z_1, \dots, z_n) := \prod_{v \in V(T)} L_{val(v)-1}(s_1, \dots, s_{val(v)-1})$. The functions L_n are extensions over the branchcuts of the iterated integrals M_n (see (1.3.2)). They are complicated functions (see [11, Section 4.6] for the definition) and coincides with M_n in the open subsets where they are holomorphic

$$(z_1, \dots, z_n) \in (\mathbb{C}^*)^n \text{ such that } z_1 + \dots + z_i \notin [0, z_1 + \dots + z_n] \text{ for } 0 < i < n.$$

For example

$$J_2(w_1, w_2) = -\frac{1}{(2\pi i)^2}\Big(\frac{1}{2\pi i} M_2(w_1, w_2)\Big) \tag{2.4.9}$$

$$\begin{aligned} J_3(w_1, w_2, w_3) = &-\frac{1}{(2\pi i)^4} M_3(w_1, w_2, w_3) + \frac{1}{(2\pi i)^5} M_2(w_1, w_2) M_2(w_1 + w_2, w_3) + \\ &+ \frac{1}{(2\pi i)^5} M_2(w_1, w_2 + w_3) M_2(w_2, w_3) \end{aligned} \tag{2.4.10}$$

in their holomorphicity domains.

Chapter 3

Formal infinite-dimensional Frobenius type structures from DT theory

The goal of this chapter is to consider (formal) Frobenius type structures and CV-structures on open subsets of the space of stability conditions on certain 3CY triangulated categories $\mathcal{C}$. Actually, these families of structures depend on a vector of parameters $\mathbf{s}$ and live on an infinite-dimensional bundle $K \to \mathrm{Stab}(\mathcal{C})$. They are defined by collections of holomorphic $(\nabla^r_{\mathbf{s}}, C, \mathcal{U}, \mathcal{V}_{\mathbf{s}}, g)$ and smooth $(D, C, \tilde{C}, \mathcal{U}, \mathcal{Q}_{\mathbf{s}}, \kappa, h)$ objects respectively (see Definitions 1.27 and 1.32), using data from a Donaldson-Thomas theory on $\mathcal{C}$. These main results are in Theorems 3.26 and 3.27.

Part of the result is just a rephrasing of the results by Joyce and Bridgeland-Toledano Laredo summarized in Section 2.4. Modulo convergence issues, ∇^J and ∇^{BTL} naturally define a Frobenius type structure on a bundle with fibers $\mathbb{C}[\mathcal{K}(\mathcal{C})]$. This is part of a more complicated structure called CV. This picture is described in Section 3.1.

In order to be fully rigorous, we work in an abstract setting. It is introduced in Section 3.2. In particular, to make sense of the Joyce generating functions in the context of triangulated categories, f^α is deformed into a well-defined formal power series in the auxiliary parameter $\mathbf{s}$. Working formally in $\mathbb{C}[\![\mathbf{s}]\!]$, one can define formal families of Frobenius type and CV- structures on K. This is proven in Section 3.3. Theorem 3.30 is a result of uniform convergence for the operator $\mathcal{Q}_{\mathbf{s}}(\lambda Z)$ and thus for the scaled deformations $f^\alpha_{\mathbf{s}}(\lambda Z)$ of the Joyce functions, for $|\lambda|$ sufficiently large. Since its proof involves techniques not related with the Frobenius manifold theory, it is given in Appendix A.

3.1 An infinite-dimensional picture

Fix a category $\mathcal{C}$ with well-defined numerical Donaldson-Thomas type invariants $\mathrm{DT}(\alpha, Z)$ enumerating objects with class $\alpha \in \mathcal{K}(\mathcal{C})$, semistable with respect to a choice of stability condition. Assume that either

- $\mathcal{C}$ is finite abelian, with n simple objects $S_1, \dots, S_n$, or
- $\mathcal{C}$ is abelian but not finite, or

- $\mathcal{C}$ is a 3CY triangulated category which admits a bounded $t-$structure with a finite heart $\mathcal{A}$ with n simple objects $S_1, \dots, S_n$. In particular $\mathcal{K}(\mathcal{C}) \simeq \mathcal{K}(\mathcal{A})$.

Denote by $\langle -, - \rangle$ the Euler pairing on $\mathcal{K}(\mathcal{C})$ and by Z a point in $\mathrm{Stab}(\mathcal{C})$. When $\mathcal{C}$ is abelian Z is a stability function $\mathcal{K}_{>0}(\mathcal{C}) \to \mathbb{H}$. When $\mathcal{C}$ is triangulated we write Z for the central charge of a stability condition $\sigma = (\mathcal{A}, Z)$, $Z : \mathcal{K}(\mathcal{C}) \to \mathbb{C}$.

Consider the Kontsevich-Soibelman Lie algebra $\mathbb{C}[\mathcal{K}(\mathcal{C})]$ generated by elements x_α. Recall that a central charge Z induces a derivation of $\mathbb{C}[\mathcal{K}(\mathcal{C})]$ acting by $Z(x_\alpha) = Z(\alpha) x_\alpha$.

Definition 3.1. Set $K \to \mathrm{Stab}(\mathcal{C})$ the trivial infinite-dimensional vector bundle with fibers

- $\mathbb{C}[\widehat{\mathcal{K}_{>0}(\mathcal{C})}]$, when $\mathcal{C}$ is finite abelian, where $\mathbb{C}[\widehat{\mathcal{K}_{>0}(\mathcal{C})}]$ is the completion of $\mathbb{C}[\mathcal{K}_{>0}(\mathcal{C})]$ along the ideal generated by the classes of simple objects $[S_1], \dots, [S_n]$;
- $\prod_{\alpha \in \mathcal{K}_{>0}(\mathcal{C}) \setminus \{0\}} \mathbb{C} x_\alpha$ when $\mathcal{C}$ is abelian not finite; and
- $\prod_{\alpha \in \mathcal{K}(\mathcal{C}) \setminus \{0\}} \mathbb{C} x_\alpha$, otherwise.

Remark. In the rest of this section, when summing over $\alpha \in \mathcal{K}(\mathcal{C})$, we will always assume $\alpha \neq 0$.

Although the following constructions in Proposition 3.2 and 3.5 are ill-defined because of the convergence problems pointed out in 2.4, they motivate the results of Section 3.3.

Proposition 3.2. *Let $K \to \mathrm{Stab}(\mathcal{C})$ be the trivial infinite-dimensional vector bundle of Definition 3.1 (in particular we have $\bar{\partial}_K x_\alpha = 0$). Fix a constant $g_0 \in \mathbb{C}^*$. Then*

$$\begin{aligned} \nabla^r &:= \mathrm{d} + \sum_\alpha \mathrm{ad}\, f^\alpha(Z) \frac{\mathrm{d}\, Z(\alpha)}{Z(\alpha)}, \\ C &:= -\,\mathrm{d}\, Z, \\ \mathcal{U} &:= Z, \\ \mathcal{V} &:= \mathrm{ad}\, f(Z) \end{aligned}$$

satisfy the conditions (1.6.1) *of definition 1.27*

$$\nabla^r(C) = 0, \quad [C, \mathcal{U}] = 0, \quad \nabla^r(\mathcal{V}) = 0,$$
$$\nabla^r(\mathcal{U}) - [C, \mathcal{V}] + C = 0.$$

If moreover $\mathcal{C}$ is triangulated we can complete these to a Frobenius type structure with the choice

$$g(x_\alpha, x_\beta) = g_0 \delta_{\alpha\beta}.$$

The proof is based on [31] and [10]. The function $Z(\alpha)^{-1} f^\alpha(Z)$ extends across the locus where $Z(\alpha) = 0$, see [31, Section 5].

Proof. Let us first consider the choice for the Higgs field C. For all $\gamma \in \mathcal{K}(\mathcal{C})$ the function $Z \mapsto Z(\gamma)$ is a local holomorphic function on $\mathrm{Stab}(\mathcal{C})$ [31]. So we can define a 1-form with values in endomorphisms by

$$\mathrm{d}\, Z(X) x_\gamma = (X Z(\gamma)) x_\gamma$$

for all local holomorphic vector fields X. One checks that $\mathrm{d}\, Z \wedge dZ = 0$.

To check (1.6.1), (1.6.2) one uses repeatedly a PDE on the functions $f^{\alpha}(Z)$ (see [31, Equation (4)]),

$$\mathrm{d}\, f^{\alpha}(Z) = \sum_{\beta,\gamma\in\mathcal{K}(\mathcal{C})\setminus\{0\},\, \alpha=\beta+\gamma} [f^{\beta}, f^{\gamma}]\, \mathrm{d}\log Z(\gamma). \tag{3.1.1}$$

Flatness of ∇^r and covariant constancy of $\mathcal{V}$ follow from the same computations as in [31] section 4 (in particular equations (71) - (73)). The other conditions follow from straightforward computations. As an example we have

$$\begin{aligned}\nabla^r(\mathrm{d}\, Z) = \mathrm{d}^2\, Z + \mathrm{ad}\sum_{\alpha} f^{\alpha}(Z)\frac{\mathrm{d}\, Z(\alpha)}{Z(\alpha)} \wedge \mathrm{d}\, Z\\ + \mathrm{d}\, Z \wedge \mathrm{ad}\sum_{\alpha} f^{\alpha}(Z)\frac{\mathrm{d}\, Z(\alpha)}{Z(\alpha)}\end{aligned}$$

where $\wedge$ denotes the composition of endomorphisms combined with the wedge product of forms. Now $\mathrm{d}^2\, Z = 0$, and evaluating on a section x_{β} gives a 2-form with values in K

$$\begin{aligned}\nabla^r(dZ)x_{\beta} &= \sum_{\alpha}[f^{\alpha}(Z), x_{\beta}](Z(\alpha))^{-1}\, \mathrm{d}\, Z(\alpha)\wedge \mathrm{d}\, Z(\beta)\\ &+ \sum_{\alpha}[f^{\alpha}(Z), x_{\beta}](Z(\alpha))^{-1}\, \mathrm{d}\, Z(\alpha+\beta)\wedge \mathrm{d}\, Z(\alpha).\end{aligned}$$

But we have $\mathrm{d}\, Z(\alpha+\beta) = \mathrm{d}\, Z(\alpha) + \mathrm{d}\, Z(\beta)$ and the vanishing $\nabla^r(\mathrm{d}\, Z)x_{\beta} = 0$ follows for all β.

As an example of a condition involving the quadratic form g in the triangulated case we check skew-symmetry of $\mathcal{V}$. We have

$$\begin{aligned}g(\mathcal{V}x_{\alpha}, x_{\beta}) &= \sum_{\gamma}\hat{f}^{\gamma}(Z)(-1)^{\langle\gamma,\alpha\rangle}\langle\gamma,\alpha\rangle g_{\alpha+\gamma,\beta}\\ &= g_0\sum_{\gamma}\hat{f}^{\gamma}(Z)\langle\gamma,\alpha\rangle\delta_{\alpha+\gamma,\beta}\\ &= g_0(-1)^{\langle\beta,\alpha\rangle}\langle\beta,\alpha\rangle\hat{f}^{\beta-\alpha}(Z).\end{aligned}$$

Similarly

$$g(x_{\alpha}, \mathcal{V}x_{\beta}) = g_0(-1)^{\langle\alpha,\beta\rangle}\langle\alpha,\beta\rangle\hat{f}^{\alpha-\beta}(Z).$$

In the 3CY case we have $\hat{f}^{\alpha-\beta}(Z) = \hat{f}^{\beta-\alpha}(Z)$ because of the shift functor. □

Recall that there is a standard construction of a flat structure connection from a Frobenius type structure (Theorem 1.28). In the Donaldson-Thomas case this has a further scale invariance property.

Lemma 3.3. *Let* $p\colon \mathbb{P}^1_z\times\mathrm{Stab}(\mathcal{C})\to\mathrm{Stab}(\mathcal{C})$ *denote the projection. Let* $\lambda\in\mathbb{R}^+$ *denote a scaling parameter. The meromorphic connection on* p^*K *given by*

$$\nabla^r + \frac{C}{z} + \left(\frac{1}{z^2}\mathcal{U} - \frac{1}{z}\mathcal{V}\right)\mathrm{d}\, z$$

is flat and invariant under the rescaling $Z\mapsto\lambda Z$, $z\mapsto\lambda z$. *In particular the Joyce function* $f(Z)$ *has the "conformal invariance" property* $f(\lambda Z) = f(Z)$.

Proof. Flatness of the connection follows from the conditions (1.6.1). Invariance under the rescaling is equivalent to the property $f(\lambda Z) = f(Z)$ which is established in [31]. □

The Frobenius type structure of Proposition 3.2 is part of a more complicated (formal) CV-structure. This point of view is also suggested naturally by [19]. Recall the notions of $(DC\widetilde{(C)})-$ and $CV-$structures presented in Section 1.7.

Lemma 3.4. *Let $K \to \mathrm{Stab}(\mathcal{C})$ be the vector bundle of Definition 3.1. Then there is a $(DC\widetilde{C})$-structure on K given by*

$$D' := \nabla^r, \quad D'' := \bar{\partial}_K,$$
$$C := -\,\mathrm{d}\,Z, \quad \widetilde{C} := \mathrm{d}\,\bar{Z}.$$

Proof. Let $\bar{\partial}_K$ denote our fixed (trivial) complex structure on K, with $\bar{\partial}_K(x_\alpha) = 0$. The condition $(D'' + C)^2 = 0$ says that K is holomorphic and C is a holomorphic Higgs bundle on it, which we know already from Proposition 3.2. Then $D'(C) = 0$ says that C is flat with respect to ∇^r, which we also know already. The condition $(D' + \widetilde{C})^2 = 0$ says that ∇^r is flat (known), $(\mathrm{d}\,\bar{Z})^2 = 0$ and $\nabla^r(\mathrm{d}\,\bar{Z}) = 0$ (easily checked). The condition $D''(\widetilde{C}) = 0$ becomes $\bar{\partial}_K(\mathrm{d}\,\bar{Z}) = 0$ and can be checked e.g. in local coordinates on $\mathrm{Stab}(\mathcal{C})$ given by $z_k = Z(\alpha_k)$ where $\alpha_1, \dots, \alpha_k$ is a basis for $\mathcal{K}(\mathcal{C})$. Finally in our case one checks that we have separately $C\widetilde{C} + \widetilde{C}C = 0$ and $D'D'' + D''D' = 0$. □

Denote by ι the involution of K acting as complex conjugation, combined with $x_\alpha \mapsto x_{-\alpha}$ in the triangulated case. Let ψ be a fixed endomorphism of K. Then we can make the following ansatz on part of the data of a CV-structure on K:

- κ is the conjugate involution $\mathrm{Ad}_\psi(\iota)$,
- the pseudo-hermitian metric h is given by $h(a, b) = g(a, \kappa(b))$ where g is the quadratic form of Proposition 3.2, when $\mathcal{C}$ is triangulated,
- $\mathcal{U}$ is the endomorphism $-Z$ as in Proposition 3.2,
- the Higgs field C is given by $-dZ$ as in Proposition 3.2, and the anti-Higgs $\widetilde{C}$ is given by $\kappa C \kappa$.

Proposition 3.5. *Let $K \to \mathrm{Stab}(\mathcal{C})$ be the vector bundle of Definition 3.1.*

(a) There exist endomorphisms $\psi(Z)$, $\mathcal{Q}(Z)$ and a connection D on K such that the choices of C, $\widetilde{C}$, κ, h, $\mathcal{U}$ above, together with D and $\mathcal{Q}$, give a CV-structure on K (in the abelian case only the conditions not involving h are satisfied). Moreover ψ and $\mathcal{Q}$ induce fibrewise derivations of $\mathbb{C}[\mathcal{K}(\mathcal{C})]$ as a commutative algebra.

(b) Fix Z and let $\lambda \in \mathbb{R}^+$ denote a scaling parameter. Then

$$\lim_{\lambda \to 0} \mathcal{Q}(\lambda Z) = \mathcal{V},$$

where $\mathcal{V} = \mathrm{ad}\, f(Z)$ is the endomorphism of Proposition 3.2 (i.e. essentially the Joyce holomorphic generating function).

Proof. We will explain a rigorous approach and prove a rigorous result (which applies to sufficiently simple abelian and triangulated categories) in section 3.3 and Theorem 3.27. The present formal statement can be "proved" (in the same sense as Proposition 3.2) by the same arguments provided we work with formal infinite sums, ignoring convergence issues. □

In the light of Proposition 3.5 (b) it is natural to make the following definition.

Definition 3.6. The *CV-deformation* of the Joyce holomorphic generating function $f(Z)$ is the operator $\mathcal{Q}(Z)$ given by Proposition 3.5 (a).

There is an analogue of Lemma 3.3, which gives a new point of view on the conformal invariance property $f(\lambda Z) = f(Z)$. It follows from the proof of Proposition 3.27.

Lemma 3.7. *Let $(D, C, \widetilde{C}, \kappa, h, \mathcal{U}, \mathcal{Q})$ be the CV-structure of Proposition 3.5. Let $p\colon \mathbb{P}^1_z \times \mathrm{Stab}(\mathcal{C}) \to \mathrm{Stab}(\mathcal{C})$ denote the projection, and suppose $\lambda \in \mathbb{R}^+$ is a scaling parameter. The meromorphic connection on $p^*K \to \mathbb{P}^1_z \times \mathrm{Stab}(\mathcal{C})$ given by*

$$D + \frac{C}{z} + z\widetilde{C} + \left(\frac{1}{z^2}\mathcal{U} - \frac{1}{z}\mathcal{Q} - \kappa\mathcal{U}\kappa\right) dz$$

is flat. Under the scaling $Z \mapsto \lambda Z$, $z = \lambda t$, in the limit $\lambda \to 0$ it flows to the flat connection of Lemma 3.3.

3.2 Abstract setting

We define an abstract setting modeled on the case of a triangulated 3CY category with a finite heart. It has the advantage of being fully rigorous, independently of the fundational problems of Donaldson-Thomas theory for 3CY categories. The analogies with the category language is easily identified. However it is made explicit at the end of the Section.

3.2.1 Stability data

Let Γ be a finite rank lattice with a skew-symmetric bilinear form $\langle -, - \rangle$. We denote by n its rank.

Definition 3.8. A *central charge* Z is a group homomorphism $\Gamma \to \mathbb{C}$.

Definition 3.9. A *spectrum* is a function of the form

$$(\alpha, Z) \mapsto \Omega(\alpha, Z) \in \mathbb{Q}$$

for all $\alpha \in \Gamma$ and Z varying in an open subset U of a linear subspace of $\mathrm{Hom}(\Gamma, \mathbb{C})$.
We say that the spectrum Ω is

- *positive* if there exists a $\mathbb{Z}$-basis $\{\gamma_i\}$ of Γ such that $\Omega(\alpha, Z)$ vanishes unless α is a positive integral combination of the γ_i. In this case we say that $\{\gamma_i\}$ is a positive basis for Ω;
- *symmetric* if
$$\Omega(\alpha, Z) = \Omega(-\alpha, Z)$$
for all $\alpha \in \Gamma$, $Z \in U$.
- the *double of a positive spectrum* if Ω is symmetric and there is a positive spectrum $\widetilde{\Omega}$ such that $\Omega(\alpha, Z) = \widetilde{\Omega}(\pm\alpha, Z)$ for all $\alpha \in \Gamma$, $Z \in U$.

A *distinguished ray* $\ell_\alpha(Z) \subset \mathbb{C}^*$ is a ray of the form $\mathbb{R}_{>0} Z(\alpha)$ such that $\Omega(\alpha, Z) \neq 0$.

Definition 3.10. Let $\{\gamma_1, \dots, \gamma_n\}$ be a fixed basis for Γ. The locus of *positive central charges* $\mathrm{Hom}^+(\Gamma, \mathbb{C}) \subset \mathrm{Hom}(\Gamma, \mathbb{C})$ is given by central charges mapping $\{\gamma_i\}$ to the open upper half plane $\mathbb{H} \subset \mathbb{C}$.
By Γ^+ we denote the "effective cone" given by positive linear combinations of the $\{\gamma_i\}$.

Thus, $\mathrm{Hom}^+(\Gamma, \mathbb{C})$ consists of central charges mapping Γ^+ to $\mathbb{H}$.

Definition 3.11. We say that $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$ and Ω satisfy the *support condition* if there exists a constant $c > 0$ such that picking a norm $|| - ||$ on $\Gamma \otimes \mathbb{C}$ we have

$$|Z(\alpha)| > c||\alpha|| \tag{3.2.1}$$

for all $\alpha \in \Gamma$ with $\Omega(\alpha, Z) \neq 0$. The condition does not depend on the specific choice of norm.

Note that if Ω is positive or the double of a positive spectrum parametrised by $\mathrm{Hom}^+(\Gamma, \mathbb{C})$ then the support condition is automatically satisfied on $\mathrm{Hom}^+(\Gamma, \mathbb{C})$. It holds uniformly on all subsets of $\mathrm{Hom}^+(\Gamma, C)$ where Z is bounded away from zero on the elements of a positive basis $\{\gamma_i\}$.

Definition 3.12. We say that a spectrum Ω *grows at most exponentially at* Z if there is a $\lambda > 0$ such that

$$\sum_{\alpha \in \Gamma} |\Omega(\alpha, Z)| \exp(-|Z(\alpha)|\lambda) < \infty. \tag{3.2.2}$$

Definition 3.13. The *Kontsevich-Soibelman Poisson algebra* $\mathfrak{g}_\Gamma$ is the (associative, commutative) group algebra $\mathbb{C}[\Gamma]$ endowed with the Lie bracket induced by $\langle -, - \rangle$: $\mathfrak{g}_\Gamma$ is generated by x_α, $\alpha \in \Gamma$, with bracket $[x_\alpha, x_\beta] = (-1)^{\langle \alpha, \beta \rangle} \langle \alpha, \beta \rangle x_{\alpha+\beta}$, and product $x_\alpha x_\beta = (-1)^{\langle \alpha, \beta \rangle} x_{\alpha+\beta}$.

One checks that $\mathfrak{g}_\Gamma$ is indeed Poisson, i.e. inner Lie algebra derivations are in fact commutative algebra derivations. To avoid confusion we write $\exp_*$ for the commutative algebra exponential in $\mathfrak{g}_\Gamma$.

Lemma 3.14. *A central charge Z defines an endomorphism of $\mathfrak{g}_\Gamma$ by $Z(x_\alpha) = Z(\alpha) x_\alpha$. This is in fact a commutative algebra derivation.*

Definition 3.15. Fix a basis $\{\gamma_i\}$ as above. We write $\mathfrak{g}_{>0} \subset \mathfrak{g}_\Gamma$ for the monoid generated by x_α where α is nonzero and has nonnegative coefficients with respect to the basis. We let $\widehat{\mathfrak{g}}_{>0}$ be the completion of $\mathfrak{g}_{>0}$ along the ideal $(x_{\gamma_1}, \dots, x_{\gamma_n})$.

Let $\mathrm{DT}(\alpha, Z)$ denote the Möbius transform of Ω,

$$\mathrm{DT}(\alpha, Z) = \sum_{k>0, k|\alpha} \frac{1}{k^2} \Omega(k^{-1}\alpha, Z). \tag{3.2.3}$$

For a choice of a strictly convex cone $\Sigma \subset \mathbb{C}^*$ and a central charge Z satisfying the support condition, there is a Poisson Lie algebra $\mathfrak{g}_{\Gamma,\Sigma,Z}$ topologically generated by elements x_α with $Z(\alpha) \in \Sigma$, and its completion $\widehat{\mathfrak{g}}_{\Gamma,\Sigma,Z}$. Assume that $Z(\alpha) \notin \partial\Sigma$ for any $\alpha \in \Gamma$. We denote by $\exp(\mathfrak{g}_{\Gamma,\Sigma,Z})$ and $\exp(\widehat{\mathfrak{g}}_{\Gamma,\Sigma,Z})$ the corresponding formal Lie groups (i.e. the group law is defined formally by the usual Baker-Campbell-Hausdorff formula). Let $l \in \mathbb{R}$.

We define $\mathfrak{g}^{l}_{\Gamma,\Sigma,Z}$ the subalgebra generated by elements x_α corresponding to points in Γ of length greater or equal than l

$$\mathfrak{g}^{l}_{\Gamma,\Sigma,Z} = \oplus_{\substack{\alpha\in\Gamma, Z(\alpha)\in\Sigma,\\ l(\alpha)\geq l}} \mathbb{C}\cdot x^{\alpha} \subset g_{\Gamma,\Sigma,Z}.$$

Assume that for any $l \in \mathbb{R}_{>0}$, there exists an open neighborhood U' of Z, $U' \subseteq U \subset \mathrm{Hom}(\Gamma,\mathbb{C})$, $U' \neq \emptyset$, such that for any central charge $Z' \in U'$, there are no points γ of length $l(\gamma) \leq l$ and $Z'(\gamma) \in \partial\Sigma$. Then the quotient

$$\mathfrak{g}^{\leq l}_{\Gamma,\Sigma} := \mathfrak{g}_{\Gamma,\Sigma,Z}/\mathfrak{g}^{l}_{\Gamma,\Sigma,Z} \tag{3.2.4}$$

does not change when varying Z as long as no distinguished ray $\ell_\alpha(Z)$ crosses the boundary $\partial\Sigma$.

Definition 3.16. The *family of stability data on* $\mathfrak{g}_\Gamma$ *parametrised by* U corresponding to the spectrum Ω is the $\mathfrak{g}_\Gamma$-valued function given by

$$(\alpha, Z) \mapsto \mathrm{DT}(\alpha, Z)x_\alpha.$$

This family of stability data on $\mathfrak{g}_\Gamma$ is continuous in the sense of [35] if the condition above holds, and if all $Z \in U'$ satisfy the support condition, and for all fixed strictly convex cone $\Sigma \subset \mathbb{C}^*$ the group element

$$\prod^{\curvearrowright, Z}_{\ell\subset\Sigma} \exp\left(\sum_{Z(\alpha)\in\ell} \mathrm{DT}(\alpha, Z)x_\alpha\right) \in \exp(\mathfrak{g}^{\leq l}_{\Gamma,\Sigma}) \tag{3.2.5}$$

is constant as long as no distinguished ray $\ell_\alpha(Z)$ crosses the boundary $\partial\Sigma$, where $\prod^{\curvearrowright, Z}$ denotes the operator writing the ensuing group elements from left to right according to the clockwise Z-order.

We say that the spectrum Ω is continuous is the corresponding family of stability data on $\mathfrak{g}_\Gamma$ is. We say that the family of stability data $\mathrm{DT}(\alpha, Z)$ is positive, symmetric, or the double of a positive family if the corresponding condition is satisfied by the underlying spectrum $\Omega(\alpha, Z)$ given by inverting (3.2.3),

$$\Omega(\alpha, Z) = \sum_{k|\alpha} \frac{1}{k^2} m(k)\,\mathrm{DT}(k^{-1}\alpha, Z)$$

where m denotes the Möbius function.

It will be important for us to regard the group element in (3.2.5), under suitable conditions, as a product of explicit "symplectomorphisms".

Definition 3.17. A central charge $Z \in \mathrm{Hom}(\Gamma,\mathbb{C})$ is *generic* if elements x_α, x_β with $Z(\alpha), Z(\beta)$ lying on the same ray ℓ have vanishing Lie bracket (i.e. $\langle\alpha,\beta\rangle = 0$). We say that Z is *strongly generic* if $Z(\alpha), Z(\beta)$ lying on the same ray ℓ implies that α, β are linearly dependent. We write $\mathrm{Hom}^{sg}(\Gamma,\mathbb{C})$ for the locus of strongly generic central charges.

Let $\mathrm{Aut}(\widehat{\mathfrak{g}}_{\Gamma,\Sigma,Z})$ and $D(\widehat{\mathfrak{g}}_{\Gamma,\Sigma,Z})$ denote the group of automorphisms of $\widehat{\mathfrak{g}}_{\Gamma,\Sigma,Z}$ as a commutative, associative algebra, respectively the $\widehat{\mathfrak{g}}_{\Gamma,\Sigma,Z}$-module of commutative algebra derivations. For $\Omega \in \mathbb{Q}$, $Z(\beta) \in \Sigma$ let T^{Ω}_{β} denote the element of $\mathrm{Aut}(\widehat{\mathfrak{g}}_{\Gamma,\Sigma,Z})$ given by

$$T^{\Omega}_{\beta}(x_\alpha) = x_\alpha(1 - x_\beta)^{\langle\beta,\alpha\rangle\Omega}$$

(the right hand side denoting a formal power series expansion). In fact T_β^Ω is a Poisson automorphism (it preserves the Lie bracket). This follows from the identity

$$T_\beta^\Omega = \exp_{D(\widehat{\mathfrak{g}}_{\Gamma,\Sigma,Z})}\left(-\Omega\sum_{k\geq 1}\frac{[x_{k\beta},-]}{k^2}\right).$$

Kontsevich-Soibelman noticed that for generic Z there is a factorization in $\mathrm{Aut}(\widehat{\mathfrak{g}}_{\Gamma,\Sigma,Z})$

$$\exp_{D(\widehat{\mathfrak{g}}_{\Gamma,\Sigma,Z})}\left(\sum_{Z(\alpha)\in\ell}-\mathrm{DT}(\alpha,Z)[x_\alpha,-]\right) = \prod_{Z(\beta)\in\ell} T_\beta^{\Omega(\beta,Z)}. \tag{3.2.6}$$

The continuity condition becomes the constraint that the product of Poisson automorphisms $\prod^{\curvearrowright,Z}_{\ell\subset\Sigma}\prod_{Z(\alpha)\in\ell} T_\alpha^{\Omega(\alpha,Z)}$ remains constant (where the equalities have to be meant in the sense the quotients (3.2.4)) in the locus of generic central charges (even when crossing the nongeneric locus) as long as no rays supporting a nonvanishing factor enter or leave Σ.

Definition 3.18. Let Ω be a positive, continuous spectrum parametrised by $\mathrm{Hom}^+(\Gamma,\mathbb{C})$ and fix a positive basis. The corresponding Joyce function $f(Z)$ is the $\widehat{\mathfrak{g}}_{>0}$-valued function with graded components $\hat{f}^\alpha(Z)x_\alpha$ given by the expression (2.4.4). This is well-defined because there are only finitely many possible decompositions in (2.4.4) for each fixed $\alpha\in\Gamma^+$.

3.2.2 Formal data

When $\mathcal{C}$ is a triangulated category, the Joyce generating functions and the objects ∇^r, $\mathcal{V}$, $\mathcal{Q}$ are always formal infinite sums. The convergence question is a priori ill-posed if no specific summation order is fixed. In this Section the problem is reformulated by transforming infinite sums into formal power series.

Introduce an auxiliary vector of parameters $\mathbf{s}=(s_1,\dots,s_n)$, $n=\mathrm{rk}(\Gamma)$. The idea of working with such formal parameters and creating then formal families of objects comes from works about scattering diagrams, see e. g. [23]. In this way, we can make sense of Joyce functions f^α as well-defined formal power series. In particular, they are a tool to make indistinguishable elements referred to variables x_α and $x_{-\alpha}$.

Let $\{\gamma_i\}$ be a basis for Γ and introduce formal parameters $\mathbf{s}=(s_1,\dots,s_n)$. We decompose $\alpha\in\Gamma$ as

$$\alpha=\sum_i a_i\gamma_i$$

and set

$$[\alpha]_\pm := \sum_i [a_i]_\pm$$

where $[a_i]_\pm$ denote the positive and negative parts, that is $[a_i]_+=\max\{a_i,0\}$, $[a_i]_-=\min\{a_i,0\}$. For all $\alpha\in\Gamma$ we write $\mathbf{s}^\alpha$ for the Laurent monomial

$$\mathbf{s}^\alpha=\prod_i s_i^{a_i}.$$

In particular $\mathbf{s}^{[\alpha]_+-[\alpha]_-}=\prod_i s_i^{|a_i|}$ is a monomial (not just a Laurent monomial) and one can replace x_α with $\mathbf{s}^{[\alpha]_+-[\alpha]_-}x_\alpha$. It will be useful to introduce the following definition.

Definition 3.19. $\mathfrak{g}_\Gamma[\![s_1,\dots,s_n]\!]$ is the algebra generated by formal elements x_α, $\alpha \in \mathcal{K}(\mathcal{C})$, over $\mathbb{C}[\![s_1,\dots,s_n]\!]$, endowed with the Poisson Lie bracket extending $[-,-]$ of $\mathfrak{g}_\Gamma$ by $\mathbb{C}[\![s_1,\dots,s_n]\!]$-linearity.

We call $J \subset \mathfrak{g}_\Gamma[\![\mathbf{s}]\!]$ the ideal generated by $s_1,\dots,s_n$.

Definition 3.20. Call the integer $[\alpha]_+ - [\alpha]_-$ the *length* of α and denote it by $l(\alpha)$.

The original ill-defined expression is recovered for $\mathbf{s} = (1,\dots,1)$, modulo convergence.

Definition 3.21. With a fixed choice of basis as above we write $T^{\Omega}_{\beta,\mathbf{s}}$ for the element of $\mathrm{Aut}(\mathfrak{g}_\Gamma[\![\mathbf{s}]\!])$ given by

$$T^{\Omega}_{\beta,\mathbf{s}}(x_\alpha) = x_\alpha(1 - \mathbf{s}^{[\beta]_+ - [\beta]_-} x_\beta)^{\langle\beta,\alpha\rangle\Omega}$$

Lemma 3.22. *Let Ω be the double of a positive spectrum. Fix a positive basis $\{\gamma_i\}$. Suppose that Ω is continuous, parametrised by $\mathrm{Hom}^+(\Gamma,\mathbb{C})$. Then the family of automorphisms $T^{\Omega}_{\beta,\mathbf{s}} \in \mathrm{Aut}(\mathfrak{g}_\Gamma[\![\mathbf{s}]\!])$ comes from a continuous family of stability data with values in $\mathfrak{g}_\Gamma[\![\mathbf{s}]\!]$ via the construction in* (3.2.6).

In particular the products $\prod^{\curvearrowright, Z}_{\ell\subset\Sigma} \prod_{Z(\alpha)\in\ell} T^{\Omega(\alpha,Z)}_{\alpha,\mathbf{s}}$ *for all fixed strictly convex sectors Σ remain constant in the locus of generic central charges in* $\mathrm{Hom}^+(\Gamma,\mathbb{C})$ *(even when crossing the nongeneric locus) as long as no rays supporting a nonvanishing factor enter or leave Σ.*

Proof. Suppose that Ω is the double of a positive, continuous spectrum parametrised by $\mathrm{Hom}^+(\Gamma,\mathbb{C})$. Then the continuity condition given by constancy of the formal Lie group element (3.2.5) holds if and only if it holds for all strictly convex cones Σ contained in the open upper half-plane $\mathbb{H}$. On such a cone $\Sigma \subset \mathbb{H}$ the constancy condition for (3.2.5) is compatible with the extra grading by $\mathbf{s}$ by the Baker-Campbell-Hausdorff formula. □

Definition 3.23. Let Ω be the double of a positive, continuous spectrum parametrised by $\mathrm{Hom}^+(\Gamma,\mathbb{C})$. The corresponding Joyce function $f_{\mathbf{s}}(Z)$ is the function with values in $\mathfrak{g}_\Gamma[\![\mathbf{s}]\!]$ with Γ-graded components $\hat{f}^{\alpha}_{\mathbf{s}}(Z)x_\alpha$, where

$$\begin{aligned}\hat{f}^{\alpha}_{\mathbf{s}}(Z) = &\sum_{\alpha_1+\dots+\alpha_k=\alpha,\, Z(\alpha_i)\neq 0} c(\alpha_1,\dots,\alpha_k) J(Z(\alpha_1),\dots,Z(\alpha_k)) \\ &\prod_i \mathbf{s}^{[\alpha_i]_+ - [\alpha_i]_-}\,\mathrm{DT}(\alpha_i, Z).\end{aligned} \tag{3.2.7}$$

This is well-defined because there are only finitely many decompositions in (3.2.7) modulo J^N for $N \gg 1$.

3.2.3 The categorical setup

The parallelism with the categorical setup is easily described. Suppose that $\mathcal{C}$ is a 3CY triangulated category and assume that $\mathcal{C}$ admits a bounded t-structure with a finite heart $\mathcal{A}$ with n simple objects $S_1,\dots,S_n$ up to isomorphism. Then there exists an isomorphism of Grothendieck groups $\mathcal{K}(\mathcal{C}) \simeq \mathcal{K}(\mathcal{A})$. Moreover $\mathcal{K}(\mathcal{C})$ has finite rank n. On $\mathcal{K}(\mathcal{C})$, the Euler pairing is skew-symmetric and integral.

Any stability condition $\sigma = (\mathcal{A}, Z)$ in the sense of Bridgeland (Definition 2.20) in the space $\mathrm{Stab}(\mathcal{C})$ define a central charge Z (Definition 3.8) with the support property (Definition 3.2.1).

Call $\mathcal{K}_{>0}(\mathcal{A})$ the convex cone of positive linear combinations of $[S_1], \ldots, [S_n]$. Let $\mathring{U}(\mathcal{A}) \subset \mathrm{Stab}(\mathcal{C})$ denote the interior of the set of stability conditions supported on $\mathcal{A}$. The set $\mathring{U}(\mathcal{A})$ is given by stability conditions with heart $\mathcal{A}$ and whose central charge Z maps $\mathcal{K}_{>0}(\mathcal{A})$ to the open upper half-plane $\mathbb{H}$. $\mathcal{K}_{>0}(\mathcal{A})$ is an "effective cone" for $\Gamma = \mathcal{K}(\mathcal{C})$, with natural positive basis given by the classes of simple objects in $\mathcal{A}$, and $\mathring{U}(\mathcal{A}) = \mathrm{Hom}^+(\Gamma, \mathbb{C})$.

Assume moreover that there are well-defined numerical Donaldson-Thomas invariants $DT(\alpha, \sigma)$ enumerating objects in $\mathcal{C}$ with class $\alpha \in \mathcal{K}(\mathcal{C})$ which are semistable with respect to a choice of a stability condition σ. This is a family of stability data with symmetric spectrum. Restricting to $U(\mathcal{A}) \subset \mathrm{Stab}(\mathcal{C})$, we may write $DT(\alpha, Z)$ for $DT(\alpha, \sigma)$, where $\sigma = (\mathcal{A}, Z)$ and α denotes both the class of an object in $\mathcal{K}(\mathcal{C})$ and of its image in $\mathcal{K}(\mathcal{A})$.

3.3 Formal families of structures

It is now possible to state and prove the main results in the abstract setting defines above.

Fix a basis $\{\gamma_i\}$ of Γ and consider the infinite dimensional vector bundle below.

Definition 3.24. Introduce a holomorphic bundle $K \to \mathrm{Hom}^+(\Gamma, \mathbb{C})$ given by:

- if Ω is positive, K is the trivial bundle with fiber $\widehat{\mathfrak{g}}_{>0}$;
- if Ω is the double of a positive spectrum, K is the trivial bundle with fiber $\mathfrak{g}_\Gamma[\![\mathbf{s}]\!]$.

For the sake of completeness we summarise the results in the case of a positive spectrum in the following Proposition. The part concerning the Frobenius type structure follows from the results of [11], while the claims about the CV-structure are proved exactly as in Proposition 3.26 below, working with $\widehat{\mathfrak{g}}_{>0}$ rather than $\mathfrak{g}_\Gamma[\![\mathbf{s}]\!]$.

Proposition 3.25. *Let Ω be a positive, continuous spectrum parametrised by $\mathrm{Hom}^+(\Gamma, \mathbb{C})$. Let $K \to \mathrm{Hom}^+(\Gamma, \mathbb{C})$ be the vector bundle of Definition 3.24. Then the obvious analogues of Propositions 3.2, 3.5 and Lemmas 3.3, 3.7 hold.*

The construction of a Frobenius type structure for the double of a positive spectrum is similar, so only a sketch of the proof is given.

Theorem 3.26. *Let Ω be the double of a positive, continuous spectrum parametrised by $\mathrm{Hom}^+(\Gamma, \mathbb{C})$. Let $K \to \mathrm{Hom}^+(\Gamma, \mathbb{C})$ be the vector bundle of Definition 3.24, with fiber $\mathfrak{g}_\Gamma[\![\mathbf{s}]\!]$. Then there is a $\mathbb{C}[\![\mathbf{s}]\!]$-linear Frobenius type structure on K with flat holomorphic connection given by*

$$\nabla^r_{\mathbf{s}} = d + \sum_\alpha \mathrm{ad}\, f^\alpha_{\mathbf{s}}(Z) \frac{dZ(\alpha)}{Z(\alpha)},$$

with residue endomorphism

$$\mathcal{V}_{\mathbf{s}} = -\,\mathrm{ad}\, f_{\mathbf{s}}(Z)$$

and with $C, \mathcal{U}, g$ given by $-dZ, Z$ and the quadratic form of Proposition 3.2, extended by $\mathbb{C}[\![\mathbf{s}]\!]$-linearity. In other words the equations $(\nabla^r_{\mathbf{s}})^2 = 0$ and (1.6.1) - (1.6.2) hold as identities of formal power series in the formal parameters $s_1, \ldots, s_n$.

In particular the coefficients of the formal power series (3.2.7) in $\mathbf{s}$ are well-defined holomorphic functions on $\mathrm{Hom}^+(\Gamma, \mathbb{C})$.

Proof (sketch). It follows from Lemma 3.22 and the results of [11] that the functions $f^{\alpha}_{\mathbf{s}}(Z)$ satisfy the PDE (3.1.1) as formal power series in $\mathbf{s}$. Then the corresponding Frobenius type structure is constructed as in the proof of Proposition 3.2. □

Let ι denote the involution of K acting as complex conjugation combined with $x_{\alpha} \mapsto x_{-\alpha}$. Note that ι is an anti-linear commutative algebra automorphism. Let $\psi_{\mathbf{s}}$ be a fixed invertible endomorphism of K. Again one can make the following ansatz on part of the data of a $\mathbb{C}[\![\mathbf{s}]\!]$-linear CV-structure on K:

- $\kappa_{\mathbf{s}}$ is the conjugate involution $\mathrm{Ad}_{\psi_{\mathbf{s}}}(\iota)$,
- the pseudo-hermitian metric $h_{\mathbf{s}}$ is given by $h(a,b) = g(a, \kappa_{\mathbf{s}}(b))$ where g is the quadratic form in Proposition 3.2,
- $\mathcal{U}$ is the endomorphism Z extended by $\mathbb{C}[\![\mathbf{s}]\!]$-linearity,
- the Higgs field C is given by $-dZ$ extended by $\mathbb{C}[\![\mathbf{s}]\!]$-linearity, and the anti-Higgs field $\widetilde{C}_{\mathbf{s}}$ by $\kappa_{\mathbf{s}} C \kappa_{\mathbf{s}}$.

Theorem 3.27. *Suppose we are in the situation of Theorem 3.26.*

(a) There exist $\mathbb{C}[\![\mathbf{s}]\!]$-linear endomorphisms $\psi_{\mathbf{s}}$ and $\mathcal{Q}_{\mathbf{s}}$ and a connection $D_{\mathbf{s}}$ on K such that the choices of C, $\widetilde{C}_{\mathbf{s}}$, $\kappa_{\mathbf{s}}$, $h_{\mathbf{s}}$, $\mathcal{U}_{\mathbf{s}}$ above together with $\mathcal{Q}_{\mathbf{s}}$ give a $\mathbb{C}[\![\mathbf{s}]\!]$-linear CV-structure on K. In other words the equations 1.7.1 and 1.7.2 hold as identities of formal power series in $\mathbf{s}$. Moreover $\psi_{\mathbf{s}}$ and $\mathcal{Q}_{\mathbf{s}}$ induce fibrewise $\mathbb{C}[\![\mathbf{s}]\!]$-linear derivations of $\mathfrak{g}_{\Gamma}[\![\mathbf{s}]\!]$ as a commutative algebra.

(b) We have

$$\lim_{\lambda\to 0} \mathcal{Q}_{\mathbf{s}}(\lambda Z) = \mathcal{V}_{\mathbf{s}},$$

where $\mathcal{V}_{\mathbf{s}} = \mathrm{ad}\, f_{\mathbf{s}}(Z)$ is the endomorphism of Theorem 3.26 (i.e. essentially the formal family of Joyce holomorphic generating functions given by (3.2.7)*).*

Proof. We consider the family of automorphisms of the commutative algebra $\mathfrak{g}_{\Gamma}[\![\mathbf{s}]\!]$ induced by $T^{\Omega(\alpha,Z)}_{\alpha,\mathbf{s}}$ for $Z \in \mathrm{Hom}^{sg}(\Gamma,\mathbb{C}) \cap \mathrm{Hom}^{+}(\Gamma,\mathbb{C})$. Fix $Z \in \mathrm{Hom}^{sg}(\Gamma,\mathbb{C}) \cap \mathrm{Hom}^{+}(\Gamma,\mathbb{C})$. In [18] section 3 the corresponding Riemann-Hilbert factorization problem for a map $X : \mathbb{C}^* \to \mathrm{Aut}(\mathfrak{g}_{\Gamma}[\![\mathbf{s}]\!])$ is studied. This is the problem of finding a holomorphic function with $X(z)$ on $\mathbb{C}^*$, with values in $\mathrm{Aut}(\mathfrak{g}_{\Gamma}[\![\mathbf{s}]\!])$, such that, for all $N \geq 1$ and $\alpha \in \Gamma$, the class of $X(z)(x_{\alpha})$ in $\mathfrak{g}_{\Gamma}[\![\mathbf{s}]\!]/J^N$ is a holomorphic function of z in the complement of the distinguished rays ℓ with $\ell \neq \ell_{\pm\alpha(Z)}$, and for $z_0 \in \ell$ we have

$$X(z_0^{+})(x_{\alpha}) = X(z_0^{-}) \circ \prod_{Z(\beta)\in\ell} T^{\Omega(\beta,Z)}_{\beta,\mathbf{s}}(x_{\alpha}) \mod J^N$$

where $z_0^{\pm}$ denote the limits in the counterclockwise, respectively clockwise directions. Note that by working modulo J^N there are only finitely many branch-cuts. In [18] Lemma 3.10 a distinguished explicit solution $X(z)$ is constructed, satisfying some additional properties. We denote this distinguished family of solutions as Z varies in $\mathrm{Hom}^{sg}(\Gamma,\mathbb{C}) \cap \mathrm{Hom}^{+}(\Gamma,\mathbb{C})$ by $X(z,Z)$, and also set

$$\tilde{X}(z,Z) = X(z,Z) \circ \exp_*(-z^{-1}Z - z\bar{Z}).$$

Consider the flat connection on $\mathbb{P}^1_z \times \mathrm{Hom}^+(\Gamma, \mathbb{C})$ given by

$$\nabla^{tr} = d - \frac{dZ}{z} + z d\bar{Z} + \left(-\frac{1}{z^2} Z + \bar{Z}\right) dz.$$

We may regard ∇^{tr} as a flat connection on the trivial vector bundle with fiber $\mathfrak{g}_\Gamma[\![\mathbf{s}]\!]/J^N$. Together with $g(a, \iota(b))$ it defines a CV-structure on the trivial vector bundle over $\mathrm{Hom}^+(\Gamma, \mathbb{C})$ with fiber $\mathfrak{g}_\Gamma[\![\mathbf{s}]\!]/J^N$. We pull back ∇^{tr} locally on a sector Σ between consecutive branch-cuts by $X(z, Z) \mod J^N$ to the locally defined flat connection

$$\nabla^{str}|_\Sigma = d - \frac{1}{z}\tilde{X} \cdot dZ + z\tilde{X} \cdot d\bar{Z} + d_Z \tilde{X} \circ \tilde{X}^{-1} + \left(-\frac{1}{z^2}\tilde{X} \cdot Z + \tilde{X} \cdot \bar{Z} + \partial_z \tilde{X} \circ \tilde{X}^{-1}\right) dz.$$

By [18] sections 3.7 and 3.9 ∇^{str} glues over different sectors Σ and is induced by a well-defined real-analytic flat connection on $\mathfrak{g}_\Gamma[\![\mathbf{s}]\!] \to \mathbb{P}^1_z \times \mathrm{Hom}^+(\Gamma, \mathbb{C})$ of the form

$$\nabla^{str}(Z) = d + \mathcal{B}^{(0)}(Z) + \frac{1}{z}\mathcal{B}^{(-1)}(Z) + z\mathcal{B}^{(1)}(Z) + \left(\frac{1}{z^2}\mathcal{A}^{(-1)} + \frac{1}{z}\mathcal{A}^{(0)} + \mathcal{A}^{(1)}\right) dz.$$

Moreover $\mathcal{A}^{(i)}$, $\mathcal{B}^{(i)}$ are derivations of $\mathfrak{g}_\Gamma[\![\mathbf{s}]\!]$ and we have

$$\begin{aligned} \mathcal{A}^{(1)}(Z) &= -\iota\mathcal{A}^{(-1)}(Z)\iota, \quad & \mathcal{A}^{(0)}(Z) &= -\iota\mathcal{A}^{(0)}(Z)\iota, \\ \mathcal{B}^{(1)}(Z) &= \iota\mathcal{B}^{(-1)}(Z)\iota, & \mathcal{B}^{(0)}(Z) &= \iota\mathcal{B}^{(0)}(Z)\iota. \end{aligned} \tag{3.3.1}$$

By [18] section 3.7 the limit $\tilde{X}_0(Z) = \lim_{z\to 0} \tilde{X}(z, Z)$ is well-defined, and we have

$$\begin{aligned} \tilde{X}_0^{-1} \cdot \nabla^{str}(Z) = d &+ \mathrm{Ad}_{\tilde{X}_0^{-1}} \mathcal{B}^{(0)}(Z) - \frac{1}{z} dZ + z \,\mathrm{Ad}_{\tilde{X}_0^{-1}} \mathcal{B}^{(1)}(Z) \\ &+ \left(-\frac{1}{z^2} Z + \frac{1}{z}\mathrm{Ad}_{\tilde{X}_0^{-1}} \mathcal{A}^{(0)} + \mathrm{Ad}_{\tilde{X}_0^{-1}} \mathcal{A}^{(1)}\right) dz. \end{aligned} \tag{3.3.2}$$

Notice that by (3.3.1) and (3.3.2) we have

$$\begin{aligned} \mathrm{Ad}_{\tilde{X}_0^{-1}} \mathcal{A}^{(1)} &= -\mathrm{Ad}_{\tilde{X}_0^{-1}} \mathrm{Ad}_\iota \mathcal{A}^{(-1)} \\ &= -\mathrm{Ad}_{\tilde{X}_0^{-1}} \mathrm{Ad}_\iota \mathrm{Ad}_{\tilde{X}_0}(-Z), \\ \mathrm{Ad}_{\tilde{X}_0^{-1}} \mathcal{B}^{(1)} &= \mathrm{Ad}_{\tilde{X}_0^{-1}} \mathrm{Ad}_\iota \mathcal{B}^{(-1)} \\ &= \mathrm{Ad}_{\tilde{X}_0^{-1}} \mathrm{Ad}_\iota \mathrm{Ad}_{\tilde{X}_0}(-dZ), \end{aligned}$$

so using the conjugate involution $\kappa = \mathrm{Ad}_{\tilde{X}_0^{-1}}(\iota)$ we may rewrite (3.3.2) as

$$\begin{aligned} \tilde{X}_0^{-1} \cdot \nabla^{str}(Z) = d &+ \mathrm{Ad}_{\tilde{X}_0^{-1}} \mathcal{B}^{(0)}(Z) - \frac{1}{z} dZ + z\kappa(-dZ)\kappa \\ &+ \left(-\frac{1}{z^2} Z + \frac{1}{z}\mathrm{Ad}_{\tilde{X}_0^{-1}} \mathcal{A}^{(0)} + \kappa Z \kappa\right) dz. \end{aligned}$$

Then the flat connection $\tilde{X}_0^{-1} \cdot \nabla^{str}(Z)$ together with κ define the required $\mathfrak{g}_\Gamma[\![\mathbf{s}]\!]$-linear CV-structure, with $D = d + \mathrm{Ad}_{\tilde{X}_0^{-1}} \mathcal{B}^{(0)}(Z)$, $C = -dZ$, $\tilde{C} = \kappa(-dZ)\kappa$, $\mathcal{U} = -dZ$, $\mathcal{Q} = -\mathrm{Ad}_{\tilde{X}_0^{-1}} \mathcal{A}^{(0)}$, $h(a, b) = g(a, \kappa b)$. The automorphism in the statement of the Proposition is given by $\psi_{\mathbf{s}}(Z) = \tilde{X}_0^{-1}(Z)$.

The limit

$$\lim_{\lambda\to 0}\mathcal{Q}(\lambda Z)=\mathcal{V}(Z)$$

is proved in [18] Theorem 4.2. We provide a sketch of the argument. Since they are constructed from a solution to the Riemann-Hilbert factorization problem, the family of connections on $\mathbb{P}^1_z$

$$d+\left(-\frac{1}{z^2}Z-\frac{1}{z}\mathcal{Q}_{\mathbf{s}}(\lambda Z)+\lambda^2\kappa_{\mathbf{s}}(\lambda Z)\,Z\kappa_{\mathbf{s}}(\lambda Z)\right)dz \tag{3.3.3}$$

parametrised by $\mathrm{Hom}^+(\Gamma,\mathbb{C})$ are isomonodromic, with constant generalized monodromy at $z=0$ for generic Z given by rays ℓ with Stokes factors $\prod_{Z(\beta)\in\ell}T^{\Omega(\beta,Z)}_{\beta,\mathbf{s}}(x_\alpha)$. One checks that the limit as $\lambda\to 0$ is well-defined and equals

$$d+\left(-\frac{1}{z^2}Z-\frac{1}{z}\lim_{\lambda\to 0}\mathcal{Q}_{\mathbf{s}}(\lambda Z)\right)dz.$$

The result follows from a uniqueness result proved in [11]. $\square$

Definition 3.28. We write $\nabla_{\mathbf{s}}(Z,\lambda)$ for the family of meromorphic connections on $\mathbb{P}^1$ given by (3.3.3).

Corollary 3.29. *The statement of Lemma 3.7 holds for the Frobenius type and CV-structures constructed in Theorems 3.26 and 3.27.*

We may give an explicit formula for the operator $\mathcal{Q}_{\mathbf{s}}(\lambda Z)$, depending on integrals attached to rooted trees with vertices decorated by elements of Γ. It is based on the formula for $\mathcal{Q}(\lambda Z)$ in [18] and is expressed in Corollary A.6 in Appendix A.

Theorem 3.27 b), says that the operator $\mathcal{Q}_{\mathbf{s}}$ may be regarded as a deformation of the formal family $\mathcal{V}_{\mathbf{s}}$. It is related to Joyce holomorphic generating functions when $\mathbf{s}=(1,\dots,1)$ because $\lim_{\lambda\to 0}\mathcal{Q}(\lambda Z)=\mathcal{V}(Z)$. Part of the Frobenius type structure is specified by $\mathcal{V}(Z)=\operatorname{ad}f(Z)$ and the quadratic form g. In particular the graded components of the Joyce function are given by the matrix elements $g(x_\alpha,\mathcal{V}(x_\beta))$ over the natural basis of sections of K.

We considered the convergence problem for the components $g(x_\alpha,\mathcal{Q}_{(1,\dots,1)}(\lambda Z)(x_\beta))$ and we obtained the following result.

Theorem 3.30. *Suppose that we are in the situation of Theorem 3.26 and that* $\mathrm{DT}(\alpha,Z_0)$ *grows at most exponentially for* $\alpha\in\Gamma$ *(in the sense of Definition 3.12). Fix a central charge* $Z_0\in\mathrm{Hom}^+(\Gamma,\mathbb{C})$*. Then for all* $\rho>0$ *there exists* $\bar\lambda$ *such that for* $\lambda>\bar\lambda$ *all the formal power series* $g(x_\alpha,\mathcal{Q}_{\mathbf{s}}(\lambda Z_0)(x_\beta))$ *converge for* $||\mathbf{s}||<\rho$*. Let* $U\subset\mathrm{Hom}^+(\Gamma,\mathbb{C})$ *denote an open subset such that the exponential growth condition for* $\mathrm{DT}(\alpha,Z)$ *holds uniformly and all* $Z\in U$ *are uniformly bounded away from zero on elements of the cone* Γ^+*. Then for all sufficiently large* λ *the CV-deformations of the Joyce functions, given by* $g(x_\alpha,\mathcal{Q}_{(1,\dots,1)}(\lambda Z)(x_\beta))$*, are well defined and real-analytic on* U*, and uniformly bounded as* α *varies in* Γ *for fixed* β*.*

The proof involves techniques from functional analysis and it is given in Appendix A.

Chapter 4

Costruction of a Frobenius structure on $\mathrm{Stab}(\mathcal{A})$

The purpose of this Chapter is to present an approach for endowing the spaces of stability conditions on some abelian categories with a Frobenius manifold structure. The approach considered is based on the construction of a Frobenius type structure on an auxiliary vector bundle to be mapped to a genuine Frobenius structure on the tangent bundle. The idea is applying Hertling's Theorem 1.30. It states some conditions under which a Frobenius type structure on a vector bundle $K \to M$ can be pulled back to a structure on the tangent bundle which makes M into a Frobenius manifold.

In this Chapter we work on the space of stability conditions $\mathrm{Stab}(\mathcal{A})$ of a finite abelian category $\mathcal{A}$ and we suppose $\mathcal{A}$ is the finite heart of a 3CY triangulated category $\mathcal{D}$ attached to a quiver with potential. This has the advantage of turning some computations into combinatorical calculations. Moreover, we do know the wall and chamber structure of $\mathrm{Stab}(\mathcal{D})$ and the hope is that one can deduce a Frobenius structure on $\mathrm{Stab}(\mathcal{D})$ canonically from $\mathrm{Stab}(\mathcal{A})$.

The first problem we face, following this approach, it that the Frobenius type structure constructed in the previous Chapter is defined over an infinite dimensional vector bundle $K \to \mathrm{Stab}(\mathcal{A})$. Moreover we actually have a family of structures parameterized by a vector of formal parameters $\mathbf{s}$.

Under suitable hypotheses, the family of structures of Theorem 3.26 may induce a (new) family of Frobenius type structures on a finite dimensional vector subbundle $K(\zeta)$ of $K \to U \subset \mathrm{Stab}(\mathcal{A})$. This process is not simply a restriction, and the original family turns out to be just tangent to the new finite-dimensional family. Sections 4.1 ad 4.2 are devoted to the construction of finite-dimensional Frobenius type structures. In Section 4.3 we consider conditions under which the pull-back to T_U is Frobenius. The main result consists in Theorems 4.26 and 4.27.

The Chapter ends with the explicit study of the construction when Q is the Dynkin quiver A_2, A_3, A_n, with zero potential, in Section 4.4.

4.1 Approximate finite-dimensional Frobenius type structure

Let (Q, W) a finite quiver with reduced potential and no 2-cycles. Denote by $\mathcal{D} = \mathcal{D}(Q, W)$ the CY3 associated category. It has a finite heart $\mathcal{A} = \mathcal{A}(Q, W)$, such that Q is the Ext-quiver $\mathrm{Ext}(\mathcal{A})$. In particular the simple objects of $\mathcal{A}$ are in natural bijection with vertices of Q, and the extensions space between them are based on the adjacency matrix η of Q:

$$\mathcal{K}(\mathcal{A}) \simeq \mathbb{Z}^{|Q_0|}, \quad \eta_{ij} = \mathrm{ext}^1_{\mathcal{A}}(S_i, S_j).$$

The natural embedding $\mathcal{A} = \mathcal{A}(Q, W) \subset \mathcal{D}(Q, W) = \mathcal{D}$ induces an isomorphism $\mathcal{K}(\mathcal{A}) \simeq \mathcal{K}(\mathcal{D})$, and the submonoid of positive combinations of $[S_i]$, $i = 1, \ldots, n$, is the effective cone $\mathcal{K}_{>0}(\mathcal{A}) \subset \mathcal{K}(\mathcal{A})$. It is mapped to $\bar{\mathbb{H}}$ by central charges Z on $\mathcal{A}$, thus $\mathrm{Stab}(\mathcal{A}) \simeq \bar{\mathbb{H}}$. One can also view $\mathrm{Stab}(\mathcal{A})$ as embedded in $\mathrm{Stab}(\mathcal{D})$.

In fact, for the general construction described in the Chapter, we will only need to assume that $\mathcal{D}$ is a finite CY3 triangulated category with a finite abelian heart $\mathcal{A}$, with $\mathcal{K}(\mathcal{D}) \simeq \mathcal{K}(\mathcal{A}) \simeq \mathbb{Z}^{\oplus n}$. However, the examples we have in mind come from the theory of quivers.

Assume that on $\mathcal{D}$ there is a well-defined enumerative theory for semistable objects. It can be restricted to $\mathcal{A}$.

Definition 4.1. Denote by $\mathrm{DT}_{\mathcal{A}}(\alpha, Z) \in \mathbb{Q}$ the Donaldson-Thomas type invariant virtually enumerating objects in $\mathcal{D}$ of class $\alpha \in \mathcal{K}(\mathcal{D}) \cong \mathcal{K}(\mathcal{A})$ which are semistable with respect to the stability condition $(\mathcal{A}, Z)$ consisting of the heart $\mathcal{A}$ and central charge $Z \in \mathrm{Stab}(\mathcal{A})$.

The shift functor $[1] \in \mathrm{Aut}(\mathcal{D})$ preserves the class of semistable objects and acts on $\mathcal{K}(\mathcal{D})$ as $-I$, so we have

$$\mathrm{DT}_{\mathcal{A}}(\alpha, Z) = \mathrm{DT}_{\mathcal{A}}(-\alpha, Z).$$

The Kontsevich-Soibelman Lie algebra $\mathbb{C}[\mathcal{K}(\mathcal{A})]$ is extended by $\mathbb{C}[\![\mathbf{s}]\!]$-linearity to $\mathbb{C}[\mathcal{K}(\mathcal{A})][\![\mathbf{s}]\!]$, $\mathbf{s} = (s_1 \ldots, s_n)$. Let $K \to \mathrm{Stab}(\mathcal{A})$ be the vector bundle with fiber $\mathbb{C}[\mathcal{K}(\mathcal{A})][\![\mathbf{s}]\!]$. By Theorem 3.26, there is a $\mathbb{C}[\![\mathbf{s}]\!]$-linear Frobenius type structure $(\nabla^r_{\mathbf{s}}, C, \mathcal{U}, \mathcal{V}_{\mathbf{s}}, g)$, given by:

- a connection

$$\nabla^r_{\mathbf{s}} = \mathrm{d} + \sum_{\alpha \neq 0} \mathrm{ad}\, \hat{f}^{\alpha}_{\mathbf{s}}(Z) x_\alpha \frac{dZ(\alpha)}{Z(\alpha)},$$

- a 1-form with values in endomorphisms

$$C = -\,\mathrm{d}\, Z, \tag{4.1.1}$$

- endomorphisms

$$\mathcal{U} = Z, \tag{4.1.2}$$

$$\mathcal{V}_{\mathbf{s}} = \mathrm{ad} \sum_{\alpha \neq 0} \hat{f}^{\alpha}_{\mathbf{s}}(Z) x_\alpha, \tag{4.1.3}$$

- a quadratic form

$$g(x_\alpha, x_\beta) = \delta_{\alpha\beta}. \tag{4.1.4}$$

Restrict, if necessary, to a suitable open subset $U \subset \mathrm{Stab}(\mathcal{A})$, which will be explicitly described later in the Chapter. The idea is to apply Theorem 1.30 to a finite rank subbudle of $K \to U$, with a Frobenius type structure.

A holomorphic section ζ of K takes the form

$$\zeta = \sum_{\alpha \in I'} \zeta_\alpha(Z, \mathbf{s}) x_\alpha \tag{4.1.5}$$

where $I' \subset K(\mathcal{A})$ and the $\zeta_\alpha(Z, \mathbf{s})$, $\alpha \in I'$, are formal power series which do not vanish identically. The contraction of a holomorphic section ζ of K with the Higgs field C is the map

$$-C_\bullet(\zeta) = \mathrm{d}\, Z(\zeta) : T_U \to K.$$

Let X be a holomorphic vector field. Then

$$\begin{aligned} dZ(X)(\zeta) &= \sum_{\alpha \in I'} \zeta_\alpha(Z, \mathbf{s}) dZ(X)(x_\alpha) \\ &= \sum_{\alpha \in I'} \zeta_\alpha(Z, \mathbf{s}) X(Z(\alpha)) x_\alpha. \end{aligned}$$

Call

$$K(\zeta) = \mathrm{im}(\mathrm{d}\, Z(\zeta)) \subset K.$$

It is a finite rank subbundle of K. It is natural to ask when ζ is in fact a section of the bundle $K(\zeta)$.

Lemma 4.2. *We have $\zeta \in \mathcal{O}(K(\zeta))$ if and only if there are elements $\alpha_1, \ldots, \alpha_r \in \mathcal{K}(\mathcal{A})$, linearly independent over $\mathbb{R}$, such that*

$$\zeta = \sum_{i=1}^{r} \zeta_i(Z, \mathbf{s}) x_{\alpha_i} + \sum_{\substack{a_1 + \cdots + a_r = 1 \\ a_1\alpha_1 + \cdots + a_r\alpha_r \neq \alpha_i,\, i=1,\ldots,r}} \zeta_{a_1, \cdots, a_r}(Z, \mathbf{s}) x_{a_1\alpha_1 + \cdots + a_r\alpha_r}$$

where $\zeta_i(Z, \mathbf{s})$, $\zeta_{a_1, \cdots, a_n}(Z, \mathbf{s})$ are formal power series in the variables $\mathbf{s}$ with holomorphic coefficients.

Proof. Take a holomorphic section ζ of K of the form (4.1.5). Let X be a holomorphic vector field. We compute

$$dZ(X)(\zeta) = \sum_{\alpha \in I'} \zeta_\alpha(Z, \mathbf{s}) X(Z(\alpha)) x_\alpha.$$

So $\zeta \in \mathcal{O}(K(\zeta))$ if and only if there exists a holomorphic vector field X such that for all $\alpha \in I'$ we have

$$X(Z(\alpha)) = 1.$$

Choose a maximal set of elements $\alpha_1, \ldots, \alpha_r$ of I' which are linearly independent over $\mathbb{R}$. The functions $Z(\alpha_1), \ldots, Z(\alpha_r)$ are part of a local coordinate system $u_1, \ldots, u_n$ on $\mathrm{Stab}(\mathcal{A})$ with $u_i = Z(\alpha_i)$ for $i = 1, \ldots, r$. The general solution X to $X u_i = 1$, $i = 1, \ldots, r$ is a vector field

$$X = \sum_{i=1}^{r} \partial_{u_i} + \sum_{j=r+1}^{n} b_j \partial_{u_j}$$

for arbitrary b_j. All the other $\alpha \in I = I' \setminus \{\alpha_1, \ldots, \alpha_r\}$ are linear combinations $\alpha = \sum_{i=1}^{r} a_i \alpha_i$. The condition $X Z(\alpha) = 1$ holds if and only if $\sum_{i=1}^{r} a_i = 1$. □

Corollary 4.3. *Suppose $\zeta \in \mathcal{O}(K(\zeta))$, $U' \subseteq \mathrm{Stab}(\mathcal{A})$. The map $-C_\bullet(\zeta) = dZ(\zeta) \colon T_{U'} \to K(\zeta)$ is injective (and so an isomorphism) if and only if in Lemma 4.2 we have $r = n$ and the functions $\zeta_1(Z, \mathbf{s}), \ldots, \zeta_n(Z, \mathbf{s})$ are nowhere vanishing on U'. In this case $K(\zeta) \subset K$ is the subbundle generated by*

$$\mathrm{d}\, Z(\partial_{Z(\alpha_i)})(\zeta) = x_{\alpha_i} + \sum_{\substack{a_1 + \cdots + a_r = 1 \\ a_1\alpha_1 + \cdots + a_r\alpha_r \neq \alpha_j,\, j=1,\ldots,n}} a_i \zeta_{a_1, \cdots, a_n}(Z, \mathbf{s}) x_{a_1\alpha_1 + \cdots + a_n\alpha_n}$$

for $i = 1, \ldots, n$ (following the notation of Lemma 4.2).

From now on, we assume that U' is an open subset where $\zeta_i(Z, \mathbf{s})$, $i = 1, \ldots, n$, are holomorphic functions with no zeroes, and that the map $-C_\bullet(\zeta)\colon T_{U'} \to K(\zeta)$ is an isomorphism.

Definition 4.4. Denote by $\pi^\zeta : K \to K(\zeta)$ the orthogonal projection onto $K(\zeta)$ with respect to the quadratic form g of (4.1.4). Define

$$\begin{aligned} \nabla^{r,\zeta}_{\mathbf{s}} &:= \pi^\zeta \circ \nabla^r_{\mathbf{s}}|_{K(\zeta)}, \quad C^\zeta := \pi^\zeta \circ C|_{K(\zeta)}, \\ \mathcal{U}^\zeta &:= \pi^\zeta \circ \mathcal{U}|_{K(\zeta)}, \quad \mathcal{V}^\zeta_{\mathbf{s}} := \pi^\zeta \circ \mathcal{V}_{\mathbf{s}}|_{K(\zeta)}, \\ g^\zeta &:= g|_{K(\zeta)}. \end{aligned}$$

The holomorphic data $(K(\zeta), \nabla^{r,\zeta}_{\mathbf{s}}, C^\zeta, \mathcal{U}^\zeta, \mathcal{V}^\zeta_{\mathbf{s}}, g^\zeta)$ give a formal family of structures on $K(\zeta)$ parametrised by $\mathbf{s}$. In general it is not a family of Frobenius type structures.

However, one can ask if the conditions of Frobenius type structure hold modulo some power $(\mathbf{s})^p$ with $p \geq 3$. If the equations defining a Frobenius type manifold are satisfied modulo terms which are at least cubic, then the formal family $(K(\zeta), \nabla^{r,\zeta}_{\mathbf{s}}, C^\zeta, \mathcal{U}^\zeta, \mathcal{V}^\zeta_{\mathbf{s}}, g^\zeta)$ will be *tangent* to a family of Frobenius type structures on $K(\zeta)$. The rest of the section is devoted to the study of this problem.

Even if it makes sense more generally we will restrict to the case when the bundle $K(\zeta)$ is preserved by the Higgs field and endomorphism $\mathcal{U}$. This condition is clarified by the following result.

Lemma 4.5. *Let ζ be a holomorphic section of K (we do not assume a priori that ζ is a section of $K(\zeta)$). The following are equivalent:*

- *$K(\zeta)$ is preserved by the Higgs field $C = -\,\mathrm{d}\,Z$,*
- *$K(\zeta)$ is preserved by the endomorphism $\mathcal{U} = Z$,*
- *the section ζ has the form*

$$\zeta = \sum_{i=1}^{r} \zeta_{\alpha_i}(Z, \mathbf{s}) x_{\alpha_i}$$

where $\alpha_1, \ldots, \alpha_r \in \mathcal{K}(\mathcal{A})$ are linearly independent over $\mathbb{R}$.

Proof. Suppose $K(\zeta)$ is preserved by C. Let us write

$$\zeta = \sum_{\alpha \in I} \zeta_\alpha(Z, \mathbf{s}) x_\alpha$$

where $I \subset K(\mathcal{A})$ and the $\zeta_\alpha(Z, \mathbf{s})$, $\alpha \in I$ are formal power series which do not vanish identically. Then by construction sections of the bundle $K(\zeta)$ have the form

$$dZ(X)(\zeta) = \sum_{\alpha \in I} \zeta_\alpha(Z, \mathbf{s}) X(Z(\alpha)) x_\alpha$$

as X varies in the space of holomorphic vector fields on U. In order to simplify the notation we set $\zeta_X = \mathrm{d}\,Z(X)(\zeta)$. Acting with the Higgs field $C = -\,\mathrm{d}\,Z$ contracted with a holomorphic field Y we find

$$C_Y \zeta_X = -\sum_{\alpha \in I} \zeta_\alpha(Z, \mathbf{s}) X(Z(\alpha)) Y(Z(\alpha)) x_\alpha.$$

So $C_Y\zeta_X$ is a section of $K(\zeta)$ if and only if there exists a holomorphic field $W = W(X,Y)$ such that for all $\alpha \in I$ we have

$$W(X,Y)(Z(\alpha)) = -X(Z(\alpha))Y(Z(\alpha)). \tag{4.1.6}$$

Let $\alpha_1, \dots, \alpha_r$ denote a maximal set of $\mathbb{R}$-linearly independent elements of I. Suppose there is a nontrivial $\alpha \in I \setminus \{\alpha_1, \dots, \alpha_r\}$. Decomposing $\alpha = a_1\alpha_1 + \cdots + a_r\alpha_r$ we find

$$\begin{aligned} W(X,Y)(Z(\alpha)) &= \sum_{i=1}^{r} a_i W(X,Y)(Z(\alpha_i)) \\ &= -\sum_{i=1}^{r} a_i X(Z(\alpha_i))Y(Z(\alpha_i)) \end{aligned} \tag{4.1.7}$$

where the second equality follows from applying (4.1.6) to each α_i. On the other hand applying (4.1.6) to α gives

$$W(X,Y)(Z(\alpha)) = -\sum_{i,j=1}^{r} a_i a_j X(Z(\alpha_i))Y(Z(\alpha_j)). \tag{4.1.8}$$

By (4.1.7) for all $k \neq l$ we have

$$W(\partial_{Z(\alpha_k)}, \partial_{Z(\alpha_l)})(Z(\alpha)) = 0.$$

On the other hand (4.1.8) gives for all $k \neq l$

$$W(\partial_{Z(\alpha_k)}, \partial_{Z(\alpha_l)})(Z(\alpha)) = -a_k a_l.$$

It follows that a_k or a_l vanish for all $k \neq l$, i.e. α must be a multiple of one of $\alpha_1, \dots, \alpha_r$. By (4.1.7) for all k we have

$$W(\partial_{Z(\alpha_k)}, \partial_{Z(\alpha_k)})(Z(\alpha)) = -a_k.$$

On the other hand (4.1.8) gives for all k

$$W(\partial_{Z(\alpha_k)}, \partial_{Z(\alpha_k)})(Z(\alpha)) = -a_k^2.$$

It follows that we must have $a_k = 0$ or $a_k = 1$ for all k. Since we already know that at most one a_k does not vanish we see that α must be one of $\alpha_1, \dots, \alpha_r$, a contradiction. Thus the section ζ must take the form

$$\zeta = \sum_{i=1}^{r} \zeta_i(Z, \mathbf{s}) x_{\alpha_i}$$

where $\alpha_1, \dots, \alpha_r \in K(\mathcal{A})$ are linearly independent over $\mathbb{R}$.

Conversely a straightforward computation shows that for a section ζ of this form and arbitrary fields X, Y we can find a vector field $W(X,Y)$ as above, so $K(\zeta)$ is preserved by C.

The argument for the endomorphism $\mathcal{U}$ is almost identical. □

From now on, we restrict to the open subset $U \subset U'$ where $Z(\alpha_i) \neq Z(\alpha_j)$ for the chosen basis $\{\alpha_1, \dots, \alpha_n\}$ of $\mathcal{K}(\mathcal{A}) \otimes \mathbb{R}$, and we consider the bundle $K(\zeta) \to U$. We assume that it is preserved by the Higgs field C and the endomorphism $\mathcal{U}$, and that the map $-C_\bullet(\zeta)$ is an isomorphism

$$-C_\bullet(\zeta)\colon T_U \xrightarrow{\cong} K(\zeta).$$

According to Corollary 4.3 and Lemma 4.5 this holds precisely when the section ζ takes the form

$$\zeta = \sum_{i=1}^{n} \zeta_i(Z, \mathbf{s}) x_{\alpha_i} \tag{4.1.9}$$

where $\alpha_1, \dots, \alpha_n \in \mathcal{K}(\mathcal{A})$ are a basis over $\mathbb{R}$ and the functions $\zeta_i(Z, \mathbf{s})$ are nowhere vanishing on U. Then

$$C^\zeta = C|_{K(\zeta)} \text{ and } \mathcal{U}^\zeta = \mathcal{U}|_{K(\zeta)}.$$

Lemma 4.6. *Pick a section ζ of the form* (4.1.9) *(so ζ is a section of $K(\zeta)$ and the latter is preserved by C and $\mathcal{U}$). Fix $i, j = 1, \dots n$. Suppose the following conditions hold:*

1. *for all $k \neq i, j$ we have either*

$$\langle \alpha_j, \alpha_i \rangle \langle \alpha_j - \alpha_k, \alpha_k - \alpha_i \rangle = \langle \alpha_j, \alpha_k \rangle \langle \alpha_k, \alpha_i \rangle,$$

or $\hat{f}_{\mathbf{s}}^{\alpha_j - \alpha_k} \hat{f}_{\mathbf{s}}^{\alpha_k - \alpha_i} \in (\mathbf{s}^3)$,

2. *for all nontrivial decompositions $\alpha_j - \alpha_i = \beta + \gamma$ with β, γ not equal to $\alpha_j - \alpha_k$, $\alpha_k - \alpha_i$ the product*

$$\langle \beta, \gamma \rangle \hat{f}_{\mathbf{s}}^{\beta}(Z) \hat{f}_{\mathbf{s}}^{\gamma}(Z)$$

is at least cubic in $\mathbf{s}$.

Then the curvature component $g(x_{\alpha_j}, F(\nabla_{\mathbf{s}}^{r,\zeta}) x_{\alpha_i})$ vanishes modulo terms which are at least cubic in $\mathbf{s}$.

Proof. Under the assumptions the bundle $K(\zeta)$ is the subbundle generated by the sections $x_{\alpha_1}, \dots, x_{\alpha_n}$. Let us write the connection $\nabla_{\mathbf{s}}^{r,\zeta}$ with respect to this local trivialization. We compute

$$\begin{aligned}
\nabla_{\mathbf{s}}^{r,\zeta}(x_{\alpha_i}) &= \sum_{\alpha \neq 0} \pi^\zeta \left(\hat{f}_{\mathbf{s}}^{\alpha}(Z) (-1)^{\langle \alpha, \alpha_i \rangle} \langle \alpha, \alpha_i \rangle x_{\alpha + \alpha_i} \right) \mathrm{d} \log Z(\alpha) \\
&= \sum_{j=1}^{n} (-1)^{\langle \alpha_j, \alpha_i \rangle} \langle \alpha_j, \alpha_i \rangle \hat{f}_{\mathbf{s}}^{\alpha_j - \alpha_i}(Z) x_{\alpha_j} \, \mathrm{d} \log Z(\alpha_j - \alpha_i).
\end{aligned}$$

So the connection matrix of 1-forms A in this local trivialization is given by

$$A_{ji} = (-1)^{\langle \alpha_j, \alpha_i \rangle} \langle \alpha_j, \alpha_i \rangle \hat{f}_{\mathbf{s}}^{\alpha_j - \alpha_i}(Z) \, \mathrm{d} \log Z(\alpha_j - \alpha_i)$$

and the curvature form $dA + A \wedge A$ is the matrix of 2-forms

$$\begin{aligned}
&(-1)^{\langle \alpha_j, \alpha_i \rangle} \langle \alpha_j, \alpha_i \rangle \, \mathrm{d} \hat{f}_{\mathbf{s}}^{\alpha_j - \alpha_i}(Z) \wedge \mathrm{d} \log Z(\alpha_j - \alpha_i) \\
&+ \sum_{k=1}^{n} (-1)^{\langle \alpha_j, \alpha_k \rangle + \langle \alpha_k, \alpha_i \rangle} \langle \alpha_j, \alpha_k \rangle \langle \alpha_k, \alpha_i \rangle \hat{f}_{\mathbf{s}}^{\alpha_j - \alpha_k}(Z) \hat{f}_{\mathbf{s}}^{\alpha_k - \alpha_i}(Z) \\
&\qquad \mathrm{d} \log Z(\alpha_j - \alpha_k) \wedge \mathrm{d} \log Z(\alpha_k - \alpha_i).
\end{aligned}$$

Flatness of the connection $\nabla^r_{\mathbf{s}}$ on K is expressed by the Joyce PDE

$$\mathrm{d}\,\hat{f}^{\alpha}_{\mathbf{s}}(Z) = -\sum_{\alpha=\beta+\gamma}(-1)^{\langle\beta,\gamma\rangle}\langle\beta,\gamma\rangle\hat{f}^{\beta}_{\mathbf{s}}(Z)\hat{f}^{\gamma}_{\mathbf{s}}(Z)\,\mathrm{d}\log Z(\beta)$$

for all $\alpha \neq 0$ (summing over decompositions with $\beta, \gamma \neq 0$). In our case we choose $\alpha = \alpha_j - \alpha_i$ and write the Joyce PDE in the form

$$\begin{aligned}\mathrm{d}\,\hat{f}^{\alpha_j-\alpha_i}_{\mathbf{s}}(Z) = &-\sum_{k\neq i,j}(-1)^{\langle\alpha_j-\alpha_k,\alpha_k-\alpha_i\rangle}\langle\alpha_j-\alpha_k,\alpha_k-\alpha_i\rangle\hat{f}^{\alpha_j-\alpha_k}_{\mathbf{s}}(Z)\hat{f}^{\alpha_k-\alpha_i}_{\mathbf{s}}(Z)\\ &\Big(\mathrm{d}\log Z(\alpha_j-\alpha_k)-\mathrm{d}\log Z(\alpha_k-\alpha_i)\Big)\\ &-\sum_{\alpha_j-\alpha_i=\beta'+\gamma'}(-1)^{\langle\beta',\gamma'\rangle}\langle\beta',\gamma'\rangle\hat{f}^{\beta'}_{\mathbf{s}}(Z)\hat{f}^{\gamma'}_{\mathbf{s}}(Z)\,\mathrm{d}\log Z(\beta')\end{aligned}$$

where in the last term we sum over decompositions with β', γ' not equal to $\alpha_j - \alpha_k, \alpha_k - \alpha_i$ for $k \neq i, j$. Note that we have

$$\begin{aligned}&\Big(\mathrm{d}\log Z(\alpha_j-\alpha_k)-\mathrm{d}\log Z(\alpha_k-\alpha_i)\Big)\wedge\mathrm{d}\log Z(\alpha_j-\alpha_i) =\\ &= \mathrm{d}\log Z(\alpha_j-\alpha_k)\wedge\mathrm{d}\log Z(\alpha_k-\alpha_i).\end{aligned}$$

It follows that

$$\begin{aligned}&(-1)^{\langle\alpha_j,\alpha_i\rangle}\langle\alpha_j,\alpha_i\rangle\,\mathrm{d}\,\hat{f}^{\alpha_j-\alpha_i}_{\mathbf{s}}(Z)\wedge\mathrm{d}\log Z(\alpha_j-\alpha_i) =\\ &= -\sum_{k\neq i,j}(-1)^{\langle\alpha_j,\alpha_i\rangle+\langle\alpha_j-\alpha_k,\alpha_k-\alpha_i\rangle}\langle\alpha_j,\alpha_i\rangle\langle\alpha_j-\alpha_k,\alpha_k-\alpha_i\rangle\\ &\qquad \hat{f}^{\alpha_j-\alpha_k}_{\mathbf{s}}(Z)\hat{f}^{\alpha_k-\alpha_i}_{\mathbf{s}}(Z)\,\mathrm{d}\log Z(\alpha_j-\alpha_k)\wedge\mathrm{d}\log Z(\alpha_k-\alpha_i)\\ &\quad -\sum_{\alpha_j-\alpha_k=\beta'+\gamma'}(-1)^{\langle\alpha_j,\alpha_i\rangle+\langle\beta',\gamma'\rangle}\langle\alpha_j,\alpha_i\rangle\langle\beta',\gamma'\rangle\hat{f}^{\beta'}_{\mathbf{s}}(Z)\hat{f}^{\gamma'}_{\mathbf{s}}(Z)\\ &\qquad \mathrm{d}\log Z(\beta')\wedge\mathrm{d}\log Z(\alpha_j-\alpha_i),\end{aligned}$$

where in the last term the sum is over decompositions with β', γ' not equal to $\alpha_j - \alpha_k, \alpha_k - \alpha_i$ for $k = 1, \ldots, n$. Thus if $\langle\beta',\gamma'\rangle\hat{f}^{\beta'}_{\mathbf{s}}(Z)\hat{f}^{\gamma'}_{\mathbf{s}}(Z)$ is at least cubic in $\mathbf{s}$ and

$$\begin{aligned}&(-1)^{\langle\alpha_j,\alpha_i\rangle+\langle\alpha_j-\alpha_k,\alpha_k-\alpha_i\rangle}\langle\alpha_j,\alpha_i\rangle\langle\alpha_j-\alpha_k,\alpha_k-\alpha_i\rangle\\ &= (-1)^{\langle\alpha_j,\alpha_k\rangle+\langle\alpha_k,\alpha_i\rangle}\langle\alpha_j,\alpha_k\rangle\langle\alpha_k,\alpha_i\rangle\end{aligned}$$

for at least all $k \neq i, j$ such that $\hat{f}^{\alpha_j-\alpha_k}_{\mathbf{s}}\hat{f}^{\alpha_k-\alpha_i}_{\mathbf{s}} \notin (\mathbf{s}^3)$ then the x_j component of $F(\nabla^{r,\zeta}_{\mathbf{s}})(x_{\alpha_i})$ vanishes modulo terms which are at least cubic in $\mathbf{s}$. □

Taking a closer look at the quadratic equations appearing in Lemma 4.6, one can find constrains on the basis $\alpha_1, \ldots, \alpha_n$ and on the Euler pairing.

Lemma 4.7. *Let α_i be a basis of $K(\mathcal{A}) \otimes \mathbb{R}$. The quadratic equations of the condition 1 in Lemma 4.6 hold for all $i, j = 1, \ldots, n$, that is for all pairwise distinct i, j, k we have*

$$\langle\alpha_j,\alpha_i\rangle\langle\alpha_j-\alpha_k,\alpha_k-\alpha_i\rangle = \langle\alpha_j,\alpha_k\rangle\langle\alpha_k,\alpha_i\rangle,$$

if and only if for all $i, j = 1, \ldots, n$ we have

$$\langle\alpha_i,\alpha_j\rangle = \epsilon_{ij}\lambda$$

where $\epsilon_{ij} = \pm 1$ is a skew-symmetric tensor and λ is a fixed arbitrary constant, such that for all pairwise distinct i, j, k we have

$$1 + \epsilon_{ij}\epsilon_{jk} + \epsilon_{ji}\epsilon_{ik} + \epsilon_{ik}\epsilon_{kj} = 0.$$

A particular solution is given by choosing $\epsilon_{ij} = 1$ for all $i < j$.

Proof. We set $x_{ij} = \langle \alpha_i, \alpha_j \rangle$, so $x_{ij} = -x_{ji}$. The quadratic equations hold if and only if

$$x_{ij}^2 + x_{ij}x_{jk} + x_{ji}x_{ik} + x_{ik}x_{kj} = 0 \tag{4.1.10}$$

for all pairwise distinct i, j, k. Cyclically permuting $i \to j \to k$ in (4.1.10) and subtracting from (4.1.10) gives

$$x_{ij}^2 - x_{jk}^2 = 0$$

for all pairwise distinct i, j, k, so we must have $x_{ij} = \epsilon_{ij}\lambda$ for a skew-symmetric tensor ϵ_{ij} and a fixed, arbitrary constant λ. Plugging this into (4.1.10) turns it into

$$1 + \epsilon_{ij}\epsilon_{jk} + \epsilon_{ji}\epsilon_{ik} + \epsilon_{ik}\epsilon_{kj} = 0. \tag{4.1.11}$$

Direct computation shows that a skew-symmetric index ϵ_{ij} with $\epsilon_{ij} = 1$ for all $i < j$ is a solution. □

Example 4.8. One may regard (4.1.11) abstractly as a system of quadratic constraints on a skew-symmetric tensor $\epsilon_{ij} = \pm 1$, without reference to a basis α_i for $K(\mathcal{A}) \otimes \mathbb{R}$. Note that when $\mathrm{rk}(\mathcal{K}(\mathcal{A})) = 2$ the condition (4.1.11) is empty. Many solutions are possible, e.g. when $\mathrm{rk}(\mathcal{K}(\mathcal{A})) = 3$ the possibilities are

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

up to overall multiplication by ± 1.

Similarly, one wishes to investigate on the other vanishing condition in Lemma 4.6.

Lemma 4.9. *Let $[S_i]$ be the basis of $\mathcal{K}(\mathcal{A})$ given by classes of simple objects. Let α_i be another basis of $K(\mathcal{A}) \otimes \mathbb{R}$. Suppose that for all $i, j = 1, \dots, n$, $i \neq j$ we have either*

1. *$\alpha_j - \alpha_i$ is the class of a simple object or its shift: $\alpha_j - \alpha_i = \pm[S]$, or*
2. *$\alpha_j - \alpha_i$ is the sum of two classes of simple objects or their shifts of the form $\alpha_j - \alpha_k$, $\alpha_k - \alpha_i$; $\alpha_j - \alpha_i = (\pm[S]) + (\pm[T])$ where, for some k, $\pm[S_p] = \alpha_i - \alpha_k$, $\pm[S_q] = \alpha_k - \alpha_j$, or*
3. *$\alpha_j - \alpha_i$ is not the sum of two classes of simple objects or their shifts.*

Then the vanishing condition 2 in Lemma 4.6 holds for all $i, j = 1, \dots, n$, that is for all nontrivial decompositions $\alpha_j - \alpha_i = \beta + \gamma$ with β, γ not equal to $\alpha_j - \alpha_k$, $\alpha_k - \alpha_i$ the product $\langle \beta, \gamma \rangle \hat{f}^{\beta}_{\mathbf{s}}(Z) \hat{f}^{\gamma}_{\mathbf{s}}(Z)$ is at least cubic in $\mathbf{s}$.

Proof. Recall (3.2.7). If $[\alpha]_+ - [\alpha]_- = l > 2$ then for any decomposition $\beta + \gamma = \alpha$ $\hat{f}^{\beta}_{\mathbf{s}}(Z)\hat{f}^{\gamma}_{\mathbf{s}}(Z) \in (\mathbf{s}^3)$. Thus, in case 3) there is nothing to prove. If $\alpha_j - \alpha_i = \pm[S]$ is of type 1), then for any nontrivial decomposition $\beta + \gamma$ either $[\beta_+] - [\beta_-]$ or $[\gamma_+] - [\gamma_-]$ is equal to 2 and $\hat{f}^{\beta}_{\mathbf{s}}(Z)\hat{f}^{\gamma}_{\mathbf{s}}(Z) \in (\mathbf{s}^3)$. Last, if $\alpha_j - \alpha_i = \pm[S] \pm [T]$, $[S], [T]$ classes of simples, and $\beta + \gamma = \alpha_j - \alpha_i$, then $\hat{f}^{\beta}_{\mathbf{s}}(Z)\hat{f}^{\gamma}_{\mathbf{s}}(Z) \notin (\mathbf{s}^3)$ if and only of $\{\beta, \gamma\} = \{\pm S, \pm T\}$. □

Example 4.10. Let $rk(\mathcal{K}(\mathcal{A})) = n$. A basis satisfying the Lemma above is given, for instance, by $\alpha_j = \sum_{i=j}^{n} [S_i]$, $j = 1, \dots, n$. Other solutions are possible and are considered later.

Lemma 4.11. *Suppose that the conditions of Lemma 4.6 hold for all $i, j = 1, \dots, n$ (so $F(\nabla^{r,\zeta}_{\mathbf{s}})$ vanishes modulo terms which are at least cubic in $\mathbf{s}$). Then we have $\nabla^{r,\zeta}_{\mathbf{s}}(\mathcal{V}^{\zeta}_{\mathbf{s}}) = 0$ modulo terms which are at least cubic in $\mathbf{s}$.*

Proof. We compute

$$\pi^{\zeta}(\mathcal{V}_{\mathbf{s}}(x_{\alpha_l})) = \pi^{\zeta}\Big(\sum_{\alpha \neq 0} \hat{f}^{\alpha}_{\mathbf{s}}(Z)(-1)^{\langle \alpha, \alpha_l \rangle}\langle \alpha, \alpha_l \rangle x_{\alpha+\alpha_l}\Big)$$
$$= \sum_{k=1}^{n} \hat{f}^{\alpha_k - \alpha_l}_{\mathbf{s}}(Z)(-1)^{\langle \alpha_k, \alpha_l \rangle}\langle \alpha_k, \alpha_l \rangle x_{\alpha_k}. \tag{4.1.12}$$

So in the local trivialization of $K(\zeta)$ given by $x_{\alpha_1}, \dots, x_{\alpha_n}$ the endomorphism $\mathcal{V}^{\zeta}_{\mathbf{s}}$ is given by the skew-symmetric matrix

$$(\mathcal{V}^{\zeta}_{\mathbf{s}})_{kl} = (-1)^{\langle \alpha_k, \alpha_l \rangle}\langle \alpha_k, \alpha_l \rangle \hat{f}^{\alpha_k - \alpha_l}_{\mathbf{s}}(Z).$$

We have

$$\begin{aligned}
\nabla^{r,\zeta}_{\mathbf{s}}(\mathcal{V}^{\zeta}_{\mathbf{s}}) &= \mathrm{d}\,\mathcal{V}^{\zeta}_{\mathbf{s}} + [A, \mathcal{V}^{\zeta}_{\mathbf{s}}] \\
&= \mathrm{d}(\mathcal{V}^{\zeta}_{\mathbf{s}})_{kl} + \sum_{p=1}^{n}(A_{kp}(\mathcal{V}^{\zeta}_{\mathbf{s}})_{pl} - (\mathcal{V}^{\zeta}_{\mathbf{s}})_{kp}A_{pl}) \\
&= (-1)^{\langle \alpha_k, \alpha_l \rangle}\langle \alpha_k, \alpha_l \rangle\, \mathrm{d}\, \hat{f}^{\alpha_k - \alpha_l}_{\mathbf{s}}(Z) \\
&+ \sum_{p=1}^{n}(-1)^{\langle \alpha_k, \alpha_p \rangle + \langle \alpha_p, \alpha_l \rangle}\langle \alpha_k, \alpha_p \rangle \langle \alpha_p, \alpha_l \rangle \hat{f}^{\alpha_k - \alpha_p}_{\mathbf{s}}(Z) \hat{f}^{\alpha_p - \alpha_l}(Z) \\
&\Big(\mathrm{d} \log Z(\alpha_k - \alpha_p) - \mathrm{d} \log Z(\alpha_p - \alpha_l)\Big)
\end{aligned}$$

using the explicit form of A_{ij}, $(\mathcal{V}^{\zeta}_{\mathbf{s}})_{ij}$ found in Lemma 4.6 and above. So if the conditions of Lemma 4.6 are satisfied for all $i, j = 1, \dots, n$, the same arguments show that $\nabla^{r,\zeta}_{\mathbf{s}}(\mathcal{V}^{\zeta}_{\mathbf{s}})$ vanishes modulo terms which are at least cubic in $\mathbf{s}$. $\square$

The following result is straightforward.

Lemma 4.12. *Suppose that ζ is of the form* (4.1.9) *(so ζ is a section of $K(\zeta)$ and the latter is preserved by C and $\mathcal{U}$). Then we have*

$$\begin{aligned}
\nabla^{r,\zeta}_{\mathbf{s}}(C|_{K(\zeta)}) &= 0, \\
[C|_{K(\zeta)}, \mathcal{U}|_{K(\zeta)}] &= 0, \\
\nabla^{r,\zeta}_{\mathbf{s}}(\mathcal{U}|_{K(\zeta)}) - [C|_{K(\zeta)}, \mathcal{V}^{\zeta}_{\mathbf{s}}] + C|_{K(\zeta)} &= 0.
\end{aligned}$$

Moreover $g|_{K(\zeta)}$ is covariant constant with respect to $\nabla^{r,\zeta}_{\mathbf{s}}$, and $C|_{K(\zeta)}, \mathcal{U}|_{K(\zeta)}$ are symmetric and $\mathcal{V}^{\zeta}_{\mathbf{s}}$ is skew-symmetric with respect to g.

The Lemmas 4.5, 4.6, 4.12 proven in this Section show that the infinite-dimensional Frobenius type structures $(\nabla^r_{\mathbf{s}}, C, \mathcal{U}, \mathcal{V}_{\mathbf{s}}, g)$ on K induce approximate finite-dimensional Frobenius type structures $(\nabla^{r,\zeta}_{\mathbf{s}}, C|_{K(\zeta)}, \mathcal{U}|_{K(\zeta)}, \mathcal{V}^{\zeta}_{\mathbf{s}}, g|_{K(\zeta)})$ on $K(\zeta)$. The result is summarized in the following Corollary.

Corollary 4.13. *Pick a section ζ of the form* (4.1.9) *and suppose that the conditions of Lemma 4.6 hold for all $i, j = 1, \dots, n$. Then*

- $-C_{\bullet}(\zeta) \colon T_U \to K(\zeta)$ *is an isomorphism,*
- *the structure on $K(\zeta)$ given by*

$$(\nabla^{r,\zeta}_{\mathbf{s}}, C|_{K(\zeta)}, \mathcal{U}|_{K(\zeta)}, \mathcal{V}^{\zeta}_{\mathbf{s}}, g|_{K(\zeta)})$$

  is a Frobenius type structure modulo terms which are at least cubic in $\mathbf{s}$. *More precisely the conditions $F(\nabla^{r,\zeta}_{\mathbf{s}}) = 0$ and $\nabla^{r,\zeta}_{\mathbf{s}}(\mathcal{V}^{\zeta}_{\mathbf{s}}) = 0$ hold as identities of formal power series in* $\mathbf{s}$, *modulo terms in* $\mathbf{s}$ *which are at least cubic, while the remaining conditions* (1.6.1) *and those on the metric $g|_{K(\zeta)}$ of Definition 1.27 hold automatically to all orders in* $\mathbf{s}$.

It is worth pointing out that the family of structure on $K(\zeta)$ given by $(\nabla^{r,\zeta}_{\mathbf{s}}, C|_{K(\zeta)}, \mathcal{U}|_{K(\zeta)}, \mathcal{V}^{\zeta}_{\mathbf{s}}, g|_{K(\zeta)})$ depends on the choice of a section ζ of the form (4.1.9) such that the conditions of Lemma 4.6 hold for all $i, j = 1, \dots, n$. In turn, the section ζ encodes moduli given by the choice of basis α_i for $K(\mathcal{A}) \otimes \mathbb{R}$ (satisfying the strong constraints of Lemma 4.6), as well as those given by the choice of holomorphic functions $\zeta_i(Z, \mathbf{s})$. However it is clear from arguments above that the structures only depends on the the choice of basis.

The choice of $\zeta = \sum_i \zeta_i x_{\alpha_i}$ will be crucial for the construction of the Frobenius manifold structures, by means of the Theorem 1.30. They will depend on the $\zeta_i(Z, \mathbf{s})$ moduli as well, through the pullback along $-C_{\bullet}(\zeta) \colon T_U \to K(\zeta)$).

In order to apply Hertling's result (Theorem 1.30) it is necessary to consider the problem of lifting them to genuine Frobenius type structures. This problem will be solved in the next section.

4.2 Lifting to a finite-dimensional Frobenius type structure

In order to apply Hertling's result (Theorem 1.30) it is necessary to consider a genuine Frobenius type structures on the finite rank bundle $K(\zeta)$. The current Section is devoted to proving the following result.

Proposition 4.14. *The approximate Frobenius type structure given by Corollary 4.13 can be lifted canonically to a genuine Frobenius type structure. In other words the solutions defined modulo* $(\mathbf{s})^3$ *to the equations $F(\nabla^{r,\zeta}_{\mathbf{s}}) = 0$ and* (1.6.1) *given by Corollary 4.13 can be lifted canonically to solutions to all orders in* $\mathbf{s}$, *and these lifted formal power series solutions converge provided $||\mathbf{s}||$ is sufficiently small. Moreover the conditions on the metric $g|_{K(\zeta)}$ are also preserved.*

Proving this fact makes use of the general theory of isomonodromy for a family of meromorphic connections on $\mathbb{P}^1$ with poles divisor $2 \cdot 0 + 1 \cdot \infty$, recalled in Section 1.3.

Consider the family of meromorphic connections on the holomorphically trivial vector bundle on $\mathbb{P}^1$ with the same fiber as $K(\zeta)$ given by

$$\nabla^{\zeta}_{\mathbf{s}}(Z) = \mathrm{d} + \Big(\frac{\mathcal{U}(Z)}{z^2} - \frac{\mathcal{V}^{\zeta}_{\mathbf{s}}(Z)}{z}\Big)\,\mathrm{d}\, z$$

with parameter space the suitable opens subset $U \subseteq \mathrm{Stab}(\mathcal{A})$ defined at page 55. This induces a family of connections on the holomorphically trivial *principal* bundle P on $\mathbb{P}^1$ with fiber $GL(K(\zeta)[\![\mathbf{s}]\!])$.

Definition 4.15. Let P be the holomorphically trivial principal bundle on $\mathbb{P}^1$ with fiber the complex affine algebraic group $GL(K(\zeta)[\![\mathbf{s}]\!]/(\mathbf{s})^3)$ corresponding to the $GL(K(\zeta)[\![\mathbf{s}]\!])$-bundle described above. The family of connections $\nabla^{\zeta}_{\mathbf{s},3}(Z)$ on P is defined as the reduction modulo $(\mathbf{s})^3$ of the connections $\nabla^{\zeta}_{\mathbf{s}}(Z)$, that is

$$\nabla^{\zeta}_{\mathbf{s},3}(Z) := \mathrm{d} + \Big(\frac{\mathcal{U}(Z)}{z^2} - \frac{\mathcal{V}^{\zeta}_{\mathbf{s},3}(Z)}{z}\Big)\,\mathrm{d}\, z$$

where $\mathcal{V}^{\zeta}_{\mathbf{s},3} \in \mathfrak{gl}(K(\zeta)[\![\mathbf{s}]\!]/(\mathbf{s})^3)$ is the reduction modulo $(\mathbf{s})^3$.

By a generalization due to Boalch and Bridgeland - Toledano Laredo of the theory of Jimbo, Miwa and Ueno (Section 1.2.1) to algebraic groups, the isomonodromy of this family is equivalent to the flatness of an extended connection on the pullback of P to $\mathbb{P}^1 \times U$.

Lemma 4.16. *The family of connections on P given by $\nabla^{\zeta}_{\mathbf{s},3}(Z)$ is isomonodromic as Z varies in U.*

Proof. By Corollary 4.13 we have $F(\nabla^{r,\zeta}_{\mathbf{s},3}) = 0$ and the equations (1.6.1) hold for $\mathrm{d}\, Z, \mathcal{U}, \mathcal{V}^{\zeta}_{\mathbf{s},3}$ in the bundle P. This can be stated equivalently by introducing a connection on the pullback of P to $\mathbb{P}^1 \times U$, given by

$$\nabla^{r,\zeta}_{\mathbf{s},3} - \frac{1}{z}dZ + \Big(\frac{\mathcal{U}(Z)}{z^2} - \frac{\mathcal{V}^{\zeta}_{\mathbf{s},3}(Z)}{z}\Big)\,\mathrm{d}\, z,$$

which is then flat (Theorem 1.28). Flatness of this connection is precisely the isomonodromy condition for $\nabla^{r,\zeta}_{\mathbf{s},3}(Z)$. $\square$

Recall that the generalized monodromy of a differential operator of the form of $\nabla^{\zeta}_{\mathbf{s},3}(Z)$ is determined by its Stokes rays and the corresponding Stokes factors (see Section 1.3 and in particular 1.3.1). Before to compute the Stokes data of $\nabla^{\zeta}_{\mathbf{s},3}(Z)$ let us make a remark.

For generic Z, the Stokes factors of $\nabla^{\zeta}_{\mathbf{s},3}(Z)$ can be computed explicitly thanks to the formula by Bridgeland and Toledano-Laredo (Theorem 1.22, page 14). Theorem 1.22 refers to a connection of the form $\nabla = \mathrm{d} - (\frac{U}{z^2} + \frac{V}{z})\,\mathrm{d}\, z$, while $\nabla^{r,\zeta}_{\mathbf{s},3}$ has the form

$$\mathrm{d} + \Big(\frac{U(Z)}{z^2} - \frac{V(Z)}{z}\Big)\,\mathrm{d}\, z \tag{4.2.1}$$

with $U = Z$. This is responsible for a minor modification of formula 1.3.3, which can be deduced immediately revisiting the proof of the Theorem, given in [11, Sections 8.4, 9.1]. The Stokes factor $\mathcal{S}_\ell$ of (4.2.1) attached to a Stokes ray $\ell = \mathbb{R}_{>0}(u_i - u_j)$ is given by

$$\mathcal{S}_\ell = \mathrm{Id} - \sum_{\substack{\gamma = u_i - u_j \\ \gamma \in \ell}} \sum_{k \geq 1} \sum_{\substack{\gamma_1, \ldots, \gamma_k \in \Phi^U \\ \gamma_1 + \cdots + \gamma_k = \gamma}} M_k(\gamma_1, \ldots, \gamma_k) V_{\gamma_1} \cdots V_{\gamma_k} E_{ij}, \tag{4.2.2}$$

E_{ij} being the elementary matrices. Notice also that Z-stability of an object of class α in $\mathcal{A}$ induces the stability of its shift in $\mathcal{A}[1]$, and the DT invariants have the symmetry property $\mathrm{DT}(\alpha, Z) = \mathrm{DT}(-\alpha, Z)$. Moreover $M_n(z_1, \ldots, z_n) = M_n(-z_1, \ldots, -z_n)$, [22].

Lemma 4.17. *The generalized monodromy is given by*

- *the Stokes rays*
$$\ell_{ij}(Z) = \mathbb{R}_{>0} Z(\alpha_i - \alpha_j) \subset \mathbb{C}^*$$
for $i \neq j$,
- *the corresponding Stokes factors*
$$\begin{aligned} \mathcal{S}_{\ell_{ij}}(Z) = I &- 2\pi i\, (\mathcal{V}^{\zeta}_{\mathbf{s},3})_{ij} E_{ij} \\ &+ \sum_k M_2(Z(\alpha_i - \alpha_k), Z(\alpha_k - \alpha_j))(\mathcal{V}^{\zeta}_{\mathbf{s},3})_{ik}(\mathcal{V}^{\zeta}_{\mathbf{s},3})_{kj} E_{ij}, \end{aligned} \tag{4.2.3}$$
where E_{ij} are the elementary matrices.

Proof. The eigenvalues of $\mathrm{ad}(\mathcal{U})$ form a set of roots
$$\Phi^Z := \{(Z(\alpha_i) - Z(\alpha_j)), i \neq j\} \subset \mathbb{C}$$
which are distinct for generic $Z \in U$. The Stokes rays emanate from directions in Φ^Z and are
$$\ell_{ij}(Z) = \mathbb{R}_{>0} Z(\alpha_i - \alpha_j).$$
The corresponding Stokes factors in $GL(K(\zeta)[\![\mathbf{s}]\!]/(\mathbf{s})^3)$ are given by (4.2.2). For order reasons they depend only on the functions $M_1(z_1) = 2\pi i$ and $M_2(z_1, z_2) = -2\pi i \int_{[0, z_1+z_2]} \frac{dt}{t - z_1}$. $\square$

The monodromy modulo $(\mathbf{s})^3$ can be computed explicitly in the situation of Lemma 4.9.

Corollary 4.18. *Suppose $\alpha_i - \alpha_j$ is the class of a simple object or the sum of classes of simple objects of the form $\alpha_i - \alpha_k$, $\alpha_k - \alpha_j$. Then the Stokes factor corresponding to ℓ_{ij} is given by*
$$\mathcal{S}_{\ell_{ij}}(Z) = I - (-1)^{\langle \alpha_i, \alpha_j \rangle} \langle \alpha_i, \alpha_j \rangle\, \mathrm{DT}_{\mathcal{A}}(\alpha_i - \alpha_j, Z) \mathbf{s}^{\alpha_i - \alpha_j} E_{ij}.$$

Proof. If $\alpha_i - \alpha_j$ is the class of a simple object or its shift then according to (4.2.3) we have modulo $(\mathbf{s})^3$
$$\begin{aligned} \mathcal{S}_{\ell_{ij}}(Z) &= I - 2\pi i (-1)^{\langle \alpha_i, \alpha_j \rangle} \langle \alpha_i, \alpha_j \rangle \hat{f}^{\alpha_i - \alpha_j}_{\mathbf{s}}(Z) E_{ij} \\ &= I - (-1)^{\langle \alpha_i, \alpha_j \rangle} \langle \alpha_i, \alpha_j \rangle\, \mathrm{DT}_{\mathcal{A}}(\alpha_i - \alpha_j, Z) \mathbf{s}^{\alpha_i - \alpha_j} E_{ij}. \end{aligned}$$

In the other case we have similarly modulo $(\mathbf{s})^3$
$$\begin{aligned} \mathcal{S}_{\ell_{ij}}(Z) = I &- 2\pi i (-1)^{\langle \alpha_i, \alpha_j \rangle} \langle \alpha_i, \alpha_j \rangle \hat{f}^{\alpha_i - \alpha_j}_{\mathbf{s}}(Z) E_{ij} \\ &+ (-1)^{\langle \alpha_i, \alpha_k \rangle + \langle \alpha_k, \alpha_j \rangle} \langle \alpha_i, \alpha_k \rangle \langle \alpha_k, \alpha_j \rangle \\ &M_2(Z(\alpha_k - \alpha_i), Z(\alpha_j - \alpha_k)) \hat{f}^{\alpha_i - \alpha_k}_{\mathbf{s}}(Z) \hat{f}^{\alpha_k - \alpha_j}_{\mathbf{s}}(Z) E_{ij}. \end{aligned}$$

Let $\log(z)$ denote the branch of the complex logarithm branched along $[0,+\infty)$. According to the formulae for holomorphic generating functions in [10] we have modulo $(\mathbf{s})^3$

$$\begin{aligned}\hat{f}_{\mathbf{s}}^{\alpha_i-\alpha_j}(Z) =& \frac{1}{2\pi i}\,\mathrm{DT}(\alpha_i-\alpha_j,Z)\mathbf{s}^{\alpha_i-\alpha_j}\\ &+\frac{1}{(2\pi i)^2}(-1)^{\langle\alpha_i-\alpha_k,\alpha_k-\alpha_j\rangle}\langle\alpha_i-\alpha_k,\alpha_k-\alpha_j\rangle\\ &M_2(Z(\alpha_i-\alpha_k),Z(\alpha_k-\alpha_j))\\ &\mathrm{DT}_{\mathcal{A}}(\alpha_i-\alpha_k,Z)\,\mathrm{DT}_{\mathcal{A}}(\alpha_k-\alpha_j,Z)\mathbf{s}^{\alpha_i-\alpha_j}.\end{aligned}$$

On the other hand we have modulo $(\mathbf{s})^3$

$$\hat{f}_{\mathbf{s}}^{\alpha_i-\alpha_k}(Z)\hat{f}_{\mathbf{s}}^{\alpha_k-\alpha_j}(Z)=\frac{1}{(2\pi i)^2}\,\mathrm{DT}_{\mathcal{A}}(\alpha_i-\alpha_k,Z)\,\mathrm{DT}_{\mathcal{A}}(\alpha_k-\alpha_j,Z)\mathbf{s}^{\alpha_i-\alpha_j}.$$

Moreover the quadratic condition gives

$$\begin{aligned}&(-1)^{\langle\alpha_i,\alpha_j\rangle}\langle\alpha_i,\alpha_j\rangle(-1)^{\langle\alpha_i-\alpha_k,\alpha_k-\alpha_j\rangle}\langle\alpha_i-\alpha_k,\alpha_k-\alpha_j\rangle\\ &=(-1)^{\langle\alpha_i,\alpha_k\rangle+\langle\alpha_k,\alpha_j\rangle}\langle\alpha_i,\alpha_k\rangle\langle\alpha_k,\alpha_j\rangle.\end{aligned}$$

The claim follows. □

Corollary 4.19. *Let $\mathcal{S}_\ell(Z)$ denote the matrices* (4.2.3) *corresponding to a choice of central charge Z, and let $\Sigma\subset\mathbb{C}^*$ be a convex open sector.*

- *The clockwise ordered product*

$$\prod_{\ell\subset\Sigma}^{\curvearrowright}\mathcal{S}_\ell(Z)\in GL(K(\zeta)[\![\mathbf{s}]\!]/\mathbf{s}^3)$$

is constant as a function of Z as long as the rays $\ell(Z)$ do not cross $\partial\Sigma$.

- *The Stokes multiplier of the connection $\nabla^{\zeta}_{\mathbf{s},3}(Z)$ with respect to the admissible ray $\mathbb{R}_{>0}$, given by the clockwise ordered product*

$$\mathcal{S}=\prod_{\ell\subset\bar{\mathbb{H}}}^{\curvearrowright}\mathcal{S}_\ell(Z)\in GL(K(\zeta)[\![\mathbf{s}]\!]/\mathbf{s}^3),\tag{4.2.4}$$

is in fact constant as a function of $Z\in U\subset\mathrm{Stab}(\mathcal{A})$.

Proof. Both statements are characterizations of isomonodromy (Section 1.3). □

Formulae by Bridgeland and Toledano-Laredo [11] allow to invert the process described above and to define an isomonodromic family of connections on $\mathbb{P}^1$ with structure group $GL(K(\zeta)[\![\mathbf{s}]\!]$, given its Stokes data.

Definition 4.20. The canonical lift $\tilde{\mathcal{S}}^{(0)}$ of the Stokes multiplier $\mathcal{S}$ given by (4.2.4) to $GL(K(\zeta)[\mathbf{s}])$ is $\mathcal{S}(Z)$ regarded as an element of $GL(K(\zeta)[\mathbf{s}])$. Note that we have $\tilde{\mathcal{S}}|_{\mathbf{s}=0}=I$. A general lift $\tilde{\mathcal{S}}$ of $\mathcal{S}$ is a matrix in $GL(K(\zeta)[\mathbf{s}])$ which agrees with $\tilde{\mathcal{S}}$ modulo $(\mathbf{s})^3$.

The Stokes data of Definition 4.20 defines a new isomonodromic family of connections corresponding to a family of genuine Frobenius type structure supported by the finite-dimensional bundle $K(\zeta)$. Its data will be distinguished from the previous ones by a $\sim$.

Proposition 4.21. *For fixed Z and sufficiently small $||\mathbf{s}||$ there is a canonical choice of a connection $\tilde{\nabla}^{\zeta}_{\mathbf{s}}(Z)$ on the trivial principal $GL(K(\zeta))$-bundle, of the form*

$$\tilde{\nabla}^{\zeta}_{\mathbf{s}}(Z) = \mathrm{d} + \Big(\frac{\mathcal{U}(Z)}{z^2} - \frac{\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}(Z)}{z}\Big)\,\mathrm{d}\,z$$

with Stokes multiplier with respect to the admissible ray $\mathbb{R}_{>0}$ given by the canonical lift $\tilde{\mathcal{S}}^{(0)}$. The connection matrix $\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}(Z)$ is skew-symmetric and depends holomorphically on both Z and $\mathbf{s}$. The reduction of $\tilde{\nabla}^{\zeta}_{\mathbf{s}}(Z)$ modulo $(\mathbf{s})^3$ is $\nabla^{\zeta}_{\mathbf{s},3}(Z)$. The same holds for any other (non-canonical) choice of a lift $\tilde{\mathcal{S}}$.

Proof. The result follows from Proposition 1.23 applied to the bundle P. □

Definition 4.22. The canonical lift of the approximate Frobenius type structure on $K(\zeta)$ $(\nabla^{r,\zeta}_{\mathbf{s}}, C|_{K(\zeta)}, \mathcal{U}|_{K(\zeta)}, \mathcal{V}^{\zeta}_{\mathbf{s}}, g|_{K(\zeta)})$ is defined as the collection of holomorphic objects

- endomorphism $\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}$, of Proposition 4.21,
- connection $\tilde{\nabla}^{r,\zeta}_{\mathbf{s}}$
$$\tilde{\nabla}^{r,\zeta}_{\mathbf{s}} = \mathrm{d} + \tilde{A}$$
with connection form $\tilde{A}_{ij} = (\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}})_{ij}\ \mathrm{d}\log Z(\alpha_i - \alpha_j)$,
- Higgs field $C|_{K(\zeta)}$, endomorphism $\mathcal{U}|_{K(\zeta)}$ and metric $g|_{K(\zeta)}$.

Of course one can give an identical definition for any other choice of a lift $\tilde{\mathcal{S}}$.

Corollary 4.23. *The collection*

$$(\tilde{\nabla}^{r,\zeta}_{\mathbf{s}}, C|_{K(\zeta)}, \mathcal{U}|_{K(\zeta)}, \tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}, g|_{K(\zeta)}) \tag{4.2.5}$$

is a Frobenius type structure on the bundle $K(\zeta) \to U$, depending holomorphically on $\mathbf{s}$ for $||\mathbf{s}||$ sufficiently small. The same holds for any other choice of a lift $\tilde{\mathcal{S}}$.

Proof. For fixed $\mathbf{s}$, with $||\mathbf{s}||$ sufficiently small, the family of connections $\tilde{\nabla}^{\zeta}_{\mathbf{s}}(Z)$ has constant generalized monodromy as Z varies in U. By a characterization of isomonodromy (see 1.2.1), the family of connections on P pulled back to $\mathbb{P}^1 \times U$

$$\tilde{\nabla}^{r,\zeta}_{\mathbf{s}} - \frac{1}{z}\,\mathrm{d}\,Z + \Big(\frac{\mathcal{U}(Z)}{z^2} - \frac{\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}(Z)}{z}\Big)\,\mathrm{d}\,z$$

is flat. This is equivalent to the equations $F(\tilde{\nabla}^{r,\zeta}_{\mathbf{s}}) = 0$ and (1.6.1). The conditions on g can be checked directly. □

4.3 Pull-back to Frobenius manifolds

So far we have discussed when the finite-dimensional vector bundle $K(\zeta)$ is isomorphic to the tangent bundle over $U \subset \mathrm{Stab}(\mathcal{A})$ via the map $-C_{\bullet}(\zeta)$,

$$K(\zeta) \xrightarrow{\simeq} T_U,$$

and when it is endowed with a Frobenius type structure $(\tilde{\nabla}^{r,\zeta}_{\mathbf{s}}, C|_{K(\zeta)}, \mathcal{U}|_{K(\zeta)}, \tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}, g|_{K(\zeta)})$. To apply Theorem 1.30, one has to check $\tilde{\nabla}^{r,\zeta}_{\mathbf{s}}$–flatness of ζ and the conformal condition $\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}(\zeta) = \frac{d}{2}\zeta, d \in \mathbb{C}$.

Lemma 4.24. *The endomorphism $\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}$ acts on the space of flat sections of $\tilde{\nabla}^{r,\zeta}_{\mathbf{s}}$; a solution of $\tilde{\nabla}^{r,\zeta}_{\mathbf{s}}(\zeta) = 0$ is an eigenvector of $\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}$ and the spectrum of $\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}$ is constant on U, that is in the Z direction (it depends highly nontrivially on $\mathbf{s} \in \Delta$).*

Proof. Let ψ be a fundamental solution of $\tilde{\nabla}^{r,\zeta}_{\mathbf{s}}$, then

$$\begin{aligned}\tilde{\nabla}^{r,\zeta}_{\mathbf{s}}(\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}\psi) &= (\mathrm{d} + \tilde{A})(\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}\psi) = (\mathrm{d}\,\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}})\psi + [\tilde{A}, \tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}]\psi + \tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}(\mathrm{d}\,\psi) + \tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}\tilde{A}\psi = \\ &= (\mathrm{d}\,\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}})\psi + [\tilde{A}, \tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}]\psi - \tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}\tilde{A}\psi + \tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}\tilde{A}\psi = \\ &= \tilde{\nabla}^{r,\zeta}_{\mathbf{s}}(\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}})\psi = 0.\end{aligned}$$

By $\tilde{\nabla}^{r,\zeta}_{\mathbf{s}}-$flatness of $\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}$, $\tilde{A}$ and $\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}$ form a "Lax pair" with respect to the complex variable Z. Let $\overline{\zeta}$ be a solution of $\tilde{\nabla}^{r,\zeta}_{\mathbf{s}}$, i.e. $\mathrm{d}\,\overline{\zeta} = -\tilde{A}\overline{\zeta}$, and suppose it is an eigenvector of the endomorphism $\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}$. Differentiating $\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}(\zeta) = D(\mathbf{s}, Z)\zeta$ one gets

$$(\mathrm{d}\,\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}})\overline{\zeta} + \tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}(\mathrm{d}\,\overline{\zeta}) = (\mathrm{d}\,D(\mathbf{s}, Z))\overline{\zeta} + D(\mathbf{s}, Z)(\mathrm{d}\,\overline{\zeta})$$

and

$$(\mathrm{d}\,\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}})\overline{\zeta} + [\tilde{A}, \tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}]\overline{\zeta} = (\mathrm{d}\,D(\mathbf{s}, Z))\overline{\zeta}.$$

$\mathrm{d}_Z\, D(\mathbf{s}, Z) = 0$ follows from the $\tilde{\nabla}^{r,\zeta}_{\mathbf{s}}-$flatness of $\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}$. □

The following Proposition holds as a natural corollary.

Proposition 4.25. *Fix the choice of basis α_i for $K(\mathcal{A}) \otimes \mathbb{R}$. Let $\frac{d}{2}$ be an eigenvalue of $\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}$. Then we can find a section ζ of $K \to U$ such that*

$$\begin{cases}\tilde{\nabla}^{r,\zeta}_{\mathbf{s}}(\zeta) = 0 \\ \tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}(\zeta) = \frac{d}{2}\zeta\end{cases}. \tag{4.3.1}$$

Restricting U if necessary we may assume that $-dZ(\zeta)$ is still an isomorphism.

The results proven so far lead to conclude that U admits a Frobenius manifold structure.

Theorem 4.26. *Let $d(\mathbf{s})$ be an eigenvalue of $\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}$. There exists a semisimple Frobenius manifold structure on $U \subset \mathrm{Stab}(\mathcal{A})$ such that*

- *the canonical coordinates are given by*
$$u_i = Z(\alpha_i),$$
- *the Euler field is*
$$E = \sum_i Z(\alpha_i)\partial_{Z(\alpha_i)}$$
- *the flat metric is given by*
$$g_{\mathbf{s}}(u) = \sum_i \zeta_i^2(Z, \mathbf{s})\,\mathrm{d}\,u_i^2$$
- *the conformal dimension is $2 - d(\mathbf{s})$.*

It is given by pulling back the Frobenius type structure (4.2.5) *along $-\,\mathrm{d}\,Z(\zeta)$, where ζ is a section of $K \to U$ as in Proposition 4.25.*

Moreover if we have $\alpha_i - \alpha_j \in \pm K(\mathcal{A})_{>0}$ for all $i \neq j$ then this can be analytically continued to a Frobenius manifold structure on all $\mathrm{Stab}(\mathcal{A})$, *without monodromy.*

Proof. The Theorem follows from the statements 4.23, 4.25 and Theorem 1.30, setting $v = \mathrm{d}\, Z_{\bullet}(\zeta)$. The multiplication and the unit field are given respectively by (1.6.4) and (1.6.5): $X \circ Y = v^{-1}(C_X v(Y))$ and $e = v^{-1}(\zeta) = \sum_i \frac{\partial}{\partial Z(\alpha_i)}$. Flatness of e comes from flatness of ζ. The inverse image of $\zeta_i(Z)x_{\alpha_i}$ is $\frac{\partial}{\partial Z(\alpha_j)}$. It is easy to verify that $\left\{\frac{\partial}{\partial Z(\alpha_i)}\right\}_{i=1,\dots,n}$ is a local frame which satisfy $\frac{\partial}{\partial Z(\alpha_i)} \circ \frac{\partial}{\partial Z(\alpha_j)} = \delta_{ij}\frac{\partial}{\partial Z(\alpha_i)}$, therefore $u_i := Z(\alpha_i)$, $i = 1, \dots, n$, are canonical coordinates.

$E := \mathcal{U}(e) = \sum_i Z(\alpha_i)\frac{\partial}{\partial Z(\alpha_i)}$ is an Euler field. The metric entries are

$$
\begin{aligned}
g_{\mathbf{s}}(u_i, u_j) &= g_{|K(\zeta)}(-\mathrm{d}\, Z_{\partial/\partial Z(\alpha_i)}(\zeta), -\mathrm{d}\, Z_{\partial/\partial Z(\alpha_j)}(\zeta)) \\
&= \delta_{ij}\zeta_i^2(Z, \mathbf{s}).
\end{aligned}
$$

The Stokes data of the Frobenius manifold structure are functions of the coordinates $u_i \neq u_j$. They can be analytically continued over the universal covering of $\mathbb{C}^n \setminus \{u_i = u_j\}_{i \neq j}$. The continuation extends the structure to all $\mathrm{Stab}(\mathcal{A}) \supset U$. □

Canonical coordinates are in general not flat and flat coordinates are obtained by solving a differential equation. Call them x_k, $k = 0, \dots, n-1$, and write the corresponding coordinate vector fields $X_{(k)} := \frac{\partial}{\partial x_k}$ as $X_{(k)} = \sum_j a_j^{(k)}(\underline{u})\frac{\partial}{\partial u_j}$, where $\underline{u}$ denotes the vector $u_1, \dots, u_n$. We can set $X_0 = e$. Say M_ζ is the matrix $diag\,\{\zeta_1(\mathbf{s}, Z), \dots, \zeta_n(\mathbf{s}, Z)\}$, corresponding to the isomorphism v. Then $X_{(k)}$ are solutions of the system

$$
\begin{cases}
\mathrm{d}\, X_{(k)} = -M_\zeta^{-1}(\mathrm{d}\, M_\zeta + \tilde{A}M_\zeta)X_{(k)} \\
\underline{u} \cdot \nabla a_i^{(k)}(\underline{u}) = (1 - d_{(k)})a_i^{(k)}(\underline{u}) \\
\forall i = 0, \dots, n-1
\end{cases}
\tag{4.3.2}
$$

The first equation of (4.3.2) comes from $\tilde{\nabla}_{\mathbf{s}}^{r,\zeta}$–flatness, the second is equivalent to semisimplicity of $-\operatorname{ad} E$. The spectrum of the semisimple Euler field is the datum of the conformal dimension $D(\mathbf{s}) = 2 - d(\mathbf{s})$ and the eigenvalues of $-\operatorname{ad} E$, that is $(D(\mathbf{s}), d_{(0)} = 1, d_{(1)}(\mathbf{s}), \dots, d_{(n-1)}(\mathbf{s}))$.

Notice that, if $d \neq 0$, then $g(e, e) = 0$. In fact $g(e, e) = \sum_i g_{ii} = \sum_i \zeta_i^2$. If $d(\mathbf{s}) \neq 0$, from $\tilde{\mathcal{V}}_{\mathbf{s}}^{\zeta}(\zeta) = \frac{d}{2}\zeta$ we have

$$
\sum_{j \neq i} (\tilde{\mathcal{V}}_{\mathbf{s}}^{\zeta})_{ij}\zeta_j = \frac{d}{2}\zeta_i
$$

and

$$
\frac{d}{2}\sum_i \zeta_i^2 = \sum_i \left(\sum_{j \neq i} (\tilde{\mathcal{V}}_{\mathbf{s}}^{\zeta})_{ij}\zeta_j\right)\zeta_i = \sum_{i,j} (\tilde{\mathcal{V}}_{\mathbf{s}}^{\zeta})_{ij}\zeta_j\zeta_i = \sum_{i<j} (\tilde{\mathcal{V}}_{\mathbf{s}}^{\zeta})_{ij}\zeta_j\zeta_i + \sum_{i>j} (\tilde{\mathcal{V}}_{\mathbf{s}}^{\zeta})_{ij}\zeta_j\zeta_i = 0.
$$

Thus the flat basis can be normalized in such a way that the metric in flat coordinates has form (1.1.5).

Finally, one can characterized the Frobenius manifold structure defined in Theorem 4.26 by its monodromy. Denote the connection $v^*\tilde{\nabla}_{\mathbf{s}}^{r,\zeta}$ simply by ∇.

Theorem 4.27. *The Stokes matrix of the semisimple Frobenius manifold structure on* $\mathrm{Stab}(\mathcal{A})$ *of Theorem 4.26 coincides with the Stokes multiplier* $\tilde{\mathcal{S}}$ *(given by Definition 4.20) of the Frobenius type structure* (4.2.5).

Proof. It is enough to compute the connection ∇ in the basis $\partial_{\tilde{u}_i} = \left(\zeta_i(Z,\mathbf{s})\right)^{-1}\partial_{u_i}$. $\square$

Remark. The main object of interest seems to be the section ζ of the bundle K. It determines the conformal dimension and the metric of the Frobenius manifold structure on $\mathrm{Stab}(\mathcal{A})$. Computing $\zeta = \zeta(\tilde{\mathcal{S}})$ as a (multi-valued) function of the Stokes multiplier is an instance of the (hard) inverse problem for semisimple Frobenius manifolds (see e.g. [24]).

4.4 Some case studies

We can now apply the general theory discussed above to the case of $\mathcal{A} = \mathrm{Rep}(A_n, 0)$. The construction is studied in details for the representation category of A_2. For $n \geq 3$ explicit formulae are hard to compute and we classify the resulting structure depending on its Stokes matrix.

4.4.1 A_2 quiver

The category $\mathcal{A} = \mathrm{Rep}(A_2)$ was the model for the construction described in the previous Sections. The low dimension of the lattice $\mathcal{K}(\mathcal{A})$ makes some steps of the machinery trivial or unnecessary. However, I think it is instructive to have a closer look to the actual construction of the Frobenius structure. We first look for a basis $\{\alpha_1, \alpha_2\}$ of $K(\mathcal{A}) \otimes \mathbb{R}$ satisfying Lemma 4.6. The first requirement of Lemma 4.6 (involving the Euler form $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathcal{A}}$ on $K(\mathcal{A})$) is empty. The second condition requires that for all nontrivial decompositions $\alpha_1 - \alpha_2 = \beta + \gamma$ the term $\langle \beta, \gamma \rangle \hat{f}^{\beta}_{\mathbf{s}}(Z) \hat{f}^{\gamma}_{\mathbf{s}}(Z)$ with $(\beta, \gamma) \notin \{\pm(\alpha_1 - \alpha_2)\}^2$, is zero or of order at least cubic in $\mathbf{s}$. If we assume that $\{\alpha_1, \alpha_2\}$ is an integral basis and write $\alpha_1 - \alpha_2 = p[S_1] + q[S_2]$, $p, q \in \mathbb{Z}$, then by (3.2.7), the condition is verified if p or $q > 1$ or one of the following holds: $(p, q) = (\pm 1, 0)$ and $\alpha_1 - \alpha_2 = \pm[S_1]$ or $(p, q) = (0, \pm 1)$ and $\alpha_1 - \alpha_2 = \pm[S_2]$. We chose the basis of the classes

$$\begin{aligned} \alpha_1 &= [S_1] + [S_2] \\ \alpha_2 &= \phantom{[S_1] + {}} [S_2] \end{aligned}$$

Notice that we are mostly interested in $\pm(\alpha_1 - \alpha_2) = \pm[S_1]$.

Let U be an open subset of the space of stability conditions over $\mathcal{A}$. The trivial vector bundle $K \to U$ with fiber $\mathbb{C}[K(\mathcal{A})][\![\mathbf{s}]\!]$ has a $\mathbb{C}[\![\mathbf{s}]\!]$-linear Frobenius type structure

$$\nabla^r_{\mathbf{s}}, = \mathrm{d} + \sum_{\alpha \neq 0} \mathrm{ad}\, f^{\alpha}_{\mathbf{s}} \frac{\mathrm{d}\, Z(\alpha)}{Z(\alpha)}, C = -\mathrm{d}\, Z,\ \mathcal{U} = Z,\ \mathcal{V}_{\mathbf{s}} = \mathrm{ad}\, f_{\mathbf{s}}(Z)),$$

where Z is a point of U acting as a derivation on the fibers of K. Let $K(\zeta)$ be the rank two sub-bundle of K generated by $Z(\alpha_1), Z(\alpha_2)$, and $\pi^{\zeta} : K \to K(\zeta)$ the projection. We define

$$\mathcal{V}^{\zeta}_{\mathbf{s}} := \pi^{\zeta} \mathcal{V}_{\mathbf{s}}, \quad \nabla^{r,\zeta}_{\mathbf{s}} := \pi^{\zeta} \nabla^r_{\mathbf{s}}.$$

Flatness of $\mathcal{V}^{\zeta}_{\mathbf{s}}$ and of $\nabla^{r,\zeta}_{\mathbf{s}}$ are equivalent to the ODE

$$\mathrm{d}(\mathcal{V}^{\zeta}_{\mathbf{s}})_{ij} = -\sum_k (\mathcal{V}^{\zeta}_{\mathbf{s}})_{ik} (\mathcal{V}^{\zeta}_{\mathbf{s}})_{kj} \big(\mathrm{d} \log(Z(\alpha_i - \alpha_k)) - \mathrm{d} \log(Z(\alpha_k - \alpha_j)) \big)$$

that is trivial for $\mathrm{rk}(\mathcal{K}(\mathcal{A})) = 2$, since $\mathrm{d}\, f^{\alpha_1 - \alpha_2}_{\mathbf{s}} = 0 \mod (\mathbf{s}^3)$. This implies that the projected Frobenius type structure is still a Frobenius type structure modulo $(\mathbf{s}^3)$ and one can just set

$$
\begin{aligned}
(\mathcal{V}_{\mathbf{s}}^{\zeta})_{12} &= (-1)^{\langle \alpha_1, \alpha_2\rangle} \langle \alpha_1, \alpha_2\rangle \hat{f}_{\mathbf{s}}^{\alpha_1-\alpha_2}/(\mathbf{s}^3) \\
&= (-1)^{\langle [S_1],[S_2]\rangle} \langle [S_1],[S_2]\rangle s_1 \, \mathrm{DT}([S_1], Z) \frac{1}{2\pi i} \\
&= \frac{s_1}{2\pi i}.
\end{aligned}
$$

Therefore $\mathcal{V}_{\mathbf{s}}^{\zeta}$ is a constant endomorphism depending on the complex parameter $s := s_1 \in \mathbb{C}^*$ and $\nabla_{\mathbf{s}}^{r,\zeta}$ has connection matrix A

$$
\begin{aligned}
A = &\begin{pmatrix} 0 & \frac{s}{2\pi i}\, \mathrm{d}\log(Z(\alpha_1-\alpha_2)) \\ -\frac{s}{2\pi i}\, \mathrm{d}\log(Z(\alpha_1-\alpha_2)) & 0 \end{pmatrix} = \\
= &\begin{pmatrix} 0 & \frac{s}{2\pi i}\, \mathrm{d}\log Z([S_1]) \\ -\frac{s}{2\pi i}\, \mathrm{d}\log Z([S_1]) & 0 \end{pmatrix}.
\end{aligned}
$$

For any value of s, $\mathcal{V}_{\mathbf{s}}^{\zeta}, \nabla_{\mathbf{s}}^{r,\zeta}$ together with a bilinear pairing $g = \mathrm{Id}$ endow U with a Frobenius type structure.

The main interesting object is the section $\zeta = \zeta_1(\mathbf{s}, Z)x_{\alpha_1} + \zeta_2(\mathbf{s}, Z)x_{\alpha_2}$, to be determined. We require that

- ζ is $\nabla_{\mathbf{s}}^{r,\zeta}$-flat
- ζ is an eigenvector for the automorphism $\mathcal{V}_{\mathbf{s}}^{\zeta}$, i. e. there exists a scalar $d \in \mathbb{C}$, with $\mathcal{V}_{\mathbf{s}}^{\zeta}\zeta = \frac{d}{2}\zeta$.

We obtain two families of solutions to this problem:

$$
\begin{aligned}
d &= \frac{s}{\pi} \qquad \text{and} \qquad \zeta(s,Z) = BZ(\alpha_1-\alpha_2)^{-s/2\pi}(x_{\alpha_1} + i x_{\alpha_2}), \\
d &= -\frac{s}{\pi} \qquad \text{and} \qquad \zeta(s,Z) = BZ(\alpha_1-\alpha_2)^{s/2\pi}(x_{\alpha_1} - i x_{\alpha_2}).
\end{aligned}
$$

Applying Hertling's Theorem 1.30 one gets a family of Frobenius structures on T_U parametrized by $s \in \mathbb{C}^*$ with

- canonical coordinates $u_1 = Z(\alpha_1), u_2 = Z(\alpha_2)$,
- unit field $e = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}$,
- Euler field $E = u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2}$
- conformal dimensions $D = 2 - d = 2 \mp s\pi^{-1}$
- diagonal metric $\tilde{g}$

$$
\begin{aligned}
\tilde{g}\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_1}\right) &= B^2\,(u_1-u_2)^{\mp s/\pi} \\
\tilde{g}\left(\frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_2}\right) &= -B^2\,(u_1-u_2)^{\mp s/\pi}
\end{aligned}
$$

- metric potential $\eta = B^2\left(1 \mp \frac{s}{\pi}\right)^{-1}(u_1-u_2)^{1\mp s/\pi}$

- extended connection

$$\overline{\nabla} = \mathrm{d}_{\mathbb{P}^1 \times U(\mathcal{A})} - \frac{1}{z}\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} + $$
$$+ \left[\frac{1}{z^2}\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} - \frac{1}{z}\begin{pmatrix} 0 & \pm i \frac{s}{2\pi i} \\ \mp i \frac{s}{2\pi i} & 0 \end{pmatrix} \right] \mathrm{d}\, z$$

since $\zeta_1^{-1}(\mathbf{s}, Z)\zeta_2(\mathbf{s}, Z) = \pm i$.

The Stokes data of the Frobenius structure above is the generalized monodromy data of

$$\nabla(Z) = \mathrm{d}_{\mathbb{P}^1} + \left(\frac{\mathcal{U}}{z^2} - \frac{\mathcal{V}}{z}\right) \mathrm{d}\, z,$$

that is the datum of Stokes rays $l_\pm = \pm\mathbb{R}_{>0}(u_2 - u_1)$, and the Stokes matrix coincides with the factor $\mathcal{S}_{\ell_{12}}$

$$\begin{aligned} \mathcal{S}(A_2) &= I - (-1)^{\langle \alpha_1, \alpha_2 \rangle} \langle \alpha_1, \alpha_2 \rangle\, \mathrm{DT}_{\mathcal{A}}(\alpha_1 - \alpha_2, Z) \mathbf{s}^{\alpha_1 - \alpha_2} E_{12} \\ &= I - (-1)^{\langle [S_1], [S_2] \rangle} \langle [S_1], [S_2] \rangle s E_{12} \\ &= I - s E_{12} \\ &= \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Notice that the canonical coordinates u_1, u_2 are not flat. A flat frame corresponds to flat sections of $K(\zeta) \to U$ and depends on the choice of the conformal dimension. We compute them explicitly for $d = s/\pi$. We already know that e is flat and we seek for $Y = a_1(Z)c_1(Z)x_{\alpha_1} + ia_2(Z)c_1(Z)x_{\alpha_2}$ such that $\nabla^r(Y) = 0$. The flatness condition implies either $a_1(Z) = a_2(Z) = const$ or

$$\begin{cases} \mathrm{d}(a_1 + a_2) = 0 \\ \mathrm{d}\log(a_1(Z) - a_2(Z)) = \mathrm{d}\log Z(\alpha_1 - \alpha_2)^{s/\pi} \end{cases},$$

We also require that E is semisimple on T_U, i.e. that v^*Y is an eigenvector of $-\operatorname{ad} E$. It is equivalent to

$$\begin{cases} a_1 - u_1 \frac{\partial a_1}{\partial u_1} - u_2 \frac{\partial_1}{\partial u_2} = d_1 a_1 \\ a_2 - u_1 \frac{\partial a_2}{\partial u_1} - u_2 \frac{\partial a_2}{\partial u_2} = d_1 a_2 \end{cases}$$

Therefore a suitable flat bases consists of the unit field $\partial_0 := e = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}$, and $\partial_1 := C(u_1 - u_2)^{s/\pi}\frac{\partial}{\partial u_1} - C(u_1 - u_2)^{s/\pi}\frac{\partial}{\partial u_2}$, $C \in \mathbb{C}^*$.

For any $s \neq \pi$, we may invert the Jacobian matrix and get flat coordinates $x_0 = \frac{1}{2}(u_1 + u_2)$ and $x_1 = \frac{1}{2}\left(\frac{\pi - s}{\pi}\right)^{-1} C^{-1}(u_1 - u_2)^{(\pi - s)/\pi}$.

In these flat coordinates $\{x_0, x_1\}$, the Euler field is semisimple $E = x_0\partial_0 + \left(\frac{\pi - s}{\pi}\right) x_1\partial_1$, with spectrum $(d_0, d_1) = (1, \frac{\pi - s}{\pi})$. The conformal dimension is $D = 1 + d_1 = 2 - \frac{s}{\pi}\ (= 2 - d)$, and the metric is represented by the (hermitian) symmetric real matrix $\begin{pmatrix} 0 & 2B^2C \\ 2B^2C & 0 \end{pmatrix}$. We can make it equal $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ riscaling the coordinates by chosing $C = 2^{-1/2}B^{-2}$. The description in terms of flat coordinates is completed by the potential $\Phi(x_0, x_1)$. For a two-dimensional Frobenius manifold, and in presence of flat identity and semisimple Euler field, the spectrum of $-\operatorname{ad} E$ (equivalently the potential) determines the classification of the Frobenius structure (see [37, Chapter I, Section 4]), with critical values $d_1 = 0, 1, \pm 2$. For generic d_1 the potential is $\Phi(x_0, x_1) = \frac{1}{2}x_0^2 x_1 + c x_1^{(2+d_1)/d_1}$. The corresponding critical values for $s \in \mathbb{C}^* \setminus \{\pi\}$ are $s = -\pi, 3\pi$. Therefore the potential in flat coordinates is reduced to

$$
\begin{aligned}
&s = -\pi: && E = x_0\partial_0 - 2x_1\partial_1 && \phi(x_0, x_1) = \frac{1}{2}x_0^2 x_1 + c\log x_1 \\
&s = 3\pi: && E = x_0\partial_0 + 2x_1\partial_1 && \phi(x_0, x_1) = \frac{1}{2}x_0^2 x_1 + c x_1^2 \log x_1 \\
&s \neq 0, \pm\pi, 3\pi: && E = x_0\partial_0 + \frac{\pi - s}{s} x_1\partial_1 && \phi(x_0, x_1) = \frac{1}{2}x_0^2 x_1 + c x_1^{(3\pi - s)/(\pi - s)}
\end{aligned}
$$

Another critical value is $s = \pi$. In this case flat coordinates depending on $C \in \mathbb{C}^*$

$$
\begin{cases} x_0 = \frac{1}{2}(u_1 + u_2) \\ x_1 = \frac{1}{2C}\log(u_1 - u_2) \end{cases}
$$

are defined, with spectrum $(d_0, d_1) = (1, 0)$ and conformal dimension $D = 1 + d_1 = 1$. The metric in flat coordinates has matrix $\begin{pmatrix} 0 & 2B^2C \\ 2B^2C & 0 \end{pmatrix}$. Moreover

$$
E = x_0\partial_0 + \frac{1}{2C}\partial_1, \text{ and } \Phi(x_0, x_1) = B^2 C x_0^2 x_1 + e^{4Cx_1}.
$$

One may wish to compare the affine family of Frobenius structures for the A_2 quiver, with the one defined over the unfolding space of singularities of z^3, which is relevant for this case study. Identify a subspace M of the affine space $\mathbb{C}^2$ with the space of polynomials $\{p(z) = z^3 + az + b \,|a \neq 0\}$. Call $\rho_- = -\left(-\frac{a}{3}\right)^{1/2}$ and $\rho_+ = \left(-\frac{a}{3}\right)^{1/2}$ the two distinct roots of $p'(z)$. Then

Theorem 4.28 ([37, Chpt. I, Sec. 4.5])**.** *M is a semisimple Frobenius manifold with the following structure data:*

- *canonical coordinates $u_i = p(\rho_-), u_2 = p(\rho_+)$, identity $e = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}$, Euler field $E = u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}$.*
- *flat metric $g := \frac{(\mathrm{d}\,u_1)^2}{p''(\rho_-)} + \frac{(\mathrm{d}\,u_2)^2}{p''(\rho_+)}$ with metric potential $\eta = -\frac{1}{2}(\rho_-^2 + \rho_+^2)$*

Specifically, the metric potential equals $\eta = \frac{a}{3} = \left(\frac{1}{4}\right)^{2/3}(u_i - u_2)^{2/3}$. Therefore, for $s = \frac{\pi}{3}$ and $B = \pm(54)^{-1/6}$ the two structures totally agree.

Similar computations may be performed when $d = -\frac{s}{\pi}$.

4.4.2 A_3 quiver

The construction for $\mathcal{A} = \mathrm{Rep}(A_3)$ is slightly more involved but still comprehensible. We can use this example to compute a more interesting Stokes matrix.

$$
A_3 = \bullet_{[S_1]} \longrightarrow \bullet_{[S_2]} \longrightarrow \bullet_{[S_3]}
$$

Consider the classes of objects

$$
\begin{aligned}
\alpha_1 &= [S_1] + [S_2] + [S_3] \\
\alpha_2 &= [S_2] + [S_3] \\
\alpha_3 &= [S_3]
\end{aligned}
$$

as a basis for $\mathcal{K}(\mathcal{A})$. Let $K(\zeta)$ denote the finite dimensional bundle isomorphic to the tangent bundle $T_{\mathrm{Stab}(\mathcal{A})}$ via the isomorphism $\mathrm{d}\, Z_\bullet(\zeta)$ for a suitable section ζ of K (Definition 3.1) and let U be an open subset of $\mathrm{Stab}(\mathcal{A})$. Notice that the reduction modulo $(\mathbf{s})^3$ of the Joyce functions $\hat{f}^{[S_1]}$ relative to $\alpha_1 - \alpha_2$ and $\hat{f}^{[S_2]}$ relative to $\alpha_2 - \alpha_3$ consist only in one linear term. $\alpha_1 - \alpha_3 = ([S_1] + [S_2])$ has a non-trivial decomposition as $(\alpha_1 - \alpha_2) + (\alpha_2 - \alpha_3)$ producing terms of order two in $\mathbf{s}$ in $\hat{f}^{\alpha_1 - \alpha_3}$. The truncation modulo $\mathbf{s}^3$ of the Joyce functions are (Theorems 2.25, 2.28)

$$\begin{aligned}
\hat{f}^{[S_1]} =& \frac{1}{2\pi i} s_1 \\
\hat{f}^{[S_2]} =& \frac{1}{2\pi i} s_2 \\
\hat{f}^{[S_1]+[S_2]} =& \frac{1}{2\pi i} \,\mathrm{DT}([S_1] + [S_2], Z) s_1 s_2 + \\
& + \frac{1}{(2\pi i)^2} \left(\log \frac{Z(\alpha_2 - \alpha_3)}{Z(\alpha_1 - \alpha_2)} - \pi i \right) s_1 s_2
\end{aligned}$$

The trivial lift to $GL_3[\![\mathbf{s}]\!]$ of the truncated objects in $GL_n[\![\mathbf{s}]\!]/(\mathbf{s}^3)$ depending on the Joyce functions doesn't define a family of Frobenius type structure on $K(\zeta)$. Consider the family of connections

$$\nabla^{\zeta}_{\mathbf{s},3} = \mathrm{d} + \left(\frac{Z}{z^2} - \frac{\mathcal{V}^{\zeta}_{\mathbf{s},3}(Z)}{z} \right) \mathrm{d}\, z$$

The skewsymmetric $\mathcal{V}^{\zeta}_{\mathbf{s},3} \in \mathfrak{gl}(K(\zeta)[\![\mathbf{s}]\!]/(\mathbf{s})^3)$ has components (4.1.12)

$$(\mathcal{V}^{\zeta}_{\mathbf{s},3})_{ij} = (-1)^{\langle \alpha_i, \alpha_j \rangle} \langle \alpha_i, \alpha_j \rangle \hat{f}^{\alpha_i - \alpha_j}$$

which do not satisfy the flatness condition for $\nabla^{\zeta}_{\mathbf{s},3}$

$$\mathrm{d}(\mathcal{V}^{\zeta}_{\mathbf{s},3})_{ij} = (\mathcal{V}^{\zeta}_{\mathbf{s},3})_{ik} (\mathcal{V}^{\zeta}_{\mathbf{s},3})_{kj} \big(\mathrm{d} \log Z(\alpha_k - \alpha_j) - \mathrm{d} \log(Z(\alpha_i - \alpha_k) \big).$$

For example

$$\mathrm{d}(\mathcal{V}^{\zeta}_{\mathbf{s},3})_{12} = 0 \neq (\mathcal{V}^{\zeta}_{\mathbf{s},3})_{13} (\mathcal{V}^{\zeta}_{\mathbf{s},3})_{32} \big(\mathrm{d} \log(Z(\alpha_3 - \alpha_2) - \mathrm{d} \log Z(\alpha_1 - \alpha_3) \big).$$

We proceed as in Section 4.2. Let Z be a central charge sending the simple classes $[S_i]$ in the upper half plane with ordered phase $\phi(Z([S_1])) < \phi(Z([S_2])) < \phi(Z([S_3]))$. According to Corollary 4.18 we have

$$\begin{aligned}
\mathcal{S}_{\ell_{12}} &= I - (-1)^{\langle [S_1],[S_2] \rangle} \langle [S_1], [S_2] \rangle s_1 E_{12}, \\
&= I - s_1 E_{12}, \\
\mathcal{S}_{\ell_{23}} &= I - (-1)^{\langle [S_2],[S_3] \rangle} \langle [S_2], [S_3] \rangle s_2 E_{23} \\
&= I - s_2 E_{23}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{S}_{\ell_{13}}(Z) &= I - (-1)^{\langle [S_2],[S_3] \rangle} \langle [S_2], [S_3] \rangle \,\mathrm{DT}_{\mathcal{A}}([S_1] + [S_2], Z) s_1 s_2 E_{13} \\
&= I - \mathrm{DT}_{\mathcal{A}}([S_1] + [S_2], Z) s_1 s_2 E_{13}.
\end{aligned}$$

For the particular choice of the stability function Z, the class $[S_2+S_1]$ is unstable and the ordered product of Stokes factor can be canonically lifted to

$$\tilde{\mathcal{S}}^{(0)} = \begin{pmatrix} 1 & -s_1 & 0 \\ 0 & 1 & -s_2 \\ 0 & 0 & 1 \end{pmatrix} \in GL_3[\![\mathbf{s}]\!].$$

$\tilde{\mathcal{S}}^{(0)}$ defines a new family of isomonodromic connections

$$\tilde{\nabla}^{\zeta}_{\mathbf{s}}(Z) = \mathrm{d} + \left(\frac{Z}{z^2} - \frac{\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}}{z}\right)\mathrm{d}\, z$$

whose order three approximation coincides with $\nabla^{\zeta}_{\mathbf{s},3}$. We set moreover $\tilde{\nabla}^{r,\zeta}_{\mathbf{s}} = \mathrm{d} + \tilde{A}$, $\tilde{A}_{ij} = (\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}})_{ij}\, \mathrm{d}\log Z(\alpha_i - \alpha_j)$, and $g = \mathrm{Id}$. $(\tilde{\nabla}^{r,\zeta}_{\mathbf{s}}, Z, \mathrm{d}\, Z, \tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}, g)$ define a Frobenius type structure over $K(\zeta)$.

We write for simplicity $\mathcal{V}$ instead of $\tilde{\mathcal{V}}^{\zeta}_{\mathbf{s}}$. Solutions of (4.3.1) are

$$\frac{d}{2} = 0, \pm i\sqrt{{\mathcal{V}_{12}}^2 + {\mathcal{V}_{23}}^2 + {\mathcal{V}_{13}}^2} \tag{4.4.1}$$

By Lemma 4.24 we know that that $d/2 \in \mathbb{C}$. It is easy to verify that ${\mathcal{V}_{12}}^2 + {\mathcal{V}_{23}}^2 + {\mathcal{V}_{13}}^2$ is constant in Z and lives in $\mathbb{C}[\![\mathbf{s}]\!]$, by using the fact that flatness of $\nabla(Z)$ is equivalent to the ODE

$$\mathrm{d}\, \mathcal{V}_{ij} = \mathcal{V}_{ik}\mathcal{V}_{kj}\big(\mathrm{d}\log Z(\alpha_k - \alpha_j) - \mathrm{d}\log(Z(\alpha_i - \alpha_k)\big).$$

For any choice of d in (4.4.1), the corresponding section $\zeta : U \to K(\zeta)$ may be expressed as $\zeta(Z) = \zeta_1(Z)x_{\alpha_1} + \zeta_2(Z)x_{\alpha_2} + \zeta_3(Z)x_{\alpha_3}$, where

$$\begin{aligned} \zeta_2(Z) =& \frac{\mathcal{V}_{23}\mathcal{V}_{13} - \frac{d}{2}\mathcal{V}_{12}}{\frac{d^2}{4} + (\mathcal{V}_{23})^2}\zeta_1(Z) =: \phi_2(Z)\zeta_1(Z) \\ \zeta_3(Z) =& \frac{\mathcal{V}_{12}\mathcal{V}_{23} - \frac{d}{2}\mathcal{V}_{13}}{\frac{d^2}{4} + (\mathcal{V}_{23})^2}\zeta_1(Z) =: \phi_3(Z)\zeta_1(Z) \end{aligned}$$

and $\zeta_1(\mathbf{s}, Z)$ solve the differential equation

$$\mathrm{d}\log\zeta_1(Z) = -\mathcal{V}_{12}\,\mathrm{d}\log Z(\alpha_1 - \alpha_2)\phi_2(Z) - \mathcal{V}_{13}\,\mathrm{d}\log Z(\alpha_1 - \alpha_3)\phi_3(Z)$$

For each choice of d, a family of semisimple Frobenius structures with conformal dimension $D = 2 - d$ is well defined on the tangent bundle to $U \subset \mathrm{Stab}(\mathcal{A})$, via pull back. They share the same canonical coordinates $u_i = Z(\alpha_i)$, $i = 1, 2, 3$, (flat) unit field $e = v^{-1}(\zeta) = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_3}$ and Euler field $E = \mathcal{U}(e) = Z(\alpha_1)\frac{\partial}{\partial u_1} + Z(\alpha_2)\frac{\partial}{\partial u_2} + Z(\alpha_3)\frac{\partial}{\partial u_3}$. They have diagonal metric $\tilde{g} = v^*(g)$, $\tilde{g}_{ii} = \zeta_i^2(\mathbf{s}, Z, d)$, with metric potential $\eta(d, Z)$

$$\begin{aligned} \eta(d, Z) &= \int ({\mathcal{V}_{12}}^2 + {\mathcal{V}_{13}}^2)^2\,\mathrm{d}\, u_1 \\ &= \left(\frac{d}{2}\right)^2 \int {\mathcal{V}_{12}}^2\,\mathrm{d}\, u_2 + \int {\mathcal{V}_{13}}^2{\mathcal{V}_{23}}^2\,\mathrm{d}\, u_2 + d\int \mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{23}\,\mathrm{d}\, u_2 \\ &= \left(\frac{d}{2}\right)^2 \int {\mathcal{V}_{13}}^2\,\mathrm{d}\, u_3 + \int {\mathcal{V}_{12}}^2{\mathcal{V}_{23}}^2\,\mathrm{d}\, u_3 + d\int \mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{23}\,\mathrm{d}\, u_3 \end{aligned}$$

in terms of canonical coordinates. Their Stokes matrix is

$$\mathcal{S}(A_3) = \begin{pmatrix} 1 & -s_1 & 0 \\ 0 & 1 & -s_2 \\ 0 & 0 & 1 \end{pmatrix}$$

4.4.3 A_n quiver

The example $\mathcal{A} = \mathrm{Rep}(A_3)$ can be generalised. Let $\mathcal{A}$ be $\mathrm{Rep}(A_n, 0)$, that is the abelian category of representations of the quiver

$$\bullet_{[S_1]} \longrightarrow \bullet_{[S_2]} \cdots \longrightarrow \bullet_{[S_n]}$$

with n vertices labelled by the ordered classes of simple objects

$$S_i = \cdots \longrightarrow 0 \longrightarrow \overset{i}{\mathbb{C}} \longrightarrow 0 \longrightarrow \cdots, \quad i = 1, \dots, n$$

$\Gamma = \mathcal{K}(\mathcal{A})$ is a lattice of rank n endowed with the symmetric bilinear pairing $\langle -, - \rangle$ associated to the Euler form and prescribed by the adjacency matrix of the quiver:

$$\langle [S_i], [S_j] \rangle = \begin{cases} -1 & \text{if } j = i+1 \\ 1 & \text{if } j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

The basis $\alpha_i = \sum_{r=i}^{n} [S_r]$ satisfies the assumptions of Lemma 4.6. This follows from Lemma 4.9 and the fact that the quadratic condition is identically satisfied. Indeed for $j = i+h+1$, $k = i+s$,

$$\langle \alpha_i, \alpha_{i+h+1} \rangle \langle \alpha_i - \alpha_{i+s}, \alpha_{i+s} - \alpha_{i+h+1} \rangle = \langle \alpha_i, \alpha_{i+s} \rangle \langle \alpha_{i+s}, \alpha_{i+h+1} \rangle$$

if and only if

$$\langle S_{i+h}, S_{i+h+1} \rangle \langle S_{i+s-1}, S_{i+s} \rangle = \langle S_{i+s-1}, S_{i+s} \rangle \langle S_{i+h}, S_{i+h+1} \rangle.$$

Since $\alpha_i - \alpha_j \in K_{>0}(\mathcal{A})$ for all $i \neq j$, this basis gives a canonical family of Frobenius manifold structures which is well-defined on all $\mathrm{Stab}(\mathcal{A}(A_n, 0))$ (i.e. the monodromy here is trivial).

We can classify these structures depending on their Stokes matrix, that coincides (Theorem 4.27) with the ordered product of Stokes factors $\mathcal{S}_{\ell_{ij}}$ of $\nabla_{\mathbf{s}}^{r,\zeta}$.

Because of isomonodromy property, we can chose any stability condition sending $[S_i]$ in the upper half plane $\mathbb{H}$. Choose Z such that

$$\phi(Z([S_1])) < \cdots < \phi(Z([S_n]))$$

where ϕ denotes the phases of a complex number in $\mathbb{C}^*$. The only stable objects, in this configuration, are the simples $S_1, \dots, S_n$. Then

$$\mathcal{S}_{\ell_{ij}} = \mathrm{Id} - (-1)^{\langle \alpha_i, \alpha_j \rangle} \langle \alpha_i, \alpha_j \rangle \, \mathrm{DT}(\alpha_i - \alpha_j, Z) \mathbf{s}^{\alpha_i - \alpha_j} E_{ij}$$

implies that

$$\mathcal{S}_{\ell_{ij}} = \begin{cases} \mathrm{Id} - s_i E_{i,i+1} & \text{if } j = i+1 \\ \mathrm{Id} & \text{if } j \neq i+1 \end{cases}.$$

We choose the canonical lift, thus for any $n \geq 2$ the Stokes matrix of the Frobenius structure on $\mathrm{Stab}(\mathcal{A}(A_n, 0))$ is $\mathcal{S}(A_n) = \prod_{j=1}^{n-1} \mathcal{S}_{\ell_{n-j,n-j+1}} = I - \sum_{i=1}^{n-1} s_i E_{i,i+1}$,

$$\mathcal{S}(A_n) = \begin{pmatrix} 1 & -s_1 & & & \\ & 1 & -s_2 & & \\ & & \ddots & & \\ & & & 1 & -s_n \\ & & & & 1 \end{pmatrix}.$$

Chapter 5

Mutations and analytic continuations

If Q and Q' are mutation equivalent quivers to which the theory of Chapter 4 applies, it is natural to ask whether the corresponding Frobenius structures are related. The question is motivated by the fact that mutation-equivalent finite quivers with potential have equivalent associated CY3 triangulated categories and define finite hearts related by tilts.

The aim of this Chapter is two-fold. On one hand, we discuss several examples to which we may apply the general theory developed in Chapter 4. We classify Frobenius structures on the space $\mathrm{Stab}(\mathcal{A})$, for $\mathcal{A}$ the abelian category $\mathrm{Rep}(\mu A_n)$ and μ a finite sequence of mutations. On the other hand, we observe that the corresponding Stokes matrices, evaluated at the special point $\mathbf{s} = \mathbf{1}$, are related by a sequence of braids (acting as in (1.5.1)). This means that the corresponding Frobenius manifolds are related by analytic continuation in the sense of Section 1.5. In order to do that, we observe that we obtain a better result if we truncate $f^{\alpha}_{\mathbf{s}}$ at order dependent on the length of α. Section 5.1 is devoted to this refinement. In section 5.2 and 5.3 the construction is applied to many examples and the braid action is verified.

The work is still in progress and a general picture true for A_n for any $n \geq 2$ is not reached: our result concerns all the mutations of A_n, $n \leq 5$, and most of the quivers of type $Q = \mu A_n$ for $n > 2$.

5.1 A refinement of the construction

In the previous chapter we set

$$\nabla^{r,\zeta}_{\mathbf{s}} = \pi^{\zeta} \cdot \nabla^{r}_{\mathbf{s}}|_{K(\zeta)},\ C^{\zeta} = \pi^{\zeta} \cdot C|_{K(\zeta)},$$
$$\mathcal{U}^{\zeta} = \pi^{\zeta} \cdot \mathcal{U}|_{K(\zeta)},\ \mathcal{V}^{\zeta} = \pi^{\zeta} \cdot \mathcal{V}_{\mathbf{s}}|_{K(\zeta)},\ g^{\zeta} = g|_{K(\zeta)}$$

and proved that under suitable assumptions they define a family of structures osculating a Frobenius type structures

$$(\tilde{\nabla}^{r,\zeta}_{\mathbf{s}}, \tilde{C}^{\zeta}, \tilde{\mathcal{U}}^{\zeta}, \tilde{\mathcal{V}}^{\zeta}, \tilde{g}^{\zeta})$$

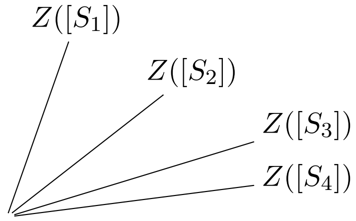
on $\mathrm{Stab}(\mathcal{A})$. The genuine Frobenius type structures coincide with the previous structures modulo terms of order ≥ 3 in $\mathbf{s}$. The approximation may be improved.

The following example motivates the try of “refining” the construction. It demonstrates that in general (unlike the example of A_n given in the previous Chapter) the product of the Stokes factors of $\nabla^{\zeta}_{\mathbf{s},3}$ is not in $GL_n(K(\zeta)[\![\mathbf{s}]\!]/\mathbf{s}^3)$.

Example 5.1. We consider the quiver

$$Q_1 = \bullet_{[S_1]} \longleftarrow \bullet_{[S_2]} \longleftarrow \bullet_{[S_3]} \longleftarrow \bullet_{[S_4]}$$

and the admissible basis $\alpha_i = \sum_{r\geq i}[S_r]$ of $\mathcal{K}(\mathrm{Rep}(Q_1, 0))$. Similarly to A_n, we can run the theory of Chapter 4 and define a Frobenius structure on an open set of $\mathrm{Stab}(\mathrm{Rep}(Q_1, 0))$. We chose the stability function Z sending the classes of the simple objects in $\mathbb{H}$



The only stable objects are S_1, S_2, S_3, S_4 and the product of the Stokes factors

$$\mathcal{S}_{\ell_{ij}} = \mathrm{Id} - (-1)^{\langle\alpha_i,\alpha_j\rangle}\langle\alpha_i, \alpha_j\rangle DT(\alpha_i - \alpha_j, Z)\mathbf{s}^{\alpha_i-\alpha_j}E_{ij}$$

of $\nabla^{\zeta}_{\mathbf{s},3}(Z)$ (Definition 4.15) is $(\mathrm{Id} + s_1E_{12})(\mathrm{Id} + s_2E_{23})(\mathrm{Id} + s_3E_{34}) = \mathrm{Id} + s_1E_{12} + s_2E_{23} + s_3E_{34} + s_1s_2E_{13} + s_2s_3E_{24} + s_1s_2s_3E_{14}$ and equals

$$\begin{pmatrix} 1 & s_1 & s_1s_2 & s_1s_2s_3 \\ 0 & 1 & s_2 & s_2s_3 \\ 0 & 0 & 1 & s_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.1.1}$$

which is not in $GL_4(\mathbb{C}[\![\mathbf{s}]\!]/\mathbf{s}^3)$. The Stokes matrix of the genuine Frobenius type structure induced on U is instead its truncation modulo $(\mathbf{s}^3)$

$$\mathcal{S} = \begin{pmatrix} 1 & s_1 & s_1s_2 & 0 \\ 0 & 1 & s_2 & s_2s_3 \\ 0 & 0 & 1 & s_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It seems natural to choose as a lift $\tilde{\mathcal{S}} \in GL_4(\mathbb{C}[\![\mathbf{s}]\!])$ the product (5.1.1) instead of the canonical lift $\tilde{\mathcal{S}}^{(0)} = \mathcal{S}$.

In this section, we want to show that, at least when $\mathcal{D} = \mathcal{D}(A_n, 0)$, we may define on its hearts a Frobenius type structure depending on approximations of Joyce functions at order higher than 3 (Corollary 5.5).
We consider the truncation

$$\hat{f}^{\alpha_i-\alpha_j}_{\mathbf{s},L} := \hat{f}^{\alpha_i-\alpha_j}_{\mathbf{s}}/(\mathbf{s}^{l(\alpha_i-\alpha_j)+1}),$$

where $l(\alpha_i - \alpha_j)$ is the length of the class $\alpha_i - \alpha_j \in \mathcal{K}(\mathcal{A}) \otimes \mathbb{R}$ (Definition 3.20), and we revisit the construction of Chapter 4. We set $\Phi^\alpha = \{\alpha_i - \alpha_j\}_{i\neq j}$. Arguing as in Lemmas 4.6 and 4.11 we can prove analogous results, replacing 3 with any $p \in \mathbb{N}_0$.

Lemma 5.2 (Lemma 4.6)**.** *Pick a section ζ of the form* (4.1.9) *(so ζ is a section of $K(\zeta)$ and the latter is preserved by C and $\mathcal{U}$). Fix $i, j = 1, \ldots n$ and $p_{ij} \in \mathbb{N}_0$. Suppose the following conditions hold:*

1. for all $k \neq i, j$ we have either

$$\langle \alpha_j, \alpha_i \rangle \langle \alpha_j - \alpha_k, \alpha_k - \alpha_i \rangle = \langle \alpha_j, \alpha_k \rangle \langle \alpha_k, \alpha_i \rangle,$$

or $\hat{f}_{\mathbf{s}}^{\alpha_j - \alpha_k} \hat{f}_{\mathbf{s}}^{\alpha_k - \alpha_i} \in (\mathbf{s}^{p_{ij}})$,

2. for all nontrivial decompositions $\alpha_j - \alpha_i = \beta + \gamma$ with β, γ not equal to $\alpha_j - \alpha_k$, $\alpha_k - \alpha_i$ the product

$$\langle \beta, \gamma \rangle \hat{f}_{\mathbf{s}}^{\beta}(Z) \hat{f}_{\mathbf{s}}^{\gamma}(Z) \in (\mathbf{s}^{p_{ij}})$$

Then the curvature component $g(x_{\alpha_j}, F(\nabla_{\mathbf{s}}^{r,\zeta}) x_{\alpha_i})$ vanishes modulo terms in $(\mathbf{s}^{p_{ij}})$.

Lemma 5.3 (Lemma 4.11)**.** *Suppose that the conditions of Lemma 5.2 hold for all $i, j = 1, \ldots, n$. Then the component $\left(\nabla_{\mathbf{s}}^{r,\zeta}(\mathcal{V}_{\mathbf{s}}^{\zeta})\right)_{ij}$ vanishes modulo terms of order at least p_{ij} in $\mathbf{s}$.*

Example 5.4. Let $\mathcal{A} = \operatorname{Rep}(Q, 0)$, where Q is any quiver whose underlying unoriented graph is the same ad A_n. The basis $\alpha_i = \sum_{j \geq i} [S_j]$ of $\mathcal{K}(\mathcal{A})$ satisfies the hypothesis of the Lemma 5.2 with $p_{ij} = l(\alpha_i - \alpha_j) + 1$.

With this choice $\max l(\alpha_i - \alpha_j) = n - 1$.

Proof. Q is $\bullet_{[S_1]} \text{———} \bullet_{[S_2]} \cdots \text{———} \bullet_{[S_n]}$ with vertices labelled by the classes of simple objects in $\mathcal{K}(\mathcal{A})$ and any choice of orientation of the edges. Then $\langle [S_i], [S_j] \rangle = \pm 1$ if $j = i \pm 1$ and vanishes otherwise. We have verified the first condition in Section 4.4.3. Set $p_{ij} = l(\alpha_i - \alpha_j) + 1$ and let $\beta + \gamma = \alpha_i - \alpha_j$, $\beta, \gamma \notin \Phi^{\alpha 2}$, then either $l(\beta) + l(\gamma) \geq l(\alpha_i - \alpha_j) + 1$ or β, γ generalizes the following situation: $\beta = [S_i] + \cdots + \widehat{[S_t]} + \cdots + [S_{j-1}]$, $\gamma = [S_t] + [S_j]$, but $\operatorname{ext}(S_t, S_j) = 0$.This implies that $\langle \beta, \gamma \rangle \hat{f}_{\mathbf{s}}^{\beta} \hat{f}_{\mathbf{s}}^{\gamma} = 0$ modulo terms of order p_{ij}. □

From now to the end of the Section, suppose that $\alpha_1, \ldots, \alpha_n$ is a basis of $\mathcal{K}(\mathcal{A}) \otimes \mathbb{R}$ and satisfies the hypothesis of Lemma 5.2. Call it *admissible.* We define

$$\begin{aligned}
(\mathcal{V}_{\mathbf{s},L}^{\zeta})_{ij} &= (-1)^{\langle \alpha_i, \alpha_j \rangle} \langle \alpha_i, \alpha_j \rangle \hat{f}_{\mathbf{s},L}^{\alpha_i - \alpha_j}, \\
\nabla_{\mathbf{s},L}^{\zeta}(Z) &= \mathrm{d} + \Big(\frac{\mathcal{U}(Z)}{z^2} - \frac{\mathcal{V}_{\mathbf{s},L}^{\zeta}}{z} \Big) \mathrm{d}\, z, \\
A_{ij}^{L} &= (-1)^{\langle \alpha_i, \alpha_j \rangle} \langle \alpha_i, \alpha_j \rangle \hat{f}_{\mathbf{s},L}^{\alpha_i - \alpha_j} \,\mathrm{d} \log Z(\alpha_i - \alpha_j), \\
\nabla_{\mathbf{s},L}^{r,\zeta} &= \mathrm{d} - A^{L}.
\end{aligned}$$

Corollary 5.5. *Say $l := \max_{ij} l(\alpha_i - \alpha_j) + 1$. $\nabla_{\mathbf{s},L}^{r,\zeta}(\mathcal{V}_{\mathbf{s},L}^{\zeta})$ and the curvature $F(\nabla_{\mathbf{s},L}^{r,\zeta})$ vanish modulo $\mathbf{s}^l$. Moreover*

a) The structure on $K(\zeta)$ given by

$$(\nabla_{\mathbf{s},L}^{r,\zeta}, C|_{K(\zeta)}, \mathcal{U}|_{K(\zeta)}, \mathcal{V}_{\mathbf{s},L}^{\zeta}, g|_{K(\zeta)}) \tag{5.1.2}$$

is a Frobenius type structure modulo terms which have order at least l in $\mathbf{s}$.

b) The family of connections on the holomorphically trivial principal bundle P on $\mathbb{P}^1$ with fiber the complex affine algebraic group $GL(K(\zeta)[\![\mathbf{s}]\!]/(\mathbf{s})^l)$ given by $\nabla_{\mathbf{s},L}^{\zeta}(Z)$ is *isomonodromic as Z varies in U. The generalized monodromy is given by the Stokes rays $\ell_{ij}(Z) = \mathbb{R}_{>0}Z(\alpha_i - \alpha_j) \subset \mathbb{C}^*$, for $i \neq j$, and the corresponding Stokes factors*

$$\begin{aligned}\mathcal{S}_{\ell_{ij}}(Z) = I - &2\pi i(\mathcal{V}^{\zeta}_{\mathbf{s},L})_{ij}E_{ij} + \\ &- \sum_{m\geq 1}\sum_{k_1\neq\cdots\neq k_m} M_{m+1}(Z(\alpha_i - \alpha_{k_1}), \ldots, Z(\alpha_{k_m} - \alpha_j)) \\ &(\mathcal{V}^{\zeta}_{\mathbf{s},L})_{ik_1}\cdots(\mathcal{V}^{\zeta}_{\mathbf{s},L})_{k_m j}E_{ij},\end{aligned} \tag{5.1.3}$$

where E_{ij} are the elementary matrices, and the second sum is over decompositions such that $l(\alpha_i - \alpha_{k_1}) + \cdots + l(\alpha_{k_m} - \alpha_j) < l(\alpha_i - \alpha_j)$. $\mathcal{S}_{\ell_{ij}} \in GL_n[\![\mathbf{s}]\!]/(\mathbf{s}^{l(\alpha_i-\alpha_j)+1})$.

c) The connection

$$\nabla^{r,\zeta}_{\mathbf{s},L} - \frac{1}{z}dZ + \Big(\frac{\mathcal{U}(Z)}{z^2} - \frac{\mathcal{V}^{\zeta}_{\mathbf{s},L}(Z)}{z}\Big)\,\mathrm{d}\,z,$$

on the pullback of P to $\mathbb{P}^1 \times U$, is flat.

d) $\mathcal{V}^{\zeta}_{\mathbf{s},L}$ is skew-symmetric and the Stokes matrix of $\nabla^{\zeta}_{\mathbf{s},L}(Z)$ is

$$\mathcal{S} := \prod^{\curvearrowright}_{\ell_{ij}\subset\bar{\mathbb{H}}} \mathcal{S}_{\ell_{ij}}(Z) \in GL(K(\zeta)[\![\mathbf{s}]\!]/\mathbf{s}^l).$$

Proof. Each point of the Corollary is a direct consequence of Lemmas above. Some ideas are developed in the previous Chapter. The only point which requires some carefulness is point d). In order to have a contribution in $(\mathbf{s}^l)$ among the entries of $\mathcal{S}$, in its ordered factorization $\prod^{\curvearrowright}_{\ell_{ij}\subset\bar{\mathbb{H}}} \mathcal{S}_{\ell_{ij}}(Z)$ should appear $\mathcal{S}_{\ell_{ik}}\mathcal{S}_{\ell_{kj}}$, such that $l(\alpha_i - \alpha_k) + l(\alpha_k - \alpha_j) > l$, but these decompositions don't contribute to the product on the positive half-plane. $\square$

Corollary 5.6 (Theorem 4.27). *For $||\mathbf{s}||$ small, the Stokes matrix $\mathcal{S} = \prod^{\curvearrowright}_{\ell\subset\bar{\mathbb{H}}} \mathcal{S}_\ell(Z)$ of the connection (over $\mathbb{P}^1$) $\nabla^{r,\zeta}_{\mathbf{s},L}$ regarded as an element of $GL(K(\zeta)[\![\mathbf{s}]\!])$ is the Stokes matrix of a genuine Frobenius manifold structure on the space* $\mathrm{Stab}(\mathcal{A})$.

Example 5.7. Choose $\mathcal{A} = \mathcal{A}(A_n, 0)$ and the admissible basis of Example 5.4. The Stokes matrix of the Frobenius structure over $\mathrm{Stab}(\mathcal{A})$ is the same computed in Section 4.4.3:

$$\mathcal{S} = \begin{pmatrix} 1 & -s_1 & & & \\ & 1 & -s_2 & & \\ & & \ddots & & \\ & & & 1 & -s_n \\ & & & & 1 \end{pmatrix}.$$

From Corollary 4.18 we can deduce that, when $l(\alpha_i - \alpha_j) \leq 2$,

$$\mathcal{S}_{\ell_{ij}}(Z) = I - (-1)^{\langle\alpha_i,\alpha_j\rangle}\langle\alpha_i, \alpha_j\rangle\,\mathrm{DT}_{\mathcal{A}}(\alpha_i - \alpha_j, Z)E_{ij}.$$

Example 5.8. We can give analogous explicit expression also for $\mathcal{S}_{\ell_{ij}}$ when

$$\alpha_i - \alpha_j = [S_{p_1}] + [S_{p_2}] + [S_{p_3}],\ l(\alpha_i - \alpha_j) = 3,$$

decomposes as $(\alpha_i - \alpha_{k_1}) + (\alpha_{k_1} - \alpha_j) = [S_{p_1}] + ([S_{p_2}] + [S_{p_3}])$ and $(\alpha_i - \alpha_{k_2}) + (\alpha_{k_2} - \alpha_j) = ([S_{p_1}] + [S_{p_2}]) + [S_{p_3}]$, and the following important assumption holds

$$\langle [S_{p_1}], [S_{p_3}] \rangle = 0.$$

Notice that it is exactly the case of a length-three difference $\alpha_i - \alpha_j$, for a basis as in the Example 5.4 for any quiver whose underlying graph is the same as A_n. The computation involves the iterated integrals M_m and J_m which are very hard to compute already for $m = 3$. However, we won't compute them explicitly and we use the fact that the Joyce coefficients J_m can be expressed in terms of M_k in a dense set (see (2.4.9)). According to Formula (4.2.2), $\mathcal{S}_{\ell_{ij}}(Z)$ is given by

$$\begin{aligned}
\mathcal{S}_{\ell_{ij}} =& \operatorname{Id} - M_1(\mathcal{V}^{\zeta}_{\mathbf{s},L})_{ij} E_{ij} - \\
& - M_2(Z(\alpha_i - \alpha_{k_1}), Z(\alpha_{k_1} - \alpha_j)(\mathcal{V}^{\zeta}_{\mathbf{s},L})_{ik_1}(\mathcal{V}^{\zeta}_{\mathbf{s},L})_{k_1 j} E_{ik_1} E_{k_1 j} - \\
& - M_2(Z(\alpha_i - \alpha_{k_2}), Z(\alpha_{k_2} - \alpha_j)(\mathcal{V}^{\zeta}_{\mathbf{s},L})_{ik_2}(\mathcal{V}^{\zeta}_{\mathbf{s},L})_{k_2 j} E_{ik_2} E_{k_2 j} - \\
& - M_3(Z(\alpha_i - \alpha_{k_1}), Z(\alpha_{k_1} - \alpha_{k_2}), Z(\alpha_{k_2} - \alpha_j)) \\
& (\mathcal{V}^{\zeta}_{\mathbf{s},L})_{ik_1}(\mathcal{V}^{\zeta}_{\mathbf{s},L})_{k_1 k_2}(\mathcal{V}^{\zeta}_{\mathbf{s},L})_{k_2 j} E_{ik_1} E_{k_1 k_2} E_{k_2 j} = \\
=& \operatorname{Id} - 2\pi i (-1)^{\langle \alpha_i, \alpha_j \rangle} \langle \alpha_i, \alpha_j \rangle \\
& \hat{f}^{\alpha_i - \alpha_j}_{\mathbf{s},L}(Z) E_{ij} - \\
& - M_2(Z(\alpha_i - \alpha_{k_1}), Z(\alpha_{k_1} - \alpha_j)(-1)^{\langle \alpha_i, \alpha_{k_1} \rangle + \langle \alpha_{k_1}, \alpha_j \rangle} \langle \alpha_i, \alpha_{k_1} \rangle \langle \alpha_{k_1}, \alpha_j \rangle \\
& \hat{f}^{\alpha_i - \alpha_{k_1}}_{\mathbf{s},L}(Z) \hat{f}^{\alpha_{k_1} - \alpha_j}_{\mathbf{s},L} E_{ij} - \\
& - M_2(Z(\alpha_i - \alpha_{k_2}), Z(\alpha_{k_2} - \alpha_j)(-1)^{\langle \alpha_i, \alpha_{k_2} \rangle + \langle \alpha_{k_2}, \alpha_j \rangle} \langle \alpha_i, \alpha_{k_2} \rangle \langle \alpha_{k_2}, \alpha_j \rangle \\
& \hat{f}^{\alpha_i - \alpha_{k_2}}_{\mathbf{s},L}(Z) \hat{f}^{\alpha_{k_2} - \alpha_j}_{\mathbf{s},L} E_{ij} - \\
& - M_3(Z(\alpha_i - \alpha_{k_1}), Z(\alpha_{k_1} - \alpha_{k_2}), Z(\alpha_{k_2} - \alpha_j))(-1)^{\langle \alpha_i, \alpha_{k_1} \rangle + \langle \alpha_{k_1}, \alpha_{k_2} \rangle + \langle \alpha_{k_2}, \alpha_j \rangle} \\
& \langle \alpha_i, \alpha_{k_1} \rangle \langle \alpha_{k_1}, \alpha_{k_2} \rangle \langle \alpha_{k_2}, \alpha_j \rangle \hat{f}^{\alpha_i - \alpha_{k_1}}_{\mathbf{s},L}(Z) \hat{f}^{\alpha_{k_1} - \alpha_{k_2}}_{\mathbf{s},L}(Z) \hat{f}^{\alpha_{k_2} - \alpha_j}_{\mathbf{s},L} E_{ij}
\end{aligned}$$

The truncated Joyce functions appearing in the calculation are

$$\begin{aligned}
\hat{f}^{\alpha_i - \alpha_j}_{\mathbf{s},L}(Z) =& \frac{1}{2\pi i} \operatorname{DT}(\alpha_i - \alpha_j) s_{p_1} s_{p_2} s_{p_3} + J_2(\alpha_i - \alpha_{k_1}, \alpha_{k_1} - \alpha_j)(-1)^{\langle \alpha_i - \alpha_{k_1}, \alpha_{k_1} - \alpha_j \rangle} \\
& \langle \alpha_i - \alpha_{k_1}, \alpha_{k_1} - \alpha_j \rangle s_{p_1} s_{p_2} s_{p_3} \\
& + J_2(\alpha_i - \alpha_{k_2}, \alpha_{k_2} - \alpha_j)(-1)^{\langle \alpha_i - \alpha_{k_2}, \alpha_{k_2} - \alpha_j \rangle} \langle \alpha_i - \alpha_{k_2}, \alpha_{k_2} - \alpha_j \rangle \\
& \operatorname{DT}(\alpha_i - \alpha_{k_2}) \operatorname{DT}(\alpha_{k_2} - \alpha_j) s_{p_1} s_{p_2} s_{p_3} + \\
& + \sum_{\substack{\{w_1, w_2, w_3\} = \\ \{\alpha_i - \alpha_{k_1}, \alpha_{k_1} - \alpha_{k_2}, \alpha_{k_2} - \alpha_j\}}} J_3(w_1, w_2, w_3)\ c(w_1, w_2, w_3) \\
& \operatorname{DT}(\alpha_i - \alpha_{k_1}) \operatorname{DT}(\alpha_{k_1} - \alpha_{k_2}) \operatorname{DT}(\alpha_{k_2} - \alpha_j) s_{p_1} s_{p_2} s_{p_3},
\end{aligned}$$

$$\hat{f}_{\mathbf{s},L}^{\alpha_{k_1}-\alpha_j}(Z) = \frac{1}{2\pi i}\operatorname{DT}(\alpha_{k_1}-\alpha_j)s_{p_2}s_{p_3} + (-1)^{\langle \alpha_{k_1}-\alpha_{k_2},\alpha_{k_2}-\alpha_j\rangle}\langle \alpha_{k_1}-\alpha_{k_2},\alpha_{k_2}-\alpha_j\rangle,$$
$$J_2(\alpha_{k_1}-\alpha_{k_2},\alpha_{k_2}-\alpha_j)\operatorname{DT}(\alpha_{k_1}-\alpha_{k_2})\operatorname{DT}(\alpha_{k_2}-\alpha_j)s_{p_2}s_{p_3}$$
$$\hat{f}_{\mathbf{s},L}^{\alpha_i-\alpha_{k_2}}(Z) = \frac{1}{2\pi i}\operatorname{DT}(\alpha_i-\alpha_{k_2})s_{p_1}s_{p_2} + (-1)^{\langle \alpha_i-\alpha_{k_1},\alpha_{k_1}-\alpha_{k_2}\rangle}\langle \alpha_i-\alpha_{k_1},\alpha_{k_1}-\alpha_{k_2}\rangle,$$
$$J_2(\alpha_i-\alpha_{k_1},\alpha_{k_1}-\alpha_{k_2})\operatorname{DT}(\alpha_i-\alpha_{k_1})\operatorname{DT}(\alpha_{k_1}-\alpha_{k_2})s_{p_1}s_{p_2},$$

and

$$\hat{f}_{\mathbf{s},L}^{\alpha_i-\alpha_{k_1}}(Z) = \frac{s_{p_1}}{2\pi i}, \quad \hat{f}_{\mathbf{s},L}^{\alpha_{k_1}-\alpha_{k_1}}(Z) = \frac{s_{p_2}}{2\pi i}, \quad \hat{f}_{\mathbf{s},L}^{\alpha_{k_2}-\alpha_j}(Z) = \frac{s_{p_3}}{2\pi i}.$$

The last piece of data to be computed is the coefficient $\overline{c}$ (2.4.1). For instance

$$\begin{aligned}
c([S_{p_1}],[S_{p_2}],[S_{p_3}]) =& c([S_{p_3}],[S_{p_2}],[S_{p_1}] = \\
=& \frac{1}{2}(-1)^{\langle [S_{p_1}],[S_{p_2}]\rangle}\langle [S_{p_1}],[S_{p_2}]\rangle(-1)^{\langle [S_{p_2}],[S_{p_3}]\rangle}\langle [S_{p_2}],[S_{p_3}]\rangle + \\
&+ \frac{1}{2}(-1)^{\langle [S_{p_1}],[S_{p_2}]\rangle}\langle [S_{p_1}],[S_{p_3}]\rangle(-1)^{\langle [S_{p_1}],[S_{p_3}]\rangle}\langle [S_{p_2}],[S_{p_3}]\rangle = \\
=& (-1)^{\langle [S_{p_1}],[S_{p_2}]\rangle}\langle [S_{p_1}],[S_{p_2}]\rangle(-1)^{\langle [S_{p_2}],[S_{p_3}]\rangle}\langle [S_{p_2}],[S_{p_3}]\rangle \\
c([S_{p_2}],[S_{p_1}],[S_{p_3}]) =& \frac{1}{2}(-1)^{\langle [S_{p_2}],[S_{p_1}]\rangle}\langle [S_{p_2}],[S_{p_1}]\rangle(-1)^{\langle [S_{p_2}],[S_{p_3}]\rangle}\langle [S_{p_2}],[S_{p_3}]\rangle
\end{aligned}$$

Using the quadratic relation of Lemma 5.2, it follows that

$$\mathcal{S}_{l_{ij}} = \operatorname{Id} + (-1)^{\langle \alpha_i,\alpha_j\rangle}\langle \alpha_i,\alpha_j\rangle\operatorname{DT}(\alpha_i-\alpha_j,Z)s_{p_1}s_{p_2}s_{p_3}E_{ij}$$

Corollary 5.9 (Theorem 4.27)**.** *When $\alpha_i-\alpha_j = [S_{p_1}]$, or $[S_{p_1}]+[S_{p_2}]$, or $[S_{p_1}]+[S_{p_2}]+[S_{p_3}]$ with $\langle [S_{p_1}],[S_{p_3}]\rangle = 0$, then*

$$\mathcal{S}_{l_{ij}} = \operatorname{Id} + (-1)^{\langle \alpha_i,\alpha_j\rangle}\langle \alpha_i,\alpha_j\rangle\operatorname{DT}(\alpha_i-\alpha_j,Z)\mathbf{s}^{\alpha_i-\alpha_j}E_{ij}.$$

The feeling is that 5.9 can be generalized to any length, under suitable assumptions on bracket relations.

5.2 Mutations of A_n

5.2.1 Admissible bases

We consider here the possible configurations of a finite quiver obtained from A_n applying a finite number of simple mutations, and admissible bases for such configurations. They can be reduced to three cases.

For the sake of simplicity, we label the vertices with integers i corresponding to the class of the simples $[S_i]$.

1. If the unoriented graph underlying μA_n is the same as A_n, then choose the basis

$$\alpha_i = \sum_{r=i}^{n}[S_r], \ i = 1,\dots,n. \tag{5.2.1}$$

2. If a clockwise oriented triangle appears,

$$\begin{tikzcd}
\cdots \arrow[r] & \bullet_{k-1} \arrow[rr] & & \bullet_{k+1} \arrow[r] \arrow[dl] & \cdots \\
& & \bullet_{k} \arrow[ul] & &
\end{tikzcd}$$

then consider the basis

$$\begin{cases} \alpha_i = \sum_{j=i}^{k-1}[S_j] + \alpha_{k-2} \quad \text{for } i < k-2 \\ \alpha_{k-1} = [S_{k-1}] + [S_{k+1}] + \alpha_{k+2} \\ \alpha_k = -[S_k] + [S_{k+1}] + \alpha_{k+2} \\ \alpha_{k+1} = [S_{k+1}] + \alpha_{k+2} \\ \alpha_i = \sum_{j=i}^{n}[S_j] \quad \text{for } i \geq k+2. \end{cases} \tag{5.2.2}$$

If the triangle is counter-clockwise oriented, then consider the same basis, or read the labels of the vertices from right to the left if necessary.

3.

$$\begin{tikzcd}
\bullet_{k-1} \arrow[rr] & & \bullet_{k+1} \arrow[rr] \arrow[dl] & & \bullet_{k+3} \arrow[dl] \\
& \bullet_{k} \arrow[ul] & & \bullet_{k+2} \arrow[ul] &
\end{tikzcd}
\qquad
\begin{cases} \alpha_{k-1} = [S_{k-1}] + [S_{k+1}] + [S_{k+3}] \\ \alpha_k = -[S_k] + [S_{k+1}] + [S_{k+3}] \\ \alpha_{k+1} = [S_{k+1}] + [S_{k+3}] \\ \alpha_{k+2} = -[S_{k+2}] + [S_{k+3}] \\ \alpha_{k+3} = [S_{k+3}] \end{cases}$$

4. The last possible configuration to be studied is

$$\begin{tikzcd}
\cdots \bullet_{l_i} \arrow[r] & \bullet_{k-1} \arrow[rr] & & \bullet_{k+1} \arrow[r] \arrow[dl] & \bullet_{r_i} \cdots \\
& & \bullet_{k} \arrow[ul] & & \\
& & \bullet_{d_i} \arrow[u] & & \\
& & \vdots \arrow[u, dash, dotted] & &
\end{tikzcd} \tag{5.2.3}$$

An admissible basis is

$$\begin{cases} \alpha_{r_i} &= \sum_{j\geq i}[S_{r_i}] \\ \alpha_{k+1} &= [S_{k+1}] + \alpha_{r_1} \\ \alpha_{k-1} &= [S_{k-1}] + [S_{k+1}] + \alpha_{r_1} \\ \alpha_k &= -[S_k] + [S_{k+1}] + \alpha_{r_1} \\ \alpha_{l_i} &= \sum_{j\geq i}[S_{l_j}] + \alpha_{k-1} \\ \alpha_{d_i} &= \sum_{j\geq i}[S_{d_j}] + \alpha_k \end{cases} \tag{5.2.4}$$

Lemma 5.10. *The bases of* $\mathcal{K}(\mathcal{A}) \otimes \mathbb{R}$ (5.2.1), (5.2.2) *and* (5.2.4) *satisfy the conditions of Lemma 5.2.*

The first case was considered in Example 5.4. The other configurations are very similar and can be directly checked with a simple computer program, for any choice of orientations of the arrows.

Remark. Of course, those listed above constitute only a special class of admissible bases and there exist many other examples of generating elements which satisfy the hypotheses of Lemma 5.2 for the same quivers.

Example 5.11. Other admissible bases for the configuration 4) are given by different choices of α_6

$$\begin{aligned} \alpha_6' &= [S_{l_i}] + [S_{k-1}] + [S_{d_i}], \\ \alpha_6'' &= [S_{l_i}] + [S_{k-1}] + [S_{d_i}] \pm [S_k], \\ \alpha_6''' &= [S_{l_i}] + [S_{k-1}] + [S_{d_i}] \pm [S_k] \pm [S_{r_i}]). \end{aligned}$$

For all the examples above, one may compute Stokes factors $\mathcal{S}_{\ell_{ij}}$ of

$$\nabla^r_{\mathbf{s},L} = \mathrm{d} + \Big(\hat{f}^{\alpha_i - \alpha_j}_{\mathbf{s},L} \, \mathrm{d}\log(Z(\alpha_i - \alpha_j)) \Big)_{ij}$$

and classify the corresponding Frobenius structure on $U \subseteq \mathrm{Stab}(\mathcal{A})$ according to their ordered product.

5.2.2 Stokes matrices and braid action

We give here the Stokes matrices of the Frobenius structures we obtain on $\mathrm{Stab}(\mathcal{A})$ with the refined construction, when $\mathcal{A} = \mathrm{Rep}(Q, 0)$ and $Q = \mu A_n$ is a heart of $\mathcal{D}(A_n, 0)$, with the choice of admissible bases given above. Q is obtained from A_n via finite sequences of mutations. We denote them as $\mathcal{S}(\mu A_n)$. ϵ is the adjacency matrix in the new basis. We focus on A_2, A_3, A_4 and A_5.

Mutation classes of A_2

$$A_2 = \bullet_1 \longrightarrow \bullet_2 \,, \ \mathcal{S}(A_2) = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}; \quad \mu_1 A_2 = \bullet_1 \longleftarrow \bullet_2 \,, \ \mathcal{S}(\mu_1 A_2) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

Mutation classes of A_3

$$A_3 = \bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3 \qquad \mathcal{S}(A_3) = \begin{pmatrix} 1 & -s_1 & 0 \\ 0 & 1 & -s_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.2.5}$$

$$\mu_3 A_3 = \bullet_1 \longrightarrow \bullet_2 \longleftarrow \bullet_3 \qquad \mathcal{S}(\mu_3 A_3) = \begin{pmatrix} 1 & -s_1 & 0 \\ 0 & 1 & s_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.2.6}$$

$$\mu_1 A_3 = \bullet_1 \longleftarrow \bullet_2 \longrightarrow \bullet_3 \qquad \mathcal{S}(\mu_1 A_3) = \begin{pmatrix} 1 & s_1 & -s_1 s_2 \\ 0 & 1 & -s_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.2.7}$$

$$\mu_1 \mu_3 A_2 = \bullet_1 \longleftarrow \bullet_2 \longleftarrow \bullet_3 \qquad \mathcal{S}(\mu_1 \mu_3 A_2) = \begin{pmatrix} 1 & s_1 & s_1 s_2 \\ 0 & 1 & s_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.2.8}$$

$$\mu_2 A_3 = \begin{tikzcd} \bullet_1 \arrow[rr] & & \bullet_3 \arrow[dl] \\ & \bullet_2 \arrow[ul] & \end{tikzcd} \qquad \mathcal{S}(\mu_2 A_3) = \begin{pmatrix} 1 & -s_1 s_2 & -s_1 \\ 0 & 1 & 0 \\ 0 & s_2 & 1 \end{pmatrix} \tag{5.2.9}$$

$$\mu_2\mu_1\mu_3 A_3 = \begin{tikzcd} \bullet_1 \arrow[dr] & & \bullet_3 \arrow[ll] \\ & \bullet_2 \arrow[ur] & \end{tikzcd} \qquad \mathcal{S}(\mu_2\mu_1\mu_3 A_3) = \begin{pmatrix} 1 & s_1 s_2 & s_1 \\ 0 & 1 & 0 \\ 0 & -s_2 & 1 \end{pmatrix} \tag{5.2.10}$$

Mutation classes of A_4

$\mu_1 A_4 = \bullet_1 \longleftarrow \bullet_2 \longrightarrow \bullet_3 \longrightarrow \bullet_4$

$$\mathcal{S}(\mu_1 A_4) = \begin{pmatrix} 1 & s_1 & -s_1 s_2 & 0 \\ 0 & 1 & -s_2 & 0 \\ 0 & 0 & 1 & -s_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2.11}$$

$\mu_4 A_4 = \bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3 \longleftarrow \bullet_4$

$$\mathcal{S}(\mu_4 A_4) = \begin{pmatrix} 1 & -s_1 & 0 & 0 \\ 0 & 1 & -s_2 & 0 \\ 0 & 0 & 1 & s_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2.12}$$

$\mu_4\mu_2\mu_1 A_4 = \bullet_1 \longrightarrow \bullet_2 \longleftarrow \bullet_3 \longleftarrow \bullet_4$

$$\mathcal{S}(\mu_4\mu_2\mu_1 A_4) = \begin{pmatrix} 1 & -s_1 & 0 & 0 \\ 0 & 1 & s_2 & s_2 s_3 \\ 0 & 0 & 1 & s_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2.13}$$

$\mu_1\mu_4 A_4 \; \bullet_1 \longleftarrow \bullet_2 \longrightarrow \bullet_3 \longrightarrow \bullet_4$

$$\mathcal{S}(\mu_1\mu_4 A_4) = \begin{pmatrix} 1 & s_1 & -s_1 s_2 & 0 \\ 0 & 1 & -s_2 & 0 \\ 0 & 0 & 1 & s_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2.14}$$

$\mu_2\mu_1 A_4 = \bullet_1 \longrightarrow \bullet_2 \longleftarrow \bullet_3 \longrightarrow \bullet_4$

$$\mathcal{S}(\mu_2\mu_1) = \begin{pmatrix} 1 & -s_1 & 0 & 0 \\ 0 & 1 & s_2 & -s_2 s_3 \\ 0 & 0 & 1 & -s_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2.15}$$

$\mu_1\mu_2\mu_1 A_4 = \bullet_1 \longleftarrow \bullet_2 \longleftarrow \bullet_3 \longrightarrow \bullet_4$

$$\mathcal{S}(\mu_1\mu_2\mu_1 A_4) = \begin{pmatrix} 1 & s_1 & s_1 s_2 & -s_1 s_2 s_3 \\ 0 & 1 & s_2 & -s_2 s_3 \\ 0 & 0 & 1 & -s_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2.16}$$

$$\mu_4\mu_1\mu_2\mu_1 A_4 = \bullet_1 \longleftarrow \bullet_2 \longleftarrow \bullet_3 \longleftarrow \bullet_4$$

$$\mathcal{S}(\mu_4\mu_1\mu_2\mu_1 A_4) = \begin{pmatrix} 1 & s_1 & s_1s_2 & s_1s_2s_3 \\ 0 & 1 & s_2 & s_2s_3 \\ 0 & 0 & 1 & s_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2.17}$$

$$\mu_2 A_4 = \begin{tikzcd} \bullet_1 \arrow[rr] & & \bullet_3 \arrow[r] \arrow[dl] & \bullet_4 \\ & \bullet_2 \arrow[ul] & & \end{tikzcd}$$

$$\mathcal{S}(\mu_2 A_4) = \begin{pmatrix} 1 & -s_1s_2 & -s_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & s_2 & 1 & -s_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2.18}$$

$$\mu_4\mu_2 A_4 = \begin{tikzcd} \bullet_1 \arrow[rr] & & \bullet_3 \arrow[dl] & \bullet_4 \arrow[l] \\ & \bullet_2 \arrow[ul] & & \end{tikzcd}$$

$$\mathcal{S}(\mu_4\mu_2 A_4) = \begin{pmatrix} 1 & -s_1s_2 & -s_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & s_2 & 1 & s_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2.19}$$

$$\mu_3 A_4 = \begin{tikzcd} \bullet_1 \arrow[r] & \bullet_2 \arrow[rr] & & \bullet_4 \arrow[dl] \\ & & \bullet_3 \arrow[ul] & \end{tikzcd}$$

$$\mathcal{S}(\mu_3 A_4) = \begin{pmatrix} 1 & -s_1 & 0 & 0 \\ 0 & 1 & -s_2s_3 & -s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & s_3 & 1 \end{pmatrix} \tag{5.2.20}$$

$$\mu_1\mu_3 A_4 = \begin{tikzcd} \bullet_1 & \bullet_2 \arrow[l] \arrow[rr] & & \bullet_4 \arrow[dl] \\ & & \bullet_3 \arrow[ul] & \end{tikzcd}$$

$$\mathcal{S}(\mu_1\mu_3 A_4) = \begin{pmatrix} 1 & s_1 & -s_1s_2s_3 & -s_1s_2 \\ 0 & 1 & -s_2s_3 & -s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & s_3 & 1 \end{pmatrix} \tag{5.2.21}$$

Mutation classes of A_5

Given that a mutation at vertex i is a local operation involving only arrows incoming or outgoing from i, computations for μA_4 can be generalized to similar configurations of μA_5

and μA_n, $n > 5$. For instance:

$\mu_1 A_5 =$ $\bullet_1 \longleftarrow \bullet_2 \longrightarrow \bullet_3 \longrightarrow \bullet_4 \longrightarrow \bullet_5$

$$\mathcal{S}(\mu_1 A_5) = \begin{pmatrix} 1 & s_1 & -s_1 s_2 & 0 & 0 \\ 0 & 1 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & -s_3 & 0 \\ 0 & 0 & 0 & 1 & -s_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2.22}$$

$\mu_5 A_5 =$ $\bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3 \longleftarrow \bullet_4$

$$\mathcal{S}(\mu_5 A_5) = \begin{pmatrix} 1 & s_1 & 0 & 0 & 0 \\ 0 & 1 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & -s_3 & 0 \\ 0 & 0 & 0 & 1 & s_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2.23}$$

$\mu_3 A_5 =$
$$\begin{tikzcd} \bullet_1 \arrow[r] & \bullet_2 \arrow[rr] & & \bullet_4 \arrow[r] \arrow[dl] & \bullet_5 \\ & & \bullet_3 \arrow[ul] & & \end{tikzcd}$$

$$\mathcal{S}(\mu_3 A_5) = \begin{pmatrix} 1 & -s_1 & 0 & 0 & 0 \\ 0 & 1 & -s_2 s_3 & -s_2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & s_3 & 1 & -s_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2.24}$$

We consider also some other interesting configurations (they do not complete all the mutation classes of A_5 which are 19).

$\mu_1 \mu_4 A_5 =$
$$\begin{tikzcd} \bullet_1 & \bullet_2 \arrow[l] \arrow[r] & \bullet_3 \arrow[rr] & & \bullet_5 \arrow[dl] \\ & & & \bullet_4 \arrow[ul] & \end{tikzcd}$$

$$\mathcal{S}(\mu_1 \mu_4 A_5) = \begin{pmatrix} 1 & s_1 & -s_1 s_2 & 0 & 0 \\ 0 & 1 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & -s_3 s_4 & -s_3 \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & s_4 & 1 \end{pmatrix} \tag{5.2.25}$$

$\mu_2 \mu_4 A_5 =$
$$\begin{tikzcd} \bullet_1 \arrow[rr] & & \bullet_3 \arrow[dl] \arrow[rr] & & \bullet_5 \arrow[dl] \\ & \bullet_2 \arrow[ul] & & \bullet_4 \arrow[ul] & \end{tikzcd}$$

$$\mathcal{S}(\mu_4 \mu_2 A_5) = \begin{pmatrix} 1 & -s_1 s_2 & -s_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & s_2 & 1 & -s_3 s_4 & -s_3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s_4 & 1 \end{pmatrix} \tag{5.2.26}$$

$$\mu_2\mu_1\mu_4 A_5 = \begin{tikzcd} \bullet_1 \arrow[r] & \bullet_2 & \bullet_3 \arrow[l] \arrow[rr] & & \bullet_5 \arrow[dl] \\ & & & \bullet_4 \arrow[ul] & \end{tikzcd}$$

$$\mathcal{S}(\mu_2\mu_1\mu_4 A_5) = \begin{pmatrix} 1 & -s_1 & 0 & 0 & 0 \\ 0 & 1 & s_2 & -s_2s_3s_4 & -s_2s_3 \\ 0 & 0 & 1 & -s_3s_4 & -s_3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s_4 & 1 \end{pmatrix} \tag{5.2.27}$$

$$\mu_1\mu_2\mu_1\mu_4 A_5 = \begin{tikzcd} \bullet_1 & \bullet_2 \arrow[l] & \bullet_3 \arrow[l] \arrow[rr] & & \bullet_5 \arrow[dl] \\ & & & \bullet_4 \arrow[ul] & \end{tikzcd}$$

$$\mathcal{S}(\mu_1\mu_2\mu_1\mu_4 A_5) = \begin{pmatrix} 1 & s_1 & s_1s_2 & -s_2s_3s_4 & -s_1s_2s_3s_4 \\ 0 & 1 & s_2 & -s_2s_3s_4 & -s_2s_3 \\ 0 & 0 & 1 & -s_3s_4 & -s_3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s_4 & 1 \end{pmatrix} \tag{5.2.28}$$

$$\mu_1\mu_3 A_5 = \begin{tikzcd} \bullet_1 & \bullet_2 \arrow[l] \arrow[rr] & & \bullet_4 \arrow[r] \arrow[dl] & \bullet_5 \\ & & \bullet_3 \arrow[ul] & & \end{tikzcd}$$

$$\mathcal{S}(\mu_1\mu_3 A_5) = \begin{pmatrix} 1 & s_1 & -s_1s_2s_3 & -s_1s_2 & 0 \\ 0 & 1 & -s_2s_3 & -s_2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & s_3 & 1 & -s_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2.29}$$

$$\mu_5\mu_1\mu_3 A_5 = \begin{tikzcd} \bullet_1 & \bullet_2 \arrow[l] \arrow[rr] & & \bullet_4 \arrow[dl] & \bullet_5 \arrow[l] \\ & & \bullet_3 \arrow[ul] & & \end{tikzcd}$$

$$\mathcal{S}(\mu_5\mu_1\mu_3 A_5) = \begin{pmatrix} 1 & s_1 & -s_1s_2s_3 & -s_1s_2 & 0 \\ 0 & 1 & -s_2s_3 & -s_2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & s_3 & 1 & s_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2.30}$$

5.3 Braid action

We may evaluate the Stokes matrices at the special point $s_1 = \cdots = s_n = 1$ and compare them. Recall that analytic continuations of the same germ of Frobenius structures are related by the action of permutation and diagonal matrices (1.4.2) or that of the braid group (1.5.1).

In this Section, we write $\mathcal{S}$ for $\mathcal{S}_{|\mathbf{s}=\mathbf{1}}(A_n)$ and $\mathcal{S}(\mu A_n)$ for $\mathcal{S}_{|\mathbf{s}=\mathbf{1}}(\mu A_n)$. We observe that, when μ is a simple mutation, $\mathcal{S}(\mu A_n)$ and $\mathcal{S}$ are actually related by the action of permutation and diagonal matrices $\mathcal{I}$ or that of the braid group. Specifically:

$$\begin{gathered}
\text{if } \mu=\mu_1 \text{ then } \mathcal{S}(\mu A_n)=\beta^{1,2}(\mathcal{S}),\\
\text{if } \mu=\mu_k,\ k=2,\dots,n-1, \text{ then } \mathcal{S}(\mu A_n)=\beta^{k,k+1}(P_{k,k+1}\mathcal{S}P_{k,k+1}),\\
\text{if } \mu=\mu_n \text{ then } \mathcal{S}(\mu A_n)=\mathcal{I}_n\mathcal{S}\mathcal{I}_n,
\end{gathered}$$

where $\mathcal{I}_k$, $k=1,\dots,n$, is the matrix which differs from the identity only for the sign of the (k,k) entry.

Given that mutations are local transformation of the quiver, when $\mu'=\mu_j\mu$ and none of the arrow outgoing/incoming to the vertex labelled with j have been affected by μ, then the action which transforms $\mathcal{S}$ in $\mathcal{S}(\mu' A_n)$ is the ordered composition of the actions associated to the single mutations μ and μ_j respectively.

The considerations above concern examples (5.2.6)-(5.2.9), (5.2.11)-(5.2.14), (5.2.18)-(5.2.21) and (5.2.22)-(5.2.26), (5.2.29)-(5.2.30). However we can find similar sequences of actions for all the examples given. For the mutation of A_3:

(5.2.10) $\mathcal{S}(\mu_2\mu_1\mu_3A_3)=P_{2,3}\mathcal{I}_1\mathcal{S}(\mu1\mu3A_3)\mathcal{I}_1P_{2,3};$

those of A_4

(5.2.15) $\mathcal{S}(\mu_2\mu_1A_4)=\beta_{34}^{-1}(\mathcal{S}),$

(5.2.16) $\mathcal{S}(\mu_1\mu_2\mu_1A_4)=\beta_{12}(\beta_{23}(\beta_{12}(\mathcal{S}))),$

(5.2.17) $\mathcal{S}(\mu_4\mu_1\mu_2\mu_1A_4)=P_{2,3}\mathcal{S}(\mu_1\mu_2\mu_1A_4)P_{2,3};$

and of A_5

(5.2.27) $\mathcal{S}(\mu_2\mu_1\mu_4A_5)=\beta_{23}(\beta_{12}\mathcal{S}(\mu_4A_5)),$

(5.2.28) $\mathcal{S}(\mu_1\mu_2\mu_1\mu_4A_5)=\beta_{12}(\mathcal{S}(\mu_2\mu_1\mu_4A_5))=\mathcal{I}_5(\mathcal{S}(\mu_1\mu_2\mu_1A_5)\mathcal{I}_5.$

Corollary 5.12. *For A_n, $n\leq 5$, all the Frobenius structures on* $\mathrm{Stab}(\mathcal{A}(\mu(A_n,0)))$ *are analytic continuations of the same germ of Frobenius manifold structure.*

5.4 Further examples

The refined machinery can be applied to many other cases. The examples we give below admits the basis of Example 5.4.

$$Q_2=\begin{tikzcd} 1 \arrow[rr, "2"] & & 3 \arrow[dl] \\ & 2 \arrow[ul] & \end{tikzcd} \qquad \mathcal{S}(Q_2)=\begin{pmatrix} 1 & -s_1 & -s_1s_2\\ & 1 & s_2\\ & & 1\end{pmatrix},$$

$$Q_3=\begin{tikzcd} 1 \arrow[rr, "2"] & & 3 \arrow[dl] \arrow[r] & 4 \arrow[r, dashed] & n \\ & 2 \arrow[ul] & & & \end{tikzcd} \qquad \mathcal{S}(Q_3)=\begin{pmatrix} 1 & -s_1 & -s_1s_2 & & & \\ & 1 & s_2 & -s_2s_3 & & \\ & & 1 & -s_3 & & \\ & & & \ddots & & \\ & & & & 1 & -s_n\\ & & & & & 1\end{pmatrix}$$

$$Q_4 = \begin{tikzcd} 1 \arrow[dr] & & 3 \arrow[ll, "2"'] \arrow[r] & 4 \arrow[r, dashed] & n \\ & 2 \arrow[ur] & & & \end{tikzcd} \qquad \mathcal{S}(Q_4) = \begin{pmatrix} 1 & s_1 & & & & \\ & 1 & -s_2 & & & \\ & & 1 & -s_3 & & \\ & & & \ddots & & \\ & & & & 1 & -s_n \\ & & & & & 1 \end{pmatrix},$$

$$Q_5 = \begin{tikzcd} & 3 \arrow[dl] & 2 \arrow[l] & \\ 4 \arrow[dr] & & & 1 \arrow[ul] \\ & 5 \arrow[r, dashed] & n \arrow[ur, dashed, "2"'] & \end{tikzcd} \qquad \mathcal{S}(Q_5) = \begin{pmatrix} 1 & s_1 & & & & \\ & 1 & -s_2 & & & \\ & & 1 & -s_3 & & \\ & & & & \ddots & \\ & & & & 1 & -s_n \\ & & & & & 1 \end{pmatrix},$$

$$Q_6 = \begin{tikzcd} 1 \arrow[rr, "2"] & & 3 \arrow[dl] \arrow[rr, "2"] & & 5 \arrow[dl] \arrow[rr, dashed, "2"] & & 2n \arrow[dl, dashed] \\ & 2 \arrow[ul] & & 4 \arrow[ul] & & 2n-1 \arrow[ul, dashed] & \end{tikzcd}$$

$$\mathcal{S}(Q_6) = \begin{pmatrix} 1 & -s_1 & -s_1 s_2 & & & & & \\ & 1 & s_2 & -s_2 s_3 & & & & \\ & & 1 & -s_3 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & -s_{2n-1} & -s_{2n-1} s_{2n} \\ & & & & & & 1 & s_{2n} \\ & & & & & & & 1 \end{pmatrix}.$$

Appendix A

A convergence result

In this Chapter we give an explicit formula for the operator $\mathcal{Q}_{\mathbf{s}}(Z)$ of Theorem 3.27 and prove a convergence property of its coefficients. The result is part of the preprint [2] and it is based on [18] due to Garcia-Fernandez, Filippini, Stoppa. The techniques used are due to them. It could be regarded as the natural deformation as formal power series of some results in [18] depending on functions f^{α}.

We keep the same notation as in Chapter 3. From now on, we always assume that Γ is a finite rank lattice and that a positive basis $\{\gamma_i\}$ has been fixed. Moreover, we fix a continuous symmetric spectrum Ω parametrized by $\mathrm{Hom}^{+}(\Gamma, \mathbb{C})$ which is the double of a positive spectrum. The main result is the following.

Theorem A.1. *Fix a central charge $Z_0 \in \mathrm{Hom}^{+}(\Gamma, \mathbb{C})$. Suppose that $\mathrm{DT}(\alpha, Z_0)$ grows at most exponentially for $\alpha \in \Gamma$ (in the sense of Definition 3.12). Then for all $\rho > 0$ there exists $\bar{\lambda}$ such that for $\lambda > \bar{\lambda}$ all the formal power series $g(x_\alpha, \mathcal{Q}_{\mathbf{s}}(\lambda Z_0)(x_\beta))$ converge for $||\mathbf{s}|| < \rho$. Let $U \subset \mathrm{Hom}^{+}(\Gamma, \mathbb{C})$ denote an open subset such that the exponential growth condition for $\mathrm{DT}(\alpha, Z)$ holds uniformly and all $Z \in U$ are uniformly bounded away from zero on elements of the cone Γ^{+}. Then for all sufficiently large λ the CV-deformations of the Joyce functions, given by $g(x_\alpha, \mathcal{Q}_{(1,\dots,1)}(\lambda Z)(x_\beta))$, are well defined and real-analytic on U, and uniformly bounded as α varies in Γ for fixed β.*

The proof is very much inspired by the work of Gaiotto, Moore and Neitzke in mathematical physics [19]. In [19, Appendix C] an integral operator is studied, and the proof of a convergence property for its iterations is sketched using functional analytic techniques. I am not confident with their work in the context of $\mathcal{N} = 2$ supersymmetric gauge theories on $\mathbb{R}^3 \times S^1_R$ (a circle of radius R) and with functional-analytic tools. To work through this problem I learned some methods for Sobolev embedding and estimates on the Hilbert transform operator.

Write T for a finite *rooted* tree, with vertices decorated by elements of Γ. We assume that T is connected unless we state explicitly otherwise. Denote the root decoration by α_T. The operation of removing the root produces a finite number of new connected, Γ-decorated trees $T \mapsto \{T_j\}$.

The goal is to discuss explicit formulae for th coefficients of the family $\nabla_{\mathbf{s}}(z, Z, \lambda)$ of meromorphic connections on $\mathbb{P}^1$ given by (3.3.3). In the meanwhile some formulae from [18] are recalled and transformed into formal power series in $\mathbf{s}$.

A.1 Explicit formula

We introduce holomorphic functions with branch-cuts

$$H_T\colon \mathbb{C}^* \times \mathrm{Hom}^+(\Gamma, C) \cap \mathrm{Hom}^{sg}(\Gamma, \mathbb{C}) \times \mathbb{R}_{\geq 0} \to \mathbb{C}^*$$

attached to trees T by the recursion

$$H_T(z, Z, \lambda) = \frac{1}{2\pi i} \int_{\ell_{\alpha_T}} \frac{dw}{w} \frac{z}{w - z} \exp(-Z(\alpha_T) w^{-1} - \lambda^2 \bar{Z}(\alpha_T) w) \prod_j H_{T_j}(w), \tag{A.1.1}$$

with the initial condition $H_\emptyset = 1$. We also introduce weights $W_T(Z) \in \Gamma \otimes \mathbb{Q}$ attached to trees by

$$W_T(Z) = \frac{1}{|\operatorname{Aut}(T)|} \mathrm{DT}(\alpha_T, Z)\alpha_T \prod_{\{v \to w\} \subset T} \langle \alpha(v), \alpha(w) \rangle \, \mathrm{DT}(\alpha(w), Z). \tag{A.1.2}$$

We can pair $W_T(Z)$ with $\beta \in \Gamma$ to obtain $\langle \beta, W_T(Z) \rangle \in \mathbb{Q}$. We extend this pairing to possibly *disconnected* trees T with finitely many connected components T_i by setting

$$\langle \beta, W_T(Z) \rangle = \prod_i \langle \beta, W_{T_i}(Z) \rangle.$$

Definition A.2. A *distinguished sector* Σ is the inverse system under inclusion of sectors Σ_N between consecutive distinguished rays ℓ such that

$$\sum_{Z(\alpha) \in \ell} \mathrm{DT}(\alpha, Z) \mathbf{s}^{[\alpha]_+ - [\alpha]_-} x_\alpha \notin J^N.$$

This is well defined because for each N there are only finitely many distinguished rays for which the above sum does not vanish modulo J^N.

Proposition A.3. *The automorphism $Y_{\mathbf{s}}(z, Z, \lambda)$ of $\mathfrak{g}_\Gamma [\![\mathbf{s}]\!]$ acting by*

$$\begin{aligned} Y_{\mathbf{s}}(z, Z, \lambda)(x_\beta) &= x_\beta \exp_* \sum_T \langle \beta, W_T(Z) \rangle H_T(z, Z, \lambda) \prod_{v \in T} \mathbf{s}^{[\alpha(v)]_+ - [\alpha(v)]_-} x_{\alpha(v)} \\ &= x_\beta \sum_{\text{disconnected } T} \langle \beta, W_T(Z) \rangle H_T(z, Z, \lambda) \prod_{v \in T} \mathbf{s}^{[\alpha(v)]_+ - [\alpha(v)]_-} x_{\alpha(v)} \end{aligned} \tag{A.1.3}$$

induces a flat section of $\nabla_{\mathbf{s}}(Z, \lambda)$ on each distinguished sector Σ.

Proof. This is proved in [18] section 4 (see in particular section 4.3). Note that in the notation of the proof of Proposition 3.27 we have $Y_{\mathbf{s}}(z, Z, \lambda) = \tilde{X}_0^{-1}(\lambda Z) \circ X(\lambda z, \lambda Z)$. □

Remark. For $\lambda = 0$, the formula above for $Y_{\mathbf{s}}(z, Z, 0)$ define a flat section of ∇_s^{BTL}, [18].

Let $A_{\mathbf{s}} \in D(\mathfrak{g}_\Gamma [\![\mathbf{s}]\!])$ denote the opposite of the connection 1-form of $\nabla_{\mathbf{s}}(z, Z, \lambda)$, so

$$\partial_z Y_{\mathbf{s}}(z, Z, \lambda) = A_{\mathbf{s}} \, Y_{\mathbf{s}}(z, Z, \lambda)$$

(where the right hand side is given by the composition of linear maps). Locally $A_{\mathbf{s}}$ is given by the composition of linear maps $(\partial_z Y_{\mathbf{s}}) Y_{\mathbf{s}}^{-1}$, where

$$\begin{aligned} \partial_z Y_{\mathbf{s}}(x_\alpha) &= \partial_z (Y_{\mathbf{s}}(z, Z, \lambda)(x_\alpha)) \\ &= Y_{\mathbf{s}}(z, Z, \lambda)(x_\alpha) \sum_T \langle \alpha, W_T(Z) \rangle \partial_z H_T(z, Z, \lambda) \prod_{v \in T} \mathbf{s}^{[\alpha(v)]_+ - [\alpha(v)]_-} x_{\alpha(v)}. \end{aligned}$$

Notice that a map of the form $(\partial_z Y)Y^{-1}$ where Y takes values in automorphisms of a commutative algebra is automatically a derivation.

Because of its specific form $Y_{\mathbf{s}}$ can be inverted explicitly via multivariate Lagrange inversion. Recall that this gives a concrete way to invert self-maps of a ring of formal power series $R[\![\xi_1,\dots,\xi_m]\!]$ of the form $\xi_i \mapsto \xi_i \exp(-\Phi_i(\xi_1,\dots,\xi_m))$ for some $\Phi_i(\xi_1,\dots,\xi_m) \in R[\![\xi_1,\dots,\xi_m]\!]$, where R is a ground $\mathbb{C}$-algebra.

To reduce the problem of explicitly inverting $Y_{\mathbf{s}}$ to a multivariate Lagrange inversion we notice that since $Y_{\mathbf{s}}$ is a commutative algebra automorphism it is enough to calculate $Y_{\mathbf{s}}^{-1}(x_{\gamma_i})$ for $i = 1,\dots,n$. We may then try to apply a Lagrange inversion formula over the base ring $R = \mathbb{C}[\![\mathbf{s}]\!]$. A further technical difficulty arises since $Y_{\mathbf{s}}$ is a self-map of a ring of Laurent polynomials $\mathbb{C}[\![\mathbf{s}]\!][x_{\gamma_1}^{\pm 1},\dots,x_{\gamma_n}^{\pm 1}]$ over $\mathbb{C}[\![\mathbf{s}]\!]$ rather than formal power series. To remedy this we introduce $2n$ auxiliary parameters $\xi = (\xi_1,\dots,\xi_{2n})$ and set for $\alpha \in \Gamma$

$$\xi^{\alpha} = \prod_{i=1}^{n} \xi_i^{[\alpha_i]_+} \prod_{j=n+1}^{2n} \xi_j^{-[\alpha_j]_-}.$$

Consider the auxiliary problem of inverting the self-map of $\mathbb{C}[\![\mathbf{s}]\!][\![\xi]\!]$ given by

$$(\xi_1,\dots,\xi_{2n}) \mapsto (F_1(\xi),\dots,F_{2n}(\xi)), \quad F_i(\xi) = \xi_i \exp(-\Phi_i(\xi))$$

where we choose

$$\Phi_i(\xi) = -\sum_T \langle \gamma_i, W_T(Z)\rangle H_T(z,Z,\lambda) \prod_{v\in T} \mathbf{s}^{[\alpha(v)]_+ - [\alpha(v)]_-} \xi^{\alpha(v)}$$

for $i = 1,\dots,n$, respectively

$$\Phi_i(\xi) = \sum_T \langle \gamma_i, W_T(Z)\rangle H_T(z,Z,\lambda) \prod_{v\in T} \mathbf{s}^{[\alpha(v)]_+ - [\alpha(v)]_-} \xi^{\alpha(v)}$$

for $i = n+1,\dots,2n$. If we can solve this then specialising $\xi_i = x_{\gamma_i}$ for $i = 1,\dots,n$, respectively $\xi_i = x_{\gamma_i}^{-1}$ for $i = n+1,\dots,2n$ determines the inverse $Y_{\mathbf{s}}^{-1}$ completely. Going back to the auxiliary problem, suppose that we can solve the equations

$$E_i(\xi) = \xi_i \exp\left(\Phi_i(E_1(\xi),\dots,E_{2m}(\xi))\right). \tag{A.1.4}$$

Then we have

$$F_i(E_1,\dots,E_{2m}) = E_i \exp\left(-\Phi_i(E_1,\dots,E_{2m})\right) = \xi_i,$$

so the inverse is given by $(\xi_1,\dots,\xi_{2m}) \mapsto (E_1(\xi),\dots,E_{2m}(\xi))$.

Lemma A.4. *There exist unique $E_i(\xi) \in \mathbb{C}[\xi][\![\mathbf{s}]\!]$ solving* (A.1.4). *Moreover for each multi-index $\mathbf{k} \in \mathbb{Z}^{2m}_{>0}$ the coefficient of $\xi^{\mathbf{k}}$ in $E_i(\xi)$ is given by*

$$[\xi^{\mathbf{k}}]E_i(\xi) = [\xi^{\mathbf{k}}] \det(\delta_{pq} + \xi_p \partial_q \Phi_p(\xi)) \xi_i \exp\left(-\sum_j k_j \Phi_j(\xi)\right). \tag{A.1.5}$$

Proof. Regard $\Phi_i(\xi)$ as formal power series in $\xi_1,\dots,\xi_{2m}$ with coefficients in $\mathbb{C}[\![\mathbf{s}]\!]$. Applying the multivariate Lagrange inversion formula in a version due to Good (see e.g. [20] Theorem 3, equation (4.5)) over the ground ring $\mathbb{C}[\![\mathbf{s}]\!]$ shows that there exists a unique solution $(E_1,\dots,E_{2m})$ of (A.1.4) where $E_i \in \mathbb{C}[\![\mathbf{s}]\!][\![\xi]\!]$ are given by (A.1.5). That we have in fact $E_i(\xi) \in \mathbb{C}[\xi][\![\mathbf{s}]\!]$ follows from the definition of $\Phi_i(\xi)$. □

For a multi-index $\mathbf{k} \in \mathbb{Z}^{2m}_{>0}$, $\mathbf{k} = (k_1, \ldots, k_{2m})$ we set $[\mathbf{k}] = \sum_{i=1}^{m}(k_i - k_{m+i})\gamma_i \in \Gamma$. Note that we have $\prod_{i=1}^{m} x_{\gamma_i}^{k_i} \prod_{j=1}^{m} x_{\gamma_j}^{-k_{j+m}} = \pm x_{[\mathbf{k}]}$ for a unique choice of sign, depending only on $\mathbf{k}$. We denote this sign by $(-1)^{\mathbf{k}}$.

Corollary A.5. *For $i = 1, \ldots, m$ and $\alpha \in \Gamma$ we have*

$$\begin{aligned} g(x_\alpha, Y_{\mathbf{s}}^{-1}(x_{\gamma_i})) &= g_0 \sum_{[\mathbf{k}]=\alpha} (-1)^{\mathbf{k}}[\xi^{\mathbf{k}}]E_i(\xi) \\ &= g_0 \sum_{[\mathbf{k}]=\alpha} (-1)^{\mathbf{k}}[\xi^{\mathbf{k}}] \det(\delta_{pq} + \xi_p \partial_q \Phi_p(\xi))\xi_i \exp\left(-\sum_j k_j \Phi_j(\xi)\right) \in \mathbb{C}[\![\mathbf{s}]\!]. \end{aligned}$$

Corollary A.6. *For $i = 1, \ldots, m$ we have*

$$\begin{aligned} A_{\mathbf{s}}(z, Z, \lambda)(x_{\gamma_i}) = &\sum_{\alpha \in \Gamma} \sum_{[\mathbf{k}]=\alpha} (-1)^{\mathbf{k}}[\xi^{\mathbf{k}}] \det(\delta_{pq} + \xi_p \partial_q \Phi_p(\xi))\xi_i \exp\left(-\sum_j k_j \Phi_j(\xi)\right) \\ &Y(x_\alpha) \sum_T \langle \alpha, W_T(Z)\rangle \partial_z H_T(z, Z, \lambda) \prod_{v \in T} s^{[\alpha(v)]_+ - [\alpha(v)]_-} x_{\alpha(v)} \in \mathfrak{g}_\Gamma[\![\mathbf{s}]\!]. \end{aligned} \tag{A.1.6}$$

In particular the CV-deformation $\mathcal{Q}_{\mathbf{s}}(\lambda Z)$ is the derivation of $\mathfrak{g}_\Gamma[\![\mathbf{s}]\!]$ determined by

$$\mathcal{Q}_{\mathbf{s}}(\lambda Z)(x_{\gamma_i}) = \operatorname{Res}_{z=0} A_{\mathbf{s}}(x_{\gamma_i}).$$

A.2 Estimates on graph integrals

Fix a tree T and $z^* \in \mathbb{C}^*$ which does not belong to any of the rays $\ell_{\alpha(v)}$ for $v \in T$. We study the graph integral

$$H_T(Z, \lambda) := H(z^*, Z, \lambda).$$

Proposition A.7. *Let T be a Γ-labelled rooted tree with n vertices. Then there exist universal constants $\bar{\lambda}, C_1, C_2 > 0$, depending only on the constant in the support condition* (3.2.1) *(in particular, independent of n, z^*), such that*

$$|H_T(Z, \lambda)| \leq C_1^n \exp(-C_2 \sum_{v \in T} |Z(\alpha(v))|\lambda) \tag{A.2.1}$$

for all $\lambda > \bar{\lambda}$.

The crucial point is that the estimate (A.2.1) holds up to the boundary of $\operatorname{Hom}^{sg}(\Gamma, \mathbb{C})$ where some distinguished rays collide, and irrespective of the presence of accumulation points for the set of distinguished rays for a fixed central charge Z.

We now collect some necessary preliminaries to the proof of Proposition A.7. For nonzero $\alpha \in \Gamma$, $\lambda > 0$ we introduce a function

$$u_{\alpha,\lambda}(s) = \frac{1}{s} \exp(-\lambda |Z(\alpha)|(s^{-1} + s))\chi_{(0,+\infty)}.$$

Notice that $u_{\alpha,\lambda} \in C^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$ for all $1 \leq p \leq \infty$.

Definition A.8. We denote by $\mathcal{H}$ the *Hilbert transform* on the real line, a bounded linear operator mapping $L^p(\mathbb{R})$ to itself for $1 < p < \infty$ (by a theorem of M. Riesz, see e.g. [36] section 3.2). In particular we have by definition

$$\mathcal{H}[u_{\alpha,\lambda}](s) = \mathrm{pv}\int_0^\infty \frac{dw}{w}\frac{1}{s-w}\exp[-\lambda|Z(\alpha)|(w^{-1}+w)].$$

By the Riesz theorem $\mathcal{H}[u_{\alpha,\lambda}](s)$ lies in $L^p(\mathbb{R})$ for $1 < p < \infty$. Standard regularity results imply that $\mathcal{H}[u_{\alpha,\lambda}](s)$ is in $C^1(\mathbb{R}_s \times \mathbb{R}_{\lambda>0})$ and that we can differentiate under the $\mathcal{H}$ operator. One can check by explicit computation that $\mathcal{H}[u_{\alpha,\lambda}]$ as well as $\partial_s\mathcal{H}[u_{\alpha,\lambda}]$ lie in $L^\infty(\mathbb{R}_s \times \mathbb{R}_{\lambda>0})$.

We consider a class of functions defined iteratively by

$$\tau s^l u_{\alpha,\lambda}(s)\prod_{i=1}^k \mathcal{H}[v_i](s) \tag{A.2.2}$$

where $\tau \in \mathbb{C}^*$, $l = 0, 1$ and each v_i is again of the form (A.2.2) for some $\alpha_i \in \Gamma$. Examples include $u_{\alpha_0,\lambda}\prod_{i=1}^k \mathcal{H}[u_{\alpha_i,\lambda}]$ as well as $u_{\alpha_0,\lambda}\mathcal{H}[u_{\alpha_1,\lambda}\mathcal{H}[u_{\alpha_2,\lambda}\cdots]]$.

Lemma A.9. *Let u be a function of the form* (A.2.2)*, with m corresponding lattice elements $\alpha_1, \dots, \alpha_m$ (not necessarily distinct). Then there are constants $C_1, C_2, \bar{\lambda}_1$, independent of m, depending only on τ and a common lower bound on $|Z(\alpha_1)|, \dots, |Z(\alpha_m)|$, such that for all $\lambda > \bar{\lambda}_1$ we have*

$$||u(s)||_{L^1} \leq C_1^m \prod_{i=1}^m \exp(-C_2|Z(\alpha_i)|\lambda).$$

Proof. We will argue by induction on m. Using the specific form (A.2.2) of u we find

$$||u(s)||_{L^1} \leq \prod_{i=1}^k ||\mathcal{H}[v_i](s)||_\infty ||\tau s^l u_{\alpha,\lambda}(s)||_{L^1}$$

provided all the $\mathcal{H}[v_i]$ are bounded. By explicit computation (for example using the Laplace approximation for exponential integrals) the factor $||\tau s^l u_{\alpha,\lambda}(\tau s)||_{L^1}$ has the required uniform exponential decay dominated by $C_1\exp(-C_2|Z(\alpha)|\lambda)$ for some fixed uniform C_2 and all sufficiently large C_1. So we focus on $||\mathcal{H}[v_i](s)||_\infty$. By an elementary Sobolev embedding we have

$$||\mathcal{H}[v_i](s)||_\infty \leq c_1||\mathcal{H}[v_i]||_{W^{1,2}}$$

so we start by controlling the L^2 norms $||\mathcal{H}[v_i]||_{L^2}$, $||\partial_s\mathcal{H}[v_i]||_{L^2}$. By L^2 boundedness of $\mathcal{H}$ and the fact that it commutes with ∂_s we find

$$||\mathcal{H}[v_i]||_{L^2} \leq c_2||v_i||_{L^2}, \quad ||\partial_s\mathcal{H}[v_i]||_{L^2} \leq c_2||\partial_s v_i||_{L^2},$$

that is

$$||\mathcal{H}[v_i]||_\infty \leq c_1c_2||v_i||_{W^{1,2}}.$$

We have reduced the problem to finding exponential bounds on $||v_i||_{L^2}$ and $||\partial_s v_i||_{L^2}$. Writing

$$v_i = \tau_i s^{l_i} u_{\beta,\lambda}(s)\prod_{j=1}^{k_i}\mathcal{H}[w_j](s)$$

we get

$$\begin{aligned}
||v_i||_{L^2} &\leq \prod_{j=1}^{k_i} ||\mathcal{H}[w_j](s)||_\infty ||\tau_i s^{l_i} u_{\beta,\lambda}(s)||_{L^2},\\
||\partial_s v_i||_{L^2} &\leq \sum_{r=1}^{k_i} ||\mathcal{H}[\partial_s w_r](s)||_{L^2} \prod_{j\neq r} ||\mathcal{H}[w_j](s)||_\infty ||\tau_i s^{l_i} u_{\beta,\lambda}(s)||_\infty\\
&+ \prod_{j=1}^{k_i} ||\mathcal{H}[w_j](s)||_\infty ||\partial_s(\tau_i s^{l_i} u_{\beta,\lambda}(s))||_{L^2}\\
&\leq c_3 \left(\sum_{r=1}^{k_i} ||\partial_s w_r(s)||_{L^2} \prod_{j\neq r} ||\mathcal{H}[w_j](s)||_\infty ||\tau_i s^{l_i} u_{\beta,\lambda}(s)||_\infty \right.\\
&\left. + \prod_{j=1}^{k_i} ||\mathcal{H}[w_j](s)||_\infty ||\partial_s(\tau_i s^{l_i} u_{\beta,\lambda}(s))||_{L^2} \right).
\end{aligned}$$

Notice that we chose the L^2 norm for the factor $\mathcal{H}[\partial_s w_r](s)$ rather than the supremum norm so that no further derivatives are required to control this. By explicit computation (e.g. Laplace approximation) the factors $||\tau_i s^{l_i} u_{\beta,\lambda}(s)||_{L^2}$, $||\tau_i s^{l_i} u_{\beta,\lambda}(s)||_\infty$ and $||\partial_s(\tau_i s^{l_i} u_{\beta,\lambda}(s))||_{L^2}$ are all dominated by $C_1 \exp(-C_2|Z(\beta)|\lambda)$ for some fixed uniform C_2 and all large C_1. Assuming inductively that we have the required exponential bounds on the norms $||w_j||_{L^2}$, $||\partial_s w_j||_{L^2}$ for all $j = 1, \dots, k_i$ the inequalities above imply a bound (denoting by m_i the number of lattice elements α^i_j attached to v_i, counted with their multiplicities)

$$||v_i||_{W^{1,2}} \leq c_4^{m_i} \prod_{j=1}^{m_i} \exp(-C_2|Z(\alpha^i_j)|\lambda).$$

Taking the product over $i = 1, \dots, m$ yields the result, with $C_1 = c_4$. □

Proof of Proposition A.7. In the course of the proof we use the notation s_v for $v \in T$ to denote positive real integration variables. Hopefully these will not be confused with the parameters $\mathbf{s}$ of our formal families; the latter never appear in the present section. Parametrising the ray $\ell_{\alpha(v)}$ for $v \in T$ by

$$\lambda^{-1}(|Z(\alpha(v))|)^{-1} Z(\alpha(v)) s_v,\ s_v \in \mathbb{R}_{>0}$$

for each $v \in T$ turns $H_T(Z, \lambda)$ into an iterated integral along the positive real line $(0, +\infty)$. Pick a vertex $w \in T$ with unique incoming vertex v distinct from the root. There is a corresponding factor in $H_T(Z, \lambda)$ given by

$$(2\pi i)^{-1} \int_0^\infty ds_w \frac{\tau_w s_v}{\tau_w s_v - s_w} u_{\alpha(w),\lambda}(s_w),$$

with

$$\tau_w = \frac{|Z(\alpha(w))|}{Z(\alpha(w))} \frac{Z(\alpha(v))}{|Z(\alpha(v))|}.$$

Let $c_1, \delta > 0$ denote positive constants to be determined independently of T (in particular, independently of n). Suppose that there is an edge $\{v \to w\} \subset T$ such that $|\operatorname{Im}(\tau_w)| < \delta$. Choose the edge for which $\operatorname{Im}(\tau_w)$ is the smallest possible in T (that is, such that the sine of the convex positive angle between the corresponding rays $\ell_{\alpha(v)}$, $\ell_{\alpha(w)}$ is less than δ, and the smallest among edges in T). Notice that by our minimal choice of $v \to w$ there are no further rays $\ell_{\alpha(w')}$ with $w \to w'$ between $\ell_{\alpha(v)}$ and $\ell_{\alpha(w)}$. We claim that for sufficiently small δ there is a uniform c_1 such that

$$|H_T(Z,\lambda)| \leq c_1(|H_{T,1}(Z,\lambda)| + |H_{T,2}(Z,\lambda)|),$$

where the iterated integrals $H_{T,1}(Z,\lambda)$ and $H_{T,2}(Z,\lambda)$ are obtained by replacing the factor

$$(2\pi i)^{-2}\int_0^{\infty} ds_v \frac{\tau_v s_o}{\tau_v s_o - s_v} u_{\alpha(v),\lambda}(s_v) \int_0^{\infty} ds_w \frac{\tau_w s_v}{\tau_w s_v - s_w} u_{\alpha(w),\lambda}(s_w) \tag{A.2.3}$$

attached to the subgraph $\{o \to v \to w\} \subset T$ (denoting by o the unique vertex mapping to v) by the Hilbert transform

$$(2\pi i)^{-2}\int_0^{\infty} ds_v \frac{\tau_v s_o}{\tau_v s_o - s_v} u_{\alpha(v),\lambda}(s_v) s_v \mathcal{H}[u_{\alpha(w),\lambda}](s_v) \tag{A.2.4}$$

in the case of $H_{T,1}(Z,\lambda)$, respectively by

$$(2\pi i)^{-1}\int_0^{\infty} ds_v \frac{\tau_v s_o}{\tau_v s_o - s_v} u_{\alpha(v),\lambda}(s_v) u_{\alpha(w),\lambda}(s_v) \tag{A.2.5}$$

in the case of $H_{T,2}(Z,\lambda)$. This holds because by the classical Sokhotski-Plemelj theorem in complex analysis (see e.g. [36] section 3.2) the limit of the factor (A.2.3) as $\tau_w \to 1$ is given by the sum of the principal value part (A.2.4), and the residue part (A.2.5), with suitable signs (determined by whether $\mathrm{Im}(\tau_w) \to 0$ from below or above). The $\tau_w \to 1$ limit holds uniformly for all $\alpha(v)$, $\alpha(w)$, so the claim follows.

Notice that we can estimate the residue part (A.2.5) by

$$||u_{\alpha(w),\lambda}||_{\infty} \left|(2\pi i)^{-1}\int_0^{\infty} ds_v \frac{\tau_v s_o}{\tau_v s_o - s_v} u_{\alpha(v),\lambda}(s_v)\right|.$$

Let T_2 be the rooted, Γ-labelled tree obtained from T by contracting the edge $\{v \to w\} \subset T$ to a single vertex decorated by $\alpha(v)$. By the estimate above we have

$$|H_{T,2}(Z,\lambda)| \leq ||u_{\alpha(w),\lambda}||_{\infty}|H_{T_2}(Z,\lambda)|),$$

so

$$|H_T(Z,\lambda)| \leq c_2(|H_{T,1}(Z,\lambda)| + ||u_{\alpha(w),\lambda}||_{\infty}|H_{T,2}(Z,\lambda)|). \tag{A.2.6}$$

On the other hand edges $\{v \to w\} \subset T$ for which we have a fixed lower bound $|\mathrm{Im}(\tau_w)| \geq \delta > 0$ can be "integrated out": let $T_3 \subset T$ be the (rooted, Γ-labelled) subtree obtained by chopping out the (rooted, Γ-labelled) subtree $T_4 \subset T$ with root w. Then there is a constant c_3, depending only on δ, such that

$$|H_T(Z,\lambda)| \leq c_3|H_{T_3}(Z,\lambda)||H_{T,4}(Z,\lambda)|,$$

where $H_{T,4}(Z,\lambda)$ equals essentially $H_{T_4}(Z,\lambda)$, but with root factor in the integral replaced with

$$\int_0^{\infty} ds_w u_{\alpha(w),\lambda}(s_w).$$

We can now proceed inductively applying the two steps described above, decreasing the number of vertices of T or increasing the number of $\mathcal{H}$ operators inserted. The process stops in a finite number of steps, yielding residual functions $H_i(Z,\lambda)$ for a finite set of indices $i \in I$, with cardinality $|I| \leq 2^n$, such that

$$|H_T(Z,\lambda)| \leq c_4^n \left(\sum_{i\in I} |H_i(Z,\lambda)|\right)$$

where $c_4 > 0$ does not depend on T. By construction each $|H_i(Z,\lambda)|$ is bounded by a finite product of factors of the form $||u_{\alpha(w),\lambda}||_\infty$ or $||u(s)||_{L^1}$, where u belongs to the class of functions (A.2.2). So by Lemma A.9 and repeated application of (A.2.6) each $|H_i(Z,\lambda)|$ is bounded by $C_1^n \exp(-C_2 \sum_{v\in T} |Z(\alpha(v))|\lambda)$ for absolute constants C_1, C_2 and all $\lambda > \bar{\lambda}$ (independently of T). The bound (A.2.1) now follows with that same $C_2, \bar{\lambda}$ and taking the constant C_1 in the statement to be $2C_1c_4$ in our present notation. □

A.3 Functional equation and convergence

In this section we complete the proof of Theorem 4.26. Fix a continuous symmetric spectrum Ω parametrised by $\mathrm{Hom}^+(\Gamma,\mathbb{C})$ which is the double of a positive spectrum.

Definition A.10. Fix constants $c_1, c_2, \lambda > 0$ and a collection of formal power series $S_\alpha(\mathbf{s}) \in \mathbb{C}[\![\mathbf{s}]\!]$ for $\alpha \in \Gamma$. Define a new collection $\mathcal{F}[S]_\beta(\mathbf{s}) \in \mathbb{C}[\![\mathbf{s}]\!]$ for $\beta \in \Gamma$ by

$$\mathcal{F}[S]_\beta(\mathbf{s}) = \prod_{\alpha\in\Gamma} (1 - c_1 \exp(-c_2|Z(\alpha)|\lambda)\mathbf{s}^{[\alpha]_+ - [\alpha]_-} S_\alpha(\mathbf{s}))^{|\langle\beta,\alpha\rangle||\Omega(\alpha,Z)|}.$$

Let us write $S^{(0)}$ for the family of constant formal power series

$$S^0_\beta(\mathbf{s}) = 1 \in \mathbb{C}[\![\mathbf{s}]\!].$$

for all $\beta \in \Gamma$. We define inductively for $i \geq 0$

$$S^{(i+1)}_\beta(\mathbf{s}) = \mathcal{F}[S^{(i)}]_\beta(\mathbf{s}).$$

Lemma A.11. *Fix $\bar{\rho} > 0$. There exists $\bar{\lambda} > 0$, depending only on $\bar{\rho}$ and the constants in the support and exponential growth conditions* (3.2.1), (3.2.2)*, such that for $\lambda \geq \bar{\lambda}$ all the formal power series $S^{(i)}_\beta(\mathbf{s})$ converge for $||\mathbf{s}|| < \bar{\rho}$, uniformly for $i \geq 0$.*

Proof. We argue by induction on i. For $r > 0$ we write $B_r = \{\mathbf{s} \in \mathbb{C}^n : ||\mathbf{s}|| < r\}$ for the open ball. Pick a norm $||-||$ on $\Gamma \otimes \mathbb{C}$. Suppose that $\bar{\rho} > 0$, $\bar{\lambda} > 0$ and $c_3 > 0$ are constants such that $S^{(i)}_\alpha(\mathbf{s})$ converges absolutely and uniformly in compact subsets of $B_{\bar{\rho}}$ and moreover we have

$$|S^{(i)}_\alpha(\mathbf{s})| < c_3 e^{||\alpha||}. \tag{A.3.1}$$

for all $\mathbf{s} \in B_{\bar{\rho}}$, $\lambda > \bar{\lambda}$, $\alpha \in \Gamma$. In the case of S_0 we can choose the constants $\bar{\rho}, \lambda > 0$ arbitrarily, while c_3 is a positive constant that only depends on the choice of norm $||-||$.

The infinite product

$$\prod_{\alpha\in\Gamma} (1 - c_1 \exp(-c_2|Z(\alpha)|\lambda)\mathbf{s}^{[\alpha]_+ - [\alpha]_-} S^{(i)}_\alpha(\mathbf{s}))^{|\langle\beta,\alpha\rangle||\Omega(\alpha,Z)|}$$

converges absolutely and uniformly in compact subsets of $B_{\bar{\rho}}$ if and only if this happens for the series

$$\sum_{\alpha\in\Gamma} |\langle\beta,\alpha\rangle||\Omega(\alpha,Z)| \log(1 - c_1 \exp(-c_2|Z(\alpha)|\lambda)\mathbf{s}^{[\alpha]_+ - [\alpha]_-} S^{(i)}_\alpha(\mathbf{s})). \tag{A.3.2}$$

There is a uniform constant $c_4 > 0$ such that for all sufficiently large λ, depending only on the constant in the support condition (3.2.1) and the inductive bound (A.3.1), the series (A.3.2) is bounded by

$$c_4||\beta|| \sum_{\alpha\in\Gamma} ||\alpha|||\Omega(\alpha, Z)|c_1 \exp(-c_2|Z(\alpha)|\lambda)c_3\bar{\rho}^{[\alpha]_+-[\alpha]_-}e^{||\alpha||}. \tag{A.3.3}$$

This bound is independent of i. If the spectrum $\Omega(\alpha, Z)$ has at most exponential growth then the series (A.3.3) converges for all sufficiently large λ, depending only on $\bar{\rho}$, the support condition (3.2.1) and the exponential growth condition (3.2.2). Moreover for all sufficiently large λ, depending only on (3.2.1), (3.2.2), the sum of the series is bounded by $||\beta|| \log c_3$, from which we get

$$|S^{(i+1)}_\beta(\mathbf{s})| < c_3 e^{||\beta||}$$

in $B_{\bar{\rho}}$. So if we choose our initial $\bar{\lambda}$ sufficiently large, depending only on $\bar{\rho}$ and the conditions (3.2.1), (3.2.2), the induction goes through. $\square$

Let T denote a Γ-labelled rooted tree as usual. We write $\mathrm{depth}(T)$ for the length of the longest oriented path in T. Let us denote by $\mu|\Omega|(\alpha, Z)$ the Möbius transform of the function $|\Omega(\alpha, Z)|$,

$$\mu|\Omega|(\alpha, Z) = \sum_{k>0,\, k|\alpha} \frac{1}{k^2}|\Omega(k^{-1}\alpha, Z)|.$$

Note that in general $\mu|\Omega|(\alpha, Z)x_\alpha$ is not a continuous family of stability data in $\mathfrak{g}_\Gamma$, and $|\Omega(\alpha, Z)|$ is not a continuous spectrum. This is completely irrelevant for our purposes, since we will only use the obvious bound

$$|\,\mathrm{DT}(\alpha, Z)| \leq \mu|\Omega|(\alpha, Z).$$

Let us introduce weights $\widetilde{W}_T(Z) \in \Gamma \otimes \mathbb{Q}$ by

$$\widetilde{W}_T(Z) = \frac{1}{|\,\mathrm{Aut}(T)|}\mu|\Omega|(\alpha_T, Z)\alpha_T \prod_{\{v\to w\}\subset T} \langle\alpha(v), \alpha(w)\rangle \mu|\Omega|(\alpha(w), Z).$$

Lemma A.12. *We have*

$$S^{(i)}_\beta(\mathbf{s}) = \sum_{\text{disconnected}\, T,\, \mathrm{depth}(T)\leq i} c_1^{|T|}|\langle\beta, \widetilde{W}_T(Z)\rangle| \exp(-c_2 \sum_{v\in T} |Z(\alpha(v))|\lambda) \prod_{v\in T} \mathbf{s}^{[\alpha(v)]_+-[\alpha(v)]_-}.$$

Proof. We write

$$S^{(i+1)}_\beta = \exp \sum_{\alpha\in\Gamma} |\langle\beta, \alpha\rangle||\Omega(\alpha, Z)| \log(1 - c_1 \exp(-c_2|Z(\alpha)|\lambda)\mathbf{s}^{[\alpha]_+-[\alpha]_-} S^{(i)}_\alpha(\mathbf{s})).$$

The result follows from expanding $\log(1 - c_1 \exp(-c_2|Z(\alpha)|\lambda)\mathbf{s}^{[\alpha]_+-[\alpha]_-} S^{(i)}_\alpha(\mathbf{s}))$ as a formal power series and arguing by induction, starting from $S^{(0)}_\alpha = 1$ for all α, precisely as in [18] section 3.6. $\square$

Corollary A.13. *Fix $c_1, c_2, \bar{\rho} > 0$. There exists $\bar{\lambda} > 0$, depending only on $\bar{\rho}$ and the constants in the support and exponential growth conditions* (3.2.1), (3.2.2)*, such that for all $\lambda \geq \bar{\lambda}$ the formal power series*

$$\sum_{\text{disconnected}\, T} c_1^{|T|}|\langle\beta, \widetilde{W}_T(Z)\rangle| \exp(-c_2 \sum_{v\in T} |Z(\alpha(v))|\lambda) \prod_{v\in T} \mathbf{s}^{[\alpha(v)]_+-[\alpha(v)]_-}$$

converges for $||\mathbf{s}|| < \bar{\rho}$.

We may now prove A.1

Proof of Theorem A.1. We show first that, under the assumptions of the Theorem, for all sufficiently large λ, depending only on $\bar{\rho}$ and the constants in the support condition (3.2.1) and the exponential bound (3.2.2) all the formal power series $g(x_\alpha, Y(z, Z, \lambda)(x_\beta))$ converge absolutely and uniformly for $||\mathbf{s}|| < \bar{\rho}$.

By our explicit formula (A.1.3) for the action of $Y(z, Z, \lambda)(x_\beta)$ it remains to prove that there exists $\bar{\lambda} > 0$ as above such that for all $\lambda > \bar{\lambda}$ and $\beta \in \Gamma$ the complex-valued formal power series

$$\sum_{\text{disconnected}\, T\,:\, \sum_{v\in T}\alpha(v)=\alpha} \langle\beta, W_T(Z)\rangle H_T(z, Z, \lambda) \prod_{v\in T} \mathbf{s}^{[\alpha(v)]_+ - [\alpha(v)]_-}$$

converges for $||\mathbf{s}|| < \bar{\rho}$.

We will in fact prove a statement which is independent of α: we claim that there exists $\bar{\lambda} > 0$ as above such that for all $\lambda > \bar{\lambda}$ and $\beta \in \Gamma$ the complex-valued formal power series

$$\sum_{\text{disconnected}\, T} \langle\beta, W_T(Z)\rangle H_T(z, Z, \lambda) \prod_{v\in T} \mathbf{s}^{[\alpha(v)]_+ - [\alpha(v)]_-}$$

(summing over all decorated trees, without the constraint that $\sum_{v\in T}\alpha(v)$ is fixed) converges for $||\mathbf{s}|| < \bar{\rho}$. By Proposition A.7 and the comparison principle it is enough to prove the claim for the formal power series

$$\sum_{\text{disconnected}\, T} C_1^{|T|} |\langle\beta, \widetilde{W}_T(Z)\rangle| \exp(-C_2 \sum_{v\in T} |Z(\alpha(v))|\lambda) \prod_{v\in T} \mathbf{s}^{[\alpha(v)]_+ - [\alpha(v)]_-} \tag{A.3.4}$$

for all β, where C_1, C_2 are the constants in (A.2.1). By Corollary A.13 we can ensure that this converges for $||\mathbf{s}|| < \bar{\rho}$ by choosing $\bar{\lambda}$ large enough, depending only on $\bar{\rho}$ and (3.2.1), (3.2.2) as required.

To extend the convergence statement to the matrix elements of the connection 1-form $A_{\mathbf{s}}$ we rely on our explicit formula (A.1.6). Plugging the expansion for $Y_{\mathbf{s}}(x_{\gamma_i})$ in (A.1.6) one checks that each Γ-graded component of $A_{\mathbf{s}}(x_{\gamma_i})$ is given by a finite product of factors which are infinite sums over decorated, disconnected trees and are all dominated by a sum of the form (A.3.4) for possibly larger but fixed constants C_1, C_2. $\square$

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