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Earthquakes on hyperbolic surfaces with geodesic boundary and Anti de Sitter geometry

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Introduction

Let S be a topological surface of genus g with $\mathfrak{n}$ points $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{n}$ removed, called punctures of S, with negative Euler characteristic $\chi(S) = 2 - 2g - \mathfrak{n}$. Denote by $\mathcal{T}_S$ the Teichmüller space of S, namely

$$\mathcal{T}_S = \{\text{hyperbolic complete metrics on } S \text{ of finite area}\} / \operatorname{Diff}_0(S)$$

where $\operatorname{Diff}_0(S)$ denotes the group of homotopically trivial diffeomorphisms of S.

Hyperbolic earthquakes constitute a well known class of deformations of elements of $\mathcal{T}_S$. Given $h \in \mathcal{T}_S$, a closed geodesic c on (S, h) and a positive number ω, an *elementary hyperbolic left* (respectively *right*) *earthquake* on (S, h) along c with shearing amount ω cuts (S, h) along c and glues back along c shearing towards the left (respectively right) by a factor ω, obtaining a new element $h' \in \mathcal{T}_S$. In general, a *hyperbolic left* (respectively *right*) *earthquake* on (S, h) is the limit of elementary hyperbolic left (respectively right) earthquakes on (S, h). Any earthquake on S determines a subset of (S, h) foliated by geodesics, called the *fault locus* of the earthquake, which is the subset along which the shearing occurs. The shearing amounts give rise to a transverse measure on the arcs on S with the fault locus as support, giving raise to what is called a *measured geodesic lamination* on S. We will denote the space of measured geodesic laminations on S by $\mathcal{ML}_S$. It turns out that any measured geodesic lamination on S is the fault locus of a left/right earthquake ([26]). W. P. Thurston in [34] proved the following celebrated result.

Thurston's Earthquake Theorem, [34]. *If S is a closed surface of genus greater than 1 then for every $(h, h') \in \mathcal{T}_S \times \mathcal{T}_S$ there is a unique couple of measured geodesic laminations (λ, μ) such that*

$$h' = E_l^\lambda(h) = E_r^\mu(h). \tag{1}$$

This result still holds in the case $\mathfrak{n} \neq 0$ (see S. P. Kerckhoff, [28]).

For any $h \in \mathcal{T}_S$, the surface (S, h) is isometric to the quotient of $\mathbb{H}^2$ by the action of a discrete subgroup Γ of $\operatorname{Isom}_0(\mathbb{H}^2) \cong PSL(2, \mathbb{R})$, which extends uniquely to a subgroup $\hat{\Gamma}$ of the isometries of $\mathbb{H}^3$, so that $\hat{\Gamma} \backslash \mathbb{H}^3$ is

isomorphic to $S \times \mathbb{R}$. Such hyperbolic metrics on $S \times \mathbb{R}$ are called *Fuchsian.* Denote by $\mathcal{F}_S$ the space of Fuchsian metrics on $S \times \mathbb{R}$; it is called the *Fuhsian locus* of S. A *quasi-Fuchsian metric* on $S \times \mathbb{R}$ is a hyperbolic metric obtained by a quasi-conformal deformation of a Fuchsian metric. A classical result of L. Bers (see [9]) assures that quasi-Fuchsian manifolds are determined by their conformal structures at infinity. It turns out that the space $\mathcal{QF}_S$ of quasi-Fuchsian metrics on $S \times \mathbb{R}$ is parametrized by $\mathcal{T}_S \times \mathcal{T}_S$. For every $\eta \in \mathcal{QF}_S$ there exists a minimal non-empty η-convex subset $\mathrm{CC}(\eta)$ called the *convex core* of η (see [36], [21]). If η is Fuchsian, then $\mathrm{CC}(\eta)$ is a totally geodesic surface isometric to S endowed with a hyperbolic metric. Otherwise, $\mathrm{CC}(\eta)$ is a 3-dimensional subset of $S \times \mathbb{R}$ with two boundary components $\partial_\pm \mathrm{CC}(\eta)$, each homeomorphic to S. As seen in $S \times \mathbb{R}$, they are surfaces bent along a family of geodesics, a couple of measured geodesic laminations $(\lambda_-, \lambda_+) = \Psi(\eta) \in \mathcal{ML}_\mathcal{S} \times \mathcal{ML}_\mathcal{S}$. Notice that the preimage through $\Psi : \mathcal{QF}_S \to \mathcal{ML}_\mathcal{S} \times \mathcal{ML}_\mathcal{S}$ of the couple of void laminations is the Fuchsian locus $\mathcal{F}_S$.
It is simple to check that if $(\lambda_-, \lambda_+) \in \mathcal{ML}_S \times \mathcal{ML}_S$ are bending laminations of the convex core $\mathrm{CC}(\eta)$ for a certain $\eta \in \mathcal{QF}_S$, then they fill up S, in the sense that any other lamination on S must intersect at least one of them. Moreover, the weight of any closed curve c in $\lambda_\pm$ cannot exceed π, since it corresponds to the bending angle along c of $\partial_\pm \mathrm{CC}(\eta)$. Denoting by $\mathcal{FML}_S^{(\pi)}$ the subset of $\mathcal{ML}_S \times \mathcal{ML}_S$ of pairs of lamination satisfying those conditions, Thurston conjectured that any couple of laminations in $\mathcal{FML}_S^{(\pi)}$ is uniquely realized as the bending laminations of the convex core of a quasi-Fuchsian manifold. F. Bonahon and J.-P. Otal proved in [12] the existence of such $\eta \in \mathcal{QF}_S$, providing that the image of Ψ is exactly the space $\mathcal{FML}_S^{(\pi)}$. In fact they also proved the uniqueness of the preimage, when the laminations are weighted multicurves. Moreover, with a different approach, Bonahon in [11] proved the uniqueness of the preimage if the laminations are sufficiently small: there exists a neighbourhood $\mathcal{V}$ of the Fuchsian locus such that the restriction of Ψ on $\mathcal{V} \setminus \mathcal{F}$ is a homeomorphism onto its image.

Quasi-Fuchsian metrics have a connection with Anti de Sitter 3-dimensional manifolds, which are locally modelled by the space

$$AdS_3 = \{[\underline{x}] \in P\mathbb{R}^3 \mid \langle \underline{x}, \underline{x} \rangle_{(2,2)} < 0\}$$

where $\langle \underline{x}, \underline{x} \rangle_{(2,2)} = x_1^2 + x_2^2 - x_3^2 - x_4^2$ for every $\underline{x} \in \mathbb{R}^4$. Such space inherits a Lorentzian metric, induced from $\langle *, * \rangle_{(2,2)}$. An AdS_3-spacetime M is *globally hyperbolic* if it contains a *Cauchy surface*, i.e. a space-like surface S intersecting each inextensible time-like geodesic exactly once. As a consequence, there is a homeomorphism $\tau : \mathbb{R} \times S \to M$ with $\tau(\{0\} \times S) = S$. Also, the metric on M induced on $\mathbb{R} \times S$ has the form $-\mathrm{d}t^2 + h_t$ where h_t is a continuous family of Riemannian metrics on S. See [6], [24].

A globally hyperbolic AdS-spacetime is *globally hyperbolic compact* (GHC) if it admits a closed Cauchy surface. In [29] G. Mess studied the space of *globally hyperbolic maximal compact* (GHMC) AdS_3-spacetimes (maximality of GHC is considered for isometric embeddings), pointing out striking analogies with the quasi-Fuchsian case. See also [2]. It turns out that GHMC AdS_3-spacetimes are determined by a couple of hyperbolic metrics on a surface S, showing that the relevant moduli space is $\mathcal{T}_S \times \mathcal{T}_S$. However, the correspondence is not based on an asymptotic compactification. The point here is that the orientation-preserving isometry group of AdS_3 is the product $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$; if (ρ_L, ρ_R) is the holonomy of a MGHC AdS_3-spacetime $M = \mathbb{R} \times S$, Mess showed that ρ_L and ρ_R are Fuchsian representations. Let us denote by h_L and h_R the hyperbolic metrics on S associated respectively with ρ_L and ρ_R. As in the quasi-Fuchsian case, a GHMC AdS_3-manifold contains a convex core, whose boundary components are two hyperbolic surfaces $(\mathbb{S}_+, h_+)$, $(\mathbb{S}_-, h_-)$ bent along two pleated laminations λ_- and λ_+ respectively. Mess discovered a simple relation in terms of earthquakes between the immersion data of the boundary of the convex core and the left and right hyperbolic metrics:

$$h_R = E_r^{\lambda_+}(h_+) = E_r^{\lambda_+}\big((E_l^{\lambda_+})^{-1}(h_L)\big)$$
$$h_R = E_l^{\lambda_-}(h_-) = E_l^{\lambda_-}\big((E_r^{\lambda_-})^{-1}(h_L)\big).$$

The inverse of a right earthquake is a left earthquake, and vice versa. Therefore, Mess actually gave a proof of Thurston's Earthquake Theorem in the GHMC AdS_3-manifold language.
Furthermore, a question on bending loci was arisen in this AdS setting: which couples of laminations on S can be realized as the bending laminations of a GHMC AdS_3-spacetime $\mathbb{R} \times S$? As in the hyperbolic setting, it can be easily checked that if λ, μ are such bending laminations, then they must fill up S (whereas the condition on weights of simple curves is not meaningful in this setting). Conjecturally, every pair of filling laminations can be uniquely realized as the bending locus of a GHMC AdS_3-spacetime. Several generalization of Mess results have been considered in the study of Anti de Sitter geometry in dimension 3. In particular the case of non compact surfaces has been considered by different authors (see for instance: [15] for surfaces with closed geodesic boundaries, [16] for closed surfaces with cone singularities,[20] for ideal polygons in $\mathbb{H}^2$).

Multi-black holes (MBH) manifolds, which take their name from physics literature, are analogous to GHMC AdS_3-manifolds. However, they are foliated by non-compact space-like surfaces, and yet they contain inextendible causal curves which do not meet all the surfaces of the foliation. Those spacetimes admit a causal bordification that is the union of time-like annuli (one for each boundary component). Denoting by $\mathcal{T}'_S$ the Teichmüller space of the metrics on S for which the completion has closed geodesic boundary, T. Barbot in [3] and [4] showed that MBH manifolds are parametrized by $\mathcal{T}'_S \times \mathcal{T}'_S$. R. Benedetti and F. Bonsante in [7] proved that for every $(h_l, h_r) \in \mathcal{T}'_S \times \mathcal{T}'_S$ and for every right (respectively left) earthquake between h_l and h_r there is a unique space-like, convex, bent, inextendible surface in the MBH manifold M associated with (h_l, h_r). Bonsante, Krasnov and Schlenker proved that bent surfaces in M associated with earthquakes between h_l and h_r do not meet "at infinity" the asymptotic boundary of M, and conversely any bent surface in a MBH manifold that does not accumulate on the asymptotic boundary is associated with an earthquake between (S, h_l) and (S, h_r). Using this characterization they get the following earthquake theorem for hyperbolic surfaces with closed geodesic boundary.

Theorem (F. Bonsante, K. Krasnov, J.-M. Schlenker, [15]). *For every $(h_l, h_r) \in \mathcal{T}'_S \times \mathcal{T}'_S$ there are exactly 2^k right earthquakes sending h_l to h_r, where k is the number of punctures of S which correspond to a closed geodesic boundary component of $\overline{S}$.*

The result clearly holds also for left earthquakes. In general, k can be different from $\mathfrak{n}$, since metrics with some cusps are elements of $\mathcal{T}'_S$. See also [14]. The number 2^k comes by counting the number of bent surfaces, in a given MBH manifold M, which do not meet the asymptotic boundary. On the universal covering AdS_3 of M, the preimages of the asymptotic boundary of those surfaces are contained in $\pi_1(S)$-invariant achronal meridians in the boundary of AdS_3 and do not meet the preimages of the asymptotic regions of M. It turns out that those meridians must be contained in the closure of the boundaries of such asymptotic regions; this set contains exactly 2^k invariant achronal meridians.

The first aim of this work is to provide an earthquake theorem to ciliated surfaces. As defined in [23], a *ciliated surface* is the data of a surface S, topologically obtained by removing from a closed surface a finite number of mutually disjoint disks Δ_i, and of a finite subset Q of $\bigcup \partial\Delta_i$, whose elements are called cilia. The associated Teichmüller space $\mathcal{T}^\star_S(\mathfrak{C})$ is given by the space of hyperbolic metrics on $\overline{S} = (S \cup \bigcup \Delta_i) \smallsetminus Q$ of finite area such that each connected component of $\bigcup \partial\Delta_i \smallsetminus Q$ is totally geodesic, up to diffeomorphisms of S fixing Q which are isotopic to the identity. We assume that $Q \cap \Delta_i \neq \emptyset$ for every i. Chapter 2 is devoted to the proof of the following result.

Theorem A. *For every $h_l, h_r \in \mathcal{T}^\star_S$ there exists a unique right earthquake sending h_l to h_r.*

Again, the same holds if we consider left earthquakes. The argument makes large use of the techniques used in [15] for the MBH case. Each

pair of ciliated surfaces (S_l, S_r) gives rise to a MBH manifold M with some marked points on the asymptotic boundary (one for each cilium of S). The key point is to look for bent surfaces in M associated with a right earthquake between S_l and S_r. If $\mathbb{S}$ is such a surface, we will prove that the asymptotic boundary of $\mathbb{S}$ contains the marked points and that $\mathbb{S}$ is contained in the closure of the past of those points. We will also show the converse: every bent surface having these properties is associated with a right earthquake from S_l to S_r. As the asymptotic boundary of $\mathbb{S}$ is achronal, we will deduce that it must coincide with the boundary of the union of the past of the marked points. From there, the existence and the uniqueness of the right earthquake between the two ciliated surfaces immediately follows.
As a particular example, we will show how the construction of such an earthquake can be applied to the case of ideal polygons of $\mathbb{H}^2$, strongly related to the inscribability of ideal polyhedra in AdS_3 (see the proof J. Danciger, S. Maloni, J.-M. Schlenker give in [20] of the earthquake theorem for ideal polygons).

AdS bending lamination conjecture: The problem of giving a characterization of which couples of measured laminations can be realized as the bending laminations of MGHC AdS_3-spacetimes was solved by Bonsante and Schlenker in the closed case.

Theorem (F. Bonsante and J.-M. Schlenker, [17]). *If S is a closed surface and (λ, μ) is a couple of filling laminations on S, then there is a couple $(h, h') \in \mathcal{T}_S \times \mathcal{T}_S$ for which* (1) *holds.*

This result is achieved in two steps: first – following the same ideas of Bonahon in the quasi-Fuchsian case – they proved that any pair of small laminations filling up S is uniquely realized, and then showed that the bending lamination map

$$\Phi : \mathcal{T}_S \times \mathcal{T}_S \to \mathcal{FML}_S = \{(\lambda, \mu) \in \mathcal{ML}_S \times \mathcal{ML}_S \mid (\lambda, \mu) \text{ fills up S}\} \qquad (2)$$

is proper. As a consequence, the topological degree of Φ is defined. The local injectivity close the Fuchsian locus D (the diagonal of $\mathcal{T}_S \times \mathcal{T}_S$) implies that $\deg \Phi$ must be 1, so the map must be surjective.
Local injectivity of Φ close the Fuchsian locus is achieved by using that earthquakes along $t\lambda$ form the Hamiltonian flow of the length function $L_\lambda : \mathcal{T}_S \to \mathbb{R}$ of λ (see [38]). Indeed the convexity properties of L_λ allows to solve the "infinitesimal version" of the problem in the normal bundle of the Fuchsian locus D, and then, by an application of the implicit function theorem, one gets the local injectivity of the map in a neighbourhood of D. The properness is instead proved by showing that the length of the bending laminations of a MGHC AdS_3-spacetime, say with respect to the left metric, can be bounded only in terms of their intersection.

In this work we consider an analogous bending lamination problem in the setting of hyperbolic metrics on a surface S with $\mathfrak{n}$ closed geodesic boundary, whose Teichmüller space is denoted by $\mathcal{T}_S^\circ$. Notice that since a MBH manifold M contains many past-convex and future-convex surfaces, the formulation of the problem must be adapted to this setting. The key point is that M contains a convex core – meaning a minimal convex subset – whose boundary is the union of two bent space-like surfaces, which can be determined as those with "minimal" bending around the boundary curves. In fact bending laminations for a bent surface in M may contain leaves spiralling around boundary components. In [15] a signed intersection (or *signed mass*) $m(c, \lambda)$ of the bending lamination λ with any boundary component c is defined, so that the sign determines the spiralling sense. If λ is a bending lamination of a surface that is a boundary component of a convex core in M associated with an earthquake between h_l and h_r, then for every boundary curve c the relation

$$|\ell_{h_l}(c) \pm 2m(c, \lambda)| = \ell_{h_r}(c)$$

holds, where $\pm$ depends on the past/future convexity of the surface. The upper bending lamination λ_+ of the upper boundary $\mathbb{S}_+$ of the convex core is determined by requiring that $2m(c, \lambda_+) < \ell_{h_l}(c)$, while the lower bending lamination λ_- verifies $m(c, \lambda_-) = -m(c, \lambda_+)$. In particular it turns out that the signed masses of the bending laminations of the boundary of a convex core are opposite.
We will prove the following theorem.

Theorem B. *Let λ and μ be two filling measured geodesic laminations on S, such that for any boundary component $c_1, \dots, c_{\mathfrak{n}}$ of S the corresponding signed masses $m(\lambda, c_i)$ and $m(\mu, c_i)$ are opposite. Let us fix a positive number $b_i > 2m(c_i, \lambda)$ for each $i = 1, \dots, \mathfrak{n}$. Then there exists (h_l, h_r) in $\mathcal{T}_S^\circ \times \mathcal{T}_S^\circ$ such that*

1. *λ and μ are the upper and lower bending laminations of the boundary of the convex core of the MBH manifold associated with (h_l, h_r);*

2. *the length of c_i with respect to h_l is b_i.*

An immediate remark is that, in the setting of multi-black holes, the spacetime M is not determined by the bending laminations: one must at least fix the boundary lengths. However, a dimensional computation shows that this requirement is not baseless: both $\mathcal{T}_S^\circ \times \mathcal{T}_S^\circ$ and the space of filling laminations $\mathcal{FML}_S^\circ$ have dimension $12g - 12 + 6\mathfrak{n}$, however the image of Φ (the analogue of the map in (2) with domain $\mathcal{T}_S^\circ \times \mathcal{T}_S^\circ$) is contained in the subspace $\mathcal{FML}_S^{(=)} \subset \mathcal{FML}_S^\circ$ of laminations having opposite masses at each boundary component. Thus, $\mathcal{FML}_S^{(=)}$ has dimension $12g - 12 + 5\mathfrak{n}$, making

it reasonable that the fibre of Φ has dimension $\mathfrak{n}$.
Fixed $\mathbf{b} = (b_1, \ldots, b_{\mathfrak{n}}) \in (\mathbb{R}_{>0})^{\mathfrak{n}}$, set

$$\mathcal{T}_S^\circ(\mathbf{b}) = \{h \in \mathcal{T}_S^\circ \mid \ell_h(c_i) = b_i, \text{ for } i = 1 \ldots, \mathfrak{n}\}$$
$$\mathcal{FML}_S^\circ(\mathbf{b}) = \{(\lambda, \mu) \in \mathcal{FML}_S^\circ \mid -m(c_i, 2\mu) = m(c_i, 2\lambda) < b_i \text{ for } i = 1, \ldots, \mathfrak{n}\}.$$

For $(\lambda, \mu) \in \mathcal{FML}_S^\circ(\mathbf{b}) \subset \mathcal{FML}_S^{(=)}$, the composition $E_l^\lambda \circ E_l^\mu$ preserves the space $\mathcal{T}_S^\circ(\mathbf{b})$. Using the relations between bending laminations and holonomies of a MBH spacetime, the result of Theorem B can be also expressed in terms of fixed points of $E_l^\lambda \circ E_l^\mu$.

Theorem. *If $(\lambda, \mu) \in \mathcal{FML}_S^\circ(\mathbf{b})$, then the composition of hyperbolic earthquakes $E_l^{t\lambda} \circ E_l^{t\mu} : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathcal{T}_S^\circ(\mathbf{b})$ admits a fixed point.*

The proof of Theorem B is carried out following the same steps as in the closed case: we prove first that small laminations can be uniquely realized as bending laminations of convex cores in MBH manifolds with assigned boundary lengths, then that the restriction

$$\Phi^{\mathbf{b}} = \Phi\big|_{\mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ} : \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ \to \mathcal{FML}_S^\circ(\mathbf{b})$$

is proper. Notice however that in order to prove those facts we cannot rely on the theory of length of laminations, because there is no simple extension of the notion of length in the case of spiralling leaves. Using a normalization procedure we are able to construct for any pair of measured laminations – under the condition that boundary masses are opposite – a function $\mathbb{L}_{(\lambda,\mu)} : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathbb{R}$ that satisfies the following properties.

1. $\mathbb{L}_{(\lambda,\mu)}$ is proper.
2. $\mathbb{L}_{(\lambda,\mu)}$ is convex along earthquake deformations on compactly supported laminations.
3. $\mathbb{L}_{(\lambda,\mu)}$ has positive definite Hessian in its critical points.
4. Using a natural Weil-Petersson symplectic form on $\mathcal{T}_S^\circ(\mathbf{b})$, the symplectic gradient of $\mathbb{L}_{(\lambda,\mu)}$ is the sum of the two infinitesimal earthquakes along λ and μ (that turns out to be tangent to $\mathcal{T}_S^\circ(\mathbf{b})$ by the condition on the masses).

Another tricky point in following the argument of the proof of Bonsante and Schlenker's Theorem is the computation of the main estimate, bounding the length (in the above sense) of the bending laminations (λ, μ) for the left metric h_l in terms of the intersection number. In [17] it was obtained studying a time-like section A of the convex core K; the region A was defined starting from a closed leaf γ of the bending locus of K, resulting as an annulus of finite area, with a boundary component isometric to γ and a piece-wise geodesic boundary component with well known angles at the vertices. In our context, however, the spiralling leaves of the laminations are not approximable by closed weighted curves, so we will spend a little more effort to get an analogous estimate for lengths of spiralling laminations.

Plan

In order to fix some notations and introduce the spaces and environments mostly considered in the following, in Chapter 1 we first recall well known formulas about the upper half-plane model and the hyperboloid model of $\mathbb{H}^2$, and define the Teichmüller space $\mathcal{T}^\star_S(\mathfrak{C})$ of hyperbolic metrics on a ciliated surface (S, Q) and the Teichmüller space $\mathcal{T}^\circ_S$ of hyperbolic metrics on a punctured surface S with closed geodesic boundary (Section 1.1). Definitions and basic properties regarding measured geodesic laminations and hyperbolic earthquakes are shown in Section 1.2. In particular, we will describe the topology of the spaces of measured laminations on ciliated surfaces and on hyperbolic surfaces with closed geodesic boundary. Finally, Section 1.3 deals with AdS_3 structures and convex subsets associated with earthquakes, with a detailed description of achronal meridians in the asymptotic boundary of AdS_3.

In Chapter 2, after showing the correspondence between complex structures on a ciliated surface and hyperbolic metrics of finite area with totally geodesic boundary (Section 2.1), we characterize convex subsets associated with earthquakes between ciliated surfaces (Section 2.2). The last part of the chapter applies the constructions to ideal polygons in $\mathbb{H}^2$.

The first section of Chapter 3 recalls the structure of measured laminations on hyperbolic surfaces with closed geodesic boundary. Section 3.2 is devoted to the definition of a length map $\mathbb{L}_{(\lambda,\mu)}$ for $(\lambda, \mu) \in \mathcal{FML}^\circ_S(\mathbf{b})$ which is a Hamiltonian for the sum of the two infinitesimal earthquakes along λ and μ, with respect to a Weil-Petersson symplectic form on $\mathcal{T}^\circ_S(\mathbf{b})$. The computation of the first order variation of $\mathbb{L}_{(\lambda,\mu)}$ along earthquakes with compactly supported fault locus will show that this condition holds. In Section 3.3 we verify that $\mathbb{L}_{(\lambda,\mu)} : \mathcal{T}^\circ_S(\mathbf{b}) \to \mathbb{R}$ is a proper map and with positive definite Hessian at its critical point (which is unique), as required.

The estimate (analogous to the main one in [17]) that allows to bound $\mathbb{L}_{(\lambda,\mu)}(h)$ in terms of the intersection between λ and μ, if they are bending laminations of a convex core associated with $(h, h') \in \mathcal{T}^\circ_S(\mathbf{b}) \times \mathcal{T}^\circ_S$, is computed in Section 3.4. As outlined before, the presence of spiralling leaves in the laminations requires some specific modifications in the argument for the compactly supported bending laminations. Because of such adjustments, the proof in Section 3.5 of the properness of $\Phi^{\mathbf{b}} : \mathcal{T}^\circ_S(\mathbf{b}) \times \mathcal{T}^\circ_S \to \mathcal{FML}^\circ_S(\mathbf{b})$ will turn up more technical than in the closed case. Again, this implies that

the topological degree of $\Phi^{\mathbf{b}}$ is defined.
Finally, Section 3.6 concludes with the proof of Theorem B, following the argument in [17]: showing the existence of preimages of small laminations in $\mathcal{FML}_S^\circ(\mathbf{b})$, we will deduce that $\deg \Phi^{\mathbf{b}} = 1$, so that $\Phi^{\mathbf{b}}$ is surjective.

Chapter 1

Preliminaries

1.1 Models of $\mathbb{H}^2$

Several formulas occuring in this work will be computed on the hyperbolic plane $\mathbb{H}^2$, namely in the upper half-plane model either in the hyperboloid model. First we therefore want to fix some notations and introduce general formulas related to such models. The Poincaré disk model will be only used in some figures, in order to make easier the visualization of certain ideas. In the last subsection we introduce the Teichmüller spaces we will consider in Chapters 2 and 3.

1.1.1 The upper half-plane model

The upper half-plane $H_+ = \{z \in \mathbb{C} \,|\, \Im z > 0\}$ endowed with the metric $(\Im z)^{-2}|dz|^2$ is one of the most used model of $\mathbb{H}^2$. See [8] for details. The formula for the distance between points is the following:

$$\tanh d(z,w) = \left|\frac{z-w}{z-\overline{w}}\right|.$$

The geodesics of H_+ are supported by vertical rays and semicirles with endpoints in $\mathbb{R}$. The boundary at infinity $\partial\mathbb{H}^2$ of $\mathbb{H}^2$ is clearly visualizable here as $\mathbb{R} \cup \{\infty\}$, thus compactifying H_+.
An isometry on H_+ has the form

$$z \mapsto \frac{az+b}{cz+d} \quad \text{with} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2).$$

We will widely use this form when computations are based on a fixed hyperbolic isometry. Recall that a hyperbolic isometry A has a fixed repulsive point in $\partial\mathbb{H}^2$, denoted by $\mathrm{x}^-(A)$, and an attractive one, denoted by $\mathrm{x}^+(A)$. The *axis* $\mathrm{ax}(A)$ of A is the geodesic going from $\mathrm{x}^-(A)$ to $\mathrm{x}^+(A)$. The positive number $\mathrm{T}(A)$ such that every point $p \in \mathrm{ax}(A)$ is sent by A to the point $A(p) \in \mathrm{ax}(A)$ distant $\mathrm{T}(A)$ from p is called *translation length* of A. If $M_A \in PSL(2,\mathbb{R})$ is a matrix representing A, we have

$$\cosh \frac{\mathrm{T}(A)}{2} = \frac{|\operatorname{tr}(M_A)|}{2} > 1.$$

We will use the upper half-plane model when the computations involve a fixed hyperbolic isometry A: it is always possible in fact to conjugate A with an isometry B so that $BAB^{-1} : z \mapsto e^{\mathrm{T}(A)} z$. Therefore, a change of coordinate will allow us to suppose that A has a very simple form.

1.1.2 The hyperboloid model

Another useful model of $\mathbb{H}^2$ that will allow us to make easier computations is the hyperboloid model. Let $\mathbb{R}^{2,1}$ be $\mathbb{R}^3$ endowed with the scalar product

$$\langle \underline{x}, \underline{y} \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2.$$

An element $\underline{x} \in \mathbb{R}^{1,2}$ is called time-like vector if $\langle \underline{x}, \underline{x} \rangle < 0$, light-like vector if $\langle \underline{x}, \underline{x} \rangle = 0$ and space-like vector if $\langle \underline{x}, \underline{x} \rangle > 0$. Define $\|\underline{x}\|_{2,1} = \sqrt{|\langle \underline{x}, \underline{x} \rangle|}$. Consider the hyperboloid of equation $\langle \underline{x}, \underline{x} \rangle = -1$ and take its connected component

$$\mathbb{I} = \{\underline{x} \in \mathbb{R}^{2,1} \mid \langle \underline{x}, \underline{x} \rangle = -1, x_0 > 0\}$$

with the inherited metric, which has constant curvature -1. Its isometries are the elements preserving $\mathbb{I}$ of

$$SO(2,1) = \{A \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) \mid A^T J_3 A = J_3\}$$

where

$$J_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Mat}_{3\times 3}(\mathbb{R}).$$

A vector v is tangent to $\mathbb{I}$ at $\underline{x}$ if and only if $\langle \underline{x}, v \rangle = 0$. Geodesics are intersections of linear subspaces of $\mathbb{R}^{2,1}$ with $\mathbb{I}$, namely for every geodesic γ there is a space-like vector n such that

$$\gamma = \{\underline{x} \in \mathbb{I} \mid \langle \underline{x}, n \rangle = 0\}.$$

The unitary vector normal to γ (pointing towards the same half-plane bounded by γ) is thus constant along γ.
The boundary at infinity of $\mathbb{I}$ is identified with

$$\partial_\infty \mathbb{I} = {}^{\displaystyle\{\underline{x} \in \mathbb{R}^{2,1} \mid \langle \underline{x}, \underline{x} \rangle = 0\}} \Big/ {}_{\displaystyle \underline{x} \sim a\underline{x}, a \in \mathbb{R}^*}$$

and its elements will be written within square brackets. The compactification of $\mathbb{I}$ is actually $\mathbb{I} \cup \partial_\infty \mathbb{I}$: a sequence $\underline{x}_n \in \mathbb{I}$ tends to $[\underline{z}] \in \partial_\infty \mathbb{I}$ if and only if the class of $\underline{x}_n$ in $\mathbb{R}P^3$ tends to the class of $\underline{z}$. See also [8].
The following formulas will be useful in Chapter 3.

1. Denoting by d the hyperbolic distance on $\mathbb{I}$, for every $\underline{x}, \underline{y} \in \mathbb{I}$
$$\cosh d(\underline{x}, \underline{y}) = -\langle \underline{x}, \underline{y} \rangle.$$

2. If $v \in T_{\underline{x}}\mathbb{I}$ has unitary norm, then the geodesic through $\underline{x}$ with tangent vector v at $\underline{x}$ is parametrized by
$$\mathbb{R} \ni s \mapsto (\cosh s)\underline{x} + (\sinh s)v \in \mathbb{I}.$$

3. If a geodesic has normal unitary vector n and ideal endpoints $[z_1]$ and $[z_2]$ then it is parametrized by
$$\mathbb{R} \ni s \mapsto \frac{e^{-s} z_1 + e^{s} z_2}{-2\langle z_1, z_2 \rangle} \in \mathbb{I};$$
moreover,
$$\langle n, z_1 \rangle = \langle n, z_2 \rangle = 0.$$

4. for every $\underline{x} \in \mathbb{I}$ and every geodesic γ, if n is a unitary vector n normal to γ pointing towards $\underline{x}$ then
$$\sinh d(\underline{x}, \gamma) = \langle \underline{x}, n \rangle.$$

The cross product in $\mathbb{R}^{2,1}$

There is a notion of cross product in $\mathbb{R}^{2,1}$, analogous to the Euclidean environment: if $\mathrm{d}V$ denotes the volume form in $\mathbb{R}^{2,1}$, the cross product between $\underline{x} \in \mathbb{R}^{2,1}$ and $\underline{y} \in \mathbb{R}^{2,1}$ is the vector $\underline{x} \boxtimes \underline{y} \in \mathbb{R}^{2,1}$ such that for every $\underline{z} \in \mathbb{R}^{2,1}$
$$\langle \underline{x} \boxtimes \underline{y}, \underline{z} \rangle = \mathrm{d}V(\underline{x}, \underline{y}, \underline{z}).$$

An explicit expression of the cross product is
$$\underline{x} \boxtimes \underline{y} = \begin{pmatrix} x_2 y_1 - x_1 y_2 \\ x_2 y_0 - x_0 y_2 \\ x_0 y_1 - x_1 y_0 \end{pmatrix}.$$

The following are basic formulas of the cross product:
$$\begin{gathered}
\underline{x} \boxtimes \underline{y} = -\underline{y} \boxtimes \underline{x} \\
\underline{x} \boxtimes \underline{x} = 0 \\
\langle \underline{x}, \underline{x} \boxtimes \underline{y} \rangle = 0 \\
\langle \underline{x}, \underline{y} \boxtimes \underline{z} \rangle = \langle \underline{z}, \underline{x} \boxtimes \underline{y} \rangle \\
(\underline{x} \boxtimes \underline{y}) \boxtimes \underline{z} = \langle \underline{y}, \underline{z} \rangle \underline{x} - \langle \underline{x}, \underline{z} \rangle \underline{y} \\
\langle \underline{x} \boxtimes \underline{y}, \underline{x} \boxtimes \underline{y} \rangle = \langle \underline{x}, \underline{y} \rangle^2 - \langle \underline{x}, \underline{x} \rangle \langle \underline{y}, \underline{y} \rangle
\end{gathered}$$
for every $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^{2,1}$.

Remark 1.1.1. If γ is a geodesic in $\mathbb{I}$ passing through $\underline{x}$ with tangent vector v, then $\underline{x} \boxtimes v$ lies in $T_{\underline{x}}\mathbb{I}$, being normal to $\underline{x}$, and is orthogonal to γ, being normal also to v. Moreover,

$$\langle \underline{x} \boxtimes v, \underline{x} \boxtimes v\rangle = \langle \underline{x}, v\rangle^2 - \langle \underline{x}, \underline{x}\rangle\langle v, v\rangle = 1,$$

so $\underline{x}\boxtimes v$ is one of the two unitary vectors normal to γ. Since $\mathrm{d}V(\underline{x}, v, \underline{x}\boxtimes v)$ is equal to 1, $\underline{x} \boxtimes v$ is the one such that $(\underline{x}, v, \underline{x} \boxtimes v)$ is an orthonormal positive basis of $\mathbb{R}^{2,1}$. Actually, the map $v \mapsto \underline{x} \boxtimes v$ from $T_{\underline{x}}\mathbb{I}$ to itself corresponds to the complex structure induced on $\mathbb{I}$.

1.1.3 Teichmüller spaces

Fix once and for all a topological surface S obtained by removing $\mathfrak{n}$ points $\mathfrak{p}_1, \dots, \mathfrak{p}_{\mathfrak{n}}$ (called punctures of S) from a compact surface of genus g, and suppose that the Euler characteristic $\chi(S) = 2 - 2g - \mathfrak{n}$ of S is negative.
In this work, subspaces of the set of the hyperbolic metrics on S having the following properties will be considered:

1. the universal covering is isometric to an open subset $\mathcal{H}$ of $\mathbb{H}^2$ with geodesic boundary;

2. the area of S is finite.

Classically, the most studied subspace of this set is the space of complete

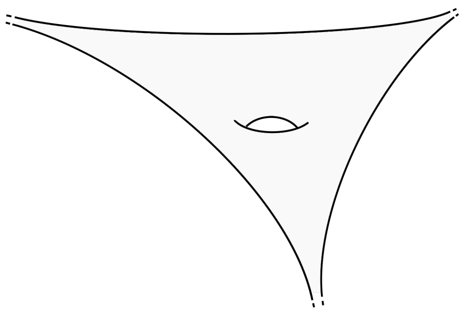


Figure 1.1

hyperbolic metrics on S of finite area, that we will denote by $\mathrm{Met}_{-1}(S)$. For its elements, at the $\mathfrak{n}$ punctures of S only cusps occur (see Figure 1.1) and the universal covering is the whole $\mathbb{H}^2$. Actually the space considered is

$$\mathcal{T}_S = \mathrm{Met}_{-1}(S)/ \mathrm{Diff}_0(S), \tag{1.1}$$

where the action on $\mathrm{Met}_{-1}(S)$ of the group $\mathrm{Diff}_0(S)$ of homotopically trivial diffeomorphisms of S is quite obvious.

Definition 1.1.1. A surface $\mathcal{W}$, homeomorphic for a certain $r \in (0,1)$ to the one obtained by removing from the closed annulus

$$\overline{A}(0,r,1) = \{z \in \mathbb{C} \ : \ r \leq |z| \leq 1\}$$

a finite subset $\{z_1, \dots, z_J\}$ of $\{z \in \mathbb{C} \ : |z| = 1\}$, provided with a complete hyperbolic metric with geodesic boundary and finite area, is called crown. The points at infinity corresponding to $z_1, \dots, z_J$ are called tips of $\mathcal{W}$.

Consider S now as a surface topologically obtained by removing $\mathfrak{n}$ closed mutually disjoint disks $\Delta_1, \dots, \Delta_{\mathfrak{n}}$ from a closed surface of genus g so that $2 - 2g - \mathfrak{n} < 0$. Fix once and for all a finite subset $\mathfrak{C}$ of $\bigcup \partial\Delta_i$, such that $\mathfrak{C} \cap \partial\Delta_i \neq \emptyset$ for every $i = 1, \dots, \mathfrak{n}$. Denote by $\mathrm{Met}^{\star}_{-1}(S, \mathfrak{C})$ the set of hyperbolic metrics on S of finite area whose completion $\overline{S}$ is topologically $S \cup \left(\bigcup \partial\Delta_i\right) \smallsetminus \mathfrak{C}$ and so that each boundary component is totally geodesic. In Chapter 2 we will focus on the space

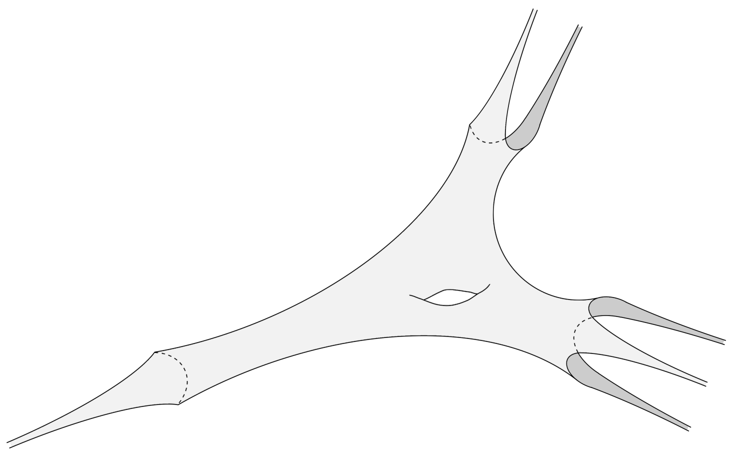


Figure 1.2

$$\mathcal{T}^{\star}_S(\mathfrak{C}) = \mathrm{Met}^{\star}_{-1}(S, \mathfrak{C}) / \mathrm{Diff}_0(S|\mathfrak{C}). \tag{1.2}$$

Here $\mathrm{Diff}_0(S|\mathfrak{C})$ denotes the identity component of the subgroup of elements $F \in \mathrm{Diff}_0(S)$ extendible to homeomorphisms of $\overline{S} \cup \mathfrak{C}$ that pointwise fix $\mathfrak{C}$. Notice that for every $h \in \mathcal{T}^{\star}_S(\mathfrak{C})$ the surface (S, h) has $\mathfrak{n}$ crown-shaped punctures, as in Figure 1.2. Moreover, a Dehn twist along a peripheral loop (see for instance [35]) is not an element of $\mathrm{Diff}_0(S|\mathfrak{C})$.

On the other hand, fix once and for all $\mathbf{b} = (b_1, \dots, b_{\mathfrak{n}}) \in (\mathbb{R}_{>0})^{\mathfrak{n}}$. Let $\mathrm{Met}^{\circ}_{-1}(S)$ be the set of hyperbolic metrics on S of finite area where $\overline{S}$ is complete and has $\mathfrak{n}$ closed geodesic boundary components. In Chapter 3 we will consider the spaces $\mathcal{T}_S^\circ = \mathrm{Met}_{-1}^\circ(S)/\operatorname{Diff}_0(\overline{S})$ and

$$\mathcal{T}_S^\circ(\mathbf{b}) = \{h \in \mathrm{Met}_{-1}^\circ(S) \,|\, \text{each puncture } \mathfrak{p}_i \text{ corresponds to a closed geodesic boundary component of } \overline{S} \text{ of } h\text{-length } b_i\}/\operatorname{Diff}_0(\overline{S}). \tag{1.3}$$

Here the $\mathfrak{n}$ boundary components of S are homeomorphic to S^1 (see Figure

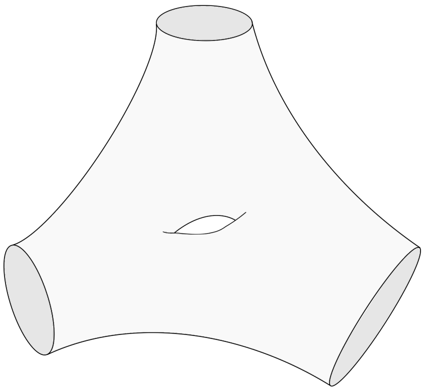


Figure 1.3

1.3).

1.2 Measured laminations and earthquakes

1.2.1 The space of measured laminations

Definition 1.2.1. Given a hyperbolic metric h on a surface S, a geodesic lamination on (S,h) is the data λ of a family of mutually disjoint complete simple geodesics (called the leaves of λ) whose union is a closed subset (called the support of λ and denoted by $\mathrm{supp}(\lambda)$) of S. Every connected component of $S \smallsetminus \lambda$ is called gap or plaque; a stratum of λ is either a leaf or a gap. A measured geodesic lamination of S is the data of a geodesic lamination λ and a transverse measure meas_λ, that is a measure defined on the arcs on S transverse to each leaf of λ and with endpoints in $S \smallsetminus \mathrm{supp}(\lambda)$ such that

- $\mathrm{meas}_\lambda(c) \neq 0$ if and only if $c \cap \mathrm{supp}(\lambda) \neq \emptyset$;
- if there exists an isotopy between two arcs c_1 and c_2 realized through arcs transverse to λ then $\mathrm{meas}_\lambda(c_1) = \mathrm{meas}_\lambda(c_2)$.

It is known (see [19]) that the Lebesgue measure of the support of a geodesic lamination is zero.

Example 1.2.1. Weighted multicurves are the simplest examples of measured geodesic lamination on S. The support is the finite union of simple closed mutually disjoint non trivial geodesics γ_i. Chosen real positive numbers ω_i (called weights) respectively assigned to γ_i, the transverse measure is given by

$$c \mapsto \sum_i \omega_i \cdot \#(\gamma_i \cap c)$$

for any arc c transverse to $\bigcup \gamma_i$.

If $h \in \mathcal{T}_S$ (see (1.1) in Subsection 1.1.3) then any measured geodesic lamination λ on (S, h) has a maximal compact sublamination $\lambda^{(0)}$, in the sense that if μ is a sublamination of λ with compact support in S then μ is a sublamination of $\lambda^{(0)}$ too. Each leaf of $\mathrm{supp}(\lambda) \smallsetminus \mathrm{supp}(\lambda^{(0)})$ is homeomorphic

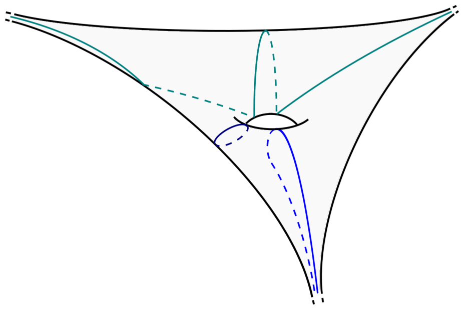


Figure 1.4: A measured geodesic lamination in $\mathcal{ML}_S$ with two leaves homeomorphic to $\mathbb{R}$ and a closed leaf

to $\mathbb{R}$ and, roughly speaking, its ideal endpoints are in the points at infinity of the cusps (see Figure 1.4).
If we denote by $\mathcal{ML}_{(S,h)}$ the measured geodesic laminations on (S, h) with $h \in \mathcal{T}_S$, being a space of measures it seems natural to provide it with the topology of the weak-convergence of measures (sometimes also called weak*-convergence). It is known (see Section 1.7 of [31]) that for every h_1, h_2 in $\mathcal{T}_S$ there is a homeomorphism $F : \mathcal{ML}_{(S,h_1)} \to \mathcal{ML}_{(S,h_2)}$ so that, roughly speaking, $\mathrm{supp}(F(\lambda))$ is obtained straightening with respect to h_2 the leaves of $\mathrm{supp}(\lambda)$. This suggests that it makes sense to associate $\mathcal{T}_S$ with the space $\mathcal{ML}_S$ of measured laminations, whose support is only a topological data; this space inherits the weak convergence topology. Finally, define

$$\mathcal{CML}_S = \{\lambda \in \mathcal{ML}_S \mid \lambda = \lambda^{(0)}\},$$

the space of laminations with compact support. The following theorem is a well known result (see [31]).

Theorem 1.2.1. *The space of weighted multicurves on S is dense in $\mathcal{CML}_S$.*

We can analogously associate $\mathcal{T}^\star_S(\mathfrak{C})$ and $\mathcal{T}^\circ_S$ (see (1.2) and (1.3) in Subsection 1.1.3) with the relative spaces of measured laminations.

The space $\mathcal{ML}^\star_S(\mathfrak{C})$

Let us start from $h \in \mathcal{T}^\star_S(\mathfrak{C})$. Any element λ of $\mathcal{ML}^\star_{(S,h)}(\mathfrak{C})$ has again a maximal compact sublamination $\lambda^{(0)}$. The leaves in $\mathrm{supp}(\lambda) \smallsetminus \mathrm{supp}(\lambda^{(0)})$ are homeomorphic to $\mathbb{R}$ and, roughly speaking, their ideal endpoints are in the tips of the crowns (see Figure 1.5). For every $h_1, h_2 \in \mathcal{T}^\star_S(\mathfrak{C})$ there exists the analogous homeomorphism between $\mathcal{ML}^\star_{(S,h_1)}(\mathfrak{C})$ and $\mathcal{ML}^\star_{(S,h_2)}(\mathfrak{C})$, so again

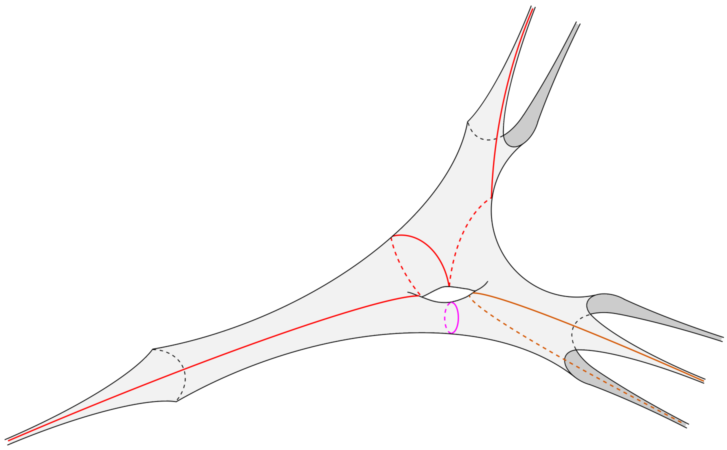


Figure 1.5: A measured geodesic lamination in $\mathcal{ML}^\star_S(\mathfrak{C})$ with two leaves homeomorphic to $\mathbb{R}$ and a closed leaf

we can associate $\mathcal{T}^\star_S(\mathfrak{C})$ with the space $\mathcal{ML}^\star_S(\mathfrak{C})$ of measured laminations and the space $\mathcal{CML}^\star_S(\mathfrak{C})$ of measured laminations with compact support.
In order to give to this space a proper topology, pick $h \in \mathcal{T}^\star_S(\mathfrak{C})$ and denote by $2S$ the surface obtained glueing two copies S_+, S_- of the h-completion of S through the identification of their boundaries isomorphic to $\mathbb{R}$, which are geodesic; this fact assures that the double (S^d, h^d) of (S,h) is a hyperbolic surface of genus $2g$ with $\#\mathfrak{C}$ punctures. Also, $h^d \in \mathcal{T}_{S^d}$ and (S^d, h^d) has finite area (twice the area of S).
More precisely, consider $\mathrm{id}_\pm : S \to S_\pm$; then

$$S^d = \left(S_+ \coprod S_-\right)/\sim \tag{1.4}$$

where

$$x \sim y \iff x = y \text{ or } \mathrm{id}_+^{-1}(x) = \mathrm{id}_-^{-1}(y) \in \partial S.$$

Consider the maps $i_\pm : S \to S^d$ defined by $i_\pm(x) = [\mathrm{id}_\pm(x)]_\sim$. The metric h^d is the one such that $(i_\pm)^*(h^d) = h$. We can define the involution map $\mathcal{I} : S^d \to S^d$ such that $\mathcal{I}(i_\pm(x)) = i_\mp(x)$; roughly speaking, $\mathcal{I}$ interchanges S_+ and S_-. Note that $\mathcal{I} \in \mathrm{Isom}(S^d)$.
In order to give a topology to $\mathcal{ML}^\star_S(\mathfrak{C})$, consider the map M from $\mathcal{ML}^\star_S(\mathfrak{C})$ to $\mathcal{ML}_{S^d}$ that associates every $\lambda \in \mathcal{ML}^\star_S(\mathfrak{C})$ with the doubled lamination

λ^d in $\mathcal{ML}_{S^d}$; it is injective and its image is strictly contained in the subspace of $\mathcal{ML}_{S^d}$ of $\mathcal{I}$-invariant measured laminations. We equip $\mathcal{ML}^\star_S(\mathfrak{C})$ with the unique topology such that the map $M : \mathcal{ML}^\star_S(\mathfrak{C}) \to M(\mathcal{ML}^\star_S(\mathfrak{C}))$ is a homeomorphism onto its image.

The space $\mathcal{ML}^\circ_S$

Let us fix for a moment $h \in \mathcal{T}^\circ_S$ and consider a measured geodesic lamination λ on (S, h). If a leaf of λ is not contained contained in a compact subset of S, then, in order to be a complete geodesic with no self-intersections, it must spiral along one or two connected components of ∂S. There are two possible senses of spiralization, as shown in Figure1.7.
Fix a ε-collar $N_\varepsilon(\partial)$ of a boundary component ∂ such that every leaf of λ intersecting $N_\varepsilon(\partial)$ spirals near ∂ (see Lemma 3.2.3 for the existence of ε). If a leaf δ of λ spirals near ∂ and $\tilde\partial$ is a lift of ∂ in the boundary of the universal

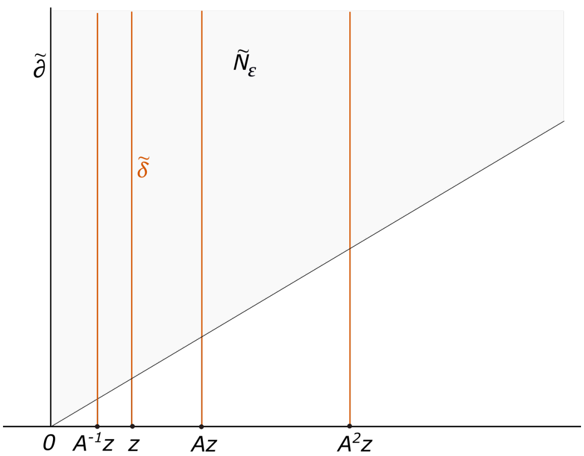


Figure 1.6

covering $\mathcal{H}$ of (S, h), let A be the generator of $\mathrm{Stab}(\tilde\partial) < \mathrm{hol}(\pi_1(S))$, where hol is the holonomy of h. The liftings of δ meeting the ε-collar $\tilde N_\varepsilon$ of $\tilde\partial$ share an ideal endpoint with $\tilde\partial$ (in Figure 1.6 is ∞) and their union coincides with the $\langle A\rangle$-orbit of $\tilde\delta$, one of such lifts of δ. The ideal endpoint that $\tilde\partial$ has in common with those liftings of δ depends on the sense of spiralization of δ itself near ∂.

It is possible to define the *mass* $\iota(\partial, \lambda)$ of ∂ with respect to λ, a positive number that indicates how much the measure of λ is concentrated near ∂. It is constructed as follows. For every $x \in N_\varepsilon(\partial)$ denote by c_x the loop with vertex at x parallel at ∂ such that $c_x \setminus \{x\}$ is an open geodesic arc. Since $\mathrm{meas}_\lambda(c_x) = \mathrm{meas}_\lambda(c_y)$ for every $x, y \in N_\varepsilon$, as shown in [17], Subsection 2.3, it is well defined the mass $\iota(\partial, \lambda) = \text{meas}_\lambda(c_x)$. Moreover, $\iota(\partial, \lambda) = 0$ if and only if $\text{supp}(\lambda) \cap N_\varepsilon = \emptyset$. The mass of ∂ does not take in account in which sense λ spirals. Fix once for all an orientation of ∂S. Such choice defines a positive and a negative sense of spiralization around ∂, as in Figure 1.7. It

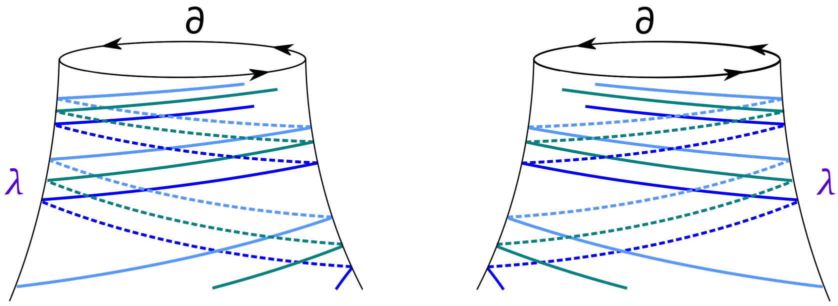


Figure 1.7: Respectively, positive and negative sense of spiralling

is now possible to define the *signed mass* $m(\partial, \lambda)$ of ∂ with respect to λ as

$$m(\partial, \lambda) = \begin{cases} +\iota(\partial, \lambda) \text{if } \lambda \text{ spirals in the positive sense around } \partial \\ -\iota(\partial, \lambda) \text{if } \lambda \text{ spirals in the negative sense around } \partial \end{cases}. \tag{1.5}$$

Remark 1.2.1. The signed mass of ∂ with respect to λ is positive (respectively negative) if and only if for every oriented lift of ∂ on $\mathcal{H}$ its ending (respectively starting) ideal endpoint is contained in the set of the ideal points of the whole preimage of λ.

Now take $h \in \mathcal{T}_S^\circ$. Any element λ of $\mathcal{ML}^\circ_{(S,h)}$ has again a maximal compact sublamination $\lambda^{(0)}$. The leaves in $\text{supp}(\lambda) \setminus \text{supp}(\lambda^{(0)})$ are homeomorphic to $\mathbb{R}$ and spiral near two boundary components (possibly coincident) of S (see Figure 1.8). They cannot spiral near a boundary component in one direction and stay in the compact part in the other direction, since geodesic laminations spiralling inside the surface cannot carry a transverse measure. See also [17]. As before, we can associate $\mathcal{T}_S^\circ$ with the space $\mathcal{ML}_S^\circ$ of measured laminations and the subspace $\mathcal{CML}_S^\circ$ of measured laminations with compact support.

Now we are going to give to $\mathcal{ML}_S^\circ$ a manifold structure. First let us introduce the straightening ν^R of a measured lamination $\nu \in \mathcal{ML}_S^\circ$. If γ is a spiralling geodesic between two connected components ∂_i and ∂_j of ∂S, consider its preimage Γ on the universal cover $\mathcal{H} \subset \mathbb{H}^2$. Every connected component of Γ is a geodesic $\tilde{\gamma}$ with endpoints in the (ideal closure) of certain lifts $\tilde{\partial}_i$ and $\tilde{\partial}_j$ of ∂_i and ∂_j respectively. If we replace each $\tilde{\gamma}$ with the geodesic arc $\tilde{\gamma}^R$ with endpoints on $\tilde{\partial}_i$ and $\tilde{\partial}_j$ perpendicular to $\tilde{\partial}_i$ and $\tilde{\partial}_j$ and we project $\tilde{\gamma}^R$ on S, we obtain a geodesic arc γ^R on S normal to ∂_i and ∂_j with endpoints on ∂_i and ∂_j. For each $\nu \in \mathcal{ML}_S^\circ$ denote by ν^R the set of geodesic (weighted) arcs obtained by ν replacing each spiralling geodesic γ

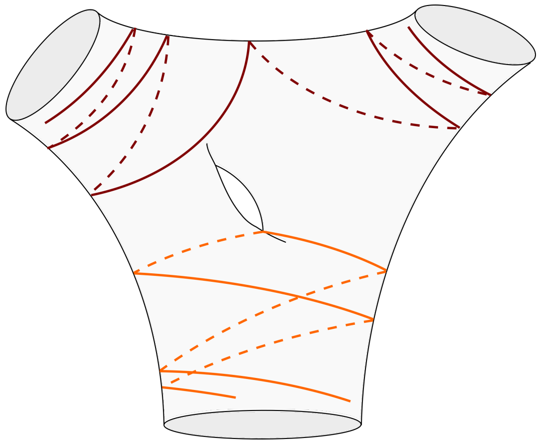


Figure 1.8: A measured geodesic lamination in $\mathcal{ML}_S^\circ$ with two spiralling leaves

of ν with γ^R.
Consider the set $\{\nu^R \,|\, \nu \in \mathcal{ML}_S^\circ\}$. This space is a submanifold of the space of measured laminations (that we denote by $\mathcal{ML}_S^\#$) studied in [1]; we will mention only the necessary details. Using the notations of [1], fixed a pant decomposition

$$P = \{C_1, \ldots, C_{3g-3+\mathfrak{n}}, B_1 = \partial_1, \ldots, B_\mathfrak{n} = \partial_\mathfrak{n}\}$$

of S with internal curves $C_1, \ldots, C_{3g-3+\mathfrak{n}}$ and boundary curves $B_1 = \partial_1, \ldots,$ $B_\mathfrak{n} = \partial_\mathfrak{n}$, every lamination $\sigma \in \mathcal{ML}_S^\#$ has coordinates

$$\big(DT(\sigma, C_1), \ldots, DT(\sigma, C_{3g-3+\mathfrak{n}}), \hat\theta(\sigma, B_1), \ldots, \hat\theta(\sigma, B_\mathfrak{n})\big)$$

where $DT(\sigma, C_i) \in \mathbb{R}^2$ depends on the behaviour of σ with respect to the internal decomposition curves C_i and $\hat\theta(\sigma, \partial_i) \in \mathbb{R}$ depends on the behaviour with respect to the boundary component ∂_i. Following their constructions, it turns out that, for every $\nu \in \mathcal{ML}_S^\circ$, $\hat\theta(\nu^R, \partial_i) = \iota(\nu, \partial_i) \geq 0$. So if we consider the coordinates $\Theta_P : \mathcal{ML}^\circ \to \mathbb{R}^{6g-6+3\mathfrak{n}}$ such that

$$\Theta_P(\nu) = \big(DT(\nu^R, C_1), \ldots, DT(\nu^R, C_{3g-3+\mathfrak{n}}), m(\nu, \partial_1), \ldots, m(\nu, \partial_\mathfrak{n})\big) \quad (1.6)$$

for $\nu \in \mathcal{ML}_S^\circ$, where $m(\nu, \partial_i)$ is the signed mass defined by (1.5), we provide $\mathcal{ML}_S^\circ$ with a manifold structure. Such coordinates depend on the pant decomposition P; however, if P' is another pant decomposition, notice that the last $\mathfrak{n}$ coordinates does not depend on the pant decomposition, whereas applying the results in [1] the change of coordinates of the other components is smooth.
Even if the projection $\mathcal{ML}_S^\circ \to \mathcal{ML}_S^R$ is not injective, the map Θ_P is injective, since we have avoided the ambiguity given by the spiralling senses

around ∂S.
For every $\lambda, \mu \in \mathcal{ML}^\circ_S$ we can now define their intersection number $\iota(\lambda, \mu)$ as half the intersection number between the double of λ^R and the double of μ^R in the double S. For the definition of the intersection number between two measured laminations in a closed surface, see [30]. If λ_i is a ω_i-weighted curve c_i on a closed surface Σ for $i = 1, 2$, then $\iota(\lambda_1, \lambda_2) = \omega_1\omega_2\#(c_1 \cap c_2)$. The extension of ι to multicurves on Σ is quite obvious, while the general case requires a little more attention.
The topology on $\mathcal{ML}^\#_S$ actually coincides with the topology of the weak*-convergence of measures. We are interested to show that also for $\mathcal{ML}^\circ_S$ the topology is the one of weak*-convergence of measures.

Lemma 1.2.2. *Consider a sequence λ_n converging to λ. If $\lambda^{[s]}$ is the sublamination of λ made by spiralling leaves, then the support of $\lambda^{[s]}$ is contained in λ_n for n sufficiently big. In particular, there exist decompositions*

$$\begin{aligned}\lambda_n &= \lambda_n^{[c]} \cup \lambda_n^{[s]} \cup \lambda_n^{[v]},\\ \lambda &= \lambda^{[cc]} \cup \lambda^{[s]} \cup \lambda^{[cv]}\end{aligned}$$

such that, up to passing to a subsequence,

- *$\lambda_n^{[c]}$ is the maximal compact sublamination of λ_n, and $\lambda_n^{[c]}$ converges to $\lambda^{[cc]}$;*
- *$\lambda^{[s]}$ is the sublamination of λ whose support consists of the spiralling leaves of λ, and $\lambda_n^{[s]}$ is the maximal sublamination of λ_n such that $\mathrm{supp}(\lambda_n^{[s]}) = \mathrm{supp}(\lambda^{[s]})$; moreover, $\lambda_n^{[s]}$ tends to $\lambda^{[s]}$;*
- *$\lambda_n^{[v]}$ is the complementary of $\lambda_n^{[s]}$ in the spiralling part of λ_n, so that $\lambda_n^{[v]}$ converges to the compact lamination $\lambda^{[cv]}$.*

Proof. A sequence ν_n converges to ν in $\mathcal{ML}^\circ_S$ when $\Theta(\nu_n)$ (see (1.6)) tends to $\Theta(\nu)$, i.e. when ν_n^R converges to ν^R in $\mathcal{ML}^\#_S$ and $\mathrm{sign}(m(\mathfrak{n}_n, \partial_i)$ converges to $\mathrm{sign}(m(\nu, \partial_i))$ for every $i = 1, \ldots, \mathfrak{n}$. The convergence of λ_n^R to λ^R in $\mathcal{ML}^\#_S$ is equivalent, as shown in [1], to the weak*-convergence of Λ_n, the double of λ_n^R, to Λ, the double of λ^R, in $\mathcal{ML}_{2S}$. We say that Λ_n is the doubled straightening of λ_n and Λ is the doubled straightening of λ. Analogously define the double straightenings $\Lambda_n^{[s]}$, $\Lambda_n^{[v]}$, $\Lambda^{[s]}$ and $\Lambda^{[cv]}$; every leaves of them inherits the weight of their corresponding ones in respectively $\lambda_n^{[s]}$, $\lambda_n^{[v]}$, $\lambda^{[s]}$ and $\lambda^{[cv]}$.
First let us show that any leaf of $\Lambda^{[s]}$ is contained in λ_n for big n. Consider a leaf l of $\lambda^{[s]}$, going say between the boundary components ∂_i and ∂_j of S. On the universal covering $\mathcal{H} \subset \mathbb{H}^2$ of S, consider a lift $\tilde{l}$ of l, going from $\tilde{\partial}_i$ and $\tilde{\partial}_j$, the boundary components of $\partial\mathcal{H}$ who projects onto ∂_i and ∂_j respectively. The straightening $\tilde{l}^R$ of $\tilde{l}$ has an endpoint $z_i \in \tilde{\partial}_i$. There is

a δ-neighbourhood D of $\tilde{l}^R$ in $\mathcal{H}$ such that for every $z \in (\overline{D} \cap \tilde{\partial}_i) \smallsetminus \{z_i\}$ the complete geodesic of $\mathbb{H}^2$ normal to $\tilde{\partial}_1$ passing through z must intersect $\tilde{\partial}_j$, but this intersection cannot be orthogonal, so if a lamination $\nu \in \mathcal{ML}_S^{\#}$ meets $D \cap \tilde{\partial}_i$, then it must contain the leaf l. Thus, leaves of $\Lambda^{[s]}$ must be contained in $\Lambda_n^{[s]}$ for big n, and in fact $\Lambda^{[s]}$ must be the limit of $\Lambda_n^{[s]}$. It follows that up to subsequence $\lambda_n^{[v]}$ must converge to a compact lamination $\lambda^{[cv]}$ and $\lambda_n^{[c]}$ to a compact lamination $\lambda^{[cc]}$. Using that λ_n converges to λ we get the result. $\square$

Proposition 1.2.3. *If $\lambda_n \to \lambda$ in $\mathcal{ML}_S^{\circ}$ then for every arc α on S with endpoints in $S \smallsetminus \big(\mathrm{supp}(\lambda) \cup \bigcup \mathrm{supp}(\lambda_n)\big)$ and for every $\varphi \in C_c^\infty(\alpha)$*

$$\int_\alpha \varphi \, \mathrm{d}(\mathrm{meas}_{\lambda_n}) \xrightarrow{n\to\infty} \int_\alpha \varphi \, \mathrm{d}(\mathrm{meas}_\lambda).$$

Proof. From now on, for simplicity we will write $\mathrm{d}\lambda_n$ and $\mathrm{d}\lambda$ respectively for $\mathrm{d}(\mathrm{meas}_{\lambda_n})$ and $\mathrm{d}(\mathrm{meas}_\lambda)$.
Take the decomposition

$$\lambda_n = \lambda_n^{[c]} \cup \lambda_n^{[s]} \cup \lambda_n^{[v]},$$
$$\lambda = \lambda^{[cc]} \cup \lambda^{[s]} \cup \lambda^{[cv]}$$

provided by Lemma 1.2.2, and consider the induced decomposition on the double straightenings Λ_n, Λ of λ_n, λ respectively:

$$\Lambda_n = \Lambda_n^{[c]} \cup \Lambda_n^{[s]} \cup \Lambda_n^{[v]},$$
$$\Lambda = \Lambda^{[cc]} \cup \Lambda^{[s]} \cup \Lambda^{[cv]}.$$

Notice that the weights of the leaves of $\Lambda_n^{[v]}$ are going to 0, since the masses

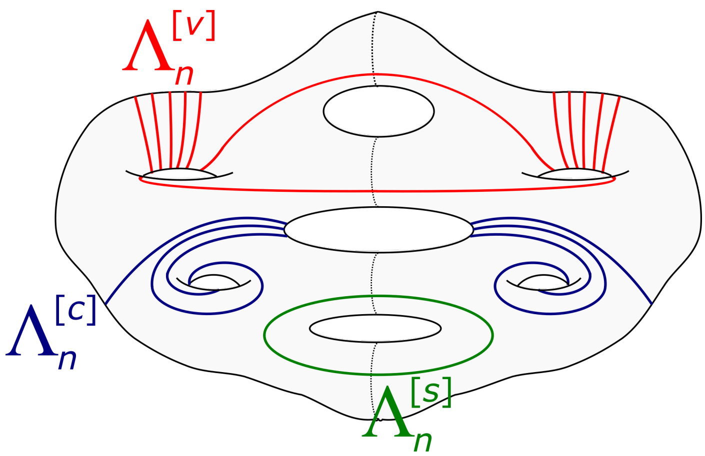


Figure 1.9

of $\Lambda_n^{[v]}$ at the boundary of S are vanishing.
Fixed $\epsilon > 0$ and denoting by

$$T_1 = \left| \int_\alpha \varphi \, \mathrm{d}\lambda_n^{[c]} - \int_\alpha \varphi \, \mathrm{d}\lambda^{[cc]} \right|$$
$$T_2 = \left| \int_\alpha \varphi \, \mathrm{d}\lambda_n^{[s]} - \int_\alpha \varphi \, \mathrm{d}\lambda^{[s]} \right|$$
$$T_3 = \left| \int_\alpha \varphi \, \mathrm{d}\lambda_n^{[v]} - \int_\alpha \varphi \, \mathrm{d}\lambda^{[cv]} \right|$$

it suffices to show that for n sufficiently large $T_1 + T_2 + T_3 \leq 6\epsilon$. The term T_1 is easy to estimate: since $(\lambda_n^{[c]})^R = \lambda_n^{[c]}$ and $(\lambda^{[cc]})^R = \lambda^{[cc]}$, actually

$$T_1 = \left| \int_\alpha \varphi \, \mathrm{d}\Lambda_n^{[c]} - \int_\alpha \varphi \, \mathrm{d}\Lambda^{[cc]} \right|,$$

where with a slight abuse of notation we denote by α also the copy of the arc α itself lying in the orientation-preserving copy of S included in $2S$, and also by φ the obvious function induced on the copy of α by φ. Since $\Lambda_n^{[c]} \to \Lambda^{[cc]}$ in the weak* sense, T_1 is not greater than ϵ for n large enough.
As stated in Lemma 1.2.2, the leaves of $\Lambda_n^{[s]}$ are the leaves of $\Lambda^{[s]}$, so for every leaf Γ of $\Lambda^{[s]}$ the weight of Γ in $\Lambda_n^{[s]}$ tends to weight of Γ itself in $\Lambda^{[s]}$. If Γ is the doubled straightening of γ, recall that the weight of Γ coincides with the weight of γ. Thus, the weight of any leaf γ of $\lambda^{[s]}$ is the limit of its weight as a leaf of $\lambda_n^{[s]}$. It is then clear then $T_2 \leq \epsilon$ for n large enough.
The term T_3 requires more attention. First of all, let us split is as

$$\begin{aligned} T_3 &\leq \left| \int_\alpha \varphi \, \mathrm{d}\lambda_n^{[v]} - \int_\alpha \varphi \, \mathrm{d}\Lambda_n^{[v]} \right| + \left| \int_\alpha \varphi \, \mathrm{d}\Lambda_n^{[v]} - \int_\alpha \varphi \, \mathrm{d}\lambda^{[cv]} \right| = \\ &= \left| \int_\alpha \varphi \, \mathrm{d}\lambda_n^{[v]} - \int_\alpha \varphi \, \mathrm{d}\Lambda_n^{[v]} \right| + \left| \int_\alpha \varphi \, \mathrm{d}\Lambda_n^{[v]} - \int_\alpha \varphi \, \mathrm{d}\Lambda^{[cv]} \right|. \end{aligned}$$

The second term of the last member is not greater then ϵ for n large enough, since $\Lambda_n^{[v]} \to \Lambda^{[cv]}$. Let us consider the first one. Fix a lift $\tilde{\alpha}$ of α in the universal covering of S. For every leaf $\tilde{\delta}$ of the preimage of a leaf δ of $\Lambda_n^{[v]}$ denote by $D_{\tilde{\alpha}}(\tilde{\delta})$ the minimum between the lengths of the two connected components of $\tilde{\delta}^R \smallsetminus \tilde{\alpha}$ if $\tilde{\delta}^R \cap \tilde{\alpha}$ is non empty. See also Figure 1.10. There is a constant $M = M(\alpha, \epsilon) > 0$ such that if $D_{\tilde{\alpha}}(\tilde{\delta}) > M$ then the ideal endpoints of $\tilde{\delta}$ are close to the ones of the prolongation of $\tilde{\delta}^R$, in the Euclidean sense, so that

$$\left| \int_\alpha \varphi \, \mathrm{d}\lambda_n^{[v]+} - \int_\alpha \varphi \, \mathrm{d}\Lambda_n^{[v]+} \right| \leq \epsilon$$

for n sufficiently large, where $\lambda_n^{[v]+}$ is the sublamination of $\lambda_n^{[v]}$ of the leaves δ whose straightening meets α having $D_{\tilde{\alpha}}(\tilde{\delta}) > M$, while $\Lambda_n^{[v]+}$ is the doubled

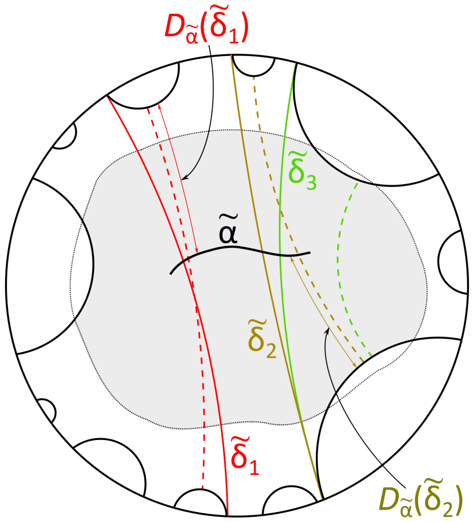


Figure 1.10: The points in the grey region have distance from $\tilde{\alpha}$ less than $M(\alpha, \epsilon)$; the leaf $\tilde{\delta}_1$ of $\tilde{\lambda}_n^{[v]}$ is contained in $\tilde{\lambda}_n^{[v]+}$, while $\tilde{\delta}_2$ and $\tilde{\delta}_3$ are contained in $\tilde{\lambda}_n^{[v]-}$

straightening of $\lambda_n^{[v]+}$. Set $\lambda_n^{[v]-} = \lambda_n^{[v]} \setminus \lambda_n^{[v]+}$ and $\Lambda_n^{[v]-} = \Lambda_n^{[v]} \setminus \Lambda_n^{[v]+}$. Now

$$\left| \int_\alpha \varphi \, \mathrm{d}\lambda_n^{[v]} - \int_\alpha \varphi \, \mathrm{d}\Lambda_n^{[v]} \right| \leq \left| \int_\alpha \varphi \, \mathrm{d}\lambda_n^{[v]+} - \int_\alpha \varphi \, \mathrm{d}\Lambda_n^{[v]+} \right| +$$
$$+ \left| \int_\alpha \varphi \, \mathrm{d}\lambda_n^{[v]-} - \int_\alpha \varphi \, \mathrm{d}\Lambda_n^{[v]-} \right| \leq \epsilon + \left| \int_\alpha \varphi \, \mathrm{d}\lambda_n^{[v]-} \right| + \left| \int_\alpha \varphi \, \mathrm{d}\Lambda_n^{[v]-} \right|.$$

Actually, $\Lambda_n^{[v]-}$ (and consequently $\lambda_n^{[v]-}$) is vanishing, since its number of leaves is bounded from above by a constant depending only on the geometry of S: on its universal covering $\mathcal{H}$, it is easy to see that the number of connected components of $\partial\mathcal{H}$ distant at most M from $\tilde{\alpha}$, which has compact support, are finite. Morever, the weights of the leaves of $\Lambda_n^{[v]}$ are going to 0, as $\lambda_n^{[v]}$ converges to a compact lamination. Thus, for n big,

$$\left| \int_\alpha \varphi \, \mathrm{d}\lambda_n^{[v]-} \right| + \left| \int_\alpha \varphi \, \mathrm{d}\Lambda_n^{[v]-} \right| \leq 2\varepsilon.$$

□

1.2.2 Hyperbolic earthquakes

Let $\mathcal{H}$ be a convex subset of $\mathbb{H}^2$ with geodesic boundary.

Definition 1.2.2. Given a geodesic lamination λ in $\mathcal{H}$, a left (respectively right) hyperbolic earthquake on $\mathcal{H}$ along λ is an injective (possibly discontinuous) map $E : \mathcal{H} \to \mathbb{H}^2$ such that

- the restriction of E on a stratum of λ is an isometry;
- denoting by $A_F \in PSL(2,\mathbb{R})$ the isometry of $\mathbb{H}^2$ extending $E_{|F}$ for every stratum F, the *comparison map*

$$\mathrm{cmp}(F,G) = A_F^{-1} \circ A_G : \mathbb{H}^2 \to \mathbb{H}^2$$

  between two different strata F and G of λ is a hyperbolic transformation whose axis weakly separates F and G and which translates to the left (respectively right), as viewed from F.

The lamination λ is called *fault locus* of the earthquake E.
It turns out that $E(\mathcal{H})$ is still a convex subset of $\mathbb{H}^2$ with geodesic boundary, as a consequence of Lemma 8.4 in [15].
Given a surface S and two hyperbolic metrics h_1, h_2 on S, set $S_i = (S, h_i)$ for $i = 1, 2$. Suppose that the universal covering $\mathcal{H}_i \subset \mathbb{H}^2$ of S_i is convex with geodesic boundary. A bijective map $E : S_1 \to S_2$ is a left (respectively right) hyperbolic earthquake if it has a lifting $\tilde{E} : \mathcal{H}_1 \to \mathcal{H}_2$ which is a left (respectively right) hyperbolic earthquake on $\mathcal{H}_1$.

Example 1.2.2. Given a close geodesic c on a closed surface S and a real positive $\omega > 0$, a left hyperbolic earthquake on S along c can be performed as follows: cut S along c, shift an edge along c to the left of a distance ω and then glue along c.

In this example it is clear that the fault locus can be endowed with a transverse measure encoding the shearing of the earthquake, obtaining a measured geodesic lamination: the ω-weighted curve c. This can be done in general, as stated in the following ([34], Proposition 6.1).

Proposition 1.2.4. *A measured geodesic lamination λ is associated to any earthquake so that* $\mathrm{supp}(\lambda)$ *coincides with the fault locus; if $a : [0,1] \to \mathbb{H}^2$ is an arc with endpoints in two gaps of λ then*

$$\mathrm{meas}_\lambda(a) = \sup_{P \text{ partition of } [0,1]} \sum_{i=1}^{I_P} \mathrm{T}(\mathrm{cmp}(A_{F_{i-1}}, A_{F_i}))$$

where for every partition $P = (0 = t_0, t_1, t_2, \ldots, t_{I_P} = 1)$ of $[0,1]$ the stratum F_i of λ is the one containing t_i. Here $\mathrm{T}(B)$ denotes the translation length of a hyperbolic transformation B.

Moreover, Thurston showed that different earthquakes produce different measured geodesic laminations (see [34]). The converse holds, since we did not suppose that E is surjective:

Proposition 1.2.5. *For every measured geodesic lamination λ on $\mathcal{H}$ there is a left earthquake $E : \mathcal{H} \to \mathbb{H}^2$ with shearing lamination λ itself. Moreover, two earthquakes on $\mathcal{H}$ with the same shearing lamination differ by precomposition by an isometry of $\mathbb{H}^2$.*

How earthquakes work on closed surfaces is well known. Given instead $h \in \mathcal{T}_S^\star(\mathfrak{C})$, with holonomy $\mathrm{hol} : \pi_1(S) \to PSL(2,\mathbb{R})$, consider the universal covering $\mathcal{H}$ of (S,h). It is the convex hull of the limit set of hol and of the preimages of the tips. A measured geodesic lamination $\lambda \in \mathcal{ML}_S^\star(\mathfrak{C})$ has a hol-invariant lift $\tilde{\lambda}$ in $\mathcal{H}$. Denote by $\tilde{E} : \mathcal{H} \to \mathbb{H}^2$ the left earthquake along $\tilde{\lambda}$. For every $\gamma \in \pi_1(S)$ the map $\tilde{E} \circ \mathrm{hol}(\gamma) : \mathcal{H} \to \mathbb{H}^2$ is again a left earthquake with shearing lamination $\tilde{\lambda}$. By Proposition 1.2.5, there exists $\rho(\gamma) \in PSL(2,\mathbb{R})$ such that

$$\tilde{E} \circ \mathrm{hol}(\gamma) = \rho(\gamma) \circ \tilde{E}. \tag{1.7}$$

Proposition 1.2.6. *The representation $\rho : \pi_1(S) \to PSL(2,\mathbb{R})$ is faithful and discrete. The quotient $\rho(\pi_1(S))\backslash\mathbb{H}^2$ is homeomorphic to S. The hyperbolic metric induced on $\rho\backslash\tilde{E}(\mathcal{H})$ lies in $\mathcal{T}_S^\star$. The map $\tilde{E}$ descends to the quotient as an earthquake map*

$$E : \mathrm{hol}(\pi_1(S))\backslash\mathcal{H} \to \rho(\pi_1(S))\backslash\tilde{E}(\mathcal{H}).$$

Proof. Since hol is faithful and $\tilde{E}$ is injective, ρ is faithful. Also, ρ is discrete since for every x lies in a gap of $\tilde{\lambda}$ the ρ-orbit of $\tilde{E}(x)$ accumulates at $\tilde{E}(p)$ if and only if the hol-orbit of p accumulates at p, by (1.7). Thus, being hol discrete, ρ is discrete too.
Since the family ρ_t, the faithful and discrete representation associated to $\tilde{E}^{t\lambda}$ for $t \in [0,1]$, is a path from hol to ρ, the surface $\rho(\pi_1(S))\backslash\tilde{E}(\mathcal{H})$ is homeomorphic to $\mathrm{hol}(\pi_1(S))\backslash\mathcal{H} \cong S$.
Finally, we have to see that $\tilde{E}(\mathcal{H})$ is the convex hull of the limit set of ρ and of a discrete set of $\partial\mathbb{H}^2$ which is ρ-invariant. First of all, $\tilde{E}(\mathcal{H})$ is convex with geodesic boundary (see [7]) and it contains the convex core of ρ (see [15]). Consider two consecutive element q and q' in the preimage $\tilde{\mathfrak{C}}$ of the tips on (S,h), in the sense that one of the open arc on $\partial\mathbb{H}^2$ between q and q' is disjoint from $\partial_\infty\mathcal{H}$. There are elements $p, p' \in \tilde{\mathfrak{C}}$ distinct from q and q' such that p is consecutive to q and p' is consecutive to q'. Denoting by $[q,q']$ the geodesic with ideal endpoints q and q', there is a bidimensional stratum F of $\tilde{\lambda}$ containing $[q,q']$. In fact, bidimensional strata of $\tilde{\lambda}$ can not accumulate on $[q,q']$: otherwise, they would have ideal vertices (that are elements of $\partial_\infty\mathcal{H}$) accumulating near q or q', which is impossible, as there is no element of $\partial_\infty\mathbb{H}^2$ between p and q, q and q', q' and p'. Thus $\tilde{E}([q,q'])$ is a complete geodesic of $\mathbb{H}^2$ contained in $\partial\tilde{E}(\mathcal{H})$, being F isometric to $\tilde{E}(F)$; moreover the images $\tilde{E}(q), \tilde{E}(q')$ are well defined isolated consecutive points of $\partial_\infty\tilde{E}(\mathcal{H})$. The discrete set $\tilde{E}(\tilde{\mathfrak{C}})$ is ρ-invariant, since $\tilde{\mathfrak{C}}$ is hol-invariant. □

Remark 1.2.2. The first part of the argument of the previous theorem is the one used to show in [15] that for every $h \in \mathcal{T}_S^\circ$ any measured geodesic lamination $\lambda \in \mathcal{ML}_S^\circ$ induces a left earthquake $E : (S,h) \to (S,h')$ with $h' \in \mathcal{T}_S^\circ$. In that case, $\mathcal{H}$ coincides with the convex hull of the limit set of hol.

This results allow to introduce the following maps between Teichmüller spaces:

Definition 1.2.3. For every measured geodesic lamination $\lambda \in \mathcal{ML}_S^\star(\mathfrak{C})$ denote by

$$E_l^\lambda, E_r^\lambda : \mathcal{T}_S^\star(\mathfrak{C}) \to \mathcal{T}_S^\star(\mathfrak{C})$$

respectively the left and right earthquake map associated to λ that sends $h \in \mathcal{T}_S^\star(\mathfrak{C})$ to the metric on the image of the left and right earthquake along λ respectively applied on (S, h).
Analogously define

$$E_l^\lambda, E_r^\lambda : \mathcal{T}_S^\circ \to \mathcal{T}_S^\circ$$

for $\lambda \in \mathcal{ML}_S^\circ$.

There will be no misinterpretations in this work, since the context will be always clear. In particular, Chapter 2 will focus on ciliated surfaces and Chapter 3 on closed geodesic boundaries.

1.3 The space AdS_3

1.3.1 Definition and properties of AdS_3

Consider the space $\mathbb{R}^{2,2}$, i.e. $\mathbb{R}^4$ with the bilinear symmetric form

$$\langle \underline{x}, \underline{y} \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3 - x_4 y_4.$$

Let $\widehat{AdS_3}$ be $\{\underline{x} \in \mathbb{R}^{2,2} \; : \; \langle \underline{x}, \underline{x} \rangle = -1\}$ and consider the restriction q to $\widehat{AdS_3}$ of the projection $(\mathbb{R}^{2,2})^* \to \mathbb{R}P^3$; the Klein model of AdS_3 is by definition $\mathrm{q}(\widehat{AdS_3})$ together with the Lorentzian structure induced by the bilinear form $\langle \cdot, \cdot \rangle$. We can thus write $AdS_3 = \{[\underline{x}] \in \mathbb{R}P^3 \; : \; \langle \underline{x}, \underline{x} \rangle < 0\}$.
Notice that $\mathrm{q} : \widehat{AdS_3} \to AdS_3$ is a 2:1 covering.
The homeomorphism $AdS_3 \to D^2 \times S^1$ defined by

$$[\underline{x}] \mapsto \frac{1}{\sqrt{x_3^2 + x_4^2}} \big((x_1, x_2), (x_3, x_4) \big)$$

shows that AdS_3 is not simply connected.
The space AdS_3 can also be identified with $PSL(2, \mathbb{R})$ via

$$[\underline{x}] \mapsto A_{[\underline{x}]} = \begin{bmatrix} x_3 - x_1 & x_2 - x_4 \\ x_2 + x_4 & x_3 + x_1. \end{bmatrix}$$

Defining $\langle A_{[\underline{x}]}, A_{[\underline{y}]} \rangle = \langle \underline{x}, \underline{y} \rangle$, it turns out that

$$\langle A_{[\underline{x}]}, A_{[\underline{y}]} \rangle = -\frac{\mathrm{tr}(A_{[\underline{x}]} A_{[\underline{y}]}^{-1})}{2}.$$

In $\mathbb{R}P^3$, we can consider $\partial_\infty AdS_3 = \{[\underline{x}] \in \mathbb{R}P^3 \ : \ \langle \underline{x}, \underline{x} \rangle = 0\}$. There are two foliations $\mathcal{F}_l, \mathcal{F}_r$ of $\partial_\infty AdS_3$ such that every leaf is a projective line and the intersection of any leaf of $\mathcal{F}_r$ with any leaf of $\mathcal{F}_l$ is a point (see [29], [2]). Denoting by φ_l and φ_r respectively the homology class of the elements of $\mathcal{F}_l$ and $\mathcal{F}_r$, the assignment of an orientation with φ_l and φ_r induces an orientation to the leaves of the two foliations. We fix once for all the orientation on $\partial_\infty AdS_3$ defined as follows: if $p \in \partial_\infty AdS_3$ and $\mathbf{e}_l, \mathbf{e}_r \in T_p\partial_\infty AdS_3$ are positive vectors tangent respectively to $\mathcal{F}_l$ and $\mathcal{F}_r$, then $(\mathbf{e}_l, \mathbf{e}_r)$ is a positive basis of $T_p\partial_\infty AdS_3$. Finally, orient AdS_3 compatibly with such orientation on $\partial_\infty AdS_3$.
We can also fix on any space-like plane $\mathcal{P}$ the orientation such that the homology class of $\partial_\infty \mathcal{P}$ with the orientation inherited by $\mathcal{P}$ is $\varphi_l - \varphi_r$. A time-orientation on AdS_3 is induced as follows: a time-like vector $\mathbf{v} \in T_pAdS_3$ with $p \in AdS_3$ is future-pointing if it induces on the space-like plane through p normal to $\mathbf{v}$ the positive orientation.
One of the reasons to use the Klein model of AdS_3 is that its geodesics are

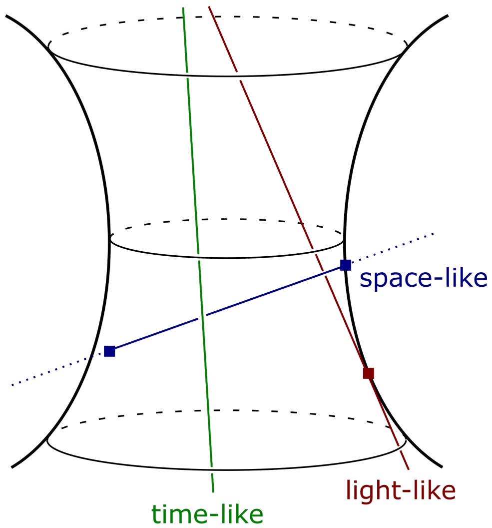


Figure 1.11: Geodesics in AdS_3 in an affine chart

the projective lines. In particular, considering lines that intersect AdS_3 as in Figure 1.11,

- those that do not intersect $\partial_\infty AdS_3$ are time-like geodesics; they are closed, entirely contained in AdS_3 and have length π;
- those that intersect $\partial_\infty AdS_3$ in two distinct points are space-like geodesics;
- those that intersect $\partial_\infty AdS_3$ in one single point are light-like geodesics; they are tangent to $\partial_\infty AdS_3$.

Totally geodesic planes in AdS_3 are intersection of projective planes with

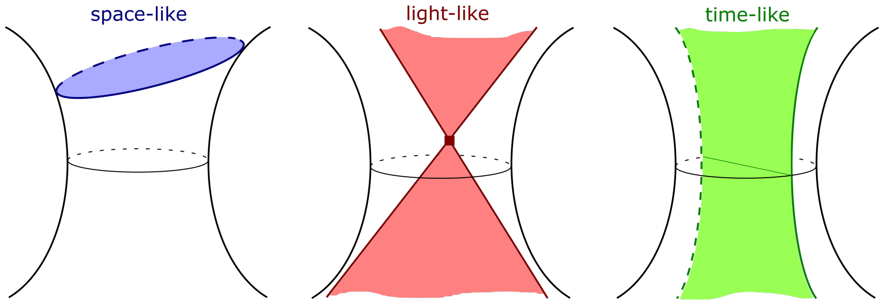


Figure 1.12: Totally geodesic planes in AdS_3 in an affine chart

AdS_3. In particular, as shown in Figure 1.12:

- time-like planes are topologically Moebius bands;
- space-like planes are topologically disks;
- light-like planes are tangent to $\partial_\infty AdS_3$.

One can associate every point p of AdS_3 with the space-like plane $\mathcal{S}(p)$ consisting of the midpoints of all time-like geodesics starting from p; that plane

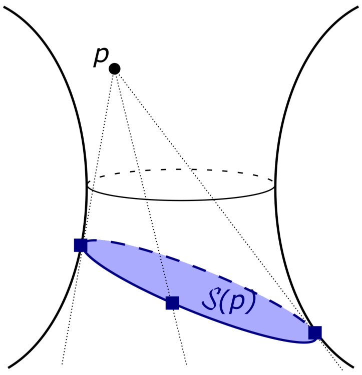


Figure 1.13: Duality between points and space-like planes in AdS_3

is such that $\partial_\infty \mathcal{S}(p) = \partial_\infty \mathcal{I}^0(p)$ where $\mathcal{I}^0(p)$ is union of light-like geodesics passing through p. Conversely, all time-like geodesics orthogonal to a space-like plane $\mathcal{S}$ meet in a single point p and it is such that $\mathcal{S} = \mathcal{S}(p)$. So $p \mapsto \mathcal{S}(p)$ is a bijective map from AdS_3 to the set of space-like planes of AdS_3, as in Figure 1.13; let $\mathcal{S}(p)$ be called the dual plane of p. One can also define a correspondence $\mathcal{L}$ between $\partial_\infty AdS_3$ and the set of light-like planes of AdS_3 by putting $\mathcal{L}(p)$ the plane tangent to $\partial_\infty AdS_3$ at p.

Now fix a space-like plane $\mathcal{P}_0$. It is a simply connected complete hyperbolic surface, so it is isometric to $\mathbb{H}^2$. In particular, we can identify $\partial_\infty \mathcal{P}_0$

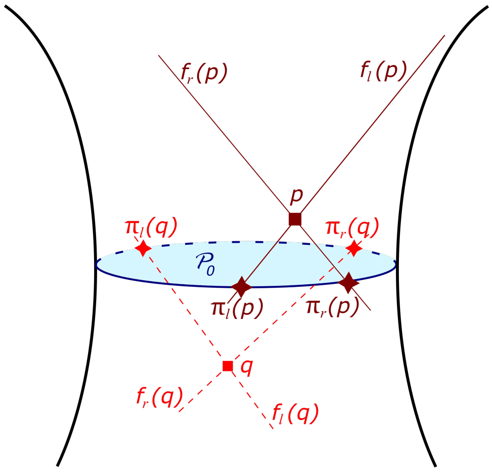


Figure 1.14: Foliations of $\partial_\infty AdS_3$

with $\partial\mathbb{H}^2$. For every point $p \in \partial_\infty AdS_3$ there is a unique leaf $f_l(p) \in \mathcal{F}_l$ that passes through p; denote by $\pi_l(p)$ the intersection between $f_l(p)$ and $\partial_\infty \mathcal{P}_0 \cong \partial\mathbb{H}^2$, as in Figure 1.14. Analogously, define $\pi_r(p)$. The obtained map $(\pi_l, \pi_r) : \partial_\infty AdS_3 \to \partial\mathbb{H}^2 \times \partial\mathbb{H}^2$ is a correspondence (notice in fact that $(\pi_l, \pi_r)(f_l(x) \cap f_r(y)) = (x, y)$ for every $(x, y) \in \partial\mathbb{H}^2 \times \partial\mathbb{H}^2$).
It is known (see [2]) that under the identification between $\partial_\infty AdS_3$ and $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$, for every space-like plane $\mathcal{S}$ there exists $A \in PSL(2, \mathbb{R})$ such that $\partial_\infty \mathcal{S} = \mathrm{graph}(A)$. Conversely, for every $A \in PSL(2, \mathbb{R})$, $\mathrm{graph}(A)$ is the boundary at infinity of a space-like plane $\mathcal{P}_A$, namely $\alpha(\mathcal{P}_0)$ where α is the extension to AdS_3 of the isometry (id, A) of $\partial_\infty AdS_3$.
Notice that the correspondence $A \leftrightarrow \mathcal{P}_A$ between $PSL(2, \mathbb{R})$ and space-like planes of AdS_3 coincides, under the identification between $PSL(2, \mathbb{R})$ and AdS_3, with the duality $p \leftrightarrow \mathcal{S}(p)$ between points and space-like planes of AdS_3. Moreover, if $(A, B) \in PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R}) \cong \mathrm{Isom}_0(AdS_3)$ then $(A, B)(X) = AXB^{-1}$ for every $X \in PSL(2, \mathbb{R}) \cong AdS_3$.

Remark 1.3.1. Since $\mathcal{P}_A = \mathrm{graph}(A)$ for every $A \in PSL(2, \mathbb{R})$, two space-like planes $\mathcal{P}_A$ and $\mathcal{P}_B$

- meet transversely if and only if AB^{-1} has two ideal fixed points, i.e. AB^{-1} is hyperbolic;

- are tangent at one point in $\partial_\infty AdS_3$ if and only if AB^{-1} has one ideal fixed point, i.e. AB^{-1} is parabolic;

- have disjoint ideal boundary if and only if AB^{-1} has no ideal fixed points, i.e. AB^{-1} is elliptic.

1.3.2 Bent surfaces and convex hulls

Even if AdS_3 is time-oriented, the notion of future of a point is not useful: it is in fact the whole AdS_3. It can be only a local notion. Then say that an embedded topological surface $\mathbb{S}$ is locally achronal in AdS_3 if there are open subsets V_j of AdS_3 covering $\mathbb{S}$ such that for every j any two distinct points $x, y \in \mathbb{S} \cap V_j$ are not joint by a time-like arc in $\mathbb{S} \cap V_j$. Say also that $\mathbb{S}$ is past convex (respectively future convex) if it is locally achronal and if there exists an open covering $\{U_i\}$ of $\mathbb{S}$ in AdS_3 such that for every i the geodesic connecting two points of $\mathbb{S} \cap U_i$ does not intersect the future (respectively the past) of $\mathbb{S}$ in U_i.
A support plane for $\mathbb{S}$ in $p \in \mathbb{S}$ turns out to be a space-like plane $\mathcal{P}$ such that $\mathbb{S} \cap \mathcal{P}$ is convex and contains p. Notice that $\mathbb{S}$ is past convex if and only if for every i and for every $p \in U_i$ there is a support plane $\mathcal{P}$ of $\mathbb{S}$ in p that does not intersect $\mathbb{S} \cap U_i$ if slightly moved in the future.

Definition 1.3.1. A past (respectively future) bent surface is a topological embedding $\mathrm{b} : \mathcal{H} \to AdS_3$ where:

- $\mathcal{H}$ is an open convex subset of $\mathbb{H}^2$ with geodesic boundary;
- there is a lamination λ of $\mathcal{H}$ such that the restriction of b to any stratum of λ is isometric and totally geodesic;
- $\mathrm{b}(\mathcal{H})$ is a past (respectively future) convex surface.

Example 1.3.1. Consider two space-like planes $\mathcal{P}$ and $\mathcal{Q}$ meeting along l: taken $p \in l$, consider the future-pointing unit vectors $n_{\mathcal{P}}$, $n_{\mathcal{Q}}$ orthogonal in p respectively in $\mathcal{P}$ and $\mathcal{Q}$ and define the bending angle between $\mathcal{P}$ and $\mathcal{Q}$ as $\vartheta(\mathcal{P}, \mathcal{Q}) = \cosh^{-1} |\langle n_{\mathcal{P}}, n_{\mathcal{Q}} \rangle|$. If l is oriented, we can assign a signed bending angle $\overline{\vartheta}(\mathcal{P}, \mathcal{Q})$: it is positive (respectively negative) if $v, n_{\mathcal{P}}, n_{\mathcal{Q}}$, where v is the positive unit tangent vector of l in p, is a positive (respectively negative) basis of $T_p AdS_3$.
Notice that if $l = \mathcal{P}_{\mathrm{id}} \cap \mathcal{P}_B$ is oriented from the repulsive point of B to the attractive one then $\overline{\vartheta}(\mathcal{P}_{\mathrm{id}}, \mathcal{P}_B) > 0$.
Since the translation length $\mathrm{T}(X)$ of a hyperbolic element X in $PSL(2, \mathbb{R})$ is given by the formula $\mathrm{T}(X) = 2\cosh^{-1}(|\operatorname{tr}(X)|/2)$, the bending angle between $\mathcal{P}_A, \mathcal{P}_B$ is

$$\cosh^{-1} |\langle A, B \rangle| = \cosh^{-1} \frac{|\operatorname{tr}(AB^{-1})|}{2} = \frac{\mathrm{T}(AB^{-1})}{2}.$$

Consider $\mathbb{S} = \mathcal{P}_r \cup l \cup \mathcal{Q}_l$ where $\mathcal{P}_r$ is the right component of $\mathcal{P} \setminus l$ and $\mathcal{Q}_l$ is the left component of $\mathcal{Q} \setminus l$ with respect to l. Then $\mathbb{S}$ is an achronal surface. Moreover, it is a past bent surface if $\overline{\vartheta}(\mathcal{P}, \mathcal{Q}) > 0$.

Remark 1.3.2. It is possible to associate to any past bent surface a transverse measure of λ. If $c : [0,1] \to \mathcal{H}$ is an arc transverse to λ and

$$I = (0 = t_0 < t_1 < \ldots < t_k = 1)$$

is a partition of $[0,1]$, chosen a support plane for every $\mathrm{b}(c(t_i))$ let $\nu(c,I)$ be the sum of the bending angles between the support planes of $\mathrm{b}(c(t_i))$ and $\mathrm{b}(c(t_{i+1}))$. Notice that if three space-like planes $\mathcal{P}_A, \mathcal{P}_B, \mathcal{P}_C$ with non-empty mutual intersections are such that $\mathcal{P}_A \cap \mathcal{P}_C$ lies above $\mathcal{P}_B$ then $\mathrm{ax}(AC^{-1})$ lies between $\mathrm{ax}(AB^{-1})$ and $\mathrm{ax}(BC^{-1})$ so the inequality

$$\vartheta(\mathcal{P}_A,\mathcal{P}_B)+\vartheta(\mathcal{P}_B,\mathcal{P}_C) = \frac{\mathrm{T}(AB^{-1})}{2} + \frac{\mathrm{T}(BC^{-1})}{2} \leq$$
$$\leq \frac{\mathrm{T}(AB^{-1}BC^{-1})}{2} = \vartheta(\mathcal{P}_A,\mathcal{P}_C)$$

holds. This property of bending angles leads to the monotonicity of $\nu(c,\cdot)$; more precisely, if J is finer than I then $\nu(c,J) \leq \nu(c,I)$. Therefore

$$\mu(c) = \lim_{|I|\to 0} \nu(c,I)$$

is well defined.

Remark 1.3.3. If $\mathrm{supp}(\lambda)$ is locally finite, then the transverse measure constructed in the remark above simply assigns to each leaf a weight: if l is a leaf adjacent to two gaps F_1 and F_2, then its weight coincides with the bending angle between $\mathrm{b}(F_1)$ and $\mathrm{b}(F_2)$.

Achronal meridians

In this subsection we will consider a particular class of meridians in $\partial_\infty AdS_3$, whose properties are better understood on the universal covering of $\partial_\infty AdS_3$. Under the identification of it with $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$ and fixed from now on the universal covering $\mathbb{R} \to \partial\mathbb{H}^2 \cong S^1$ given by $t \mapsto e^{2\pi i t}$, we will consider the universal convering $\Upsilon : \mathbb{R}^2 \to \partial\mathbb{H}^2 \times \partial\mathbb{H}^2$ given by $\Upsilon(x,y) = (e^{2\pi i x}, e^{2\pi i y})$. Notice that the preimage by Υ of a leaf of the left foliation has the form $(\mathbb{Z}+x)\times\mathbb{R}$ for some $x \in \mathbb{R}$ and the preimage of a right leaf has the form $\mathbb{R}\times(\mathbb{Z}+y)$ for some $y \in \mathbb{R}$.

Remark 1.3.4. A lifting of a space-like curve C in $\partial_\infty AdS_3$ is the graph of a continuous and strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+n) = f(x) + m,$$

where $n, m \in \mathbb{Z}$ are such that the homotopy class $[C]$ of C is $n[f_l] + m[f_r]$ with $f_l \in \mathcal{F}_l$ and $f_r \in \mathcal{F}_r$. In particular, liftings of space-like meridians are graphs of orientation-preserving homeomorphisms $f : \mathbb{R} \to \mathbb{R}$ such that

$f(x+1) = f(x) + 1$.
The closure of space-like meridians contains also locally achronal meridians, i.e. meridians C where every $p \in C$ has a neighbourhood U in $\partial_\infty AdS_3$ such that no pair of points $q, r \in U \cap C$ is joint by a time-like arc in U. Therefore, locally achronal meridians are associated to limits of graphs of orientation-preserving $(1,1)$-periodic homeomorphisms $f : \mathbb{R} \to \mathbb{R}$; such limits are of the form

$$\mathrm{Gr}_f = \left\{ (x,y) \in \mathbb{R}^2 \mid \lim_{t \to x^-} f(t) \leq y \leq \lim_{t \to x^+} f(t) \right\}$$

with $f : \mathbb{R} \to \mathbb{R}$ not decreasing such that $f(\cdot + 1) = f(\cdot) + 1$.
The converse does not hold: there are non-decreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(\cdot + 1) = f(\cdot) + 1$ but $\Upsilon(\mathrm{Gr}_f)$ is not an achronal meridian. Namely, they are of the form $f_{\alpha,\beta}(x) = \lfloor x + \alpha \rfloor + \beta$ with $(\alpha, \beta) \in \mathbb{R}^2$. It is not difficult to see that $\Upsilon(\mathrm{Gr}_{f_{\alpha,\beta}})$ is the union of the two leaves $f_l(\Upsilon(\alpha,\beta))$ and $f_r(\Upsilon(\alpha,\beta))$ of the left and right foliations $\mathcal{F}_l$ and $\mathcal{F}_r$ of $\partial_\infty AdS_3$ passing through the point $\Upsilon(\alpha, \beta)$.

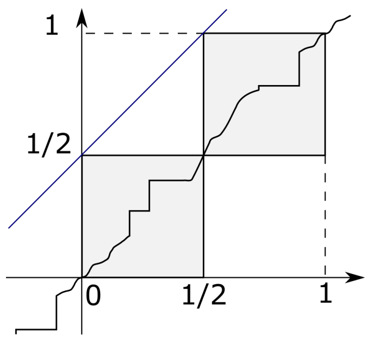
The following lemmas and remarks will be useful in the next section, where we consider the relation between locally achronal meridians and bent surfaces in AdS_3.

Lemma 1.3.1. *If C is an achronal meridian, then there exists a space-like plane $\mathcal{P}$ in AdS_3 such that $C \cap \partial_\infty \mathcal{P} = \emptyset$.*

Proof. Consider a function $f : \mathbb{R} \to \mathbb{R}$ such that Gr_f projects to C; up to isometries, we can suppose that $f(0) = 0$ (and so $f(1) = 1$), $f(1/2) = 1/2$ and f is continuous in 0 and $1/2$. The function f is not decreasing so

$$\mathrm{Gr}_f \cap [0,1]^2 \subset [0,1/2]^2 \cup [1/2,1]^2$$

and the continuity of f in 0, 1 and $1/2$ prevents that the points $(0, 1/2)$ and $(1/2, 1)$ lie on Gr_f.
Therefore, graph$(x \mapsto x + 1/2)$ is disjoint from Gr_f; notice that $x \mapsto x + 1/2$



is a lifting of the rotation of angle π of $\partial \mathbb{H}^2$, so it is the lifting of the trace at infinity of an isometry $A \in PSL(2, \mathbb{R})$. It follows that $C \cap \partial_\infty \mathcal{P}_A = \emptyset$. $\square$

The previous lemma implies that considering the convex hull $\mathcal{K}$ of a locally achronal meridian C makes sense: it is well defined in the affine chart $\mathbb{R}^3 = \mathbb{R}P^3 \setminus \mathcal{P}$. That convex hull has the properties that its interior is contained in AdS_3 and its boundary in $\mathbb{R}^3$ meets $\partial_\infty AdS_3$ exactly in C (see Lemma 6.3 in [15]).

Remark 1.3.5. Support planes of $\mathcal{K}$ cannot be time-like, because for homological reasons time-like planes always meet C (transversely) . If C is not the boundary at infinity of a space-like plane then $\mathcal{K}$ is topologically a closed ball and $\partial\mathcal{K}$ in AdS_3 has two connected components (separated by C), which are achronal surfaces; we will refer to the past convex one as $\partial_+\mathcal{K}$, the upper boundary of $\mathcal{K}$, and to the future convex one as $\partial_-\mathcal{K}$, the lower boundary of $\mathcal{K}$.
If $\mathcal{P}$ is a space-like support plane then $\mathcal{P} \cap \partial\mathcal{K} = \mathrm{CH}(\mathcal{P} \cap C)$ is either a geodesic or a hyperbolic ideal polygon (also with an infinite number of edges); if $\mathcal{P}$ is a light-like support plane then it is tangent to $\partial_\infty AdS_3$ at a certain $p \in C$ and $\mathcal{P} \cap \partial\mathcal{K} = \mathrm{CH}(\mathcal{P} \cap C)$ is a (light-like) triangle with a vertex in p and two edges lying in $f_l(p)$ and $f_r(p)$. We will refer to the set of points of $\partial_\pm\mathcal{K}$ which admit only space-like support planes as the space-like part of $\partial_\pm\mathcal{K}$.

Remark 1.3.6. If $f : \mathbb{R} \to \mathbb{R}$ is an increasing function such that $\Upsilon(\mathrm{Gr}_f) = C$ and $A \in PSL(2,\mathbb{R})$ then $\mathcal{P}_A$ is an upper support plane for C if and only if there is a lifting $\tilde{A} : \mathbb{R} \to \mathbb{R}$ of $A_{|\partial\mathbb{H}^2}$ such that $\tilde{A}^{-1}(f(x)) \le x$ for every $x \in \mathbb{R}$ and $\tilde{A}^{-1} \circ f$ admits two fixed points in $[0,1)$. In fact $\mathcal{P}_A = \mathrm{CH}\{(x, A(x))\}$ does not disconnect $C = \Upsilon(\mathrm{Gr}_f)$ if and only if there is a lifting $\tilde{A}$ such that $f(t) \le \tilde{A}(t)$ for every $t \in \mathbb{R}$ (since $\mathcal{P}_A$ is an upper support plane) and $\mathcal{P}_A \cap \mathcal{K}$ is a geodesic or an ideal polygon if and only if at least two points lie in $\partial_\infty\mathcal{P}_A \cap C$, meaning that $\tilde{A}(t) = f(t)$ for at least two point $t \in [0,1)$.

Lemma 1.3.2. *If $\mathcal{P}$ and $\mathcal{Q}$ are space-like upper support planes of $\mathcal{K}$ then they intersect along a line a. Moreover, if a is oriented so that $\overline{\vartheta}(\mathcal{P},\mathcal{Q}) > 0$, then $\mathcal{P} \cap \mathcal{K}$ is contained in the right side $\mathcal{P}_r$ of a in $\mathcal{P}$ and $\mathcal{Q} \cap \mathcal{K}$ is contained in the left side $\mathcal{Q}_l$ of a in $\mathcal{Q}$.*

Proof. If by contradiction $\mathcal{P}$ and $\mathcal{Q}$ are disjoint, then you can slightly move them in the future to get space-like planes $\mathcal{P}'$ and $\mathcal{Q}'$ such that $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{P}'$ and $\mathcal{Q}'$ are mutually disjoint and $\mathcal{P}' \cap \mathcal{K} = \mathcal{Q}' \cap \mathcal{K} = \emptyset$.
Notice that $\mathcal{P}$ and $\mathcal{Q}$ are contained in two different connected components of $AdS_3 \setminus (\mathcal{P}' \cup \mathcal{Q}')$. However, $\mathcal{K}$ is connected, since it is convex, and both its intersections with $\mathcal{P}$ and $\mathcal{Q}$ are not empty, leading to a contradiction.
Moreover, any geodesic segment joining a point of $\mathcal{P} \cap \mathcal{K}$ with a point of $\mathcal{Q} \cap \mathcal{K}$ must be contained in $\mathcal{K}$ and can not intersect space-like planes obtained slighty moving $\mathcal{P}$ and $\mathcal{Q}$ in the future. Then the only possibility is that the endpoints of the segment lie respectively in $\mathcal{P}_r$ and $\mathcal{Q}_l$. $\square$

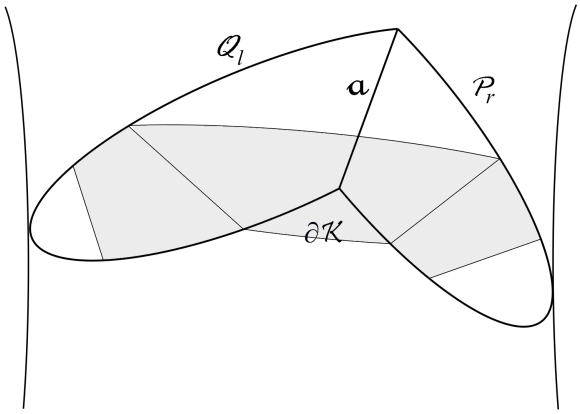


Figure 1.15

Remark 1.3.7. Consider $\mathcal{P}$, $\mathcal{Q}$, a, $\mathcal{P}_r$ and $\mathcal{Q}_l$ as in Lemma 1.3.2 and Figure 1.15. Let $p_- = (x_-, y_-)$ be the starting point of a and $p_+ = (x_+, y_+)$ the ending one. Let s be the geodesic in $\mathbb{H}^2$ with starting point x_- and ending point x_+. Since $\partial_\infty(\mathcal{P} \cap \mathcal{K}) = \mathcal{P} \cap C = \{(x_\alpha, y_\alpha)\}$ is contained in $\partial_\infty \mathcal{P}_r$, all the points x_α lie on the right side of s, and analogously all the points x'_β lie on the left side of s, where $\mathcal{Q} \cap C = \{(x'_\beta, y'_\beta)\}$. In particular, s weakly separates $\mathrm{CH}\{x_\alpha\}$ from $\mathrm{CH}\{x'_\beta\}$.

Sets connectible by achronal meridians

For any points $x, y, z \in \partial\mathbb{H}^2$ such that $x \neq z$, we write $x \leq y \leq z$ if y lies in the positive segment in $\partial\mathbb{H}^2$ with first endpoint x and second endpoint z. We write $x < y < z$ if $x \leq y \leq z$ and $x \neq y \neq z$.

Definition 1.3.2. A subset Ω of $\partial_\infty AdS_3$ not contained in a right or in a left leaf is said to be connectible by an achronal meridian if given three points $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \Omega$ with $x_1 < x_2 < x_3$ then either $y_1 = y_2 = y_3$ or $y_1 < y_2 < y_3$.

Any achronal meridian C is connectible by an achronal meridian: it is the projection of Gr_f for a certain increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+1) = f(x) + 1$.

Lemma 1.3.3. *For every set Ω connectible by an achronal meridian there exists an achronal meridian containing Ω. Moreover, there are two extremal achronal meridians $C_-(\Omega), C_+(\Omega)$ containing Ω such that every achronal meridian containing Ω lies between them.*

Proof. Let $\tilde{\Omega} = \Upsilon^{-1}(\Omega) \cap [0,1]^2$, where $\Upsilon : \mathbb{R}^2 \to \partial\mathbb{H}^2 \times \partial\mathbb{H}^2$ is the covering map described at the beginning of Subsection 1.3.2. By definition, if

$(\tilde{x},\tilde{y}),(\tilde{x}',\tilde{y}') \in \tilde{\Omega}$ and $\tilde{x} < \tilde{x}'$ then $\tilde{y} \leq \tilde{y}'$. Define $f_- : [0,1] \to [0,1]$ by setting $f_-(0) = 0$, $f_-(1) = 1$ and

$$f_-(t) = \sup\{\tilde{y} \mid \exists(\tilde{x},\tilde{y}) \in \tilde{\Omega} \text{ s.t. } \tilde{x} \leq t\}$$

where we consider $\sup \emptyset = 0$. Extend this function to the increasing map $f_- : \mathbb{R} \to \mathbb{R}$ such that $f_-(t+1) = f_-(t)+1$. If $(\tilde{x},\tilde{y}) \in \tilde{\Omega}$ then by construction $f_-(\tilde{x}) \geq \tilde{y}$; on the other hand, by the property of $\tilde{\Omega}$, $\lim f_-(t) \leq \tilde{y}$ as $t \to \tilde{x}^-$. Therefore, $(\tilde{x},\tilde{y}) \in \mathrm{Gr}_{f_-}$. Notice also that if $\Upsilon(t,0) \in \Omega$ then $(0,t) \times \{0\} \subset \mathrm{Gr}_{f_-}$.
The curve $C_-(\Omega) = \Upsilon(G_{f_-})$ contains Ω so it cannot be contained in a right or left leaf of $\partial_\infty AdS_3$; hence, $C_-(\Omega)$ is an achronal meridian, containing Ω.
Analogously, define $f_+ : [0,1] \to [0,1]$ by setting $f_+(0) = 0$, $f_+(1) = 1$ and

$$f_+(t) = \inf\{\tilde{y} \mid \exists(\tilde{x},\tilde{y}) \in \tilde{\Omega} \text{ s.t. } \tilde{x} \geq t\}$$

where we consider $\inf \emptyset = 1$, and extend this function to the increasing map $f_+ : \mathbb{R} \to \mathbb{R}$ such that $f_+(t+1) = f_+(t) + 1$. Also $C_+(\Omega) = \Upsilon(\mathrm{Gr}_{f_+})$ is an achronal meridian containing Ω.
Moreover, any achronal meridian C containing Ω has to be $\Upsilon(\mathrm{Gr}_f)$ for some increasing map $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+1) = f(x) + 1$ and $f(0) = 0$. By construction, $f_- \leq f \leq f_+$. $\square$

Remark 1.3.8. If $\Omega \subset \partial_\infty AdS_3$ is connectible by an achronal meridian C, then $\overline{\Omega} \subseteq \overline{C} = C$. Thus $\overline{\Omega}$ is connectible by an achronal meridian.

Relating to the notations of the previous proof, one can consider the region

$$\mathcal{B}(\Omega) = \Upsilon(\{(x,y) \in [0,1]^2 \mid f_-(x) \leq y \leq f_+(x)\}) \tag{1.8}$$

which turns out to be the union of all the achronal meridians containing Ω. It is the union of $\overline{\Omega}$ and some (possibly degenerate) rectangles of the form $[x,x'] \times [y,y']$ with light-like edges. Rectangles of the form $[x,x'] \times [y,y']$ such that $x \neq x'$ correspond to connected components of $\partial\mathbb{H}^2 \setminus \pi_l(\Omega)$ and viceversa.

Definition 1.3.3. An achronal meridian is said Ω-extremal if it is contained in the boundary of $\mathcal{B}(\Omega)$.

Remark 1.3.9. The lower meridian and the upper meridian passing through $\partial_\infty\mathbb{S}$ are Ω-extremal meridians. Every meridian contained in the union of the upper and the lower meridian is an Ω-extremal meridian. There are no other Ω-extremal meridians.

Chapter 2

Earthquakes between ciliated surfaces

2.1 Ciliated surfaces and $\mathcal{T}_S^\star(\mathfrak{C})$

Definition 2.1.1. A ciliated surface is the data of:

- a surface S, topologically obtained by removing $\mathfrak{n}$ mutually disjoint open disks $\Delta_1, \ldots, \Delta_\mathfrak{n}$ from a compact connected oriented surface;
- m distinct points $p_1, \ldots, p_m$ on $S \smallsetminus \bigcup \partial\Delta_i$;
- $\mathfrak{s}$ points $\mathfrak{q}_1, \ldots, \mathfrak{q}_\mathfrak{s}$, called cilia, on $\bigcup \partial\Delta_i$.

Let us call $\mathfrak{C} = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_\mathfrak{s}\}$. In this section we make the hypothesis that $m = 0$ and that $\partial\Delta_i \cap \mathfrak{C} \neq \emptyset$ for every $i = 1, \ldots, \mathfrak{n}$. At the end, we will show what happens otherwise.
Consider the Teichmüller space $\mathcal{T}_S^\star(\mathfrak{C}) = \mathrm{Met}_{-1}^\star(S, \mathfrak{C}) / \operatorname{Diff}_0(S|\mathfrak{C})$ introduced in 1.2. In this section we will show that there is a correspondence between $\mathcal{T}_S^\star(\mathfrak{C})$ and the space $\mathcal{J}_S(\mathfrak{C})$ of complex structures on $\overline{S} = S \cup \bigcup \partial\Delta_i$ up to the action of $\operatorname{Diff}_0(S|\mathfrak{C})$. For the generalization of complex structures on surfaces with boundaries, see [23]
Given $J \in \mathcal{J}_S(\mathfrak{C})$, take its corresponding Fuchsian representation $\rho \in \mathcal{T}_S^\star(\mathfrak{C})$ given by the Uniformisation Theorem. The developing map dev (see [8]) from the universal cover of S to $\mathbb{H}^2$ induces an isometric immersion

$$(\overline{S}, J) \to \rho(\pi_1(S)) \backslash (\overline{\mathbb{H}}^2 \smallsetminus \Lambda_\rho),$$

where Λ_ρ is the limit set of ρ.
Notice that if $\psi : (S, J) \to (S, J')$ is an isometric element of $\operatorname{Diff}_0(S|\mathfrak{C})$, we get a lift $\hat{\psi} : \hat{S} \to \hat{S}'$ on the universal covers which extends to the ideal boundary of $\hat{S}$ so that its restriction Ψ from $\partial\hat{S}$ onto $\partial\hat{S}'$ is an equivariant homeomorphism: for every $\gamma \in \pi_1(S)$ and $x \in \partial\hat{S}$,

$$\Psi(\gamma.x) = \gamma.\Psi(x).$$

Now the developing maps dev and dev$'$ are defined up to post-composition by an element of $\mathrm{Isom}_0(\mathbb{H}^2)$, so we can suppose $\mathrm{dev}\,\big|_{\partial\hat{S}} = \mathrm{dev}' \circ \Psi$. If $\hat{\mathfrak{C}} \subset \hat{S}$ and $\hat{\mathfrak{C}}' \subset \hat{S}'$ are the preimages of Q, then $\Psi(\hat{\mathfrak{C}}) = \hat{\mathfrak{C}}'$. Thus,

$$\mathrm{dev}(\hat{\mathfrak{C}}) = \mathrm{dev}'(\Psi(\hat{\mathfrak{C}})) = \mathrm{dev}'(\hat{\mathfrak{C}}').$$

Therefore, on $\overline{\mathbb{H}}^2$ the closure $\Lambda_{\mathfrak{C}}$ in $\overline{\mathbb{H}}^2$ of the set $\overline{\mathrm{dev}(\hat{\mathfrak{C}})}$ is independent on the choice of the developing map, and well defined up to elements of $PSL(2,\mathbb{R})$. Also, we have

$$\Lambda_{\mathfrak{C}} = \Lambda_\rho \cup \tilde{\mathfrak{C}}, \tag{2.1}$$

being $\tilde{\mathfrak{C}} = \mathrm{dev}(\hat{\mathfrak{C}})$ discrete in $\overline{\mathbb{H}}^2 \setminus \Lambda_\rho$. Consider the convex hull $\mathcal{H} \subset \mathbb{H}^2$ of $\Lambda_{\mathfrak{C}}$. The hyperbolic metric $h \in \mathcal{T}_S^\star(\mathfrak{C})$ on $\rho(\pi_1(S))\backslash\mathcal{H}$ is the element we associate with the class of J.

Remark 2.1.1. In Subsection 1.1 the double S^d of S was defined (see Equation (1.4)). The orientation-preserving immersion $i = i_+ : (S,h) \to (S^d, h^d)$ can be lifted to $\tilde{i} : \mathcal{H} \to \mathbb{H}^2$ so that $\tilde{i}(\mathcal{H}) = \mathcal{H}$.
Now take $(h_l, h_r) \in \mathcal{T}_S^\star(\mathfrak{C}) \times \mathcal{T}_S^\star(\mathfrak{C})$ and consider $(h_l^d, h_r^d) \in \mathcal{T}_{S^d} \times \mathcal{T}_{S^d}$ and

$$\begin{aligned}\tilde{i}_l : \mathcal{H}_l = \mathrm{CH}(\Lambda_{\mathfrak{C}_l}) \to \tilde{S}_l^d = \mathbb{H}^2\\ \tilde{i}_r : \mathcal{H}_r = \mathrm{CH}(\Lambda_{\mathfrak{C}_r}) \to \tilde{S}_r^d = \mathbb{H}^2.\end{aligned}$$

If $F : (S, h_l) \to (S, h_r)$ is an element of $\mathrm{Diff}(S|\mathfrak{C})$, let $F^d : (S^d, h_l^d) \to (S^d, h_r^d)$ be the induced diffeomorphism. It is known (see [21]) that F^d can be lifted to $\tilde{F}^d : \mathbb{H}^2 \to \mathbb{H}^2$ so that the restriction $\tilde{F}^d_\infty : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ of its extension to $\overline{\mathbb{H}^2}$ is a homeomorphism.

$$\begin{tikzcd} S_l \arrow[r, "F"] \arrow[d, hook, "i_l"] & S_r \arrow[d, hook, "i_r"] \\ S_l^d \arrow[r, "F^d"] & S_r^d \end{tikzcd} \qquad \begin{tikzcd} \mathbb{H}^2 \supset \mathcal{H}_l \arrow[r, "\tilde{F}"] \arrow[d, hook, "\tilde{i}_l"] & \mathcal{H}_r \subset \mathbb{H}^2 \arrow[d, hook, "\tilde{i}_r"] \\ \mathbb{H}^2 = \tilde{S}_l^d \arrow[r, "\tilde{F}^d"] & \tilde{S}_r^d = \mathbb{H}^2 \end{tikzcd}$$

In particular, the restriction of $\tilde{F}^d_\infty$ to $\tilde{i}_l(\Lambda_{\mathfrak{C}_r})$ is a homeomorphism onto $\tilde{i}_r(\Lambda_{\mathfrak{C}_l})$. Thus, we get a homeomorphism $\varphi = \tilde{F}^d_\infty : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ whose restriction on $\Lambda_{\mathfrak{C}_l}$ extends $\tilde{F} : \mathcal{H}_l \to \mathcal{H}_r$.

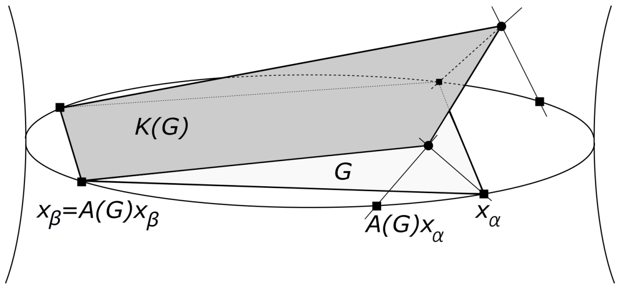
Remark 2.1.2. In [23], also a notion of complex structures that in $p_1, \ldots, p_m$ (the points removed in Definition 2.1.1) have cusps is studied. We could have considered the set of metrics with crowns and cusps instead of $\mathrm{Met}^\star_{-1}(S,\mathfrak{C})$ (which has only crowns) not assuming $m = 0$, and all the conclusions would still hold, but for coherence and simplicity we chose to focus on $\mathcal{T}_S^\star(\mathfrak{C})$.
On the contrary, the assumption that $\partial\Delta_i \cap \mathfrak{C} \neq \emptyset$ for every i will be a necessary condition to get uniqueness in the statement of the earthquake theorem for ciliated surfaces, as it will be pointed out in Remark 2.2.2.

2.2 Earthquakes and bent surfaces

In this section we will first show how to construct bent surfaces in AdS_3 that encode hyperbolic earthquakes between two given hyperbolic metrics $h_l, h_r \in \mathcal{T}_S^\star(\mathfrak{C})$. Afterwards, we will show that the boundary at infinity of such bent surfaces are sets connectible by achronal meridians.
Finally, we will prove Theorem A using such constructions.

2.2.1 Bent surfaces associated with earthquakes

Consider two surfaces $S_l = (S, h_l)$ and $S_r = (S, h_r)$ with $(h_l, h_r) \in \mathcal{T}_S^\star(\mathfrak{C})$ and holonomies hol_l, hol_r respectively, related by a right hyperbolic earthquake $E_r^\lambda : S_l \to S_r$ associated with $\lambda \in \mathcal{ML}_S(S)$. Consider its $\pi_1(S)$-invariant lifting to the universal coverings $\tilde{E} : \mathcal{H}_l \to \mathcal{H}_r$. The aim of this subsection is to show how to construct a bent surface $\mathbb{S}$ in AdS_3 that encodes $\tilde{E}$, as in [29], [2], [15].
Given a gap G of $\tilde{\lambda}_l$, the preimage of λ in $\mathcal{H}_l$, let $A(G) \in PSL(2, \mathbb{R})$ be the extension of $\tilde{E}_{|G}$ and $\{x_\alpha\}$ the set of ideal vertices of G. On the space-like



plane $\mathcal{P}_G = \mathcal{P}_{A(G)}$ whose boundary in $\partial_\infty AdS_3$ is $\mathrm{graph}(A(G))$ consider the convex hull $K(G)$ of $\{(x_\alpha, A(G)(x_\alpha))\}$. Let also $r_G : \mathbb{H}^2 \to AdS_3$ be the isometric embedding with image $\mathcal{P}_{A(G)}$ such that the trace at infinity of r_G is $(\mathrm{id}, A(G))$.

Proposition 2.2.1.

$$\mathbb{S} = \overline{\bigcup_{G \,\mathrm{gap}} K(G)}$$

is a $\pi_1(S)$-invariant past bent surface in AdS_3.

Proof. Assume that λ is locally finite, so that we are in the situation described in Remark 1.3.3. The general case is obtained through an approximation argument, using past bent surfaces (along locally finite laminations) in the future of $\mathbb{S}$ tending to $\mathbb{S}$ (see the construction of the transverse measure in Remark 1.3.2).
By construction (see Subsection 1.3.1), the trace at infinity of the space-like

plane $\mathcal{P}_0$ is graph(id). Given two gaps F and G of $\tilde{\lambda}_l$, consider the isometries $(\mathrm{id}, A(F)) : \mathcal{P}_0 \to \mathcal{P}_F$ and $(\mathrm{id}, A(G)) : \mathcal{P}_0 \to \mathcal{P}_G$. Then

$$r_G = (\mathrm{id}, A(G)) \circ (\mathrm{id}, A(F))^{-1} \circ r_F = (\mathrm{id}, B) \circ r_F$$

where $B = A(G) \circ A(F)^{-1}$. So $K(G) = r_G(G) = (\mathrm{id}, B)\big(r_F(G)\big)$.
Let l be the image of the axis of $B^* = \mathrm{cmp}(F, G) = A(F)^{-1} \circ A(G)$ through r_F; remembering that the attractive and repulsive points of hyperbolic elements verify

$$\mathrm{x}^{\pm}(A_1 \circ A_2 \circ A_1^{-1}) = A_1(\mathrm{x}^{\pm}(A_2))$$

for every transformation $A_1, A_2 \in PSL(2, \mathbb{R})$, the endpoints of l are

$$\begin{aligned} p_{\pm} &= \big(\mathrm{x}^{\pm}(B^*), A(F)\, \mathrm{x}^{\pm}(B^*)\big) = \\ &= \big(\mathrm{x}^{\pm}(B^*), \mathrm{x}^{\pm}(A(F) \circ B^* \circ A(F)^{-1})\big) = \\ &= \big(\mathrm{x}^{\pm}(B^*), \mathrm{x}^{\pm}(B)\big). \end{aligned}$$

For $(\mathrm{id}, B)(p_{\pm}) = \big(\mathrm{x}^{\pm}(B^*), B(\mathrm{x}^{\pm}(B))\big) = p_{\pm}$, l is (id, B)-invariant. If $\mathcal{P}_l$ and $\mathcal{P}_r$ are the half-planes on $\mathcal{P}_F$ (the space-like plane whose trace at infinity is the graph of the isometry $A(F)$ that extends $\tilde{E}_{|F}$) bounded by l such that

$$K(F) = \mathrm{CH}(\{(x_\alpha, A(G)(x_\alpha))\}) = r_F(F) \subset \mathcal{P}_l$$

and $r_F(G) \subset \mathcal{P}_r$, then

$$K(G) = r_G(G) = (\mathrm{id}, B)\big(r_F(G)\big) \subset (\mathrm{id}, B)(\mathcal{P}_r).$$

Since $K(F) \cup K(G) \subset \mathcal{P}_l \cup (\mathrm{id}, B)\mathcal{P}_r$ which is achronal, $\mathbb{S}$ is achronal.
Notice that, since $\tilde{E}$ is $\pi_1(S)$-invariant (denoting by $\mathrm{CH}(*)$ the convex hull of its argument),

$$\begin{aligned} K(\mathrm{hol}_l(\gamma)(F)) &= \mathrm{CH}\left\{\big(\mathrm{hol}_l(\gamma)(x_\alpha), A(F)\, \mathrm{hol}_l(\gamma)(x_\alpha)\big)\right\} = \\ &= \mathrm{CH}\left\{\big(\mathrm{hol}_l(\gamma)(x_\alpha), \mathrm{hol}_r(\gamma) A(F)(\gamma)(x_\alpha)\big)\right\} = \\ &= (\mathrm{hol}_l(\gamma), \mathrm{hol}_r(\gamma))\big(\mathrm{CH}\left\{\big(x_\alpha, A(F)(x_\alpha)\big)\right\}\big) = \\ &= (\mathrm{hol}_l(\gamma), \mathrm{hol}_r(\gamma))(K(F)) \end{aligned}$$

and so $\mathbb{S}$ is $\pi_1(S)$-invariant.
It is not true that glueing the maps r_F gives a bending map $\mathrm{b} : \mathcal{H} \to \mathbb{S}$, because it is not well defined on $\tilde{\lambda}$: if $p \in F \cap G$ and $F \neq G$ then

$$r_G(p) = (\mathrm{id}, B) \circ r_F(p).$$

The transformation (id, B) acts on l as a translation of length

$$\frac{\mathrm{T}(B)}{2} = \frac{\mathrm{T}(B^*)}{2} = \frac{\mathrm{T}(\mathrm{cmp}(F, G))}{2}.$$

So if we fix a gap F_0, translate on the right its adjacent ones of a factor half the weight of their separating leaf and iterate (that is, a right earthquake on $\mathcal{H}_l$ associated with $\tilde{\lambda}_l/2$), we get a domain $\mathcal{H}$ on which is induced and well defined the bending map $\mathrm{b} : \mathcal{H} \to \mathbb{S}$.
Every internal point of $K(F)$ has a neighbourhood where it is immediate to check that $\mathbb{S}$ is past convex. In the case where a point q lies in $K(F) \cap K(G)$, notice that $K(F) \cup K(G) \subset \mathcal{P}_r \cup (\mathrm{id}, B)\mathcal{P}_l$ and

$$\overline{\vartheta}(\mathcal{P}_{A(F)}, (\mathrm{id}, B)\mathcal{P}_{A(F)}) = \overline{\vartheta}(\mathcal{P}_{\mathrm{id}}, (\mathrm{id}, B)\mathcal{P}_{\mathrm{id}}) = \overline{\vartheta}(\mathcal{P}_{\mathrm{id}}, \mathcal{P}_B) > 0$$

so by Remark 1.3.1 also q has a neighbourhood where the past convexity of $\mathbb{S}$ is verified. □

2.2.2 Achronal meridians and convex sets associated with bent surfaces

Now we will see that the boundary at infinity of the bent surface $\mathbb{S}$ in AdS_3 associated previously with $E_r^\lambda : S_l \to S_r$ is connectible by achronal meridians; among them, there is one whose convex hull in AdS_3 has $\mathbb{S}$ itself contained in the boundary. .
Since $\tilde{E} : \mathcal{H}_l \to \mathcal{H}_r$ can be equivariantly extended on $\partial G \subset \partial \mathbb{H}^2$ for every stratum G, the set

$$\partial_\infty \mathbb{S} = \overline{\{(x, \tilde{E}(x)) \in \partial_\infty AdS_3 : x \in \partial G \text{ with } G \text{ stratum}\}}$$

is well defined. Notice that $\partial_\infty \mathbb{S} = \bigcup \partial_\infty K(G)$. Moreover, any point of ∂G is an element of the limit set of $\mathcal{H}_l$ or the preimage of a tip of S_l.

Lemma 2.2.2. *The set $\partial_\infty \mathbb{S}$ is connectible by an achronal meridian.*

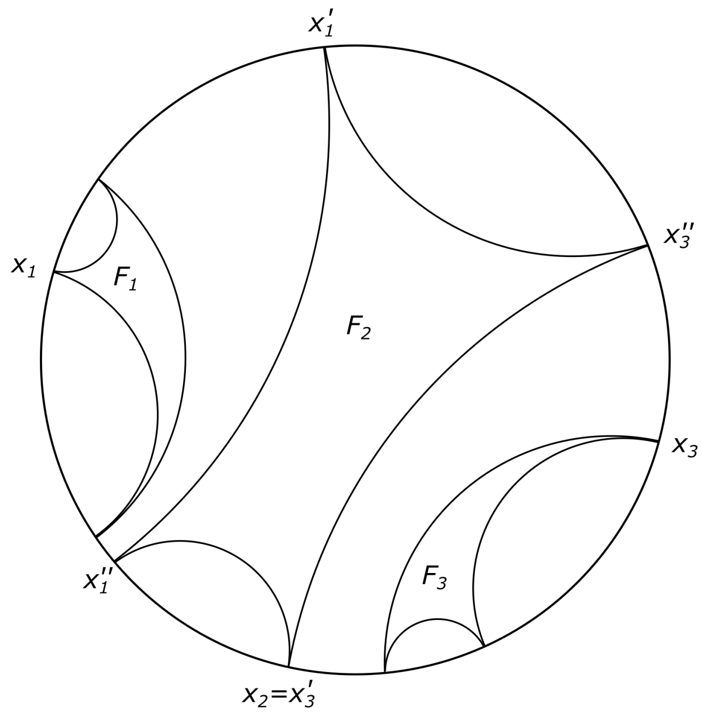
Proof. By Remark 1.3.8, it is sufficient to show that the set

$$T = \{(x, \tilde{E}(x)) \in \partial_\infty AdS_3 : x \in \partial G \text{ with } G \text{ stratum}\}$$

is connectible by an achronal meridian, being $\partial_\infty \mathbb{S} = \overline{T}$. By definition, we must check that if (x_1, y_1), (x_2, y_2), (x_3, y_3) are points in T such that $x_1 < x_2 < x_3$ then $y_1 < y_2 < y_3$. For every $i = 1, 2, 3$ there is a stratum F_i of $\tilde{\lambda}_r$ such that $(x_i, y_i) \in K(F_i)$. Suppose that $F_1 \neq F_2 \neq F_3$ (it is easy to check the other cases, once this one is considered). We can suppose without loss of generality that F_2 separates F_1 and F_3 or there exists a stratum F_4 separating F_1, F_2 and F_3. In the first case, there are points x_1', x_1'', x_3', x_3'' in $\partial_\infty F_2$ such that

$$x_1' \le x_1 \le x_1'' \quad \text{and} \quad x_3' \le x_3 \le x_3''$$

and the intervals (x_1', x_1'') and (x_3', x_3'') are disjoint and do not contain points of $\partial_\infty F_2$. Notice that either $x_1'' = x_2 = x_3'$ or $x_1'' \le x_2 \le x_3'$.



Now take the extension $A \in PSL(2,\mathbb{R})$ of $E_r^{\tilde{\lambda}_r}\big|_{F_2}$. Since $A^{-1} \circ E_r^{\tilde{\lambda}_r}$ fixes F_2, $A^{-1}\big(E_r^{\tilde{\lambda}_r}(x_i)\big) \in [x_i', x_i'']$ for $i = 1, 3$. Therefore,

$$A^{-1}y_1 = A^{-1}\big(E_r^{\tilde{\lambda}_r}(x_1)\big) < x_2 < A^{-1}\big(E_r^{\tilde{\lambda}_r}(x_3)\big) = A^{-1}y_3.$$

Since the trace at infinity of A preserves the orientation of $\partial\mathbb{H}^2$, we conclude that $y_1 < y_2 < y_3$.
The latter case is similar (considering the earthquake $B^{-1} \circ E_r^{\tilde{\lambda}_r}$, where $B \in PSL(2,\mathbb{R})$ extends $E_r^{\tilde{\lambda}_r}\big|_{F_4}$). □

Remark 2.2.1. Since $\mathbb{S}$ is invariant by the action of $\pi_1(S)$ (see Proposition 2.2.1), it is easy to check that also $C_-(\partial_\infty\mathbb{S})$ and $C_+(\partial_\infty\mathbb{S})$, the extremal meridians introduced in Lemma 1.3.3, are $\pi_1(S)$-invariant.

The following proposition shows how to recover $\mathbb{S}$ from $C_-(\partial_\infty\mathbb{S})$.

Proposition 2.2.3. *The bent surface $\mathbb{S}$ is the space-like part of the upper boundary of $\mathcal{K} = \mathrm{CH}(C_-(\partial_\infty\mathbb{S}))$, called the future boundary of $\mathcal{K}$ and denoted by $\partial_+\mathcal{K}$.*

Proof. The first part of the argument follows [15]. Recall that, for every stratum G of $\tilde{\lambda}_l$ with ideal vertices $\{x_\alpha\}$, $K(G)$ is the convex hull of $\{(x_\alpha, \tilde{E}(x_\alpha))\} \subset C_-(\partial_\infty\mathbb{S})$; then $\mathbb{S} \subset \mathcal{K}$.
In order to prove that $\mathbb{S} \subset \partial_+\mathcal{K}$, it is sufficient to show that for every gap

F of $\tilde{\lambda}_l$ the space-like plane $\mathcal{P}_F$ containing $K(F)$ is an upper support plane for $\mathcal{K}$.
Without loss of generality, suppose that $\tilde{E}_{|F} = A(F) = \mathrm{id}_F$. Then $\mathcal{P}_F = \mathcal{P}_0$. If G is another gap then there are $x, x' \in \partial_\infty F$ such that the geodesic with endpoints x, x' is a component of ∂F and $x \leq y \leq x'$ for every $y \in \partial_\infty G$. Then $x < \tilde{E}(y) < y$ or $y = \tilde{E}(y) \in \{x, x'\}$ for every $y \in \partial_\infty G$, being $\tilde{E}$ a right earthquake. Therefore, $f_-(t) \leq t$ for every $t \in [0,1]$; using Remark 1.3.6 and noticing that $\overline{x}, \overline{x}'$ are fixed points of f_-, $\mathcal{P}_F$ turns out to be an upper support plane for $\mathcal{K}$.
In order to prove that $\partial_+\mathcal{K} \smallsetminus \mathbb{S}$ is not space-like, it is sufficient to show that for every $p \in \partial_+\mathcal{K} \smallsetminus \mathrm{b}(\mathcal{H})$ there exists a support light-like plane for $\mathcal{K}$ in p. Take a support plane $\mathcal{P}$ for $\mathcal{K}$ in p. It can not be time-like. If it is light-like, we have finished. If it is space-like instead, consider

$$\mathcal{P} \cap C_-(\partial_\infty \mathbb{S}) = \{(x_\alpha, y_\alpha)\}$$

and the convex hull $D_\mathcal{P} \subset \mathbb{H}^2$ of $\{x_\alpha\}$. By Remark 1.3.7, $D_\mathcal{P}$ is weakly separated from all the strata of $\tilde{\lambda}_l$, and so from $\mathcal{H}_l$. Then there is a component $I = (x', x'')$ of $\partial\mathbb{H}^2 \smallsetminus \partial\mathcal{H}_l$ such that $\{x_\alpha\} \subset I$. It follows that $\mathcal{P} \cap C_-(\partial_\infty\mathbb{S}) \subset \bar{I} \times \bar{J}$ where $J = (\tilde{E}(x'), \tilde{E}(x'')) = (y', y'')$. More precisely,

$$\mathcal{P} \cap C_-(\partial_\infty\mathbb{S}) \subset (\bar{I} \times \bar{J}) \cap C_-(\partial_\infty\mathbb{S}) = (\bar{I} \times \{y'\}) \cup (\{x''\} \times \bar{J}) = Z.$$

If $q' = (x', y')$ and $q'' = (x'', y'')$, the only way for the space-like plane $\mathcal{P}$ to be an upper support plane for $\mathrm{CH}(C_-(\partial_\infty\mathbb{S}))$ is to meet Z only in q' and q''. Moreover, since $\mathcal{P}$ is a support plane in $p \notin \mathrm{b}(\mathcal{H})$, p lies on the geodesic s with endpoints q', q''.
By Remark 1.3.5, the light-like plane $\mathcal{L}$ dual to $d = (x', y'')$ is a support plane for $\mathcal{K}$. It contains $f_l(d), f_r(d)$, so in particular q', q''. Thus, $p \in s \subset \mathcal{L}$. $\square$

2.2.3 Existence and uniqueness of $E : S_l \to S_r$

This subsection is devoted to the proof of Theorem A.

Theorem 2.2.4. *For every $h_l, h_r \in \mathcal{T}_S^\star(\mathfrak{C})$ there exists a unique right earthquake between $S_l = (S, h_l)$ and (S, h_r).*

Proof. Consider the universal covers

$$\mathcal{H}_l \to S_l = \mathrm{hol}_l(\pi_1(S)) \backslash \mathcal{H}_l$$
$$\mathcal{H}_r \to S_r = \mathrm{hol}_r(\pi_1(S)) \backslash \mathcal{H}_r$$

with $\mathcal{H}_l, \mathcal{H}_r \subset \mathbb{H}^2$, defined in Section 2.1. Recall that the developing maps $\mathrm{dev}_l : \tilde{S}_l \to \mathbb{H}^2$ and $\mathrm{dev}_r : \tilde{S}_r \to \mathbb{H}^2$ can be extended on the preimages of the tips and determine the same set in $\mathbb{H}^2$ up to post-composition by elements of $PSL(2,\mathbb{R})$. This fact lead us in Remark 2.1.1 to find a homeomorphism $\varphi : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ such that its graph contains

$$\Big\{ \big(\mathrm{x}^+(\mathrm{hol}_l(\gamma)), \mathrm{x}^+(\mathrm{hol}_r(\gamma)) \big) \,:\, \gamma \in \pi_1(S) \Big\},$$

where $\mathrm{x}^+(A)$ denotes the attractive point of any hyperbolic element A in $PSL(2,\mathbb{R})$. Also, the restriction of φ to $\tilde{\mathfrak{C}}_l$, the ideal set in $\partial_\infty\mathcal{H}_l$ corresponding to the preimage of $\tilde{\mathfrak{C}}$, is an equivariant homeomorphism onto $\tilde{\mathfrak{C}}_r$, so that $\tilde{\mathfrak{q}}$ and $\varphi(\tilde{\mathfrak{q}})$ project to the same cilium for every $\tilde{\mathfrak{q}} \in \tilde{\mathfrak{C}}_l$. Since φ is a homeomorphism, its graph is an achronal meridian in $\partial_\infty AdS_3$, by Remark 1.3.4. In particular, $\Omega = \mathrm{graph}(\varphi\big|_{\Lambda_{\mathcal{Q}_l}})$ is a set connectible by an achronal meridian.

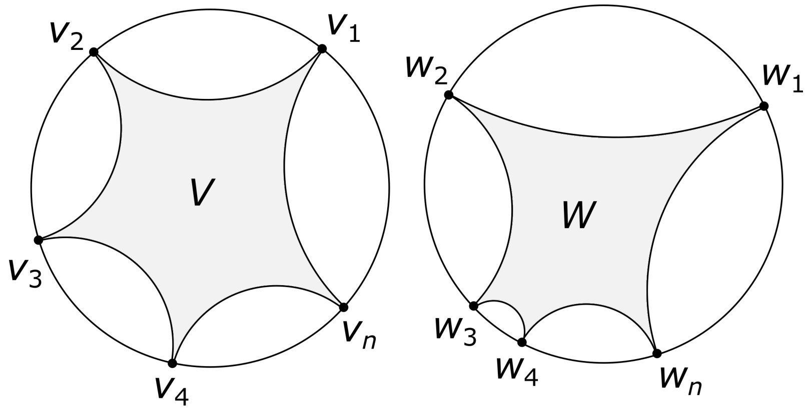
By the work of Benedetti and Bonsante, [**?**] earthquakes between surfaces with geodesic boundary with holonomy hol_l and hol_r, are in bijective correspondence with $(\mathrm{hol}_l, \mathrm{hol}_r)$-invariant bent surfaces in AdS^3, through the construction given in the previous sections. In particular a bijective correspondence exists between those earthquakes and $(\mathrm{hol}_l, \mathrm{hol}_r)$-invariant achronal meridians, up to isometries of AdS_3. To prove the theorem we will look for achronal meridians corresponding to earthquakes between ciliated surfaces which fix the tips (that is $E(\tilde{\mathfrak{q}}) = \varphi(\tilde{\mathfrak{q}})$ for any $\tilde{\mathfrak{q}} \in \Lambda_{\mathfrak{C}_l}$).
Recall that $\partial_\infty AdS_3$ has a left foliation $\mathcal{F}_l$ and a right one $\mathcal{F}_r$. Fixed a space-like plane $\mathcal{P}_0$, identified with $\mathbb{H}^2$, consider the left and right projections $\pi_l, \pi_r : \partial_\infty AdS_3 \to \partial_\infty\mathcal{P}_0 \cong \partial\mathbb{H}^2$ through the leaves of $\mathcal{F}_l$ and $\mathcal{F}_r$. Now pick an achronal meridian C in $\partial_\infty AdS_3$. Given a face or a bending line F of the space-like part $\mathbb{S}^+(C)$ of the future boundary of the convex hull of C we can consider the ideal convex sets or geodesics F_l and F_r in $\mathbb{H}^2$ obtained as the convex hull of the left and right projections of endpoints of F. The domain H_l of the right earthquake associated with $\mathbb{S}^+(C)$ is the convex domain of $\mathbb{H}^2$ obtained as the union of the faces F_l, and the earthquake sends F_l to F_r. Also, from $\mathbb{S}^-(C)$, we get a left earthquake between H_l and H_r. See [7] for details. We are looking for those achronal curves C for which $H_l = \mathcal{H}_l$ and $H_r = \mathcal{H}_r$, which means that $\pi_l(\mathbb{S}^+(C)) = \Lambda_{\mathfrak{C}_l}$ and $\pi_r(\mathbb{S}^+(C)) = \Lambda_{\mathfrak{C}_r}$.
Consider $\tilde{\mathfrak{q}} \in \mathfrak{C}_l$. It is an isolated vertex of $\mathcal{H}_l$. As seen before, if a vertex of a stratum of $\mathbb{S}^+$ lies on $[\tilde{\mathfrak{q}}, \tilde{\mathfrak{q}}'] \times [\varphi(\tilde{\mathfrak{q}}), \varphi(\tilde{\mathfrak{q}}')]$, then it must be one of the vertices. As we are requiring that $\tilde{E}(\tilde{\mathfrak{q}}) = \varphi(\tilde{\mathfrak{q}})$, any stratum meeting $[\tilde{\mathfrak{q}}, \tilde{\mathfrak{q}}'] \times [\varphi(\tilde{\mathfrak{q}}), \varphi(\tilde{\mathfrak{q}}')]$ must have a vertex either at $(\tilde{\mathfrak{q}}, \varphi(\tilde{\mathfrak{q}}))$ or at $(\tilde{\mathfrak{q}}', \varphi(\tilde{\mathfrak{q}}'))$. Imposing that $\tilde{E}(\tilde{\mathfrak{q}}) = \varphi(\tilde{\mathfrak{q}})$, we also have that $(\tilde{\mathfrak{q}}, \varphi(\tilde{\mathfrak{q}}))$ must be either a point in a stratum or an accumulation point of strata. However, the latter case cannot hold, since the left (or the right) projections of ideal points of accumulating strata would be elements of H_l (or H_r) tending to $\tilde{\mathfrak{q}}'$ (or $\varphi(\tilde{\mathfrak{q}})'$), leading to a contradiction. Thus we get that there is a stratum passing through $(\tilde{\mathfrak{q}}, \varphi(\tilde{\mathfrak{q}}))$. However, we need that $\tilde{\mathfrak{q}}$ lies in $\partial_\infty H_l$ and $\varphi(\tilde{\mathfrak{q}})$ lies

in $\partial_\infty H_r$, so there must be a strata of $\mathbb{S}^+$ whose trace contains $(\tilde{\mathfrak{q}}, \varphi(\tilde{\mathfrak{q}}))$. Therefore, C has to be an achronal meridian containing Ω.
In order to see that C has also to be Ω-extremal, take the light-like plane $\mathcal{L}$ dual to $(\tilde{\mathfrak{q}}', \varphi(\tilde{\mathfrak{q}}))$. Suppose that it transversally meet $\mathbb{S}^+(C)$; then it decomposes $\mathbb{S}^+(C)$ in two regions, and the asymptotic boundary of one of the two regions must be contained in $(\tilde{\mathfrak{q}}, \tilde{\mathfrak{q}}') \times (\varphi(\tilde{\mathfrak{q}}), \varphi(\tilde{\mathfrak{q}}'))$. As we know that no stratum has endpoints in this region we get a contradiction. Thus $\mathcal{L}$ must be a light-like support plane, so the curve C is a past Ω-extremal curve. Thus, C can be only the achronal meridian $C_-(\Omega)$. This shows the uniqueness part of the statement.
We now have to show that the curve $C_-(\Omega)$ is associated with an earthquake between the ciliated surfaces. First notice that as the light-like plane dual to $(\tilde{\mathfrak{q}}, \varphi(\tilde{\mathfrak{q}}'))$ is a future support plane for $\mathbb{S}^+(C)$, so the space-like line between $(\tilde{\mathfrak{q}}, \varphi(\tilde{\mathfrak{q}}))$ and $(\tilde{\mathfrak{q}}', \varphi(\tilde{\mathfrak{q}}'))$ disconnects $\mathbb{S}^+(C)$ in two regions, one of them being a light-like triangle. So space-like strata of $\mathbb{S}^+(C)$ can meet $[\tilde{\mathfrak{q}}, \tilde{\mathfrak{q}}'] \times [\varphi(\tilde{\mathfrak{q}}), \varphi'(\tilde{\mathfrak{q}})]$ only at $(\tilde{\mathfrak{q}}, \varphi(\tilde{\mathfrak{q}})), (\tilde{\mathfrak{q}}', \varphi(\tilde{\mathfrak{q}}'))$. It follows that $\mathcal{H}_l$ contains points $\tilde{\mathfrak{q}}$, but no other points between two tips. So H_l coincides with $\mathcal{H}_l$, and $\tilde{E}$ sends $\tilde{\mathfrak{q}}$ to $\varphi(\tilde{\mathfrak{q}})$. $\square$

Remark 2.2.2. As outlined in Section 2.1, this result can be easily generalized to all types of ciliated surfaces. We chose S as the topological data of a closed surface with $m = 0$ points and $\mathfrak{n} - m$ closed disjoint disks Δ_i removed and of a finite set $\mathfrak{C}$ in $\bigcup \partial\Delta_i$ so that $\mathfrak{C} \cap \partial\Delta_i \neq \emptyset$ for every i, in order to have only crowns at the $\mathfrak{n}$ punctures. If we drop the restriction $m = 0$, at the punctures in the removed points $p_1, \dots, p_m$ cusps occur. If we also eliminate the condition that $\partial\Delta_i$ must have at least one cilium for every i, the disks without cilia give raise to closed geodesic boundary components. All the arguments in [7] and [15] apply also in this case, with a convenient Teichmüller space (where punctures can have cusp, crown or closed geodesic structure), and again given two admissible hyperbolic metrics (h_l, h_r) on S there will be 2^k right earthquakes sending h_l to h_r, as in Theorem , where k is the number of boundary components of S having a closed geodesic structure both in h_l and h_r.

2.3 Ideal polygons

In this section we are dealing with ideal polygons of $\mathbb{H}^2$, considered as simply connected hyperbolic surfaces of genus 0 with one puncture and as many cilia as their ideal vertices. The existence and uniqueness of earthquakes between two ideal n-gons is already known (see for example [20], ore use an argument of double). Here we show the constructive proof, using bent surfaces in AdS_3 as in [20]. That is meant to be an illustrative example of the arguments involved in the previous section.



Proposition 2.3.1. *Let V and W two ideal n-gons in $\mathbb{H}^2$. Suppose the vertices of V and W numbered as $v_1, \dots, v_n$ and $w_1, \dots, w_n$ respectively, consecutively and counterclockwise. Then there exists a unique right earthquake $E : V \to W$ such that $E(v_j) = w_j \; \forall j$.*

Proof. Let U be the set $\{u_j = (v_j, w_j) \in \partial_\infty AdS_3 \mid j = 1, \dots, n\}$. There exists $\psi \in \mathrm{Homeo}^+(\partial\mathbb{H}^2)$ such that $\psi(v_j) = w_j$ for $j = 1, \dots, n$. Then we can find a space-like plane $\mathcal{Q}$ disjoint from graph$(\psi) \in \partial_\infty AdS_3$. Consider $\mathcal{K} = \mathrm{CH}(U)$ in (an affine chart of) AdS_3 and his future boundary

$$\partial_+\mathcal{K} = \{x \in \partial\mathcal{K} \mid \mathcal{K} \subset I^-(\mathcal{P}) \text{ for every support plane } \mathcal{P} \text{ of } \mathcal{K} \text{ with } x \in \mathcal{P}\}.$$

Denote by C the union of the geodesics with ideal endpoints u_j and u_{j+1}, for $j = 0, \dots, n$ (where $u_0 = u_n$). Notice that $\mathcal{K}$ can be thought as an ideal

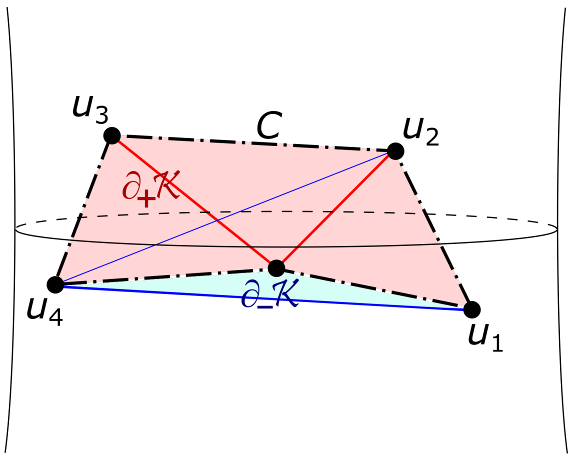


Figure 2.1: The convex hull of U

polyhedron in AdS_3 and that $\partial\mathcal{K}$ topologically is S^2 with n punctures joint by a close curve C. The two connected components of $S^2 \setminus C$ correspond

to the future and the past boundary of $\mathcal{K}$.
No faces of $\partial_+\mathcal{K}$ can be light-like (in the sense that they can not lie on light-like planes). In order to check this, by contradiction suppose there are three distinct vertices u_a, u_b, u_c of a face contained in a light-like plane $\mathcal{L}$; then

$$u_a, u_b, u_c \in \mathcal{L} \cap \partial_\infty AdS_3 = f_l(q) \cup f_r(q)$$

for a certain $q \in \partial_\infty AdS_3$ (by $f_l(q)$ and $f_r(q)$ we respectively refer to the left and the right leaf of $\partial_\infty AdS_3$ passing through q). Therefore, two vertices lie in the same leaf, e.g. u_a e u_b. However, if $u_a, u_b \in f_l(q)$ then $v_a = v_b$ and if instead $u_a, u_b \in f_r(q)$ then $w_a = w_b$, which in both cases gives a contraddiction.
Faces of $\partial_+\mathcal{K}$ can not be time-like: every time-like plane meets graph(ψ) in at most two points.
For every space-like plane $\mathcal{S}$ denote by $\phi_{\mathcal{S},l} : \mathcal{S} \to \mathcal{P}_{\mathrm{id}}$ and $\phi_{\mathcal{S},r} : \mathcal{S} \to \mathcal{P}_{\mathrm{id}}$ the isometries such that $\left(\phi_{\mathcal{S},l}\right)_{|\partial_\infty \mathcal{S}} = \pi_l$ and $\left(\phi_{\mathcal{S},r}\right)_{|\partial_\infty \mathcal{S}} = \pi_r$. For every $x \in \partial_+\mathcal{K}$ choose a space-like support plane $\mathcal{S}(x)$. Consider the earthquake maps $\mathcal{E}_l : \partial_+\mathcal{K} \to \mathcal{P}_{\mathrm{id}}$ and $\mathcal{E}_r : \partial_+\mathcal{K} \to \mathcal{P}_{\mathrm{id}}$ such that $\mathcal{E}_l(x) = \phi_{\mathcal{S}(x),l}(x)$ and $\mathcal{E}_r(x) = \phi_{\mathcal{S}(x),r}(x)$.
We claim that $\mathcal{E}_l(\partial_+\mathcal{K}) = V$ and $\mathcal{E}_r(\partial_+\mathcal{K}) = W$. It is in fact true that, for every j, $\mathcal{E}_l(u_j) = v_j$, so if a face F of $\partial_+\mathcal{K}$ has vertices $u_{j_1}, \ldots, u_{j_k}$ then $\mathcal{E}_l(F)$ is the ideal polygon with vertices $v_{j_1}, \ldots, v_{j_k}$. From the topological point of view discussed above, the faces of $\partial_+\mathcal{K}$ partition a disk and all the vertices lie on $\partial_\infty AdS_3$. Therefore the images through $\mathcal{E}_l$ of such faces partition $\mathrm{CH}(\{v_1, \ldots, v_n\})$, which coincides with V.
So the map $E = \mathcal{E}_r \circ (\mathcal{E}_l)^{-1} : V \to W$ is well defined and by construction it is

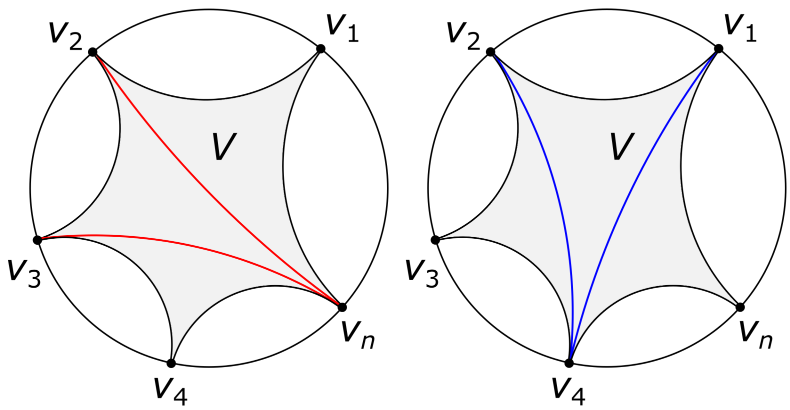


Figure 2.2: The projections of the bending loci of $\partial_+\mathcal{K}$ and $\partial_-\mathcal{K}$ on V

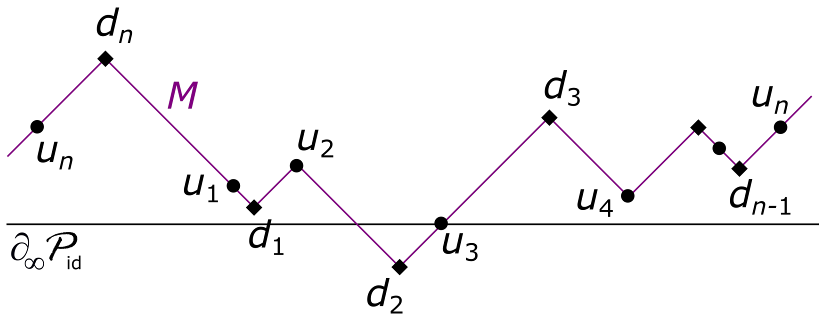
a right earthquake (see next Remark, point 1) such that $E(v_j) = w_j$ $\forall j$. $\square$

Remark 2.3.1. Since U can be contained in exactly 2^n extremal meridians, a natural question is if for each one of them there exists an earthquake induced

by the future boundaries of their convex hulls, obtaining 2^n earthquakes with the same boundary conditions.
Extremal meridians containing U are closed polygonal chains in $\partial_\infty AdS_3$ with vertices u_j for some $j \in \{0, \ldots, n\}$ and

$$d_j \in \{f_l(u_j) \cap f_r(u_{j+1}), f_r(u_j) \cap f_l(u_{j+1})\}$$

for every $j \in \{1, \ldots, n\}$ (where $u_{n+1} = u_1$).
For every $q \in \partial_\infty AdS_3$, let $\mathcal{L}(q)$ be the light-like plane tangent in q to



$\partial_\infty AdS_3$. Choose M between the 2^n considered extremal meridians and let H be $\mathrm{CH}(M)$. We say that d_j is upper if $d_j = f_l(u_j) \cap f_r(u_{j+1})$, lower if $d_j = f_r(u_j) \cap f_l(u_{j+1})$; we have three cases.

1) *Every d_j is lower.* In that case

$$\partial_+ H = \partial_+ \mathcal{K} \cup \bigcup_{j=1}^{n} \mathrm{CH}(\{u_j, d_j, u_{j+1}\}),$$

but, for every j, $\mathrm{CH}(\{u_j, d_j, u_{j+1}\}) \subset \mathcal{L}(d_j)$. Then the future boundary of H is $\partial_+ \mathcal{K}$ and so we get the earthquake of the Proposition 2.3.1.

2) *Every d_j is upper.* In that case (where we assume $d_0 = d_n$)

$$\partial_+ H = \partial_+ \mathrm{CH}(\{d_1, \ldots, d_n\}) \cup \bigcup_{j=1}^{n} \mathrm{CH}(\{u_j, d_j, d_{j-1}\})$$

but, for every j, $\mathrm{CH}(\{u_j, d_j, d_{j-1}\}) \subset \mathcal{L}(u_j)$. Then the future boundary of H is $\partial_+ \mathrm{CH}(\{d_1, \ldots, d_n\})$. However, the earthquake obtained as $\mathcal{E}_r \circ (\mathcal{E}_l)^{-1}$ has $v_j \mapsto w_{j+1}$ as boundary conditions, since $\pi_l(d_j) = v_j$ and $\pi_r(d_j) = w_{j+1}$.

3) *There are an upper d_i and a lower d_k.* In that case (where we assume $d_0 = d_n$) there exists j such that d_{j-1} is upper and d_j lower, or viceversa. Let us consider the first possibility (the second one is analogous). It turns out that u_j lies in $\partial_+ H$ but not in its space-like part: the unique face of

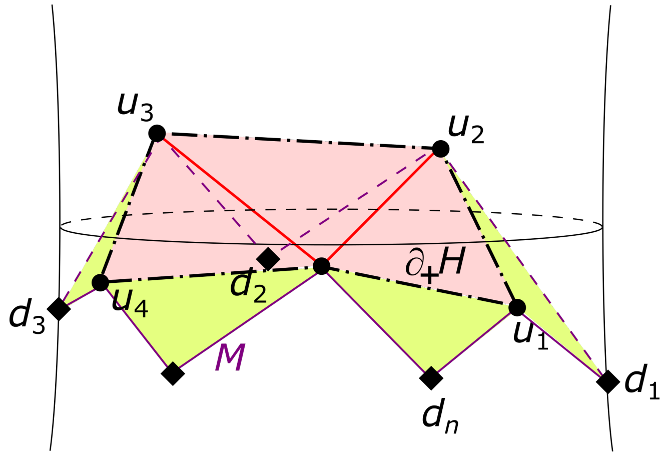


Figure 2.3: Every d_j is lower. Compare to Figure 2.1

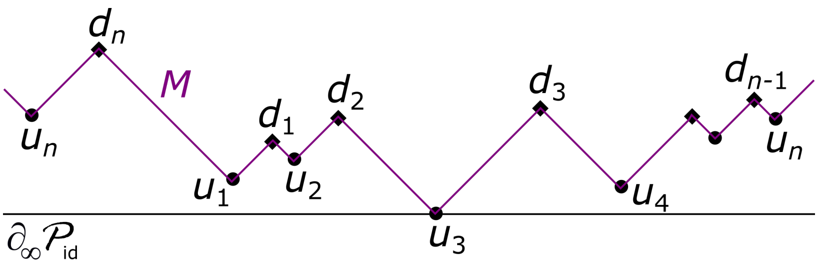


Figure 2.4: Every d_j is upper

$\partial_+ H$ where u_j lies in is the one containing $\mathrm{CH}(\{d_{j-1}, d_j, u_{j+1})$, which is itself contained in $\mathcal{L}(d_j)$.
Notice now that $\left(\pi_{l|M}\right)^{-1}(v_j) = \{u_j\}$. Thus the image through $\mathcal{E}_l$ of the future boundary of H does not contain v_j and then $\mathcal{E}_r \circ (\mathcal{E}_l)^{-1}$, (beyond the fact that a priori it may not be an hyperbolic earthquake), does not have V as domain.

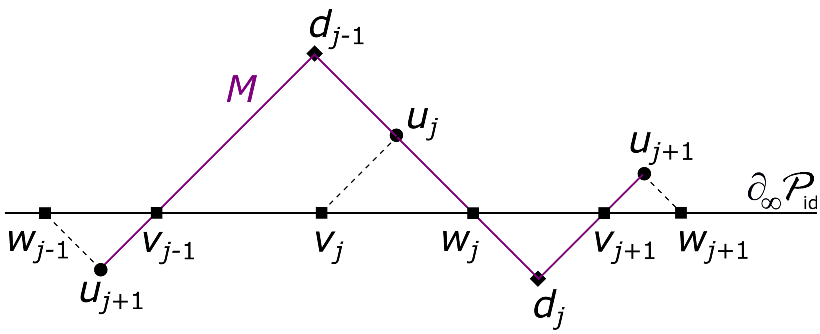


Figure 2.5: A d_j is lower and d_{j-1} is upper

Chapter 3

The map $\Phi^{\mathbf{b}} : \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ \to \mathcal{F}^0\mathcal{ML}(\mathbf{b})$

3.1 Setting

Let S be a surface of genus g with $\mathfrak{n}$ boundary components, named $\partial_1, \ldots, \partial_{\mathfrak{n}}$, with $\chi(S) = 2 - 2g - \mathfrak{n} < 0$. For every $\lambda \in \mathcal{ML}_S^\circ$ let $E_r^\lambda : \mathcal{T}_S^\circ \to \mathcal{T}_S^\circ$ be the right earthquake along λ on S and $E_l^\lambda : \mathcal{T}_S^\circ \to \mathcal{T}_S^\circ$ be the left earthquake along λ on S.
In the closed case, namely when $\mathfrak{n} = 0$, it is well known that given an element $(h, h') \in \mathcal{T}_S \times \mathcal{T}_S$ there exists a unique element $\Phi(h, h') = (\lambda, \mu)$ in $\mathcal{ML}_S \times \mathcal{ML}_S$ such that $h' = E_l^\lambda(h) = E_r^\mu(h)$ (see [34]). Moreover, λ and μ fill up S, which means that any closed curve with no intersection with λ and μ is homotopically trivial.
The inverse problem can be stated in the following form: given two filling measured geodesic laminations λ and μ on S, what can be said about $\Phi^{-1}(\lambda, \mu)$? An answer is given by Theorem 1.1 in [15], which asserts that there exists an element (h, h') in $\mathcal{T}_S \times \mathcal{T}_S$ such that $h' = E_l^\lambda(h) = E_r^\mu(h)$.
In the case where $\mathfrak{n} \neq 0$, the analogous questions can be considered. For every $(h, h') \in \mathcal{T}_S^\circ \times \mathcal{T}_S^\circ$ there are exactly $2^{\mathfrak{n}}$ couples of measured geodesic laminations such that the left earthquake along the first one and the right earthquakes along the second one take h to h'. This is Theorem 1.2 in [15]. Moreover, λ and μ fill up S, in the sense that any closed curve with no intersections with λ and μ is homotopically trivial or isotopic to a boundary component of S.
One of the main differences with respect to the closed case is that measured geodesic laminations on S can contain spiralling leaves around some boundary components of S. Fixed an orientation on S, the induced one on ∂S allows to define a positive sense of spiralling near each ∂_i and a negative one (see [15]). This two possibilities are related to the lack of uniqueness of the earthquakes. For every couple (λ, μ) such that $h' = E_l^\lambda(h) = E_r^\mu(h)$, if λ

spirals near a component ∂_i then μ spirals near ∂_i in the opposite way to λ. In order to find an analogue of the map Φ considered in the closed case, in the next subsection we will give more details on the behaviour of earthquakes near ∂S.

3.1.1 Boundary conditions

Let $\lambda \in \mathcal{ML}_S^{\circ}$ be a measured lamination on S and ∂_i a boundary component of S. Call $\iota(\partial_i, \lambda)$ the mass of λ at ∂_i (see [15], subsection 2.3) and $m(\partial_i, \lambda)$ the signed mass, such that

$$m(\partial_i, \lambda) = \begin{cases} +\iota(\partial_i, \lambda) & \text{if } \lambda \text{ spirals in the positive sense near } \partial_i \\ -\iota(\partial_i, \lambda) & \text{otherwise} \end{cases},$$

where the positive sense is determined by the orientation on ∂S induced by the orientation of S. In particular (see [15], Proposition 3.3)

$$\begin{cases} \ell_{E_r^{\lambda}(h)}(\partial_i) = |\ell_h(\partial_i) + m(\partial_i, \lambda)| \\ \ell_{E_l^{\lambda}(h)}(\partial_i) = |\ell_h(\partial_i) - m(\partial_i, \lambda)| \end{cases}$$

So there are two ways to transform the length of a boundary component ∂_i from b_i to b'_i through a left earthquake along λ; one changes the way of spiralling of λ (and roughly speaking passes through a cusp), the other one keeps the way of spiralling of λ (and does not pass through a cusp). See Subsection 3.3 in [15].

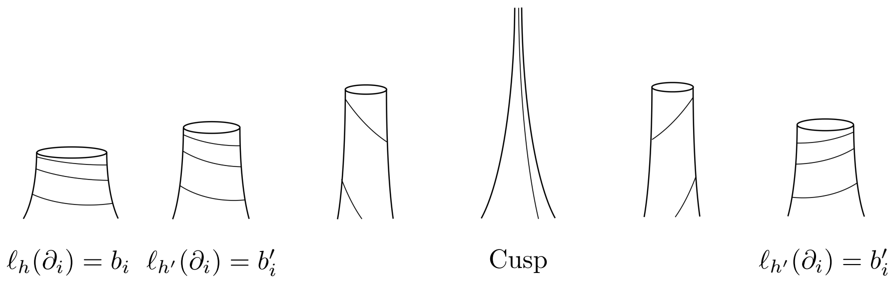


Figure 3.7: The case $b_i > b'_i$

For every $(h, h') \in \mathcal{T}_S^{\circ} \times \mathcal{T}_S^{\circ}$, Theorem shows that there are $2^{\mathfrak{n}}$ couples of filling laminations (λ, μ) such that $E_l^{2\lambda}(h) = E_r^{2\mu}(h) = h'$. Each couple (λ, μ) satisfies then

$$\ell_{h'}(\partial_i) = |\ell_h(\partial_i) + m(\partial_i, 2\mu)| = |\ell_h(\partial_i) - m(\partial, 2\lambda)|$$

for every $i = 1, \ldots, \mathfrak{n}$. There is exactly one couple for which the two earthquakes do not pass through a cusp, namely the one verifying (see [15])

$$\ell_h(\partial_i) + m(\partial_i, 2\mu) = \ell_h(\partial_i) - m(\partial, 2\lambda) > 0 \quad \forall i = 1, \ldots, \mathfrak{n}.$$

Therefore, we have the open condition $m(\partial_i, 2\mu) = -m(\partial, 2\lambda) > -\ell_h(\partial_i)$. If we fix $\mathbf{b} = (b_1, \ldots, b_{\mathfrak{n}})$ and consider

$$\mathcal{FML}^\circ_S(\mathbf{b}) = \{(\lambda, \mu) \in \mathcal{ML}^\circ_S \times \mathcal{ML}^\circ_S \mid \lambda \cup \mu \text{ fills up } S, \text{ and} \\ -m(\partial_i, 2\mu) = m(\partial_i, 2\lambda) < b_i \text{ for } i = 1, \ldots, \mathfrak{n}\}$$

then it is well defined the map

$$\Phi^{\mathbf{b}} : \mathcal{T}^\circ_S(\mathbf{b}) \times \mathcal{T}^\circ_S \to \mathcal{FML}^\circ_S(\mathbf{b}) \cup \{(0,0)\}$$

where

$$\mathcal{T}^\circ_S(\mathbf{b}) = \{h \in \mathcal{T}^\circ_S \mid \ell_h(\partial_i) = b_i, \text{ for } i = 1 \ldots, \mathfrak{n}\},$$

that associates (h, h') with the unique $(\lambda, \mu) \in \mathcal{FML}^\circ_S(\mathbf{b})$ such that

$$E^\lambda_l(h) = E^\mu_r(h) = h'.$$

We have denoted by $(0,0) \in \mathcal{ML}^\circ_S \times \mathcal{ML}^\circ_S$ the couple of void laminations; let $\mathcal{F}^0\mathcal{ML}^\circ_S(\mathbf{b})$ be $\mathcal{FML}^\circ_S(\mathbf{b}) \cup \{(0,0)\}$. The aim of this chapter is to prove Theorem B.

3.1.2 Infinitesimal earthquakes

Definition 3.1.1. Given $\lambda \in \mathcal{ML}^\circ_S$, the infinitesimal left earthquake along λ is the vector field $e^\lambda_l : \mathcal{T}^\circ_S \to T\mathcal{T}^\circ_S$ such that

$$e^\lambda_l(h) = \frac{\mathrm{d}}{\mathrm{d}t}_{|0} E^{t\lambda}_l(h).$$

A pant decomposition of S with (internal) curves κ_i induces the coordinates

$$(\mathbf{l}, \boldsymbol{\tau}, \boldsymbol{\beta}) = (l_1 \ldots, l_{3g-3+\mathfrak{n}}, \tau_1, \ldots, \tau_{3g-3+\mathfrak{n}}, \beta_1, \ldots, \beta_{\mathfrak{n}})$$

on $\mathcal{T}^\circ_S$, where l_j denotes the length of κ_j, τ_j the twist factor of κ_j, and β_i the length of the boundary component ∂_i of S. The space $\mathcal{T}^\circ_S(\mathbf{b})$ is the submanifold of $\mathcal{T}^\circ_S$ individuated by the N equations $\boldsymbol{\beta} = \mathbf{b}$.
If μ has not compact support then there exists $i \in \{1, \ldots, \mathfrak{n}\}$ such that $m_i = m(\partial_i, \mu) \neq 0$, so we have

$$\ell_{E^{t\lambda}_l(h)}(\partial_i) = |b_i - 2tm_i| \neq b_i$$

for $t \in (0, \varepsilon)$ with ε sufficiently small; such a linear behaviour shows that if $h \in \mathcal{T}^\circ_S(\mathbf{b})$ then $e^\lambda_l(h)$ does not lie in $T_h\mathcal{T}^\circ_S(\mathbf{b})$. However, for every (λ, μ) in $\mathcal{FML}^\circ_S(\mathbf{b})$, and $t \in [0,1]$, the composition $E^{t\lambda}_l \circ E^{t\mu}_l$ preserves $\mathcal{T}^\circ_S(\mathbf{b})$, so that $e^\lambda_l + e^\mu_l$ is a tangent vector field of $\mathcal{T}^\circ_S(\mathbf{b})$.
In the closed case, the following theorem (see [11]) is the key to find fixed points of $E^{t\lambda}_l \circ E^{t\mu}_l$ for (λ, μ) filling measured laminations and small t.

Theorem 3.1.1. *Let S be a closed surface of genus greater than 1 and λ and μ measured laminations on S. The intersection between e_l^λ and $-e_l^\mu$, considered as submanifolds in $T\mathcal{T}_S$, is transverse. Moreover, if λ and μ fill up S then these sections meet in exactly one point $k_0(\lambda, \mu)$. Otherwise, they are disjoint.*

In the proof, $k_0(\lambda, \mu)$ is found as the unique minimum point of the function $L_\lambda + L_\mu : \mathcal{T}_S^\circ \to [0, +\infty)$, defined as follows.

Defintion 3.1.2. Let S be a closed surface of genus greater than 1 and λ a measured lamination on S. If the support of λ is a closed curve c with weight ω, the map $L_\lambda : \mathcal{T}_S \to [0, +\infty)$ associates h with $\omega\, \ell_h(c)$. For any $\lambda \in \mathcal{ML}_S$, since the space of weighted closed curves is dense in $\mathcal{ML}_S$, if λ_n are weighted closed curves approximating λ then define $L_\lambda(h)$ as $\lim L_{\lambda_n}(h)$.

The key properties of the map $L_\lambda + L_\mu$, under the fundamental hypothesis when (λ, μ) fills up S, are that:

- It is strictly convex along earthquake paths, which means that for every $h \in \mathcal{T}_S$ and $\nu \in \mathcal{ML}_S$ the function $t \mapsto (L_\lambda + L_\mu)(E_l^{t\nu}(h))$ is convex on $[0, 1]$ (see [26]);

- It is proper (see [26]);

- $e_l^\lambda + e_l^\mu$ is the symplectic gradient of $L_\lambda + L_\mu$ with respect to the Weil-Petersson symplectic form (see [38]).

In our setting, we need to provide $\mathcal{T}_S^\circ(\mathbf{b})$ with a symplectic form ϖ. We notice that $\mathcal{T}_S^\circ$ is not in general a symplectic manifold as its dimension could be odd. However, there is a natural Weil-Petersson form on $\mathcal{T}_S^\circ(\mathbf{b})$ obtained in the following way. Let $2S$ be the double of S along its boundary. It is a closed oriented surface of genus $2g + \mathfrak{n} - 1$. Denote by $\iota^+ : S \to 2S$ the orientation-preserving natural inclusion and by $\iota^- : S \to 2S$ the orientation-reversing one. If we consider a pant decomposition on S with internal curves $\kappa_1, \dots, \kappa_{6(g-1)+2\mathfrak{n}}$ and boundary curves $\partial_1, \dots, \partial_{\mathfrak{n}}$, there is an induced pant decomposition on $2S$ invariant by the natural involution, namely the one with curves $\iota^\pm(\kappa_j)$, $\iota^+(\partial_i) = \iota^-(\partial_i)$. Let ϖ_{WP} denote the Weil-Petersson form on the Teichmüller space $\mathcal{T}_{2S}$ of $2S$. It can be written as

$$\varpi_{WP} = \sum_{j=1}^{6(g-1)+2\mathfrak{n}} (d\ell_j^+ \wedge d\tau_j^+ + d\ell_j^- \wedge d\tau_j^-) + \sum_{i=1}^{\mathfrak{n}} d\ell_i^0 \wedge d\tau_i^0$$

where $\ell_j^\pm$ and $\tau_j^\pm$ denote respectively the length coordinate and the twist coordinate relative to $\iota^\pm(\kappa_j)$, while ℓ_i^0 and τ_i^0 denote respectively the length

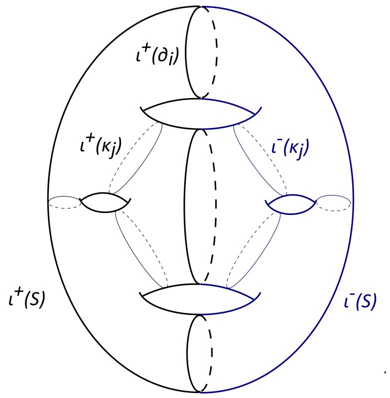


Figure 3.8

and twist coordinate relative to $\iota^+(\partial_i)$. Consider the natural immersion $f : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathcal{T}_{2S}$ that doubles a metric on S. With the 2-form

$$\varpi = f^*\varpi_{WP} = 2 \sum_{j=1}^{6(g-1)+2\mathfrak{n}} d\ell_j \wedge d\tau_j$$

it turns out that $(\mathcal{T}_S^\circ(\mathbf{b}), \varpi)$ is a symplectic manifold. Since $\varpi_{WP}(v_1, v_2)$ does not depend on the pant decomposition of $2S$, also $\varpi(w_1, w_2)$ does not depend on the chosen pant decomposition of S.

3.1.3 Plan

The first step to prove Theorem B is defining a function length

$$\mathbb{L}_{(\lambda,\mu)} : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathbb{R}$$

for $(\lambda, \mu) \in \mathcal{FML}_S^\circ(\mathbf{b})$ such that

- $\mathbb{L}_{(\lambda,\mu)}$ coincides with $L_\lambda + L_\mu$ if $(\lambda, \mu) \in \mathcal{CML}_S^\circ \times \mathcal{CML}_S^\circ$, where
$$\mathcal{CML}_S^\circ = \{\lambda \in \mathcal{ML}_S^\circ \, : \, \lambda \text{ has compact support}\};$$
- $\mathbb{L}_{(\lambda,\mu)}$ is proper and convex along earthquakes paths;
- a multiple of $\mathbb{L}_{(\lambda,\mu)}$ is a Hamiltonian of $e_l^\lambda + e_l^\mu$ with respect to the symplectic form $\varpi = \iota^*\varpi_{WP}$, which means that there exists $c \in \mathbb{R}^*$ such that $\varpi(*, e_l^\lambda + e_l^\mu) = c \cdot \mathrm{d}\mathbb{L}_{(\lambda,\mu)}(*)$.

This will be the aim of Section 3.2.
Section 3.4 is devoted to the proof of a technical estimate, analogous to the one proved in Section 4 in [17] but requiring a deeper analysis, due to the presence of spiralling leaves of λ and μ. Given $(h, h') \in \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ and

$(\lambda,\mu) = \Phi^{\mathbf{b}}(h,h')$, the estimate relates $\mathbb{L}_{(\lambda,\mu)}(h)$ with the intersection number $\iota(\lambda,\mu)$, defined as the total measure of $\lambda \times \mu$ (see for example [26], I.B). In Section 3.5 the estimate of Section 3.4 will lead to the properness of $\Phi^{\mathbf{b}}$. Again, spiralling leaves will be carefully treated.
Section 3.6.1 will be devoted to the proof of the existence of an open neighbourhood $\mathcal{U}$ of $\mathcal{D} = \{(h,h) : h \in \mathcal{T}_S^\circ(\mathbf{b})\}$ in $\mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ such that the restriction of $\Phi^{\mathbf{b}}$ to $\mathcal{U} \smallsetminus \mathcal{D}$ is a homeomorphism onto its image $V \subset \mathcal{FML}_S^\circ(\mathbf{b})$, which for any $(\lambda,\mu) \in \mathcal{FML}_S^\circ(\mathbf{b})$ and for t sufficiently small contains $(t\lambda, t\mu)$. Notice that $\Phi^{\mathbf{b}}(\mathcal{D}) = \{(0,0)\}$; this is why we have to remove $\mathcal{D}$ from $\mathcal{U}$ to get a homeomeorphism. Also, $(0,0) \notin \mathcal{FML}_S^\circ(\mathbf{b})$.
Since $\Phi^{\mathbf{b}}$ is continuous and proper, it is possible to define its degree, which, by the result of Section 3.6.1, is 1. Therefore, Theorem B will be easily proved.

3.2 The field $e_l^\lambda + e_l^\mu$ is Hamiltonian

Consider (λ,μ) in the space

$$\mathcal{ML}_S^\circ(\mathbf{b}) = \{(\nu,\nu') \in (\mathcal{ML}_S^\circ)^2 \ : \ -m(\partial_i,\nu') = m(\partial_i,\nu) < b_i \ \forall i = 1,\ldots,\mathfrak{n}\}$$

such that the support of λ and the support of μ consist both of one spiralling geodesic between two boundary components ∂ and ∂'. We will define the map $L = L_{(\lambda,\mu)} : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathbb{R}$ in this simple case in the first subsection; then in the next two subsections we will study respectively its first order variation and its convexity along earthquakes.
The last subsection will be devoted to the definition of the length function $\mathbb{L}_{(\lambda,\mu)} : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathbb{R}$ for a generic couple $(\lambda,\mu) \in \mathcal{ML}_S^\circ(\mathbf{b})$. Since it will be constructed as the sum of maps having the form of the first defined, the properties enlightened in the first two subsections still will hold.
In the next section we will study the properness and show that $\mathbb{L}_{(\lambda,\mu)}$ is a Hamiltonian for $e_l^\lambda + e_l^\mu$.

3.2.1 The condition $\varpi(e_l^\lambda + e_l^\mu, *)$

Proposition 3.2.1. *For any $h \in \mathcal{T}_S^\circ(\mathbf{b})$ the map $\psi^h : \mathcal{CML}_S^\circ \to T_h\mathcal{T}_S^\circ(\mathbf{b})$ such that $\psi^h(\nu) = e_l^\nu(h)$ is a homeomorphism.*

Proof. For every $h \in \mathcal{T}_S^\circ(\mathbf{b})$ the map ψ^h is well defined, since $\iota(\partial_i,\nu) = 0$ for every $\nu \in \mathcal{CML}_S^\circ$ and for every $i = 1,\ldots,\mathfrak{n}$.
In order to see that ψ^h is injective, consider the Teichmüller space $\mathcal{T}_{2S}$ of the double of S and the space $\mathcal{ML}_{2S}$ of the measured laminations on $2S$. Denote by $\iota_+ : S \to 2S$ the orientation-preserving natural inclusion and by $\iota_- : S \to 2S$ the orientation-reversing one. For any $h_-, h_+ \in \mathcal{T}_S^\circ(\mathbf{b})$ denote by $h_- * h_+$ the element in $\mathcal{T}_{2S}$ such that $\iota_\pm^*(h_- * h_+) = h_\pm$, say with

null twisting factor on the components of ∂S. Fixed $h_0 \in \mathcal{T}_S^\circ(\mathbf{b})$, the maps $\varphi : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathcal{T}_{2S}$ and $\eta : \mathcal{CML}_S^\circ \to \mathcal{ML}_{2S}$ such that $\varphi(h) = h_0 * h$ and $\eta(\nu) = \iota_+(\nu)$ are injections.
Notice also that the map $\varphi : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathcal{T}_{2S}$ is an immersion that for every $\nu \in \mathcal{CML}_S^\circ$ conjugates $E^\nu : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathcal{T}_S^\circ(\mathbf{b})$ with $E^{\eta(\nu)} : \mathcal{T}_{2S} \to \mathcal{T}_{2S}$. Thus, if $v \in T_h\mathcal{T}_S^\circ(\mathbf{b})$ then (by [17]) there is a unique $\nu_+ \in \mathcal{ML}_{2S}$ such that $\mathrm{d}\varphi(v) = e^{\nu_+}$, being $2S$ compact, and it must be supported only on $\iota_+(S)$, so $\nu_+ = \eta(\nu)$ for some $\nu \in \mathcal{CML}_S^\circ$. It follows that $v = e^\nu$. This shows that ψ^h is a bijection.
Since $T_h\mathcal{T}_S^\circ(\mathbf{b})$ and $\mathcal{CML}_S^\circ$ are topological manifolds of the same dimension $6(g-1)+2\mathfrak{n}$, by invariance of domain ψ^h is actually a homeomorphism. □

Consider a simple closed curve γ not isotopic to a boundary component. Choose a pant decomposition $\{\gamma, \kappa_2, \kappa_3, \ldots\}$ of S. Denoting by γ also the measured lamination supported by the curve γ with unitary weight, we have for every $h \in \mathcal{T}_S^\circ$ that

$$\begin{aligned}\varpi_h(e_l^\gamma, e_l^\lambda + e_l^\mu) &= 2\Big(\mathrm{d}\ell_\gamma \wedge \mathrm{d}\tau_\gamma + \sum_i \mathrm{d}\ell_{\kappa_i} \wedge \mathrm{d}\tau_{\kappa_i}\Big)(e_l^\gamma, e_l^\lambda + e_l^\mu) = \\ &= \mathrm{d}\ell_\gamma(e_l^\lambda + e_l^\mu) = \frac{1}{2}\mathrm{d}L_\gamma(e_l^\lambda + e_l^\mu) = \\ &= \frac{\mathrm{d}}{\mathrm{d}t}_{|0}\Big(L_\gamma(E_l^{t\lambda}(h)) + L_\gamma(E_l^{t\mu}(h))\Big).\end{aligned}$$

Kerckhoff in [26] proved that on a closed surface S if γ and ν are laminations with a closed curve as support then for every h in the Teichmüller space of S the following holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|0}L_\gamma(E_l^{t\nu}(h)) = \int_\gamma \cos\theta_{(\gamma,\nu)}(h)\mathrm{d}\nu \tag{3.1}$$

where $\theta_{(\gamma,\nu)}(h)$ denotes the angle measured counterclockwise from γ to ν in the h-realization. In the proof in [26] of Equation (3.1) the fact that ν was a closed curve was actually irrelevant. Thus, in our context, the same argument shows that for any h in $\mathcal{T}_S^\circ$ and $\nu \in \mathcal{ML}_S^\circ$

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|0}L_\gamma(E_l^{t\nu}(h)) = \int_\gamma \cos\theta_{(\gamma,\nu)}(h)\mathrm{d}\nu.$$

Therefore,

$$\begin{aligned}\varpi(e_l^\gamma, e_l^\lambda + e_l^\mu) &= \Big(\int_\gamma \cos\theta_{(\gamma,\lambda)}\mathrm{d}\lambda + \int_\gamma \cos\theta_{(\gamma,\mu)}\mathrm{d}\mu\Big) = \\ &= \Big(\mathrm{Cos}(\gamma,\lambda) + \mathrm{Cos}(\gamma,\mu)\Big)\end{aligned}$$

where, following the notation of [28], we put

$$\mathrm{Cos}(\gamma, \lambda) = \int \cos\theta_{(\gamma,\lambda)} \mathrm{d}\lambda \otimes \mathrm{d}\gamma$$
$$\mathrm{Cos}(\gamma, \mu) = \int \cos\theta_{(\gamma,\mu)} \mathrm{d}\mu \otimes \mathrm{d}\gamma.$$

Here we are using the function

$$(r_1, r_2) \mapsto \cos\theta_{(r_1,r_2)} \tag{3.2}$$

defined on the space

$$G_2(\mathbb{H}^2) \cong \left\{ \left((S^1 \times S^1) \setminus \mathrm{diag}(S^1)\right) \Big/ (x,y) \sim (y,x) \right\}^2$$

of couples of geodesics in $\mathbb{H}^2$ and supported by the subspace of couples of incident geodesics. Here again $\theta_{(r_1,r_2)}$ is measured counterclockwise from r_1 to r_2.
If a function $H : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathbb{R}$ verifies

$$\mathrm{d}H(e_l^\gamma) = \Big(\mathrm{Cos}(\gamma, \lambda) + \mathrm{Cos}(\gamma, \mu) \Big),$$

then, since the space of simple weighted closed curves is dense in $\mathcal{CML}_S^\circ$, by an approximation argument we get that for every $\nu \in \mathcal{CML}_S^\circ$

$$\mathrm{d}H(e_l^\nu) = \Big(\mathrm{Cos}(\nu, \lambda) + \mathrm{Cos}(\nu, \mu) \Big) = \varpi(e_l^\nu, e_l^\lambda + e_l^\mu).$$

Thus, by definition, H is Hamiltonian of the field $e_l^\lambda + e_l^\mu$.
If λ and μ have compact support, with the same argument one gets that $H = -(L_\lambda + L_\mu)$ is a suitable Hamiltonian. In the following sections we will show that it is always possible to construct a Hamiltonian of $e_l^\lambda + e_l^\mu$ for every $(\lambda, \mu) \in \mathcal{ML}_S^\circ(\mathbf{b})$.

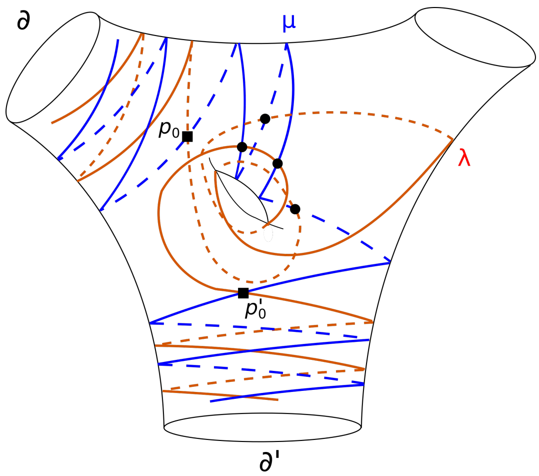
3.2.2 The map $L : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathbb{R}$

Given $(\lambda, \mu) \in \mathcal{ML}_S^\circ(\mathbf{b})$ such that the support of λ and the support of μ consist both of one geodesic spiralling between two boundary components ∂ and ∂', in this subsection we will make constructions and show properties related to the neigbourhoods of ∂ and ∂' that will be useful in this section and in the following ones.
Take a hyperbolic metric $h \in \mathcal{T}_S^\circ(\mathbf{b})$ and let b and b' be respectively $\ell_h(\partial)$ and $\ell_h(\partial')$. Orient λ and μ so that they go from ∂' to ∂. If $\partial = \partial'$, such orientation is chosen so that the ideal endpoints of any lift $\tilde{\partial}$ of ∂ are both starting or both ending ideal endpoints of the lifts of λ and μ tangent to $\tilde{\partial}$ at infinity. We claim there exists $p_0 \in \lambda \cap \mu$ with the following two properties.

1. Denote by λ_* and μ_* the rays in λ and μ respectively originating at p_0 with positive direction and enumerate consecutively on λ_* the elements of $\lambda_* \cap \mu_*$, starting from p_0, as p_1, p_2, p_3, $\ldots$. Denote by $\hat{\lambda}_k$ the arc of λ going from p_k to p_{k+1} and by $\hat{\mu}_k$ the arc of μ going from p_k to p_{k+1}. Then for every $k \in \mathbb{N}$ the piecewise geodesic loop $\hat{\lambda}_k \cup \hat{\mu}_k$ is isotopic to ∂.

2. In $\lambda \setminus \lambda_*$ there is no point with the previous property.

If such p_0 exists, it is clearly unique. Analogously there is a point $p_0' \in \lambda \cap \mu$ with the same properties relatively to ∂' (here λ_*' and μ_*' must start at p_0' with negative direction).



Proposition 3.2.2. *There exists $p_0 \in \lambda \cap \mu$ satisfying properties 1 and 2.*

Proof. On the universal cover $\mathcal{H}$ in the upper half-plane model of $\mathbb{H}^2$ choose coordinates such that a preimage of ∂ coincides with the imaginary ray and a lift $\tilde{\lambda}$ of λ is $1 + i\mathbb{R}_{>0}$. Here we are supposing that λ spirals around ∂ in say the positive sense. The following argument still works in the negative sense case, where we require that a lift of λ is $-1 + i\mathbb{R}_{>0}$.
Let $\gamma : z \mapsto e^b z$ denote the holonomy transformation corresponding to ∂. The union of the lifts of μ with an ideal endpoint in 0 is γ-invariant. Among them, there exists a unique $\tilde{\mu}$ such that $\tilde{\lambda} \cap \gamma^k(\tilde{\mu})$ is nonempty for every $k \geq 0$ and $\tilde{\lambda} \cap \gamma^k(\tilde{\mu})$ is empty for every $k < 0$. For every $k \geq 0$ let $\tilde{p}_k$ be the intersection between $\tilde{\lambda}$ and $\gamma^k(\tilde{\mu})$ and p_k the projection of $\tilde{p}_k$ on S.
The points $p_0, p_1, \ldots$ turn out to be the points in $\lambda_* \cap \mu_*$ enumerated consecutively on λ_*. To check Property 1, it suffices to show that they are also enumerated consecutively from the point of view of μ_*. Consider the lift $\tilde{\mu}_*$ of μ_* starting at $\tilde{p}_0$ with ideal endpoint in 0. Its points have real part in $(0, 1]$, so $\tilde{\mu}_*$ meets $\gamma^n(\tilde{\lambda})$ (which has ideal endpoints e^{bn} and ∞) if and only if $e^{bn} \leq 1$, i.e. $n \leq 0$. Denote by $\tilde{\lambda}_*$ the lift of λ_* contained in $\tilde{\lambda}$. It has

origin in $\tilde{p}_0$ and ideal endpoint in ∞. The rays $\gamma^n(\tilde{\lambda}_*)$ are lifts of λ_* for every $n \in \mathbb{Z}$. In particular, for every $n \geq 0$, the origin $\gamma^{-n}(\tilde{p}_0)$ of $\gamma^{-n}(\tilde{\lambda}_*)$ lies in the halfplane bounded by $\tilde{\mu}$ whose boundary at infinity contains 1, so $\tilde{\mu}_*$ meets $\gamma^{-n}(\tilde{\lambda}_*)$ for every $n \geq 0$. Therefore, μ_* meets λ_* consecutively in the projections of the points

$$\gamma^{-n}(\tilde{\lambda}_*) \cap \tilde{\mu}_* = \gamma^{-n}\big(\tilde{\lambda}_* \cap \gamma^n(\tilde{\mu}_*)\big) = \gamma^{-n}\big(\tilde{\lambda} \cap \gamma^n(\tilde{\mu})\big) = \gamma^{-n}(\tilde{p}_n)$$

which are exactly $p_0, p_1, p_2, \ldots$.
The way we found $\tilde{p}_0$ shows that property 2 is verified. □

Remark 3.2.1. Let us consider the points $\tilde{p}_k$ found in the proof of the previous proposition. They belong to $\tilde{\lambda}$, so $\Re\tilde{p}_k = 1$ for every k. The geodesic μ spirals around ∂ in the opposite sense of λ, so an ideal endpoint of μ must be 0. In order to determine the other ideal endpoint, say $2r$, denote by $\phi \in (0, \pi/2)$ the argument of the point $\tilde{p}_0$. Since $\tilde{p}_0 \in \tilde{\mu}$, it must be $|r - \tilde{p}_0| = r$. Thus,

$$r^2 + |\tilde{p}_0|^2 - 2r\,\Re\tilde{p}_0 = r^2$$
$$2r = \frac{|\tilde{p}_0|^2}{\Re\tilde{p}_0} = \frac{1 + \tan^2\phi}{1} = \cos^{-2}\phi.$$

Moreover, this implies that $\gamma^k(\tilde{\mu})$ has ideal endpoints 0 and $e^{bk}\cos^{-2}\phi$. From this, for every $k \geq 0$ we can compute the imaginary part of the points $\tilde{p}_k = \tilde{\lambda} \cap \gamma^k(\tilde{\mu})$:

$$\left|\frac{e^{bk}\cos^{-2}\phi}{2} - \tilde{p}_k\right| = \frac{e^{bk}\cos^{-2}\phi}{2}$$
$$1 + (\Im\tilde{p}_k)^2 - e^{bk}\cos^{-2}\phi = 0$$

and so

$$\tilde{p}_k = \tilde{\lambda} \cap \gamma^k(\tilde{\mu}) = 1 + i\sqrt{e^{bk}\cos^{-2}\phi - 1}. \tag{3.3}$$

Remark 3.2.2. The distance between p_k and ∂ is computed by

$$\tanh d(p_k, \partial) = \tanh d(\tilde{p}_k\,\tilde{\partial}) = \cos\arg\tilde{p}_k = \frac{\Re\tilde{p}_k}{|\tilde{p}_k|} = e^{-bk/2}\cos\phi.$$

Lemma 3.2.3. *Fix* $\mathbf{b} \in (\mathbb{R}_{>0})^N$. *For every boundary component* ∂ *of* S *there exists* $\varepsilon(\partial) > 0$ *such that for every* $h \in \mathcal{T}_S^\circ(\mathbf{b})$ *every simple complete geodesic that enters the* $\varepsilon(\partial)$*-collar* $\mathcal{N}(\partial)$ *of* ∂ *exits no more.*

Proof. Choose $h \in \mathcal{T}_S^\circ(\mathbf{b})$ and set $b = \ell(\partial)$. On the universal cover $\mathcal{H} \subset \mathbb{H}^2$ take coordinates such that the imaginary ray projects on a boundary component ∂. If $\gamma : z \mapsto e^b z$ is the corresponding holonomy transformation, consider the geodesic σ_0 in $\mathbb{H}^2$ corresponding to the Euclidean semicircumference centered in 1 such that $\gamma(\sigma_0)$ has a common ideal point with σ_0

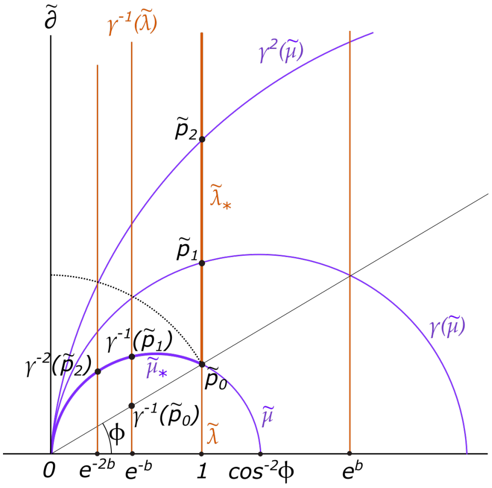
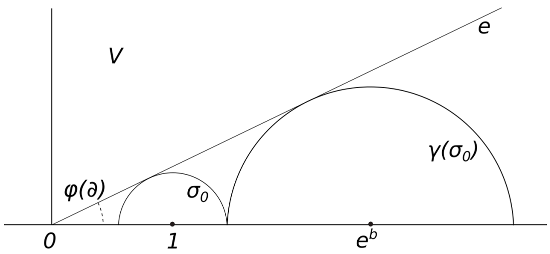


Figure 3.9



different from 0, as in figure, and consider the Euclidean ray e starting from 0 and tangent to both the geodesics. The angle $\varphi(\partial)$ that e forms with the positive x axis is such that $\sin\varphi(\partial) = \tanh(b/2)$; the ray e bounds the $\varepsilon(\partial)$-collar V of the imaginary ray, where $\cosh\varepsilon(\partial) = \big(\sin\varphi(\partial)\big)^{-1} = \big(\tanh(b/2)\big)^{-1}$. Now notice that if a geodesic σ in $\mathbb{H}^2$ enters and exits V then its projection on S is not simple, since $\gamma(\sigma)$ would meet σ. So, for every simple geodesic $\sigma : \mathbb{R} \to S$ and for every $t_* \in \mathbb{R}$, if $\sigma(t_*)$ lies in the projection U of V on S then either $\sigma(t) \in U$ for every $t \geq t_*$ or $\sigma(t) \in U$ for every $t \leq t_*$. $\square$

For every boundary component ∂_i of S, we will denote by $\mathcal{N}(\partial_i)$ the $\varepsilon(\partial_i)$-collar of ∂_i and we will call the union $\mathcal{N}$ of such collars *spiralization neighbourhood*.

Remark 3.2.3. If $k \geq 1$ then p_k lies in $\mathcal{N}(\partial)$. In fact, a point x of λ lies in $\mathcal{N}(\partial)$ if and only if the preimage of x lying in $\tilde{\lambda}$ has imaginary part greater than $\tan\varphi(\partial) = \sinh(b/2)$ (see Lemma 3.2.3). For $k \geq 1$ we have

$$\Im\tilde{p}_k \geq \Im\tilde{p}_1 = \sqrt{e^b\cos^{-2}\phi - 1} \geq \sqrt{e^b - 1} \geq \sinh(b/2).$$

It may be possible that p_0 does not lie in $\mathcal{N}(\partial)$. That is the reason why the definition of L will involve p_1 and not p_0.

Denote by p_1' the point near ∂' analogous to p_1. Now we can define a continuous map $L = L_{(\lambda,\mu)} : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathbb{R}$, that will turn out to be the opposite of a Hamiltonian of $e_l^\lambda + e_l^\mu$.

Definition 3.2.1. Take $(\lambda, \mu) \in \mathcal{ML}_S^\circ(\mathbf{b})$ such that the support of λ and the support of μ consist respectively of a single ω-weighted geodesic spiralling between two boundary components ∂ and ∂', and consider the points p_1 and p_1' introduced above. Let ρ be the union of the geodesic arc in λ with endpoints p_1 and p_1' and the the geodesic arc in μ with endpoints p_1 and p_1'. For every $h \in \mathcal{T}_S^\circ(\mathbf{b})$, set

$$L(h) = \omega\ell_h(\rho) + 2\omega \log\big(\cosh d_h(p_1, \partial) \cdot \cosh d_h(p_1', \partial')\big).$$

Remark 3.2.4. In the definition above, if we choose p_k and p_k' instead of p_1 and p_1' then we get a map

$$L_k = \omega\ell_h(\rho_k) + 2\omega \log\big(\cosh d_h(p_k, \partial) \cdot \cosh d_h(p_k', \partial')\big), \tag{3.4}$$

where ρ_k is the loop made by the truncations of λ and μ at p_k and p_k'. The function L_k is different from $L = L_1$, but has the property that the difference is constant; indeed we have

$$L_k - L \equiv \omega(k-1)(b + b'), \tag{3.5}$$

so $-L$ is a Hamiltonian of $e_l^\lambda + e_l^\mu$ if and only if $-L_k$ is. In order to see that $L_k - L$ is constant, denote by $\hat\lambda_k$ the projection on $\lambda \subset S$ of the geodesic arc $[\tilde p_k, \tilde p_{k+1}] \subset \tilde\lambda$ and by $\hat\mu_k$ the projection on $\mu \subset S$ of the geodesic arc $[\gamma^{-k}(\tilde p_k), \gamma^{-k}(\tilde p_{k+1})] \subset \tilde\mu$. Let ρ_k be the union of the geodesic arc in λ with endpoints p_k and p_k' and the the geodesic arc in μ with endpoints p_k and p_k'. Notice that

$$\rho_k = \rho \cup \bigcup_{m=1}^{k-1} (\hat\lambda_m \cup \hat\mu_m \cup \hat\lambda_m' \cup \hat\mu_m').$$

The map L_k takes the form

$$L_k(h) = \omega\ell_h(\rho_k) + 2\omega \log\big(\cosh d_h(p_k, \partial) \cdot \cosh d_h(p_k', \partial')\big).$$

Using Equation (3.3), we get

$$\begin{aligned} \cosh d(p_k, \partial) &= \cosh d(\tilde p_k, \tilde\partial) = \sin^{-1} \arg \tilde p_k = \frac{|\tilde p_k|}{\Im \tilde p_k} = \\ &= \frac{e^{bk/2}\cos^{-1}\phi}{\sqrt{e^{bk}\cos^{-2}\phi - 1}} = \frac{1}{\sqrt{1 - e^{bk}\cos^{-2}\phi}} \end{aligned} \tag{3.6}$$

and

$$\ell(\hat\lambda_k) = d(\tilde p_k, \tilde p_{k+1}) = \frac{1}{2}\log\frac{e^{b(k+1)} - \cos^2\phi}{e^{2k} - \cos^2\phi}$$

$$\ell(\hat\mu_k) = d(\gamma^{-k}(\tilde p_k), \gamma^{-k}(\tilde p_{k+1})) = d(\tilde p_k, \tilde p_{k+1}),$$

so

$$d_k = \ell(\hat{\lambda}_k) = \ell(\hat{\mu}_k) = \frac{1}{2}\log\frac{e^{b(k+1)} - \cos^2\phi}{e^{2k} - \cos^2\phi}. \tag{3.7}$$

Similarly subarcs $\hat{\lambda}'_k$ and $\hat{\mu}'_k$ have length

$$d'_k = \frac{1}{2}\log\frac{e^{b'(k+1)} - \cos^2\phi'}{e^{b'k} - \cos^2\phi'}.$$

Now consider the following limit, for $m \geq 1$.

$$\lim_{n\to\infty}\Big(\ell_h(\rho_n) - nb - nb'\Big) =$$
$$=\ell_h(\rho_m) + \lim_{n\to\infty}\left(\sum_{k=m}^{n-1}\Big(\ell_h(\hat{\lambda}_k) + \ell_h(\hat{\mu}_k) + \ell_h(\hat{\lambda}'_k) + \ell_h(\hat{\mu}'_k)\Big) - nb - nb'\right) =$$
$$=\ell_h(\rho_m) + \lim_{n\to\infty}\left(\sum_{k=m}^{n-1}\Big(2d_k + 2d'_k\Big) - nb - nb'\right) =$$
$$=\ell_h(\rho_m) + \lim_{n\to\infty}(S_n^{(m)} + S_n'^{(m)})$$

where $S_n^{(m)} = -nb + \sum 2d_k$ and $S_n'^{(m)} = -nb' + \sum 2d'_k$. By Equation (3.7),

$$\begin{aligned}
S_n^{(m)} &= -nb + \sum_{k=m}^{n-1}\left[\log\frac{e^{b(k+1)} - \cos^2\phi}{e^{2k} - \cos^2\phi}\right] = \\
&= -nb + \sum_{k=m}^{n-1}\left[\log e^b\frac{1 - e^{-b(k+1)}\cos^2\phi}{1 - e^{-bk}\cos^2\phi}\right] = \\
&= -nb + (n-m)b + \sum_{k=m}^{n-1}\log\frac{1 - e^{-b(k+1)}\cos^2\phi}{1 - e^{-bk}\cos^2\phi} = \\
&= -mb + \log\prod_{k=m}^{n-1}\frac{1 - e^{-b(k+1)}\cos^2\phi}{1 - e^{-bk}\cos^2\phi} = \\
&= -mb + \log\frac{1 - e^{-bn}\cos^2\phi}{1 - e^{-bm}\cos^2\phi}
\end{aligned}$$

and so, by Equation (3.6),

$$\lim_{n\to\infty} S_n^{(m)} = -mb + \log\frac{1}{1 - e^{-bm}\cos^2\phi} = -mb + 2\log\cosh d(p_m, \partial)$$

and analogously

$$\lim_{n\to\infty} S_n^{(m)} = -mb' + 2\log\cosh d(p'_m, \partial').$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\Big(\ell_h(\rho_n) - nb - nb'\Big) &= \ell_h(\rho_m) - mb - mb' + \\ &= + 2\log\Big(\cosh d_h(p_m, \partial)\cdot\cosh d_h(p'_m, \partial')\Big) = \\ &= \omega^{-1}L_m(h) - m(b+b')\end{aligned}$$

for every $m \geq 1$. In particular, $\omega^{-1}L_k(h) - k(b+b') = \omega^{-1}L_1(h) - (b+b')$, so

$$L_k(h) - L(h) = \omega(k-1)(b+b')$$

for every $h \in \mathcal{T}_S^\circ(\mathbf{b})$ and $k \geq 1$.
Actually, this works also for $k = 0$, providing $\rho_0 = \rho \smallsetminus (\hat{\lambda}_0 \cup \hat{\mu}_0)$. Therefore, we can conclude that for every $k \geq 0$ the map L_k differs from L by a constant depending only on $\mathbf{b}$ and k.

Now we will give bounds to the distance between p_k and ∂ depending only on k and the lengths of the boundary components of S and not on the choice of $h \in \mathcal{T}_S^\circ(\mathbf{b})$. These estimates will be useful in Section 3.5. Clearly, there exist analogous bounds for $d(p'_k, \partial')$.

Remark 3.2.5. In the setting of the proof of Proposition 3.2.2 and Remark 3.2.1, p_0 was chosen such that $\tilde{\lambda} \cap \gamma^k(\tilde{\mu})$ is nonempty for every $k \geq 0$ and $\tilde{\lambda} \cap \gamma^k(\tilde{\mu})$ is empty for every $k < 0$. Since $\tilde{\mu}$ has ideal endpoints 0 and $\cos^{-2}\phi$ (where ϕ depends on $h \in \mathcal{T}_S^\circ(\mathbf{b})$), it must be

$$\gamma^{-1}(\cos^{-2}\phi) = e^{-b}\cos^{-2}\phi < 1 < \cos^{-2}\phi$$

and so $\cos\phi > e^{-b/2}$. Now

$$\begin{aligned}\tanh d(p_k, \partial) &= \tanh d(\tilde{p}_k, \tilde{\partial}) = \cos\arg\tilde{p}_k = \\ &= \frac{\Re\tilde{p}_k}{|\tilde{p}_k|} = e^{-bk/2}\cos\phi > e^{-b(k+1)/2} > 0,\end{aligned}$$

and on the other hand

$$\tanh d(p_k, \partial) = e^{-bk/2}\cos\phi < e^{-bk/2} < 1.$$

Remark 3.2.6. Consider the arc $\hat{\lambda}_1$ from p_1 to p'_1. Consider the arc τ in $\lambda \smallsetminus \hat{\lambda}_1$ with an endpoint in p_1 of length $\log\cosh d(p_1, \partial)$ and the arc τ' in $\lambda \smallsetminus \hat{\lambda}_1$ with an endpoint in p'_1 of length $\log\cosh d(p'_1, \partial')$. Denote by $\lambda_\bullet$ the arc $\tau \cup \hat{\lambda}_1 \cup \tau' \subset \operatorname{supp}(\lambda)$. The distances of the endpoints of $\lambda_\bullet$ from ∂ and ∂' are still bounded from above and below by positive constants depending only on $\mathbf{b}$. If $\mu_\bullet$ is the analogous arc in $\operatorname{supp}(\mu)$, then we have

$$L_{(\lambda,\mu)}(h) = \omega\,\ell_h(\lambda_\bullet \cup \mu_\bullet).$$

3.2.3 The first order variation of L

The goal of this Subsection is to prove the following proposition:

Proposition 3.2.4. *Take $(\lambda, \mu) \in \mathcal{ML}_S^\circ(\mathbf{b})$ such that the support of λ and the support of μ consist respectively of a single ω-weighted geodesic spiralling between two boundary components ∂ and ∂', and consider the map $L = L_{(\lambda,\mu)} : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathbb{R}$ given by definition 3.2.1. For every non-peripheral and nontrivial simple close curve γ on S and for every $h \in \mathcal{T}_S^\circ(\mathbf{b})$ the equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} L(E_l^{t\gamma}(h)) = \int \cos\theta_{(\lambda,\gamma)}(t)\, \mathrm{d}\gamma \otimes \mathrm{d}\lambda + \int \cos\theta_{(\mu,\gamma)}(t)\, \mathrm{d}\gamma \otimes \mu \tag{3.8}$$

holds, where $\theta_{(\lambda,\gamma)}(t)$ is the angle measured counterclockwise from the support of λ to γ and $\theta_{(\mu,\gamma)}(t)$ is the angle measured counterclockwise from the support of μ to γ, in the $E_l^{t\gamma}(h)$-realization of γ, λ and μ.

Notice that we are slightly abusing the notation, denoting by γ also the measured lamination supported by the curve γ with unitary weight. This proposition will be true more in general, replacing γ with a measured lamination ν with compact support, as shown at the end of the Subsection. Since

$$L(h) = \omega \ell_h(\rho) + 2\omega \log[\cosh d_h(p_1, \partial) \cdot \cosh d_h(p_1', \partial')],$$

we will first compute the derivative in $t = 0$ of $\ell_{E_l^{t\gamma}(h)}(\rho)$, which will turn out to be

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \ell_{E_l^{t\gamma}(h)}(\rho) = &\int \cos\theta_{(\lambda,\gamma)}(0)\, \mathrm{d}\gamma \otimes \mathrm{d}\lambda + \int \cos\theta_{(\mu,\gamma)}(0)\, \mathrm{d}\gamma \otimes \mathrm{d}\mu + \\ &+ \mathcal{R}(0) + \mathcal{R}'(0)\end{aligned}$$

where $\mathcal{R}, \mathcal{R}'$ are terms due to the presence of the two vertices p_1 and p_1' in ρ. The strategy follows Kerckhoff's proof of the Nielsen Realization Problem ([**?**]).
After that, setting $F(d) = 2 \log \cosh d$, we will show that

$$\mathcal{R}(0) + \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} F\big(d_{E_l^{t\gamma}(h)}(p_1, \partial)\big) = 0 \tag{3.9}$$

$$\mathcal{R}'(0) + \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} F\big(d_{E_l^{t\gamma}(h)}(p_1', \partial')\big) = 0 \tag{3.10}$$

thus proving Equation (3.8).

Let us start to compute the derivative of $\ell_{E_l^{t\gamma}(h)}(\rho)$. Notice that the loop ρ is piecewise geodesic and has exactly two vertices, which are p_1 and p_1'.

If $\iota(\gamma, \lambda) = \iota(\gamma, \mu) = 0$ then $\ell(\rho)$ is constant. Otherwise, γ meets at least one between λ and μ. Notice that $\gamma \cap \rho = \gamma \cap (\lambda \cup \mu)$, since p_1 and p_1'

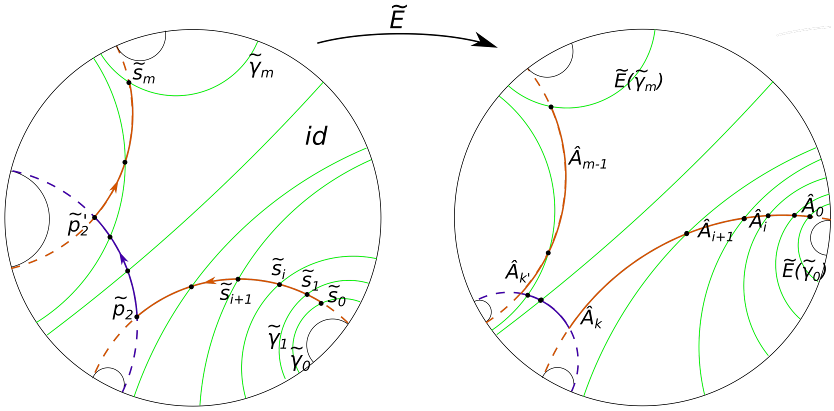


Figure 3.10: Determination of $\hat{\rho}$ and $\hat{A}_i$

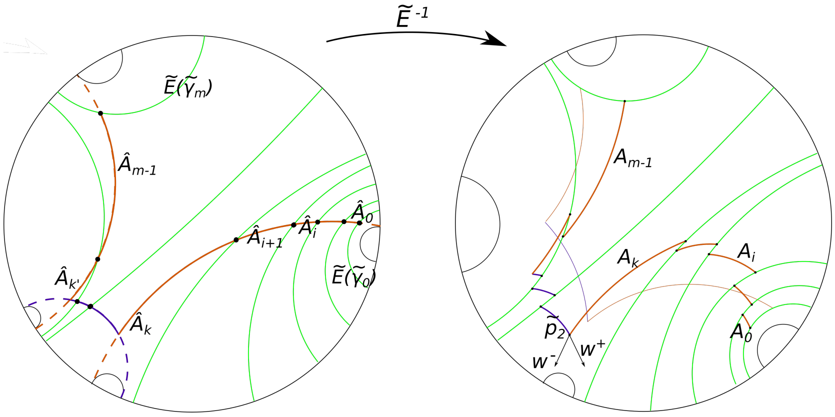


Figure 3.11: Determination of A_i

lie in the spiralization neighbourhood (see Lemma 3.2.3 and Remark 3.2.3). Choosing an orientation of ρ, enumerate consecutively its intersections with γ as $s_0, s_1, \ldots, s_{m-1}$. Pick a preimage $\tilde{s}_0$ of s_0 on the universal cover $\mathcal{H}$ of S. If $r : [0,1] \to S$ is a parametrization of the loop ρ such that $r(0) = r(1) = s_0$, take the lift $\tilde{r} : [0,1] \to \mathcal{H}$ with $\tilde{r}(0) = \tilde{s}_0$. The preimages of γ determine the strata of the lifting $\tilde{E}$ of $E_l^{t\gamma}$. In particular, denote by $\tilde{\gamma}_i$ the preimage of γ that meets $\tilde{r}([0,1])$ at a preimage of s_i, for $i = 1, \ldots, m-1$; denote by $\tilde{\gamma}_0$ the preimage of γ passing through $\tilde{s}_0$ and $\tilde{\gamma}_m$ the one passing through $\tilde{r}(1) = \tilde{s}_m$.

The path $\tilde{r}$ is piecewise geodesic, with vertices $\tilde{p}_1$ and $\tilde{p}_1'$. The prolongations of the three geodesic arcs of $\tilde{r}$ end at six points on $\partial_\infty\mathcal{H}$. Applying $\tilde{E}$, the images of these six points and of $\tilde{\gamma}_0$ and $\tilde{\gamma}_m$ are sufficient to define the arc $\hat{\rho}$ (which does not coincide with the image of $\tilde{r}$) starting from $\tilde{E}(\tilde{\gamma}_0)$ and ending in $\tilde{E}(\tilde{\gamma}_m)$ such that its length is equal to $\ell_{E_l^{t\gamma}(h)}(\rho)$. The arc $\hat{\rho}$ is divided in m subarcs $\hat{A}_0, \ldots, \hat{A}_{m-1}$ such that $\hat{A}_i$ has endpoints on $\tilde{E}(\tilde{\gamma}_i)$

and $\tilde{E}(\tilde{\gamma}_{i+1})$. These subarcs are all geodesic, except for those containing the vertices $\hat{p}_1$ and $\hat{p}_1'$ of $\hat{\rho}$. Let k be the index of the subarc of $\tilde{r}$ containing the vertex $\hat{p}_2$ and k' the index of the subarc containing the vertex $\hat{p}_2'$. Notice that k can be different from k', as in Figure 3.10, or equal, as in Figure 3.12. In any case, the preimage A_i under $\tilde{E}$ of $\hat{A}_i$ is a geodesic arc for $i \neq k, k'$

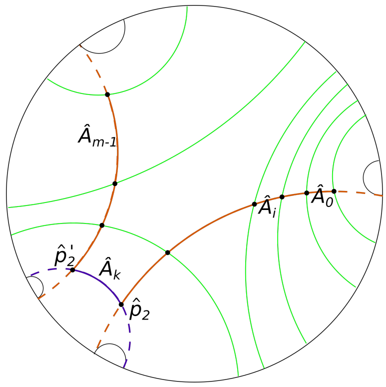


Figure 3.12: Case $k = k'$

and a piecewise geodesic for $i = k, k'$, with endpoints on $\tilde{\gamma}_i$ and $\tilde{\gamma}_{i+1}$ with the same length as $\hat{A}_i$. This leads to

$$\ell_{E_l^{t\gamma}(h)}(\rho) = \sum_{i=0}^{m-1} \ell_h(A_i(t)).$$

Denote with θ_i the angle in $\tilde{s}_i$ measured counterclockwise from $\tilde{r}$ to $\tilde{\gamma}_i$, by v_i the unitary tangent vector to κ at $\tilde{s}_i$ and by u_i the unitary tangent vector to $\tilde{\gamma}_i$ at $\tilde{s}_i$ such that $\pi - \theta_i$ is the angle between v_i and u_i. Notice that

$$\int \cos\theta_{(\lambda,\gamma)}\, \mathrm{d}\gamma \otimes \mathrm{d}\lambda + \int \cos\theta_{(\mu,\gamma)}\, \mathrm{d}\gamma \otimes \mathrm{d}\mu = \omega \sum_{i=1}^{m} \cos\theta_i.$$

Recall that in the statement of Proposition 3.2.4 λ and μ have weight ω while γ has weight 1. We want to prove first the following result.

Proposition 3.2.5.

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|0} \sum_{i=0}^{m-1} \ell_h(A_i(t)) = \sum_{i=1}^{m} \cos\theta_i + \mathcal{R}(0) + \mathcal{R}'(0) \tag{3.11}$$

where $\mathcal{R}$ and $\mathcal{R}'$ are terms related to the two vertices of κ (explicitly computed in the proof, see equations (3.14) *and* (3.15)*).*

Proof. Consider in $\mathbb{R}^{2,1} = (\mathbb{R}^3, \langle *, * \rangle)$ (where $\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2$) the hyperboloid model of $\mathbb{H}^2$, namely $\{x \in \mathbb{R}^{2,1} : \langle x, x\rangle = -1, x_0 > 0\}$. See

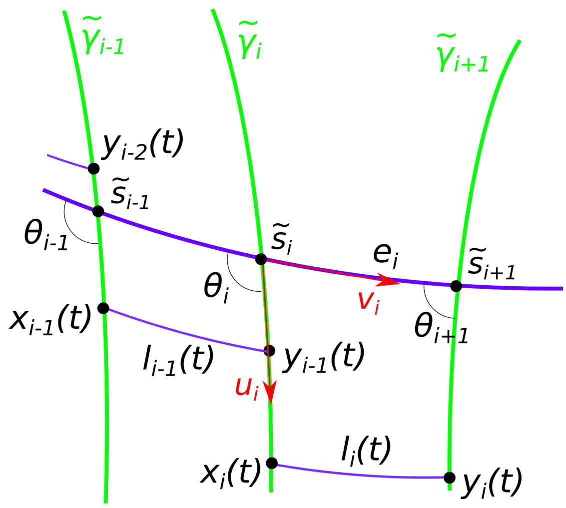


Figure 3.13

Subsection 1.1.2.
For every $i \neq k, k'$, the arcs $A_i(t)$ have endpoints $x_i(t) \in \tilde{\gamma}_i$ and $y_i(t) \in \tilde{\gamma}_{i+1}$ and length $l_i(t)$. Since $\cosh l_i(t) = -\langle x_i(t), y_i(t)\rangle$, denoting by $e_i = \ell(A_i(0))$ and differentiating in $t = 0$ we get

$$(\sinh e_i)\dot{l}_i(0) = -\langle \dot{x}_i(0), y_i(0)\rangle - \langle x_i(0), \dot{y}_i(0)\rangle.$$

From

$$\begin{aligned} y_i(0) &= x_i(0)\cosh e_i + v_i \sinh e_i \\ x_i(0) &= y_i(0)\cosh e_i - v_{i+1}\sinh e_i \end{aligned}$$

we get

$$\begin{aligned} (\sinh e_i)\dot{l}_i(0) = &- \langle \dot{x}_i(0), x_i(0)\rangle \cosh e_i - \langle \dot{x}_i(0), v_i\rangle \sinh e_i - \\ &- \langle \dot{y}_i(0), y_i(0)\rangle \cosh e_i + \langle \dot{y}_i(0), v_{i+1}\rangle \sinh e_i, \end{aligned}$$

which gives

$$\dot{l}_i(0) = -\langle \dot{x}_i(0), v_i\rangle + \langle \dot{y}_i(0), v_{i+1}\rangle.$$

Denote by w^- the unitary tangent vector to the first geodesic piece of A_k at $\tilde{p}_1$ and by w^+ the unitary tangent vector to the second geodesic piece of A_k at $\tilde{p}_1$, both pointing towards $\partial\mathcal{H}$. Analogously define w'^- and w'^+ at $\tilde{p}'_1$. With the same argument, if $k \neq k'$ we get

$$\begin{aligned} \dot{l}_k(0) = &\Big(- \langle \dot{x}_k(0), v_k\rangle + \langle \dot{\tilde{p}}_2(0), w^-\rangle\Big) + \Big(- \langle \dot{\tilde{p}}_2(0), -w^+\rangle + \langle \dot{y}_k(0), v_{k+1}\rangle\Big) = \\ = &- \langle \dot{x}_k(0), v_k\rangle + \langle \dot{\tilde{p}}_2(0), w^- + w^+\rangle + \langle \dot{y}_k(0), v_{k+1}\rangle \end{aligned}$$

and

$$\dot{l}_{k'}(0) = -\langle \dot{x}_{k'}(0), v_{k'}\rangle + \langle \dot{\tilde{p}}_2'(0), w'^- + w'^+\rangle + \langle \dot{y}_{k'}(0), v_{k'+1}\rangle,$$

while if $k = k'$ then

$$\begin{aligned}\dot{l}_k(0) =& \Big(- \langle \dot{x}_k(0), v_k\rangle + \langle \dot{\tilde{p}}_2(0), w^-\rangle\Big) + \Big(- \langle \dot{\tilde{p}}_2(0), -w^+\rangle + \langle \dot{\tilde{p}}_2'(0), w'^-\rangle\Big) + \\ &+ \Big(- \langle \dot{\tilde{p}}_2'(0), -w'^+\rangle + \langle \dot{y}_{k'}(0), v_{k'+1}\rangle\Big) = -\langle \dot{x}_k(0), v_k\rangle + \\ &+ \langle \dot{\tilde{p}}_2(0), w^- + w^+\rangle + \langle \dot{\tilde{p}}_2'(0), w'^- + w'^+\rangle + \langle \dot{y}_{k'}(0), v_{k'+1}\rangle.\end{aligned}$$

In both cases, the sum of the derivatives of the length of all A_i's gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \ell_{E_l^{t\gamma}(h)}(\rho_0) = \sum_{i=0}^{m-1} \dot{l}_i(0) = \sum_{i=0}^{m-1} \Big(- \langle \dot{x}_i(0), v_i\rangle + \langle \dot{y}_i(0), v_{i+1}\rangle\Big) + \\ + \langle \dot{\tilde{p}}_2(0), w^- + w^+\rangle + \langle \dot{\tilde{p}}_2'(0), w'^- + w'^+\rangle. \qquad (3.12)\end{aligned}$$

We claim that for $i = 1, \ldots, m-1$, the following identity holds:

$$\dot{y}_{i-1}(0) = \dot{x}_i(0) - u_i, \qquad (3.13)$$

where u_i is the unitary tangent vector to $\tilde{\gamma}_i$ at $x_i(0)$ so that $-\cos\theta_i = \langle u_i, v_i\rangle$ (see Figure 3.13). Denote by $d_i(t)$ the signed distance between $y_{i-1}(0) = x_i(0)$ and $y_{i-1}(t)$ on $\tilde{\gamma}_i$ oriented as u_i. Then

$$\begin{aligned}y_{i-1}(t) &= x_i(0)\cosh d_i(t) + u_i \sinh d_i(t)\\ x_i(t) &= x_i(0)\cosh\big(d_i(t) + t\big) + u_i \sinh\big(d_i(t) + t\big).\end{aligned}$$

Therefore,

$$\begin{aligned}\dot{y}_{i-1}(0) &= u_i \dot{d}_i(0)\\ \dot{x}_i(0) &= u_i\big(\dot{d}_i(0) + 1\big)\end{aligned}$$

leading to (3.13).

Since s_0 is a point were ρ_0 is smooth and $\tilde{s}_0 = x_0(0)$ and $\tilde{s}_m = y_{m-1}(0)$ are preimages of s_0, there exists a covering transformation T such that $\dot{y}_{m-1}(0) = T\dot{x}(0)$ and $v_m = Tv_0$.

Using (3.13), now we can write the sum in (3.12) as

$$\begin{aligned}&\sum_{i=0}^{m-1} \Big(- \langle \dot{x}_i(0), v_i\rangle + \langle \dot{y}_i(0), v_{i+1}\rangle\Big) = \\ &= -\sum_{i=0}^{m-1} \langle \dot{x}_i(0), v_i\rangle + \sum_{i=1}^{m-1} \langle \dot{y}_{i-1}(0), v_i\rangle + \langle \dot{y}_{m-1}(0), v_m\rangle = \\ &= -\sum_{i=0}^{m-1} \langle \dot{x}_i(0), v_i\rangle + \sum_{i=1}^{m-1} \langle \dot{x}_i(0) - u_i, v_i\rangle + \langle T\dot{x}_0(0), Tv_0\rangle = \\ &= -\langle \dot{x}_0(0), v_0\rangle - \sum_{i=1}^{m} \langle u_i, v_i\rangle + \langle \dot{x}_0(0), v_0\rangle = \sum_{i=1}^{m} \cos\theta_i.\end{aligned}$$

Setting

$$\mathcal{R}(0) = \langle \dot{\tilde{p}}_2(0), w^- + w^+ \rangle \tag{3.14}$$

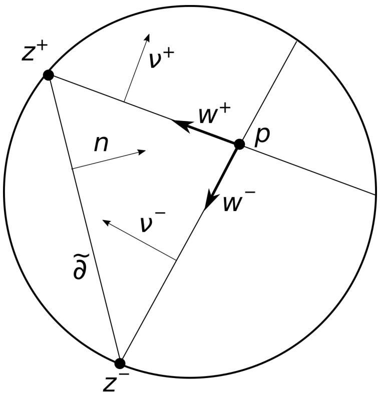
$$\mathcal{R}'(0) = \langle \dot{\tilde{p}}_2'(0), w'^- + w'^+ \rangle. \tag{3.15}$$

we get that (3.11) holds. □

Now we have to show that Equations (3.9) and (3.10) hold. The latter equation is analogous to the former one, so we will prove only Equation (3.9).
In the hyperboloid model of $\mathbb{H}^2$, keep the notations of the proof of Proposition 3.2.5 and denote by $[z^+]$ and $[z^-]$ the ideal endpoints of $\tilde{\partial}$, so that $w^\pm$ is pointing towards $[z^\pm]$. The unitary vector

$$n = \frac{z^- \boxtimes z^+}{\|z^- \boxtimes z^+\|_{2,1}}$$

is the normal unitary vector of $\tilde{\partial}$ pointing towards $\tilde{p}_1$. Up to precomposing



by a proper isometry, we can suppose that $[z^+]$ and $[z^-]$ are kept fixed by $\tilde{E}$, thus $\tilde{E}(n) = n$. If $\underline{p} = \tilde{p}_1$ and $d = d(p_1, \partial) = d(\tilde{p}_1, \tilde{\partial})$, then $\sinh d = \langle \underline{p}, n \rangle$. Therefore

$$\dot{d} = \frac{\langle \underline{\dot{p}}, n \rangle}{\cosh d}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} F(d) = 2 \frac{\sinh d}{\cosh^2 d} \langle \underline{\dot{p}}, n \rangle,$$

where we have set $F(d) = 2 \log \cosh d$, so that the expression of L is

$$L(h) = \ell_h(\rho) + F(d_h(p_1, \partial)) + F(d_h(p_1', \partial')).$$

Now Equation (3.9) becomes

$$\left\langle \underline{\dot{p}}, w^+ + w^- + 2 \frac{\sinh d}{\cosh^2 d} n \right\rangle = 0.$$

The following proposition will prove such equation computing $w^\pm$ in terms of $\underline{p}$ and n.

Proposition 3.2.6.

$$\langle \dot{\underline{p}}, w^+ + w^- \rangle = -2\frac{\sinh d}{\cosh^2 d}\langle \dot{\underline{p}}, n\rangle.$$

Proof. The vector $w^\pm$ can be written as $\underline{p} \boxtimes \nu^\pm$, where $\nu^\pm$ is the unitary vector tangent to $\mathbb{H}^2$ and normal to $w^\pm$ (i.e. to λ/μ) oriented in the proper way; namely,

$$\nu^\pm = -\frac{z^\pm \boxtimes \underline{p}}{\|z^\pm \boxtimes \underline{p}\|_{2,1}}.$$

Thus,

$$w^\pm = -\underline{p} \boxtimes \frac{z^\pm \boxtimes \underline{p}}{\|z^\pm \boxtimes \underline{p}\|_{2,1}} = -\frac{-\langle \underline{p}, \underline{p}\rangle z^\pm + \langle z^\pm, \underline{p}\rangle \underline{p}}{\langle z^\pm, \underline{p}\rangle} = -\frac{z^\pm + \langle z^\pm, \underline{p}\rangle \underline{p}}{\langle z^\pm, \underline{p}\rangle}.$$

We claim we can suppose that

$$z^\pm = \underline{p} - (\sinh d)n \pm \underline{p} \boxtimes n. \tag{3.16}$$

First, we have to see that the second term of (3.16) is a null vector; let us compute the square norm of $\underline{p} \boxtimes n$:

$$\langle \underline{p} \boxtimes n, \underline{p} \boxtimes n\rangle = \langle \underline{p}, n\rangle^2 - \langle \underline{p}, \underline{p}\rangle\langle n, n\rangle = \sinh^2 d + 1 = \cosh^2 d$$

Now

$$\begin{aligned}
&\langle \underline{p} - (\sinh d)n \pm \underline{p} \boxtimes n, \underline{p} - (\sinh d)n \pm \underline{p} \boxtimes n\rangle = \\
=&\langle \underline{p}, \underline{p}\rangle - (\sinh d)\langle \underline{p}, n\rangle - (\sinh d)\langle n, \underline{p}\rangle + (\sinh^2 d)\langle n, n\rangle + \langle \underline{p} \boxtimes n, \underline{p} \boxtimes n\rangle = \\
=& -1 - \sinh^2 d - \sinh^2 d + \sinh^2 d + \cosh^2 d = 0.
\end{aligned}$$

On the other hand, we have to check that $\underline{p} - (\sinh d)n \pm \underline{p} \boxtimes n$ are the ideal endpoints of $\tilde{\partial}$ (or equivalently $\langle \underline{p} - (\sinh d)n \pm \underline{p} \boxtimes n, n\rangle = 0$) such that

$$\Big(\langle \underline{p} - (\sinh d)n - \underline{p} \boxtimes n,\ \underline{p} - (\sinh d)n + \underline{p} \boxtimes n,\ n\Big)$$

forms a negative basis of $\mathbb{R}^{2,1}$. Now

$$\langle \underline{p} - (\sinh d)n \pm \underline{p} \boxtimes n, n\rangle = \langle \underline{p}, n\rangle - (\sinh d)\langle n, n\rangle = 0$$

and

$$\begin{aligned}
&\langle \underline{p} - (\sinh d)n - \underline{p} \boxtimes n, (\underline{p} - (\sinh d)n + \underline{p} \boxtimes n) \boxtimes n\rangle = \\
=&\langle \underline{p} - (\sinh d)n - \underline{p} \boxtimes n, \underline{p} \boxtimes n + \underline{p}\rangle = -1 - \cosh^2 d < 0.
\end{aligned}$$

Thus, we can compute

$$\langle z^\pm, \underline{p} \rangle = \langle \underline{p} - (\sinh d)n \pm \underline{p} \boxtimes n, \underline{p} \rangle = -\cosh^2 d$$

and

$$\begin{aligned} w^\pm &= -\frac{z^\pm + \langle z^\pm, \underline{p}\rangle \underline{p}}{\langle z^\pm, \underline{p}\rangle} = \\ &= -\frac{\underline{p} - (\sinh d)n \pm \underline{p} \boxtimes n - (\cosh^2 d)\underline{p}}{-\cosh^2 d} = \\ &= -\frac{(\sinh^2 d)\underline{p} + (\sinh d)n \mp \underline{p} \boxtimes n}{\cosh^2 d}. \end{aligned}$$

Now

$$\langle \dot{\underline{p}}, w^+ + w^- \rangle = \left\langle \dot{\underline{p}}, -\frac{(2\sinh^2 d)\underline{p} + (2\sinh d)n}{\cosh^2 d} \right\rangle = -2\frac{\sinh d}{\cosh^2 d}\langle \dot{\underline{p}}, n\rangle.$$

□

Finally, let us consider the general case: the first order variation of $t \mapsto L(E^{t\nu_l(h)})$ when $\nu \in \mathcal{CML}_S^\circ$.

Proposition 3.2.7. *Take $(\lambda, \mu) \in \mathcal{ML}_S^\circ(\mathbf{b})$ such that the supports of λ and μ consist respectively of a single weighted geodesic spiralling between two boundary components ∂ and ∂'. For every $h \in \mathcal{T}_S^\circ(\mathbf{b})$ and $\nu \in \mathcal{CML}_S^\circ$ the following formula holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{(\lambda,\mu)}(E_l^{t\nu}(h)) = \int \cos\theta_{(\lambda,\nu)}(t)\mathrm{d}\nu \otimes \mathrm{d}\lambda + \int \cos\theta_{(\mu,\nu)}(t)\mathrm{d}\nu \otimes \mathrm{d}\mu.$$

Proof. The space of weighted curves on S is dense in $\mathcal{CML}_S^\circ$ (see [31]), so take a sequence (γ_n) of weighted curves converging to ν. With the notation used in [28] and recalled in Subsection 3.2.1, we have seen that

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{(\lambda,\mu)}(E_l^{t\gamma_n}(h)) = \mathrm{Cos}(\lambda, \gamma_n)(t) + \mathrm{Cos}(\mu, \gamma_n)(t).$$

Clearly $L_{(\lambda,\mu)}(E_l^{t\gamma_n}(h))_{|t=0} = L_{(\lambda,\mu)}(E_l^{t\nu}(h))_{|t=0}$ for every n, so if we prove that $\mathrm{Cos}(\lambda, \gamma_n) + \mathrm{Cos}(\mu, \gamma_n)$ tends uniformly to $\mathrm{Cos}(\lambda, \nu) + \mathrm{Cos}(\mu, \nu)$ then $L_{(\lambda,\mu)}(E_l^{t\gamma_n}(h))$ tends to $L_{(\lambda,\mu)}(E_l^{t\nu}(h))$ and (3.17) holds. Kerckhoff showed in [28] itself that $\mathrm{Cos}(\delta, \gamma_n)$ tends uniformly to $\mathrm{Cos}(\delta, \nu)$ for every δ closed curve in S, but his argument still works if δ is a spiralling leaf of a lamination on S, so we can conclude. □

3.2.4 The map $\mathbb{L}_{(\lambda,\mu)} : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathbb{R}$

We have defined $L_{(\lambda,\mu)}$ in the simple case where λ and μ are made of a unique spiralling leaf with weight ω and we have seen that for any $\nu \in \mathcal{ML}_0(S)$ the following formula holds:

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{(\lambda,\mu)}(E_l^{t\nu}(h)) = \int \cos\theta_{(\lambda,\nu)} \mathrm{d}\nu \otimes \mathrm{d}\lambda + \int \cos\theta_{(\mu,\nu)} \mathrm{d}\nu \otimes \mathrm{d}\mu. \tag{3.17}$$

Using this result, we construct here a function $\mathbb{L}_{(\lambda,\mu)}$ on $\mathcal{T}_S^\circ(\mathbf{b})$ for a general (λ,μ) in

$$\mathcal{ML}_S^\circ(\mathbf{b}) = \{(\lambda,\mu) \in (\mathcal{ML}_S^\circ)^2 \, : \, -2m(\partial_i,\mu) = 2m(\partial_i,\lambda) < b_i \\ \forall i = 1,\ldots,\mathfrak{n}\}$$

that satisfies (3.17).

If λ_1 and λ_2 are measured laminations with empty transverse intersection, their sum $\lambda_1 \oplus \lambda_2$ is defined by putting $\mathrm{supp}(\lambda_1 \oplus \lambda_2) = \mathrm{supp}(\lambda_1) \cup \mathrm{supp}(\lambda_2)$ and $\mathrm{meas}_{\lambda_1 \oplus \lambda_2} = \mathrm{meas}_{\lambda_1} + \mathrm{meas}_{\lambda_2}$. By example, if $\lambda = (\delta,\omega)$ is a weighted curve and $\omega = \omega_1 + \omega_2$ then λ is the sum of $\lambda_1 = (\delta,\omega_1)$ and $\lambda_2 = (\delta,\omega_2)$.

Remark 3.2.7. If $\lambda = \lambda_1 \oplus \lambda_2$, consider the relative left earthquakes E_l^λ, $E_l^{\lambda_1}$ and $E_l^{\lambda_2}$. Add to λ_1 the leaves of $\lambda_2 \setminus \lambda_1$ and provide them with weight 0; do the same to λ_2 with the leaves of $\lambda_1 \setminus \lambda_2$. The shearing amount of E^λ along any leaf δ of λ is now the sum of the shearing amounts along δ of E^{λ_1} and of E^{λ_2}, so that $E^\lambda = E^{\lambda_1} \circ E^{\lambda_2}$. More in general, $E^{t\lambda} = E^{t\lambda_1} \circ E^{t\lambda_2}$for every $t \in [0,1]$. It follows that $e_l^\lambda = e_l^{\lambda_1} + e_l^{\lambda_2}$.

Definition 3.2.2. We say that a couple of laminations (λ,μ) is a circuit of laminations if it can be expressed as

$$(\lambda,\mu) = \left(\bigoplus_{i=1}^{I} \lambda_i, \, \bigoplus_{i=1}^{I} \mu_i \right)$$

where $I \in \mathbb{N}$, such that

- λ_i and μ_i are single spiralling weighted leaves for every $i = 1,\ldots,I$, all with the same weight ω;
- for every $i = 1,\ldots,I$ there are boundary components $D_\pm^i \in \{\partial_1,\ldots,\partial_N\}$ of S such that λ_i spirals between D_-^i and D_+^i while μ_i spirals between D_+^i and D_-^{i+1}, providing $D_-^{I+1} = D_-^1$.

Notice that there can be distinct i and j such that $\lambda_i = \lambda_j$ or $\mu_i = \mu_j$, as in the following example.

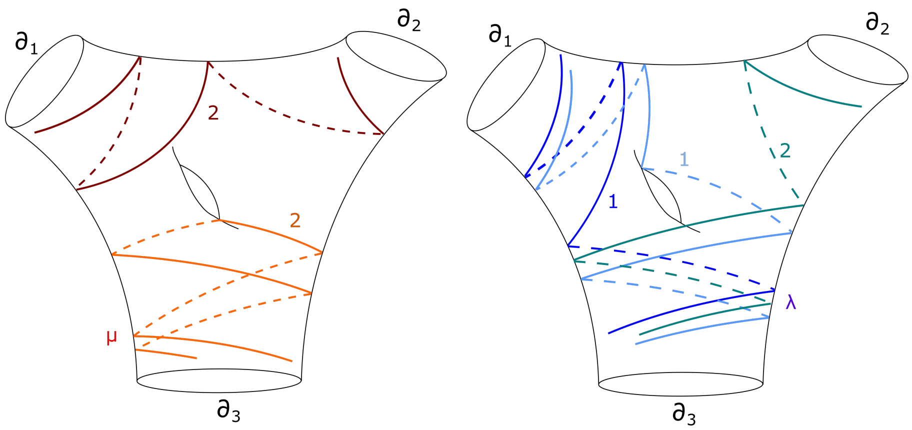


Figure 3.14

Example 3.2.1. Consider two laminations as in Figure 3.14, realized in a certain metric h.
They can be outlined in the first scheme of Figure 3.15, where it is easy to see that (λ, μ) can be decomposed in the second scheme of the same figure, which is a 1-weighted circuit.
Figure 3.16 shows that the decomposition is not unique and that (λ, μ) can

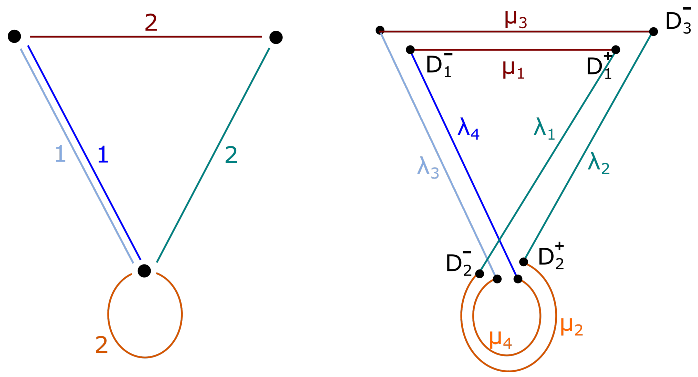


Figure 3.15

also be seen as the sum of two circuital laminations.

Remark 3.2.8. If

$$\left(\bigoplus_{i=1}^{I} \lambda_i, \ \bigoplus_{i=1}^{I} \mu_i \right)$$

is a circuit of laminations, then for every boundary component ∂ of S

$$\iota\left(\bigoplus_{i=1}^{I} \lambda_i, \ \partial \right) = \iota\left(\bigoplus_{i=1}^{I} \mu_i, \ \partial \right).$$

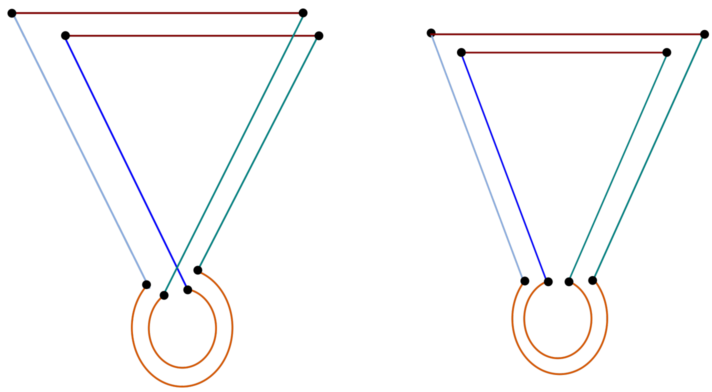


Figure 3.16

An important result on circuits of laminations is the following:

Proposition 3.2.8. *If $(\lambda, \mu) = \left(\bigoplus_{i=1}^I \lambda_i, \bigoplus_{i=1}^I \mu_i\right)$ is a circuit of laminations with weight ω, there is a truncation $\rho = \lambda_1 \cup \mu_1 \cup \lambda_2 \cup \ldots \cup \mu_I$, analogous to ρ_1 in the case $I = 1$ considered in Subsection 3.2.2, which is a loop with $2I$ vertices $q_1, \ldots, q_{2I}$ lying in the spiralization neighbourhood. Denote by D_i the boundary component such that $q_i \in \mathcal{N}(D_i)$. Define*

$$L_{(\lambda,\mu)}(h) = \omega\, \ell_h(\rho) + 2\omega \log \prod_{i=1}^{2I} \cosh d_h(q_i, D_i). \tag{3.18}$$

Then Equation (3.17) *still holds.*

Proof. In Subsection 3.2.3 Equation (3.17) was proven in the simple case where $I = 1$ in Proposition 3.2.4, where exactly two vertices (called p_1 and p_1') of the truncation ρ of $\operatorname{supp}(\lambda) \cup \operatorname{supp}(\mu)$ occurred. However, the number of vertices of the truncation was irrelevant: doing again the computations, we would get analogously

$$\omega \frac{\mathrm{d}}{\mathrm{d}t} \ell_{E_l^{t\gamma}(h)}(\rho) = \operatorname{Cos}(\lambda, \gamma) + \operatorname{Cos}(\mu, \gamma) + \omega \sum_{i=1}^{2I} \mathcal{R}_i$$

and, for every $i = 1, \ldots, 2I$,

$$\mathcal{R}_i + \frac{\mathrm{d}}{\mathrm{d}t} \log \cosh d_{E_l^{t\gamma}(h)}(q_i, D_i) = 0,$$

proving that Equation (3.17) still holds if (λ, μ) is a circuit of lamination. $\square$

Therefore, in order to extend the definition of $L_{(\lambda,\mu)}$ to the general case, if we prove that (λ, μ) is decomposable into the sum of their compact parts

$(\lambda^{(0)}, \mu^{(0)})$ and of circuits of laminations $(\lambda^{(j)}, \mu^{(j)})$ for $j = 1, \ldots, J$, then we can define a function

$$\mathbb{L} = \mathbb{L}_{(\lambda,\mu)} = \sum_{j=0}^{J} L^{(j)} \tag{3.19}$$

where $L^{(0)} = L_{\lambda^{(0)}} + L_{\mu^{(0)}}$ and $L^{(j)}$ is the length map of $(\lambda^{(j)}, \mu^{(j)})$ pointed out in Proposition 3.2.8, for $j \neq 0$. Since Equation (3.17) holds for every $L^{(j)}$, we can deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{L}(E_l^{t\nu}(h)) = \int \cos\theta_{(\lambda,\nu)} \mathrm{d}\nu \otimes \mathrm{d}\lambda + \int \cos\theta_{(\mu,\nu)} \mathrm{d}\nu \otimes \mathrm{d}\mu \tag{3.20}$$

for every $\nu \in \mathcal{CML}_S^\circ$ and $h \in \mathcal{T}_S^\circ(\mathbf{b})$.
That (λ, μ) can be decomposed into compact and circuital laminations is the statement of the following result.

Proposition 3.2.9. *Every $(\lambda, \mu) \in \mathcal{FML}_S^\circ(\mathbf{b})$ can be decomposed as*

$$(\lambda, \mu) = \left(\lambda^{(0)} \oplus \bigoplus_{j=1}^{J} \lambda^{(j)}, \ \mu^{(0)} \oplus \bigoplus_{j=1}^{J} \mu^{(j)} \right)$$

where $(\lambda^{(0)}, \mu^{(0)})$ is the compact part of (λ, μ) and $(\lambda^{(j)}, \mu^{(j)})$ is for every $j = 1, \ldots, J$ a circuit of lamimations.

Proof. If $(\lambda, \mu) = (\lambda^{(0)}, \mu^{(0)})$ there is nothing to prove. Otherwise, consider the graph G associated with $(\lambda_s, \mu_s) = (\lambda \smallsetminus \lambda^{(0)}, \mu \smallsetminus \mu^{(0)})$ where the vertices correspond to the boundary components of ∂S and the edges correspond to the leaves of λ and μ. The edges are coloured, say blue if it is a leaf of λ and red otherwise. Two vertices of an edge can coincide; also, there can be two vertices bounding more than one or two edges of any colour.
We start to look for a circuit of laminations $(\lambda^{(1)}, \mu^{(1)}) = (\oplus\lambda_i, \oplus\mu_i)$ which is a sublamination of (λ_s, μ_s); this is equivalent to find a cycle in the graph G with alternating colours. Such cycle can pass through an edge more than once, since, as we noticed, in the circuit $i \neq j$ can occur such that $\lambda_i = \lambda_j$ or $\mu_i = \mu_j$.
Since $(\lambda, \mu) \in \mathcal{FML}_S^\circ(\mathbf{b})$, if a vertex is reached by a red edge then it is also a vertex of a blue edge, and vice versa.
Let us start from a vertex D_0 reached by a blue edge δ_1 and denote by D_1 the other vertex (maybe coincident with D_0) of δ_1. There must be a red edge η_1 starting from D_1 and ending in a certain vertex D_2. If $D_2 = D_0$ then (δ_1, η_1) is an alternating cycle and we therefore found a circuital sublamination of (λ, μ). Otherwise, again there must be a blue edge δ_2 starting in D_2 and ending in a certain vertex D_3. If $D_3 = D_1$ then we found the cycle (δ_2, η_1), otherwise we can exit from D_3 following a blue edge η_2. Continuing with

these steps, we find an alternating path on G (in the sense that consecutive edges in this path have different colours).
Iterating this construction, if we can find M such that there is $N < M$ and the subpath from D_N to D_M is an alternating cycle, then we have finished. We claim that if we visit a vertex D_j for the third time then either we have already found such M (and it is less than j) or there is $N < j$ such that the path from D_N to D_j is an alternating cycle (so j is the M we were looking for). There are only two possibilities when we visit a D_j for the third time.

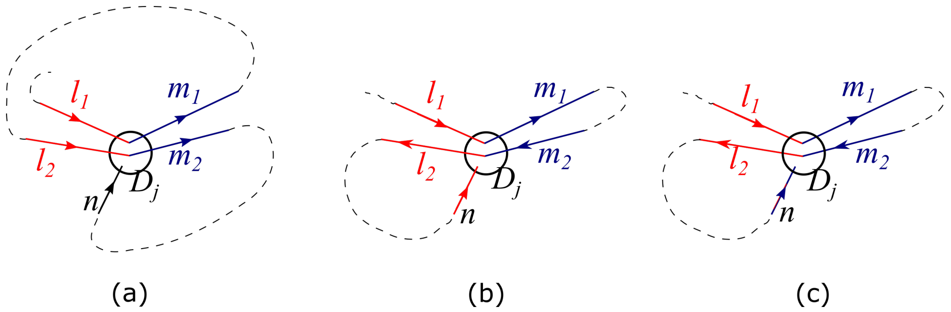


Figure 3.17

- Two edges l_1 and l_2 of the same colour have already entered D_j (with l_1 walked before l_2) and consequently two edges of the other colour m_1 and m_2 have exited, as in Figure 3.17 (a). But then we already found an alternating cycle: the one starting with m_1 and ending with l_2.

- Two edges l_1 and m_2 of different colours have already entered D_j and consequently two edges m_1, l_2 of the colour respectively of m_2 and l_1 have already exited. If we enter D_j for the third time with an edge n of the same colour of l_1 and l_2, as in Figure 3.17 (b), then take the alternating cycle starting from m_1 and ending with n; otherwise, as in Figure 3.17 (c), take the alternating cycle starting with l_2 and ending with n.

So there exists an alternating cycle in G made by the sequence of edges

$$(r_1, s_1, \ldots, r_K, s_K).$$

Up to cycling rename the edges, we can suppose r_1 is red. Each edge r_k corresponds to a leaf λ_k of λ, while each edge s_k corresponds to a leaf μ_k of μ.
We want now to endow the circuit of lamination

$$(\lambda^{(1)}, \mu^{(1)}) = \left(\bigoplus_{k=1}^{K} \lambda_k, \bigoplus_{k=1}^{K} \mu_k \right)$$

with a weight $\omega^{(1)}$ so that if (Λ, M) is the couple of laminations such that

$$(\lambda, \mu) = (\lambda^{(0)}, \mu^{(0)}) \oplus (\lambda^{(1)}, \mu^{(1)}) \oplus (\Lambda, M)$$

then $(\lambda^{(1)}, \mu^{(1)})$ has at least one leaf not contained in the support of (Λ, M). For every spiralling leaf δ of λ, denote by ω_δ its weight. Define

$$W(\delta) = \frac{\omega_\delta}{\#\{k \in \{1, \dots, K\} \mid \lambda_k = \delta\}}.$$

Analogously, define

$$W(\eta) = \frac{\omega_\eta}{\#\{k \in \{1, \dots, K\} \mid \mu_k = \eta\}}.$$

for every spiralling leaf η of μ. Now set

$$\omega^{(1)} = \min\{W(\zeta) \mid \zeta \text{ is a leaf of } \lambda \text{ or a leaf of } \mu\}.$$

In this way, the leaf of λ or μ where such minimum is achieved does not appear in the support of (Λ, M).
If (Λ, M) is the couple of void laminations, we have finished. Otherwise, notice that again $(\Lambda, M) \in \mathcal{FML}_S^\circ(\mathbf{b})$ (it depends on the fact that $\lambda^{(1)}, \mu^{(1)}$) lies in $\mathcal{FML}_S^\circ(\mathbf{b})$; see Remark 3.2.8). Moreover (Λ, M) has less leaves than (λ, μ). By a simple inductive argument we get the decomposition of (λ, μ) in circuital sublaminations

$$(\lambda, \mu) = \left(\bigoplus_{j=0}^{J} \lambda^{(j)}, \bigoplus_{j=0}^{J} \mu^{(j)} \right).$$

□

Remark 3.2.9. The decomposition described in the proposition above is not unique: different choices of edges of G exiting from a vertex produce different cycles; also, the choice of the vertex D_0 was made arbitrarily. The length function $\mathbb{L}$ defined in Equation (3.19) depends on the resulting decomposition of (λ, μ). However, for every $\nu \in \mathcal{CML}_S^\circ$ and $h \in \mathcal{T}_S^\circ(\mathbf{b})$, by Remark 3.2.7 (forgetting for a moment of writing k)

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathbb{L}_{(\lambda,\mu)}(E_l^{t\nu}(h)) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \left[\left(L_{\lambda^{(0)}} + L_{\mu^{(0)}} + \sum_{j=1}^{J} L^{(j)} \right)(E_l^{t\nu}(h)) \right] =$$

$$= \sum_{j=0}^{J} \left(\int \cos\theta_{(\lambda^{(j)},\nu)}(h) \mathrm{d}\nu \otimes \mathrm{d}\lambda^{(j)} + \int \cos\theta_{(\mu^{(j)},\nu)}(h) \mathrm{d}\nu \otimes \mathrm{d}\mu^{(j)} \right) =$$

$$= -\sum_{j=0}^{J} \varpi_h\big(e_l^\nu, e_l^{\lambda^{(j)}} + e_l^{\mu^{(j)}}\big) = -2\varpi_h\big(e_l^\nu, e_l^\lambda + e_l^\mu\big)$$

and this holds for any other length map constructed with different allowed choices. Therefore, all these maps are Hamiltonians of $-(e_l^\lambda + e_l^\mu)$, differing one from the other by a constant.

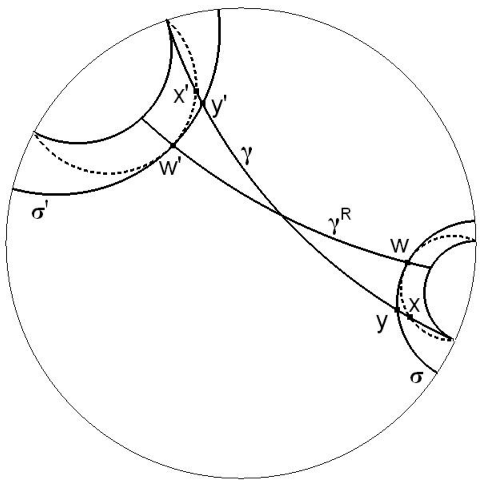
3.3 Properties of $\mathbb{L}_{(\lambda,\mu)}$

3.3.1 The properness of $\mathbb{L}_{(\lambda,\mu)}$

As explained in section 1.2 any spiralling geodesic γ of a measured geodesic lamination can be replaced by a geodesic arc γ^R orthogonal to the boundary. For each $\nu \in \mathcal{ML}_S$ denote by ν^R the set of geodesic arcs obtained by ν replacing each spiralling geodesic γ of ν with γ^R.
Now consider $(\lambda,\mu) \in \mathcal{FML}_S^\circ(\mathbf{b})$. Then (λ^R,μ^R) still fills up S, in the sense that every simple closed non trivial curve on S intersects $\lambda^R \cup \mu^R$ or is isotopic to a boundary component of S. In fact if c is a simple closed non trivial curve not isotopic to a boundary component then it meets a leaf γ of $\lambda \cup \mu$ in a certain point p. Take a preimage $\tilde{p}$ on the universal cover $\mathcal{H}$ in the Poincaré disk model and denote by $\tilde{\gamma}$ and $\tilde{c}$ the lifts of γ and c respectively that pass through $\tilde{p}$. Now $\tilde{\gamma}$ and the prolongation of its replacement ρ separate the same two maximal subsets of $\partial_\infty\mathcal{H}$. One endpoint of $\tilde{c}$ lies in one of these subsets and the other endpoint in the other one, so c still intersects γ^R.

Lemma 3.3.1. *Consider two disjoint geodesics ∂ and ∂' in $\mathbb{H}^2$, a geodesic γ going from an endpoint of ∂ to an endpoint of ∂', the geodesic arc γ^R with endpoints on ∂ and ∂' normal to ∂ and ∂', two positive real numbers $\epsilon,\epsilon' \leq \ell(\gamma^R)/2$, the ϵ-collars N of ∂ and the ϵ'-collar N' of ∂'. Then*

$$\ell(\gamma \setminus (N \cup N')) \geq \ell(\gamma^R \setminus (N \cup N')) = \ell(\gamma^R) - \epsilon - \epsilon'$$



Proof. Denote by w and w' the intersections of γ^R with ∂N and $\partial N'$ respectively and with x and x' the intersections of γ with ∂N and $\partial N'$ respectively. Take σ and σ', the geodesics normal to γ^R passing respectively through w and w'. Denote by y and y' the intersections of γ with σ and

σ' respectively. Notice that N and N' are included respectively in the region between ∂ and σ and in the region between ∂' and σ'. Therefore, $d(w, w') \leq d(y, y') \leq d(x, x')$. □

Proposition 3.3.2. *The map* $\mathbb{L}_{(\lambda,\mu)} : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathbb{R}$ *is proper.*

Proof. Choose a pant decomposition of S with curves $\kappa_1, \ldots, \kappa_{3(g-1)+\mathfrak{n}}$, $\partial_1, \ldots, \partial_{\mathfrak{n}}$ and consider the related coordinates

$$(l_1, \ldots, l_{3(g-1)+\mathfrak{n}}, \tau_1, \ldots, \tau_{3(g-1)+\mathfrak{n}})$$

on $\mathcal{T}_S^\circ(\mathbf{b})$, where l_i is the length of κ_i and τ_i is the twist factor on κ_i. Choose also for every κ_i two dual curves κ_i^* and κ_i^{**} whose lengths can reconstruct τ_i (as explained in [22]; see Figure 3.18).
We have seen at the beginning of this subsection that if $(\lambda, \mu) \in \mathcal{FML}_S^\circ(\mathbf{b})$ then $\lambda^R \cup \mu^R$ fills up S; this implies that every simple closed non-trivial

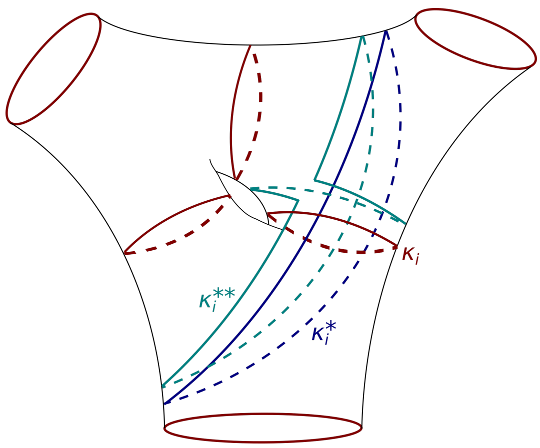


Figure 3.18

curve in S is isotopic to a curve on $G = \lambda^R \cup \mu^R \cup D$, where $D = \bigcup \partial_j$.
We claim that

$$\begin{aligned} \mathbb{L}_G : &\mathcal{T}_S^\circ(\mathbf{b}) \to \mathbb{R} \\ &h \mapsto \ell_h(\lambda^R) + \ell_h(\mu^R) + b_1 + \ldots + b_{\mathfrak{n}} \end{aligned}$$

is a proper map. Pick a divergent sequence $\{h_n\}$ in $\mathcal{T}_S^\circ(\mathbf{b})$; then the sequence

$$\{(l_1, \ldots, l_{3(g-1)+\mathfrak{n}}, \tau_1, \ldots, \tau_{3(g-1)+\mathfrak{n}})(h_n)\}$$

is divergent in $\mathbb{R}^{6(g-1)+2\mathfrak{n}}$. This implies that

$$S_n = \sum_{i=1}^{3(g-1)+\mathfrak{n}} \left[\ell_{h_n}([\kappa_i]) + \ell_{h_n}([\kappa_i^*]) + \ell_{h_n}([\kappa_i^{**}])\right] \xrightarrow{n\to\infty} +\infty,$$

where for any closed curve κ and hyperbolic metric h we denote by $\ell_h([\kappa])$ the h-length of the geodesic h-realization of κ.
Each κ_i (and κ_i^* and κ_i^{**}) is isotopic to many (not necessarily simple) curves in G, but for every i the number

$$m_i = \min\left\{ \max_{p\in G}\left\{\#\big(\pi^{-1}(p)\cap([0,1]\times\{0\})\big)\right\} \,\Big|\, \pi:[0,1]\times[0,1]\to S \text{ isotopy} \right.$$
$$\left. \text{between } \pi(*,0)=\kappa_i \text{ and } \pi(*,1) \text{ closed curve in } G\right\},$$

which denotes a sort of minimum of the degrees of the isotopies between κ_i and any curve in G, does not depend on the metric. The same holds for m_i^* and m_i^{**} (the analogous numbers for κ_i^* and κ_i^{**} respectively). If m_0 is the maximum among all m_i's, m_i^*'s and m_i^{**}'s, then $S_n \leq m_0\mathbb{L}_G(h_n)$. Therefore, $\{\mathbb{L}_G(h_n)\}$ is going to infinity as $\{h_n\}$ is diverging.
Since $\mathbb{L}_G(h_n) = \ell_{h_n}(\lambda^R\cup\mu^R)+\sum b_i$ is diverging, two possibilities occur:

- a compact sublamination ν^R of $\lambda^R\cup\mu^R$ has divergent length; but since $\nu^R=\nu$, also $\mathbb{L}_{(\lambda,\mu)}(h_n)$ is diverging;
- no closed leaf of $\lambda^R\cup\mu^R$ has divergent length; then an arc γ^R in $\lambda^R\cup\mu^R$ (replacement of a spiralling leaf γ of $\lambda\cup\mu$ between ∂ and ∂') has divergent length. Also $\ell_{h_n}\big(\gamma\smallsetminus(\mathcal{N}(\partial)\cup\mathcal{N}(\partial'))\big)$ diverges, by Lemma 3.3.1, where $\mathcal{N}(\partial)$ is the $\varepsilon(\partial)$-collar introduced in Section 3.3. From the definition,

  $$\mathbb{L}_{(\lambda,\mu)}(h_n) > \omega\ell_{h_n}(\gamma-\mathcal{N}(\partial)-\mathcal{N}(\partial')) > \omega\Big(\ell_{h_n}(\gamma^R)-\varepsilon(\partial)-\varepsilon(\partial')\Big),$$

  implying that $\mathbb{L}_{(\lambda,\mu)}(h_n)$ is diverging.

□

3.3.2 The second order variation of $\mathbb{L}_{(\lambda,\mu)}$

Now we want to show that $\mathbb{L}_{(\lambda,\mu)}$ is convex along any left earthquake along $\nu\in\mathcal{CML}_S^\circ$.

Lemma 3.3.3. *Take counterclockwise six distinct points Q, Q', M, P, P', N consecutively on $\partial_\infty\mathbb{H}^2$, as in Figure 3.19. Denoting by $[z_1,z_2]$ the geodesic with ideal endpoints z_1 and z_2, let O and O'' be the points where $[M,N]$ meets respectively $[P,Q]$ and $[P',Q']$. Then $\measuredangle(M\hat{O}Q) > \measuredangle(M\hat{O}''Q')$.*

Proof. Let O' be the point where $[M,N]$ meets $[P',Q]$. The area $\mathcal{A}$ of the triangle MOQ is smaller than the area $\mathcal{A}'$ of the triangle $MO'Q$, so

$$\pi-\measuredangle(M\hat{O}Q)=\mathcal{A}<\mathcal{A}'=\pi-\measuredangle(M\hat{O}'Q),$$

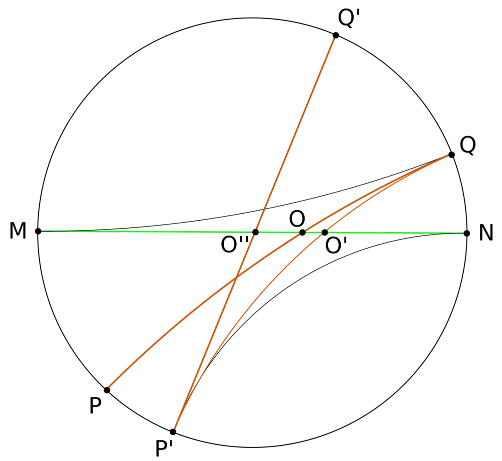


Figure 3.19

that is $\measuredangle(M\hat{O}Q) > \measuredangle(M\hat{O}'Q)$.
Analogously, the area $\mathcal{S}$ of the triangle $NO''P'$ is greater than the area $\mathcal{S}'$ of the triangle $NO'P'$, so

$$\pi - \measuredangle(N\hat{O}''P') = \mathcal{S} > \mathcal{S}' = \pi - \measuredangle(N\hat{O}'P'),$$

that is $\measuredangle(M\hat{O}''P') > \measuredangle(M\hat{O}'P)$. □

Remark 3.3.1. The strict inequality of Lemma 3.3.3 still holds if $P = P'$ and $Q \neq Q'$ or viceversa.

Proposition 3.3.4. *The map $\mathbb{L}_{(\lambda,\mu)} : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathbb{R}_{\geq 0}$ is strictly convex along left earthquakes; equivalently, $t \mapsto \mathbb{L}(E_l^{t\nu}(h))$ is strictly convex for every ν in $\mathcal{CML}_S^\circ$ and every $h \in \mathcal{T}_S^\circ(\mathbf{b})$.*

Proof. We already know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{L}(\mathcal{E}_l^{t\nu}(h)) = \int_\lambda \cos\theta_{(\lambda,\nu)}(t)\mathrm{d}\nu + \int_\mu \cos\theta_{(\mu,\nu)}(t)\mathrm{d}\nu,$$

so we have to check that $\theta(t)$ is a strictly decreasing function of t. It is sufficient to check the discrete case.
On the universal cover, pick a lift l of a leaf of ν and without loss of generality suppose it is kept fixed by the left earthquake. Choose a lift c of a leaf $\lambda \cup \mu$ that intersects l at a certain point O, preimage of a point $x \in (\lambda \cup \mu) \cap \nu$, and denote by θ_x the angle taken counterclockwise from l to c, by M and N the endpoints of l and by P and Q the endpoints of c in such a way that the counterclocwise order is M, P, N, Q.
The left earthquake $E_l^{t\gamma}$ will move P to a point P' between P and N and will move Q to a point Q' between Q and M. The angle θ_x becomes the

angle $\theta_x(t)$ that l forms with $[P', Q']$, taken counterclockwise. So we are in the conditions of Lemma 3.3.3 (more precisely, of Remark 3.3.1) and we can say that $\theta_x > \theta_x(t)$.
Since the first order variation of $\mathbb{L}_{(\lambda,\mu)}$ is the sum of $\cos\theta_x(t)$ taken on all $x \in (\lambda \cup \mu) \cap \nu$, we can conclude that $t \mapsto \theta(t)$ is strictly decreasing. □

Corollary 3.3.5. *The map $\mathbb{L}_{(\lambda,\mu)} : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathbb{R}$ admits exactly one point of minimum.*

Proof. Since $\mathbb{L}_{(\lambda,\mu)} : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathbb{R}_{\geq 0}$ is a continuous proper map, it has a minimum. Pick two points of minimum h_1 and h_2. From the main theorem of [15], there exists 2^{n} left earthquakes between h_1 and h_2; however, since the boundary lengths of S are the same with respect to h_1 and h_2, one of such earthquakes has a compact lamination ν as fault locus. Therefore, $\phi : t \mapsto E_l^{t\nu}(h_1)$ is a continuous path in $\mathcal{T}_S^\circ(\mathbf{b})$ from $\phi(0) = h_1$ and $\phi(1) = h_2$. By Proposition 3.3.4, $t \mapsto \mathbb{L}_{(\lambda,\mu)}(\phi(t))$ is a strictly convex map. But $\mathbb{L}_{(\lambda,\mu)}(\phi(0)) = \mathbb{L}_{(\lambda,\mu)}(\phi(1))$. Then it must be $h_1 = h_2$. □

3.3.3 The Hessian of $\mathbb{L}_{(\lambda,\mu)}$

The goal of this subsection is to show that the Hessian of $\mathbb{L}_{(\lambda,\mu)}$ is positive definite on a critical point $h \in \mathcal{T}_S^\circ(\mathbf{b})$ of $\mathbb{L}_{(\lambda,\mu)}$. If λ and μ have compact discrete support, then the result is already known (and can be easily extended to the compact support case) through explicit formulas (see [38], [18]), which however involve quantities that are not definible in our setting.
Let us consider $\nu \in \mathcal{CML}_S^\circ$. We already know from Subsections 3.2.3 and 3.2.4 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{L}_{(\lambda,\mu)}(E_l^{t\nu}(h)) = \int \cos\theta_{(\lambda,\nu)}(t)\,\mathrm{d}\nu \otimes \mathrm{d}\lambda + \int \cos\theta_{(\mu,\nu)}(t)\,\mathrm{d}\gamma \otimes \mu$$

holds, where $\theta_{(\lambda,\nu)}(t)$ is the angle measured counterclockwise from the support of λ to ν and $\theta_{(\mu,\nu)}(t)$ is the angle measured counterclockwise from the support of μ to ν, in the $E_l^{t\nu}(h)$-realization of ν, λ and μ.
The compact part of ν is approximated by closed weighted curves, so let us consider first a unitary closed curve γ. If δ is a weighted spiralling leaf of $\lambda \cup \mu$, we will first compute

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|0} \int \cos\theta_{(\delta,\gamma)}(t)\,\mathrm{d}\gamma \otimes \mathrm{d}\delta = \sum_{i=1}^{m} \frac{\mathrm{d}}{\mathrm{d}t}_{|0} \cos\theta_i,$$

where, enumerating consecutively along δ the points $x_1, \ldots, x_m$ in $\zeta \cap \gamma$, θ_i is the angle measured counterclockwise from δ to γ at x_i. Then we will deduce an estimate which guarantees that even passing at the limit of closed curves the second derivative stays positive.

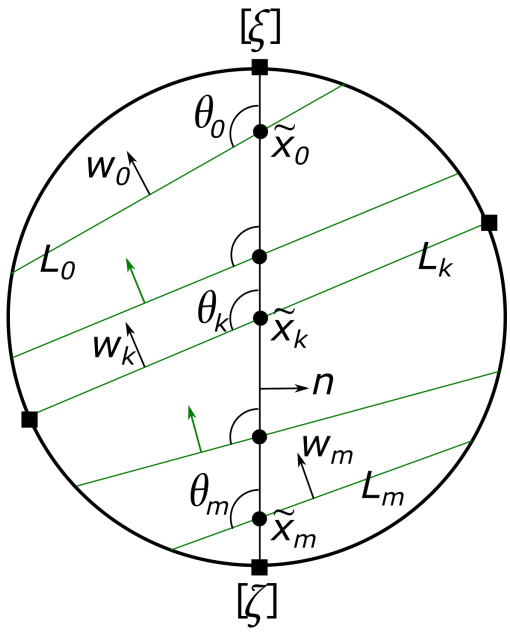


Figure 3.20

Let us transfer the problem in the universal covering $\mathcal{H} \subset \mathbb{H}^2$ of S in the hyperboloid model (see Subsection 1.1.2). Fix a lift $\tilde{\delta}$ of δ; denote by $\tilde{x}_1, \dots, \tilde{x}_m$ the preimages of $x_1, \dots, x_m$ on $\tilde{\delta}$ and by $L_1, \dots, L_m$ the liftings of γ passing respectively through $\tilde{x}_1, \dots, \tilde{x}_m$. Denote by $[\xi]$ and $[\zeta]$ the ideal endpoints of $\tilde{\delta}$ so that $\tilde{x}_1, \dots, \tilde{x}_m$ are enumerated from $[\xi]$ to $[\zeta]$ and $\xi_0 = \zeta_0 = 1$, if we write vectors $\underline{x}$ in $\mathbb{R}^{2,1}$ as $\underline{x} = (x_0, x_1, x_2)$. We can choose coordinates such that $\langle \xi, \zeta \rangle = -1$. Fix $k \in \{1, \dots, m\}$ and consider the lift $\tilde{E}^t$ of $E_l^{t\gamma}$ which fixes L_k and L_{k-1} (if $k = 1$ take the earthquake that fixes the gap adjacent with L_1 whose ideal boundary contains $[\xi]$). Choose unitary vectors $w_1, \dots, w_m$ normal respectively to $L_1, \dots, L_m$ so that $\cos\theta_i = \langle w_k, n \rangle$ for every i. See Figure (3.20). Now, since we are in the hyperboloid model of $\mathbb{H}^2$, let us identify $\mathbb{R}^{2,1}$ with the Lie algebra $\mathfrak{so}(2,1)$. Now

$$
\begin{aligned}
\xi(t) &= \tilde{E}^t(\xi) = \exp(-tw_1) \cdots \exp(-tw_{k-1})\xi, \\
\zeta(t) &= \tilde{E}^t(\zeta) = \exp(+tw_k) \cdots \exp(+tw_m)\zeta, \\
n(t) &= \frac{\xi(t) \boxtimes \zeta(t)}{\|\xi(t) \boxtimes \zeta(t)\|_{2,1}} = \frac{\xi(t) \boxtimes \zeta(t)}{-\langle \xi(t), \zeta(t) \rangle}
\end{aligned}
$$

so

$$\dot{\xi}(0) = -\sum_{i=1}^{k-1} w_i \boxtimes \xi, \tag{3.21}$$

$$\dot{\zeta}(0) = \sum_{i=k}^{m} w_i \boxtimes \zeta. \tag{3.22}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t_{|0}} \cos\theta_k(t) = \frac{\mathrm{d}}{\mathrm{d}t_{|0}} \langle w_k, n(t)\rangle = \langle w_k, \dot{n}(0)\rangle,$$

let us compute $\dot{n}(0)$. In general,

$$\begin{aligned}\dot{n}(0) =& \frac{\dot{\xi}(0) \boxtimes \zeta + \xi \boxtimes \dot{\zeta}(0)}{-\langle \xi \boxtimes \zeta\rangle} + \frac{\xi \boxtimes \zeta}{\langle \xi \boxtimes \zeta\rangle^2} \frac{\mathrm{d}}{\mathrm{d}t_{|0}} \langle \xi(t), \zeta(t)\rangle = \\ =& \dot{\xi}(0) \boxtimes \zeta + \xi \boxtimes \dot{\zeta}(0) + n \cdot \frac{\mathrm{d}}{\mathrm{d}t_{|0}} \langle \xi(t), \zeta(t)\rangle.\end{aligned}$$

Setting $z = \dot{\xi}(0) \boxtimes \zeta + \dot{\xi}(0) \boxtimes \zeta$, we deduce that there is $\beta \in \mathbb{R}$ such that $\dot{n}(0) = z + \beta n$. So from

$$0 = \langle \dot{n}(0), n\rangle = \langle z, n\rangle + \beta\langle n, n\rangle = \langle z, n\rangle + \beta$$

we get

$$\dot{n}(0) = z - \langle z, n\rangle n.$$

Shortly writing $\dot{\xi}$ for $\dot{\xi}(0)$ and $\dot{\zeta}$ for $\dot{\zeta}(0)$, setting for every i

$$w_i = a_i\xi + b_i\zeta + c_i n,$$

and using (3.21), (3.22), we compute z as

$$\begin{aligned} z =& \dot{\xi} \boxtimes \zeta + \xi \boxtimes \dot{\zeta} = -\sum_{i=1}^{k-1} (w_i \boxtimes \xi) \boxtimes \zeta + \sum_{i=k}^{m} \xi \boxtimes (w_i \boxtimes \zeta) = \\ =& -\left(\sum_{i=1}^{k-1} \big(\langle \xi, \zeta\rangle w_i - \langle w_i, \zeta\rangle \xi\big) + \sum_{i=k}^{m} \big(\langle \zeta, \xi\rangle w_i - \langle w_i, \xi\rangle \zeta\big)\right) = \\ =& -\left(\sum_{i=1}^{k-1} \big(- w_i + a_i\xi\big) + \sum_{i=k}^{m} \big(- w_i + b_i\zeta\big)\right) = \\ =& -\left(\sum_{i=1}^{k-1} (-b_i\zeta - c_i n) + \sum_{i=k}^{m} (-a_i\xi - c_i n)\right) = \\ =& \sum_{i=1}^{k-1} b_i\zeta + \sum_{i=k}^{m} a_i\xi + \sum_{i=1}^{m} c_i n. \end{aligned}$$

Now

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|0} \cos\theta_k(t) = \langle w_k, \dot{n}(0)\rangle = \langle w_k, z\rangle - \langle z, n\rangle\langle w_k, n\rangle.$$

The three products take values

$$\begin{aligned}
\langle w_k, z\rangle &= \left\langle a_k\xi + b_k\zeta + c_k n \ , \ \sum_{i=1}^{k-1} b_i\zeta + \sum_{i=k}^{m} a_i\xi + \sum_{i=1}^{m} c_i n \right\rangle = \\
&= -\sum_{i=1}^{k-1} a_k b_i - \sum_{i=k}^{m} a_i b_k + \sum_{i=1}^{m} c_i c_k \\
\langle z, n\rangle &= \left\langle \sum_{i=1}^{k-1} b_i\zeta + \sum_{i=k}^{m} a_i\xi + \sum_{i=1}^{m} c_i n \ , \ n \right\rangle = \sum_{i=1}^{m} c_i \\
\langle w_k, n\rangle &= \langle a_k\xi + b_k\zeta + c_k n \ , \ n\rangle = c_k
\end{aligned}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|0} \cos\theta_k(t) = -\sum_{i=1}^{k-1} a_k b_i - \sum_{i=k}^{m} a_i b_k.$$

The sum over k gives

$$\sum_{k=1}^{m} \frac{\mathrm{d}}{\mathrm{d}t}_{|0} \cos\theta_k(t) = -\sum_{k=1}^{m}\sum_{i=1}^{k-1} a_k b_i - \sum_{k=1}^{m}\sum_{i=k}^{m} a_i b_k = -\sum_{k=1}^{m} a_k b_k - 2\sum_{i<k} a_i b_k.$$

Notice that $c_k = \langle w_k, n\rangle = \cos\theta_k$ and

$$1 = \langle w_k, w_k\rangle = -2a_k b_k + c_k^2,$$

which implies $-a_k b_k = (\sin^2\theta_k)/2$. The terms $r_{ik} = -a_i b_k > 0$ have the property that $r_{ik} r_{ki} = (\sin^2\theta_i \sin^2\theta_k)/4$; moreover,

$$\cosh d(L_i, L_k) = \langle w_i, w_k\rangle = -a_i b_k - a_k b_i + c_i c_k = r_{ik} + r_{ki} + \cos\theta_i \cos\theta_k.$$

Since $d(L_i, L_k)$ is bounded by the maximal length of a curve in $\big(\mathrm{supp}(\lambda) \cup \mathrm{supp}(\mu)\big) \smallsetminus \mathcal{N}$, there is $M_0 > 0$ such that

$$r_{ik} + r_{ki} = \cosh d(L_i, L_k) - \cos\theta_i \cos\theta_k \leq \cosh M_0 + 1.$$

Now

$$r_{ik} = \frac{r_{ik} r_{ki}}{r_{ki}} \geq \frac{r_{ik} r_{ki}}{r_{ik} + r_{ki}} \geq \frac{\sin^2\theta_i \sin^2\theta_k}{4(\cosh M_0 + 1)}.$$

We finally get

$$\sum_{k=1}^{m} \frac{\mathrm{d}}{\mathrm{d}t}_{|0} \cos\theta_k(t) \geq \frac{1}{2}\left(\sum_{k=1}^{m} \sin^2\theta_k + \sum_{i<k} \frac{\sin^2\theta_i \sin^2\theta_k}{2(\cosh M_0 + 1)} \right).$$

This holds for a spiralling leaf δ in $\lambda \cup \mu$. Considering all the leaves of λ and μ, there is $M_1 > 0$ such that we obtain

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbb{L}_{(\lambda,\mu)}(E_l^{t\gamma}(h)) \geq & M_1 \int_\lambda \int_\lambda \sin^2\theta_{(\lambda,\gamma)}(x)\sin^2\theta_{(\lambda,\gamma)}(y)\mathrm{d}\gamma(x)\mathrm{d}\gamma(y) + \\ & + M_1 \int_\mu \int_\mu \sin^2\theta_{(\mu,\gamma)}(x)\sin^2\theta_{(\mu,\gamma)}(y)\mathrm{d}\gamma(x)\mathrm{d}\gamma(y)\end{aligned}$$

or equivalently

$$\begin{aligned}\mathrm{Hess}_h\,\mathbb{L}_{(\lambda,\mu)}(e_l^\gamma(h), e_l^\gamma(h)) \geq & M_1 \iint \sin^2\theta_{(\lambda,\gamma)}(x)\sin^2\theta_{(\lambda,\gamma)}(y)\mathrm{d}\gamma(x)\mathrm{d}\gamma(y) + \\ & + M_1 \iint \sin^2\theta_{(\mu,\gamma)}(x)\sin^2\theta_{(\mu,\gamma)}(y)\mathrm{d}\gamma(x)\mathrm{d}\gamma(y).\end{aligned}$$

Now let us consider a generic $\nu \in \mathcal{CML}_S^\circ$. It is the limit of weighted closed curves γ_n. As for the first order variation of $\mathbb{L}_{(\lambda,\mu)}$, with an approximation argument we get that

$$\begin{aligned}\mathrm{Hess}_h\,\mathbb{L}_{(\lambda,\mu)}(e_l^\nu(h), e_l^\nu(h)) \geq & M_1 \iint \sin^2\theta_{(\lambda,\nu)}(x)\sin^2\theta_{(\lambda,\nu)}(y)\mathrm{d}\nu(x)\mathrm{d}\nu(y) + \\ & + M_1 \iint \sin^2\theta_{(\mu,\nu)}(x)\sin^2\theta_{(\mu,\nu)}(y)\mathrm{d}\nu(x)\mathrm{d}\nu(y).\end{aligned}$$

Therefore, $\mathrm{Hess}_h\,\mathbb{L}_{(\lambda,\mu)}$ is definite positive.

3.4 The estimate

For the purposes of the next two sections, given $(h, h') \in \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ and $(\lambda, \mu) = \Phi^{\mathbf{b}}(h, h') \in \mathcal{FML}_S^\circ(\mathbf{b})$, we need that the weighted length of an arc c in $\mathrm{supp}(\lambda)$ can be controlled by the intersection number between λ and μ. More precisely, the statement is the following.

Theorem 3.4.1. *Given $(h, h') \in \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ and $(\lambda, \mu) = \Phi^{\mathbf{b}}(h, h')$, consider the associated convex $\mathcal{K}$ in AdS_3 with $\mathbb{S}^\pm = \partial_\pm \mathcal{K}$ bent over $\tilde{\lambda}^\pm$. Suppose λ^+ is a ω-weigthed spiralling curve. Consider a leaf L of $\tilde{\lambda}^+$, a face F of $\mathbb{S}^+$ whose boundary contains L and the time-like plane Π perpendicular to F containing L. Take a geodesic arc c on L. Denote by U the arc $\mathbb{S}^- \cap \bigcup_{z\in c} \tau_z$. Denote by m and m' the positive masses of (λ, μ) near the boundaries between which the projection of L is spiralling. Then*

$$\ell(c)\min\{\bar{\kappa}_0, \bar{K}_0\omega\,\ell(c)\} \leq \iota(U, \mu) + \bar{C}_0\Big(m(\bar{M} + F(m)) + m'(\bar{M} + F(m'))\Big). \tag{3.23}$$

The constants in (3.23) *depend only on* $\mathbf{b}$ *and on the distances of the endpoints of c from the boundary components, while $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a universal function which is increasing, differentiable and with $F(0) = 0$.*

The inequality that will be found in the last part of this section is linear for large weighted lengths and quadratic for small ones; those two properties will be used respectively in Section 3.5 and Section 3.6.
The estimate is computed in the AdS_3 environment: following [17], we will take a region R in a time-like section of the convex core $\mathcal{K}$ of (h, h'), so that $\partial_+\mathcal{K} \cap R$ is the lift of the arc c. First, we bound $\mathrm{Area}(R)$ from below by a function of the length of c and its weight, which corresponds to the bending angle along the lift of c of $\partial_+\mathcal{K}$. Unlike the closed case, the accumulation of the measure of (λ, μ) near ∂S will force us to look for a sort of control on the support planes near the boundary of $\mathbb{S}^+$, depending only on the distance between the endpoints of c and ∂S. A Gauss-Bonnet formula in the Anti de Sitter context will then relate $\mathrm{Area}(R)$ with the angles occurring in $\partial_-\mathcal{K} \cap R$, leading to an upper bound for $\mathrm{Area}(R)$ depending in a certain way on the intersection number between c and μ.

3.4.1 Bounding $\mathrm{Area}(R)$ from below

The following lemma was proved in the Appendix of [17]:

Lemma 3.4.2. *There exists $\Delta > 0$ as follows. Let a and b be two disjoint lines in $\mathbb{H}^2$ and let x be a point in the connected component of $\mathbb{H}^2 \setminus (a \cup b)$ with boundary $a \cup b$. Suppose that $d(x, a) \leq \Delta$ and $d(x, b) \leq \Delta$. Then the geodesic segment σ of length 1 starting orthogonally from a and containing x intersects b.*

If λ is a measured lamination formed by a unique leaf with weight ω and spiralling, take a geodesic arc c in the support of λ with at least an endpoint in the spiralization neighbourhood $\mathcal{N}$. Choose an orientation on c and for every $x \in c$ denote by σ_x the (length-arc parametrized) geodesic segment of length 1 starting orthogonally from x towards say the left side of c.
Take some $\mathbb{B} > 0$ and consider

$$c^* = \{x \in c \, : \, \#(\sigma_x \cap c) \leq \mathbb{B}\}.$$

Lemma 3.4.3.

$$\Delta \cdot \ell(c^*) \leq \mathbb{B} \cdot \mathrm{Area}(S).$$

Proof. Consider the normal exponential map

$$\begin{aligned} \exp : c^* \times [0, \Delta] &\to S \\ (x, r) &\mapsto \sigma_x(r). \end{aligned}$$

Pick $y \in S$ and consider $\#\exp^{-1}(y) = \{(x_1, r_1), \dots, (x_n, r_n)\}$. If $\tilde{y}$ is a preimage of y on the universal cover $\mathcal{H} \subset \mathbb{H}^2$ of S, choose for every x_i a preimage $\tilde{x}_i$ such that the lift of $\exp(\{x_i\} \times [0, \Delta])$ passing through $\tilde{y}$ contains $\tilde{x}_i$ too. Let also $\tilde{c}_i$ be the lifts of c containing $\tilde{x}_i$ respectively and denote by

C_i the complete geodesics containing $\tilde{c}_i$ respectively.
Since the segment $[\tilde{y}, \tilde{x}_i]$ is orthogonal to C_i, for any $j \neq i$ the line C_j is disjoint from C_i.
Up to changing the indices, we can suppose that there are half-planes P_1 and P_2, bounded by C_1 and C_2 respectively, that do not meet any other C_i and such that $\tilde{y} \notin P_1$.
If C_i separates $\tilde{y}$ and C_1 then the segment $\tilde{\sigma}^1$ of length 1 starting from $\tilde{x}_1$ and passing through $\tilde{y}$ intersects C_i. Otherwise, $d(\tilde{y}, C_i) \leq \Delta$ and $d(\tilde{y}, C_1) \leq \Delta$; by Lemma 3.4.2, $\tilde{\sigma}^1$ meets C_i.
Now, since $x_1 \in c^*$,

$$n = \#\left(\tilde{\sigma}^1 \cap \bigcup C_i\right) = \#(\sigma_{x_1} \cap c) \leq \mathbb{B}.$$

This implies that $\#\exp^{-1}(y) \leq \mathbb{B}$ for every $y \in S$. Therefore,

$$\Delta \cdot \ell(c^*) \leq \mathrm{Area}(c^* \times [0, \Delta]) \leq \mathbb{B} \cdot \mathrm{Area}(S).$$

□

Corollary 3.4.4. *Providing every σ_x with unitary weight, if*

$$c^\dagger = \left\{ x \in c \, : \, \iota(\sigma_x, c) \geq \omega \frac{\Delta \cdot \ell(c)}{2\,\mathrm{Area}(S)} \right\}$$

then $\ell(c^\dagger) \geq \ell(c)/2$.

Proof. The intersection number $\iota(\sigma_x, c)$ has value $\omega \cdot \#(\sigma_x \cap c)$. Choosing $\mathbb{B} = \frac{\Delta \cdot \ell(c)}{2\,\mathrm{Area}(S)}$ we have that $\ell(c^*) \leq \frac{\ell(c)}{2}$. Therefore

$$\ell(c \setminus c^*) \geq \frac{\ell(c)}{2}.$$

□

The following Lemma is proved in [17].

Lemma 3.4.5. *Let Σ_+ be a convex surface obtained by bending a space-like plane in AdS_3 along a finite measured lamination $\tilde{\lambda}_+$. Let σ be any geodesic path in Σ_+ joining a point $x \in \tilde{\lambda}_+$ to some point y. Let $\mathcal{P}$ be the space-like plane through x extending the face of Σ_+ that does not meet $\sigma \setminus \{x\}$. If y lies on a bending line, let $\mathcal{Q}$ be the space-like plane through y extending the face of Σ_+ that does not meet $\sigma \setminus \{y\}$; otherwise, let $\mathcal{Q}$ be the space-like plane extending the face of Σ_+ containing y.*
Then $\mathcal{P}$ and $\mathcal{Q}$ meet along a space-like line r. Moreover, the following two properties hold:

- $\overline{\vartheta}(\mathcal{P}, \mathcal{Q}) \geq \iota(\sigma, \tilde{\lambda}_+)$*, where also the intersection in x is counted;*

- *if $d_{\mathcal{P}}(x,r)$ is the distance from x to r on the plane $\mathcal{P}$ and $d_{\mathcal{Q}}(y,r)$ is the distance from y to r on the plane $\mathcal{Q}$, then $d_{\mathcal{P}}(x,r) + d_{\mathcal{Q}}(y,r) \leq \ell(\sigma)$.*

If $(h,h') \in \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ and $(\lambda,\mu) = \Phi(h,h')$ are such that λ has a spiralling curve L, say with weight ω, consider the associated convex set $\mathcal{K}$ in AdS_3 with upper bending lamination $\tilde{\lambda}_+$. Choose a geodesic arc c in a leaf $\tilde{L}$ of $\tilde{\lambda}_+$, which is a preimage of L, and let δ be the minimum of the distances of the endpoints of c from $\partial\mathbb{S}^+$, the one-dimensional boundary of the space-like part of the two-dimensional future boundary of $\mathcal{K}$. For every x in c denote by $\sigma_{x,\delta}$ the segment on $\mathbb{S}^+$ starting from x orthogonal to c of length δ and by τ_x the time-like geodesic through x orthogonal to the face F of $\mathbb{S}^+$ such that $F \cap \sigma_{x,\delta} = \{x\}$. Let y be the endpoint of $\sigma_{x,\delta}$ distinct from x. Notice that

$$d_{\mathbb{S}^+}(y, \partial\mathbb{S}^+) \geq \delta.$$

Denote by $\mathcal{P}$ the space-like plane containing the face F, by $\mathcal{Q}$ the one containing the face of $\mathbb{S}^+$ containing y (if y lies on a leaf of $\tilde{\lambda}^+$, take as $\mathcal{Q}$ the space-like plane containing the face F' of $\mathbb{S}^+$ such that $F' \cap \sigma_{x,\delta} = \{y\}$). Such planes meet along a space-like geodesic l. Denote by $\mathcal{R}$ the union of the closed half-planes of $\mathcal{P}$ and $\mathcal{Q}$, determined by l, intersecting $\mathbb{S}^+$; we say that $\mathcal{R}$ is a *roof* of $\mathbb{S}^+$, since $\mathbb{S}^+$ lies entirely in the past of $\mathcal{R}$.
The time-like plane Π' containing y and τ_x meets $\partial_+\mathcal{K}$ along a curve with endpoints x' and y' at distance at least δ from x or y. The roof R determines on Π' a triangle with angle α at the future vertex w.

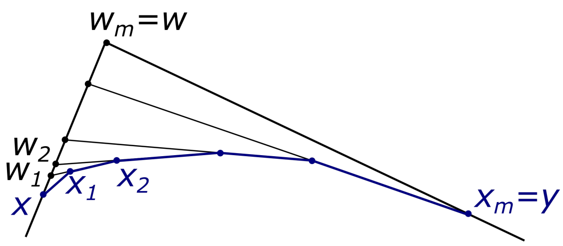


Figure 3.21

Lemma 3.4.6. *Let Π be a time-like plane in AdS_3, and let $\mathcal{P}$ and $\mathcal{Q}$ be two space-like planes, such that Π, $\mathcal{P}$ and $\mathcal{Q}$ meet exactly at one point. Then the angle in Π between $\Pi \cap \mathcal{P}$ and $\Pi \cap \mathcal{Q}$ is smaller than the angle between $\mathcal{P}$ and $\mathcal{Q}$.*

For the proof, see [17]. In particular, $\alpha < \theta = \vartheta(\mathcal{P},\mathcal{Q})$.

If u_x is the length of $\tau_x \cap \mathcal{K}$, we have the following results.

Lemma 3.4.7. *The distance $d(x,w)$ is not greater than δ.*

Proof. For every $r \in \Pi' \cap \mathbb{S}^+$, denote by Γ_r the piecewise geodesic arc in $\Pi' \cap \mathbb{S}^+$ from x to r. The arc $\Gamma_y = \sigma_{x,\delta}$ has vertices $x = x_0$, x_1, ..., $x_m = y$. Suppose that $y \in \tilde{\lambda}^+$. The other case can be proved with same argument as the following. Denote, for $k = 1, \dots, m$, by w_k the future vertex of the triangle in Π' individuated by the intersection of Π' with the past of the roof of Γ_{x_k}, as in Figure 3.21. We claim that

$$d(x, w_k) + d(w_k, x_k) \leq \ell(\Gamma_{x_k}) \text{ for every } k. \tag{3.24}$$

We proceed by induction on k. If $k = 1$ then

$$d(x, w_1) + d(w_1, x_1) \leq d(x, x_1) = \ell(\Gamma_{x_1}).$$

If $k > 1$, suppose that $d(x, w_k) + d(w_k, x_k) \leq \ell(\Gamma_{x_k})$. Moreover,

$$d(w_k, w_{k+1}) + d(w_{k+1}, x_{k+1}) \leq d(w_k, x_{k+1}) = d(w_k, x_k) + d(x_k, x_{k+1}).$$

Now

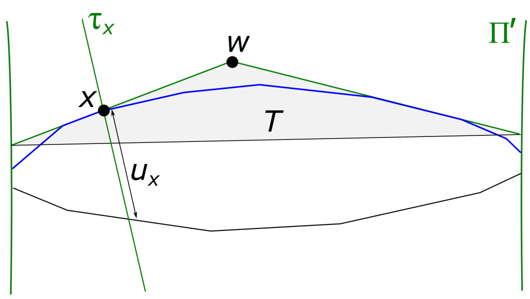
$$\begin{aligned} d(x, w_k) + d(w_k, x_k) + &d(w_k, w_{k+1}) + d(w_{k+1}, x_{k+1}) \leq \\ &\leq \ell(\Gamma_{x_k}) + d(w_k, x_k) + d(x_k, x_{k+1}) \\ d(x, w_{k+1}) + &d(w_{k+1}, x_{k+1}) \leq \ell(\Gamma_{x_{k+1}}). \end{aligned}$$

Thus, (3.24) is proved. In particular, it holds for $k = m$. So

$$d(x, w) = \leq d(x, w) + d(w, y) \leq \ell(\Gamma_y).$$

□

Lemma 3.4.8. *For every $\kappa > 0$ there is $K_1 > 0$ depending only on δ and κ such that if $\iota(\tilde{\lambda}^+, \sigma_{x,\delta}) \geq \kappa$ then $u_x \geq K_1$.*



Proof. Notice first that $u_x \geq \ell(\tau_x \cap T)$ where T is the triangle in Π' with future vertex w and two edges contained in $\mathcal{R} \cap \Pi'$. Since

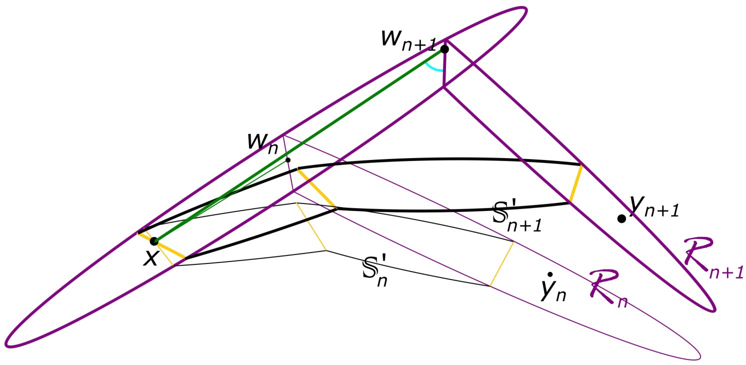
$$\theta \geq \iota(\tilde{\lambda}^+, \sigma_{x,\delta}) \geq \kappa,$$

the worst configuration (which means the one that minimizes $\ell(\tau_x \cap T)$) happens when $\theta = \kappa$ and the l is as far from x as possible. However, $w \in l$ and $d(x, w) \leq \delta$, so in such configuration there is K_1 such that $u_x \geq K_1$. □

Lemma 3.4.9. *For every $\kappa > 0$ there is $K_2 = K_2(\delta) > 0$ such that if $\iota(\tilde{\lambda}^+, \sigma_{x,\delta}) \leq \kappa$ then $\theta \leq K_2$.*

Proof. Assume that there exists a sequence $(\mathbb{S}_n^+, x_n, \mathcal{R}_n)$ of surfaces, points and roofs (with $R_n \subset \mathcal{P}_n \cup \mathcal{Q}_n$ and $y_n \in \mathcal{Q}_n$) such that $\iota(\lambda_n^+, \sigma_{x_n,\delta}) \leq \kappa$ but $\theta_n > n$. Up to translating by an isometry we can assume that x_n and $\mathcal{P}_n$ are fixed. Now $\mathbb{S}_n^+$ is the image of a bending map b_n^+ and we may suppose that $x_n = \mathrm{b}_n^+(z_0)$ and $y_n = \mathrm{b}_n^+(z_1)$ for every n, where z_0 and z_1 are fixed points at distance δ. As we are assuming that $\iota(\lambda_n^+, \sigma_{x_n,\delta}) < \kappa$, then the bending maps converge to a bent space-like immersion in a neighbourhood of $[z_0, z_1]$. So the tangent plane at $\mathrm{b}_n^+(z_1)$ cannot become light-like and then the angle θ_n is bounded. $\square$

Lemma 3.4.10. *Consider a family $(\mathbb{S}_n^+, \mathcal{R}_n, x_n, y_n)$ of bent surfaces, roofs and points as usually. If there is $\kappa > 0$ such that $\iota(\tilde{\lambda}_n^+, \sigma_{x_n,\delta}) \leq \kappa$ then there is $\varepsilon = \varepsilon(\delta) > 0$ not depending on x_n such that the angle ψ_n in w_n between $[x_n, w_n]$ and l_n is greater than ε.*

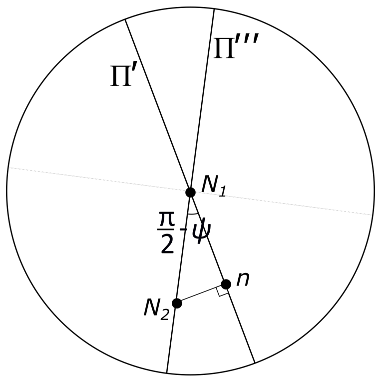


Proof. By contradiction suppose that, up to subsequences, the angles ψ_n go to 0. We can assume that $x_n \equiv x$ and $\mathcal{P}_n \equiv \mathcal{P}$. Notice that by the assumption on the intersection, the subsurface $\mathbb{S}'_n$ made of strata separating x from y_n converge to a bent surface $\mathbb{S}' \subset \mathbb{S}_n^+$ (up to a subsequence). Analogously $\mathcal{R}_n$ converges to a roof $\mathcal{R}$. Notice that R and $\mathbb{S}'$ must contain the limit of points y_n (that are at bounded distance from x). If x is not on the bending line of R, the distance from x to the bending line of $\mathcal{R}_n$ is uniformly bounded from above and as w_n is at bounded distance from x the angle ψ_n cannot converge to 0. If x lies on the bending line of $\mathcal{R}$, then also $\mathbb{S}'$ must be contained in a face of $\mathcal{R}$. Notice however that, as on $\mathbb{S}'_n$ x and y_n are related by a geodesic orthogonal to the bending line through x, the line between x and y on $\mathcal{R}$ must be orthogonal to the bending line. But then the plane Π_n is almost orthogonal to the bending line, so that $\psi_n \to \pi/2$. $\square$

Lemma 3.4.11. *For every κ there is $K_3 = K_3(\delta) > 0$ such that if*

$$\iota(\tilde{\lambda}^+, \sigma_{x,\delta}) \leq \kappa$$

then α, the angle at the future vertex of the triangle individuated by the roof $\mathcal{R}$ on Π', is greater than or equal to $K_3\iota(\tilde{\lambda}^+, \sigma_{x,\delta})$.



Proof. The value of α is the distance of the orthogonal projections of N_1 and N_2, the future-pointing unitary vectors at w normal respectively to $\mathcal{P}$ and $\mathcal{Q}$, on $T_w\Pi'$. Being $\mathcal{P}$ perpendicular to Π', $N_1 \in T_w\Pi$. On the other hand, $N_2 \in T_w\Pi'''$ with Π''' time-like, passing through w and orthogonal to l. Identify $\mathbb{H}^2$ with the unitary future-pointing vectors at w. We get that N_1 and N_2 lie in the geodesic corresponding to Π''', which forms with the geodesic corresponding to Π' an angle of $\pi/2 - \psi$. Lemma 3.4.5 assures that $\theta = d(N_{\mathcal{P}}, N_{\mathcal{Q}}) \geq \iota(\tilde{\lambda}^+, \sigma_{x,\delta})$, while Lemma 3.4.10 says that $\psi \geq \varepsilon$. Now if n is the projection of N_2 on the Π', we get

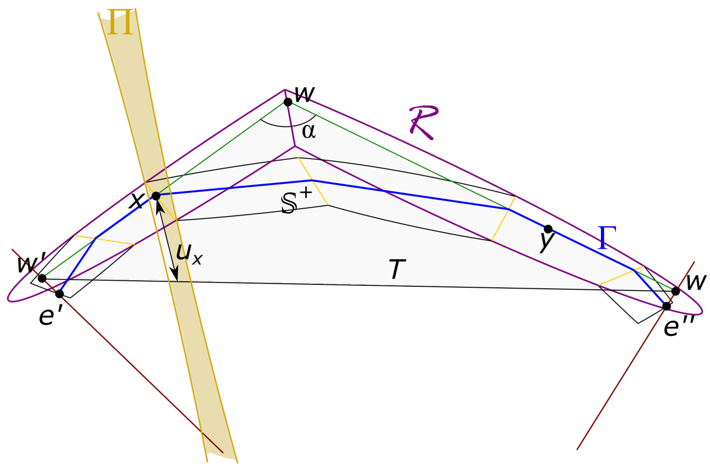
$$\frac{\sin(\pi/2 - \psi)}{\sinh d(n, N_2)} = \frac{\sin(\pi/2)}{\sinh \theta}$$
$$\sinh d(n, N_2) = \cos\psi \sinh\theta \leq \cos\varepsilon \sinh\theta$$
$$\alpha \geq \theta - d(n, N_2) \geq \theta - \operatorname{arcsinh}(\cos\varepsilon \sinh\theta) \geq K_3\theta \geq K_3\iota(\tilde{\lambda}^+, \sigma_{x,\delta}).$$

□

Lemma 3.4.12. *If $(h, h') \in \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ and $(\lambda, \mu) = \Phi(h, h')$ are such that λ has a spiralling curve l, say with weight ω, consider the associated convex set $\mathcal{K}$ in AdS_3 with upper bending lamination $\tilde{\lambda}_+$. Choose a number $\delta \in (0, \varepsilon(\mathbf{b}))$. There exist $\kappa_0, K_0 > 0$ depending only on δ such that for every geodesic arc c in a leaf of $\tilde{\lambda}_+$, which is a preimage of l, if both endpoints of c have distance from $\partial\mathbb{S}^+$ greater than δ, where $\mathbb{S}^+$ is the space-like part of $\partial_+\mathcal{K}$, then for every $x \in c$*

$$u_x = \ell(\tau_x \cap \mathcal{K}) > K_0 \min\{\kappa_0, \iota(\sigma_{x,\delta}, \lambda_+)\}$$

where $\sigma_{x,\delta}$ is the segment on $\mathbb{S}^+$ starting from x orthogonal to c of length δ and τ_x is the time-like geodesic through x in the time-like plane Π orthogonal to the face F of $\mathbb{S}^+$ such that $F \cap \sigma_{x,\delta} = \{x\}$.



Proof. Let us construct the roof $\mathcal{R}$ bent along l using the plane $\mathcal{P}$ extending F and a support plane $\mathcal{Q}$ at y, the other endpoint of $\sigma_{x,\delta}$. Let Π' be the plane orthogonal to $\mathcal{P}$ containing x and y. The intersection of Π' with $\mathbb{S}^+$ is a curve Γ with two endpoints e', e'' that could be at the interior of AdS_3. Consider the light-like lines through e' and e'' pointing in the direction opposite to Γ. Consider the intersection points w', w'' of the roof R with this lines. Notice that the line $[w', w'']$ is in the future of the line $[e', e'']$, so the length of $\tau_x \cap \mathcal{K}$ is bigger than the length of $\tau_x \cap T$, where T is the triangle with vertices w', w, w'', where $\{w\} = l \cap \Pi'$. As the interval of the curve Γ with endpoints e' and x can be orthogonally projected on a subinterval of $[x, w']$, then the length of $[x, w']$ is bigger than δ. Analogously the length of $[w, w']$ is bigger than δ.

By previous lemmas we also have that

(a) the distance $D = d(x, w)$ is not bigger than δ;

(b) there are constants K_1, κ_1 depending only on δ such that

$$\alpha \geq K_1 \min\{\kappa_1, \iota(\tilde{\lambda}^+, \sigma_{x,\delta})\}.$$

Now the triangle T has $d(w, w') > \delta + D$ and $d(w'', w) > \delta$. The length of the segment starting at x orthogonally to $[w', w]$ and ending in $[w', w'']$ is greater than a function $F(\delta, \alpha)$ (which can easily be computed in convenient coordinates on AdS_2) such that $F(\delta, \alpha)$ tends linearly to 0 as $\alpha \to 0$.

We get

$$u_x \geq F(\delta, \alpha) \geq K_2 \min\{\kappa_2, \alpha\} \geq K_0 \min\{\kappa_0, \iota(\tilde{\lambda}^+, \sigma_{x,\delta})\}.$$

$\square$

Lemma 3.4.13. *In the hypothesis of Lemma 3.4.12, if*

$$R = \bigcup_{x\in c} (\tau_x \cap \mathcal{K}),$$

then

$$\text{Area}(R) \geq \ell(c) \min\{\bar{\kappa}_0, \bar{K}_0 \omega\, \ell(c)\},$$

where $\bar{\kappa}_0$, and $\bar{K}_0$ depend only on δ.

Proof. We know from Corollary 3.4.4 that

$$\ell(c^\dagger) \geq \frac{\ell(c)}{2}.$$

For every $x \in c^\dagger$, $\iota(\sigma_{x,\delta}, \lambda_+) \geq \mathbb{B}^\dagger \cdot \omega\Delta\, \ell(c)$. Then, by Lemma 3.4.12,

$$\ell(\tau_x \cap R) \geq K_0 \min\{\kappa_0, \iota(\sigma_{x,\delta}, \lambda_+)\} \geq K_0 \min\{\kappa_0, \mathbb{B}^\dagger \cdot \omega\Delta\, \ell(c)\}. \tag{3.25}$$

Denoting by G the latter member of inequality (3.25), the map

$$\begin{aligned} \tau : c^\dagger \times [0, G] &\to R \\ (x, s) &\mapsto \tau_x(s) \end{aligned}$$

has the Jacobian at (x, s) equal to $\cos s$ (see [25]), and so at least 1/2. This leads to

$$\text{Area}(R) \geq \text{Area}(\tau(c^\dagger \times [0, G])) \geq \frac{G}{2}\ell(c^\dagger) \geq \frac{G}{4}\ell(c)$$

□

3.4.2 Bounding Area(R) from above

We are going to show that Area(R) can be estimated in terms of $\iota(\lambda, \mu)$ and of the weights of the spiralling leaves. Gauss-Bonnet formula for regions in time-like planes applied to R, which has piecewise geodesic boundary, implies that

$$\text{Area}(R) \leq \phi_2 - \phi_1 + \sum_{n=1}^{m} \theta_n$$

where $\theta_1, \ldots, \theta_m$ are the external angles at the m vertices of the $m+1$ space-like edges of $\partial R \cap \mathbb{S}^-$, while ϕ_1 and ϕ_2 are angles that $\partial R \cap \mathbb{S}^-$ forms with the two time-like edges of ∂R starting orthogonally at the endpoints of c. Since R is contained in

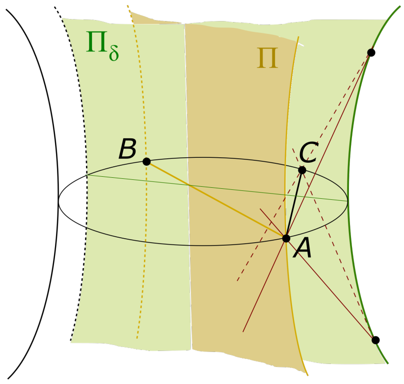
$$\mathcal{A} = \mathcal{K} \cap \Pi,$$

where Π was defined introduced in the statement of Lemma 3.4.12, if we show that the sum of the infinitely many angles θ_k on $\mathbb{S}^- \cap \partial\mathcal{A}$ is finite, then Area(R) $\leq$ Area($\mathcal{A}$) and it suffices to bound the area of $\mathcal{A}$ in terms of $\iota(\lambda, \mu)$

and of the weights of the spiralling leaves.

We first have to check that if a point p in the past boundary of $\mathcal{K}$ is not too close to $\partial\mathbb{S}^-$ (in a sense that we are going to specify in the statement of Lemma 3.4.16) then a past support plane passing through p is far from being a space-light plane (again, in a suitable sense). Some sublemmas are required for this purpose.

Sublemma 3.4.14. *Fixed $A, B, C \in \partial_\infty AdS_3$ and a time-like plane Π with $A, B \in \partial_\infty \Pi$, denote by $\mathcal{L}_1, \mathcal{L}_2$ the light-like planes through $[A, C]$ such that $B \in V = \mathcal{I}^-(\mathcal{L}_1) \cap \mathcal{I}^+(\mathcal{L}_2)$. Then for every $\delta > 0$ there is time-like plane Π_δ orthogonal to $[A, C]$, such that if $p \in \Pi \cap \mathcal{I}^-(\mathcal{L}_1) \cap \mathcal{I}^+(\mathcal{L}_2)$ has distance in AdS_3 from $[A, C]$ greater than or equal to δ, then Π_δ separates p and A, in the sense that the ray from p to A meets Π_δ.*



Proof. Consider the dual line $[D, E]$ of $[A, C]$. We can suppose (confusing elements of $\mathbb{R}^{2,2}$ with their projections of AdS_3) that

$$\langle A, C \rangle = \langle D, E \rangle = -\frac{1}{\sqrt{2}}.$$

Fix coordinates (s, t, u) on V so that

$$(s, t, u) \leftrightarrow \frac{1}{\sqrt{2}}\Big((\cosh u)(e^t A + e^{-t} C) + (\sinh u)(e^s E - e^{-s} D)\Big)$$

Notice that $d\big((s, t, u), [A, C]\big) = u$ and $\{t = T\}$ is for every T a plane orthogonal to $[A, C]$. The unitary vector N normal to Π has the form $aA + (e^b E - e^{-b} D)/\sqrt{2}$ with $a > 0$, so from $p \in \Pi$ we get

$$0 = \langle N, p \rangle = -\frac{ae^{-t}\cosh u}{2} + \frac{\sinh u}{2\sqrt{2}}(e^{-s+b} + e^{s-b})$$

$$e^{-t} = \frac{\sqrt{2}}{a}(\tanh u)\cosh(s - b) \geq \frac{\sqrt{2}}{a}\tanh u$$

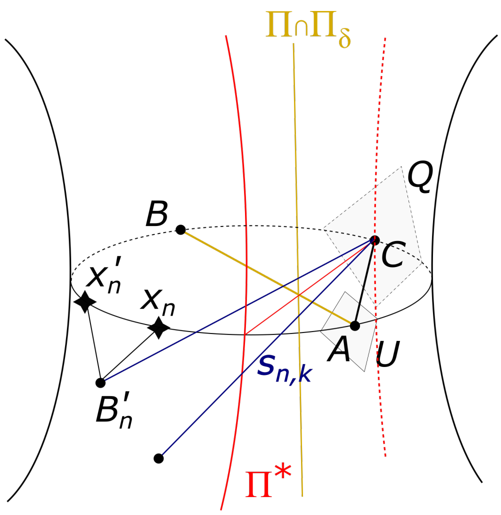
$$t \leq -\log\left(\frac{\sqrt{2}}{a}\tanh u\right) = T(a, u).$$

Thus, take $\Pi_\delta = \{t = T(a,\delta)\}$ in order to conclude. $\square$

Sublemma 3.4.15. *Consider bending maps*

$$\mathrm{b}_n : H^+ = \{z \in \mathbb{H}^2 \,|\, \Re z > 0\} \to AdS_3$$

along laminations λ_n with leaves ending in ∞ invariant by a hyperbolic subgroup $\langle g_n \rangle < PSL(2,\mathbb{R})$ with axis $[0,\infty]$, such that moreover $[1,\infty]$ is a leaf of λ_n for every n. Suppose there is $\Omega > 0$ such that $\iota(\lambda_n, \partial_i) \le \Omega < \mathrm{T}(g_n)$, where $\mathrm{T}(g_n)$ is the translation length of g_n. Also, assume that b_n converge to a bending map b_∞ and that $\mathrm{b}_n(\infty) = C$ e $\mathrm{b}_n(0) = A$.
Let $B'_n = \mathrm{b}_n(1)$ and $s_{n,k} = \mathrm{hol}(g_n^k)([C, B'_n])$. Then there exists $\overline{k}$ (independent on n) such that $s_{n,k}$ does not meet Π if $k \ge \overline{k}$, and $s_{n,k}$ meets Π_δ before Π (following the orientation from C towards B'_n) if $k < -\overline{k}$, where Π_δ is the time-like plane defined in Sublemma 3.4.14.



Proof. The points B'_n lie in a compact region of

$$\partial_\infty AdS_3 \setminus (f_l(A) \cup f_r(A) \cup f_l(C) \cup f_r(C))$$

where $f_l(X)$ and $f_r(X)$ denote the leaves of respectively the left and right foliation of $\partial_\infty AdS_3$ passing through $X \in \partial_\infty AdS_3$. Under the identification $\partial_\infty AdS_3 = \partial\mathbb{H}^2 \times \partial\mathbb{H}^2$, if $A = (a,a')$, $C = (c,c')$ and $B'_n = (x_n, x'_n)$, then x_n lies in a compact subset of $S^1 \setminus \{a,c\}$ while x'_n lies in a compact subset $S^1 \setminus \{a',c'\}$.
Given $\epsilon > 0$ there exists L such that if Γ, Γ' are hyperbolic transformations with axis $[a,c]$ and $[a',c']$ and repulsive points a and a' have translation length greater than L then $\Gamma(x_n)$ e $\Gamma'(x'_n)$ have Euclidean distance from c and c' less than ϵ. Taking k such that $\mathrm{hol}(g_n)^k$ has translation length bigger than L for every n, we get that for every ϵ there is $\overline{k}_1$ such that, for every

$k \geq \overline{k}_1$ and for every n, $\mathrm{hol}(g_n)^k(x_n)$ e $\mathrm{hol}(g_n)^k(x'_n)$ have Euclidean distance from c and c' respectively less than ϵ.
Now fix ϵ such that the region

$$Q = (c - \epsilon, c + \epsilon) \times (c' - \epsilon, c' + \epsilon) \subset \partial_\infty AdS_3$$

does not meet Π. If $k > \overline{k}_1$ then the endpoint of $s_{n,k} = \mathrm{hol}(g_n)^k([C, B'_n])$ different from C lies in Q for every n, so $s_{n,k}$ does not intersect Π.
On the other hand, consider the plane Π^* through C and $\Pi \cap \Pi_\delta$. Notice that $A \notin \Pi^*$. So there is a neighbourhood U of A disjoint from Π^*.
If a geodesic from C ends in U then it must meet Π_δ before Π. With an argument analogous to the previous one, there is $\overline{k}_2$ independent on n such that, for every n and for every $k \leq \overline{k}_2$, the endpoint of $s_{n,-k}$ different from C lies in U.
Finally, take $\overline{k}_0 = \max\{\overline{k}_1, \overline{k}_2\}$. □

Now fix in $\mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ the family T_Ω of representations $(\mathrm{hol}_l, \mathrm{hol}_r)$ such that

- the masses of the associated bending laminations (λ, μ) have modulus bounded by Ω;
- the hol_r-lengths of the boundary components are greater than Ω^{-1}.

Notice that this is equivalent to asking that

$$-\Omega \leq -m(\mu, \partial_i) = m(\lambda, \partial_i) \leq \min\{\Omega, b_i - \Omega^{-1}\}.$$

This is a condition that holds in the case where (λ, μ) lies in a compact subset of $\mathcal{FML}_S^\circ(\mathbf{b})$.

Lemma 3.4.16. *Fix A, B, C and Π as in Sublemma 3.4.14. For every $\delta > 0$ there exists a compact family Z of space-like planes depending only on Ω, $\mathbf{b}$ and δ veirifying the following property.*
For every couple $(\mathrm{hol}_l, \mathrm{hol}_r) \in T_\Omega$, consider the universal covering of S and take the convex core $\mathcal{K}$ associated with $(\mathrm{hol}_l, \mathrm{hol}_r)$ in AdS_3 such that $[A, B]$ is a leaf of the upper bending lamination and $[A, C]$ is a boundary component of $\mathbb{S}_n^-$. If $p \in \Pi$ lies in spiralization neighbourhood of the space-like lower part of $\partial\mathcal{K}$ and has distance from $[A, C]$ greater than δ, then the support plane through p lies in Z.

Proof. Suppose there is a sequence of convex cores $\mathcal{K}_n$ as in the statement and points p_n in the spiralization neighbourhood of $\mathbb{S}_n^-$ with distance from $[A, C]$ in (δ, ε) but such that the support plane through p_n tends to a light-like plane.
Let $\bar{\mathbb{S}}_n$ the surface obtained bending $H^+ = \{z \in \mathbb{H}^2 \,|\, \Re z > 0\}$ following

the sublamination $\bar{\mu}_n$ of the leaves of $\tilde{\mu}_n$ ending in ∞. Since $\bar{\mu}_n$ varies in a compact subset, there is a subsequence of $\bar{\mathbb{S}}_n$ tending to a space-like bent surface $\bar{\mathbb{S}}_\infty$; moreover, we can suppose the bending maps b_n^- are converging to $\mathrm{b}_\infty^- : H^+ \to \mathbb{S}_\infty^-$ on the spiralization neighbourhood.
From Sublemma 3.4.15 we know that $x_n = (\mathrm{b}_n^-)^{-1}(p_n)$ lies in the region of $\mathbb{H}^2$ between two geodesics $[M^{-k_0}, \infty]$ and $[M^{k_0}, \infty]$, with $M > 0$. Since $d(x_n, [0, \infty]) \in [\delta, \varepsilon(\mathbf{b})]$, the point x_n actually lies in a compact domain of $\mathbb{H}^2$, thus (up to subsequence) p_n converges to a point of $\bar{\mathbb{S}}_\infty$. This contradicts the fact that the support planes of p_n tend to a light-like plane.

□

Lemma 3.4.17. *Consider the curve*

$$U = \bigcup_{x \in [A,B]} \tau_x \cap \partial_- \mathcal{K}$$

Choose in $U \cap \mathcal{N}(\tilde{\partial}) \cap \tilde{\lambda}^-$ a vertex of U at distance δ from $[A, C]$. It belongs to a leaf l of $\tilde{\lambda}^-$. If γ is the hyperbolic transformation associated to $\tilde{\partial}$, with attractive point in A, denote by p_k the points $U \cap \gamma^k(l)$. Then there are constants a_0, d_0 depending only on the masses of μ and on the weights of its spiralling leaves such that the angle θ_k at p_k of U is less than or equal to $d_0 e^{-a_0 k}\zeta$, where ζ is the weight of l.

Proof. Notice that $\gamma^k(l)$ has C as ideal endpoint for every k. Clearly $p_k \to A$. Denote by $\mathcal{P}$ and $\mathcal{Q}$ the support planes containing l such that $\theta_1 = \vartheta(\mathcal{P}, \mathcal{Q})$. The angle θ_k can be computed as follows:

$$\theta_k = \vartheta(\gamma^k(\mathcal{P}) \cap \Pi, \gamma^k(\mathcal{Q}) \cap \Pi) = \vartheta(\mathcal{P} \cap \gamma^{-k}(\Pi), \mathcal{Q} \cap \gamma^{-k}(\Pi)).$$

Let us consider the 1-parameter group γ_s such that $\gamma_1 = \gamma$ and study the dependence of θ_s on s. Fix the basis (A, C, E, D) of $\mathbb{R}^{2,2}$ (we are confusing elements of $\mathbb{R}^{2,2}$ with the corresponding points in the projective space) so that A, C, E, D are light-like vectors with E and D lying in the line dual to the one passing through A and C. Let us rescale so that:
(i) $\langle A, C\rangle = \langle D, E\rangle = -1/\sqrt{2}$;
(ii) a multiple of p is the center of mass of A, C, E, D;
(iii) $\gamma_s(A) = e^{bs}A$, $\gamma_s(C) = e^{-bs}C$, $\gamma_s(D) = e^{b's}$, $\gamma_s(E) = e^{-b's}E$.
The numbers b and b' depend on $\mathbf{b}$ but are bounded and positive.
We have that the normal of Π is $aA + cC + dD + fE$ with $c = 0$, being $A \in \Pi$, so with $d, e \neq 0$ (Π would be light-like otherwise). Thus, the normal $n(s)$ to $\gamma_s(\Pi)$ is $ae^{bs}A + de^{b's}D + fe^{-b's}E$. Denote by $N_\mathcal{P}$ and $N_\mathcal{Q}$ the normal vectors to $\mathcal{P}$ and $\mathcal{Q}$ respectively. For $s > 0$,

$$|\langle N_\mathcal{P}, n(-s)\rangle| \geq e^{b's} d |\langle N_\mathcal{P}, D\rangle| - k_0$$
$$|\langle N_\mathcal{Q}, n(-s)\rangle| \geq e^{b's} d |\langle N_\mathcal{Q}, D\rangle| - k_0.$$

Imposing that C lies in $\mathcal{P}$ and $\mathcal{Q}$ we have that $N_{\mathcal{P}} = cC + (aD + a^{-1}E)$ and analogously $N_{\mathcal{Q}} = c'C + (a'D + (a')^{-1}E)$. By Lemma 3.4.16, $\mathcal{N}_{\mathcal{P}}$ and $N_{\mathcal{Q}}$ lie in some compact region of space-like planes which depend only on Ω, a constant that bounds the masses of (λ, μ) at the components of ∂S. So there are constant depending only on Ω such that

$$|\langle N_{\mathcal{P}}, n(-s)\rangle| \geq k_1 e^{b's}$$
$$|\langle N_{\mathcal{Q}}, n(-s)\rangle| \geq k_1 e^{b's}.$$

If $\{z\} = \mathcal{P} \cap \mathcal{Q} \cap \gamma_{-s}(\Pi)$, then $T_z\gamma_{-s}(\Pi) = n(-s)^\perp$; moreover, the normal vector to $\mathcal{P} \cap \gamma_{-s}(\Pi)$ is the orthogonal projection of $N_{\mathcal{P}}$ on $n(-s)^\perp$. The same considerations holds for $\mathcal{Q} \cap \gamma_{-s}(\Pi)$.
Identifying $\mathbb{H}^2$ with UT_z, we have that $n(-s)^\perp$ is a line and the distance between $N_{\mathcal{P}}$ and $N_{\mathcal{Q}}$ from that line is greater than $\operatorname{arcsinh}(k_1 e^{bs})$. Since $N_{\mathcal{P}}$ and $N_{\mathcal{Q}}$ have distance ζ, the segment with endpoints $N_{\mathcal{P}}$ and $N_{\mathcal{Q}}$ does not meet $n(-s)^\perp$ and the projections of $N_{\mathcal{P}}$ and $N_{\mathcal{Q}}$ on $n(-s)^\perp$ have distance η verifying

$$-1 + \cosh\eta < e^{-s}(-1 + \cosh\zeta).$$

We claim that there is $K = K(\zeta)$ with $K(0) = 1$ such that

$$a^2(-1 + \cosh\zeta) < -1 + \cosh(Ka\zeta).$$

Write $-1+\cosh x = x^2 g(x)$, where $g(0) = 1/2$ and g is smooth and increasing on $\mathbb{R}_{>0}$. Consider the function

$$\begin{aligned} F(a,\zeta) =& \frac{-1 + \cosh(Ka\zeta)}{a^2(-1+\cosh\zeta)} = \frac{(Ka\zeta)^2 g(ka\zeta)}{a^2(\zeta)^2 g(\zeta)} = \\ =& \frac{K^2 g(ka\zeta)}{g(\zeta)} > \frac{K^2 g(0)}{g(\zeta)}. \end{aligned}$$

We can choose $K(\zeta)$ so that $K(\zeta)^2 > g(\zeta)/g(0)$, which gives

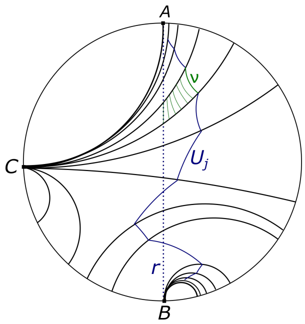
$$-1 + \cosh(K(\zeta)a\zeta) > a^2(-1 + \cosh\zeta)$$

and so $\eta < K(\zeta)e^{-s/2}\zeta$. Now take $s = k$: we have found $\theta_k \leq K(\zeta)e^{-k/2}\zeta$. □

Remark 3.4.1. In the last part of the proof of the previous lemma, $K(\zeta)$ was required to be greater than an increasing function of ζ which has value 1 at $\zeta = 0$. Therefore, we can also suppose that K is a differentiable increasing function on $\mathbb{R}_{\geq 0}$. Now, Lemma 3.4.17 says that there is a differentiable increasing function F of ζ such that $\theta_k \leq F(\zeta)e^{-a_0 k}$ with $F(0) = 0$.

The path

$$U_j = \bigcup_{z \in]A,B[} (\tau_j(z) \cap \mathbb{S}^-_j) \supset c^-_j \tag{3.26}$$



on $\mathbb{S}_j^-$ is a piecewise geodesic space-like curve from A to B. Moreover, there is a homotopy $H : [0,1]\times U_j \to \mathbb{S}_j^-$ between U_j and the h_j^--realization r such that, for every connected component ν of $U_j \setminus \tilde{\lambda}_j^-$, the restriction $H_{|[0,1]\times\overline{\nu}}$ is an isotopy between $\overline{\nu}$ and a subarc of r realized through arcs with endpoints in the two leaves of $\tilde{\lambda}_j$ containing $\partial\nu$. In this sense, we can say that U_j meets $\tilde{\lambda}_j^-$ as r does.
Gauss-Bonnet formula gives us that

$$\begin{aligned}\mathcal{A} = {} & \iota(U,\mu) + \sum_{k\geq 1}\theta_k + \sum_{k\geq 1}\theta'_k \leq \\ \leq & \iota(U,\mu) + C(a_0)\cdot \sum_{\substack{\zeta_i \text{ weights of the} \\ \text{spiralling leaves} \\ \text{near } \partial \text{ of } \mu}} F(\zeta_i) + C(a'_0)\cdot \sum_{\substack{\zeta'_j \text{ weights of the} \\ \text{spiralling leaves} \\ \text{near } \partial' \text{ of } \mu}} F(\zeta'_j)\end{aligned}$$

where $C(a_0) = (1-e^{-a_0})^{-1}$ and $C(a'_0) = (1-e^{-a'_0})^{-1}$. Moreover, we know that $\sum\zeta_i = |m(\lambda,\partial)| = |m(\mu,\partial)|$ and $\sum\zeta'_j = |m(\lambda,\partial')| = |m(\mu,\partial')|$. Denoting by m and m' these two quantities and setting

$$\bar{M} = \sup_{[0,1]}\frac{F(t)}{t}$$

it turns out that $\sum F(\zeta_i) \leq m(\bar{M}+F(m))$ and $\sum F(\zeta'_i) \leq m'(\bar{M}+F(m'))$. Therefore, we can say there are constants $\bar{C}_0$ and $\bar{M}$, depending only on $\mathbf{b}$ and on δ, such that

$$\text{Area}(R) = \mathcal{A} \leq \iota(U,\mu) + \bar{C}_0\Big(m(\bar{M}+F(m)) + m'(\bar{M}+F(m'))\Big).$$

Using also Lemma 3.4.13, Theorem 3.4.1 immediately follows.

3.5 The properness of $\Phi^{\mathbf{b}}$

Taken a sequence (h_j, h'_j) in $\mathcal{T}^\circ_S(\mathbf{b}) \times \mathcal{T}^\circ_S$ such that the sequence

$$(\lambda_j, \mu_j) = \Phi^{\mathbf{b}}(h_j, h'_j)$$

is converging in $\mathcal{FML}_S^\circ(\mathbf{b})$ to a certain (λ_0, μ_0), we want to show that there exists $\lim(h_j, h'_j) \in (\Phi^{\mathbf{b}})^{-1}(\lambda_0, \mu_0) \subset \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$. This is precisely the statement of the last theorem of this section. Estimate (3.23) plays his first role in the proof of the properness of the map $\Phi^{\mathbf{b}} : \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ \to \mathcal{FML}_S^\circ(\mathbf{b})$. Another key ingredient to get such result is the fact that the length map $\mathbb{L}_{(\lambda,\mu)}$ is proper for every $(\lambda, \mu) \in \mathcal{FML}_S^\circ(\mathbf{b})$, as stated in Proposition 3.3.2. If $(\lambda_0, \mu_0) \in \mathcal{CML}_S^\circ \times \mathcal{CML}_S^\circ$, then the existence of $\lim(h_j, h'_j)$ is already proved slightly adapting the argument for the case of S closed in [17], since all the estimates involved there still hold if $(h_j, h'_j) \in \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$. Otherwise, in order to find a compact subset of $\mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ where (h_j, h'_j) live, we will try to uniformly bound $\mathbb{L}_{(\lambda_j,\mu_j)}(h_j)$ in $\mathbb{R}$. Since (λ_j, μ_j) is converging, we will see that $\iota(\lambda_j, \mu_j)$ is uniformly bounded: using such a limitation we will bound $\mathbb{L}_{(\lambda_j,\mu_j)}$. For every (λ, μ) the construction of the terms of $\mathbb{L}_{(\lambda,\mu)}$ was based on $\ell(\hat{\lambda}_\bullet) + \ell(\hat{\mu}_\bullet)$, where $\hat{\lambda}_\bullet$ and $\hat{\mu}_\bullet$ (see Subsection 3.2.2) are arcs in λ and μ whose endpoints are in the spiralization neighbourhood of S and have distance from the boundary components of ∂S bounded above and below by positive constants depending only on $\mathbf{b}$ (see Remark 3.2.6). In particular, $\ell(\hat{\lambda}_\bullet)$ and $\ell(\hat{\mu}_\bullet)$ are those terms of $\mathbb{L}_{(\lambda,\mu)}$ that we are interested to bound. In inequality (3.23) the weighted length of an arc c in the bending locus of a past convex bent surface is involved and actually is bounded from an intersection number: taking $c_j \supseteq (\hat{\lambda}_\bullet)_j$, we will bound each $\mathbb{L}_{(\lambda_j,\mu_j)}(h_j)$ by limiting each term of them through estimate (3.23) and using the uniform bound on $\iota(\lambda_j, \mu_j)$.
With respect to the closed case we need to add some technical work, due to the fact that spiralling leaves of (λ, μ) are not approximated by weighted closed curves.

First of all we fix j and focus on a spiralling leaf γ_j of λ_j such that γ_j tends to a leaf of λ_0. Notice that ω_j does not converge to 0, since γ_j tends to a spiralling leaf of λ_0 (moreover, $\mathrm{supp}(\gamma_j)$ is definitevly constant). Since $(\lambda_j, \mu_j) \to (\lambda_0, \mu_0)$, there exists $\Omega > 0$ such that the positive masses of λ_j and μ_j near any boundary component of ∂S are less than or equal to Ω.
Let $\mathcal{K}_j$ be the convex core in AdS_3 associated to (h_j, h'_j) and $\mathrm{b}_j^\pm : \mathcal{H} \to AdS_3$ be the bending maps with image $\mathbb{S}_j^\pm = \partial_\pm \mathcal{K}_j$, with fault locus $\tilde{\lambda}_j^\pm \subset \mathbb{S}_j^\pm$.
Fix three distinct points A, B, C on $\partial_\infty AdS_3$ lying in the ideal boundary of a space-like plane $\mathcal{P}_0$. As in Section 3.2, take on $\mathcal{H}$ a lift $\tilde{\gamma}_j$ of γ_j and pick on $\tilde{\gamma}_j$ the preimages $\tilde{p}, \tilde{p}'$ of p_1, p'_1 respectively (in Proposition 3.2.8 they correspond to q_i, q_{i+1} for a certain i). Up to post-composing b_j^+ with an isometry we may assume $\mathrm{b}_j^+(\tilde{\gamma}_j) = [A, B]$ and $[A, C] \subset \partial \mathcal{K}_j$. In this way, $[A, C]$ is the image through b_j^+ of the lift $\tilde{\partial}$ of a boundary component ∂ of S near which γ is spiralling. Let Π_j a time-like plane in AdS_3 containing $[A, B]$ and normal to a face of $\mathbb{S}_j^+$ with an edge coinciding with $[A, B]$. We are interested in the arc $c_j = [\mathrm{b}_j^+(\tilde{p}), \mathrm{b}_j^+(\tilde{p}')] \subset [A, B]$. Now, from Theorem

3.4.1, we have

$$\begin{aligned}\omega_j\ell(c_j)\min\{\bar\kappa_0,\bar K_0\omega_j\,\ell(c_j)\}\le\\ \le\iota(\lambda_j,\mu_j)+\bar C_0\omega_j\Big(m_j(\bar M+F(m_j))+m'_j(\bar M+F(m'_j))\Big).\end{aligned}\quad(3.27)$$

Recall that there is Ω such that all the positive masses of (λ_j,μ_j) near the boundary components of S are not bigger than Ω; moreover, F is increasing. Estimate (3.27) becomes then

$$\omega_j\ell(c_j)\min\big\{\bar\kappa_0,\bar K_0\omega_j\ell(c_j)\big\}\le\iota(\lambda_j,\mu_j)+\bar\Omega.\quad(3.28)$$

for a certain constant $\bar\Omega$ independent on j.

Now we have all the elements to prove that the sequence (h_j,h'_j) considered in the beginning of this section converges in $\mathcal{T}^\circ_S(\mathbf{b})\times\mathcal{T}^\circ_S$ to a couple $(h_0,h'_0)\in(\Phi^{\mathbf{b}})^{-1}\big(\lim\Phi^{\mathbf{b}}(h_j,h'_j)\big)$. We show first that $\mathbb{L}_{\Phi^{\mathbf{b}}(h_j,h'_j)}(h_j)$ is a bounded sequence in $\mathbb{R}$.

Lemma 3.5.1. *If $(h_j,h'_j)\in\mathcal{T}^\circ_S(\mathbf{b})\times\mathcal{T}^\circ_S$ are such that $(\lambda_j,\mu_j)=\Phi^{\mathbf{b}}(h_j,h'_j)$ converge to $(\lambda_0,\mu_0)\in\mathcal{FML}^\circ_S(\mathbf{b})$, then there exists a constant $C_0>0$ such that $\mathbb{L}_{(\lambda_j,\mu_j)}(h_j)\le C_0$ for every j.*

Proof. Since $\iota(*\,,*)$ is continuous, there exists $\bar\iota>0$ such that $\iota(\lambda_j,\mu_j)\le\bar\iota$. Notice that λ_j and μ_j cannot have more than N spiralling leaves, with N depending only on the genus g and the number of bondary components $\mathfrak{n}$. The map $\mathbb{L}_{\lambda_j,\mu_j}$, following the construction in Subsection 3.2.4, can actually be rewritten as the sum of the compact part $L^{(0)}_j$ with the spiralling part. Slightly adapting the argument for the closed case, we know that there is $C_1>0$ such that, for every j,

$$L^{(0)}_j(h_j)=L_{\lambda^{(0)}_j}(h_j)+L_{\mu^{(0)}_j}(h_j)\le C_1.$$

The compact parts $(\lambda^{(0)}_j,\mu^{(0)}_j)$ could not fill up S, but that was not a necessary hypothesis in [17] to get that estimate.
The spiralling part of $\mathbb{L}_{(\lambda_j,\mu_j)}$, looking at how it is defined, is actually less than

$$\sum_i\omega_{j,i}\ell(c_{j,i})+\sum_k\zeta_{j,k}\ell(d_{j,k})\quad(3.29)$$

where $c_{j,i}$ are arcs on $\gamma_{j,i}$ (assuming that the spiralling part of λ_j is $\bigcup_i\gamma_{j,i}$) whose endpoints have distance from the boundary components bounded from above and below by positive constants depending only on $\mathbf{b}$, and $\omega_{j,i}$ is the weight of the leaf $\gamma_{j,i}$. The second term in (3.29) refers analogously

to μ_j.
From (3.28), there is a constant C_2 independent on j such that

$$\sum_i \omega_{j,i}\ell(c_{j,i}) + \sum_k \zeta_{j,k}\ell(d_{j,k}) \leq NC_2 + NC_2.$$

Now we have that

$$\mathbb{L}_{(\lambda_j,\mu_j)}(h_j) \leq C_1 + 2NC_2 = C_0.$$

□

Theorem 3.5.2. *If $(h_j, h'_j) \in \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ are such that $(\lambda_j, \mu_j) = \Phi^{\mathbf{b}}(h_j, h'_j)$ converge to $(\lambda_0, \mu_0) \in \mathcal{FML}_S^\circ(\mathbf{b})$, then there exists a subsequence of (h_j, h'_j) converging to (h_0, h'_0) in $\mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ and $(\lambda_0, \mu_0) = \Phi^{\mathbf{b}}(h_0, h'_0)$.*

Proof. The previous lemma states that for every j

$$h_j \in K_j = \mathbb{L}^{-1}_{(\lambda_j,\mu_j)}([0, C_0]).$$

We noticed in Remark 3.2.9 that in the construction of the map $\mathbb{L}_{(\lambda_j,\mu_j)}$ certain choices were made. However, the convergence of the 1-forms $\mathrm{d}\mathbb{L}_{(\lambda_j,\mu_j)}$ to $\mathrm{d}\mathbb{L}_{(\lambda_0,\mu_0)}$ does not depend on such choices. On the other hand, fixed h_* in $\mathcal{T}_S^\circ(\mathbf{b})$, since (λ_j, μ_j) tends to (λ_0, μ_0) we can make such choices so that $\mathbb{L}_{(\lambda_j,\mu_j)}(h_*)$ is convergent. Therefore, there exists $\mathbb{F} : \mathcal{T}_S^\circ(\mathbf{b}) \to \mathbb{R}$ such that $\mathbb{L}_{(\lambda_j,\mu_j)}(h)$ tends to $\mathbb{F}(h)$ for every $h \in \mathcal{T}_S^\circ(\mathbf{b})$ and $\mathrm{d}\mathbb{F} = \mathrm{d}\mathbb{L}_{(\lambda_0,\mu_0)}$. Also, being $\mathbb{F} - \mathbb{L}_{(\lambda_0,\mu_0)}$ constant, $\mathbb{F}$ is proper.
Since $\mathbb{L}_{(\lambda_j,\mu_j)}$ is proper, the subsets K_j are compact and convex for earthquake paths, so $K_j \to K = \mathbb{F}^{-1}([0, C_0])$ (see [17]).
So h_j remains in a compact subset of $\mathcal{T}_S^\circ(\mathbf{b})$ and, after taking a subsequence, h_j converges to a limit h_0. Take $h'_0 = E_l^{\lambda_0}(h_0)$ to conclude. □

3.6 The map $\Phi^{\mathbf{b}}$ is surjective

3.6.1 Existence for small laminations

Let $\mathcal{D}$ be the space $\{(h, h) \in \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ \;:\; h \in \mathcal{T}_S^\circ(\mathbf{b})\}$. The aim of this subsection is to find a neighbourhood $\mathcal{U}$ of $\mathcal{D}$ in $\mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ such that the restriction of $\Phi^{\mathbf{b}}$ to $\mathcal{U} \setminus \mathcal{D}$ is a homeomorphism onto its image $V \subset \mathcal{FML}_S^\circ(\mathbf{b})$ with the property that for any $(\lambda, \mu) \in \mathcal{FML}_S^\circ(\mathbf{b})$ there is $t > 0$ sufficiently small such that $(t\lambda, t\mu) \in V$.

Proposition 3.6.1. *Let $e_l^\lambda + e_l^\mu$ denote the vector field $\frac{\mathrm{d}}{\mathrm{d}t}_{|0}(E_l^{t\lambda} \circ E_l^{t\mu})$, given $(\lambda, \mu) \in \mathcal{FML}_S^\circ(\mathbf{b})$. Then $e_l^\lambda + e_l^\mu$ is a smooth vector field on $\mathcal{T}_S^\circ(\mathbf{b})$.*

Proof. Let us suppose (λ, μ) has a non empty compact sublamination. Decompose $(\lambda, \mu) = (\lambda_c, \mu_c) \oplus (\lambda_s, \mu_s)$ as the sum of the compact maximal sublaminations with the spiralling sublamination. Then $e_l^{\lambda} + e_l^{\mu}$ can be decomposed as $e_l^{\lambda_c} + e_l^{\lambda_s} + e_l^{\mu_c} + e_l^{\mu_s}$. By classical results, $e_l^{\lambda_c} + e_l^{\mu_c}$ is smooth. So we can suppose $(\lambda, \mu) = (\lambda_s, \mu_s)$ and consider only this case.
It is convenient to see $\mathcal{T}_S^{\circ}(\mathbf{b})$ as the space of faithful discrete representations $\rho : \pi_1(S) \to PSL(2, \mathbb{R})$ with conditions that fix the images of peripheral loops, up to conjugacy. For every $\rho \in \mathcal{T}_S^{\circ}(\mathbf{b})$, consider the universal covering $\mathcal{H}$ of S such that $\rho(\pi_1(S))\backslash\mathcal{H} \cong S$ and fix a point $z \in \mathcal{H}$; the infinitesimal earthquake, regarded as an element of the cohomology $H^1(\pi_1(S), \mathbb{R}^{2,1})$, is represented by the element $e_l^{\lambda}(\rho) : \pi_1(S) \to \mathfrak{so}(2,1) \cong \mathbb{R}^{2,1}$ has the form

$$\gamma \mapsto \int_{\mathcal{G}} v(r)\chi_{\mathcal{G}(\gamma)}(r)\,\mathrm{d}\lambda$$

where

- the space
$$\mathfrak{so}(2,1) = \left\{M \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) \,|\, M^T \cdot J_3 + J_3 \cdot M = 0\right\}$$
is the Lie algebra of $SO(2,1)$ and $J_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

- the space
$$\mathcal{G} \cong (S^1 \times S^1) \smallsetminus \mathrm{diag}(S^1)$$
is the set of oriented geodesics on $\mathbb{H}^2$,

- the map
$$v : \mathcal{G} \to \mathfrak{so}(2,1)$$
sends $r \in \mathcal{G}$ to the infinitesimal generator of the hyperbolic transformations on $\mathbb{I} \subset \mathbb{R}^{2,1}$ (see Section 1.1.2) with r as oriented axis,

- the set $\mathcal{G}(\gamma) \subset \mathcal{G}$ is the subset containing the leaves of $\mathrm{supp}(\lambda)$, oriented consistently with the λ-earthquake whose lifting $\tilde{\lambda}$ on $\mathcal{H}$ fixes z, that meet the geodesic arc $[z, \rho(\gamma)(z)]$,

- $\mathrm{d}\lambda$ denotes $\mathrm{d}\,\mathrm{meas}_{\lambda}$.

Given a smooth family $(\rho_t)_{t\in I} \subset \mathcal{T}_S^{\circ}(\mathbf{b})$, where I is an interval of $\mathbb{R}$ containing 0, we want to show that for every $\gamma \in \pi_1(S)$ the map $t \mapsto e_l^{\lambda}(\rho_t)(\gamma)$ is smooth. Consider the covers $\mathcal{H}_t$ and subsets $\mathcal{G}_t(\gamma) \subset \mathcal{G}$. Denote by $\tilde{\lambda}_t$ the realization of $\tilde{\lambda}$ in $\mathcal{H}_t$. Now

$$e_l^{\lambda}(\rho_t)(\gamma) = \int_{\mathcal{G}} v(r)\chi_{\mathcal{G}_t(\gamma)}(r)\,\mathrm{d}\lambda_t.$$

For every $t \in I$ there exists a homeomorphism $\zeta_t : \partial\mathcal{H}_0 \to \partial\mathcal{H}_t$ which is ρ_t-equivariant, i.e.

$$\zeta_t(\rho_t(\beta)(x)) = \rho_t(\beta)(\zeta_t(x)) \ \ \forall x \in \partial\mathcal{H}_0 \ \forall \beta \in \pi_1(S),$$

and such that for every x that is an endpoint of an axis of $\rho_0(\alpha)$ for some $\alpha \in \pi_1(S)$ the map $t \mapsto \zeta_t(x)$ is smooth. It induces a map

$$Z_t = (\zeta_t)_* : \mathcal{G} \xrightarrow{\sim} \mathcal{G}.$$

The function $t \mapsto Z_t(r)$ is smooth for every geodesic r with endpoints in Λ_0, while $t \mapsto Z_t^{-1}(r)$ is smooth for every geodesic r with endpoints that are also endpoints of axis of some transformation of $\rho_0(\pi_1(S))$. It turns out that $\lambda_t = Z_t(\lambda_0)$, in the obvious sense. Notice that the endpoints of the leaves of λ_t are also endpoints of boundary components for every $t \in I$. Also, $\mathcal{G}_t(\gamma)(Z_t(s)) = \mathcal{G}_0(\gamma)(s)$ for every $s \in \mathcal{G}$. Now we have

$$e_l^\lambda(\rho_t)(\gamma) = \int_{\mathcal{G}} v(r)\chi_{\mathcal{G}_t(\gamma)}(r)\,\mathrm{d}Z_t(\lambda_0) = \int_{\mathcal{G}} v(Z_t(s))\chi_{\mathcal{G}_0(\gamma)}(s)\,\mathrm{d}\lambda_0.$$

The integrand of the latter member is a smooth function of t, so we get that $t \mapsto e_l^\lambda(\rho_t)(\gamma)$ is smooth for every $\gamma \in \pi_1(S)$. $\square$

The following theorem is the generalization of Theorem 3.1.1 for surfaces with $\mathfrak{n}$ closed geodesic boundary components.

Theorem 3.6.2. *Let S be a surface with $\mathfrak{n}$ punctures and negative Euler characteristic and let λ and μ be measured laminations on S. The intersection between e_l^λ and $-e_l^\mu$, considered as submanifolds in $T\mathcal{T}_S^\circ$, is transverse. Moreover, if λ and μ fill up S then these sections meet in exactly one point $k_0(\lambda,\mu)$. Otherwise, they are disjoint.*

Proof. The proof works like in [11], where the key points are that there exists a unique critical point h_0 of $\mathbb{L}_{(\lambda,\mu)}$ (see Corollary 3.3.5), which also has the property that $\mathrm{Hess}_{h_0}(\mathbb{L}_{(\lambda,\mu)})$ is positive definite (see Subsection 3.3.3), and that $\mathbb{L}_{(\lambda,\mu)}$ is the symplectic gradient of $e_l^\lambda + e_l^\mu$. You only have to decompose $\mathbb{L}_{(\lambda,\mu)}$ as $\mathbb{L}_\lambda + \mathbb{L}_\mu$, where, following the notations of Subsection 3.2.4,

$$\mathbb{L}_\lambda(h) = L_{\lambda^{(0)}}(h) + \sum_{j=1}^{J} \omega^{(j)} \left[\ell_h(\rho^{(j)} \cap \lambda) + \log \prod_{i=1}^{2I} \cosh d_h(q_i^{(j)}, D_i^{(j)}) \right]$$

$$\mathbb{L}_\mu(h) = L_{\mu^{(0)}}(h) + \sum_{j=1}^{J} \omega^{(j)} \left[\ell_h(\rho^{(j)} \cap \mu) + \log \prod_{i=1}^{2I} \cosh d_h(q_i^{(j)}, D_i^{(j)}) \right]$$

and work on $\mathcal{T}_S^\circ(\mathbf{b})$ instead of $\mathcal{T}_S^\circ$. $\square$

Proposition 3.6.5 finds its analogous in the closed case. Actually, the argument is quite the same one, but paying attention to the spaces we are working on: we are now considering earthquakes between different Teichmüller spaces, as pointed out during the proof, and the space

$$\mathcal{F}^0\mathcal{ML}_S^\circ(\mathbf{b}) = \mathcal{FML}_S^\circ(\mathbf{b}) \cup \{(0,0)\}$$

is not a cone, since it is not true that for every $(\lambda,\mu) \in \mathcal{F}^0\mathcal{ML}_S^\circ(\mathbf{b})$ the couple $(t\lambda, t\mu)$ lies in $\mathcal{F}^0\mathcal{ML}_S^\circ(\mathbf{b})$ for any $t \in [0,\infty)$; however, it is true for any $t \in [0,1]$. First we state some known results of Riemannian geometry.

Lemma 3.6.3. *Let M be a differentiable manifold, $J \subset \mathbb{R}$ an interval containing 0, $X : M \times J \to TM$ a vector field of class $\mathcal{C}^k$ ($k \geq 1$) continuously depending on the real variable. Then there exists W open neighbourhood of $M \times \{0\}$ in $M \times J$ and $\phi : W \to M$ of class $\mathcal{C}^k$ such that, for every $(p_0, 0) \in W$, $\phi(p_0, \cdot)$ is a $\mathcal{C}^{k+1}$ solution of*

$$\begin{cases} \dot{f}(t) = X(f(t), t) \\ f(0) = p_0 \end{cases}.$$

Lemma 3.6.4. *Let Y be a topological space, M a differentiable n-dimensional manifold, $\{\psi_y : M \to \mathbb{R}^n\}_{y \in Y}$ a family of maps of class $\mathcal{C}^k$ ($k \geq 1$) so that*

$$Y \ni y \mapsto \psi_y \in \mathcal{C}^k(M, \mathbb{R}^n)$$

is a continuous map. Suppose there exist $y_0 \in Y$ and $p_0 \in M$ such that $\psi_{y_0}(p_0) = 0$ and $\mathrm{d}_{p_0}\psi_{y_0}$ is not singular. Then there exist a neighbourhood V of y_0 and a neighbourhood U of p_0 such that for any $y \in V$ there exists a unique solution $p = u(y) \in U$ of

$$\psi_y(p) = 0.$$

Moreover, the map

$$V \ni y \mapsto u(y) \in U$$

is continuous.

Proposition 3.6.5. *Consider $(\lambda_0, \mu_0) \in \mathcal{FML}_S^\circ(\mathbf{b})$ and let $h_0 = k_0(\lambda_0, \mu_0)$ be the unique critical point of $\mathbb{L}_{(\lambda^{(0)}, \mu^{(0)})}$. There exist $\epsilon > 0$, a neighbourhood V of (λ_0, μ_0) in $\mathcal{FML}_S^\circ(\mathbf{b})$, a neighbourhood U of h_0 in $\mathcal{T}_S^\circ(\mathbf{b})$ and a continuous map $k : V \times [0, \epsilon) \to U$ such that:*

- *for $(\lambda, \mu, t) \in V \times (0, \epsilon)$, $k(\lambda, \mu, t)$ is the unique fixed point of $E_l^{t\lambda} \circ E_l^{t\mu}$ lying in in U;*
- *for $(\lambda, \mu) \in V$, $k(\lambda, \mu, 0) = k_0(\lambda, \mu)$.*

Proof. If $(\lambda,\mu)\in\mathcal{FML}_S^\circ(\mathbf{b})$ then $E_l^{t\lambda}$ and $E_r^{t\mu}$ go from $\mathcal{T}_S^\circ(\mathbf{b})$ to $\mathcal{T}_S^\circ(\mathbf{b}-t\mathbf{m}_\lambda)$ where $\mathbf{m}_\lambda = (m(\partial_1,\lambda),\dots,m(\partial_{\mathfrak{n}},\lambda))$. There exist coordinate maps

$$\underline{x}_{(\lambda,\mu,t)} : \mathcal{T}_S^\circ(\mathbf{b}-t\mathbf{m}_\lambda) \to \mathbb{R}^{6(g-1)+2\mathfrak{n}}$$

such that

$$(\lambda,\mu,t)\mapsto \mathcal{E}_l^{t\lambda}-\mathcal{E}_r^{t\mu} = \underline{x}_{(\lambda,\mu,t)}\circ E_l^{t\lambda} - \underline{x}_{(\lambda,\mu,t)}\circ E_r^{t\mu}$$

is a continuous map from $\mathcal{FML}_S^\circ(\mathbf{b})\times[0,1]$ to $\mathcal{C}^1(\mathcal{T}_S^\circ(\mathbf{b}),\mathbb{R}^{6(g-1)+2\mathfrak{n}})$, equipped with the $\mathcal{C}^1$-topology.
Now define

$$\varphi_{(\lambda,\mu,t)}(h) = \frac{\mathcal{E}_l^{t\lambda}(h)-\mathcal{E}_r^{t\mu}(h)}{t} \quad \text{for } (\lambda,\mu,t)\in\mathcal{FML}_S^\circ(\mathbf{b})\times(0,1] \tag{3.30}$$

$$\varphi_{(\lambda,\mu,0)}(h) = d\underline{x}_{(\lambda,\mu,0)}(e_l^\lambda(h)+e_l^\mu(h)) \quad \text{for } (\lambda,\mu)\in\mathcal{FML}_S^\circ(\mathbf{b}) \tag{3.31}$$

where implicitly we are identifying $T_{\underline{x}(h)}\mathbb{R}^{6(g-1)+2\mathfrak{n}}$ with $\mathbb{R}^{6(g-1)+2\mathfrak{n}}$. Such $\varphi : \mathcal{FML}_S^\circ(\mathbf{b})\times[0,1]\to\mathcal{C}^1(\mathcal{T}_S^\circ(\mathbf{b}),\mathbb{R}^{6(g-1)+2\mathfrak{n}})$ is continuous, by Lemma 3.6.3, and

$$\varphi_{(\lambda_0,\mu_0,0)}(h_0) = d\underline{x}_{(\lambda_0,\mu_0,0)}(e_l^{\lambda_0}(h_0)+e_l^{\mu_0}(h_0)) = d\underline{x}_{(\lambda_0,\mu_0,0)}(0) = \underline{0}.$$

By Theorem 3.6.2, $d_{h_0}\big(\varphi_{(\lambda_0,\mu_0,0)}\big)$ is non-singular. So applying Theorem 3.6.4 there exist V neighbourhood of (λ_0,μ_0) in $\mathcal{FML}_S^\circ(\mathbf{b})$, $\epsilon>0$, U neighbourhood of h_0 in $\mathcal{T}_S^\circ(\mathbf{b})$ and a continuous function $k : V\times[0,\epsilon)\to U$ such that $k(\lambda,\mu,t)$ is the unique point in U such that $\varphi_{(\lambda,\mu,t)}(k(\lambda,\mu,t))=0$. If $t>0$ then, by equation (3.30), $k(\lambda,\mu,t)$ is the unique fixed point in U of $E_l^{t\lambda}\circ E_l^{t\mu}$. If $t=0$ then, by equation (3.31), $k(\lambda,\mu,0)=k_0(\lambda,\mu)$. □

Now consider a hypersurface $\mathcal{S}$ in $\mathcal{FML}_S^\circ(\mathbf{b})$ such that for every (λ,μ) in $\mathcal{FML}_S^\circ(\mathbf{b})$ there exists a unique $t\in(0,+\infty)$ such that $(t\lambda,t\mu)\in\mathcal{S}$. The previous Proposition shows that there exist an open covering $\{V^{(i)}\}_{i\in I}$ of $\mathcal{S}$ and maps $k^{(i)} : V^{(i)}\times[0,\epsilon^{(i)})\to U^{(i)}\subset\mathcal{T}_S^\circ(\mathbf{b})$ such that for all $i\in I$

1. for $(\lambda,\mu,t)\in V^{(i)}\times(0,\epsilon^{(i)})$, $k^{(i)}(\lambda,\mu,t)$ is the unique fixed point of $E_l^{t\lambda}\circ E_l^{t\mu}$ lying in in $U^{(i)}$;

2. for $(\lambda,\mu)\in V^{(i)}$, $k^{(i)}(\lambda,\mu,0)=k_0(\lambda,\mu)$.

These maps can be glued to a global map, as shown in the following lemma.

Lemma 3.6.6. *There exist an open neighbourhood W of $\mathcal{S}\times\{0\}$ in $\mathcal{S}\times[0,1]$ and a continuous map $k^* : W\to\mathcal{T}_S^\circ(\mathbf{b})$ such that*

- *for $t>0$ and $(\lambda,\mu,t)\in W$, $k^*(\lambda,\mu,t)$ is a fixed point of $E_l^{t\lambda}\circ E_l^{t\mu}$;*

- *for* $(\lambda, \mu, 0) \in W$, $k^*(\lambda, \mu, 0) = k_0(\lambda, \mu)$*;*
- *for* $(\lambda, \mu, t) \in W$, $\{(\lambda, \mu)\} \times [0, t] \subset W$.

The proof of Lemma 3.6.6 follows the argument of Lemma 3.6 in [17]. Now consider the diffeomorphism

$$\pi : \mathcal{S} \times (0, 1] \to \pi\big(\mathcal{S} \times (0, 1]\big) \subset \mathcal{FML}_S^\circ(\mathbf{b})$$

defined by $\pi(\lambda, \mu, t) = (t\lambda, t\mu)$. Set $V = \pi\big(W \setminus (\mathcal{S} \times \{0\})\big)$, where W is the open subset defined in Lemma 3.6.6, and set $k = k^* \circ (\pi^{-1}) : V \to \mathcal{T}_S^\circ(\mathbf{b})$. This map leads to the construction on V of a right inverse of the map $\Phi^{\mathbf{b}}$.

Corollary 3.6.7. *The open set V verifies the following properties.*

- *For every $(\lambda, \mu) \in \mathcal{FML}_S^\circ(\mathbf{b})$ there is $t > 0$ such that $(t\lambda, t\mu) \in V$.*
- *If $(\lambda, \mu) \in V$ then $(t\lambda, t\mu) \in V$ for every $t \in (0, 1]$.*
- *There is a continuous map $\sigma : V \to \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ that is a right inverse for $\Phi^{\mathbf{b}}$. Moreover*

$$\lim_{t \to 0^+} \sigma(t\lambda, t\mu) = \big(k_0(\lambda, \mu), k_0(\lambda, \mu)\big).$$

Proof. The first two properties follow directly from the construction of V. Since $k(\lambda, \mu)$ is by construction a fixed point of $E_l^\lambda \circ E_l^\mu$, the map σ can be defined by putting

$$\sigma(\lambda, \mu) = \big(k(\lambda, \mu), E^\lambda(k(\lambda, \mu))\big).$$

□

Existence near $\mathcal{D}$

Denote by U the image of the injective map $\sigma : V \to \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ introduced in Corollary 3.6.7. By the Theorem of the Invariance of Domain, U is an open subset of $\mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$. In this subsection we will prove that $\mathcal{U} = U \cup \mathcal{D}$ is an open neighbourhood in $\mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ of the set $\mathcal{D}$ defined at the beginning of the current section.

Lemma 3.6.8. *Let $\{(h_k, h'_k)\} \subset \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ be a sequence converging to $(h, h) \in \mathcal{D} \subset \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ and $\{t_k\} \subset (0, +\infty)$ a sequence such that, putting $\Phi^{\mathbf{b}}(h_k, h'_k) = (t_k\lambda_k, t_k\mu_k)$, the sequence $\{\lambda_k\}$ converges to a measured lamination $\lambda \neq 0$. Then the sequence $\{\mu_k\}$ also converges to a measured lamination μ. Moreover $(\lambda, \mu) \in \mathcal{FML}_S^\circ(\mathbf{b})$ and $h = k_0(\lambda, \mu)$.*

Proof. The argument of the proof of this lemma is the same of Lemma 3.9 in [17], where it was proved that

$$\lim_{n\to\infty} e_r^{\mu_k}(h_k) = e_l^{\lambda}(h).$$

Being in the case where S was a closed surface, the map

$$\begin{aligned}\mathcal{ML}_S^\circ \times \mathcal{T}_S^\circ &\to T\mathcal{T}_S^\circ\\ (\mu, h) &\mapsto e_r^{\mu}(h)\end{aligned}$$

was a homeomorphism onto its image and they could conclude.
In our case, we have to consider this modification: since we are considering $\mathcal{T}_S^\circ(\mathbf{b})$, let us define

$$\mathcal{ML}_S^\circ(\mathbf{b})_r = \{\mu \in \mathcal{ML}_S \,:\, -m(\partial_i, \mu) < b_i \text{ for } i = 1, \dots, \mathfrak{n}\}.$$

We need that also $F : \mathcal{ML}_S^\circ(\mathbf{b})_r \times \mathcal{T}_S^\circ(\mathbf{b}) \ni (\mu, h) \mapsto e_r^{\mu}(h) \in T\mathcal{T}_S^\circ$ is a homeomorphism onto its image. Equipping $\mathcal{ML}_S^\circ(\mathbf{b})_r$ with the topology induced by the bijection with $\mathcal{T}_S^\circ$ given by $\mu \mapsto E_r^{\mu}(h)$, where $h \in \mathcal{T}_S^\circ(\mathbf{b})$, and since $\mathcal{T}_S^\circ(\mathbf{b})$ is a submanifold of $\mathcal{T}_S^\circ$ (with codimension $\mathfrak{n}$), we get that the continuous injective map F is actually a homeomorphism onto its image. □

Corollary 3.6.9. *If $\{(h_k, h'_k)\} \subset \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ converges to an element $(h, h) \in \mathcal{D} \subset \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ then there exists a sequence $\{t_k\} \subset (0, +\infty)$ such that, putting $\Phi^{\mathbf{b}}(h_k, h'_k) = (t_k\lambda_k, t_k\mu_k)$, the sequence $\{(\lambda_k, \mu_k)\}$ is pre-compact in $\mathcal{FML}_S^\circ(\mathbf{b})$.*

Lemma 3.6.10. *The set $\mathcal{U}$ is an open neighbourhood of $\mathcal{D}$ in $\mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$.*

3.6.2 Surjectivity of $\Phi^{\mathbf{b}}$

In this subsection we will prove Theorem B. The results of Subsection 3.6.1, combined with the estimate (3.23), will lead to the proof that there exists an open subset U of $\mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$ such that the restriction of $\Phi^{\mathbf{b}}$ from U to $\Phi^{\mathbf{b}}(U)$ is a homeomorhpism. Then the result in Section 3.5 about the properness of $\Phi^{\mathbf{b}}$ will be the last step of the proof of Theorem B.

Proposition 3.6.11. *If*

$$X = \big\{(h, h') \in \mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ \,:\, (\Phi^{\mathbf{b}})^{-1}\big(\Phi^{\mathbf{b}}(h, h')\big) = \{(h, h')\}\big\}$$

then $X \cup \mathcal{D}$ is a neighbourhood of $\mathcal{D}$ in $\mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ$.

Proof. By contradiction, suppose there exists a sequence

$$\{(h_n, h'_n)\} \subset (\mathcal{T}_S^\circ(\mathbf{b}) \times \mathcal{T}_S^\circ) \smallsetminus X$$

converging to $(h,h) \in \mathcal{D}$. By Proposition 3.6.10 there exist other elements $(g_n, g'_n) \in U = \mathcal{U} \setminus \mathcal{D}$ such that $\Phi^{\mathbf{b}}(g_n, g'_n) = \Phi^{\mathbf{b}}(h_n, h'_n)$. Moreover, by Lemma 3.6.9, there exists an infinitesimal sequence $\{t_n\} \subset \mathbb{R}_{>0}$ such that

$$\Phi^{\mathbf{b}}(g_n, g'_n) = \Phi^{\mathbf{b}}(h_n, h'_n) = (t_n\lambda_n, t_n\mu_n)$$

with $\{(\lambda_n, \mu_n)\}$ precompact in $\mathcal{FML}^\circ_S(\mathbf{b})$. Taking a subsequence we can suppose $(\lambda_n, \mu_n) \to (\lambda, \mu)$; so h is the critical point $k_0(\lambda, \mu)$ of $e^\lambda_l + e^\mu_l$. Also, there is $\Omega > 0$ such that for every n the positive masses of (λ_n, μ_n) near ∂S are less than Ω.
Estimate (3.28) about $(t_n\lambda_n, t_n\mu_n)$, remembering how it descends from (3.27), for n sufficiently large, becomes

$$\bar{K}_0\Big(t_n\omega_n\ell_{g_n}(c_n)\Big)^2 \leq \iota(t_n\lambda_n, t_n\mu_n) + t_n^2\bar{\Omega}$$

which gives

$$\bar{K}_0\Big(\omega_n\ell_{g_n}(c_n)\Big)^2 \leq \iota(\lambda_n, \mu_n) + \bar{\Omega}.$$

So it turns out that $\mathbb{L}_{(\lambda_n,\mu_n)}(g_n)$ (and analogously g'_n) is bounded by a constant L_0 not depending on n. Since $\{(\lambda_n, \mu_n)\}$ is converging, the compact subsets $\mathbb{L}^{-1}_{(\lambda_n,\mu_n)}([0, L_0])$, which respectively contain g_n and g'_n, are converging to the compact subset $\mathbb{L}^{-1}_{(\lambda,\mu)}([0, L_0])$. Therefore g_n and g'_n range in a compact subset. By Lemmas 3.6.8 and 3.6.9, any convergent subsequence of g_n or g'_n must converge to $k_0(\lambda, \mu) = h$. Thus we deduce $(g_n, g'_n) \to (h, h)$.
Now by Corollary 3.6.7 the neighbourhood $\mathcal{U}$ of (h,h) has the property that $\Phi^{\mathbf{b}}$ restricted to $U = \mathcal{U} \setminus \mathcal{D}$ is a homeomorphism onto its open image. However, both (h_n, h'_n) and (g_n, g'_n) lie in $\mathcal{U}$ for n sufficiently large, leading to a contradiction. □

Corollary 3.6.12. *There is an open set $W \subset \mathcal{FML}^\circ_S(\mathbf{b})$ such that the restriction of $\Phi^{\mathbf{b}}$ to $(\Phi^{\mathbf{b}})^{-1}(W)$ is a homeomorphism onto W.*

Proof. There is an open subset Y in X such that $Y \cup \mathcal{D}$ is an open neighbourhood of $\mathcal{D}$ in $\mathcal{T}^\circ_S(\mathbf{b}) \times \mathcal{T}^\circ_S$. Put $W = \Phi^{\mathbf{b}}(Y)$. By Corollary 3.6.7 we can actually choose Y such that $\Phi^{\mathbf{b}}$ restricted to Y is injective, so that W is an open set. Moreover, since $Y \subset X$, $(\Phi^{\mathbf{b}})^{-1}(w) = \{w\}$ for every $w \in W$; thus $(\Phi^{\mathbf{b}})^{-1}(W) = Y$. □

Corollary 3.6.13. *The degree of the map $\Phi^{\mathbf{b}}$ is 1.*

Proof. Since a continuous proper map which restricts to an homeomorphism from an open subset to its image has degree 1, the proof follows directly from Corollary 3.6.12. □

Therefore we can conclude that $\Phi^{\mathbf{b}} : \mathcal{T}^\circ_S(\mathbf{b}) \times \mathcal{T}^\circ_S \to \mathcal{FML}^\circ_S(\mathbf{b})$ is a surjective map, thus proving Theorem B.

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