UNIVERSITÀ DEGLI STUDI DI PAVIA

DOTTORATO DI RICERCA IN MATEMATICA E STATISTICA

TESI DI DOTTORATO

Galois covers, Gaussian maps and totally geodesic submanifolds in the Jacobian locus

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XXX CICLO

Ringraziamenti

Al termine di questi tre anni di dottorato desidero ringraziare tutte le persone che, in modi diversi, mi hanno accompagnata in questo percorso e senza le quali non avrei potuto realizzare questo lavoro.

Prima tra tutti vorrei ringraziare la mia relatrice, Paola Frediani, per avermi ispirata a intraprendere la strada della ricerca quando ancora ero studentessa, e per avermi seguita pazientemente durante gli anni di dottorato. Grazie per essere sempre stata gentile e disponibile, sorridente anche nei momenti di sconforto, quando il mio lavoro procedeva a stenti. Grazie per tutte le preziose spiegazioni, i consigli e l'umanità senza i quali questo lavoro non sarebbe mai stato possibile.

Assieme a lei vorrei ringraziare Alessandro Ghigi per essere stato sempre presente e pronto ad aiutare, e per aver portato ogni giorno il buonumore in dipartimento.

Ringrazio Pietro Pirola, Coordinatore del dottorato di Pavia, per incarnare l'amore per la matematica e mostrare ogni giorno quanto possa essere un'esperienza sublime e passionale. Grazie anche per aver reso possibile l'esperienza alla scuola PRAGMATIC di Catania e per essere sempre stato d'ispirazione in questi anni.

Un rigraziamento speaciale và a Carlos Naranjo, per avermi seguita al PRAGMATIC e nella bellissima esperienza all'università di Barcellona. Grazie per tutto il tempo dedicato, per i preziosi consigli matematici e turistici, e per tutta la cordialità e la simpatia che hanno reso fondamentale per il mio lavoro e umanamente splendida un'esperienza che mi spaventava.

Grazie a Giovanni per essere il mio risultato migliore.

Contents

Introduction

The purpose of this thesis is to investigate the existence of totally geodesic submanifolds of $\mathcal{A}_g$ lying in the Jacobian locus. Totally geodesic submanifolds constitute a class of subvarieties of $\mathcal{A}_g$ containing the so called *special* or *Shimura subvarieties*. The motivation for our study comes from Coleman-Oort's conjecture, that predicts that no special subvarieties should exist in the Jacobian locus if the genus is big enough.

More precisely, denote by $\mathcal{A}_g$ the moduli space of principally polarized abelian varieties of dimension g over $\mathbb{C}$, by $\mathcal{M}_g$ the moduli space of smooth complex algebraic curves of genus g and by $j\colon \mathcal{M}_g \to \mathcal{A}_g$ the period map or Torelli map. Set $\mathsf{T}_g^0 := j(\mathcal{M}_g)$ and call it the open Torelli locus. The closure of T_g^0 in $\mathcal{A}_g$ is called the *Torelli locus* (see e.g.[61]) and is denoted by T_g. The expectation formulated by Oort ([68]) is that for large enough genus g there should not exist any positive-dimensional special subvariety Z of $\mathcal{A}_g$, such that $\mathsf{Z} \subset \mathsf{T}_g$ and $\mathsf{Z} \cap \mathsf{T}_g^0 \neq \emptyset$.

One reason for this expectation coming from differential geometry is that a special (or Shimura) subvariety of $\mathcal{A}_g$ is totally geodesic with respect to the (orbifold) metric of $\mathcal{A}_g$ induced by the symmetric metric on the Siegel space $\mathfrak{H}_g$, of which $\mathcal{A}_g$ is a quotient by the group action of $Sp(2g, \mathbb{Z})$. One expects the Torelli locus to be very curved and a way of stating this is to say that it should not contain totally geodesic submanifolds. Important results in this direction were achieved in [50], [34], [82], [54], [55], [21], [49]. In [30], a study of the second fundamental form of the period map was used to give an upper bound for the possible dimension of a totally geodesic submanifold of $\mathcal{A}_g$ contained in the Torelli locus. This study was based on previous works on the second fundamental form of the period map done in [32], [27], [29]. Moreover, an important theorem of Mumford [64] (see [59] for a more general result) says that an algebraic totally geodesic subvariety of $\mathcal{A}_g$ is

special if and only if it contains a CM point, so the expectation formulated by Oort is both of geometric and arithmetic nature. See [61, §4] for more details.

On the other hand, as we mentioned above, for low genus ($g \leq 7$) there are examples of special subvarieties in T_g, and they are all constructed as families of Jacobians of Galois covers of the line (see [80, 63, 33, 77, 60], [61, §5] for the abelian Galois covers, [44] for the non abelian case and for a complete list).

All the examples of families of Galois covers constructed so far satisfy a sufficient condition to yield a Shimura subvariety that we briefly explain. Consider a Galois cover $f : C \to C' = C/G$, where $G \subset Aut(C)$ is the Galois group and C' is a curve of genus g'. Set $g = g(C)$, then one has a monomorphism of G in the mapping class group $\mathrm{Map}_g := \pi_0(\mathrm{Diff}^+(C))$. The fixed point locus $\mathcal{T}_g^G$ of the action of G on the Teichmüller space $\mathcal{T}_g$ is a complex submanifold of dimension $3g' - 3 + r$. We consider its image M in $\mathcal{M}_g$ and then the closure Z of the image of M in $\mathcal{A}_g$ via the Torelli map.

Set $N := \dim(S^2 H^0(C, K_C))^G$, then the condition that we will denote by $(*)$ is that N must be equal to the dimension of Z, that is:

$$(*) \quad N = 3g' - 3 + r.$$

In [30], it is proven that this condition implies that the subvariety Z is totally geodesic and in [44] it is proven that, in fact, it gives a Shimura subvariety in case $g' = 0$, and the same proof also works if $g' > 0$. Moonen proved using arithmetic methods that condition $(*)$ is also necessary in the case of cyclic Galois covers of $\mathbb{P}^1$. Results in this direction can also be found in [58]. In [44] the authors gave the complete list of all the families of Galois covers of $\mathbb{P}^1$ of genus $g \leq 9$ satisfying condition $(*)$ and hence yielding Shimura subvarieties of $\mathcal{A}_g$ contained in the Torelli locus.

Totally geodesic submanifolds of $\mathcal{A}_g$ are related to the second Gaussian map. In particular in [32] it is proven that the second fundamental form of the orbifold immersion $j : \mathcal{M}_g \to \mathcal{A}_g$ (the immersion holds outside the hyperelliptic locus, see [70] for details) lifts the second Gaussian map of the canonical bundle, as stated in an unpublished paper of Green and Griffiths (see [48]). In [32], an explicit expression for the second fundamental form when evaluated on Schiffer variations is provided (see also [30, Theorem 2.6]). More precisely, $\rho(\xi_p \odot \xi_p)$ reduces, up to a constant, to the evaluation of the second Gaussian map at the point p. However, it is much more difficult

to use the expression given in [32] to compute the second fundamental form on $\xi_p \odot \xi_q$, when $p \neq q$.

Colombo and Frediani in [29] used the explicit expression of the second fundamental form to compute the curvature of the restriction to $\mathcal{M}_g$ of the Siegel metric: in particular, the authors give an explicit formula for the holomorphic sectional curvature of $\mathcal{M}_g$ in the direction ξ_p in terms of the holomorphic sectional curvature of $\mathcal{A}_g$ and the second Gaussian map. The same expression is also used by Colombo, Frediani and Ghigi in [30] to obtain an upper bound for the dimension of totally geodesic germs passing through $[C] \in \mathcal{M}_g$, depending on the gonality of C.

The relation of the second Gaussian map with curvature properties of $\mathcal{M}_g$ in $\mathcal{A}_g$ suggests that its rank could give informations on the geometry of $\mathcal{M}_g$. Driven by this geometrical motivation in [27] the authors compute the rank of this map on both the hyperelliptic and trigonal locus. Moreover Calabri, Ciberto and Miranda in [17] proved that the second Gaussian map has maximal rank for the general curve of any genus, and Colombo, Frediani and Pareschi in [31] proved that it can never be surjective for a curve lying on an abelian surface. Not much more is known about the rank of the second Gaussian map.

Overview of the results

In order to present our results, we introduce the general setting in a slightly more detailed way. Denote, as before, by $\mathcal{M}_g$ the moduli space of smooth complex algebraic curves of genus g, and by $\mathcal{A}_g$ the moduli space of principally polarized abelian varieties of dimension g. Consider the Torelli map, or period map,

$$j : \mathcal{M}_g \to \mathcal{A}_g, \tag{0.0.1}$$

associating to a smooth projective curve $[C] \in \mathcal{M}_g$ its Jacobian variety, as a principally polarized abelian variety: $j([C]) = ([JC], \Theta_C)$. Recall that the image of the Torelli map is called *open Torelli locus* or *open Jacobian locus*. We will denote it by T_g^0 and we will simply denote by T_g its closure, called *Torelli locus* or *Jacobian locus*. Both $\mathcal{M}_g$ and $\mathcal{A}_g$ are complex orbifolds, and $\mathcal{A}_g$ is endowed with a locally symmetric metric, the so-called Siegel metric. We denote the corresponding metric connection by ∇. It is well known that the Torelli map is an orbifold immersion outside the hyperelliptic locus (see

e.g. [70]). Since for $g \geq 4$ the dimension of $\mathcal{M}_g$ is strictly smaller than the dimension of $\mathcal{A}_g$, it makes sense to study the metric properties of $\mathcal{M}_g$ with respect to the Siegel metric. More precisely, fix a non hyperelliptic curve $[C] \in \mathcal{M}_g$. Outside the hyperelliptic locus consider the short exact sequence of tangent bundles associated to the (orbifold) immersion $\mathcal{M}_g \to \mathcal{A}_g$, evaluated in $[C]$:

$$0 \to T_{[C]}\mathcal{M}_g \xrightarrow{dj} T_{([JC],\Theta)}\mathcal{A}_g \xrightarrow{\pi} N_{\mathcal{A}_g \setminus \mathcal{M}_g,([JC],\Theta)} \to 0. \tag{0.0.2}$$

For simplicity, we will denote as $N_{[JC],\Theta}$ the normal bundle of the short exact sequence when evaluated in $([JC],\Theta)$. The following definition comes from Riemannian geometry.

Definition 0.0.1. *The second fundamental form relative to the Torelli map is the following:*

$$\begin{array}{rcl} II : S^2 T_{[C]}\mathcal{M}_g & \longrightarrow & N_{([JC],\Theta)}, \\ v \odot w & \longmapsto & \pi(\nabla_v(w)). \end{array} \tag{0.0.3}$$

Notice that the map is well defined, since the Siegel connection ∇ is symmetric. Using the second fundamental form, we define the totally geodesic submanifolds contained in T_g. With a little abuse of notation, we will identify a submanifold of $\mathcal{M}_g$ with its image in $\mathcal{A}_g$ via the Torelli map.

Definition 0.0.2. *A submanifold $X \subset \mathcal{M}_g$ is totally geodesic with respect to the Siegel metric if for all $[C] \in \mathcal{M}_g$ and for all $u, v \in T_{[C]}\mathcal{M}_g$*

$$II_X(u,v) = 0,$$

where II_X is the second fundamental form of the inclusion $j : X \subset \mathcal{M}_g \to \mathcal{A}_g$.

Since $dj^\vee : S^2 H^0(C, K_C) \to H^0(C, 2K_C)$ is the multiplication map of sections, the dual of the second fundamental form is $\rho := II^\vee : I_2(K) \to S^2 H^0(C, 2K_C) \cong S^2 H^0(C, T_C)^\vee$. Notice that $II \equiv 0$ if and only if $\rho \equiv 0$. When there is no risk of ambiguity, we will refer to ρ as second fundamental form as well.

The second fundamental form gives information about the curvature of the embedded submanifold with respect to the metric properties of the greater one. We expect the Torelli locus to be very curved in $\mathcal{A}_g$, i.e. that very few totally geodesic submanifolds should exist. More precisely, the conjecture is the following:

Conjecture 0.0.1. *For large genus there does not exist any totally geodesic submanifold contained in the Jacobian locus, such that its intersection with the open Jacobian locus is nonempty.*

We point out that this conjecture is a bit stronger than Coleman-Oort's conjecture on the non-existence of special subavarieties in the Torelli locus for high genus. In fact, it is possible to characterize special subvarieties as totally geodesic subvarieties plus a condition of arithmetic nature (see Theorem 1.6.2).

As mentioned above, many partial results about this conjecture have been obtained. For low genus $g \leq 7$, there are examples of totally geodesic submanifolds in T_g, all of them constructed from families of Galois covers of the projective line. Every known example satisfies condition $(*)$ and in [44] is proven that condition $(*)$ is also sufficient for a subvariety to be totally geodesic (indeed, special). The condition is the following: consider a family of Galois covers of $\mathbb{P}^1$ with fixed Galois group and monodromy. Denote by r the number of ramification values and call $N := \dim(S^2H^0(C, K_C))^G$. Assume that

$$N = r - 3. \tag{$*$}$$

Then the family is a special subvariety of $\mathcal{A}_g$ that is contained in the Torelli locus, and intersect the open Torelli locus.

The results contained in the first three chapters of this thesis can be grouped in three strongly connected areas: *examples of totally geodesic submanifolds (in low genus)*, *non-totally geodesic loci* and *computation of the rank of the first and second Gaussian map*. Since the last chapter of this work is somehow split from the rest, we will overview it separately at the end of this section.

In Chapter 1 we construct new examples of totally geodesic submanifolds contained in the Jacobian locus, which are obtained as Jacobians of families of Galois covers of curves of genus $g' = 1$. All these examples satisfy a condition that extends condition $(*)$ for covers of curves of higher genus $g' > 0$, and that we still denote by $(*)$. We also prove that if $g' \geq 1$, and the family satisfies $(*)$, then $g \leq 6g' + 1$. This immediately implies that if $g' = 1$ there are no examples satisfying condition $(*)$ for $g \geq 8$. More precisely, we have the following:

Theorem 0.0.3. *For all $g \geq 2$ and $g' = 1$ there exist exactly 6 positive dimensional families of Galois covers satisfying condition $(*)$, hence yielding Shimura subvarieties of $\mathcal{A}_g$ contained in the Torelli locus. Two of the 6 families yield new Shimura subvarieties, while the others yield Shimura subvarieties which have already been obtained as families of Galois covers of $\mathbb{P}^1$ in [44]. Moreover:*

- *For all $g > 3$ and $g' = 2$, there do not exist positive dimensional families of Galois covers satisfying condition $(*)$.*

- *For $g \leq 9$ and $g' > 2$ there do not exist positive dimensional families of Galois covers satisfying condition $(*)$.*

- *If $g' \geq 1$ and we have a positive dimensional family of Galois covers $f : C \to C'$ with $g' = g(C')$ and $g = g(C)$ which satisfies condition $(*)$, then $g \leq 6g' + 1$.*

The 6 families with $g' = 1$ satisfying $(*)$, that is $N = r$, are the following:

(1) $g = 2$, $G = \mathbb{Z}/2\mathbb{Z}$, $N = r = 2$.
(2) $g = 3$, $G = \mathbb{Z}/2\mathbb{Z}$, $N = r = 4$.
(3) $g = 3$, $G = \mathbb{Z}/3\mathbb{Z}$, $N = r = 2$.
(4) $g = 3$, $G = \mathbb{Z}/4\mathbb{Z}$, $N = r = 2$.
(5) $g = 3$, $G = Q_8$, $N = r = 1$.
(6) $g = 4$, $G = \mathbb{Z}/3\mathbb{Z}$, $N = r = 3$.

Family (2) and family (6) give two new Shimura subvarieties, while the others yield Shimura subvarieties which have already been obtained as families of Galois covers of $\mathbb{P}^1$ in [44].

More precisely:

(1) gives the same subvariety as family (26) of Table 2 in [44] (this family was already found in [61]).

(3) gives the same subvariety as family (31) of Table 2 in [44].

(4) gives the same subvariety as family (32) of Table 2 in [44].

(5) gives the same subvariety as family (34) of Table 2 in [44].

All the above families with $g \geq 3$ are not contained in the hyperelliptic locus. A complete description of the families is given in Section 1.7. All

original results in Chapter 1 have been published in [45].

In Chapter 2 we investigate on condition $(*)$, wondering if it is necessary, in general, for a submanifold $X \subset \mathcal{A}_g$ defined as a family of Galois cover to be special (until now, no examples of totally geodesic submanifolds not satisfying the condition are known). We prove the following:

Theorem 0.0.4. *The bielliptic locus is not totally geodesic if $g \geq 4$. The bi-hyperelliptic locus, i.e. the locus of curves covering a hyperelliptic curve of genus g' with a $2:1$ map, is not totally geodesic if $g \geq 3g'$.*

In [32] it is proven that the second fundamental form lifts the second Gaussian map $\mu_2 : I_2(K_C) \to H^0(C, 4K_C)$. This relation between the two maps is our motivation to compute the rank of the second Gaussian map on some loci. For what concerns the bielliptic locus, we will show that every quadric in the $I_2(K)$ is actually invariant, providing an upper bound for the rank of the second Gauss-Wahl map. Moreover, we will find a lower bound for it. The result is the following:

Theorem 0.0.5. *The rank of the second Gauss-Wahl map on the general curve of the bielliptic locus satisfies the following bounds, depending on genus:*

(1) If g is odd then: $\quad 2g - 10 \leq \operatorname{rank} \mu_2 \leq 5g - 5.$

(2) If g is even then: $\quad 2g - 9 \leq \operatorname{rank} \mu_2 \leq 5g - 5.$

In Chapter 3, we continue the investigation on the rank of the Gaussian maps with the support of the `MAPLE` computer software. We write a script allowing us to compute a lower bound for the rank of both the first and the second Gaussian map of curves that are cyclic Galois covers of $\mathbb{P}^1$ (see Appendix A for details on the code). We produce a list of results in low genus ($g \leq 30$) on the general point of the bielliptic locus, tetragonal locus, and bi-hyperelliptic locus:

Theorem 0.0.6. *The second Gauss-Wahl map on the bielliptic locus is generically injective if $5 \leq g \leq 8$, moreover it cannot be surjective for genus $g \geq 14$. For $8 \leq g \leq 30$, it satisfies $\operatorname{rank} \mu_2 \geq 2g - 1$ for the general curve.*

The general tetragonal curve of genus $8 \leq g \leq 30$ has $\operatorname{rank}\ \mu_1 \ = \ 5g - 14$.

The general tetragonal curve has injective second Gauss-Wahl map for genus $5 \leq g \leq 9$. For $12 \leq g \leq 30$ it satisfies: $\operatorname{rank} \mu_2 \geq 6g - 31$.

The general curve C covering a hyperelliptic curve C' of genus $g' = 2$ has injective first Gauss-Wahl map in genus $g = 6$ and injective second Gauss-Wahl map in genus $6 \leq g \leq 9$. Moreover $\operatorname{rank} \mu_1 \geq 4g - 5$ *for* $9 \leq g \leq 30$ *and* $\operatorname{rank} \mu_2 \geq 3g$ *for* $12 \leq g \leq 30$.

In particular, the general lower bound for the rank of both the first and the second Gauss-Wahl map on the tetragonal locus, is attained for the maximum possible value of g', that is $g' = \lfloor g/3 \rfloor$. To better explain what g' is, notice that every non-hyperelliptic curve C which is a cyclic $4:1$ cover of $\mathbb{P}^1$ also covers with a $2:1$ map either an elliptic ($g' = 1$) or a hyperelliptic curve C' of genus $g' \geq 2$. By varying the monodromy of the $4:1$ cover we proved that all integer values of $g' \leq 1/3\, g$ can be obtained (see Remark 2.5.4).

From `MAPLE` computations, the following expectations arise:

Expectation. *The rank of the second Gauss-Wahl map for a bielliptic curve of genus $g \geq 8$ is $2g - 1$.*

Expectation. *The rank of the first Gauss-Wahl map on the generic tetragonal curve of genus $g \geq 8$ is equal to $5g - 14$. For every genus g this bound is attained for for $g(C') = \lfloor g/3 \rfloor$.*

Expectation. *The rank of the second Gauss-Wahl map on the generic tetragonal curve of genus $g \geq 12$ is equal to $6g - 31$. For every genus g this bound is attained for $g(C') = \lfloor g/3 \rfloor$.*

Expectation. *Let C be a tetragonal curve covering $2:1$ a curve of genus 2. Then* $\operatorname{rank} \mu_1 = 4g - 5$ *for $g \geq 9$ and* $\operatorname{rank} \mu_2 = 3g$ *for $g \geq 12$.*

Finally, as mentioned before, Chapter 4 is a bit detached from the other parts of the thesis. It relies on the study of the geometry of the moduli space $\mathcal{A}_4^{(1,1,2,2)}$, parametrizing isomorphism classes of 4-dimensional abelian varieties with polarization of type $(1,1,2,2)$. More precisely, we present the construction of two divisors in the moduli space $\mathcal{A}_4^{(1,1,2,2)}$, and check their invariance under the natural involution $\iota : \mathcal{A}_4^{(1,1,2,2)} \to \mathcal{A}_4^{(1,1,2,2)}$ (for details on the involution see [10]).

Our geometrical motivation comes from birational geometry. In fact the Kodaira dimension of $\mathcal{A}_4^{(1,1,2,2)}$ is still unknown, and the study of its Picard group could be useful to understand it. In general, while the rationality

properties of the principally polarized case are fairly well understood (see [26, 38, 62, 83, 65, 81]) less is known in the non-principally polarized setting. The only result about unirationality of such moduli spaces is due to Bardelli, Ciliberto and Verra [7], who proved that $\mathcal{A}_4^{(1,2,2,2)}$ is unirational (so is its dual $\mathcal{A}_4^{(1,1,1,2)}$). Nevertheless, nothing is known about neither the unirationality of $\mathcal{A}_4^{(1,1,2,2)}$ nor its Kodaira dimension.

In order to state our result more neatly, we introduce here the basic objects that are part of our main statement. Let C be an irreducible smooth complex projective curve of genus g and let $\pi : D \to C$ a smooth double cover ramified in $r > 0$ points. These covers are parametrized by the moduli space $\mathcal{R}_{g,r} := \{\pi : D \xrightarrow{2:1} C \,|\, g(C) = g, \pi \text{ has } r \text{ ramification values}\}$. The Prym variety associated to some cover $\pi \in \mathcal{R}_{g,r}$ is defined as $P(D, C) = \ker\{Nm(\pi) : JD \to JC\}$. It is an abelian variety of dimension $g - 1 + r/2$ with polarization Ξ of type $\delta = (\underbrace{1, \ldots, 1}_{r/2-1}, \underbrace{2, \ldots, 2}_{g})$. The main result in this chapter is the following:

Theorem 0.0.7. *The image of the Prym map $P : \mathcal{R}_{2,6} \to \mathcal{A}_4^{(1,1,2,2)}$ which sends a cover $\pi : D \to C$ to its Prym variety $P(D, C) = \ker\{Nm(\pi) : JD \to JC\}$, defines a divisor in $\mathcal{A}_4^{(1,1,2,2)}$.*

Let $\tilde{\mathcal{A}}_4$ parametrize triples (X, L_X, H) such that (X, L_X) is a principally polarized abelian variety of dimension 4 and $H \subset X_2$ is a 2-torsion totally isotropic subgroup of four elements. Consider $(X, L_X, H) \in \tilde{A}_4$ and take the quotient map $f : X \to A := X/H$. Choose over A a polarization L_A such that its pullback by f is L_X^2. Then (A, L_A) defines a divisor in $\mathcal{A}_4^{(1,1,2,2)}$.

Moreover the first divisor is invariant via the natural involution defined over $\mathcal{A}_4^{(1,1,2,2)}$, while the second divisor is not.

The last Chapter is part of a work which was started in PRAGMATIC, research school in algebraic geometry and commutative algebra at the University of Catania. This work has been accepted from Le Matematiche [76].

Structure of the thesis

This thesis consists of four chapters and two appendices. The reader can find a specific introduction at the beginning of each chapter. While the first three chapters are interconnected, Chapter 4 is independent and can be read

separately. In the following we briefly describe the contents of each chapter.

In Chapter 1, after providing a survey on several well known results on Galois covers of Riemann surfaces, most of all without proofs, we explain how Riemann's existence theorem can be extended to families of Galois covers, following [44]. We recall very briefly the definitions and results on special subvarieties of $\mathcal{A}_g$, and we explain the condition $(*)$ as presented in [44], sufficient for a variety to be special. In Section 1.7, using condition $(*)$, we construct 6 examples of special subvarietes contained in the Torelli locus from families of Galois covers of elliptic curves. For all of them we explicitly check that condition $(*)$ is satisfied and we control whether the corresponding special subvarieties have already been found from families of Galois covers of the projective line. In Section 1.8, we study families of covers of higher genus curves.

In Chapter 2, after recalling some results on cyclic Galois covers of $\mathbb{P}^1$ needed in our analysis, we give a brief overview on Gaussian maps, illustrating the classical results on the first and second Gauss-Wahl maps. We mention the remarkable connection between the second Gauss-Wahl map and the second fundamental form and we recall an expression useful for the computation of the second fundamental form in term of the Gaussian map (see Theorem 2.3.2). The main result in this chapter is the proof that both the bielliptic ($g \geq 4$) and bi-hyperelliptic ($g \geq 3g'$) locus do not yield totally geodesic subvarieties of $\mathcal{A}_g$ (see Theorem 2.4.5, Theorem 2.4.7, Theorem 2.5.6 and Theorem 2.5.7). Lastly, in Section 2.4.2, we managed to find a bound for the rank of the second Gauss-Wahl map for the general curve of the bielliptic locus.

Chapter 3 consists in a series of results found via the computer software `MAPLE` on the rank of Gaussian maps on the bielliptic, the tetragonal, and the bi-hyperelliptic loci. The script is presented in Appendix A, and provides a lower bound for those ranks. In this chapter we present several tables listing a lower bound for the rank of the first and second Gauss-Wahl maps for some fixed curves which are cyclic Galois covers of $\mathbb{P}^1$. For convenience, we report some of the obtained tables in Appendix B.

Chapter 4 is a bit more detached from the other chapters. After a short overview on some classical results about Prym varieties, in Section 4.2 we present the construction of two divisors in $\mathcal{A}_4^{(1,1,2,2)}$ and in Section 4.3 we check their invariance under the natural involution.

Chapter 1

Shimura varieties via Galois covers of elliptic curves

In this chapter we exhibit new examples of Shimura subvarieties contained in the Torelli locus, arising from curves that are Galois cover of an elliptic curve.

We explain here the general setting. Denote by $\mathcal{A}_g$ the moduli space of principally polarized abelian varieties of dimension g over $\mathbb{C}$, by $\mathcal{M}_g$ the moduli space of smooth complex algebraic curves of genus g and by $j\colon \mathcal{M}_g \to \mathcal{A}_g$ the period map or Torelli map. Set $\mathsf{T}_g^0 := j(\mathcal{M}_g)$ and call it the open Torelli locus. The closure of T_g^0 in $\mathcal{A}_g$ is called the *Torelli locus* (see e.g.[61]) and is denoted by T_g.

As we mentioned in the introduction, the expectation formulated by Oort ([69]) is that for large enough genus g there should not exist a positive-dimensional special subvariety Z of $\mathcal{A}_g$, such that $\mathsf{Z} \subset \mathsf{T}_g$ and $\mathsf{Z} \cap \mathsf{T}_g^0 \neq \emptyset$ (see Conjecture 0.0.1). On the other hand for low genus $g \leq 7$ there are examples of such Z and they are all constructed as families of Jacobians of Galois covers of the line (see [80, 63, 33, 77, 60], [61, §5] for the abelian Galois covers, [44] for the non abelian case and for a complete list).

All the examples of families of Galois covers constructed so far satisfy a sufficient condition to yield a Shimura subvariety that we briefly explain. Consider a Galois cover $f : C \to C' = C/G$, where $G \subset Aut(C)$ is the Galois group, C' is a curve of genus g'. Set $g = g(C)$, then one has a

monomorphism of G in the mapping class group $\mathrm{Map}_g := \pi_0(\mathrm{Diff}^+(C))$. The fixed point locus $\mathcal{T}_g^G$ of the action of G on the Teichmüller space $\mathcal{T}_g$ is a complex submanifold of dimension $3g'-3+r$ (see Section 1.6). We consider its image M in $\mathcal{M}_g$ and then the closure Z of the image of M in $\mathcal{A}_g$ via the Torelli morphism.

Set $N := \dim(S^2H^0(C, K_C))^G$, then the condition that we will denote by $(*)$ is that N must be equal to the dimension of Z, that is:

$$N = 3g' - 3 + r. \tag{$*$}$$

In [30] it is proven that condition $(*)$ implies that the subvariety Z is totally geodesic and in [44] it is proven that in fact it gives a Shimura subvariety in the case $g' = 0$ and the same proof also works if $g' > 0$ as we remark in Section 1.6. Moonen proved using arithmetic methods that condition $(*)$ is also necessary in the case of cyclic Galois covers of $\mathbb{P}^1$. Results in this direction can also be found in [58].

In [44] the authors gave the complete list of all the families of Galois covers of $\mathbb{P}^1$ of genus $g \leq 9$ satisfying condition $(*)$ and hence yielding Shimura subvarieties of $\mathcal{A}_g$ contained in the Torelli locus. Here we do the same for Galois covers of curves of higher genus g' and we find new examples when $g' = 1$ (a complete description of the families is given in Section 1.7). We also prove that if $g' \geq 1$, and the family satisfies $(*)$, then $g \leq 6g' + 1$. This immediately implies that if $g' = 1$ there are no examples satisfying condition $(*)$ for $g \geq 8$ (see Section 1.8).

This chapter is organized as follows.

In Section 1.1 we recall some basic facts on Galois covers of Riemann surfaces which will be used through the whole dissertation.

In Section 1.2 we explain the classical correpondence between classes of Galois covers and the numerical datum $(\mathbf{m}, G, \theta)$ ($m_i \in \mathbb{Z}$, $m_i \geq 2$, G finite group $\theta : \Gamma_{g',r} \to G$ epimorphism) via Riemann's existence theorem, focusing on Galois cover of $\mathbb{P}^1$ before, and studying the general situation $C \to C' \cong C/G$ later.

In Section 1.3 we show that Riemann's existence theorem also holds with families of Galois covers, namely that to any datum $(\mathbf{m}, G, \theta)$ is associated a family of Galois covers of a compact Riemann surface. Here we give some necessary formalism on Teichmüller spaces, and we introduce the *mapping*

class group acting on it.

In Section 1.4 we show how the mapping class group acts on Teichmüller space, and we write explicitely *Hurwitz moves*, considering separately the case of Galois covers of $\mathbb{P}^1$ and the case of Galois covers of a curve of genus g'.

In Section 1.5 we recall some basic facts on representation theory, focusing on the action of some finite group G over the space of holomorphic 1-forms. We will give a simple expression useful to compute the number of invariant elements in $S^2H^0(C, K_C)$.

In Section 1.6 we recall very briefly the definitions and results (mostly without proofs) on special subvarieties of $\mathcal{A}_g$, focusing on PEL special subvarieties, and we show how the condition $(*)$ implies that a family of Galois covers yields a special subvariety following [44].

In Section 1.7 we give the explicit description of the new examples of special subvarieties obtained as Galois covers of a genus 1 curve. In particular we will give the complete list of Galois covers of elliptic curves satisfying condition $(*)$ up to genus 7. We will show that two of them provide new examples of Shimura subvariaties in the Torelli locus.

In Section 1.8, we deal with the natural question of Shimura varieties obtained from families covering higher genus curves. We will show that if $g' \geq 1$ and the family satisfies $(*)$, then $g \leq 6g'+1$. This immediately implies that if $g' = 1$ there are no examples satisfying condition $(*)$ for $g' \geq 8$ (see Theorem 1.8.1), hence our list is complete. Finally we briefly describe the `MAGMA` script used to find the families and and we give the link to the script.

1.1 covers of Riemann surfaces

This section gathers several known results on Galois covers of Riemann surfaces, which will be used through the whole thesis. We start with some preliminary notions and definitions on group actions on Riemann surfaces. We mainly follow the definitions and notations of Miranda [57].

Definition 1.1.1. *Let G be a group and X a Riemann surface. An action of G on X is a map $\Phi : G \times X \to X$ sending the pair (g, p) to $g \cdot p$, such*

that (1) if $h \in G$, $g \cdot (h \cdot p) = (gh) \cdot p$, and (2) if e is the neutral element of G, then $e \cdot p = p$.

We will suppose that the group G is finite and that the map Φ is holomorphic. Moreover we assume that the action of G on X is effective, i.e. there is no non-trivial element in G fixing the whole curve X. Notice that without loss of generality we can restrict ourselves to the effective case by taking the quotient of X with respect to the kernel of Φ.

Definition 1.1.2. *The set* $\mathrm{orb}(p) = \{p' \in X$ *such that* $\exists\, g \in G \mid g \cdot p = p'\}$ *is called orbit of the point p; the group* $\mathrm{stab}(p) = \{g \in G$ *such that* $g \cdot p = p\}$ *is called stabilizer of the point p.*

In our hypothesis, every stabilizer subgroup is cyclic and points with non-trivial stabilizer are finite. Also, recall that there is a unique structure of Riemann surface on the quotient X/G such that the quotient map $\pi : X \to X/G$ is holomorphic. Moreover $\deg(\pi) = |G|$ and $\mathrm{mult}_p(\pi) = |\,\mathrm{stab}(p)|$.

Remark 1.1.3. In the above discussion we have focused on case G finite since it is our case of interest. Actually it is possible to put a complex structure on X/G also when G is an infinite group. In that case it is necessary to require that the action of the group G on X is properly discontinuous, i.e. that for all p, q in X there exist some neighbourhoods, respectively U_p and U_q, such that the set $\{g \in G$ such that $g \cdot U_p \cap \mathrm{U}_q \neq \emptyset\}$ is finite. With this extra hypothesis, the quotient X/G is an Hausdorff space, the set of points with non-trivial stabilizer is discrete and all stabilizers are cyclic groups.

We have the following interesting result, describing how the group G acts on the Riemann surface X, locally.

Theorem 1.1.4 (Linearization of the action)**.** *Let G be a finite group acting holomorphically and effectively on the Riemann surface X, and consider a point $p \in X$ with non-trivial stabilizer. Let $g \in \mathrm{stab}(p)$ be a generator of the stabilizer group. Then there is a local coordinate z on p such that, locally $g \cdot z = \xi_m z$, where $\xi_m = e^{\frac{2\pi i}{m}}$ is a primitive m-th root of unity.*

A map $\pi : X \to X/G$ as above is called *Galois cover* with *Galois group* G. If the group G is cyclic, we will call $\pi : X \to X/G$ *cyclic cover.*

Consider a branch point $y \in X/G$ for the cover $\pi : X \to X/G$. Consider than the set of ramification points on its fiber, that is $\pi^{-1}(y) =$

$\{x_1, x_2, \ldots, x_r\}$. Notice that ramification points are exactly points with non-trivial stabilizer. Moreover since ramification points lying in the same fiber have conjugate stabilizer subgroups, we have that $|\operatorname{stab}(x_i)|$ is the same for $i = 1 \ldots, r$. Then:

$$r = \frac{|G|}{|\operatorname{stab} x_i|}, \quad \forall i = 1, \ldots, r.$$

In this special case Riemann-Hurwitz formula can be written as follows:

Theorem 1.1.5. *Let G be a finite group acting holomorphically and effectively on a Riemann surface X with quotient map $\pi : X \to X/G$. Suppose that there are k branch points $y_1, \ldots, y_k \in X/G$, with π having multiplicity r_i at the $|G|/r_i$ points over y_i. Then*

$$2g(X) - 2 = |G|\Big(2g(X/G) - 2 + \sum_{i=1}^{k} \frac{1}{r_i}(r_i - 1)\Big). \tag{1.1.1}$$

For a compact Riemann surface of genus 2 or more, the previous formula allows to compute a bound on the cardinality of $|G|$, that is $|G| \leq 84(g-1)$. Up to prove the finiteness of $\operatorname{Aut}(X)$ (see [57, Sec VII, Theorem 4.18]), this implies that for every compact Riemann surface with $g \geq 2$,

$$|\operatorname{Aut}(X)| \leq 84(g-1).$$

1.2 Riemann's existence theorem

In the following, we will explain the classical correspondence between classes of branched ramified covers and a purely algebraic data, stating the well-known Riemann's existence theorem. In order to do that, we have to recall some basic facts about covers and monodromies. We mainly refer to Miranda [57].

Take a topological cover $f : X \to Y$. Fix a point $y \in Y$ and take the fiber over y: $f^{-1}(y) = \{x_1, \ldots, x_d\}$. Every loop γ based in y can be lifted to d path $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_d$, where $\tilde{\gamma}_i$ is the only lifting such that $\tilde{\gamma}_i(0) = x_i$. Notice that endpoints of $\tilde{\gamma}_i$ should also stay on the fiber over y, so for every i there exists one j such that $\tilde{\gamma}_i(1) = x_j = x_{\sigma(i)}$.

The function σ is a permutation of the indices $\{1, 2 \ldots, d\}$, and it is easy to see that depends only on the class of the loop $[\gamma] \in \pi_1(Y, y)$. This motivates the following.

Definition 1.2.1. *The monodromy representation of a topological cover $f : X \to Y$ of finite degree d is the group homomorphism*

$$\rho : \ \pi_1(Y, y) \to S_d, \tag{1.2.1}$$

associating to every class of closed paths based on $y \in Y$ the induced permutation of points over y. The map ρ is called monodromy map.

Notice that, since the domain X is connected, it is straightforward to prove that the image of the map ρ is transitive in S_d, i.e. for every pair on indices i, j in $\{1, \ldots, d\}$ there exists a permutation $\sigma \in \rho(\pi_1(Y, y))$ such that $\sigma(i) = j$.

Since our case of interest are not topological covers but branched ones, we should extend this definition to the ramified case. So consider a holomorphic map between compact Riemann surfaces $f : X \to Y$ of degree d, call $B \subseteq Y$ the set of branches of f and $R = f^{-1}(B)$ its preimage (which is finite by the previous discussion). Observe that removing branches and ramifications we obtain a topological cover $f|_{X-R} : X - R \to Y - B$.

Definition 1.2.2. *The monodromy representation of a ramified cover $f : X \to Y$ of a finite degree d is the monodromy representation of the induced topological cover, obtained removing branches and ramifications:*

$$\rho : \pi_1(Y - B, y) \to S_d, \tag{1.2.2}$$

where $B \subseteq Y$ is the set of branches of f.

Using the monodromy map we have associated a purely algebraic data to a topological one. Riemann proved that the converse holds as well. More precisely we have the following.

Theorem 1.2.3 (Riemann's existence theorem)**.** *There is a $1 : 1$ correspondence between these two sets:*

$$\left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{connected ramified covers} \\ f : X \to Y \textit{ of degree } d \\ \textit{whose branch points lie in } B \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{group isomorphisms} \\ \rho : \pi_1(Y - B, q) \to S_d \\ \textit{with transitive image} \\ (\textit{up to conjugacy in } S_d) \end{array} \right\}.$$

Proof. For the proof see for example [57, Sec. III, Proposition 4.9]. □

Moreover at a point $b \in B$, if γ is a small loop in $Y - B$ around b based at q, and if $\rho([\gamma])$ has cycle structure $(m_1, \dots, m_k)$, then there are k preimages $u_1, \dots, u_k$ of b in the corresponding cover $F_\rho : X_\rho \to Y$, with $\mathrm{mult}_{u_j}(F_\rho) = m_j$ *for each j.*

1.2.1 Riemann's existence theorem for Galois covers of $\mathbb{P}^1$

We specialize the previous proposition in case $f : X \to \mathbb{P}^1$ ramified cover of the Riemann sphere. Call $t = \{t_1, \dots, t_r\} \subset \mathbb{P}^1$ the branch locus of f. The fundamental group of $U_t = \mathbb{P}^1 - t$, with basepoint in some $t_0 \in U_t$, by elementary topology, is isomorphic to the abstract group with r generators and a single relation between them:

$$\pi_1(U_t, t_0) \cong \Gamma_{0,r} = \langle [\gamma_1], [\gamma_2], \dots, [\gamma_r] \ : \ [\gamma_1][\gamma_2] \dots [\gamma_r] = 1 \rangle, \tag{1.2.3}$$

where the element $[\gamma_i]$ corresponds to a simple closed loop winding around the point t_i counter-clockwise.

Therefore it is clear that to give a homomorphism $\rho : \pi_1(U_t, q) \to S_d$ is equivalent (up to isomorphisms) to choose r permutations $\sigma_1, \sigma_2, \dots, \sigma_r$ such that $\sigma_1 \sigma_2 \cdots \sigma_r = 1$. Since the image of ρ is generated by the σ_i's, the vector $(\sigma_1, \dots, \sigma_r)$ is often called *generating vector.*

Consider the special case $f \colon C \longrightarrow \mathbb{P}^1$ Galois cover with branch locus t, call $U_t = \mathbb{P}^1 - t$ and set $V := f^{-1}(U_t)$. Then $f|_V : V \to U_t$ is a Galois cover. Let G denote the group of deck transformations of $f|_V$. Then there is a surjective homomorphism $\pi_1(U_t, t_0) \twoheadrightarrow G$, which is well-defined up to composition by an inner automorphism of G. Since $\Gamma_{0,r} \cong \pi_1(U_t, t_0)$ we get an epimorphism $\theta : \Gamma_{0,r} \twoheadrightarrow G$. If m_i is the local monodromy around t_i, set $\mathbf{m} = (m_1, \dots, m_r)$. We give the following definition:

Definition 1.2.4. *A* datum *is a triple $(\mathbf{m}, G, \theta)$, where $\mathbf{m} := (m_1, \dots, m_r)$ is an r-tuple of integers $m_i \geq 2$, G is a finite group and $\theta : \Gamma_{0,r} \twoheadrightarrow G$ is an epimorphism such that $\theta(\gamma_i)$ has order m_i for each i.*

Thus a Galois cover of $\mathbb{P}^1$ branched over t gives rise – up to some choices – to a datum. The Riemann's existence theorem ensures that the process can

be reversed: a branch locus t and a datum determine a cover of $\mathbb{P}^1$ up to isomorphism. The genus g of the Riemann surface X is given by Riemann-Hurwitz formula (1.1.1).

1.2.2 Riemann's existence theorem for Galois covers

We can generalize the previous construction in case $f : X \to Y$, where Y is a compact Riemann surface of genus $g' \geq 0$. As before, let $t := (t_1, \ldots, t_r)$ be an r-tuple of distinct points in Y, set $U_t := Y - \{t_1, \ldots, t_r\}$ and choose a base point $t_0 \in U_t$. There exists an isomorphism

$$\pi_1(U_t, t_0) \cong \Gamma_{g',r} := \langle \alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'}, \gamma_1, \ldots, \gamma_r \mid \prod_1^r \gamma_i \prod_1^{g'} [\alpha_j, \beta_j] = 1 \rangle \tag{1.2.4}$$

given by the choice of a geometric basis of $\pi_1(U_t, t_0)$ as follows:

$\alpha_1, \beta_1, ..., \alpha_{g'}, \beta_{g'}$ are simple loops in $Y - \{t_1, \ldots, t_r\}$ which only intersect in t_0, whose homology classes in $H_1(Y, \mathbb{Z})$ form a symplectic basis.

Let $\tilde{\gamma}_i$ be an arc connecting t_0 with t_i contained in $(Y - \{\alpha_1, \beta_1, ..., \alpha_{g'}, \beta_{g'}\}) \cup t_0$ and such that for $i \neq j$, $\tilde{\gamma}_i$ and $\tilde{\gamma}_j$ only intersect in t_0. We also assume that $\tilde{\gamma}_1, ...\tilde{\gamma}_r$ stem out of t_0 with distinct tangents following each other in counter-clockwise order. The loops $\gamma_1, ..., \gamma_r$ are defined as follows: γ_i starts at t_0, goes along $\tilde{\gamma}_i$ to a point near t_i, makes a small simple loop counter-clockwise around t_i and goes back to t_0 following $\tilde{\gamma}_i$.

Set $V := f^{-1}(U_t)$. Then $f|_V : V \to U_t$ is an unramified Galois cover with Galois group G. Since $\Gamma_{g',r} \cong \pi_1(U_t, t_0)$, we get an epimorphism $\theta : \Gamma_{g',r} \to G$. If m_i is the local monodromy around t_i, set $\mathbf{m} = (m_1, \ldots, m_r)$. We can define a datum identically as before:

Definition 1.2.5. *A* datum *is a triple* $(\mathbf{m}, G, \theta)$*, where* $\mathbf{m} := (m_1, \ldots, m_r)$ *is an* r*-tuple of integers* $m_i \geq 2$*,* G *is a finite group and* $\theta : \Gamma_{g',r} \to G$ *is an epimorphism such that* $\theta(\gamma_i)$ *has order* m_i *for each* i*.*

Also in this case a Galois cover of Y branched over t gives rise to a datum and the Riemann's existence theorem ensures that the process can be reversed.

Remark 1.2.6. From the previous discussion, once chosen x_0 fixed point

in $f^{-1}(t_0)$, we have the following short exact sequence:

$$1 \to \pi_1(V, x_0) \to \pi_1(U_t, t_0) \to G \to 1, \tag{1.2.5}$$

where each $[\gamma_i] \in \pi_1(U_t, t_0)$ maps to an element g_i whose order is m_i.

From isomorphism (1.2.4) one can identify the fundamental group $\pi_1(U_t, t_0)$ with the abstract group $\Gamma_{g',r}$, getting:

$$1 \to \pi_1(V, x_0) \to \Gamma_{g',r} \to G \to 1.$$

We have the following theorem (see [19]):

Theorem 1.2.7. *In the same setting as before, there exists a subgroup $\Gamma_{g',r,\mathbf{m}} < \Gamma_{g',r}$, depending on the monodromy data $\mathbf{m} = (m_1, \ldots, m_r)$, such that the following short exact sequence holds:*

$$1 \to \pi_1(X, x_0) \to \Gamma_{g',r,\mathbf{m}} \to G \to 1. \tag{1.2.6}$$

Moreover the subgroup $\Gamma_{g',r,\mathbf{m}}$ is given by the quotient of $\Gamma_{g',r}$ by the minimal normal subgroup containing the elements $(\gamma_i)^{m_i}$:

$$\begin{aligned} \Gamma_{g',r,\mathbf{m}} = \langle \alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'}, \gamma_1, \ldots, \gamma_r \mid \prod_1^r \gamma_i \prod_1^{g'} [\alpha_j, \beta_j] = 1, \\ \gamma_1^{m_1} = \cdots = \gamma_r^{m_r} = 1 \rangle. \end{aligned} \tag{1.2.7}$$

The group $\Gamma_{g,r,\mathbf{m}}$ is usually called *orbifold fundamental group* (see [19], [20]).

Notice that if $r = 0$, the corresponding group Γ_g is isomorphic to the fundamental group of a compact Riemann surface of genus g. Using the isomorphism $\pi_1(X, x_0) \cong \Gamma_g$ we can write again short exact sequence (1.2.6) as:

$$1 \to \Gamma_g \to \Gamma_{g',r,\mathbf{m}} \to G \to 1. \tag{1.2.8}$$

The following is another riformulation of Riemann's existence theorem involving the orbifold fundamental group $\Gamma_{g',r,\mathbf{m}}$ (see [75]):

Proposition 1.2.8. *A finite group G acts as a group of automorphisms of some compact Riemann surface X of genus $g \geq 2$ if and only if there exist a group of type $\Gamma_{g',r,\mathbf{m}}$ and an epimorphism $\theta : \Gamma_{g',r,\mathbf{m}} \twoheadrightarrow G$ such that $\ker \theta \cong \Gamma_g$.*

We will call an epimorphism $\theta : \Gamma_{g',r,\mathbf{m}} \twoheadrightarrow G$ as in the proposition *admissible epimorphism.* Notice that, since Γ_g is torsion-free, every admissible epimorphism should preserve the order of generators: to give an admissible epimorphism $\theta : \Gamma_{g',r,\mathbf{m}} \twoheadrightarrow G$ is actually equivalent to give an epimorphism $\tilde{\theta} : \Gamma_{g',r} \twoheadrightarrow G$ such that $\tilde{\theta}(\gamma_i)$ has order m_i for all i.

1.3 Families of Galois covers

We will show that the process can be reversed also in families, namely that to any datum is associated a family of Galois covers of a compact Riemann surface Y of genus g'.

We start with some necessary formalism on Teichmüller theory, mainly following [40], [47]. In the following, let S be a real closed oriented surface of genus $g \geq 2$.

Definition 1.3.1. *A marked surface is a pair (X, φ) where X is a real closed oriented surface of genus $g \geq 2$ and $\varphi : S \to X$ is an orientation preserving diffeomorphism* (marking).

Definition 1.3.2. *Two marked surfaces (X, φ), (X', φ') are equivalent if there exists a diffeomorphism $g : X \to X'$ such that φ is isotopic to $g \circ \varphi$. The space of equivalence classes is called the* Teichmüller space *$\mathcal{T}_g$ of a surface of genus g.*

In the previous notation, let $t = (t_1, \ldots, t_r)$ be a finite subset of S, and $U_t = S - t$ be the surface punctured at t. Call $p_i = \varphi(t_i)$ and $q_i = \varphi'(t_i)$ for $i = 1, \ldots, r$. We have the following definition.

Definition 1.3.3. *Two marked surfaces (X, φ) and (X', φ') are equivalent with respect to t if there exists a diffeomorphism $g : X \to X'$ such that φ is isotopic to $g \circ \varphi$ and $q_i = g(p_i)$ for $i = 1, \ldots, r$. The space of equivalence classes is called the* Teichmüller space *$\mathcal{T}_{g,r}$ of a surface of genus g with $r \geq 1$ marked points.*

We are going to briefly define an important group acting on the Teichmüller space, that is the mapping class group Map_g (we will meet again this group in the next section). The mapping class group is the group of

isotopy classes of orientation preserving diffeomorphisms of S:

$$\mathrm{Map}_g = \frac{\mathrm{Diff}^+(S)}{\mathrm{Diff}_0(S)}. \tag{1.3.1}$$

Remark 1.3.4. The Dehn–Nielsen–Baer theorem states that the mapping class group of S is isomorphic to the outer automorphisms group of the fundamental group of S (see, e.g., [41, Theorem 8.1]):

$$\mathrm{Map}_g \cong \mathrm{Out}^+(\pi_1(S, x_0)),$$

where x_0 is a point in S.

The mapping class group Map_g acts on Teichmüller space $\mathcal{T}_g$ by precomposition of marking. If ζ is a diffeomorphism of S then $\zeta(X, \varphi)$ is the point in $\mathcal{T}_g$ which is given by the same surface X, but where the diffeomorphism φ has been replaced by $\varphi \circ \zeta^{-1}$. If η is isotopic to ζ then the marked structures $\eta(X, \varphi)$ and $\zeta(X, \varphi)$ are equivalent and hence this definition indeed defines an action of the mapping class group on $\mathcal{T}_g$ (see [78] for a complete description).

This shows that the mapping class group acts on the Teichmüller space. It is well known that the action of Map_g on $\mathcal{T}_g$ is properly discontinuous (so all stabilizers are finite) but, in general, is not free. One can consider the projection $\mathcal{T}_g \to \mathcal{T}_g / \mathrm{Map}_g$, which corresponds to the forgetful map $(X, \varphi) \mapsto X$. We have the following (see e.g. [20], [47]).

Theorem 1.3.5. *The quotient of the Teichmüller space with respect to the mapping class group is isomorphic to the moduli space of curves:*

$$\mathcal{T}_g / \mathrm{Map}_g \cong \mathcal{M}_g.$$

Since the Teichmüller space is topologically a ball, and the mapping class group is a discrete group acting on it via a properly discontinuous action, moduli spaces of curves inherit the structure of *topological orbifold.*

We apply basics of Teichmüller theory to prove that Riemann's existence theorem applies also in families. In fact let $(\mathbf{m}, G, \theta)$ be a datum. Choose a point $[Y, t = (t_1, ..., t_r), \psi]$ in the Teichmüller space $\mathcal{T}_{g',r}$. This means that Y is a compact Riemann surface of genus g', $t = (t_1, ..., t_r)$ is an r-tuple of points in Y such that $t_i \neq t_j$ for $i \neq j$ and $\psi : \pi_1(U_t, t_0) \cong \Gamma_{g',r}$ is an isomorphism, where t_0 is a base point in U_t. Using $\theta \circ \psi$, by the above, we

get a G-cover $C_t \to Y$ branched at the points t_i with local monodromies $m_1, \dots, , m_r$.

Observe that we have a monomorphism of G into the mapping class group Map_g, in fact $G < \mathrm{Aut}(C_t) \subseteq \mathrm{Diff}_+(C_t)$, and every elements of G corresponding to a complex automorphism acting as the identity on the first homology group has to be the identity[1].

There is a correspondence between the fixed point locus of G on the Teichmüller space $\mathcal{T}_g$ and the Teichmüller space $\mathcal{T}_{g',r}$, being g and g' related by Riemann-Hurwitz formula (1.1.1). For a proof of the next theorem see for example [20].

Theorem 1.3.6. *$\mathcal{T}_g^G$ is a complex submanifold of dimension $3g' - 3 + r$, isomorphic to the Teichmüller space $\mathcal{T}_{g',r}$.*

This isomorphism can be described as follows: if (C, φ) is a curve with a marking such that $[(C, \varphi)] \in \mathcal{T}_g^G$, the corresponding point in $\mathcal{T}_{g',r}$ is $[(C/G, \psi, b_1, \dots, b_r)]$, where ψ is the induced marking (see [47]) and $b_1, \dots, b_r$ are the critical values of the projection $C \to C/G$.

We remark that on $\mathcal{T}_g^G$ there is a universal family $\mathcal{C} \to \mathcal{T}_g^G$ of curves with a G–action, that is simply the restriction of the universal family on $\mathcal{T}_g$.

Denote by $\mathsf{M}(\mathbf{m}, G, \theta)$ the image of $\mathcal{T}_g^G$ in $\mathcal{M}_g$. It is an irreducible algebraic subvariety of the same dimension as $\mathcal{T}_g^G \cong \mathcal{T}_{g',r}$, i.e. $3g' - 3 + r$. Applying the Torelli map to $\mathsf{M}(\mathbf{m}, G, \theta)$ one gets a subset of $\mathcal{A}_g$. We let $\mathsf{Z}(\mathbf{m}, G, \theta)$ denote the closure of this subset in $\mathcal{A}_g$. By the above it is an algebraic subvariety of dimension $3g' - 3 + r$.

1.4 Mapping class group and Hurwitz moves

Different data $(\mathbf{m}, G, \theta)$ and $(\mathbf{m}, G, \theta')$ may give rise to the same subvariety of $\mathcal{M}_g$. This is related to the choice of the isomorphism $\pi_1(U_t, t_0) \cong \Gamma_{g',r}$.

[1] One can see the injection also considering the exact sequence (1.2.8): it gives a homomorphism $\rho : G \to \mathrm{Out}^+(\pi_1(C_t, t_0))$, associating to every element g of G the action on $\Gamma_g \cong \pi_1(C_t, t_0)$ obtained by conjugation with a lifting of g. This action is well defined then only up to inner automorphisms. Again, ρ has to be injective because acts as the identity on the first homology group. Finally one can conclude using Remark 1.3.4.

The change from one choice to another can be described using an action of the mapping class group. It is possible to define it in a purely algebraic way as well as with a geometrical approach, as we have done in the previous section. We start with the algebraic setting and definition.

1.4.1 Algebraic approach

According to Proposition 1.2.8, if θ is an admissible epimorphism, there exists a short exact sequence

$$1 \to \Gamma_g \xrightarrow{i_\theta} \Gamma_{g',r,\mathbf{m}} \xrightarrow{\theta} G \to 1. \tag{1.4.1}$$

It is clear that the Riemann surface C, uniquely determined by the group $\Gamma_{g',r,\mathbf{m}}$ and the epimorphism θ, is defined (up to automorphisms) not by the specific θ, but rather by its kernel $i_\theta(\Gamma_g)$; this motivates the following definition.

Definition 1.4.1. *We set*

$$Epi(\Gamma_{g',r,\mathbf{m}}, G) = \left\{ \begin{array}{l} \textit{Admissible epimorphism } \theta : \Gamma_{g',r,\mathbf{m}} \twoheadrightarrow G \\ \textit{such that } \ker\theta \cong \Gamma_g \end{array} \right\} \Big/ \sim$$

where $\theta_1 \sim \theta_2$ *if and only if* $\ker\theta_1 = \ker\theta_2$.

We are going to write this equivalence relation in a more concrete way. To do this, we define a discrete group of symmetry of $\Gamma_{g',r,\mathbf{m}}$, that is, the (algebraic) mapping class group.

Recall that an automorphism $\eta \in \mathrm{Aut}(\Gamma_{g',r,\mathbf{m}})$ is said to be orientation-preserving if, for all $i \in \{1, \ldots, r\}$, there exists j such that $\eta(\gamma_i)$ in conjugated to γ_j. This of course implies that $o(\gamma_i) = o(\gamma_j)$.

Definition 1.4.2. *The subgroup of orientation-preserving automorphisms of* $\Gamma_{g',r,\mathbf{m}}$ *is denoted by* $\mathrm{Aut}^+(\Gamma_{g',r,\mathbf{m}})$*, and the group of inner automorphisms of* $\Gamma_{g',r,\mathbf{m}}$ *is denoted by* $\mathrm{Inn}(\Gamma_{g',r,\mathbf{m}})$*. The quotient*

$$\mathrm{Map}_{g',r,\mathbf{m}} = \mathrm{Aut}^+(\Gamma_{g',r,\mathbf{m}}) \big/ \mathrm{Inn}(\Gamma_{g',r,\mathbf{m}})$$

is called the (algebraic) mapping class group *of* $\Gamma_{g',r,\mathbf{m}}$.

There is a natural action of $\mathrm{Aut}(G) \times \mathrm{Map}_{g',r,\mathbf{m}}$ on $Epi(\Gamma_{g',r,\mathbf{m}}, G)$, namely

$$(\lambda, \eta) \cdot \theta = \lambda \circ \theta \circ \eta.$$

We have the following correspondence (see [75, Proposition 1.6]).

Theorem 1.4.3. *Two admissible epimorphisms θ_1, $\theta_2 \in Epi(\Gamma_{g',r,\mathbf{m}}, G)$ define the same equivalence class of G-actions if and only if they lie in the same* $\mathrm{Aut}(G) \times \mathrm{Map}_{g',r,\mathbf{m}}$*-orbit.*

1.4.2 Geometric approach

Using Riemann's existence theorem is it possible to see the mapping class group also with a differentiable, more geometric, approach, as in Definition (1.3.1) of the previous section. More in general one can define the mapping class group marked in r points looking at the differentiable structure of the curve. Let S be a differentiable model of a compact Riemann surface of genus g' and $p_1, \ldots, p_r \in S$, we have the following:

Definition 1.4.4. *The (geometric)* mapping class group *is the quotient between orientation-preserving diffeomorphisms of $S - \{p_1, \ldots, p_r\}$ and the ones isotopic to the identity:*

$$\mathrm{Map}_{g',[r]} = \pi_0 \,\mathrm{Diff}^+(S - \{p_1, \ldots, p_r\}) := \frac{\mathrm{Diff}^+(S - \{p_1, \ldots, p_r\})}{\mathrm{Diff}_0(S - \{p_1, \ldots, p_r\})}.$$

We have the following correspondence:

Theorem 1.4.5. *The algebraic mapping class group and the geometric mapping class group are isomorphic:*

$$\mathrm{Map}_{g',r,\mathbf{m}} = \frac{\mathrm{Aut}^+(\Gamma_{g',r,\mathbf{m}})}{\mathrm{Inn}(\Gamma_{g',r,\mathbf{m}})} \cong \frac{\mathrm{Diff}^+(S - \{p_1, \ldots, p_r\})}{\mathrm{Diff}_0(S - \{p_1, \ldots, p_r\})} = \mathrm{Map}_{g',[r]}\,.$$

Proof. See Maclachlan [56, Chap. 4]. □

Notice that orientation-preserving automorphisms correspond to orientation-preserving diffeomorphisms of the underlying differentiable manifold.

Remark 1.4.6. Following the same ideas behind Theorem 1.3.5, one can find the obvious correspondence between the Teichmüller space with r marked points and the moduli space of curves with r marked points.

Both algebraic and geometric approaches motivate the following:

Definition 1.4.7. *The orbits of the group* $\langle \mathrm{Aut}(G), \mathrm{Map}_{g',r,\mathbf{m}} \rangle$*–action (*Hurwitz's moves*) are called* Hurwitz equivalence classes.

In the following we will describe Hurwitz moves first in case $g' = 0$, then for any $g' \geq 0$.

1.4.3 Hurwitz moves in case $g' = 0$

By the result mentioned before, the algebraic mapping class group $\mathrm{Map}_{0,r,\mathbf{m}} = \mathrm{Aut}^+(\Gamma_{0,r,\mathbf{m}})/\mathrm{Inn}(\Gamma_{0,r,\mathbf{m}})$ is isomorphic to the geometric one $\mathrm{Map}_{0,[r]} = \pi_0 \mathrm{Diff}^+(\mathbb{P}^1 - \{p_1, \ldots, p_r\})$. The last one is in turn isomorphic to the Braid group $\mathbf{B}_r$ via the following theorem (see [13, Theorem 1.1]).

Theorem 1.4.8. *The mapping class group* $\mathrm{Map}_{0,[r]} = \pi_0 \mathrm{Diff}^+(\mathbb{P}^1 - \{p_1, \ldots, p_r\})$ *is isomorphic to the Braid group* $\mathbf{B}_r$ *on* r *strands, which can be presented as*

$$\mathbf{B}_r = \langle \sigma_1, \ldots, \sigma_{r-1} \,\big|\, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \rangle.$$

We can describe Hurwitz moves in this case.

Theorem 1.4.9. *Up to inner automorphism, the action of* $\mathrm{Map}_{0,[r]}$ *on the group* $\Gamma_{0,r,\mathbf{m}}$ *is given by*

$$\sigma_i : \begin{cases} \gamma_i \mapsto \gamma_{i+1}, & \\ \gamma_{i+1} \mapsto \gamma_{i+1}^{-1}\gamma_i\gamma_{i+1}, & \\ \gamma_j \mapsto \gamma_j, & \text{if } j \neq i,\, i+1. \end{cases} \tag{1.4.2}$$

Remark that, when the group G is abelian, the mapping class group $\mathrm{Map}_{0,[r]}$ acts as a permutation:

Corollary 1.4.10. *Let* G *be a finite abelian group and let* $\mathcal{V} = (g_1, \ldots, g_r)$ *be a generating vector of* G *with respect to* $\Gamma_{0,r,\mathbf{m}}$. *Then the Hurwitz moves coincide with the group of permutations of* g_i*'s.*

In this way Artin's standard generators σ_i's of $\mathbf{B}_r$ can be represented by the so-called half-twists. We give the following geometric description for half-twists [18].

Definition 1.4.11. *The half-twist* σ_j *is a diffeomorphism of* $\mathbb{P}^1 - \{p_1, \ldots, p_r\}$ *isotopic to the homeomorphism given by:*

- *A rotation of* 180 *degrees on the disk with center* $j+1/2$ *and radius* $1/2$*;*

- *on a circle with the same center and radius* $(2+t)/4$ *the map* σ_j *is the identity if* $t \geq 1$ *and a rotation of* $180(1-t)$ *degrees, if* $t \leq 1$.

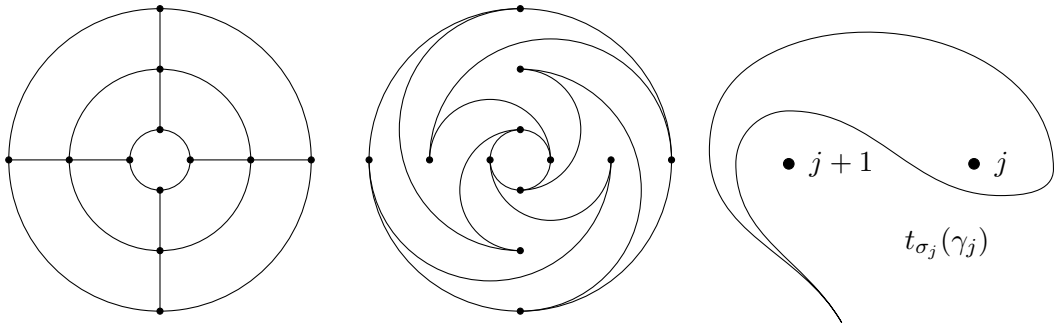

Figure 1.1: Half twist. For the original figure, see [73].

1.4.4 Hurwitz moves in case $g' \geq 1$

In case $g' \geq 1$ and no marked points the mapping class group $\mathrm{Map}_{g'}$ acts on $\mathcal{T}_{g'}$ via Dehn twists. A Dehn twist corresponds geometrically to the diffeomorphism of a truncated cylinder which is the identity on the boundary, a rotation by 180 degrees on the equator, and on each parallel at height t is a rotation by $t \cdot 360$ degrees (where $t \in [0,1]$).

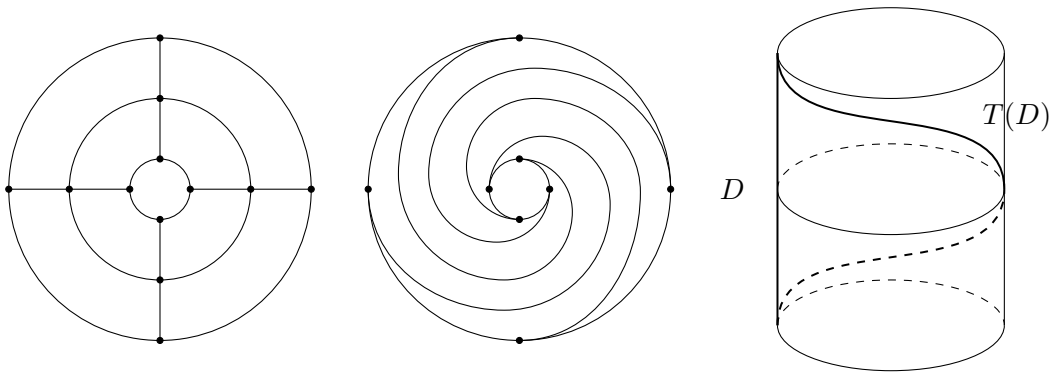

Figure 1.2: Dehn twist. On the right, the action of a Dehn twist T on the segment D (see [18],[73]).

One can define in the same way a Dehn twist with respect to any closed curve in C:

Definition 1.4.12. *Let C be an oriented Riemann surface. Then a positive Dehn twist t_α with respect to a simple closed curve α on C is an isotopy class of a diffeomorphism h of C which is equal to the identity outside a neighbourhood of α orientedly homeomorphic to to an annulus in the plan, while inside the annulus h rotates the inner boundary of the annulus by $360°$ to the right and damps the rotation down to the identity at the outer boundary.*

The following classical result is due to Dehn [36].

Theorem 1.4.13. *The mapping class group $\mathrm{Map}_{g'}$ is generated by Dehn twists.*

More precisely, Dehn twists generating the mapping class group are Dehn twists with respect to the curves represented in Figure 1.3.

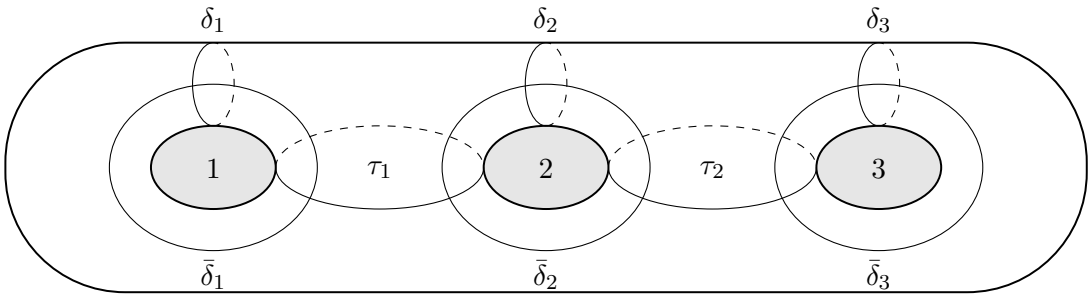

Figure 1.3

To study the more general situation, in which we consider the mapping class group with marked points, $\mathrm{Map}_{g',[r]}$, $r > 0$, we have to introduce a third type of twist, the ξ-twists, which link the holes with the marked points.

Let us describe a ξ-twist. Consider the annulus $A := \{z = \rho e^{i\theta} \in \mathbb{C} \big| 1 \leq \rho \leq 2$, and define $h : A \to A$ as follows:

$$h(\rho, \theta) = \begin{cases} (\rho, \theta - 4\pi(\rho - 1)), & 1 \leq \rho \leq 3/2; \\ (\rho, \theta - 4\pi(2 - \rho)), & 3/2 \leq \rho \leq 2. \end{cases} \tag{1.4.3}$$

Definition 1.4.14. *Let C be a Riemann surface, and α a simple closed curve on C. Let ι be a diffeomorphism between A and a tubular neighbour-*

hood of α. Then the ξ-twist t_α with respect to α is defined as $\iota \circ h \circ \iota^{-1}\Big|_{\iota(A)}$ extended to the whole C as the identity on $C - \iota(A)$.

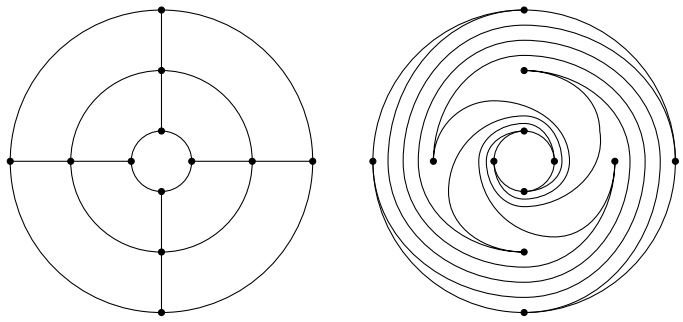

Figure 1.4: ξ twist. For the original figure, see [73].

We have the following result, due to Birman [12]:

Theorem 1.4.15. *Let $g(C') \neq 0$ and either $g(C') > 0$ or $r > 1$. Then the group $\mathrm{Map}_{g',[r]}$ is generated by the $3g' - 1$ Dehn twists with respect to the curves δ_j, $\bar{\delta}_j$ and $\tau_j m$ by the $2rg'$ ξ-twists with respect to the curves $\xi^l_{j,d}$ and the $r-1$ half twists about the points $p_1, \ldots, p_r$ in Figure 1.5.*

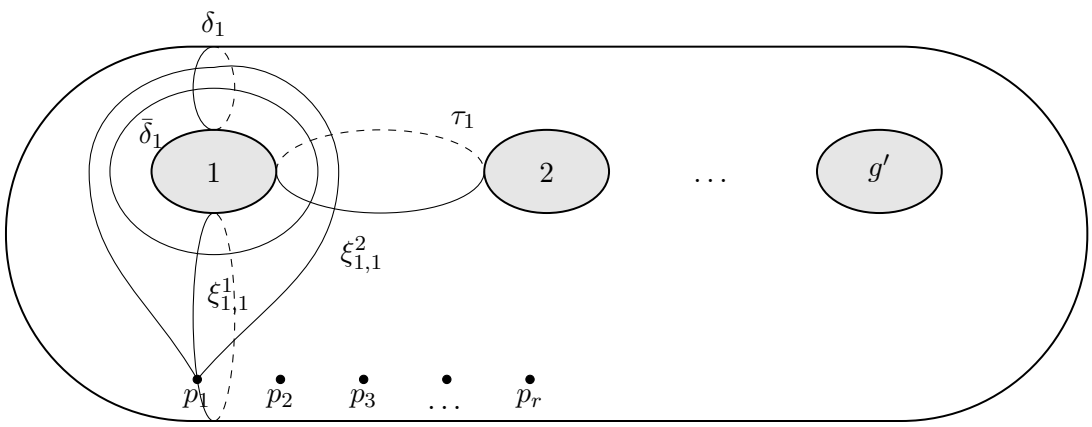

Figure 1.5

We have the following complete description of the action of the mapping class group $\mathrm{Map}_{g',[r]}$ on the group $\Gamma_{g',r,\mathbf{m}}$ (see [73] for the proof).

Theorem 1.4.16. *Let C' be a curve of genus g', $B = \{p_1, \dots, p_r\}$, and with $g' \neq 0$ and $g' > 1$ or $r > 1$. Up to inner automorphisms, the action of $\mathrm{Map}_{g',[r]}$ on $\Gamma_{g',r,\mathbf{m}}$ is induced by the following action on a geometric basis of $\pi_1(C' - B, p_0)$:*

$$t_{\delta_j} : \begin{cases} \alpha_j \mapsto \alpha_j \beta_j^{-1}, & \\ \alpha_i \mapsto \alpha_i, & \text{for all } i \neq j, \\ \beta_i \mapsto \beta_i, & \text{for all } i, \\ \gamma_i \mapsto \gamma_i, & \text{for all } i. \end{cases} \qquad t_{\bar{\delta}_j} : \begin{cases} \alpha_i \mapsto \alpha_i, & \text{for all } i \\ \beta_j \mapsto \beta_j \alpha_j, & \\ \beta_i \mapsto \beta_i, & \text{for all } i \neq j, \\ \gamma_i \mapsto \gamma_i, & \text{for all } i. \end{cases}$$

$$t_{\sigma_h} : \begin{cases} \alpha_i \mapsto \alpha_i, & \text{for all } i \\ \beta_i \mapsto \beta_i & \text{for all } i, \\ \gamma_{h+1} \mapsto \gamma_{h+1}^{-1} \gamma_h \gamma_{h+1}, & \\ \gamma_i \mapsto \gamma_i & \text{for all } i \neq h,\ h+1 \end{cases} \qquad t_{\tau_k} : \begin{cases} \alpha_k \mapsto \alpha_k \eta_k^{-1}, & \\ \beta_k \mapsto \beta_k^{\eta_k}, & \\ \alpha_{k+1} \mapsto \eta_k \alpha_k & \\ \alpha_i \mapsto \alpha_i, & \text{for all } i \neq k,\ k+1, \\ \beta_i \mapsto \beta_i & \text{for all } i \neq k, \\ \gamma_i \mapsto \gamma_i & \text{for all } i. \end{cases}$$

$$t_{\xi^1_{j,d}} : \begin{cases} \alpha_j \mapsto \chi_{j,d} \alpha_j, & \\ \alpha_i \mapsto \alpha_i, & \text{for all } i \neq j, \\ \beta_i \mapsto \beta_i & \text{for all } i, \\ \gamma_d \mapsto \gamma_d^{\epsilon_{j,d}}, & \\ \gamma_i \mapsto \gamma_i & \text{for all } j \neq d. \end{cases} \qquad t_{\xi^2_{j,d}} : \begin{cases} \alpha_i \mapsto \alpha_i, & \text{for all } i, \\ \beta_j \mapsto \alpha_j^{-1} \chi_{j,d} \alpha_j \beta_j, & \\ \beta_i \mapsto \beta_i, & \text{for all } i \neq j, \\ \gamma_d \mapsto \gamma_d^{\epsilon'_{j,d}}, & \\ \gamma_i \mapsto \gamma_i & \text{for all } j \neq d. \end{cases}$$

for $i \leq j \leq g'$, $1 \leq k \leq (g'-1)$, $1 \leq h \leq (r-1)$, and $1 \leq d \leq r$. Where

$$\begin{cases} \eta_k = \beta_k^{-1} \alpha_{k+1} \beta_{k+1} \alpha_{k+1}^{-1}, \\ \chi_{j,d} = (\prod_{k=1}^{j-1} [\alpha_k, \beta_k])^{-1} \gamma_d \prod_{k=1}^{j-1} [\alpha_k, \beta_k], \\ \epsilon_{j,d} = \gamma_d (\prod_{k=1}^{j} [\alpha_k, \beta_k]) \alpha_j \beta_j \alpha_j^{-1} (\prod_{k=1}^{j} [\alpha_k, \beta_k)^{-1}, \\ \epsilon'_{j,d} = \gamma_d (\prod_{k=1}^{j} [\alpha_k, \beta_k]) \alpha_j^{-1} (\prod_{k=1}^{j} [\alpha_k, \beta_k])^{-1}. \end{cases}$$

Since the notation of the previous theorem is a bit hard, we try to explain better how Hurwitz moves work with a simple example.

Example 1. Consider the family of Galois cover of an elliptic curve E via the Galois group $\mathbb{Z}/2\mathbb{Z}$ branched over two points. To determine whether

such a cover is realizable and define it properly, we have to check if there exists any admissible epimorphism

$$\theta : \Gamma_{1,2,\mathbf{m}} = \langle \gamma_1, \gamma_2, \alpha, \beta \big| \gamma_1^2 = \gamma_2^2 = \gamma_1 \cdot \gamma_2 \cdot [\alpha, \beta] = 1 \rangle \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}.$$

Since $\mathbb{Z}/2\mathbb{Z} \cong \langle z \rangle$ has only one element of order 2, this forces $\theta(\gamma_i)$ to be equal to z for $i = 1, 2$. Since G is abelian, every possible choice for $\theta(\alpha)$ and $\theta(\beta)$ satisfies the relation $\theta(\gamma_1) \cdot \theta(\gamma_2) \cdot [\theta(\alpha), \theta(\beta)] = 1$. There are four possible generating vectors:

$$\langle \theta(\alpha), \theta(\beta);\, \theta(\gamma_1), \theta(\gamma_2) \rangle = \begin{cases} \langle z, z;\, z, z \rangle \text{ or} \\ \langle z, 1;\, z, z \rangle \text{ or} \\ \langle 1, z;\, z, z \rangle \text{ or} \\ \langle 1, 1;\, z, z \rangle. \end{cases}$$

We claim that all of them lie in the same Hurwitz equivalence class. In fact, since the action of the mapping class group $\mathrm{Map}_{1,[2]}$ on $\Gamma_{1,2,\mathbf{m}}$ is generated (up to inner automorphism) by the seven moves described in Theorem 1.4.16 and $\mathbb{Z}/2\mathbb{Z}$ is abelian, the move induced by $t_{\xi^1_{1,1}}$ is given by $\theta(\alpha) \mapsto \theta(\gamma_1\alpha)$ and the identity on the other generators, while the move induced by $t_{\xi^2_{1,1}}$ is given by $\theta(\beta) \mapsto \theta(\gamma_1\beta)$ and the identity on the other generators. This moves yield at once that systems of generators $\langle z, z;\, z, z \rangle$, $\langle 1, z;\, z, z \rangle$, $\langle z, 1;\, z, z \rangle$, $\langle 1, 1;\, z, z \rangle$ are Hurwitz equivalent.

1.5 Representation of G on $H^0(C, K_C)$

In this section we will study the action of the group G on the space of holomorphic 1-forms $H^0(C, K_C)$. We will use some basic tools from representation theory in order to count the dimension of the G-invariant part of $S^2H^0(C, K_C)$, which will be denoted by $(S^2H^0(C, K_C))^G$. As we will see in Section 1.6, the dimension $h^0(S^2(C_t, K_{C_t}))^G$ of the space of invariant forms over the family $\{C_t\}$ gives informations about the variety in $\mathcal{A}_g$ associated to the family. We start with some preliminary definitions and results, mainly following Serre [79].

Let V be a finite vector space over $\mathbb{C}$ and fix a basis of it. Consider the space $GL(V)$ of automorphisms of V, corresponding to the space of matrices $n \times n$ with non-vanishing determinant. We have the following:

Definition 1.5.1. *Let G be a finite group. A* representation *of G over V is a homomorphism*

$$\rho : G \to GL(V). \tag{1.5.1}$$

When ρ is given, we call V representation space *relative to the group G and we call the dimension of V* degree *of the representation.*

Consider now two vector spaces V_1 and V_2, and call $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ two representations over V_1 and V_2 respectively. These representations are said to be isomorphic if there exists an isomorphism $\tau : V \to V'$ which "transforms" ρ_1 into ρ_2. More precisely:

Definition 1.5.2. *Two representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are isomorphic if there exists an isomorphism of vector spaces $\tau : V \to V'$ such that*

$$\tau \circ \rho_1(g) = \rho_2(g) \circ \tau, \tag{1.5.2}$$

for all $g \in G$.

Notice that thinking of $GL(V_i)$ as a space of matrices ($i = 1, 2$), relation (1.5.2) is equivalent to say that the matrices involved are simultaneously similar.

Consider a subspace $W < V$. We call W *stable* under the action of G if $\rho(g)(w) \in W$ for every $w \in W$. In this case $\rho^W(g) := \rho(g)\big|_W \in GL(W)$ for every $g \in G$, meaning that $\rho^W : G \to GL(W)$ is a subrepresentation of ρ. We have the following:

Definition 1.5.3. *A representation $\rho : G \to GL(V)$ is* irreducible *if there does not exist any subspace $W < V$ which is stable under the action of G (i.e. if ρ does not admit any proper subrepresentation).*

We remark that every representation of degree 1 is irreducible. Moreover one can prove that every representation can always be decomposed as a (non unique) direct sum of irreducible representations.

Definition 1.5.4. *We call* character *of a representation ρ the map*

$$\begin{aligned} \chi_\rho : G &\to \mathbb{C}, \\ g &\mapsto \mathrm{Tr}(\rho(g)). \end{aligned} \tag{1.5.3}$$

A character χ_ρ is said to be irreducible *if ρ is irreducible.*

Notice that characters are function on G constant over conjugacy classes.

The following property holds:

Lemma 1.5.5. *Let $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ be two representations of the group G, whose characters are χ_{ρ_1} and χ_{ρ_2} respectively. Then:*

$$\begin{aligned} &\rho_1 \oplus \rho_2 : G \to GL(V_1 \oplus V_2) \quad \textit{has character } \chi_{\rho_1} + \chi_{\rho_2}, \\ &\rho_1 \otimes \rho_2 : G \to GL(V_1 \otimes V_2) \quad \textit{has character } \chi_{\rho_1} \cdot \chi_{\rho_2}. \end{aligned} \tag{1.5.4}$$

Since it will be useful in the following, we give a formula for the computation of the character $\chi_{S^2\rho}$ of the symmetric square S^2V induced by the representation on V.

Consider a representation $\rho : G \to GL(V)$. One can define the action induced by ρ on the tensor product $V \otimes V$ as

$$\begin{aligned} \rho \otimes \rho : G &\to GL(V \otimes V), \\ g &\mapsto \rho(g) \otimes \rho(g). \end{aligned} \tag{1.5.5}$$

It is easy to check that the subspace $S^2V < V \otimes V$ (as well as its complementary $\bigwedge^2 V < V \otimes V$) is stable under the action of $\rho \otimes \rho$. Since ρ^{S^2V} is a well defined subrepresentation of ρ, we can consider its character $\chi_{S^2\rho} := \chi_{\rho^{S^2V}}$. We have the following:

Corollary 1.5.6. *Let $\rho : G \to GL(V)$. Then*

$$\chi_{S^2\rho}(g) = \frac{1}{2}\big(\chi_\rho(g)^2 + \chi_\rho(g^2)\big). \tag{1.5.6}$$

It is possible to define an Hermitian scalar product on the space of class functions over G (i.e. functions constant over conjugacy classes of G, see [43]). In particular this scalar product is well defined over characters.

Theorem 1.5.7. *Consider $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ representations of the group G of order n, and call χ_1 and χ_2, respectively, their characters. Then*

$$\big(\chi_1 \big| \chi_2\big) := \frac{1}{n} \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)} \tag{1.5.7}$$

is a well defined Hermitian scalar product on the space of characters.

The importance of this scalar product is that irreducible characters form an orthonormal basis for the space of all characters. This is straightforward from the next theorem (see [79, Theorem 2.3.3] for the proof).

Theorem 1.5.8. *Keep notations as above. Then:*

1. *if χ_1 is the character of an irreducible representation, then $\big(\chi_1|\chi_1\big) = 1$ (i.e. χ_1 has norm 1);*
2. *if χ_1 and χ_2 are characters of two non-isomorphic irreducible representations than $\big(\chi_1|\chi_2\big) = 0$ (i.e. χ_1 and χ_2 are orthogonal).*

Remark 1.5.9. In a more general setting, it is possible to prove that the set of irreducible characters of a finite group G is an orthonormal basis of the vector space of class functions $CF(G)$ with respect to the Hermitian product (1.5.7) (see [79]).

Theorem 1.5.10. *Let $\rho : G \to GL(V)$ be a linear representation of G with character χ, and suppose that V decomposes as a direct sum of irreducible representations as:*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_r.$$

Take an irreducible representation $\rho' : G \to GL(W)$, and consider the associated character χ'. Then the scalar $\big(\chi|\chi'\big)$ gives the number of W_i isomorphic to W.

Keep the notation of the previous theorem and call χ_i the character of W_i $(i = 1, \ldots, r)$. From Lemma 1.5.5 it follows that $\chi = \chi_1 + \chi_2 + \cdots + \chi_r$. Then $\big(\chi_i|\chi'\big)$ is 1 if ρ' is isomorphic to ρ^{W_i} and 0 otherwise. The result follows:

Corollary 1.5.11. *The number of W_i isomorphic to W does not depend on the chosen decomposition.*

Although the decomposition of a representation in irreducible components is not unique, its factors are determined up to isomorphisms. Theorems above show that every representation is uniquely determined by its character:

Corollary 1.5.12. *Two representations with the same character are isomorphic.*

Summarizing, every character χ_φ admits a decomposition

$$\chi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} \big(\chi_\varphi | \chi\big) \cdot \chi. \tag{1.5.8}$$

We apply now representation theory to our case of interest. Let C be a compact Riemann surface of genus $g \geq 2$ and $\mathrm{Aut}(C)$ be the group of automorphisms of C. We want to study the representation of a group $G < \mathrm{Aut}(C)$ on the space of holormorphic 1-forms $H^0(C, K_C)$.

Consider a Galois cover $\pi : C \to C' = C/G$ with Galois group G and call $\varphi : G \to \mathrm{Aut}(C)$ the corresponding group action. Let ψ be the representation of G associated to φ via pullback of holomorphic 1-forms in the following way:

$$\begin{aligned} \psi : G &\to H^0(C, K_C), \\ g &\mapsto [\omega \mapsto \varphi(g^{-1})^*\omega]. \end{aligned} \tag{1.5.9}$$

Notice that it is necessary to put g^{-1} instead of g in the definition to make ψ a homomorphism.

The character χ_ψ, according to formula (1.5.8), decomposes as a linear combination of irreducible characters. In this setting, the Chevalley-Weil formula provides a way to compute the integral coefficients $\big(\chi_\psi | \chi\big)$. The result is a consequence of the following theorem (see e.g. [42, Thm. V.2.9, p. 264] for the proof.)

Theorem 1.5.13 (Eichler Trace Formula)**.** *Consider a compact Riemann surface C of genus $g \geq 2$. Take $G < \mathrm{Aut}(C)$, and consider $g \in G$ of order $m > 1$. Call $\mathsf{Fix}(g)$ the set of points fixed by g. Then:*

$$\chi_\rho(g) = \mathrm{Tr}(\rho(g)) = 1 + \sum_{P \in \mathsf{Fix}(g)} \frac{\overline{\zeta_P(g)}}{1 - \overline{\zeta_P(g)}}, \tag{1.5.10}$$

We need a technical lemma which allows us to prove a different version of Eichler trace formula, that will be the content of Corollary 1.5.15. In order to state it we need some preliminary discussion.

Recall that near the points fixed by G there is a local coordinate such that generators of the stabilizer group act as a multiplication by a primitive root of unity. More precisely if P is critical and g is a generator of $\mathrm{stab}(P)$

of order m, then the differential of the action of g in P acts on T_PC by multiplication by an m-th root of unity $\zeta_P(g)$ (see Theorem 1.1.4).

Set $\zeta_m := e^{2\pi i/m}$, $I(m) := \{\nu \in \mathbb{Z} : 1 \leq \nu < m, \gcd(\nu, m) = 1\}$, and for $\nu \in I(m)$ consider $\mathsf{Fix}_\nu(g) := \{P \in C : g \cdot P = P,\ \zeta_P(g) = \zeta_m^{-\nu}\}$.

We have the following lemma (the result is a consequence of [51, Theorem 7], see also [16, Lemma 11.5]).

Lemma 1.5.14. *If $G \subseteq \mathrm{Aut}(C)$ and $g \in G$ has order m, then:*

$$|\mathsf{Fix}_\nu(g)| = |C_G(g)| \cdot \sum_{\substack{1 \leq i \leq r, \\ m | m_i, \\ g \sim_G x_i^{m_i \nu/m}}} \frac{1}{m_i}, \tag{1.5.11}$$

where $C_G(g)$ denotes the centralizer of g in G and $\sim_G$ denotes the equivalence relation given by conjugation in G.

Using Lemma 1.5.14 we obtain a second version of Eichler trace formula:

Corollary 1.5.15. *Keep notations as above. Then:*

$$\chi_\rho(g) = 1 + |C_G(g)| \sum_{\nu \in I(m)} \left\{ \sum_{\substack{1 \leq i \leq r, \\ m | m_i, \\ g \sim_G x_i^{m_i \nu/m}}} \frac{1}{m_i} \right\} \frac{\zeta_m^\nu}{1 - \zeta_m^\nu}. \tag{1.5.12}$$

The Chevalley-Weil formula is an important consequence of Eichler trace formula. It gives the multiplicity of a given irreducible representation of G in $H^0(C, K_C)$, i.e. the integral coefficients in expression (1.5.8). More precisely, for $\chi \in \mathrm{Irr}(G)$, let $\sigma := \sigma_\chi$ be the corresponding irreducible representation and call d_χ its degree. Moreover, let x_i be an element of order m_i in G that represents the local monodromy of the cover $C \to C'$ at the branch point t_i, and call $E_{i,\alpha}$ the number of eigenvalues of $\sigma(x_i)$ that are equal to $\zeta_{m_i}^\alpha$, where $\zeta_{m_i} = e^{2\pi i/m_i}$ as usual.

Theorem 1.5.16 (Chevalley-Weil formula [22])**.** *Consider a Galois cover $\pi : C \to C'$ with Galois group G and r branches, and call ρ the representation induced on the $H^0(C, K_C)$. Let $m = (m_1, \ldots, m_r)$ be the monodromy and take $E_{i,\alpha}$ and d_χ as above. Then the following holds:*

$$\big(\chi_\rho | \chi\big) = d_\chi(g(C') - 1) + \sum_{i=1}^{r} \sum_{\alpha=1}^{m_i - 1} \frac{\alpha \cdot E_{i,\alpha}}{m_i} + \big(\chi | \chi_{triv}\big), \tag{1.5.13}$$

where χ_{triv} is the character of the trivial representation $\rho_{triv}(g) = 1$, $\forall\, g \in G$.

Proof. A proof of this result can be found in [42]. Se also [43], [66]. □

To conclude this section, we state a formula to compute the multiplicity N of the trivial representation inside $S^2\rho$, that is the number of invariant elements in $S^2H^0(C, K_C)$.

Corollary 1.5.17. *Keep the same notation as before. Then:*

$$N = (\chi_{S^2\rho}, 1) = \frac{1}{|G|}\sum_{x\in G}\chi_{S^2\rho}(x) = \frac{1}{2|G|}\sum_{x\in G}\big(\chi_\rho(x^2) + \chi_\rho(x)^2\big). \quad (1.5.14)$$

Proof. The result follows from Theorem 1.5.8 and Corollary 1.5.6. □

Remark that the representation ρ only depends on the datum $(\mathbf{m}, G, \theta)$: the theory applies identically for a family of curves with the same datum. This implies that also N depends on the datum only.

1.6 Special subvarieties

In this section we discuss the notion of *special subvarieties* or *Shimura subvarieties* of $\mathcal{A}_g$. Since the abstract formalism of Shimura subvarieties is rather cumbersome, we will give a brief introduction functional to our purposes. In particular, despite their original definition involves Hodge classes, we will define Shimura subvarieties via the characterization relating them to totally geodesic submanifolds of $\mathcal{A}_g$. Moreover we will focus on a concrete class of Shimura subvarieties, *PEL Shimura subvarieties*, which is the only class of Shimura we will actually use.

To hint, special subvarieties are the Hodge loci of certain natural variations of Hodge structures. They are defined by the existence of certain Hodge classes; in particular they are the maximal closed irreducible subvarieties of $\mathcal{A}_g$ on which certain given classes are Hodge classes. In the following we will give some sketch about this approach, mainly referring to the survey of Moonen and Oort for details on the involved Hodge theory as well as for a deeper discussion on Shimura varieties [61].

Fix a rank $2g$ lattice Λ and an alternating form $Q : \Lambda \times \Lambda \to \mathbb{Z}$ of type $(1, \dots, 1)$. For F a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$, set $\Lambda_F := \Lambda \otimes_{\mathbb{Z}} F$. The Siegel

upper half-space is defined as follows [52, Thm. 7.4]:

$$\mathfrak{H}_g := \{J \in \mathrm{GL}(\Lambda_{\mathbb{R}}) : J^2 = -I, J^*Q = Q, Q(x, Jx) > 0, \ \forall x \neq 0\}. \quad (1.6.1)$$

The group $\mathrm{Sp}(\Lambda, Q)$ acts on $\mathfrak{H}_g$ by conjugation and $\mathcal{A}_g = \mathrm{Sp}(\Lambda, Q)\backslash\mathfrak{H}_g$. This space has the structure of a smooth algebraic stack and also of a complex analytic orbifold. Denote by A_J the quotient $\Lambda_{\mathbb{R}}/\Lambda$ endowed with the complex structure J and the polarization Q. On $\mathfrak{H}_g$ there is a natural variation of rational Hodge structure, that descends to a variation of Hodge structure on $\mathcal{A}_g$ over $\mathbb{Q}$ (in the orbifold sense), whose fiber over A is $H^1(A, \mathbb{Q})$. The Hodge loci for this variation of Hodge structure are called *special subvarieties* or *Shimura subvarieties.*

The zero dimensional special subvarieties are precisely the *CM (Complex Multiplication) points*, corresponding to abelian varieties where $\mathrm{End}(A)_{\mathbb{Q}} := \mathrm{End}(A) \otimes \mathbb{Q}$ contains a commutative, semi-simple subfield F such that $[F : \mathbb{Q}] = 2\dim(A) = 2g$. In case A simple abelian variety, this is the same to ask that $\mathrm{End}(A)_{\mathbb{Q}}$ is a quadratic extension of a totally real field. For example, in case $g = 1$, abelian varieties whose lattices are $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$, where ω is the 3-th root of unity, have complex multiplication (actually, there are more, see e.g. [37, Lecture 3]). The arithmetical properties of CM points are fairy well known. Since they are behond our purposes, we only mention the following useful result (see e.g. [61]):

Proposition 1.6.1. *Special subvarieties of $\mathcal{A}_g$ contain a subset of CM points which is dense for the Zariski topology.*

CM points play a key role in our study: they allows us to give a characterization of special subvarieties of $\mathcal{A}_g$ not involving the Shimura machinery. In particular Moonen found a correspondence between Shimura subvarieties of $\mathcal{A}_g$ and totally geodesic submanifolds of $\mathcal{A}_g$ admitting a CM point [64] (see [59] for a more general result). The result is the following:

Theorem 1.6.2. *An algebraic totally geodesic subvariety of $\mathcal{A}_g$ is special if and only if it is totally geodesic and contains a CM point.*

As stated in the very beginning of this section, in the following we will use this characterization as a definition of Shimura subvarieties of $\mathcal{A}_g$. In particular, the reader should think about Shimura subvarieties of $\mathcal{A}_g$ as totally geodesic submanifolds with an extra arithmetic condition. This makes the

connection between totally geodesic submanifolds of $\mathcal{A}_g$ and Shimura subvarieties of $\mathcal{A}_g$ clear. In this setting, Conjecture 0.0.1, which motivates the study done in this dissertation, is a stronger version of the next conjecture, originally stated by Oort.

Conjecture 1.6.1 (Oort [68])**.** *For large g, there does not exist a special subvariety $Z \subset \mathcal{A}_g$ with $\dim(Z) \geq 1$ such that $Z \subseteq T_g$ and $Z \cap T_g^0$ is nonempty.*

As pointed out in the introduction, one reason for this expectation coming from differential geometry is that a special (or Shimura) subvariety of $\mathcal{A}_g$ is totally geodesic with respect to the (orbifold) metric of $\mathcal{A}_g$ induced by the symmetric metric on the Siegel space $\mathfrak{H}_g$ of which $\mathcal{A}_g$ is a quotient by $Sp(2g, \mathbb{Z})$. One expects the Torelli locus to be very curved and a way of expressing this is to say that it should not contain totally geodesic subvarieties. Moreover in view of Theorem 1.6.2, the expectation formulated by Oort is both of geometric and arithmetic nature. See [61, §4] for more details.

In this thesis we will deal with a single class of special subvarieties of $\mathcal{A}_g$, that are the special subvarieties of PEL type. The name comes from the fact that PEL special subvarieties have a modular interpretation in terms of abelian varieties with a *p*olarization, given *e*ndomorphisms and a *l*evel structure. Given $J \in \mathfrak{H}_g$, set

$$\mathrm{End}_{\mathbb{Q}}(A_J) := \{f \in \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) : Jf = fJ\}.$$

We define PEL Shimura subvarieties as follows (see [61, §3.9] for details).

Definition 1.6.3. *Fix a point $J_0 \in \mathfrak{H}_g$ and set $D := \mathrm{End}_{\mathbb{Q}}(A_{J_0})$. The* PEL type *special subvariety $\mathsf{Z}(D)$ is defined as the image in $\mathcal{A}_g$ of the connected component of the set $\{J \in \mathfrak{H}_g : D \subseteq \mathrm{End}_{\mathbb{Q}}(A_J)\}$ that contains J_0.*

We conclude this section giving a sufficient condition for a family of Galois covers to yield a Shimura subvariety of $\mathcal{A}_g$ of PEL type. We recall some preliminary result proven in [44, Section 3], useful to discuss Theorem 1.6.8.

Proposition 1.6.4. *Let $G \subseteq \mathrm{Sp}(\Lambda, E)$ be a finite subgroup. Denote by $\mathfrak{H}_g^G$ the set of points of $\mathfrak{H}_g$ that are fixed by G. Then $\mathfrak{H}_g^G$ is a connected complex submanifold of $\mathfrak{H}_g$.*

Set $D_G := \{f \in \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) : Jf = fJ,\ \forall J \in \mathfrak{H}_g^G\}$. Then:

Lemma 1.6.5. *If $J \in \mathfrak{H}_g^G$, then $D_G \subseteq \mathrm{End}_{\mathbb{Q}}(A_J)$ and the equality holds for J in a dense subset of $\mathfrak{H}_g^G$.*

Proposition 1.6.6. *The image of $\mathfrak{H}_g^G$ in $\mathcal{A}_g$ coincides with the PEL subvariety $\mathsf{Z}(D_G)$.*

Lemma 1.6.7. *If $J \in \mathfrak{H}_g^G$, then $\dim \mathfrak{H}_g^G = \dim \mathsf{Z}(D_G) = \dim(S^2\Lambda_{\mathbb{R}})^G$ where $\Lambda_{\mathbb{R}}$ is endowed with the complex structure J.*

Recall that $N = \dim \left(S^2 H^0(C, K_C)\right)^G$ and that $\mathsf{Z}(\mathbf{m}, G, \theta)$ is defined in Section 1.3.

Theorem 1.6.8 (see Theorem 3.9 in [44])**.** *Fix a datum $(\mathbf{m}, G, \theta)$ and assume that*

$$N = 3g' - 3 + r. \tag{$*$}$$

Then $\mathsf{Z}(\mathbf{m}, G, \theta)$ is a special subvariety of PEL type of $\mathcal{A}_g$ that is contained in T_g and such that $\mathsf{Z}(\mathbf{m}, G, \theta) \cap \mathsf{T}_g^0 \neq \emptyset$.

Proof. Let $\mathcal{C} \to \mathcal{T}_g^G$ be the universal family as in Section 1.3. For any $t \in \mathcal{T}_g^G$, G acts holomorphically on C_t, so it maps injectively into $\mathrm{Sp}(\Lambda, Q)$, where $\Lambda = H_1(C_t, \mathbb{Z})$ and Q is the intersection form. Denote by G' the image of G in $\mathrm{Sp}(\Lambda, Q)$. It does not depend on t since it is purely topological. Recall that the Siegel upper half-space $\mathfrak{H}_g$ parametrizes complex structures on the real torus $\Lambda_{\mathbb{R}}/\Lambda = H_1(C_t, \mathbb{R})/H_1(C_t, \mathbb{Z})$ which are compatible with the polarization Q. The period map associates to the curve C_t the complex structure J_t on $\Lambda_{\mathbb{R}}$ obtained from the splitting $H^1(C_t, \mathbb{C}) = H^{1,0}(C_t) \oplus H^{0,1}(C_t)$ and the isomorphism $H_1(C_t, \mathbb{R})^*_{\mathbb{C}} = H^1(C_t, \mathbb{C})$. The complex structure J_t is invariant by G', since the group G acts holomorphically on C_t. This shows that $J_t \in \mathfrak{H}_g^{G'}$, so the Jacobian $j(C_t)$ lies in $\mathsf{Z}(D_{G'})$. This shows that $\mathsf{Z}(\mathbf{m}, G, \theta) \subseteq \mathsf{Z}(D_{G'})$. Since $\mathsf{Z}(D_{G'})$ is irreducible (see e.g. Proposition 1.6.4), to conclude the proof it is enough to check that they have the same dimension. The dimension of $\mathsf{Z}(\mathbf{m}, G, \theta)$ is $3g' - 3 + r$, see Section 1.3. By Lemma 1.6.7, if $J \in \mathfrak{H}_g^{G'}$, then $\dim \mathsf{Z}(D_{G'}) = \dim \mathfrak{H}_g^{G'} = \dim(S^2\Lambda_{\mathbb{R}})^{G'}$, where $\Lambda_{\mathbb{R}}$ is endowed with the complex structure J. If J corresponds to the Jacobian of a curve C in the family, then $(S^2\Lambda_{\mathbb{R}})^{G'}$ is isomorphic to the dual of $(S^2 H^0(C, K_C))^G$. Thus $\dim \mathsf{Z}(D_{G'}) = N$ and $(*)$ yields the result. $\square$

Condition $(*)$ has been originally stated by Colombo, Frediani and Ghigi in [30, Proposition 5.4] as a sufficient condition that makes the variety

$Z(\mathbf{m}, G, \theta)$ totally geodesic. We remark that condition $(*)$ is an equality condition on the dimensions of $S^2H^0(K_{C_t})^G$ and $H^0(2K_{C_t})^G$, implying that the kernel N^{*G} is empty. Moreover, since the multiplication map m is G-equivariant, any G-invariant element in N^* lies in $I_2(K_{C_t})$: condition $(*)$ is equivalent to ask

$$I_2(K_{C_t})^G = \emptyset. \tag{1.6.2}$$

Frediani, Ghigi and Penegini used condition $(*)$ for a systematic search of special subvarieties of the form $\mathsf{Z}(\mathbf{m}, G, \theta)$, obtained as Galois cover of the projective line (see [44]). Using the computer algebra program `MAGMA` [1], they determined all the families $\mathsf{Z}(\mathbf{m}, G, \theta)$ with genus $g \leq 9$, and computed the number N checking which families satisfies the condition $(*)$. The result they obtained is the following:

Theorem 1.6.9. *For genus $g \leq 9$ there are exactly 40 data $(\mathbf{m}, G, \theta)$ such that $N = r - 3 > 0$. For these 40 data the image $Z(\mathbf{m}, G, \theta)$ is a special subvariety of $\mathcal{A}_g$ of positive dimension, which is contained in T_g and intersects T_g^0. Among these data there are 20 cyclic ones and 7 abelian non-cyclic ones. The remaining 13 have non-abelian Galois group. All these data occur in genus $g \leq 7$.*

In the following section we will generalize this result, studying Galois covers of curves of higher genus g'. We will find new examples when $g' = 1$, and we will also prove that if $g' \geq 1$, and the family satisfies $(*)$, then $g \leq 6g' + 1$. The last condition immediately implies that if $g' = 1$ there are no examples satisfying condition $(*)$ for $g \geq 8$.

1.7 Examples of special subvarieties in the Torelli locus

This is the very central part of this chapter: here we explicitly construct examples of Shimura subvarieties contained in the Torelli locus from families of Galois covers of elliptic curves.

More precisely, we constructed all families of curves (up to genus 7) covering an elliptic curve E, using systematically Riemann's existence theorem (see Section 1.2), i.e. we found all suitable data giving rise to Galois covers of E. Then, with Eichler trace formula (1.5.10) and Corollary (1.5.17), we

checked whether some of these families satisfy the sufficient condition $(*)$, hence give rise to special subvarieties of $\mathcal{A}_g$. Finally we controlled if the obtained Shimura subvarieties are actually new with considerations on groups involved, dimensions of the families and monodromies: we will show that two of them yield new Shimura subvarieties of $\mathcal{A}_g$ (namely, family (2) and family (6)), while the other examples arise from certain Shimura subvarieties of $\mathcal{A}_g$ already obtained as families of Galois covers of $\mathbb{P}^1$ in [44].

For all families ($g \leq 7$) of Galois covers of E satisfying $(*)$ we now give a presentation of the Galois group G and an explicit description of a representative of an epimorphism

$$\theta : \Gamma_{1,r} = \langle \alpha, \beta, \gamma_1, ..., \gamma_r \mid \gamma_1 ... \gamma_r \alpha\beta\alpha^{-1}\beta^{-1} = 1 \rangle \twoheadrightarrow G.$$

Genus 2

(1) $G = \mathbb{Z}/2\mathbb{Z} = \langle z \mid z^2 = 1 \rangle$.
$\mathbf{m} = (2, 2)$ $\theta : \Gamma_{1,2} \to \mathbb{Z}/2\mathbb{Z}$,
$\theta(\gamma_1) = \theta(\gamma_2) = z$, $\theta(\alpha) = \theta(\beta) = 1$.

Genus 3

(2) $G = \mathbb{Z}/2\mathbb{Z} = \langle z \mid z^2 = 1 \rangle$.
$\mathbf{m} = (2, 2, 2, 2)$ $\theta : \Gamma_{1,4} \to \mathbb{Z}/2\mathbb{Z}$,
$\theta(\gamma_i) = z$, $\forall i = 1, ..., 4$, $\theta(\alpha) = \theta(\beta) = 1$.

(3) $G = \mathbb{Z}/3\mathbb{Z} = \langle z \mid z^3 = 1 \rangle$.
$\mathbf{m} = (3, 3)$ $\theta : \Gamma_{1,2} \to \mathbb{Z}/3\mathbb{Z}$,
$\theta(\gamma_1) = z, \theta(\gamma_2) = z^2$, $\theta(\alpha) = \theta(\beta) = 1$.

(4) $G = \mathbb{Z}/4\mathbb{Z} = \langle z \mid z^4 = 1 \rangle$.
$\mathbf{m} = (2, 2)$ $\theta : \Gamma_{1,2} \to \mathbb{Z}/4\mathbb{Z}$,
$\theta(\gamma_1) = \theta(\gamma_2) = z^2$, $\theta(\alpha) = \theta(\beta) = 1$.

(5) $G = Q_8 = \langle g_1, g_2, g_3 : g_1^2 = g_2^2 = g_3, g_3^2 = 1, g_1^{-1} g_2 g_1 = g_2 g_3 \rangle$
$\mathbf{m} = (2)$ $\theta : \Gamma_{1,1} \to Q_8$,
$\theta(\gamma_1) = g_3$, $\theta(\alpha) = g_2$, $\theta(\beta) = g_1$.

Genus 4

(6) $G = \mathbb{Z}/3\mathbb{Z} = \langle z \mid z^3 = 1 \rangle$.
$\mathbf{m} = (3, 3, 3)$ $\theta : \Gamma_{1,3} \to \mathbb{Z}/3\mathbb{Z}$,
$\theta(\gamma_i) = z$, $\forall i = 1, 2, 3$, $\theta(\alpha) = \theta(\beta) = 1$.

1.7.1 Independence from the epimorphism θ

First of all we will show that in each of the above cases, once we fix the group G and the ramification $\mathbf{m}$, all the possible data $(\mathbf{m}, G, \theta)$ belong to the same Hurwitz equivalence class and hence give rise to the same subvariety of $\mathcal{A}_g$. Let us prove this for all the cases, giving the details only for some cases, being the others very similar.

Case (1)

This case is exactly the cover studied in Example 1, Section 1.4. For convenience of the reader we report here all computations. The group $\mathbb{Z}/2\mathbb{Z} \cong \langle z \rangle$ has only one element of order 2. This forces $\theta(\gamma_i)$ to be equal to z for $i = 1, 2$. Recall that the action of the mapping class group $\mathrm{Map}_{1,[2]}$ on $\Gamma_{1,2}$ is generated (up to inner automorphism) by the seven moves described in Theorem 1.4.16. In particular, since $\mathbb{Z}/2\mathbb{Z}$ is abelian, the move induced by $t_{\xi^1_{1,1}}$ is given by $\theta(\alpha) \mapsto \theta(\gamma_1 \alpha)$ and the identity on the other generators, while the move induced by $t_{\xi^2_{1,1}}$ is given by $\theta(\beta) \mapsto \theta(\gamma_1 \beta)$ and the identity on the other generators. These moves yields at once that systems of generators $\langle \theta(\alpha), \theta(\beta);\, \theta(\gamma_1), \theta(\gamma_2) \rangle = \langle z, z;\, z, z \rangle$, $\langle 1, 1;\, z, z \rangle$, $\langle 1, z;\, z, z \rangle$ and $\langle z, 1;\, z, z \rangle$ are Hurwitz equivalent.

Case (2)

The proof is the same as the one for case (1) with obvious changes.

Case (3)

Since $\mathbb{Z}/3\mathbb{Z}$ is abelian its commutator subgroup is trivial, thus up to automorphisms one can choose $\theta(\gamma_1) = z$ and $\theta(\gamma_2) = z^2$. Then we proceed as in case (1) with obvious changes.

Case (4)

The proof is the same as the one for case (1) with obvious changes.

Case (5)

Since there is only one element of order 2 in Q_8, $\theta(\gamma_1) = g_3$. Up to simultaneous conjugation, the image of the pair (α, β) by θ is one of the following (g_1, g_2), (g_2, g_1), (g_1, g_1g_2), (g_1g_2, g_1), (g_1g_2, g_2) and (g_2, g_1g_2). Using only the moves t_{δ_1} and $t_{\tilde{\delta}_1}$, described above, and the automorphisms of Q_8 we see that all the pairs are equivalent to (g_2, g_1). Therefore all the systems of generators are Hurwitz equivalent to $\langle g_2, g_1; g_3\rangle$. Notice that this proof can be found also in [72, Proposition 5.9]. Indeed, in that article this very cover is used to construct a new surface of general type with $p_g = q = 2$.

Case (6)

This case is similar to case (3). In fact $\theta(\gamma_i)$ can be either z or z^3. The only possibility to get $\theta(\gamma_1)\theta(\gamma_2)\theta(\gamma_3) = 1$ are either $\theta(\gamma_i) = z$ for all i, or $\theta(\gamma_i) = z^2$ for all i. These choices are equivalent via automorphisms of G. Moreover, with the same idea behind case (1), we can conclude that every choice of $\theta(\alpha)$ and $\theta(\beta)$ is equivalent.

1.7.2 Sufficient condition $(*)$ is satisfied

We will now show that the families listed above do in fact verify condition $(*)$ and hence yield special subvarieties of $\mathcal{A}_g$.

Case (1)

Clearly θ is an epimorphism. By Eichler trace formula (1.5.10) we find $\chi_\rho(z) = 0$, and using Corollary 1.5.17 we obtain $N = 2$, which coincides with the number of critical values, and so with the dimension of the family, therefore our family is special.

Case (2)

Clearly θ is an epimorphism. Now we want to show that the sufficient condition $(*)$ is satisfied, so we compute the number N using Corollary 1.5.17. Eichler trace formula (1.5.10) immediately yields $\chi_\rho(z) = -1$, and since $\chi_\rho(1) = g = 3$ we have $N = 4$, which coincides with the number of critical values, and so with the dimension of the family. This proves that our family of Galois covers is special.

Case (3)

It is clear that θ is an epimorphism. Using Eichler trace formula (1.5.10) we immediately obtain $\chi_\rho(z) = \chi_\rho(z^2) = 0$ and by Corollary 1.5.17 we obtain $N = 2$, which coincides with the number of critical values, and so with the dimension of the family, therefore the family is special.

Case (4)

It is clear that θ is an epimorphism. Using Eichler trace formula (1.5.10) we obtain $\chi_\rho(z) = \chi_\rho(z^3) = 1$, $\chi_\rho(z^2) = -1$ and by Corollary 1.5.17 we obtain $N = 2$, which coincides with the number of critical values, and so with the dimension of the family. The family is special.

Case (5)

One easily checks that θ is an epimorphism. Using Eichler trace formula (1.5.10) we find that the trace of every non zero element different from g_3 is equal to 1, and that $\chi_\rho(g_3) = -1$. By Corollary 1.5.17 we obtain $N = 1$,

which coincides with the number of critical values, and so with the dimension of the family, therefore the family is special.

Case (6)

Clearly θ is an epimorphism. Eichler trace formula (1.5.10) immediately yields $\chi_\rho(z) = \zeta_3$, $\chi_\rho(z^2) = \bar{\zeta}_3$ and since $\chi_\rho(1) = g = 4$, by Corollary 1.5.17 we have $N = 3$, which coincides with the number of critical values, and so with the dimension of the family. This proves that our family is special.

1.7.3 New examples and already known ones

In this section we will show that only two of the examples listed in Section 1.7, namely family (2) and (6), give rise to new special subvarieties of $\mathcal{A}_g$, while the others have already been obtained as families of Galois covers of $\mathbb{P}^1$.

Let us explain this. Assume that a family of Galois covers of genus 1 curves with Galois group G satisfying $(*)$ yields the same Shimura subvariety of $\mathcal{A}_g$ of dimension s as one of those obtained via a family of Galois covers of $\mathbb{P}^1$ satisfying condition $(*)$. Then each cover $\varphi : X \to X/G$ of the family of covers of genus one curves has the property that the curve X also admits an action of a group $K \subset Aut(X)$ such that $X/K \cong \mathbb{P}^1$ and we have: $dim(S^2H^0(K_X)^G) = dim(S^2H^0(K_X)^K) = s$. So each curve X of the family admits an action of a group $\tilde{G} \subset Aut(X)$ containing both G and K such that $X/\tilde{G} \cong \mathbb{P}^1$ and since $S^2H^0(K_X)^{\tilde{G}} \subset S^2H^0(K_X)^K$, also the family of covers $\psi : X \to X/\tilde{G} \cong \mathbb{P}^1$ satisfies condition $(*)$ and we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & X/G \arrow[d, "\sigma"] \\ & X/\tilde{G} \cong \mathbb{P}^1 \end{tikzcd} \tag{1.7.1}$$

So we can assume that $G \subset K$.

Since all the families of Galois covers of $\mathbb{P}^1$ satisfying $(*)$ in genus less than 10 have already been found in [44] Table 2, it will suffice to compare

our families with the ones listed there.

Case (1)

We show that this family yields the same subvariety in $\mathcal{A}_g$ as family (26) in Table 2 of [44] (this family was already found in [61]). Let us recall the description of this family. It is a family of Galois covers of $\mathbb{P}^1$ with Galois group $\tilde{G} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle x, y |\, x^2 = y^2 = 1,\, xy = yx \rangle$, with ramification data $(2,2,2,2,2)$ and with epimorphism $\tilde{\theta} : \Gamma_{0,5} = \langle \delta_1, ..., \delta_5 \mid \prod_{i=1,...,5} \delta_i = 1 \rangle \to \tilde{G}$ given by

$$\tilde{\theta}(\delta_1) = x,\ \tilde{\theta}(\delta_2) = x,\ \tilde{\theta}(\delta_3) = x,\ \tilde{\theta}(\delta_4) = y,\ \tilde{\theta}(\delta_5) = xy.$$

We want to prove that for any cover of this family $\psi : X \to X/\tilde{G} \cong \mathbb{P}^1$, X also admits an action by a subgroup $G \cong \mathbb{Z}/2\mathbb{Z}$ of $\tilde{G}$ such that the quotient map $\varphi : X \to E \cong X/G$, belongs our family (1) and we have a diagram as in (1.7.1).

Consider the cyclic subgroup $G \cong \langle y |\, y^2 = 1 \rangle < \tilde{G}$. Looking at ramification data, we see that all stabilizer subgroups of $\tilde{G}$ have order 2 and looking at the epimorphism $\tilde{\theta}$ we see that the stabilizer subgroup associated to the fourth branch point q_4 is $\langle y \rangle = G$. This implies that points in $\psi^{-1}(q_4) = \{p_1, p_2\}$ are critical points for the action of G as well. Moreover they cannot belong to the same fiber with respect to the action of G since every element of G stabilizes both p_1 and p_2. To conclude note that every other stabilizer subgroup for the critical points of ψ does not contain any non trivial element of G, so q_4 is the only branch point of φ. This proves that the map φ has exactly 2 critical values which are the images of p_1 and p_2 by φ and the ramification is $\mathbf{m} = (2,2)$. So we have a family with the same group and the same ramification as in family (1) and hence by the unicity argument given in Section 1.7, we conclude that it gives the same special subvariety in $\mathcal{A}_g$ as the one given by family (1).

Concluding, the special subvariety given by family (1) gives the same special subvariety obtained as a family of Galois covers of $\mathbb{P}^1$ via $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ corresponding to family (26) of Table 2 of [44] and already studied in [61].

Case (2)

Since the family has dimension 4, it has bigger dimension than any possibile family given as Galois cover of $\mathbb{P}^1$ satisfying $(*)$. In fact looking at the table of all possible special varieties presented as Galois covers of $\mathbb{P}^1$ satisfying $(*)$ of genus $g \leq 9$ we see that none of these has dimension greater than 3 (see Table 2 of [44]). This proves that the family gives a new special subvariety contained in the Torelli locus. It also follows that the family is not contained in the hyperelliptic locus. In fact, if every curve C of the family were hyperelliptic, one could consider the group H generated by G and the hyperelliptic involution σ. The quotient $C/H \cong \mathbb{P}^1$, so we would obtain a family of Galois covers of $\mathbb{P}^1$ which clearly still verifies condition $(*)$, and this does not exist, as we just pointed out.

Case (3)

We claim that this family yields the same Shimura subvariety of $\mathcal{A}_g$ as family (31) in Table 2 of [44]. Let us describe this family of Galois covers of $\mathbb{P}^1$ as in 4.1 of [44]. The Galois group $\tilde{G}$ is isomorphic to the symmetric group S_3, $\tilde{G} = \langle x, y | \, y^2 = x^3 = 1, \, y^{-1}xy = x^2 \rangle$, $\mathbf{m} = (2, 2, 2, 2, 3)$ and the epimorphism $\tilde{\theta} : \Gamma_{0,5} = \langle \delta_1, ..., \delta_5 \mid \prod_{i=1,...,5} \delta_i = 1 \rangle \to \tilde{G}$ is given by

$$\tilde{\theta}(\delta_1) = xy, \; \tilde{\theta}(\delta_2) = x^2y, \; \tilde{\theta}(\delta_3) = y, \; \tilde{\theta}(\delta_4) = xy, \; \tilde{\theta}(\delta_5) = x^2.$$

We will show that every cover $\psi : X \to X/\tilde{G} \cong \mathbb{P}^1$ of this family also admits a $G = \mathbb{Z}/3\mathbb{Z}$-action such that X/G has genus 1, the map $\varphi : X \to X/G$ is one of the covers of our family (3) and we have a factorisation as in (1.7.1). In fact, set $G = \langle x | \, x^3 = 1 \rangle < \tilde{G}$ that is the only cyclic subgroup of order 3 on $\tilde{G}$. Looking at the stabilisers of the action of $\tilde{G}$, we see that the two critical points of ψ in the fibre over the critical value q_5 have both G as stabiliser, hence they are critical points also for the action of φ and they are mapped by φ in two different critical values. All the other critical points of ψ have stabilisers of order 2, hence they are not critical values for the map φ. So the map φ has ramification $(3, 3)$ and by the unicity argument of Section 1.7 we can assume that $\varphi : X \to X/G$ belongs to our family (3). Concluding, the special subvariety given by family (3) is the same special subvariety obtained from the family (31) of Galois covers of $\mathbb{P}^1$ via S_3 found in [44]. This family is not contained in the hyperelliptic locus (see the proof of Theorem 5.3 of [44]).

Case (4)

We show that this family yields the same Shimura subvariety of $\mathcal{A}_g$ as family (32) in Table 2 of [44]. Let us describe this family of Galois covers of $\mathbb{P}^1$ as in 4.1 of [44]. The Galois group $\tilde{G}$ is isomorphic to the dihedral group D_4, $\tilde{G} = \langle x, y |\, y^2 = x^4 = 1,\, y^{-1}xy = x^3 \rangle$, $\mathbf{m} = (2,2,2,2,2)$ and the epimorphism $\tilde{\theta} : \Gamma_{0,5} = \langle \delta_1, ..., \delta_5 \mid \prod_{i=1,...,5} \delta_i = 1 \rangle \to \tilde{G}$ is given by

$$\tilde{\theta}(\delta_1) = xy,\ \tilde{\theta}(\delta_2) = x^2y,\ \tilde{\theta}(\delta_3) = x^2,\ \tilde{\theta}(\delta_4) = x^2y,\ \tilde{\theta}(\delta_5) = x^3y.$$

As above we want to show that every cover $\psi : X \to X/\tilde{G} \cong \mathbb{P}^1$ of this family also admits a $G = \mathbb{Z}/4\mathbb{Z}$-action such that X/G has genus 1, the map $\varphi : X \to X/G$ is one of the covers of our family (4) and we have a factorisation as in (1.7.1).

We can identify $G \cong \langle x |\, x^4 = 1 \rangle < \tilde{G}$. The stabilizer subgroups for the action of $\tilde{G}$ of the critical points over the third branch point q_3 are all given by the center $H = \langle x^2 \rangle$ of $\tilde{G}$ which is contained in G. This implies that points in $\psi^{-1}(q_3) = \{p_1, p_2, p_3, p_4\}$ are critical points for the action of G as well.

Moreover the four points $p_1, .., p_4$ are partitioned in exactly two orbits for the action of G, hence they give rise to two critical values of the map φ. Finally, observing that the other 4 conjugacy classes of stabilizers for D_4 do not contain nontrivial elements belonging to G, we conclude that the action of $G < D_4$ has ramification data $(2,2)$ and by the unicity argument in Section 1.7 we can assume that it gives a cover belonging to our family (4). Concluding, the special variety given by family (4) is the same special variety obtained as the family (32) of Galois covers of $\mathbb{P}^1$ via D_4 found in [44]. This family is not contained in the hyperelliptic locus (see the proof of Theorem 5.3 of [44]).

Case (5)

We show that this family yields the same Shimura subvariety of $\mathcal{A}_g$ as family (34) in Table 2 of [44]. Let us describe this family of Galois covers of $\mathbb{P}^1$ as in 4.1 of [44]: the Galois group is

$$\tilde{G} = (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z}) \cong$$

$$\langle y_1, y_2, y_3 |\, y_1^2 = y_2^2 = y_3^4 = 1,\ y_2y_3 = y_3y_2, y_1^{-1}y_2y_1 = y_2y_3^2,\ y_1^{-1}y_3y_1 = y_3\rangle,$$

$\mathbf{m} = (2, 2, 2, 4)$ and the epimorphism

$$\tilde{\theta} : \Gamma_{0,4} = \langle \delta_1, ..., \delta_4 \mid \prod_{i=1,...,4} \delta_i = 1\rangle \to \tilde{G}$$

is given by

$$\tilde{\theta}(\delta_1) = y_1,\ \tilde{\theta}(\delta_2) = y_1y_2y_3^3,\ \tilde{\theta}(\delta_3) = y_2y_3^2,\ \tilde{\theta}(\delta_4) = y_3^3.$$

Observe that the conjugacy classes of the non-trivial elements of $\tilde{G}$ are:

$$\begin{aligned}
&\text{order } 2: \ \{y_1, y_3^2y_1\},\ \{y_2, y_3^2y_2\},\ \{y_3^2\},\ \{y_2y_3y_1, y_2y_3^3y_1\},\\
&\text{order } 4: \ \{y_3\},\ \{y_3^3\}\ \{y_2y_3, y_2y_3^3\}\ \{y_2y_1, y_2y_3^2y_1\}\ \{y_3y_1, y_3^3y_1\},
\end{aligned}$$

As above we want to show that every cover $\psi : X \to X/\tilde{G} \cong \mathbb{P}^1$ of this family also admits a $G = Q_8$-action such that X/G has genus 1, the map $\varphi : X \to X/G$ is one of the covers of our family (5) and we have a factorisation as in (1.7.1). In order to prove that the factorization holds, first of all we have to check that G is isomorphic to a subgroup of $\tilde{G}$. One easily checks that the following map

$$i : Q_8 \to (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$$

$$g_1 \mapsto y_2y_3,\ g_2 \mapsto y_2y_1,\ g_3 \mapsto y_3^2.$$

yields an injective homomorphism that identifies $G = Q_8$ with a proper subgroup of $(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$.

As before, to conclude that the two families are in fact the same one, we have to study their stabilizer subgroups. Looking at the epimorphism $\tilde{\theta} : \Gamma_{0,4} \to \tilde{G}$ we see that the stabilizer subgroup associated to the fourth branch point q_4 is the normal subgroup $K := \langle y_3^3\rangle = \{1, y_3, y_3^2, y_3^3\}$. The subgroup $H = \{1, y_3^2\}$ of G is clearly contained in $K = \langle y_3^3\rangle$. This implies that points in $\psi^{-1}(q_4) = \{p_1, p_2, p_3, p_4\}$ are critical points for the action of G as well. Up to a permutations of the p_i's, we see that $\tilde{G}$ acts this way on the fiber:

$$\begin{aligned}
p_2 &= y_1(p_1) = y_1y_3(p_1) = y_1y_3^2(p_1) = y_1y_3^3(p_1),\\
p_3 &= y_2(p_1) = y_2y_3(p_1) = y_2y_3^2(p_1) = y_2y_3^3(p_1),\\
p_4 &= y_2y_1(p_1) = y_2y_1y_3(p_1) = y_2y_1y_3^2(p_1) = y_2y_1y_3^3(p_1).
\end{aligned}$$

If we consider the action of G we get:

$$\begin{aligned}
p_2 &= y_1y_3(p_1) = i(g_2^{-1}g_1)(p_1) = y_1y_3^3(p_1) = i(g_1^{-1}g_2)(p_1),\\
p_3 &= y_2y_3(p_1) = i(g_1)(p_1) = y_2y_3^3(p_1) = i(g_1^{-1})(p_1),\\
p_4 &= y_2y_1(p_1) = i(g_2)(p_1) = y_2y_1y_3^2(p_1) = i(g_2g_3)(p_1).
\end{aligned}$$

So p_1, p_2, p_3, p_4 are 4 ramification points that map to the same critical value for the map φ and they have multiplicity 2. To conclude we have to prove that these are all the critical points of φ. But this is actually true, because none of the stabilizer subgroups of the critical points of ψ that are mapped to the first three critical values includes any subgroup of G. Concluding, by the unicity argument in Section 1.7, the special variety given by the family (5) is the same special variety obtained as the family (34) of Galois covers of $\mathbb{P}^1$ via $(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$ found in [44]. This family is not contained in the hyperelliptic locus (see the proof of Theorem 5.3 of [44]).

Case (6)

We want to check that the family does not yield the same Shimura subvariety of $\mathcal{A}_g$ as one obtained via a family of Galois covers of $\mathbb{P}^1$ already known to be special. Nonetheless, looking at Table 2 in [44] one checks there are no families of Galois covers of $\mathbb{P}^1$ satisfying condition $(*)$ with dimension greater or equal than 3 admitting $\mathbb{Z}/3\mathbb{Z}$ as a proper subgroup of the Galois group. This proves that the family gives a new special subvariety contained in the Torelli locus. It also follows by the same argument as in the previous example that it is not contained in the hyperelliptic locus.

Remark 1.7.1. We observe that family (6) is interesting also for another reason, in fact it is the same family used by Pirola in [74] to construct a counterexample to a conjecture of Xiao on the relative irregularity of a fibration of a surface on a curve.

Remark 1.7.2. Notice that in [53] a classification of all the representations of the actions of the possible groups G on the space of holomorphic one forms of a curve of genus $g = 3, 4$ is given and also using their description one can verify that in genus $3, 4$ our families are the only ones satisfying condition $(*)$ if $g' = 1$.

1.8 Constraints in higher genus

In this section we deal with the natural question of Shimura varieties obtained from families covering higher genus curves. Consider a family of Galois covers $f : C \to C'$. The following result shows that there is a bound for $g(C)$ depending on $g(C')$, which is necessary to make condition $(*)$ possible.

Theorem 1.8.1. *If $g' \geq 1$ and we have a positive dimensional family of Galois covers $f : C \to C'$ with $g' = g(C')$ and $g = g(C)$ which satisfies condition $(*)$, then $g \leq 6g' + 1$. In particular, for $g \geq 8$ (resp. 14) there do not exist positive dimensional families of Galois covers with $g' = 1$ (resp. 2) and which satisfy condition $(*)$.*

Proof. The idea of the proof is the following: if such a family exists, with the same method used by Pirola in Section 2 of [74] one constructs a fibration $S \to B$ of a surface S on a curve B, whose general fibre has genus g and whose relative irregularity is at least $g - g'$. Then we apply Corollary 3 of [89].

In fact, assume that $\mathsf{M} := \mathsf{M}(\mathbf{m}, G, \theta)$ is as usual the variety parametrising elements of such a family for a given datum $(\mathbf{m}, G, \theta)$. Every point $\mathsf{p} \in \mathsf{M}$ corresponds to an isomorphism class of a curve C of genus g admitting G as a subgroup of $Aut(C)$, whose quotient $C' := C/G$ has genus $g' \geq 1$ and whose monodromy is given by θ. Denoting by $f : C \to C'$ the Galois cover, to such a point one can associate the abelian variety $W = J(C)/f^*(J(C'))$, which is isogenous to the Prym variety of the cover. The abelian variety W has a polarisation Θ and we denote by $\mathsf{A}_{g-g'}(\Theta)$ the moduli space of polarised abelian varieties of dimension $g - g'$ with the given type of polarisation. Denote by $\Psi : \mathsf{M} \to \mathsf{A}_{g-g'}(\Theta)$ the map associating to p the polarised variety $[W, \Theta]$. The variety W inherits from C the automorphism group G. The differential of the map Ψ at the point $\mathsf{p} \in \mathsf{M}$ is a map

$$d\Psi_{\mathsf{p}} : H^1(C, T_C)^G \to S^2 H^{0,1}(W)$$

and its image is contained in $(S^2 H^{0,1}(W))^G$, since this is the space of infinitesimal deformations of (W, Θ) that preserve the action of G. So if we denote by $\mathsf{P} \subset \mathsf{A}_{g-g'}(\Theta)$ the image of Ψ , the tangent space $T_{[W,\Theta]}\mathsf{P}$ of P at $[W, \theta]$ is contained in $(S^2 H^{0,1}(W))^G$. The dual of the differential gives a map:

$$d\Psi^*_{\mathsf{p}} : (S^2 H^{1,0}(W))^G \to H^0(C, 2K_C)^G.$$

Observe that $H^0(K_C) = H^0(K_C)^G \oplus H^0(K_C)^-$, where the space of invariants $H^0(K_C)^G \cong H^0(C', K_{C'})$ has dimension g', and the complement $H^0(K_C)^- \cong H^{1,0}(W)$. Therefore we have

$$(S^2H^0(K_C))^G \cong S^2H^0(K_{C'}) \oplus (S^2H^0(K_C)^-)^G \cong \\ \cong S^2H^0(K_{C'}) \oplus (S^2H^{1,0}(W))^G \quad (1.8.1)$$

The dual of the differential $d\Psi_{\mathsf{p}}^* : (S^2H^0(K_C)^-)^G \to H^0(C, 2K_C)^G$ is given by the multiplication map and since $(S^2H^0(K_C)^-)^G \subset (S^2H^0(K_C))^G$ and by our assumption $(*)$ the multiplication map

$$(S^2H^0(K_C))^G \to H^0(C, 2K_C)^G$$

is an isomorphism, we conclude that $d\Psi_{\mathsf{p}}^*$ is injective. Hence $d\Psi_{\mathsf{p}}$ is surjective and $T_{[W,\Theta]}\mathsf{P} = (S^2H^{0,1}(W))^G$. By (1.8.1) its dimension is equal to $N - \frac{g'(g'+1)}{2}$, where $N = \dim(S^2H^0(K_C))^G$ is the dimension of M, by our condition $(*)$. So for a general point $[W, \Theta] \in \mathsf{P}$, $\dim \Psi^{-1}(W, \Theta) = \frac{g'(g'+1)}{2} \geq 1$, thus we can find a curve $Y \subset \Psi^{-1}(W, \Theta)$ contained in M_g. Denote by $\overline{Y}$ its closure in $\overline{\mathsf{M}}_g$. So we get a family of curves of genus g, $h' : S' \to B'$ such that $\overline{Y}$ is the image of the modular map $B' \to \overline{\mathsf{M}}_g$, $b' \mapsto [h'^{-1}(b')]$. By resolving singularities and taking pullbacks we get a smooth surface S, a smooth curve B and a map $h : S \to B$. Up to a base change we can assume that h has a section η. So if we take the Zariski open subset U of B of points having non-singular fibres, we can use the section η to take the Abel-Jacobi maps $A_{\eta(t)} : C_t \to J(C_t)$, $t \in U$, compose them with the projections $J(C_t) \to W$ and obtain mappings: $\varphi_t : C_t \to W$, $\forall t \in U$. Using the pull-backs $\varphi_t^* : H^1(W, \mathbb{Q}) \to H^1(C_t, \mathbb{Q})$ we get an injection of $H^1(W, \mathbb{Q})$ in $H^0(B, R^1h_*\mathbb{Q})$. By the Leray spectral sequence we identify $H^0(B, R^1h_*\mathbb{Q})$ with the cokernel of the map $h^* : H^1(B, \mathbb{Q}) \to H^1(S, \mathbb{Q})$, thus we have

$$\dim H^1(S, \mathbb{Q}) - \dim H^1(B, \mathbb{Q}) \geq \dim H^1(W, \mathbb{Q}) = 2(g - g').$$

So if we denote by $q = h^0(S, \Omega_S^1)$ and by b the genus of the curve B we have $q - b \geq g - g'$. Since by construction the family is not isotrivial, we can apply Corollary 3 of [89], which says that $q - b \leq \frac{5g+1}{6}$ and so we get $g - g' \leq q - b \leq \frac{5g+1}{6}$, hence $g \leq 6g' + 1$.

Clearly if $g' = 1$ this implies $g \leq 7$.

□

Using the above theorem, we can summarize the results obtained in this chapter as follows:

Theorem 1.8.2. *For all $g \geq 2$ and $g' = 1$ there exist exactly 6 positive dimensional families of Galois covers satisfying condition $(*)$, hence yielding Shimura subvarieties of $\mathcal{A}_g$ contained in the Torelli locus.*

Two of the 6 families yield new Shimura subvarieties (i.e. case (2) and case (6) of the list in Section 1.7), while the others yield Shimura subvarieties which have already been obtained as families of Galois covers of $\mathbb{P}^1$ in [44].

For all $g > 3$ and $g' = 2$ there do not exist positive dimensional families of Galois covers satisfying condition $(*)$.

For $g \leq 9$ and $g' > 2$ there do not exist positive dimensional families of Galois covers satisfying condition $(*)$.

Proof. It only remains to show that if $g \leq 7$ (resp. 13) and $g' = 1$ (resp. 2) there does not exist any other family satisfying $(*)$ except for the 6 families described above and if $g \leq 9$ and $g' > 1$ there do not exist families satisfying property $(*)$. To do this we used the computer software `MAGMA`.

A slightly modified version of the `MAGMA` script used in [44] enables us to check that the families given is Section 1.7 are the only ones under the following conditions. The covering curve has genus $g \leq 9$ and the quotient is a curve of genus $g' \geq 1$, moreover for the case $g' = 2$ we extended the calculation up to $g = 13$. By Proposition (1.8.1) we know that if $g' = 1$ these are all the families satisfying $(*)$. It is not bold to conjecture that these are all also in the case $g' > 1$. The `MAGMA` script that we used is available at:

```
users.mat.unimi.it/users/penegini/

publications/PossGruppigFix_Elliptic_v2.m
```

This script differs from the one in [44] essentially for the fact that it does not return a representative up to Hurwitz equivalence of a datum. But it gives all possible ramification data. This is because the Hurwitz's moves for the data we have found could be easily handled by hand as we have seen in Section 1.7. In addition, this helped to speed up the finding-example process as well. The other changes in the script are the obvious ones related to the fact that the genus of the base is not 0 anymore. It is important

to notice that the `MAGMA` script works perfectly fine for covering curves of genera $g > 9$, we simply did not include other results for time reasons.

$\square$

Chapter 2

Bielliptic and bi-hyperelliptic loci

In this chapter we study some particular loci inside the moduli space $\mathcal{M}_g$, namely the bielliptic locus (i.e. the locus of curves admitting a $2:1$ cover over an elliptic curve E) and the bi-hyperelliptic locus (i.e. the locus of curves admitting a $2:1$ cover over a hyperelliptic curve C', $g(C') \geq 2$). We will show that these loci do not provide totally geodesic submanifolds of $\mathcal{A}_g$. We will also find a bound for the rank of the second Gauss-Wahl map when computed over the bielliptic locus.

Our purpose is to investigate the condition $(*)$: as far as we know condition $(*)$ is sufficient but not necessary, in general, for $\mathsf{Z}(\mathbf{m}, G, \theta)$ to be special. Even if there is no evidence for condition $(*)$ to be necessary, no examples of totally geodesic submanifolds not satisfying the condition are known. An important result in this direction has been obtained by Moonen [60]: using deep techniques in arithmetic geometry he proved that condition $(*)$ is both sufficient and necessary in case of cyclic covers of $\mathbb{P}^1$. For the general case, the problem is still unsolved.

Totally geodesic submanifolds of $\mathcal{A}_g$ are related to the second Gauss-Wahl map via the diagram originally given in [32, Theorem 3.1]. In that paper, it is given an explicit expression for the second fundamental form of the orbifold immersion $j : \mathcal{M}_g \to \mathcal{A}_g$ (the immersion holds outside the hyperelliptic locus, see [70]), and it is proven that the second fundamental form lifts the second Gaussian map $\mu_2 : I_2(K_C) \to H^0(C, 4K_C)$, as stated

in an unpublished paper of Green and Griffiths (see [48]).

More precisely an explicit expression for the second fundamental form when evaluated on Schiffer variations is provided (see Theorem (2.3.2)). In particular, $\rho(\xi_p \odot \xi_p)$ reduces to the evaluation of the second Gauss-Wahl map at the point p. It is much more difficult to use the expression given in [32] to compute the second fundamental form on $\xi_p \odot \xi_q$, when $p \neq q$. In this case, in fact, the formula contains the evaluation at q of a meromorphic 1-form on the curve, called η_p, which has a double pole at p and is defined by Hodge theory. Since the form η_p is implicitly defined, it seems to be hard to compute it in general.

Nevertheless, Colombo and Frediani in [29] used Theorem (2.3.2) to compute the curvature of the restriction to $\mathcal{M}_g$ of the Siegel metric: in particular the authors give an explicit formula for the holomorphic sectional curvature of $\mathcal{M}_g$ in direction ξ_p in terms of the holomorphic sectional curvature of $\mathcal{A}_g$ and the second Gauss-Wahl map.

Moreover, Colombo, Frediani and Ghigi in [30] used the same formula to get some constraints on the existence of totally geodesic submanifolds in $\mathcal{A}_g$ contained in the Jacobian locus. In particular they found an upper bound for the dimension of totally geodesic germs passing through $[C] \in \mathcal{M}_g$ depending on the gonality of C. As a straightforward consequence, they found a bound depending on the genus only: every totally geodesic submanifold $Y \subset \mathcal{A}_g$ contained in the Jacobian locus should satisfy $\dim Y \leq 5/2(g-1)$. The proof is based on the fact that, for a quadric Q of rank at most equal to 4, the second Gauss-Wahl map can be written as a product of the first Gauss maps relative to the two adjoint line bundles that define the quadric (see Theorem 2.2.10). This simplifies the computations considerably.

Here we use the same trick to prove that both the bielliptic ($g \geq 4$) and bi-hyperelliptic ($g \geq 3g'$) loci are not totally geodesic. In both cases we use some induced $k:1$ map over $\mathbb{P}^1$, and its adjoint map as well, to construct a suitable invariant quadric Q. We avoid the problem of computing the form η by choosing a setting such that $Q(u,v)$ vanishes (see expression (2.3.8)). In the spirit of checking whether condition $(*)$ is necessary for $\mathsf{Z}(\mathbf{m}, G, \theta)$ to be special, here we find some examples of non-special subvarieties not satisfying condition $(*)$ (see Theorem 1.8.2).

On top, we include another application of Theorem 2.2.10 at the end of Section 2.4: we transform the computation of the second Gaussian map on

the bielliptic locus to the easier computation of the first Gaussian map on a line bundle over C, and compute its rank. Remark that this is possible if the quadric Q is constructed using two adjoint line bundles. Since quadrics of this type do not cover the whole $I_2(K)$, we will find just a lower bound.

This chapter is organized as follows.

In Section 2.1 we will analyse deeply cyclic covers of $\mathbb{P}^1$ reporting some classical results. In particular, using the decomposition in eigenspaces $H^0(C, K_C) = \bigoplus_{n=0}^{m-1} V_n$ we will recall a very explicit expression for holormorphic 1-forms in $H^0(C, K_C)$ (see expression (2.1.3)).

In Section 2.2 we will introduce Gauss-Wahl maps. After giving their general definition, we will focus on the Gauss-Wahl map of first and second order, recalling rank properties of these maps on some particular loci.

In Section 2.3 we will explain the link between the second fundamental form of $j(\mathcal{M}_g) \subset \mathcal{A}_g$ and the second Gauss-Wahl map. In particular we will show that the second fundamental form $\rho : I_2(K) \to S^2H^0(C, 2K)$ is the lifting of the second Gauss-Wahl map, as proved in [32]. Moreover, in Theorem 2.3.2, we recall the explicit expression of the second fundamental form when evaluated on the product of two Schiffer variations, and we use it to find constraints on the dimension of totally geodesic submanifold of $\mathcal{A}_g$ contained in the Jacobian locus, following [30].

In Section 2.4 we study the locus of bielliptic curves of genus g. First we will make some general considerations on the bielliptic locus, then we will prove that it is not totally geodesic if $g \geq 4$ (while it is for $g = 3$, see example (2) in Section 1.7). We will conclude this section performing a computation for the second Gauss-Wahl map on this locus, and giving a bound for it.

In Section 2.5, finally, we study the locus of bi-hyperelliptic curves, i.e. curves C admitting a $2:1$ cover over a hyperelliptic curve C', $g(C') \geq 2$. As in the bielliptic case, we use Theorem 2.3.2 to write the second fundamental form explicitly and then to prove that the locus is not totally geodesic if $g(C) \geq 3g(C')$.

2.1 Galois cyclic covers of the projective line

In this section we analyse cyclic covers of $\mathbb{P}^1$, giving for the datum $(\mathbf{m}, G, \theta)$ an equivalent expression, which will be useful to explicitly write down a basis for the space of holomorphic 1-forms.

Here we give the general set-up. Recall that by Riemann's existence theorem (see Section 1.2) there is a correspondence between a datum $(\mathbf{m}, G, \theta)$ and Galois covers with Galois group G, monodromy defined by $\mathbf{m} = (m_1, \ldots, m_r)$ and by the epimorphism $\theta : \Gamma_{g',r} \twoheadrightarrow C/G$. Notice that in case $C/G \cong \mathbb{P}^1$ with G cyclic, to give a datum is equivalent to give a triple $(m, r, \mathbf{a})$ as follows:

Corollary 2.1.1. *Let $m, r \in \mathbb{N}$, $m \geq 2$, $r \geq 4$. Let $\mathbf{a} = (a_1, \ldots, a_r)$ be a vector of integers $a_i \in \mathbb{N}$ such that:*

(1) $a_i \not\equiv 0$ *mod* m *for all* $i = 1, \ldots, r$*;*

(2) $\mathrm{gdc}(m, a_1, \ldots, a_r) = 1$*;*

(3) $\sum_{i=1,\ldots,r} a_i \equiv 0$ *mod* m*.*

Then there is a $1:1$ correspondence between the datum $(\mathbf{m}, G, \theta)$ associated to a family of Galois cover of $\mathbb{P}^1$ with cyclic group G and a triple $(m, r, \mathbf{a})$ as above.

The correspondence is given as follows:

- $m \geq 2$ is the integer representing the order of the cyclic group G;
- $r \geq 4$ represents the number of critical values of the cover;
- $\mathbf{a} = (a_1, \ldots, a_r)$ encodes all informations about the monodromy. In particular $[a_i]$ (defined as the class of a_i modulo m) corresponds to $\theta(\gamma_i)$. Since $\mathrm{ord}([a_i]) = m_i$, vector $\mathbf{a}$ allows to reconstruct $\mathbf{m}$, and gives all informations about θ. When there is no risk of confusion, we will simply refer to the vector $\mathbf{a} = ([a_1], \ldots, [a_r])$ as the monodromy of the cover.

Notice that condition (1) corresponds to the condition $m_i \geq 2$ for all $i = 1, \ldots, r$; conditions (2) and (3) correspond, respectively, to the surjectivity

of θ, and to the condition $\prod_{i=1,\dots,r}\theta(\gamma_i)\equiv 0 \bmod m$. Also, observe that using Riemann-Hurwitz formula, we can determine the genus of the covering curve.

Consider now a triple $(m,r,\mathbf{a})$, and identify the group $G=\mathbb{Z}/m\mathbb{Z}$ with the group of m-th roots of unity. From the previous discussion, once r points $t_1,\dots,t_r\in\mathbb{P}^1$ are fixed, there is a well defined curve C_t covering $\mathbb{P}^1$ with Galois group G and monodromy $m_i=\mathrm{ord}(a_i)$ over the point t_i. Varying the branch points $t=(t_1,\dots,t_r)$ we obtain a $(r-3)$-dimensional family of curves $\mathcal{C}\to B$, as discussed in Section 1.3, where B parametrizes all admissible vectors of branch points, i.e. all r-uples of distinct points in $\mathbb{P}^1$.

The fiber over a fixed point $t\in B$ is the normalization of the affine curve (see [60]):

$$y^m=\prod_{i=1,\dots,r}(x-t_i)^{a_i}. \tag{2.1.1}$$

There is an action of G on $\mathcal{C}$ given by the rule $\zeta\cdot(x,y,t)=(x,\zeta\cdot y,t)$, $\zeta\in G$. This motivates the following:

Definition 2.1.2. *Two triples $(m,r,\mathbf{a})$ and $(m',r',\mathbf{a}')$ are equivalent if $m=m'$, $r=r'$ and if the class of a and a' in $\left(\mathbb{Z}/m\mathbb{Z}\right)^r$ are in the same orbit under $(\mathbb{Z}/m\mathbb{Z})^*\times S_r$, where $(\mathbb{Z}/m\mathbb{Z})^*$ acts diagonally by multiplication, and the symmetric group S_r acts by permutations of indices.*

Now, once a triple $(m,r,\mathbf{a})$ is fixed, we want to give an explicit expression for a basis of the space of holormorphic one forms $H^0(C_t,K_{C_t})$. First of all, notice that since the group G acts on C_t, there is a decomposition $H^0(C_t,K_{C_t})=\bigoplus_{n=0}^{m-1}V_n$, where V_n's are the subspaces of 1-forms ω such that $\zeta\cdot\omega=\zeta^{-n}\omega$, being ζ the primitive m-th root of unity. Take $n\in\mathbb{Z}/m\mathbb{Z}$, $i\in\{1,\dots,r\}$ and consider the following data:

$$d_n=-1+\sum_{i=1}^{N}\left\langle\frac{-na_i}{m}\right\rangle,\qquad l(i,n)=\left\lfloor\frac{-na_i}{m}\right\rfloor, \tag{2.1.2}$$

where $\langle x\rangle$ and $\lfloor x\rfloor$ denote, respectively, the fractional and integral part of x.

The integer d_n is exactly the dimension of the n-th eigenspace V_n, while the combinatorial data $l(i,n)$ is useful to define the following differential forms (expressed in model (2.1.1)):

$$\omega_{n,\nu}=y^n(x-t_1)^\nu\prod_{i=1}^{N}(x-t_i)^{l(i,n)}dx. \tag{2.1.3}$$

Note that these only depend on the pair $(n \bmod m, \nu)$. We have the following result (see [60], [30]).

Theorem 2.1.3. *Let $n \in \mathbb{Z}/m\mathbb{Z}$, with $n \neq 0$. The forms $\omega_{n,\nu}$ for $0 \leq \nu \leq d_n - 1$ are regular 1-forms on C_t, and they are a basis of*

$$V_n = \{\omega \in H^0(C_t, K_{C_t}) \text{ such that } \zeta \cdot \omega = \zeta^{-n}\omega\}.$$

Notice that if two forms lies in the same eigenspace, then they differ from a factor $(x - t_1)^k$. More precisely $\omega_{n,\nu} = (x - t_1)^\nu \omega_{n,0}$.

In the following, we want to write down forms $\omega_{n,\nu}$ in a different way, using the relation between x and y given in expression (2.1.1). We show that, under some weak hypothesis, y is a local coordinate, so using equation (2.1.1) it is possible to cut out the dependence from the variable x. We start differentiating expression (2.1.1): if $y^m = g(x) := \prod_{i=1}^r (x - t_i)^{a_i}$ describes the affine curve, then

$$my^{m-1}dy = g'(x)dx. \tag{2.1.4}$$

We claim that if there exists at least one $a_i = 1$ (assume $a_1 = 1$ and $t_1 = 0$), then y is local coordinate around zero. In fact in that case:

$$g(x) = \prod_{i=1}^r (x - t_i)^{a_i} = x \prod_{i=2}^r (x - t_i)^{a_i} = xh(x).$$

Since $h(0) \neq 0$ (because t_i's are all distinct) and $g'(x) = h(x) + xh'(x)$, necessarily $g'(0) \neq 0$.

In this case, since y is a local coordinate around 0, one can write, locally, $x = \varphi(y) = g^{-1}(y^m)$. Substituting in expression (2.1.3) and using equation (2.1.4), locally around 0 we get:

$$\begin{aligned} \omega_{n,\nu} &= y^n x^\nu \prod_{i=1}^N (x - t_i)^{l(i,n)}\, dx = \\ &= y^n \varphi(y)^\nu \prod_{i=1}^N (\varphi(y) - t_i)^{l(i,n)} \left(\frac{my^{m-1}}{g'(\varphi(y))} \right) dy = \\ &= \frac{m}{g'(\varphi(y))}\, y^{n-1} \varphi(y)^\nu \prod_{i=1}^N (\varphi(y) - t_i)^{l(i,n)+a_i}\, dy. \end{aligned} \tag{2.1.5}$$

From now on, we will always assume $t_1 = 0$ without loss of generality.

2.2 Gauss-Wahl maps

The first Gaussian map was introduced in [85] to study deformations of the cone over the canonical curves. The name *Gaussian* derives from the correspondence between the first Gaussian map of a non-hyperelliptic curve relative to the canonical bundle and the classical Gauss map, as we will see in Remark 2.2.4. The first Gaussian map lies in a hierarchy of maps, called (generalized) Gaussians. Although there are several results about the first Gaussian map, very little is known about Gaussian maps of higher order. We start this section giving the general definitions, mainly following Wahl [88].

Let C be a smooth projective curve, and let $\Delta := \{(x,x) \in C \times C \,:\, x \in C\} \subset S := C \times C$ be the diagonal. Consider $\mathcal{I}_\Delta$ to be the ideal sheaf of the diagonal and take its powers $\mathcal{I}^k$. It is possible to prove that $\mathcal{I}^k$ sits in the short exact sequence

$$0 \to \mathcal{I}_\Delta^{i+1} \to \mathcal{I}_\Delta^i \to S^i\Omega_C^1 \to 0, \tag{2.2.1}$$

where $S^i\Omega_C^1$ is identified to its image via the diagonal map.

Consider $p_1 : S \to C$ and $p_2 : S \to C$ the projections on the first and the second component respectively, take L and M line bundles on C and define $L \boxtimes M := p_1^*L \otimes p_2^*M$: it is a well defined line bundle on S. Tensoring short exact sequence (2.2.1) with $L \boxtimes M$ and taking global sections we obtain:

$$0 \to H^0(S, \mathcal{I}_\Delta^{i+1} \otimes L \boxtimes M) \to H^0(S, \mathcal{I}_\Delta^i \otimes L \boxtimes M) \to H^0(S, S^i\Omega_C^1 \otimes L \otimes M).$$

The Künneth formula gives $H^0(S, L \boxtimes M) \cong H^0(C, L) \otimes H^0(C, M)$. Set $R_i(L, M) := H^0(S, \mathcal{I}_\Delta^i \otimes L \boxtimes M)$. We have:

$$0 \to R_{i+1}(L, M) \to R_i(L, M) \xrightarrow{\mu_{i,L,M}} H^0(C, S^iK_C \otimes L \otimes M). \tag{2.2.2}$$

Definition 2.2.1. *The map $\mu_{i,L,M}$ is called i-th* generalised Gaussian map.

From exact sequence (2.2.2) it is clear that the domain of the i-th Gaussian map is the kernel of the previous one:

$$\mu_{i,L,M} : \ker \mu_{i-1,L,M} \to H^0(S^iK_C \otimes L \otimes M). \tag{2.2.3}$$

Moreover Gaussian maps of even, respectively odd, order vanish identically on skew-symmetric, respectively symmetric, tensors (see e.g. [88]). We will

exclusively deal with Gaussian maps of order one and two, assuming also that line bundles L and M coincide. Denote $\mu_{i,L} := \mu_{i,L,L}$. We give the following definition.

Definition 2.2.2. *The map $\mu_{i,K}$ is called i-th* Gauss-Wahl map.

Notice that the Gaussian map $\mu_{0,L,M}$ is the classical multiplication map:

$$\begin{aligned} \mu_{0,L,M} : H^0(C,L) \otimes H^0(C,M) &\to H^0(C, L \otimes M), \\ s \otimes t &\mapsto st. \end{aligned} \tag{2.2.4}$$

Many surjectivity results about this map are known, relating the rank of this particular Gaussian map to the geometry of the curve C. These include (see e.g. [5, Section 3.2]):

1. (Noether) If C is non-hyperelliptic, then $\mu_{0,K}$ is surjective;
2. (Petri) If L is very ample ($g > 0$) then $\mu_{0,K,L}$ is surjective;
3. (Castelnuovo) If $\deg L \geq 2g$ and $\deg M \geq 2g+1$, then $\mu_{0,L,M}$ is surjective.

In the following we focus on the *first Gauss-Wahl map* and the *second Gauss-Wahl map* respectively.

2.2.1 The first Gauss-Wahl map

As previously said, the domain of the first Gauss-Wahl map coincides with the kernel of the multiplication map of sections of the canonical bundle. Since $\mu_{0,K}$ identically vanishes on skew-symmetric tensors, the following decomposition holds:

$$R_1(K) = \Lambda^2 H^0(C,K) \oplus I_2(K),$$

where $I_2(K)$ is the kernel of $S^2H^0(C,K) \to H^0(C,K^2)$. Since the first Gauss-Wahl map vanishes on symmetric tensors, one can write:

$$\mu_1 := \mu_{1,K} : \Lambda^2 H^0(C,K) \to H^0(C,K^3), \tag{2.2.5}$$

that is essentially the map associating $f \otimes g \mapsto f\,dg - g\,df$. More precisely, fix a basis $\{\omega_i\}$ of $H^0(C, K)$. In local coordinates assume that $\omega_i = f_i(z)dz$. Then the local expression of $\mu_1(\omega_i \wedge \omega_j)$ is the following:

$$\mu_1(\omega_i \wedge \omega_j) = (f_i'(z)f_j(z) - f_i(z)\,f_j'(z))\,dz^3. \tag{2.2.6}$$

Consequently the zero divisor of $\mu_1(\omega_i \wedge \omega_j)$ is twice the base locus of the pencil $\langle \omega_i, \omega_j \rangle$ plus the ramification divisor of the associated morphism (see for example [24], [88]). One can prove that this definition agrees with the general one given in line (2.2.2). It is easy to check that the first Gauss-Wahl map is independent of the choice of local generators, and that if $\omega_i \wedge \omega_j = 0$ then $\mu_1(\omega_i \wedge \omega_j) = 0$ (see e.g. Wahl [86]).

From the definition follows that the first Gauss-Wahl map is injective on decomposable vectors, since $f\,dg - g\,df = f^2 d(g/f) = 0$ implies $f \wedge g = 0$. Moreover we point out that it is G-equivariant:

Theorem 2.2.3. *Let $g : C \to C$ be an automorphism of the curve C. Then the following diagram commutes:*

$$\begin{array}{ccc} \Lambda^2 H^0(C, K) & \xrightarrow{\mu_1} & H^0(C, 3K) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g^*} \\ \Lambda^2 H^0(C, K) & \xrightarrow{\mu_1} & H^0(C, 3K) \end{array} \tag{2.2.7}$$

Proof. Consider g forms $\omega_i \in H^0(C, K)$, $i = 1, \dots g$, and their local expressions $\omega_i = f_i(z)dz$. Take an automorphism $g : C \to C$ and call $z = g(w)$ the transformed coordinate. We will check directly that $\mu_1 \circ g^* = g^* \circ \mu_1$. We start computing $\mu_1 \circ g^*$. Observe that g^* acts on forms ω_i as $g^*\omega_i = f_i(g(w))g'(w)dw$. We compute:

$$\begin{aligned} \mu_1\Big(\sum_{i,j} a_{ij}\, g^*\omega_i \wedge g^*\omega_j\Big) &= \sum_{i,j} a_{ij}\Big[(f_i(g(w))g'(w))' f_j(g(w))g'(w) - \\ &- (f_i(g(w))g'(w))(f_j(g(w))g'(w))'\Big](dz)^3 = \\ &= \sum_{i,j} a_{ij}\Big[\Big(f_i'(g(w))(g'(w))^2 + f_i(g(w))g''(w)\Big) f_j(g(w))g'(w) - \\ &- f_i(g(w))g'(w)\Big(f_j'(g(w))(g'(w))^2 + f_j(g(w))g''(w)\Big)\Big](dw)^3 = \\ &= \sum_{i,j} a_{ij}(g'(w))^3\Big(f_i'(g(w))f_j(g(w)) - f_i(g(w))f_j'(g(w)\Big)(dz)^3. \end{aligned}$$

We claim that one obtains the same expression considering $g^* \circ \mu_1$. In fact if $\mu_1(\sum_{ij} a_{ij}\, \omega_i \wedge \omega_j) = \alpha(z)(dz)^3$, then:

$$g^* : \alpha(z)(dz)^3 \mapsto \alpha(g(w))(g'(w))^3(dw)^3, \tag{2.2.8}$$

which is the same as expression (2.2.1). This concludes the proof. □

Remark 2.2.4. Observe that, in the non-hyperelliptic case, the first Gauss-Wahl map corresponds to the Gauss map in the usual sense[1] (see Wahl [88]). To see this, consider a non-hyperelliptic curve C embedded in $\mathbb{P}^{g-1}$ via the canonical bundle K. The (classical) Gauss map sends C to the Grassmannian $\mathbb{G}(1, g-1)$ of lines in $\mathbb{P}^{g-1}$, associating to each point its tangent line followed by the Plücker embedding into $\mathbb{P}^N$, where $N = g(g-1)/2 - 1$. Locally, take $P \in C$, with local coordinate t. If $H^0(K) = \langle \omega_0, \dots \omega_{g-1} \rangle$, and $\omega_i = f_i(t)dt$, where $f_i(t)$ are regular functions vanishing at P and $f_0 \equiv 1$, the Plücker embedding of the point of $\mathbb{G}(1, g-1)$ corresponding to the tangent line in P is found considering the 2×2 minors of the matrix:

$$\begin{pmatrix} 1 & f_1(t) & f_2(t) & \dots & f_{g-1}(t) \\ 0 & f_1'(t) & f_2'(t) & \dots & f_{g-1}'(t) \end{pmatrix},$$

which is the same as the first Gauss-Wahl map defined in (2.2.6).

To introduce the techniques used in the following, we deal here with the special problem of computing the rank of the first Gauss-Wahl map on the hyperelliptic locus ($g \geq 2$), see [24], [88].

Theorem 2.2.5. *If C is a hyperelliptic curve of genus $g \geq 2$, the rank of the first Gauss-Wahl map is $2g - 3$.*

Proof. Let C be a hyperelliptic curve of genus g, whose affine equation is $y^2 = f(x)$, where f is a polynomial of degree $2g + 1$ with simple roots. Observe that differentiating the affine equation one gets $2y\, dy = f'(x)\, dx$, that implies:

$$\frac{dx}{y} = \frac{2dy}{f'(x)}. \tag{2.2.9}$$

Since y and $f'(x)$ have disjoint vanishing loci, this shows that dx/y is a holomorphic 1-form on C. In particular, since forms $x^i dx/y$ and $x^j dx/y$ are

[1] Actually, this is true for every first Gaussian map $\mu_{1,|L|}$ with L very ample.

independent for every $i \neq j$, one can find a basis for the space of holomorphic 1-forms as follows:

$$H^0(C, K) = \left\langle \omega_i := x^i \frac{dx}{y} \,\middle|\, 0 \leq i \leq g-1 \right\rangle. \tag{2.2.10}$$

Applying the first Gauss-Wahl map to $\omega_i \wedge \omega_j$ one gets:

$$\begin{aligned} \mu_1(\omega_i \wedge \omega_j) &= (ix^{i+j-1} - jx^{i+j-1}) \left(\frac{dx}{y}\right)^3 = \\ &= (i-j)x^{i+j-1} \left(\frac{dx}{y}\right)^3. \end{aligned} \tag{2.2.11}$$

We want to compute the dimension of the span of $\{\mu_1(\omega_i \wedge \omega_j) \,|\, 0 \leq i < j \leq g-1\}$. Notice that $k = i+j-1$ takes all values between 0 and $g-4$: there are exactly $2g-3$ distinct powers of x. Hence our claim. □

Actually it is possible to prove that $2g-3$ is a lower bound for the rank of μ_1, which is reached if and only if the curve is hyperelliptic (see [46], [87]).

Theorem 2.2.6. *If $g \geq 4$, then* $\operatorname{rank} \mu_1 \geq 2g-3$, *with equality if and only if C is hyperelliptic.*

For high genus the rank of μ_1 is bounded above by the dimension of the target space, which is $5g-5$, and by the previous theorem is bounded below by $2g-3$. As remarked in [24], there are precisely $3g-3$ values in this interval, number which coincides with the dimension of the moduli space $\mathcal{M}_g$. Therefore one can wonder if all values are reached.

Another remarkable result is due to Wahl [85]. It concerns Gauss-Wahl maps of curves over $K3$-surfaces:

Theorem 2.2.7. *If C is a smooth curve which lies on a $K3$-surface, then its Gauss-Wahl map is not surjective.*

There are many other results on the first Gauss-Wahl map: Ciliberto and Miranda found that it is generically injective for $g \leq 8$ (see [25]). Later Ciliberto, Harris and Miranda proved that it is surjective for the general curve of genus $g = 10$ and $g \geq 12$ ([23], see also Voisin's proof in [84]). Recalling that the general curve of genus $g = 11$ lies on a $K3$-surface, this result is the best possible.

Table 2.1: Generic behaviour of $\mu_1 : \Lambda^2 H^0(C, K) \to H^0(K^3)$

	$3 \leq g \leq 8$	$g = 9$	$g = 10$	$g = 11$	$g \geq 12$
μ_1	injective	1-dim kernel	isomorphism	corank 1	surjective

In the following we give a list of results concerning the rank of the first Gauss-Wahl map on some specific loci: in 1992 Ciliberto and Miranda computed the rank of the first Gauss-Wahl map on the generic curve in the trigonal locus ($\operatorname{rank} \mu_1 = 4g - 10$), bielliptic locus ($\operatorname{rank} \mu_1 = 3g - 3$) and over smooth plane quintics ($\operatorname{rank} \mu_1 = 4g - 9$) [23]. The general result on the trigonal locus has been generalized to every trigonal curve by Brawner [15]. In its Ph.D. thesis [14] Brawner also computed a bound for the rank of the first Gauss-Wahl map on the tetragonal locus ($\operatorname{rank} \mu_1 \leq 5g - 14$). More recently, Ballico and Fontanari proved the surjectivity of the first Gauss-Wahl map on curves that are complete intersections [6].

2.2.2 The second Gauss-Wahl map

Recalling exact sequence (2.2.2), the second Gauss-Wahl map is:

$$\mu_2 := \mu_{2,K} : R_2(K) \to H^0(C, K^4), \tag{2.2.12}$$

where $R_2(K) = \ker(\mu_1 : \Lambda^2 H^0(C, K) \oplus I_2(K) \to H^0(K^3))$. Since μ_1 identically vanishes on $I_2(K)$ and since μ_2 identically vanishes on skew-symmetric tensors, we can consider:

$$\mu_2 : I_2(K) \to H^0(C, K^4). \tag{2.2.13}$$

Let us describe it in local coordinates: fix a basis $\{\omega_i\}$ of $H^0(C, K)$, and assume that, locally, $\omega_i = f_i(z)dz$. Take a linear combination $\sum_{i,j} a_{ij}\omega_i \otimes \omega_j$ lying in the $I_2(K)$, so that $\sum_{i,j} a_{ij}\omega_i\,\omega_j = 0$. Then the local expression of $\mu_2(\omega_i \odot \omega_j)$ is the following:

$$\mu_2(\sum_{i,j} a_{ij}\,\omega_i \otimes \omega_j) = \sum_{i,j} a_{ij} f_i'(z) f_j'(z)(dz)^4. \tag{2.2.14}$$

Remark 2.2.8. By the very definition of the $I_2(K)$ the sum $\sum_{i,j} a_{ij} f_i(z) f_j(z)$ is equal to zero. By symmetry, this implies that the sum $\sum_{i,j} a_{ij} f_i'(z) f_j(z)$ vanishes as well. Hence differentiating $\sum_{i,j} a_{ij} f_i'(z) f_j(z) \equiv 0$ one gets

$$\sum_{i,j} a_{ij} f_i''(z) f_j(z) + \sum_{i,j} a_{ij} f_i'(z) f_j'(z) \equiv 0.$$

This shows that the local definition given in expression (2.2.14) is equivalent to $\mu_2(\sum_{i,j} a_{ij}\,\omega_i \otimes \omega_j) := -\sum_{i,j} a_{ij} f_i''(z) f_j(z)(dz)^4$.

As in the case of the first Gauss-Wahl map, one can easily prove that this definition agrees with Definition 2.2.2, and that is independent from the choice of local generators. Moreover, it is G-equivariant as well:

Theorem 2.2.9. *Let $g : C \to C$ be an automorphism of the curve C. Then the following diagram commutes:*

$$\begin{tikzcd} I_2(K) \arrow[r, "\mu_2"] \arrow[d, "g^*"] & H^0(C, 4K) \arrow[d, "g^*"] \\ I_2(K) \arrow[r, "\mu_2"] & H^0(C, 4K) \end{tikzcd} \tag{2.2.15}$$

Proof. Consider forms $\omega_i \in H^0(C, K)$, $i = 1, \ldots, g$, and their local expressions $\omega_i = f_i(z)dz$. Take an automorphism $g : C \to C$ and call $z = g(w)$ the transformed coordinate. As in Theorem 2.2.3, we will check directly that $\mu_2 \circ g^* = g^* \circ \mu_2$. We start computing $\mu_2 \circ g^*$. Observe that g^* acts on forms ω_i as $g^*\omega_i = f_i(g(w))g'(w)dw$. We compute:

$$\begin{aligned}
\mu_2\Big(\sum_{i,j} a_{ij} g^*\omega_i \otimes g^*\omega_j\Big) &= \mu_2\Big(\sum_{i,j} a_{ij} f_i(g(w))g'(w)dw \otimes f_j(g(w))g'(w)dw\Big) = \\
&= \sum_{i,j} a_{ij}\Big((f_i'(g(w))(g'(w))^2 + f_i(g(w))g''(w)\Big)\cdot \\
&\cdot \Big(f_j'(g(w))(g'(w))^2 + f_j(g(w))g''(w)\Big)(dw)^4 = \\
&= \sum_{i,j} a_{ij} f_i'(g(w)) f_j'(g(w))(g'(w))^4 (dw)^4 + \\
&+ \sum_{i,j} a_{ij} f_i'(g(w) f_j(g(w))(g'(w))^2 g''(w)(dw)^4 + \\
&+ \sum_{i,j} a_{ij} f_i(g(w)) f_j'(g(w))(g'(w))^2 g''(w)(dw)^4 + \\
&+ \sum_{i,j} a_{ij} f_i(g(w)) f_j(g(w))(g''(w))^2 (dw)^4 = \\
&= \sum_{i,j} a_{ij} f_i'(g(w)) f_j'(g(w))(g'(w))^4 (dw)^4.
\end{aligned}$$

We claim that one obtains the same expression considering $g^* \circ \mu_2$. In fact if $\mu_2(\omega_1 \wedge \omega_2) = \alpha(z)(dz)^4$, then:

$$g^* : \alpha(z)(dz)^4 \mapsto \alpha(g(w))(g'(w))^4(dw)^4, \tag{2.2.16}$$

which is the same as expression (2.2.2). This concludes the proof. □

Since elements in the domain have a more complicated structure and higher order derivatives appear, dealing with Gauss-Wahl maps of greater order is much more difficult then dealing with the first Gauss-Wahl map. In the following we will show a trick allowing, in some cases, to translate the problem of studying the second Gauss-Wahl map on a problem on first Gauss-Wahl maps. We borrow ideas and notations from [27]. We start recalling the following useful bijection (see e.g. [5, p. 261]):

$$\{[Q] \in \mathbb{P}(I_2(K)) \mid \operatorname{rank}(Q) \leq 4\}$$

$$\updownarrow$$

$$\{\{L, K-L, V, W\} \mid V \subset H^0(L), \dim V = 2,\ W \subset H^0(K-L), \dim W = 2\}.$$

One can construct a quadric in the $I_2(K)$ from two adjoint line bundles L, $K-L$, with at least two global sections as follows: let $V = \langle x_1, x_2 \rangle \subset H^0(L)$ and $W = \langle t_1, t_2 \rangle \subset H^0(K-L)$. Then the quadric $Q = x_1t_1 \odot x_2t_2 - x_1t_2 \odot x_2t_1$ lies in $S^2H^0(K)$ by construction. Moreover it is immediate to see that it lies in the $I_2(K)$, in fact $m(Q) = x_1t_1x_2t_2 - x_1t_2x_2t_1 = 0$.

This correspondence has the following application:

Theorem 2.2.10. *If a quadric Q of rank at most 4 corresponds to $\{L, K-L, V, W\}$ and $V = \langle s_0, s_1 \rangle$, $W = \langle t_0, t_1 \rangle$, then*

$$\mu_2(Q) = \mu_{1,L}(s_0 \wedge s_1)\mu_{1,K-L}(t_0 \wedge t_1). \tag{2.2.17}$$

In particular $\mu_2(Q) \neq 0$.

Proof. By construction, $Q = (s_0t_0) \otimes (s_1t_1) - (s_0t_1) \otimes (s_1t_0) \in I_2(K)$. Locally $s_i = g_il$, where l is a local section of L, $t_i = h_il^{-1}dz$, so

$$\mu_2(Q) = ((g_0h_0)'(g_1h_1)' - (g_0h_1)'(h_0g_1)')(dz)^4 =$$

$$= (g_1g_0' - g_0g_1')(h_1h_0' - h_0h_1')(l^2dz)((l^{-1}dz)^2dz) = \mu_{1,L}(s_0 \wedge s_1)\mu_{1,K-L}(t_0 \wedge t_1).$$

□

This theorem is used in [27] to compute the rank of the first Gauss-Wahl map for hyperelliptic and trigonal curves. More precisely, the following holds:

Theorem 2.2.11. *Let C be a hyperelliptic curve of genus $g \geq 3$. Then*

$$\operatorname{rank} \mu_2 = 2g - 5,$$

and its image has the Weierstrass points as base points.

Theorem 2.2.12. *Let C be a trigonal non-hyperelliptic curve of genus $g \geq 8$. Then*

$$\operatorname{rank} \mu_2 = 4g - 18,$$

and its image has the ramification points of the $\mathfrak{g}^1_3$ as base points.

Here, $p \in C$ is a basepoint for $\operatorname{Im}(\mu_2)$ if $\mu_2(Q)(p) = 0$ for every $Q \in I_2(K)$. In the same paper, the authors proved that for any non-hyperelliptic, non-trigonal curve of genus $g \geq 5$ the image of μ_2 has no base points.

Differently from the case of the first Gauss-Wahl map, there is not any obstruction to the surjectivity of the second Gauss-Wahl map for a curve on a $K3$-surface[2]. Nevertheless it is possible to prove that every curve lying on an abelian surface has non-surjective second Gauss-Wahl map [31].

Theorem 2.2.13. *Let C be a curve contained in an abelian surface. Then the corank of μ_2 is at least 2.*

As in case of μ_1, we expect that the general curve has maximal rank in most of the cases. Notice that, for dimensional reasons, surjectivity can be expected for curves of genus $g \geq 18$. Calabri, Ciliberto and Miranda proved that the second Gauss-Wahl map has maximal rank for general curves of every genus [17]:

Theorem 2.2.14. *The second Gauss-Wahl map for a general curve of every genus g has maximal rank, namely it is injective for $g \leq 17$ and surjective for $g \geq 18$.*

We conclude this section stating a very general result, that gives a lower bound for the second Gauss-Wahl map of every curve of genus g.

Theorem 2.2.15 (Proposition 2.5, [27])**.** *For any curve of genus $g \geq 4$,*

$$\operatorname{rank} \mu_2 \geq g - 3. \tag{2.2.18}$$

[2]It is possible to prove that the general curve on a polarized $K3$-surface of degree $2g-2$ has surjective μ_2, if the genus is sufficiently high ($g > 280$) [28].

2.3 Second fundamental form and second Gaussian maps

There is a remarkable link between the second fundamental form of $j(\mathcal{M}_g) \subset \mathcal{A}_g$ and the second Gauss-Wahl map: as we will see in this section, the second Gaussian map lifts to the second fundamental form. This property allows one to study the curvature of the moduli space $\mathcal{M}_g$ and totally geodesic submanifolds, using results on the more tractable Gaussian maps.

Let C be a curve. We start considering the map dual to the second fundamental form, which we have already seen in Section 1.6:

$$\rho : I_2(K) \to S^2 H^0(C, 2K). \tag{2.3.1}$$

The following result has been stated in an unpublished paper of Green and Griffiths [48], and later proved[3] by Colombo, Pirola and Tortora [32]:

Theorem 2.3.1. *Let $m : S^2 H^0(C, 2K) \to H^0(C, 4K)$ be the multiplication map. Then the following diagram*

$$\begin{tikzcd} I_2(K) \arrow[r, "\rho"] \arrow[d, "\mu_2"'] & S^2 H^0(C, 2K) \arrow[d, "m"] \\ H^0(C, 4K) \arrow[r, equal] & H^0(C, 4K) \end{tikzcd} \tag{2.3.2}$$

is commutative up to a constant.

The theorem shows that the map ρ is the lifting of the second Gauss-Wahl map, and suggests the possibility of making explicit computations, at least in case of curves. This seemed, from the beginning, a step towards understanding the curvature of the moduli space.

The strategy of the proof of Theorem 2.3.1 relies in comparing the values of $\mu_2(Q)$ and values of $(m \circ \rho)(Q)$ over all points p in some open subset $U \subset C$. While $\mu_2(Q)(p)$ can be easily computed using explicit expression (2.2.14), to evaluate $(m \circ \rho)(Q)(p)$ it is convenient to define the dual map m^* in term of particular elements in $H^1(C, T_C)$, called Schiffer variations. We briefly recall their definition.

[3] Actually the result in [32] holds in a more general setting: it concerns not only the canonical bundle, but any line bundle on a curve.

Schiffer variations Fix a point p over the curve C and consider the short exact sequence of tangent bundle: $0 \to T_C \to T_C(p) \to T_C(p)\big|_p \to 0$. The coboundary gives an injection $H^0(T_C(p)\big|_p) \cong \mathbb{C} \hookrightarrow H^1(C, T_C)$. Elements in the image are called *Schiffer variations* at p. More precisely, fix a chart (U, z) centered in p and take a bump function $b \in C_0^\infty(U)$ which is equal to 1 in a neighbourhood of p. Define

$$\theta := \frac{\bar{\partial} b}{z} \cdot \frac{\partial}{\partial z}. \tag{2.3.3}$$

θ is a Dolbeault representative of a Schiffer variation at p. The map

$$\xi : TC \to H^1(C, T_C), \qquad u = \lambda \frac{\partial}{\partial z}(p) \mapsto \xi_u := \lambda^2[\theta] \tag{2.3.4}$$

does not depend on the choice of the coordinates. It is well known that Schiffer variations generate $H^1(C, T_C)$ (see [4, p.175]). Sometimes we will write ξ_p instead of ξ_u.

Coming back to the idea behind Theorem 2.3.1, if ν_p is the evaluation map at p, then, up to a constant, $m^*(\nu_p) = \xi_p \odot \xi_p$. Thus $(m \circ \rho)(Q)(p) = (\xi_p \odot \xi_p)(\rho(Q))$, and the right term is computed using the explicit representation (2.3.3) of the Schiffer variation at p.

In particular it is possible to give an explicit expression for the second fundamental form when evaluated on the product of two Schiffer variations. While the computation of $\rho(Q)(\xi_p \odot \xi_p)$ goes through a computation of the second Gaussian map, it is much harder to compute the second fundamental form on $\xi_p \odot \xi_q$ when $p \neq q$. The formula contains, in fact, the evaluation at q of a meromorphic 1-form on the curve, called η_p, which has a double pole at p and is defined by Hodge theory. We explain how this form arises in the following paragraph.

The form η_p Consider a curve C of genus $g \geq 4$, and take a point $p \in C$. Consider the space $H^0(C, K_C(2p))$ of meromorphic 1-forms on C with a double pole on p, and notice that it goes injectively into $H^1(C - p, \mathbb{C})$. By the Mayer-Vietoris sequence, the isomorphism $H^1(C, \mathbb{C}) \cong H^1(C - p, \mathbb{C})$ holds, thus there is an injection:

$$j_p : \ H^0(C, K_C(2p)) \hookrightarrow H^1(C, \mathbb{C}). \tag{2.3.5}$$

Remark that the holomorphic part $H^{1,0}(C)$ is inside the image of j_p. Moreover, since $h^0(C, K_C(2p)) = g+1$, the preimage of the anti-holomorphic part,

$j_p^{-1}(H^{0,1}(C))$, has dimension 1. Now, fix a local chart (U, z) centered in p. Then there exists a unique element φ on this line such that its expression on $U - p$ is

$$\varphi := \left(\frac{1}{z^2} + h(z)\right) dz, \tag{2.3.6}$$

where h is a holomorphic function. One can finally define the map η_p as follows:

$$\begin{aligned} \eta_p : T_pC &\longrightarrow H^0(C, K_C(2p)), \\ u = \lambda \frac{\partial}{\partial z}(p) &\longmapsto \eta_p(u) = \lambda\varphi. \end{aligned} \tag{2.3.7}$$

An easy computation shows that η_p does not depend on the choice of the local coordinate.

The following is the key theorem for the whole chapter, due to Colombo, Pirola and Tortora [32]. See also [30].

Theorem 2.3.2. *Let C be a non-hyperelliptic curve of genus $g \geq 4$. Let $p, q \in C$ and $u \in T_pC$, $v \in T_qC$. Then we have*

$$\begin{aligned} \rho(Q)(\xi_u \odot \xi_v) &= -4\pi i \eta_p(u, v) Q(u, v), \\ \rho(Q)(\xi_u \otimes \xi_u) &= -2\pi i \mu_2(Q)(u^{\otimes 4}). \end{aligned} \tag{2.3.8}$$

Using this theorem, the computation of the second fundamental form on the product of Schiffer variations reduces to the evaluation of the second Gaussian map and to the study of the form η_p. Although this important result holds, in general it seems rather hard to control the behaviour of η_p in a way to get constraints on the second fundamental form. Moreover Theorem 2.3.2 concerns the second fundamental form of the period map $j : \mathcal{M}_g \to \mathcal{A}_g$. To study whether a submanifold $X \subset \mathcal{M}_g$ is totally geodesic, one needs to study the second fundamental form ρ_X of the inclusion $X \to \mathcal{A}_g$. The following diagram clarifies the connection between the two maps, ρ and ρ_X, when X is the variety described by a family $\{C_t\}_t$ of curves covering $\mathbb{P}^1$ with Galois group G.

$$\begin{tikzcd} 0 \arrow[r] & I_2(K_{C_t}) \arrow[r] \arrow[d, hook] & S^2H^0(K_{C_t}) \arrow[r, "m"] \arrow[d, "="] & H^0(2K_{C_t}) \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & N^* \arrow[r] & S^2H^0(K_{C_t}) \arrow[r] & H^0(2K_{C_t})^G \arrow[r] & 0 \end{tikzcd} \tag{2.3.9}$$

Here $0 \to I_2(K_{C_t}) \to S^2H^0(K_{C_t}) \to H^0(2K_{C_t}) \to 0$ is the cotangent exact sequence of the period map $j : \mathcal{M}_g \to \mathcal{A}_g$ at the point $[C_t]$, whereas $0 \to N^* \to S^2H^0(K_{C_t}) \to H^0(2K_{C_t})^G \to 0$ is cotangent exact sequence relative to the immersion $i : X \to \mathcal{A}_g$. Since the second fundamental form is G-equivariant (see [30, Theorem 5.3]), diagram (2.3.9) implies that:

$$\rho(Q)(v_1 \odot v_2) = \rho_X(Q)(v_1 \odot v_2), \qquad \begin{array}{l} \forall\, v_1, v_2 \in H^1(TC)^G, \\ \forall\, Q \in I_2(K)^G. \end{array} \tag{2.3.10}$$

Theorem 2.3.2 has been used in [29] to compute the curvature of the restriction to $\mathcal{M}_g$ of the Siegel metric. The authors also gave an explicit formula for the holomorphic sectional curvature of $\mathcal{M}_g$ in the direction ξ_p in terms of the holomorphic sectional curvature of $\mathcal{A}_g$ and the second Gaussian map.

We conclude this section observing that it is possible to give a more intrinsic description of the form η_p: Colombo, Frediani and Ghigi proved that as p varies on the curve C the form η_p glues to give a holomorphic section $\hat{\eta}$ of the line bundle $K_S(2\Delta)$, being $S = C \times C$ and $\Delta \subset S$ the diagonal [30]. The authors also proved that the second fundamental form coincides with the multiplication by $\hat{\eta}$. The result is the following (notice that, by Künneth formula $H^0(S, 2K_S) \cong H^0(C, 2K_C) \otimes H^0(C, 2K_C)$; in particular $I_2(K_C) \subset H^0(S, K_S(-2\Delta))$):

Theorem 2.3.3. *The following diagram commutes:*

$$\begin{tikzcd} I_2(K) \arrow[r, "\rho"] \arrow[d, hook] & S^2H^0(C, 2K) \arrow[d, "m"] \\ H^0(S, K_S(-2\Delta)) \arrow[r, "Q \mapsto Q\cdot\hat{\eta}"] & H^0(S, 2K_S) \end{tikzcd} \tag{2.3.11}$$

In [30] the theorem just stated is used to find constraints on the dimension of totally geodesic submanifold of $\mathcal{A}_g$ contained in the Jacobian locus. We mention the following results:

Theorem 2.3.4. *Assume that C is a k-gonal curve of genus g, with $g \geq 4$ and $k \geq 3$. Let Y be a germ of a totally geodesic submanifold of $\mathcal{A}_g$ which is contained in the Jacobian locus and passes through $J([C]) = [J(C)]$. Then*

$$\dim Y \leq 2g + k - 4. \tag{2.3.12}$$

Since the gonality always satisfies $k \leq \lfloor \frac{g+3}{2} \rfloor$, this immediately yields a bound depending only on the genus g:

Corollary 2.3.5. *If $g \geq 4$ and Y is a germ of a totally geodesic submanifold of $\mathcal{A}_g$ contained in the Jacobian locus, then*

$$\dim Y \leq \frac{5}{2}\Big(g-1\Big). \tag{2.3.13}$$

The last results allow to conclude that the k-gonal locus is not totally geodesic in $\mathcal{A}_g$, if $g \geq 4$, $k \geq 3$. In fact the dimension of the k-gonal locus in this case is $2g + 2k - 5 > 2g + k - 4$: the statement immediately follows from Theorem 2.3.4 [30].

2.4 Bielliptic locus

A curve C is called bielliptic if it is double cover of an elliptic curve E:

$$C \xrightarrow{2:1} E \xrightarrow{2:1} \mathbb{P}^1. \tag{2.4.1}$$

In this section we will study the locus of bielliptic curves of genus g, which we will denote by $\mathcal{B}_g$. We will first make some general considerations on $\mathcal{B}_g$, then we will prove that the locus is not totally geodesic if $g \geq 4$ (while it is for $g = 3$, see example (2) in the list of Section 1.7). We will conclude this section performing a computation for the rank of the second Gauss-Wahl map on this loci, and giving a bound for it.

We start recalling some elementary results on $\mathcal{B}_g$, which will be useful to clarify the set-up and basic properties of this locus. First of all we point out that from Castelnuovo-Severi inequality (see for instance [3]) it follows immediately that every bielliptic curve with genus $g \geq 4$ can not be hyperelliptic (see [8]). We compute the dimension of the bielliptic locus.

Lemma 2.4.1. *The bielliptic locus has dimension $2g - 2$.*

Proof. Using Riemann Hurwitz formula (1.1.1) we get that the cover $C \to E$ has $2g - 2$ ramification points. Since the moduli space of elliptic curves has dimension 1 and we can fix, using automorphisms of E, a point on E, the dimension of the family is exactly $2g - 2$. □

It is known that the bielliptic locus $\mathcal{B}_g$ is irreducible (see e.g. [8]). In the following, we give a different proof.

Lemma 2.4.2. *The bielliptic locus is irreducible.*

Proof. The action of the mapping class group $\mathrm{Map}_{1,[2g-2]}$ on $\Gamma_{1,2g-2}$ is completely described in Theorem 1.4.16. Notice that the ξ-twist $t_{\xi^1_{1,d}}$ maps $\alpha \mapsto \alpha\gamma_d$, and let other generators invariant. So it induces the action given by $\theta(\alpha) \mapsto \theta(\gamma_d\alpha)$, and the identity on the other generators. In the same way, the ξ-twist $t_{\xi^2_{1,d}}$ induces the action given by $\theta(\beta) \mapsto \theta(\gamma_d\beta)$, and the identity on the other generators. This gives that the system of generators $\langle\theta(\gamma_1),\dots,\theta(\gamma_r);\theta(\alpha),\theta(\beta)\rangle = \langle z,\dots,z;1,1\rangle$ and systems of generators $\langle z,\dots,z;z,z\rangle = \langle z,\dots,z;1,z\rangle = \langle z,\dots,z;z,1\rangle$ are Hurwitz equivalent. So all possible choices for the epimorphism $\Gamma_{1,2g-2} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ are equivalent. This implies that the locus is irreducible. $\square$

Now we prove, with a simple dimension count, that every quadric in the bielliptic locus is invariant via the bielliptic involution.

Lemma 2.4.3. *Call σ the bielliptic involution. Then in the bielliptic locus every quadric is invariant, that is:*

$$I_2(K) = I_2(K)^\sigma. \tag{2.4.2}$$

Proof. If C is a bielliptic curve, $H^0(C,K_C)$ splits in the invariant part, which satisfies $H^0(C,K_C)^\sigma \cong H^0(E,K_E)$, and the anti-invariant one:

$$H^0(C,K_C) \cong H^0(E,K_E) \oplus H^0(C,K_C)^-. \tag{2.4.3}$$

We decompose also the symmetric product $S^2H^0(C,K_C)$ as the sum of its invariant and anti-invariant parts. More precisely:

$$\begin{aligned} &S^2H^0(C,K_C) = \big(S^2H^0(C,K_C)\big)^\sigma \oplus \big(S^2H^0(C,K_C)\big)^-, \text{ where:} \\ &\quad \big(S^2H^0(C,K_C)\big)^\sigma \cong S^2H^0(E,K_E) \oplus S^2H^0(C,K_C)^-; \\ &\quad \big(S^2H^0(C,K_C)\big)^- \cong H^0(E,K_E) \otimes H^0(C,K_C)^-. \end{aligned} \tag{2.4.4}$$

We can directly compute: $\dim\big(S^2H^0(C,K_C)\big)^- = h^0(C,K_C)^- = g-1$. Since $h^0(C,2K_C)^\sigma$ is the dimension of the bielliptic locus, we have $h^0(C,2K_C)^- =$

$h^0(C, 2K_C) - h^0(C, 2K_C)^\sigma = 3g - 3 - (2g - 2) = g - 1$. From short exact sequence

$$0 \to I_2(K_C)^- \to S^2H^0(C, K_C)^- \to H^0(C, 2K_C)^- \to 0,$$

we get $I_2(K_C)^- = (0)$, so $I_2(K_C) = I_2(K_C)^\sigma$. Hence the claim. □

The previous lemma allows us to find an upper bound for the rank of the second Gauss-Wahl map when evaluated on the bielliptic locus. This is the content of the following lemma.

Lemma 2.4.4. *On the bielliptic locus the second Gauss-Wahl map has rank at most* $5g - 5$.

Proof. Recall from diagram (2.2.15) that the second Gauss-Wahl map is G-equivariant. This implies that on the bielliptic locus, the second Gauss-Wahl map takes values on the invariant part of $H^0(C, 4K_C)$:

$$\mu_2 : I_2(K_C)^\sigma \to H^0(C, 4K_C)^\sigma. \tag{2.4.5}$$

Therefore, in order to bound the corank of μ_2, we try to compute $h^0(C, 4K_C)^-$.

First of all, notice that we can consider elements of $H^0(C, 4K_C)^-$ obtained via the multiplication of an invariant 1-form with a section of $H^0(C, 3K_C)^-$, that is:

$$H^0(K_C)^\sigma \otimes H^0(3K_C)^- \hookrightarrow H^0(4K_C)^-. \tag{2.4.6}$$

Since $H^0(C, K_C)^\sigma = 1$, this map is injective. This shows that the dimension of $H^0(C, 4K_C)^-$ is bounded below by the dimension of $H^0(C, 3K_C)^-$. We try to compute the last one.

Analogously as before, consider elements in $H^0(3K_C)^-$ obtained as a product between an anti-invariant 1-form with a section of $H^0(C, 2K_C)^\sigma$, that is, consider the map $H^0(K_C)^- \otimes H^0(2K_C)^\sigma \to H^0(3K_C)^-$. If we fix an element $\eta \in H^0(K_C)^-$ and we consider the restriction of the multiplication map to

$$\langle \eta \rangle \otimes H^0(2K_C)^\sigma \hookrightarrow H^0(3K_C)^-, \tag{2.4.7}$$

we get an injective map, since one of the two vector spaces in the tensor product has dimension 1. Moreover $h^0(2K_C)^\sigma = 2g-2$, that is the dimension of the bielliptic locus: we obtain

$$h^0(4K_C)^- \geq h^0(3K_C)^- \geq 2g - 2.$$

This implies $\operatorname{corank} \mu_2 \geq 2g - 2$, so $\operatorname{rank} \mu_2 \leq 5g - 5$ as required. □

In the next part we will prove that the moduli space of bielliptic curves is not totally geodesic in $\mathcal{A}_g$ if the genus $g(C) \geq 4$.

2.4.1 The bielliptic locus is not totally geodesic

In this section we will prove that the bielliptic locus is not totally geodesic. We will consider separately case $g = 4$ and case $g \geq 5$. Notice that in case $g = 3$ we already know that the bielliptic locus is totally geodesic, since it is one of the examples arising in Section 1.7.

We start studying case $g \geq 5$. The trick for the proof is analogous to the one used by Colombo, Frediani and Ghigi in [30] to bound the dimension of germs of totally geodesic submanifolds contained in the Jacobian locus (see Corollary 2.3.5): maps $k : 1$ over $\mathbb{P}^1$ are used to construct two adjoint line bundles on the curve C and consequently a quadric $Q \in I_2(K)$. Theorem 2.3.2 translates the computation of the second fundamental form in Q in the product of two first Gauss maps relative to the adjoint line bundles (see Theorem 2.2.10). The problem of computing η in expression (2.3.2) is avoided choosing a setting in which $Q(u, v)$ vanishes.

Theorem 2.4.5. *The bielliptic locus is not totally geodesic if $g \geq 5$.*

Proof. Consider a bielliptic curve C of genus $g \geq 5$. It admits a $\mathfrak{g}^1_4$ which we call $|F|$. Fix a basis for $H^0(F) = \langle x_1, x_2 \rangle$ and consider the adjoint $\mathfrak{g}^1_4$ given by $|K - F|$. If we compute, using Riemann-Hurwitz formula, the dimension of its space of global sections we obtain:

$$h^0(K - F) = h^0(F) + \deg(K) - \deg(F) - g + 1 = g - 3.$$

Here the importance of the hypothesis $g \geq 5$: in this case we can find a pencil $\langle t_1, t_2 \rangle \subseteq H^0(K - F)$. Using both the $\mathfrak{g}^1_4$'s, we can construct a quadric containing the canonical curve in the following way:

$$Q := x_1 t_1 \odot x_2 t_2 - x_1 t_2 \odot x_2 t_1 \in I_2(K_C). \tag{2.4.8}$$

Notice that $Q \in I_2(K_C)$ by construction. Also, since by Lemma 2.4.3 every quadric in the bielliptic locus is invariant, $Q \in I_2(K_C)^\sigma$.

Looking at equality (2.3.10), to prove that the subvariety of $\mathcal{M}_g$ made of bielliptic curves is not totally geodesic it is enough to find a pair of tangent

vectors $v, w \in H^1(C, TC)^\sigma$ such that $\rho(Q)(v \odot w) \neq 0$. We will use Theorem 2.3.2 to evaluate $\rho(Q)$ over Schiffer variations.

Consider two points $p_1, p_2 \in C$ lying on the same fiber over a point $p \in E$ sufficiently general, that is non critical for $|F|$, $|K - F|$ neither fixed for the involution σ. We notice that since the bielliptic involution switches ξ_{p_1} and ξ_{p_2}, the element $\xi_{p_1} + \xi_{p_2} \in H^1(C, TC)^\sigma$. Compute:

$$\rho(Q)((\xi_{p_1} + \xi_{p_2}) \odot (\xi_{p_1} + \xi_{p_2})) = \rho(Q)(\xi_{p_1} \odot \xi_{p_1}) + \rho(Q)(\xi_{p_2} \odot \xi_{p_2}) + 2\rho(Q)(\xi_{p_1} \odot \xi_{p_2}).$$

Since the map ρ is σ-equivariant, and the quadric Q also is, one has:

$$\rho(Q)(\xi_{p_2} \odot \xi_{p_2}) = \rho(Q)(\xi_{\sigma(p_1)} \odot \xi_{\sigma(p_1)}) = \rho(\sigma^* Q)(\xi_{p_1} \odot \xi_{p_1}) = \rho(Q)(\xi_{p_1} \odot \xi_{p_1}).$$

Putting together the previous equations one gets:

$$\rho(Q)((\xi_{p_1} + \xi_{p_2}) \odot (\xi_{p_1} + \xi_{p_2})) = 2\rho(Q)(\xi_{p_1} \odot \xi_{p_1}) + 2\rho(Q)(\xi_{p_1} \odot \xi_{p_2}).$$

Consider term $\rho(Q)(\xi_{p_1} \odot \xi_{p_2})$: from Theorem 2.3.2, $\rho(Q)(\xi_{p_1} \odot \xi_{p_2}) = -4\pi i\, \eta_{p_1}(p_2) Q(p_1, p_2)$. Notice that, by construction, the quadric Q vanishes when evaluated over point lying over the same fiber. In fact since p_1 and p_2 are in the same fiber for $|F|$, then $[x_1(p_1) : x_2(p_1)] = [x_1(p_2) : x_2(p_2)]$. So there exists $\lambda \in \mathbb{C}^*$ such that $x_i(p_2) = \lambda x_i(p_1)$ for $i = 1, 2$:

$$\begin{aligned} Q(p_1, p_2) =& x_1(p_1)t_1(p_1)x_2(p_2)t_2(p_2) + x_1(p_2)t_1(p_2)x_2(p_1)t_2(p_1) - \\ & - x_1(p_1)t_2(p_1)x_2(p_2)t_1(p_2) - x_1(p_2)t_2(p_2)x_2(p_1)t_1(p_1) = \\ & = \lambda x_1(p_1)t_1(p_1)x_2(p_1)t_2(p_2) + \lambda x_1(p_1)t_1(p_2)x_2(p_1)t_2(p_1) - \\ & - \lambda x_1(p_1)t_2(p_1)x_2(p_1)t_1(p_2) - \lambda x_1(p_1)t_2(p_2)x_2(p_1)t_1(p_1) = \\ & = \lambda x_1(p_1)x_2(p_1)\big(t_1(p_1)t_2(p_2) + t_1(p_2)t_2(p_1) - \\ & \qquad - t_1(p_2)t_2(p_1) - t_1(p_1)t_2(p_2)\big) = 0. \end{aligned} \tag{2.4.9}$$

This immediately implies that the mixed term $\rho(Q)(\xi_{p_1} \odot \xi_{p_2})$ is equal to 0. Consider now $\rho(Q)(\xi_{p_1} \odot \xi_{p_1})$. It corresponds, up to scalar multiplication, to the second Gauss-Wahl map $\mu_2(Q)$. From Theorem 2.2.10 we know that the second Gauss-Wahl map in our case decomposes as:

$$\mu_2(Q) = \mu_{1,F}(x_1 \wedge x_2)\mu_{1,K-F}(t_1 \wedge t_2).$$

Furthermore $\mu_{1,F}(x_1 \wedge x_2)$ and $\mu_{1,K-F}(t_1 \wedge t_2)$ vanish, respectively, only over critical points of $|F|$ and $|K-F|$: since point p_1 is non-critical for the two maps we can conclude that

$$\rho(Q)(\xi_{p_1} \odot \xi_{p_1}) = -2\pi i \mu_2(Q)(p_1) \neq 0.$$

We found a tangent vector $\xi_{p_1} + \xi_{p_2} \in H^1(C, TC)^\sigma$ such that $\rho(Q)((\xi_{p_1} + \xi_{p_2}) \odot (\xi_{p_1} + \xi_{p_2})) \neq 0$ where Q is an invariant quadric. So the bielliptic locus is not totally geodesic if $g \geq 5$. □

Summarizing, we used the $\mathfrak{g}^1_4$'s $|F|$ and $|K-F|$ of the bielliptic curve to construct an invariant quadric in the $I_2(K)$. Then we found a pair of elements in $H^1(C, TC)^\sigma$ on which we are able to compute the second fundamental form. Since we have an explicit expression for the second fundamental form when evaluated on the product of two Schiffer variations, we find a suitable combination of them that is invariant and annihilates the quadric Q. Finally, the computation reduces to a product of two first Gaussian maps, that we know to be different from 0. This prevents the possibility of being totally geodesic.

Notice that we have considered separately cases $g \geq 5$ and $g = 4$ because in the latter case $h^0(K-F) = 1$, so one cannot construct a quadric of rank 4 using $|F|$ and $|K-F|$. In case $g = 4$ we will fix the problem by using the two $\mathfrak{g}^1_3$'s of C: from Brill-Noether theory a linear serie $|M|$ of degree 3 with two global sections exists. First of all we prove that we obtain a second $\mathfrak{g}^1_3$ by the adjoint linear system. It is different from $|M|$, and is switched with $|M|$ via the bielliptic involution.

Theorem 2.4.6. *Let C be a smooth bielliptic curve of genus $g = 4$. Then C admits two different $\mathfrak{g}^1_3$'s switched by the bielliptic involution.*

Proof. We start recalling that a bielliptic curve of genus $g = 4$ is non-hyperelliptic. Its canonical model is a smooth complete intersection between a quadric and a cubic in $\mathbb{P}^3$. From Brill-Noether theory follows that there are exactly two $\mathfrak{g}^1_3$'s over C, cut out by the lines of a ruling of the quadric containing the canonical model of C (see e.g. [5, page 206]). We need to prove that the two $\mathfrak{g}^1_3$'s are switched by the bielliptic involution.

Call $|F|$ a $\mathfrak{g}^1_3$, and call φ the corresponding map $\varphi : C \to \mathbb{P}^1$. Consider the adjoint linear serie $|K-F|$ and the induced map $\psi : C \to \mathbb{P}^1$. Let σ be

the bielliptic involution. We want to prove that $\varphi \circ \sigma = \psi$. Since $\varphi \circ \sigma$ is a $\mathfrak{g}^1_3$ as well, it is enough to prove that $\varphi \circ \sigma \neq \varphi$. Assume by contradiction that $\varphi \circ \sigma = \varphi$, so that the following diagram holds:

$$\begin{tikzcd} C \arrow[rr, "\sigma"] \arrow[dr, "\varphi"'] & & C \arrow[dl, "\varphi"] \\ & \mathbb{P}^1 & \end{tikzcd} \tag{2.4.10}$$

If $\varphi \circ \sigma = \varphi$, then φ induces:

$$\begin{tikzcd} C \arrow[rr, "\pi"] \arrow[dr, "\varphi"'] & & C/\langle\sigma\rangle \arrow[dl, "\bar{\varphi}"] \\ & \mathbb{P}^1 & \end{tikzcd} \tag{2.4.11}$$

Here $\bar{\varphi}$ if defined by $\bar{\varphi}(\pi(y)) = \varphi(y)$. Remark that $\bar{\varphi}$ is well defined, since whenever $\pi(y) = \pi(y')$ we have $y' = \sigma(y)$. We conclude the proof observing that such a situation can not occur, since φ is $3:1$ and π is $2:1$. $\square$

We are ready to prove that the bielliptic locus is not totally geodesic also in genus $g = 4$.

Theorem 2.4.7. *The bielliptic locus is not totally geodesic if $g = 4$.*

Proof. Let C be a bielliptic curve of genus 4. From Theorem 2.4.6 C admits two different $\mathfrak{g}^1_3$'s, $|M|$ and $|K - M|$, switched by the bielliptic involution. That is, the following diagram holds:

$$\begin{tikzcd} C \arrow[rr, "\sigma"] \arrow[dr, "\varphi"'] & & C \arrow[dl, "\psi"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

where φ and ψ are, respectively, maps induced by $|M|$ and $|K - M|$ over $\mathbb{P}^1$.

Choose a basis x_1, x_2 for $H^0(M)$, and a basis t_1, t_2 for $H^0(K - M)$, where $t_i = x_i \circ \sigma$. We need to pick a point which is fixed by the involution σ and regular for the $\mathfrak{g}^1_3$. It exists via the following claim.

Claim. *There exists at most one fixed point for σ which is critical for the maps φ and ψ as well.*

To prove the claim, argue by contradiction: suppose that there are two different points, p and p' fixed by σ and critical for φ and ψ. Call $w = \varphi(p) = \psi(p)$ and $w' = \varphi(p') = \psi(p')$. Consider the pullbacks:

$$\begin{aligned} |L| &\equiv \varphi^*(w) = |2p+q|, & |K-L| &\equiv \psi^*(w) = |2p+q'|, \\ |L| &\equiv \varphi^*(w') = |2p'+r|, & |K-L| &\equiv \psi^*(w') = |2p'+r'|. \end{aligned} \tag{2.4.12}$$

Equations above imply that $|K-2L| \equiv |q'-q| \equiv |r'-r|$. Reordering, one gets $|q'+r| \equiv |q+r'|$. This is absurd since the curve C is not hyperelliptic.

Using the claim, we can pick a point $p_1 \in C$ which is fixed by the involution σ and regular for φ and ψ. Call p its image in $\mathbb{P}^1$.

Claim. *The Schiffer variation ξ_{p_1} is invariant under the bielliptic involution σ.*

Proof. Using Theorem 1.1.4, we can find a local chart (U, z) in a neighbourhood of p_1 such that $\sigma(z) = -z$. Set $w = -z$ and call $\sigma : C \to C$ the involution. By hypothesis, $\sigma(p_1) = p_1$. Without any restriction, we can choose the bump function b_{p_1} to be invariant by σ. A Schiffer variation in p_1 is (we choose the coefficient $\lambda = 1$):

$$\xi_{p_1} = \frac{\bar{\partial} b_{p_1}}{z} \cdot \frac{\partial}{\partial z} = \frac{1}{z} \cdot \frac{\partial b_{p_1}}{\partial \bar{z}} \cdot d\bar{z} \cdot \frac{\partial}{\partial z}. \tag{2.4.13}$$

Notice that locally:

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{\partial w}{\partial z} \cdot \frac{\partial}{\partial w}, & \frac{\partial w}{\partial z} &= \frac{\partial \sigma}{\partial z} = -1; \\ \frac{\partial}{\partial \bar{z}} &= \frac{\partial \bar{w}}{\partial \bar{z}} \cdot \frac{\partial}{\partial \bar{w}}, & \frac{\partial \bar{w}}{\partial \bar{z}} &= \frac{\partial \bar{\sigma}}{\partial \bar{z}} = -1; \end{aligned}$$

implying

$$\frac{\partial}{\partial z} = -\frac{\partial}{\partial w}, \qquad \frac{\partial}{\partial \bar{z}} = -\frac{\partial}{\partial \bar{w}}.$$

Moreover the following holds:

$$\begin{aligned} \frac{1}{z} &= \frac{w}{z} \cdot \frac{1}{w} = -\frac{1}{w}, \\ \frac{d}{dz} &= \frac{dw}{dz} \cdot \frac{d}{dw} = -\frac{d}{dw}. \end{aligned}$$

Studying separately the pullback of each term of equation (2.4.13) via the involution σ, one gets:

$$\frac{1}{z} \mapsto -\frac{1}{w} \qquad \frac{\partial b_{p_1}}{\partial \bar{z}} \mapsto -\frac{\partial b_{p_1}}{\partial \bar{w}} \qquad d\bar{z} \mapsto -d\bar{w} \qquad \frac{\partial}{\partial z} \mapsto -\frac{\partial}{\partial w}. \tag{2.4.14}$$

The Schiffer variation ξ_{p_1} is invariant via σ^*, in fact:

$$\xi_{p_1} = \frac{1}{z} \cdot \frac{\partial b_{p_1}}{\partial \bar{z}} \cdot dz \cdot \frac{\partial}{\partial z} \mapsto \Big(-\frac{1}{w} \Big) \cdot \Big(-\frac{\partial b_{p_1}}{\partial \bar{w}} \Big) \cdot \Big(-d\bar{w} \Big) \cdot \Big(-\frac{\partial}{\partial w} \Big) = \xi_{p_1}.$$

□

We have a ready-to-use invariant Schiffer variation. To apply Theorem 2.3.2, we just need to find an appropriate quadric. Notice that in genus 4 the space of quadrics containing the canonical curve has dimension 1, and recall that every quadric in the bielliptic locus is invariant, so every nonzero quadric $Q \in I_2(K_C) = I_2(K_C)^\sigma$ generates the whole space.

Similarly to case $g \geq 5$, we construct an invariant quadric using the $\mathfrak{g}^3_1$'s:

$$Q := x_1 t_1 \odot x_2 t_2 - x_1 t_2 \odot x_2 t_1 \in I_2(K_C)^\sigma,$$

where $H^0(M) = \langle x_1, x_2 \rangle$ and $H^0(K - M) = \langle t_1, t_2 \rangle$. Again, using Theorem 2.3.2, it is possible to perform an explicit computation for $\rho(Q)$ when evaluated over the invariant Schiffer variation ξ_{p_1}. We immediately get that this computation reduces to the computation of the second Gauss-Wahl map:

$$\rho(Q)(\xi_{p_1} \odot \xi_{p_1}) = -2\pi i \mu_2(Q)(p_1),$$

which we know to be

$$\mu_2(Q) = \mu_{1,|M|}(x_1 \wedge x_2)\mu_{1,|K-M|}(t_1 \wedge t_2).$$

Since $|M|$ and $|K - M|$ are base point free, by construction $\mu_{1,|M|}(x_1 \wedge x_2)$ and $\mu_{1,|K-M|}(t_1 \wedge t_2)$ vanish, respectively, only over critical points of $|M|$ and $|K - M|$. Since point p_1 is non-critical for the two maps, we can conclude that

$$\rho(Q)(\xi_{p_1} \odot \xi_{p_1}) = -2\pi i \mu_2(Q)(p_1) \neq 0.$$

This shows that the bielliptic locus is not totally geodesic also when $g = 4$.

□

Despite these ideas seem quite easy to be generalized, it turns to be hard to perform similar computations in a general setting. The simplicity of the bielliptic case relies in the existence of the bielliptic involution σ: using σ one can construct an appropriate combination of Schiffer variations that is invariant and annihilates the quadric. We anticipate that in the next section we will generalize this result a little, studying the locus of curves admitting a $2:1$ cover over a hyperelliptic curve. The involution will play a key role also in this setting.

2.4.2 Second Gauss-Wahl map on the bielliptic locus

In this part we will use some results and properties stated before to find a bound for the rank of the second Gauss-Wahl map on the general curve of the bielliptic locus ($g \geq 5$). Recall that the rank of the first Gauss-Wahl map of a bielliptic curve is known to be $3g-3$ (see [24, Section 3]).

The trick in the following calculus is to use Theorem 2.2.10 to reduce the computation of the second Gauss-Wahl map on a quadric Q to the easier computation of the first one. Remark that this is possible if the quadric is constructed using two adjoint line bundles. Since quadrics of this type do not cover the whole $I_2(K)$, we will find just a lower bound.

We start giving some details: call $|F|$ and $|K-F|$ the adjoint $\mathfrak{g}^1_4$'s of a bielliptic curve C. Fix a basis for $H^0(F) = \langle x, y\rangle$, and a basis for $H^0(K-F) = \langle t_1, t_2, \ldots, t_{g-4}\rangle$. Picking two independent sections in $t_i, t_j \in H^0(K-F)$, it is possible to construct an invariant quadric in $I_2(K_C)$:

$$Q_{i,j} = xt_i \odot yt_j - xt_j \odot yt_i \in I_2(K_C) = I_2(K_C)^\sigma. \tag{2.4.15}$$

For every quadric of this type, the second Gauss-Wahl map splits as:

$$\mu_2(Q_{i,j}) = \mu_{1,|F|}(x \wedge y)\mu_{1,|K-F|}(t_i \wedge t_j). \tag{2.4.16}$$

Since $\mu_{1,|F|}(x \wedge y)$ is a fixed non-zero section in $H^0(K+2F)$, then

$$\operatorname{rank} \mu_2 \geq \operatorname{rank} \mu_2|_{\langle Q_{i,j}\rangle} = \operatorname{rank} \mu_{1,|K-F|}. \tag{2.4.17}$$

In order to bound the rank of the second Gauss-Wahl map μ_2, we study the first Gauss map $\mu_{1,|K-F|} : \bigwedge^2 H^0(K-F) \to H^0(3K-2F)$.

To simplify the computation, make the restrictive assumption that there exists a Galois cover $C \xrightarrow{\mathbb{Z}/4\mathbb{Z}} \mathbb{P}^1$ that factors via the bielliptic curve:

$$\begin{tikzcd} C \arrow[rr, "\mathbb{Z}/2\mathbb{Z}"] \arrow[dr, "\mathbb{Z}/4\mathbb{Z}"'] & & E \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd} \tag{2.4.18}$$

First of all we prove that for every genus g there exists a Galois cover $C \to \mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$ such that diagram (2.4.18) holds. In general, we

want to study the number of connected components of the locus:

$$\mathcal{B}_{g,Gal} := \left\{ ([C],[E]) \in \mathcal{M}_g \times \mathcal{M}_1 \mid C \text{ satisfies:} \quad \begin{tikzcd} C \arrow[rr, "\mathbb{Z}/2\mathbb{Z}"] \arrow[dr, "\psi"'] & & E \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd} \right.$$

$$\left. \text{where } \psi : C \to \mathbb{P}^1 \text{ is a Galois cover with group } \mathbb{Z}/4\mathbb{Z} \right\}.$$

Theorem 2.4.8. *The locus $\mathcal{B}_{g,Gal}$, $g \geq 3$, is non empty. Moreover it is irreducible if g is even and it has 2 connected components in case g odd.*

Proof. Consider the tower of covers:

$$C \xrightarrow{\varphi} E \xrightarrow{\pi} \mathbb{P}^1.$$

By Riemann-Hurwitz formula we get that φ has $r_{2,\varphi} = 2(g-1)$ double branch points, and π has $r_{2,\pi} = 4$ double branch points. We want to study possible ramification points for the composed map $C \to \mathbb{P}^1$ and find monodromies such that the following diagram holds:

$$\begin{tikzcd} C \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & E \arrow[dl, "\pi"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

where $\psi : C \to \mathbb{P}^1$ is a Galois cover with group $\mathbb{Z}/4\mathbb{Z}$.

Notice that if $p \in C$ is a point of multiplicity 4 with respect to the map ψ, then φ ramifies in p, and π ramifies over its image $\varphi(p) \in E$. Let $\tilde{r}_2 = r_{2,\varphi} + r_{2,\psi}$. Then the following holds:

$$r_{2,\psi} = \frac{\tilde{r}_2 - 2r_{4,\psi}}{2},$$

where $r_{2,\psi}$ and $r_{4,\psi}$ are respectively the number of $2:1$ and $4:1$ branch points for the cover ψ (notice that we divided by two since we are interested in double branch points of a $4:1$ cover). Substituting, we get a relation between the number of double and total branch points of ψ:

$$r_{2,\psi} = g + 1 - r_{4,\psi}. \tag{2.4.19}$$

Applying Riemann-Hurwitz formula directly to the cover $\psi : C \to \mathbb{P}^1$, we find a second relation between $r_{2,\psi}$ and $r_{4,\psi}$:

$$\begin{aligned} 2g - 2 &= 4(-2) + \sum (m_i - 1) = \\ &= -8 + r_{4,\psi}(4-1) + 2 \cdot r_{2,\psi}(2-1) = -8 + 3 \cdot r_{4,\psi} + 2 \cdot r_{2,\psi}. \end{aligned}$$

Solving the linear system

$$\begin{cases} r_{2,\psi} + r_{4,\psi} = g+1 \\ 2r_{2,\psi} + 3r_{4,\psi} = 2g+6 \end{cases} \tag{2.4.20}$$

we obtain that the only possible monodromy for the composed map $C \to \mathbb{P}^1$ giving a Galois cover with group $\mathbb{Z}/4\mathbb{Z}$ is:

$$r_2 = g-3; \qquad\qquad r_4 = 4.$$

This proves that there is only one possible choice for the order of branch points of $C \to \mathbb{P}^1$, that is given by $m = [4^4 : 2^{g-3}]$. To define a cover $C \to \mathbb{P}^1$ properly, we have to define an epimorphism $\theta : \Gamma_{0,m} \twoheadrightarrow G$, where:

$$\Gamma_{0,m} = \left\langle \gamma_1, \dots, \gamma_{g+1} \quad | \quad \gamma_i^{m_i} = 1,\ \gamma_1 \cdot \dots \cdot \gamma_r = 1 \right\rangle. \tag{2.4.21}$$

Representing $G = \mathbb{Z}/4\mathbb{Z} = \langle z \rangle$, if $\operatorname{ord}\gamma_1 = \dots = \operatorname{ord}\gamma_{r_2} = 2$, necessarily $\theta(\gamma_1) = \dots = \theta(\gamma_{r_2}) = z^2$. We have to check how θ can act on γ_i, $i \geq r_2+1$, that is on γ_i's of order 4.

We will consider separately cases g odd and g even.

When the genus is odd. If g is odd, r_2 is even. Condition $\gamma_1 \dots \gamma_{r_2} \cdot \gamma_{r_2+1} \dots \gamma_{r_2+4} = 1$ gives:

$$\begin{aligned} \theta(\gamma_1) \cdot \ldots \theta(\gamma_{r_2}) \cdot \theta(\gamma_{r_2+1}) \dots \theta(\gamma_{r_2+4}) &= 1, \\ \text{that is:} \qquad (z^2)^{r_2} \cdot \theta(\gamma_{r_2+1}) \dots \theta(\gamma_{r_2+4}) &= 1, \\ \text{that is:} \qquad \theta(\gamma_{r_2+1}) \dots \theta(\gamma_{r_2+4}) &= 1. \end{aligned}$$

So we get:

$$\theta(\gamma_{r_2+1}) \cdot \theta(\gamma_{r_2+2}) \cdot \theta(\gamma_{r_2+3}) \cdot \theta(\gamma_{r_2+4}) = 1. \tag{2.4.22}$$

Since $\operatorname{ord}(\gamma_{r_i}) = 4$ for $i \geq r_2+1$, for these i's there are two possibilities: either $\theta(\gamma_i) = z$ or $\theta(\gamma_i) = z^3$. From condition (2.4.22) follows that the number of γ_i's such that $\theta(\gamma_i) = z$ is necessarily even (the same for γ_i's such that $\theta(\gamma_i) = z^3$). In the end, there are three possibilities, up to permutations of γ_i's:

1. $\theta(\gamma_{r_2+1}) = \theta(\gamma_{r_2+2}) = \theta(\gamma_{r_2+3}) = \theta(\gamma_{r_2+4}) = z$;
2. $\theta(\gamma_{r_2+1}) = \theta(\gamma_{r_2+2}) = z$ and $\theta(\gamma_{r_2+3}) = \theta(\gamma_{r_2+4}) = z^3$;

3. $\theta(\gamma_{r_2+1}) = \theta(\gamma_{r_2+2}) = \theta(\gamma_{r_2+3}) = \theta(\gamma_{r_2+4}) = z^3$.

Finally, notice that one can identify case (1) and case (3) via the diagonal action induced by the automorphism of G switching z and z^3. Moreover, there exist Hurwitz moves permuting γ_i's in every possible way. Since there is at least one element of order $4 = \mathrm{ord}(G)$ we have immediately surjectivity for θ. This implies that, if g is odd, $\mathcal{B}_{g,Gal}$ is non empty and has exactly 2 connected components.

When the genus is even. If g is even, r_2 is odd. With the same argument as in case (1) we obtain:

$$\theta(\gamma_{r_2+1}) \cdot \theta(\gamma_{r_2+2}) \cdot \theta(\gamma_{r_2+3}) \cdot \theta(\gamma_{r_2+4}) = z^2. \tag{2.4.23}$$

This equation implies that the number of γ_i's such that $\theta(\gamma_i) = z$ is necessarily odd. In this case we have only two possibilities, up to permutations:

1. $\theta(\gamma_{r_2+1}) = \theta(\gamma_{r_2+2}) = \theta(\gamma_{r_2+3}) = z$ and $\theta(\gamma_{r_2+4}) = z^3$;
2. $\theta(\gamma_{r_2+1}) = z$ and $\theta(\gamma_{r_2+2}) = \theta(\gamma_{r_2+3}) = \theta(\gamma_{r_2+4}) = z^3$.

As in the previous case, these monodromy are equivalent under the automorphism of G switching z and z^3. Since θ is surjective, this shows that in case g is even the locus $\mathcal{B}_{g,Gal}$ is non empty and irreducible. □

Using the notation of Section 1.2, Theorem 2.4.8 shows that the following are all possible monodromies for the Galois cover $C \to \mathbb{P}^1$ with $[C] \in \mathcal{B}_{g,Gal}$:

1. If g is odd, then:
   (a) $a = [1^4 : 2^{g-3}]$ or
   (b) $a = [1^2 : 3^2 : 2^{g-3}]$.
2. If g is even, then:
   (a) $a = [1^3 : 3 : 2^{g-3}]$.

In the following we compute the rank of the second Gauss-Wahl map, considering separately cases g odd and g even.

Rank of the second Wahl map when the genus is odd

Recall that using inequality (2.4.17), we have turned the computation of the rank of the second Gauss-Wahl map when evaluated over a quadric $Q_{i,j}$ into the computation of the rank of $\mu_{1,|K-F|} : \Lambda^2 H^0(K-F) \to H^0(3K-2F)$. In the following, we will use the results given in Section 2.1 to write down explicitly the forms lying in $\Lambda^2 H^0(K-F)$. We start computing the dimension of the eigenspaces in all connected components:

$$(a) \qquad d_1 = \frac{g+1}{2}; \qquad d_2 = 1; \qquad d_3 = \frac{g-3}{2};$$

$$(b) \qquad d_1 = \frac{g-1}{2}; \qquad d_2 = 1; \qquad d_3 = \frac{g-1}{2}.$$

In both connected components, one can write $H^0(K_C) = V_1 \oplus V_2 \oplus V_3$, where eigenspaces are composed as follows:

$$\begin{aligned} V_1 &= \langle \omega_{1,0}, x\,\omega_{1,0}, \dots, x^{d_1-1}\,\omega_{1,0} \rangle; \\ V_2 &= \langle \omega_{2,0} \rangle; \\ V_3 &= \langle \omega_{3,0}, x\,\omega_{3,0}, \dots, x^{d_3-1}\,\omega_{3,0} \rangle. \end{aligned}$$

If we choose $|F|$ as the $\mathfrak{g}^1_4$ given by be the fiber over 0 of the projection $(x,y) \mapsto x$, it is possible to write down explicitly $H^0(K-F)$ as the direct sum of the two eigenspaces:

$$\begin{aligned} W_1 &= \langle x\,\omega_{1,0}, \dots, x^{d_1-1}\,\omega_{1,0} \rangle, \\ W_3 &= \langle x\,\omega_{3,0}, \dots, x^{d_3-1}\,\omega_{3,0} \rangle. \end{aligned}$$

We are interested in the space $\wedge^2 H^0(K-F)$, which is the domain of the Gauss map we want to study. Notice that it decomposes in three summands:

$$\wedge^2 H^0(K-F) = \wedge^2 W_1 \oplus \wedge^2 W_3 \oplus \Big(W_1 \otimes W_3 \Big). \tag{2.4.24}$$

We want to check how the group G acts on these summands. We claim that, if $H^0(K)$ decomposes as a direct sum of eigenspaces $V_1 \oplus \cdots \oplus V_{n-1}$ (same notation as before), then the tensor product $\omega_i \otimes \omega_j$ lies in the $(i+j)$-th eigenspace of $V_i \otimes V_j$, where ω_k is an element in V_k.

Lemma 2.4.9. *Keep notations as above. Identify the group G with $\mathbb{Z}/n\mathbb{Z}$, and call ξ_n the primitive n-th root of unity. Then, for every $g \in G$:*

$$g^*(\omega_i \wedge \omega_j) = \xi_n^{-(i+j)}\,\omega_i \wedge \omega_j. \tag{2.4.25}$$

Proof. The proof is straightforward:

$$g^*(\omega_i \wedge \omega_j) = \xi_n^{-i}\,\omega_i \wedge \xi_n^{-j}\omega_j = \xi_n^{-(i+j)}\,\omega_i \wedge \omega_j.$$

□

Lemma 2.4.9 shows that, in decomposition (2.4.24), the group G acts as a multiplication by the unit imaginary number i on the subspace $\wedge^2 W_1 \oplus \wedge^2 W_3$ and is the identity on every element in $W_1 \otimes W_3$.

Since the first Gauss map is G-equivariant (see diagram (2.2.7)), we can consider the map acting separately on eigenspaces relative to different eigenvalues. Call Z_0 and Z_2 the eigenspaces in $H^0(3K-2F)$ relative to the eigenvalues 0 and 2 respectively. The following diagram holds:

$$\begin{array}{rcccc} \mu_{1,|K-F|} & : & \wedge^2 H^0(K-F) & \longrightarrow & H^0(3K-2F) \\ & & \| & & \cup \\ & & W_1 \otimes W_3 & \longrightarrow & Z_0 \\ & & \oplus & & \oplus \\ & & \wedge^2 W_1 \oplus \wedge^2 W_3 & \longrightarrow & Z_2. \end{array},$$

In the following we will bound, separately, the rank of $\mu_{1,|K-F|}$ when restricted to $\wedge^2 W_1 \oplus \wedge^2 W_3$ and to $W_1 \otimes W_3$. The rank of the non-restricted map will be the sum of the two. We start computing the rank of the first Gauss map when restricted to any of the two summands of $\wedge^2 W_1 \oplus \wedge^2 W_3$. We remark that every part of the proof holds identically in the two connected components.

Theorem 2.4.10. *Keep the same notations as before. In every connected component the following holds:*

$$\operatorname{rank}(\mu_{1,|K-F|})\big|_{\wedge^2 W_1} = 2d_1 - 5; \qquad \operatorname{rank}(\mu_{1,|K-F|})\big|_{\wedge^2 W_3} = 2d_3 - 5.$$

Proof. Since the procedure is the same, we focus on W_1. Choose a local chart and write $\omega_{1,0} = f_{1,0}(y)dy$. Call $v_i = \varphi(y)^i f_{1,0}(y)dy$, that implies $W_1 = \langle v_1, \ldots, v_{d_1-1}\rangle$. Differentiating v_i one gets:

$$dv_i = (i\varphi(y)^{i-1}\varphi'(y)f_{1,0}(y) + \varphi(y)^i f'_{1,0}(y))dy. \tag{2.4.26}$$

Performing the computation for the first Wahl map we get:

$$\begin{aligned}\mu_1(v_i \wedge v_j) &= (i\varphi(y)^{i-1}\varphi'(y)f_{1,0}(y) + \varphi(y)^i f'_{1,0}(y))\varphi(y)^j f_{1,0}(y) - \\ &\quad - \varphi(y)^i f_{1,0}(y)(j\varphi(y)^{j-1}\varphi'(y)f_{1,0}(y) + \varphi(y)^j f'_{1,0}(y)) = \\ &= i\varphi(y)^{i+j-1}\varphi'(y)f_{1,0}(y)^2 - j\varphi(y)^{i+j-1}\varphi'(y)f_{1,0}(y)^2 = \\ &= (i-j)(\varphi(y)^{i+j-1}\varphi'(y)f_{1,0}(y)^2).\end{aligned} \tag{2.4.27}$$

Considering any linear combination of v_i's one obtains:

$$\begin{aligned}\mu_1\Big(\sum_{i,j} a_{ij}v_i \wedge v_j\Big) &= \sum_{i<j} a_{ij}(i-j)(\varphi(y)^{i+j-1}\varphi'(y)f_{1,0}(y)^2) = \\ &= \varphi'(y)f_{1,0}(y)^2 \sum_{i<j} a_{ij}(i-j)\varphi(y)^{i+j-1} = \\ &= \varphi'(y)f_{1,0}(y)^2 \sum_{k=3}^{2d_1-3} \sum_{i<j,\, i+j=k} a_{ij}(i-j)\varphi(y)^{k-1} = \\ &= F(y) \sum_{k=3}^{2d_1-3} A_k\varphi(y)^{k-1}.\end{aligned} \tag{2.4.28}$$

Since k may vary in the set $\{3, 4, \ldots, 2d_1 - 3\}$, there are $2d_1 - 5$ possible exponents for $\varphi(y)$. Moreover, remark that for every (i, j), (h, l) such that $k = i + j = h + l$ the corresponding vectors $\mu_1(v_i \wedge v_j)$ and $\mu_1(v_h \wedge v_l)$ are linearly dependent. This implies that $\operatorname{rank}(\mu_{1,|K-F|})|_{\wedge^2 W_1} = 2d_1 - 5$. In the same way one obtains $\operatorname{rank}(\mu_{1,|K-F|})|_{\wedge^2 W_3} = 2d_3 - 5$. □

From the previous theorem, substituting the dimension of d_i's depending on the chosen monodromy we have:

(a) $\quad \operatorname{rank}(\mu_{1,|K-F|})|_{\wedge^2 W_1} = g - 4; \qquad \operatorname{rank}(\mu_{1,|K-F|})|_{\wedge^2 W_3} = g - 8;$

(b) $\quad \operatorname{rank}(\mu_{1,|K-F|})|_{\wedge^2 W_1} = g - 6; \qquad \operatorname{rank}(\mu_{1,|K-F|})|_{\wedge^2 W_3} = g - 6.$

Since $\operatorname{rank} \mu_1\big|_{\wedge^2 W_1 \otimes \wedge^2 W_2} \geq \max\left(\operatorname{rank} \mu_1\big|_{\wedge^2 W_1}, \operatorname{rank} \mu_1\big|_{\wedge^2 W_3}\right)$, this computation bounds the rank of the first Gauss map when evaluated on $\wedge^2 W_1 \otimes \wedge^2 W_3$. While it seems to be numerically hard to find a better estimate on $\wedge^2 W_1 \otimes \wedge^2 W_3$, there is no reason for our bound to be sharp.

In the following, we compute the rank of $\mu_{1,|K-F|}$ when restricted to the eigenspace $W_1 \otimes W_3$. We start with an elementary but useful remark.

Remark 2.4.11. Recall the explicit expression of forms $\omega_{n,\nu}$ given in equation (2.1.3), and notice that all forms have the common factor

$$F(y) := \frac{m}{g'(\varphi(y))}. \tag{2.4.29}$$

It is convenient, in some cases, to consider the simplified forms obtained dividing all forms by $F(y)$. In particular, we claim that the simplification does not affect the value of the rank of the first Gauss map. More precisely, for $n = 1, \dots, m-1$, $\nu = 0, \dots, d_n - 1$, call:

$$\tilde{\omega}_{n,\nu} = \varphi(y)^n \, y^\nu \prod (\varphi(y) - t_i)^{l(i,n)+a_i} dy. \tag{2.4.30}$$

Forms $\tilde{\omega}_{n,\nu}$ are defined as $\omega_{n,\nu}/F(y)$. Call their local expression $\tilde{\omega}_{n,\nu} = \tilde{f}_{n,\nu}(y)\, dy$.

Comparing the first Gauss map when evaluated on the wedge product $\omega_{n,\nu} \wedge \omega_{l,\eta}$ with the one evaluated on $\tilde{\omega}_{n,\nu} \wedge \tilde{\omega}_{l,\eta}$, we see that they differ for the positive value $F(y)^2$:

$$\begin{aligned}
\mu_1(\omega_{n,\nu} \wedge \omega_{l,\eta}) &= \frac{d}{dy}\Big(F(y)\, \tilde{f}_{n,\nu}(y)\Big) F(y) \tilde{f}_{l,\eta}(y) - \tilde{f}_{n,\nu}(y) \frac{d}{dy}\Big(F(y)\, \tilde{f}_{l,\eta}(y)\Big) dy = \\
&= F(y)F'(y)\tilde{f}_{n,\nu}(y)\tilde{f}_{l,\eta}(y) + F(y)^2 \frac{d}{dy}\Big(\tilde{f}_{n,\nu}(y)\Big)\tilde{f}_{l,\eta}(y) - \\
&- F(y)F'(y)\tilde{f}_{n,\nu}(y)\tilde{f}_{l,\eta}(y) - F(y)^2 \tilde{f}_{n,\nu}(y) \frac{d}{dy}\Big(\tilde{f}_{l,\eta}(y)\Big) = \\
&= F(y)^2 \mu_1(\tilde{\omega}_{n,\nu} \wedge \tilde{\omega}_{l,\eta}).
\end{aligned}$$

This shows that one can compute the rank of the first Gauss map on the simplified forms $\tilde{\omega}_{n,\nu}$ instead then on the complete ones. In the remaining part of this chapter, we will always use these simplified forms. By abuse of notation we will simply denote $\tilde{\omega}_{n,\nu}$ as $\omega_{n,\nu}$, and we will call $f_{n,\nu}(y)dy$ their local expressions.

Theorem 2.4.12. *Keep notations as above. In every connected component the following holds:*

$$\operatorname{rank}(\mu_{1,|K-F|})|_{W_1 \otimes W_3} \geq g - 4. \tag{2.4.31}$$

Proof. The proof relies in a direct computation of the first Gauss map, that is performed using the explicit expression of 1-forms described in Section 2.1. More precisely, we will use the simplified expression (2.4.30), calling

forms $\omega_{n,\nu}$ simply. Via Remark 2.4.11, the simplification does not affect the rank of the map.

Choose a local coordinate y around 0. Then:

$$\omega_{1,0} = f_{1,0}(y)dy, \qquad \omega_{3,0} = f_{3,0}(y)dy. \tag{2.4.32}$$

As before, choosing the line bundle F given by the fiber over 0 of the projection $(x, y) \mapsto x$, the vector space $H^0(K - F)$ splits as a direct sum of the two eigenspaces:

$$\begin{aligned} W_1 &= \langle x\,\omega_{1,0}, \dots, x^{d_1-1}\,\omega_{1,0}\rangle = \langle v_1, \dots, v_{d_1-1}\rangle, \\ W_3 &= \langle x\,\omega_{3,0}, \dots, x^{d_3-1}\,\omega_{3,0}\rangle = \langle w_1, \dots, w_{d_3-1}\rangle, \end{aligned} \tag{2.4.33}$$

where $v_i = \varphi(y)^i f_{1,0}(y)dy$, and $w_j = \varphi(y)^j f_{3,0}(y)dy$.

Differentiating we get:

$$\begin{aligned} dv_i &= (i\varphi(y)^{i-1}\varphi'(y)f_{1,0}(y) + \varphi(y)^i f'_{1,0}(y))dy, \\ dw_j &= (j\varphi(y)^{j-1}\varphi'(y)f_{3,0}(y) + \varphi(y)^j f'_{3,0}(y))dy. \end{aligned} \tag{2.4.34}$$

We compute explicitly the first Gauss map on the wedge product $v_i \wedge w_j$. Although the computation is similar to the one performed in (2.4.27), since forms in the domain lies in two different eigenspaces we will not be able to compute the precise rank in this case, but we still give a bound for it. The first computation gives:

$$\begin{aligned} \mu_1(v_i \wedge w_j) &= (i\varphi(y)^{i-1}\varphi'(y)f_{1,0}(y) + \varphi(y)^i f'_{1,0}(y))\varphi(y)^j f_{3,0}(y) - \\ &\quad - \varphi(y)^i f_{1,0}(y)(j\varphi(y)^{j-1}\varphi'(y)f_{3,0}(y) + \varphi(y)^j f'_{3,0}(y)) = \\ &= (i-j)\varphi(y)^{i+j-1}\varphi'(y)f_{1,0}(y)f_{3,0}(y) + \\ &\quad + \varphi(y)^{i+j}(f'_{1,0}(y)f_{3,0}(y) - f_{1,0}(y)f'_{3,0}(y)). \end{aligned} \tag{2.4.35}$$

Considering any linear combination of v_i's and w_j's one gets:

$$\begin{aligned}\mu_1\Big(\sum_{i=1}^{d_1-1}\sum_{j=1}^{d_3-1} a_{ij}v_i\wedge w_j\Big) &= \sum_{i=1}^{d_1-1}\sum_{j=1}^{d_3-1} a_{ij}\Big[(i-j)\varphi'(y)f_{1,0}(y)f_{3,0}(y)+\\ &+\varphi(y)(f'_{1,0}(y)f_{3,0}(y)-f_{1,0}(y)f'_{3,0}(y))\Big]\varphi(y)^{i+j-1} =\\ &= \sum_{k=2}^{d_1+d_3-2}\sum_{i+j=k} a_{ij}\Big[(i-j)\varphi'(y)f_{1,0}(y)f_{3,0}(y)+\\ &+\varphi(y)(f'_{1,0}(y)f_{3,0}(y)-f_{1,0}(y)f'_{3,0}(y))\Big]\varphi(y)^{k-1}.\end{aligned}$$

Let $A_k = \sum_{i+j=k} a_{ij}(i-j)$ and $B_k = \sum_{i+j=k} a_{ij}$. Substituting, equation above becomes:

$$\begin{aligned}\mu_1\Big(\sum_{i=1}^{d_1-1}\sum_{j=1}^{d_3-1} a_{ij}v_i\wedge w_j\Big) &= \sum_{k=2}^{d_1+d_3-2}\Big[A_k\varphi'(y)f_{1,0}(y)f_{3,0}(y)+\\ &+B_k\Big(\varphi(y)(f'_{1,0}(y)f_{3,0}(y)-f_{1,0}(y)f'_{3,0}(y))\Big)\Big]\varphi(y)^{k-1}.\end{aligned} \tag{2.4.36}$$

To go further with the computation, we need to write explicitly what coefficients $f_{1,0}(y)$ and $f_{3,0}(y)$ are. We will do that using expression (2.4.30). Recalling the content of Section 2.1, we need the combinatorial data $l(i,n) = \big[-n\cdot a_i/4\big]$ to write the local expression for $\omega_{n,\nu}$. These data depend on the chosen monodromy, and 1-forms a priori do as well. Nevertheless we will find that, in our case, while $f_{3,0}(y)$ will depend on the monodromy, $f_{1,0}(y)$ will not.

We write here the list of combinatorial data with monodromy (a) and (b):

$$
\begin{aligned}
(a)\quad & l(i,1) = \Big[-\frac{a_i}{4}\Big] = -1 \quad \text{for all } i;\\
& l(i,3) = \Big[-\frac{3a_i}{4}\Big] = -1 \quad \text{if } i \le 4;\\
& l(i,3) = \Big[-\frac{3a_i}{4}\Big] = -2 \quad \text{if } i \ge 5;\\
(b)\quad & l(i,1) = \Big[-\frac{a_i}{4}\Big] = -1 \quad \text{for all } i;\\
& l(i,3) = \Big[-\frac{3a_i}{4}\Big] = -1 \quad \text{if } i \le 2;\\
& l(i,3) = \Big[-\frac{3a_i}{4}\Big] = -3 \quad \text{if } 3 \le i \le 4;\\
& l(i,3) = \Big[-\frac{3a_i}{4}\Big] = -2 \quad \text{if } i \ge 5.
\end{aligned}
\tag{2.4.37}
$$

Substituting $l(i,n)$ in expression (2.4.30), we see that, in every connected components, $\omega_{3,0}$ has the same coefficient, while the coefficient of $\omega_{1,0}$ changes:

$$
\begin{aligned}
(a)\qquad & f_{1,0}(y) = \prod_{i\ge 5}(\varphi(y)-t_i),\\
(b)\qquad & f_{1,0}(y) = \prod_{3\le i\le 4}(\varphi(y)-t_i)^2 \prod_{i\ge 5}(\varphi(y)-t_i),\\
(a),(b)\qquad & f_{3,0}(y) = y^2.
\end{aligned}
\tag{2.4.38}
$$

We write for short, both for cases (a) and (b):

$$
\begin{aligned}
& f_{1,0}(y) = \prod_{1\le i\le 4}(\varphi(y)-t_i)^I \prod_{i\ge 5}(\varphi(y)-t_i),\\
& f_{3,0}(y) = y^2,
\end{aligned}
\tag{2.4.39}
$$

where I is the multi-index defined as follows:

$$
I = \begin{cases} [0,0,0,0], & \text{if monodromy is (a);}\\ [0,0,2,2] & \text{if monodromy is (b).}\end{cases}
$$

Notice that in both cases $t_i \neq 0\ \forall i \neq 1$ since $f_{1,0}(0) \neq 0$.

Consider Taylor expansions centered in 0 for all factors appearing in equation (2.4.36). For convenience of the reader, we write down the Taylor ex-

pansion of each term separately:

$$\begin{aligned}
\varphi(y) &= \frac{1}{h'(0)}y^4 + \frac{1}{2h''(0)}y^8 + \text{ h.o.t.}\\
\varphi'(y) &= \frac{4}{h'(0)}y^3 + \frac{8}{2h''(0)}y^7 + \text{ h.o.t.}\\
f_{1,0}(y) &= \prod_{1\le i\le 4}(-t_i)^I \prod_{i\ge 5}(-t_i) + \text{ h.o.t.}\\
f'_{1,0}(y) &= F_I(y)\varphi'(y) + \text{ h.o.t. } =\\
&= F_I(y)\Big(\frac{4}{h'(0)}y^3 + \frac{8}{2h''(0)}y^7 + \text{ h.o.t.}\Big) + \text{ h.o.t.}\\
f_{3,0}(y) &= y^2,\\
f'_{3,0}(y) &= 2y,
\end{aligned}$$

where $F_I(y)$ is a function depending on the monodromy. In particular, there are the following possibilities for $F_I(y)$, respectively when (a) $I = [0,0,0,0]$ or (b) $I = [0,0,2,2]$:

$$\begin{aligned}
(a)\qquad F_1(y) &= \sum_{j\ge 5}\prod_{i\ne j}(\varphi(y)-t_i),\\
(b)\qquad F_1(y) &= \sum_{j\ge 5}\prod_{i\ne j}(\varphi(y)-t_i)\prod_{i=3,4}(\varphi(y)-t_i)^2 +\\
&\quad + \prod_{i\ge 5}(\varphi(y)-t_i)\sum_{j=3,4}\prod_{i\ne j}2(\varphi(y)-t_i)^2(\varphi(y)-t_j).
\end{aligned} \tag{2.4.40}$$

Notice that in all cases $f'_{1,0}(y)$ has vanishing order at least equal to 3, that is the vanishing order of $\varphi'(y)$. Substituting in expression (2.4.36), we get:

$$\begin{aligned}
\mu_1\Big(\sum_{i=1}^{d_1-1}\sum_{j=1}^{d_3-1} a_{ij}v_i\wedge w_j\Big) &= \sum_{k=2}^{d_1+d_3-2}\left(\frac{1}{h'(0)}y^4 + \frac{1}{2h''(0)}y^8 + \text{ h.o.t.}\right)^{k-1}\cdot\\
&\cdot\Bigg\{A_k\left(\frac{4}{h'(0)}y^3 + \frac{8}{2h''(0)}y^7 + \text{ h.o.t.}\right)\Bigg[\prod_{1\le i\le 4}(-t_i)^I\prod_{i\ge 5}(-t_i) + \text{ h.o.t}\Bigg]\cdot\\
&\cdot y^2 + B_k\left(\frac{1}{h'(0)}y^4 + \frac{1}{2h''(0)}y^8 + \text{ h.o.t.}\right)\Bigg[F_I(y)\Big(\frac{4}{h'(0)}y^3 + \frac{8}{2h''(0)}y^7 +\\
&+ \text{ h.o.t.}\Big)y^2 - 2y\left(\prod_{1\le i\le 4}(-t_i)^I\prod_{i\ge 5}(-t_i) + \text{ h.o.t}\right)\Bigg]\Bigg\}.
\end{aligned} \tag{2.4.41}$$

In order to simplify the computation, we study the rank of the first Gauss map not on the whole $W_1\otimes W_3$ but on a certain subspace.

Claim. *Let W be any subspace of $W_1 \otimes W_3$ such that for every integer $k \in \{2, \dots, d_1 + d_3 - 2\}$ there exists exactly one pair (i, j) such that $v_i \wedge w_j \in W$ and $i + j = k$. Then $\mu_1\big|_W$ has rank $g - 4$.*

Fix W and call $J := \{(i, j)$ such that $v_i \wedge w_j \in W\}$. Notice that the double sum $\sum_{i=1}^{d_1-1} \sum_{j=1}^{d_3-1} a_{ij} v_i \wedge w_j$ is a linear combination of all elements in W if we impose $a_{ij} = 0$ for all pairs (i, j) not occurring in J. Keeping notation as before, this implies that $A_k = (i - j)a_k$ and $B_k = a_k$, where we called a_k the only coefficient a_{ij} such that $(i, j) \in J$. The existence of a single coefficient a_k in both A_k and B_k, instead than a linear combination of a_{ij} simplifies considerably the computation.

Come back to expression (2.4.41), impose $a_{ij} = 0$ for all $(i, j) \notin J$, and consider the vanishing orders in 0. In both cases (a) and (b) the lowest vanishing order is 9, reached when $k = 2$. The coefficient of the term of order 9 in the Taylor expansion gives the same equation with both monodromies:

$$(a), (b): \quad \frac{2}{h'(0)^2} \prod_{1 \le i \le 4} (-t_i)^I \prod_{i \ge 5} (-t_i)(2A_2 - B_2) = 0. \tag{2.4.42}$$

Notice that $A_2 = (1 - 1)a_2 = 0$, and $B_2 = a_2$. Equation (2.4.42) implies $2A_2 = B_2 = 0$, so that $a_2 = a_{1,1} = 0$.

Consider the next vanishing order, that is 13. Since $A_2 = B_2 = 0$ the only part of order 13 appears when $k = 3$. In this case the coefficient (again, the same for all monodromies) is:

$$\frac{2}{h'(0)^3} \prod_{1 \le i \le 4} (-t_i)^I \prod_{i \ge 5} (-t_i)(2A_3 - B_3) = F_I(y) \cdot B_3 = 0. \tag{2.4.43}$$

Also in this case $B_3 = A_3 = 0$, that implies $a_3 = 0$. One can proceed by induction: the coefficient of the term vanishing in 0 with order k will be:

$$\frac{2}{h'(0)^k} \prod_{1 \le i \le 4} (-t_i)^I \prod_{i \ge 5} (-t_i)(2A_k - B_k) = F_I(y) \cdot B_k = 0, \tag{2.4.44}$$

implying $A_k = B_k = 0$, so $a_k = 0$.

This proves that $\operatorname{rank} \mu_1\big|_W$ is exactly the cardinality of J. It is straightforward to check that the cardinality of J is the same for every possible choice for W. We count it considering, for example, the following instance:

$$J = \{(1, 1),\ (1, 2),\ \dots (1, d_3 - 1),\ (2, d_3 - 1),\ (3, d_3 - 1),\ \dots (d_1 - 1, d_3 - 1)\}.$$

Counting elements, one sees that J has cardinality $d_3-1+d_1-2 = d_1+d_3-3$. For the previous discussion, this value coincide with $\operatorname{rank} \mu_1\big|_W$. This proves the claim.

The proof of the theorem is a direct consequence of the claim. Since $\operatorname{rank} \mu_1\big|_{W_1\otimes W_3} \geq \operatorname{rank} \mu_1\big|_W$, it is straightforward that there are at least $d_1 + d_3 - 3$ independent sections in $\mu_1(W_1 \otimes W_3)$. This implies:

$$\operatorname{rank} \mu_1\big|_{W_1\otimes W_3} \geq d_1 + d_3 - 3. \tag{2.4.45}$$

□

From the previous theorem, substituting the d_i's, one can find a bound depending on the genus. Moreover, since $d_1 + d_3 = g - 1$ in every connected component, the final bound does not depend on the monodromy:

$$(a), (b): \quad \operatorname{rank} \mu_1\big|_{W_1\otimes W_3} \geq g - 4. \tag{2.4.46}$$

This concludes the computation in case g odd. Putting together results of Theorem 2.4.10 and Theorem 2.4.12, one gets a general bound for the rank of the second Gauss-Wahl map (recall equation (2.4.17)) depending on the monodromy. Moreover Lemma 2.4.4 ensures that $5g - 5$ is always an upper bound. Summarizing:

$$\begin{aligned} (a): &\quad 2g - 8 \leq \operatorname{rank} \mu_2 \leq 5g - 5, \\ (b): &\quad 2g - 10 \leq \operatorname{rank} \mu_2 \leq 5g - 5. \end{aligned} \tag{2.4.47}$$

Rank of the second Wahl map when the genus is even

The strategy is identical both in cases g odd and g even. One can apply directly results obtained in the previous section just changing monodromies and dimensions when necessary. We remark here that since in Theorem 2.4.10 we did not use monodromy and dimensions in any part of the proof, we can apply the theorem straightforward. Nevertheless we used the monodromy in Theorem 2.4.12, when we wrote explicitly forms in $H^0(K - F)$. We will prove that a modification of the monodromy datum does not affect the result.

We start recalling that, from Theorem 2.4.8, the locus $\mathcal{B}_{g,Gal}$ is irreducible if g is even, and the only admissible monodromy for $C \to \mathbb{P}^1$ is given by:

$$(a): \quad a = [1^3 : 3 : 2^{g-3}]. \tag{2.4.48}$$

As discussed in Section 2.1.3, this datum corresponds to a decomposition of the space of holomorphic 1-forms into eigenspaces: $H^0(K) = V_1 \oplus V_2 \oplus V_3$. Using formula (2.1.2) one can check that the dimensions of these eigenspaces are, respectively:

$$(a): \quad d_1 = \frac{g}{2}; \qquad d_2 = 1; \qquad d_3 = \frac{g-2}{2}. \tag{2.4.49}$$

As remarked in the beginning of this part, one can apply directly Theorem 2.4.10, since in its proof we didn't use neither the monodromy or the dimensions of the eigenspaces, obtaining the following:

$$(a): \quad \operatorname{rank} \mu_{1,|K-F|}\big|_{\wedge^2 W_1} = g-5; \qquad \operatorname{rank} \mu_{1,|K-F|}\big|_{\wedge^2 W_3} = g-7. \tag{2.4.50}$$

As in the odd case, we wish to find a bound for the first Gauss map when restricted on the product $W_1 \otimes W_3$. We are going to show that although new monodromies and dimensions affect the explicit expression of the 1-forms v_1's and w_j's, every consideration we made on the Taylor expansion of $\mu_1\Big(\sum_{i=1}^{d_1-1}\sum_{j=1}^{d_3-1} a_{ij} v_i \wedge w_j\Big)$ is still true. In particular, the claim inside the proof of Theorem 2.4.10 identically holds, and the final bound, depending on dimensions, turns to be the same:

$$\operatorname{rank} \mu_1\big|_{W_1 \otimes W_3} \geq d_1 + d_3 - 3. \tag{2.4.51}$$

We give some details. Following Section 2.1, we need to compute the combinatorial data $l(i,n)$ to write 1-forms explicitly. Via a direct computation one gets:

$$\begin{aligned} (a) \quad l(i,1) &= \Big[-\frac{a_i}{4}\Big] = -1 \quad \text{for all } i; \\ l(i,3) &= \Big[-\frac{3a_i}{4}\Big] = -1 \quad \text{if } i \leq 3; \\ l(i,3) &= \Big[-\frac{3a_i}{4}\Big] = -3 \quad \text{if } i = 4; \\ l(i,3) &= \Big[-\frac{3a_i}{4}\Big] = -2 \quad \text{if } i \geq 5. \end{aligned} \tag{2.4.52}$$

Using these data, one can write explicitly the local coefficients $f_{1,0}(y)$ and $f_{3,0}(y)$ for the 1-forms $\omega_{1,0}$ and $\omega_{3,0}$ respectively. We do it, using expression (2.4.30). Despite the coefficient $f_{3,0}(y)$ turns to be equal to the corresponding coefficient obtained in the odd case, $f_{1,0}(y)$ is not:

$$\begin{aligned} f_{1,0}(y) &= (\varphi(y) - t_4)^2 \prod_{i \geq 5} (\varphi(y) - t_i), \\ f_{3,0}(y) &= y^2. \end{aligned} \tag{2.4.53}$$

Notice that even if $f_{1,0}(y)$ is different from the corresponding coefficient when computed in odd genus, they both satisfy $f_{1,0}(0) \neq 0$ and $\operatorname{ord}_0 f'_{1,0} \geq 3$, that are the only properties of $f_{1,0}(y)$ we used in the claim: one can apply straightforward Theorem 2.4.12 obtaining the following bound:

$$(a): \quad \operatorname{rank} \mu_1 \geq d_1 + d_3 - 3 = g - 4. \tag{2.4.54}$$

From equation (2.4.51) and bound (2.4.54) we obtain a lower bound for the rank of the first Gauss map in case g even. Recalling that Theorem 2.4.4 gives an upper bound for it, we can sum up everything as follows:

$$(a): \quad 2g - 9 \leq \operatorname{rank} \mu_2 \leq 5g - 5. \tag{2.4.55}$$

We summarize the results obtained in this section with the following corollary. Here we put together bounds found in case of odd genus (2.4.47), with bound (2.4.55) for curves of even genus.

Corollary 2.4.13. *The rank of the second Gauss-Wahl map on the general curve of the bielliptic locus satisfies the following bounds, depending on genus and monodromy:*

(1) If g is odd then:

$$\begin{aligned} (a): &\quad 2g - 8 \leq \operatorname{rank} \mu_2 \leq 5g - 5; \\ (b): &\quad 2g - 10 \leq \operatorname{rank} \mu_2 \leq 5g - 5. \end{aligned} \tag{2.4.56}$$

(2) If g is even then:

$$(a): \quad 2g - 9 \leq \operatorname{rank} \mu_2 \leq 5g - 5. \tag{2.4.57}$$

Moreover, while the lower bounds hold for the general curve, the upper ones hold for every bielliptic curve of genus g.

We conclude this section remarking that, even if our result is not sharp (see also Theorem 3.1.1), it is not a consequence of Theorem 2.2.15, that gives the lower bound $g - 3$ for the second Gauss-Wahl map of all curves of genus $g \geq 4$.

2.5 Bi-hyperelliptic locus

A curve C is called bi-hyperelleptic if it is a double cover of a hyperelliptic curve C', with $g' = g(C') \geq 2$:

$$C \xrightarrow{2:1} C' \xrightarrow{2:1} \mathbb{P}^1. \tag{2.5.1}$$

We will denote with $\mathcal{BH}_{g,g'}$ the locus of bi-hyperelliptic curves $C \to C'$ where $g(C) = g$ and $g(C') = g'$. As in the bielliptic case, we start making some general considerations on $\mathcal{BH}_{g,g'}$. Then, involving the sublocus $\mathcal{BH}_{g,g',Gal}$ (that is the locus of bi-hyperelliptic curves $C \to C'$ such that the composition $C \to C' \to \mathbb{P}^1$ is Galois with group $\mathbb{Z}/4\mathbb{Z}$), we will prove that $\mathcal{BH}_{g,g'}$ is not totally geodesic if $g \geq 3g'$. We remark that since the last one is a structural condition that is necessary to make the space $\mathcal{BH}_{g,g',Gal}$ non empty, it is unavoidable using our approach.

We introduce the general set-up, stating some elementary results of $\mathcal{BH}_{g,g'}$ that will be useful in the following. We start computing its dimension.

Lemma 2.5.1. *The bi-hyperelliptic locus is non-empty if* $2g - 4g' + 2 \geq 0$. *Moreover it has dimension* $N = 2g - 2g' + 1$ *if* $N \geq 0$.

Proof. If C' is fixed, the cover $C \to C'$ has $2(g - 2g' + 1)$ branch points. By Riemann's existence theorem, once the set of branch points is fixed and the curve C' is fixed as well, there are finite possible covers $C \to C'$ with the given branch points. Letting the curve C' vary in the space of hyperelliptic curves of genus g', we obtain that the bi-hyperelliptic locus has dimension $2(g - 2g' + 1) + 2g' - 1$. Hence the claim. □

With the following lemma we remark that $\mathcal{BH}_{g,g'}$ is irreducible.

Lemma 2.5.2. *The bi-hyperelliptic locus is irreducible.*

Proof. From Theorem 1.4.16, we can show that the moduli space of curves C of genus g such that there exists a $2:1$ map $C \to C'$ with $g(C') = g'$ is irreducible. In fact, fixed a datum $(\mathbf{m}, G, \theta)$ we have $T_{g',r}$ and two such data are exchanged by Hurwitz moves (see the discussion in Section 1.3). Differently from case $g' = 1$, here C' is not automatically hyperelliptic: we have to check that the hyperelliptic part is irreducible as well.

Denote by $\mathsf{T}(\mathbf{m}, G, \theta)$ the fixed point locus $\mathcal{T}_g^G$ in the Teichmüller space, and recall that $\varphi : \mathcal{T}_{g',r} \cong \mathsf{T}(\mathbf{m}, G, \theta)$ (see Theorem 1.3.6). Denote by $\mathsf{M}(\mathbf{m}, G, \theta)$ the image of $\mathcal{T}_g^G$ in $\mathcal{M}_g$ and consider the following diagram:

$$\begin{tikzcd} \mathsf{T}(\mathbf{m}, G, \theta) \arrow[r, "\varphi^{-1}"] \arrow[d, "\pi"] & \mathcal{T}_{g',r} \arrow[d, "p"] \\ \mathsf{M}(\mathbf{m}, G, \theta) \arrow[r, "\psi"] & \mathcal{M}_{g'} \end{tikzcd} \tag{2.5.2}$$

Here, π is the restriction of the projection map, ψ associates to a class $[C] \in \mathsf{M}(\mathbf{m}, G, \theta)$ its quotient via the group G, and p is the composition of the forgetful map $\alpha : \mathcal{T}_{g',r} \to \mathcal{T}_{g'}$ with the quotient map $\pi'; \mathcal{T}_{g'} \to \mathcal{M}_{g'}$.

We want to prove that the preimage of the hyperelliptic locus $\mathcal{H}_{g'} \subset \mathcal{M}_{g'}$ via ψ is connected inside $\mathsf{M}(\mathbf{m}, G, \theta)$. Since diagram (2.5.2) is commutative, $\pi^{-1}(\psi^{-1}(\mathcal{H}_{g'})) = \psi(p^{-1}(\mathcal{H}_{g'}))$. Moreover, since π is surjective, the equality implies $\psi^{-1}(\mathcal{H}_{g'}) = \pi(\psi(p^{-1}(\mathcal{H}_{g'}))$.

Consider the map p, associating to every element in the Teichmüller space $T_{g',r}$ its quotient via the action of the mapping class group. Call $T_{g',r}^{\mathcal{H}} := \pi^{-1}(\mathcal{H}_{g'})$ the preimage of the hyperelliptic locus. We claim that the mapping class group preserves $T_{g',r}^{\mathcal{H}}$. In fact, if $[f] \in \mathrm{Map}_{g',r}$, then

$$f : ([C' - B], \varphi) \to ([C' - B], f \circ \varphi).$$

Here we check whether the connected component in $\mathcal{M}_{g'}$ changes if the involution σ is transformed to a different involution σ' via f. It is known (see e.g. [41, Proposition 7.15]) that every couple of involutions σ and σ' over a hyperelliptic curve are related via an element $[f]$ of the mapping class group as follows: $\sigma' = f^{-1} \circ \sigma \circ f$. This immediately shows that, if x is fixed by σ', then its image $f(x)$ is fixed by σ, so that $T_{g'}^{\sigma'} \to T_{g'}^{\sigma}$.

Call $\mathcal{S}$ the set of all possible hyperelliptic involutions. Then $\pi'^{-1}(\mathcal{H}_{g'}) = \bigcup_{\sigma \in \mathcal{S}} T_{g'}^{\sigma}$ and for any $\sigma \in \mathcal{S}$, $\pi'(\mathcal{T}_{g'}^{\sigma}) = \mathcal{H}_{g'}$ and $\mathcal{T}_{g'}^{\sigma}$ is a smooth complex submanifold of $\mathcal{T}_{g'}$. Denote by $\mathcal{T}_{g',r}^{\sigma} := \alpha^{-1}(\mathcal{T}_{g'}^{\sigma})$. The map α is a holomorphic submersion with irreducible fibers (see e.g. [5] p. 471). So the fibers are irreducible, the set $\mathcal{T}_{g'}^{\sigma}$ is irreducible, hence $\mathcal{T}_{g',r}^{\sigma} = \alpha^{-1}(\mathcal{T}_{g'}^{\sigma})$ is also irreducible.

Since for all σ and σ' the sets $\mathcal{T}_{g',r}^{\sigma}$ and $\mathcal{T}_{g',r}^{\sigma'}$ are switched via an element of

the mapping class group $Map_{g',r}$, we have: $\psi^{-1}(\mathcal{H}_{g'}) = \pi(\varphi(p^{-1}(\mathcal{H}_{g'}))) = \pi(\varphi(\bigcup_{\sigma\in\mathcal{S}} \mathcal{T}^{\sigma}_{g',r})) = \pi(\varphi(\mathcal{T}^{\sigma}_{g',r}))$. This concludes the proof.

□

Unfortunately, there is not an analogous of Lemma 2.4.3 in the bi-hyperelliptic case, in fact $I_2(K) \neq I_2(K)^{\sigma}$. This difference will make the computation a bit harder, but still affordable using the bi-hyperelliptic involution and working over $\mathbb{P}^1$. More precisely we make the further assumption that the composition $C \to C' \to \mathbb{P}^1$ gives a cyclic Galois cover of $\mathbb{P}^1$:

$$\begin{tikzcd} C \arrow[rr, "\mathbb{Z}/2\mathbb{Z}"] \arrow[dr, "\mathbb{Z}/4\mathbb{Z}"'] & & C' \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd} \tag{2.5.3}$$

We will show that this extra hypothesis will not affect the generality of the result (see Remark 2.5.5).

We start proving that, for every $g \geq 3g'$, there exists a Galois cover $C \to \mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$ such that it factors through:

$$\begin{tikzcd} C \arrow[rr] \arrow[dr] & & C' \arrow[dl, "2:1"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

where $g' = g(C')$. In general, we want to count the number of connected components of the locus:

$$\mathcal{BH}_{g,g',Gal} := \left\{ ([C],[C']) \in \mathcal{M}_g \times \mathcal{H}_{g'} \mid C \text{ satisfies: } \begin{tikzcd} C \arrow[rr, "\mathbb{Z}/2\mathbb{Z}"] \arrow[dr, "\psi"'] & & C' \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd} \right.$$

$$\left. \text{where } \psi : C \to \mathbb{P}^1 \text{ is a Galois cover with group } \mathbb{Z}/4\mathbb{Z} \right\}.$$

Theorem 2.5.3. *The locus $\mathcal{BH}_{g,g',Gal}$ is non empty if $g \geq 3g'$. Moreover it has k connected components, where*

$$k = \begin{cases} (2g'+4)/4 & \text{if } g' \text{ is even,} \\ (2g'+2)/4 & \text{if } g' \text{ is odd and } g \text{ is even,} \\ (2g'+6)/4 & \text{if } g' \text{ and } g \text{ are odd.} \end{cases}$$

Proof. Consider the tower of covers:

$$C \xrightarrow{\varphi} C' \xrightarrow{\pi} \mathbb{P}^1.$$

Using Riemann-Hurwitz formula (1.1.1), map $\pi : C' \to \mathbb{P}^1$ has $r_{2,\pi} = 2(g'+1)$ ramification points and $\varphi : C \to C'$ has $r_{2,\varphi} = 2(g-2g'+1)$ ramification points. Notice that if $p \in C$ is a point of multiplicity 4 with respect to the map ψ, then φ ramifies in p, and π ramifies over its image $\varphi(p) \in C'$. Let $\tilde{r}_2 = r_{2,\varphi} + r_{2,\pi}$. Then the following holds:

$$r_{2,\psi} = \frac{\tilde{r}_2 - 2r_{4,\psi}}{2}, \tag{2.5.4}$$

where $r_{2,\psi}$ and $r_{4,\psi}$ are respectively the number of $2:1$ and $4:1$ branch points for the cover ψ. Substituting $\tilde{r}_2$ we get a relation between the number of double and total branch points of ψ:

$$r_{2,\psi} = g - g' + 2 - r_{4,\psi}. \tag{2.5.5}$$

We find a second relation between $r_{2,\psi}$ and $r_{4,\psi}$ applying Riemann-Hurwitz formula directly to the cover $\psi : C \to \mathbb{P}^1$:

$$\begin{aligned} 2g-2 &= 4(-2) + \sum (m_i - 1) = \\ &= -8 + r_{4,\psi}(4-1) + 2 \cdot r_{2,\psi}(2-1) = -8 + 3 \cdot r_{4,\psi} + 2 \cdot r_{2,\psi}. \end{aligned}$$

Solving the linear system

$$\begin{cases} r_{2,\psi} + r_{4,\psi} = g - g' + 2, \\ 2r_{2,\psi} + 3r_{4,\psi} = 2g + 6, \end{cases} \tag{2.5.6}$$

one obtains that the only possible monodromy for the composed map $C \to \mathbb{P}^1$ giving a Galois cover with group $\mathbb{Z}/4\mathbb{Z}$ is:

$$r_2 = g - 3g' \geq 0; \qquad r_4 = 2g' + 2. \tag{2.5.7}$$

Notice that it is necessary to require

$$g \geq 3g'. \tag{2.5.8}$$

This proves that for every possible pair (g, g') such that $g \geq 3g'$ there is only one possible choice for the multiplicity of branch points of the composed map $C \to \mathbb{P}^1$, that is $m = [4^{2g'+2} : 2^{g-3g'}]$. Remark that the locus $\mathcal{BH}_{g,g',Gal}$ is empty if condition (2.5.8) is not satisfied.

To define properly the cover $C \to \mathbb{P}^1$ we have to define an epimorphism $\theta : \Gamma_{0,m} \twoheadrightarrow \mathbb{Z}/4\mathbb{Z}$, where:

$$\Gamma_{0,m} = \langle \gamma_1, \dots, \gamma_{r_4}, \gamma_{r_4+1}, \dots, \gamma_{r_4+r_2} \,|\gamma_i^{m_i} = 1,\ \gamma_1 \cdots \gamma_r = 1 \rangle. \tag{2.5.9}$$

Representing $\mathbb{Z}/4\mathbb{Z} = \langle z \rangle$, necessarily $\theta(\gamma_i) = z^2$ for $r_4 + 1 \leq i \leq r_4 + r_2$, since z^2 is the only element of order 2 in $\mathbb{Z}/4\mathbb{Z}$. For $1 \leq i \leq r_4$ there are two possibilities: they can either be z or z^3. We are going to count the number of i's such that $\theta(\gamma_i) = z$ (or, analogously, $\theta(\gamma_i) = z^3$). The number of different possibilities, a part from automorphisms of G, will correspond to the number of connected components of the locus.

Condition $\gamma_1 \cdots \gamma_{r_4} \cdot \gamma_{r_4+1} \cdots \gamma_{r_4+r_2} = 1$ gives:

$$\begin{aligned} &\theta(\gamma_1) \cdots \theta(\gamma_{r_4}) \cdot \theta(\gamma_{r_4+1}) \cdots \theta(\gamma_{r_4+r_2}) = 1, \\ \text{that is:} \quad &\theta(\gamma_1) \cdots \theta(\gamma_{r_4}) \cdot (z^2)^{r_2} = 1. \end{aligned} \tag{2.5.10}$$

Notice that r_4 is always even and r_2 is even if and only if g and g' are both even or both odd. We have to distinguish the two cases.

When the genera have the same parity Since in this case r_2 is even, condition (2.5.10) implies:

$$\theta(\gamma_1) \cdot \dots \cdot \theta(\gamma_{r_4}) = 1,$$

where either $\theta(\gamma_i) = z$ or $\theta(\gamma_i) = z^3$, for $1 \leq i \leq r_4$.

We have to study separately cases $r_4 \equiv 2 \bmod 4$ and $r_4 \equiv 0 \bmod 4$. In the first case the number of γ_i's such that $\theta(\gamma_i) = z$ is necessarily odd (the same for γ_i's such that $\theta(\gamma_i) = z^3$). Up to permutations of γ_i's, we have the following possibilities:

(1) $\theta(\gamma_1) = \cdots = \theta(\gamma_{r_4-1}) = z$ and $\theta(\gamma_{r_4}) = z^3$;

(2) $\theta(\gamma_1) = \cdots = \theta(\gamma_{r_4-3}) = z$ and $\theta(\gamma_{r_4-2}) = \theta(\gamma_{r_4-1}) = \theta(\gamma_{r_4}) = z^3$;

$\vdots$

($r_4/2$) $\theta(\gamma_1) = z$ and $\theta(\gamma_2) = \cdots = \theta(\gamma_{r_4}) = z^3$.

Notice that the previous choices are equivalent via $\mathrm{Aut}(G)$ two by two: in fact we have just listed all possibilities having an odd number of $\gamma_i's$ equal to z (and so, since r_4 is even, also an odd number of $\gamma_i's$ equal to z^3). Since via automorphisms of $G = \mathbb{Z}/4\mathbb{Z}$ one can switch z and z^3, the epimorphism θ numbered with (i) is always equivalent to the epimorphism numbered with $(r_4/2 - i + 1)$. Moreover, there exist Hurwitz moves permuting γ_i's in every possible way: there are $(r_4 + 2)/4$ possibilities for θ, and since there exists at least one element of order $4 = \mathrm{ord}\, G$, the map is surjective.

This concludes the case in which g and g' have the same parity, and $r_4 \equiv 2$ mod 4 as well. Consider now case $r_4 \equiv 0$ mod 4. Up to permutations of γ_i's the following holds:

(1) $\theta(\gamma_1) = \cdots = \theta(\gamma_{r_4}) = z$;

(2) $\theta(\gamma_1) = \cdots = \theta(\gamma_{r_4-2}) = z$ and $\theta(\gamma_{r_4-1}) = \theta(\gamma_{r_4}) = z^3$;

$\vdots$

$((r_4 + 2)/2)$ $\theta(\gamma_1) = \cdots = \theta(\gamma_{r_4}) = z^3$.

Also in this case, choices (i) and $((r_4 + 2)/2 - i + 1)$ are equivalent via automorphism of G. Moreover since there are Hurwitz moves permuting γ_i's in every possible way, there are exactly $(r_4 + 4)/4$ possibilities. Again, since there is at least one element of order $4 = \mathrm{ord}(G)$, we immediately deduce surjectivity for θ.

This proves that, if g and g' have the same parity (and $g \geq 3g'$), $\mathcal{BH}_{g,g',Gal}$ has $(r_4 + 2)/4$ connected components in case $r_4 \equiv 2$ mod 4, and $(r_4 + 4)/4$ connected components in case $r_4 \equiv 0$ mod 4.

When the genera have distinct parity When g and g' have different parity, relation (2.5.10) implies:

$$\theta(\gamma_1) \cdot \cdots \cdot \theta(\gamma_{r_4}) = z^2.$$

As in the previous paragraph, we have to distinguish whether $r_4 \equiv 2$ mod 4 or $r_4 \equiv 0$ mod 4. If $r_4 \equiv 2$ mod 4 the number of γ_i's such that $\theta(\gamma_i) = z$ is even. Up to permutations of γ_i's we have:

(1) $\theta(\gamma_1) = \cdots = \theta(\gamma_{r_4}) = z$;

(2) $\theta(\gamma_1) = \cdots = \theta(\gamma_{r_4-2}) = z$ and $\theta(\gamma_{r_4-1}) = \theta(\gamma_{r_4}) = z^3$;

$\vdots$

$((r_4+2)/2)$ $\theta(\gamma_1) = \cdots = \theta(\gamma_{r_4}) = z^3$.

Since choices (i) and $((r_4+2)/2 - i + 1)$ are equivalent via automorphism of G, there are $(r_4+2)/4$ possibility for γ. Moreover, since there is at least one element of order $4 = \mathrm{ord}(G)$, surjectivity is automatic.

Consider the last case: $r_4 \equiv 0 \bmod 4$. In this case the number of γ_i's such that $\theta(\gamma_i) = z$ is odd. We have:

(1) $\theta(\gamma_1) = \cdots = \theta(\gamma_{r_4-1}) = z$ and $\theta(\gamma_{r_4}) = z^3$;

(2) $\theta(\gamma_1) = \cdots = \theta(\gamma_{r_4-3}) = z$ and $\theta(\gamma_{r_4-2}) = \theta(\gamma_{r_4-1}) = \theta(\gamma_{r_4}) = z^3$;

$\vdots$

$(r_4/2)$ $\theta(\gamma_1) = z$ and $\theta(\gamma_2) = \cdots = \theta(\gamma_{r_4}) = z^3$.

Since also these choices are equivalent via $\mathrm{Aut}(G)$ two by two, there are $r_4/4$ possibilities for γ. Moreover θ is surjective.

This proves that, if g and g' have different parity (and $g \geq 3g'$), $\mathcal{BH}_{g,g',Gal}$ has $(r_4+2)/4$ connected components in case $r_4 \equiv 2 \bmod 4$, and $r_4/4$ connected components in case $r_4 \equiv 0 \bmod 4$.

Substituting $r_4 = 2g' + 2$ in all cases, we conclude the proof. □

Remark 2.5.4. In the proof of Theorem 2.5.3 we have shown that if condition (2.5.8) is satisfied, than there exist a curve C of genus g, a hyperelliptic curve C' of genus g' and a $2:1$ cover $C \to C'$ such that

$$\begin{tikzcd} C \arrow[rr, "\mathbb{Z}/2\mathbb{Z}"] \arrow[dr, "\mathbb{Z}/4\mathbb{Z}"'] & & C' \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd} \tag{2.5.11}$$

Moreover we claim that every cyclic cover $C \to \mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$ induces over C a bi-hyperelliptic structure. In particular we claim that for

every $g' \leq 1/3\, g$ (see condition (2.5.8)) there exists a monodromy for $C \to \mathbb{P}^1$ such that diagram (2.5.11) commutes, with either C' elliptic or hyperelliptic. In fact applying Riemann-Hurwitz formula to $C \xrightarrow{4:1} \mathbb{P}^1$ one gets:

$$2g + 6 = 3r_4 + 2r_2,$$

where r_4 and r_2 are the numbers of ramification values of order 4 and 2 respectively. Notice that r_4 is necessarily even, and call $r_4 = 2k$. We get $r_2 = g - 3(k-1) \geq 0$. Comparing this expression with expression (2.5.7), we obtain that for every value of $k \geq 1$ the curve C admits a $2:1$ map over either an elliptic or a hyperelliptic curve C' of genus $g' := k - 1$. Condition $r_2 \geq 0$ translates in condition (2.5.8). This proves that every curve C which is a cyclic $4:1$ cover of $\mathbb{P}^1$ also covers with a $2:1$ map an elliptic or a hyperelliptic curve C' of genus g'. By varying the monodromy of the $4:1$ cover we obtain all possible values of g', i.e. $g' \leq 1/3\, g$.

In the next part we will prove that the moduli space of bi-hyperelliptic curves is not totally geodesic in $\mathcal{A}_g$ as soon as $g \geq 3g'$.

2.5.1 The bi-hyperelliptic locus is not totally geodesic

In this section we will prove that the bi-hyperelliptic locus is not totally geodesic when condition (2.5.8) is satisfied. We will consider separately cases $g' = 2$ and $g' \geq 3$.

The idea, as in the bielliptic case, is to apply Theorem 2.3.2 to write the second fundamental form explicitly. As remarked in equality (2.3.10), if the quadric is G-invariant, the existence of a pair of invariant Schiffer variations (ξ_p, ξ_q) such that $\rho(Q)(\xi_p \odot \xi_q) \neq 0$, allows to conclude that the locus is not totally geodesic.

Remark 2.5.5. We need some extra considerations to prove that the restriction to the locus $\mathcal{BH}_{g,g',Gal}$ does not affect the generality of the result. Take a connected component $Z \subset \mathcal{BH}_{g,g',Gal}$ and consider the following chain of inclusions:

$$Z \subseteq \mathcal{BH}_{g,g'} \hookrightarrow \mathcal{A}_g. \tag{2.5.12}$$

As previously said, $\mathcal{BH}_{g,g'}$ is the set of curves covering a hyperelliptic curve C' with group $G_1 = \mathbb{Z}/2\mathbb{Z}$, and $\mathcal{BH}_{g,g',Gal}$ is the subset of $\mathcal{BH}_{g,g'}$ made of

curves covering $\mathbb{P}^1$ with group $G_2 = \mathbb{Z}/4\mathbb{Z}$. Call, respectively,

$$\begin{aligned} \rho_1 :\ & N^*_{\mathcal{BH}_{g,g'}/\mathcal{A}_g} \to S^2H^0(2K)^{G_1} && \text{the second fundamental form} \\ & && \text{relative to } \mathcal{BH}_{g,g'} \subseteq \mathcal{A}_g, \\ \rho_2 :\ & N^*_{Z/\mathcal{A}_g} \to S^2H^0(2K)^{G_2} && \text{the second fundamental form} \\ & && \text{relative to } \mathcal{BH}_{g,g',Gal} \subseteq \mathcal{A}_g. \end{aligned}$$

We are interested in ρ_1. To conclude that it is not identically zero, using equality (2.3.10), it is enough to find some quadric $Q \in I_2(K)^{G_1}$ and a pair of elements $v_1, v_2 \in H^1(TC)^{G_1}$ such that $\rho(Q)(v_1 \odot v_2) \neq 0$. We will do it passing through ρ_2: take a quadric $Q \in I_2(K)^{G_2}$. Since it is invariant via the whole G_2 it is invariant also via $G_1 < G_2$. Take two elements in $H^1(TC)^{G_2}$: again, they are also invariant via G_1. Using the same idea behind equality (2.3.10), this implies that

$$\rho(Q)(v_1 \odot v_2) = \rho_2(Q)(v_1 \odot v_2) = \rho_1(Q)(v_1 \odot v_2), \tag{2.5.13}$$

where ρ is the second fundamental form of $\mathcal{M}_g \to \mathcal{A}_g$: it is sufficient to prove $\rho_2(Q)(v_1 \odot v_2) \neq 0$ to conclude that the whole locus $\mathcal{BH}_{g,g'}$ is not totally geodesic. This is what we will check in the proof of the following theorems.

The main difference with respect to the bielliptic case is that in the bi-hyperelliptic case $I_2(K) \neq I_2(K)^G$. Nevertheless, once restricted to $\mathcal{BH}_{g,g',Gal}$, we can construct an invariant quadric using the decomposition of $H^0(K_C)$ given by the structure of cyclic Galois cover of $\mathbb{P}^1$.

We start studying case $g' \geq 3$.

Theorem 2.5.6. *If $g \geq 3g'$ and $g' \geq 3$, the bi-hyperelliptic locus is not totally geodesic.*

Proof. Consider a curve $C \in \mathcal{BH}_{g,g',Gal}$, and take the Galois cover $\psi : C \to \mathbb{P}^1$. Using the same notation as in Section 2.1, consider the decomposition of of the space of holomorphic one forms in eigenspaces, $H^0(K_C) = V_1 \oplus V_2 \oplus V_3$, and observe that, independently from the chosen monodromy,

$$\dim V_2 = g' \geq 3. \tag{2.5.14}$$

This inequality implies that forms $\omega_{2,0}, \omega_{2,1}, \omega_{2,2} \in V_2$ exist (see Section 2.1 for notations). With the following claim, using these forms we construct a quadric invariant for the whole $G = \mathbb{Z}/4\mathbb{Z}$.

Claim. *Keep notations as above. The quadric*

$$Q := \omega_{2,0} \odot \omega_{2,2} - \omega_{2,1} \odot \omega_{2,1} \tag{2.5.15}$$

is invariant via $G = \mathbb{Z}/4\mathbb{Z}$.

Notice that the quadric Q lies in $I_2(K_C)$ by construction (see Subsection 2.2.2). It is straightforward that it is also invariant: let ζ_4 be the fourth root of unity, and recall from Section 2.1 that in general $\omega_{n,\nu} = x^\nu \omega_{n,0}$. Then:

$$g^*(Q) = \zeta_4^2 \cdot \omega_{2,0} \odot \zeta_4^2 \cdot \omega_{2,2} - \zeta_4^2 \cdot \omega_{2,1} \odot \zeta_4^2 \cdot \omega_{2,1} = \zeta_4^4 \cdot Q = Q.$$

Remark that evaluating the quadric Q over two points p_1, p_2, lying in the same fiber over $\mathbb{P}^1$ we obtain zero. In fact, let $\omega_{2,\nu} = x^\nu f_{2,0}(y)dy$, then:

$$\begin{aligned} Q(p_1,p_2) &= \omega_{2,0}(p_1)\omega_{2,2}(p_2) + \omega_{2,0}(p_2)\omega_{2,2}(p_1) \\ &\quad - \omega_{2,1}(p_1)\omega_{2,1}(p_2) - \omega_{2,1}(p_2)\omega_{2,1}(p_1) = \\ &= f_{2,0}(p_1)x(p_2)^2 f_{2,0}(p_2) + f_{2,0}(p_2)x(p_1)^2 f_{2,0}(p_1) - \\ &\quad - x(p_1)f_{2,0}(p_1)x(p_2)f_{2,0}(p_2) - x(p_2)f_{2,0}(p_2)x(p_1)f_{2,0}(p_1) = 0, \end{aligned}$$

since $x(p_1) = x(p_2)$.

In order to apply Theorem 2.3.2, we need a pair of invariant (combination of) Schiffer variations. To construct them observe that $\mu_2(Q)$ is not the zero map. In fact if $\omega_{2,0} = f(y)dy$, then $\omega_{2,1} = \varphi(y)\omega_{2,0}$ and $\omega_{2,2} = (\varphi(y))^2\omega_{2,0}$. We have that $Q = f(y)dy \odot (\varphi(y))^2 f(y)dy - \varphi(y)f(y)dy \odot \varphi(y)f(y)dy$. This implies:

$$\begin{aligned} \mu_2(Q) &= f'(y)(\varphi(y)^2 f(y))' - (\varphi(y)f(y))'(\varphi(y)f(y))' = \\ &= f'(y)(2\varphi(y)\varphi'(y)f(y) + \varphi(y)^2 f'(y)) - \\ &\quad - (\varphi'(y)f(y) + \varphi(y)f'(y))(\varphi'(y)f(y) + \varphi(y)f'(y)) = \\ &= 2f'(y)f(y)\varphi(y)\varphi'(y) + \varphi(y)^2 f'(y)^2 - \\ &\quad - \varphi'(y)^2 f(y)^2 - \varphi(y)^2 f'(y)^2 - 2\varphi(y)\varphi'(y)f(y)f'(y) = \\ &= -\varphi'(y)^2 f(y)^2 \neq 0. \end{aligned} \tag{2.5.16}$$

So $\mu_2(Q) \neq 0$ is a section of $H^0(4K)$, hence it has a finite number of zeros. Take p_1 such that $\mu_2(Q)(p_1) \neq 0$ and such that $\sigma(p_1) = p_2 \neq p_1$, being σ the bi-hyperelliptic involution. The element $\xi_{p_1} + \xi_{p_2}$ is invariant, in fact: $\sigma_*(\xi_{p_1} + \xi_{p_2}) = \xi_{\sigma(p_1)} + \xi_{\sigma(p_2)} = \xi_{p_1} + \xi_{p_2}$. Finally, using Theorem 2.3.2,

one sees that since $Q(p_1, p_2) = 0$, necessarily $\rho(Q)(\xi_{p_1} \odot \xi_{p_2}) = 0$. Moreover, since $\rho(Q)(\xi_{p_1} \odot \xi_{p_1}) = \rho(Q)(\xi_{p_2} \odot \xi_{p_2})$ the following holds:

$$\rho(Q)((\xi_{p_1} + \xi_{p_2}) \odot (\xi_{p_1} + \xi_{p_2})) = 2\rho(Q)(\xi_{p_1} \odot \xi_{p_1}) = -4\pi i \mu_2(Q)(p_1) \neq 0.$$

Using Remark (2.5.5), the last equation implies that the whole bi-hyperellptic locus $\mathcal{BH}_{g,g'}$ $(g \geq 3g')$ is not totally geodesic in $\mathcal{A}_g$ if $g' \geq 3$. □

We point out that the hypothesis $g' \geq 3$ is used in the very beginning of the proof of Theorem 2.5.6 to construct the invariant quadric involving three forms in V_2. Nonetheless, there are several possibilities to construct a quadric using the eigenspaces V_1, V_2 and V_3:

(1) for every quadruple (i, j, k, h) such that $i + j = k + h$ the quadric $Q = x^i\omega_{n,0} \odot x^j\omega_{n,0} - x^k\omega_{n,0} \odot x^h\omega_{n,0}$ lies in the $I_2(K)$ for $n = 1, 2, 3$;

(2) for every pair (i, j) such that $i \neq j$, the quadric $Q = x^i\omega_{n,0} \odot x^j\omega_{m,0} - x^j\omega_{n,0} \odot x^i\omega_{m,0}$ lies in the $I_2(K)$ for $n \neq m$.

Notice that we need to check that the dimension of the appropriate V_i's is big enough to perform the construction. That is, in case (1) we need that V_1 (respectively V_2, V_3) has dimension al least equal to 3. In case (2) we need that both the eigenspaces involved have dimension at least 2. Once the quadric is constructed, we need to control its invariance.

Case (1) To understand whether the previous quadrics are invariant via the bi-hyperelliptic involution, consider a general element $g \in G$ and compute g^*Q. Recall that V_n's are subspaces of 1-forms ω such that $g \cdot \omega = \zeta_4^{-n}\omega$. It is easy to check that all quadrics of type (1) are invariant under the bi-hyperelliptic involution:

$$\begin{aligned} g^*(Q) &= x^i\zeta_4^{-n} \cdot \omega_{n,0} \odot x^j\zeta_4^{-n} \cdot \omega_{n,0} - x^k\zeta_4^{-n} \cdot \omega_{n,0} \odot x^h\zeta_4^{-n} \cdot \omega_{n,0} = \\ &= \zeta_4^{-2n} \cdot Q = (e^{\pi i})^n Q = (-1)^n Q. \end{aligned} \tag{2.5.17}$$

There are two possibilities for $g^*(Q)$: either Q is invariant via the whole $\mathbb{Z}/4Z$ (case $n = 2$), or Q is invariant via $\mathbb{Z}/2\mathbb{Z} < \mathbb{Z}/4\mathbb{Z}$ (cases $n = 1, 3$). In all three cases Q is invariant via the bi-hyperelliptic involution. Observe that quadric (2.5.15) is a quadric of this type ($n = 2$, $i = 0$, $j = 2$, $k = h = 1$).

Case (2) Consider Q quadric of type (2). Let $g \in G$ and compute $g^*(Q)$ in this case as well:

$$\begin{aligned} g^*(Q) &= x^i\zeta_4^{-n} \cdot \omega_{n,0} \odot x^j\zeta_4^{-m} \cdot \omega_{m,0} - x^j\zeta_4^{-n} \cdot \omega_{n,0} \odot x^i\zeta_4^{-m} \cdot \omega_{m,0} = \\ &= \zeta_4^{-(m+n)} \cdot Q = (e^{2\pi i/4})^{-(m+n)}Q. \end{aligned} \tag{2.5.18}$$

There are three possibilities for $g^*(Q)$: $m+n=3$ ($m=1$, $n=2$ or viceversa), $m+n=4$ ($m=1$, $n=3$ or viceversa) and $m+n=5$ ($m=2$, $n=3$ or viceversa). If $m+n=3$ the quadric Q is not invariant under $\mathbb{Z}/2\mathbb{Z}$, since $\mathbb{Z}/4\mathbb{Z}$ acts on the quadric as a multiplication by i. If $m+n=4$ then $g^*Q=Q$: the quadric is invariant via the whole $\mathbb{Z}/4\mathbb{Z}$ (and in particular via $\mathbb{Z}/2\mathbb{Z}$). Finally, if $m+n=5$ we get $g^*Q=-iQ$: the quadric is not invariant via $\mathbb{Z}/2\mathbb{Z}$ since the group $\mathbb{Z}/4\mathbb{Z}$ acts as a multiplication by $-i$.

Summarizing, all quadrics of type (1) and quadrics of type (2) with $m=1$, $n=3$ are invariant via $\mathbb{Z}/2\mathbb{Z}$. We remind the reader that all invariant quadrics of type (2) have already been found in [30, Proposition 5.7]

In the following, in order to improve the restrictive condition $g' \geq 3$, we try to apply the same argument of the proof of Theorem 2.5.6 with different quadrics. Notice that since the condition $g \geq 3g'$ is essential, the only missing case is $g'=2$.

Theorem 2.5.7. *If $g \geq 3g'$ and $g'=2$, the bi-hyperelliptic locus is not totally geodesic.*

Proof. We start computing possible monodromies for $C \to \mathbb{P}^1$ in case $g'=2$. Applying result (2.5.7), one obtains that the orders of ramifications are necessarily $m=[4^6 : 2^{g-6}]$. Moreover, looking at the proof of Theorem 2.5.7, one immediately checks that:

1. If g is even then:

$$\begin{aligned} &(a) \quad a=[1^5:3:2^{g-6}] \text{ or} \\ &(b) \quad a=[1^3:3^3:2^{g-6}], \end{aligned} \tag{2.5.19}$$

and the dimension of the eigenspaces V_1, V_2, V_3 are the following:

$$(a) \qquad d_1=\frac{g}{2}; \qquad d_2=2; \qquad d_3=\frac{g}{2}-2;$$

$$(b) \qquad d_1=\frac{g}{2}-1; \qquad d_2=2; \qquad d_3=\frac{g}{2}-1.$$

2. If g is odd then:

$$\begin{aligned} &(a) \quad a = [1^6 : 2^{g-6}] \text{ or} \\ &(b) \quad a = [1^4 : 3^2 : 2^{g-6}], \end{aligned} \tag{2.5.20}$$

and the dimension of the eigenspaces V_1, V_2, V_3 are the following:

$$(a) \qquad d_1 = \frac{g+1}{2}; \qquad d_2 = 2; \qquad d_3 = \frac{g-1}{2} - 2;$$

$$(b) \qquad d_1 = \frac{g-1}{2}; \qquad d_2 = 2; \qquad d_3 = \frac{g-1}{2} - 1.$$

We claim that for every genus $g \geq 6$, it is possible to construct an invariant quadric in every connected component.

Consider first case g even. In the connected component denoted by (a), if $g \geq 6$ then $d_1 \geq 3$. One can construct the invariant quadric of type (1): $Q = x^2\, \omega_{1,0} \odot \omega_{1,0} - x\, \omega_{1,0} \odot x\, \omega_{1,0}$. In the connected component denoted by (b), instead, condition $g \geq 6$ implies $d_1 \geq 2$ and $d_3 \geq 2$. In this case one can construct the quadric of type (2): $Q = x\, \omega_{1,0} \odot \omega_{3,0} - x\, \omega_{3,0} \odot \omega_{1,0}$. Condition $g \geq 6$ is sufficient in case g odd as well, in fact it implies $d_1 \geq 3$ both in monodromy (a) and (b). One can construct the invariant quadric $Q = x^2\, \omega_{2,0} \odot \omega_{2,0} - x\, \omega_{2,0} \odot x\, \omega_{2,0}$.

The previous argument proves that if $g' = 2$, $g \geq 6$, for every possible monodromy, we can construct an invariant quadric. Notice that for our structural condition $g \geq 3g'$, the hypothesis $g \geq 6$ is the best possible.

We use these quadrics to prove that the locus of bi-hyperelliptic curves is not totally geodesic even in case $g' = 2$, with an argument identical to the one used in Theorem 2.5.7. It is straightforward to prove that $\mu_2(Q) \neq 0$ for both quadrics of type (1) and (2): the first case is analogous to the one analysed in expression (2.5.16), while the second one is as follows. Call $\omega_{1,0} = f_{1,0}(y)dy$ and $\omega_{3,0} = f_{3,0}(y)dy$. Let $Q = x\omega_{1,0} \odot \omega_{3,0} - x\omega_{3,0} \odot \omega_{1,0}$ as before, where $x = \varphi(y)$ (for notations see Section 2.1). Then:

$$\begin{aligned} \mu_2(Q) &= (\varphi(y) f_{1,0}(y))' f'_{3,0}(y) - (\varphi(y) f_{3,0}(y))' f'_{1,0}(y) = \\ &= (\varphi'(y) f_{1,0}(y) + \varphi(y) f'_{1,0}(y)) f'_{3,0}(y) - \\ &\quad - (\varphi'(y) f_{3,0}(y) + \varphi(y) f'_{3,0}(y)) f'_{1,0}(y) = \\ &= \varphi'(y) f_{1,0}(y) f'_{3,0}(y) + \varphi(y) f'_{1,0}(y) f'_{3,0}(y) - \\ &\quad - \varphi'(y) f_{3,0}(y) f'_{1,0}(y) - \varphi(y) f'_{3,0}(y) f'_{1,0}(y) = \\ &= \varphi'(y)(f_{1,0}(y) f'_{3,0}(y) - f_{3,0}(y) f'_{1,0}(y)) = \varphi'(y) \mu_1(\omega_{1,0} \wedge \omega_{3,0}). \end{aligned}$$

We conclude that $\mu_2(Q)$ is different from zero, since $\varphi' \neq 0$, and the first Gauss-Wahl map in injective on decomposable vectors (see the discussion at the beginning of Subsection 2.2.1).

We can take p_1 in C such that $\mu_2(Q) \neq 0$ and $p_2 = \sigma(p_1)$, being σ the bi-hyperelliptic involution. Then $Q(p_1, p_2) = 0$ and Theorem 2.3.2 gives in all connected components:

$$\rho(Q)((\xi_{p_1} + \xi_{p_2}) \odot (\xi_{p_1} + \xi_{p_2})) = 2\rho(Q)(\xi_{p_1} \otimes \xi_{p_1}) = -4\pi i \mu_2(Q)(p_1) \neq 0.$$

Using Remark 2.5.5, this concludes the proof. $\square$

Chapter 3

Computations

In this chapter we use the computer software `MAPLE` to obtain more informations on the rank of the first and second Gauss-Wahl map on some loci.

The results cited in Subsection 2.2.1 and Subsection 2.2.2 suggest that these map should be regular in some loci. We recall that the rank of both the first and the second Gauss-Wahl map on the hyperelliptic locus depends on the genus only ($\operatorname{rank}\mu_1 = 2g-3$, see [46], [87]; $\operatorname{rank}\mu_2 = 2g-5$, see [27]). Moreover we recall that the rank of the first and second Gauss-Wahl map on the trigonal locus depends only on g as well ($\operatorname{rank}\mu_1 = 4g-10$, see [23], [15]; $\operatorname{rank}\mu_2 = 4g-18$, see [27]). The same holds for the rank of the first Gauss-Wahl map on the bielliptic locus ($\operatorname{rank}\mu_1 = 3g-3$, see [23]) and for smooth plane quintics ($\operatorname{rank}\mu_1 = 4g-9$, see [23]).

In the following, using the computer software `MAPLE`, we will perform some computations for the rank of the first and second Gauss-Wahl map on some particular loci. More precisely we will study the rank of the second Gauss-Wahl map on the bielliptic locus, the rank of both the first and the second Gauss-Wahl map on the tetragonal locus, and the rank of the first and the second Gauss-Wahl map on the bi-hyperelliptic locus (see Section 2.5 for notation).

The `MAPLE` script, explained in Appendix A, provides a lower bound for the rank of both the first and the second Gauss-Wahl map of every curve C which is a cyclic cover of $\mathbb{P}^1$, once is fixed the Galois group G, the genus g,

the monodromy $a = [a_1, \dots a_N]$, and the set of branch points $t = [t_1, \dots t_N]$. For simplicity, we will use the following notation for the monodromy:

$$[a_1^{M_1} : a_2^{M_2} : \dots a_N^{M_N}] := [\underbrace{a_1 : \cdots : a_1}_{M_1} : \underbrace{a_2 : \cdots : a_2}_{M_2} : \cdots : \underbrace{a_N : \cdots : a_N}_{M_N}].$$

This chapter is organized as follows:

In Section 3.1 we study the rank of the second Gauss-Wahl map on the tetragonal locus. Recall from Lemma 2.4.4 that on this locus the second Gauss-Wahl map has rank at most $5g-5$. Via the `MAPLE` script we will prove that the general bielliptic curve of genus $8 \leq g \leq 30$ satisfies $\operatorname{rank} \mu_2 \geq 2g-1$. Also, our result suggests that the rank of any bielliptic curve of genus $g \geq 8$ should be $2g-1$. We point out that this result is not so far from the bound given in Corollary 2.4.13.

In Section 3.2 we study the rank of the first Gauss-Wahl map on the tetragonal locus. We use the `MAPLE` script to compute a lower bound for the rank of the first Gauss-Wahl map when evaluated on a curve C which covers $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$. Recall that Brawner has already studied this locus, obtaining $\operatorname{rank} \mu_1 \leq 5g-14$. We will prove that this bound is attained for any genus up to 30, and our computations suggest that $5g-14$ is the generic bound for every genus $g \geq 8$.

In Section 3.3 we compute the second Gauss-Wahl map on the tetragonal locus. We will list examples up to genus 30 e we prove that the general tetragonal curve has $\operatorname{rank} \mu_2 \geq 6g-31$ in those cases.

In Section 3.4 we study the rank of the first and second Gauss-Wahl map on the bi-hyperelliptic locus using the `MAPLE` script. We will start analysing case $g' = 2$, and we conclude the section as well as the chapter reporting the complete table of cyclic Galois covers $C \to \mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$ ($g \leq 16$), listing for all of them the lower bound for the rank of the first and second Gauss-Wahl map given by `MAPLE`.

3.1 Rank of the second Gauss-Wahl map on the bielliptic locus

In this section we will use the `MAPLE` code reported in Appendix A to gain some extra information about the rank of the second Gauss-Wahl map when evaluated over the bielliptic locus. The code gives us a lower bound for both the rank of the first and the second Gauss-Wahl map.

Recall that the second Gauss-Wahl map is:

$$\mu_2 : I_2(K_C) \to H^0(C, 4K_C). \tag{3.1.1}$$

Computing the dimensions of the vector spaces involved, one gets:

$$\dim I_2(K_C) = \frac{(g-2)(g-3)}{2}, \qquad \dim H^0(C, 4K_C) = 7g - 7. \tag{3.1.2}$$

For dimensional reasons, the map can be injective for genus $g \leq 17$ only. Moreover, recall from Theorem 2.4.4 that the second Gauss-Wahl map has corank at least $2g-2$ over the bielliptic locus. In particular, if $g \geq 18$ the map can not be surjective, and it can neither be injective when $14 \leq g \leq 17$. In the last cases, one can deduce the following lower bounds for the kernel dimensions:

$$\begin{aligned} &\bullet\, g = 14, \quad \dim \ker \mu_2 \geq 1, \\ &\bullet\, g = 15, \quad \dim \ker \mu_2 \geq 8, \\ &\bullet\, g = 16, \quad \dim \ker \mu_2 \geq 16, \\ &\bullet\, g = 17, \quad \dim \ker \mu_2 \geq 25. \end{aligned} \tag{3.1.3}$$

In the following, we use the `MAPLE` code to study the rank of the second Gauss-Wahl map on the bielliptic locus (recall from the end of Subsection 2.2.1 that the rank of the first Gauss-Wahl map on this locus is already known to be $3g-3$). In particular we report here a lower bound for the rank of the map when evaluated over bielliptic curves of genus $5 \leq g \leq 30$ that are defined as cyclic covers of $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$. Recall that we have already studied which monodromies for a cyclic $4:1$ cover of $\mathbb{P}^1$ give a bielliptic curve (see Theorem 2.4.8).

Table 3.1: Rank for the second Gauss-Wahl map of bielliptic curves obtained as cyclic cover of $\mathbb{P}^1$ with group $\mathbb{Z}/4\mathbb{Z}$. The rank is computed using MAPLE; the maximal rank is given by the minimum between $\dim I_2(K)$ and $5g-5$.

genus	monodromy	rank μ_2	max rank μ_2
5	$[1^4 : 2^2]$	3	3
6	$[1^3 : 3 : 2^3]$	6	6
7	$[1^4 : 2^4]$	10	10
8	$[1^3 : 3 : 2^5]$	15	15
9	$[1^4 : 2^6]$	≥ 17	21
10	$[1^3 : 3 : 2^7]$	≥ 19	28
11	$[1^4 : 2^8]$	≥ 21	36
12	$[1^3 : 3 : 2^9]$	≥ 23	45
13	$[1^4 : 2^{10}]$	≥ 25	55
14	$[1^3 : 3 : 2^{11}]$	≥ 27	65
15	$[1^4 : 2^{12}]$	≥ 29	70
16	$[1^3 : 3 : 2^{13}]$	≥ 31	75
17	$[1^4 : 2^{14}]$	≥ 33	80
18	$[1^3 : 3 : 2^{15}]$	≥ 35	85
19	$[1^4 : 2^{16}]$	≥ 37	90
20	$[1^3 : 3 : 2^{17}]$	≥ 39	95
21	$[1^4 : 2^{18}]$	≥ 41	100
22	$[1^3 : 3 : 2^{19}]$	≥ 43	105
23	$[1^4 : 2^{20}]$	≥ 45	110
24	$[1^3 : 3 : 2^{21}]$	≥ 47	115
25	$[1^4 : 2^{22}]$	≥ 49	120
26	$[1^3 : 3 : 2^{23}]$	≥ 51	125
27	$[1^4 : 2^{24}]$	≥ 53	130
28	$[1^3 : 3 : 2^{25}]$	≥ 55	135
29	$[1^4 : 2^{26}]$	≥ 57	140
30	$[1^3 : 3 : 2^{27}]$	≥ 59	145

In Table 3.1 we exhibited one example of bielliptic curve for every genus $5 \leq g \leq 30$, and we reported a lower bound for the rank of the second

Gauss-Wahl map for any of them. Remark that, from Table 3.1, we find $\operatorname{rank}\mu_2 \geq 2g-1$ for every $g \geq 8$, so the general bielliptic curve of genus $8 \leq g \leq 30$ has the same property. We point out that, in those genera, we obtain the same lower bound for all possible monodromies of $C \xrightarrow{\mathbb{Z}/4\mathbb{Z}} \mathbb{P}^1$ such that C is bielliptic. Moreover, we found the same bound for the bielliptic curves giving a Galois cover a $\mathbb{P}^1$ via Galois group $\mathbb{Z}/8\mathbb{Z}$ or $\mathbb{Z}/12\mathbb{Z}$, i.e. curves factorizing via the following diagrams:

$$\begin{tikzcd} C \arrow[rr, "2:1"] \arrow[dr, "6:1"'] & & E \arrow[dl, "3:1"] \\ & \mathbb{P}^1 & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "2:1"] \arrow[dr, "8:1"'] & & E \arrow[dl, "4:1"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

We report in Appendix B the complete list of bielliptic curves obtained as Galois cover of $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/12\mathbb{Z}$ for genus up to 30. We stress that the `MAPLE` code gives $\operatorname{rank}\mu_2 \approx 2g-1$ for any bielliptic curve of this type ($g \geq 8$, `prec`=500, see Appendix A for details on the script). This motivates the following:

Conjecture 3.1.1. *The rank of the second Gauss-Wahl map for a bielliptic curve of genus $g \geq 8$ is $2g-1$.*

Despite this conjecture follows from an approximate computation on some explicit examples of bielliptic curves, it reflects what it seems to be quite natural to happen: as for the first Gauss-Wahl map, we suspected some regularity of the second Gauss-Wahl map on some loci (cfr. the end of Subsection 2.2.1). In particular the rank over the bielliptic locus is likely to depend on the genus only. Moreover we point out that the expectation is not so far from the bound found in Corollary 2.4.13 considering only quadrics constructed using two adjoint line bundles, as in (2.4.15). We conclude this section summarizing the obtained results in the following theorem:

Theorem 3.1.1. *The second Gauss-Wahl map on the bielliptic locus is generically injective if $5 \leq g \leq 8$, moreover it cannot be surjective for genera $g \geq 14$. The general bielliptic curve of genus $8 \leq g \leq 30$ satisfies:*

$$\operatorname{rank}\mu_2 \geq 2g-1. \tag{3.1.4}$$

3.2 Rank of the first Gauss-Wahl map on the tetragonal locus

In this section we study the rank of the first Gauss-Wahl map on the tetragonal locus. We start recalling from the end of Subsection 2.2.1 that Brawner in its Ph.D. thesis [14] has already studied this rank, obtaining:

$$\operatorname{rank} \mu_1 \leq 5g - 14. \tag{3.2.1}$$

Here we report a table consisting in some explicit computation of the rank of the first Gauss-Wahl map performed by `MAPLE`. In particular we computed the rank of the first Gauss-Wahl map on curves C of genus $5 \leq g \leq 17$ covering $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$. We write down for every genus the maximal rank arising from our computations and the monodromy leading to it. We recall (see Remark 2.5.4) that every non hyperelliptic cyclic cover of $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$ covers $2:1$ either an elliptic or a hyperelliptic curve of genus g', and that varying the monodromy of the $4:1$ cover we obtain all $g' \leq 1/3\, g$. In the table we list the genera g' as well.

Table 3.2: Rank for the first Gauss-Wahl map of tetragonal curves obtained as cyclic cover of $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$. The maximal rank is given by $5g - 14$ (see Brawner [14]).

g	g'	monodromy	$\operatorname{rank} \mu_1$	max $\operatorname{rank} \mu_1$
5	1	$[1^2 : 3^2 : 2^2]$	≥ 10	11
6	2	$[1^5 : 3]$	≥ 15	16
7	2	$[1^6 : 2]$	≥ 20	21
8	2	$[1^5 : 3 : 2^2]$	26	26
9	2	$[1^6 : 2^3]$	31	31
10	3	$[1^7 : 3 : 2]$	36	36
11	3	$[1^8 : 2^2]$	41	41
12	4	$[1^9 : 3]$	46	46
13	4	$[1^{10} : 2]$	51	51
14	4	$[1^9 : 3 : 2^2]$	56	56
15	5	$[1^{12}]$	61	61
16	5	$[1^{11} : 3 : 2]$	66	66

From Table 3.2 we find that generically $\operatorname{rank}\mu_1 \geq 5g-14$ for all $8 \leq g \leq 16$. We point out that this generic lower bound coincides with the upper bound found by Brawner [14]. Moreover, for $10 \leq g \leq 16$, the bound is attained by the maximum possible g', that is:

$$g' = \left\lfloor \frac{g}{3} \right\rfloor. \tag{3.2.2}$$

We managed to go on with computations on higher genera, considering (for time reasons) only monodromies for $C \to \mathbb{P}^1$ such that $C \xrightarrow{2:1} C' \to \mathbb{P}^1$ with C' hyperelliptic of genus g' satisfying condition (3.2.2). We obtained in this way one example of curve of rank $5g-14$ for all genus up to 30. We report our results in the following table.

Table 3.3: Rank of the second Gauss-Wahl map of tetragonal curves obtained as cyclic cover of $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$ and $g' = \lfloor g/3 \rfloor$.

g	g'	monodromy	$\operatorname{rank}\mu_2$
5	1	$[1^2 : 3^2 : 2^2]$	≥ 10
6	2	$[1^5 : 3]$	≥ 15
7	2	$[1^6 : 2]$	≥ 20
8	2	$[1^5 : 3 : 2^2]$	$5g-14$
9	3	$[1^6, 3^2]$	$5g-14$
10	3	$[1^7 : 3 : 2]$	$5g-14$
11	3	$[1^8 : 2^2]$	$5g-14$
12	4	$[1^9 : 3]$	$5g-14$
13	4	$[1^{10} : 2]$	$5g-14$
14	4	$[1^9 : 3 : 2^2]$	$5g-14$
15	5	$[1^{12}]$	$5g-14$
16	5	$[1^{11} : 3 : 2]$	$5g-14$
17	5	$[1^{12} : 2^2]$	$5g-14$
18	6	$[1^{13} : 3]$	$5g-14$
19	6	$[1^{14} : 2]$	$5g-14$
20	6	$[1^3 : 3 : 2^2]$	$5g-14$
21	7	$[1^{14} : 3^2]$	$5g-14$

22	7	$[1^{15} : 3 : 2]$	$5g - 14$
23	7	$[1^{16} : 2^2]$	$5g - 14$
24	8	$[1^{15} : 3^3]$	$5g - 14$
25	8	$[1^{18} : 2]$	$5g - 14$
26	8	$[1^{17} : 3 : 2^2]$	$5g - 14$
27	9	$[1^{18} : 3^2]$	$5g - 14$
28	9	$[1^{17} : 3^3 : 2]$	$5g - 14$
29	9	$[1^{20} : 2^2]$	$5g - 14$
30	10	$[1^{21} : 3]$	$5g - 14$

Our result, combined with the result of Brawner, leads to the following expectation:

Expectation. *The rank of the first Gauss-Wahl map on the generic tetragonal curve of genus $g \geq 8$ is equal to $5g - 14$. For every genus g this bound is attained for some curve C such that diagram*

$$\begin{tikzcd} C \arrow[rr, "2:1"] \arrow[dr, "4:1"'] & & C' \arrow[dl, "2:1"] \\ & \mathbb{P}^1 & \end{tikzcd} \tag{3.2.3}$$

holds, with $g(C') = \lfloor g/3 \rfloor$.

As for the bielliptic case, it seems quite natural to expect that the rank of the first Gauss-Wahl map on some loci is in some sense regular. In the spirit of this, we point out that (at least for high genus) the rank seems to be dependent on g and g' only, and not the specific monodromy, i.e. it remains the same by varying the connected component in $\mathcal{BH}_{g,g',\mathrm{Gal}}$ (see Section 2.5 for notations). To justify this assertion we report in Appendix B the complete table of all the obtained results for the rank of the first Gauss-Wahl map of a curve C which covers $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$ ($5 \leq g \leq 16$), highlighting g' as well. We conclude this section summarizing in the following theorem the obtained results:

Theorem 3.2.1. *The general tetragonal curve of genus $8 \leq g \leq 30$ satisfies:*

$$\operatorname{rank} \mu_1 = 5g - 14. \tag{3.2.4}$$

Moreover this bound is attained by the maximum possible g' such that diagram (3.2.3) holds, that is $g(C') = \lfloor g/3 \rfloor$.

3.3 Rank of the second Gauss-Wahl map on the tetragonal locus

In this section we study the rank of the second Gauss-Wahl map on the tetragonal locus. The results, obtained using `MAPLE`, suggest that the scenery is very similar to the one analysed for the first Gauss-Wahl map. In the following, we report the list of curves $C \xrightarrow{\mathbb{Z}/4\mathbb{Z}} \mathbb{P}^1$ providing the greater rank we found for the second Gauss-Wahl map for $5 \leq g \leq 16$. Since the rank seems to grow up as soon as g' grows, we computed the ranks up to $g = 30$ for g' such that (3.2.2) holds and for some fixed monodromy. As in Table 3.2, for all examples we report the genus g' as well.

Table 3.4: Rank of the second Gauss-Wahl map of tetragonal (non hyperelliptic) curves obtained as cyclic cover of $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$. The maximal rank is given by the minimum between $\dim I_2(K_C)$ and $h^0(4K_C)$ (see Section 3.1 for detailes).

g	g'	monodromy	rank μ_2	max rank μ_2
5	1	$[1^2 : 3^2 : 2^2]$	3	3
6	2	$[1^5 : 3]$	3	6
7	2	$[1^6 : 2]$	10	10
8	2	$[1^5 : 3 : 2^2]$	15	15
9	2	$[1^4 : 3^2 : 2^3]$	21	21
10	3	$[1^7 : 3 : 2]$	≥ 27	28
11	3	$[1^8 : 2^2]$	≥ 34	36
12	4	$[1^7 : 3 : 2^3]$	≥ 41	45
13	4	$[1^8 : 2^4]$	≥ 47	55
14	4	$[1^9 : 3 : 2^2]$	≥ 53	66
15	5	$[1^{10} : 3^2]$	≥ 59	78
16	5	$[1^{11} : 3 : 2]$	≥ 65	91
17	5	$[1^{12} : 2^2]$	≥ 71	105
18	6	$[1^{13} : 3]$	≥ 77	119
19	6	$[1^{14} : 2]$	≥ 83	126

20	6	$[1^3 : 3 : 2^2]$	≥ 89	133
21	7	$[1^{14} : 3^2]$	≥ 95	140
22	7	$[1^{15} : 3 : 2]$	≥ 101	147
23	7	$[1^{16} : 2^2]$	≥ 107	154
24	8	$[1^{15} : 3^3]$	≥ 113	161
25	8	$[1^{18} : 2]$	≥ 119	168
26	8	$[1^{17} : 3 : 2^2]$	≥ 125	175
27	9	$[1^{18} : 3^2]$	≥ 131	182
28	9	$[1^{17} : 3^3 : 2]$	≥ 137	189
29	9	$[1^{20} : 2^2]$	≥ 143	196
30	10	$[1^{21} : 3]$	≥ 149	203

Also in this case, notice that for genus big enough ($12 \leq g \leq 30$), the generic tetragonal curve satisfies $\operatorname{rank} \mu_1 \geq 6g - 31$, and this bound is achieved when g' is the greatest possible, i.e. $g' = \lfloor g/3 \rfloor$. We point out that, from our computations, this rank is the biggest one in the complete list of curves C obtained as cover of $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$, and $12 \leq g \leq 16$ (`prec` $= 150$, see Section A for details; we did not perform the computation in all possible monodromies for genus up to 30 for time reasons). Our result suggests the following:

Expectation. *The rank of the second Gauss-Wahl map on the generic tetragonal curve of genus $g \geq 12$ is equal to $6g - 31$. For every genus g this bound is attained for some curve C such that diagram*

$$\begin{tikzcd} C \arrow[rr, "2:1"] \arrow[dr, "4:1"'] & & C' \arrow[dl, "2:1"] \\ & \mathbb{P}^1 & \end{tikzcd} \tag{3.3.1}$$

holds, with $g(C') = \lfloor g/3 \rfloor$.

We conclude this section summarizing all the obtained results in the following theorem:

Theorem 3.3.1. *The general tetragonal curve has injective second Gauss-Wahl map for genus $5 \leq g \leq 9$. For $12 \leq g \leq 30$ the general tetragonal*

curve of genus g satisfies:

$$\operatorname{rank} \mu_2 \geq 6g - 31. \tag{3.3.2}$$

Moreover this bound is attained by the maximum possible g' such that diagram 3.2.3 holds, that is $g(C') = \lfloor g/3 \rfloor$.

3.4 Gauss-Wahl map on the bi-hyperelliptic loci

In this section, we study the rank of the first and second Gauss-Wahl map on the bi-hyperelliptic loci using the `MAPLE` script. We will see that the rank of both maps seems to depend on g and g' only, not on the chosen monodromy of $C \to \mathbb{P}^1$, i.e. these ranks seem to remain the same by varying the connected component in $\mathcal{BH}_{g,g',\mathrm{Gal}}$ (see Section 2.4 for notations). We will start our analysis from $g' = 2$.

Case $g' = 2$

In the following, we report the rank of the first and the second Gauss-Wahl map of curves that are covers of $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$ and factorizing via a genus $g' = 2$ curve (see diagram 3.2.3). We will list every examples up to genus 30.

Table 3.5: Rank of the first and second Gauss-Wahl map of curves obtained as cyclic cover of $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$ such that $C \to C' \to \mathbb{P}^1$ with $g(C') = 2$.

g	monodromy	$\operatorname{rank} \mu_1$	$\operatorname{rank} \mu_2$
6	$[1^5 : 3]$	15	6
6	$[1^3 : 3^3]$	≥ 14	6
7	$[1^6 : 2]$	≥ 20	10
7	$[1^4 : 3^2 : 2]$	≥ 20	10
8	$[1^5 : 3 : 2^2]$	≥ 26	15
8	$[1^3 : 3^3 : 2^2]$	≥ 26	15
9	$[1^6 : 2^3]$	≥ 31	≥ 20
9	$[1^4 : 3^2 : 2^3]$	≥ 31	21

10	$[1^5 : 3 : 2^4]$	≥ 35	≥ 27
10	$[1^3 : 3^3 : 2^4]$	≥ 35	≥ 27
11	$[1^6 : 2^5]$	≥ 39	≥ 32
11	$[1^4 : 3^2 : 2^5]$	≥ 39	≥ 32
12	$[1^5 : 3 : 2^6]$	≥ 43	≥ 36
12	$[1^3 : 3^3 : 2^6]$	≥ 43	≥ 36
13	$[1^6 : 2^7]$	≥ 47	≥ 39
13	$[1^4 : 3^2 : 2^7]$	≥ 47	≥ 39
14	$[1^5 : 3 : 2^8]$	≥ 51	≥ 42
14	$[1^3 : 3^3 : 2^8]$	≥ 51	≥ 42
15	$[1^6 : 2^9]$	≥ 55	≥ 45
15	$[1^4 : 3^2 : 2^9]$	≥ 55	≥ 45
16	$[1^5 : 3 : 2^{10}]$	≥ 59	≥ 48
16	$[1^3 : 3^3 : 2^{10}]$	≥ 59	≥ 48
17	$[1^6 : 2^{11}]$	≥ 63	≥ 51
17	$[1^4 : 3^2 : 2^{11}]$	≥ 63	≥ 51
18	$[1^5 : 3 : 2^{12}]$	≥ 67	≥ 54
18	$[1^3 : 3^3 : 2^{12}]$	≥ 67	≥ 54
19	$[1^6 : 2^{13}]$	≥ 71	≥ 57
19	$[1^4 : 3^2 : 2^{13}]$	≥ 71	≥ 57
20	$[1^5 : 3 : 2^{14}]$	≥ 75	≥ 60
20	$[1^3 : 3^3 : 2^{14}]$	≥ 75	≥ 60
21	$[1^6 : 2^{15}]$	≥ 79	≥ 63
21	$[1^4 : 3^2 : 2^{15}]$	≥ 79	≥ 63
22	$[1^5 : 3 : 2^{16}]$	≥ 83	≥ 66
22	$[1^3 : 3^3 : 2^{16}]$	≥ 83	≥ 66
23	$[1^6 : 2^{17}]$	≥ 87	≥ 69
23	$[1^4 : 3^2 : 2^{17}]$	≥ 87	≥ 69
24	$[1^5 : 3 : 2^{18}]$	≥ 91	≥ 72
24	$[1^3 : 3^3 : 2^{18}]$	≥ 91	≥ 72
25	$[1^6 : 2^{19}]$	≥ 95	≥ 75
25	$[1^4 : 3^2 : 2^{19}]$	≥ 95	≥ 75
26	$[1^5 : 3 : 2^{20}]$	≥ 99	≥ 78
26	$[1^3 : 3^3 : 2^{20}]$	≥ 99	≥ 78
27	$[1^6 : 2^{21}]$	≥ 103	≥ 81
27	$[1^4 : 3^2 : 2^{21}]$	≥ 103	≥ 81
28	$[1^5 : 3 : 2^{22}]$	≥ 107	≥ 84
28	$[1^3 : 3^3 : 2^{22}]$	≥ 107	≥ 84

29	$[1^6 : 2^{23}]$	≥ 111	≥ 87
29	$[1^4 : 3^2 : 2^{23}]$	≥ 111	≥ 87
30	$[1^5 : 3 : 2^{24}]$	≥ 115	≥ 90
30	$[1^3 : 3^3 : 2^{24}]$	≥ 115	≥ 90

From the table arises that for all cyclic examples with $9 \leq g \leq 30$ the rank of the first Gauss-Wahl map is at least $4g - 5$. Also, when $12 \leq g \leq 30$, the rank of the second Gauss-Wahl map is at least $3g$. We point out that, for those genera the monodromy does not affect the rank of the two considered map. This leads to the following:

Expectation. *Let C be a general tetragonal curve of genus g covering $2:1$ a curve of genus 2. Then the following holds:*

- $\operatorname{rank} \mu_1 = 4g - 5$ *for all* $g \geq 9$,
- $\operatorname{rank} \mu_2 = 3g$ *for all* $g \geq 12$.

We conclude this section with a theorem summarizing our results:

Theorem 3.4.1. *The general curve C covering $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$ and such that diagram (3.2.3) commutes with $g' = 2$ has injective first Gauss-Wahl map in genus $g = 6$ and injective second Gauss-Wahl map in genus $6 \leq g \leq 9$. Moreover it satisfies:*

- $\operatorname{rank} \mu_1 \geq 4g - 5$ *for* $9 \leq g \leq 30$,
- $\operatorname{rank} \mu_2 \geq 3g$ *for* $12 \leq g \leq 30$.

General case

In this last part we analyse the locus of bi-hyperelliptic curves for $g' \geq 3$. We simply report the complete table of cyclic Galois covers of $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$ ($5 \leq g \leq 16$), highlighting g', the monodromy, and the rank of the first and second Gauss-Wahl map computed by `MAPLE`. As the reader could notice, also in the general case it seems to be possible to make expectations similar to the ones made in cases $g' = 1$ and $g' = 2$.

Table 3.6: Rank of the first and second Gauss-Wahl map evaluated on the locus of curves C covering $\mathbb{P}^1$ with group $\mathbb{Z}/4\mathbb{Z}$ and $g' \geq 3$.

g	g'	monodromy	rank μ_2	rank μ_2
9	3	$[1^8]$	≥ 30	≥ 20
9	3	$[1^6, 3^2]$	≥ 30	≥ 21
9	3	$[1^4, 3^4]$	≥ 30	≥ 20
10	3	$[1^7 : 3 : 2]$	≥ 36	≥ 27
10	3	$[1^5 : 3^3 : 2]$	≥ 36	≥ 27
11	3	$[1^8 : 2^2]$	≥ 41	≥ 33
11	3	$[1^6 : 3^2 : 2^2]$	≥ 41	≥ 34
11	3	$[1^4 : 3^4 : 2^2]$	≥ 41	≥ 34
12	3	$[1^7 : 3 : 2^3]$	≥ 46	≥ 41
12	3	$[1^5 : 3^3 : 2^3]$	≥ 46	≥ 41
12	4	$[1^9 : 3]$	≥ 46	≥ 39
12	4	$[1^7 : 3^3]$	≥ 46	≥ 39
12	4	$[1^5 : 3^5]$	≥ 46	≥ 39
13	3	$[1^8 : 2^4]$	≥ 50	≥ 47
13	3	$[1^6 : 3^2 : 2^4]$	≥ 50	≥ 47
13	3	$[1^4 : 3^4 : 2^4]$	≥ 50	≥ 47
13	4	$[1^{10} : 2]$	≥ 51	≥ 45
13	4	$[1^8 : 3^2 : 2]$	≥ 51	≥ 46
13	4	$[1^6 : 3^4 : 2]$	≥ 51	≥ 46
14	3	$[1^7 : 3 : 2^5]$	≥ 54	≥ 53
14	3	$[1^5 : 3^3 : 2^5]$	≥ 54	≥ 53
14	4	$[1^9 : 3 : 2^2]$	≥ 56	≥ 53
14	4	$[1^7 : 3^3 : 2^2]$	≥ 56	≥ 53
14	4	$[1^5 : 3^5 : 2^2]$	≥ 56	≥ 53
15	3	$[1^8 : 2^6]$	≥ 58	≥ 58
15	3	$[1^6 : 3^2 : 2^6]$	≥ 58	≥ 58
15	3	$[1^4 : 3^4 : 2^6]$	≥ 58	≥ 58
15	4	$[1^{10} : 2^3]$	≥ 61	≥ 59
15	4	$[1^8 : 3^2 : 2^3]$	≥ 61	≥ 59
15	4	$[1^6 : 3^4 : 2^3]$	≥ 61	≥ 59
15	5	$[1^{12}]$	≥ 61	≥ 57
15	5	$[1^{10} : 3^2]$	≥ 61	≥ 58
15	5	$[1^8 : 3^4]$	≥ 61	≥ 58
15	5	$[1^6 : 3^6]$	≥ 61	≥ 58

16	3	$[1^7 : 3 : 2^7]$	≥ 62	≥ 63
16	3	$[1^5 : 3^3 : 2^7]$	≥ 62	≥ 63
16	4	$[1^9 : 3 : 2^4]$	≥ 65	≥ 65
16	4	$[1^7 : 3^3 : 2^4]$	≥ 65	≥ 65
16	4	$[1^5 : 3^5 : 2^4]$	≥ 65	≥ 65
16	5	$[1^{11} : 3 : 2]$	≥ 66	≥ 65
16	5	$[1^9 : 3^3 : 2]$	≥ 66	≥ 65
16	5	$[1^7 : 3^5 : 2]$	≥ 66	≥ 65

We conclude this chapter commenting briefly Table 3.6. Notice that in almost all cases the rank of the first and second Gauss-Wahl map depends on g and g' only. The only exceptions are given by cases $(g, g', a) = (9, 3, [1^6 : 3^2])$, $(11, 3, [1^8 : 2^2])$ and $(13, 4, [1^{10}, 2])$. This makes one suspect that either the ranks stabilize for higher genus or the precision set in the `MAPLE` script is not enough for these particular monodromies (for those three cases we made the script run up to `prec`=1000).

Chapter 4

The geometry of $\mathcal{A}_4^{(1,1,2,2)}$

The purpose of this chapter is to study the geometry of the moduli space $\mathcal{A}_4^{(1,1,2,2)}$, parametrizing isomorphism classes of 4-dimensional abelian varieties with polarization of type $(1,1,2,2)$. More precisely, our aim is to study its Picard group $\mathrm{Pic}(\mathcal{A}_4^{(1,1,2,2)})$ in order to get informations about its Kodaira dimension.

In general, the problem of computing the Kodaira dimension of the moduli spaces $\mathcal{A}_g^{(d_1,\dots,d_g)}$ has been a topic of intense study in the last years. Since the direct calculation of the Kodaira dimension $\kappa(X) := \dim \bigoplus_i H^0(X, K_X^{\otimes i})$ of a variety X is often very hard to perform, the majority of results about $\kappa(\mathcal{A}_g^{(d_1,\dots,d_g)})$ have been obtained as a consequence of generality and rationality properties: it is well known in fact that every variety of general type X has $\kappa(X) = \dim(X)$, maximal, and that every variety X which is unirational (i.e. that admits a rational dominant map $\mathbb{P} \dashrightarrow X$) has $\kappa(X) = -\infty$.

The case $\mathcal{A}_g := \mathcal{A}_g^{(1,\dots,1)}$ of principally polarized abelian varieties, has been almost solved: it has been shown that the moduli space $\mathcal{A}_g$ is unirational if $g \leq 5$, that implies that its Kodaira dimension $\kappa(\mathcal{A}_g) = -\infty$ (see [26], [38], [62], [83]). Is has also been shown that the moduli spaces $\mathcal{A}_g$ are of general type for $g \geq 7$, so their Kodaira dimension turns out to be maximal (see [65], [81]). The only unsolved case is $\mathcal{A}_6$, whose Kodaira dimension is yet unknown.

Less is known about the Kodaira dimension of the case of non-principally polarized abelian varieties: we recall the result of Tai, who proved that the

moduli space $\mathcal{A}_g^{(d_1,\dots,d_g)}$ is of general type when $g \geq 16$ for every choice of the polarization, and when $g \geq 8$ but for certain polarizations [81]. The only result about unirationality of such moduli spaces is due to Bardelli, Ciliberto and Verra [7], who proved that $\mathcal{A}_4^{(1,2,2,2)}$ is unirational. Moreover, since this space is isomorphic to $\mathcal{A}_4^{(1,1,1,2)}$ (see Birkenhake and Lange [10]), this also implies the unirationality of $\mathcal{A}_4^{(1,1,1,2)}$. Nevertheless nothing is known about neither the unirationality of $\mathcal{A}_4^{(1,1,2,2)}$ nor its Kodaira dimension. In this chapter, in order to better understand the geometry of $\mathcal{A}_4^{(1,1,2,2)}$, we try to get more information on its Picard group.

The chapter is organized as follows:

In Section 4.1 we recall some basic theory on polarized abelian varieties, Prym maps and Prym varieties. Moreover we explain the bigonal construction ([71]): a procedure that associates to a tower of double covers $D \to C \to K$ another tower of double covers, whose Prym is dual to the Prym of $D \to C$.

In Section 4.2 we construct explicit divisors of that moduli space following two different approaches: the first divisor is constructed as the image of the Prym map $P : \mathcal{R}_{2,6} \to \mathcal{A}_4^{(1,1,2,2)}$, sending a cover $\pi : D \to C$ in $\mathcal{R}_{2,6}$ to its Prym. The second divisor is constructed from the map $\tilde{\mathcal{A}}_4 \to \mathcal{A}_4^{(1,1,2,2)}$, sending a principally polarized abelian variety X of dimension 4 together with a fixed totally isotropic order 4 subgroup H of 2-torsion elements to the quotient X/H, and then considering the image of the Jacobian locus by this map.

In Section 4.3, to get more informations about these divisors, we check if they are invariant under the natural involution defined on the moduli space $\mathcal{A}_4^{(1,1,2,2)}$ by Birkenhake and Lange, sending a polarized abelian variety (A, L_A) to its dual $(A^\vee, L_A^\vee)$ (see [10]). We almost immediately obtain that the divisor constructed with the Prym procedure is fixed by the involution, by using the result of Pantazis stating that two bigonally related covers have dual Prym varieties (see [71]). On the other hand, with a bit more work, we obtain that the second divisor is not invariant under the involution: the clue here is a theorem due to Bardelli and Pirola, stating that if there exists an isogeny between two Jacobians JC and JC' (JC generic, with dimension at least 4), then the two Jacobians are isomorphic, and the isogeny is the multiplication by an integer (see [9]). Since the involution does not preserve this divisor, we get a very explicit description of a different divisor

in $\mathcal{A}_4^{(1,1,2,2)}$, obtained by duality.

4.1 Notation and preliminaries

We start by stating some well known results about complex polarized abelian varieties and Pryms, then we recall the main ideas of the bigonal construction, which will be used in the next section. Our main reference for this preliminary section is Birkenhake and Lange's book [11].

4.1.1 Polarized abelian varieties

A polarized abelian variety (A, H) of dimension g is the datum of a complex torus $A = \mathbb{C}^g/\Lambda$ and a non-degenerate positive definite hermitian form $H \in H^2(A, \mathbb{Z})$ satisfying $\mathrm{Im}(H)(\Lambda \times \Lambda) \subset \mathbb{Z}$. The form H is the first Chern class of an ample line bundle L over A. Sometimes, when the polarization is not needed, we will use A to refer to it.

The following proposition is necessary to define the type of a polarization H of a g-dimensional abelian variety.

Proposition 4.1.1. *Let $E : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ be an alternating and non-degenerate form. Then there exist positive integers $d_1, \ldots, d_g$ such that $d_i \mid d_{i+1}$ for all $i = 1, \ldots, g$, and there exists a basis of Λ such that the matrix associated to E in this basis is :*

$$E = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}, \text{ where } D = \begin{pmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_n \end{pmatrix}.$$

We define the type of the polarization H to be the g-tuple $(d_1, \ldots, d_g)$ of integers from the proposition above (where $E = \mathrm{Im}(H)$). A polarization H whose type is $(1, \ldots, 1)$ is called principal. The main example of principally polarized abelian varieties are Jacobians of curves.

A morphism $f : A \longrightarrow A'$ between two abelian varieties A and A' is called an isogeny if and only if it satisfies the following properties :

1. A and A' have the same dimension,
2. f is surjective,
3. $Ker(f)$ is finite.

We remark that any two of the above properties imply the remaining one, so to define an isogeny it actually suffices to have only two of the properties.

Let (A, H) be a polarized abelian variety. Fix a line bundle $L \in \mathrm{Pic}(A)$ satisfying $c_1(L) = H$. The morphism $\lambda_L : A \longrightarrow A^\vee$ given by $a \mapsto \tau_a^* L \otimes L^{-1}$ is an isogeny. Here, $A^\vee = \mathrm{Pic}^0(A)$ is the dual of A, and τ_a^* is the translation by a in A. We get the following result, describing the kernel $K(L)$ of λ_L:

Theorem 4.1.2. *If L is a polarization of type $(d_1, \ldots, d_g)$, then $d_i \mid d_{i+1}$ for all $i = 1, \ldots, g$, and*

$$K(L) \cong \left(\mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_g\mathbb{Z}\right)^2.$$

It is useful to remark that $\deg(\lambda_L) = |K(L)| = d_1^2 \times \ldots \times d_g^2$.

In order to understand better the relation between line bundles over isogenous abelian varieties, we introduce the Riemann bilinear form associated to a line bundle: if $K(L)$ is the kernel of a line bundle L over $A = V/\Lambda$, we define the Riemann bilinear form as the bilinear alternating form

$$\begin{aligned} e^L : K(L) \times K(L) &\longrightarrow \mathbb{C}^* \\ (x, y) &\longmapsto \exp^{-2i\pi H(\tilde{x}, \tilde{y})}, \end{aligned}$$

where $\tilde{x}$, $\tilde{y}$ are the liftings of respectively x, y to the vector space V. Also, we recall that $H = c_1(L)$ and we have $K(L) = \{x \in A \,|\, H(l, \tilde{x}) \in \mathbb{Z}$, for all $l \in \Lambda\}$ (see [35] Chapter VI Section 4).

Notice that if the line bundle L is ample then the form e^L is non-degenerate. To appreciate the importance of this pairing, we state a useful result relating line bundles of isogenous abelian varieties, whose proof can be found in Birkenhake and Lange's book ([11], Corollary 6.3.5). Before stating the theorem, recall that a subgroup $K < K(L)$ is totally isotropic with respect to e^L if for all $x, y \in K$ we have $e^L(x, y) = 1$.

Proposition 4.1.3. *Let $f : X \to Y$ be an isogeny of abelian varieties and let L be a line bundle on X. Then the following statements are equivalent:*

1. $L = f^*(M)$ *for some* $M \in \mathrm{Pic}(Y)$,

2. $\ker(f)$ *is a totally isotropic subgroup of* $K(L)$ *with respect to* e^L.

To conclude this section we recall the construction of Birkenhake and Lange's involution: denote by $\mathcal{A}_g^{(d_1,\ldots,d_g)}$ the coarse moduli space parametrizing isomorphism classes of g-dimensional polarized abelian varieties of type $(d_1,\ldots,d_g)$; it is a quasi-projective variety of dimension $\frac{g(g+1)}{2}$. In [10], Birkenhake and Lange have shown that there is an isomorphism of the coarse moduli spaces

$$\mathcal{A}_g^{(d_1,\ldots,d_g)} \xrightarrow{\cong} \mathcal{A}_g^{\left(\frac{d_1 d_g}{d_g},\frac{d_1 d_g}{d_{g-1}},\ldots,\frac{d_1 d_g}{d_2},\frac{d_1 d_g}{d_1}\right)}.$$

In case $g = 4$ and polarization $(1,1,2,2)$, the isomorphism is actually an automorphism $\rho : \mathcal{A}_4^{(1,1,2,2)} \longrightarrow \mathcal{A}_4^{(1,1,2,2)}$, associating to a polarized abelian variety (A, L_A) its dual variety $(A^\vee, L_{A^\vee})$. The polarization $L_{A^\vee}$ on $A^\vee$ is constructed in order to satisfy $(L_{A^\vee})^\vee = L_A$ (see [10], Proposition 2.3). Since $(A^\vee)^\vee = A$ we get $\rho^2((A, L_A)) = (A, L_A)$, hence ρ is an involution on the moduli space $\mathcal{A}_4^{(1,1,2,2)}$.

4.1.2 Prym maps and Prym varieties

Let $C \in \mathcal{M}_g$, $D \in \mathcal{M}_{g'}$, and let $D \xrightarrow{\pi} C$ be a finite morphism of degree 2 branched on a divisor $B = q_1 + \ldots + q_r$, with $q_i \in C$ and $q_i \neq q_j$ for all $i \neq j$. The curve D is obtained as $Spec\,(\mathcal{O}_C \oplus \eta^{-1})$ with $\eta \in \mathrm{Pic}(C)$ such that $\eta^{\otimes 2} \cong \mathcal{O}_C(B)$. Observe that Riemann-Hurwitz formula gives that the genus of D is $2g - 1 + \frac{r}{2}$ (notice that r has to be even).

We get the following diagram:

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, hook, "\pi"] & \mathrm{Alb}(D) = JD \\ C \arrow[r, hook] & JC \end{tikzcd}$$

It is possible to complete it using the universal property of the Albanese of D. Hence the following commutative diagram holds:

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, hook, "\pi"] & \mathrm{Alb}(D) = JD \arrow[d, "Nm_\pi"] \\ C \arrow[r, hook] & JC \end{tikzcd}$$

The map $JD \xrightarrow{Nm_\pi} JC$ is called norm map. It is well known that this map is surjective if the cover is branched. We are ready to define the Prym variety attached to a cover.

Definition 4.1.4. *The Prym variety attached to the cover $D \xrightarrow{\pi} C$ is the connected component containing the origin of the kernel of the norm map:*

$$P(D, C) = \ker(Nm_\pi)^0.$$

The Prym variety $(P(D, C), \Xi)$ turns to be a polarized abelian variety of dimension $g - 1 + \frac{r}{2}$: the polarization Ξ is obtained as the first Chern class of the restriction on $P(D, C)$ of the line bundle $\mathcal{O}_{JD}(\Theta_D)$, where Θ_D is the principal polarization of JD. Notice that Ξ is of type $(\underbrace{1, \ldots, 1}_{\frac{r}{2}-1}, \underbrace{2, \ldots, 2}_{g})$.

4.1.3 The bigonal construction

The bigonal construction is a procedure that associates to a tower of double covers $D \to C \to K$ another tower of double covers, whose Prym is dual to $P(D \to C)$. Since the duality result of Pryms will be useful later in our discussion, we give some details (see Pantazis for an accurate description [71]).

Let $\varphi : C \to K$ be a cover of degree 2 (hence the "bi" in bigonal) and let $\pi : D \to C$ be a branched cover of degree 2. The curve D is equipped with an involution ι that switches the two elements of the fiber of a generic point $c \in C$. The two given covers determine a degree 2^2 cover $\Gamma \longrightarrow K$, whose fiber over a generic point $k \in K$ consists of 4 sections s_k of π over k:

$$s_k : \varphi^{-1}(k) \longrightarrow \pi^{-1}\varphi^{-1}(k), \qquad \pi \circ s_k = id_K.$$

The curve Γ is defined set-theoretically as

$$\Gamma = \{B \in \mathrm{Pic}^2(D) \mid F_{Nm(B)} = \varphi^{-1}(k),\ k \in K\},$$

where $Nm : \mathrm{Pic}^2(D) \longrightarrow \mathrm{Pic}^2(C)$ sends $[x_1 + x_2] \in \mathrm{Pic}^2(D)$ to $[\pi(x_1) + \pi(x_2)]$, and $F_B = \{x_1, x_2 \,|\, B = [x_1 + x_2]\}$. With this definition, one can view a point $p \in \Gamma$ which belongs to the fiber of some $k \in K$ as a section s_k.

There is an involution on Γ defined by $\tilde{\iota}(s_k) = \iota \circ s_k$, $k \in K$, which in turn gives an equivalence relation where two points $s_1, s_2 \in \Gamma$ are said to be equivalent if $s_1 = \tilde{\iota}(s_2)$. Considering the quotient $\Gamma_0 = \Gamma/\tilde{\iota}$ one obtains a tower of degree 2 covers $\Gamma \longrightarrow \Gamma_0 \longrightarrow K$.

The two towers $D \xrightarrow{\pi} C \xrightarrow{\varphi} K$ and $\Gamma \xrightarrow{\tilde{\pi}} \Gamma_0 \xrightarrow{\tilde{\varphi}} K$ are said to be bigonally related (see Donagi for details [39]). Since φ and π are branched, this implies that $\tilde{\varphi}$ and $\tilde{\pi}$ are branched as well. Moreover, observe that the bigonal construction switches branch loci.

The following result, due to Pantazis [71], claims that bigonally related Prym varieties are dual :

Theorem 4.1.5. *Consider a pair of maps of degree 2, $D \to C \to \mathbb{P}^1$, and the bigonally related tower $\Gamma \to \Gamma_0 \to \mathbb{P}^1$. Consider the Pryms:*

$$P(D, C) := \ker^0(Nm : J(D) \to J(C)),$$
$$P(\Gamma, \Gamma_0) := \ker^0(Nm : J(\Gamma) \to J(\Gamma_0)).$$

Then $(P(D, C), \theta)$ and $(P(\Gamma, \Gamma_0), \theta')$ are dual polarized abelian varieties.

We conclude this introductory section by briefly defining some notions and fixing some notation which we shall use throughout the rest of this chapter:

- $X_m < X$ is the kernel of $\cdot m : X \longrightarrow X$, the multiplication by m. We will usually refer to X_m as the subgroup of m-torsion elements of X;
- $\mathcal{R}_{g,r}$ will denote the moduli space of double covers of a curve of genus g with r ramifications;
- we denote as $P : \mathcal{R}_{g,r} \to \mathcal{A}^{\delta}_{g-1-\frac{r}{2}}$ the Prym map, associating to a cover its Prym variety;
- if C is a curve, Θ_C will denote the principal polarization of the Jacobian JC. If A is a polarized abelian variety, we will use the line bundle L_A to refer to the polarization of A, instead of the hermitian form $H = c_1(L_A)$.

4.2 Construction of divisors in $\mathcal{A}_4^{(1,1,2,2)}$

In this section we construct two divisors of the moduli space $\mathcal{A}_4^{(1,1,2,2)}$: the first one will be constructed as the image of $\mathcal{R}_{2,6}$ by the Prym map P, the other one will be obtained as the image of $\mathcal{M}_g$ in $\mathcal{A}_4^{(1,1,2,2)}$ via the Torelli map and a quotient construction.

4.2.1 Prym construction

The first construction of a divisor in $\mathcal{A}_4^{(1,1,2,2)}$ immediately follows from the Prym map $P : \mathcal{R}_{2,6} \to \mathcal{A}_4^{(1,1,2,2)}$ which sends a cover $\pi : D \to C$ in $\mathcal{R}_{2,6}$ to its Prym variety. From the general theory of Pryms, since the cover π ramifies, the kernel of the norm map is connected, thus $P(D, C) = \ker\{Nm(\pi) : JD \to JC\}$. It is a Prym variety of dimension 4 and polarization $(1, 1, 2, 2)$.

The Prym map $P : \mathcal{R}_{2,6} \to \mathcal{A}_4^{(1,1,2,2)}$ has been studied in a recent work of J. C. Naranjo and A. Ortega [67]: the two authors show that it is injective. Since $\mathcal{R}_{2,6}$ has dimension $3g - 3 + r = 9$, its image by P is a divisor of the 10-dimensional moduli space $\mathcal{A}_4^{(1,1,2,2)}$. We name the obtained divisor $\mathcal{P}$.

4.2.2 Quotient construction

We start defining the moduli space of principally polarized abelian varieties of dimension 4 with a fixed totally isotropic subgroup of 2-torsion elements:

$$\tilde{\mathcal{A}}_4 = \{(X, L_X, H) | (X, L_X) \text{ is a principally polarized abelian variety of dimension 4,} \\ H \subset X_2 \text{ is a totally isotropic subgroup of four elements}\}$$

Let $(X, L_X, H) \in \tilde{\mathcal{A}}_4$, and take the quotient $A := X/H$. It gives a degree 4 isogeny $f : X \to A$. Via Proposition 4.1.3, one can choose over A a polarization L_A whose pullback by f is $L_X^{\otimes 2}$. Considering the isogenies

induced by the polarizations, we get the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\lambda_{2\Theta}"'] & A \arrow[d, "\lambda_{L_A}"] \\ X^\vee & A^\vee \arrow[l, "f^\vee"] \end{tikzcd} \tag{4.2.1}$$

Observe that $X_2 = f^{-1}(\ker(f^\vee \circ \lambda_{L_A}))$. Computing the degree of the involved maps one gets that $\deg(\lambda_{2\Theta}) = |X_2| = 2^8$ has to be equal to $\deg(f^\vee \circ \lambda_{L_A} \circ f) = 2^2 \cdot |\ker(\lambda_{L_A})| \cdot 2^2$, meaning that $|\ker(\lambda_{L_A})| = 2^4$. Since $\ker(\lambda_{L_A})$ is a commutative subgroup, then

$$\ker(\lambda_{L_A}) \cong (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^2, \tag{4.2.2}$$

so $(A, L_A) \in \mathcal{A}_4^{(1,1,2,2)}$.

This construction gives a finite cover $\varpi : \tilde{\mathcal{A}}_4 \longrightarrow \mathcal{A}_4^{(1,1,2,2)}$ which takes a triple (X, L_X, H) and sends it to (A, L_A).

Consider the Torelli map $\tau : \mathcal{M}_4 \to \mathcal{A}_4$. Since this map is generically injective, the Jacobian locus is a subvariety of dimension 9 of $\mathcal{A}_4$. Consider the Jacobian locus inside $\tilde{\mathcal{A}}_4$ in the natural way: its image by the finite cover ϖ defines a divisor in $\mathcal{A}_4^{(1,1,2,2)}$. Call it $\mathcal{J}$.

4.3 Invariance of divisors via the involution

In the previous section we have obtained the divisor $\mathcal{P}$ via the Prym construction, and the divisor $\mathcal{J}$ via the quotient construction. In this section we check how these two divisors behave under the involution ρ. We state here our main result:

Theorem 4.3.1. *Let $\mathcal{P}$ and $\mathcal{J}$ be the divisors of $\mathcal{A}_4^{(1,1,2,2)}$ constructed in Section 4.2. Let $\rho : \mathcal{A}_4^{(1,1,2,2)} \to \mathcal{A}_4^{(1,1,2,2)}$ be the Birkenhake and Lange's involution. Then we have the following:*

1. *$\mathcal{P} = \rho(\mathcal{P})$: $\mathcal{P}$ is invariant under the involution;*
2. *$\mathcal{J} \neq \rho(\mathcal{J})$: $\mathcal{J}$ is not invariant under the involution.*

Proof of point (1). To prove point (1) of Theorem 4.3.1, we need to show that the dual of a Prym variety inside the Prym divisor $\mathcal{P}$ is also a Prym variety. This will follow from the bigonal construction and Theorem 4.1.5. In fact, let $D \xrightarrow{\pi} C$ be a general branched cover in $\mathcal{R}_{2,6}$. C is a hyperelliptic curve since it is of genus two, so it admits a $2:1$ cover of $\mathbb{P}^1$, which we call φ. There are six Weierstrass points on C, which by generality one can suppose to be different than the branch locus of the cover $D \xrightarrow{\pi} C$. Applying the bigonal construction to the tower $D \xrightarrow{\pi} C \xrightarrow{\varphi} \mathbb{P}^1$, we get a corresponding tower $\Gamma \xrightarrow{\tilde{\pi}} \Gamma_0 \xrightarrow{\tilde{\varphi}} \mathbb{P}^1$, where $\Gamma \xrightarrow{\tilde{\pi}} \Gamma_0$ is a degree two cover with 6 branch points. We need to check whether $P(\Gamma, \Gamma_0)$ is in $\mathcal{P}$. Let us count the genera of the curves Γ and Γ_0: the ramification divisor of the degree 4 covering $\Gamma \longrightarrow \mathbb{P}^1$ is

$$R = w_1 + \ldots + w_6 + b_1 + \ldots + b_6 + b_1' + \ldots + b_6',$$

where w_i is in the fiber over $k_{w_i} \in \mathbb{P}^1$, which is the image of a Weierstrass point by φ, whereas b_i's and b_i''s are the elements of the fiber over $k_{b_i} \in \mathbb{P}^1$ which is the image by φ of a branch point in C. Hence $\deg(R) = 18$. By Riemann-Hurwitz formula the genus of Γ is 6. The ramification divisor of the degree 2 covering $\Gamma \xrightarrow{\gamma} \Gamma_0$ is

$$R' = w_1 + \ldots + w_6 :$$

the points w_i are as described above and are those fixed by the involution $\tilde{\iota}$, so $\deg(R') = 6$. Using Riemann-Hurwitz formula again we get that the genus of Γ_0 is 2. Thus the cover $\Gamma \xrightarrow{\tilde{\pi}} \Gamma_0$ lies in $\mathcal{R}_{2,6}$, meaning that $P(\Gamma, \Gamma_0)$ is indeed in the divisor $\mathcal{P}$. Using Theorem 4.1.5, we obtain that $P(D, C)$ and $P(\Gamma, \Gamma_0)$ are dual, which concludes the proof.

□

Part (2) of Theorem 4.3.1 requires more work.

From now on, let $(X, L_X) = (JC, \Theta_C)$ for some curve C, and $(A, L_A) = (\frac{JC}{\langle \alpha_1, \alpha_2 \rangle}, L_A)$ where α_1 and α_2 are 2-torsion elements in JC such that $e^{L_A}(\alpha_1, \alpha_2) = 1$, where e^{L_A} is the Riemann bilinear form associated to $K(L_A)$. Recall that elements in $\mathcal{J}$ are polarized abelian varieties (A, L_A) with an isogeny of degree 4 from a Jacobian $f : JC \longrightarrow A$, such that $f^*(L_A) = \Theta_C^2$. The dual divisor $\mathcal{J}' = \rho(\mathcal{J})$ is a variety whose elements are polarized abelian varieties $(A', L_{A'})$ with an isogeny of degree 4 to a Jacobian $f' : A' \longrightarrow JC'$, such that $f'^*(\Theta_{C'}) = L_{A'}$.

The following lemma is useful to find a more explicit description of $\mathcal{J}'$:

Lemma 4.3.2. *In the previous setting, the kernel $K(L_A)$ is*

$$K(L_A) = \frac{\langle \alpha_1, \alpha_2 \rangle^{\perp}}{\langle \alpha_1, \alpha_2 \rangle} \subset \frac{JC}{\langle \alpha_1, \alpha_2 \rangle} = A.$$

Proof. Since both groups have the same cardinality (16 elements), it is enough to prove one inclusion. Let's see that $K(L_A)$ is contained in

$$\frac{\langle \alpha_1, \alpha_2 \rangle^{\perp}}{\langle \alpha_1, \alpha_2 \rangle}.$$

Let $\tilde{a} \in K(L_A)$, then $\tilde{a} = f(a)$ for some $a \in JC$. Therefore:

$$1 = e^{L_A}(f(a), 0) = e^{L_A}(f(a), f(\alpha_i)) = e^{2\Theta}(a, \alpha_i) = e_2(a, \alpha_i).$$

Hence $\tilde{a} \in \langle \alpha_1, \alpha_2 \rangle^{\perp}$. □

Define the following moduli space:

$$\tilde{\mathcal{A}}_4' = \{(X, L_X, H) | (X, L) \text{ is a principally polarized 4-dimensional abelian variety, } H \subset X_2 \text{ and } H^{\perp} \text{ is an isotropic subgroup of four elements}\}$$

We define the new divisor $\mathcal{J}'$ using a construction analogous to the quotient one: let $(X, L_X, H) \in \tilde{\mathcal{A}}_4'$, and put $A' = X/H$. This gives a degree 4 isogeny $f' : A' \longrightarrow X/X_2 \cong X$. A' is polarized by $L_{A'} = f'^*(L_X)$, which is of the desired type $(1, 1, 2, 2)$. The moduli space $\tilde{\mathcal{A}}_4'$ also gives a finite cover ϖ' for $\mathcal{A}_4^{(1,1,2,2)}$. As before, the image of the Jacobian locus by ϖ' defines a divisor which is in fact $\mathcal{J}'$.

Set $A = JC/\langle \alpha_1, \alpha_2 \rangle$. Then $A^{\vee} = \rho(A)$. By definition $\lambda_{L_A} : A \to A^{\vee}$: using Lemma 4.3.2 and the third isomorphism theorem one can write $A^{\vee}$ explicitly as a quotient of JC:

$$A^{\vee} \cong \frac{A}{\ker(\lambda_{L_A})} \cong \frac{JC/\langle \alpha_1, \alpha_2 \rangle}{\langle \alpha_1, \alpha_2 \rangle^{\perp}/\langle \alpha_1, \alpha_2 \rangle} \cong \frac{JC}{\langle \alpha_1, \alpha_2 \rangle^{\perp}}.$$

Moreover, $(A' = \frac{JC}{\langle \alpha_1, \alpha_2 \rangle^{\perp}}, L_{A'})$ is the image of $(JC, \Theta_C, \langle \alpha_1, \alpha_2 \rangle^{\perp})$ by ϖ'. This duality argument gives a correspondence between the two divisors $\mathcal{J}$. It is explained in the following diagram:

$$\begin{tikzcd}
\tilde{\mathcal{A}}_4 \arrow[r, "\perp"] \arrow[d, "\varpi"'] & \tilde{\mathcal{A}'}_4 \arrow[d, "\varpi'"] \\
\mathcal{A}_4^{(1,1,2,2)} \arrow[r, "\rho"] & \mathcal{A}_4^{(1,1,2,2)}
\end{tikzcd}$$

The map $\perp$ takes the triple (X, L_X, H) to $(X^\vee, L_X^\vee, H^\perp)$; the maps ϖ, ϖ' are the two finite covers of $\mathcal{A}_4^{(1,1,2,2)}$ defined above, and the map ρ is the Birkenhake and Lange's involution. The diagram commutes via the following lemma:

Lemma 4.3.3. *The pullback by $f^\vee : A^\vee \longrightarrow JC^\vee$ of $\Theta_C^\vee$ is algebraically equivalent to $L_{A^\vee}$.*

Proof. The statement is equivalent to $f \circ \lambda_\Theta^{-1} \circ f^\vee = \lambda_{L_A^\vee}$, then it is enough to prove that the following diagram commutes:

$$\begin{tikzcd}
JC^\vee \arrow[d, "\lambda_\Theta^{-1}"'] & A^\vee \arrow[l, "f^\vee"'] \arrow[d, "\lambda_{L_A^\vee}"] \\
JC \arrow[r, "f"] & A
\end{tikzcd}$$

Consider the diagram:

$$\begin{tikzcd}
JC \arrow[r, "f"] \arrow[d, "\lambda_{2\Theta}"'] & A \arrow[d, "\lambda_{L_A}"] \\
JC^\vee \arrow[d, "\lambda_\Theta^{-1}"'] & A^\vee \arrow[l, "f^\vee"] \arrow[d, "\lambda_{L_A^\vee}"] \\
JC & A
\end{tikzcd}$$

Observe that $\lambda_\Theta^{-1} \circ \lambda_{2\Theta} = 2_{JC}$, and also $\lambda_{L_A} \circ \lambda_{L_A^\vee} = 2_A$. Taking an element $x \in JC$, we have:

$$\lambda_\Theta^{-1} \circ \lambda_{2\Theta}(x) = 2x, \qquad \lambda_{L_A} \circ \lambda_{L_A^\vee} \circ f = 2f(x).$$

By linearity, one can complete the diagram with $f : JC \to A$.

$\square$

Elements of $\mathcal{J}'$ are pairs $(A' = JC/\langle\alpha_1,\alpha_2\rangle^\perp, L_{A'})$ together with a degree 4 isogeny $f' : A' \longrightarrow JC$ such that $L_{A'} = f'^*(\Theta_C)$. The commutativity of the above diagram implies that $\rho(\mathcal{J}) = \mathcal{J}'$: indeed, given $(A = JC/\langle\alpha_1,\alpha_2\rangle, L_A) \in \mathcal{J}$, we have $\rho(A) = A^\vee = JC/\langle\alpha_1,\alpha_2\rangle^\perp = A'$. To see that $\rho(L_A) = L_{A'}$, observe that $f' = f^\vee$ and use Lemma 4.3.3.

It is useful to note that the following result holds:

Lemma 4.3.4. *The pullback by λ_{L_A} of $L_{A^\vee}$ is algebraically equivalent to L_A^2.*

Proof. The proof is analogous to the previous one: the statement is equivalent to

$$(\lambda_{L_A})^\vee \circ \lambda_{L_A^\vee} \circ \lambda_{L_A} = 2\lambda_{L_A}.$$

But since $(\lambda_{L_A})^\vee : A \to A^\vee$ is the same as $\lambda_{L_A} : A \to A^\vee$, and $\lambda_{L_A} \circ \lambda_{L_A^\vee} = 2_A$ the equality is straightforward:

$$(\lambda_{L_A})^\vee \circ \lambda_{L_A^\vee} \circ \lambda_{L_A} = \lambda_{L_A} \circ \lambda_{L_A^\vee} \circ \lambda_{L_A} = 2\lambda_{L_A}.$$

□

Proof of Theorem 4.3.1 point (2). Suppose $\mathcal{J} = \mathcal{J}'$. Since elements in $\mathcal{J}$ are of type $\frac{JC}{\langle\alpha_1,\alpha_2\rangle}$ for some curve C and some 2-torsion elements α_1 and α_2, and elements in J' are of type $JD/\langle\beta_1,\beta_2\rangle^\perp$ for some curve D and some 2-torsion elements β_1 and β_2, the equality implies that for every pair $(JC, \langle\alpha_1,\alpha_2\rangle)$ in $\mathcal{J}$ one can find another pair $(JD, \langle\beta_1,\beta_2\rangle)$ such that

$$\frac{JC}{\langle\alpha_1,\alpha_2\rangle} = \frac{JD}{\langle\beta_1,\beta_2\rangle^\perp}.$$

Consider the diagram

$$\begin{tikzcd}
JC \arrow[r, "f_C"] \arrow[d, "2_{JC}"] & \frac{JC}{\langle\alpha_1,\alpha_2\rangle} = \frac{JD}{\langle\beta_1,\beta_2\rangle^\perp} \arrow[d, "\lambda_{L_A}"] & JD \arrow[l, "f_D"'] \arrow[d, "2_{JD}"] \\
JC^\vee & \frac{JC^\vee}{\langle\alpha_1,\alpha_2\rangle^\perp} = \frac{JD^\vee}{\langle\beta_1,\beta_2\rangle} \arrow[l, "f_C^\vee"'] \arrow[r, "f_D^\vee"'] & JD^\vee
\end{tikzcd}$$

Composing $f_C^\vee \circ \lambda_{L_A} \circ f_D$, we obtain an isogeny from JD to $JC^\vee \cong JC$, where the isomorphism is given using the principal polarization of JC.

Computing the degree of this map we find that it is 2^{12}, that is the product of the degree of the three factorizing maps ($\deg f_D = 2^6$, $\deg \lambda_{L_A} = 2^4$, $\deg f_C^\vee = 2^2$). We use the following result:

Theorem 4.3.5 (Bardelli, Pirola)**.** *If χ is an isogeny between two Jacobians of dimension $g \geq 4$, and $\mathcal{J}$ is generic, than $\mathcal{J} \cong \mathcal{J}'$ and χ is the multiplication by an integer.*

Applying Theorem 4.3.5, we find that the Jacobians JC and JD are isomorphic as principally polarized abelian varieties, so, using Torelli theorem, we get that the curves C and D have to be isomorphic as well. Moreover, the isogeny $\chi = f_C^\vee \circ \lambda_{L_A} \circ f_D$ has to be the multiplication by an integer map. But this cannot be, since the multiplication by m has degree $m^{2\times 4} = m^8$, thus can never be equal to 2^{12}. Hence $\mathcal{J} \neq \mathcal{J}'$, which completes the proof.

$\square$

Appendices

Appendix A

MAPLE script

This appendix describes the `MAPLE` code used to generate the computational results from Chapter 3 of the dissertation (see [2]).

Before going deeper in technicalities, we explain here the simple strategy behind the code. The purpose is to provide a lower bound for the rank of the first and second Gauss-Wahl map when evaluated over a (generic) curve which is a cyclic Galois cover of $\mathbb{P}^1$. Following the notation introduced in Section 2.1, we fix the `genus` of the curve, the order `m` of the Galois group, and the monodromy datum `a=<a_1,..,a_N>`. Moreover we fix the branch points `t=<t_1,...,t_N>` as well. Since we want to use expression (2.1.5), we consider only monodromies such that $a_1 = 1$, and we always choose $t_1 = 0$.

For convenience of the reader, we recall that the fiber over a fixed point $t_i \in t$ is the normalization of the affine curve:

$$y^m = g(x) := \prod_{i=1}^{N} (x - t_i)^{a_i}. \tag{A.0.1}$$

Call φ the local inverse of g around 0, that is:

$$\varphi(y) = g^{-1}(y^m) = x. \tag{A.0.2}$$

We write forms $\omega_{n,\nu} \in H^0(K)$ concretely using expression:

$$\omega_{n,\nu} = \frac{m}{g'(\varphi(y))}\, y^{n-1} \varphi(y)^{\nu} \prod_{i=1}^{N} (\varphi(y) - t_i)^{l(i,n)+a_i}\, dy. \tag{A.0.3}$$

We point out that `MAPLE` can find the local inverse of g around 0, which we called `phi`, as well as the derivative of g with respect to x, obtained via `diff(g(x),x))`. Every form in (A.0.3) is explicitly computable locally around 0. It is easy to compute the first and the second Gauss-Wahl map just using their definitions. We do it, and we consider their Taylor expansion truncated at some fixed precision. We put all coefficients of the Taylor expansions in two matrices: `MatM1` is the matrix relative to the first Gauss-Wahl map and `MatM2` is the matrix relative to the second one. Finally, computing the rank of these matrices, we obtain an approximate value for the rank of μ_1 and μ_2 respectively.

The precision in the approximations depends on the parameter `prec`, which we set at the beginning. It determines at which order all Taylor series stop. The results we obtain are lower bounds for the rank of the first and second Gauss-Wahl map: it is possible, in fact, that a pair of vectors which are dependent when truncated at the n-th entry becomes independent if truncated at level $n+1$.

In the following, we include and comment the `MAPLE` source in case of a curve of genus 5 which covers $\mathbb{P}^1$ with Galois group $G = \mathbb{Z}/4\mathbb{Z}$ and monodromy data $a = [1 : 1 : 3 : 3 : 2 : 2]$ over the branch points $t = [0\ ,\ 1\ ,\ -1\ ,\ 2\ ,\ -2\ ,\ 3]$. Remark that we have already seen this cover in Section 3.1 as an example of bielliptic curve with maximal second Gauss-Wahl map (see Table 3.1).

```
restart;

Typesetting:-Settings(functionassign = false);
with(PolynomialTools); with(LinearAlgebra);
```

We use the `PolynomialTools` package, which provides a collection of commands useful to work with polynomial. In particular we use in the code the `CoefficientVector(p, x)` calling sequence, which returns a vector of coefficients from a polynomial `p` in the variable `x`. We use the `LinearAlgebra` package as well, which offers routines to construct and manipulate matrices and vectors and solve linear algebra problems. Commands `Dimension(M)`, which computes the dimension of the matrix M, and command `NullSpace(A)`, which compute a basis for the kernel of a matrix, are inside this package.

```
genus := 5; m := 4;
a := <1, 1, 3, 3, 2, 2>; t := <0, 1, -1, 2, -2, 3>;

r := Dimension(a);
l := Matrix(r, m-1);
d := Vector(m-1);
forma := Vector(genus);
prec := 150;

#Dimension of \Lambda^2 H^0(K)
L := (1/2)*genus*(genus-1);

#Dimension of I_2(K)
N := (1/2)*genus*(genus+1)-3*(genus-1);

k := 1;
Max1 := 0; Max2 := 0; Max3 := 0;


g := x -> mul((x-t(i))^a(i), i = 1 .. r);
phi := solve(g(x) = y^m, x)[1];

phiTay := y -> taylor(phi, y = 0, prec);

eq0 := x = convert(phiTay(y), polynom);
eq1 := y^m = g(x);
```

In the previous lines we have initialized all variables. We fixed the `prec` parameter to 150. We called `g` the function defined in (A.0.1), and `phi` is local inverse around 0. `phiTay` is the polynomial version of `phi` truncated at order `prec`. Equations `eq1` and `eq0` are respectively equation (A.0.1) and equation (A.0.2) in polynomial form. We used the command `convert(phiTay(y),polynom)` to get rid of the infinitesimal term $o(y^{\text{prec}})$ in the Taylor series.

```
for n from 1 to m-1 do
   for i from 1 to r do
      l[i, n] := floor(-n*a(i)/m);
      d[n] := -1+add(-n*a(j)/m-floor(-n*a(j)/m), j = 1 .. r);
```

```
   end do;
end do;

for n from 1 to m-1 do
   for v from 0 to d[n]-1 do
      wpar[n, v] := m*y^(n-1)*x^v*mul((x-t(i))^(l[i, n]+a(i)),
         i = 1 .. r);
      fnum[n, v] := convert(taylor(algsubs(eq0, wpar[n, v]),
         y = 0, prec), polynom);
      fden[n, v] := convert(taylor(algsubs(eq0, diff(g(x), x)),
         y = 0, prec), polynom);
      forma[k] := convert(taylor(fnum[n, v]/fden[n, v], y = 0,
         prec), polynom);
      k := k+1;
   end do;
end do;
```

Here we computed the combinatorial data `l[i,n]` and `d[n]` to construct all forms in $H^0(K)$ using expression (A.0.3). Then we converted them in polynomials, using the Taylor expansion of numerator and denominator separately, and considering the Taylor expansion of the quotient truncated at level `prec`. Finally, we put all forms in vector `forma`. Using the same notation of Section 2.1, we point out that we ordered forms as follows:

$$\{\texttt{forma[1]},\ldots,\texttt{forma[genus]}\}$$

$$\|$$

$$\{\omega_{1,0},\ldots,\omega_{1,d_1-1},\omega_{2,0},\ldots,\omega_{2,d_2-1},\omega_{3,0},\ldots,\omega_{3,d_3-1}\}.$$

```
#First Gauss-Wahl map

for i from 1 to genus do
   for j from i+1 to genus do
      M1[i, j] := convert(taylor((diff(forma[j], y))*forma[i]-
         (diff(forma[i], y))*forma[j], y = 0, prec-1), polynom);
```

```
      C1[i, j] := CoefficientVector(M1[i, j], y);
      Max1 := max(Max1, Dimension(C1[i, j]));
   end do;
end do;
```

In the previous lines, we evaluated the first Gauss-Wahl map on forms `forma(i)` $\wedge$ `forma(j)`, and we truncated the obtained polynomials at order `prec-1`, that is the greater significant one. We put these polynomials in the entry (i, j) of the matrix `M1`. Finally we put the coefficients of these polynomial in the multimatrix `C1`. Observe that `C1[i,j]` is the coefficients vector of μ_1(`forma(i)` $\wedge$ `forma(j)`).

```
#Multiplication map and multiplication map of derivatives

k := 1;
for i from 1 to genus do
   for j from 1 to genus do
      M[i, j] := convert(taylor(forma[i]*forma[j], y = 0, prec),
         polynom);
      CM[i, j] := CoefficientVector(M[i, j], y);
      Max2 := max(Max2, Dimension(CM[i, j]));
      MD[i, j] := convert(taylor((diff(forma[i], y))*(diff(forma[j],
         y)), y = 0, prec-1), polynom);
      if i <= j then
         M2[k] := MD[i, j];
         k := k+1;
      end if;
   end do;
end do;
```

Here we followed the same procedure used in the case of the first Gauss-Wahl map. We constructed the matrix `M`, having in each entry the Taylor serie for `forma[i]*forma[j]`, and then we isolated the coefficients in the multimatrix `CM`. Finally we constructed the matrix `MD`, such that the entry `MD[i,j]` contains the Taylor serie for `forma'[i]*forma'[j]`. In the last *if* cycle, we ordered vectors contained in the `MD` in the simpler matrix `M2`.

```
#Set the right length of coefficient vectors
```

```
for i from 1 to genus do
   for j from i+1 to genus do
      for q from Dimension(C1[i, j])+1 to Max1 do
         C1[i, j](q) := 0;
      end do;
   end do;
   for j from 1 to genus do
      for q from Dimension(CM[i, j])+1 to Max2 do
         CM[i, j](q) := 0;
      end do;
   end do;
end do;
```

This is a technical *for* cycle, useful to guarantee that all columns in `C1` and `CM` have the same length. To be more precise, until now `C1` and `CM` were not matrices, but vectors whose entry were other vectors of a-priori different length. Here we homogenize all lengths adding zeros when necessary.

```
#Matrix of coefficients of the first Gauss-Wahl map

MatM1 := Vector(prec-1);
for i from 1 to genus-1 do
   for j from i+1 to genus do
      MatM1 := <MatM1, C1[i, j]>;
   end do;
end do;
MatM1 := DeleteColumn(MatM1, 1);

Rank(MatM1);

                                10
```

As before, here we ordered the vectors contained in the multimatrix `C1` in the simpler matrix `MatM1`. Finally, we computed `Rank(MatM1)`, which is a lower bound for the rank of the second Gauss-Wahl map, for the previous discussion.

```
#Matrix of coefficients of the multiplication map
```

```
MatM := Vector(prec);
for i from 1 to genus do
   for j from i to genus do
     MatM := <MatM, CM[i, j]>;
   end do;
end do;
MatM := DeleteColumn(MatM, 1);

Rank(MatM);
```

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Here we use for the multiplication map the same strategy as before: we ordered the vectors contained in `CM` in matrix `MatM` (we will use this matrix later to describe quadrics in the $I_2(K)$). Finally, we computed `Rank(MatM)`, which is a lower bound for the rank of the multiplication map, allowing us to check whether the curve is hyperelliptic or not.

```
#Computation of the second Gauss-Wahl map:

K := NullSpace(MatM);
Max3 := 0;
for k from 1 to N do
   Omega[k] := add(K[k][i]*M2[i], i = 1 .. L+genus);
   F[k] := CoefficientVector(Omega[k], y);
   Max3 := max(Max3, Dimension(F[k]));
end do;

for k from 1 to N do
   for q from Dimension(F[k])+1 to Max3 do
      F[k](q) := 0;
   end do;
end do;
```

Here we computed the second Gauss-Wahl map starting from the matrix `MatM`. The output of command `NullSpace(MatM)` is a matrix whose columns are vectors in the kernel of `MatM`. We call it `K`. In the first *for* cycle we

computed the second Gauss-Wahl map and isolate the coefficients. In the last *for* cycle we adjusted the dimensions adding zeros to make `F` a matrix, as before.

```
#Matrix of coefficients of the second Gauss-Wahl map

Mat := Vector(prec-1);
for i from 1 to N do
   Mat := <Mat, F[i]>;
end do;
Mat := DeleteColumn(Mat, 1);

Rank(Mat);
                                3
```

In the last few lines we computed the approximate rank of the second Gauss-Wahl map. As in the case of the first Gauss-Wahl map, `Rank(Mat)` is a lower bound for rank μ_2. Nevertheless, in this case the approximate values coincide with the maximal one: we can conclude that the map is injective.

Appendix B

Tables for bielliptic curves

This appendix is intended as a support for Section 2.4. Here we attach the complete list of all bielliptic curves of genus $5 \leq g \leq 30$ covering $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/8\mathbb{Z}$ and $\mathbb{Z}/12\mathbb{Z}$, that is:

$$\begin{tikzcd} C \arrow[rr, "2:1"] \arrow[dr, "4:1"'] & & E \arrow[dl, "2:1"] \\ & \mathbb{P}^1 & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "2:1"] \arrow[dr, "6:1"'] & & E \arrow[dl, "3:1"] \\ & \mathbb{P}^1 & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "2:1"] \arrow[dr, "8:1"'] & & E \arrow[dl, "4:1"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

For all of them we report the lower bound for the rank of the second Gauss-Wahl map provided by `MAPLE`. We point out that, for $8 \leq g \leq 30$, we obtain the same lower bound for all possible monodromies of $C \to \mathbb{P}^1 \cong C/G$ such that C is bielliptic, and for all possible choices for the Galois group G.

The `MAPLE` code gives rank $\mu_2 \approx 2g-1$ for any bielliptic curve of this type ($g \geq 8$, `prec`=500). We point out that this bound is quiet similar to the bound analytically found in Subsection 2.4.2, which we report.

Theorem B.0.1 (See Corollary 2.4.13)**.** *Let C be a bielliptic curve covering $\mathbb{P}^1$ with Galois group $\mathbb{Z}/4\mathbb{Z}$. As shown in Theorem 2.4.8, possible monodromies for $C \to \mathbb{P}^1$ are, depending on the genus:*

1. *In case g is odd:*
   (a) $a = [1^4 : 2^{g-3}]$ *or*
   (b) $a = [1^2 : 3^2 : 2^{g-3}]$.
2. *In case g is even:*
   (a) $a = [1^3 : 3 : 2^{g-3}]$.

Then the following bounds holds, depending on genus and monodromy:

(1) If g is odd:

$$\begin{aligned} (a): &\quad 2g-8 \leq \operatorname{rank}\mu_2 \leq 5g-5; \\ (b): &\quad 2g-10 \leq \operatorname{rank}\mu_2 \leq 5g-5. \end{aligned} \tag{B.0.1}$$

(2) If g is even then:

$$(a): \quad 2g-9 \leq \operatorname{rank}\mu_2 \leq 5g-5. \tag{B.0.2}$$

In the following, we list the complete table of bielliptic curves for $g \leq 30$.

Table B.1: Table obtained when $C \to \mathbb{P}^1$ has Galois group $G = \mathbb{Z}/4\mathbb{Z}$. We list all possible monodromies up to $g = 30$.

group	genus	monodromy	rank μ_2
$\mathbb{Z}/4\mathbb{Z}$	5	$[1^2 : 3^2 : 2^2]$	3
$\mathbb{Z}/4\mathbb{Z}$	5	$[1^4 : 2^2]$	3
$\mathbb{Z}/4\mathbb{Z}$	6	$[1^3 : 3 : 2^3]$	6
$\mathbb{Z}/4\mathbb{Z}$	6	$[1 : 3^3 : 2^3]$	6
$\mathbb{Z}/4\mathbb{Z}$	7	$[1^4 : 2^4]$	10
$\mathbb{Z}/4\mathbb{Z}$	7	$[1^2 : 3^2 : 2^4]$	10
$\mathbb{Z}/4\mathbb{Z}$	8	$[1^3 : 3 : 2^5]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	8	$[1 : 3^3 : 2^5]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	9	$[1^4 : 2^6]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	9	$[1^2 : 3^2 : 2^6]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	10	$[1^3 : 3 : 2^7]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	10	$[1 : 3^3 : 2^7]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	11	$[1^4 : 2^8]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	11	$[1^2 : 3^2 : 2^8]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	12	$[1^3 : 3 : 2^9]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	12	$[1 : 3^3 : 2^9]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	13	$[1^4 : 2^{10}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	13	$[1^2 : 3^2 : 2^{10}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	14	$[1^3 : 3 : 2^{11}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	14	$[1 : 3^3 : 2^{11}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	15	$[1^4 : 2^{12}]$	$2g-1$

$\mathbb{Z}/4\mathbb{Z}$	15	$[1^2 : 3^2 : 2^{12}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	16	$[1^3 : 3 : 2^{13}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	16	$[1 : 3^3 : 2^{13}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	17	$[1^4 : 2^{14}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	17	$[1^2 : 3^2 : 2^{14}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	18	$[1^3 : 3 : 2^{15}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	18	$[1 : 3^3 : 2^{15}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	19	$[1^4 : 2^{16}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	19	$1^2 : 3^2 : 2^{16}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	20	$[1^3 : 3 : 2^{17}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	20	$[1 : 3^3 : 2^{17}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	21	$[1^4 : 2^{18}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	21	$[1^2 : 3^2 : 2^{18}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	22	$[1^3 : 3 : 2^{19}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	22	$[1 : 3^3 : 2^{19}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	23	$[1^4 : 2^{20}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	23	$[1^2 : 3^2 : 2^{20}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	24	$[1^3 : 3 : 2^{21}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	24	$[1 : 3^3 : 2^{21}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	25	$[1^4 : 2^{22}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	25	$[1^2 : 3^2 : 2^{22}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	26	$[1^3 : 3 : 2^{23}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	26	$[1 : 3^3 : 2^{23}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	27	$[1^4 : 2^{24}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	27	$[1^2 : 3^2 : 2^{24}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	28	$[1^3 : 3 : 2^{25}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	28	$[1 : 3^3 : 2^{25}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	29	$[1^4 : 2^{26}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	29	$[1^2 : 3^2 : 2^{26}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	30	$[1^3 : 3 : 2^{27}]$	$2g-1$
$\mathbb{Z}/4\mathbb{Z}$	30	$[1 : 3^3 : 2^{27}]$	$2g-1$

Table B.2: Table obtained when $C \to \mathbb{P}^1$ has Galois group $G = \mathbb{Z}/8\mathbb{Z}$. We list all possible monodromies up to $g = 30$.

group	genus	monodromy	rank μ_2
$\mathbb{Z}/8\mathbb{Z}$	5	$[1^2 : 2 : 4]$	3
$\mathbb{Z}/8\mathbb{Z}$	5	$[1 : 5 : 6 : 4]$	3
$\mathbb{Z}/8\mathbb{Z}$	7	$[1^2 : 6 : 4^2]$	10
$\mathbb{Z}/8\mathbb{Z}$	7	$[1 : 5 : 2 : 4^2]$	10
$\mathbb{Z}/8\mathbb{Z}$	9	$[1^2 : 2 : 4^3]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	9	$[1 : 5 : 6 : 4^3]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	11	$[1^2 : 6 : 4^4]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	11	$[1 : 5 : 2 : 4^4]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	13	$[1^2 : 2 : 4^5]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	13	$[1 : 5 : 6 : 4^5]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	15	$[1^2 : 6 : 4^6]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	15	$[1 : 5 : 2 : 4^6]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	17	$[1^2 : 2 : 4^7]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	17	$[1 : 5 : 6 : 4^7]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	19	$[1^2 : 6 : 4^8]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	19	$[1 : 5 : 2 : 4^8]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	21	$[1^2 : 2 : 4^9]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	21	$[1 : 5 : 6 : 4^9]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	23	$[1^2 : 6 : 4^{10}]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	23	$[1 : 5 : 2 : 4^{10}]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	25	$[1^2 : 2 : 4^{11}]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	25	$[1 : 5 : 6 : 4^{11}]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	27	$[1^2 : 6 : 4^{12}]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	27	$[1 : 5 : 2 : 4^{12}]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	29	$[1^2 : 2 : 4^{13}]$	$2g-1$
$\mathbb{Z}/8\mathbb{Z}$	29	$[1 : 5 : 6 : 4^{13}]$	$2g-1$

Table B.3: Table obtained when $C \to \mathbb{P}^1$ has Galois group $G = \mathbb{Z}/12\mathbb{Z}$. We list all possible monodromies up to $g = 30$.

group	genus	monodromy	rank μ_2
$\mathbb{Z}/12\mathbb{Z}$	6	$[1, 9, 8, 6]$	6
$\mathbb{Z}/12\mathbb{Z}$	7	$[1, 2, 3, 6]$	10
$\mathbb{Z}/12\mathbb{Z}$	9	$[1, 8, 3, 6^2]$	$2g - 1$
$\mathbb{Z}/12\mathbb{Z}$	10	$[1, 2, 9, 6^2]$	$2g - 1$
$\mathbb{Z}/12\mathbb{Z}$	12	$[1, 9, 8, 6^3]$	$2g - 1$
$\mathbb{Z}/12\mathbb{Z}$	13	$[1, 2, 3, 6^3]$	$2g - 1$
$\mathbb{Z}/12\mathbb{Z}$	15	$[1, 8, 3, 6^4]$	$2g - 1$
$\mathbb{Z}/12\mathbb{Z}$	16	$[1, 2, 9, 6^4]$	$2g - 1$
$\mathbb{Z}/12\mathbb{Z}$	18	$[1, 9, 8, 6^5]$	$2g - 1$
$\mathbb{Z}/12\mathbb{Z}$	19	$[1, 2, 3, 6^5]$	$2g - 1$
$\mathbb{Z}/12\mathbb{Z}$	21	$[1, 8, 36^6]$	$2g - 1$
$\mathbb{Z}/12\mathbb{Z}$	22	$[1, 2, 9, 6^6]$	$2g - 1$
$\mathbb{Z}/12\mathbb{Z}$	24	$[1, 9, 8, 6^7]$	$2g - 1$
$\mathbb{Z}/12\mathbb{Z}$	25	$[1, 2, 3, 6^7]$	$2g - 1$
$\mathbb{Z}/12\mathbb{Z}$	27	$[1, 8, 3, 6^8]$	$2g - 1$
$\mathbb{Z}/12\mathbb{Z}$	28	$[1, 2, 9, 6^8]$	$2g - 1$
$\mathbb{Z}/12\mathbb{Z}$	30	$[1, 9, 8, 6^9]$	$2g - 1$

We conclude stressing that the results provide evidence for the following:

Expectation. *The rank of the second Gauss-Wahl map for a bielliptic curve of genus $g \geq 8$ is $2g - 1$.*

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