UNIVERSITÀ DEGLI STUDI DI PAVIA

TESI DI DOTTORATO

Phase field systems with maximal monotone nonlinearities related to sliding mode control problems

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Al mio papà, alla mia mamma e a don Primo.

Ai miei cari nonni: Nina e Mario, Maddalena e Mario.

Contents

Chapter 1

Introduction

The present thesis focuses on the mathematical analysis of a class of phase field systems involving partial differential equations and arising from thermodynamic models.

In the first chapter, we deal with some phase–field systems that perturbed by a maximal monotone nonlinearity, proving existence, uniqueness and longtime behavior of the strong solution. The second chapter is concerned with the study of Cahn–Hilliard systems characterized by the presence of a maximal monotone term: we prove existence, uniqueness and regularity of the solution; moreover, we consider the related sliding mode control problem and we can discuss the sliding mode property. In the third chapter, we analyze a singular phase–field system containing a logarithmic nonlinearity and by a possibly nonlocal maximal monotone operator: the resulting problem is highly nonlinear and difficult to handle, so that we are able to prove only the existence of solutions.

1.1 Phase–field systems

In the first chapter, we consider the phase–field system

$$\partial_t(\vartheta + \ell\varphi) - k\Delta\vartheta + \zeta = f \quad \text{a.e. in } Q := (0,T)\times\Omega, \tag{1.1.1}$$

$$\partial_t\varphi - \upsilon\Delta\varphi + \xi + \pi(\varphi) = \gamma\vartheta \quad \text{a.e. in } Q, \tag{1.1.2}$$

$$\zeta(t) \in A(\vartheta(t) + \alpha\varphi(t) - \eta^*) \text{ for a.e. } t \in (0,T), \tag{1.1.3}$$

$$\xi \in \beta(\varphi) \text{ a.e. in } Q, \tag{1.1.4}$$

where Ω is the domain in which the evolution takes place, T is some final time, ϑ denotes the relative temperature around some critical value that is taken to be 0 without loss of generality, and φ is the order parameter which can represent the local proportion of one of the two phases. As usual, to ensure thermomechanical consistency, suitable physical constraints on φ are considered: if it is assumed, e.g., that the two phases may coexist at each point with different proportions, it turns out to be reasonable to require that φ

lies between 0 and 1, with $1-\varphi$ representing the proportion of the second phase. In particular, the values $\varphi = 0$ and $\varphi = 1$ may correspond to the pure phases, while φ is between 0 and 1 in the regions when both phases are present. Clearly, the the system provides an evolution for φ that has to comply with the previous physical constraint. Moreover, ℓ, k, υ, γ and α are positive constants, η^* is a given function in $H^2(\Omega)$ with suitable regularity properties and f is a source term. The above system is complemented by homogeneous Neumann boundary conditions for both ϑ and φ, that is,

$$\partial_\nu \vartheta = 0, \qquad \partial_\nu \varphi = 0 \quad \text{on } \Sigma := (0,T) \times \Gamma, \tag{1.1.5}$$

where Γ is the boundary of Ω and ∂_ν is the outward normal derivative, and by the initial conditions

$$\vartheta(0) = \vartheta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{1.1.6}$$

The term $\xi + \pi(\varphi)$, appearing in (1.1.2), represents the derivative (or the subdifferential) of a double–well potential $\mathcal{W}$ defined as the sum

$$\mathcal{W} = \tilde{\beta} + \tilde{\pi}, \tag{1.1.7}$$

where

$$\tilde{\beta} : \mathbb{R} \longrightarrow [0, +\infty] \text{ is proper, l.s.c. and convex with } \tilde{\beta}(0) = 0, \tag{1.1.8}$$

$$\tilde{\pi} : \mathbb{R} \to \mathbb{R},\ \tilde{\pi} \in C^1(\mathbb{R}) \text{ with } \pi := \tilde{\pi}' \text{ Lipschitz continuous.} \tag{1.1.9}$$

Since $\tilde{\beta}$ is proper, l.s.c. and convex, the subdifferential $\partial\tilde{\beta} =: \beta$ is well defined and is a maximal monotone graph. In our problem we also consider a maximal monotone operator

$$A : \ H := L^2(\Omega) \longrightarrow 2^H \tag{1.1.10}$$

such that

$$0 \in A(0), \qquad \|y\|_H \le C(1 + \|x\|_H) \qquad \text{for all } x \in H,\ y \in Ax, \tag{1.1.11}$$

for some constant $C > 0$. For a comprehensive presentation of the theory of subdifferentials and maximal monotone operators, we refer, e.g., to [1, 11, 61].

The problem (1.1.1)–(1.1.6), thoroughly discussed in [32], is an interesting development of the following simple version of the phase–field system of Caginalp type (see [13]):

$$\partial_t(\vartheta + \ell\varphi) - k\Delta\vartheta = f \quad \text{in } Q, \tag{1.1.12}$$

$$\partial_t \varphi - \upsilon\Delta\varphi + \mathcal{W}'(\varphi) = \gamma\vartheta \quad \text{in } Q. \tag{1.1.13}$$

As already noticed, $\mathcal{W}' \cong \xi + \pi$ is related to a double–well potential $\mathcal{W}$. Typical examples for $\mathcal{W}$ are

$$\mathcal{W}_{reg}(r) = \frac{1}{4}(r^2 - 1)^2, \qquad r \in \mathbb{R}, \tag{1.1.14}$$

$$\mathcal{W}_{log}(r) = ((1 + r)\ln(1 + r) + (1 - r)\ln(1 - r)) - c_0 r^2, \quad r \in (-1, 1), \tag{1.1.15}$$

where $c_0 > 1$ in (1.1.15) in order to produce a double–well. The potentials (1.1.14) and (1.1.15) are the usual classical regular potential and the so–called logarithmic potential, respectively.

The well–posedness, the long-time behavior of solutions, and also the related optimal control problems concerning Caginalp-type systems have been widely studied in the literature. We refer, without any sake of completeness, e.g., to [12, 13, 29, 38, 43, 53, 54, 56, 63] and references therein for the well–posedness and long time behavior results and to [22, 23, 30, 48, 49] for the treatment of optimal control problems.

The paper [2] is related to control problems, but it goes in the direction of designing sliding mode controls (SMC) for a particular phase–field system. The main objective of the authors is to find some state-feedback control laws $(\vartheta, \varphi) \mapsto u(\vartheta, \varphi)$ that, that, once inserted into the equations, can force the solution to reach some submanifold of the phase space, in finite time, then slide along it. The first analytical difficulty consists in deriving the equations governing the sliding modes and the conditions for this motion to exist. The problem needs the development of special methods, since the conventional theorems regarding existence and uniqueness of solutions are not directly applicable. Moreover, the authors need to manipulate the system through the control in order to constrain the evolution on the desired sliding manifold.

In particular, in the paper [2] the authors consider the operator $\text{Sign} : H \longrightarrow 2^H$ defined as $\text{Sign}(v) = \frac{v}{\|v\|}$, if $v \neq 0$, and $\text{Sign}(0) = B_1(0)$, if $v = 0$, where $B_1(0)$ is the closed unit ball of H. Sign is a maximal monotone operator on H and is a nonlocal counterpart of the operator $\text{sign} : \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ defined as $\text{sign}(r) = \frac{r}{|r|}$, if $r \neq 0$, and $\text{sign}(0) = [-1, 1]$, if $r = 0$. Let us point out the system dealt with in [2]:

$$\partial_t(\vartheta + \ell\varphi) - k\Delta\vartheta = f - \rho\sigma \quad \text{a.e. in } Q, \tag{1.1.16}$$

$$\partial_t\varphi - \upsilon\Delta\varphi + \xi + \pi(\varphi) = \gamma\vartheta \quad \text{a.e. in } Q, \tag{1.1.17}$$

$$\sigma(t) \in \text{Sign}(\vartheta(t) + \alpha\varphi(t) - \eta^*) \quad \text{for a.e. } t \in (0, T), \tag{1.1.18}$$

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q, \tag{1.1.19}$$

$$\partial_\nu\vartheta = 0, \qquad \partial_\nu\varphi = 0 \quad \text{on } \Sigma, \tag{1.1.20}$$

$$\vartheta(0) = \vartheta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega, \tag{1.1.21}$$

which turns out to be a particular case of (1.1.1)–(1.1.6) with $A = \rho\,\text{Sign}$. The paper [2] is mostly concerned with the sliding mode property for (1.1.16)–(1.1.21).

In the first chapter of this thesis we deal with (1.1.1)–(1.1.6), which is rather a generalization of the problem (1.1.16)–(1.1.21) since we only require (1.1.10)–(1.1.11) for the maximal monotone operator A. We prove existence and regularity of the solutions for the problem (1.1.1)–(1.1.6), as well as the uniqueness and the continuous dependence on the initial data in case $\alpha = \ell$. In order to obtain our results, we first make a change of variable. We set:

$$\eta = \vartheta + \alpha\varphi - \eta^*. \tag{1.1.22}$$

Consequently, the previous system (1.1.1)–(1.1.6) becomes

$$\partial_t(\eta + (\ell - \alpha)\varphi) - k\Delta\eta + k\alpha\Delta\varphi + \zeta = f - k\Delta\eta^* \quad \text{a.e. in } Q, \tag{1.1.23}$$

$$\partial_t\varphi - \upsilon\Delta\varphi + \xi + \pi(\varphi) = \gamma(\eta - \alpha\varphi + \eta^*) \quad \text{a.e. in } Q, \tag{1.1.24}$$

$$\zeta(t) \in A(\eta(t)) \quad \text{for a.e. } t \in (0,T), \tag{1.1.25}$$

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q. \tag{1.1.26}$$

$$\partial_\nu\eta = 0, \qquad \partial_\nu\varphi = 0 \quad \text{on } \Sigma, \tag{1.1.27}$$

$$\eta(0) = \eta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{1.1.28}$$

In order to prove the existence of solutions to (1.1.23)–(1.1.28), we first consider the approximating problem (P_ε), obtained from problem (P) by approximating A and β by their Yosida regularizations. Then we construct a further approximating problem $(P_{\varepsilon,n})$, obtained from (P_ε) by a Faedo-Galerkin scheme based on a system of eigenfunctions $\{v_n\} \subseteq W$, where

$$W = \{u \in H^2(\Omega) : \ \partial_\nu u = 0 \text{ on } \partial\Omega\}. \tag{1.1.29}$$

Then, we prove the existence of a local solution for $(P_{\varepsilon,n})$ and, passing to the limit as $n \to +\infty$, we infer that the limit of some subsequence of solutions for $(P_{\varepsilon,n})$ yields a solution of (P_ε). Finally, we pass to the limit as $\varepsilon \searrow 0$ and show that some limit of a subsequence yields a solution of (P).

Next, we let $\alpha = \ell$ and write problem (P) for two different sets of initial data f_i, η_i^*, η_{0_i} and φ_{0_i}, $i = 1, 2$. By performing suitable contracting estimates for the difference of the corresponding solutions, we deduce the continuous dependence result whence the uniqueness property is also achieved.

1.2 Cahn–Hilliard systems

The Cahn–Hilliard equation, originally introduced in [14] and first studied mathematically in the seminal paper [37], yields a description of the evolution phenomenon of the solid–solid phase separation. In general, an evolution process goes on diffusively. However, the phenomenon of the solid–solid phase separation does not seem to follow this structure: e.g., when a binary alloy is cooled down sufficiently, each phase concentrates and the material quickly becomes inhomogeneous, forming a fine-grained structure in which each of the two components appears more or less alternatively (see, e.g., [54]). The Cahn–Hilliard equation is a celebrated model which describes this process (usually known as spinodal decomposition) by the simple framework of partial differential equations. The mathematical literature concerning this problem is rather vast. Let us quote [15,20,24,42, 45,55,59,60,66] and also refer to [19] in which a forced mass constraint on the boundary is considered.

In the second chapter, we consider the following Cahn–Hilliard system perturbed by the presence of an additional maximal monotone nonlinearity:

$$\partial_t(\vartheta + \ell\varphi) - \Delta\vartheta + \zeta = f \quad \text{a.e. in } Q, \tag{1.2.1}$$

$$\partial_t\varphi - \Delta\mu = 0 \quad \text{a.e. in } Q, \tag{1.2.2}$$

$$\mu = -\upsilon\Delta\varphi + \xi + \pi(\varphi) - \gamma\vartheta \quad \text{a.e. in } Q, \tag{1.2.3}$$

$$\zeta(t) \in A(a\vartheta(t) + b\varphi(t) - \eta^*) \quad \text{for a.e. } t \in (0,T), \tag{1.2.4}$$

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q, \tag{1.2.5}$$

where ϑ, φ and μ denote the temperature, the order parameter and the chemical potential, respectively. We point out that here ϑ does not represent the absolute temperature, but it is related to it by

$$\vartheta = \Theta - \Theta_c, \tag{1.2.6}$$

where Θ_c denotes a critical temperature. Moreover, η^* is a function in $H^2(\Omega)$ with null outward normal derivative on the boundary of Ω, f is a source term and a, b, ℓ, γ are constants. In particular, let ℓ and γ be positive. The above system is complemented by homogeneous Neumann boundary conditions for ϑ, φ and μ, that is,

$$\partial_\nu\vartheta = \partial_\nu\varphi = \partial_\nu\mu = 0 \quad \text{on } \Sigma, \tag{1.2.7}$$

and by the initial conditions

$$\vartheta(0) = \vartheta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{1.2.8}$$

The term $\xi + \pi(\varphi)$, appearing in (1.2.3), represents the derivative of the double–well potential $\mathcal{W}$ defined as in (1.1.7) and satisfying (1.1.8)–(1.1.9), while A is the maximal monotone operator described by (1.1.10)–(1.1.11).

As usual for Cahn–Hilliard system, the integral mean value of $\varphi(t)$ remains constant during the whole evolution. Indeed, fixing an arbitrary $t \in (0,T)$ and integrating (1.2.2) over Ω, we infer that

$$\frac{d}{dt}\int_\Omega \varphi(t) = 0, \tag{1.2.9}$$

whence it immediately follows that

$$m(\varphi(t)) := \frac{1}{|\Omega|}\int_\Omega \varphi(t) = \frac{1}{|\Omega|}\int_\Omega \varphi_0 \qquad \text{for every } t \in (0,T). \tag{1.2.10}$$

We also observe that the system (1.2.1)–(1.2.8) is a fourth-order problem constructed as the conserved version of the phase–field system (1.1.1)–(1.1.6).

In this thesis (see also [33]), we first show the existence of solutions for Problem (P) (see (1.2.1)–(1.2.8)). In order to carry out this purpose, we consider the approximating problem (P_ε), obtained from (P) by approximating A and β by their Yosida regularizations. In performing our uniform estimates we often refer to [21], where the authors propose the study of a nonlinear diffusion problem as an asymptotic limit of a particular Cahn–Hilliard system. Then, we pass to the limit as $\varepsilon \searrow 0$ and show that some limit of a subsequence of solutions for (P_ε) yields a solution of (P). Next, we let $a\ell = b$ which is, in some sense, a physical restriction since the argument of the variable in the operator A is thus proportional to the internal energy of the system. We also write Problem (P) for two different sets of data f_i, η_i^*, ϑ_{0_i} and φ_{0_i}, $i = 1, 2$. By suitably performing contracting estimates for the difference of the corresponding solutions, we deduce the continuous dependence result whence the uniqueness property is also achieved.

Then, we consider a sliding mode control (SMC) problem. Hence, the main idea behind this scheme is first to identify a manifold of lower dimension (called the sliding manifold) where the control goal is fulfilled and such that the original system restricted to this sliding manifold has a desired behavior, and then to act on the system through the control in order to constrain the evolution on it, that is, to design a SMC-law that forces the trajectories of the system to reach the sliding surface and maintains them on it (see, e.g., [51, 57]). The main advantage of sliding mode control is that it allows the separation of the motion of the overall system in independent partial components of lower dimensions, and consequently it reduces the complexity of the control problem. In particular, we prove the existence of sliding modes for the solutions of our system (P) for a suitable choice of the operator A and of the coefficients a and b. We take $a = 1$, $b = \ell$ and $A = \rho\,\mathrm{Sign}$, where ρ is a positive coefficient and $\mathrm{Sign} : H \longrightarrow 2^H$ is the maximal monotone operator defined in the previuos Subsection. Thus we prescribe a state-feedback control law acting on the rescaled internal energy $(\vartheta + \ell\varphi)$ of the system in order that the dynamics of the system modified in this way forces the value $(\vartheta(t) + \ell\varphi(t))$ to reach a manifold of the phase space in a finite time and then lie there with a sliding mode (cf. [2, 27]).

Concerning the study of optimal control problems for phase–field systems, we quote [22, 23, 30, 49]. Recent investigations have been also addressed to the optimal control problem for Cahn-Hilliard systems: let us mention [17, 18, 24–26, 46]. We also refer to [67, 68] which deals with the convective Cahn–Hilliard equation, and to [47, 65], where some discretized versions of the general Cahn–Hilliard systems are studied.

Then, assuming $a = 1$, $b = \ell$ and $A = \rho\,\mathrm{Sign}$ in (1.2.1)–(1.2.8), we prove the existence of sliding modes for Problem (P) by identifying $\rho^* > 0$ such that the following property is fulfilled: for every $\rho > \rho^*$, there exists a solution $(\vartheta, \varphi, \mu)$ to Problem (P) and a time T^* such that, for every $t \in [T^*, T]$

$$\vartheta(t) + \ell\varphi(t) = \eta^* \qquad \text{a.e. in } \Omega. \tag{1.2.11}$$

It is curious and interesting that we are able to handle a feedback law and prove the mentioned property just for the internal energy of the system, which is a special linear combination of the variables ϑ and φ. However, for a discussion of the SMC laws, linear and nonlinear, that can be considered for phase–field systems, we refer to the Introduction of [2].

1.3 Singular phase–field systems

In the third chapter of the Thesis, we consider a system of partial differential equations (PDE) arising from a thermodynamic model describing phase transitions. The system is written in terms of a rescaled balance of energy and of a balance law for the microforces that govern the phase transition. Moreover, the first equation of the system is perturbed by the presence of an additional maximal monotone nonlinearity. The third chapter will focus only on analytical aspects and, in particular, will investigate the existence of solutions. In order to make the presentation clear from the beginning, we briefly introduce the main ingredients of the PDE system and give some comments on the physical meaning.

We deal with a two–phase system located in a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$ and let $T > 0$ denote some final time. The unknowns of the problem are the absolute temperature ϑ and an order parameter φ.

Now, let us state precisely the equations as well as the initial and boundary conditions. The equations governing the evolution of ϑ and φ are recovered as balance laws. The first equation comes from a reduction of the energy balance equation divided by the absolute temperature ϑ (see [5, formulas (2.33)–(2.35)]). Therefore, the so-called entropy balance can be written in $\Omega \times (0, T)$ as follows:

$$\partial_t(\ln \vartheta + \ell\varphi) - k\Delta\vartheta = f, \tag{1.3.1}$$

where ℓ is a positive parameter, $k > 0$ is a thermal coefficient for the entropy flux $\mathbf{Q}$, which is related to the heat flux vector $\mathbf{q}$ by $\mathbf{Q} = \mathbf{q}/\vartheta$, and f stands for an external entropy source.

Here, we assume that the entropy balance equation (1.3.1) is perturbed by the presence of an additional maximal monotone nonlinearity, i.e.,

$$\partial_t(\ln \vartheta + \ell\varphi) - k\Delta\vartheta + \zeta = f, \tag{1.3.2}$$

where

$$\zeta(t) \in A(\vartheta(t) - \vartheta^*) \qquad \text{for a.e. } t \in (0, T). \tag{1.3.3}$$

Here, ϑ^* is a positive and smooth function ($\vartheta^* \in H^2(\Omega)$ with null outward normal derivative on the boundary) and A, i.e. the maximal monotone operator described by (1.1.10)–(1.1.11), is the subdifferential of a proper, convex and lower semicontinuous function $\Upsilon : L^2(\Omega) \to \mathbb{R}$. In order to explain the role of this further nonlinearity, we refer to [2], where a class of sliding mode control problems is considered: a state-feedback control $(\vartheta, \varphi) \mapsto u(\vartheta, \varphi)$ is added in the balance equations with the purpose of forcing the trajectories of the system to reach the sliding surface (i.e., a manifold of lower dimension where the control goal is fulfilled and such that the original system restricted to this manifold has a desired behavior) in finite time and maintains them on it. As widely described in [2], this study is physically meaningful in the framework of phase transition processes.

In the first two chapters of this thesis (see also [32,33]) the existence of strong solutions, the global well–posedness of the system and the sliding mode property can be proved;

unfortunately, here the problem we consider is rather more delicate due to the doubly nonlinear character of equation (1.3.2) and it turns out that we cannot perform a so complete analysis. On the other hand, we observe that, due to the presence of the logarithm of the temperature in the entropy equation (1.3.2), in the system we investigate here the positivity of the variable representing the absolute temperature follows directly from solving the problem, i.e., from finding a solution component ϑ to which the logarithm applies. This is an important feature and avoids the use of other methods or the setting of special assumptions, in order to guarantee the positivity of ϑ in the space-time domain.

The second equation of the system under study describes the phase dynamics and is deduced from a balance law for the microscopic forces that are responsible for the phase transition process. According to [40, 44], this balance reads

$$\partial_t\varphi - \Delta\varphi + \beta(\varphi) + \pi(\varphi) \ni \ell\vartheta, \tag{1.3.4}$$

where $\beta+\pi$ represents the derivative, or the subdifferential, of the double–well potential $\mathcal{W}$ defined as in (1.1.7) and satisfying (1.1.8)–(1.1.9). We recall that many different choices of $\tilde{\beta}$ and $\tilde{\pi}$ have been introduced in the literature (see, e.g., [3, 6, 39, 58]). In case of a solid-liquid phase transition, $\mathcal{W}$ may be taken in a way that the full potential (cf. (1.3.4))

$$\varphi \mapsto \tilde{\beta}(\varphi) + \tilde{\pi}(\varphi) - \ell\vartheta\varphi$$

exhibits one of the two minima $\varphi = 0$ and $\varphi = 1$ as global minimum for equilibrium, depending on whether ϑ is below or above a critical value ϑ_c, which may represent a phase change temperature. A sample case is given by $\tilde{\pi}(\varphi) = \ell\vartheta_c\varphi$ and by the $\tilde{\beta}$ that coincides with the indicator function $I_{[0,1]}$ of the interval $[0, 1]$, that is,

$$\tilde{\beta}(\rho) = I_{[0,1]}(\rho) = \begin{cases} 0 & \text{if } 0 \le \rho \le 1 \\ +\infty & \text{elsewhere} \end{cases}$$

so that $\beta = \partial I_{[0,1]}$ is specified by

$$r \in \beta(\rho) \quad \text{if and only if} \quad r \begin{cases} \le 0 & \text{if } \rho = 0 \\ = 0 & \text{if } 0 < \rho < 1 \\ \ge 0 & \text{if } \rho = 1 \end{cases}.$$

Of course, this yields a singular case for the potential $\mathcal{W}$, in which $\tilde{\beta}$ is not differentiable, and it is known in the literature as the double obstacle case (cf. [3, 6, 40])

In the last decades phase–field models have attracted a number of mathematicians and applied scientists to describe many different physical phenomena. Let us just recall some results in the literature that are related to our system. Some key references are the papers [4–6]. Besides, we quote [8], where a first simplified version of the entropy system is considered, and [7, 9] for related analyses and results. About special choices of the heat flux and phase–field models ensuring positivity of the absolute temperature, we aim to quote the papers [28, 29, 31], where some Penrose–Fife models have been addressed.

The full problem investigated in the third chapter consists of equations (1.3.2)–(1.3.4) coupled with suitable boundary and initial conditions. In particular, we prescribe a no-flux condition on the boundary for both variables:

$$\partial_\nu \vartheta = 0, \quad \partial_\nu \varphi = 0 \quad \text{on } \Sigma. \tag{1.3.5}$$

Besides, in the light of (1.3.3), initial conditions are stated for $\ln \vartheta$ and φ:

$$\ln \vartheta(0) = \ln \vartheta_0, \quad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{1.3.6}$$

The resulting system is highly nonlinear. The main difficulties lie in the treatment of the doubly nonlinear equation (1.3.2). The expert reader can realise that it is not trivial to recover some coerciveness and regularity for ϑ from (1.3.2), (1.3.3) and (1.3.5); morever, the presence of both $\ln \vartheta$ under time derivative and the selection ζ from $A(\vartheta - \vartheta^*)$ complicates possible uniqueness arguments. For the moment, we are just able to prove the existence of solutions for the described problem (see [16]). To this aim, we introduce a backward finite differences scheme and first examine the solvability of it, for which we have to introduce another approximating problem based on the use of Yosida regularizations for the maximal monotone operators. We prove several uniform estimates which allow us to pass to the limit by means of compactness and monotonicity arguments.

Chapter 2

Preliminary assumptions

2.1 Notations

We assume $\Omega \subseteq \mathbb{R}^3$ to be open, bounded, connected, of class C^1 and we write $|\Omega|$ for its Lebesgue measure. Moreover, Γ and ∂_ν stand for the boundary of Ω and the outward normal derivative, respectively. Given a finite final time $T > 0$, for every $t \in (0, T]$ we set

$$Q_t = \Omega \times (0,t), \quad Q = Q_T, \qquad \Sigma_t = \Gamma \times (0,t), \quad \Sigma = \Sigma_T.$$

We also introduce the spaces

$$H = L^2(\Omega), \quad V = H^1(\Omega), \quad V_0 = H_0^1(\Omega), \quad W = \{u \in H^2(\Omega) : \ \partial_\nu u = 0 \text{ on } \Gamma\}, \tag{2.1.1}$$

with usual norms $\|\cdot\|_H$, $\|\cdot\|_V$, $\|\cdot\|_W$ and related inner products $(\cdot,\cdot)_H$, $(\cdot,\cdot)_V$, $(\cdot,\cdot)_W$, respectively. We identify H with its dual space H^*, so that $W \subset V \subset H \subset V^* \subset W^*$ with dense and compact embeddings. Let $\langle \cdot,\cdot \rangle$ denote the duality pairing between V^* and V. The notation $\|\cdot\|_p$ $(1 \le p \le \infty)$ stands for the standard norm in $L^p(\Omega)$. For short, in the notation of norms we do not distinguish between a space and a power thereof.

Moreover, in the following the small–case symbol c stands for different constants which depend only on Ω, on the final time T, on the shape of the nonlinearities and on the constants and the norms of the functions involved in the assumptions of our statements. On the contrary, we use different symbols to denote precise constants to which we could refer. It is important to point out that the meaning of c might change from line to line and even in the same chain of inequalities.

Finally, from now on, we interpret the operator $-\Delta$ as the Laplacian operator from the space W to H, then including the Neumann homogeneous boundary condition. Moreover, we extend $-\Delta$ to an operator from V to V^* by setting

$$\langle -\Delta u, v \rangle := \int_\Omega \nabla u \cdot \nabla v, \quad u, v \in V. \tag{2.1.2}$$

2.2 Inequalities

In the sequel, we account for the continuous embeddings $V \subset L^q(\Omega)$, with $1 \le q \le 6$, $W \subset C^0(\overline{\Omega})$ and for the related Sobolev inequalities:

$$\|v\|_q \le C_s \|v\|_V \qquad \text{and} \qquad \|v\|_\infty \le C_s \|v\|_W \tag{2.2.1}$$

for $v \in V$ and $v \in W$, respectively, where C_s depends on Ω only, since sharpness is not needed. Now, let us recall a variant of the Poincaré inequality, i.e., there exists a positive constant C_p such that

$$\|v\|_V \le C_p\Big(\|v\|_{L^1(\Omega)} + \|\nabla v\|_H\Big), \qquad v \in V. \tag{2.2.2}$$

Moreover, we will use an inequality deduced from the compactness of the embedding $V \subset H \subset V^*$ (see [62, Lemma 8, p. 84]): for all $\delta > 0$ there exists a constant $K > 0$ such that

$$\|z\|_H \le \delta \|z\|_V + K\|z\|_{V^*} \qquad \text{for all } z \in H. \tag{2.2.3}$$

Furthermore, we often employ the Hölder inequality, and the Young's inequalities, i.e., for every $a > 0$, $b > 0$, $\alpha \in (0,1)$ and $\delta > 0$ we have that

$$ab \le \alpha a^{\frac{1}{\alpha}} + (1-\alpha) b^{\frac{1}{1-\alpha}}, \tag{2.2.4}$$

$$ab \le \delta a^2 + \frac{1}{4\delta} b^2. \tag{2.2.5}$$

Finally, let us point out that for every a, $b \in \mathbb{R}$ we have that

$$(a-b)a = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2. \tag{2.2.6}$$

2.3 Preliminary results

In this section, we state some useful results.

Lemma 2.3.1. *Assume that a, $b \in \mathbb{R}$ are strictly positive. Then*

$$(a-b) \le \big(\ln a^2 - \ln b^2\big)(a+b). \tag{2.3.1}$$

Proof. We consider $a > b$ (if $b > a$ the technique of the proof is analogous) and obtain

$$(a-b) \le (\ln a^2 - \ln b^2)(a+b) = 2(\ln a - \ln b)(a+b) = 2\ln\left(\frac{a}{b}\right)(a+b).$$

Then, dividing by b, we have that

$$\left(\frac{a}{b} - 1\right) \le 2\ln\left(\frac{a}{b}\right)\left(\frac{a}{b} + 1\right). \tag{2.3.2}$$

Letting $x = a/b$, we can rewrite (2.3.2) as

$$(x-1) \leq 2(x+1)\ln x \qquad \text{for } x \geq 1.$$

Now, we observe that (2.3.1) is verified if and only if the function

$$f(x) := 2(x+1)\ln x - x + 1 \quad \text{is nonnegative for every } x \geq 1. \tag{2.3.3}$$

Since $f(1) = 0$ and $f'(x) > 0$ for every $x \geq 1$, we conclude that (2.3.3) holds. Then, the proof of the lemma is complete. □

Lemma 2.3.2. *Let a_0, b_0, ψ_0, $\rho \in \mathbb{R}$ be such that*

$$a_0,\ b_0,\ \psi_0 \geq 0 \quad \textit{and} \quad \rho > a_0^2 + 2b_0 + 2\frac{\psi_0}{T} \tag{2.3.4}$$

and let $\psi :\ [0,T] \to [0,+\infty)$ be an absolutely continuous function satisfying $\psi(0) = \psi_0$ and

$$\psi' + \rho \leq a_0\rho^{1/2} + b_0 \quad \textit{a.e. in the set } P := \{t \in (0,T) : \psi(t) > 0\}. \tag{2.3.5}$$

Then, the following conditions hold true:

1. *If $\psi_0 = 0$, then ψ vanishes identically.*
2. *If $\psi_0 > 0$, then there exists $T^* \in (0,T)$ satisfying $T^* \leq 2\psi_0/(\rho - a_0^2 - 2b_0)$ such that ψ is strictly decreasing in $(0,T^*)$ and ψ vanishes in $[T^*,T]$.*

Proof. See [2, Lemma 4.1]. □

Finally, let us recall the discrete version of the Gronwall lemma.

Lemma 2.3.3. *If $(a_0,\ldots,a_N) \in [0,+\infty)^{N+1}$ and $(b_1,\ldots,b_N) \in [0,+\infty)^N$ satisfy*

$$a_m \leq a_0 + \sum_{n=1}^{m-1} a_n b_n \qquad \textit{for} \quad m = 1,\ldots,N, \tag{2.3.6}$$

then

$$a_m \leq a_0 \ \textit{exp}\left(\sum_{n=1}^{m-1} b_n\right) \qquad \textit{for } m = 1,\ldots,N. \tag{2.3.7}$$

Proof. See [52, Prop. 2.2.1]. □

2.4 Operators

In this section, we describe the operators appearing in the systems under study.

The double–well potential $\mathcal{W}$. We introduce the double–well potential $\mathcal{W}$ as the sum

$$\mathcal{W} = \tilde{\beta} + \tilde{\pi}, \tag{2.4.1}$$

where

$$\tilde{\beta} : \mathbb{R} \longrightarrow [0, +\infty] \text{ is proper, l.s.c. and convex with } \tilde{\beta}(0) = 0, \tag{2.4.2}$$

$$\tilde{\pi} : \mathbb{R} \to \mathbb{R},\ \tilde{\pi} \in C^1(\mathbb{R}) \text{ with } \pi := \tilde{\pi}' \text{ Lipschitz continuous.} \tag{2.4.3}$$

Since $\tilde{\beta}$ is proper, l.s.c. and convex, the subdifferential $\beta := \partial\tilde{\beta}$ is well–defined. We denote by $D(\beta)$ and $D(\tilde{\beta})$ the effective domains of β and $\tilde{\beta}$, respectively, and also assume that $\mathrm{int}(D(\beta)) \neq \emptyset$. Thanks to these assumptions, β is a maximal monotone graph. Moreover, as $\tilde{\beta}$ takes its minimum in 0, we have that $0 \in \beta(0)$. Now, we observe that β induces the operator $\mathcal{B}$ on $L^2(Q)$ in the following way:

$$\mathcal{B} : L^2(Q) \longrightarrow L^2(Q) \tag{2.4.4}$$

$$\xi \in \mathcal{B}(\varphi) \Longleftrightarrow \xi(x,t) \in \beta(\varphi(x,t)) \quad \text{for a.e. } (x,t) \in Q. \tag{2.4.5}$$

We notice that

$$\beta = \partial\tilde{\beta}, \qquad\qquad \mathcal{B} = \partial\Psi, \tag{2.4.6}$$

where

$$\Psi :\ L^2(Q) \longrightarrow (-\infty, +\infty] \tag{2.4.7}$$

$$\Psi(u) = \begin{cases} \int_Q \tilde{\beta}(u) & \text{if } u \in L^2(Q) \text{ and } \tilde{\beta}(u) \in L^1(Q), \\ +\infty & \text{elsewhere, with } u \in L^2(Q). \end{cases} \tag{2.4.8}$$

The maximal monotone operator A. We consider the maximal monotone operator

$$A : H \longrightarrow H \tag{2.4.9}$$

and we assume that

$$\begin{gathered} A \text{ is the subdifferential of a convex and l.s.c. function } \Upsilon : H \longrightarrow \mathbb{R} \\ \text{which takes its minimum in 0 and has at most a quadratic growth.} \end{gathered} \tag{2.4.10}$$

These properties are related to our assumptions on $A = \partial\Upsilon$, which read

$$0 \in A(0), \quad \exists\, C_A > 0 \text{ such that } \|y\|_H \leq C_A(1 + \|x\|_H) \quad \forall\, x \in H,\ \forall\, y \in Ax. \tag{2.4.11}$$

We also introduce the operator $\mathcal{A}$ induced by A on $L^2(0,T;H)$ in the following way

$$\mathcal{A} : L^2(0,T;H) \longrightarrow L^2(0,T;H) \tag{2.4.12}$$

$$\zeta \in \mathcal{A}(\eta) \Longleftrightarrow \zeta(t) \in A(\eta(t)) \quad \text{for a.e. } t \in (0,T). \tag{2.4.13}$$

We notice that $\mathcal{A}$ is a maximal monotone operator.

The operator Sign**.** Let us consider the operator

$$\operatorname{sign} \ : \mathbb{R} \longrightarrow 2^{\mathbb{R}}, \quad \operatorname{sign}(r) = \begin{cases} \frac{r}{|r|} & \text{if } r \neq 0, \\ [-1,1] & \text{if } r = 0, \end{cases} \tag{2.4.14}$$

and its nonlocal counterpart in H, that is,

$$\operatorname{Sign} : H \longrightarrow 2^{H}, \quad \operatorname{Sign}(v) = \begin{cases} \frac{v}{\|v\|_H} & \text{if } v \neq 0, \\ B_1(0) & \text{if } v = 0, \end{cases} \tag{2.4.15}$$

where $B_1(0)$ denotes the closed unit ball of H. It is straightforward to check that Sign satisfies (2.4.10)–(2.4.11) and turns out to be the subdifferential of the norm function $v \mapsto \|v\|_H$. Concerning the graph sign, it is well known that it induces a maximal monotone operator in H which is the the subdifferential of the convex function $v \mapsto \int_\Omega |v|$. In the following two paragraphs, we consider other interesting operators satisfying (2.4.10)–(2.4.11).

Example 1. We consider the operator

$$A_1 : \ \mathbb{R} \longrightarrow \mathbb{R} \tag{2.4.16}$$

$$A_1(r) = \begin{cases} \alpha_1 r & \text{if } r < 0, \\ 0 & \text{if } 0 \leq r \leq 1, \\ \alpha_2 r & \text{if } r > 1, \end{cases} \tag{2.4.17}$$

where α_1 and α_2 are positive coefficients. We observe that A_1 is a maximal monotone operator on $\mathbb{R}$, whose graph consists of an horizontal line segment and two rays of slope α_1, α_2. Moreover, $0 \in A_1(0)$ and

$$|v| \leq C(1+|r|) \qquad \text{for all } r \in \mathbb{R}, \ v \in A_1(r), \tag{2.4.18}$$

with $C = \max(\alpha_1, \alpha_2)$. Then A_1 satisfies (2.4.11)–(2.4.11). We notice that A_1 corresponds to the graph which correlates the enthalpy to the temperature in the Stefan problem (see, e.g., [34, 36, 41]).

Example 2. We consider the operator

$$A_2 : \ H \longrightarrow H \tag{2.4.19}$$

$$A_2(v) = \alpha |v|^{q-1} v, \tag{2.4.20}$$

where $0 < q < 1$ and α is a function in $L^\infty(\Omega)$ with $\alpha(x) \geq 0$ for a.e. $x \in \Omega$. We observe that A_2 induces a (nonlocal) multivalued maximal monotone operator on H, with $0 \in A_2(0)$. Moreover, A_2 can be considered a weighted perturbation of the operator appearing in the porous media equation and in the fast diffusion equation (see, e.g., [35, 50, 64]).

The operator m. We consider the operator $m : V^* \to \mathbb{R}$ defined by

$$m(z^*) := \frac{1}{|\Omega|}\langle z^*, 1\rangle_{V^*,V} \quad \text{for all } z^* \in V^*. \tag{2.4.21}$$

We observe that, if $z^* \in H$, then

$$m(z^*) = \frac{1}{|\Omega|}\int_\Omega z^* dx. \tag{2.4.22}$$

The operator $\mathcal{N}$. We also consider the operator

$$\mathcal{N} : D(\mathcal{N}) \subseteq V^* \to V, \tag{2.4.23}$$

defined on its domain

$$D(\mathcal{N}) := \{w \in V^* : m(w^*) = 0\}. \tag{2.4.24}$$

For every $w^* \in D(\mathcal{N})$, we define $w = \mathcal{N}w^*$ if $w \in V$, $m(w) = 0$ and w is a solution of the following variational equation

$$\int_\Omega \nabla w \cdot \nabla z dx = \langle w^*, z\rangle_{V^*,V} \quad \text{for all } z \in V. \tag{2.4.25}$$

If $w^* \in D(\mathcal{N}) \cap H$, then w is the unique solution to the elliptic problem

$$\begin{cases} -\Delta w = w^* & \text{a.e. in } \Omega, \\ \partial_\nu w = 0 & \text{a.e. in } \Gamma, \\ m(w) = 0. \end{cases} \tag{2.4.26}$$

We observe that, due to elliptic regularity, $w \in W$. Moreover, for every $v^*, w^* \in D(\mathcal{N})$, $v = \mathcal{N}v^*$ and $w = \mathcal{N}w^*$ we have that

$$\begin{aligned} \langle w^*, \mathcal{N}v^*\rangle_{V^*,V} = \langle w^*, v\rangle_{V^*,V} &= \int_\Omega \nabla w \cdot \nabla v dx \\ &= \langle v^*, w\rangle_{V^*,V} = \langle v^*, \mathcal{N}w^*\rangle_{V^*,V}. \end{aligned}$$

Consequently, by defining

$$\|w^*\|_{V^*}^2 := \big\|\nabla\mathcal{N}\big(w^* - m(w^*)\big)\big\|_{H^3}^2 + \big|m(w^*)\big|^2 \quad \text{for all } w^* \in V^*, \tag{2.4.27}$$

it turns out that $\|\cdot\|_{V^*}$ is a norm in V^*.

2.5 Moreau–Yosida regularization

Moreau–Yosida regularization of β and $\tilde{\beta}$. We introduce the Yosida regularization of the operator β (see (2.4.2)). For $\varepsilon > 0$ let

$$\beta_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}, \qquad \beta_\varepsilon = \frac{I - (I + \varepsilon\beta)^{-1}}{\varepsilon}. \tag{2.5.1}$$

We remark that β_ε is Lipschitz continuous (with Lipschitz constant $1/\varepsilon$) and satisfies the following properties: denoting by $R_\varepsilon = (I + \varepsilon\beta)^{-1}$ the resolvent operator, we have that

$$\beta_\varepsilon(x) \in \beta(R_\varepsilon x) \quad \text{for all } x \in \mathbb{R}, \tag{2.5.2}$$

$$|\beta_\varepsilon(x)| \le |\beta^\circ(x)|, \quad \lim_{\varepsilon \searrow 0} \beta_\varepsilon(x) = \beta^\circ(x) \quad \text{for all } x \in D(\beta), \tag{2.5.3}$$

where $\beta^\circ(x)$ is the element of the range of $\beta(x)$ having minimal modulus. We also introduce the Moreau–Yosida regularization of $\tilde{\beta}$. For $\varepsilon > 0$ and $x \in \mathbb{R}$ we set

$$\tilde{\beta}_\varepsilon : \mathbb{R} \longrightarrow [0, +\infty], \qquad \tilde{\beta}_\varepsilon(x) := \min_{y \in \mathbb{R}} \left\{ \tilde{\beta}(y) + \frac{1}{2\varepsilon}|x - y|^2 \right\}$$

and recall that

$$\tilde{\beta}_\varepsilon(x) \le \tilde{\beta}(x) \quad \text{for every } x \in \mathbb{R}. \tag{2.5.4}$$

We also observe that β_ε is the derivative of $\tilde{\beta}_\varepsilon$. Then, for every $x_1, x_2 \in \mathbb{R}$ we have that

$$\tilde{\beta}_\varepsilon(x_2) = \tilde{\beta}_\varepsilon(x_1) + \int_{x_1}^{x_2} \beta_\varepsilon(s)\, ds. \tag{2.5.5}$$

Yosida regularization of A. We introduce the Yosida regularization of A (see). For $\varepsilon > 0$ we define

$$A_\varepsilon : H \longrightarrow H, \qquad A_\varepsilon = \frac{I - (I + \varepsilon A)^{-1}}{\varepsilon}. \tag{2.5.6}$$

Note that A_ε is Lipschitz-continuous (with Lipschitz constant $1/\varepsilon$) and maximal monotone in H. Moreover, A satisfies the following properties: denoting by $J_\varepsilon = (I + \varepsilon A)^{-1}$ the resolvent operator, for all $\delta > 0$ and for all $x \in H$, we have that

$$A_\varepsilon x \in A(J_\varepsilon x), \tag{2.5.7}$$

$$\|A_\varepsilon x\|_H \le \|A^\circ x\|_H, \quad \lim_{\varepsilon \searrow 0} \|A_\varepsilon x - A^\circ x\|_H = 0, \tag{2.5.8}$$

where $A^\circ x$ is the element of the range of A having minimal norm. Let us point out a key property of A_ε, which is a consequence of (2.4.11): indeed, there holds

$$\|A_\varepsilon x\|_H \le C_A(1 + \|x\|_H) \quad \text{for all } x \in H. \tag{2.5.9}$$

Notice that $0 \in A(0)$ and $0 \in I(0)$: consequently, for every $\varepsilon > 0$ we infer that $J_\varepsilon(0) = 0$. Moreover, since A is maximal monotone, J_ε is a contraction. Then, from (2.4.11) and (2.5.7), for every $x \in H$ we have that

$$\|A_\varepsilon x\|_H \le C_A(\|J_\varepsilon x\|_H + 1) \le C_A(\|J_\varepsilon x - J_\varepsilon 0\|_H + 1) \le C_A(\|x\|_H + 1).$$

Yosida regularization of Sign**.** Let us introduce the operator $\mathrm{Sign}_\varepsilon : H \to H$ as the Yosida regularization at level $\varepsilon > 0$ of the operator Sign. We observe that $\mathrm{Sign}_\varepsilon(v)$ is the gradient at v of the C^1 functional $\|\cdot\|_{H,\varepsilon}$ defined as

$$\|v\|_{H,\varepsilon} := \min_{w\in H}\{\frac{1}{2\varepsilon}\|w-v\|_H^2 + \|w\|_H\} = \int_0^{\|v\|_H} \min\{s/\varepsilon, 1\}\, ds \quad \text{for every } v \in H. \tag{2.5.10}$$

We also recall that

$$\mathrm{Sign}_\varepsilon(v) = \frac{v}{\max\{\varepsilon, \|v\|_H\}} \quad \text{for every } v \in H. \tag{2.5.11}$$

Yosida regularization of ln**.** We introduce the Yosida regularization of ln. For $\varepsilon > 0$ we set

$$\ln_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}, \qquad \ln_\varepsilon := \frac{I - (I + \varepsilon \ln)^{-1}}{\varepsilon}. \tag{2.5.12}$$

where I denotes the identity. We point out that $\ln_\varepsilon$ is monotone, Lipschitz continuous (with Lipschitz constant $1/\varepsilon$) and satisfies the following properties: denoting by $L_\varepsilon = (I + \varepsilon\ln)^{-1}$ the resolvent operator, we have that

$$\ln_\varepsilon(x) \in \ln(L_\varepsilon x) \quad \text{for all } x \in \mathbb{R}, \tag{2.5.13}$$

$$|\ln_\varepsilon(x)| \le |\ln(x)|, \quad \lim_{\varepsilon \searrow 0} \ln_\varepsilon(x) = \ln(x) \quad \text{for all } x > 0. \tag{2.5.14}$$

We also introduce the nonnegative and convex functions

$$\Lambda(x) = \int_1^x \ln r\, dr, \quad \Lambda_\varepsilon(y) = \int_1^y \ln_\varepsilon r\, dr \quad \text{for all } x > 0 \text{ and } y \in \mathbb{R}. \tag{2.5.15}$$

Note that the graph $x \mapsto \ln x$ is nothing but the subdifferential of the convex function Λ extended by lower semicontinuity in 0 and with value $+\infty$ for $x < 0$. On the other hand, Λ_ε coincides with the Moreau–Yosida regularization of Λ and, in particular, we have that

$$0 \le \Lambda_\varepsilon(x) \le \Lambda(x) \quad \text{for every } x > 0. \tag{2.5.16}$$

Regularization of f. Assume that $f \in L^2(0,T;H)$. We denote by f_ε the regularization of f, constructed in such a way that

$$f_\varepsilon \in C^1([0,T];H) \text{ for all } \varepsilon > 0, \qquad \lim_{\varepsilon\searrow 0} \|f_\varepsilon - f\|_{L^2(0,T;H)} = 0. \tag{2.5.17}$$

For example, we can consider f_ε as the solution of the following system:

$$\begin{cases} -\varepsilon f_\varepsilon''(t) + f_\varepsilon(t) = f(t), \quad t \in (0,T), \\ f_\varepsilon(0) = f_\varepsilon(T) = 0. \end{cases} \tag{2.5.18}$$

Thanks to Sobolev immersions and elliptic regularity, (2.5.17) is achieved.

Chapter 3

Solvability of a class of phase–field systems related to a sliding mode control problem

In this chapter we investigate a phase–field system of Caginalp type perturbed by the presence of an additional maximal monotone nonlinearity. Such a system arises from a recent study of a sliding mode control problem. We prove existence of strong solutions. Moreover, under further assumptions, we show the continuous dependence on the initial data and the uniqueness of the solution.

3.1 Setting of the problem and results

We assume that

$$\ell,\ \alpha,\ k,\ \upsilon,\ \gamma \in (0,+\infty), \tag{3.1.1}$$

$$f \in L^2(Q), \qquad \eta^* \in W, \tag{3.1.2}$$

$$\eta_0 \in V, \quad \varphi_0 \in V, \quad \tilde{\beta}(\varphi_0) \in L^1(\Omega). \tag{3.1.3}$$

We look for a pair (η, φ) satisfying at least the regularity requirements

$$\eta,\ \varphi\ \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \tag{3.1.4}$$

and solving the problem (P):

$$\partial_t(\eta + (\ell - \alpha)\varphi) - k\Delta\eta + k\alpha\Delta\varphi + \zeta = f - k\Delta\eta^* \ \text{ a.e. in } Q, \tag{3.1.5}$$

$$\partial_t\varphi - \upsilon\Delta\varphi + \xi + \pi(\varphi) = \gamma(\eta - \alpha\varphi + \eta^*) \ \text{ a.e. in } Q, \tag{3.1.6}$$

$$\zeta(t) \in A(\eta(t)) \ \text{ for a.e. } t \in (0,T), \tag{3.1.7}$$

$$\xi \in \beta(\varphi) \ \text{ a.e. in } Q, \tag{3.1.8}$$

$$\partial_\nu \vartheta = 0, \qquad \partial_\nu \varphi = 0 \quad \text{on } \Sigma, \tag{3.1.9}$$

$$\eta(0) = \eta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{3.1.10}$$

We notice that the homogeneous Neumann boundary conditions for both η and φ required by (3.1.9) follow from (3.1.4), due to the definition of W (see (2.1.1)).

Theorem - (Existence) 3.1.1. *Assume* (3.1.1)–(3.1.3), (2.4.2)–(2.4.3) *and* (2.4.9)–(2.4.11). *Then problem* (P) *(see* (3.1.5)–(3.1.10)*) has at least a solution* (η, φ) *satisfying the regularity requirements* (3.1.4).

Theorem - (Uniqueness and continuous dependence) 3.1.2. *Assume* (3.1.1)–(3.1.3), (2.4.2)–(2.4.3) *and* (2.4.9)–(2.4.11). *If* $\alpha = \ell$, *the solution* (φ, η) *of problem* (P) *(see* (3.1.5)–(3.1.10)*) is unique. Moreover, if* f_i, η_i^*, η_{0_i}, φ_{0_i}, $i = 1, 2$, *are given as in* (3.1.2)–(3.1.3) *and* (φ_i, η_i), $i = 1, 2$, *are the corresponding solutions, then the estimate*

$$\|\eta_1 - \eta_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\varphi_1 - \varphi_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}$$

$$\leq C(\|f_1 - f_2\|_{L^2(Q)} + \|\eta_1^* - \eta_2^*\|_W + \|\eta_{0_1} - \eta_{0_2}\|_H + \|\varphi_{0_1} - \varphi_{0_2}\|_H) \tag{3.1.11}$$

holds true for some constant C *depending only on* Ω, T *and the parameters* ℓ, α, k, υ, γ.

3.2 Proof of the existence theorem

3.2.1 The approximating problem (P_ε)

This section is devoted to the proof of Theorem 3.1.1. In order to obtain this result, we look for a pair $(\eta_\varepsilon, \varphi_\varepsilon)$ satisfying at least the regularity requirements

$$\eta_\varepsilon,\ \varphi_\varepsilon \ \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \tag{3.2.1}$$

and solving the approximating problem (P_ε):

$$\partial_t(\eta_\varepsilon + (\ell - \alpha)\varphi_\varepsilon) - k\Delta\eta_\varepsilon + k\alpha\Delta\varphi_\varepsilon + \zeta_\varepsilon = f_\varepsilon - k\Delta\eta^* \ \text{ a.e. in } Q, \tag{3.2.2}$$

$$\partial_t\varphi_\varepsilon - \upsilon\Delta\varphi_\varepsilon + \xi_\varepsilon + \pi(\varphi_\varepsilon) = \gamma(\eta_\varepsilon - \alpha\varphi_\varepsilon + \eta^*) \ \text{ a.e. in } Q, \tag{3.2.3}$$

$$\zeta_\varepsilon(t) = A_\varepsilon\eta_\varepsilon(t) \ \text{ for a.e. } t \in (0,T), \tag{3.2.4}$$

$$\xi_\varepsilon = \beta_\varepsilon(\varphi_\varepsilon) \ \text{ a.e. in } Q, \tag{3.2.5}$$

$$\partial_\nu\eta_\varepsilon = 0, \qquad \partial_\nu\varphi_\varepsilon = 0 \quad \text{on } \Sigma, \tag{3.2.6}$$

$$\eta_\varepsilon(0) = \eta_0, \qquad \varphi_\varepsilon(0) = \varphi_0 \quad \text{in } \Omega, \tag{3.2.7}$$

where A_ε and β_ε are the Yosida regularizations of A and β defined in (2.5.6) and (2.5.1), respectively. We notice that the homogeneous Neumann boundary conditions for both η_ε and φ_ε required by (3.2.6) follow from (3.2.1) due to the definition of W (see (2.1.1)).

3.2.2 The approximating problem $(P_{\varepsilon,n})$

Now, we apply the Faedo-Galerkin method to the approximating problem (P_ε). We consider the orthonormal basis $\{v_i\}_{\{i\geq 1\}}$ of V formed by the normalized eigenfunctions of the Laplace operator with homogeneous Neumann boundary condition, that is

$$\begin{cases} -\Delta v_i = \lambda_i v_i & \text{in } \Omega, \\ \partial_\nu v_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2.8}$$

Note that, owing to the regularity of Ω, $v_i \in W$ for all $i \geq 1$. Then, for any integer $n \geq 1$, we denote by V_n the n-dimentional subspace of V spanned by $\{v_1, \cdots, v_n\}$. Hence, $\{V_n\}$ is a sequence of finite dimensional subspaces such that $\bigcup_{n=1}^{+\infty} V_n$ is dense in V and $V_k \subseteq V_n$ for all $k \leq n$.

Definition of the approximating problem $(P_{\varepsilon,n})$**.** We first approximate the initial data η_0 and φ_0. We set

$$\eta_{0,n} = P_{V_n}\eta_0, \qquad \varphi_{0,n} = P_{V_n}\varphi_0. \tag{3.2.9}$$

We notice that

$$\lim_{n\to+\infty} \|\eta_{0,n} - \eta_0\|_V = 0 \quad \text{and} \quad \lim_{n\to+\infty} \|\varphi_{0,n} - \varphi_0\|_V = 0. \tag{3.2.10}$$

Note that the convergence provided by (3.2.10) assures that $\eta_{0,n}$ and $\varphi_{0,n}$ are bounded in V. Now, we introduce the new approximating problem $(P_{\varepsilon,n})$. We look for $t_n \in]0,T]$ and a pair $(\eta_{\varepsilon,n}, \varphi_{\varepsilon,n})$ (in the following we will write (η_n, φ_n) instead of $(\eta_{\varepsilon,n}, \varphi_{\varepsilon,n})$) such that

$$\eta_n \in C^1([0,t_n];V_n), \qquad \varphi_n \in C^1([0,t_n];V_n), \tag{3.2.11}$$

and, for every $v \in V_n$ and for every $t \in [0,t_n]$, solving the approximating problem $(P_{\varepsilon,n})$:

$$(\partial_t[\eta_n(t) + (\ell-\alpha)\varphi_n(t)] - k\Delta\eta_n(t) + k\alpha\Delta\varphi_n(t) + A_\varepsilon\eta_n(t), v)_H$$
$$= (f_\varepsilon(t) - k\Delta\eta^*, v)_H, \tag{3.2.12}$$
$$(\partial_t\varphi_n(t) - \upsilon\Delta\varphi_n(t) + \beta_\varepsilon(\varphi_n(t)) + \pi(\varphi_n(t)), v)_H$$
$$= (\gamma[\eta_n(t) - \alpha\varphi_n(t) + \eta^*], v)_H, \tag{3.2.13}$$
$$\partial_\nu\eta_n = 0, \qquad \partial_\nu\varphi_n = 0 \quad \text{on } \Sigma, \tag{3.2.14}$$
$$\eta_n(0) = \eta_{0,n}, \qquad \varphi_n(0) = \varphi_{0,n} \quad \text{in } \Omega. \tag{3.2.15}$$

This is a Cauchy problem for a system of nonlinear ordinary differential equations. In the next section we will show by a change of variable that this system admits a local solution (η_n, φ_n), which is of the form

$$\varphi_n(t) = \sum_{i=1}^{n} a_{in}(t)v_i, \tag{3.2.16}$$

$$\eta_n(t) = \sum_{i=1}^{n} b_{in}(t)v_i, \tag{3.2.17}$$

for some $a_{in} \in C^1([0,t_n])$ and $b_{in} \in C^1([0,t_n])$.

Remark. We point out that

$$\int_\Omega \tilde{\beta}_\varepsilon(\varphi_{0,n}) \le C + \frac{1}{2\varepsilon}\|\varphi_0 - \varphi_{0,n}\|_H(\|\varphi_0\|_H + \|\varphi_{0,n}\|_H), \tag{3.2.18}$$

where

$$C = \|\tilde{\beta}(\varphi_0)\|_{L^1(\Omega)}. \tag{3.2.19}$$

Indeed, for every $\varepsilon \in (0,1]$, thanks to the property (2.5.4) of $\tilde{\beta}_\varepsilon$, we have that

$$0 \le \tilde{\beta}_\varepsilon(\varphi_0) \le \tilde{\beta}(\varphi_0). \tag{3.2.20}$$

Since $\tilde{\beta}(\varphi_0) \in L^1(\Omega)$ (see (3.1.3)), we obtain that

$$\int_\Omega \tilde{\beta}_\varepsilon(\varphi_0) \le C, \tag{3.2.21}$$

where $C = \|\tilde{\beta}(\varphi_0)\|_{L^1(\Omega)}$. From (2.5.5), using the Lipschitz continuity of β_ε, we have

$$\begin{aligned}
\tilde{\beta}_\varepsilon(\varphi_{0,n}) &\le \tilde{\beta}_\varepsilon(\varphi_0) + \Big| \int_{\varphi_0}^{\varphi_{0,n}} \beta_\varepsilon(s)\, ds \Big| \\
&\le \tilde{\beta}_\varepsilon(\varphi_0) + \frac{1}{\varepsilon}\int_{\varphi_0}^{\varphi_{0,n}} |s|\, ds \\
&\le \tilde{\beta}_\varepsilon(\varphi_0) + \frac{1}{2\varepsilon}|\varphi_0 - \varphi_{0,n}|(|\varphi_0| + |\varphi_{0,n}|).
\end{aligned} \tag{3.2.22}$$

By integrating (3.2.22) over Ω, we obtain that

$$\int_\Omega \tilde{\beta}_\varepsilon(\varphi_{0,n}) \le Q_\varepsilon(n), \tag{3.2.23}$$

where

$$Q_\varepsilon(n) = C + \frac{1}{2\varepsilon}\|\varphi_0 - \varphi_{0,n}\|_H(\|\varphi_0\|_H + \|\varphi_{0,n}\|_H).$$

Remark 3.2.1. Thanks to (3.2.17) and the Lipschitz continuity of A_ε, we obtain that

$$A_\varepsilon(\eta_n) \in C^{0,1}([0,t_n]; H). \tag{3.2.24}$$

Indeed, $\|v_i\|_H \le \|v_i\|_V = 1$, for all $i \in \mathbb{N}$. Then we choose $t, t' \in [0, t_n]$ and we have the following inequality:

$$\begin{aligned}
\|A_\varepsilon(\eta_n(t)) - A_\varepsilon(\eta_n(t'))\|_H &= \left\| A_\varepsilon\left(\sum_{i=1}^n b_{in}(t)v_i\right) - A_\varepsilon\left(\sum_{i=1}^n b_{in}(t')v_i\right)\right\|_H \\
&\le \frac{1}{\varepsilon}\left\|\sum_{i=1}^n (b_{in}(t) - b_{in}(t'))v_i\right\|_H \\
&\le \frac{1}{\varepsilon}\sum_{i=1}^n |b_{in}(t) - b_{in}(t')|\, \|v_i\|_H \\
&= \frac{1}{\varepsilon}\sum_{i=1}^n |b_{in}(t) - b_{in}(t')|.
\end{aligned}$$

Since b_{in} are continuous, we obtain (3.2.24).

Existence of a local solution for $(P_{\varepsilon,n})$**.** In order to prove the existence of a local solution (η_n, φ_n) for the approximating problem $(P_{\varepsilon,n})$, we make a change of variable. We set

$$\vartheta_n = \eta_n + (\ell - \alpha)\varphi_n, \qquad \vartheta_{0,n} = \eta_{0,n} + (\ell - \alpha)\varphi_{0,n}, \tag{3.2.25}$$

and we prove that there exists a local solution (ϑ_n, φ_n) of the problem

$$\begin{aligned}
&(\partial_t\vartheta_n - k\Delta\vartheta_n + k\ell\Delta\varphi_n + A_\varepsilon(\vartheta_n - (\ell-\alpha)\varphi_n), v)_H = (f_\varepsilon - k\Delta\eta^*, v)_H,\\
&(\partial_t\varphi_n - \upsilon\Delta\varphi_n + \beta_\varepsilon(\varphi_n) + \pi(\varphi_n), v)_H = (\gamma[\vartheta_n - \ell\varphi_n + \eta^*], v)_H,\\
&\varphi_n(0) = \varphi_{0,n}, \qquad \vartheta_n(0) = \vartheta_{0,n},
\end{aligned} \tag{3.2.26}$$

whenever $v \in V_n$. Re-arranging the above system in explicit form, we have

$$\begin{aligned}
&(\partial_t\vartheta_n, v)_H = (k\Delta\vartheta_n - k\ell\Delta\varphi_n - A_\varepsilon(\vartheta_n - (\ell-\alpha)\varphi_n) + f_\varepsilon - k\Delta\eta^*, v)_H,\\
&(\partial_t\varphi_n, v)_H = (\upsilon\Delta\varphi_n - \beta_\varepsilon(\varphi_n) - \pi(\varphi_n) + \gamma[\vartheta_n - \ell\varphi_n + \eta^*], v)_H,\\
&\varphi_n(0) = \varphi_{0,n}, \qquad \vartheta_n(0) = \vartheta_{0,n},
\end{aligned} \tag{3.2.27}$$

whenever $v \in V_n$. Thanks to the initial hypotheses (3.1.1)–(3.1.3), (2.4.2)–(2.4.3) and to the regularity of A_ε shown in (3.2.24), the right-hand side of (3.2.27) is a Lipschitz continuous function from $[0, t_n]$ to $\mathbb{R}^n$. Consequently, there exists a local solution for the approximating problem $(P_{\varepsilon,n})$.

3.2.3 Global a priori estimates

In this section we obtain four a priori estimates inferred from the main equations of the approximating problem $(P_{\varepsilon,n})$ (see (3.2.12)–(3.2.15)).

First a priori estimate. We add $\upsilon\varphi_n$ to both sides of (3.2.13) and we test (3.2.12) by η_n and (3.2.13) by $\partial_t\varphi_n$, respectively. Then we sum up and integrate over Q_t, $t \in (0, T]$. We obtain that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\eta_n(t)|^2 + (\ell-\alpha)\int_{Q_t} \partial_t\varphi_n\eta_n + k\int_{Q_t} |\nabla\eta_n|^2 - k\alpha\int_{Q_t} \nabla\varphi_n\cdot\nabla\eta_n + \int_{Q_t} A_\varepsilon\eta_n\eta_n\\
&\quad + \int_{Q_t} |\partial_t\varphi_n|^2 + \frac{\upsilon}{2}\int_\Omega |\varphi_n(t)|^2 + \frac{\upsilon}{2}\int_\Omega |\nabla\varphi_n(t)|^2 + \int_{Q_t} \partial_t\widehat{\beta}_\varepsilon(\varphi_n)\\
&= \frac{1}{2}\int_\Omega |\eta_{0,n}|^2 + \frac{\upsilon}{2}\int_\Omega |\varphi_{0,n}|^2 + \frac{\upsilon}{2}\int_\Omega |\nabla\varphi_{0,n}|^2 + \int_{Q_t} (f_\varepsilon - k\Delta\eta^*)\eta_n\\
&\quad + \int_{Q_t} [\gamma\eta_n + (\upsilon - \alpha\gamma)\varphi_n + \gamma\eta^*]\partial_t\varphi_n - \int_{Q_t} \pi(\varphi_n)\partial_t\varphi_n.
\end{aligned} \tag{3.2.28}$$

To estimate the last integral on the right-hand side of (3.2.28), we observe that π is a Lipschitz continuous function with Lipschitz constant C_π. Consequently we have that

$$\begin{aligned}
|\pi(\varphi_n)| &\leq |\pi(\varphi_n) - \pi(0)| + |\pi(0)|\\
&\leq C_\pi|\varphi_n| + |\pi(0)|\\
&\leq C_1(|\varphi_n| + 1),
\end{aligned} \tag{3.2.29}$$

where $C_1 = \max\{C_\pi; |\pi(0)|\}$. Due to (2.2.5) and (3.2.29), we obtain that

$$\begin{aligned}
-\int_{Q_t} \pi(\varphi_n)\partial_t\varphi_n &\leq \int_{Q_t} |\pi(\varphi_n)\partial_t\varphi_n| \\
&\leq \int_{Q_t} C_1(|\varphi_n|+1)|\partial_t\varphi_n| \\
&\leq \frac{1}{8}\int_{Q_t} |\partial_t\varphi_n|^2 + 2C_1^2\int_{Q_t}(|\varphi_n|+1)^2 \\
&= \frac{1}{8}\int_{Q_t} |\partial_t\varphi_n|^2 + 4C_1^2\int_{Q_t}|\varphi_n|^2 + c. \qquad (3.2.30)
\end{aligned}$$

Now, we recall that A_ε is a maximal monotone operator and $A_\varepsilon(0) = 0$. Hence we have that

$$\int_{Q_t} A_\varepsilon\eta_n\eta_n \geq 0. \qquad (3.2.31)$$

Using (3.2.30)–(3.2.31), from (3.2.28) we obtain that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\eta_n(t)|^2 + k\int_{Q_t}|\nabla\eta_n|^2 + \int_{Q_t}|\partial_t\varphi_n|^2 + \frac{\upsilon}{2}\int_\Omega|\varphi_n(t)|^2 + \frac{\upsilon}{2}\int_\Omega|\nabla\varphi_n(t)|^2 + \int_\Omega\widehat{\beta}_\varepsilon(\varphi_n(t)) \\
&\leq c + \frac{1}{2}\int_\Omega|\eta_{0,n}|^2 + \frac{\upsilon}{2}\int_\Omega|\varphi_{0,n}|^2 + \frac{\upsilon}{2}\int_\Omega|\nabla\varphi_{0,n}|^2 + \int_\Omega\widehat{\beta}_\varepsilon(\varphi_{0,n}) \\
&-(\ell-\alpha)\int_{Q_t}\partial_t\varphi_n\eta_n + k\alpha\int_{Q_t}\nabla\varphi_n\cdot\nabla\eta_n + \frac{1}{8}\int_{Q_t}|\partial_t\varphi_n|^2 + 4C_1^2\int_{Q_t}|\varphi_n|^2 \\
&+\int_{Q_t}(f_\varepsilon - k\Delta\eta^*)\eta_n + \int_{Q_t}[\gamma\eta_n + (\upsilon-\alpha\gamma)\varphi_n + \gamma\eta^*]\partial_t\varphi_n. \qquad (3.2.32)
\end{aligned}$$

We notice that the convergence provided by (3.2.10) assures that $\eta_{0,n}$ and $\varphi_{0,n}$ are bounded in V. Consequently, thanks to (3.2.23), the first four integrals on the right-hand side of (3.2.32) are estimated as follows:

$$\frac{1}{2}\int_\Omega|\eta_{0,n}|^2 + \frac{\upsilon}{2}\int_\Omega|\varphi_{0,n}|^2 + \frac{\upsilon}{2}\int_\Omega|\nabla\varphi_{0,n}|^2 + \int_\Omega\widehat{\beta}_\varepsilon(\varphi_{0,n}) \leq c + Q_\varepsilon(n). \qquad (3.2.33)$$

We also notice that

$$\begin{aligned}
k\alpha\int_{Q_t}\nabla\varphi_n\cdot\nabla\eta_n &= \frac{k}{2}\Big(2\alpha\int_{Q_t}\nabla\varphi_n\cdot\nabla\eta_n\Big) \\
&\leq \frac{k}{2}\int_{Q_t}|\nabla\eta_n|^2 + \frac{k\alpha^2}{2}\int_{Q_t}|\nabla\varphi_n|^2 \\
&= \frac{k}{2}\int_{Q_t}|\nabla\eta_n|^2 + \frac{k\alpha^2}{\upsilon}\int_{Q_t}\frac{\upsilon}{2}|\nabla\varphi_n|^2. \qquad (3.2.34)
\end{aligned}$$

We re-arrange the right-hand side of (3.2.32) using (2.2.5), (3.2.33) and (3.2.34). Then we have that

$$\frac{1}{2}\|\eta_n(t)\|_H^2 + k\int_{Q_t}|\nabla\eta_n|^2 + \int_{Q_t}|\partial_t\varphi_n|^2 + \frac{\upsilon}{2}\|\varphi_n(t)\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t))$$
$$\leq c + Q_\varepsilon(n) + 2(\ell-\alpha)^2\int_{Q_t}|\eta_n|^2 + \frac{1}{8}\int_{Q_t}|\partial_t\varphi_n|^2 + \frac{k}{2}\int_{Q_t}|\nabla\eta_n|^2 + \frac{k\alpha^2}{\upsilon}\int_{Q_t}\frac{\upsilon}{2}|\nabla\varphi_n|^2$$
$$+\frac{1}{8}\int_{Q_t}|\partial_t\varphi_n|^2 + 4C_1^2\int_{Q_t}|\varphi_n|^2 + 2\int_{Q_t}|f_\varepsilon - k\Delta\eta^*|^2 + \frac{1}{8}\int_{Q_t}|\eta_n|^2$$
$$+2\int_{Q_t}|\gamma\eta_n + (\upsilon-\alpha\gamma)\varphi_n + \gamma\eta^*|^2 + \frac{1}{8}\int_{Q_t}|\partial_t\varphi_n|^2. \tag{3.2.35}$$

According to (2.5.17), f_ε is bounded in $L^2(0,T;H)$ uniformly with respect to ε. Consequently, due to (3.1.2)–(3.1.2), the seventh integral on the right-hand side of (3.2.35) is under control and similarly the third addendum in the ninth integral on the right-hand side. Then we infer that

$$\frac{1}{2}\|\eta_n(t)\|_H^2 + k\int_{Q_t}|\nabla\eta_n|^2 + \int_{Q_t}|\partial_t\varphi_n|^2 + \frac{\upsilon}{2}\|\varphi_n(t)\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t))$$
$$\leq c + Q_\varepsilon(n) + \Big[2(\ell-\alpha)^2 + \frac{1}{8}\Big]\int_{Q_t}|\eta_n|^2 + \frac{1}{2}\int_{Q_t}|\partial_t\varphi_n|^2 + \frac{k\alpha^2}{\upsilon}\int_{Q_t}\frac{\upsilon}{2}|\nabla\varphi_n|^2$$
$$+4C_1^2\int_{Q_t}|\varphi_n|^2 + 8\gamma^2\int_{Q_t}|\eta_n|^2 + 8(\upsilon-\alpha\gamma)^2\int_{Q_t}|\varphi_n|^2. \tag{3.2.36}$$

Now, we recollect the constants in (3.2.36) and obtain that

$$\frac{1}{2}\|\eta_n(t)\|_H^2 + k\int_{Q_t}|\nabla\eta_n|^2 + \int_{Q_t}|\partial_t\varphi_n|^2 + \frac{\upsilon}{2}\|\varphi_n(t)\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t))$$
$$\leq c + Q_\varepsilon(n) + C_2\frac{1}{2}\int_0^t\|\eta_n(s)\|_H^2\,ds + C_3\frac{\upsilon}{2}\int_0^t\|\nabla\varphi_n(s)\|_H^2\,ds + C_4\frac{\upsilon}{2}\int_0^t\|\varphi_n(s)\|_H^2\,ds, \tag{3.2.37}$$

where

$$C_2 = 2[2(\ell-\alpha)^2 + \frac{1}{8} + 8\gamma^2], \quad C_3 = \frac{k\alpha^2}{\upsilon}, \quad C_4 = \frac{2[4C_1^2 + 8(\upsilon-\alpha\gamma)^2]}{\upsilon}.$$

Consequently, from (3.2.37) we have that

$$\frac{1}{2}\|\eta_n(t)\|_H^2 + k\int_{Q_t}|\nabla\eta_n|^2 + \int_{Q_t}|\partial_t\varphi_n|^2 + \frac{\upsilon}{2}\|\varphi_n(t)\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t))$$
$$\leq c + Q_\varepsilon(n) + C_5\left(\frac{1}{2}\int_0^t\|\eta_n(s)\|_H^2\,ds + \frac{\upsilon}{2}\int_0^t\|\varphi_n(s)\|_V^2\,ds\right), \tag{3.2.38}$$

where

$$C_5 = \max (C_2,\ C_3,\ C_4).$$

Then, from (3.2.38) we conclude that

$$\frac{1}{2}\|\eta_n(t)\|_H^2 + k\int_{Q_t} |\nabla\eta_n|^2 + \int_{Q_t} |\partial_t\varphi_n|^2 + \frac{\upsilon}{2}\|\varphi_n(t)\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t))$$
$$\leq c_\varepsilon\left(1 + \frac{1}{2}\int_0^t \|\eta_n(s)\|_H^2\, ds + \frac{\upsilon}{2}\int_0^t \|\varphi_n(s)\|_V^2\, ds\right). \tag{3.2.39}$$

Now, we apply the Gronwall lemma to (3.2.39) and infer that

$$\frac{1}{2}\|\eta_n(t)\|_H^2 + k\int_{Q_t} |\nabla\eta_n|^2 + \int_{Q_t} |\partial_t\varphi_n|^2 + \frac{\upsilon}{2}\|\varphi_n(t)\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t)) \leq c_\varepsilon. \tag{3.2.40}$$

As (3.2.40) holds true for any $t \in [0, t_n)$, we conclude that

$$\|\varphi_n\|_{H^1(0,t_n;H)\cap L^\infty(0,t_n;V)} \leq c_\varepsilon, \tag{3.2.41}$$

$$\|\eta_n\|_{L^\infty(0,t_n;H)\cap L^2(0,t_n;V)} \leq c_\varepsilon, \tag{3.2.42}$$

$$\|\widehat{\beta}_\varepsilon(\varphi_n)\|_{L^\infty(0,t_n;L^1(\Omega))} \leq c_\varepsilon. \tag{3.2.43}$$

Second a priori estimate. First of all, we notice that $\pi(\varphi_n)$ is bounded in $L^2(0, t_n; H)$ owing to (2.4.3) and (3.2.41). Thanks to (3.2.41)–(3.2.43), we can rewrite (3.2.13) as

$$(-\upsilon\Delta\varphi_n + \beta_\varepsilon(\varphi_n), v)_H = (g_1, v)_H, \qquad \text{for all } v \in V_n, \tag{3.2.44}$$

with $\|g_1\|_{L^2(0,t_n;H)} \leq c_\varepsilon$. The choice of the basis v_i as in (3.2.8) allows us to test (3.2.44) by $-\Delta\varphi_n$. Integrating over $(0, t)$, we obtain that

$$\upsilon\int_{Q_t} |\Delta\varphi_n|^2 + \int_{Q_t} \nabla\varphi_n \cdot \nabla\beta_\varepsilon(\varphi_n) = -\int_{Q_t} g_1\Delta\varphi_n. \tag{3.2.45}$$

Using inequalities (2.2.4)–(2.2.5), from (3.2.45) we have that

$$\frac{\upsilon}{2}\int_{Q_t} |\Delta\varphi_n|^2 + \int_{Q_t} \beta_\varepsilon'(\varphi_n)|\nabla\varphi_n|^2 \leq \frac{1}{2\upsilon}\int_{Q_t} |g_1|^2. \tag{3.2.46}$$

Due to (3.2.41) and the monotonicity of β_ε, from (3.2.46) we obtain that

$$\|\Delta\varphi_n\|_{L^2(0,t;H)} \leq c_\varepsilon. \tag{3.2.47}$$

We observe that (3.2.47) holds true for any $t \in [0, t_n)$. Then, using elliptic regularity, from (3.2.41) and (3.2.47) we infer that

$$\|\varphi_n\|_{L^2(0,t_n;W)} \leq c_\varepsilon. \tag{3.2.48}$$

Third a priori estimate. Thanks to the previous a priori estimates, from (3.2.12) it follows that

$$(\partial_t\eta_n - k\Delta\eta_n + A_\varepsilon\eta_n, v)_H = (g_2, v)_H \qquad \text{for all } v \in V_n, \tag{3.2.49}$$

with $\|g_2\|_{L^2(0,t_n;H)} \le c_\varepsilon$. We test (3.2.49) by $\partial_t\eta_n$ and integrate over $(0,t)$; we obtain that

$$\int_{Q_t} |\partial_t\eta_n|^2 + \frac{k}{2}\int_\Omega |\nabla\eta_n(t)|^2 + \int_{Q_t} A_\varepsilon\eta_n\partial_t\eta_n = \frac{k}{2}\int_\Omega |\nabla\eta_{0,n}|^2 + \int_{Q_t} g_2\partial_t\eta_n. \tag{3.2.50}$$

Then, using the property (2.5.9) of A_ε and inequalities (2.2.4)–(2.2.5), from (3.2.50) we infer that

$$\int_{Q_t} |\partial_t\eta_n|^2 + \frac{k}{2}\int_\Omega |\nabla\eta_n(t)|$$

$$\begin{aligned}
&\le \frac{k}{2}\int_\Omega |\nabla\eta_{0,n}|^2 + \int_{Q_t} |A_\varepsilon\eta_n\partial_t\eta_n| + 2\int_{Q_t} |g_2|^2 + \frac{1}{8}\int_{Q_t} |\partial_t\eta_n|^2 \\
&\le \frac{k}{2}\int_\Omega |\nabla\eta_{0,n}|^2 + 2\int_{Q_t} |A_\varepsilon\eta_n|^2 + \frac{1}{8}\int_{Q_t} |\partial_t\eta_n|^2 + 2\int_{Q_t} |g_2|^2 + \frac{1}{8}\int_{Q_t} |\partial_t\eta_n|^2 \\
&= \frac{k}{2}\int_\Omega |\nabla\eta_{0,n}|^2 + 2\int_0^t \|A_\varepsilon\eta_n(s)\|_H^2\, ds + \frac{1}{4}\int_{Q_t} |\partial_t\eta_n|^2 + 2\int_{Q_t} |g_2|^2 \\
&\le \frac{k}{2}\int_\Omega |\nabla\eta_{0,n}|^2 + 2\int_0^t [C(\|\eta_n(s)\|_H + 1)]^2\, ds + \frac{1}{4}\int_{Q_t} |\partial_t\eta_n|^2 + 2\int_{Q_t} |g_2|^2 \\
&\le c + \frac{k}{2}\int_\Omega |\nabla\eta_{0,n}|^2 + 4C^2\int_0^t \|\eta_n(s)\|_H^2\, ds + \frac{1}{2}\int_{Q_t} |\partial_t\eta_n|^2 + 2\int_{Q_t} |g_2|^2.
\end{aligned} \tag{3.2.51}$$

Due to (3.1.3), the first integral on the right-hand side of (3.2.51) is under control. Then, from (3.2.51) we infer that

$$\frac{1}{2}\int_{Q_t} |\partial_t\eta_n|^2 + \frac{k}{2}\int_\Omega |\nabla\eta_n(t)| \le c + 4C^2\int_0^t \|\eta_n(s)\|_H^2\, ds + 2\int_{Q_t} |g_2|^2. \tag{3.2.52}$$

We observe that (3.2.52) holds true for any $t \in [0, t_n)$, Then, due to the previous estimates (3.2.41)–(3.2.42), we conclude that

$$\|\eta_n\|_{H^1(0,t_n;H)\cap L^\infty(0,t_n;V)} \le c_\varepsilon. \tag{3.2.53}$$

Fourth a priori estimate. Due to the previous estimates (3.2.41)–(3.2.43), (3.2.48) and (3.2.53), by comparison in (3.2.49), we infer that

$$\|\Delta\eta_n\|_{L^2(0,t_n;H)} \le c_\varepsilon. \tag{3.2.54}$$

Consequently, we conclude that

$$\|\eta_n\|_{L^2(0,t_n;W)} \le c_\varepsilon. \tag{3.2.55}$$

Summary of the a priori estimates. Since the constants appearing in the a priori estimates are all independent of t_n, the local solution can be extended to a solution defined on the whole interval $[0,T]$, i.e., we can assume $t_n = T$ for any n. Hence, due to (3.2.41)–(3.2.43), (3.2.48), (3.2.53) and (3.2.55), we conclude that

$$\|\varphi_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le c_\varepsilon, \tag{3.2.56}$$
$$\|\eta_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le c_\varepsilon. \tag{3.2.57}$$

3.2.4 Passage to the limit as $n \to +\infty$

Now, we let $n \to +\infty$ and show that the limit of some subsequences of solutions for $(P_{\varepsilon,n})$ (see (3.2.12)–(3.2.15)) yields a solution of (P_ε) (see (3.2.2)–(3.2.7)). Estimates (3.2.56)–(3.2.57) for φ_n and η_n and the well–known weak or weak* compactness results ensure the existence of a pair $(\varphi_\varepsilon, \eta_\varepsilon)$ such that, at least for a subsequence,

$$\varphi_n \rightharpoonup \varphi_\varepsilon \quad \text{in} \quad H^1(0,T;H)\cap L^2(0,T;W), \tag{3.2.58}$$
$$\varphi_n \rightharpoonup^* \varphi_\varepsilon \quad \text{in} \quad L^\infty(0,T;V), \tag{3.2.59}$$
$$\eta_n \rightharpoonup \eta_\varepsilon \quad \text{in} \quad H^1(0,T;H)\cap L^2(0,T;W), \tag{3.2.60}$$
$$\eta_n \rightharpoonup^* \eta_\varepsilon \quad \text{in} \quad L^\infty(0,T;V), \tag{3.2.61}$$

as $n \to +\infty$. We notice that W, V, H are Banach spaces and $W \subset V \subset H$ with dense and compact embeddings. Then, we are under the assumptions of [62, Prop. 4, Sec. 8] and this fact implies the following strong convergences:

$$\varphi_n \to \varphi_\varepsilon \quad \text{in} \quad C^0([0,T];H)\cap L^2(0,T;V), \tag{3.2.62}$$
$$\eta_n \to \eta_\varepsilon \quad \text{in} \quad C^0([0,T];H)\cap L^2(0,T;V), \tag{3.2.63}$$

as $n \to +\infty$. Since π, A_ε and β_ε are Lipschitz continuous, we infer that

$$|\pi(\varphi_n) - \pi(\varphi_\varepsilon)| \le C_\pi |\varphi_n - \varphi_\varepsilon| \qquad \text{a.e. in } Q, \tag{3.2.64}$$

$$\|A_\varepsilon \eta_n - A_\varepsilon \eta_\varepsilon\|_H \le \frac{1}{\varepsilon}\|\eta_n - \eta_\varepsilon\|_H \qquad \text{a.e. in } [0,T], \tag{3.2.65}$$

$$|\beta_\varepsilon(\varphi_n) - \beta_\varepsilon(\varphi_\varepsilon)| \le \frac{1}{\varepsilon}|\varphi_n - \varphi_\varepsilon| \qquad \text{a.e. in } Q. \tag{3.2.66}$$

Due to (3.2.64)–(3.2.64), we conclude that

$$\pi(\varphi_n) \to \pi(\varphi_\varepsilon) \quad \text{in} \quad C^0([0,T];H), \tag{3.2.67}$$
$$A_\varepsilon \eta_n \to A_\varepsilon \eta_\varepsilon \quad \text{in} \quad C^0([0,T];H), \tag{3.2.68}$$
$$\beta_\varepsilon(\varphi_n) \to \beta_\varepsilon(\varphi_\varepsilon) \quad \text{in} \quad C^0([0,T];H), \tag{3.2.69}$$

as $n \to +\infty$. Now, we fix $k \le n$ and we observe that, for every $v \in V_k$ and for every $t \in [0,T]$, the solution (η_n, φ_n) of problem $(P_{\varepsilon,n})$ satisfies

$$(\partial_t[\eta_n(t) + (\ell - \alpha)\varphi_n(t)] - k\Delta\eta_n(t) + k\alpha\Delta\varphi_n(t) + A_\varepsilon\eta_n(t), v)_H$$

$$= (f_\varepsilon(t) - k\Delta\eta^*, v)_H, \tag{3.2.70}$$

$$(\partial_t\varphi_n(t) - \upsilon\Delta\varphi_n(t) + \beta_\varepsilon(\varphi_n(t)) + \pi(\varphi_n(t)), v)_H$$

$$= (\gamma[\eta_n(t) - \alpha\varphi_n(t) + \eta^*], v)_H. \tag{3.2.71}$$

If k is fixed and $n \to +\infty$, we have the convergence of every term of (3.2.70)–(3.2.71) to the corresponding one with η_ε, φ_ε whenever $v \in V_k$, i.e.,

$$(\partial_t[\eta_\varepsilon(t) + (\ell - \alpha)\varphi_\varepsilon(t)] - k\Delta\eta_\varepsilon(t) + k\alpha\Delta\varphi_\varepsilon(t) + A_\varepsilon\eta_\varepsilon(t), v)_H$$

$$= (f_\varepsilon(t) - k\Delta\eta^*, v)_H, \tag{3.2.72}$$

$$(\partial_t\varphi_\varepsilon(t) - \upsilon\Delta\varphi_\varepsilon(t) + \beta_\varepsilon(\varphi_\varepsilon(t)) + \pi(\varphi_\varepsilon(t)), v)_H$$

$$= (\gamma[\eta_\varepsilon(t) - \alpha\varphi_\varepsilon(t) + \eta^*], v)_H. \tag{3.2.73}$$

As k is arbitrary, the limit equalities hold true for every $v \in \bigcup_{k=1}^\infty V_k$, which is dense in V. Then the limit equalities actually hold for every $v \in V$, i.e.,

$$\partial_t(\eta_\varepsilon + (\ell - \alpha)\varphi_\varepsilon) - k\Delta\eta_\varepsilon + k\alpha\Delta\varphi_\varepsilon + A_\varepsilon\eta_\varepsilon = f_\varepsilon - k\Delta\eta^* \quad \text{a.e. in } Q, \tag{3.2.74}$$

$$\partial_t\varphi_\varepsilon - \upsilon\Delta\varphi_\varepsilon + \beta_\varepsilon(\varphi_\varepsilon) + \pi(\varphi_\varepsilon) = \gamma(\eta_\varepsilon - \alpha\varphi_\varepsilon + \eta^*) \quad \text{a.e. in } Q. \tag{3.2.75}$$

Now, we prove the convergence of the initial data. We recall that

$$\eta_{0,n} = P_{V_n}\eta_0, \qquad \varphi_{0,n} = P_{V_n}\varphi_0. \tag{3.2.76}$$

If ε is fixed, then

$$\lim_{n\to+\infty} \eta_{0,n} = \eta_0 \quad \text{in } V, \tag{3.2.77}$$

$$\lim_{n\to+\infty} \varphi_{0,n} = \varphi_0 \quad \text{in } V, \tag{3.2.78}$$

and then also in H. These observations and (3.2.62)–(3.2.63) show that the weak limit of some subsequences of solutions for $(P_{\varepsilon,n})$ (see (3.2.12)–(3.2.15)) yields a solution for (P_ε) (see (3.2.2)–(3.2.7)). We also notice that taking the limit as $n \to +\infty$ in (3.2.23) entails that $Q_\varepsilon(n) \to C$, with

$$\int_\Omega \tilde{\beta}_\varepsilon(\varphi_0) \le C. \tag{3.2.79}$$

Then, after the first passage to the limit, we conclude that estimates (3.2.56)–(3.2.57) still hold for the limiting functions with constants independent of ε, i.e.,

$$\|\varphi_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le c, \tag{3.2.80}$$

$$\|\eta_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le c. \tag{3.2.81}$$

3.2.5 Passage to the limit as $\varepsilon \searrow 0$

Now, we let $\varepsilon \searrow 0$ and show that the limit of some subsequences of solutions for (P_ε) (see (3.2.2)–(3.2.7)) tends to a solution of the initial problem (P) (see (3.1.5)–(3.1.10)). First of all, due to (3.2.58)–(3.2.63), (3.2.69) and (3.2.79), we have that the constants in (3.2.80)–(3.2.81) do not depend on ε. Moreover, thanks to (3.2.80)–(3.2.81), by comparison in (3.2.75), we infer that

$$\|\beta_\varepsilon(\varphi_\varepsilon)\|_{L^2(Q)} \le c. \tag{3.2.82}$$

The well–known weak or weak* compactness results and the useful theorem [62, Prop. 4, Sec. 8] ensure the existence of a pair (φ, η) such that, at least for a subsequence,

$$\varphi_\varepsilon \rightharpoonup^* \varphi \quad \text{in} \quad H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{3.2.83}$$

$$\eta_\varepsilon \rightharpoonup^* \eta \quad \text{in} \quad H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{3.2.84}$$

$$\varphi_\varepsilon \to \varphi \quad \text{in} \quad C^0([0,T];H) \cap L^2(0,T;V), \tag{3.2.85}$$

$$\eta_\varepsilon \to \eta \quad \text{in} \quad C^0([0,T];H) \cap L^2(0,T;V), \tag{3.2.86}$$

as $\varepsilon \searrow 0$. Now, we observe that (3.2.85) implies that

$$\varphi_\varepsilon \to \varphi \qquad \text{in } L^2(0,T;H) \equiv L^2(Q) \tag{3.2.87}$$

as $\varepsilon \searrow 0$. We set $\xi_\varepsilon = \beta_\varepsilon(\varphi_\varepsilon)$ and remark that

$$\|\xi_\varepsilon\|_{L^2(Q)} = \|\beta_\varepsilon(\varphi_\varepsilon)\|_{L^2(Q)} \le c. \tag{3.2.88}$$

Thus, we may suppose that, as $\varepsilon \searrow 0$, at least for a subsequence,

$$\xi_\varepsilon \rightharpoonup \xi \qquad \text{in } L^2(Q), \tag{3.2.89}$$

for some $\xi \in L^2(Q)$. Now, we introduce the operator $\mathcal{B}_\varepsilon$ induced by β_ε on $L^2(Q)$ in the following way:

$$\mathcal{B}_\varepsilon : L^2(Q) \longrightarrow L^2(Q) \tag{3.2.90}$$

$$\xi_\varepsilon \in \mathcal{B}_\varepsilon(\varphi_\varepsilon) \Longleftrightarrow \xi_\varepsilon(x,t) \in \beta_\varepsilon(\varphi_\varepsilon(x,t)) \quad \text{for a.e. } (x,t) \in Q. \tag{3.2.91}$$

Due to (3.2.87) and (3.2.89), we have that

$$\begin{cases} \mathcal{B}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \xi & \text{in } L^2(Q), \\ \varphi_\varepsilon \to \varphi & \text{in } L^2(Q), \end{cases} \tag{3.2.92}$$

$$\limsup_{\varepsilon \searrow 0} \int_Q \xi_\varepsilon \varphi_\varepsilon = \int_Q \xi \varphi. \tag{3.2.93}$$

Thanks to (3.2.92)–(3.2.93) and to the useful results proved in [1, Prop. 2.2, p. 38], we conclude that

$$\xi \in \mathcal{B}(\varphi) \quad \text{in } L^2(Q), \tag{3.2.94}$$

where $\mathcal{B}$ is defined by (2.4.4)–(2.4.5). This is equivalent to say that

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q. \tag{3.2.95}$$

Moreover, we pass to the limit in A_ε by repeating the previous arguments and conclude that

$$\zeta \in \mathcal{A}(\eta) \quad \text{in } L^2(0,T;H), \tag{3.2.96}$$

with obvious definition for $\mathcal{A}$ (see (2.4.12)–(2.4.13)), and this is equivalent to say that

$$\zeta \in A(\eta) \quad \text{a.e. in } [0,T]. \tag{3.2.97}$$

Conclusion of the proof. Thanks to the previous steps, we conclude that, as $\varepsilon \searrow 0$, the limit of some subsequences of solutions $(\eta_\varepsilon, \varphi_\varepsilon)$ to (P_ε) (see (3.2.2)–(3.2.7)) yields a solution (η, φ) of the initial boundary value problem (P), i.e.,

$$\partial_t(\eta + (\ell - \alpha)\varphi) - k\Delta\eta + k\alpha\Delta\varphi + \zeta = f - k\Delta\eta^* \;\text{ a.e. in } Q, \tag{3.2.98}$$

$$\partial_t\varphi - \upsilon\Delta\varphi + \xi + \pi(\varphi) = \gamma(\eta - \alpha\varphi + \eta^*) \;\text{ a.e. in } Q, \tag{3.2.99}$$

$$\zeta(t) \in A(\eta(t)) \;\text{ for a.e. } t \in (0,T), \tag{3.2.100}$$

$$\xi \in \beta(\varphi) \;\text{ a.e. in } Q, \tag{3.2.101}$$

$$\partial_\nu\eta = 0, \qquad \partial_\nu\varphi = 0 \quad \text{on } \Sigma, \tag{3.2.102}$$

$$\eta(0) = \eta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{3.2.103}$$

We notice that the homogeneous Neumann boundary conditions for both η and φ follow from (3.1.4), due to the definition of W (see (2.1.1)).

3.3 Proof of the continuous dependence theorem

This section is devoted to the proof of Theorem 3.1.2.

Assume $\alpha = \ell$. If f_i, η_i^*, η_{0_i}, φ_{0_i}, $i = 1,2$, are given as in (3.1.2)–(3.1.3) and (φ_i, η_i), $i = 1,2$, are the corresponding solutions, we can write problem (3.1.5)–(3.1.10) for both (φ_i, η_i), $i = 1,2$, obtaining

$$\partial_t\eta_i - k\Delta\eta_i + k\ell\Delta\varphi_i + \zeta_i = f_i - k\Delta\eta_i^* \;\text{ a.e. in } Q, \tag{3.3.1}$$

$$\partial_t\varphi_i - \upsilon\Delta\varphi_i + \xi_i + \pi(\varphi_i) = \gamma(\eta_i - \ell\varphi_i + \eta_i^*) \;\text{ a.e. in } Q, \tag{3.3.2}$$

$$\zeta_i(t) \in A(\eta_i(t)) \;\text{ for a.e. } t \in (0,T), \tag{3.3.3}$$

$$\xi_i \in \beta(\varphi_i) \;\text{ a.e. in } Q, \tag{3.3.4}$$

$$\partial_\nu\eta_i = 0, \qquad \partial_\nu\varphi_i = 0 \quad \text{on } \Sigma, \tag{3.3.5}$$

$$\eta_i(0) = \eta_{0_i}, \qquad \varphi_i(0) = \varphi_{0_i}. \tag{3.3.6}$$

First of all, we set

$$\varphi = \varphi_1 - \varphi_2, \quad \eta = \eta_1 - \eta_2, \tag{3.3.7}$$
$$f = f_1 - f_2, \quad \eta^* = \eta_1^* - \eta_2^*, \tag{3.3.8}$$
$$\varphi_0 = \varphi_{0_1} - \varphi_{0_2}, \quad \eta_0 = \eta_{0_1} - \eta_{0_2}. \tag{3.3.9}$$

We write (3.3.1) for both (φ_1, η_1) and (φ_2, η_2) and we take the difference. We obtain that

$$\partial_t \eta - k\Delta\eta + k\ell\Delta\varphi + \zeta_1 - \zeta_2 = f - k\Delta\eta^*. \tag{3.3.10}$$

We write (3.3.2) for both (φ_1, η_1) and (φ_2, η_2) and we take the difference. We obtain that

$$\partial_t \varphi - \upsilon\Delta\varphi + \xi_1 - \xi_2 + \pi(\varphi_1) - \pi(\varphi_2) = \gamma(\eta - \ell\varphi + \eta^*). \tag{3.3.11}$$

We multiply (3.3.10) by η and (3.3.11) by $\frac{kl^2}{\upsilon}\varphi$. Then we sum up and integrate over Q_t, $t \in (0, T]$. We have that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\eta(t)|^2 + \frac{k\ell^2}{2\upsilon}\int_\Omega |\varphi(t)|^2 + k\int_{Q_t}(|\nabla\eta|^2 - \ell\nabla\varphi\nabla\eta + \ell^2|\nabla\varphi|^2) \\
&+ \int_{Q_t}(\zeta_1 - \zeta_2)(\eta_1 - \eta_2) + \frac{k\ell^2}{\upsilon}\int_{Q_t}(\xi_1 - \xi_2)(\varphi_1 - \varphi_2) \\
&= \frac{1}{2}\|\eta_0\|_H^2 + \frac{k\ell^2}{2\upsilon}\|\varphi_0\|_H^2 - \frac{k\ell^2}{\upsilon}\int_{Q_t}[\pi(\varphi_1) - \pi(\varphi_2)](\varphi_1 - \varphi_2) \\
&+ \int_{Q_t}(f - k\Delta\eta^*)\eta + \frac{\gamma k\ell^2}{\upsilon}\int_{Q_t}\eta\varphi - \frac{\gamma k\ell^3}{\upsilon}\int_{Q_t}|\varphi|^2 + \frac{\gamma k\ell^2}{\upsilon}\int_{Q_t}\eta^*\varphi.
\end{aligned} \tag{3.3.12}$$

Since A and β are maximal monotone, we have that

$$\int_{Q_t}(\zeta_1 - \zeta_2)(\eta_1 - \eta_2) \geq 0, \tag{3.3.13}$$

$$\int_{Q_t}(\xi_1 - \xi_2)(\varphi_1 - \varphi_2) \geq 0. \tag{3.3.14}$$

Moreover, thanks to the Lipschitz continuity of π, we infer that

$$\begin{aligned}
-\frac{k\ell^2}{\upsilon}\int_{Q_t}[\pi(\varphi_1) - \pi(\varphi_2)](\varphi_1 - \varphi_2) &\leq \frac{k\ell^2}{\upsilon}\int_{Q_t}|\pi(\varphi_1) - \pi(\varphi_2)||\varphi_1 - \varphi_2| \\
&\leq \frac{k\ell^2 C_\pi}{\upsilon}\int_{Q_t}|\varphi|^2.
\end{aligned} \tag{3.3.15}$$

We notice that the integral involving the gradients in (3.3.12) is estimated from below in this way:

$$\int_{Q_t}(|\nabla\eta|^2 - \ell\nabla\varphi\nabla\eta + \ell^2|\nabla\varphi|^2) \geq \frac{1}{2}\int_{Q_t}(|\nabla\eta|^2 + \ell^2|\nabla\varphi|^2). \tag{3.3.16}$$

We also observe that

$$-\frac{\gamma k\ell^3}{\upsilon}\int_{Q_t}|\varphi|^2 \le 0. \tag{3.3.17}$$

Then, due to (3.3.13)–(3.3.17), from (3.3.12) we infer that

$$\frac{1}{2}\int_\Omega |\eta(t)|^2 + \frac{k\ell^2}{2\upsilon}\int_\Omega |\varphi(t)|^2 + \frac{1}{2}\int_{Q_t}(|\nabla\eta|^2 + \ell^2|\nabla\varphi|^2)$$

$$\le \frac{1}{2}\|\eta_0\|_H^2 + \frac{k\ell^2 C_\pi}{\upsilon}\int_{Q_t}|\varphi|^2 + \frac{k\ell^2}{2\upsilon}\|\varphi_0\|_H^2 + \int_{Q_t}(f - k\Delta\eta^*)\eta + \frac{\gamma k\ell^2}{\upsilon}\int_{Q_t}\eta\varphi + \frac{\gamma k\ell^2}{\upsilon}\int_{Q_t}\eta^*\varphi.$$

By applying the inequality (2.2.4) to the last three terms of the right-hand side of the previous equation, we obtain that

$$\frac{1}{2}\|\eta(t)\|_H^2 + \frac{k\ell^2}{2\upsilon}\|\varphi(t)\|_H^2 + \frac{1}{2}\int_{Q_t}(|\nabla\eta|^2 + \ell^2|\nabla\varphi|^2)$$

$$\le \frac{1}{2}\|\eta_0\|_H^2 + \frac{k\ell^2}{2\upsilon}\|\varphi_0\|_H^2 + \frac{1}{8}\int_{Q_t}|\eta|^2 + 2\int_{Q_t}|f - k\Delta\eta^*|^2 + \frac{1}{8}\int_{Q_t}|\eta|^2$$

$$+2\Big(\frac{\gamma k\ell^2}{\upsilon}\Big)^2\int_{Q_t}|\varphi|^2 + \frac{1}{8}\int_{Q_t}|\eta^*|^2 + 2\Big(\frac{\gamma k\ell^2}{\upsilon}\Big)^2\int_{Q_t}|\varphi|^2 + \frac{k\ell^2 C_\pi}{\upsilon}\int_{Q_t}|\varphi|^2. \tag{3.3.18}$$

From (3.3.18) we infer that

$$\frac{1}{2}\|\eta(t)\|_H^2 + \frac{k\ell^2}{2\upsilon}\|\varphi(t)\|_H^2 + \frac{1}{2}\int_{Q_t}(|\nabla\eta|^2 + \ell^2|\nabla\varphi|^2)$$

$$\le \frac{1}{2}\|\eta_0\|_H^2 + \frac{k\ell^2}{2\upsilon}\|\varphi_0\|_H^2 + 4\|f\|_{L^2(Q)}^2 + 4k^2T\|\eta^*\|_W^2 + \frac{1}{8}T\|\eta^*\|_H^2$$

$$+M\int_0^t\left(\frac{1}{2}\|\eta(s)\|_H^2 + \frac{k\ell^2}{2\upsilon}\|\varphi(s)\|_H^2 + \frac{1}{2}\int_{Q_s}(|\nabla\eta|^2 + \ell^2|\nabla\varphi|^2)\right)ds, \tag{3.3.19}$$

where

$$M = \max\left(\frac{4\gamma^2k\ell^2 + 2\upsilon C_\pi}{\upsilon}; \frac{1}{2}\right).$$

From (3.3.19), by applying the Gronwall lemma, we conclude that

$$\frac{1}{2}\|\eta(t)\|_H^2 + \frac{k\ell^2}{2\upsilon}\|\varphi(t)\|_H^2 + \frac{1}{2}\|\nabla\eta\|_{L^2(0,t;H)}^2 + \frac{\ell^2}{2}\|\nabla\varphi\|_{L^2(0,t;H)}^2$$

$$\le C_1\left[4\|f\|_{L^2(Q)}^2 + 4k^2T\|\eta^*\|_W^2 + \frac{1}{8}T\|\eta^*\|_W^2 + C_0\Big(\|\eta_0\|_H^2 + \|\varphi_0\|_H^2\Big)\right], \tag{3.3.20}$$

where

$$C_0 = \max\left(\frac{1}{2};\ \frac{k\ell^2}{2\upsilon}\right), \qquad C_1 = e^{TM}.$$

From (3.3.20), we infer that

$$C_3\Big(\|\eta(t)\|_H^2+\|\varphi(t)\|_H^2+\|\nabla\eta\|_{L^2(0,t;H)}^2+\|\nabla\varphi\|_{L^2(0,t;H)}^2\Big)$$

$$\leq \frac{1}{2}\|\eta(t)\|_H^2+\frac{k\ell^2}{2\upsilon}\|\varphi(t)\|_H^2+\frac{1}{2}\|\nabla\eta\|_{L^2(0,t;H)}^2+\frac{\ell^2}{2}\|\nabla\varphi\|_{L^2(0,t;H)}^2$$

$$\leq C_2\bigg(\|f\|_{L^2(Q)}^2+\|\eta^*\|_W^2+\|\eta_0\|_H^2+\|\varphi_0\|_H^2\bigg)$$

$$\leq C_2\bigg(\|f\|_{L^2(Q)}+\|\eta^*\|_W+\|\eta_0\|_H+\|\varphi_0\|_H\bigg)^2, \tag{3.3.21}$$

where

$$C_2=\max\bigg(4C_1;\ 4k^2TC_1;\ \frac{1}{8}TC_1;\ C_1C_0\bigg),\qquad C_3=\min\bigg(\frac{1}{2};\ \frac{k\ell^2}{2\upsilon};\ \frac{\ell^2}{2}\bigg).$$

From (3.3.21) we obtain that

$$\|\eta(t)\|_H^2+\|\varphi(t)\|_H^2+\|\nabla\eta\|_{L^2(0,t;H)}^2+\|\nabla\varphi\|_{L^2(0,t;H)}^2$$

$$\leq C_4\bigg(\|f\|_{L^2(Q)}+\|\eta^*\|_W+\|\eta_0\|_H+\|\varphi_0\|_H\bigg)^2, \tag{3.3.22}$$

where $C_4=\frac{C_2}{C_3}$. From (3.3.22) we conclude that there exists a constant $C>0$ which depends only on Ω, T depends only on Ω, T and the parameters ℓ, α, k, υ, γ of the system, such that

$$\|\eta_1-\eta_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}+\|\varphi_1-\varphi_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}$$

$$\leq C\big(\|f_1-f_2\|_{L^2(Q)}+\|\eta_1^*-\eta_2^*\|_W+\|\eta_{0_1}-\eta_{0_2}\|_H+\|\varphi_{0_1}-\varphi_{0_2}\|_H\big). \tag{3.3.23}$$

To infer the uniqueness of the solution, we choose $f_1=f_2$, $\eta_1^*=\eta_2^*$, $\varphi_{0_1}=\varphi_{0_2}$, $\eta_{0_1}=\eta_{0_2}$. Then, replacing the corresponding values in (3.3.23), we obtain that

$$\|\eta_1-\eta_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}+\|\varphi_1-\varphi_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}=0. \tag{3.3.24}$$

Hence $\eta_1=\eta_2$ and $\varphi_1=\varphi_2$. Then the solution of problem (P) (see (3.3.1)–(3.3.6)) is unique.

Chapter 4

On a class of conserved phase–field systems with a maximal monotone perturbation

In this chapter we prove existence and regularity for the solutions to a Cahn–Hilliard system describing the phenomenon of phase separation for a material contained in a bounded and regular domain. Since the first equation of the system is perturbed by the presence of an additional maximal monotone operator, we show our results using suitable regularization of the nonlinearities of the problem and performing some a priori estimates which allow us to pass to the limit thanks to compactness and monotonicity arguments. Next, under further assumptions, we deduce a continuous dependence estimate whence the uniqueness property is also achieved. Then, we consider the related sliding mode control (SMC) problem and show that the chosen SMC law forces a suitable linear combination of the temperature and the phase to reach a given (space-dependent) value within finite time.

4.1 Setting of the problem and results

We assume that

$$\ell,\ \upsilon,\ \gamma \in (0,+\infty), \qquad a,\ b \in \mathbb{R}, \tag{4.1.1}$$

$$f \in L^2(0,T,H), \tag{4.1.2}$$

$$\eta^* \in W, \quad \vartheta_0 \in H, \quad \varphi_0 \in V, \quad \tilde{\beta}(\varphi_0) \in L^1(\Omega), \quad m(\varphi_0) =: m_0 \in \operatorname{int}(D(\beta)). \tag{4.1.3}$$

We look for a triplet $(\vartheta, \varphi, \mu)$ satisfying at least the regularity requirements

$$\vartheta \in H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V), \tag{4.1.4}$$

$$\varphi \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{4.1.5}$$

$$\mu \in L^2(0,T;V), \tag{4.1.6}$$

and solving the Problem (P), that is,

$$\partial_t(\vartheta + \ell\varphi) - \Delta\vartheta + \zeta = f \ \text{ a.e. in } Q, \tag{4.1.7}$$

$$\partial_t\varphi - \Delta\mu = 0 \ \text{ a.e. in } Q, \tag{4.1.8}$$

$$\mu = -\upsilon\Delta\varphi + \xi + \pi(\varphi) - \gamma\vartheta \ \text{ a.e. in } Q, \tag{4.1.9}$$

$$\zeta(t) \in A(a\vartheta(t) + b\varphi(t) - \eta^*) \text{ for a.e. } t \in (0,T), \tag{4.1.10}$$

$$\xi \in \beta(\varphi) \text{ a.e. in } Q, \tag{4.1.11}$$

$$\partial_\nu\vartheta = \partial_\nu\varphi = \partial_\nu\mu = 0 \quad \text{on } \Sigma, \tag{4.1.12}$$

$$\vartheta(0) = \vartheta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{4.1.13}$$

Theorem 4.1.1 (Existence)**.** *If* (2.4.2)–(2.4.3), (2.4.9)–(2.4.11) *and* (4.1.1)–(4.1.3) *hold, then Problem* (P) *(see* (4.1.7)–(4.1.13)*) has at least one solution* (ϑ,φ,μ) *satisfying* (4.1.4)–(4.1.6)*.*

Theorem 4.1.2 (Regularity)**.** *Assume* (2.4.2)–(2.4.3), (2.4.9)–(2.4.11), (4.1.1)–(4.1.2),

$$\eta^* \in W, \quad \vartheta_0 \in V, \quad \varphi_0 \in W, \quad \beta^0(\varphi_0) \in H, \quad m_0 \in \operatorname{int}(D(\beta)) \tag{4.1.14}$$

and that there exists $\varepsilon_0 \in (0,1]$ *such that*

$$\|-\upsilon\Delta\varphi_0 + \beta_\varepsilon(\varphi_0) + \pi(\varphi_0) - \gamma\vartheta_0\|_V \le c \qquad \text{for every } \varepsilon \in (0,\varepsilon_0], \tag{4.1.15}$$

for some positive constant c*, where* β_ε *is the Yosida regularization of* β *(see* (2.5.1)*). Then Problem* (P) *(see* (4.1.7)–(4.1.13)*) has at least one solution* (ϑ,φ,μ) *satisfying*

$$\vartheta \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{4.1.16}$$

$$\varphi \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{4.1.17}$$

$$\mu \in L^\infty(0,T;V) \cap L^2(0,T;W). \tag{4.1.18}$$

Remark. We fix $t \in (0,T)$ and integrate (4.1.8) over Ω. We infer that

$$\int_\Omega \partial_t\varphi(t) - \int_\Omega \Delta\mu(t) = 0. \tag{4.1.19}$$

Integrating by parts the second term of the left-hand side of (4.1.19), we obtain that

$$\frac{d}{dt}\int_\Omega \varphi(t) = 0. \tag{4.1.20}$$

Consequently, recalling the definition of m stated by (2.4.21)–(2.4.22), we conclude that

$$m(\varphi(t)) = \frac{1}{|\Omega|}\int_\Omega \varphi(t) = \frac{1}{|\Omega|}\int_\Omega \varphi_0 = m(\varphi_0) =: m_0 \quad \text{for every } t \in (0,T). \tag{4.1.21}$$

Change of variables. In the following it we will be useful to consider the equivalent modified form of the initial Problem (P) (see (4.1.7)–(4.1.13)). We make a change of variables and set

$$\eta = a\vartheta + b\varphi - \eta^*, \qquad \eta_0 = a\vartheta_0 + b\varphi_0 - \eta^*. \tag{4.1.22}$$

Due to (4.1.22), from (4.1.7)–(4.1.13) we obtain the modified problem $(\widetilde{P})$:

$$\partial_t(\eta + (a\ell - b)\varphi) - \Delta\eta + b\Delta\varphi - \Delta\eta^* + a\zeta = af \quad \text{a.e. in } Q, \tag{4.1.23}$$

$$\partial_t\varphi - \Delta\mu = 0 \quad \text{a.e. in } Q, \tag{4.1.24}$$

$$\mu = -\upsilon\Delta\varphi + \xi + \pi(\varphi) - \frac{\gamma}{a}(\eta - b\varphi + \eta^*) \quad \text{a.e. in } Q, \tag{4.1.25}$$

$$\zeta(t) \in A(\eta(t)) \text{ for a.e. } t \in (0,T), \tag{4.1.26}$$

$$\xi \in \beta(\varphi) \text{ a.e. in } Q, \tag{4.1.27}$$

$$\partial_\nu\eta = \partial_\nu\varphi = \partial_\nu\mu = 0 \quad \text{on } \Sigma, \tag{4.1.28}$$

$$\eta(0) = \eta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{4.1.29}$$

Theorem 4.1.3 (Uniqueness and continuous dependence)**.** *Assume* (2.4.2)–(2.4.3), (2.4.9)–(2.4.11) *and* (4.1.1)–(4.1.3)*. If a, $b > 0$ and $a\ell = b$, then the solution (η, φ, μ) of problem $(\widetilde{P})$ (see* (4.1.23)–(4.1.29)*) is unique. Moreover, we assume that f_i, η_i^*, η_{0_i}, φ_{0_i}, $i = 1, 2$, are given as in* (4.1.2)–(4.1.3) *and $(\eta_i, \varphi_i, \mu_i)$, $i = 1, 2$, are the corresponding solutions. If*

$$m(\varphi_{0_1}) = m(\varphi_{0_2}), \tag{4.1.30}$$

then the estimate

$$\|\eta_1 - \eta_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\varphi_1 - \varphi_2\|_{L^\infty(0,T;V^*)\cap L^2(0,T;V)}$$
$$\leq c\big(\|\varphi_{0_1} - \varphi_{0_2}\|_{V^*} + \|\eta_{0_1} - \eta_{0_2}\|_H + \|f_1 - f_2\|_{L^2(0,T;H)} + \|\eta_1^* - \eta_2^*\|_W\big) \tag{4.1.31}$$

holds true for some constant c that depends only on Ω, T and the structure (2.4.2)–(2.4.3), (2.4.9)–(2.4.11) *and* (4.1.1)–(4.1.3) *of the system.*

Theorem 4.1.4 (Sliding mode control)**.** *Assume* (2.4.2)–(2.4.3), (2.4.9)–(2.4.11), (4.1.1), *$a = 1$, $b = \ell$ and*

$$f \in L^\infty(0,T,H), \tag{4.1.32}$$

$$\eta^* \in W, \quad \vartheta_0 \in V, \quad \varphi_0 \in W, \quad \beta^0(\varphi_0) \in H, \quad m_0 \in \operatorname{int}(D(\beta)). \tag{4.1.33}$$

We consider $A = \rho\,\mathrm{Sign}$, where ρ is a positive coefficient, Sign *is defined as in* (2.4.15) *and σ is an element of the range of* Sign*, i.e.,*

$$\sigma(t) \in \mathrm{Sign}(\vartheta(t) + \ell\varphi(t) - \eta^*) \textit{ for a.e. } t \in (0,T), \tag{4.1.34}$$

Then, for some $\rho^* > 0$ and for every $\rho > \rho^*$, there exists a solution $(\vartheta, \varphi, \mu)$ to Problem (P) (see (4.1.7)–(4.1.13)*) and a time T^* such that, for every $t \in [T^*, T]$*

$$\vartheta(t) + \ell\varphi(t) = \eta^* \qquad \textit{a.e. in } \Omega. \tag{4.1.35}$$

4.2 Existence - The approximating problem (P_ε)

The following three sections are devoted to the proof of the existence Theorem 4.1.1.

Regularization of the initial data. We denote by $\vartheta_{0\varepsilon}$ and $\varphi_{0\varepsilon}$ the regularization of the initial data ϑ_0 and φ_0, respectively, obtained solving the following elliptic problems:

$$\begin{cases} \vartheta_{0\varepsilon} - \varepsilon\Delta\vartheta_{0\varepsilon} = \vartheta_0 & \text{in } \Omega, \\ \partial_\nu \vartheta_{0\varepsilon} = 0 & \text{on } \Gamma. \end{cases} \tag{4.2.1}$$

$$\begin{cases} \varphi_{0\varepsilon} - \varepsilon\Delta\varphi_{0\varepsilon} = \varphi_0 & \text{in } \Omega, \\ \partial_\nu \varphi_{0\varepsilon} = 0 & \text{on } \Gamma. \end{cases} \tag{4.2.2}$$

Since $\vartheta_0 \in H$ and $\varphi_0 \in V$, by elliptic regularity we infer that $\vartheta_{0\varepsilon} \in W$ and $\varphi_{0\varepsilon} \in W \cap H^3(\Omega)$. Moreover, integrating over Ω the first equation of (4.2.2), we obtain that

$$m_0 = \frac{1}{|\Omega|}\int_\Omega \varphi_0 = \frac{1}{|\Omega|}\int_\Omega \varphi_{0\varepsilon} =: m_{0\varepsilon}. \tag{4.2.3}$$

From (4.1.3) and (4.1.21) it immediately follows that $m_{0\varepsilon} \in \operatorname{int}(D(\beta))$. Since β is maximal monotone, testing the first equation of (4.2.2) by $\beta_\varepsilon(\varphi_{0\varepsilon})$ and integrating over Ω, we have that

$$\int_\Omega (\varphi_{0\varepsilon} - \varphi_0)\beta_\varepsilon(\varphi_{0\varepsilon}) = -\varepsilon \int_\Omega |\nabla\varphi_{0\varepsilon}|^2 \beta'_\varepsilon(\varphi_{0\varepsilon}) \le 0. \tag{4.2.4}$$

Recalling that β_ε is the subdifferential of $\tilde\beta_\varepsilon$, from (4.2.4) we infer that

$$\int_\Omega \tilde\beta_\varepsilon(\varphi_{0\varepsilon}) - \int_\Omega \tilde\beta_\varepsilon(\varphi_0) \le \int_\Omega (\varphi_{0\varepsilon} - \varphi_0)\beta_\varepsilon(\varphi_{0\varepsilon}) \le 0. \tag{4.2.5}$$

Consequently, due to (4.1.3), (2.5.4), (4.2.5) and the definition of $\tilde\beta_\varepsilon$, we conclude that

$$0 \le \int_\Omega \tilde\beta_\varepsilon(\varphi_{0\varepsilon}) \le \int_\Omega \tilde\beta_\varepsilon(\varphi_0) \le \int_\Omega \tilde\beta(\varphi_0) < +\infty, \tag{4.2.6}$$

whence there exists a positive constant c, independent of ε, such that $\|\tilde\beta(\varphi_{0\varepsilon})\|_{L^1(\Omega)} \le c$. Now, we test (4.2.1) by $\vartheta_{0\varepsilon}$ and integrate over Ω. We obtain that

$$\int_\Omega |\vartheta_{0\varepsilon}|^2 + \varepsilon\int_\Omega |\nabla\vartheta_{0\varepsilon}|^2 = \int_\Omega \vartheta_0\vartheta_{0\varepsilon} \le \frac{1}{2}\int_\Omega |\vartheta_0|^2 + \frac{1}{2}\int_\Omega |\vartheta_{0\varepsilon}|^2. \tag{4.2.7}$$

Since $\vartheta_0 \in H$, from (4.2.7) it immediately follows that $\varepsilon\vartheta_{0\varepsilon} \longrightarrow 0$ in V as $\varepsilon \searrow 0$. Besides, there exists a positive constant c, independent of ε, such that $\|\vartheta_{0\varepsilon}\|_H \le c$. Then, testing the first equation of the system (4.2.1) by an arbitrary function $v \in V$ and passing to the limit as $\varepsilon \searrow 0$, we obtain that

$$\lim_{\varepsilon\searrow 0}\left(\int_\Omega \vartheta_{0\varepsilon} v + \varepsilon\int_\Omega \nabla\vartheta_{0\varepsilon}\cdot\nabla v - \int_\Omega \vartheta_0 v\right) = 0 \qquad \text{for all } v \in V, \tag{4.2.8}$$

whence $\vartheta_{0\varepsilon} \rightharpoonup \vartheta_0$ in H. Moreover, from (4.2.7) and (4.2.8) we infer that

$$\int_\Omega |\vartheta_0|^2 \le \liminf_{\varepsilon\searrow 0} \int_\Omega |\vartheta_{0\varepsilon}|^2 \le \limsup_{\varepsilon\searrow 0} \int_\Omega |\vartheta_{0\varepsilon}|^2 \le \int_\Omega |\vartheta_0|^2. \tag{4.2.9}$$

Thanks to (4.2.9), $\|\vartheta_{0\varepsilon}\|_H \longrightarrow \|\vartheta_0\|_H$ and this ensures, due to the weak convergence already proved, that $\vartheta_{0\varepsilon} \longrightarrow \vartheta_0$ in H.

With a similar technique, testing (4.2.2) by $\varphi_{0\varepsilon}$ and integrating over Ω, we obtain that $\varphi_{0\varepsilon} \longrightarrow \varphi_0$ in H. Now, we test (4.2.2) by $-\Delta\varphi_{0\varepsilon}$ and integrate over Ω. We obtain that

$$\int_\Omega |\nabla\varphi_{0\varepsilon}|^2 + \varepsilon \int_\Omega |\Delta\varphi_{0\varepsilon}|^2 = \int_\Omega \nabla\varphi_0 \cdot \nabla\varphi_{0\varepsilon} \le \frac{1}{2}\int_\Omega |\nabla\varphi_0|^2 + \frac{1}{2}\int_\Omega |\nabla\varphi_{0\varepsilon}|^2. \tag{4.2.10}$$

Since $\varphi_0 \in V$, from (4.2.10) it immediately follows that $\varepsilon\varphi_{0\varepsilon} \longrightarrow 0$ in W as $\varepsilon \searrow 0$. Furthermore, there exists a positive constant c, independent of ε, such that $\|\nabla\varphi_{0\varepsilon}\|_H \le c$. Recalling that $\|\varphi_{0\varepsilon}\|_H \le c$, we conclude that $\|\varphi_{0\varepsilon}\|_V \le c$. Then, testing the the first equation of the system (4.2.2) by $-\Delta w$, where w is an arbitrary function in W, and passing to the limit as $\varepsilon \searrow 0$, we obtain

$$\lim_{\varepsilon\searrow 0}\left(\int_\Omega \nabla\varphi_{0\varepsilon}\cdot\nabla w + \varepsilon\int_\Omega \Delta\varphi_{0\varepsilon}\cdot\Delta w - \int_\Omega \nabla\varphi_0\cdot\nabla w\right) = 0 \qquad \text{for all } w \in W, \tag{4.2.11}$$

whence $\varphi_{0\varepsilon} \rightharpoonup \varphi_0$ in V. Moreover, from (4.2.10)–(4.2.11) we infer that

$$\int_\Omega |\nabla\varphi_0|^2 \le \liminf_{\varepsilon\searrow 0} \int_\Omega |\nabla\varphi_{0\varepsilon}|^2 \le \limsup_{\varepsilon\searrow 0} \int_\Omega |\nabla\varphi_{0\varepsilon}|^2 \le \int_\Omega |\nabla\varphi_0|^2. \tag{4.2.12}$$

Thanks to (4.2.12), $\|\nabla\varphi_{0\varepsilon}\|_H \longrightarrow \|\nabla\varphi_0\|_H$ and this ensures, due to the weak convergence already proved, that $\varphi_{0\varepsilon} \longrightarrow \varphi_0$ in V. Now, let us summarize the main properties of $\vartheta_{0\varepsilon}$ and $\varphi_{0\varepsilon}$. For every $\varepsilon \in (0,1)$ we have that

$$\vartheta_{0\varepsilon} \in W, \quad \varphi_{0\varepsilon} \in W \cap H^3(\Omega), \quad m_{0\varepsilon} \in \operatorname{int}(D(\beta)), \quad \|\tilde\beta(\varphi_{0\varepsilon})\|_{L^1(\Omega)} \le c, \tag{4.2.13}$$

$$\lim_{\varepsilon\searrow 0} \|\vartheta_0 - \vartheta_{0\varepsilon}\|_H = 0, \qquad \lim_{\varepsilon\searrow 0} \|\varphi_0 - \varphi_{0\varepsilon}\|_V = 0, \tag{4.2.14}$$

$$-\upsilon\Delta\varphi_{0\varepsilon} + \beta_\varepsilon(\varphi_{0\varepsilon}) + \pi(\varphi_{0\varepsilon}) - \gamma\vartheta_{0\varepsilon} \in V. \tag{4.2.15}$$

Approximating problem (P_ε). We look for a triplet $(\vartheta_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon)$ satisfying at least the regularity requirements

$$\vartheta_\varepsilon \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{4.2.16}$$

$$\varphi_\varepsilon \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{4.2.17}$$

$$\mu_\varepsilon \in L^\infty(0,T;V) \cap L^2(0,T;W), \tag{4.2.18}$$

and solving the approximating problem (P_ε):

$$\partial_t(\vartheta_\varepsilon + \ell\varphi_\varepsilon) - \Delta\vartheta_\varepsilon + \zeta_\varepsilon = f_\varepsilon \text{ a.e. in } Q, \tag{4.2.19}$$

$$\partial_t\varphi_\varepsilon - \Delta\mu_\varepsilon = 0 \text{ a.e. in } Q, \tag{4.2.20}$$

$$\mu_\varepsilon = -\upsilon\Delta\varphi_\varepsilon + \xi_\varepsilon + \pi(\varphi_\varepsilon) - \gamma\vartheta_\varepsilon \text{ a.e. in } Q, \tag{4.2.21}$$

$$\zeta_\varepsilon(t) \in A_\varepsilon(a\vartheta_\varepsilon(t) + b\varphi_\varepsilon(t) - \eta^*) \text{ for a.e. } t \in (0,T), \tag{4.2.22}$$

$$\xi_\varepsilon \in \beta_\varepsilon(\varphi_\varepsilon) \text{ a.e. in } Q, \tag{4.2.23}$$

$$\partial_\nu\vartheta_\varepsilon = \partial_\nu\varphi_\varepsilon = \partial_\nu\mu_\varepsilon = 0 \text{ on } \Sigma, \tag{4.2.24}$$

$$\vartheta_\varepsilon(0) = \vartheta_{0\varepsilon}, \qquad \varphi_\varepsilon(0) = \varphi_{0\varepsilon} \text{ in } \Omega, \tag{4.2.25}$$

where β_ε and A_ε are the Yosida regularizations of β and A defined in (2.5.6) and (2.5.1). We notice that the homogeneous Neumann boundary conditions are already contained in the conditions (4.2.16)–(4.2.18) due to the definition of W (see (2.1.1)).

We observe that, for almost every $t \in (0,T)$, we can re-write the approximating problem (P_ε) in the following way:

$$\langle\partial_t(\vartheta_\varepsilon + \ell\varphi_\varepsilon)(t), z\rangle_{V^*,V} + \int_\Omega \nabla\vartheta_\varepsilon(t)\cdot\nabla z + \langle\zeta_\varepsilon(t), z\rangle_{V^*,V} = \langle f_\varepsilon(t), z\rangle_{V^*,V} \text{ for all } z \in V, \tag{4.2.26}$$

$$\langle\partial_t\varphi_\varepsilon(t), z\rangle_{V^*,V} + \int_\Omega \nabla\mu_\varepsilon(t)\cdot\nabla z = 0 \text{ for all } z \in V, \tag{4.2.27}$$

$$\mu_\varepsilon(t) = -\upsilon\Delta\varphi_\varepsilon(t) + \xi_\varepsilon(t) + \pi(\varphi_\varepsilon(t)) - \gamma\vartheta_\varepsilon(t) \text{ in } H, \tag{4.2.28}$$

$$\zeta_\varepsilon(t) \in A_\varepsilon(a\vartheta_\varepsilon(t) + b\varphi_\varepsilon(t) - \eta^*), \tag{4.2.29}$$

$$\xi \in \beta_\varepsilon(\varphi_\varepsilon) \text{ a.e. in } Q, \tag{4.2.30}$$

$$\partial_\nu\varphi_\varepsilon = 0 \quad \text{a.e. on } \Sigma, \tag{4.2.31}$$

$$\vartheta_\varepsilon(0) = \vartheta_{0\varepsilon}, \qquad \varphi_\varepsilon(0) = \varphi_{0\varepsilon} \quad \text{in } \Omega. \tag{4.2.32}$$

Since $m_{0\varepsilon} = m_0$, recalling the definition of $\mathcal{N}$ (see (2.4.23)–(2.4.26)), we have that $\partial_t\varphi_\varepsilon(t) \in D(\mathcal{N})$. Hence, (4.2.27) can be written as

$$\mathcal{N}\partial_t\varphi_\varepsilon(t) = m(\mu_\varepsilon(t)) - \mu_\varepsilon(t) \text{ in } V, \tag{4.2.33}$$

and this and (4.2.27) entail

$$m(\mu_\varepsilon(t)) - \mathcal{N}\partial_t\varphi_\varepsilon(t) = -\upsilon\Delta\varphi_\varepsilon(t) + \xi_\varepsilon(t) + \pi(\varphi_\varepsilon(t)) - \gamma\vartheta_\varepsilon(t) \text{ in } H. \tag{4.2.34}$$

4.3 Existence - Global a priori estimates

In this section, we will deduce some a priori estimates inferred from (4.2.26)–(4.2.34).

First a priori estimate. According to (4.2.3), $m(\partial_t\varphi_\varepsilon) = 0$. Consequently, $\partial_t\varphi_\varepsilon \in D(\mathcal{N})$ and we can test (4.2.27) by $\mathcal{N}\partial_t\varphi_\varepsilon$. Integrating over $(0,t)$, $t \in (0,T]$, we obtain that

$$\int_0^t \|\partial_t\varphi_\varepsilon(s)\|_{V^*}^2\, ds + \int_{Q_t} \nabla\mu_\varepsilon \cdot \nabla\mathcal{N}\partial_t\varphi_\varepsilon = \int_0^t \|\partial_t\varphi_\varepsilon\|_{V^*}^2 + \int_{Q_t} \mu_\varepsilon\partial_t\varphi_\varepsilon = 0. \tag{4.3.1}$$

Recalling that

$$\upsilon\int_{Q_t} \varphi_\varepsilon\partial_t\varphi_\varepsilon = \frac{\upsilon}{2}\int_\Omega |\varphi_\varepsilon(t)|^2 - \frac{\upsilon}{2}\int_\Omega |\varphi_{0\varepsilon}|^2, \tag{4.3.2}$$

we combine (4.2.26) tested by $\frac{\gamma}{\ell}\vartheta_\varepsilon$, (4.3.1) and (4.3.2). Then we subtract (4.2.28) tested by $\partial_t\varphi_\varepsilon$ and integrate over $(0,t)$. We have that

$$\frac{\gamma}{2\ell}\int_\Omega |\vartheta_\varepsilon(t)|^2 + \frac{\gamma}{\ell}\int_{Q_t} |\nabla\vartheta_\varepsilon|^2 + \int_0^t \|\partial_t\varphi_\varepsilon(s)\|_{V^*}^2\, ds + \frac{\upsilon}{2}\|\varphi_\varepsilon(t)\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_\varepsilon(t))$$
$$= \frac{\gamma}{2\ell}\|\vartheta_{0\varepsilon}\|_H^2 + \frac{\upsilon}{2}\|\varphi_{0\varepsilon}\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_{0\varepsilon}) + \frac{\gamma}{\ell}\int_{Q_t} f_\varepsilon\vartheta_\varepsilon - \frac{\gamma}{\ell}\int_{Q_t} \zeta_\varepsilon\vartheta_\varepsilon + \int_{Q_t} (\upsilon\varphi_\varepsilon - \pi(\varphi_\varepsilon))\partial_t\varphi_\varepsilon. \tag{4.3.3}$$

As π is a Lipschitz continuous function with Lipschitz constant $C_\pi = \|\pi'\|_\infty$, we obtain that

$$|\pi(\varphi_\varepsilon(s))| \le |\pi(\varphi_\varepsilon(s)) - \pi(0)| + |\pi(0)| \le C_\pi|\varphi_\varepsilon(s)| + |\pi(0)|. \tag{4.3.4}$$

Consequently, thanks to (4.3.4), we infer that

$$\|\upsilon\varphi_\varepsilon(s) - \pi(\varphi_\varepsilon(s))\|_V^2 = \int_\Omega |\upsilon\varphi_\varepsilon(s) - \pi(\varphi_\varepsilon(s))|^2 + \int_\Omega |\upsilon\nabla\varphi_\varepsilon(s) - \pi'(\varphi_\varepsilon(s))\nabla\varphi_\varepsilon(s)|^2$$
$$\le 2\int_\Omega \Big(\upsilon^2|\varphi_\varepsilon(s)|^2 + |\pi(\varphi_\varepsilon(s))|^2\Big) + 2\int_\Omega \Big(\upsilon^2|\nabla\varphi_\varepsilon(s)|^2 + \|\pi'\|_\infty^2|\nabla\varphi_\varepsilon(s)|^2\Big)$$
$$\le 2\upsilon^2\int_\Omega |\varphi_\varepsilon(s)|^2 + 4C_\pi^2\int_\Omega |\varphi_\varepsilon(s)|^2 + 4|\Omega||\pi(0)|^2 + 2\upsilon^2\int_\Omega |\nabla\varphi_\varepsilon(s)|^2 + 2C_\pi^2\int_\Omega |\nabla\varphi_\varepsilon(s)|^2$$
$$= (2\upsilon^2 + 4C_\pi^2)\int_\Omega |\varphi_\varepsilon(s)|^2 + (2\upsilon^2 + 2C_\pi^2)\int_\Omega |\nabla\varphi_\varepsilon(s)|^2 + 4|\pi(0)|^2|\Omega| \le c(\|\varphi_\varepsilon(s)\|_V^2 + 1),$$

whence we obtain that the last term on the right-hand side of (4.3.3) is estimated as follows

$$\int_{Q_t} (\upsilon\varphi_\varepsilon - \pi(\varphi_\varepsilon))\partial_t\varphi_\varepsilon \le \frac{1}{2}\int_0^t \|\partial_t\varphi_\varepsilon(s)\|_{V^*}^2\, ds + \frac{1}{2}\int_0^t \|\upsilon\varphi_\varepsilon(s) - \pi(\varphi_\varepsilon(s))\|_V^2\, ds$$
$$\le \frac{1}{2}\int_0^t \|\partial_t\varphi_\varepsilon(s)\|_{V^*}^2\, ds + c\int_0^t (\|\varphi_\varepsilon(s)\|_V^2 + 1)\, ds. \tag{4.3.5}$$

Due to the liear growth of A_ε stated by (2.5.9), we have that

$$-\frac{\gamma}{\ell}\int_{Q_t} \zeta_\varepsilon\vartheta_\varepsilon \le \frac{\gamma}{\ell}\int_{Q_t} |\zeta_\varepsilon(s)||\vartheta_\varepsilon(s)|\, ds \le \frac{\gamma}{\ell}\int_0^t \|\zeta_\varepsilon(s)\|_H^2\, ds + \frac{\gamma}{\ell}\int_0^t \|\vartheta_\varepsilon(s)\|_H^2\, ds$$

$$
\begin{aligned}
&\leq \frac{\gamma}{\ell}\int_0^t C_A^2(1+\|a\vartheta_\varepsilon(s)+b\varphi_\varepsilon(s)-\eta^*\|_H)^2\,ds+\frac{\gamma}{\ell}\int_0^t\|\vartheta_\varepsilon(s)\|_H^2\,ds\\
&\leq \frac{\gamma}{\ell}\int_0^t 4C_A^2(1+|a|^2\|\vartheta_\varepsilon(s)\|_H^2+|b|^2\|\varphi_\varepsilon(s)\|_H^2+\|\eta^*\|_H^2)\,ds+\frac{\gamma}{\ell}\int_0^t\|\vartheta_\varepsilon(s)\|_H^2\,ds\\
&\leq \frac{\gamma}{\ell}4C_A^2T+\frac{\gamma}{\ell}4C_A^2|a|^2\int_0^t\|\vartheta_\varepsilon(s)\|_H^2\,ds+\frac{\gamma}{\ell}4C_A^2|b|^2\int_0^t\|\varphi_\varepsilon(s)\|_H^2\,ds\\
&\quad+\frac{\gamma}{\ell}4C_A^2T\|\eta^*\|_H^2+\frac{\gamma}{\ell}\int_0^t\|\vartheta_\varepsilon(s)\|_H^2\,ds\\
&\leq c\left(\int_0^t\|\vartheta_\varepsilon(s)\|_H^2\,ds+\int_0^t\|\varphi_\varepsilon(s)\|_H^2\,ds+1\right).
\end{aligned}\tag{4.3.6}
$$

Moreover, by applying (2.2.5) to the fourth term on the right-hand side of (4.3.3), we have that

$$
\frac{\gamma}{\ell}\int_{Q_t}f_\varepsilon\vartheta_\varepsilon\leq\frac{\gamma}{\ell}\int_{Q_t}|f_\varepsilon|^2+\frac{\gamma}{4\ell}\int_{Q_t}|\vartheta_\varepsilon|^2=\frac{\gamma}{\ell}\int_{Q_t}|f_\varepsilon|^2+\frac{\gamma}{4\ell}\int_0^t\|\vartheta_\varepsilon(s)\|_H^2\,ds. \tag{4.3.7}
$$

We rearrange the right-hand side of (4.3.3) using (4.3.5)–(4.3.7) and obtain that

$$
\begin{aligned}
&\frac{\gamma}{2\ell}\int_\Omega|\vartheta_\varepsilon(t)|^2+\frac{\gamma}{\ell}\int_{Q_t}|\nabla\vartheta_\varepsilon|^2+\frac12\int_0^t\|\partial_t\varphi_\varepsilon(s)\|_{V^*}^2\,ds+\frac{\upsilon}{2}\|\varphi_\varepsilon(t)\|_V^2+\int_\Omega\widehat{\beta}_\varepsilon(\varphi_\varepsilon(t))\\
&\leq\frac{\gamma}{2\ell}\|\vartheta_{0\varepsilon}\|_H^2+\frac{\upsilon}{2}\|\varphi_{0\varepsilon}\|_V^2+\int_\Omega\widehat{\beta}_\varepsilon(\varphi_{0\varepsilon})+\frac{\gamma}{\ell}\int_0^t\|f_\varepsilon(s)\|_H^2\,ds\\
&\quad+c\left(\int_0^t\|\varphi_\varepsilon(s)\|_V^2\,ds+\int_0^t\|\vartheta_\varepsilon(s)\|_H^2\,ds+1\right)+\frac{\gamma}{4\ell}\int_0^t\|\vartheta_\varepsilon(s)\|_H^2\,ds.
\end{aligned}\tag{4.3.8}
$$

Due to (4.2.13)–(4.2.14), the first three terms of the right-hand side of (4.3.8) are bounded and similarly the fourth term, thanks to (2.5.17). Then, applying the Gronwall lemma, we conclude that there exists a positive constant c, independent of ε, such that

$$
\frac{\gamma}{2\ell}\int_\Omega|\vartheta_\varepsilon(t)|^2+\int_{Q_t}|\nabla\vartheta_\varepsilon|^2+\frac12\int_0^t\|\partial_t\varphi_\varepsilon(s)\|_{V^*}^2\,ds+\frac{\upsilon}{2}\|\varphi_\varepsilon(t)\|_V^2+\int_\Omega\widehat{\beta}_\varepsilon(\varphi_\varepsilon(t))\leq c, \tag{4.3.9}
$$

whence it immediately follows that

$$
\|\vartheta_\varepsilon\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}\leq c, \tag{4.3.10}
$$

$$
\|\varphi_\varepsilon\|_{H^1(0,T;V^*)\cap L^\infty(0,T;V)}\leq c, \tag{4.3.11}
$$

$$
\|\widehat{\beta}_\varepsilon(\varphi_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))}\leq c. \tag{4.3.12}
$$

Due to (4.3.10)–(4.3.12), by (2.5.9) we have that

$$
\|\zeta_\varepsilon\|_{L^\infty(0,T;H)}\leq c, \tag{4.3.13}
$$

and, consequently, by comparison in (4.2.26) we infer that

$$
\|\partial_t\vartheta_\varepsilon\|_{L^2(0,T;V^*)}\leq c. \tag{4.3.14}
$$

Second a priori estimate. Recalling that $m_{0\varepsilon} = m_0$ due to (4.2.27), we have that $\varphi_\varepsilon(s) - m_0 \in D(\mathcal{N})$ for every $s \in (0,T)$. We test (4.2.34) at time s by $(\varphi_\varepsilon(s) - m_0) \in D(\mathcal{N})$ and we infer that

$$(\xi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H = -(\mathcal{N}\partial_t\varphi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H + (m(\mu_\varepsilon(s)), \varphi_\varepsilon(s) - m_0)_H$$

$$+ \upsilon(\Delta\varphi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H - (\pi(\varphi_\varepsilon(s)), \varphi_\varepsilon(s) - m_0)_H + \gamma(\vartheta_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H. \quad (4.3.15)$$

We recall that there exists a positive constant c such that $\|z\|_{V^*} \leq c\|z\|_H$ for all $z \in H$. Consequently the first term of the right-hand side of (4.3.15) is estimated as follows:

$$\begin{aligned} -(\mathcal{N}\partial_t\varphi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H &= -(\partial_t\varphi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_{V^*} \\ &\leq \|\partial_t\varphi_\varepsilon(s)\|_{V^*}(\|\varphi_\varepsilon(s)\|_{V^*} + |m_0||\Omega|) \\ &= c\|\partial_t\varphi_\varepsilon(s)\|_{V^*}(\|\varphi_\varepsilon(s)\|_H + 1). \end{aligned} \quad (4.3.16)$$

Recalling (4.2.3), we have that

$$(m(\mu_\varepsilon(s)), \varphi_\varepsilon(s) - m_0)_H = m(\mu_\varepsilon(s))\left(\int_\Omega \varphi_\varepsilon(s) - |\Omega| m_0\right) = 0. \quad (4.3.17)$$

Due to the Neumann homogeneous boundary conditions for φ_ε, we have that

$$\int_\Omega \Delta\varphi_\varepsilon(s) = 0. \quad (4.3.18)$$

Thanks to (4.3.18), we infer that

$$\upsilon(\Delta\varphi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H = -\upsilon\|\nabla\varphi_\varepsilon(s)\|_H^2 - m_0\int_\Omega \Delta\varphi_\varepsilon(s) = -\upsilon\|\nabla\varphi_\varepsilon(s)\|_H^2 \leq 0. \quad (4.3.19)$$

As π is a Lipschitz continuous function with Lipschitz constant C_π, we obtain that

$$\begin{aligned} -(\pi(\varphi_\varepsilon(s)), \varphi_\varepsilon(s) - m_0)_H &\leq \int_\Omega |\pi(\varphi_\varepsilon(s))||\varphi_\varepsilon(s) - m_0| \\ &\leq \int_\Omega \Big(|\pi(\varphi_\varepsilon(s)) - \pi(0)| + |\pi(0)|\Big)\Big(|\varphi_\varepsilon(s)| + |m_0|\Big) \\ &\leq \int_\Omega \Big(C_\pi|\varphi_\varepsilon(s)| + |\pi(0)|\Big)\Big(|\varphi_\varepsilon(s)| + |m_0|\Big) \\ &\leq C_\pi\|\varphi_\varepsilon(s)\|_H^2 + \Big(C_\pi|m_0| + |\pi(0)|\Big)\|\varphi_\varepsilon(s)\|_H^2 + c \\ &\leq c(\|\varphi_\varepsilon(s)\|_H^2 + 1). \end{aligned} \quad (4.3.20)$$

Moreover, we have that

$$\begin{aligned} \gamma(\vartheta_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H &\leq \gamma\int_\Omega |\vartheta_\varepsilon(s)||\varphi_\varepsilon(s)| + \gamma|m_0|\int_\Omega |\vartheta_\varepsilon(s)| \\ &\leq \gamma\|\vartheta_\varepsilon(s)\|_H^2 + \gamma\|\varphi_\varepsilon(s)\|_H^2 + \gamma|m_0|\|\vartheta_\varepsilon(s)\|_H^2 + \gamma|m_0||\Omega| \\ &\leq c(\|\vartheta_\varepsilon(s)\|_H^2 + \|\varphi_\varepsilon(s)\|_H^2 + 1). \end{aligned} \quad (4.3.21)$$

Consequently, rearranging the right-hand side of (4.3.15) using (4.3.16)–(4.3.17) and (4.3.19)–(4.3.21), we obtain that

$$(\xi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H \leq c\Big(\|\partial_t\varphi_\varepsilon(s)\|_{V^*} + \|\varphi_\varepsilon(s)\|_H^2 + \|\vartheta_\varepsilon(s)\|_H^2 + 1 \Big). \tag{4.3.22}$$

Due to a useful inequality stated in [42, Section 5], it turns out that

$$|\xi_\varepsilon(s)| \leq c[\xi_\varepsilon(s)(\varphi_\varepsilon(s) - m_0) + 1]. \tag{4.3.23}$$

We integrate (4.3.23) over Ω and, due to (4.3.22), we infer that

$$\|\xi_\varepsilon(s)\|_{L^1(\Omega)} \leq c\Big[(\xi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H + 1 \Big]$$

$$\leq c\Big(\|\partial_t\varphi_\varepsilon(s)\|_{V^*} + \|\varphi_\varepsilon(s)\|_H^2 + \|\vartheta_\varepsilon(s)\|_H^2 + 1 \Big). \tag{4.3.24}$$

Due to (4.3.10)–(4.3.11), from (4.3.24) we conclude that there exists a positive constant c, independent of ε, such that

$$\|\xi_\varepsilon\|_{L^2(0,T;L^1(\Omega))} \leq c. \tag{4.3.25}$$

Third a priori estimate. As π is a Lipschitz continuous function with Lipschitz constant C_π, for every $s \in (0,T)$ we have that

$$\begin{aligned} |\pi(\varphi_\varepsilon(s))|^2 &\leq \big(|\pi(\varphi_\varepsilon(s)) - \pi(0)| + |\pi(0)| \big)^2 \\ &\leq \big(C_\pi|\varphi_\varepsilon(s)| + |\pi(0)| \big)^2 \\ &\leq c\big(|\varphi_\varepsilon(s)|^2 + 1 \big). \end{aligned} \tag{4.3.26}$$

Now, integrating (4.2.34) over Ω, squaring the resultant and using (4.3.10)–(4.3.14) and (4.3.26), we obtain that

$$\begin{aligned} |m(\mu_\varepsilon(s))|^2 &\leq \frac{3}{|\Omega|^2}\Big(\|\xi_\varepsilon(s)\|_{L^1(\Omega)}^2 + |\Omega| \|\pi(\varphi_\varepsilon(s))\|_H^2 + \gamma\|\vartheta_\varepsilon(s)\|_H^2 \Big) \\ &\leq c\Big(\|\xi_\varepsilon(s)\|_{L^1(\Omega)}^2 + \|\varphi_\varepsilon(s)\|_H^2 + \|\vartheta_\varepsilon(s)\|_H^2 + 1 \Big). \end{aligned} \tag{4.3.27}$$

Consequently, integrating (4.3.27) over $(0,T)$ and recalling the previous a priori estimates (4.3.10)–(4.3.11) and (4.3.25), we conclude that there exists a positive constant c, independent of ε, such that

$$\|m(\mu_\varepsilon)\|_{L^2(0,T)} \leq c. \tag{4.3.28}$$

Fourth a priori estimate. We recall that the Poincaré inequality states that there exists a positive constant c_p such that

$$\|z\|_V^2 \leq c_p \|\nabla z\|_H^2 \quad \text{for all } z \in V \text{ with } m(z) = 0. \tag{4.3.29}$$

We integrate over $(0, T)$ the square of the norms in V of each term of (4.2.33). Then, applying (4.3.28) and (4.3.29), we obtain that

$$\begin{aligned} \int_0^T \|\mu_\varepsilon(s)\|_V^2 \, ds &\leq 2\int_0^T \|m(\mu_\varepsilon(s))\|_V^2 \, ds + 2\int_0^T \|\mathcal{N}\partial_t\varphi_\varepsilon(s)\|_V^2 \, ds \\ &\leq 2\int_0^T |m(\mu_\varepsilon(s))|^2 \, ds + 2c_p \int_0^T \|\nabla\mathcal{N}\partial_t\varphi_\varepsilon(s)\|_H^2 \, ds \\ &\leq c + 2c_p \int_0^T \|\partial_t\varphi_\varepsilon(s)\|_{V^*}^2 \, ds. \end{aligned} \tag{4.3.30}$$

Due to (4.3.11), we conclude that there exists a positive constant c, independent of ε, such that

$$\|\mu_\varepsilon\|_{L^2(0,T;V)} \leq c. \tag{4.3.31}$$

Fifth a priori estimate. We test (4.2.28) at time $s \in (0, T)$ by $\xi_\varepsilon(s) \in V$ and integrate the resultant over Ω. We obtain that

$$\|\xi_\varepsilon(s)\|_H^2 = \big(\mu_\varepsilon(s) + \upsilon\Delta\varphi_\varepsilon(s) - \pi(\varphi_\varepsilon(s)) + \gamma\vartheta_\varepsilon(s), \xi_\varepsilon(s)\big)_H. \tag{4.3.32}$$

Due to the monotonicity of β_ε, we have that

$$\begin{aligned} (\upsilon\Delta\varphi_\varepsilon(s), \xi_\varepsilon(s))_H &= \upsilon\int_\Omega \Delta\varphi_\varepsilon(s)\xi_\varepsilon(s) \\ &= -\upsilon\int_\Omega \nabla\varphi_\varepsilon(s)\cdot\nabla\xi_\varepsilon(s) \\ &= -\upsilon\int_\Omega |\nabla\varphi_\varepsilon(s)|^2\beta_\varepsilon'(\varphi_\varepsilon(s)) \leq 0. \end{aligned} \tag{4.3.33}$$

Using (4.3.33) and the Young inequality, we can estimate (4.3.32) as follows

$$\begin{aligned} \|\xi_\varepsilon(s)\|_H^2 &\leq \big(\mu_\varepsilon(s) - \pi(\varphi_\varepsilon(s)) + \gamma\vartheta_\varepsilon(s), \xi_\varepsilon(s)\big)_H \\ &\leq \|\mu_\varepsilon(s) - \pi(\varphi_\varepsilon(s)) + \gamma\vartheta_\varepsilon(s)\|_H \|\xi_\varepsilon(s)\|_H \\ &\leq \frac{1}{2}\|\xi_\varepsilon(s)\|_H^2 + 2\big(\|\mu_\varepsilon(s)\|_H^2 + \|\pi(\varphi_\varepsilon(s))\|_H^2 + \gamma^2\|\vartheta_\varepsilon(s)\|_H^2\big). \end{aligned} \tag{4.3.34}$$

Due to (4.3.26), from (4.3.34) we infer that

$$\|\xi_\varepsilon(s)\|_H^2 \leq c\big(\|\mu_\varepsilon(s)\|_H^2 + \|\varphi_\varepsilon(s)\|_H^2 + \|\vartheta_\varepsilon(s)\|_H^2 + 1\big). \tag{4.3.35}$$

Then, integrating (4.3.35) over $(0, T)$ with respect to s and using (4.3.10)–(4.3.11) and (4.3.31), we have that

$$\|\xi_\varepsilon\|_{L^2(0,T;H)} \leq c, \tag{4.3.36}$$

for some positive constant c, independent of ε.

Sixth a priori estimate. We integrate over $(0,T)$ the square of the norms in H of each term of (4.2.28). Then, using (4.3.26), (4.3.31) and (4.3.36), we obtain that

$$\upsilon^2 \int_0^T \|\Delta\varphi_\varepsilon(s)\|_H^2 \, ds$$

$$\leq 4\int_0^T \|\mu_\varepsilon(s)\|_H^2 \, ds + 4\int_0^T \|\xi_\varepsilon(s)\|_H^2 \, ds + 4\int_0^T \|\pi(\varphi_\varepsilon(s))\|_H^2 \, ds + 4\gamma^2\int_0^T \|\vartheta_\varepsilon(s)\|_H^2 \, ds$$

$$\leq c\Bigg(\int_0^T \|\varphi_\varepsilon(s)\|_H^2 \, ds + \int_0^T \|\vartheta_\varepsilon(s)\|_H^2 \, ds + 1\Bigg). \tag{4.3.37}$$

Thanks to (4.3.10)–(4.3.11), we conclude that there exists a positive constant c, independent of ε, such that

$$\|\varphi_\varepsilon\|_{L^2(0,T;W)} \leq c. \tag{4.3.38}$$

Summary of the a priori estimates. Let us summarize the a priori estimates. From (4.3.10)–(4.3.14), (4.3.31), (4.3.36) and (4.3.38) we conclude that there exists a constant $c > 0$, independent of ε, such that

$$\|\vartheta_\varepsilon\|_{H^1(0,T;V^*)\cap L^\infty(0,T;H)\cap L^2(0,T;V)} \leq c, \tag{4.3.39}$$

$$\|\varphi_\varepsilon\|_{H^1(0,T;V^*)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq c, \tag{4.3.40}$$

$$\|\zeta_\varepsilon\|_{L^\infty(0,T;H)} \leq c, \tag{4.3.41}$$

$$\|\xi_\varepsilon\|_{L^2(0,T;H)} \leq c, \tag{4.3.42}$$

$$\|\mu_\varepsilon\|_{L^2(0,T;V)} \leq c. \tag{4.3.43}$$

4.4 Existence - Passage to the limit as $\varepsilon \searrow 0$

Based on available results (cf., e.g., [19]), it turns out that there exists a solution $(\vartheta_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon)$ of (P_ε) satisfying the regularity requirements (4.2.16)–(4.2.18) and solving (4.2.19)-(4.2.25). In this section we pass to the limit as $\varepsilon \searrow 0$ and prove that the limit of subsequences of solutions $(\vartheta_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon)$ for (P_ε) (see (4.2.19)–(4.2.25)) yields a solution $(\vartheta, \varphi, \mu)$ of (P) (see (4.1.7)–(4.1.13)).

Thanks to the uniform estimates (4.3.39)–(4.3.43), there exists a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\varepsilon_k \searrow 0$ as $k \to +\infty$ and some limit functions $\vartheta \in H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V)$, $\varphi \in H^1(0,T;V^*)\cap L^\infty(0,T;H)\cap L^2(0,T;W)$, $\mu \in L^2(0,T;V)$, $\xi \in L^2(0,T;H)$

and $\zeta \in L^\infty(0,T;H)$ such that

$$\vartheta_{\varepsilon_k} \rightharpoonup^* \vartheta \quad \text{in} \quad H^1(0,T;V^*) \cap L^\infty(0,T;H), \tag{4.4.1}$$
$$\vartheta_{\varepsilon_k} \rightharpoonup \vartheta \quad \text{in} \quad L^2(0,T;V), \tag{4.4.2}$$
$$\varphi_{\varepsilon_k} \rightharpoonup^* \varphi \quad \text{in} \quad H^1(0,T;V^*) \cap L^\infty(0,T;V), \tag{4.4.3}$$
$$\varphi_{\varepsilon_k} \rightharpoonup \varphi \quad \text{in} \quad L^2(0,T;W), \tag{4.4.4}$$
$$\mu_{\varepsilon_k} \rightharpoonup \mu \quad \text{in} \quad L^2(0,T;V), \tag{4.4.5}$$
$$\xi_{\varepsilon_k} \rightharpoonup \xi \quad \text{in} \quad L^2(0,T;H), \tag{4.4.6}$$
$$\zeta_{\varepsilon_k} \rightharpoonup^* \zeta \quad \text{in} \quad L^\infty(0,T;H), \tag{4.4.7}$$

as $k \to +\infty$. From (4.4.1)–(4.4.4) and the well–known Ascoli–Arzelá theorem (see, e.g., [62, Sect. 8, Cor. 4]), we infer that

$$\vartheta_{\varepsilon_k} \longrightarrow \vartheta \quad \text{in} \quad C^0([0,T];V^*) \cap L^2(0,T;H), \tag{4.4.8}$$
$$\varphi_{\varepsilon_k} \longrightarrow \varphi \quad \text{in} \quad C^0([0,T];H) \cap L^2(0,T;V), \tag{4.4.9}$$

as $k \to +\infty$. As π is a Lipschitz continuous function, for a.e. $s \in [0,T]$ we have that

$$|\pi(\varphi_{\varepsilon_k}(s)) - \pi(\varphi(s))| \leq C_\pi |\varphi_{\varepsilon_k}(s) - \varphi(s)|. \tag{4.4.10}$$

Thanks to (4.4.9), we conclude that

$$\pi(\varphi_{\varepsilon_k}(s)) \longrightarrow \pi(\varphi(s)) \quad \text{in } L^2(0,T;H), \tag{4.4.11}$$

as $k \to +\infty$.

Passage to the limit on ξ_ε. In this paragraph we check that $\xi \in \beta(\varphi)$ a.e. in Q. To this aim, we recall that

$$\varphi_{\varepsilon_k} \to \varphi \qquad \text{in } L^2(0,T;H) \equiv L^2(Q), \tag{4.4.12}$$

$$\xi_{\varepsilon_k} \rightharpoonup \xi \qquad \text{in } L^2(0,T;H), \tag{4.4.13}$$

as $k \to +\infty$. Now, we introduce the operator $\mathcal{B}_\varepsilon$ induced by β_ε on $L^2(Q)$ in the following way

$$\mathcal{B}_\varepsilon : L^2(Q) \longrightarrow L^2(Q)$$
$$\xi_\varepsilon \in \mathcal{B}_\varepsilon(\varphi_\varepsilon) \Longleftrightarrow \xi_\varepsilon(x,t) \in \beta_\varepsilon(\varphi_\varepsilon(x,t)) \quad \text{for a.e. } (x,t) \in Q. \tag{4.4.14}$$

Due to (4.4.12)–(4.4.13), as $k \to +\infty$, we have that

$$\begin{cases} \mathcal{B}_{\varepsilon_k}(\varphi_{\varepsilon_k}) \rightharpoonup \xi & \text{in } L^2(Q), \\ \varphi_{\varepsilon_k} \to \varphi & \text{in } L^2(Q), \end{cases} \tag{4.4.15}$$

$$\limsup_{k \to +\infty} \int_Q \xi_{\varepsilon_k} \varphi_{\varepsilon_k} = \int_Q \xi \varphi. \tag{4.4.16}$$

Thanks to (4.4.15)–(4.4.16) and to the general result [1, Proposition 2.2, p. 38], we conclude that

$$\xi \in \mathcal{B}(\varphi) \quad \text{in } L^2(Q), \tag{4.4.17}$$

with analogous definition for $\mathcal{B}$ (see (2.4.4)–(2.4.5)). This is equivalent to saying that

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q. \tag{4.4.18}$$

Passage to the limit on ζ_ε. In this paragraph we check that $\zeta(t) \in A(a\vartheta(t)+b\varphi(t)-\eta^*)$ for a.e. $t \in [0,T]$. Let us recall that

$$\vartheta_{\varepsilon_k} \to \vartheta \quad \text{in} \quad L^2(0,T;H), \tag{4.4.19}$$

$$\varphi_{\varepsilon_k} \to \varphi \quad \text{in} \quad L^2(0,T;H), \tag{4.4.20}$$

$$\zeta_{\varepsilon_k} \rightharpoonup \zeta \quad \text{in} \quad L^2(0,T;H), \tag{4.4.21}$$

as $k \to +\infty$. Setting

$$\eta_{\varepsilon_k} := a\vartheta_{\varepsilon_k} + b\varphi_{\varepsilon_k} - \eta^*, \qquad \eta := a\vartheta + b\varphi - \eta^*,$$

thanks to (4.4.19)–(4.4.20), we have that

$$\eta_{\varepsilon_k} \longrightarrow \eta \quad \text{in } L^2(0,T;H), \tag{4.4.22}$$

as $k \to +\infty$. Now, we introduce the operator $\mathcal{A}_\varepsilon$ induced by A_ε on $L^2(0,T;H)$ in the following way

$$\mathcal{A}_\varepsilon : L^2(0,T;H) \longrightarrow L^2(0,T;H)$$

$$\zeta_\varepsilon \in \mathcal{A}_\varepsilon(\eta_\varepsilon) \Longleftrightarrow \zeta_\varepsilon(t) \in A_\varepsilon(\eta_\varepsilon(t)) \quad \text{for a.e. } t \in [0,T]. \tag{4.4.23}$$

Due to (4.4.19)–(4.4.21), we have that

$$\begin{cases} \mathcal{A}_{\varepsilon_k}(\eta_{\varepsilon_k}) \rightharpoonup \zeta & \text{in } L^2(0,T;H), \\ \eta_{\varepsilon_k} \to \eta & \text{in } L^2(0,T;H), \end{cases} \tag{4.4.24}$$

$$\limsup_{k\to+\infty} \int_Q \zeta_{\varepsilon_k}\eta_{\varepsilon_k} = \int_Q \zeta\eta. \tag{4.4.25}$$

Thanks to (4.4.24)–(4.4.25) and the convergence result [1, Proposition 2.2, p. 38], we conclude that

$$\zeta \in \mathcal{A}(\eta) \quad \text{in } L^2(0,T;H), \tag{4.4.26}$$

with obvious definition for $\mathcal{A}$ (see (2.4.12)–(2.4.13)). This is equivalent to saying that

$$\zeta(t) \in A(a\vartheta(t) + b\varphi(t) - \eta^*) \quad \text{for a.e. } t \in [0,T]. \tag{4.4.27}$$

Conclusion of the proof Using (4.4.1)–(4.4.11), (4.4.18) and (4.4.27), we can pass to the limit as $\varepsilon \searrow 0$ in (4.2.19)–(4.2.25) obtaining (4.1.7)–(4.1.13) for the limiting functions ϑ, φ and μ.

4.5 Regularity

This section is devoted to the proof of Theorem 4.1.2. In order to obtain additional regularity for the solutions, we need further a priori estimates obtained from the approximating problem (P_ε) (see (4.2.19)–(4.2.25)) in which we take $\vartheta_{0\varepsilon} = \vartheta_0$ and $\varphi_{0\varepsilon} = \varphi_0$.

Seventh a priori estimate. We test (4.2.19) by $\partial_t \vartheta_\varepsilon$ and integrate over Q_t, $t \in (0, T]$. We have that

$$\int_{Q_t} |\partial_t \vartheta_\varepsilon|^2 + \ell \int_{Q_t} \partial_t \varphi_\varepsilon \partial_t \vartheta_\varepsilon + \frac{1}{2} \int_\Omega |\nabla \vartheta_\varepsilon(t)| + \int_{Q_t} \zeta_\varepsilon \partial_t \vartheta_\varepsilon = \int_{Q_t} f_\varepsilon \partial_t \vartheta_\varepsilon + \frac{1}{2} \int_\Omega |\nabla \vartheta_0|. \quad (4.5.1)$$

We now proceed with a formal estimate since we have to differentiate (4.2.20) and (4.2.21) with respect to time. For a rigorous approach, one can argue, e.g., as in [20, Subsection 4.4]. By time differentiation of (4.2.20) and (4.2.21) we have

$$\partial_{tt} \varphi_\varepsilon - \Delta \partial_t \mu_\varepsilon = 0, \quad (4.5.2)$$

$$\partial_t \mu_\varepsilon = -\upsilon \Delta \partial_t \varphi_\varepsilon + \beta'_\varepsilon(\varphi_\varepsilon) \partial_t \varphi_\varepsilon + \pi'(\varphi_\varepsilon) \partial_t \varphi_\varepsilon - \gamma \partial_t \vartheta_\varepsilon. \quad (4.5.3)$$

According to (4.2.3), $m(\partial_t \varphi_\varepsilon) = 0$. Consequently, $\partial_t \varphi_\varepsilon \in D(\mathcal{N})$ and we can test (4.5.2) by $\frac{\ell}{\gamma} \mathcal{N}(\partial_t \varphi_\varepsilon)$. Integrating the resultant over Q_t, we obtain that

$$-\frac{\ell}{\gamma} \int_{Q_t} \partial_t \mu_\varepsilon \partial_t \varphi_\varepsilon = \frac{\ell}{2\gamma} \|\partial_t \varphi_\varepsilon(t)\|^2_{V^*} - \frac{\ell}{2\gamma} \|\partial_t \varphi_\varepsilon(0)\|^2_{V^*}. \quad (4.5.4)$$

We test (4.5.3) by $\frac{\ell}{\gamma} \partial_t \varphi_\varepsilon$ and integrate over Q_t. We have that

$$\frac{\ell}{\gamma} \int_{Q_t} \partial_t \mu_\varepsilon \partial_t \varphi_\varepsilon$$

$$= \frac{\upsilon \ell}{\gamma} \int_{Q_t} |\nabla \partial_t \varphi_\varepsilon|^2 + \frac{\ell}{\gamma} \int_{Q_t} \beta'_\varepsilon(\varphi_\varepsilon) |\partial_t \varphi_\varepsilon|^2 + \frac{\ell}{\gamma} \int_{Q_t} \pi'(\varphi_\varepsilon) |\partial_t \varphi_\varepsilon|^2 - \ell \int_{Q_t} \partial_t \varphi_\varepsilon \partial_t \vartheta_\varepsilon. \quad (4.5.5)$$

By combining (4.5.1), (4.5.4) and (4.5.5), we infer that

$$\int_{Q_t} |\partial_t \vartheta_\varepsilon|^2 + \frac{1}{2} \int_\Omega |\nabla \vartheta_\varepsilon(t)| + \frac{\upsilon \ell}{\gamma} \int_{Q_t} |\nabla \partial_t \varphi_\varepsilon|^2 + \frac{\ell}{2\gamma} \|\partial_t \varphi_\varepsilon(t)\|^2_{V^*} = \frac{1}{2} \int_\Omega |\nabla \vartheta_0|$$

$$+ \int_{Q_t} f_\varepsilon \partial_t \vartheta_\varepsilon + \frac{\ell}{2\gamma} \|\partial_t \varphi_\varepsilon(0)\|^2_{V^*} - \int_{Q_t} \zeta_\varepsilon \partial_t \vartheta_\varepsilon - \frac{\ell}{\gamma} \int_{Q_t} \beta'_\varepsilon(\varphi_\varepsilon) |\partial_t \varphi_\varepsilon|^2 - \frac{\ell}{\gamma} \int_{Q_t} \pi'(\varphi_\varepsilon) |\partial_t \varphi_\varepsilon|^2. \quad (4.5.6)$$

By applying inequality (2.2.5) to the second term on the right-hand side of (4.5.6), we infer that

$$\int_{Q_t} f_\varepsilon \partial_t \vartheta_\varepsilon \leq \|f_\varepsilon\|^2_{L^2(0,T;H)} + \frac{1}{4} \int_{Q_t} |\partial_t \vartheta_\varepsilon|^2. \quad (4.5.7)$$

Moreover, as β_ε is a maximal monotone operator, we have that $\beta_\varepsilon' > 0$ and consequently

$$-\frac{\ell}{\gamma}\int_{Q_t}\beta_\varepsilon'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2 \leq 0. \tag{4.5.8}$$

Due to (4.3.13), we have that

$$-\int_{Q_t}\zeta_\varepsilon\partial_t\vartheta_\varepsilon \leq \int_{Q_t}|\zeta_\varepsilon|^2 + \frac{1}{4}\int_{Q_t}|\partial_t\vartheta_\varepsilon|^2 \leq c + \frac{1}{4}\int_{Q_t}|\partial_t\vartheta_\varepsilon|^2. \tag{4.5.9}$$

As π is a Lipschitz continuous function with Lipschitz constant C_π, we have that

$$-\frac{\ell}{\gamma}\int_{Q_t}\pi'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2 \leq \frac{\ell}{\gamma}\int_{Q_t}|\pi'(\varphi_\varepsilon)||\partial_t\varphi_\varepsilon|^2 \leq \frac{C_\pi\ell}{\gamma}\int_{Q_t}|\partial_t\varphi_\varepsilon|^2. \tag{4.5.10}$$

Adding $\frac{\upsilon\ell}{\gamma}\int_{Q_t}|\partial_t\varphi_\varepsilon|^2$ to both side of (4.5.6) and rearranging the right-hand side of (4.5.6) using (4.5.7)–(4.5.10), we obtain that

$$\begin{aligned}&\frac{1}{2}\int_{Q_t}|\partial_t\vartheta_\varepsilon|^2 + \frac{1}{2}\int_\Omega|\nabla\vartheta_\varepsilon(t)| + \frac{\upsilon\ell}{\gamma}\int_0^t\|\partial_t\varphi_\varepsilon(s)\|_V^2\,ds + \frac{\ell}{2\gamma}\|\partial_t\varphi_\varepsilon(t)\|_{V^*}^2\\ &\leq \frac{1}{2}\|\vartheta_0\|_V^2 + \frac{\ell}{2\gamma}\|\partial_t\varphi_\varepsilon(0)\|_{V^*}^2 + \|f_\varepsilon\|_{L^2(0,T;H)}^2 + \left(\frac{C_\pi\ell}{\gamma}+\frac{\upsilon\ell}{\gamma}\right)\int_0^t\|\partial_t\varphi_\varepsilon(s)\|_{V^*}^2\,ds + c.\end{aligned} \tag{4.5.11}$$

Thanks to the compactness of the embedding $V \subset H \subset V^*$, the inequality stated by [62, Lemma 8, p. 84] ensures that, choosing

$$\delta = \left(\frac{\upsilon\ell}{4\gamma}\left(\frac{C_\pi\ell}{\gamma}+\frac{\upsilon\ell}{\gamma}\right)^{-1}\right)^{\frac{1}{2}},$$

we can estimate the fourth term on the right-hand side of (4.5.11) as follows

$$\left(\frac{C_\pi\ell}{\gamma}+\frac{\upsilon\ell}{\gamma}\right)\int_0^t\|\partial_t\varphi_\varepsilon(s)\|_H^2\,ds \leq \frac{\upsilon\ell}{2\gamma}\int_0^t\|\partial_t\varphi_\varepsilon(s)\|_V^2\,ds + c\int_0^t\|\partial_t\varphi_\varepsilon(s)\|_{V^*}^2\,ds. \tag{4.5.12}$$

Due to (4.5.12), from (4.5.11) we have that

$$\begin{aligned}&\frac{1}{2}\int_{Q_t}|\partial_t\vartheta_\varepsilon|^2 + \frac{1}{2}\int_\Omega|\nabla\vartheta_\varepsilon(t)| + \frac{\upsilon\ell}{2\gamma}\int_0^t\|\partial_t\varphi_\varepsilon(s)\|_V^2\,ds + \frac{\ell}{2\gamma}\|\partial_t\varphi_\varepsilon(t)\|_{V^*}^2\\ &\leq \frac{1}{2}\|\vartheta_0\|_V^2 + \frac{\ell}{2\gamma}\|\partial_t\varphi_\varepsilon(0)\|_{V^*}^2 + \|f_\varepsilon\|_{L^2(0,T;H)}^2 + c\int_0^t\frac{\ell}{2\gamma}\|\partial_t\varphi_\varepsilon(s)\|_{V^*}^2\,ds\\ &\leq \frac{1}{2}\|\vartheta_0\|_V^2 + \frac{\ell}{2\gamma}\|\partial_t\varphi_\varepsilon(0)\|_{V^*}^2 + \|f_\varepsilon\|_{L^2(0,T;H)}^2 + c\|\varphi_\varepsilon\|_{H^1(0,T;V^*)}^2 + c.\end{aligned} \tag{4.5.13}$$

Since $(-\upsilon\Delta\varphi_0 + \beta_\varepsilon(\varphi_0) + \pi(\varphi_0) - \gamma\vartheta_0)$ is bounded in V uniformly with respect to ε according to (4.1.15), we deduce, by comparison in (4.2.20)–(4.2.21), that the second term on the right-hand side of (4.5.13) is estimated by a positive constant. Hence, due to (4.1.14), (4.2.14)–(4.2.18) and (4.3.40), the right-hand side of (4.5.13) is bounded and we conclude that there exists a positive constant c, independent of ε, such that

$$\|\vartheta_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\varphi_\varepsilon\|_{W^{1,\infty}(0,T;V^*)\cap H^1(0,T;V)} \leq c. \tag{4.5.14}$$

Eighth a priori estimate. From (4.2.19), we have that

$$\Delta\vartheta_\varepsilon = \partial_t(\vartheta_\varepsilon + \ell\varphi_\varepsilon) + \zeta_\varepsilon - f_\varepsilon =: h_\varepsilon. \tag{4.5.15}$$

We observe that (4.5.14) ensures that h_ε is bounded in $L^2(0,T;H)$ uniformly with respect to ε. Then we infer that there exists a constant $c > 0$, independent of ε, such that

$$\|\vartheta_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le c. \tag{4.5.16}$$

Ninth a priori estimate. Due to (4.5.14)–(4.5.16), from (4.3.24) we deduce that

$$\|\xi_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \le c. \tag{4.5.17}$$

Now, using (4.3.27), we infer that $\|m(\mu_\varepsilon)\|_{L^\infty(0,T)} \le c$. By comparison in (4.2.17) and (4.2.33), it follows that

$$\|\mu_\varepsilon\|_{L^\infty(0,T;V)} \le c. \tag{4.5.18}$$

Moreover, from (4.3.35), we obtain that $\|\xi_\varepsilon\|_{L^\infty(0,T;H)} \le c$. Then, by comparison in (4.2.21), we conclude that

$$\|\Delta\varphi_\varepsilon\|_{L^\infty(0,T;H)\cap L^2(0,T;W)} \le c. \tag{4.5.19}$$

Conclusion of the proof. As (4.5.14), (4.5.16) and (4.5.17)–(4.5.19) follow uniformly with respect to ε, the same estimates hold true for the limiting functions ϑ, φ and μ. Hence, (4.1.16)–(4.1.18) are fulfilled and

$$\|\vartheta\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le c, \tag{4.5.20}$$

$$\|\varphi\|_{W^{1,\infty}(0,T;V^*)\cap H^1(0,T;V)\cap L^\infty(0,T;W)} \le c, \tag{4.5.21}$$

$$\|\mu\|_{L^\infty(0,T;V)} \le c. \tag{4.5.22}$$

4.6 Uniqueness and continuous dependence

This section is devoted to the proof of Theorem 4.1.3.

Assume $a\ell = b$. If f_i, η_i^*, η_{0_i}, φ_{0_i}, $i = 1,2$, are given as in (4.1.2)–(4.1.3) and (η_i, φ_i), $i = 1,2$, are the corresponding solutions of problem $(\widetilde{P})$ (see (4.1.23)–(4.1.29)), then we can write problem $(\widetilde{P})$ for both (η_i, φ_i), $i = 1,2$ and take the difference between the respective equations. Setting $\eta := \eta_1 - \eta_2$, $\varphi := \varphi_1 - \varphi_2$, $\mu := \mu_1 - \mu_2$, $f := f_1 - f_2$, $\eta^* := \eta_1^* - \eta_2^*$, $\eta_0 := \eta_{0_1} - \eta_{0_2}$, $\varphi_0 := \varphi_{0_1} - \varphi_{0_2}$, we obtain that

$$\partial_t\eta - \Delta\eta + b\Delta\varphi - \Delta\eta^* + a(\zeta_1 - \zeta_2) = af, \tag{4.6.1}$$

$$\partial_t\varphi - \Delta\mu = 0, \tag{4.6.2}$$

$$\mu = -\upsilon\Delta\varphi + \xi_1 - \xi_2 + \pi(\varphi_1) - \pi(\varphi_2) - \frac{\gamma}{a}(\eta - b\varphi + \eta^*). \tag{4.6.3}$$

We observe that, due to (4.1.30), $m(\varphi_0) = 0$. Consequently, thanks to (4.1.21), $m(\varphi) = 0$ and $\varphi \in D(\mathcal{N})$ a.e. in $(0, T)$ (see (2.4.24)). Now, we test (4.6.1) by η. Integrating over Q_t, $t \in (0, T]$, we have that

$$\frac{1}{2}\int_\Omega |\eta(t)|^2 + \int_{Q_t} |\nabla\eta|^2 - b\int_{Q_t} \nabla\varphi\cdot\nabla\eta + a\int_{Q_t} (\zeta_1 - \zeta_2)(\eta_1 - \eta_2)$$
$$= \frac{1}{2}\int_\Omega |\eta_0|^2 + \int_{Q_t} (af + \Delta\eta^*)\eta. \tag{4.6.4}$$

We test (4.6.2) by $\frac{b^2}{\upsilon}\mathcal{N}\varphi$. Integrating over $(0, t)$, we obtain that

$$\frac{b^2}{\upsilon}\int_0^t \langle\partial_t\varphi(s), \mathcal{N}\varphi(s)\rangle_{V^*,V}\, ds + \frac{b^2}{\upsilon}\int_{Q_t} \nabla\mu\cdot\nabla\mathcal{N}\varphi = 0,$$
$$\frac{b^2}{2\upsilon}\|\varphi(t)\|^2_{V^*} + \frac{b^2}{\upsilon}\int_{Q_t} \mu\varphi = \frac{b^2}{2\upsilon}\|\varphi_0\|^2_{V^*}. \tag{4.6.5}$$

Testing (4.6.3) by $-\frac{b^2}{\upsilon}\varphi$ and integrating over Q_t, we have that

$$-\frac{b^2}{\upsilon}\int_{Q_t} \mu\varphi = -b^2\int_{Q_t} |\nabla\varphi|^2 - \frac{b^2}{\upsilon}\int_{Q_t} (\xi_1 - \xi_2)(\varphi_1 - \varphi_2)$$
$$-\frac{b^2}{\upsilon}\int_{Q_t} [\pi(\varphi_1) - \pi(\varphi_2)](\varphi_1 - \varphi_2) + \frac{\gamma b^2}{a\upsilon}\int_{Q_t} (\eta - b\varphi + \eta^*)\varphi. \tag{4.6.6}$$

Then, we combine (4.6.4)–(4.6.6) and infer that

$$\frac{1}{2}\|\eta(t)\|^2_H + \int_{Q_t} (|\nabla\eta|^2 - b\nabla\varphi\cdot\nabla\eta + b^2|\nabla\varphi|^2) + \frac{b^2}{2\upsilon}\|\varphi(t)\|^2_{V^*}$$
$$+a\int_{Q_t} (\zeta_1 - \zeta_2)(\eta_1 - \eta_2) + \frac{b^2}{\upsilon}\int_{Q_t} (\xi_1 - \xi_2)(\varphi_1 - \varphi_2)$$
$$= -\frac{b^2}{\upsilon}\int_{Q_t} [\pi(\varphi_1) - \pi(\varphi_2)](\varphi_1 - \varphi_2) + \frac{\gamma b^2}{a\upsilon}\int_{Q_t} (\eta - b\varphi + \eta^*)\varphi$$
$$+\frac{b^2}{2\upsilon}\|\varphi_0\|^2_{V^*} + \frac{1}{2}\|\eta_0\|^2_H + \int_{Q_t} (af + \Delta\eta^*)\eta. \tag{4.6.7}$$

Since A and β are maximal monotone, we have that

$$a\int_{Q_t} (\zeta_1 - \zeta_2)(\eta_1 - \eta_2) \geq 0, \tag{4.6.8}$$

$$\frac{b^2}{\upsilon}\int_{Q_t} (\xi_1 - \xi_2)(\varphi_1 - \varphi_2) \geq 0. \tag{4.6.9}$$

Moreover, thanks to the Lipschitz continuity of π, we infer that

$$\begin{aligned}-\frac{b^2}{\upsilon}\int_{Q_t}[\pi(\varphi_1)-\pi(\varphi_2)](\varphi_1-\varphi_2) &\le \frac{b^2}{\upsilon}\int_{Q_t}|\pi(\varphi_1)-\pi(\varphi_2)||\varphi_1-\varphi_2|\\ &\le \frac{C_\pi b^2}{\upsilon}\int_{Q_t}|\varphi|^2. \qquad (4.6.10)\end{aligned}$$

We also notice that the integral involving the gradients is estimated from below in this way:

$$\int_{Q_t}(|\nabla\eta|^2-b\nabla\varphi\cdot\nabla\eta+b^2|\nabla\varphi|^2)\ge\frac{1}{2}\int_{Q_t}(|\nabla\eta|^2+b^2|\nabla\varphi|^2). \qquad (4.6.11)$$

Recalling that

$$-\frac{\gamma b^3}{a\upsilon}\int_{Q_t}|\varphi|^2\le 0, \qquad (4.6.12)$$

applying inequality (2.2.5) to the second and fifth term on the right-hand side of (4.6.7), using (4.6.8)–(4.6.11) and adding to both sides $b^2\int_0^t\|\varphi(s)\|_H^2\,ds$, we infer that

$$\begin{aligned}&\frac{1}{2}\|\eta(t)\|_H^2+\int_{Q_t}|\nabla\eta|^2+b^2\int_0^t\|\varphi(s)\|_V^2\,ds+\frac{b^2}{2\upsilon}\|\varphi(t)\|_{V^*}^2\\ &\le (K+b^2)\int_0^t\|\varphi(s)\|_H^2\,ds+\frac{1}{2}\int_{Q_t}|\eta|^2+\frac{b^2}{2\upsilon}\|\varphi_0\|_{V^*}^2+\frac{1}{2}\|\eta_0\|_H^2+2a^2\|f\|_{L^2(0,T;H)}^2+3T\|\eta^*\|_W^2,\end{aligned} \qquad (4.6.13)$$

where

$$K=\left[\frac{C_\pi b^2}{\upsilon}+2\Big(\frac{\gamma b^2}{a\upsilon}\Big)^2\right].$$

We observe that, for every $\delta>0$,

$$\|\varphi(t)\|_H^2=\langle\varphi(t),\varphi(t)\rangle_{V^*,V}\le\|\varphi(t)\|_{V^*}\|\varphi(t)\|_V\le\frac{\delta}{2}\|\varphi(t)\|_V^2+\frac{1}{2\delta}\|\varphi(t)\|_{V^*}^2. \qquad (4.6.14)$$

Choosing $\delta=\frac{b^2}{K+b^2}$ in (4.6.14), we can estimate the first term of the right-hand side of (4.6.13) as follows:

$$(K+b^2)\int_0^t\|\varphi(s)\|_H^2\,ds\le\frac{b^2}{2}\int_0^t\|\varphi(s)\|_V^2\,ds+\frac{(K+b^2)^2\upsilon}{b^4}\int_0^t\frac{b^2}{2\upsilon}\|\varphi(s)\|_{V^*}^2\,ds. \qquad (4.6.15)$$

Then, due to (4.6.15), from (4.6.13) we obtain that

$$\begin{aligned}&\frac{1}{2}\|\eta(t)\|_H^2+\int_{Q_t}|\nabla\eta|^2+\frac{b^2}{2}\int_0^t\|\varphi(s)\|_V^2\,ds+\frac{b^2}{2\upsilon}\|\varphi(t)\|_{V^*}^2\\ &\le c\int_0^t\left(\frac{1}{2}\|\eta(s)\|_H^2+\frac{b^2}{2\upsilon}\|\varphi(s)\|_{V^*}^2\right)ds+\frac{b^2}{2\upsilon}\|\varphi_0\|_{V^*}^2+\frac{1}{2}\|\eta_0\|_H^2+2a^2\|f\|_{L^2(0,T;H)}^2+3T\|\eta^*\|_W^2.\end{aligned} \qquad (4.6.16)$$

Due to (4.1.2)–(4.1.6), the last four terms on the right-hand side of (4.6.16) are bounded uniformly with respect to ε . Then, by applying the Gronwall lemma, we conclude that

$$\|\eta(t)\|_H + \|\nabla\eta\|_{L^2(0,T;H)} + \|\varphi\|_{L^2(0,T;V)} + \|\varphi(t)\|_{V^*}$$
$$\leq c\Big(\|\varphi_0\|_{V^*} + \|\eta_0\|_H + \|f\|_{L^2(0,T;H)} + \|\eta^*\|_W \Big) \tag{4.6.17}$$

for some positive constant c which depends only on Ω, T and the structure (2.4.2)–(2.4.3), (2.4.9)–(2.4.11) and (4.1.1)–(4.1.3) of the system. Now, we recall that (4.6.17) is equivalent to

$$\|\eta_1 - \eta_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\varphi_1 - \varphi_2\|_{L^\infty(0,T;V^*)\cap L^2(0,T;V)}$$
$$\leq c\big(\|\varphi_{0_1} - \varphi_{0_2}\|_{V^*} + \|\eta_{0_1} - \eta_{0_2}\|_H + \|f_1 - f_2\|_{L^2(0,T;H)} + \|\eta_1^* - \eta_2^*\|_W\big). \tag{4.6.18}$$

If $f_1 = f_2$, $\eta_1^* = \eta_2^*$, $\eta_{0_1} = \eta_{0_2}$ and $\varphi_{0_1} = \varphi_{0_2}$, from (4.6.18) we conclude that $\eta_1 = \eta_2$ and $\varphi_1 = \varphi_2$, i.e., the solution of problem $(\widetilde{P})$ (see (4.1.23)–(4.1.29)) is unique. From this fact, we immediately infer the uniqueness of the solution for our initial Problem (P) (see (4.1.7)–(4.1.13)).

4.7 Sliding mode control

This section is devoted to the proof of Theorem 4.1.4. The argument we use in the proof relies in Lemma 2.3.2 (see [2, Lemma 4.1, p. 20]). We assume $a = 1$, $b = \ell$ and $A = \rho\,\mathrm{Sign}$ and consider the approximating problem $(\widetilde{P}_\varepsilon)$ obtained from (P_ε) (see (4.2.19)–(4.2.25)) with the usual change of variables

$$\eta_\varepsilon = \vartheta_\varepsilon + \ell\varphi_\varepsilon - \eta^*, \qquad \eta_{0\varepsilon} = \vartheta_{0\varepsilon} + \ell\varphi_{0\varepsilon} - \eta^*. \tag{4.7.1}$$

We have that

$$\partial_t\eta_\varepsilon - \Delta\eta_\varepsilon + \ell\Delta\varphi_\varepsilon - \Delta\eta^* + \rho\sigma_\varepsilon = f_\varepsilon \ \text{ a.e. in } Q, \tag{4.7.2}$$

$$\partial_t\varphi_\varepsilon - \Delta\mu_\varepsilon = 0 \ \text{ a.e. in } Q, \tag{4.7.3}$$

$$\mu_\varepsilon = -\upsilon\Delta\varphi_\varepsilon + \xi_\varepsilon + \pi(\varphi_\varepsilon) - \gamma(\eta_\varepsilon - \ell\varphi_\varepsilon + \eta^*) \ \text{ a.e. in } Q, \tag{4.7.4}$$

$$\sigma_\varepsilon(t) \in \mathrm{Sign}_\varepsilon(\eta_\varepsilon(t)) \text{ for a.e. } t \in (0,T), \tag{4.7.5}$$

$$\xi_\varepsilon \in \beta_\varepsilon(\varphi_\varepsilon) \text{ a.e. in } Q, \tag{4.7.6}$$

$$\partial_\nu\eta_\varepsilon = \partial_\nu\varphi_\varepsilon = \partial_\nu\mu_\varepsilon = 0 \ \text{ on } \Sigma, \tag{4.7.7}$$

$$\eta_\varepsilon(0) = \eta_{0\varepsilon}, \qquad \varphi_\varepsilon(0) = \varphi_{0\varepsilon} \quad \text{in } \Omega. \tag{4.7.8}$$

Further a priori uniform estimates. We test (4.7.2) by $\partial_t \eta_\varepsilon$ and integrate over Q_t. Recalling that

$$\int_{Q_t} \rho \sigma_\varepsilon \partial_t \eta_\varepsilon = \rho \|\eta_\varepsilon(t)\|_{H,\varepsilon} - \rho \|\eta_0\|_{H,\varepsilon}, \tag{4.7.9}$$

we have that

$$\int_{Q_t} |\partial_t \eta_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla \eta_\varepsilon(t)|^2 + \rho \|\eta_\varepsilon(t)\|_{H,\varepsilon} = \frac{1}{2}\int_\Omega |\nabla \eta_0|^2$$

$$+\rho \|\eta_0\|_{H,\varepsilon} + \int_{Q_t} \Delta \eta^* \partial_t \eta_\varepsilon + \int_{Q_t} f_\varepsilon \partial_t \eta_\varepsilon - \int_{Q_t} \ell \Delta \varphi_\varepsilon \partial_t \eta_\varepsilon. \tag{4.7.10}$$

We observe that $\|\eta_0\|_{H,\varepsilon} \leq \|\eta_0\|_H$ (cf. (2.5.10)). Then, thanks to (4.1.3) and (4.1.15), the first two terms on the right-hand side of (4.7.10) are estimated as follows:

$$\frac{1}{2}\int_\Omega |\nabla \eta_0|^2 + \rho \|\eta_0\|_{H,\varepsilon} \leq c(1+\rho). \tag{4.7.11}$$

Due to (4.1.3) and (2.5.17), applying (2.2.5) to the third and fourth term on the right-hand side of (4.7.10), we have that

$$\int_{Q_t} \Delta \eta^* \partial_t \eta_\varepsilon \leq \frac{1}{4}\int_{Q_t} |\partial_t \eta_\varepsilon|^2 + \int_{Q_t} |\Delta \eta^*|^2 = \frac{1}{4}\int_{Q_t} |\partial_t \eta_\varepsilon|^2 + c, \tag{4.7.12}$$

$$\int_{Q_t} f_\varepsilon \partial_t \eta_\varepsilon \leq \frac{1}{4}\int_{Q_t} |\partial_t \eta_\varepsilon|^2 + \int_{Q_t} |f_\varepsilon|^2 \leq \frac{1}{4}\int_{Q_t} |\partial_t \eta_\varepsilon|^2 + c. \tag{4.7.13}$$

Moreover, integrating by parts the last term of (4.7.10), we formally have that

$$-\int_{Q_t} \ell \Delta \varphi_\varepsilon \partial_t \eta_\varepsilon = \ell \int_{Q_t} \nabla \varphi_\varepsilon \cdot \nabla(\partial_t \eta_\varepsilon)$$

$$= \ell \int_\Omega \nabla \varphi_\varepsilon(t) \cdot \nabla \eta_\varepsilon(t) - \ell \int_\Omega \nabla \varphi_0 \cdot \nabla \eta_0 - \ell \int_{Q_t} \nabla(\partial_t \varphi_\varepsilon) \cdot \nabla \eta_\varepsilon. \tag{4.7.14}$$

Using (2.2.5) and the Hölder inequality, the first term on the right-hand side of (4.7.14) is estimated as follows:

$$\left| \ell \int_\Omega \nabla \varphi_\varepsilon(t) \cdot \nabla \eta_\varepsilon(t) \right| \leq \frac{1}{4}\int_\Omega |\nabla \eta_\varepsilon(t)|^2 + \ell^2 \int_\Omega |\nabla \varphi_\varepsilon(t)|^2$$

$$= \frac{1}{4}\int_\Omega |\nabla \eta_\varepsilon(t)|^2 + \ell^2 \int_\Omega \left| \nabla\left(\varphi_0 + \int_0^t \partial_t \varphi_\varepsilon(s)\, ds \right) \right|^2$$

$$\leq \frac{1}{4}\int_\Omega |\nabla \eta_\varepsilon(t)|^2 + 2\ell^2 \int_\Omega |\nabla \varphi_0|^2 + 2\ell^2 \int_\Omega \left| \int_0^t \nabla(\partial_t \varphi_\varepsilon(s))\, ds \right|^2$$

$$\leq \frac{1}{4}\int_{\Omega}|\nabla\eta_{\varepsilon}(t)|^{2}+2\ell^{2}\int_{\Omega}|\nabla\varphi_{0}|^{2}+2T\ell^{2}\int_{Q_{t}}|\nabla(\partial_{t}\varphi_{\varepsilon})|^{2}. \tag{4.7.15}$$

Due to (4.1.3), the second term on the right-hand side of (4.7.14) and similarly the second term on the right-hand side of (4.7.15) are estimated by a positive constant c independent of ρ and ε. Indeed

$$-\ell\int_{\Omega}\nabla\varphi_{0}\cdot\nabla\eta_{0}\leq \ell^{2}\int_{\Omega}|\nabla\varphi_{0}|^{2}+\frac{1}{4}\int_{\Omega}|\nabla\eta_{0}|^{2}\leq c. \tag{4.7.16}$$

Applying inequality (2.2.5) to the last term on the right-hand side of (4.7.14) we obtain that

$$-\ell\int_{Q_{t}}\nabla(\partial_{t}\varphi_{\varepsilon})\cdot\nabla\eta_{\varepsilon}\leq \frac{1}{4}\int_{Q_{t}}|\nabla\eta_{\varepsilon}|^{2}+\ell^{2}\int_{Q_{t}}|\nabla(\partial_{t}\varphi_{\varepsilon})|^{2}. \tag{4.7.17}$$

Then, thanks to (4.7.11)–(4.7.17), from (4.7.10) we infer that

$$\frac{1}{2}\int_{Q_{t}}|\partial_{t}\eta_{\varepsilon}|^{2}+\frac{1}{4}\int_{\Omega}|\nabla\eta_{\varepsilon}(t)|^{2}+\rho\|\eta_{\varepsilon}(t)\|_{H,\varepsilon}$$

$$\leq c(1+\rho)+\ell^{2}(1+2T)\int_{Q_{t}}|\nabla(\partial_{t}\varphi_{\varepsilon})|^{2}+\frac{1}{4}\int_{Q_{t}}|\nabla\eta_{\varepsilon}|^{2}. \tag{4.7.18}$$

Now, we formally differentiate (4.7.3) and (4.7.4) with respect to time and obtain that

$$\partial_{tt}\varphi_{\varepsilon}-\Delta\partial_{t}\mu_{\varepsilon}=0, \tag{4.7.19}$$

$$\partial_{t}\mu_{\varepsilon}=-\upsilon\Delta\partial_{t}\varphi_{\varepsilon}+\beta_{\varepsilon}'(\varphi_{\varepsilon})\partial_{t}\varphi_{\varepsilon}+\pi'(\varphi_{\varepsilon})\partial_{t}\varphi_{\varepsilon}-\gamma(\partial_{t}\eta_{\varepsilon}-\ell\partial_{t}\varphi_{\varepsilon}). \tag{4.7.20}$$

According to (4.2.3), $m(\partial_{t}\varphi_{\varepsilon})=0$. Consequently, $\partial_{t}\varphi_{\varepsilon}\in D(\mathcal{N})$ and we can test (4.7.19) by $\mathcal{N}(\partial_{t}\varphi_{\varepsilon})$ and (4.7.20) by $\partial_{t}\varphi_{\varepsilon}$, respectively. Integrating over Q_{t}, we have that

$$-\int_{Q_{t}}\partial_{t}\mu_{\varepsilon}\partial_{t}\varphi_{\varepsilon}=\frac{1}{2}\|\partial_{t}\varphi_{\varepsilon}(t)\|_{V^{*}}^{2}-\frac{1}{2}\|\partial_{t}\varphi_{\varepsilon}(0)\|_{V^{*}}^{2}, \tag{4.7.21}$$

$$\int_{Q_{t}}\partial_{t}\mu_{\varepsilon}\partial_{t}\varphi_{\varepsilon}=\upsilon\int_{Q_{t}}|\nabla\partial_{t}\varphi_{\varepsilon}|^{2}+\int_{Q_{t}}\beta_{\varepsilon}'(\varphi_{\varepsilon})|\partial_{t}\varphi_{\varepsilon}|^{2}$$

$$+\int_{Q_{t}}\pi'(\varphi_{\varepsilon})|\partial_{t}\varphi_{\varepsilon}|^{2}-\gamma\int_{Q_{t}}\partial_{t}\varphi_{\varepsilon}\partial_{t}\eta_{\varepsilon}+\ell\gamma\int_{Q_{t}}|\partial_{t}\varphi_{\varepsilon}|^{2}. \tag{4.7.22}$$

Combining (4.7.21) and (4.7.22) we obtain that

$$\frac{1}{2}\|\partial_{t}\varphi_{\varepsilon}(t)\|_{V^{*}}^{2}+\upsilon\int_{Q_{t}}|\nabla\partial_{t}\varphi_{\varepsilon}|^{2}+\ell\gamma\int_{Q_{t}}|\partial_{t}\varphi_{\varepsilon}|^{2}=\frac{1}{2}\|\partial_{t}\varphi_{\varepsilon}(0)\|_{V^{*}}^{2}$$

$$-\int_{Q_{t}}\beta_{\varepsilon}'(\varphi_{\varepsilon})|\partial_{t}\varphi_{\varepsilon}|^{2}-\int_{Q_{t}}\pi'(\varphi_{\varepsilon})|\partial_{t}\varphi_{\varepsilon}|^{2}+\gamma\int_{Q_{t}}\partial_{t}\eta_{\varepsilon}\partial_{t}\varphi_{\varepsilon}. \tag{4.7.23}$$

Thanks to (4.1.3) and (4.1.15), the first term on the right-hand side of (4.7.23) is bounded by a positive constant c independent of ρ and ε (cf. the analogous bound discussed below

(4.5.13)). Since β_ε is maximal monotone, the second term on the right-hand side of (4.7.23) is non-positive. As π is a Lipschitz continuous function with Lipschitz constant C_π, we have that

$$-\int_{Q_t} \pi'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2 \le C_\pi \int_{Q_t} |\partial_t\varphi_\varepsilon|^2. \tag{4.7.24}$$

Finally, using (2.2.5), the last term on the right-hand side of (4.7.23) is estimated as follows:

$$\gamma\int_{Q_t} \partial_t\eta_\varepsilon\partial_t\varphi_\varepsilon \le \frac{1}{4}\left(\frac{\upsilon}{\ell^2(1+2T)+1}\right)\int_{Q_t}|\partial_t\eta_\varepsilon|^2 + \gamma^2\frac{\ell^2(1+2T)+1}{\upsilon}\int_{Q_t}|\partial_t\varphi_\varepsilon|^2, \tag{4.7.25}$$

where the reason of such involved constants will be clear in a moment. Due to (4.7.24)–(4.7.25) and the previous observations, from (4.7.23) we infer that

$$\frac{1}{2}\|\partial_t\varphi_\varepsilon(t)\|^2_{V^*} + \upsilon\int_{Q_t}|\nabla\partial_t\varphi_\varepsilon|^2 + \ell\gamma\int_{Q_t}|\partial_t\varphi_\varepsilon|^2$$

$$\le c + \frac{1}{4}\left(\frac{\upsilon}{\ell^2(1+2T)+1}\right)\int_{Q_t}|\partial_t\eta_\varepsilon|^2 + \left(\gamma^2\frac{\ell^2(1+2T)+1}{\upsilon} + C_\pi\right)\int_{Q_t}|\partial_t\varphi_\varepsilon|^2. \tag{4.7.26}$$

Multiplying (4.7.26) by $(\ell^2(1+2T)+1)/\upsilon$ and adding it to (4.7.18), we infer that

$$\frac{1}{4}\int_{Q_t}|\partial_t\eta_\varepsilon|^2 + \frac{1}{4}\int_\Omega|\nabla\eta_\varepsilon(t)|^2 + \rho\|\eta_\varepsilon(t)\|_{H,\varepsilon} + \int_{Q_t}|\nabla\partial_t\varphi_\varepsilon|^2 + C_1\int_{Q_t}|\partial_t\varphi_\varepsilon|^2$$

$$+C_2\|\partial_t\varphi_\varepsilon(t)\|^2_{V^*} \le c(1+\rho) + \frac{1}{4}\int_{Q_t}|\nabla\eta_\varepsilon|^2 + C_3\int_{Q_t}|\partial_t\varphi_\varepsilon|^2, \tag{4.7.27}$$

where

$$C_1 = \frac{\ell^3\gamma(1+2T)+\ell\gamma}{\upsilon}, \qquad C_2 = \frac{\ell^2(1+2T)+1}{2\upsilon},$$

$$C_3 = \gamma^2\left(\frac{\ell^2(1+2T)+1}{\upsilon}\right)^2 + C_\pi\frac{\ell^2(1+2T)+1}{\upsilon} + \ell^2(1+2T).$$

Denoting by C_4 the minimum between 1 and C_1, and applying the inequality (2.2.3) with $\delta = \sqrt{C_4}/\sqrt{2C_3}$ to the last term on the right-hand side of (4.7.27), we obtain that

$$C_3\int_{Q_t}|\partial_t\varphi_\varepsilon|^2 \le \frac{C_4}{2}\int_0^t\|\partial_t\varphi_\varepsilon(s)\|^2_V\,ds + 2K^2C_3\int_0^t\|\partial_t\varphi_\varepsilon(s)\|^2_{V^*}\,ds. \tag{4.7.28}$$

Thanks to (4.7.28), from (4.7.27) we infer that

$$\frac{1}{4}\int_{Q_t}|\partial_t\eta_\varepsilon|^2 + \frac{1}{4}\int_\Omega|\nabla\eta_\varepsilon(t)|^2 + \rho\|\eta_\varepsilon(t)\|_{H,\varepsilon} + \frac{C_4}{2}\int_0^t\|\partial_t\varphi_\varepsilon(s)\|^2_V + C_2\|\partial_t\varphi_\varepsilon(t)\|^2_{V^*}$$

$$\le c(1+\rho) + \frac{1}{4}\int_{Q_t}|\nabla\eta_\varepsilon|^2 + 2K^2C_3\int_0^t\|\partial_t\varphi_\varepsilon(s)\|^2_{V^*}. \tag{4.7.29}$$

From (4.7.29), by applying the Gronwall lemma, we conclude that

$$\|\partial_t\eta_\varepsilon\|_{L^2(0,T;H)} + \|\eta_\varepsilon\|_{L^\infty(0,T;V)} + \|\partial_t\varphi_\varepsilon\|_{L^\infty(0,T;V^*)} + \|\partial_t\varphi_\varepsilon\|_{L^2(0,T;V)} \le c(1+\rho^{1/2}), \quad (4.7.30)$$

whence

$$\|\eta_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c(1+\rho^{1/2}), \quad (4.7.31)$$

$$\|\varphi_\varepsilon\|_{W^{1,\infty}(0,T;V^*)\cap H^1(0,T;V)} \le c(1+\rho^{1/2}). \quad (4.7.32)$$

Due to (4.7.31)–(4.7.32) and the change of variables stated by (4.7.1), we have that

$$\|\vartheta_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c(1+\rho^{1/2}). \quad (4.7.33)$$

Proceeding as in the second a priori estimate (cf. (4.3.15)–(4.3.23)) and recalling (4.7.32)–(4.7.33), from (4.3.24) we infer that

$$\|\xi_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \le c(1+\rho^{1/2}). \quad (4.7.34)$$

Now, with the analogous technique applied in the third a priori estimate, thanks to (4.7.32)–(4.7.34), from (4.3.27) we obtain that

$$\|m(\mu_\varepsilon)\|_{L^\infty(0,T)} \le c(1+\rho^{1/2}). \quad (4.7.35)$$

Then, due to (4.7.35) and the Poincaré inequality, by comparison in (4.7.3) we deduce that

$$\|\mu_\varepsilon\|_{L^\infty(0,T;V)} \le c(1+\rho^{1/2}). \quad (4.7.36)$$

Finally, with the same computations as explained in the fifth a priori estimate (cf. (4.3.32)–(4.3.34)), thanks to (4.7.32)–(4.7.33) and (4.7.36), from (4.3.35) we infer that

$$\|\xi_\varepsilon\|_{L^\infty(0,T;H)} \le c(1+\rho^{1/2}), \quad (4.7.37)$$

whence, by comparison of every term in (4.7.4), we conclude that

$$\|\Delta\varphi_\varepsilon\|_{L^\infty(0,T;H)} \le c(1+\rho^{1/2}). \quad (4.7.38)$$

Existence of sliding mode. Due to (4.1.3), (2.5.17) and (4.7.38), we can rewrite (4.7.2) in the form

$$\partial_t\eta_\varepsilon - \Delta\eta_\varepsilon + \rho\sigma_\varepsilon = g_\varepsilon := f_\varepsilon - \ell\Delta\varphi_\varepsilon + \Delta\eta^*, \quad (4.7.39)$$

with

$$\|g_\varepsilon\|_{L^\infty(0,T;H)} \le c(1+\rho^{1/2}), \quad (4.7.40)$$

where c depends only on the structure and the data involved in the statement. In order to prove the existence of sliding mode, we fix the constant c appearing in (4.7.40) and set

$$\rho^* := c^2 + 2c + \frac{2}{T}\|\vartheta_0 + \ell\varphi_0 - \eta^*\|_H \quad (4.7.41)$$

and assume $\rho > \rho^*$. We also set

$$\psi_\varepsilon(t) := \|\eta_\varepsilon(t)\|_H \quad \text{for } t \in [0,T]. \tag{4.7.42}$$

By assuming $h \in (0,T)$ and $t \in (0, T-h)$, we multiply (4.7.39) by $\sigma_\varepsilon = \mathrm{Sign}_\varepsilon(\eta_\varepsilon)$ and integrate over $(t, t+h) \times \Omega$. We have that

$$\int_t^{t+h} (\partial_t \eta_\varepsilon(s), \sigma_\varepsilon(s))_H \; ds + \int_t^{t+h} \int_\Omega \nabla \eta_\varepsilon \cdot \nabla \sigma_\varepsilon + \rho \int_t^{t+h} \|\sigma_\varepsilon(s)\|_H^2 \; ds$$
$$= \int_t^{t+h} (g_\varepsilon(s), \sigma_\varepsilon(s))_H \; ds. \tag{4.7.43}$$

Recalling that $\mathrm{Sign}_\varepsilon(v)$ is the gradient at v of the C^1 functional $\|\cdot\|_{H,\varepsilon}$, from (2.5.10)–(2.5.11) we deduce that

$$(\partial_t \eta_\varepsilon(s), \sigma_\varepsilon(s))_H = \frac{d}{dt} \int_0^{\psi_\varepsilon(t)} \min\{s/\varepsilon,\ 1\} \; ds \quad \text{for a.a. } t \in (0,T).$$

Then, for the first term on the right-hand side of (4.7.43) we have that

$$\int_t^{t+h} (\partial_t \eta_\varepsilon(s), \sigma_\varepsilon(s))_H \; ds = \int_{\psi_\varepsilon(t)}^{\psi_\varepsilon(t+h)} \min\{s/\varepsilon, 1\} \; ds.$$

We also notice that (2.5.11) implies that

$$\nabla \eta_\varepsilon(t) \cdot \nabla \sigma_\varepsilon(t) = \frac{|\nabla \eta_\varepsilon(t)|^2}{\max\{\varepsilon, \|\eta_\varepsilon(t)\|_H\}} \geq 0 \quad \text{a.e. in } \Omega, \text{ for a.e. } t \in (0,T),$$

whence the second integral on the left-hand side of (4.7.43) is nonnegative. Moreover, as $\|\sigma_\varepsilon(s)\|_H \leq 1$ for every s (see (2.4.15)) and (4.7.40) holds, we infer from (4.7.43) that

$$\int_{\psi_\varepsilon(t)}^{\psi_\varepsilon(t+h)} \min\{s/\varepsilon, 1\} \; ds + \rho \int_t^{t+h} \|\sigma_\varepsilon(s)\|_H^2 \; ds \leq hc(\rho^{1/2} + 1). \tag{4.7.44}$$

At this point, we let $\varepsilon \searrow 0$. Due to (4.4.8)–(4.4.9), (4.7.1) and the uniqueness of the solution of the limit Problem (4.1.23)–(4.1.29) (cf. Theorem 4.1.3) we have that

$$\eta_\varepsilon \to \eta \qquad \text{in } C^0(0,T;H). \tag{4.7.45}$$

Besides, using standard weak, weakstar and compactness results, from (4.7.44) we infer that

$$\sigma_\varepsilon \rightharpoonup^* \sigma \qquad \text{in } L^\infty(0,T;H). \tag{4.7.46}$$

Then, taking the limit as $\varepsilon \searrow 0$ in (4.7.44) and denoting by

$$\psi(t) := \|\eta(t)\|_H \qquad \text{for } t \in [0,T], \tag{4.7.47}$$

we obtain that

$$\psi(t+h) - \psi(t) + \rho \int_t^{t+h} \|\sigma(s)\|_H^2 \, ds$$

$$\leq \lim_{\varepsilon \searrow 0} \int_{\psi_\varepsilon(t)}^{\psi_\varepsilon(t+h)} \min\{s/\varepsilon, 1\} \, ds + \rho \liminf_{\varepsilon \searrow 0} \int_t^{t+h} \|\sigma_\varepsilon(s)\|_H^2 \, ds \leq hc(\rho^{1/2} + 1) \qquad (4.7.48)$$

for every $h \in (0, T)$ and $t \in (0, T - h)$. Finally, we multiply (4.7.48) by $1/h$ and let h tend to zero. We conclude that

$$\psi'(t) + \rho \|\sigma(t)\|_H^2 \leq c(\rho^{1/2} + 1) \quad \text{for a.a. } t \in (0, T). \qquad (4.7.49)$$

As $\|\sigma(t)\|_H = 1$ if $\|\eta(t)\|_H > 0$ (see (2.4.15)), we can apply Lemma 2.3.2 with $a_0 = b_0 = c$ and we observe that our condition $\rho > \rho^*$ completely fits the assumptions by (4.7.41). Thus, we find $T^* \in [0, T)$ such that $\eta(t) = 0$ for every $t \in [T^*, T]$, i.e., (4.1.35).

Chapter 5

Singular system related to a sliding mode control problem

In this chapter we consider a singular phase–field system located in a smooth and bounded three-dimensional domain. The entropy balance equation is perturbed by a logarithmic nonlinearity and by the presence of an additional term involving a possibly nonlocal maximal monotone operator and arising from a class of sliding mode control problems. The second equation of the system accounts for the phase dynamics, and it is deduced from a balance law for the microscopic forces that are responsible for the phase transition process. The resulting system is highly nonlinear; the main difficulties lie in the contemporary presence of two nonlinearities, one of which under time derivative, in the entropy balance equation. Consequently, we are able to prove only the existence of solutions. To this aim, we will introduce a backward finite differences scheme and argue on this by proving uniform estimates and passing to the limit on the time step.

5.1 Statement of the problem and results

As far as the data of our problem are concerned, let ℓ and $k > 0$ be two real constants. We also consider the data f, ϑ^*, ϑ_0 and φ_0 such that

$$f \in H^1(0,T;H) \cap L^1(0,T;L^\infty(\Omega)), \tag{5.1.1}$$

$$\vartheta^* \in W, \quad \vartheta^* > 0 \ \text{ in } \Omega, \tag{5.1.2}$$

$$\vartheta_0 \in V, \quad \vartheta_0 > 0 \ \text{ a.e. in } \Omega, \quad \ln \vartheta_0 \in H, \quad \varphi_0 \in W. \tag{5.1.3}$$

We also assume that

$$\varphi_0 \in D(\beta) \ \text{ a.e. in } \Omega, \text{ and there exists } \xi_0 \in H \text{ such that } \xi_0 \in \beta(\varphi_0) \ \text{ a.e. in } \Omega, \tag{5.1.4}$$

whence

$$\tilde\beta(\varphi_0) \in L^1(\Omega). \tag{5.1.5}$$

Indeed, thanks to the definition of the subdifferential and to (2.4.2), we have that

$$0 \leq \int_\Omega \tilde{\beta}(\varphi_0) \leq (\xi_0, \varphi_0) \leq \|\xi_0\|_H \|\varphi_0\|_H.$$

Our aim is to find a quadruplet $(\vartheta, \varphi, \zeta, \xi)$ satisfying the regularity conditions

$$\vartheta \in L^2(0,T;V), \tag{5.1.6}$$
$$\vartheta > 0 \quad \text{a.e. in } Q \quad \text{and} \quad \ln\vartheta \in H^1(0,T;V^*) \cap L^\infty(0,T;H), \tag{5.1.7}$$
$$\varphi \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W), \tag{5.1.8}$$
$$\zeta \in L^2(0,T;H), \qquad \xi \in L^2(0,T;H), \tag{5.1.9}$$

and solving the Problem (P) defined by

$$\partial_t(\ln\vartheta(t) + \ell\varphi(t)) - k\Delta\vartheta(t) + \zeta(t) = f(t) \quad \text{in } V^*, \text{ for a.e. } t \in (0,T), \tag{5.1.10}$$
$$\partial_t\varphi - \Delta\varphi + \xi + \pi(\varphi) = \ell\vartheta \quad \text{a.e. in } Q, \tag{5.1.11}$$
$$\zeta(t) \in A(\vartheta(t) - \vartheta^*) \quad \text{for a.e. } t \in (0,T), \tag{5.1.12}$$
$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q, \tag{5.1.13}$$
$$\partial_\nu\vartheta = 0, \quad \partial_\nu\varphi = 0 \quad \text{in the sense of traces on } \Sigma, \tag{5.1.14}$$
$$\ln\vartheta(0) = \ln\vartheta_0, \qquad \varphi(0) = \varphi_0 \quad \text{a.e. in } \Omega. \tag{5.1.15}$$

Here, we pointed out the boundary conditions (5.1.14) although they are already contained in the specified meaning of $-\Delta$ (cf. (2.1.2)). By the way, a variational formulation of (5.1.10) reads

$$\langle \partial_t(\ln\vartheta(t) + \ell\varphi(t)) + \zeta(t), v\rangle + k\int_\Omega \nabla\vartheta(t)\cdot\nabla v = \int_\Omega f(t)v \tag{5.1.16}$$

for all $v \in V$ and for a.e. $t \in (0,T)$. About the initial conditions in (5.1.15), note that from (5.1.7) it follows that $\ln\vartheta$ is at least weakly continuous from $[0,T]$ to H.

The following result is concerned with the existence of solutions to Problem (P).

Theorem 5.1.1. *Assume* (5.1.1)–(2.4.11). *Then the Problem* (P) *stated by* (5.1.10)–(5.1.15) *has at least a solution* $(\vartheta, \varphi, \zeta, \xi)$ *satisfying* (5.1.6)–(5.1.9) *and the regularity properties*

$$\vartheta \in L^\infty(0,T;V), \qquad \zeta \in L^\infty(0,T;H), \tag{5.1.17}$$

$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \qquad \xi \in L^\infty(0,T;H). \tag{5.1.18}$$

The proof of Theorem 5.1.1 will be given in the subsequent three sections.

5.2 The approximating problem (P_τ)

First of all, let us underline that, for simplicity, in this chapter the same symbol β and A will be used for the maximal monotone operators induced by β and A on $H \equiv L^2(\Omega)$ and $L^2(0,T;H) \equiv L^2(Q)$.

In order to prove the existence theorem, first we introduce a backward finite differences scheme. Assume that N is a positive integer and let Z be any normed space. By fixing the time step

$$\tau = T/N, \quad N \in \mathbb{N},$$

we introduce the interpolation maps from Z^{N+1} into either $L^\infty(0,T;Z)$ or $W^{1,\infty}(0,T;Z)$. For $(z^0, z^1, \dots, z^N) \in Z^{N+1}$, we define the piecewise constant functions $\overline{z}_\tau$ and the piecewise linear functions $\widehat{z}_\tau$, respectively:

$$\overline{z}_\tau \in L^\infty(0,T;Z), \quad \overline{z}((i+s)\tau) = z^{i+1}, \quad \widehat{z}_\tau \in W^{1,\infty}(0,T;Z),$$
$$\widehat{z}((i+s)\tau) = z^i + s(z^{i+1} - z^i), \qquad \text{if } 0 < s < 1 \text{ and } i = 0, \dots, N-1. \tag{5.2.1}$$

By a direct computation, it is straightforward to prove that

$$\|\overline{z}_\tau - \widehat{z}_\tau\|_{L^\infty(0,T;Z)} = \max_{i=0,\dots,N-1} \|z_{i+1} - z_i\|_Z = \tau \|\partial_t \widehat{z}_\tau\|_{L^\infty(0,T;Z)}, \tag{5.2.2}$$

$$\|\overline{z}_\tau - \widehat{z}_\tau\|^2_{L^2(0,T;Z)} = \frac{\tau}{3} \sum_{i=0}^{N-1} \|z_{i+1} - z_i\|_Z^2 = \frac{\tau^2}{3} \|\partial_t \widehat{z}_\tau\|^2_{L^2(0,T;Z)}, \tag{5.2.3}$$

$$\begin{aligned} \|\overline{z}_\tau - \widehat{z}_\tau\|^2_{L^\infty(0,T;Z)} &= \max_{i=0,\dots,N-1} \|z_{i+1} - z_i\|_Z^2 \\ &\le \sum_{i=0}^{N-1} \tau^2 \left\| \frac{z_{i+1} - z_i}{\tau} \right\|_Z^2 \le \tau \|\partial_t \widehat{z}_\tau\|^2_{L^2(0,T;Z)}. \end{aligned} \tag{5.2.4}$$

Then, we consider the approximating problem (P_τ). We set

$$f^i := \frac{1}{\tau} \int_{(i-1)\tau}^{i\tau} f(s)\, ds, \qquad \text{for } i = 1, \dots, N, \tag{5.2.5}$$

and we look for two vectors $(\vartheta^0, \vartheta^1, \dots, \vartheta^N) \in V^{N+1}$, $(\varphi^0, \varphi^1, \dots, \varphi^N) \in W^{N+1}$ satisfying, for $i = 1, \dots, N$, the system

$$\vartheta^i > 0 \quad \text{a.e. in } \Omega, \quad \ln \vartheta^i \in H, \quad \exists\, \zeta^i,\, \xi^i \in H \quad \text{such that} \tag{5.2.6}$$

$$\tau^{1/2}\vartheta^i + \ln \vartheta^i + \ell\varphi^i + \tau\zeta^i - \tau k \Delta \vartheta^i = \tau f^i + \tau^{1/2}\vartheta^{i-1} + \ln \vartheta^{i-1} + \ell \varphi^{i-1} \quad \text{a.e. in } \Omega, \tag{5.2.7}$$

$$\varphi^i - \tau\Delta\varphi^i + \tau\xi^i + \tau\pi(\varphi^i) = \varphi^{i-1} + \tau\ell\vartheta^i \quad \text{a.e. in } \Omega, \tag{5.2.8}$$

$$\zeta^i \in A(\vartheta^i - \vartheta^*) \quad \text{a.e. in } \Omega, \tag{5.2.9}$$

$$\xi^i \in \beta(\varphi^i) \quad \text{a.e. in } \Omega, \tag{5.2.10}$$

$$\partial_\nu \vartheta^i = \partial_\nu \varphi^i = 0 \quad \text{a.e. on } \Gamma, \tag{5.2.11}$$

$$\vartheta^0 = \vartheta_0, \quad \varphi^0 = \varphi_0 \quad \text{a.e. in } \Omega. \tag{5.2.12}$$

In view of (5.1.1)–(5.1.3), we infer that for $i = 1$ the right-hand side of (5.2.7) is an element of H, and for any given φ^1 (present in the left-hand side) we have to find the corresponding ϑ^1, along with ξ^1, fulfilling (5.2.6)–(5.2.7) and (5.2.9); in case we succeed, from a comparison in (5.2.7) it will turn out that $\vartheta^1 \in W$. Then, we insert ϑ^1, depending on φ^1, in the right-hand side of (5.2.8) and we seek somehow a fixed point φ^1 , together with $\xi^1 \in H$, satisfying (5.2.8) and (5.2.10). Once we recover φ^1 and the related ϑ^1, we can start again our procedure, and so on. Then, it is important to show that, for a fixed i and known data f^i, ϑ^{i-1}, $\ln \vartheta^{i-1}$, φ^{i-1} we are able to find a pair (ϑ^i, φ^i) solving (5.2.6)–(5.2.11).

Theorem 5.2.1. *There exists some fixed value $\tau_1 \le \min\{1, T\}$, depending only on the data, such that for any time step $0 < \tau < \tau_1$ the approximating problem (P_τ) stated by* (5.2.6)–(5.2.12) *has a unique solution*

$$(\vartheta^0, \vartheta^1, \dots, \vartheta^N) \in V \times W^N, \qquad (\varphi^0, \varphi^1, \dots, \varphi^N) \in W^{N+1}.$$

Let us now rewrite the discrete equation (5.2.7)–(5.2.12) by using the piecewise constant and piecewise linear functions defined in (5.2.1), with obvious notation, and obtain that

$$\tau^{1/2}\partial_t\widehat{\vartheta}_\tau + \partial_t\widehat{\ln\vartheta}_\tau + \ell\partial_t\widehat{\varphi}_\tau + \overline{\zeta}_\tau - k\Delta\overline{\vartheta}_\tau = \overline{f}_\tau \quad \text{a.e. in } Q, \tag{5.2.13}$$

$$\partial_t\widehat{\varphi}_\tau - \Delta\overline{\varphi}_\tau + \overline{\xi}_\tau + \pi(\overline{\varphi}_\tau) = \ell\overline{\vartheta}_\tau \quad \text{a.e. in } Q, \tag{5.2.14}$$

$$\overline{\zeta}_\tau(t) \in A(\overline{\vartheta}_\tau(t) - \vartheta^*) \quad \text{for a.e. } t \in (0,T), \tag{5.2.15}$$

$$\overline{\xi}_\tau \in \beta(\overline{\varphi}_\tau) \quad \text{a.e. in } Q, \tag{5.2.16}$$

$$\partial_\nu\overline{\vartheta}_\tau = \partial_\nu\overline{\varphi}_\tau = 0 \quad \text{a.e. on } \Sigma, \tag{5.2.17}$$

$$\widehat{\vartheta}_\tau(0) = \vartheta_0, \quad \widehat{\varphi}_\tau(0) = \varphi_0 \quad \text{a.e. in } \Omega. \tag{5.2.18}$$

5.2.1 The auxiliary approximating problem (AP_ε)

In this subsection we introduce the auxiliary approximating problem (AP_ε) obtained by considering the approximating problem (P_τ) at each step $i = 1, \dots, N$ and replacing the monotone operators appearing in (5.2.6)–(5.2.12) with their Yosida regularizations. About general properties of maximal monotone operators and subdifferentials of convex functiions, we refer the reader to [1, 11].

Definition of the auxiliary approximating problem (AP_ε). We fix τ and specify an auxiliary approximating problem (AP_ε), which is obtained by considering (5.2.6)–(5.2.11) for a fixed i and introducing the regularized operators defined above. We set

$$g := \tau f^i + \tau^{1/2}\vartheta^{i-1} + \ln\vartheta^{i-1} + \ell\varphi^{i-1}, \qquad h := \varphi^{i-1}, \tag{5.2.19}$$

and note that both g and h are prescribed elements of H (cf. (5.2.5), (5.1.1), (5.1.3), (5.1.3) and (5.2.6)). We look for a pair $(\Theta_\varepsilon, \Phi_\varepsilon)$ such that

$$\tau^{1/2}\Theta_\varepsilon + \ln_\varepsilon \Theta_\varepsilon + \tau A_\varepsilon(\Theta_\varepsilon - \vartheta^*) - \tau k \Delta\Theta_\varepsilon = -\ell\Phi_\varepsilon + g \quad \text{a.e. in } \Omega, \tag{5.2.20}$$

$$\Phi_\varepsilon - \tau\Delta\Phi_\varepsilon + \tau\beta_\varepsilon(\Phi_\varepsilon) + \tau\pi(\Phi_\varepsilon) = h + \tau\ell\Theta_\varepsilon \quad \text{a.e. in } \Omega, \tag{5.2.21}$$

where $\ln_\varepsilon$, A_ε and β_ε are the Yosida regularization of ln, A and β defined by (2.5.12), (2.5.6) and (2.5.1), respectively. Here, according to the extended meaning of $-\Delta$ (see (2.1.2)), we omit the specification of the boundary conditions as with (5.2.11).

Theorem 5.2.2. *Let $g, h \in H$. Then there exists some fixed value $\tau_2 \le \min\{1, T\}$, depending only on the data, such that for every time step $\tau \in (0, \tau_2)$ and for all $\varepsilon \in (0, 1]$ the auxiliary approximating problem (AP_ε) stated by (5.2.20)–(5.2.21) has a unique solution $(\Theta_\varepsilon, \Phi_\varepsilon)$.*

5.2.2 Existence of a solution for (AP_ε)

In order to prove the existence of the solution for the auxiliary approximating problem (AP_ε) we intend to apply [1, Corollary 1.3, p. 48]. To this aim, we point out that, for τ small enough, the two operators

$$[\tau^{1/2} I + \ln_\varepsilon + \tau A_\varepsilon(\,\cdot\, - \vartheta^*) - \tau k\Delta] \quad \text{appearing in (5.2.20)}, \tag{5.2.22}$$

$$[I + \tau\beta_\varepsilon + \tau\pi - \tau\Delta] \quad \text{appearing in (5.2.21)}, \tag{5.2.23}$$

both with domain W and range H, are maximal monotone and coercive. Indeed, they are the sum of a monotone, Lipschitz continuous and coercive operator:

$$\tau^{1/2} I + \ln_\varepsilon + \tau A_\varepsilon(\,\cdot\, - \vartheta^*) \quad \text{in (5.2.22), and} \quad I + \tau\beta_\varepsilon + \tau\pi \quad \text{in (5.2.23)},$$

and of a maximal monotone operator that is $-\Delta$ with a positive coefficient in front. We now check our first claim. Letting v_1, $v_2 \in H$, we have that

$$\begin{aligned} &\big((\tau^{1/2} I + \ln_\varepsilon + \tau A_\varepsilon(\,\cdot\, - \vartheta^*))(v_1) - (\tau^{1/2} I + \ln_\varepsilon + \tau A_\varepsilon(\,\cdot\, - \vartheta^*))(v_2), v_1 - v_2\big) \\ &\ge \tau^{1/2}\|v_1 - v_2\|_H^2 + \big(\ln_\varepsilon(v_1) - \ln_\varepsilon(v_2), v_1 - v_2\big) \\ &\quad + \tau\big(A_\varepsilon(v_1 - \vartheta^*) - A_\varepsilon(v_2 - \vartheta^*), (v_1 - \vartheta^*) - (v_2 - \vartheta^*)\big). \end{aligned}$$

Due to the monotonicity of $\ln_\varepsilon$ and A_ε, we have that the last two terms on the right-hand side are nonnegative, so that

$$\begin{aligned} &\big((\tau^{1/2} I + \ln_\varepsilon + \tau A_\varepsilon(\,\cdot\, - \vartheta^*))(v_1) - (\tau^{1/2} I + \ln_\varepsilon + \tau A_\varepsilon(\,\cdot\, - \vartheta^*))(v_2), v_1 - v_2\big) \\ &\ge \tau^{1/2}\|v_1 - v_2\|_H^2, \end{aligned} \tag{5.2.24}$$

i.e., the operator $\tau^{1/2} I + \ln_\varepsilon + \tau A_\varepsilon(\,\cdot\, - \vartheta^*)$ is strongly monotone, hence coercive, in H. Next, for all v_1, $v_2 \in H$ we have that

$$\begin{aligned} &((I + \tau\beta_\varepsilon + \tau\pi)(v_1) - (I + \tau\beta_\varepsilon + \tau\pi)(v_2), v_1 - v_2) \\ &\ge \|v_1 - v_2\|_H^2 + \tau(\beta_\varepsilon(v_1) - \beta_\varepsilon(v_2), v_1 - v_2) - C_\pi\tau\|v_1 - v_2\|_H^2. \end{aligned} \tag{5.2.25}$$

where C_π denotes a Lipschitz constant for π. Since β_ε is monotone, it turns out that

$$\tau(\beta_\varepsilon(v_1) - \beta_\varepsilon(v_2), v_1 - v_2) \geq 0$$

and, choosing $\tau_2 \leq 1/2C_\pi$, from (5.2.25) we infer that

$$((I + \tau\beta_\varepsilon + \tau\pi)(v_1) - (I + \tau\beta_\varepsilon + \tau\pi)(v_1), v_1 - v_2) \geq \frac{1}{2}\|v_1 - v_2\|_H^2, \tag{5.2.26}$$

whence the operator $I+\tau\beta_\varepsilon+\tau\pi$ is strongly monotone and coercive in H, for every $\tau \leq \tau_2$.

Now, in order to prove Theorem 5.2.2, we divide the proof into two steps. In the first step, we fix $\overline{\Theta}_\varepsilon \in H$ in place of Θ_ε on the right-hand side of (5.2.21) and find a solution Φ_ε for (5.2.21). In the second step, we insert on the right-hand side of (5.2.20) the element Φ_ε obtained in the first step and find a solution Θ_ε to (5.2.20). Now, let $\overline{\Theta}_{1,\varepsilon}$ and $\overline{\Theta}_{2,\varepsilon}$ be two different input data. We denote by $\Phi_{1,\varepsilon}$, $\Phi_{2,\varepsilon}$ the corresponding solutions for (5.2.21) obtained in the first step and by $\Theta_{1,\varepsilon}$, $\Theta_{2,\varepsilon}$ the related solution of (5.2.20) found in the second step.

Hence, taking the difference between the two equations (5.2.21) written for $\overline{\Theta}_{1,\varepsilon}$ and $\overline{\Theta}_{2,\varepsilon}$ and testing the result by $(\Phi_{1,\varepsilon} - \Phi_{2,\varepsilon})$, we have that

$$\begin{aligned}&\big((I + \tau\beta_\varepsilon + \tau\pi)(\Phi_{1,\varepsilon}) - (I + \tau\beta_\varepsilon + \tau\pi)(\Phi_{2,\varepsilon}), \Phi_{1,\varepsilon} - \Phi_{2,\varepsilon}\big)\\ &\quad + \tau\int_\Omega |\nabla(\Phi_{1,\varepsilon} - \Phi_{2,\varepsilon})|^2 \leq \tau\ell\big(\overline{\Theta}_{1,\varepsilon} - \overline{\Theta}_{2,\varepsilon}, \Phi_{1,\varepsilon} - \Phi_{2,\varepsilon}\big).\end{aligned} \tag{5.2.27}$$

Then, applying (5.2.26) and (2.2.5) to the first term on the left-hand side of (5.2.27) and to the right-hand side of (5.2.27), respectively, we infer that

$$\frac{1}{2}\|\Phi_{1,\varepsilon} - \Phi_{2,\varepsilon}\|_H^2 + \tau\int_\Omega |\nabla(\Phi_{1,\varepsilon} - \Phi_{2,\varepsilon})|^2 \leq \frac{1}{4}\|\Phi_{1,\varepsilon} - \Phi_{2,\varepsilon}\|_H^2 + \tau^2\ell^2\|\overline{\Theta}_{1,\varepsilon} - \overline{\Theta}_{2,\varepsilon}\|_H^2,$$

whence

$$\|\Phi_{1,\varepsilon} - \Phi_{2,\varepsilon}\|_H^2 \leq 4\tau^2\ell^2\|\overline{\Theta}_{1,\varepsilon} - \overline{\Theta}_{2,\varepsilon}\|_H^2. \tag{5.2.28}$$

Now, we take the difference between the corresponding equations (5.2.20) written for the solutions $\Phi_{1,\varepsilon}$, $\Phi_{2,\varepsilon}$ obtained in the first step and test by $(\Theta_{1,\varepsilon} - \Theta_{2,\varepsilon})$. We obtain that

$$\begin{aligned}&\big((\tau^{1/2}I + \ln_\varepsilon + \tau A_\varepsilon(\,\cdot\, - \vartheta^*))(\Theta_{1,\varepsilon}) - (\tau^{1/2}I + \ln_\varepsilon + \tau A_\varepsilon(\,\cdot\, - \vartheta^*))(\Theta_{2,\varepsilon}), \Theta_{1,\varepsilon} - \Theta_{2,\varepsilon}\big)\\ &\quad + \tau k\int_\Omega |\nabla(\Theta_{1,\varepsilon} - \Theta_{2,\varepsilon})|^2 \leq \frac{\ell^2}{2\tau^{1/2}}\|\Phi_{1,\varepsilon} - \Phi_{2,\varepsilon}\|_H^2 + \frac{\tau^{1/2}}{2}\|\Theta_{1,\varepsilon} - \Theta_{2,\varepsilon}\|_H^2.\end{aligned} \tag{5.2.29}$$

By recalling (5.2.24) and using it in the left-hand side of (5.2.29) we infer that

$$\tau^{1/2}\|\Theta_{1,\varepsilon} - \Theta_{2,\varepsilon}\|_H^2 \leq \frac{\ell^2}{\tau^{1/2}}\|\Phi_{1,\varepsilon} - \Phi_{2,\varepsilon}\|_H^2. \tag{5.2.30}$$

Then, by combining this inequality with (5.2.28), we deduce that

$$\|\Theta_{1,\varepsilon} - \Theta_{2,\varepsilon}\|_H^2 \leq 4\tau\ell^4\|\overline{\Theta}_{1,\varepsilon} - \overline{\Theta}_{2,\varepsilon}\|_H^2, \tag{5.2.31}$$

whence we obtain a contraction mapping for every $\tau \leq \tau_2$, provided that $\tau_2 \leq 1/(8\ell^4)$. Finally, by applying the Banach fixed point theorem, we conclude that there exists a unique solution $(\Theta_\varepsilon, \Phi_\varepsilon)$ to the auxiliary problem (AP_ε).

5.2.3 A priori estimates on AP_ε

In this subsection we derive a series of a priori estimates, independent of ε, inferred from the equations (5.2.20)–(5.2.21) of the auxiliary approximating problem (AP_ε).

First a priori estimate. We test (5.2.20) by $\tau(\Theta_\varepsilon - \vartheta^*)$ and (5.2.21) by Φ_ε, then we sum up. By exploiting the cancellation of the suitable corresponding terms and recalling the definition (2.5.15) of Λ_ε, we obtain that

$$\begin{aligned}
&\tau^{3/2}\|\Theta_\varepsilon - \vartheta^*\|_H^2 + \tau\Lambda_\varepsilon(\Theta_\varepsilon) + \tau^2(A_\varepsilon(\Theta_\varepsilon - \vartheta^*), \Theta_\varepsilon - \vartheta^*) + \tau^2 k \int_\Omega |\nabla(\Theta_\varepsilon - \vartheta^*)|^2 \\
&+ \big((I + \tau\beta_\varepsilon + \tau\pi)(\Phi_\varepsilon) - (I + \tau\beta_\varepsilon + \tau\pi)(0), \Phi_\varepsilon\big) + \tau \int_\Omega |\nabla\Phi_\varepsilon|^2 \\
&\leq\; -\tau^{3/2}(\vartheta^*, \Theta_\varepsilon - \vartheta^*) + \tau\Lambda_\varepsilon(\vartheta^*) - \tau^2 k \int_\Omega \nabla\vartheta^* \cdot \nabla(\Theta_\varepsilon - \vartheta^*) \\
&\quad + \ell\tau(\Phi_\varepsilon, \vartheta^*) + \tau(g, \Theta_\varepsilon - \vartheta^*) - \tau(\pi(0), \Phi_\varepsilon) + (h, \Phi_\varepsilon). \qquad (5.2.32)
\end{aligned}$$

Let us note that all terms on the left-hand side are nonnegative; in particular, recalling (5.2.26), we have that

$$\big((I + \tau\beta_\varepsilon + \tau\pi)(\Phi_\varepsilon) - (I + \tau\beta_\varepsilon + \tau\pi)(0), \Phi_\varepsilon\big) \geq \frac{1}{2}\|\Phi_\varepsilon\|_H^2, \qquad (5.2.33)$$

Due to (5.1.2) and the continuity of the positive function ϑ^*, (2.5.16) helps us in estimating the second term on the right-hand side of (5.2.32):

$$\tau\Lambda_\varepsilon(\vartheta^*) \leq \tau\Lambda(\vartheta^*) \leq c\,\tau. \qquad (5.2.34)$$

Since g, $h \in H$ and (5.1.2) holds, by applying the Young inequality (2.2.5) to the other terms on the right-hand side of (5.2.32), we find that

$$\begin{aligned}
-\tau^{3/2}(\vartheta^*, \Theta_\varepsilon - \vartheta^*) &\leq \frac{\tau^{3/2}}{4}\|\Theta_\varepsilon - \vartheta^*\|_H^2 + c\,\tau^{3/2}, && (5.2.35)\\
-\tau^2 k \int_\Omega \nabla\vartheta^* \cdot \nabla(\Theta_\varepsilon - \vartheta^*) &\leq \frac{\tau^2 k}{2} \int_\Omega |\nabla(\Theta_\varepsilon - \vartheta^*)|^2 + c\,\tau^2, && (5.2.36)\\
\ell\tau(\Phi_\varepsilon, \vartheta^*) &\leq \frac{1}{8}\|\Phi_\varepsilon\|_H^2 + c\,\tau^2, && (5.2.37)\\
\tau(g, \Theta_\varepsilon - \vartheta^*) &\leq \frac{\tau^{3/2}}{4}\|\Theta_\varepsilon - \vartheta^*\|_H^2 + c\,\tau^{1/2}, && (5.2.38)
\end{aligned}$$

$$-\,\tau(\pi(0), \Phi_\varepsilon) \leq \frac{1}{8}\|\Phi_\varepsilon\|_H^2 + c\,\tau^2, \qquad (h, \Phi_\varepsilon) \leq \frac{1}{8}\|\Phi_\varepsilon\|_H^2 + c. \qquad (5.2.39)$$

Then, in view of (5.2.33)–(5.2.39), from (5.2.32) and (5.1.2) it is not difficult to infer that

$$\tau^{3/4}\|\Theta_\varepsilon\|_H + \tau\|\nabla\Theta_\varepsilon\|_H + \|\Phi_\varepsilon\|_H + \tau^{1/2}\|\nabla\Phi_\varepsilon\|_H \leq c \qquad (5.2.40)$$

taking into account that $\tau \leq \tau_2$.

Second a priori estimate. We test (5.2.21) by $\beta_\varepsilon(\Phi_\varepsilon)$ and obtain that

$$\begin{aligned}&\big(\Phi_\varepsilon, \beta_\varepsilon(\Phi_\varepsilon)\big) + \tau \int_\Omega \beta'_\varepsilon(\Phi_\varepsilon)|\nabla\Phi_\varepsilon|^2 + \tau \int_\Omega |\beta_\varepsilon(\Phi_\varepsilon)|^2 \\ &\leq \, - \tau \int_\Omega \pi(\Phi_\varepsilon)\beta_\varepsilon(\Phi_\varepsilon) + \tau\ell \int_\Omega \Theta_\varepsilon \beta_\varepsilon(\Phi_\varepsilon) + \int_\Omega h\beta_\varepsilon(\Phi_\varepsilon). \end{aligned} \tag{5.2.41}$$

Thanks to the monotonicity of β_ε and to the condition $\beta_\varepsilon(0) = 0$, the terms on the left-hand side are nonnegative. As π is Lipschitz continuous, by applying the Young inequality (2.2.5) to every term on the right-hand side of (5.2.41) and using (5.2.40), for $0 < \tau \leq 1$ we obtain that

$$-\tau \int_\Omega \pi(\Phi_\varepsilon)\beta_\varepsilon(\Phi_\varepsilon) \;\leq\; \frac{\tau}{4} \int_\Omega |\beta_\varepsilon(\Phi_\varepsilon)|^2 + c, \tag{5.2.42}$$

$$\tau\ell \int_\Omega \Theta_\varepsilon \beta_\varepsilon(\Phi_\varepsilon) \;\leq\; \frac{\tau}{4} \int_\Omega |\beta_\varepsilon(\Phi_\varepsilon)|^2 + \frac{c}{\tau^{1/2}}, \tag{5.2.43}$$

$$\int_\Omega h\beta_\varepsilon(\Phi_\varepsilon) \;\leq\; \frac{\tau}{4} \int_\Omega |\beta_\varepsilon(\Phi_\varepsilon)|^2 + \frac{c}{\tau}. \tag{5.2.44}$$

Then, owing to (5.2.42)–(5.2.44), from (5.2.41) it follows that

$$\tau\|\beta_\varepsilon(\Phi_\varepsilon)\|_H^2 \leq c\big(1 + \tau^{-1}\big), \qquad \text{so that} \qquad \tau\|\beta_\varepsilon(\Phi_\varepsilon)\|_H \leq c. \tag{5.2.45}$$

Hence, by comparison in (5.2.21), we conclude that $\tau\|\Delta\Phi_\varepsilon\|_H \leq c$ and, from (5.2.40) and standard elliptic regularity results,

$$\tau\|\Phi_\varepsilon\|_W \leq c. \tag{5.2.46}$$

Third a priori estimate. Recalling (2.5.9), (5.1.2) and (5.2.40), we immediately deduce that

$$\tau\|A_\varepsilon(\Theta_\varepsilon - \vartheta^*)\|_H \leq \tau\, C_A(1 + \|\Theta_\varepsilon\|_H + \|\vartheta^*\|_H) \leq c. \tag{5.2.47}$$

Next, we test (5.2.20) by $\ln_\varepsilon\Theta_\varepsilon$ and obtain that

$$\begin{aligned}&\|\ln_\varepsilon\Theta_\varepsilon\|_H^2 + \tau k \int_\Omega \ln'_\varepsilon(\Theta_\varepsilon)|\nabla\Theta_\varepsilon|^2 \leq \, - \tau^{1/2}(\Theta_\varepsilon, \ln_\varepsilon\Theta_\varepsilon) \\ &- \tau\big(A_\varepsilon(\Theta_\varepsilon - \vartheta^*), \ln_\varepsilon\Theta_\varepsilon\big) - \ell(\Phi_\varepsilon, \ln_\varepsilon\Theta_\varepsilon) + (g, \ln_\varepsilon\Theta_\varepsilon). \end{aligned} \tag{5.2.48}$$

Then, by applying the Cauchy–Schwarz inequality to every term on the right-hand side and using (5.2.40) and (5.2.47), we infer that

$$\|\ln_\varepsilon\Theta_\varepsilon\|_H \leq \tau^{1/2}\|\Theta_\varepsilon\|_H + c \leq c\big(\tau^{-1/4} + 1\big), \tag{5.2.49}$$

whence

$$\tau^{1/4}\|\ln_\varepsilon\Theta_\varepsilon\|_H \leq c. \tag{5.2.50}$$

Moreover, due to (5.2.50) and (5.2.40), by comparison in (5.2.20) it is straightforward to see that $\tau^{5/4}\|\Delta\Theta_\varepsilon\|_H \leq c$ and consequently

$$\tau^{5/4}\|\Theta_\varepsilon\|_W \leq c. \tag{5.2.51}$$

5.2.4 Passage to the limit as $\varepsilon \searrow 0$

In this subsection we pass to the limit as $\varepsilon \searrow 0$ and prove that the limit of subsequences of solutions $(\Theta_\varepsilon, \Phi_\varepsilon)$ for (AP_ε) (see (5.2.20)–(5.2.21)) yields a solution (ϑ^i, φ^i) to (5.2.6)–(5.2.10); then, we can conclude that the problem (P_τ) has a solution.

Since the constants appearing in (5.2.40) and (5.2.45)–(5.2.51) do not depend on ε, we infer that, at least for a subsequence, there exist some limit functions $(\vartheta^i, \varphi^i, L^i, Z^i, B^i)$ such that

$$\Theta_\varepsilon \rightharpoonup \vartheta^i \quad \text{and} \quad \Phi_\varepsilon \rightharpoonup \varphi^i \quad \text{in } W, \tag{5.2.52}$$

$$\ln_\varepsilon(\Theta_\varepsilon) \rightharpoonup L^i, \quad A_\varepsilon(\Theta_\varepsilon - \vartheta^*) \rightharpoonup Z^i \quad \text{and} \quad \beta_\varepsilon(\Phi_\varepsilon) \rightharpoonup B^i \quad \text{in } H, \tag{5.2.53}$$

as $\varepsilon \searrow 0$. Thanks to the well–known compact embedding $W \subset V$, from (5.2.52) we infer that

$$\Theta_\varepsilon \to \vartheta^i \quad \text{and} \quad \Phi_\varepsilon \to \varphi^i \quad \text{in } V. \tag{5.2.54}$$

Besides, as π is Lipschitz continuous, we have that $|\pi(\Phi_\varepsilon) - \pi(\varphi^i)| \le C_\pi |\Phi_\varepsilon - \varphi^i|$, whence, thanks to (5.2.54), we obtain that

$$\pi(\Phi_\varepsilon) \to \pi(\varphi^i) \qquad \text{in } H, \tag{5.2.55}$$

as $\varepsilon \searrow 0$. Now, we pass to the limit on $\ln_\varepsilon(\Theta_\varepsilon)$, $A_\varepsilon(\Theta_\varepsilon - \vartheta^*)$ and $\beta_\varepsilon(\Phi_\varepsilon)$. In view of a general convergence result involving maximal monotone operators (see, e.g., [1, Proposition 1.1, p. 42]), thanks to the strong convergences in H ensured by (5.2.54) and to the weak convergences in (5.2.53), we conclude that

$$L^i \in \ln(\varphi^i), \qquad Z^i \in A(\vartheta^i - \vartheta^*), \qquad B^i \in \beta(\varphi^i). \tag{5.2.56}$$

In conclusion, using (5.2.52)–(5.2.56) and recalling (5.2.19), we can pass to the limit as $\varepsilon \searrow 0$ in (5.2.20)–(5.2.21) so to obtain (5.2.6)–(5.2.10) for the limiting functions ϑ^i and φ^i.

5.2.5 Uniqueness of the solution of (P_τ)

In this section we prove that the approximating problem (P_τ) stated by (5.2.6)–(5.2.12) has a unique solution. Then, the proof of Theorem 5.2.1 will be complete.

We write problem (P_τ) for two solutions $(\vartheta_1^i, \varphi_1^i)$, $(\vartheta_2^i, \varphi_2^i)$ and set $\vartheta^i := \vartheta_1^i - \vartheta_2^i$ and $\varphi^i := \varphi_1^i - \varphi_2^i$, $i = 1, \dots, N$. Then, we multiply by $\tau\vartheta^i$ the difference between the corresponding equations (5.2.7) and by φ^i the difference between the corresponding equations (5.2.8). Adding the resultant equations, we obtain that

$$\tau^{3/2}\|\vartheta^i\|_H^2 + \tau\big(\ln\vartheta_1^i - \ln\vartheta_2^i, \vartheta_1^i - \vartheta_2^i\big) + \tau^2\big(\zeta_1^i - \zeta_2^i, \vartheta_1^i - \vartheta^* - (\vartheta_2^i - \vartheta^*)\big) + \tau^2\int_\Omega |\nabla\vartheta^i|^2$$
$$+ \|\varphi^i\|_H^2 + \tau\int_\Omega |\nabla\varphi^i|^2 + \tau(\xi_1^i - \xi_2^i, \varphi_1^i - \varphi_2^i) = -\tau\big(\pi(\varphi_1^i) - \pi(\varphi_2^i), \varphi_1^i - \varphi_2^i\big). \tag{5.2.57}$$

Since ln, A and β are monotone, in view of (5.2.9) and (5.2.10) the second, the third and the seventh term on the left-hand side of (5.2.57) are nonnegative. Besides, if $\tau \le 1/(2C_\pi)$, thanks to the Lipschitz continuity of π, the right-hand side of (5.2.57) can be estimated as

$$-\tau\big(\pi(\varphi_1^i) - \pi(\varphi_2^i), \varphi_1^i - \varphi_2^i\big) \le \frac{1}{2}\|\varphi^i\|_H^2. \tag{5.2.58}$$

Then, due to (5.2.58), from (5.2.57) we infer that

$$\tau^{3/2}\|\vartheta^i\|_H^2 + \tau^2\int_\Omega |\nabla\vartheta^i|^2 + \frac{1}{2}\|\varphi^i\|_H^2 + \tau\int_\Omega |\nabla\varphi^i|^2 \le 0, \tag{5.2.59}$$

whence we easily conclude that $\vartheta^i = \varphi^i = 0$, i.e., $\vartheta_1^i = \vartheta_2^i$ and $\varphi_1^i = \varphi_2^i$ for $i = 1, \ldots, N$.

5.3 A priori estimates on (AP_τ)

In this section we deduce some uniform estimates, independent of τ and inferred from the equations (5.2.6)–(5.2.12) of the approximating problem (P_τ).

First uniform estimate. We test (5.2.7) by ϑ^i and (5.2.8) by $(\varphi^i - \varphi^{i-1})/\tau$, then we sum up. Adding $(\varphi^i, \varphi^i - \varphi^{i-1})$ to both sides of the resulting equality and exploiting the cancellation of the suitable corresponding terms, we obtain that

$$\begin{aligned}
&\tau^{1/2}(\vartheta^i - \vartheta^{i-1}, \vartheta^i) + \big(\ln\vartheta^i - \ln\vartheta^{i-1}, \vartheta^i\big) + \tau(\zeta^i, \vartheta^i - \vartheta^*) + \tau k\int_\Omega |\nabla\vartheta^i|^2 \\
&+ \tau\left\|\frac{\varphi^i - \varphi^{i-1}}{\tau}\right\|_H^2 + (\varphi^i, \varphi^i - \varphi^{i-1}) + (\nabla\varphi^i, \nabla\varphi^i - \nabla\varphi^{i-1}) + (\xi^i, \varphi^i - \varphi^{i-1}) \\
&= -\tau(\zeta^i, \vartheta^*) + \tau(f^i, \vartheta^i) - \big(\pi(\varphi^i) - \varphi^i, \varphi^i - \varphi^{i-1}\big).
\end{aligned} \tag{5.3.1}$$

Due to (2.2.6), we can rewrite the first, the fifth and the sixth term on the left-hand side of (5.3.1) as

$$\tau^{1/2}(\vartheta^i - \vartheta^{i-1}, \vartheta^i) = \frac{\tau^{1/2}}{2}\|\vartheta^i\|_H^2 - \frac{\tau^{1/2}}{2}\|\vartheta^{i-1}\|_H^2 + \frac{\tau^{1/2}}{2}\|\vartheta^i - \vartheta^{i-1}\|_H^2, \tag{5.3.2}$$

$$(\varphi^i, \varphi^i - \varphi^{i-1}) + (\nabla\varphi^i, \nabla\varphi^i - \nabla\varphi^{i-1}) = \frac{1}{2}\|\varphi^i\|_V^2 - \frac{1}{2}\|\varphi^{i-1}\|_V^2 + \frac{1}{2}\|\varphi^i - \varphi^{i-1}\|_V^2. \tag{5.3.3}$$

Moreover, since the function $u \longmapsto e^u$ is convex and e^u turns out to be its subdifferential, by setting $u^i = \ln\vartheta^i$ we obtain that

$$\big(\ln\vartheta^i - \ln\vartheta^{i-1}, \vartheta^i\big) = \big(u^i - u^{i-1}, e^{u^i}\big) \ge \int_\Omega e^{u^i} - \int_\Omega e^{u^{i-1}} = \|\vartheta^i\|_{L^1(\Omega)} - \|\vartheta^{i-1}\|_{L^1(\Omega)}. \tag{5.3.4}$$

Recalling that A is a maximal monotone operator and $0 \in A(0)$, by (5.2.9) the third term on the left-hand side of (5.3.1) is nonnegative. We also notice that, since β is the subdifferential of $\tilde{\beta}$, from (5.2.10) it follows that

$$(\xi^i, \varphi^i - \varphi^{i-1}) \geq \int_\Omega \tilde{\beta}(\varphi^i) - \int_\Omega \tilde{\beta}(\varphi^{i-1}), \tag{5.3.5}$$

while, due to (2.2.2), (2.2.5) and the sub-linear growth of A stated by (2.4.11), we deduce that

$$\begin{aligned} -\tau(\zeta^i, \vartheta^*) &\leq C_A \tau (1 + \|\vartheta^i - \vartheta^*\|_H) \|\vartheta^*\|_H \leq c\tau(1 + \|\vartheta^i\|_H) \leq c\tau(1 + \|\vartheta^i\|_V) \\ &\leq c\tau(1 + \|\vartheta^i\|_{L^1(\Omega)} + \|\nabla\vartheta^i\|_H) \leq c\tau + \tau C_1 \|\vartheta^i\|_{L^1(\Omega)} + \tau\frac{k}{2}\|\nabla\vartheta^i\|_H^2, \end{aligned} \tag{5.3.6}$$

where we have applied the Young inequality in the last term and where the constant C_1 depends on C_A, $\|\vartheta^*\|_H$ and C_p. Due to the the boundedness of f^i in $L^\infty(\Omega)$ and the Lipschitz continuity of π, we also infer that

$$\tau(f^i, \vartheta^i) \leq \tau \|f^i\|_{L^\infty(\Omega)} \|\vartheta^i\|_{L^1(\Omega)}, \tag{5.3.7}$$

$$-\big(\pi(\varphi^i) - \varphi^i, \varphi^i - \varphi^{i-1}\big) \leq c\tau(1 + \|\varphi^i\|_H) \left\| \frac{\varphi^i - \varphi^{i-1}}{\tau} \right\|_H \tag{5.3.8}$$

$$\leq \frac{\tau}{2} \left\| \frac{\varphi^i - \varphi^{i-1}}{\tau} \right\|_H^2 + \tau C_2 (1 + \|\varphi^i\|_H^2), \tag{5.3.9}$$

where C_2 depends on C_π, $|\pi(0)|$ and $|\Omega|$. Now, we apply the estimates (5.3.2)–(5.3.9) to the corresponding terms of (5.3.1) and sum up for $i = 1, \dots, n$, letting $n \leq N$. We obtain that

$$\begin{aligned} &\frac{\tau^{1/2}}{2}\|\vartheta^n\|_H^2 + \sum_{i=1}^n \frac{\tau^{1/2}}{2}\|\vartheta^i - \vartheta^{i-1}\|_H^2 + \|\vartheta^n\|_{L^1(\Omega)} + \frac{k}{2}\sum_{i=1}^n \tau\|\nabla\vartheta^i\|_H^2 \\ &+ \frac{1}{2}\sum_{i=1}^n \tau \left\| \frac{\varphi^i - \varphi^{i-1}}{\tau} \right\|_H^2 + \frac{1}{2}\|\varphi^n\|_V^2 + \frac{1}{2}\sum_{i=1}^n \|\varphi^i - \varphi^{i-1}\|_V^2 + \int_\Omega \tilde{\beta}(\varphi^n) \\ &\leq \frac{\tau^{1/2}}{2}\|\vartheta_0\|_H^2 + \|\vartheta_0\|_{L^1(\Omega)} + \frac{1}{2}\|\varphi_0\|_V^2 + \int_\Omega \tilde{\beta}(\varphi_0) + \tau\sum_{i=1}^n \|f^i\|_{L^\infty(\Omega)}\|\vartheta^i\|_{L^1(\Omega)} \\ &\quad + C_1 \sum_{i=1}^n \tau\|\vartheta^i\|_{L^1(\Omega)} + C_2\sum_{i=1}^n \tau\|\varphi^i\|_H^2 + c. \end{aligned} \tag{5.3.10}$$

On account of (5.1.3)–(5.1.3) and (5.1.5), the first four terms on the right-hand side of (5.3.10) are bounded. Now, recalling the definition (5.2.5) of f^i, we have that

$$\tau\sum_{i=1}^n \|f^i\|_{L^\infty(\Omega)}\|\vartheta^i\|_{L^1(\Omega)} = \|\vartheta^n\|_{L^1(\Omega)} \int_{(n-1)\tau}^{n\tau} \|f(s)\|_{L^\infty(\Omega)}\, ds + \sum_{i=1}^{n-1} \|f^i\|_{L^\infty(\Omega)}\|\vartheta^i\|_{L^1(\Omega)}.$$

Thanks to the absolute continuity of the integral, if τ is small enough (independently of n) we have that

$$\int_{(n-1)\tau}^{n\tau} \|f(s)\|_{L^\infty(\Omega)}\, ds \le \frac{1}{4}, \qquad C_1\tau \le \frac{1}{4}, \qquad C_2\tau \le \frac{1}{4}. \tag{5.3.11}$$

Then, on the basis of (5.3.11), from (5.3.10) we infer that

$$\begin{aligned} &\frac{\tau^{1/2}}{2}\|\vartheta^n\|_H^2 + \sum_{i=1}^{n} \frac{\tau^{1/2}}{2}\|\vartheta^i - \vartheta^{i-1}\|_H^2 + \frac{1}{2}\|\vartheta^n\|_{L^1(\Omega)} + \frac{k}{2}\sum_{i=1}^{n}\tau\|\nabla\vartheta^i\|_H^2 \\ &+ \frac{1}{2}\sum_{i=1}^{n}\tau\left\|\frac{\varphi^i - \varphi^{i-1}}{\tau}\right\|_H^2 + \frac{1}{4}\|\varphi^n\|_V^2 + \frac{1}{2}\sum_{i=1}^{n}\|\varphi^i - \varphi^{i-1}\|_V^2 + \int_\Omega \tilde\beta(\varphi^n) \\ &\le c + \sum_{i=1}^{n-1}\tau\Big(\|f^i\|_{L^\infty(\Omega)}\|\vartheta^i\|_{L^1(\Omega)} + C_1\|\vartheta^i\|_{L^1(\Omega)} + C_2\|\varphi^i\|_H^2\Big). \end{aligned} \tag{5.3.12}$$

Now, we observe that

$$\sum_{i=1}^{n-1}\tau C_1 \le \sum_{i=1}^{N}\tau C_1 = C_1 T, \qquad \sum_{i=1}^{n-1}\tau C_2 \le \sum_{i=1}^{N}\tau C_2 = C_2 T. \tag{5.3.13}$$

and, according to (5.1.1),

$$\sum_{i=1}^{n-1}\tau\|f^i\|_{L^\infty(\Omega)} \le \sum_{i=1}^{N}\int_{(i-1)\tau}^{i\tau}\|f(s)\|_{L^\infty(\Omega)}\, ds = \int_0^T \|f(s)\|_{L^\infty(\Omega)}\, ds \le c.$$

Then, we can apply Lemma 2.3.3 and, recalling the notations (5.2.1), we conclude that

$$\begin{aligned} &\tau^{1/2}\|\overline{\vartheta}_\tau\|_{L^\infty(0,T;H)}^2 + \tau^{3/2}\|\partial_t\widehat{\vartheta}_\tau\|_{L^2(0,T;H)}^2 + \|\overline{\vartheta}_\tau\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla\overline{\vartheta}_\tau\|_{L^2(0,T;H)}^2 \\ &+ \|\partial_t\widehat{\varphi}_\tau\|_{L^2(0,T;H)}^2 + \|\overline{\varphi}_\tau\|_{L^\infty(0,T;V)}^2 + \tau\|\partial_t\widehat{\varphi}_\tau\|_{L^2(0,T;V)}^2 + \|\tilde\beta(\overline{\varphi}_\tau)\|_{L^\infty(0,T;L^1(\Omega))} \le c. \end{aligned} \tag{5.3.14}$$

Since the third and the fourth term of the left-hand side of (5.3.14) are bounded, owing to (2.2.2) we also infer that

$$\|\overline{\vartheta}_\tau\|_{L^2(0,T;V)} \le c. \tag{5.3.15}$$

Besides, in view of (5.2.9) and due to the sub-linear growth of A stated by (2.4.11) and to (5.1.2), we deduce that

$$\|\overline{\zeta}_\tau\|_{L^2(0,T;H)} \le c. \tag{5.3.16}$$

Second uniform estimate. We formally test (5.2.8) by ξ^i and obtain

$$(\varphi^i - \varphi^{i-1}, \xi^i) + \tau\|\xi^i\|_H^2 \le \tau\big(\pi(\varphi^i) + \ell\vartheta^i, \xi^i\big). \tag{5.3.17}$$

We point out that the previous estimate (5.3.17) can be rigorously derived by testing (5.2.21) by $\beta_\varepsilon(\Phi_\varepsilon)$ and then passing to the limit as $\varepsilon \searrow 0$. Since β is the subdifferential of $\tilde{\beta}$, we have that

$$(\varphi^i - \varphi^{i-1}, \xi^i) \geq \int_\Omega \tilde{\beta}(\varphi^i) - \int_\Omega \tilde{\beta}(\varphi^{i-1}). \tag{5.3.18}$$

Moreover, due to the Lipschitz continuity of π, applying the Young inequality (2.2.5) to the right-hand side of (5.3.17), we deduce that

$$\tau\big(\pi(\varphi^i) + \ell\vartheta^i, \xi^i\big) \leq \frac{1}{2}\tau\|\xi^i\|_H^2 + c\,\tau\big(1 + \|\varphi^i\|_H^2 + \|\vartheta^i\|_H^2\big). \tag{5.3.19}$$

Now, combining (5.3.17)–(5.3.19) and summing up for $i = 1, \dots, n$, with $n \leq N$, we infer that

$$\int_\Omega \tilde{\beta}(\varphi^n) + \frac{1}{2}\sum_{i=1}^n \tau\|\xi^i\|_H^2 \leq \int_\Omega \tilde{\beta}(\varphi_0) + \sum_{i=1}^n \tau\big(1 + \|\varphi^i\|_H^2 + \|\vartheta^i\|_H^2\big), \tag{5.3.20}$$

whence, due to (5.3.14)–(5.3.15), we obtain that

$$\|\overline{\xi}_\tau\|_{L^2(0,T;H)} \leq c. \tag{5.3.21}$$

Finally, by comparison in (5.2.14), we conclude that $\|\Delta\overline{\varphi}_\tau\|_{L^2(0,T;H)} \leq c$. Then, thanks to (5.3.14) and elliptic regularity, we find that

$$\|\overline{\varphi}_\tau\|_{L^2(0,T;W)} \leq c. \tag{5.3.22}$$

Third uniform estimate. We introduce the function $\psi_n : \mathbb{R} \longmapsto \mathbb{R}$ obtained by truncating the logarithmic function in the following way:

$$\psi_n(u) = \begin{cases} \ln(u) & \text{if } u \geq 1/n, \\ -\ln(n) & \text{if } u < 1/n. \end{cases}$$

It is easy to see that ψ_n is an increasing and Lipschitz continuous function. Then, defining

$$j_n(u) = \int_1^u \psi_n(s)\,ds, \qquad u \in \mathbb{R}, \qquad \text{and} \qquad j(u) = \int_1^u \ln s\,ds, \qquad u > 0, \tag{5.3.23}$$

and testing (5.2.7) by $\psi_n(\vartheta^i)$, we obtain that

$$\begin{aligned} &\tau^{1/2}\big(\vartheta^i - \vartheta^{i-1}, \psi_n(\vartheta^i)\big) + \big(\ln\vartheta^i - \ln\vartheta^{i-1}, \psi_n(\vartheta^i)\big) + \tau k\int_{\Omega\cap\{\vartheta^i \geq 1/n\}} \frac{|\nabla\vartheta^i|^2}{\vartheta^i} \\ &= -\ell\big(\varphi^i - \varphi^{i-1}, \psi_n(\vartheta^i)\big) - \tau\big(\zeta^i, \psi_n(\vartheta^i)\big) + \tau\big(f^i, \psi_n(\vartheta^i)\big). \end{aligned} \tag{5.3.24}$$

Recalling that j_n is a convex function with derivative ψ_n, we have that

$$\tau^{1/2}\big(\vartheta^i - \vartheta^{i-1}, \psi_n(\vartheta^i)\big) \geq \tau^{1/2}\int_\Omega j_n(\vartheta^i) - \tau^{1/2}\int_\Omega j_n(\vartheta^{i-1}), \tag{5.3.25}$$

and consequently from (5.3.24) we infer that

$$\tau k \int_{\Omega\cap\{\vartheta^i\geq 1/n\}} \frac{|\nabla\vartheta^i|^2}{\vartheta^i} \leq \tau^{1/2}\int_\Omega j_n(\vartheta^{i-1}) - \tau^{1/2}\int_\Omega j_n(\vartheta^i)$$
$$-\int_\Omega (\ln\vartheta^i - \ln\vartheta^{i-1})\psi_n(\vartheta^i) - \int_\Omega \big(\ell(\varphi^i-\varphi^{i-1}) + \tau\zeta^i - \tau f^i\big)\psi_n(\vartheta^i). \tag{5.3.26}$$

Due to the properties of the subdifferential, we have that

$$0 \leq j(\vartheta^k) \leq j(1) + (\ln\vartheta^k, \vartheta^k - 1) \quad \text{for } k = 0,1,\ldots,N. \tag{5.3.27}$$

Since $\ln\vartheta^k \in H$, $\vartheta^k > 0$ a.e. in Ω and $\vartheta^k \in H$, from (5.3.27) we infer that $j(\vartheta^k) \in L^1(\Omega)$; consequently, passing to the limit as $n \to +\infty$, we obtain that

$$\begin{array}{lcll} \psi_n(\vartheta^k) & \to & \ln\vartheta^k & \text{in } H \text{ and a.e. in } \Omega, \\ j_n(\vartheta^k) & \to & j(\vartheta^k) & \text{in } L^1(\Omega) \text{ and a.e. in } \Omega, \end{array}$$

for $k = 0,1,\ldots,N$. Then, taking the lim inf in (5.3.26) as $n \to +\infty$ and applying the Fatou Lemma and (2.2.6), we have that

$$\tau k\int_\Omega \frac{|\nabla\vartheta^i|^2}{\vartheta^i} \leq \tau^{1/2}\int_\Omega j(\vartheta^{i-1}) - \tau^{1/2}\int_\Omega j(\vartheta^i) + \frac12\int_\Omega |\ln\vartheta^{i-1}|^2 - \frac12\int_\Omega |\ln\vartheta^i|^2$$
$$-\frac12\int_\Omega |\ln\vartheta^i - \ln\vartheta^{i-1}|^2 - \int_\Omega \big(\ell(\varphi^i-\varphi^{i-1}) + \tau\zeta^i - \tau f^i\big)\ln\vartheta^i. \tag{5.3.28}$$

Now, sum up (5.3.28) for $i = 1,\ldots,k$, with $k \leq N$, and obtain that

$$\tau^{1/2}\int_\Omega j(\vartheta^k) + \frac12\|\ln\vartheta^k\|_H^2 + \frac12\sum_{i=1}^k \tau^2\left\|\frac{\ln\vartheta^i - \ln\vartheta^{i-1}}{\tau}\right\|_H^2 + k\sum_{i=1}^k \tau\int_\Omega \frac{|\nabla\vartheta^i|^2}{\vartheta^i}$$
$$\leq \tau^{1/2}\int_\Omega j(\vartheta_0) + \frac12\|\ln\vartheta_0\|_H^2 + \frac14\sum_{i=1}^k \tau\|\ln\vartheta^i\|_H^2 + c\sum_{i=1}^k \tau\left\|\frac{\varphi^i-\varphi^{i-1}}{\tau}\right\|_H^2$$
$$+ c\sum_{i=1}^k \tau\|\zeta^i\|_H^2 + c\sum_{i=1}^k \tau\|f^i\|_H^2. \tag{5.3.29}$$

We observe that if $\tau \leq 1$ then

$$\frac14\sum_{i=1}^k \tau\|\ln\vartheta^i\|_H^2 \leq \frac14\sum_{i=1}^{k-1} \tau\|\ln\vartheta^i\|_H^2 + \frac14\|\ln\vartheta^k\|_H^2. \tag{5.3.30}$$

We also notice that the fourth and the fifth term on the right-hand side of (5.3.29) are bounded by a positive constant c, due to (5.3.14) and (5.3.16), respectively. Moreover, thanks to (5.1.1) and to the definition (5.2.5) of f^i, by using the Hölder inequality the last term on the right-hand side of (5.3.29) can be estimated as follows:

$$c\sum_{i=1}^{k}\tau\|f^i\|_H^2 \le c\sum_{i=1}^{k}\tau\left\|\frac{1}{\tau}\int_{(i-1)\tau}^{i\tau} f(s)\,ds\right\|_H^2$$
$$\le c\sum_{i=1}^{k}\int_{(i-1)\tau}^{i\tau}\|f(s)\|_H^2\,ds \le c\|f\|^2_{L^2(0,T;H)}. \tag{5.3.31}$$

Then, combining (5.3.29) with (5.3.30)–(5.3.31) (see also (5.1.3) and (5.3.27)), we infer that

$$\tau^{1/2}\int_\Omega j(\vartheta^k) + \frac{1}{4}\|\ln\vartheta^k\|_H^2 + \frac{1}{2}\sum_{i=1}^{k}\tau^2\left\|\frac{\ln\vartheta^i - \ln\vartheta^{i-1}}{\tau}\right\|_H^2$$
$$+ 4k\sum_{i=1}^{k}\tau\int_\Omega\left|\nabla(\vartheta^i)^{1/2}\right|^2 \le c + \frac{1}{4}\sum_{i=1}^{k-1}\tau\|\ln\vartheta^i\|_H^2,$$

whence, by applying Lemma 2.3.3, we conclude that

$$\tau^{1/2}\|j(\overline{\vartheta}_\tau)\|_{L^\infty(0,T;L^1(\Omega))} + \|\overline{\ln\vartheta}_\tau\|_{L^\infty(0,T;H)} + \left\|\nabla\overline{\vartheta^{1/2}}_\tau\right\|_{L^2(0,T;H)} \le c. \tag{5.3.32}$$

Moreover, due to (5.3.14) as well, we also infer that

$$\|\overline{\vartheta^{1/2}}_\tau\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c. \tag{5.3.33}$$

Fourth uniform estimate. We test (5.2.7) by $(\vartheta^i - \vartheta^{i-1})$. Then, we take the difference between (5.2.8) written for i and for $i-1$, and test by $(\varphi^i - \varphi^{i-1})/\tau$. Using (2.4.10) and adding, it is note difficult to obtain that

$$\tau^{1/2}\|\vartheta^i - \vartheta^{i-1}\|_H^2 + (\ln\vartheta^i - \ln\vartheta^{i-1}, \vartheta^i - \vartheta^{i-1}) + \ell(\varphi^i - \varphi^{i-1}, \vartheta^i - \vartheta^{i-1})$$
$$+ \tau\Upsilon(\vartheta^i - \vartheta^*) - \tau\Upsilon(\vartheta^{i-1} - \vartheta^*) + \tau\frac{k}{2}\left(\|\nabla\vartheta^i\|_H^2 + \|\nabla(\vartheta^i - \vartheta^{i-1})\|_H^2 - \|\nabla\vartheta^{i-1}\|_H^2\right)$$
$$+ \frac{\tau}{2}\left\|\frac{\varphi^i - \varphi^{i-1}}{\tau}\right\|_H^2 + \frac{\tau}{2}\left\|\frac{\varphi^i - \varphi^{i-1}}{\tau} - \frac{\varphi^{i-1} - \varphi^{i-2}}{\tau}\right\|_H^2 - \frac{\tau}{2}\left\|\frac{\varphi^{i-1} - \varphi^{i-2}}{\tau}\right\|_H^2$$
$$+ \tau^2\left\|\nabla\frac{\varphi^i - \varphi^{i-1}}{\tau}\right\|_H^2 + (\xi^i - \xi^{i-1}, \varphi^i - \varphi^{i-1}) - \tau\ell\left(\vartheta^i - \vartheta^{i-1}, \frac{\varphi^i - \varphi^{i-1}}{\tau}\right)$$
$$\le \tau(f^i, \vartheta^i - \vartheta^{i-1}) - \tau\left(\pi(\varphi^i) - \pi(\varphi^{i-1}), \frac{\varphi^i - \varphi^{i-1}}{\tau}\right), \tag{5.3.34}$$

for $i = 2, \dots, N$. Now, we write (5.2.7) and (5.2.8) for $i = 1$ and test the corresponding equations by $(\vartheta^1 - \vartheta^0)$ and $(\varphi^1 - \varphi^0)/\tau$, respectively. Since $\vartheta^0 = \vartheta_0$ and $\varphi^0 = \varphi_0$, we have that

$$\tau^{1/2}\|\vartheta^1-\vartheta^0\|_H^2+(\ln\vartheta^1-\ln\vartheta^0,\vartheta^1-\vartheta^0)+\ell(\varphi^1-\varphi^0,\vartheta^1-\vartheta^0)+\tau\Upsilon(\vartheta^1-\vartheta^*)$$
$$-\tau\Upsilon(\vartheta_0-\vartheta^*)+\tau\frac{k}{2}\left(\|\nabla\vartheta^1\|_H^2+\|\nabla(\vartheta^1-\vartheta^0)\|_H^2-\|\nabla\vartheta_0\|_H^2\right)+\tau\left\|\frac{\varphi^1-\varphi^0}{\tau}\right\|_H^2$$
$$+\|\nabla(\varphi^1-\varphi^0)\|_H^2+(\xi^1-\xi_0,\varphi^1-\varphi_0)\le-\tau\left(\pi(\varphi^1)-\pi(\varphi^0),\frac{\varphi^1-\varphi^0}{\tau}\right)$$
$$+\tau\ell\left(\vartheta^1-\vartheta^0,\frac{\varphi^1-\varphi^0}{\tau}\right)+\tau(f^1,\vartheta^1-\vartheta^0)+(\ell\vartheta_0+\Delta\varphi_0-\xi_0-\pi(\varphi_0),\varphi^1-\varphi^0).$$
(5.3.35)

Then, we divide (5.3.34) and (5.3.35) by τ and sum up the corresponding equations for $i=1,\dots,n$, with $n\le N$. Since β is maximal monotone and (5.2.10) and (5.1.4) hold, then the eleventh term on the left-hand side of (5.3.34) and the ninth term on the left-hand side of (5.3.35) are nonnegative. Assuming $\varphi^{-1}=\varphi_0$, we infer that

$$\tau^{1/2}\sum_{i=1}^n\tau\left\|\frac{\vartheta^i-\vartheta^{i-1}}{\tau}\right\|_H^2+\sum_{i=1}^n\frac{1}{\tau}(\ln\vartheta^i-\ln\vartheta^{i-1},\vartheta^i-\vartheta^{i-1})+\Upsilon(\vartheta^n-\vartheta^*)$$
$$+\frac{k}{2}\|\nabla\vartheta^n\|_H^2+\frac{k}{2}\tau\sum_{i=1}^n\tau\left\|\nabla\frac{\vartheta^i-\vartheta^{i-1}}{\tau}\right\|_H^2+\frac{1}{2}\left\|\frac{\varphi^n-\varphi^{n-1}}{\tau}\right\|_H^2$$
$$+\frac{1}{2}\sum_{i=1}^n\left\|\frac{\varphi^i-\varphi^{i-1}}{\tau}-\frac{\varphi^{i-1}-\varphi^{i-2}}{\tau}\right\|_H^2+\frac{1}{2}\left\|\frac{\varphi^1-\varphi^0}{\tau}\right\|_H^2+\sum_{i=1}^n\tau\left\|\nabla\frac{\varphi^i-\varphi^{i-1}}{\tau}\right\|_H^2$$
$$\le\Upsilon(\vartheta_0-\vartheta^*)+\frac{k}{2}\|\nabla\vartheta_0\|_H^2+\|\ell\vartheta_0+\Delta\varphi_0-\xi_0-\pi(\varphi_0)\|_H^2+\frac{1}{4}\left\|\frac{\varphi^1-\varphi^0}{\tau}\right\|_H^2$$
$$+(f^n,\vartheta^n)-(f^1,\vartheta_0)-\sum_{i=1}^{n-1}(f^{i+1}-f^i,\vartheta^i)+\sum_{i=1}^nC_\pi\tau\left\|\frac{\varphi^i-\varphi^{i-1}}{\tau}\right\|_H^2. \qquad (5.3.36)$$

In view of (2.4.3)–(5.1.4) and noting that $\vartheta_0\in V$, $\varphi_0\in W$ and Υ has at most a quadratic growth (see (5.1.3)–(5.1.3) and (2.4.10)), the first three terms on the right-hand side of (5.3.36) are bounded by a positive constant. Besides, using (2.2.5), (2.2.2) and the Hölder inequality and recalling (5.1.1) and (5.3.14), the fifth and the sixth term on the right-hand side of (5.3.36) can be estimated as follows:

$$|(f^n,\vartheta^n)|\le\|f^n\|_H\|\vartheta^n\|_H\le C_p\|f^n\|_H\left(\|\vartheta^n\|_{L^1(\Omega)}+\|\nabla\vartheta^n\|_H\right)$$
$$\le C_p\|f\|_{C^0([0,T];H)}\left(\|\overline{\vartheta}_\tau\|_{L^\infty(0,T;L^1(\Omega))}+\|\nabla\vartheta^n\|_H\right)\le\frac{k}{4}\|\nabla\vartheta^n\|_H^2+c, \qquad (5.3.37)$$

$$|(f^1,\vartheta_0)|\le\|f^1\|_H\|\vartheta_0\|_H\le\|f\|_{C^0([0,T];H)}\|\vartheta_0\|_H\le c. \qquad (5.3.38)$$

With the help of (2.2.5), Hölder's inequality and (5.3.15) we also infer that

$$\left|\sum_{i=1}^{n-1}(f^{i+1}-f^i,\vartheta^i)\right|\le\sum_{i=2}^n\tau\left\|\frac{f^i-f^{i-1}}{\tau}\right\|_H\|\vartheta^{i-1}\|_H$$

$$\leq \frac{1}{2}\sum_{i=2}^{n}\tau\left\|\frac{f^i-f^{i-1}}{\tau}\right\|_H^2+\frac{1}{2}\sum_{i=1}^{n-1}\tau\|\vartheta^i\|_H^2\leq \frac{1}{2}\sum_{i=2}^{n}\tau\left\|\frac{f^i-f^{i-1}}{\tau}\right\|_H^2+c. \tag{5.3.39}$$

Recalling (5.1.1) and the definition of f^i (see (5.2.5)), we have that

$$\begin{aligned}
&\left\|\frac{f^i-f^{i-1}}{\tau}\right\|_H^2=\left\|\frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}f(s)\,ds-\frac{1}{\tau^2}\int_{(i-2)\tau}^{(i-1)\tau}f(s)\,ds\right\|_H^2\\
&=\left\|\frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}\Big(f(s)-f(s-\tau)\Big)ds\right\|_H^2\\
&\leq \frac{1}{\tau^4}\left|\int_{(i-1)\tau}^{i\tau}\|f(s)-f(s-\tau)\|_H\,ds\right|^2\leq \frac{1}{\tau^3}\int_{(i-1)\tau}^{i\tau}\left\|\int_{s-\tau}^{s}\partial_t f(t)\,dt\right\|_H^2 ds\\
&\leq \frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}\left(\int_{s-\tau}^{s}\|\partial_t f(t)\|_H^2\,dt\right)ds\leq \frac{1}{\tau}\|\partial_t f\|^2_{L^2((i-2)\tau,i\tau;H)},
\end{aligned}$$

so that

$$\frac{1}{2}\sum_{i=2}^{n}\tau\left\|\frac{f^i-f^{i-1}}{\tau}\right\|_H^2\leq \|\partial_t f\|^2_{L^2(0,T;H)}. \tag{5.3.40}$$

Next, we take advantage of Lemma 2.3.1 in order to deal with the second term on the left-hand side of (5.3.36). Indeed (cf. (2.3.1)), we realize that

$$\left|(\vartheta^i)^{1/2}-(\vartheta^{i-1})^{1/2}\right|^2\leq (\ln\vartheta^i-\ln\vartheta^{i-1},\vartheta^i-\vartheta^{i-1}),$$

whence

$$\sum_{i=1}^{n}\frac{1}{\tau}(\ln\vartheta^i-\ln\vartheta^{i-1},\vartheta^i-\vartheta^{i-1})\geq \sum_{i=1}^{n}\tau\left\|\frac{(\vartheta^i)^{1/2}-(\vartheta^{i-1})^{1/2}}{\tau}\right\|_H^2. \tag{5.3.41}$$

Collecting now (5.3.37)–(5.3.41), from (5.3.36) and (5.3.14) we infer that

$$\begin{aligned}
&\tau^{1/4}\|\partial_t\widehat{\vartheta}_\tau\|_{L^2(0,T;H)}+\|\partial_t\widehat{\vartheta^{1/2}}_\tau\|_{L^2(0,T;H)}+\|\Upsilon(\overline{\vartheta}_\tau-\vartheta^*)\|_{L^\infty(0,T)}+\|\overline{\vartheta}_\tau\|_{L^\infty(0,T;V)}\\
&+\tau^{1/2}\|\partial_t\widehat{\vartheta}_\tau\|_{L^2(0,T;V)}+\|\partial_t\widehat{\varphi}_\tau\|_{L^\infty(0,T;H)}+\|\partial_t\widehat{\varphi}_\tau\|_{L^2(0,T;V)}\leq c.
\end{aligned}\tag{5.3.42}$$

Therefore, thanks to (5.2.15) and using (2.4.11) and (5.1.2), we have that

$$\|\overline{\zeta}_\tau\|_{L^\infty(0,T;H)}\leq c. \tag{5.3.43}$$

Moreover, by comparison in (5.2.13) and in view of (5.3.14)–(5.3.16), (5.3.21)–(5.3.22), (5.3.32)–(5.3.33) and (5.3.42), we obtain that

$$\begin{aligned}
&\|\partial_t\widehat{\ln\vartheta}_\tau\|_{L^2(0,T;V^*)}\leq c\tau^{1/2}\|\partial_t\widehat{\vartheta}_\tau\|_{L^2(0,T;H)}+c\|\partial_t\widehat{\varphi}_\tau\|_{L^2(0,T;H)}\\
&\quad+c\|\overline{\zeta}_\tau\|_{L^2(0,T;H)}+k\|\Delta\overline{\vartheta}_\tau\|_{L^2(0,T;V^*)}+c\|\overline{f}_\tau\|_{L^2(0,T;H)}\leq c.
\end{aligned}\tag{5.3.44}$$

Furthermore, recalling (5.2.14), a comparison of the terms yields the bound

$$\|\Delta\overline{\varphi}_\tau+\overline{\xi}_\tau\|_{L^\infty(0,T;H)}\leq c. \tag{5.3.45}$$

Hence, by arguing as in the Second uniform estimate, we can improve (5.3.21) and (5.3.22) to find out that

$$\|\overline{\xi}_\tau\|_{L^\infty(0,T;H)}+\|\overline{\varphi}_\tau\|_{L^\infty(0,T;W)}\leq c. \tag{5.3.46}$$

Summary of the uniform estimates. Let us collect the previous estimates. From (5.3.14)–(5.3.16), (5.3.21)–(5.3.22), (5.3.32)–(5.3.33) and (5.3.42)–(5.3.46) we conclude that there exists a constant $c > 0$, independent of τ, such that

$$
\begin{aligned}
&\|\overline{\vartheta}_\tau\|_{L^\infty(0,T;V)} + \|\widehat{\vartheta}_\tau\|_{L^\infty(0,T;V)} + \tau^{1/4}\|\partial_t\widehat{\vartheta}_\tau\|_{L^2(0,T;H)} \\
&+ \|\overline{\ln\vartheta}_\tau\|_{L^\infty(0,T;H)} + \|\widehat{\ln\vartheta}_\tau\|_{H^1(0,T;V^*)\cap L^\infty(0,T;H)} \\
&+ \|\overline{\vartheta^{1/2}}_\tau\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\widehat{\vartheta^{1/2}}_\tau\|_{H^1(0,T;H)\cap L^2(0,T;V)} + \|\overline{\zeta}_\tau\|_{L^\infty(0,T;H)} \\
&+ \|\overline{\varphi}_\tau\|_{L^\infty(0,T;W)} + \|\widehat{\varphi}_\tau\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W)} + \|\overline{\xi}_\tau\|_{L^\infty(0,T;H)} \leq c.
\end{aligned}
\tag{5.3.47}
$$

5.4 Passage to the limit as $\tau \searrow 0$

Thanks to (5.3.47) and to the well–known weak or weak* compactness results, we deduce that, at least for a subsequence of $\tau \searrow 0$, there exist ten limit functions ϑ, $\widehat{\vartheta}$, λ, $\widehat{\lambda}$, w, $\widehat{w}$, ζ, φ, $\widehat{\varphi}$, and ξ such that

$$\overline{\vartheta}_\tau \rightharpoonup^* \vartheta \quad \text{in} \quad L^\infty(0,T;V), \tag{5.4.1}$$
$$\widehat{\vartheta}_\tau \rightharpoonup^* \widehat{\vartheta} \quad \text{in} \quad L^\infty(0,T;V), \tag{5.4.2}$$
$$\tau^{1/4}\widehat{\vartheta}_\tau \rightharpoonup^* 0 \quad \text{in} \quad H^1(0,T;H)\cap L^\infty(0,T;V), \tag{5.4.3}$$
$$\overline{\ln\vartheta}_\tau \rightharpoonup^* \lambda \quad \text{in} \quad L^\infty(0,T;H), \tag{5.4.4}$$
$$\widehat{\ln\vartheta}_\tau \rightharpoonup^* \widehat{\lambda} \quad \text{in} \quad H^1(0,T;V^*)\cap L^\infty(0,T;H), \tag{5.4.5}$$
$$\overline{\vartheta^{1/2}}_\tau \rightharpoonup^* w \quad \text{in} \quad L^\infty(0,T;H)\cap L^2(0,T;V), \tag{5.4.6}$$
$$\widehat{\vartheta^{1/2}}_\tau \rightharpoonup \widehat{w} \quad \text{in} \quad H^1(0,T;H)\cap L^2(0,T;V), \tag{5.4.7}$$
$$\overline{\zeta}_\tau \rightharpoonup^* \zeta \quad \text{in} \quad L^\infty(0,T;H), \tag{5.4.8}$$
$$\overline{\varphi}_\tau \rightharpoonup^* \varphi \quad \text{in} \quad L^\infty(0,T;W), \tag{5.4.9}$$
$$\widehat{\varphi}_\tau \rightharpoonup^* \widehat{\varphi} \quad \text{in} \quad W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W), \tag{5.4.10}$$
$$\overline{\xi}_\tau \rightharpoonup^* \xi \quad \text{in} \quad L^\infty(0,T;H). \tag{5.4.11}$$

First, we observe that $\vartheta = \widehat{\vartheta}$: indeed, thanks to (5.2.3) and (5.4.3), we have that

$$\|\overline{\vartheta}_\tau - \widehat{\vartheta}_\tau\|_{L^2(0,T;H)} \leq \frac{\tau}{\sqrt{3}}\|\partial_t\widehat{\vartheta}_\tau\|_{L^2(0,T;H)} \leq c\tau^{3/4} \tag{5.4.12}$$

and consequently $\overline{\vartheta}_\tau - \widehat{\vartheta}_\tau \to 0$ strongly in $L^2(0,T;H)$. Moreover, it turns out that $\lambda = \widehat{\lambda}$: in fact, on account of (5.2.4) and (5.4.5) we have that

$$\|\overline{\ln\vartheta}_\tau - \widehat{\ln\vartheta}_\tau\|_{L^\infty(0,T;V^*)} \leq \tau^{1/2}\|\partial_t\widehat{\ln\vartheta}_\tau\|_{L^2(0,T;V^*)} \leq c\tau^{1/2}, \tag{5.4.13}$$

whence

$$\lim_{\tau\searrow 0}\|\overline{\ln\vartheta}_\tau - \widehat{\ln\vartheta}_\tau\|_{L^\infty(0,T;V^*)} = 0. \tag{5.4.14}$$

Similarly, thanks to (5.2.3) and (5.4.10), we see that

$$\|\overline{\vartheta^{1/2}}_\tau - \widehat{\vartheta^{1/2}}_\tau\|_{L^2(0,T;H)} \le \frac{\tau}{\sqrt{3}} \|\partial_t \widehat{\vartheta^{1/2}}_\tau\|_{L^2(0,T;H)} \le c\tau, \tag{5.4.15}$$

which entails

$$\lim_{\tau \searrow 0} \|\overline{\vartheta^{1/2}}_\tau - \widehat{\vartheta^{1/2}}_\tau\|_{L^2(0,T;H)} = 0 \tag{5.4.16}$$

and $w = \widehat{w}$. Finally, we check that $\varphi = \widehat{\varphi}$. In the light of (5.2.4), we have that

$$\|\overline{\varphi}_\tau - \widehat{\varphi}_\tau\|_{L^\infty(0,T;V)} \le \tau \|\partial_t \widehat{\varphi}_\tau\|_{L^2(0,T;V)} \le c\tau \tag{5.4.17}$$

and consequently

$$\lim_{\tau \searrow 0} \|\overline{\varphi}_\tau - \widehat{\varphi}_\tau\|_{L^\infty(0,T;V)} = 0. \tag{5.4.18}$$

Next, in view of the convergences in (5.4.5), (5.4.7), (5.4.10) and owing to the strong compactness lemma stated in [62, Lemma 8, p. 84], we have that

$$\widehat{\ln \vartheta}_\tau \to \lambda \quad \text{in} \quad C^0([0,T];V^*), \tag{5.4.19}$$

$$\widehat{\vartheta^{1/2}}_\tau \to w \quad \text{in} \quad L^2(0,T;H), \tag{5.4.20}$$

$$\widehat{\varphi}_\tau \to \varphi \quad \text{in} \quad C^0([0,T];V). \tag{5.4.21}$$

Then, by (5.4.14)–(5.4.18) we can also conclude that

$$\overline{\ln \vartheta}_\tau \to \lambda \quad \text{in} \quad L^\infty(0,T;V^*), \tag{5.4.22}$$

$$\overline{\vartheta^{1/2}}_\tau \to w \quad \text{in} \quad L^2(0,T;H), \tag{5.4.23}$$

$$\overline{\varphi}_\tau \to \varphi \quad \text{in} \quad L^\infty(0,T;V). \tag{5.4.24}$$

Thanks to (5.4.24) and to the Lipschitz continuity of π, we have that

$$\pi(\overline{\varphi}_\tau) \to \pi(\varphi) \quad \text{in} \;\; L^\infty(0,T;H). \tag{5.4.25}$$

Now, we check that $\lambda = \ln \theta$: in fact, due to the weak convergence of $\overline{\vartheta}_\tau$ ensured by (5.4.1) and to the strong convergence of $\ln(\overline{\vartheta}_\tau)$ in (5.4.22) (see (5.4.4) as well), we have that

$$\limsup_{\tau \searrow 0} \int_0^T \int_\Omega \left(\ln \overline{\vartheta}_\tau\right) \overline{\vartheta}_\tau = \lim_{\tau \searrow 0} \int_0^T \langle \ln \overline{\vartheta}_\tau, \overline{\vartheta}_\tau \rangle = \int_0^T \langle \lambda, \vartheta \rangle = \int_0^T \int_\Omega \lambda \vartheta, \tag{5.4.26}$$

so that a standard tool for maximal monotone operators (cf., e.g., [1, Lemma 1.3, p. 42]) ensure that $\lambda = \ln \vartheta$. In the light of (5.2.16) and of the convergences (5.4.11) and (5.4.24), it is even simpler to check that ξ and φ satisfy (5.1.13).

At this point, recalling also (5.4.4), (5.4.5), (5.4.10) and passing to the limit in (5.2.13) and (5.2.14), we arrive at (5.1.10) and (5.1.11). In addition, note that (5.2.12) implies that $\widehat{\ln \vartheta}_\tau(0) = \ln \vartheta_0$ and $\widehat{\varphi}_\tau(0) = \varphi_0$; thus, thanks to (5.4.22) and (5.4.24), passing to the limit as $\tau \searrow 0$ leads to the initial conditions (5.1.15).

It remains to show (5.1.12). To this aim, we point out that (5.4.23) implies that, possibly taking another subsequence, $\overline{\vartheta^{1/2}}_\tau \to w$ almost everywhere in Q. Then, using (5.4.1) and the Egorov theorem, it is not difficult to verify that

$$\overline{\vartheta}_\tau = \left(\overline{\vartheta^{1/2}}_\tau\right)^2 \to w^2 \qquad \text{a.e. in } Q \text{ and in } L^2(0,T;H), \tag{5.4.27}$$

as well as $\vartheta = w^2$. Details of this argument can be found, for instance, in [10, Exercise 4.16, part 3, p. 123]. Then, as A induces a natural maximal monotone operator on $L^2(0,T;H)$, recalling (5.2.15) and observing that (cf. (5.4.8))

$$\limsup_{\tau\searrow 0} \int_0^T \left(\overline{\zeta}_\tau, \overline{\vartheta}_\tau - \vartheta^*\right)_H = \lim_{\tau\searrow 0} \int_0^T \left(\overline{\zeta}_\tau, \overline{\vartheta}_\tau - \vartheta^*\right)_H = \int_0^T \left(\zeta, \vartheta - \vartheta^*\right)_H , \tag{5.4.28}$$

we easily recover (5.1.12). Therefore, Theorem 5.1.1 is completely proved.

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Ringraziamenti

Ringrazio il Signore per avermi tenuto per mano durante questi anni di studio e per avermi circondato di così tante persone buone: tra queste, ringrazio innanzitutto la mia mamma e il mio papà che mi sono sempre stati accanto con tanto affetto; con loro ringrazio don Primo per la sua vicinanza saggia e paterna. Ringrazio di cuore tutta la mia splendida famiglia, che immagino di abbracciare in queste righe conclusive: la super nonna Maddalena; gli affezionati zii Francy e Lory, Daniele e Ombretta, Mauro e Vittoria; l'intramontabile zio Domenico e i mitici cugini Matteo, Luca e Daniele. Un ricordo intriso di affetto al nonno Mario, alla nonna Nina e al nonno Mario, che dal cielo gioiscono con noi.

Ringrazio i miei cari amici Chiara, Valeria, Rec, Simone, Nicola e Francesco che hanno reso belli e umanamente ricchi questi anni; con loro ringrazio i miei compagni di Dottorato, in particolare Marghe, Daniele, Gio, Luca, Carlo, Andrea, Anna, Ilenia, Marco, Monica, Barbara e Alberto. Un abbraccio ad Elena, Antonia, Gianluca, don Enrico e a tutti i ragazzi dell'Oratorio di Castelverde e del Liceo Scientifico "G. Aselli" di Cremona.

Ringrazio infine i luminosi insegnanti che hanno ritmato il mio percorso di studi: la Maestra Rosa, la Maestra Grazia e la Maestra Rossella; la Prof.sa Afra Bellini, la Prof.sa Mariachiara Stercoli e il Prof. Angelo Zigliani. Un ricordo speciale per i miei severi professori del Liceo che, con la loro limpida fermezza, mi hanno trasmesso la gioia e il piacere di imparare: il Prof. Gianpietro Billi, la Prof.sa Manuela Filippi, la Prof.sa Ilaria Lasagni, il Prof. Remo Barbieri e la Prof.sa Patrizia Giarola. Ringrazio il Ch.mo Prof. Marco Degiovanni e il Ch.mo Prof. Alessandro Giacomini per avermi accompagnato durante gli studi universitari. Esprimo infine la mia gratitudine al Ch.mo Prof. Pierluigi Colli per avermi seguito in questi anni di Dottorato con estrema competenza e profonda umanità.

Castelverde, Ottobre 2017.