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DOTTORATO DI RICERCA IN
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Regularity results on two dimensional stochastic Navier-Stokes equations in vorticity form

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Introduction

Navier-Stokes equations, modeling incompressible, viscous flows in two or three space dimensions, are one of the most well-known classical systems of fluid dynamics. The amount of scientific literature on fluid dynamics is enormous and goes back a long way. A complete historical review would take us too far. We recall only the relevant informations for our purpose. In order to find out more we refer to [21, 86, 87].
Motivated by physical considerations, aiming at including perturbative effects, which cannot be modeled in a deterministic way, the equations have been modified adding a noise term. In the last years a huge amount of work has been done regarding this class of stochastic partial differential equations of fundamental importance in physics, see e.g. [1, 22, 89].

The thesis concerns the study of the stochastic Navier-Stokes equations in their so-called vorticity form (referred to hereafter as the vorticity equation), in the planar setting. The recognition of the vorticity as a central object in understanding of fluid flows dates back to the early days of this field. We refer to the classic books [51, 52]. In the past years, a growing number of researchers have turned their attention to the vorticity in their work. This increased interest is well reflected for instance in the books [55, 56], which are our main references for what concerns the deterministic results concerning the equation. Regarding the stochastic case it will suffice to mention the works [43, 59].

We briefly recall the Navier-Stokes equations and derive the vorticity form focusing in particular to the physical two dimensional setting.

We use the same notation for scalar and vector-valued functions. From the context will be clear the case we are considering. Let d denote the dimension of the space; given vectors and vector functions in $\mathbb{R}^d$, their components are labelled as $v = (v_1, ..., v_d)$. We denote the scalar product by $u \cdot v = \sum_{i=1} u_i v_i$, the Euclidean norm by $|v|^2 = \sum_{i=1}^d (v_i)^2$. Partial derivatives with respect to the time or spatial coordinates are denoted by $\frac{\partial}{\partial t}$, $\partial_{x_i} = \frac{\partial}{\partial x_i}$ respectively. By means of the gradient operator $\nabla = (\partial_{x_1}, ..., \partial_{x_d})$ we represent the divergence of a vector field v by $\nabla \cdot v$. The Laplace operator is $\Delta = \nabla \cdot \nabla = \sum_{i=1}^d (\partial_{x_i})^2$. The curl of a vector field v is $\mathrm{curl}(v) = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) \underline{\mathrm{i}} + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) \underline{\mathrm{j}} + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) \underline{\mathrm{k}}$, where $\underline{\mathrm{i}}$, $\underline{\mathrm{j}}$, $\underline{\mathrm{k}}$ are the unit vectors for the x_1-, x_2- and x_3-axes respectively. The orthogonal gradient is $\nabla^\perp = \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right)$.

The Navier-Stokes equations

Let us denote the velocity of the fluid by $v(t,x)$, the pressure by $p(t,x)$ and the constant coefficient of the kinematic viscosity by ν ($\nu > 0$). The Navier-Stokes equations in a domain

$D \subseteq \mathbb{R}^d$, $d = 2, 3$ are

$$\begin{cases} \frac{\partial v}{\partial t} + (v \cdot \nabla) v = \nu \Delta v - \nabla p + f, \\ \nabla \cdot v = 0, \\ v_{|t=0} = v_0. \end{cases} \tag{.0.0.1}$$

If $D \subset \mathbb{R}^d$, the equations are supplemented by boundary conditions for any $t \geq 0$. The notation $(v \cdot \nabla)v$ stands for the vector field with components $[(v \cdot \nabla)v]_j = \sum_{k=1}^d v_k \frac{\partial}{\partial x_k} v_j$; the operation Δv has to be understood componentwise. The equation $\nabla \cdot v = 0$ states that the fluid is not compressible. From now on we assume $\nu = 1$ for simplicity.

The vector field f is a force acting on the body of the fluid; we consider a stochastic force. Its addition can be motivated by a number of considerations. Since the Navier-Stokes equations are dissipative, if there is no external forcing, the system relaxes to the zero state where the fluid is at rest. Hence, if one is interested in probing the nonlinear dynamics, some forcing is necessary. Stochastic forcing is often proposed, particularly in the study of turbulent fluid flows, as a way to add a "generic" forcing that, from a physical point of view, seems reasonable. A more technical reason for introducing a random force is that it may improve the properties of well-posedness of the deterministic equation.
We consider a Gaussian-type perturbation white in time, colored in space by a suitable covariance operator. Since the present Section is intended as a brief introduction to the equations, we postpone the detailed description and hypothesis on the noise term to the following Chapters.

Realistic examples of fluids usually have complicated boundary conditions, for example with inflow and outflow of fluid. From a mathematical point of view is more reasonable to consider less realistic cases that are simpler to deal with. We treat two cases. The simplest one is the case of a two dimensional square with periodic boundary conditions: we can think that the fluid occupies the full space $\mathbb{R}^2$ but all the fields appearing in (.0.0.1) are periodic. Equations in spaces subjected to periodic boundary conditions can be reformulated as problems on a flat torus $\mathbb{T}^2$. Working on a torus has at least two technical advantages: it has no boundary and it is compact. Moreover, if we work in this context we have the Fourier series techniques for the computation of explicit solutions. The second case we consider is the whole space $\mathbb{R}^2$. It shares with $\mathbb{T}^2$ the advantage of no boundary but lacks compactness. Apparently this case looks much more realistic than the torus case but a lot depends on the conditions at infinity: if we assume that fields decay to zero at infinity, then we have the advantages of no boundary, but we loose most interesting physical examples. If on the contrary we accept non-zero values at infinity, even increasing and fluctuating, then the problem becomes physically more interesting, but from the mathematical prospective it is more difficult to treat. When we deal with $\mathbb{R}^2$ we always mean the simpler case of fields decay to zero at infinity.

The vorticity formulation of Navier-Stokes equations

The typical way of describing the motion of a fluid is through its velocity field. However, more than 150 years ago Helmholtz realized that in addition to the velocity, the vorticity of the fluid carries important information about the nature of the flow. The present Section is intended to briefly recall the vorticity formulation of Navier-Stokes equations. The reader is referred to the books [21, 55] for extensive discussions concerning the role of this equation in fluid dynamics.
The equation describes the evolution of the vorticity of a particle of a fluid as it moves with

its flow, that is, the local rotation of the fluid. Suppose that a small ball is located within the fluid (the centre of the ball being fixed at a certain point). If the ball has a rough surface, the fluid flowing past it will make it rotate; the rotation axis (oriented according to the right hand rule) points in the direction of the *curl* of the field at the centre of the ball. In other words, the vorticity tells how the velocity vector changes when one moves by an infinitesimal distance in a direction perpendicular to it.

Let $d = 3$, if we take the *curl* of the first equation in the Navier-Stokes system (.0.0.1) and denote by

$$\xi = \mathrm{curl}(v) \tag{.0.0.2}$$

the vorticity, exploiting the identity $\mathrm{curl}((v \cdot \nabla)v) = (v \cdot \nabla)\xi - (\xi \cdot \nabla)v$, we obtain

$$\frac{\partial \xi}{\partial t} + (v \cdot \nabla)\xi = \Delta\xi + (\xi \cdot \nabla)v + \mathrm{curl}(f). \tag{.0.0.3}$$

Let us notice that the pressure term has disappeared. The term $(\xi \cdot \nabla)v$ is called *vortex stretching* term and it describes the action on the vorticity field (for instance the elongation) due to the deformation tensor. The vortex stretching term is the main source of difficulties in the theoretical analysis of the three dimensional Navier-Stokes equations. One could develop the following picture: vorticity is not just transported and diffused, but may increase by stretching: higher vorticity may imply higher velocities locally around the axis of rotation; this may produce more intense deformation and increase the vorticity further.
The force driving the equation for the vorticity is formally obtained by taking the *curl* of the force driving the original equations. In dealing with a stochastic perturbation this will correspond, in a sense that we shall make precise in Part II, to consider the *curl* of the covariance operator of the noise driving the Navier-Stokes equations.

Let us now focus on the vorticity formulation in the two dimensional (planar) case. Since we live in a three-dimensional world, it may be less obvious why the understanding of two-dimensional fluid flows is of interest. Strictly speaking, two dimensional fluids do not exist, but some relevant physical examples can be considered two dimensional at a proper scale. In many applications, such as the atmosphere or the ocean, the fluid domain is much smaller in one direction than in the other two. In such circumstances a two-dimensional approximation to the fluid motion can provide very accurate insights into the behavior of the physical system. Formally, a fluid is two-dimensional if $v = (v_1, v_2, 0)$, with v_1 and v_2 independent of x_3, namely the flow is invariant with respect to translations in the direction given by the x_3 axis. In a two-dimensional flow, where the velocity is independent of the third coordinate and its third component is equal to zero, the vorticity vector is always perpendicular to the plane of the flow, and it is given by

$$\xi \;=\; \mathrm{curl}\; v = \nabla^{\perp} \cdot v \;=\; \left(0, 0, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right). \tag{.0.0.4}$$

We see already an important difference between two and three dimensions. In two dimension, only one component of the vorticity is nonzero, and thus we can treat the vorticity essentially as a *scalar* field. Moreover, in the two-dimensional case the vorticity equation does not contain the vortex stretching term: the difficult term $(\xi \cdot \nabla)v$ disappears since ξ has only the third component different from zero and $\frac{\partial v}{\partial x_3}$ is zero. The absence of vortex stretching means that vorticity is just transported and diffused. This is a first insight of the fact that two and three

dimensional fluids behave in qualitatively different fashions.
Equation (.0.0.3) reduces to a non linear diffusion equation for ξ:

$$\frac{\partial \xi}{\partial t} + v \cdot \nabla \xi = \Delta \xi + \operatorname{curl}(f), \tag{.0.0.5}$$

where $\nabla \cdot v = 0$ and $\xi = \nabla^{\perp} \cdot v$.
The velocity still appears in equation (.0.0.5) for the evolution of the vorticity. However, if we remember that the vorticity is the *curl* of a divergence-free vector field, then we can recover the velocity v from the vorticity ξ by means of a non local operator. This allows, on one side, to eliminate the velocity from the vorticity equation (.0.0.5) to yield a self-contained equation for ξ; on the other side to provide, for enough regular solutions, the equivalence between the Navier-Stokes equations and their vorticity stream formulation (see [55, Propositions 2.1, 2.4] for a statement of the result in the deterministic two-dimensional case). From the second equation in (.0.0.1), the vector function v is divergence-free, then there exists (assuming D simply connected) a unique (up to an additive constant) stream function Ψ such that

$$v = -\operatorname{curl}(\Psi) = -\nabla^{\perp}\Psi. \tag{.0.0.6}$$

Combining equations (.0.0.2) and (.0.0.6) we obtain the Poisson equation for Ψ:

$$\xi = -\Delta\Psi. \tag{.0.0.7}$$

In the cases $D = \mathbb{R}^2$ and $D = \mathbb{T}^2$, a solution to this equation is given by

$$\Psi = G * \xi, \tag{.0.0.8}$$

where G is the fundamental solution (the Green kernel) of the Poisson equation on $\mathbb{R}^2$ or $\mathbb{T}^2$ respectively.
We can replace expressions (.0.0.6)-(.0.0.8) with the more explicit formula

$$v(t,x) = (k * \xi)(t,x) = \int_{\mathbb{R}^2} k(x-y)\xi(t,y)\,\mathrm{d}y, \tag{.0.0.9}$$

where k is the so called Biot-Savart kernel. It is obtained by taking the orthogonal gradient of the fundamental solution G of the Poisson equation.
The fact that $\nabla \cdot k = 0$ (which can be verified directly), yields by (.0.0.9) the incompressibility condition $\nabla \cdot v = 0$. Given the vorticity, relation (.0.0.9) allows to build the velocity field. If $D = \mathbb{R}^2$, (.0.0.9) holds when $v \to 0$ as $|x| \to \infty$ (for more details see e.g. [56, Chapter 1.2]). From (.0.0.9) it is evident that the velocity v is recovered from the vorticity ξ by a *non local* operator given by the convolution with the kernel k.
If $D = \mathbb{R}^2$, then the Biot-Savart kernel is given by (see [55, Chapter 2.1])

$$k(x) = \nabla^{\perp}\left(-\frac{1}{2\pi}\ln|x|\right) = -\frac{1}{2\pi}\frac{x^{\perp}}{|x|^2}, \tag{.0.0.10}$$

with the natural notation $x^{\perp} = (-x_2, x_1)$.
Similarly, if $D = \mathbb{T}^2$, in an analogous way it is possible to find an explicit expression for $k(\cdot)$, given in terms of a series. A detailed discussion is postponed to Part I.
From (.0.0.10) it is evident (and the same consideration holds in the flat torus case) that the

Biot-Savart kernel k is singular at the origin. The vorticity ξ need to be a very nice function in order for the formula $v = k * \xi$ to makes sense mathematically. In particular if $\xi(t, \cdot)$ has compact support (e.g. in $\mathbb{T}^2$), for every fixed t, then $v(t, \cdot)$ is a well defined integrable function. Problems arise on non compact domains (e.g. $\mathbb{R}^2$). In this case, a sufficient condition for the validity of (.0.0.9) is that ξ is at least bounded and integrable. This observation will be crucial in facing the problems we address in the thesis.

Plan of the thesis

The thesis concerns the two dimensional Navier-Stokes equations in vorticity form, driven by a stochastic Gaussian perturbation. The equations are studied in the two cases $D = \mathbb{R}^2$ and $D = \mathbb{T}^2$. The very different topological proprieties of the domain (in particular compactness) lead to the use of completely different techniques in the analysis of the equations. For this reason we have decided to divide the thesis in two parts, one concerning the analysis on the flat torus and one concerning the analysis on the whole plane $\mathbb{R}^2$.

The starting point of the research is to study the regularity, in the Malliavin sense, of solutions to stochastic fluid dynamical equations in dimension bigger than one. We are interested in the problem of the existence of a density for the law of the random variable given by the solution process at fixed points in time and space. This property is important in the analysis of hitting probabilities (see [29, 31]) and concentration inequalities (see [71]). Most of the literature on this subject concerns the heat and wave equations (see e.g. [72, 4, 68, 79, 58, 57] and the references therein). The paper [20] deals with the Cahn-Hilliard equation and [66, 92] deal with the one dimensional Burgers equation. This latter equation has similar features to the Navier-Stokes equations. The results we obtain can be considered an extension of the results contained in the above cited papers concerning the Burgers equation.
We work on the two dimensional stochastic Navier-Stokes equations in vorticity form, instead of the usual formulation with respect to the vector velocity. The reasons are twofold. Firstly, in the two dimensional case the vortex stretching term disappears. Secondly, since the solution is a *scalar* random field the technique of analysis of the existence of the density by means of Malliavin calculus is much less involved than the case for a vector unknown. The kind of nonlinearity we deal with, although presents similar features to the quadratic term of the Burgers equation, is different from all the non linear terms considered in the existing literature. In fact the relation between the velocity and the vorticity is given by a non local (in space) operator and this fact considerably complicates the involved estimates. Moreover, we work in the two spatial dimension setting since, as it it is well known, the two-dimensional Navier-Stokes equations are a well posed problem. On the other hand, one of the millennium prize problems (see the presentation of Fefferman [36]) concerns the well posedness of the three dimensional Navier-Stokes system. For this reason it does not make sense to investigate the regularity of the solution in the Malliavin sense in the three dimensional case.
The Gaussian perturbation driving the equation provides the probability context where to study the equation. In particular it generates the context where to work when we perform the Malliavin analysis of the solution. Since we are in dimension greater than one, we consider a noise colored in space. This, from a technical point of view, is more difficult to treat than the space-time white noise.

Dealing with the two dimensional stochastic vorticity equation subjected to periodic boundary conditions, we address the problem of the existence of a density by means of the techniques used in [42], [66, 92] for the Burgers equation, working in the martingale measure approach introduced by Walsh. The compactness of the domain plays a crucial role in our analysis. The vorticity form of the Navier-Stokes equations on the flat torus can be rewritten as a closed equation for the vorticity by means of the Biot-Savart law. On a compact domain we can handle the singularity at the origin of the Biot-Savart kernel. Suitable estimates made on it highlight the similarity between our non linear term and that appearing in the Burgers equation. Exploiting the Biot-Savart law, we never take explicitly into account the Navier-Stokes equations for the velocity.

Regarding the study of the vorticity equation on the whole plane $\mathbb{R}^2$, the original idea was to extend the obtained results in the flat torus case to this setting. We were interested in the Malliavin analysis of the solution process when the driving perturbation was given by a spatially homogeneous Gaussian random field. Due to the lack of literature in this context, we firstly deal with the problem concerning the existence and uniqueness of a solution to the vorticity equation (with enough regularity to perform Malliavin analysis). The technique used for the flat torus case can not be readapted in the case of an unbounded domain. The lack of compactness of the domain throws up a substantial difficulty that led to use different tools and a completely different way of proceeding. On the whole plane $\mathbb{R}^2$, v can formally be given by means of the Biot-Savart law in terms of ξ. Nevertheless, due to the singularity at the origin of the Biot-Savart kernel, on a non compact domain ξ has to be a very nice function in order for this expression to make sense. In other words, we can not directly handle the closed form for the vorticity equation, where v is given by convolving ξ with the Biot-Savart kernel. Its singularity prevents us to obtain the needed estimates that allows to the treat the vorticity equation as a closed equation for ξ, as in the flat torus case. The problem has to be approached in a different way: we have to explicitly take into account the equation for the velocity. In this case we work in the Da Prato-Zabczyk functional setting and we study the regularity of v, as a solution to the Navier-Stokes equations. The stated regularity on v allows to handle and study the equation for the vorticity. The link between the assumptions on the noise driving the equation for the velocity and the equation for the vorticity becomes, in this setting, quite crucial. The existence results are obtained in a rather abstract setting and cover the particular case in which the random perturbation is a spatially homogeneous Wiener random field. This kind of noise, which plays an important role in statistical theories of turbulence, allows to consider random field solutions. At the moment we are carrying out the research in this context, studying the regularity in the Malliavin sense for the image law of the solution process at fixed points in time and space.

Outline

The thesis is dived in two parts. Listed in order of appearance, we deal with

I the analysis of the vorticity equation on the flat torus;

II the analysis of the vorticity equation on $\mathbb{R}^2$.

The structure of the two parts is such that each one can be read as a stand alone: they start with an introductory Chapter to the topic and the questions addressed. A preliminaries Chapter provides the needed notations, the mathematical setting and some analytic preliminaries. The following Chapters show the mathematical achievements. At the end of some Chapters will be present a "Note and Comments" Section: there we shall provide some bibliographical references. Topics recalled in that Section are not essential in the understanding of the proved results but provide a background on the existing literature concerning that particular topic.

Some general probabilistic preliminaries needed in the thesis are gathered in the Appendices. In Appendix A we briefly recall the two different stochastic integration theories developed on one side by Walsh, on the other side by Da Prato and Zabczyk. The latter theory has been lately extended to cover more general classes of Banach space valued processes: we shall provide some basic facts needed in the thesis. We present both the stochastic integration theories since in the thesis both are needed. In particular, in Part I Walsh stochastic theory provides the underlying context where study both the existence of a solution and its regularity in the Malliavin sense. As regards the case on the whole plane $\mathbb{R}^2$ (Part II), we use the Hilbert and Banach space integration theory in order to prove the existence and uniqueness of solutions with the desired regularity. In the same Appendix we shall also provide the tools needed in Section I.1.3.4 where we show that, with the particular noise we consider, the Hilbert space-valued integral and the martingale measure stochastic integral turns out to be equivalent. In Appendix B we recall some definitions and results on Malliavin calculus. All the recalled fact are available in literature. We choose to collect them, in a rather concise way, for the sake of clarity and completeness.

Regarding the analytic preliminaries we shall need, we choose to introduce them separately in the preliminary Chapters of Parts I and II. Some general notations used throughout the thesis are given at the end of the present Introduction.

Abstracts

We briefly summarize the addressed problems with a short abstract stating the main proved results.

Analysis on the flat torus

We consider the two dimensional stochastic Navier-Stokes equations in vorticity form

$$\begin{cases} \dfrac{\partial \xi}{\partial t}(t,x) - \nu\Delta\xi(t,x) + v(t,x)\cdot\nabla\xi(t,x) = \sigma(\xi(t,x))w(\mathrm{d}x,\mathrm{d}t) & (t,x)\in(0,T]\times[0,2\pi]^2 \\ \nabla\cdot v(t,x) = 0 & (t,x)\in[0,T]\times[0,2\pi]^2 \\ \xi(t,x) = \nabla^{\perp}\cdot v(t,x) & (t,x)\in[0,T]\times[0,2\pi]^2 \\ \xi(0,x) = \xi_0(x) & x\in[0,2\pi]^2\,. \end{cases} \tag{.0.0.11}$$

with periodic boundary conditions and a multiplicative noise. ξ_0 is the initial datum of the problem. $w(\mathrm{d}t,\mathrm{d}x)$ is the formal notation for a spatially homogeneous (periodic and with zero mean in the space variable) Gaussian noise white in time and colored in space, by a suitable covariance operator, defined on some probability space. σ is some real-valued function satisfying proper (rather classical) assumptions.
System (.0.0.11) can be rewritten as a closed equation for the vorticity, since v can be expressed in terms of ξ by means of the Biot-Savart law.
The main result we obtain concerns the absolutely continuity of the image law of the solution process evaluated at fixed points in time and space.
A good notion of solution process, which allows us to evaluate it at fixed points in time and space, is given by the (weak) random field solution in the sense of Walsh. We interpret

equation (.0.0.11) in this latter sense as an integral evolution equation written by means of the convolution with the fundamental solution of the heat equation. The heat kernel becomes less smooth as the dimension increases. For this reason, we have to consider a noise colored in space by a covariance operator with some regularizing effects to ensure the well posedness of the stochastic term that appears in the evolution formulation of (.0.0.11).
Although periodic boundary conditions are less realistic, the technical advantages they provide are a key point in our analysis. The compactness of the domain plays a crucial role. We have rather good estimates on the Biot-Savart kernel k and this allows us to work directly on (.0.0.11) (interpreted in the Walsh integral sense) without explicitly taking into account the Navier-Stokes equations for the velocity.
We address the issue of the existence and uniqueness of a solution to (.0.0.11) in the martingale measure approach introduced by Walsh. The techniques applied in our work are based on a rather classical stopping time argument. Assuming ξ_0 is a continuous function we prove the existence of a unique solution process with space-time continuous trajectory. Thanks to the regularizing effect of the heat kernel the covariance of the noise need not to be a trace class operator, we require less.
In literature there are no results on stochastic vorticity equation, based on the martingale measure approach, considering the Walsh notion of solution. The existing results use the Prato-Zabczyk functional approach. With this latter approach, under suitable assumptions on the initial datum and the covariance of the noise, it is possible to prove the existence of a mild solution taking values in a suitable Hilbert space embedded in the space of continuous functions. In the particular case of a spatially homogeneous noise it turns out that this notion of solution is equivalent to the solution in the Walsh sense. However, our result allows us to obtain in a straightforward way a solution with space-time continuous paths. Moreover, the employed techniques allow us to require the minimal hypothesis on the covariance operator. By means of the Da Prato-Zabczyk approach we would obtain the same regularity results under stronger assumptions.
The problem concerning the existence of a density is addressed, once again, by means of a localization argument, using classical tools of Malliavin calculus. If ξ_0 is a continuous function and the covariance of the noise is a trace class operator we prove that the image law of the random variable, obtained evaluating the solution process at fixed points in time and space admits a density with respect to the Lebesgue measure on $\mathbb{R}$. Proofs are based on standard techniques of Malliavin calculus, but the nonlinear term appearing in (.0.0.11) represents a non-negligible technical source of difficulty. Malliavin analysis would be even more complicated if we consider the Navier-Stokes system for the velocity. Since the velocity and vorticity formulations are equivalent in the flat torus case, we work on the vorticity form to highlight the novelties of the results, when compared with the existing literature.
In the proofs of all the above mentioned results, the estimates performed on the Biot-Savart kernel play a crucial role. Moreover, a precise knowledge of the fundamental solution of the heat kernel and on its space derivatives is required to be able to have results based on this variational formulation.

Analysis on $\mathbb{R}^2$

We consider the two-dimensional stochastic Navier-Stokes equations on the whole plane $\mathbb{R}^2$

$$\begin{cases} \mathrm{d}v(t) + [-\Delta v(t) + (v(t)\cdot\nabla)v(t) + \nabla p(t)]\,\mathrm{d}t = G(v(t))\,\mathrm{d}W(t) & t\in[0,T] \\ \nabla\cdot v(t) = 0 & t\in[0,T] \\ v(0,x) = v_0(x) & x\in\mathbb{R}^2. \end{cases} \tag{.0.0.12}$$

We are concerned with a multiplicative noise. v_0 is the initial datum. W is a cylindrical $\mathcal{H}$-Wiener process, whit $\mathcal{H}$ a real separable Hilbert space, and G is the covariance operator of the noise. Taking the *curl* on both sides of the first equation to (.0.0.12), we recover the equation for the vorticity.

$$\begin{cases} \mathrm{d}\xi(t) + [-\Delta\xi(t) + v(t)\cdot\nabla\xi(t)]\;\mathrm{d}t = \mathrm{curl}(G(v(t))\,\mathrm{d}W(t)) & t\in[0,T] \\ \nabla\cdot v(t) = 0, & t\in[0,T] \\ \xi(t) = \nabla^{\perp}\cdot v(t), & t\in[0,T] \\ \xi(0,x) = \xi_0(x) & x\in\mathbb{R}^2. \end{cases} \tag{.0.0.13}$$

We address the problem of existence and uniqueness of a solution to (.0.0.13) under different assumptions on the covariance operator G. The literature presents results on v, solution to (.0.0.12). Our contribution is to provide results involving (.0.0.13).

The first result we obtain is inspired by [18]: we work in a rather abstract setting exploiting the functional approach à la Da Prato-Zabczyk. We prove the existence of a martingale solution to (.0.0.12) by means of the Faedo-Galerkin method. The techniques used in the proofs are based on the construction of the Faedo-Galerkin type approximations of the solutions and some a priori estimates that allow one to prove compactness properties of the corresponding probability measures and finally to obtain a solution of the equations. Since on the whole $\mathbb{R}^2$ the embedding of the Sobolev space of square integrable gradient into the L^2 space is not compact, this method requires the use of spaces with weights. Then we prove pathwise uniqueness and from that and [45] we infer the existence of a unique strong (in the probability sense) solution. We work in Banach spaces (using γ-radonifying operators instead of Hilbert-Schmidt operators) and this allows to gain a better space regularity for the solution process. The existence of a strong solution to (.0.0.13) follows from the result proved for the velocity as a corollary. The assumptions made on the initial datum and the covariance operator G are quite strong, but they allows to obtain a solution ξ which is continuous in time, q-integrable in space, for every $q\geq 2$, with bounded moments of every order. Moreover results obtained in this rather abstract setting covers the case in which the equation is driven by a spatially homogeneous Gaussian random field and the covariance operator is of Nemitsky form. Starting from these results, we will face in the future the analysis of regularity of the solution $\xi(t,x)$ in the Malliavin sense.

The second result we obtain is inspired by [16]. In this case we considerably weaken the assumptions on G. In particular the covariance operator appearing in (.0.0.13) is not regular enough to allow us to use Itô formula in the space of finite energy velocity vectors, and an approximation procedure is required. In this case we work directly on the equation for the vorticity. We construct a sequence of approximating processes which solves equations with a regularized covariance operator. For their well posedness we exploit the above mentioned results obtained under the stronger assumptions on the covariance operator. We use a tightness argument to pass to the limit and obtain a solution of the equation (.0.0.13). The a priori estimates that allow to prove compactness require a certain regularity on the velocity. This is proved by studying equation (.0.0.12). Differently from before, in this case the existence and uniqueness result holds only $\mathbb{P}$-almost surely.

Notation

We present here some general notation. As regards the definition of functional spaces we remand to the "Mathematical Setting" Section at the beginning of each Part.

General mathematics

$d =$ dimension of the space ($d = 2, 3$)
$\mathbb{N} = \{0, 1, 2, 3, ...\}$
$\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$
$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
$\mathbb{Z}^2 = \{k = (k_1, k_2), k_i \in \mathbb{Z}, i = 1, 2\}$
$\mathbb{Z}^2_0 = \mathbb{Z}^2 \setminus \{0\}$
$\mathbb{Z}^2_+ = \{k = (k_1, k_2) \in \mathbb{Z}^2 : k_1 > 0\} \cup \{k = (0, k_2) \in \mathbb{Z}^2 : k_2 > 0\}$
$\mathbb{R}$, real numbers
$\mathbb{R}^d = \{x = (x_1, x_2, ..., x_d), x_i \in \mathbb{R}, i = 1, ..., d\}$
$\mathbb{C}$, complex numbers
$x \cdot y = \sum_{k=1}^d x_k y_k$, $x, y \in \mathbb{R}^d$, scalar product in $\mathbb{R}^d$
$|x| = \sqrt{x \cdot x}$, $x, y \in \mathbb{R}^d$, norm in $\mathbb{R}^d$
$\Re$, real part
$\Im$, imaginary part
$z = \Re z + i\Im z$ element in $\mathbb{C}$
$|z| = \sqrt{(\Re z)^2 + (\Im z)^2}$, absolute value of a complex number
$\bar{z} = \Re z - i\Im z$ complex coniugate of z
$k^\perp = (-k_2, k_1)$
p^*, conjugate exponent, $\frac{1}{p} + \frac{1}{p^*} = 1$
$\mathbf{1}$, indicator function
$*$, convolution
$a \wedge b = \min\{a, b\}$

Spaces

H, $\mathcal{H}$, V, Hilbert spaces
$\langle \cdot, \cdot \rangle_H$, scalar product in H
E, F, Banach spaces
E^*, dual Banach space of E
${}_{E^*}\langle \cdot, \cdot \rangle_E$, standard duality pairing
$\mathcal{L}(X, Y)$, bounded linear operators, p. 83

$\mathcal{L}(H) := \mathcal{L}(H, H)$
$\bar{\mathcal{L}}(X, Y)$, bounded bilinear operators, p. 159
$L_1(H, V)$, nuclear operators from H to V, p. 151
$L_{\mathrm{HS}}(H, V)$, Hilbert-Schmidt operators from H to V, p.151
$R(H, E)$, γ-radonyfing operators from H to E, p. 157
$L^2_\sharp(D)$, p. 21
$\dot{L}^2_\sharp(D)$, p. 21
$\dot{L}^p_\sharp(D)$, p. 21
$W^{b,p}(D)$, $W^b(D)$, $W^{-b}(D)$, p.22
H, $L_p(D)$, p. 22
$H^{b,p}(D)$, $H^b(D)$, $H^{-b}(D)$, p. 22
L^2_Q, p. 30
$\mathcal{H}_T$, p. 30
$\mathbb{L}^q(\mathbb{R}^2)$, p. 84
$C_0^\infty(\mathbb{R}^2)$, $C^\infty_{sol}(\mathbb{R}^2)$, p. 84
$C_p^\infty(\mathbb{R}^2)$, p. 163
$\mathcal{S}(\mathbb{R}^2)$, $\mathcal{S}'(\mathbb{R}^2)$, p. 84
$W^{s,q}(\mathbb{R}^2)$, p. 84
$H^{s,q}(\mathbb{R}^2)$, p. 85
$H^{1,2}_{L^2}$, p. 85
$\mathbb{L}^2_{loc}$, p. 85
$L^2(0, T; \mathbb{L}^2_{loc})$, p. 85
$L^\alpha_W(0, T; L^q)$, p. 85
$L^\infty_W(0, T; L^q)$, p. 85
$C([0, T]\,; L^2_W)$, p. 85
$C^\beta([0, T]\,; H^{\delta,2})$, p. 85
$\mathbb{U}$, $\mathbb{V}$, p. 86
$\mathbb{L}^2_\theta$, p. 102
$\mathcal{S}_s(\mathbb{R}^2)$, $\mathcal{S}'_s(\mathbb{R}^2)$, p. 117
$L^2_{(s)}(\mathbb{R}^2, \mu)$, p. 111
$\mathcal{H}_{\mathcal{W}}$, p. 111
U, p. 119

Measure and probability

r.v., random variable
a.e., almost everywhere
a.s., almost surely
i.i.d., independent identically distributed
$\mathcal{B}(X)$, Borel σ-algebra on X
$\mathcal{B}_b(X)$, Borel σ algebra of bounded sets on X
$(\Omega, \mathcal{F}, \mathbb{P})$, probability space
$\{\mathcal{F}_t\}_{t\in[0,T]}$, filtration
$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, stochastic basis
$\mathcal{L}(X)$, law of the random variable X
$\mathcal{N}(\mu, \sigma^2)$, Gaussian distribution with mean μ and variance σ^2

$\beta(t)$, Brownian motion
$\mathcal{C}$, smooth cylindrical random variables, p. 162
D^k, Malliavin derivative of order k, p. 163
$\mathbb{D}^{k,p}$, space of k-times Malliavin differentiable function in $L^p(\Omega)$, p. 163
$\mathbb{D}^{k,p}_{loc}$, localization of $\mathbb{D}^{k,p}$, p. 163

Fluid dynamics

v, velocity
p, pressure
ξ, vorticity
f, external force
k, Biot-Savart kernel

Operators

∂_k, partial derivative
Δ, Laplacian
∇, gradient operator on $\mathbb{R}^d$
$\nabla\cdot$, divergence
$\nabla^\perp = \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right)$, orthogonal gradient
$\mathcal{P}$, orthogonal projection into solenoidal spaces, p. 86
$P^{(n)}$, p. 93
T^*, adjoint operator, p. 83
$D(T)$, domain of T
TrT, trace of T, p. 151
Ker$T$, kernel of T
Id_X, identity operator on the space X
$\{S(t)\}_{t\in[0,T]}$, semigroup generated by A, pp. 24, 87
$\hat{\cdot}$, $\mathcal{F}$, Fourier transform, p. 117
$\mathcal{F}^{-1}$, inverse Fourier transform, p. 117
δ_k, delta Dirac function centered in k

Miscellaneous

w.r.t., with respect to
SDE, stochastic differential equations
PDE, partial differential equations
SPDE, stochastic partial differential equations
RKHS, reproducing kernel Hilbert space

General comments about the notation

If there is not confusion in the standard duality pairing ${}_{E^*}\langle\cdot,\cdot\rangle_E$, we shall omit the subscripts E and E^* and write $\langle\cdot,\cdot\rangle$. In the case of the scalar product $\langle\cdot,\cdot\rangle_H$ too, if no confusion seems

likely we omit the subscript H.
We shall indicate with C a constant that may varies from line to line. In certain cases, we write $C_{\alpha,\beta,\dots}$ to emphasize the dependence of the constant on the parameters $\alpha, \beta, \dots$.
Spaces over the domain D will be denoted without explicitly mentioning the domain, e.g. L^p stands for $L^p(D)$.
If not specified by $\{\mathcal{F}_t\}_{t\in[0,T]}$ we mean the natural filtration.

Part I

Stochastic Navier-Stokes equations: analysis on the flat torus

Introduction

We are concerned with the two-dimensional Navier-Stokes equations in their vorticity form:

$$\begin{cases} \dfrac{\partial \xi}{\partial t}(t,x) - \Delta\xi(t,x) + v(t,x)\cdot\nabla\xi(t,x) = \sigma(\xi(t,x))w(\mathrm{d}x,\mathrm{d}t) & (t,x)\in(0,T]\times D \\ \nabla\cdot v(t,x) = 0 & (t,x)\in[0,T]\times D \\ \xi(t,x) = \nabla^{\perp}\cdot v(t,x) & (t,x)\in[0,T]\times D \\ \xi(0,x) = \xi_0(x) & x\in D, \end{cases} \tag{I.0.0.1}$$

with $D = [0, 2\pi]^2$. We recall that the unknown of the problem is the vorticity of the flow, denoted by ξ. The data of the problem are the initial vorticity ξ_0 and the stochastic forcing term w. System (I.0.0.1) can be rewritten as a closed equation for the vorticity, since v can be explicitly expressed in terms of ξ by means of the Biot-Savart law $v = k*\xi$. The coefficient σ is some real-valued function and $w(\mathrm{d}x, \mathrm{d}t)$ is the formal notation for some Gaussian perturbation defined on some probability space; we will assume that it is white in time and weighted in space by the action of a suitable operator. We denote by $\{\mathcal{F}_t\}_{t\in[0,T]}$ the filtration generated by w.

Suitable boundary conditions are associated to system (I.0.0.1); in the present part periodic boundary conditions are assumed. Equations in two dimensional spaces subjected to periodic conditions can be reformulated as problems on a flat torus, that we shall denote by $\mathbb{T}^2$. Intuitively, in periodic boundary conditions, the square $[0, 2\pi]^2$ is replicated throughout space to form an infinite lattice. When a particle of fluid moves in the central square, its periodic image moves with exactly the same orientation in exactly the same way in every one of the other squares. Thus, as a particle of fluid leaves the central square, one of its images will enter through the opposite side. As a particle of fluid moves through a boundary, all its corresponding images move across their corresponding boundaries. The number of particles in the central square (and hence in the entire system) is conserved. When a particle leaves the square by crossing a boundary, attention may be switched to the identical particle just entering from the opposite side. In the two dimensional case it is useful to picture the single central simulation box as being rolled up to form the surface of a three dimensional torus. Each original dashed line now closes on itself. A particle following either one of these lines comes back to the same point on the torus, just as a particle in the central box of the infinite lattice comes back to the same location if it follows a straight line. With the torus configuration there is no need to consider an infinite number of replicas of the system, nor any image particles. The torus correctly represents the topology of the two-dimensional system. $\mathbb{T}^2$ can be thought as a rectangle without boundaries in which we identify the points $(x_1, 0)$ with $(x_1, 2\pi)$ and $(0, x_2)$ with $(2\pi, x_2)$. This means that the space variables are assumed to be elements of $\mathbb{T}^2$ and the periodic fields can be identified with a function on the flat torus (for more details see e.g. [83, Chapter 7]).

We interpret the first equation appearing in (I.0.0.1) in the sense of Walsh. Let $g(t,x,y)$ be the fundamental solution to the heat equation on the flat torus (see Section I.1.2.2). We shall see that a random field $\xi = \{\xi(t,x), t \in [0,T] \times D\}$ is a solution to equation (I.0.0.1) if it satisfies the evolution equation

$$\xi(t,x) = \int_D g(t,x,y)\xi_0(y)\,\mathrm{d}y + \int_0^t \int_D \nabla_y g(t-s,x,y) \cdot v(s,y)\xi(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ + \int_0^t \int_D g(t-s,x,y)\sigma(\xi(s,y))\,w(\mathrm{d}y,\mathrm{d}s) \qquad \text{(I.0.0.2)}$$

with $v = k * \xi$. We consider this notion of solution since our main aim is to prove the absolute continuity of the image law of the solution to equation (I.0.0.1) w.r.t. the Lebsgue measure on $\mathbb{R}$ at fixed points in time and space. Since we work on a spatial domain in dimension $d = 2$, we can not consider a time-space white noise, if we want to deal with real-valued solutions $\{\xi(t,x), t \in [0,T], x \in D\}$; this is because the fundamental solution of the heat equation becomes less smooth as the dimension increases. We consider a Gaussian noise white in time, colored in space by a suitable covariance operator Q (similarly as in [57]). In order to define the worthy martingale measure, w.r.t. which the stochastic integral is defined, we follow a similar approach to [28] requiring the weakest possible conditions on the weight we have to impose on the noise. In any case, we shall point out that the stochastic integral appearing in (I.0.0.2), formally defined in the Walsh sense, can be understood in the setting introduced by Da Prato and Zabczyk.

The main source of difficulty in the study of (I.0.0.2) is given by the non linear term which is non Lipschitz. We adopt a method of localization, inspired by the papers [42], [66] and [92], concerning the one dimensional stochastic Burgers equation. In [42] authors, by means of a stopping time argument prove the existence of a unique solution (in the Walsh sense) to the Burgers equation (on the real line). [66] and [92] focus on the problem concerning the Malliavin analysis of the solution process. A localization argument is needed in order to deal with the non linear term that appears in the Burgers equation. This latter equation has similar features as the Navier-Stokes equations. The work presented in this Part can be considered an extension of the results obtained in the above mentioned papers, representing a first step in the study of the regularity in Malliavin sense for solutions to stochastic fluid dynamical equations in dimension bigger than one.

Proceeding along the lines of the above mentioned papers, we shall introduce a suitable truncation factor that allows to obtain an approximated equation with a globally Lipschitz coefficient. To handle the velocity v we shall exploit the Biot-Savart law. The integrability properties of the Biot-Savart kernel and some estimates performed on it will be fundamental in our approach. We shall work on the approximated equation and then pass to the limit. By means of this localization procedure we prove the existence of a unique solution to (I.0.0.2). The method we use allows to obtain in a straightforward way a solution with a space-time continuous modification.

The same localization argument will be used in the study of the existence of a density for the image law of the solution process at fixed points in time and space. In dealing with this last problem we use classical Malliavin calculus tools proving that the r.v. given by the solution process at fixed points in time and space is locally differentiable in Malliavin sense and the norm of its derivative (in the space where it lives) is a.s. positive. As in [66] and [92] the smoothness of the density can not be obtained via the localization argument and remains an open problem.

The present Part is organized as follows. Chapter I.1 focus on the problem of the existence of a unique space-time continuous solution to equation (I.0.0.1). In Chapter I.2 we shall prove the existence of a density for the image law of the solution process at fixed points in time and space.

I.0.1 Mathematical Setting

In the present Section we introduce the notation used in this part of the thesis. We define the functional spaces and we recall some Sobolev embedding theorems.

I.0.1.1 Spaces and operators.

Let $D = [0, 2\pi]^2$, we consider the space $L^2_\sharp := L^2_\sharp(D)$ of all complex-valued 2π-periodic functions in x_1 and x_2 which are measurable and square integrable on D, endowed with the scalar product

$$\langle f, g\rangle_{L^2} = \int_D f(x)\overline{g(x)}\,\mathrm{d}x$$

and the norm $\|\cdot\|_{L^2} = \sqrt{\langle\cdot,\cdot\rangle_{L^2}}$. We also consider the space $\left[L^2_\sharp\right]^2 := \left[L^2_\sharp(D)\right]^2$ consisting of all pairs $u = (u_1, u_2)$ of complex-valued periodic functions endowed with the inner product

$$\begin{aligned}\langle u, v\rangle_{[L^2]^2} &:= \int_D u(x)\cdot\overline{v(x)}\,\mathrm{d}x\\ &= \int_D \left[u_1(x)\overline{v_1(x)} + u_2(x)\overline{v_2(x)}\right]\mathrm{d}x, \qquad u, v \in \left[L^2_\sharp\right]^2.\end{aligned}$$

By an innocuous abuse of notation, if the context is clear, the scalar product $\langle\cdot,\cdot\rangle_{[L^2]^2}$ will be denoted by $\langle\cdot,\cdot\rangle_{L^2}$ (similarly we shall denote the norm $\|\cdot\|_{[L^p]^2}$ by $\|\cdot\|_{L^p}$).
An orthonormal basis for the space $L^2_\sharp$ is given by $\{e_k\}_{k\in\mathbb{Z}^2}$, where

$$e_k(x) = \frac{1}{2\pi}e^{ik\cdot x}, \qquad x \in D,\ k \in \mathbb{Z}^2. \tag{I.0.1.1}$$

As usual in the periodic case, we deal with mean value zero vectors. This gives a simplification in the mathematical treatment but does not prevent to consider non zero mean value vectors: this can be dealt in a similar way (see [86]). We use the notation $\dot{L}^2_\sharp$ to keep tracks of the zero-mean condition. An orthonormal system for the space $\dot{L}^2_\sharp$, formed by eigenfunctions of the operator $-\Delta$ with associated eigenvalues $\lambda_k = |k|^2$, is given by $\{e_k\}_{k\in\mathbb{Z}^2_0}$ with e_k as in (I.0.1.1). The real-valued functions in $\dot{L}^2_\sharp$ can be characterized by their Fourier series expansion as follows

$$\dot{L}^2_\sharp = \{f(x) = \sum_{k\in\mathbb{Z}^2_0} f_k e_k(x) : \bar{f}_k = f_{-k} \text{ for any } k, \sum_{k\in\mathbb{Z}^2_0}|f_k|^2 < \infty\}, \tag{I.0.1.2}$$

where the terms

$$f_k = \frac{1}{2\pi}\int_D f(x)e^{-ik\cdot x}\,\mathrm{d}x, \qquad k \in \mathbb{Z}^2_0, \tag{I.0.1.3}$$

represents the Fourier coefficient of f. For every $p > 2$, with $\dot{L}^p_\sharp$ we denote the subspaces of $L^p := L^p(D)$ consisting of zero mean and periodic scalar functions. These are Banach spaces with norms inherited from L^p.
Let A denote the Laplacian operator $-\Delta$ with periodic boundary conditions. For every $b \in \mathbb{R}$, we define the powers of the operator A as follows:

$$\text{if } f = \sum_{k\in\mathbb{Z}^2_0} f_k e_k \qquad \text{then } A^b f = \sum_{k\in\mathbb{Z}^2_0} |k|^{2b} f_k e_k$$

and

$$D(A^b) = \{f = \sum_{k\in\mathbb{Z}^2_0} f_k e_k : \sum_{k\in\mathbb{Z}^2_0} |k|^{4b}|f_k|^2 < \infty\}.$$

For any $b \in \mathbb{R}_+$ and $p \geq 1$ we set

$$W^{b,p} = \{f \in \dot{L}^p_\sharp : A^{\frac{b}{2}} f \in \dot{L}^p_\sharp\}. \tag{I.0.1.4}$$

These are Banach spaces with the usual norm; when $p = 2$ they become Hilbert spaces and we denote them by W^b. For $b < 0$ we define W^b as the dual space of W^{-b} with respect to the L^2-scalar product.

Similarly, we proceed to define the space regularity of vector fields which are periodic, zero mean value and divergence free. We have the corresponding action of the Laplace operator on each component of the vector. Therefore we define the space

$$H = \{v \in [\dot{L}^2_\sharp]^2 : \nabla \cdot v = 0\}, \tag{I.0.1.5}$$

where the divergence free condition has to be understood in the distributional sense. This is an Hilbert space with the scalar product inherited from $\left[L^2\right]^2$. We denote the norm in this space by $|\cdot|_H$, $|u|^2_H := \langle u, u\rangle_H$. A basis for the space H is $\{\frac{k^\perp}{|k|} e_k\}_{k\in\mathbb{Z}^2_0}$, where $k^\perp = (-k_2, k_1)$ and e_k is given in (I.0.1.1). For $p > 2$ let us set $L_p := H \cap [L^p]^2$. These are Banach spaces with norms inherited from $[L^p]^2$. Similarly, for vector spaces we set

$$H^b_p = \{v \in L_p : A^{\frac{b}{2}} v \in L_p\}. \tag{I.0.1.6}$$

These are Banach spaces with the usual norm; when $p = 2$ they become Hilbert spaces and we denote them by H^b. For $b < 0$ we define H^b as the dual space of H^{-b} with respect to the H-scalar product.
The Poincaré inequality holds; moreover, the zero mean value assumption provides that $\|v\|_{H^b_p}$ is equivalent to $\left(\|v\|^p_{L_p} + \|v\|^p_{H^b_p}\right)^{\frac{1}{p}}$.

I.0.1.2 Some embedding theorems

In the sequel we shall use the following Sobolev inequalities (valid for the case $d = 2$).

Theorem I.0.1.1. *i. or every $2 < p < \infty$ the space H^1_p is compactly embedded in L_∞, namely there exists a constant C (depending on p such that):*

$$\|v\|_{L_\infty} \leq C\|v\|_{H^1_p}, \tag{I.0.1.7}$$

ii. the space W^a is compactly embedded in L^∞ for $a > 1$.

Proof. For statement (i) see [9, Theorem 9.16] and for statement (ii) see [9, Corollary 9.15]. □

Chapter I.1

Existence, uniqueness and regularity of the solution

I.1.1 Introduction

In the present Chapter we shall prove the existence and uniqueness of the solution to problem (I.0.0.2). We follow an approach similar to [42] for the one dimensional stochastic Burger equation and to [20] for the Cahn-Hilliard stochastic equation. The regularization property of the heat kernel as stated in Lemma I.1.5.1 plays a key role in our method. Since the non linear term that appears in (I.0.0.2) is non Lipschitz, we adopt a method of localization: by means of a contraction principle, we prove at first the result for the smoothed equation with truncated coefficient. This kind of result provides the uniqueness for the solution to (I.0.0.1) and its local existence, namely the existence on the time interval $[0, \tau]$ where τ is a stopping time. To prove the global existence we show that $\tau = T$ $\mathbb{P}$-a.s. We then study the regularity of ξ proving that if ξ_0 is a continuous function on D, then the solution admits a modification which is a space-time continuous process.

We make the following set of assumptions on the covariance function. $\sigma : \mathbb{R} \to \mathbb{R}$ is a Borel function such that:

(H1): σ satisfies a linear growth condition and it is globally Lipschitz: there exists a constant $L > 0$ such that
$$|\sigma(p) - \sigma(q)| \leq L|p - q|, \qquad \forall p, q \in \mathbb{R};$$

(H2): σ is bounded.

The main results we shall prove is the following.

Theorem I.1.1.1. *Let $b > 0$ in* (I.1.3.3) *and $p > 2$. Let us assume that Hypothesis (H1)-(H2) hold. If $\xi_0 \in L^p$, then there exists a unique solution to equation* (I.0.0.2) *which is continuous with values in L^p. Moreover, if $\xi_0 \in C(D)$ the solution admits a modification which is a space-time continuous process.*

The present Chapter is organized as follows. in Section I.1.2 we present some analytic preliminaries. In Section I.1.3 we deal with the random forcing term: we characterize the worthy martingale measure with respect to which the stochastic integral appearing in (I.0.0.2) is defined and we characterize the class of predictable processes. Moreover we briefly show how the stochastic integral (defined in Walsh sense) can be understood in the setting introduced by Da Prato and Zabczyk. Section I.1.4 concerns the well posedness and regularity of the stochastic convolution term appearing in (I.0.0.2). In Section I.1.5 we present some technical lemmas. In Section I.1.6 we establish the existence and uniqueness of the solution to (I.0.0.1) as well as its $\mathbb{P}$-a.s. space-time continuity.

I.1.2 Analytic preliminaries

In this Section we briefly present two different (equivalent) methods for writing a periodic function, we state the results concerning the needed estimates of the heat kernel and its gradient on the flat torus and we present the Biot-Savart law that exploit the relation between the velocity and the vorticity.

I.1.2.1 Fourier series, method of images and Poisson summation formula

Working in a periodic setting has some technical advantages. One is that we have (at least) two ways of representing an (appropriate) function f. We can write f by means of its Fourier series expansion (see (I.0.1.2) and (I.0.1.3)) or by means of the so called method of images. With this latter method, one starts with the expression of the function f in the whole space $\mathbb{R}^2$. Then, said $T > 0$ the period (in every direction) of the given function f, the construction is elementary: we simply write

$$\sum_{k\in\mathbb{Z}^2} f(x+Tk). \tag{I.1.2.1}$$

Since this (formal) sum is taken over the lattice points of $\mathbb{Z}^2$ it is clearly periodic (for the passage from x to $x+k'$ merely permutes the terms in (I.1.2.1)). We shall refer to the passage from f to the sum (I.1.2.1) as the periodization of f.
The two methods of writing a periodic function, the Fourier series expansion and the method of images, are equivalent. The Poisson summation formula states that the two approaches to a periodic analog of f are essentially identical. This conclusion can be formulated precisely in many ways. The most suitable for us is the following.

Theorem I.1.2.1. *Suppose $f \in L^1(\mathbb{R}^2)$. Let 2π be the period (in every direction) of the function f. Then the series $\sum_{k\in\mathbb{Z}^2} f(x+2\pi k)$ converges in the norm of $L^1(D)$. The resulting function has the Fourier expansion $\sum_{k\in\mathbb{Z}^2} f_k e_k(x)$.*

For more details on the results briefly recalled in this Section see for instance [35, Chapters 2.7§5 and 2.11§3], [83, Chapter 7§2] and [85]).

I.1.2.2 The heat kernel

We deal with the heat kernel g appearing in equation (I.0.0.2): we need suitable estimates on g since its regularizing effect (see Lemma I.1.5.1) will play a key role.

The operator $-A$ generates a semigroup $S(t)=e^{-tA}$: for $\xi \in \dot{L}^2_\sharp$ and $t\in[0,T]$ we have

$$[S(t)\xi](x) = \sum_{k\in\mathbb{Z}^2} e^{-|k|^2 t}\langle \xi, e_k\rangle_{L^2} e_k(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}^2}\langle \xi, e_k\rangle_{L^2} e^{-t|k|^2+ik\cdot x}. \tag{I.1.2.2}$$

Moreover, the action of the semigroup on the function ξ can be expressed as the convolution

$$[S(t)\xi](x) = \int_D g(t,x,y)\xi(y)\,\mathrm{d}y \tag{I.1.2.3}$$

where g is the fundamental solution (or heat kernel) to the problem

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) - \Delta u(t,x) = 0, & (t,x) \in (0,T] \times D \\ u(t,\cdot) \text{ is periodic}, & t \in [0,T] \\ u(0,x) = \delta_0(x-y), & x,y \in D. \end{cases} \tag{I.1.2.4}$$

As explained in Section I.1.2.1 we have two (equivalent) expressions for g. By means of Fourier series expansion we recover

$$g(t,x,y) = \frac{1}{(2\pi)^2} \sum_{k\in\mathbb{Z}^2} e^{-t|k|^2 + ik\cdot(x-y)}. \tag{I.1.2.5}$$

The expression of the kernel obtained by means of the method of images is more suitable for the kind of analysis we have to perform. The periodization of the expression for the heat kernel on $\mathbb{R}^2$ yields

$$g(t,x,y) = \frac{1}{4\pi t} \sum_{k\in\mathbb{Z}^2} e^{-\frac{|x-y+2k\pi|^2}{4t}}. \tag{I.1.2.6}$$

By the Poisson summation formula, (I.1.2.5) and (I.1.2.6) are equivalent. It is easy, using (I.1.2.5) or (I.1.2.6), to check the following properties

Proposition I.1.2.2. *For any $x, y \in D$ and $t > 0$ we have*

- *Symmetry: $g(t,x,y) = g(t,y,x)$,*
- *$g(t,x,y) = g(t,0,x-y)$.*

Remark I.1.2.3. *Since problem* (I.1.2.4) *can be reformulated as a problem on the flat torus as*

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) - \Delta u(t,x) = 0, & (t,x) \in (0,T] \times \mathbb{T}^2 \\ u(0,x) = \delta_0(x-y), & x \in \mathbb{T}^2 \end{cases} \tag{I.1.2.7}$$

the regularity results concerning the fundamental solutions follows from more general results valid for manifolds. For a detailed discussion see for instance [40, Theorem 7.13, Theorem 7.20, Corollary 8.12].

Following an idea of [66], we obtain estimates on the heat kernel and its gradient in the two dimensional case.

Theorem I.1.2.4. *For fixed $0 < s < t$ and $x \in D$ the following estimates hold:*

i. for every $0 < \beta < \frac{4}{3}$ there exists a constant $C_\beta > 0$ such that

$$\int_D |\nabla_y g(s,x,y)|^\beta \mathrm{d}y \le C_\beta s^{-\frac{3\beta}{2}+1} \tag{I.1.2.8}$$

and

$$\int_0^t \int_D |\nabla_y g(s,x,y)|^\beta \mathrm{d}y\mathrm{d}s \le C_\beta t^{-\frac{3\beta}{2}+2}, \tag{I.1.2.9}$$

ii. for every $0<\beta<2$ *there exists a constant* $C_\beta>0$ *such that*

$$\int_D |g(s,x,y)|^\beta \mathrm{d}y \le C_\beta s^{1-\beta} \tag{I.1.2.10}$$

and

$$\int_0^t\int_D |g(s,x,y)|^\beta \mathrm{d}y\mathrm{d}s \le C_\beta t^{2-\beta}. \tag{I.1.2.11}$$

Proof. For the estimate of the heat kernel and its gradient we use the explicit expression given by (I.1.2.6). We factorize the two-dimensional kernel into the one dimensional components. We then proceed following the idea of [66, Lemma 2.1].

$$g(t,x,y)=\frac{1}{4\pi t}\sum_{k\in\mathbb{Z}^2} e^{-\frac{|x-y+2\pi k|^2}{4t}} = g_1(t,x_1,y_1)g_2(t,x_2,y_2),$$

where we set, for $i=1,2$

$$g_i(t,x_i,y_i)=\frac{1}{\sqrt{4\pi t}}\sum_{k_i\in\mathbb{Z}} e^{\frac{-|x_i-y_i+2\pi k_i|^2}{4t}}.$$

For the one-dimensional heat kernel the following decomposition holds:

$$g_i(t,x_i,y_i)=H_i^1(t,x_i,y_i)+H_i^2(t,x_i,y_i)+H_i^3(t,x_i,y_i)+\bar{g}_i(t,x_i,y_i)$$

where

$$H_i^1(t,x_i,y_i)=\frac{1}{\sqrt{4\pi t}}e^{\frac{-|x_i-y_i|^2}{4t}},\quad H_i^2(t,x_i,y_i)=\frac{1}{\sqrt{4\pi t}}e^{\frac{-|x_i-y_i+2\pi|^2}{4t}},$$
$$H_i^3(t,x_i,y_i)=\frac{1}{\sqrt{4\pi t}}e^{\frac{-|x_i-y_i-2\pi|^2}{4t}}$$

and

$$(t,x_i,y_i)\to\bar{g}_i(t,x_i,y_i)\in C^\infty([0,T]\times\mathbb{R}^2). \tag{I.1.2.12}$$

Then we can rewrite the two dimensional heat kernel as follows

$$g(t,x,y)=\left(H_1^1(t,x_1,y_1)+H_1^2(t,x_1,y_1)+H_1^3(t,x_1,y_1)+\bar{g}_1(t,x_1,y_1)\right)\cdot$$
$$\left(H_2^1(t,x_2,y_2)+H_2^2(t,x_2,y_2)+H_2^3(t,x_2,y_2)+\bar{g}_2(t,x_2,y_2)\right).$$

We are interested in estimating the heat kernel and its gradient, more precisely in estimates of the following type:

$$\int_0^t\int_D |g(s,x,y)|^\beta \mathrm{d}y\mathrm{d}s,\qquad \int_0^t\int_D |\nabla_y g(s,x,y)|^\beta \mathrm{d}y\mathrm{d}s, \tag{I.1.2.13}$$

for $t>0$ and a suitable $\beta>0$.
Let us at first notice that the terms of the form $H_1^k\bar{g}_2$ and $H_2^k\bar{g}_1$ with $k=1,2,3$ do not give

any problems. In fact let us consider for example the case $H_1^1\bar{g}_2$ (the others are similar). We have

$$\begin{aligned}
&|\nabla_y(H_1^1\bar{g}_2)|^\beta = \left(|\nabla_y(H_1^1\bar{g}_2)|^2\right)^{\frac{\beta}{2}}\\
&\le C_\beta\left(\frac{(2|x_1-y_1|)^\beta}{\pi^{\frac{\beta}{2}}(4t)^{\frac{3\beta}{2}}}e^{-\frac{\beta|x_1-y_1|^2}{4t}}\,|\bar{g}_2(t,x_2,y_2)|^\beta+\frac{1}{(4\pi t)^{\frac{\beta}{2}}}e^{-\frac{\beta|x_1-y_1|^2}{4t}}\left|\frac{\partial}{\partial y_2}\bar{g}_2(t,x_2,y_2)\right|^\beta\right)\\
&\le C_\beta\frac{|x_1-y_1|^\beta}{t^{\frac{3\beta}{2}}}e^{-\frac{\beta|x_1-y_1|^2}{4t}}\,|\bar{g}_2(t,x_2,y_2)|^\beta+\frac{C_\beta}{t^{\frac{\beta}{2}}}e^{-\frac{\beta|x_1-y_1|^2}{4t}}\left|\frac{\partial}{\partial y_2}\bar{g}_2(t,x_2,y_2)\right|^\beta.
\end{aligned}$$

Then, using the following identity

$$\int_{\mathbb{R}}|z|^r e^{-\frac{z^2}{\sigma^2}}\,\mathrm{d}z = C_r\sigma^{r+1} \tag{I.1.2.14}$$

we get

$$\begin{aligned}
&\int_0^t\int_D|\nabla_y(H_1^1\bar{g}_2)(s,x,y)|^\beta\mathrm{d}y\,\mathrm{d}s\\
&\quad\le C_\beta\int_0^t\int_0^{2\pi}\left(\int_0^{2\pi}\frac{|x_1-y_1|^\beta}{s^{\frac{3\beta}{2}}}e^{-\frac{\beta|x_1-y_1|^2}{4s}}\,\mathrm{d}y_1\right)|\bar{g}_2(s,x_2,y_2)|^\beta\,\mathrm{d}y_2\mathrm{d}s\\
&\quad+C_\beta\int_0^t\frac{1}{s^{\frac{\beta}{2}}}\int_0^{2\pi}\left(\int_0^{2\pi}e^{-\frac{\beta|x_1-y_1|^2}{4s}}\,\mathrm{d}y_1\right)\left|\frac{\partial}{\partial y_2}\bar{g}_2(s,x_2,y_2)\right|^\beta\,\mathrm{d}y_2\,\mathrm{d}s\\
&\quad\le C_\beta\int_0^t\int_0^{2\pi}s^{\frac{1}{2}-\beta}\;|\bar{g}_2(s,x_2,y_2)|^\beta\,\mathrm{d}y_2\,\mathrm{d}s\\
&\qquad+C_\beta\int_0^t\int_0^{2\pi}s^{\frac{1-\beta}{2}}\left|\frac{\partial}{\partial y_2}\bar{g}_2(s,x_2,y_2)\right|^\beta\,\mathrm{d}y_2\mathrm{d}s
\end{aligned}$$

and we have the convergence of the integrals thanks to (B.2.2), when $\beta<\frac{3}{2}$.
Thus it follows that the behavior of integrals in (I.1.2.13) is determined by the corresponding integrals with $H_1^kH_2^l$ with $k,l=1,2,3$, instead of g. Since computations are similar we do all the required estimates only for the case $H(t,x,y):=H_1^1(t,x_1,y_1)H_2^1(t,x_2,y_2)$. We have

$$|\nabla_yH(t,x,y)|^\beta=\frac{e^{-\frac{\beta|x-y|^2}{4t}}|x-y|^\beta}{(8\pi)^\beta t^{2\beta}},$$

so we recover

$$\begin{aligned}
&\int_D|\nabla_yH(s,x,y)|^\beta\mathrm{d}y=\int_D\frac{e^{-\frac{\beta|x-y|^2}{4s}}|x-y|^\beta}{(8\pi)^\beta s^{2\beta}}\mathrm{d}y\\
&\le C_\beta\int_{\mathbb{R}^2}\frac{e^{-\frac{\beta|z|^2}{4s}}|z|^\beta}{s^{2\beta}}\mathrm{d}z=C_\beta\int_0^{2\pi}\int_0^\infty\frac{e^{-\frac{\beta\rho^2}{4s}}\rho^{\beta+1}}{s^{2\beta}}\,\mathrm{d}\rho\,\mathrm{d}\phi\\
&\le C_\beta\frac{1}{s^{2\beta}}\int_0^\infty\rho^{\beta+1}e^{-\frac{\beta\rho^2}{4s}}\,\mathrm{d}\rho.
\end{aligned}$$

Using now identity (I.1.2.14) we get

$$\int_D|\nabla_yH(s,x,y)|^\beta\mathrm{d}y\,\mathrm{d}s\le C_\beta s^{-\frac{3\beta}{2}+1}.$$

Calculating the time integral we obtain,

$$\int_0^t \int_D |\nabla_y H(s,x,y)|^\beta \mathrm{d}y\,\mathrm{d}s \le C_\beta \int_0^t s^{-\frac{3\beta}{2}+1}\mathrm{d}s \le C_\beta t^{-\frac{3\beta}{2}+2}, \tag{I.1.2.15}$$

which converges provided $\beta < \frac{4}{3}$.

Remark I.1.2.5. *Notice that estimate* (I.1.2.15) *is uniform in* x.

For estimates (I.1.2.10) and (I.1.2.11) we proceed in a similar way. Also in this case we do all the required estimates for $H(t,x,y) := H_1^1(t,x_1,y_1)H_2^1(t,x_2,y_2)$. By means of (I.1.2.14) we get

$$\begin{aligned}\int_D |H(s,x,y)|^\beta \,\mathrm{d}y &= \int_D \frac{1}{(4\pi s)^\beta} e^{-\frac{\beta|x-y|^2}{4s}}\,\mathrm{d}y \le C_\beta \int_{\mathbb{R}^2} \frac{1}{s^\beta} e^{-\frac{\beta|z|^2}{4s}}\,\mathrm{d}z \\ &= 2\pi C_\beta \int_0^\infty \frac{e^{-\frac{\beta\rho^2}{4s}}}{s^\beta}\rho\,\mathrm{d}\rho \le C_\beta s^{1-\beta}.\end{aligned}$$

Computing the time integral we obtain

$$\int_0^t \int_D |H(s,x,y)|^\beta \,\mathrm{d}y\,\mathrm{d}s \le C_\beta \int_0^t s^{1-\beta}\,\mathrm{d}s \le C_\beta t^{2-\beta},$$

which converges provided $\beta < 2$.

□

I.1.2.3 The Biot-Savart law

Now we deal with the Biot-Savart law expressing the velocity vector field v in terms of the vorticity scalar field ξ (we mainly refer to [55] and [56]). We can give an explicit representation of v in terms of ξ solving the system

$$\begin{cases} \nabla^\perp \cdot v = \xi \\ \nabla \cdot v = 0. \end{cases} \tag{I.1.2.16}$$

Since $\nabla \cdot v = 0$, there exists a (unique up to an additive constant) stream function Ψ such that

$$v = \nabla^\perp \Psi. \tag{I.1.2.17}$$

From the relation $\xi = \nabla^\perp \cdot v$ we get the Poisson equation for Ψ

$$-\Delta\Psi = \xi \tag{I.1.2.18}$$

on the flat torus $\mathbb{T}^2$. Given a periodic function ξ, equation (I.1.2.18) has a periodic solution Ψ; namely, if

$$\xi(t,x) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}^2} \xi_k e^{ix\cdot k}, \tag{I.1.2.19}$$

then

$$\Psi(t,x) = -\frac{i}{2\pi} \sum_{k\in\mathbb{Z}^2} \frac{\xi_k}{|k|^2} e^{ix\cdot k} \tag{I.1.2.20}$$

provided that compatibility condition $\int_D \xi(x)\,\mathrm{d}x = 0$ is satisfied (see for instance [55, Proposition 1.17]). In our case it is easy to check that this condition holds, since $\xi = \nabla^\perp \cdot v$ and ξ is periodic. From (I.1.2.17) and (I.1.2.20) we obtain a periodic version of the Biot-Savart law:

$$v(t,x) = -\frac{i}{2\pi} \sum_{k\in\mathbb{Z}_0^2} \xi_k \frac{k^\perp}{|k|^2} e^{ik\cdot x}. \tag{I.1.2.21}$$

Expressions (I.1.2.19) and (I.1.2.21) show that the velocity v has one order more of regularity with respect to the vorticity ξ: if $\xi \in W^{b-1,p}$ then $v \in H^b_p$. In particular, the norms $\|v\|_{H^b_p}$ and $\|\xi\|_{W^{b-1,p}}$ are equivalent.

In general (see, e.g., [56, Chapter 1]), the Biot-Savart law expresses the velocity in term of the vorticity as

$$v(x) = (k * \xi)(x) = \int_D k(x-y)\xi(y)\,\mathrm{d}y, \tag{I.1.2.22}$$

where the Biot-Savart kernel is given by

$$k = \nabla^\perp G = \left(-\frac{\partial G}{\partial x_2}, \frac{\partial G}{\partial x_1}\right) \tag{I.1.2.23}$$

and G is the fundamental solution (or Poisson kernel) of the Laplacian on the torus with mean zero. In fact the solution to (I.1.2.18) can be written as $\Psi(x) = \int_D G(x-y)\xi(y)\,\mathrm{d}y$ and thus, from (I.1.2.17) and (I.1.2.23) we get (I.1.2.22) (see [55, Lemma 1.12] for more details). Notice that from (I.1.2.22) it is evident that the relation between v and ξ is non local in space.
By the method of images we obtain an explicit expression for G. It is sufficient to take the periodization of the expression of the Poisson kernel on $\mathbb{R}^2$ to get

$$G(x) = -\frac{1}{2\pi} \sum_{k\in\mathbb{Z}_0^2} \ln|x + 2k\pi|. \tag{I.1.2.24}$$

For the function G the following regularity result holds (for more details see [56, Chapter 1] and [11, Proposition B.1]).

Proposition I.1.2.6. *The function G is in $C^\infty(\mathbb{T}^2 \setminus \{0\})$. Its behaviour in zero is given by*

$$|G(x)| \le C(-\log|x| + 1)$$

and that of its gradient by

$$|\nabla G(x)| \le C_1(|x|^{-1} + 1). \tag{I.1.2.25}$$

Proposition I.1.2.6 is a special case (at least in dimension two) of a general fact, valid for compact C^∞ Riemannian manifolds of finite dimension (see [2, Theorem 4.13]).
We summarize the basic properties of the Biot-Savart kernel in the following lemma (see [11, Lemma 2.17]), which is a consequence of Proposition I.1.2.6 and (I.1.2.23).

Lemma I.1.2.7. *For every $1 \le p < 2$ the map k, defined above, is an $[L^p(D)]^2$ divergence-free (in the distributional sense) vector field.*

Remark I.1.2.8. *In principle, for every $p < 2$, $\int_D k(x-y)\,\mathrm{d}y$ is a constant that depends on x, but it can be easily majored by a constant which does not depend on x. This is straightforward using the estimate $|\nabla G(x)| \le C(|x|^{-1} + 1)$ and recalling that* (I.1.2.23) *holds.*

In the last part of this Section we provide some useful estimates. From (I.1.2.21), using the Sobolev embedding $H^1_p \subset L_\infty$ for $p > 2$ (see Theorem I.0.1.1(i)) and the equivalence of the norms $\|v\|_{H^1_p}$ and $\|\xi\|_{L^p}$ we infer that for any $p > 2$ there exists a constant C_p such that

$$\|k * \xi\|_{L_\infty} = \|v\|_{L_\infty} \leq C_p \|\xi\|_{L^p}. \tag{I.1.2.26}$$

From (I.1.2.22) and Lemma I.1.2.7, using Young's inequality when $p \geq 1$, $1 \leq \alpha < 2$, $\beta \geq 1$ with $\frac{1}{p} + 1 = \frac{1}{\alpha} + \frac{1}{\beta}$ we infer that

$$\|k * \xi\|_{L_p} = \|v\|_{L_p} \leq \|k\|_{L_\alpha} \|\xi\|_{L^\beta}. \tag{I.1.2.27}$$

I.1.3 The random forcing term

In this Section we deal with the stochastic term that appears in (I.0.0.2). We introduce the noise as an isonormal Gaussian process on a proper Hilbert space. We show how to construct a worthy martingale measure and we characterize the class of predictable processes for which the stochastic integral in the Walsh sense is well defined. Then we show how the constructed integral can be understood in the Da Prato-Zabczyk framework (see Section A.3.2).

I.1.3.1 The isonormal Gaussian process on $\mathcal{H}_T$

Given $T > 0$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space. Let $Q : \dot{L}^2_\sharp \to \dot{L}^2_\sharp$ be a positive symmetric bounded linear operator. We define L^2_Q as the completion of the space of all square integrable, zero mean-value, periodic functions $\varphi : D \to \mathbb{R}$ with respect to the scalar product

$$\langle \varphi, \psi \rangle_{L^2_Q} = \langle Q\varphi, \psi \rangle_{L^2}.$$

Set $\mathcal{H}_T = L^2(0, T; L^2_Q)$. This space is a real separable Hilbert space with respect to the scalar product

$$\langle f, g \rangle_{\mathcal{H}_T} = \int_0^T \langle f(s), g(s) \rangle_{L^2_Q} \, \mathrm{d}s = \int_0^T \langle Qf(s), g(s) \rangle_{L^2} \, \mathrm{d}s. \tag{I.1.3.1}$$

Let us consider the isonormal Gaussian process $W = \{W(h), h \in \mathcal{H}_T\}$ (see Definition B.2.1). The map $h \to W(h)$ provides a linear isometry from $\mathcal{H}_T$ onto $\mathcal{O}$, which is a closed subset of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ whose elements are zero-mean Gaussian random variables. The isometry reads as

$$\mathbb{E}\left(W(h)\; W(g)\right) = \langle h, g \rangle_{\mathcal{H}_T}. \tag{I.1.3.2}$$

In particular, for $h \in \mathcal{H}_T$, $W(h)$ is a zero-mean Gaussian random variable with covariance $\mathbb{E}[W(h)^2] = \|h\|^2_{\mathcal{H}_T}$. Notice that, by construction, the random forcing term is periodic and with zero mean in the space variable. Since we are in a spatial domain of dimension larger than one, it is not surprising (see, e.g., [25]) that we cannot consider Q to be the indentity, but we need Q to have some regularizing effect. We choose to work with a covariance operator of the form

$$Q = (-\Delta)^{-b}, \tag{I.1.3.3}$$

for some $b > 0$. This means that

$$Qe_k = |k|^{-2b} e_k \qquad \forall k \in \mathbb{Z}^2_0$$

and a complete orthonormal basis of L^2_Q is given by $\tilde{e}_k(x) = \frac{1}{\sqrt{2\pi}}|k|^b \cos(k \cdot x)$ and $\tilde{e}_{-k}(x) = \frac{1}{\sqrt{2\pi}}|k|^b \sin(k \cdot x)$ for $k \in \mathbb{Z}^2_+$. Notice that the choice of Q as in (I.1.3.3) is made only in order to simplify some computations but it does not prevent to consider a more general operator Q which does not commute with the Laplacian operator or which has finite dimensional range. By TrQ we denote the trace of the operator Q (see (A.3.2)). If Q is as in (I.1.3.3) then $\mathrm{Tr}Q = \sum_{k\in\mathbb{Z}^2_0} |k|^{-2b}$.

Lemma I.1.3.1. *If $b > 0$ in* (I.1.3.3)*, then $g(t - \cdot, x, \cdot) \in \mathcal{H}_t$ for every $t > 0$.*

In particular, $W(g(t - \cdot, x, \cdot))$ is a well defined zero-mean Gaussian random variable with variance $\|g(t - \cdot, x, \cdot)\|^2_{\mathcal{H}_t}$.

Proof.

$$\begin{aligned}
\|g(t - \cdot, x, \cdot)\|^2_{\mathcal{H}_t} &= \int_0^t \|g(t - s, x, \cdot)\|^2_{L^2_Q}\, \mathrm{d}s = \int_0^t \langle Qg(t - s, x, \cdot), g(t - s, x, \cdot)\rangle_{L^2}\, \mathrm{d}s \\
&= \int_0^t \|Q^{\frac{1}{2}} g(t - s, x, \cdot)\|^2_{L^2}\, \mathrm{d}s = \int_0^t \sum_{k\in\mathbb{Z}^2_0} |\langle e_k, Q^{\frac{1}{2}} g(t - s, x, \cdot)\rangle_{L^2}|^2\, \mathrm{d}s \\
&= \sum_{k\in\mathbb{Z}^2_0} \int_0^t \left|\langle Q^{\frac{1}{2}} e_k, g(t - s, x, \cdot)\rangle_{L^2}\right|^2\, \mathrm{d}s \\
&= \sum_{k\in\mathbb{Z}^2_0} |k|^{-2b} \int_0^t |\langle e_k, g(t - s, x, \cdot)\rangle_{L^2}|^2\, \mathrm{d}s \qquad\qquad (\mathrm{I.1.3.4}) \\
&= \sum_{k\in\mathbb{Z}^2_0} |k|^{-2b} \int_0^t e^{-2|k|^2(t-s)} |e_k(x)|^2\, \mathrm{d}s \qquad \text{by (I.1.2.5)} \\
&= \frac{1}{(2\pi)^2} \sum_{k\in\mathbb{Z}^2_0} |k|^{-2b} \frac{(1 - e^{-2|k|^2 t})}{2|k|^2} \qquad \text{since } |e_k(x)| = \frac{1}{2\pi} \\
&\le \frac{1}{2(2\pi)^2} \sum_{k\in\mathbb{Z}^2_0} |k|^{-2-2b}.
\end{aligned}$$

The latter series is convergent if and only if $b > 0$. □

Remark I.1.3.2. *Notice that, since we work on the flat torus, we have good estimates on the norm of the normalized eigenfunctions e_k of the Laplacian. Thanks to this fact we have rather weak assumptions on the covariance operator of the noise, i.e. the exponent b in* (I.1.3.3)*. However, in a general domain of $\mathbb{R}^2$ with smooth boundary, the growth of normalized eigenfunctions is more difficult to control. Useful estimates for this case are provided for instance in [39].*

I.1.3.2 Constructing a worthy martingale measure

We work with the Gaussian process indexed by elements of $\mathcal{H}_T$ defined in Section I.1.3.1. In order to use the Walsh's theory of stochastic integration and SPDE's (see Section A.2), we need to construct from this process a worthy martingale measure. We explain the construction

in this Section.
For any $t \in [0,T]$, $A \in \mathcal{B}(D)$, let us set

$$w_t(A) := W(\mathbf{1}_{[0,t]}(\cdot)\mathbf{1}_A(\bullet)), \tag{I.1.3.5}$$

and let

$$\mathcal{F}_t = \sigma(w_s(A), s \leq t, A \in \mathcal{B}(D)) \vee \mathcal{N}, \tag{I.1.3.6}$$

where $\mathcal{B}(D)$ denotes the (bounded) Borel sets of D and $\mathcal{N}$ is the σ-field generated by the $\mathbb{P}$-null sets. It can be checked that

$$(w_t(A), \mathcal{F}_t, t \geq 0, A \in \mathcal{B}(D))$$

is a martingale measure according to [90], page 287. Its covariance functional

$$\bar{R}_t(A,B) = \langle w(A), w(B) \rangle_t$$

is deterministic and equal to $\mathbb{E}\left[w_t(A)w_t(B)\right]$ that is

$$\bar{R}_t(A,B) = t\langle \mathbf{1}_A, \mathbf{1}_B \rangle_{L^2_Q}.$$

The covariance measure coincides with the dominating measure and it is given by

$$R(A \times B \times (s,t]) = \bar{R}_t(A,B) - \bar{R}_s(A,B) = (t-s)\langle \mathbf{1}_A, \mathbf{1}_B \rangle_{L^2_Q}.$$

R is a positive definite measure on $D \times D \times [0,T]$ (see [90], page 290). Therefore the martingale measure w is worthy with dominating measure $K \equiv R$ (see [90], page 291).
The key relationship between W and w is that

$$W(\varphi) = \int_0^t \int_D \varphi(s,y)\, w(\mathrm{d}y, \mathrm{d}s),$$

where the stochastic integral on the right hand side is Walsh's martingale measure stochastic integral.

Remark I.1.3.3. *Notice that expression* (I.1.3.5) *makes sense under the assumption* $b > 0$ *in* (I.1.3.3). *In fact it is well defined if* $\mathbf{1}_{[0,t]}(\cdot)\mathbf{1}_A(\bullet) \in \mathcal{H}_T$*, namely if* $\mathbf{1}_A(\bullet) \in L^2_Q$*. By Parseval's Theorem we can write*

$$\|\mathbf{1}_A\|_{L^2_Q} = \sum_{k \in \mathbb{Z}^2_0} |k|^{-2b} |\langle e_k, \mathbf{1}_A \rangle_{L^2}|^2, \tag{I.1.3.7}$$

where $\langle e_k, \mathbf{1}_A \rangle_{L^2}$ *is the Fourier transform of* $\mathbf{1}_A$.
Let us suppose at first that A *is a rectangle of the form* $[a,b] \times [c,d]$ *for some* $0 < a < b$ *and* $0 < c < d$. *Then*

$$\begin{aligned}\langle e_k, \mathbf{1}_A \rangle_{L^2} &= \frac{1}{2\pi} \int_A e^{ik\cdot x}\, \mathrm{d}x \\ &= \frac{1}{2\pi} \Bigg[\left(\frac{e^{ik_1 b} - e^{ik_1 a}}{ik_1} \frac{e^{ik_2 d} - e^{ik_2 c}}{ik_2} \right) \mathbf{1}_{(k_1 \neq 0, k_2 \neq 0)} + \left((b-a) \frac{e^{ik_2 d} - e^{ik_2 c}}{ik_2} \right) \mathbf{1}_{(k_1 = 0)} \\ &\quad + \left((d-c) \frac{e^{ik_1 b} - e^{ik_1 a}}{ik_1} \right) \mathbf{1}_{(k_2 = 0)} \Bigg],\end{aligned}$$

thus

$$|\langle e_k, \mathbf{1}_A\rangle_{L^2}| \le \frac{1}{2\pi}\left[\frac{4}{|k_1||k_2|}\mathbf{1}_{(k_1\neq 0, k_2\neq 0)} + 2\frac{(b-a)}{k_2}\mathbf{1}_{(k_1=0)} + 2\frac{(d-c)}{k_1}\mathbf{1}_{(k_2=0)}\right].$$

Then from (I.1.3.7) *we obtain*

$$\|\mathbf{1}_A\|^2_{L^2_Q} \le C_A\left[\sum_{k\in\mathbb{Z}^2_0}\frac{|k|^{-2b}}{|k_1|^2|k_2|^2}\mathbf{1}_{(k_1\neq 0,k_2\neq 0)} + \sum_{k\in\mathbb{Z}^2_0}\frac{|k|^{-2b}}{|k_2|^2}\mathbf{1}_{(k_1=0)} + \sum_{k\in\mathbb{Z}^2_0}\frac{|k|^{-2b}}{|k_1|^2}\mathbf{1}_{(k_2=0)}\right].$$

Let us estimates separately the three terms, approximating the series by an integral and passing in polar coordinates. For the first term, fixed $0<\varepsilon<1$, *we get*

$$\begin{aligned}\sum_{k\in\mathbb{Z}^2_0}\frac{|k|^{-2b}}{|k_1|^2|k_2|^2}\mathbf{1}_{(k_1\neq 0,k_2\neq 0)} &\sim \iint_{\mathbb{R}^2\setminus([-\varepsilon,\varepsilon]\times\mathbb{R})\cup(\mathbb{R}\times[-\varepsilon,\varepsilon]))}\frac{(x^2+y^2)^{-b}}{|x|^2|y|^2}\,\mathrm{d}x\,\mathrm{d}y\\ &= \left(\int_{\varepsilon\sqrt{2}}^{\infty}\rho^{-2b-3}\,\mathrm{d}\rho\right)\sum_{i=0}^{3}\int_{\frac{\pi i}{2}+\varepsilon}^{\frac{\pi(i+1)}{2}-\varepsilon}\frac{\mathrm{d}\theta}{|\cos\theta|^2|\sin\theta|^2},\end{aligned}$$

which converges provided $b>-1$. *As regards the second term (for the third one estimates are the same)*

$$\sum_{k\in\mathbb{Z}^2_0}\frac{|k|^{-2b}}{|k_2|^2}\mathbf{1}_{(k_1=0)} \sim \int_{\varepsilon}^{\infty}\rho^{-2b-1}\,\mathrm{d}\rho,$$

which converges provided $b>0$.

Similar estimates hold true if we consider a finite union of rectangles. Finally, if A *is a generic set in* $\mathcal{B}(D)$, *then it is contained in a finite union of rectangles and thus we obtain the stated condition* $b>0$.

I.1.3.3 The class of predictable processes.

The stochastic term that appears in (I.0.0.2) is understood in the Walsh sense w.r.t the martingale measure defined in Section I.1.3.2. Let us examine the class of processes $\varphi=\varphi(t,x)$ for which the stochastic integral is defined. Recall that (see Definition A.2.7) an elementary process is defined as a process φ such that

$$\varphi(t,x;\omega) = \mathbf{1}_{(a,b]}(t)\mathbf{1}_A(x)X(\omega),$$

where $0<a<b$, $A\in\mathcal{B}(D)$ and X is a bounded and $\mathcal{F}_a$-measurable random variable. Let $\mathcal{R}$ denote the class of elementary processes such that

$$\mathbb{E}\|\varphi\|^2_{\mathcal{H}_T} = \mathbb{E}\int_0^T \|\varphi(s,\cdot)\|^2_{L^2_Q}\,\mathrm{d}s < \infty.$$

Let $\mathcal{K}$ be the set of all jointly measurable processes φ such that $\mathbb{E}\|\varphi\|^2_{\mathcal{H}_T}<\infty$. Note that $\mathcal{R}\subset\mathcal{K}$ and let $\mathcal{P}_M$ be the closure of $\mathcal{R}$ in $\mathcal{K}$. By predictable processes we mean elements in

$\mathcal{P}_M$. According to [90], the stochastic integral $\int_0^t \int_D \varphi(s,y)\, w(\mathrm{d}s, \mathrm{d}y)$ is defined for all $\varphi \in \mathcal{P}_M$. The Itô isometry holds and reads as

$$\mathbb{E}\left[\int_0^T \int_D \varphi(s,y)\, w(\mathrm{d}y, \mathrm{d}s)\right]^2 = \mathbb{E}\int_0^T \|\varphi(s,\cdot)\|^2_{L^2_Q}\, \mathrm{d}s. \tag{I.1.3.8}$$

Inspired by [28, Proposition 2] we give a checkable sufficient condition for a process to belong to $\mathcal{P}_M$.

Lemma I.1.3.4. *Suppose that a process $\varphi = \{\varphi(t,x), 0 \le t \le T, x \in D\}$ satisfies the following properties:*

(i) for all (t,x), $\varphi(t,x)$ is $\mathcal{F}_t$-measurable;

(ii) $(t,x;\omega) \to \varphi(t,x;\omega)$ is $\mathcal{B}([0,T]\times D)\times\mathcal{F}$-measurable;

(iii) for all (t,x), $\mathbb{E}\left[\varphi(t,x)^2\right] < \infty$ and the function from $[0,T]\times D$ into $L^2(\Omega,\mathcal{F},\mathbb{P})$ is continuous;

(iv) there exist $t_0 > 0$ such that

$$\mathbb{E}\int_0^{t_0} \|\varphi(s,\cdot)\|^2_{L^2_Q}\, \mathrm{d}s < \infty. \tag{I.1.3.9}$$

Then $\varphi \mathbf{1}_{[0,t_0]} \in \mathcal{P}_M$.

We recall here the proof for the sake of completeness, although it is almost the same as the proof of [28, Proposition 2] with some obvious modifications.

Proof. We aim at proving that every process satisfying the four assumptions of the lemma can be approximated by a sequence of elementary processes in the norm $L^2(\Omega;\mathcal{H}_T)$, .
Fix $\varepsilon > 0$. Then, by assumption (iii), there exists $n \in \mathbb{N}$ large enough such that, for all $s,t \in [0,t_0]$ and $x,y \in D$,

$$|t-s| + |x-y| \le \frac{2t_0}{n} \quad \Rightarrow \quad \|\varphi(t,x) - \varphi(s,y)\|_{L^2(\Omega)} < \varepsilon.$$

Set $t_j = jt_0/n$ and let $\{K_l\}$ be a finite family of disjoint subsets of D of diameter strictly less than t_0/n such that $\bigcup_l K_l = D$. Fix $x_l \in K_l$ and set

$$\varphi(t,x) = \sum_{j=0}^{n-1}\sum_l \varphi(t_j,x_l)\mathbf{1}_{(t_j,t_{j+1}]}(t)\mathbf{1}_{K_l}(x).$$

Then $\varphi_n \in \mathcal{P}_M$ and $\mathbb{E}\|\varphi_n - \varphi\mathbf{1}_{[0,t_0]}\|^2_{\mathcal{H}_T}$ is equal to

$$\begin{aligned}
\mathbb{E}\left[\int_0^{t_0} \|\varphi_n(t,\cdot) - \varphi(t,\cdot)\|^2_{L^2_Q}\, \mathrm{d}t\right] &= \mathbb{E}\left[\int_0^{t_0} \left\|\sum_{j=0}^{n-1}\sum_l \varphi(t_j,x_l)\mathbf{1}_{(t_j,t_{j+1}]}(t)\mathbf{1}_{K_l}(\cdot) - \varphi(t,\cdot)\right\|^2_{L^2_Q} \mathrm{d}t\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \sum_l \|\varphi(t_j,x_l) - \varphi(t,\cdot)\|^2_{L^2_Q(K_l)}\, \mathrm{d}t\right] \\
&= \sum_{j=0}^{n-1}\sum_l \int_{t_j}^{t_{j+1}} \mathbb{E}\,\|\varphi(t_j,x_l) - \varphi(t,\cdot)\|^2_{L^2_Q(K_l)}\, \mathrm{d}t.
\end{aligned}$$

Cauchy-Schwartz inequality implies

$$\begin{aligned}\|\varphi(t_j,x_l)-\varphi(t,\cdot)\|^2_{L^2_Q(K_l)} &= \sum_{k\in\mathbb{Z}^2_0}|k|^{-2b}|\langle e_k(\cdot),\varphi(t_j,x_l)-\varphi(t,\cdot)\rangle_{L^2(K_l)}|^2\\ &\le \sum_{k\in\mathbb{Z}^2_0}|k|^{-2b}\|e_k\|^2_{L^2}\|\varphi(t_j,x_l)-\varphi(t,\cdot)\|^2_{L^2(K_l)}\\ &= \sum_{k\in\mathbb{Z}^2_0}|k|^{-2b}\int_{K_l}|\varphi(t_j,x_l)-\varphi(t,x)|^2\,\mathrm{d}x\end{aligned}$$

and so

$$\begin{aligned}\mathbb{E}\left[\int_0^{t_0}\|\varphi_n(t,\cdot)-\varphi(t,\cdot)\|^2_{L^2_Q}\,\mathrm{d}t\right] &\le \left(\sum_{k\in\mathbb{Z}^2_0}|k|^{-2b}\right)\sum_{j=0}^{n-1}\sum_l\int_{t_j}^{t_{j+1}}\int_{K_l}\mathbb{E}|\varphi(t_j,x_l)-\varphi(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le \varepsilon^2\mathrm{Tr}Qt_0|D|.\end{aligned}$$

Therefore $\varphi\in\mathcal{P}_M$. □

Remark I.1.3.5. *Notice that* (I.1.3.9) *requires the assumption* $TrQ<\infty$ *that means* $b>1$ *if* Q *is as in* (I.1.3.3). *In what follows, when we will prove the existence of a space-time continuous global solution to equation* (I.0.0.1), *we shall work under the weaker assumption* $b>0$. *This is possible thanks to the smoothing action of the heat kernel; in fact we are concerned with a stochastic integral of the following form:* $\int_0^t\int_D g(t-s,x,y)\psi(s,y)\,w(\mathrm{d}y,\mathrm{d}s)$. *This means that, if we deal with a process* φ *of the form* $\varphi(s,y)=g(t-s,x,y)\psi(s,y)$, *then* $\varphi\in\mathcal{P}_M$ *provided* φ *satisfies requirements (i)-(iii) of Lemma I.1.3.4 and* $b>0$ *in* (I.1.3.3). *In fact we can consider a sequence of elementary processes of the form* $\varphi_n(s,y)=g(t-s,x,y)\psi_n(s,y)$. *Then* $\varphi(s,y)-\varphi_n(s,y)=g(t-s,x,y)\,[\psi(s,y)-\psi_n(s,y)]$. *Proceeding as in the proof of Lemma I.1.3.4, from Proposition I.1.4.1 (which holds under the weaker assumption* $b>0$*) we have*

$$\begin{aligned}\mathbb{E}\left[\int_0^{t_0}\|\varphi_n(t,\cdot)-\varphi(t,\cdot)\|^2_{L^2_Q}\,\mathrm{d}t\right] &\le \|g(t-\cdot,x,\cdot)\|^2_{\mathcal{H}_T}\sum_{j=0}^{n-1}\sum_l\left[\sup_{x\in K_l}\sup_{s_j\le s\le s_{j+1}}\mathbb{E}|\psi(s_j,x_l)-\psi(s,x)|^2\right]\\ &\le \varepsilon^2C_Tt_0|D|,\end{aligned}$$

and so, for a process of the form $g(t-s,x,y)\psi(s,y)$, *Lemma I.1.3.4 holds under the weaker assumption* $b>0$.

Notice that in this setting the Burkholder-Davis-Gundy's inequality reads as

$$\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_D\varphi(s,y)\,w(\mathrm{d}y,\mathrm{d}s)\right|^p\le c_p\left[\mathbb{E}\int_0^T\|\varphi(s,\cdot)\|^2_{L^2_Q}\,\mathrm{d}s\right]^{\frac{p}{2}},$$

for $p\ge2$ and a suitable constant c_p.

I.1.3.4 Interpreting the stochastic integral in the Da Prato-Zabczyk setting

The stochastic integral, formally defined above in the Walsh sense, can be understood in the setting introduced by Da Prato and Zabczyk (see Section A.3.2). We briefly explain how. We construct a cylindrical Wiener process and we relate the stochastic integral w.r.t such a process (see Section A.3.3) with the Walsh martingale measure stochastic integral (see Section A.2). In other words, the integral in the Walsh sense may be understood as a stochastic integral with respect to a cylindrical Wiener process. We make this precise in Proposition I.1.3.7. In particular, bearing in mind Proposition A.3.11 (and in general results of Section A.3.4) this is sufficient to have the equivalence between the Walsh stochastic integral and the Hilbert-valued stochastic integral. In the present Section we proceed following the approach of [30], where authors consider the case of a spatially homogeneous noise on $\mathbb{R}^2$ white in time with a spatial correlation in space. In a sense, the results of the present Section are the "discrete counterpart" of results of [30]: do not forget that the noise on the flat torus, how introduced in this Part, is spatially homogeneous (see Definition II.1.7.2).

Starting from the isonormal Gaussian process of Section I.1.3.1 we can construct a cylindrical Wiener process in the sense of Definition A.3.8. For every $h \in L^2_Q$ and for every $t \in [0,T]$ set

$$Y_t(h) := W(\mathbf{1}_{[0,t]}h), \tag{I.1.3.10}$$

It is easy to prove the following result.

Proposition I.1.3.6. *The process $Y = \{Y_t(h), t \geq 0, h \in L^2_Q\}$ is a standard (i.e. with covariance operator the identity $Id_{L^2_Q}$) cylindrical Wiener process on L^2_Q.*

We consider the filtration $\mathcal{F}_t = \sigma(W_s(h), s \leq t, h \in L^2_Q) \vee \mathcal{N}$. The class of all predictable processes is given by $L^2(\Omega \times [0,T]; L^2_Q) \equiv L^2(\Omega; \mathcal{H}_T)$. For this class of processes it is well defined the integral w.r.t. the cylindrical Wiener process given by (I.1.3.10). Recalling that $\{\tilde{e}_k\}_k$ is a complete orthonormal basis of L^2_Q, we notice that $\{Y_t(\tilde{e}_k)\}_t = \{W(\mathbf{1}_{[0,t]}\tilde{e}_k)\}_t$ defines a sequence of standard one-dimensional Brownian motions. For every $h \in L^2(\Omega; \mathcal{H}_T)$ we set

$$h \cdot Y := \sum_{k \in \mathbb{Z}_0} \int_0^t \langle h(s,\cdot), \tilde{e}_k(\cdot) \rangle_{L^2_Q} \, \mathrm{d}Y_s(\tilde{e}_k), \tag{I.1.3.11}$$

and the integral is well defined in the sense of Section A.3.3. The isometry property holds and reads as

$$\mathbb{E}\left[(h \cdot Y)^2\right] = \mathbb{E}\left[\int_0^T \|h(s,\cdot)\|^2_{L^2_Q} \, \mathrm{d}s\right]. \tag{I.1.3.12}$$

The following result relates (I.1.3.11) to the Walsh integral w.r.t. the martingale measure w.

Proposition I.1.3.7. *If $g \in L^2(\Omega; \mathcal{H}_T)$, then $g \cdot w = g \cdot Y$, where the l.h.s is a Walsh integral and the r.h.s. is the integral w.r.t. a cylindrical Wiener process as defined in* (I.1.3.11).

Proof. In order the check the equality of integrals we use the fact that the set of elementary processes is dense in $(\mathcal{P}_M, \|\cdot\|_{\mathcal{H}_T})$. Let us consider the elementary process $g(t,x;\omega) = \mathbf{1}_{[a,b]}(t)\mathbf{1}_A(x)X(\omega)$, with $0 \leq a < b \leq T$ and $A \in \mathcal{B}_b(D)$, X is a bounded $\mathcal{F}_a$-measurable r.v.. The Walsh integral of g, w.r.t. the martingale measure (I.1.3.5), is defined as

$$\begin{aligned}\int_0^T \int_D g(t,x)\, w(\mathrm{d}t, \mathrm{d}x) &= X\left(w_b(A) - w_a(A)\right) \\ &=: X\left(W(\mathbf{1}_{(0,b]}(\cdot)\mathbf{1}_A(\bullet)) - W(\mathbf{1}_{(0,a]}(\cdot)\mathbf{1}_A(\bullet))\right) = X\, W(\mathbf{1}_{[a,b]}(\cdot)\mathbf{1}_A(\bullet))\end{aligned}$$

As regards the integral with respect to the cylindrical Wiener process Y given in Proposition I.1.3.6, we have

$$
\begin{aligned}
g \cdot Y &= \sum_{k\in\mathbb{Z}_0^2} \int_0^T \langle g(t,\cdot), \tilde{e}_k\rangle_{L^2_Q} \,\mathrm{d}Y_t(\tilde{e}_k) = \sum_{k\in\mathbb{Z}_0^2} \int_0^T X\mathbf{1}_{[a,b]}(\cdot)\langle \mathbf{1}_A(\cdot), \tilde{e}_k\rangle_{L^2_Q} \,\mathrm{d}Y_t(\tilde{e}_k)\\
&= \sum_{k\in\mathbb{Z}_0^2} \int_a^b X\langle \mathbf{1}_A(\cdot), \tilde{e}_k\rangle_{L^2_Q} \,\mathrm{d}Y_t(\tilde{e}_k) = X \sum_{k\in\mathbb{Z}_0^2} \langle \mathbf{1}_A(\cdot), \tilde{e}_k\rangle_{L^2_Q} (Y_b(\tilde{e}_k) - Y_a(\tilde{e}_k))\\
&= X \sum_{k\in\mathbb{Z}_0^2} \langle \mathbf{1}_A(\cdot), \tilde{e}_k\rangle_{L^2_Q} (W(\mathbf{1}_{[0,b]}\tilde{e}_k) - W(\mathbf{1}_{[0,a]}\tilde{e}_k)) = X \sum_{k\in\mathbb{Z}_0^2} \langle \mathbf{1}_A(\cdot), \tilde{e}_k\rangle_{L^2_Q} W(\mathbf{1}_{[a,b]}\tilde{e}_k)\\
&\qquad XW\left(\mathbf{1}_{[a,b]} \sum_{k\in\mathbb{Z}_0^2} \langle \mathbf{1}_A(\cdot), \tilde{e}_k\rangle_{L^2_Q} \tilde{e}_k\right) = X\ W(\mathbf{1}_{[a,b]}(\cdot)\mathbf{1}_A(\bullet)).
\end{aligned}
$$

The thesis follows then by a density argument. Formally, let $g \in \mathcal{H}_T$, then there exists a sequence $\{g_n\}_n \in \mathcal{R}$ such that $\|g - g_n\|_{\mathcal{H}_T} \to 0$. We have proved that $g_n \cdot w = g_n \cdot Y$ for every n and so, by means of the isometry property (see (I.1.3.8) and (I.1.3.12)), we get

$$
\begin{aligned}
\mathbb{E}\left[|g\cdot w - g\cdot Y|^2\right] &\le 2\mathbb{E}\left[|g\cdot w - g_n\cdot Y|^2\right] + 2\mathbb{E}\left[|g_n\cdot Y - g\cdot Y|^2\right]\\
&\le 2\mathbb{E}\left[|g\cdot w - g_n\cdot w|^2\right] + 2\mathbb{E}\left[|g_n\cdot Y - g\cdot Y|^2\right]\\
&\le 4\|g - g_n\|_{L^2(\Omega;\mathcal{H}_T)} \to 0,
\end{aligned}
$$

from which it follows $g \cdot w = g \cdot Y$ for every $g \in L^2(\Omega; \mathcal{H}_T)$.

□

According to Proposition I.1.3.7, when one integrates a predictable process in $L^2(\Omega; \mathcal{H}_T)$, it is possible to use either the Walsh integral or the integral w.r.t. a cylindrical Wiener process. Moreover, Proposition A.3.11 allows us to associate the noise of Section I.1.3.1, viewed as a cylindrical Wiener process Y with covariance $Id_{L^2_Q}$ in Proposition I.1.3.6, with a cylindrical $Id_{L^2_Q}$-Wiener process on the Hilbert space L^2_Q, as described in Section A.3.2, and to relate the associated stochastic integrals.

I.1.4 Well posedness and regularity of the stochastic convolution term

The present Section concerns the the study of the well posedness and regularity of the stochastic convolution term. The following result gives the (weakest) conditions on the operator Q and the integrability conditions we have to require on the integrand process in order to have the well posedness of the stochastic integral. The conditions we recover are essentially the same obtained by Dalang in [27, Theorem 5] (see also [30], [58] and [73]). The difference is given by the fact that Dalang works on the whole space and imposes the needed conditions on the Fourier transforms of the fundamental solution to the considered equation. We work instead on the flat torus and the needed conditions are imposed on the Fourier series expansion of the heat kernel.

The following result provides an estimate, uniform in time and space, for a generic $p \geq 2$. Notice that for the well posedness of the stochastic integral it is sufficient to consider the case $p = 2$ with fixed $t \in [0, T]$, $x \in D$.

Proposition I.1.4.1. *Let $\{\varphi(t, x), 0 \leq t \leq T, x \in D\}$ be a predictable process such that, for some $p \geq 2$*

$$\sup_{0 \leq t \leq T} \sup_{x \in D} \mathbb{E}|\varphi(t, x)|^p < \infty. \tag{I.1.4.1}$$

Let us assume $b > 0$ in (I.1.3.3), *then*

$$\mathbb{E}\left|\int_0^t \int_D g(t-s, x, y)\varphi(s, y)\, w(\mathrm{d}y, \mathrm{d}s)\right|^p \leq \|g(t-\cdot, x, \cdot)\|_{\mathcal{H}_t}^{p-2} \int_0^t \sup_{y \in D} \mathbb{E}|\varphi(s, y)|^p \, \|g(t-s, x, \cdot)\|_{L^2_Q}^2 \,\mathrm{d}s. \tag{I.1.4.2}$$

Remark I.1.4.2. *Notice that, in general, we have to make the assumption $g(t - \cdot, x, \cdot) \in \mathcal{H}_t$. If we work with a generic operator Q, which diagonalizes on the basis of eigenfunction of the operator $-\Delta$ on $\dot{L}^2_\sharp$ with corresponding eigenvalues μ_k, this reads as $\sum_{k \in \mathbb{Z}^2_0} \frac{\mu_k}{|k|^2} < \infty$. For Q as in* (I.1.3.3), *by* (I.1.3.4) *this corresponds to assume $b > 0$.*

Proof. By means of Burkholder-Davis-Gundy's inequality we get

$$\begin{aligned}
\mathbb{E}\left|\int_0^t \int_D g(t-s, x, y)\varphi(s, y)\, w(\mathrm{d}y, \mathrm{d}s)\right|^p &\leq \mathbb{E}\left[\int_0^t \|g(t-s, x, \cdot)\varphi(s, \cdot)\|_{L^2_Q}^2 \,\mathrm{d}s\right]^{\frac{p}{2}} \\
&= \mathbb{E}\left[\int_0^t \sum_{k \in \mathbb{Z}^2_0} |\langle |k|^{-b} e_k, g(t-s, x, \cdot)\varphi(s, \cdot)\rangle_{L^2}|^2 \,\mathrm{d}s\right]^{\frac{p}{2}}.
\end{aligned}$$

Using Hölder's inequality we infer

$$\begin{aligned}
&|\langle |k|^{-b} e_k, g(t-s, x, \cdot)\varphi(s, \cdot)\rangle_{L^2}|^2 \\
&\qquad \leq |\langle |k|^{-b} e_k, g(t-s, x, \cdot)\rangle_{L^2}|^{\frac{2(p-2)}{p}} |\langle |k|^{-b} e_k, g(t-s, x, \cdot)|\varphi(s, \cdot)|^{\frac{p}{2}}\rangle_{L^2}|^{\frac{4}{p}}
\end{aligned}$$

so that, again by means of Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t \int_D g(t-s, x, y)\varphi(s, y)\, w(\mathrm{d}y, \mathrm{d}s)\right|^p \\
&\leq \mathbb{E}\left[\int_0^t \sum_{k \in \mathbb{Z}^2_0} |\langle |k|^{-b} e_k, g(t-s, x, \cdot)\rangle_{L^2}|^{\frac{2(p-2)}{p}} |\langle |k|^{-b} e_k, g(t-s, x, \cdot)|\varphi(s, \cdot)|^{\frac{p}{2}}\rangle_{L^2}|^{\frac{4}{p}} \,\mathrm{d}s\right]^{\frac{p}{2}} \\
&\leq \left(\int_0^t \sum_{k \in \mathbb{Z}^2_0} |\langle |k|^{-b} e_k, g(t-s, x, \cdot)\rangle_{L^2}|^2 \,\mathrm{d}s\right)^{\frac{p}{2}-1} \\
&\qquad \times \mathbb{E}\left[\int_0^t \sum_{k \in \mathbb{Z}^2_0} |\langle |k|^{-b} e_k, g(t-s, x, \cdot)|\varphi(s, \cdot)|^{\frac{p}{2}}\rangle_{L^2}|^2 \,\mathrm{d}s\right].
\end{aligned}$$

The first factor is equal to $\|g(t-\cdot,x,\cdot)\|_{\mathcal{H}_t}^{p-2}$, as regards the second one, by means of Hölder's inequality we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \sum_{k\in\mathbb{Z}_0^2} |\langle |k|^{-b}e_k, g(t-s,x,\cdot)|\varphi(s,\cdot)|^{\frac{p}{2}}\rangle_{L^2}|^2 \,\mathrm{d}s\right]\\
&\le \mathbb{E}\left[\int_0^t \sum_{k\in\mathbb{Z}_0^2} \langle |k|^{-b}e_k, g(t-s,x,\cdot)\rangle_{L^2}\langle |k|^{-b}e_k, g(t-s,x,\cdot)|\varphi(s,\cdot)|^{p}\rangle_{L^2} \,\mathrm{d}s\right]\\
&\le \int_0^t \sup_{y\in D}\mathbb{E}|\varphi(s,y)|^p \sum_{k\in\mathbb{Z}_0^2} |\langle |k|^{-b}e_k, g(t-s,x,\cdot)\rangle_{L^2}|^2 \,\mathrm{d}s\\
&= \int_0^t \sup_{y\in D}\mathbb{E}|\varphi(s,y)|^p \|g(t-s,x,\cdot)\|^2_{L^2_Q} \,\mathrm{d}s,
\end{aligned}$$

from which we get the thesis. □

Remark I.1.4.3. *Notice that, by applying Hölder's inequality in* (I.1.4.2) *we immediately obtain*

$$\mathbb{E}\left|\int_0^t\int_D g(t-s,x,y)\varphi(s,y)\,w(\mathrm{d}y,\mathrm{d}s)\right|^p \le \|g(t-\cdot,x,\cdot)\|^p_{\mathcal{H}_t}\sup_{0\le s\le T}\sup_{y\in D}\mathbb{E}|\varphi(s,y)|^p.$$

We now establish some estimates concerning the regularity of the stochastic convolution term. Such estimates are needed in the following sections where we prove the existence of the solution to (I.0.0.1) and its space-time continuity.

Lemma I.1.4.4. *Let $\varphi = \{\varphi(t,x), 0\le t\le T, x\in D\}$ be a predictable process such that*

$$C_p(\varphi) := \mathbb{E}\sup_{0\le t\le T}\|\varphi(t,\cdot)\|^p_{L^p} < \infty,$$

for some $p>4$. If $b>0$ in (I.1.3.3) *then*

$$\mathbb{E}\sup_{0\le t\le T}\sup_{x\in D}\left|\int_0^t\int_D g(t-s,x,y)\varphi(s,y)\,w(\mathrm{d}y,\mathrm{d}s)\right|^p \le C_{p,\alpha,Q}T^{\alpha p-1}\; C_p(\varphi), \tag{I.1.4.3}$$

where $\alpha\in(0,\frac{1}{2})$.

Proof. The proof of the result is based on the factorization method introduced in [24]. At a certain point we shall use the smoothness property of the heat kernel.

For $\varphi\in L^p(\Omega; L^\infty(0,T;L^p))$, $0\le t\le T$, $x\in D$ we write

$$(I\varphi)(t,x) := \int_0^t\int_D g(t-s,x,y)\varphi(s,y)\,w(\mathrm{d}y,\mathrm{d}s). \tag{I.1.4.4}$$

Given $\alpha\in\left(0,\frac{1}{2}\right)$ we write

$$(J_\alpha\varphi)(\sigma,z) := \int_0^\sigma\int_D(\sigma-s)^{-\alpha}g(\sigma-s,z,y)\varphi(s,y)\,w(\mathrm{d}y,\mathrm{d}s)$$

and

$$(J^{\alpha-1}Z)(t,x) = \frac{\sin(\pi\alpha)}{\pi}\int_0^t (t-\sigma)^{\alpha-1}\left(\int_D g(t-\sigma,x,z)Z(\sigma,z)\,\mathrm{d}z\right)\mathrm{d}\sigma.$$

The factorization identity reads as

$$I \equiv J^{\alpha-1}J_\alpha. \tag{I.1.4.5}$$

For simplicity we divide the proof in three steps.

Step 1. We prove at first that $J_\alpha(\varphi) \in L^p(\Omega\times[0,T]\,;L^p)$ for $\varphi \in L^p(\Omega;L^\infty(0,T;L^p))$.

From Burkholder-Davis-Gundy's inequality and Minkowki's inequality (see e.g. [84, Theorem 6.2.14]) we get

$$\begin{aligned}
\mathbb{E}\|(J_\alpha\varphi)(\sigma,\cdot)\|_{L^p}^p &= \int_D \mathbb{E}\left|\int_0^\sigma\int_D(\sigma-s)^{-\alpha}g(\sigma-s,z,y)\varphi(s,y)\,w(\mathrm{d}y,\mathrm{d}s)\right|^p \mathrm{d}z\\
&\le \int_D \mathbb{E}\left|\int_0^\sigma \left\|(\sigma-s)^{-\alpha}g(\sigma-s,z,\cdot)\varphi(s,\cdot)\right\|_{L^2_Q}^2 \mathrm{d}s\right|^{\frac{p}{2}}\mathrm{d}z\\
&= \mathbb{E}\left\|\int_0^\sigma(\sigma-s)^{-2\alpha}\left[|Q^{\frac12}g(\sigma-s,0,\cdot)|^2\star|\varphi(s,\cdot)|^2\right]\mathrm{d}s\right\|_{L^{\frac{p}{2}}}^{\frac{p}{2}}\\
&\le \mathbb{E}\left[\int_0^\sigma(\sigma-s)^{-2\alpha}\left\|\left[Q^{\frac12}g(\sigma-s,0,\cdot)\right]^2\star[\varphi(s,\cdot)]^2\right\|_{L^{\frac{p}{2}}}\mathrm{d}s\right]^{\frac{p}{2}}
\end{aligned}$$

By means of Young's inequality the inner norm can be estimated as

$$\begin{aligned}
\left\|\left[Q^{\frac12}g(\sigma-s,0,\cdot)\right]^2\star[\varphi(s,\cdot)]^2\right\|_{L^{\frac{p}{2}}} &\le \left\|\left[Q^{\frac12}g(\sigma-s,0,\cdot)\right]^2\right\|_{L^1}\left\|[\varphi(s,\cdot)]^2\right\|_{L^{\frac{p}{2}}}\\
&= \|g(\sigma-s,0,\cdot)\|_{L^2_Q}^2\|\varphi(s,\cdot)\|_{L^p}^2,
\end{aligned}$$

so, applying Hölder's inequality, we get

$$\begin{aligned}
\mathbb{E}\|(J_\alpha\varphi)(\sigma,\cdot)\|_{L^p}^p &\le \mathbb{E}\left[\int_0^\sigma(\sigma-s)^{-2\alpha}\|g(\sigma-s,0,\cdot)\|_{L^2_Q}^2\|\varphi(s,\cdot)\|_{L^p}^2\,\mathrm{d}s\right]^{\frac{p}{2}}\\
&\le \left(\int_0^\sigma(\sigma-s)^{-2\alpha}\|g(\sigma-s,0,\cdot)\|_{L^2_Q}^2\,\mathrm{d}s\right)^{\frac{p}{2}-1}\\
&\qquad\times\mathbb{E}\left[\int_0^\sigma(\sigma-s)^{-2\alpha}\|g(\sigma-s,0,\cdot)\|_{L^2_Q}^2\|\varphi(s,\cdot)\|_{L^p}^p\,\mathrm{d}s\right]\\
&\le \left(\mathbb{E}\sup_{0\le s\le\sigma}\|\varphi(s,\cdot)\|_{L^p}^p\right)\left(\int_0^\sigma(\sigma-s)^{-2\alpha}\|g(\sigma-s,0,\cdot)\|_{L^2_Q}^2\,\mathrm{d}s\right)^{\frac{p}{2}}.
\end{aligned}$$

We can estimate the second factor in the following way

$$\begin{aligned}
\int_0^\sigma(\sigma-s)^{-2\alpha}\|g(\sigma-s,0,\cdot)\|_{L^2_Q}^2\,\mathrm{d}s &= \frac{1}{(2\pi)^2}\int_0^\sigma\sum_{k\in\mathbb{Z}^2_0}|k|^{-2b}e^{-2|k|^2(\sigma-s)}(\sigma-s)^{-2\alpha}\,\mathrm{d}s\\
&= \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}^2_0}|k|^{-2b}\frac{1}{(2|k|^2)^{1-2\alpha}}\int_0^{2|k|^2\sigma}e^{-x}x^{-2\alpha}\,\mathrm{d}x\\
&\le C\,\Gamma(1-2\alpha)\sum_{k\in\mathbb{Z}^2_0}|k|^{-2b+4\alpha-2},
\end{aligned}$$

where the Gamma function and the series converge provided respectively $\alpha < \frac{1}{2}$ and $b > 2\alpha$. Thus,

$$\mathbb{E}\|(J_\alpha\varphi)(\sigma,\cdot)\|_{L^p}^p \leq C_{p,\alpha,b}\,\mathbb{E}\sup_{0\leq s\leq T}\|\varphi(s,\cdot)\|_{L^p}^p, \tag{I.1.4.6}$$

and then

$$\int_0^T \mathbb{E}\|(J_\alpha\varphi)(s,\cdot)\|_{L^p}^p\,\mathrm{d}s \leq C_{p,\alpha,b}\int_0^T \mathbb{E}\sup_{0\leq s\leq T}\|\varphi(s,\cdot)\|_{L^p}^p\,\mathrm{d}s \leq C_{p,\alpha,b}C_p(\varphi)T. \tag{I.1.4.7}$$

Step 2: $J^{\alpha-1} \in \mathcal{L}(L^p(\Omega\times[0,T]\,;L^p), L^p(\Omega;L^\infty([0,T]\times D)))$.

By means of Hölder's inequality and (I.1.2.10) we infer

$$\begin{aligned}
|(J^{\alpha-1}Z)(t,x)| &\leq \frac{\sin(\pi\alpha)}{\pi}\int_0^t (t-\sigma)^{\alpha-1}\|g(t-\sigma,\cdot,z)\|_{L^{\frac{p}{p-1}}}\|Z(\sigma,\cdot)\|_{L^p}\,\mathrm{d}\sigma\\
&\leq \frac{\sin(\pi\alpha)}{\pi}\int_0^t (t-\sigma)^{\alpha-1-\frac{1}{p}}\|Z(\sigma,\cdot)\|_{L^p}\,\mathrm{d}\sigma \qquad \text{by (I.1.2.10)}\\
&\leq C_{\alpha,p}T^{\alpha-\frac{2}{p}}\left(\int_0^T\|Z(\sigma,\cdot)\|_{L^p}^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}} \qquad\qquad (\text{I.1.4.8})
\end{aligned}$$

provided $p > \frac{2}{\alpha}$. Since the obtained estimate is uniform in $t\in[0,T]$, $x\in D$, we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\sup_{x\in D}|(J^{\alpha-1}Z)(t,x)|^p\right] \leq C_{p,\alpha}T^{\alpha p-2}\int_0^T\mathbb{E}\|Z(\sigma,\cdot)\|_{L^p}^p\,\mathrm{d}\sigma. \tag{I.1.4.9}$$

Step 3: From (I.1.4.5), (I.1.4.6) and (I.1.4.9) we finally get

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T}\sup_{x\in D}|(I\varphi)(t,x)|^p\right] &= \mathbb{E}\left[\sup_{0\leq t\leq T}\sup_{x\in D}|(J^{\alpha-1}(J_\alpha\varphi))(t,x)|^p\right]\\
&\leq C_{p,\alpha}T^{\alpha p-2}\int_0^T\mathbb{E}\|(J_\alpha\varphi)(\sigma,\cdot)\|_{L^p}^p\,\mathrm{d}\sigma\\
&\leq C_{p,\alpha,b}C_p(\varphi)T^{\alpha p-1} \qquad \text{by (I.1.4.7).}
\end{aligned}$$

□

Corollary I.1.4.5. *Under the same assumptions of Lemma I.1.4.4, the stochastic convolution term $J\varphi$ admits a space-time continuous modification.*

Proof. In the proof of Lemma I.1.4.4 (see (I.1.4.8)) we have shown, in particular, that $J^{\alpha-1}\in\mathcal{L}(L^p(0,T;L^p),L^\infty([0,T]\times D))$ $\mathbb{P}$-a.s.. If the integrand process φ belongs to the space $L^p(\Omega;L^\infty(0,T;L^p))$ with $p>4$, then $J_\alpha\varphi\in L^p(\Omega\times[0,T]\,;L^p)$ (see Step 1 in the proof of Lemma I.1.4.4). Then, by means of the factorization identity (I.1.4.5), $I\varphi$ admits a modification with a L^∞-type space-time regularity. In order to prove the existence of a space-time continuous modification of the convolution term is then sufficient to prove that actually $J^{\alpha-1}$ maps $L^p(0,T;L^p)$ into $C([0,T]\times D)$. For step functions ψ, $J^{\alpha-1}\psi$ is a space-time continuous function. This follows by the fact that the map $(t,x) \mapsto \int_0^t \int_D (t-s)^{\alpha-1} g(t-s,x,y)\,\mathrm{d}y\,\mathrm{d}s$ is continuous and the integral $\int_0^t \int_D |(t-s)^{\alpha-1} g(t-s,x,y)|\,\mathrm{d}y\,\mathrm{d}s$ is well posed. In fact, by (I.1.2.10)

$$\int_0^t \int_D |(t-s)^{\alpha-1} g(t-s,x,y)|\,\mathrm{d}y\,\mathrm{d}s \le C \int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s = \frac{C}{\alpha} t^{\alpha},$$

provided $\alpha > 0$. This kind of regularity can be extended to every $\psi \in L^p(0,T;L^p)$ by a standard approximation procedure. □

Corollary I.1.4.6. *Under the same assumptions of Lemma I.1.4.4 it holds*

$$\mathbb{E} \sup_{0\le t\le T} \left\| \int_0^t \int_D g(t-s,\cdot,y)\varphi(s,y)\, w(\mathrm{d}y,\mathrm{d}s) \right\|_{L^p}^p \le C_{p,\alpha,Q} T^{\alpha p-1}\; C_p(\varphi). \tag{I.1.4.10}$$

Proof. It follows immediately by the embedding $L^\infty(D) \subset L^p(D)$, which holds for every $p \ge 1$. □

I.1.5 Some preliminaries Lemmas

In this Section we establish some estimates showing the regularizing effect of convolution with the gradient of the kernel g or with g itself, as they appear in the formulation (I.0.0.2) à la Walsh of our problem.

Let J be the linear operator defined as

$$(J\varphi)(t,x) := \int_0^t \int_D \nabla_y g(t-s,x,y)\cdot \varphi(s,y)\,\mathrm{d}y\,\mathrm{d}s, \tag{I.1.5.1}$$

for $t \in [0,T]$, $x \in D$. We have that J is well defined in some spaces as defined in the following lemma.

Lemma I.1.5.1. *i) Let $p \ge 1$, $\alpha \ge 1$, $1 \le \beta < \frac{4}{3}$, $\gamma > \frac{2\beta}{2-\beta}$ such that $\frac{1}{\beta} = 1 + \frac{1}{p} - \frac{1}{\alpha}$. Then J is a bounded linear operator from $L^\gamma(0,T;L_\alpha)$ into $L^\infty(0,T;L^p)$. Moreover there exists a constant C_β such that*

$$\|J(\varphi)(t,\cdot)\|_{L^p} \le C_\beta \int_0^t (t-s)^{\frac{1}{\beta}-\frac{3}{2}} \|\varphi(s,\cdot)\|_{L_\alpha}\,\mathrm{d}s, \tag{I.1.5.2}$$

$$\|J(\varphi)(t,\cdot)\|_{L^p} \le C_\beta t^{\frac{1}{\beta}-\frac{3}{2}+\frac{\gamma-1}{\gamma}} \left(\int_0^t \|\varphi(s,\cdot)\|_{L_\alpha}^\gamma\,\mathrm{d}s \right)^{\frac{1}{\gamma}} \tag{I.1.5.3}$$

for all $t \in [0,T]$.

ii) Let $p > 4$ and $\gamma > \frac{2p}{p-2}$. Then the operator J maps $L^\gamma(0,T;L_p)$ into $C([0,T]\times D)$. Moreover there exists a constant $C_{T,p}$ such that

$$\sup_{0\le t\le T}\sup_{x\in D} |(J\varphi)(t,x)| \le C_{T,p} \left(\int_0^T \|\varphi(r,\cdot)\|_{L_p}^\gamma\,\mathrm{d}r \right)^{\frac{1}{\gamma}}. \tag{I.1.5.4}$$

Proof. These results are inspired by [41, Lemma 3.1], but we need to perform all the computations since now we are in a two dimensional domain.

We first prove *i)*. Using the continuous version of Minkowski's inequality, then Young's inequality with $\frac{1}{\alpha}+\frac{1}{\beta}=1+\frac{1}{p}$, and (I.1.2.8) we get

$$\begin{aligned}
&\left\| \int_0^t \int_D \nabla_y g(t-s,\cdot,y)\cdot\varphi(s,y)\,\mathrm{d}y\mathrm{d}s \right\|_{L^p} \\
&\quad\le \int_0^t \left\| \int_D \nabla_y g(t-s,\cdot,y)\cdot\varphi(s,y)\,\mathrm{d}y \right\|_{L^p} \mathrm{d}s = \int_0^t \|\nabla_y g(t-s,0,\cdot)*\varphi(s,\cdot)\|_{L^p}\,\mathrm{d}s \\
&\qquad\le \int_0^t \|\nabla_y g(t-s,0,\cdot)\|_{L^\beta}\|\varphi(s,\cdot)\|_{L_\alpha}\,\mathrm{d}s \le C_\beta \int_0^t (t-s)^{\frac{1}{\beta}-\frac{3}{2}}\|\varphi(s,\cdot)\|_{L_\alpha}\,\mathrm{d}s
\end{aligned}$$

This proves (I.1.5.2). By Hölder's inequality, provided $\gamma>\frac{2\beta}{2-\beta}$ we estimate the latter quantity by

$$C_\beta\left(\int_0^t (t-s)^{(\frac{1}{\beta}-\frac{3}{2})\frac{\gamma}{\gamma-1}}\mathrm{d}s\right)^{\frac{\gamma-1}{\gamma}}\left(\int_0^t \|\varphi(s,\cdot)\|_{L_\alpha}^\gamma\,\mathrm{d}s\right)^{\frac{1}{\gamma}}.$$

Calculating the first time integral we obtain (I.1.5.3).

As regards *ii)*, we use the factorization method (for more details see, e.g., [22, Section 2.2.1]), which is based on the equality

$$\frac{\pi}{\sin(\pi a)}=\int_s^t (t-r)^{a-1}(r-s)^{-a}\,\mathrm{d}r,\qquad a\in(0,1)\,. \tag{I.1.5.5}$$

We also use the Chapman-Kolmogorov relation for $s<r<t$

$$\int_D g(t-r,x,z)g(r-s,z,y)\,\mathrm{d}z=g(t-s,x,y)$$

which, thanks to the symmetry of the kernel g in the space variables, gives

$$\begin{aligned}
&\int_D \partial_{z_i} g(t-r,x,z)g(r-s,z,y)\,\mathrm{d}z=\int_D -\partial_{x_i} g(t-r,x,z)g(r-s,z,y)\ \mathrm{d}z \\
&\quad=-\partial_{x_i}\int_D g(t-r,x,z)g(r-s,z,y)\ \mathrm{d}z=-\partial_{x_i}g(t-s,x,y)=\partial_{y_i}g(t-s,x,y).
\end{aligned} \tag{I.1.5.6}$$

Let us show that $J\varphi$, defined in (I.1.5.1), has an equivalent expression given by

$$(J\varphi)(t,x)=\frac{\sin(\pi a)}{\pi}\int_0^t (t-r)^{a-1}\left(\int_D \nabla_z g(t-r,x,z)\cdot Y^a(r,z)\,\mathrm{d}z\right)\mathrm{d}r \tag{I.1.5.7}$$

with

$$Y^a(r,z)=\int_0^r\int_D (r-s)^{-a}g(r-s,z,y)\varphi(s,y)\,\mathrm{d}y\,\mathrm{d}s.$$

For this it is enough to check that

$$\begin{aligned}
&\int_0^t\int_D \partial_{y_i}g(t-s,x,y)\varphi_i(s,y)\,\mathrm{d}y\,\mathrm{d}s \\
&\qquad\qquad=\frac{\sin(\pi a)}{\pi}\int_0^t (t-r)^{a-1}\left(\int_D \partial_{z_i}g(t-r,x,z)Y_i^a(r,z)\,\mathrm{d}z\right)\mathrm{d}r
\end{aligned}$$

for $i = 1, 2$. Let us work on the r.h.s.; keeping in mind the definition of Y_i^a and by means of Fubini theorem we infer that

$$\begin{aligned}
&\int_0^t (t-r)^{a-1} \left(\int_D \partial_{z_i} g(t-r,x,z) Y_i^a(r,z)\,\mathrm{d}z \right) \mathrm{d}r \\
&= \int_0^t (t-r)^{a-1} \left(\int_D \partial_{z_i} g(t-r,x,z) \right. \\
&\qquad \left. \left[\int_0^r \int_D (r-s)^{-a} g(r-s,z,y) \varphi_i(s,y)\,\mathrm{d}y\,\mathrm{d}s \right] \mathrm{d}z \right) \mathrm{d}r \\
&= \int_0^t \left(\int_s^t (t-r)^{a-1} (r-s)^{-a} \right. \\
&\qquad \left. \left[\int_D \left[\int_D \partial_{z_i} g(t-r,x,z) g(r-s,z,y) \mathrm{d}z \right] \varphi_i(s,y)\,\mathrm{d}y \right] \mathrm{d}r \right) \mathrm{d}s \\
&= \int_0^t \left(\int_s^t (t-r)^{a-1} (r-s)^{-a} \left[\int_D \partial_{y_i} g(t-s,x,y) \varphi_i(s,y)\,\mathrm{d}y \right] \mathrm{d}r \right) \mathrm{d}s \quad \text{by (I.1.5.6)} \\
&= \frac{\pi}{\sin(\pi a)} \int_0^t \int_D \partial_{y_i} g(t-s,x,y) \varphi_i(s,y)\,\mathrm{d}y\,\mathrm{d}s \qquad \text{by (I.1.5.5).}
\end{aligned}$$

This proves (I.1.5.7). Therefore, by Hölder's inequality we get

$$\begin{aligned}
|(J\varphi)(t,x)| &\le \frac{\sin(\pi a)}{\pi} \int_0^t (t-r)^{a-1} \|\nabla_z g(t-r,x,\cdot)\|_{L^{\frac{p}{p-1}}} \|Y^a(r,\cdot)\|_{L_p} \mathrm{d}r \\
&\le C_p \frac{\sin(\pi a)}{\pi} \int_0^t (t-r)^{a-1-\frac{3}{2}+\frac{p-1}{p}} \|Y^a(r,\cdot)\|_{L_p} \mathrm{d}r \quad \text{by (I.1.2.8) if } p>4.
\end{aligned}$$

Now we estimate $\|Y^a(r,\cdot)\|_{L_p}$; by means of Minkowsky's and Young's inequalities and using (I.1.2.11) we infer that

$$\begin{aligned}
\|Y^a(r,\cdot)\|_{L_p} &= \left\| \int_0^r \int_D (r-s)^{-a} g(r-s,\cdot,y) \varphi(s,y)\,\mathrm{d}y\,\mathrm{d}s \right\|_{L_p} \\
&\le \int_0^r (r-s)^{-a} \left\| \int_D g(r-s,\cdot,y)\varphi(s,y)\,\mathrm{d}y \right\|_{L_p} \mathrm{d}s \\
&= \int_0^r (r-s)^{-a} \|g(r-s,0,\cdot) * \varphi(s,\cdot)\|_{L_p} \mathrm{d}s \\
&\le \int_0^r (r-s)^{-a} \|g(r-s,0,\cdot)\|_{L_1} \|\varphi(s,\cdot)\|_{L_p} \mathrm{d}s \\
&\le C \int_0^r (r-s)^{-a} \|\varphi(s,\cdot)\|_{L_p} \mathrm{d}s.
\end{aligned}$$

Collecting the above estimates, by means of Fubini theorem we obtain that

$$\begin{aligned}
|(J\varphi)(t,x)| &\le C \frac{\sin(\pi a)}{\pi} \int_0^t (t-r)^{a-\frac{3}{2}-\frac{1}{p}} \left(\int_0^r (r-s)^{-a} \|\varphi(s,\cdot)\|_{L_p} \mathrm{d}s \right) \mathrm{d}r \\
&= C \frac{\sin(\pi a)}{\pi} \int_0^t \|\varphi(s,\cdot)\|_{L^p} \left(\int_s^t (t-r)^{a-\frac{3}{2}-\frac{1}{p}} (r-s)^{-a} \mathrm{d}r \right) \mathrm{d}s.
\end{aligned}$$

With the change of variables $r = s + z(t-s)$ we can compute the inner integral as follows:

$$\begin{aligned}\int_s^t (t-r)^{a-\frac{3}{2}-\frac{1}{p}}(r-s)^{-a}\mathrm{d}r &= (t-s)^{-\frac{p+2}{2p}} \int_0^1 (1-z)^{a-\frac{3}{2}-\frac{1}{p}} z^{-a}\, dz \\ &= (t-s)^{-\frac{p+2}{2p}} \int_0^1 (1-z)^{a-1-\frac{p+2}{2p}} z^{-a}\, dz.\end{aligned}$$

The latter integral is equal to the beta function $B\left(1-a, a-\frac{p+2}{2p}\right)$, which is finite provided $\frac{p+2}{2p} < a < 1$; therefore given $p > 4$ we choose $a \in \left(\frac{p+2}{2p}, 1\right)$. Hence

$$\begin{aligned}|(J\varphi)(t,x)| &\le C_p \int_0^t (t-s)^{-\frac{p+2}{2p}} \|\varphi(s,\cdot)\|_{L_p}\, \mathrm{d}s \\ &\le C_p \left(\int_0^t (t-s)^{-\frac{p+2}{2p}\frac{\gamma}{\gamma-1}}\mathrm{d}s\right)^{\frac{\gamma-1}{\gamma}} \left(\int_0^t \|\varphi(s,\cdot)\|^\gamma_{L_p}\, \mathrm{d}s\right)^{\frac{1}{\gamma}} \\ &\le C_{T,p} \left(\int_0^T \|\varphi(s,\cdot)\|^\gamma_{L_p}\, \mathrm{d}s\right)^{\frac{1}{\gamma}}\end{aligned}$$

for $\frac{p+2}{2p}\frac{\gamma}{\gamma-1} < 1$, i.e. $\gamma > \frac{2p}{p-2}$.

The above estimate shows that $J\varphi \in L^\infty([0,T]\times D)$ for every $\varphi \in L^\gamma(0,T;L_p)$. It remains to prove that $J\varphi \in C([0,T]\times D)$. Let us notice that for step functions φ, $J\varphi$ is a space-time continuous function; this follows from the well posedness of the integral $\int_0^t \int_D \nabla_y g(t-s,x,y)\,\mathrm{d}y\,\mathrm{d}s$ (let us recall that $\int_0^t \int_D |\nabla_y g(t-s,x,y)|\,\mathrm{d}y\,\mathrm{d}s < \infty$, see (I.1.2.9)). This kind of regularity can be then extended to every $\varphi \in L^\gamma(0,T;L_p)$ by a standard approximation procedure. □

I.1.6 Existence and uniqueness of the solution

The main aim of this Section is to prove the existence and uniqueness of the solution to the SPDE (I.0.0.1) as stated in Theorem I.1.1.1. Since the derivative of w is formal, we consider the equation in a weak sense, as in [90] for the stochastic heat equation. In order to simplify the notation, recalling the relation between the vorticity scalar field ξ and velocity vector field v given by the Biot-Savart law (I.1.2.22), let us define the vector field $q(\xi) = \xi\ (k*\xi)$, i.e.

$$[q(\xi)](x) = \xi(x)\int_D k(x-y)\xi(y)\,\mathrm{d}y. \tag{I.1.6.1}$$

By means of Hölder's inequality, from (I.1.2.26) if $p > 2$ we know that

$$\|q(\xi)\|_{L_p} \le \|\xi\|_{L^p}\|k*\xi\|_{L_\infty} \le C_p\|\xi\|^2_{L^p} \tag{I.1.6.2}$$

namely $q : L^p \to L_p$ for any $p > 2$. This allows to write system (I.0.0.1) in an equivalent form, where the velocity does not appear anymore.

Since $v = k*\xi$ is divergence free, for the nonlinear term in equation (I.0.0.1) we have

$$v\cdot\nabla\xi = \nabla\cdot(v\xi) = \nabla\cdot q(\xi).$$

Therefore we give this definition of solution to system (I.0.0.1). This is a weak solution in the sense of PDE's, hence involving test functions φ.

Definition I.1.6.1. *We say that an $\dot L^2_\sharp$-valued continuous, $\mathcal{F}_t$-adapted, jointly measurable (in the variables $t, x; \omega$) stochastic process ξ is a solution to* (I.0.0.1) *if it solves* (I.0.0.1) *in the following sense: for every $t \in [0,T]$, $\varphi \in W^a$ with $a > 2$ we have*

$$\int_D \xi(t,x)\varphi(x)\,\mathrm{d}x - \int_0^t \int_D \xi(s,x)\Delta\varphi(x)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t \int_D q(\xi(s,\cdot))(x)\cdot\nabla\varphi(x)\,\mathrm{d}x\,\mathrm{d}s = \int_D \xi_0(x)\varphi(x)\,\mathrm{d}x + \int_0^t \int_D \varphi(x)\sigma(\xi(s,x))\,w(\mathrm{d}x,\mathrm{d}s) \tag{I.1.6.3}$$

$\mathbb{P}$-*a.s.*

Notice that the non linear term is well defined since, using repeatedly Hölder's inequality and the Sobolev embedding, we obtain

$$\begin{aligned}\left|\int_D q(\xi(s,\cdot))(x)\cdot\nabla\varphi(x)\,\mathrm{d}x\right| &\le \|\nabla\varphi\|_{L^\infty}\|q(\xi(s,\cdot))\|_{L_1}\\ &\le C\|\nabla\varphi\|_{W^s}\|k*\xi(s,\cdot)\|_{L_2}\|\xi(s,\cdot)\|_{L^2} \quad \text{if } s>1\\ &\le C\|\varphi\|_{W^{s+1}}\|\xi(s,\cdot)\|^2_{L^2} \ \text{ by (I.1.2.27) } (\alpha=1,\ \beta=p=2).\end{aligned}$$

Following the idea of [90] for the heat equation or of [41] for the Burgers equation one obtains that this is equivalent to ask that, for any $(t,x) \in [0,T]\times D$, $\xi = \{\xi(t,x), 0 \le t \le T, x \in D\}$ is a jointly measurable, $\mathcal{F}_t$-adapted process which satisfies

$$\xi(t,x) = \int_D g(t,x,y)\xi_0(y)\,\mathrm{d}y + \int_0^t \int_D \nabla_y g(t-s,x,y)\cdot q(\xi(s,\cdot))(y)\,\mathrm{d}y\,\mathrm{d}s + \int_0^t \int_D g(t-s,x,y)\sigma(\xi(s,y))\,w(\mathrm{d}y,\mathrm{d}s) \tag{I.1.6.4}$$

$\mathbb{P}$-a.s. The stochastic integral is understood in the Walsh sense w.r.t. the martingale measure introduced in Subsection I.1.3.2.
The non linear term $q(\xi)$ that appears in (I.1.6.4) is non Lipschitz. Therefore, we use a localization argument to prove the existence and uniqueness of the solution. By means of a fixed point argument we prove at first the existence and uniqueness result for a local solution; then the global result follows from suitable estimates on the process ξ.
Thus, we first solve the problem when the nonlinearity is truncated to be globally Lipschitz.

I.1.6.1 The case of truncated nonlinearity

Let $N \ge 1$ and denote by $\Theta_N : [0,+\infty) \to [0,1]$ a C^1 function such that $|\Theta_N'(s)| \le 2$ for any $s \ge 0$ and

$$\Theta_N(s) = \begin{cases} 1 & \text{if } 0 \le s < N \\ 0 & \text{if } s \ge N+1 \end{cases} \tag{I.1.6.5}$$

Given $\xi \in L^p$, for $p > 2$, we define

$$q_N(\xi) = q(\xi)\Theta_N(\|\xi\|_{L^p}), \tag{I.1.6.6}$$

$$\tilde q_N(\xi) = q(\xi)\Theta_N'(\|\xi\|_{L^p}). \tag{I.1.6.7}$$

By (I.1.6.2) we know that $q_N, \tilde q_N : L^p \to L_p$ for any $p > 2$. In addition we have

Lemma I.1.6.2. *Fix $N \geq 1$ and $p > 2$. Then there exist positive constants C_p and $L_{N,p}$ such that*

$$\|q_N(\xi)\|_{L_p} \leq C_p(N+1)^2 \qquad \forall \xi \in L^p, \tag{I.1.6.8}$$

$$\|\tilde{q}_N(\xi)\|_{L_p} \leq C_p(N+1)^2 \qquad \forall \xi \in L^p \tag{I.1.6.9}$$

and

$$\|q_N(\xi) - q_N(\eta)\|_{L_p} \leq L_{N,p}\|\xi - \eta\|_{L^p} \qquad \forall \xi, \eta \in L^p. \tag{I.1.6.10}$$

Proof. The global bounds comes from (I.1.6.2):

$$\|q_N(\xi)\|_{L_p} \leq C_p\|\xi\|^2_{L^p}\Theta_N(\|\xi\|_{L^p}) \leq C_p(N+1)^2,$$

$$\|\tilde{q}_N(\xi)\|_{L_p} \leq C_p\|\xi\|^2_{L^p}|\Theta'_N(\|\xi\|_{L^p})| \leq C_p(N+1)^2.$$

Let us now show that q_N is a Lipschitz continuous function. The idea is to use the mean value theorem: we show that q_N is Gâteaux differentiable in any point of L^p and its derivative is bounded. The result will follow by

$$\|q_N(\xi) - q_N(\eta)\|_{L_p} \leq \sup_{t\in[0,1]} \|Dq_N(t\xi + (1-t)\eta)\|_{\mathcal{L}(L^p;L_p)}\|\xi - \eta\|_{L^p} \tag{I.1.6.11}$$

where $Dq_N(\xi) : h \to D_h q_N(\xi)$ is a linear and bounded operator from L^p into L_p defined as

$$D_h q_N(\xi) := \lim_{\varepsilon\to 0} \frac{q_N(\xi + \varepsilon h) - q_N(\xi)}{\varepsilon} \tag{I.1.6.12}$$

and

$$\|Dq_N(\xi)\|_{\mathcal{L}(L^p;L_p)} = \sup_{\|h\|_{L^p}\leq 1} \|D_h q_N(\xi)\|_{L_p}.$$

By (I.1.6.12) we have

$$D_h q_N(\xi) = q(\xi)\ D_h\Theta_N(\|\xi\|_{L^p}) + h\ (k * \xi)\Theta_N(\|\xi\|_{L^p}) + \xi\ (k * h)\Theta_N(\|\xi\|_{L^p}).$$

Since $D_h(\|\xi\|_{L^p}) = \|\xi\|^{1-p}_{L^p}\langle \xi|\xi|^{p-2}, h\rangle$ we get

$$D_h\Theta_N(\|\xi\|_{L^p}) = \Theta'_N(\|\xi\|_{L^p})\|\xi\|^{1-p}_{L^p}\langle \xi|\xi|^{p-2}, h\rangle.$$

Therefore, bearing in mind (I.1.2.26) and (I.1.6.9) we infer that

$$\begin{aligned}
&\|D_h q_N(\xi)\|_{L_p} \\
&\quad\leq |\Theta'_N(\|\xi\|_{L^p})|\|\xi\|^{1-p}_{L^p}|\langle \xi|\xi|^{p-2}, h\rangle|\ \|q(\xi)\|_{L_p} \\
&\qquad + |\Theta_N(\|\xi\|_{L^p})|\|h(k*\xi)\|_{L_p} + |\Theta_N(\|\xi\|_{L^p})|\|\xi(k*h)\|_{L_p} \\
&\quad\leq |\Theta'_N(\|\xi\|_{L^p})|\|h\|_{L^p}\|q(\xi)\|_{L_p} \\
&\qquad + |\Theta_N(\|\xi\|_{L^p})|\|h\|_{L^p}\|k*\xi\|_{L_\infty} + |\Theta_N(\|\xi\|_{L^p})|\|\xi\|_{L^p}\|k*h\|_{L_\infty} \\
&\quad\leq \|h\|_{L^p}\|\tilde{q}_N(\xi)\|_{L_p} + 2C_p|\Theta_N(\|\xi\|_{L^p})|\|h\|_{L^p}\|\xi\|_{L^p} \\
&\quad\leq C_p(N+1)^2\|h\|_{L^p} + 2C_p(N+1)\|h\|_{L^p}.
\end{aligned}$$

Hence we get

$$\sup_{\xi} \|Dq_N(\xi)\|_{\mathcal{L}(L^p;L_p)} \leq C_p(N+1)^2 + 2C_p(N+1).$$

Thanks to (I.1.6.11) this proves (I.1.6.10). □

Remark I.1.6.3. *Notice that, since the relation between v and ξ is non local in space (see* (I.1.2.22)*), we can not consider a tuncation punctual in space, namely of the form $\Theta_N(|\xi(t,x)|)$, $(t,x) \in [0,T] \times D$. This would not provide a bound for v as* (I.1.6.8).

We aim at proving the existence and uniqueness of the solution to the smoothed version of system (I.0.0.1) that is

$$\begin{cases} \dfrac{\partial \xi_N}{\partial t}(t,x) - \Delta\xi_N(t,x) + v_N(t,x)\cdot\nabla\xi_N(t,x)\Theta_N(\|\xi_N(t,\cdot)\|_{L^p}) = \sigma(\xi_N(t,x))w(\mathrm{d}x,\mathrm{d}t) \\ \nabla\cdot v_N(t,x) = 0 \\ \xi_N(t,x) = \nabla^\perp\cdot v_N(t,x) \\ \xi_N(0,x) = \xi_0(x) \end{cases}$$

Thanks to (I.1.6.1) and (I.1.6.6) this can be written in the Walsh formulation as

$$\begin{aligned} \xi_N(t,x) = \int_D g(t,x,y)\xi_0(y)\,\mathrm{d}y + \int_0^t\int_D \nabla_y g(t-s,x,y)\cdot q_N(\xi_N(s,\cdot))(y)\,\mathrm{d}y\,\mathrm{d}s \\ + \int_0^t\int_D g(t-s,x,y)\sigma(\xi_N(t,x))\,w(\mathrm{d}y,\mathrm{d}s). \end{aligned} \tag{I.1.6.13}$$

We have the following result.

Proposition I.1.6.4. *Let $N \geq 1$, $b > 0$ in* (I.1.3.3) *and $p > 4$. Let assume that Hypothesis (H1) and (H2) hold. If $\xi_0 \in L^p$, then there exists a unique solution ξ_N to equation* (I.1.6.13) *which is an $\mathcal{F}_t$-adapted jointly measurable L^p-valued continuous process such that*

$$\mathbb{E}\sup_{0\leq t\leq T}\|\xi_N(t,\cdot)\|_{L^p}^p < \infty. \tag{I.1.6.14}$$

Remark I.1.6.5. *The stochastic convolution term appearing in* (I.1.6.13) *is well posed; the formal proof of this fact is given in Theorem I.2.2.2, where we also show that the solution process ξ_N to* (I.1.6.13) *is jointly measurable.*

Proof. Let $N \geq 1$ and $p > 4$. Let $\mathcal{B}$ denote the space of all L^p-valued $\mathcal{F}_t$-adapted continuous processes η such that

$$\|\eta\|_{\mathcal{B}}^p = \mathbb{E}\sup_{0\leq t\leq T}\|\eta(t,\cdot)\|_{L^p}^p < \infty.$$

Let us set

$$(\mathcal{M}\xi_N)(t,x) = \int_D g(t,x,y)\xi_0(y)\,\mathrm{d}y + (Jq_N(\xi_N))(t,x) + \mathcal{A}\xi_N(t,x),$$

where $(Jq_N(\xi_N))(t,x)$ is given by (I.1.5.1) and

$$(\mathcal{A}\xi_N)(t,x) := \int_0^t\int_D g(t-s,x,y)\sigma(\xi_N(s,y))\,w(\mathrm{d}y,\mathrm{d}s). \tag{I.1.6.15}$$

We shall prove that $\mathcal{M}$ defines a contraction on $\mathcal{B}$. First,

$$\|\mathcal{M}\xi_N\|_{\mathcal{B}} \leq C_p\mathbb{E}\left[\sup_{0\leq t\leq T}\left(\left\|\int_D g(t,x,y)\xi_0(y)\,\mathrm{d}y\right\|_{L^p}^p + \|(Jq_N(\xi_N))(t,\cdot)\|_{L^p}^p + \|(\mathcal{A}\xi_N)(t,\cdot)\|_{L^p}^p\right)\right].$$

Using Young's inequality and (I.1.2.10), we infer that

$$\sup_{0\le t\le T}\left\|\int_D g(t,\cdot,y)\xi_0(y)\,\mathrm{d}y\right\|_{L^p}^p \le \sup_{0\le t\le T}\left[\|g(t,0,\cdot)\|_{L^1}\ \|\xi_0\|_{L^p}\right]^p < \infty.$$

By estimates (I.1.5.2) (with $\beta = 1$, $\alpha = p$) and (I.1.6.8) we get

$$\|(Jq_N(\xi_N))(t,\cdot)\|_{L^p} \le \int_0^t (t-s)^{-\frac{1}{2}}\|q_N(\xi_N(s,\cdot))\|_{L_p}\,\mathrm{d}s \le C_p(N+1)^2 t^{\frac{1}{2}}$$

and so $\|Jq_N(\xi_N)\|_{\mathcal{B}} < C_p(N+1)^2T^{\frac{1}{2}} < \infty$. Finally by Corollary I.1.4.6 and Hypothesis (H2),

$$\|\mathcal{A}\xi_N\|_{\mathcal{B}} = \mathbb{E}\sup_{0\le t\le T}\|(\mathcal{A}\xi_N)(t,\cdot)\|_{L^p}^p \le C_{T,p,Q}\ \mathbb{E}\sup_{0\le t\le T}\|\sigma(\xi_N(t,\cdot))\|_{L^p}^p < \infty.$$

Thus $\mathcal{M}$ is an operator mapping the Banach space $\mathcal{B}$ into itself. It remains to prove that $\mathcal{M}$ defines a contraction. From (I.1.5.2) with $\alpha = p$, $\beta = 1$ and the Lipschitz result of Lemma I.1.6.2, we infer that

$$\begin{aligned}
\|(Jq_N(\xi_N^1))(t,\cdot) - (Jq_N(\xi_N^2))(t,\cdot)\|_{L^p} &\le C\int_0^t (t-s)^{-\frac{1}{2}}\|q_N(\xi_N^1(s,\cdot)) - q_N(\xi_N^2(s,\cdot))\|_{L_p} ds \\
&\le CL_{N,p}\int_0^t (t-s)^{-\frac{1}{2}}\|\xi_N^1(s,\cdot) - \xi_N^2(s,\cdot)\|_{L^p}\,\mathrm{d}s \\
&\le CL_{N,p}\left(\sup_{0\le s\le t}\|\xi_N^1(s,\cdot) - \xi_N^2(s,\cdot)\|_{L^p}\right)\int_0^t (t-s)^{-\frac{1}{2}}\mathrm{d}s \\
&= C_{N,p}T^{\frac{1}{2}}\left(\sup_{0\le t\le T}\|\xi_N^1(t,\cdot) - \xi_N^2(t,\cdot)\|_{L^p}\right)
\end{aligned}$$

Since the above estimate holds for every $t \in [0,T]$, taking the $\sup_{0\le t\le T}$ on the l.h.s. and then the expectation, we get

$$\|Jq_N(\xi_N^1) - Jq_N(\xi_N^2)\|_{\mathcal{B}} \le CL_{N,p}T^{\frac{1}{2}}\|\xi_N^1 - \xi_N^2\|_{\mathcal{B}}.$$

Using the lipschitzianity of σ (hypothesis (H1)), by Corollary I.1.4.6 we have for $\alpha \in (0, \frac{1}{2})$

$$\|\mathcal{A}\xi_N^1 - \mathcal{A}\xi_N^2\|_{\mathcal{B}} \le LC_{p,\alpha,Q}T^{\alpha p-1}\mathbb{E}\sup_{0\le t\le T}\|\xi_N^1(t,\cdot) - \xi_N^2(t,\cdot)\|_{L^p}^p = LC_{p,\alpha,Q}T^{\alpha p-1}\|\xi_N^1 - \xi_N^2\|_{\mathcal{B}}.$$

Collecting the above estimates we finally get

$$\|\mathcal{M}\xi_N^1 - \mathcal{M}\xi_N^2\|_{\mathcal{B}} \le C_{L,N,p,Q,\alpha}\max\left(T^{\frac{1}{2}}, T^{\alpha p-1}\right)\|\xi_N^1 - \xi_N^2\|_{\mathcal{B}}.$$

If T satisfies $C_{N,p,Q}\max\left(T^{\frac{1}{2}}, T^{\alpha p-1}\right) < 1$, then $\mathcal{M}$ is a contraction on $\mathcal{B}$. Hence the operator $\mathcal{M}$ admits a unique fixed point in the set $\{\xi \in \mathcal{B} : \xi(0,\cdot) = \xi_0\}$. Otherwise we choose $\tilde{t} > 0$ such that $C_{N,p,Q}\max\left(\tilde{t}^{\frac{1}{2}}, \tilde{t}^{\alpha p-1}\right) < 1$ and we conclude the existence of a unique solution on the time interval $[0,\tilde{t}]$. Since $C_{N,p,Q}$ does not depend on ξ_0, by a standard argument we construct a unique solution ξ to the SPDE (I.1.6.13) by concatenation on every interval of length $\tilde{t}$ until we recover the time interval $[0,T]$. □

In the following Section we shall see that Proposition I.1.6.4 provides uniqueness and local existence for the solution in Theorem I.1.1.1. To gain the global existence we need a uniform estimate as proved in the following lemma, inspired by [41] and [42].

Lemma I.1.6.6. *Let $p > 2$. Let $z = \{z(t,x), t \in [0,T], x \in D\}$ be a function belonging to $C([0,T];L^p)$. For every $N \geq 1$ let $\beta_N \in C([0,T];L^p) \cap L^2(0,T;W^1)$ be a solution of the integral equation*

$$\beta_N(t,x) = \int_D g(t,x,y)\xi_0(y)\,\mathrm{d}y + \int_0^t \int_D \nabla_y g(t-s,x,y) \cdot q_N(\beta_N(s,\cdot) + z(s,\cdot))(y)\,\mathrm{d}y\,\mathrm{d}s \tag{I.1.6.16}$$

where $\xi_0 \in L^p$. Then we have

$$\sup_{N\geq 1} \sup_{t\in[0,T]} \|\beta_N(t,\cdot)\|^p_{L^p} \leq \left[\|\xi_0\|^p_{L^p} + C_1(z)\right] e^{C_2(z)}$$

where $C_1(z)$ and $C_2(z)$ are given by

$$C_1(z) = C_p T \sup_{t\in[0,T]} \|z(t,\cdot)\|^{2p}_{L^p}$$

and

$$C_2(z) = C_p T \left(1 + \sup_{t\in[0,T]} \|z(t,\cdot)\|^2_{L^p}\right)$$

for some positive constant C_p.

Proof. Let us fix $N \geq 1$. As done before, we can show that a solution to (I.1.6.16) is a weak solution to the PDE

$$\frac{\partial}{\partial t}\beta_N = \Delta\beta_N - \nabla \cdot q_N(\beta_N + z) \tag{I.1.6.17}$$

with initial condition $\beta_N(0,x) = \xi_0(x)$.

We consider the time evolution of the L^p-norm of $\beta_N(t,\cdot)$. Notice that, since $\beta_N \in L^2(0,T;W^1)$; $\nabla\beta_N$ exists and we will use this fact in the following computations. We point out that, formally, the proof of the existence of the solution to (I.1.6.16) requires some Galerkin approximations β_N^k of β_N. A priori estimates, uniformly in k, are proved for β_N^k and then we pass to the limit.

From (I.1.6.17) we infer that

$$\begin{aligned}\frac{d}{dt}\|\beta_N(t,\cdot)\|^p_{L^p} &= p\int_D |\beta_N(t,x)|^{p-2}\beta_N(t,x)\frac{\partial}{\partial t}\beta_N(t,x)\;dx \\ &= p\int_D |\beta_N(t,x)|^{p-2}\beta_N(t,x)\Delta\beta_N(t,x)\;dx \\ &\quad - p\int_D |\beta_N(t,x)|^{p-2}\beta_N(t,x)\nabla \cdot q_N(\beta_N(t,\cdot) + z(t,\cdot))(x)\;dx\end{aligned}$$

Integrating by parts the two latter integrals we obtain (writing for short $\beta_N(t)$ instead of $\beta_N(t,\cdot)$)

$$\begin{aligned}&\frac{d}{dt}\|\beta_N(t)\|^p_{L^p} + p(p-1)\||\beta_N(t)|^{\frac{p-2}{2}}\nabla\beta_N(t)\|^2_{L^2} \\ &\qquad = p(p-1)\langle|\beta_N(t)|^{p-2}, \nabla\beta_N(t) \cdot q_N(\beta_N(t) + z(t))\rangle.\end{aligned}$$

We need to work on the latter term. Let us write the quadratic term $q_N(\beta_N(t)+z(t))$ in the form $\Theta_N(\|\beta_N(t)+z(t)\|_{L^p})k*(\beta_N(t)+z(t))\ (\beta_N(t)+z(t)) = \Theta_N(\|\beta_N(t)+z(t)\|_{L^p})k*(\beta_N(t)+z(t))\ \beta_N(t)+\Theta_N(\|\beta_N(t)+z(t)\|_{L^p})k*(\beta_N(t)+z(t))\ z(t)$; then using the basic property $\langle|\beta_N(t)|^{p-2}\beta_N(t), \nabla\beta_N(t)\cdot v(t)\rangle = 0$ (where v is a divergence free velocity field; this is obtained again by integration by parts, see for instance [8, Lemma 2.2]) we obtain

$$\frac{d}{dt}\|\beta_N(t)\|^p_{L^p} + p(p-1)\||\beta_N(t)|^{\frac{p-2}{2}}\nabla\beta_N(t)\|^2_{L^2}$$
$$= p(p-1)\langle\Theta_N(\|\beta_N(t)+z(t)\|_{L^p})|\beta_N(t)|^{p-2}z(t), \nabla\beta_N(t)\cdot[k*(\beta_N(t)+z(t))]\rangle. \quad \text{(I.1.6.18)}$$

Let us estimate the r.h.s., using Hölder's and Young's inequalities.

$$\begin{aligned}
&\left|\langle\Theta_N(\|\beta_N(t)+z(t)\|_{L^p})|\beta_N(t)|^{p-2}z(t), \nabla\beta_N(t)\cdot[k*(\beta_N(t)+z(t))]\rangle\right| \\
&\le |\Theta_N(\|\beta_N(t)+z(t)\|_{L^p})|\ \||\beta_N(t)|^{\frac{p-2}{2}}\nabla\beta_N(t)\|_{L^2} \\
&\qquad \||\beta_N(t)|^{\frac{p-2}{2}}z(t)\|_{L^2}\|k*(\beta_N(t)+z(t))\|_{L_\infty} \\
&\le C_p\||\beta_N(t)|^{\frac{p-2}{2}}\nabla\beta_N(t)\|_{L^2}\||\beta_N(t)|^{\frac{p-2}{2}}z(t)\|_{L^2}\|\beta_N(t)+z(t)\|_{L^p} \qquad \text{by (I.1.2.26)} \\
&\le C_p\||\beta_N(t)|^{\frac{p-2}{2}}\nabla\beta_N(t)\|_{L^2}\|\beta_N(t)\|^{\frac{p-2}{2}}_{L^p}\|z(t)\|_{L^p}\left(\|\beta_N(t)\|_{L^p}+\|z(t)\|_{L^p}\right) \\
&\le \frac{1}{2}\||\beta_N(t)|^{\frac{p-2}{2}}\nabla\beta_N(t)\|^2_{L^2} + C_p\|\beta_N(t)\|^p_{L^p}\|z(t)\|^2_{L^p} \\
&\qquad + C_p\|\beta_N(t)\|^p_{L^p} + C_p\|z(t)\|^{2p}_{L^p}.
\end{aligned}$$

Coming back to equation (I.1.6.18), we have obtained that

$$\frac{d}{dt}\|\beta_N(t)\|^p_{L^p} + \frac{p(p-1)}{2}\||\beta_N(t)|^{\frac{p-2}{2}}\nabla\beta_N(t)\|^2_{L^2}$$
$$\le C_p\left(1+\|z(t)\|^2_{L^p}\right)\|\beta_N(t)\|^p_{L^p} + C_p\|z(t)\|^{2p}_{L^p}. \quad \text{(I.1.6.19)}$$

Using Gronwall lemma on the inequality

$$\frac{d}{dt}\|\beta_N(t)\|^p_{L^p} \le C_p\left(1+\|z(t)\|^2_{L^p}\right)\|\beta_N(t)\|^p_{L^p} + C_p\|z(t)\|^{2p}_{L^p}$$

we obtain

$$\begin{aligned}
\|\beta_N(t)\|^p_{L^p} &\le \|\xi_0\|^p_{L^p}e^{C_p\int_0^t\left(1+\|z(s)\|^2_{L^p}\right)\mathrm{d}s} + C_p\int_0^t e^{C_p\int_r^t\left(1+\|z(s)\|^2_{L^p}\right)\mathrm{d}s}\|z(r)\|^{2p}_{L^p}\,\mathrm{d}r \\
&\le e^{C_pT\left(1+\sup_{0\le s\le T}\|z(s)\|^2_{L^p}\right)}\left(\|\xi_0\|^p_{L^p} + C_pT\sup_{0\le r\le T}\|z(r)\|^{2p}_{L^p}\right).
\end{aligned}$$

The r.h.s. of the above estimate does not depend on N and so we get the desired result.

I.1.6.2 Proof of the main result

We go back to the original equation (I.0.0.1) in the form given by (I.1.6.4) and prove the existence and uniqueness result stated in Theorem I.1.1.1.

Proof of Theorem I.1.1.1. Uniqueness is provided in a classical way by a stopping time argument. More precisely, suppose that ξ^1 and ξ^2 are two solutions to equation (I.0.0.1). These are L^p-valued processes (for $p > 2$), continuous in time. Both satisfy (I.1.6.4) thanks to the equivalence between the formulations (I.1.6.3) and (I.1.6.4). Let us define the stopping times

$$\tau_N^i := \inf\{t \geq 0 : \|\xi^i(t,\cdot)\|_{L^p} \geq N\} \wedge T, \qquad i = 1, 2,$$

for every $N \geq 1$ and let us set $\tau_N^* := \tau_N^1 \wedge \tau_N^2$. Setting $\xi_N^i(t) = \xi^i(t \wedge \tau_N^*)$ for $i = 1, 2$, for all $t \in [0, T]$ we have that the processes ξ_N^1 and ξ_N^2 satisfy (I.1.6.13); hence, by the uniqueness result given by Proposition I.1.6.4, $\xi_N^1 = \xi_N^2$ $\mathbb{P}$-a.s. for all $t \in [0, T]$, that is $\xi^1 = \xi^2$ on $[0, \tau_N^*)$ $\mathbb{P}$-a.s. Since τ_N^* converges $\mathbb{P}$-a.s. to T, as N tends to infinity, we deduce $\xi^1 = \xi^2$ $\mathbb{P}$-a.s for every $t \in [0, T]$.

Let us now prove the existence of the solution in $[0, T]$. Let $p > 2$; let us define the stopping time

$$\sigma_N := \inf\{t \geq 0 : \|\xi_N(t,\cdot)\|_{L^p} \geq N\} \wedge T, \tag{I.1.6.20}$$

for every $N \geq 1$. $\{\sigma_N\}_{N\geq 1}$ defines a non decreasing sequence. In Proposition I.1.6.4 we have shown the global existence and uniqueness of the solution ξ_N to the truncated problem (I.1.6.13). By uniqueness of the solution to (I.1.6.13), the local property of the stochastic integral yields, for $M > N$, $\xi_N(t,\cdot) = \xi_M(t,\cdot)$ for $t \leq \sigma_N$; so we can define a process ξ by $\xi(t,\cdot) = \xi_N(t,\cdot)$ for $t \in [0, \sigma_N]$. Set $\sigma_\infty := \sup_{N\geq 1} \sigma_N$, then Proposition I.1.6.4 tells us that we have constructed a solution to (I.1.6.13) in the random interval $[0, \sigma_\infty)$, and it is unique. To conclude, we just need to prove that

$$\sigma_\infty = T \qquad \mathbb{P} - a.s. \tag{I.1.6.21}$$

that is equivalent to verify that

$$\lim_{N\to\infty} \mathbb{P}(\sigma_N < T) = 0.$$

Set

$$z_N(t,x) = \int_0^t \int_D g(t-s,x,y)\sigma(\xi_N(s,y))\, w(\mathrm{d}y, \mathrm{d}s) \tag{I.1.6.22}$$

Applying Corollary I.1.4.6, from Hypothesis (H2), we obtain

$$\sup_{N\geq 1} \mathbb{E}\left[\sup_{0\leq t\leq T} \|z_N(t,\cdot)\|_{L^p}^p\right] < \infty. \tag{I.1.6.23}$$

For every $N \geq 1$, set $\beta_N(t,x) = \xi_N(t,x) - z_N(t,x)$ for $t < \sigma_\infty$. β_N and z_N satisfy Hypothesis of Lemma I.1.6.6. In fact for $p > 4$, $b > 0$ and Hypothesis (H2), from Corollary I.1.4.6, $z_N \in C([0,T]\,; L^p)$ $\mathbb{P}$-a.s.; from Proposition I.1.6.4 we know that $\xi_N \in C([0,T]; L^p)$ $\mathbb{P}$-a.s. Hence, for sure, $\beta_N \in C([0,T]; L^p)$ $\mathbb{P}$-a.s. for every $N \geq 1$. Actually β_N is more regular than ξ_N and z_N. Indeed, β_N satisfies the equation $\frac{\partial \beta_N}{\partial t} - \Delta\beta_N = -\nabla \cdot q_N(\xi_N)$ where $q_N(\xi_N)$ belongs at least to $L^2(0,T; L_2)$ thanks to (I.1.6.8). Hence, according to a classical regularity result for parabolic equations (see e.g. [54, Chapter 4.4, Theorem 4.1]) we have that $\beta_N \in L^2(0,T; W^1)$. Then we have that, for every $N \geq 1$,

$$\sup_{t\in[0,T]} \log \|\beta_N(t,\cdot)\|_{L^p} \leq \frac{1}{p}\log(\|\xi_0\|_{L^p}^p + C_1(z_N)) + \frac{C_2(z_N)}{p}$$

and $\mathbb{E}[C_1(z_N)]$, $\mathbb{E}[C_2(z_N)]$ are bounded by a constant that does not depend of N, according to (I.1.6.23). Hence, for all $N \geq 1$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \log \|\beta_N(t,\cdot)\|_{L^p}\right] \leq C_{p,T}\left(1+\log\|\xi_0\|^p_{L^p}\right) < \infty,$$

uniformly in N, by means of Jensen's inequality. By Chebychev's inequality, it follows that

$$\begin{aligned}
\mathbb{P}(\sigma_N < T) &= \mathbb{P}\left(\sup_{t\in[0,T]} \|\xi_N(t,\cdot)\|_{L^p} \geq N\right) \\
&\leq \mathbb{P}\left(\sup_{t\in[0,T]} \|\beta_N(t,\cdot)\|_{L^p} \geq \frac{N}{2}\right) + \mathbb{P}\left(\sup_{t\in[0,T]} \|z_N(t,\cdot)\|_{L^p} \geq \frac{N}{2}\right) \\
&\leq \frac{1}{\log\left(\frac{N}{2}\right)}\mathbb{E}\left[\sup_{t\in[0,T]} \log\|\beta_N(t,\cdot)\|_{L^p}\right] + \frac{2}{N}\mathbb{E}\left[\sup_{t\in[0,T]} \|z_N(t,\cdot)\|_{L^p}\right] \\
&\leq \frac{C_{p,T}\left(1+\log\|\xi_0\|^p_{L^p}\right)}{\log N} + \frac{\hat{C}_{p,T}}{N},
\end{aligned}$$

for some constant $C_{p,T}$ and $\hat{C}_{p,T}$, independent of N. Then we obtain that $\lim_{N\to\infty} \mathbb{P}(\sigma_N < T) = 0$.

Finally, we assume ξ_0 to be continuous; then the solution ξ given by (I.1.6.4) is the sum of three terms. The first one, $\int_D g(t,x,y)\xi_0(y)\,\mathrm{d}y$ is continuous by the properties of g (see Theorem I.1.2.4*(ii)*). As regards the second one, since $\xi_0 \in C(D)$, then $\xi_0 \in L^{\tilde{p}}$ for any $\tilde{p}$. Choosing a value of $\tilde{p} > 4$, we find that $q(\xi) \in C([0,T]\,; L_{\tilde{p}})$ and Lemma I.1.5.1*(ii)* provides that $Jq(\xi) \in C([0,T]\times D)$. Finally the third term is continuous thanks to Corollary I.1.4.5. □

Remark I.1.6.7. *Let us notice that, once we have proved the existence of a unique local solution in the space $\mathcal{B}$ (see Proposition I.1.6.4), in order to have the existence of a space-time continuous modification of the stochastic convolution term* (I.1.6.22) *it is sufficient to require σ satisfying a linear growth condition. The stronger hypothesis (H2) is made in order to obtain an uniform estimate in N. Only in this way we can pass to the limit $N \to \infty$ and prove the existence of a global solution (see proof of Theorem I.1.1.1).*

Moreover, σ satisfying a linear growth condition is sufficient for (I.1.6.22) *to be well defined. The stronger assumption (H2) is needed for the well posedness of*

$$\int_0^t \int_D g(t-s,x,y)\sigma(\xi(s,y))\,\mathrm{d}y\,\mathrm{d}s. \tag{I.1.6.24}$$

In fact, we can not perform a fixed point for ξ in the space $L^\infty([0,T]\times D; L^p(\Omega))$ gaining the regularity of the integrand process needed for the well posedness of (I.1.6.24) *(see Proposition I.1.4.1). This is due to the presence of the non linear term which is not Lipschitz continuous.*

□

Chapter I.2

Existence of a density for the image law of the solution

I.2.1 Introduction

In the present Chapter we study the regularity of the solution to (I.0.0.2) in the sense of stochastic calculus of variations, namely we prove the existence of the density of the random variable $\xi(t,x)$, solution to (I.0.0.2), for fixed $(t,x) \in [0,T] \times D$. For this we use the Malliavin calculus (see [72]) associated to the noise that appears in (I.0.0.1). We prove at first that for any fixed $(t,x) \in [0,T] \times D$ the random variable $\xi(t,x)$ belongs to the Sobolev space $\mathbb{D}^{1,p}_{loc}$ for every $p > 4$. Then we prove that the law of $\xi(t,x)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. We point out here that the localization argument we use in order to achieve this result does not provide the smoothness of the density since we do not have the boundedness of the derivatives of every order. Moreover, let us notice that the technique of analysis of the existence of the density by means of Malliavin calculus is suited for a scalar unknown; the case for a vector unknown is much more involved (see, e.g., [72]). This is the reason why we work on the Navier-Stokes equations in vorticity form (I.0.0.1) instead of the usual formulation with respect to the vector velocity.

We shall require more regularity on the covariance function σ. In addition to hypothesis (H1)-(H2) (see Chapter I.1) we make the following assumptions.

(H3): σ is of class C^1 on $\mathbb{R}$ and has first derivative bounded;

(H4): there exists $\sigma_0 > 0$ such that $|\sigma(x)| \geq \sigma_0$ for all $x \in \mathbb{R}$.

The main result we will prove is the following.

Theorem I.2.1.1. *Let* $b > 1$ *in* (I.1.3.3) *and assume that hypothesis (H1)-(H4) hold. If* $\xi_0 \in C(D)$*, then for every* $t \in [0,T]$ *and* $x \in D$ *the image law of the random variable* $\xi(t,x)$ *is absolutely continuous w.r.t. to the Lebesgue measure on* $\mathbb{R}$.

We can use the framework of the Malliavin calculus in the setting introduced in Section I.1.3.1, namely the underlying Gaussian space on which to perform Malliavin calculus is given by the isonormal Gaussian process on the Hilbert space $\mathcal{H}_T$. The basic facts about Malliavin calculus used in this Chapter are recalled in Appendix B. We use these results for the random variable $\xi(t,x)$, solution to equation (I.1.6.4) and the random variable $\xi_N(t,x)$ solution to equation (I.1.6.13). We appeal to the Bouleau-Hirsch criterium (Proposition B.3.1). More precisely, by means of Proposition B.3.3, in Section B.3, we show that $\xi_N(t,x) \in \mathbb{D}^{1,p}$; hence $\xi(t,x) \in \mathbb{D}^{1,p}_{\text{loc}}$. In Section I.2.3 we prove that $\xi_N(t,x)$ satisfies assumption (B.3.1) of Theorem B.3.1. The same condition holds for $\xi(t,x)$ as we shall see in Section I.2.4.

The present Chapter is organized as follows. In Section I.2.2 we show that, for every $n \in \mathbb{N}$, for every $(t,x) \in [0,T] \times D$, $\xi_N(t,x) \in \mathbb{D}^{1,p}$, $p > 4$. In Section I.2.3 we prove that $\xi_N(t,x)$ satisfies assumption (B.3.1) of the Bouleau-Hirsch criterium. In Section I.2.4 we construct a sequence $\{\Omega_N\}_{N\geq 1}$ such that (Ω_N, ξ_N) localizes $\xi(t,x)$ in $\mathbb{D}^{1,p}$. From that we infer the existence of a density for the image law of $\xi(t,x)$. Finally, Section I.2.5 is devoted to a brief overview on the results available in literature concerning the regularity in Malliavin sense for solutions to SPDEs.

I.2.2 Malliavin analysis of the truncated equation

In order to show that $\xi_N(t,x) \in \mathbb{D}^{1,p}$ we use Proposition B.3.3. We introduce a Picard approximation sequence $\{\xi_N^k\}_k$ for ξ_N and we show that as $k \to +\infty$, the sequence $\xi_N^k(t,x)$ converges to $\xi_N(t,x)$ in $L^p(\Omega)$ (for $N \geq 1$ fixed) and $\sup_k \|\xi_N^k(t,x)\|_{1,p} < \infty$ uniformly in $(t,x) \in [0,T] \times D$. A similar argument has been used in [20] for the Cahn-Hilliard stochastic equation and in [66] for the one dimensional Burgers equation. Let us point out that the smoothness of the density cannot be obtained via this location argument, since this procedure does not provide the boundedness of the Malliavin derivatives of every order.

First, we need to improve the result of Proposition I.1.6.4. This is done in the following theorem, whose proof provides the approximating sequence $\{\xi_N^k\}_k$ of the Picard scheme. We shall need a variation on Gronwall classical lemma (see [27, Lemma 15]), that we recall here for the sake of completeness.

Lemma I.2.2.1. ***(Extension of Gronwall's Lemma).*** *Let $g : [0,T] \to \mathbb{R}_+$ be a non-negative function such that*

$$\int_0^T g(s)\,\mathrm{d}s < \infty.$$

Then there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ of non-negative real numbers such that $\sum_{n=1}^{\infty} a_n < \infty$ with the following property. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative functions on $[0,T]$ and k_1, k_2 be non-negative numbers such that for $0 \leq t \leq T$,

$$f_n(t) \leq k_1 + \int_0^t (k_2 + f_{n-1}(s))g(t-s)\,\mathrm{d}s.$$

If $\sup_{0\leq s\leq T} f_0(s) = M$, then for $n \geq 1$,

$$f_n(t) \leq k_1 + (k_1 + k_2)\sum_{i=1}^{n-1} a_i + (k_2 + M)a_n.$$

In particular, $\sup_{n\geq 0}\sup_{0\leq t\leq T} f_n(t) < \infty$, and if $k_1 = k_2 = 0$, then $\sum_{n\geq 0} f_n(t)$ converges uniformly on $[0,T]$.

Theorem I.2.2.2. *Fix $N \geq 1$ and $p > 4$. Let assume that Hypothesis (H1) and (H2) hold. If $b > 0$ in* (I.1.3.3) *and ξ_0 is a continuous function on D, then the solution process ξ_N to* (I.1.6.13) *is $L^p(\Omega)$-continuous and satisfies*

$$\sup_{0\leq t\leq T}\sup_{x\in D}\mathbb{E}|\xi_N(t,x)|^p < \infty. \tag{I.2.2.1}$$

Proof. We shall follow a standard Picard iteration scheme. For every $(t,x) \in [0,T]\times D$, let us define

$$\xi_N^0(t,x) = \int_D \xi_0(y)g(t,x,y)\,\mathrm{d}y \tag{I.2.2.2}$$

and recursively for $k \geq 0$, assuming that $\xi_N^k(t,x)$ has been defined,

$$\xi_N^{k+1}(t,x) = \xi_N^0(t,x) + (Jq_N(\xi_N^k))(t,x) + (\mathcal{A}\xi_N^k)(t,x) \tag{I.2.2.3}$$

with $Jq_N(\xi_N^k)$ and $\mathcal{A}$, defined respectively in in (I.1.5.1) and (I.1.6.15). Assume by induction that for any $T > 0$,

$$\sup_{0\leq t\leq T}\sup_{x\in D}\mathbb{E}|\xi_N^k(t,x)|^p < \infty, \tag{I.2.2.4}$$

that $\xi_N^k(t,x)$ is $\mathcal{F}_t$-measurable for all $x \in D$ and $0 \leq t \leq T$ and that $(t,x) \to \xi_N^k(t,x)$ is $L^p(\Omega)$-continuous for $p > 4$. By Remark I.2.2.3 we see that $(t,x;\omega) \to \xi_N^k(t,x;\omega)$ has a jointly measurable version. Then the stochastic convolution term $\mathcal{A}\xi_N^k$ appearing in (I.2.2.3) is well defined. Its well posedness follows from Lemma I.1.3.4 thanks to Proposition I.1.4.1, the linear growth condition of σ, (I.2.2.4) and the inductive step. On the other hand (I.1.5.4), (I.1.6.8) and Proposition I.1.6.4 provide the well posedness of the non linear term $Jq_N(\xi_N^k)$. It follows that $\xi_N^{k+1}(t,x)$ is well defined and by (I.1.6.8) and Proposition I.1.4.1, the linear growth condition of the coefficient σ and (I.2.2.4), it holds

$$\sup_{0\leq t\leq T}\sup_{x\in D}\mathbb{E}|\xi_N^{k+1}(t,x)|^p < \infty.$$

We first prove that for $T > 0$ and $p > 4$,

$$\sup_{k\geq 0}\sup_{0\leq t\leq T}\sup_{x\in D}\mathbb{E}|\xi_N^k(t,x)|^p < \infty. \tag{I.2.2.5}$$

It holds

$$\mathbb{E}|\xi_N^{k+1}(t,x)|^p \leq C_p\left(\mathbb{E}|\xi_N^0(t,x)|^p + \mathbb{E}|Jq_N(\xi_N^k))(t,x)|^p + \mathbb{E}|\mathcal{A}(\xi_N^k)(t,x)|^p\right).$$

For every $(t,x) \in [0,T]\times D$, from Lemma I.1.5.1*(ii)* (for $\gamma = p$, provided $p > 4$) and (I.1.6.8) we get

$$\mathbb{E}|(J(q_N(\xi_N^k))(t,x)|^p \leq C_{N,T,p}. \tag{I.2.2.6}$$

By (I.1.3.4) and (I.1.4.2) it follows

$$\begin{aligned}\mathbb{E}|(\mathcal{A}\xi_N^k)(t,x)|^p &\leq \|g(t-\cdot,x,\cdot)\|_{\mathcal{H}_t}^{p-2}\int_0^t \sup_{y\in D}\mathbb{E}|\xi_N^k(s,y)|^p\ \|g(t-s,x,\cdot)\|_{L_Q^2}^2\,\mathrm{d}s\\ &\leq C_{p,T}\int_0^t \sup_{y\in D}\mathbb{E}|\xi_N^k(s,y)|^p\ \|g(t-s,x,\cdot)\|_{L_Q^2}^2\,\mathrm{d}s.\end{aligned}$$

and the above estimates is uniform in x thanks to (I.1.3.4). Since

$$\sup_{x\in D}\mathbb{E}|\xi_N^0(t,x)|^p=\sup_{x\in D}\left|\int_D\xi_0(y)g(t,x,y)\,\mathrm{d}y\right|^p\le\sup_{x\in D}[\|\xi_0\|_{L^\infty}\|g(t-s,\cdot,y)\|_{L^1}]\le\|\xi_0\|_{L^\infty}<\infty,$$

we get

$$\sup_{x\in D}\mathbb{E}|\xi_N^{k+1}(t,x)|^p\le(\|\xi_0\|_{L^\infty}+C_{p,T,N})+C_{p,T}\int_0^t\sup_{y\in D}\mathbb{E}|\xi_N^k(s,y)|^p\|g(t-s,x,\cdot)\|^2_{L^2_Q}\,\mathrm{d}s.$$

Setting

$$f_k(t):=\sup_{x\in D}\mathbb{E}|\xi_N^k(t,x)|^p,$$

since $\sup_{0\le s\le T}f_0(s)<\infty$, we conclude by Lemma I.2.2.1 that (I.2.2.5) holds. In order to conclude that the sequence $\{\xi_N^k(t,x)\}_{k\ge0}$ converges in L^p let us set

$$\varphi_k(t):=\sup_{x\in D}\mathbb{E}|\xi_N^{k+1}(s,x)-\xi_N^k(s,x)|^p.$$

Proceeding as above, by Hypothesis (H1) and Lemma I.1.6.2, we infer

$$\varphi_k(t)\le C_{p,T,N,|D|}\int_0^t\varphi_{k-1}(s)\left(1+\|g(t-s,x,\cdot)\|^2_{L^2_Q}\right)\mathrm{d}s. \tag{I.2.2.7}$$

By Hypothesis (H2), for the same considerations made for the well posedness of (I.2.2.3), $\sup_{0\le s\le T}\varphi_0(s)<\infty$. By Lemma I.2.2.1 we conclude that $\sum_{k\ge0}\varphi_k(t)$ converges uniformly on $[0,T]$. It follows that $\varphi_k(t)\to0$ uniformly on $[0,T]$, as k tends to infinity. That means that the sequence $\xi_N^k(t,x)$ converges in $L^p(\Omega)$, uniformly in $(t,x)\in[0,T]\times D$, to a limit that we denote $\xi_N(t,x)$. In order to check that ξ_N has a jointly measurable version, we have to show that it is continuous in $L^2(\Omega)$ as pointed out in Remark I.2.2.3. Once this is done it is easy to verify that, by construction, the process $\xi_N=\{\xi_N(t,x),0\le t\le T,x\in D\}$ is the solution to (I.1.6.13) and satisfies (I.2.2.1). Since the convergence of ξ_N^k to ξ_N is uniform in L^p, it is sufficient to show that each ξ_N^k is $L^p(\Omega)$-continuous, for $p>4$ (and so in particular $L^2(\Omega)$-continuous). This is proved in Lemma I.2.2.4. Uniqueness of the solution to (I.1.6.13) is checked by a standard argument: the same calculations that lead to (I.2.2.7) shall be employed. □

Remark I.2.2.3. *Given a process $X=\{X(t,x),0\le t\le T,x\in D\}$, this has a jointly measurable version if the map $(t,x)\to X(t,x)$ from $[0,T]\times D$ into the space of random variables is continuous in probability (see [33, Chapter IV, Theorem 30]). If we deal with a Gaussian process this is equivalent to show that the map $(t,x)\to X(t,x)$ is $L^2(\Omega)$-continuous.*

Lemma I.2.2.4. *Under the same assumptions of Theorem I.2.2.2, each of the processes $\xi_N^k=\{\xi_N^k(t,x),0\le t\le T,x\in D\}$ defined in the proof of that Theorem is $L^p(\Omega)$-continuous for $p>4$.*

Proof. Fix $N\ge1$ and $k\ge0$, assume by induction that ξ_N^k is $L^p(\Omega)$-continuous and (I.2.2.4) holds. Let us begin with time increments. For $0\le t\le T$, $x\in D$, $h>0$, from (I.2.2.3) it follows

$$\mathbb{E}|\xi_N^{k+1}(t+h,x)-\xi_N^{k+1}(t,x)|^p\le c_p(A_1^{(h)}+A_2^{(h)}),$$

where

$$A_1^{(h)} = \mathbb{E}\left|\int_0^{t+h}\int_D g(t+h-s,x,y)\sigma(\xi_N^k(s,y))\,w(\mathrm{d}s,\mathrm{d}y) - \int_0^t\int_D g(t-s,x,y)\sigma(\xi_N^k(s,y))\,w(\mathrm{d}s,\mathrm{d}y)\right|^p$$

and

$$A_2^{(h)} = \mathbb{E}\left|\int_0^{t+h}\int_D \nabla_y g(t+h-s,x,y)\cdot q_N(\xi_N^k(s,\cdot))(y)\,\mathrm{d}y\,\mathrm{d}s - \int_0^t\int_D \nabla_y g(t-s,x,y)\cdot q_N(\xi_N^k(s,\cdot))(y)\,\mathrm{d}y\,\mathrm{d}s\right|^p.$$

By Remark I.1.4.3 it follows

$$\begin{aligned}
A_1^{(h)} &\le c_p\mathbb{E}\left|\int_0^t\int_D [g(t+h-s,x,y)-g(t-s,x,y)]\,\sigma(\xi_N^k(s,y))\,w(\mathrm{d}s,\mathrm{d}y)\right|^p \\
&\quad + c_p\mathbb{E}\left|\int_t^{t+h}\int_D g(t+h-s,x,y)\sigma(\xi_N^k(s,y))\,w(\mathrm{d}s,\mathrm{d}y)\right|^p \\
&\le C_{T,p}\sup_{0\le t\le T}\sup_{x\in D}\mathbb{E}|\sigma(\xi_N^k(t,x))|^p \\
&\quad \times\left(\|g(t+h-\cdot,x,\cdot)-g(t-\cdot,x,\cdot)\|_{\mathcal{H}_t}^p + \|g(t+h-\cdot,x,\cdot)\|_{\mathcal{H}_{(t,t+h)}}^p\right) \\
&= C_{T,p}\sup_{0\le t\le T}\sup_{x\in D}\mathbb{E}|\sigma(\xi_N^k(t,x))|^p\left(\left[\int_0^t\|g(t+h-s,x,\cdot)-g(t-s,x,\cdot)\|_{L_Q^2}^2\,\mathrm{d}s\right]^{\frac{p}{2}}\right. \\
&\quad \left.+\left[\int_t^{t+h}\|g(t+h-s,x,\cdot)\|_{L_Q^2}^2\,\mathrm{d}s\right]^{\frac{p}{2}}\right).
\end{aligned}$$

Let us set

$$A_{11}^{(h)} = \int_0^t\|g(t+h-s,x,\cdot)-g(t-s,x,\cdot)\|_{L_Q^2}^2\,\mathrm{d}s$$

and

$$A_{12}^{(h)} = \int_t^{t+h}\|g(t+h-s,x,\cdot)\|_{L_Q^2}^2\,\mathrm{d}s$$

By (I.1.3.4) we have

$$\begin{aligned}
A_{11}^{(h)} &= \sum_{k\in\mathbb{Z}_0^2}|k|^{-2b}\int_0^t\left|e^{-|k|^2(t+h-s)}e_k(x)-e^{-|k|^2(t-s)}e_k(x)\right|^2\mathrm{d}s \\
&= \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}_0^2}|k|^{-2b}\left(e^{-|k|^2h}-1\right)^2\int_0^t e^{-2|k|^2(t-s)}\,\mathrm{d}s \\
&\le \frac{1}{2(2\pi)^2}\sum_{k\in\mathbb{Z}_0^2}|k|^{-2b-2}\left(e^{-|k|^2h}-1\right)^2\to 0\qquad\text{for } h\to 0,
\end{aligned}$$

provided $b > 0$, by the Dominated Convergence Theorem. Similarly,

$$\begin{aligned} A_{12}^{(h)} &= \frac{1}{(2\pi)^2} \sum_{k \in \mathbb{Z}_0^2} |k|^{-2b} \int_t^{t+h} e^{-2|k|^2(t+h-s)}\,\mathrm{d}s \\ &= \frac{1}{2(2\pi)^2} \sum_{k \in \mathbb{Z}_0^2} |k|^{-2b-2} \left(1 - e^{-2|k|^2 h}\right) \to 0 \qquad \text{for } h \to 0, \end{aligned}$$

provided $b > 0$. Then we conclude that $A_1^{(h)} \to 0$ for $h \to 0$.

Concerning $A_2^{(h)}$, observe that

$$A_2^{(h)} \le c_p (A_{21}^{(h)} + A_{22}^{(h)})$$

where

$$A_{21}^{(h)} = \mathbb{E}\left| \int_0^t \int_D [\nabla_y g(t+h-s,x,y) - \nabla_y g(t-s,x,y)] \cdot q_N(\xi_N^k(s,\cdot))(y)\,\mathrm{d}y\,\mathrm{d}s \right|^p$$

and

$$A_{22}^{(h)} = \mathbb{E}\left| \int_t^{t+h} \int_D \nabla_y g(t+h-s,x,y) \cdot q_N(\xi_N^k(s,\cdot))(y)\,\mathrm{d}y\,\mathrm{d}s \right|^p.$$

From Hölder's inequality, by (I.1.6.8) it follows

$$\begin{aligned} A_{21}^{(h)} &\le c_{N,p} \int_0^t \int_D \mathbb{E}|q_N(\xi_N^k(s,y))|^p\,\mathrm{d}y\,\mathrm{d}s \\ &\quad \times \left(\int_0^t \int_D |\nabla_y g(t+h-s,x,y) - \nabla_y g(t-s,x,y)|^{\frac{p}{p-1}}\,\mathrm{d}y\,\mathrm{d}s \right)^{p-1} \\ &\le c_{N,p,T} \left(\int_0^t \int_D |\nabla_y g(t+h-s,x,y) - \nabla_y g(t-s,x,y)|^{\frac{p}{p-1}}\,\mathrm{d}y\,\mathrm{d}s \right)^{p-1}, \end{aligned}$$

for a $p > 4$. By the Dominated Convergence Theorem, (I.1.2.9) and the fact that $(t,x) \to g(\cdot - s, \cdot, y)$ is $C^\infty([0,T] \times D)$,

$$\int_0^t \int_D |\nabla_y g(t+h-s,x,y) - \nabla_y g(t-s,x,y)|^{\frac{p}{p-1}}\,\mathrm{d}y\,\mathrm{d}s \to 0 \qquad \text{for } h \to 0.$$

Then it follows that $A_{21}^{(h)} \to 0$ as $h \to 0$. In a similar way we prove that $A_{21}^{(h)} \to 0$ as $h \to 0$ and so we conclude that $A_2^{(h)} \to 0$ as $h \to 0$, proving in this way the time continuity of $\xi_N^k(t,x)$ for a fixed $x \in D$.

We now consider space increments. For $0 \le t \le T$, $x \in D$, $h \in \mathbb{R}^2$, from (I.2.2.3) it follows

$$\mathbb{E}|\xi_N^{k+1}(t,x+h) - \xi_N^{k+1}(t,x)|^p \le c_p (B_1^{(h)} + B_2^{(h)}),$$

where

$$B_1^{(h)} = \mathbb{E}\left| \int_0^t \int_D [g(t-s,x+h,y) - g(t-s,x,y)]\, \sigma(\xi_N^k(s,y))\, w(\mathrm{d}s,\mathrm{d}y) \right|^p$$

and

$$B_2^{(h)} = \mathbb{E}\left|\int_0^t \int_D [\nabla_y g(t-s,x+h,y) - \nabla_y g(t-s,x,y)] \cdot q_N(\xi_N^k(s,\cdot))(y)\,\mathrm{d}y\,\mathrm{d}s\right|^p.$$

By Proposition I.1.4.1 and Hölder's inequality it follows

$$\begin{aligned} B_1^{(h)} &\le C_{T,p} \sup_{0\le t\le T} \sup_{x\in D} \mathbb{E}|\sigma(\xi_N^k(t,x))|^p \|g(t-\cdot,x+h,\cdot) - g(t-\cdot,x,\cdot)\|_{\mathcal{H}_t}^p \\ &= C_{T,p} \sup_{0\le t\le T} \sup_{x\in D} \mathbb{E}|\sigma(\xi_N^k(t,x))|^p \left[\int_0^t \|g(t-s,x+h,\cdot) - g(t-s,x,\cdot)\|_{L_Q^2}^2\,\mathrm{d}s\right]^{\frac{p}{2}}. \end{aligned}$$

Let us set

$$B_{11}^{(h)} = \int_0^t \|g(t-s,x+h,\cdot) - g(t-s,x,\cdot)\|_{L_Q^2}^2\,\mathrm{d}s$$

By (I.1.3.4) we have

$$\begin{aligned} B_{11}^{(h)} &= \sum_{k\in\mathbb{Z}_0^2} |k|^{-2b} \int_0^t \left|e^{-|k|^2(t-s)} e_k(x+h) - e^{-|k|^2(t-s)} e_k(x)\right|^2 \mathrm{d}s \\ &= \sum_{k\in\mathbb{Z}_0^2} |k|^{-2b} |e_k(x+h) - e_k(x)|^2 \int_0^t e^{-2|k|^2(t-s)}\,\mathrm{d}s \\ &= \frac{1}{(2\pi)^2} \sum_{k\in\mathbb{Z}_0^2} |k|^{-2b} \left|e^{ik\cdot(x+h)} - e^{ik\cdot x}\right|^2 \int_0^t e^{-2|k|^2(t-s)}\,\mathrm{d}s \\ &= \frac{1}{2(2\pi)^2} \sum_{k\in\mathbb{Z}_0^2} |k|^{-2b-2} \left(1 - e^{-2|k|^2 t}\right) \left|e^{ik\cdot(x+h)} - e^{ik\cdot x}\right|^2 \\ &\le \frac{1}{2(2\pi)^2} \sum_{k\in\mathbb{Z}_0^2} |k|^{-2b-2} \left|e^{ik\cdot h} - 1\right|^2 \to 0 \qquad \text{for } |h|\to 0, \end{aligned}$$

provided $b > 0$, by the Dominated Convergence Theorem. Then we conclude that $B_1^{(h)} \to 0$ for $|h| \to 0$. As regards the term $B_2^{(h)}$ proceeding similarly to $A_{21}^{(h)}$ we can show that it tends to zero as $|h| \to 0$, provided $p > 4$. □

Let us study the Malliavin derivative of the solution ξ_N to the smoothed equation (I.1.6.13). Let us recall that the underlying Gaussian space on which to perform Malliavin calculus is given by the isonormal Gaussian process on the Hilbert space $\mathcal{H}_T := L^2(0,T;L_Q^2)$ which can be associated to the noise coloured in space by the covariance Q. As pointed out in Remark B.2.4, for a random variable X, differentiable in the Malliavin sense, we shall use the notation $D_{r,\varphi}X = \langle D_{r,\bullet}X, \varphi\rangle_{L_Q^2}$, $r \in [0,T]$, $\varphi \in L_Q^2$.

In this part, to keep things as simple as possible, in some points we go back to the notation involving ξ_N and v_N instead of $q_N(\xi_N)$, with $v_N = k * \xi_N$. Keeping in mind the definition of $\tilde{q}_N(\xi)$ given (I.1.6.7) we state the following result.

Theorem I.2.2.5. *Fix $N \ge 1$. Let us assume that Hypothesis (H1)-(H3) hold. Suppose that $b > 0$ in* (I.1.3.3) *and ξ_0 is a continuous function on D. Then for all $(t,x) \in [0,T] \times D$*

the solution $\xi_N(t,x)$ *to* (I.1.6.13) *belongs to* $\mathbb{D}^{1,p}$ *for every* $p>4$ *and its Malliavin derivative satisfies the equation*

$$\begin{aligned}
&D_{r,\varphi}\xi_N(t,x) = \langle g(t-r,x,\bullet)\mathbf{1}_{[0,t]}(r)\sigma(\xi_N(r,\bullet)),\varphi\rangle_{L^2_Q}\\
&+\int_r^t\int_D \nabla_y g(t-s,x,y)\cdot\ v_N(s,y)\Theta_N(\|\xi_N(s,\cdot)\|_{L^p})\ D_{r,\varphi}\xi_N(s,y)\mathrm{d}y\,\mathrm{d}s\\
&+\int_r^t\int_D\left(\nabla_y g(t-s,x,y)\cdot\int_D k(y-\alpha)D_{r,\varphi}\xi_N(s,\alpha)\,\mathrm{d}\alpha\right)\\
&\qquad\Theta_N(\|\xi_N(s,\cdot)\|_{L^p})\xi_N(s,y)\,\mathrm{d}y\,\mathrm{d}s\\
&+\int_r^t\int_D \nabla_y g(t-s,x,y)\cdot\ \tilde q_N(\xi_N(s,\cdot))(y)\\
&\qquad\|\xi_N(s,\cdot)\|_{L^p}^{1-p}\left(\int_D|\xi_N(s,\beta)|^{p-2}\xi_N(s,\beta)D_{r,\varphi}\xi_N(s,\beta)\,\mathrm{d}\beta\right)\mathrm{d}y\,\mathrm{d}s\\
&+\int_r^t\int_D g(t-s,x,y)\sigma'(\xi_N(s,y))D_{r,\varphi}\xi_N(s,y)\,w(\mathrm{d}y,\mathrm{d}s)
\end{aligned}\tag{I.2.2.8}$$

if $r\le t$, *and* $D_{r,\varphi}\xi_N(t,x)=0$ *if* $r>t$.

Proof. The proof of this part is based on Proposition B.3.3. Let us consider the Picard approximation sequence $\left\{\xi_N^k(t,x)\right\}_k$ defined in (I.2.2.2)-(I.2.2.3); given the convergence (as $k\to+\infty$) obtained in the proof of Theorem I.2.2.2, it is sufficient to show that

$$\sup_k\sup_{0\le t\le T}\sup_{x\in D}\mathbb{E}\|D\xi_N^k(t,x)\|^p_{\mathcal{H}_T}<+\infty,\tag{I.2.2.9}$$

in order to prove that $\xi_N(t,x)\in\mathbb{D}^{1,p}$. Since ξ_N^0 is deterministic, it belongs to $\mathbb{D}^{1,p}$ and its Malliavin derivative is zero. Let us suppose that, for $k\ge 1$ and $p>4$, $\xi_N^k(t,x)\in\mathbb{D}^{1,p}$ for every $(t,x)\in[0,T]\times D$ and

$$\sup_{0\le t\le T}\sup_{x\in D}\mathbb{E}\|D\xi_N^k(t,x)\|^p_{\mathcal{H}_T}<\infty.$$

Applying the operator D to equation (I.2.2.3) we obtain that the Malliavin derivative of $\xi_N^k(t,x)$ satisfies the equation (for more details see for instance [23, Proposition 2.15 and Proposition 2.16] and [72, Proposition 1.3.2])

$$\begin{aligned}
&D_{r,\varphi}\xi_N^{k+1}(t,x) = \langle g(t-r,x,\bullet)\mathbf{1}_{[0,t]}(r)\sigma(\xi_N^k(r,\bullet)),\varphi\rangle_{L^2_Q}\\
&+\int_r^t\int_D \nabla_y g(t-s,x,y)\cdot\ v_N^k(s,y)\Theta_N(\|\xi_N^k(s,\cdot)\|_{L^p})\ D_{r,\varphi}\xi_N^k(s,y)\mathrm{d}y\,\mathrm{d}s\\
&+\int_r^t\int_D\left(\nabla_y g(t-s,x,y)\cdot\int_D k(y-\alpha)D_{r,\varphi}\xi_N^k(s,\alpha)\,\mathrm{d}\alpha\right)\\
&\qquad\Theta_N(\|\xi_N^k(s,\cdot)\|_{L^p})\xi_N^k(s,y)\,\mathrm{d}y\,\mathrm{d}s\\
&+\int_r^t\int_D \nabla_y g(t-s,x,y)\cdot\ \tilde q_N(\xi_N^k(s,\cdot))(y)\\
&\qquad\|\xi_N^k(s,\cdot)\|_{L^p}^{1-p}\left(\int_D|\xi_N^k(s,\beta)|^{p-2}\xi_N^k(s,\beta)D_{r,\varphi}\xi_N^k(s,\beta)\,\mathrm{d}\beta\right)\mathrm{d}y\,\mathrm{d}s\\
&+\int_0^t\int_D g(t-s,x,y)\sigma'(\xi_N^k(s,y))D_{r,\varphi}\xi_N^k(s,y)\,w(\mathrm{d}y,\mathrm{d}s).
\end{aligned}\tag{I.2.2.10}$$

Let us set for simplicity

$$I_1(r,\varphi) := \int_r^t \int_D \nabla_y g(t-s,x,y)\cdot\ v_N^k(s,y)\Theta_N(\|\xi_N^k(s,\cdot)\|_{L^p})\ D_{r,\varphi}\xi_N^k(s,y)\mathrm{d}y\,\mathrm{d}s \quad \text{(I.2.2.11)}$$

$$I_2(r,\varphi) := \int_r^t \int_D \left(\nabla_y g(t-s,x,y)\cdot \int_D k(y-\alpha)D_{r,\varphi}\xi_N^k(s,\alpha)\,\mathrm{d}\alpha\right) \Theta_N(\|\xi_N^k(s,\cdot)\|_{L^p})\xi_N^k(s,y)\,\mathrm{d}y\,\mathrm{d}s \quad \text{(I.2.2.12)}$$

$$I_3(r,\varphi) := \int_r^t \int_D \nabla_y g(t-s,x,y)\cdot\ \tilde{q}_N(\xi_N^k(s,\cdot))(y) \left(\|\xi_N^k(s,\cdot)\|_{L^p}^{1-p}\int_D |\xi_N^k(s,\beta)|^{p-2}\xi_N^k(s,\beta)D_{r,\varphi}\xi_N^k(s,\beta)\,\mathrm{d}\beta\right)\mathrm{d}y\,\mathrm{d}s. \quad \text{(I.2.2.13)}$$

$$I_4(r,\varphi) := \int_0^t \int_D g(t-s,x,y)\sigma'(\xi_N^k(s,y))D_{r,\varphi}\xi_N^k(s,y)\,w(\mathrm{d}y,\mathrm{d}s) \quad \text{(I.2.2.14)}$$

Then

$$\mathbb{E}\|D\xi_N^{k+1}(t,x)\|_{\mathcal{H}_T}^p \le C_p\left(\|g(t-\cdot,x,\bullet)\mathbf{1}_{[0,t]}(\cdot)\sigma(\xi_N^k(\cdot,\bullet))\|_{\mathcal{H}_T}^p + \sum_{i=1}^4 \mathbb{E}\|I_i\|_{\mathcal{H}_T}^p\right). \quad \text{(I.2.2.15)}$$

Let us estimate the various terms in (I.2.2.15).

Minkowski's and Hölder's inequalities imply that

$$\begin{aligned}
\mathbb{E}\|I_1\|_{\mathcal{H}_T}^p &\le \mathbb{E}\left[\int_0^t\int_D \left|\nabla_y g(t-s,x,y)\cdot\ v_N^k(s,y)\Theta_N(\|\xi_N^k(s,\cdot)\|_{L^p})\right|\ \|D\xi_N^k(s,y)\|_{\mathcal{H}_T}\mathrm{d}y\mathrm{d}s\right]^p\\
&\le \mathbb{E}\left[\int_0^t |\Theta_N(\|\xi_N^k(s,\cdot)\|_{L^p})|\|\nabla_y g(t-s,x,\cdot)\|_{L^{\frac{p}{p-1}}} \left(\int_D |v_N^k(s,y)|^p\ \|D\xi_N^k(s,y)\|_{\mathcal{H}_T}^p\mathrm{d}y\right)^{\frac{1}{p}}\mathrm{d}s\right]^p\\
&\le \mathbb{E}\left[\int_0^t |\Theta_N(\|\xi_N^k(s,\cdot)\|_{L^p})|\|\nabla_y g(t-s,x,\cdot)\|_{L^{\frac{p}{p-1}}} \|v_N^k(s,\cdot)\|_{L_\infty}\ \|D\xi_N^k(s,\cdot)\|_{L^p(D;\mathcal{H}_T)}\,\mathrm{d}s\right]^p\\
&\le C_N\left(\int_0^t\int_D |\nabla_y g(t-s,x,y)|^{\frac{p}{p-1}}\,\mathrm{d}y\,\mathrm{d}s\right)^{p-1} \mathbb{E}\left[\int_0^t \|D\xi_N^k(s,\cdot)\|_{L^p(D;\mathcal{H}_T)}^p\,\mathrm{d}s\right] \qquad \text{by (I.1.2.26)}\\
&\le C_N t^{\frac{p}{2}-2}\int_0^t\int_D \mathbb{E}\|D\xi_N^k(s,y)\|_{\mathcal{H}_T}^p\,\mathrm{d}y\,\mathrm{d}s \qquad \text{by (I.1.2.9) provided } p>4\\
&\le C_{N,p,T,|D|}\int_0^t \sup_{y\in D}\mathbb{E}\|D\xi_N^k(s,y)\|_{\mathcal{H}_T}^p\,\mathrm{d}s.
\end{aligned}$$

As regards the term I_2 using Fubini's Theorem, Minkowski's and Hölder's inequalities we have

$$\begin{aligned}
\mathbb{E}\|I_2\|^p_{\mathcal{H}_T} &= \mathbb{E}\left\|\int_r^t \int_D \left(\int_D \nabla_y g(t-s,x,y)\cdot k(y-\alpha)\xi^k_N(s,y)\,\mathrm{d}y\right)\right.\\
&\qquad\left.\Theta_N(\|\xi^k_N(s,\cdot)\|_{L^p})D\xi^k_N(s,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}s\right\|^p_{\mathcal{H}_T}\\
&\le \mathbb{E}\left[\int_0^t \int_D \left|\left(\int_D \nabla_y g(t-s,x,y)\cdot k(y-\alpha)\xi^k_N(s,y)\,\mathrm{d}y\right)\right.\right.\\
&\qquad\left.\left.\Theta_N(\|\xi^k_N(s,\cdot)\|_{L^p})\right|\,\|D\xi^k_N(s,\alpha)\|_{\mathcal{H}_T}\,\mathrm{d}\alpha\,\mathrm{d}s\right]^p\\
&\le \mathbb{E}\left[\int_0^t \int_D \|\nabla_y g(t-s,x,\cdot)\cdot k(\cdot-\alpha)\|_{L^{\frac{p}{p-1}}}\|\xi^k_N(s,\cdot)\|_{L^p}\right.\\
&\qquad\left.|\Theta_N(\|\xi^k_N(s,\cdot)\|_{L^p})|\;\|D\xi^k_N(s,\alpha)\|_{\mathcal{H}_T}\,\mathrm{d}\alpha\,\mathrm{d}s\right]^p\\
&\le C_N\mathbb{E}\left[\int_0^t \int_D \|\nabla_y g(t-s,x,\cdot)\cdot k(\cdot-\alpha)\|_{L^{\frac{p}{p-1}}}\|D\xi^k_N(s,\alpha)\|_{\mathcal{H}_T}\,\mathrm{d}\alpha\,\mathrm{d}s\right]^p\\
&\le C_N\mathbb{E}\left[\int_0^t \left(\int_D\int_D |\nabla_y g(t-s,x,y)\cdot k(y-\alpha)|^{\frac{p}{p-1}}\,\mathrm{d}y\,\mathrm{d}\alpha\right)^{\frac{p-1}{p}}\right.\\
&\qquad\left.\|D\xi^k_N(s,\cdot)\|_{L^p(D;\mathcal{H}_T)}\,\mathrm{d}s\right]^p.
\end{aligned}$$

By means of Fubini's Theorem, if $p>4$, we can estimate the inner integral

$$\begin{aligned}
&\int_D\int_D |\nabla_y g(t-s,x,y)\cdot k(y-\alpha)|^{\frac{p}{p-1}}\,\mathrm{d}y\,\mathrm{d}\alpha \qquad\text{(I.2.2.16)}\\
&\le \int_D\int_D |\nabla_y g(t-s,x,y)|^{\frac{p}{p-1}}\;|k(y-\alpha)|^{\frac{p}{p-1}}\,\mathrm{d}y\,\mathrm{d}\alpha\\
&= \int_D |\nabla_y g(t-s,x,y)|^{\frac{p}{p-1}}\;\left(\int_D |k(y-\alpha)|^{\frac{p}{p-1}}\,\mathrm{d}\alpha\right)\,\mathrm{d}y\\
&\le C\int_D |\nabla_y g(t-s,x,y)|^{\frac{p}{p-1}}\,\mathrm{d}y \qquad \text{by Lemma I.1.2.7 and Remark I.1.2.8}\\
&\le C_p(t-s)^{-\frac{3}{2}\left(\frac{p}{p-1}\right)+1} \qquad \text{by (I.1.2.8),} \qquad\text{(I.2.2.17)}
\end{aligned}$$

obtaining

$$\begin{aligned}
\mathbb{E}\|I_2\|^p_{\mathcal{H}_T} &\le C_{N,p}\mathbb{E}\left[\int_0^t (t-s)^{-\frac{p+2}{2p}}\|D\xi^k_N(s,\cdot)\|_{L^p(D;\mathcal{H}_T)}\,\mathrm{d}s\right]^p\\
&\le C_{N,p}\left(\int_0^t (t-s)^{\frac{p+2}{2(1-p)}}\,\mathrm{d}s\right)^{p-1}\mathbb{E}\left[\int_0^t \|D\xi^k_N(s,\cdot)\|^p_{L^p(D;\mathcal{H}_T)}\,\mathrm{d}s\right]\\
&\le C_{N,p}t^{\frac{p}{2}-2}\int_0^t\int_D \mathbb{E}\|D\xi^k_N(s,y)\|^p_{\mathcal{H}_T}\,\mathrm{d}y\,\mathrm{d}s\\
&\le C_{N,p,T,|D|}\int_0^t \sup_{y\in D}\mathbb{E}\|D\xi^k_N(s,y)\|^p_{\mathcal{H}_T}\,\mathrm{d}s,
\end{aligned}$$

provided $p > 4$.
As regards the term I_3, using as above Minkowski's and Hölder's inequalities, we have

$$\begin{aligned}\mathbb{E}\|I_3\|^p_{\mathcal{H}_T} &= \mathbb{E}\bigg\|\int_r^t \int_D \nabla_y g(t-s,x,y)\cdot \tilde{q}_N(\xi^k_N)(s,\cdot))(y) \\ &\qquad \|\xi^k_N(s,\cdot)\|^{1-p}_{L^p}\left(\int_D |\xi^k_N(s,\beta)|^{p-2}\xi^k_N(s,\beta)D\xi^k_N(s,\beta)\,\mathrm{d}\beta\right)\mathrm{d}y\,\mathrm{d}s\bigg\|^p_{\mathcal{H}_T} \\ &\le \mathbb{E}\bigg[\int_0^t \int_D |\nabla_y g(t-s,x,y)\cdot \tilde{q}_N(\xi^k_N)(s,\cdot))(y)| \\ &\qquad \|\xi^k_N(s,\cdot)\|^{1-p}_{L^p}\left(\int_D |\xi^k_N(s,\beta)|^{p-1}\|D\xi^k_N(s,\beta)\|_{\mathcal{H}_T}\,\mathrm{d}\beta\right)\mathrm{d}y\,\mathrm{d}s\bigg]^p \\ &\le \mathbb{E}\left[\int_0^t \|\nabla_y g(t-s,x,\cdot)\cdot \tilde{q}_N(\xi^k_N(s,\cdot))\|_{L^1}\|D\xi^k_N(s,\cdot)\|_{L^p(D;\mathcal{H}_T)}\,\mathrm{d}s\right]^p.\end{aligned}$$

(I.1.2.8) and (I.1.6.9) imply that

$$\begin{aligned}\|\nabla_y g(t-s,x,\cdot)\cdot \tilde{q}_N(\xi^k_N(s,\cdot))\|_{L^1} &\le \|\nabla_y g(t-s,x,\cdot)\|_{L^{\frac{p}{p-1}}}\|\tilde{q}_N(\xi^k_N(s,\cdot))\|_{L^p} \\ &\le C_p(N+1)^2(t-s)^{-\frac{p+2}{2p}} \qquad \text{provided } p>4.\end{aligned}$$

Thanks to Hölder's inequality,

$$\begin{aligned}\mathbb{E}\|I_3\|^p_{\mathcal{H}_T} &\le C_{N,p}\,\mathbb{E}\left[\int_0^t (t-s)^{-\frac{p+2}{2p}}\|D\xi^k_N(s,\cdot)\|_{L^p(D;\mathcal{H}_T)}\,\mathrm{d}s\right]^p \\ &\le C_{N,p}t^{\frac{p}{2}-2}\int_0^t\int_D \mathbb{E}\|D\xi^k_N(s,y)\|^p_{\mathcal{H}_T}\,\mathrm{d}y\,\mathrm{d}s \\ &\le C_{N,p,T,|D|}\int_0^t \sup_{y\in D}\mathbb{E}\|D\xi^k_N(s,y)\|^p_{\mathcal{H}_T}\,\mathrm{d}s,\end{aligned}$$

provided $p > 4$.

For the last term I_4, by means of Minkowski's inequality, from Proposition I.1.4.1, Hypothesis (H3) and Lemma I.1.3.1 it follows

$$\begin{aligned}\mathbb{E}\|I_4\|^p_{\mathcal{H}_T} &\le \mathbb{E}\left|\int_0^t\int_D g(t-s,x,y)\sigma'(\xi^k_N(s,y))\|D\xi^k_N(s,y)\|_{\mathcal{H}_T}\,w(\mathrm{d}y,\mathrm{d}s)\right|^p \\ &\le \|g(t-\cdot,x,\bullet)\|^{p-2}_{\mathcal{H}_t}\int_0^t \sup_{y\in D}\mathbb{E}\left[|\sigma'(\xi^k_N(s,y))\|D\xi^k_N(s,y)\|_{\mathcal{H}_T}|^p\right]\|g(t-s,x,\bullet)\|^2_{L^2_Q}\,\mathrm{d}s \\ &\le C\|g(t-\cdot,x,\bullet)\|^{p-2}_{\mathcal{H}_t}\int_0^t \sup_{y\in D}\mathbb{E}\|D\xi^k_N(s,y)\|^p_{\mathcal{H}_T}\|g(t-s,x,\bullet)\|^2_{L^2_Q}\,\mathrm{d}s \\ &\le C_{T,p}\int_0^t \sup_{y\in D}\mathbb{E}\|D\xi^k_N(s,y)\|^p_{\mathcal{H}_T}\|g(t-s,x,\bullet)\|^2_{L^2_Q}\,\mathrm{d}s.\end{aligned}$$

Finally, from Proposition I.1.4.1 and Hypothesis (H2), it follows

$$\begin{aligned}\mathbb{E}\|g(t-\cdot,x,\bullet)\mathbf{1}_{[0,t]}(\cdot)\sigma(\xi^k_N(\cdot,\bullet))\|^p_{\mathcal{H}_T} &= \mathbb{E}\left[\int_0^t \|g(t-s,x,\bullet)\sigma(\xi^k_N(s,\bullet))\|^2_{L^2_Q}\,\mathrm{d}s\right]^{\frac{p}{2}} \\ &\le \|g(t-\cdot,x,\bullet)\|^{p-2}_{\mathcal{H}_t}\int_0^t \sup_{y\in D}\mathbb{E}|\sigma(\xi^k_N(s,y))|^p\|g(t-s,x,\bullet)\|^2_{L^2_Q}\,\mathrm{d}s \\ &\le C_{T,p}\|g(t-\cdot,x,\bullet)\|^p_{\mathcal{H}_T}.\end{aligned}$$

Collecting all the above estimates we get the following inequality

$$\sup_{x\in D}\mathbb{E}\|D\xi_N^{k+1}(t,x)\|^p_{\mathcal{H}_T}\le C_{T,p}\|g(t-\cdot,x,\bullet)\|^p_{\mathcal{H}_T}$$
$$+C_{N,p,T,|D|}\int_0^t\left(1+\|g(t-s,x,\bullet)\|^2_{L^2_Q}\right)\sup_{y\in D}\mathbb{E}\|D\xi_N^k(s,y)\|^p_{\mathcal{H}_T}\,\mathrm{d}s.$$

Setting

$$\varphi_k(t):=\sup_{x\in D}\mathbb{E}\|D\xi_N^k(t,x)\|^p_{\mathcal{H}_T}$$

we get

$$\varphi_{k+1}(t)\le C_{T,p}\|g(t-\cdot,x,\bullet)\|^p_{\mathcal{H}_T}+C_{N,p,T,|D|}\int_0^t\left(1+\|g(t-s,x,\bullet)\|^2_{L^2_Q}\right)\varphi_k(s)\,\mathrm{d}s.$$

Since the Malliavin derivative of ξ_N^0 is zero,

$$\sup_{0\le t\le T}\varphi_0(t)<\infty.$$

Then we conclude that (I.2.2.9) holds by Lemma I.2.2.1. Finally, equality (I.2.2.8) is obtained by applying the operator D to both members of equation (I.1.6.13). □

I.2.3 Nondegeneracy condition

Now we check condition (B.3.1) of the Bouleau-Hirsh criterium, for the solution ξ_N to the truncated equation. Let $t\in[0,T]$ and $x\in D$. We aim at proving that

$$\|D\xi_N(t,x)\|^2_{\mathcal{H}_T}>0\qquad \mathbb{P}-a.s. \tag{I.2.3.1}$$

The following lemma is an improvement of Theorem I.2.2.5 and it is needed in order to prove Theorem I.2.3.2. We need to consider a time interval smaller than $[0,T]$ and consider the $\mathcal{H}_T$-norm of $\xi_N(\cdot,x)$ on $(t-\varepsilon,t)$ for some $\varepsilon>0$ small enough. For every $\psi\in\mathcal{H}_T$ we define the norm

$$\|\psi\|_{\mathcal{H}_{(t-\varepsilon,t)}}:=\|\mathbf{1}_{(t-\varepsilon,t)}(\cdot)\psi\|_{\mathcal{H}_T}.$$

It is straightforward to get

$$\|\psi\|_{\mathcal{H}_T}\ge\|\psi\|_{\mathcal{H}_{(t-\varepsilon,t)}}.$$

Lemma I.2.3.1. *Let $N\ge1$, $b>1$ in* (I.1.3.3) *and $p>4$. If ξ_0 is a continuous function on D, then there exists a constant $C_{N,p,Q,T}$ such that for every $0<\varepsilon<t$*

$$\sup_{\sigma\in[t-\varepsilon,t]}\sup_{x\in D}\mathbb{E}\|D\xi_N(\sigma,x)\|^p_{\mathcal{H}(t-\varepsilon,t)}\le C_{N,p,Q,T}\varepsilon^{\frac{p}{2}}.$$

Proof. For $t-\varepsilon\le\sigma\le t$, set $\eta_N^\varepsilon(\sigma,x)=\mathbb{E}\|D\xi_N(\sigma,x)\|^p_{\mathcal{H}_{(t-\varepsilon,\sigma)}}$. According to (I.2.2.8),

$$\eta_N^\varepsilon(\sigma,x)\le C_p\left(\|g(\sigma-\cdot,x,\bullet)\mathbf{1}_{[0,\sigma]}(\cdot)\sigma(\xi_N(\cdot,\bullet))\|^p_{\mathcal{H}_{(t-\varepsilon,\sigma)}}+\sum_{i=1}^4\mathbb{E}\|I_i\|^p_{\mathcal{H}_{(t-\varepsilon,\sigma)}}\right),$$

where the terms I_i, $i = 1, 2, 3, 4$, are defined in (I.2.2.11)-(I.2.2.14). From Proposition I.1.4.1 and Hypothesis (H2), it follows

$$\mathbb{E}\|g(t-\cdot,x,\bullet)\mathbf{1}_{[0,t]}(\cdot)\sigma(\xi_N^k(\cdot,\bullet))\|^p_{\mathcal{H}_{(t-\varepsilon,\sigma)}} \leq C_{T,p}\|g(t-\cdot,x,\bullet)\|^p_{\mathcal{H}_{(t-\varepsilon,\sigma)}}$$

By (I.1.3.4) and the change of variables $s = r - \sigma + \varepsilon$, we get

$$\begin{aligned}
\|g(t-\cdot,x,\bullet)\|^2_{\mathcal{H}_{(t-\varepsilon,\sigma)}} &= \int_{t-\varepsilon}^{\sigma} \|g(\sigma-r,x,\bullet)\|^2_{L^2_Q}\,\mathrm{d}r \leq \sum_{k\in\mathbb{Z}^2_0}|k|^{-2b}\int_0^{\varepsilon} e^{-2|k|^2(\varepsilon-s)}|e_k(x)|^2\,\mathrm{d}s \\
&= \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}^2_0}\frac{|k|^{-2b-2}}{2}(1-e^{-2|k|^2\varepsilon}) \leq \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}^2_0}\frac{|k|^{-2b-2}}{2}(2|k|^2\varepsilon) \\
&= \frac{\varepsilon}{(2\pi)^2}\sum_{k\in\mathbb{Z}^2_0}|k|^{-2b} \leq \frac{\varepsilon}{(2\pi)^2}\,\mathrm{Tr}Q, \qquad \text{(I.2.3.2)}
\end{aligned}$$

which is finite provided $b > 1$. So

$$\mathbb{E}\|g(t-\cdot,x,\bullet)\mathbf{1}_{[0,t]}(\cdot)\sigma(\xi_N(\cdot,\bullet))\|^p_{\mathcal{H}_{(t-\varepsilon,\sigma)}} \leq C_{T,p}\frac{(\mathrm{Tr}Q)^{\frac{p}{2}}\,\varepsilon^{\frac{p}{2}}}{(2\pi)^p} = C_{T,p,Q}\,\varepsilon^{\frac{p}{2}}. \qquad \text{(I.2.3.3)}$$

Minkowski's and Hölder's inequalities and (I.1.2.26) imply that

$$\begin{aligned}
\mathbb{E}\|I_1\|^p_{\mathcal{H}(\sigma-\varepsilon,\sigma)} &= \mathbb{E}\Bigg[\int_{t-\varepsilon}^{\sigma}\Bigg\|\int_r^{\sigma}\int_D \nabla_y g(\sigma-s,x,y)\cdot v_N(s,y) \\
&\qquad \Theta_N(\|\xi_N(s,\cdot)\|_{L^p})D_{r,\cdot}\xi_N(s,y)\,\mathrm{d}y\,\mathrm{d}s\Bigg\|^2_{L^2_Q}\,\mathrm{d}r\Bigg]^{\frac{p}{2}} \\
&= \mathbb{E}\Bigg[\int_{t-\varepsilon}^{\sigma}\Bigg\|\int_{t-\varepsilon}^{\sigma}\int_D \nabla_y g(\sigma-s,x,y)\cdot v_N(s,y) \\
&\qquad \Theta_N(\|\xi_N(s,\cdot)\|_{L^p})D_{r,\cdot}\xi_N(s,y)\,\mathrm{d}y\,\mathrm{d}s\Bigg\|^2_{L^2_Q}\,\mathrm{d}r\Bigg]^{\frac{p}{2}} \\
&\leq \mathbb{E}\Bigg[\int_{t-\varepsilon}^{\sigma}\int_D |\nabla_y g(\sigma-s,x,y)\cdot v_N(s,y)| \\
&\qquad |\Theta_N(\|\xi_N(s,\cdot)\|_{L^p})|\ \|D\xi_N(s,y)\|_{\mathcal{H}_{(t-\varepsilon,\sigma)}}\,\mathrm{d}y\,\mathrm{d}s\Bigg]^p \\
&\leq C_N\left(\int_0^T\int_D |\nabla_y g(\sigma-s,x,y)|^{\frac{p}{p-1}}\,\mathrm{d}y\,\mathrm{d}s\right)^{p-1} \\
&\qquad \int_{t-\varepsilon}^{\sigma}\int_D \mathbb{E}\|D\xi_N(s,y)\|^p_{\mathcal{H}_{(t-\varepsilon,s)}}\,\mathrm{d}y\,\mathrm{d}s \\
&\leq C_N T^{\frac{p}{2}-2}\int_{t-\varepsilon}^{\sigma}\sup_{y\in D}\mathbb{E}\|D\xi_N(s,y)\|^p_{\mathcal{H}_{(t-\varepsilon,s)}}\,\mathrm{d}s \qquad \text{by (I.1.2.9) if } p>4.
\end{aligned}$$

As regards the term I_2, proceeding in a similar way, by means of Fubini Theorem, Hölder's and Minkowski's inequalities we get

$$
\begin{aligned}
&\mathbb{E}\|I_2\|^p_{\mathcal{H}_{(t-\varepsilon,\sigma)}} \\
&= \mathbb{E}\left[\int_{t-\varepsilon}^{\sigma} \left\| \int_{t-\varepsilon}^{\sigma} \int_D \left(\nabla_y g(\sigma - s, x, y) \cdot \int_D k(y-\alpha) D_{r,\cdot}\xi_N(s,\alpha)\, \mathrm{d}\alpha \right) \right.\right. \\
&\qquad\qquad \left.\left. \Theta_N(\|\xi_N(s,\cdot)\|_{L^p})\xi_N(s,y)\, \mathrm{d}y\, \mathrm{d}s \right\|^2_{L^2_Q} \mathrm{d}r \right]^{\frac{p}{2}} \\
&\le \mathbb{E}\left[\int_{t-\varepsilon}^{\sigma} \int_D \left| \int_D \nabla_y g(\sigma - s, x, y) \cdot k(y-\alpha)\xi_N(s,y)\Theta_N(\|\xi_N(s,\cdot)\|_{L^p})\, \mathrm{d}y \right| \right. \\
&\qquad\qquad \left. \|D\xi_N(s,\alpha)\|_{\mathcal{H}_{(t-\varepsilon,\sigma)}} \mathrm{d}\alpha\, \mathrm{d}s \right]^p \\
&\le C_N \mathbb{E}\left[\int_{t-\varepsilon}^{\sigma} \int_D \|\nabla_y g(\sigma - s, x, \cdot)\cdot k(\cdot - \alpha)\|_{L^{\frac{p}{p-1}}} \|D\xi_N(s,\alpha)\|_{\mathcal{H}_{(t-\varepsilon,\sigma)}} \mathrm{d}\alpha\, \mathrm{d}s \right]^p \\
&\le C_N \left(\int_0^T (\sigma - s)^{\frac{p+2}{2(1-p)}} \mathrm{d}s \right)^{p-1} \\
&\qquad\qquad \int_{t-\varepsilon}^{\sigma} \int_D \mathbb{E}\|D\xi_N(s,y)\|^p_{\mathcal{H}_{(t-\varepsilon,s)}} \mathrm{d}y\, \mathrm{d}s \qquad \text{by (I.2.2.16) if } p > 4 \\
&\le C_N T^{\frac{p}{2}-2} \int_{t-\varepsilon}^{\sigma} \sup_{y\in D} \mathbb{E}\|D\xi_N(s,y)\|^p_{\mathcal{H}_{(t-\varepsilon,s)}} \mathrm{d}s.
\end{aligned}
$$

For the term I_3, Minkowski's any Hölder's inequalities imply that

$$
\begin{aligned}
\mathbb{E}\|I_3\|^p_{\mathcal{H}_{(t-\varepsilon,\sigma)}} &= \mathbb{E}\left[\int_{t-\varepsilon}^{\sigma} \left\| \int_{t-\varepsilon}^{\sigma} \int_D \nabla_y g(\sigma - s, x, y)\cdot \tilde{q}_N(\xi_N(s,\cdot))(y) \|\xi_N(s,\cdot)\|^{1-p}_{L^p} \right.\right. \\
&\qquad \left.\left. \left(\int_D |\xi_N(s,\beta)|^{p-2}\xi_N(s,\beta) D_{r,\cdot}\xi_N(s,\beta)\, \mathrm{d}\beta \right) \mathrm{d}y\, \mathrm{d}s \right\|^2_{L^2_Q} \mathrm{d}r \right]^{\frac{p}{2}} \\
&\le \mathbb{E}\left[\int_{t-\varepsilon}^{\sigma} \int_D |\nabla_y g(\sigma - s, x, y)\cdot \tilde{q}_N(\xi_N(s,\cdot))(y)|\; \|\xi_N(s,\cdot)\|^{1-p}_{L^p} \right. \\
&\qquad \left. \left(\int_D |\xi_N(s,\beta)|^{p-1} \|D\xi_N(s,\beta)\|_{\mathcal{H}_{(t-\varepsilon,\sigma)}} \mathrm{d}\beta \right) \mathrm{d}y\, \mathrm{d}s \right]^p \\
&\le C_{N,p} \mathbb{E}\left[\int_{t-\varepsilon}^{\sigma} \|\nabla_y g(\sigma - s, x, \cdot)\|_{L^{\frac{p}{p-1}}} \|D\xi_N(s,\cdot)\|_{L^p(D;\mathcal{H}_{(t-\varepsilon,\sigma)})} \right]^p \quad \text{by (I.1.6.9)} \\
&\le C_{N,p} \left(\int_0^T \int_D |\nabla_y g(\sigma - s, x, y)|^{\frac{p}{p-1}} \mathrm{d}y\, \mathrm{d}s \right)^{p-1} \\
&\qquad \int_{t-\varepsilon}^{\sigma} \int_D \mathbb{E}\|D\xi_N(s,y)\|^p_{\mathcal{H}_{(t-\varepsilon,s)}} \mathrm{d}y\, \mathrm{d}s \\
&\le C_{N,p} T^{\frac{p}{2}-2} \int_{t-\varepsilon}^{\sigma} \sup_{y\in D} \mathbb{E}\|D\xi_N(s,y)\|^p_{\mathcal{H}_{(t-\varepsilon,s)}} \mathrm{d}s \qquad \text{by (I.1.2.9) if } p > 4.
\end{aligned}
$$

Using Minkowski's inequality, from Proposition I.1.4.1 the linear growth condition on σ

and Hypothesis (H3), we get

$$\begin{aligned}\mathbb{E}\|I_4\|^p_{\mathcal{H}_{(t-\varepsilon,\sigma)}} &= \mathbb{E}\left[\int_{t-\varepsilon}^{\sigma}\left\|\int_{t-\varepsilon}^{\sigma}\int_D g(\sigma-s,x,y)\sigma'(\xi_N(s,y))D_{r,\cdot}\xi_N(s,y)\,w(\mathrm{d}y,\mathrm{d}s)\right\|^2_{L^2_Q}\mathrm{d}r\right]^{\frac{p}{2}}\\
&= \mathbb{E}\left[\int_{t-\varepsilon}^{\sigma}\int_D g(\sigma-s,x,y)\sigma'(\xi_N(s,y))\|D\xi_N(s,y)\|_{\mathcal{H}_{(t-\varepsilon,\sigma)}}\,w(\mathrm{d}y,\mathrm{d}s)\right]^p\\
&\le C\|g(\sigma-\cdot,x,\bullet)\|^{p-2}_{\mathcal{H}_{(t-\varepsilon,\sigma)}}\int_{t-\varepsilon}^{\sigma}\sup_{y\in D}\mathbb{E}\|D\xi_N(s,y)\|^p_{\mathcal{H}_{(t-\varepsilon,s)}}\|g(\sigma-s,x,\bullet)\|^2_{L^2_Q}\,\mathrm{d}s\\
&\le C_{T,p}\int_{t-\varepsilon}^{\sigma}\sup_{y\in D}\mathbb{E}\|D\xi_N(s,y)\|^p_{\mathcal{H}_{(t-\varepsilon,s)}}\|g(\sigma-s,x,\bullet)\|^2_{L^2_Q}\,\mathrm{d}s.\end{aligned}$$

Collecting the above estimates we get

$$\sup_{x\in D}\eta^{\varepsilon}_N(\sigma,x)\le C_{p,Q,T}\varepsilon^{\frac{p}{2}}+C_{N,p,T}\int_{t-\varepsilon}^{\sigma}\sup_{y\in D}\eta^{\varepsilon}_N(s,y)\left(1+\|g(\sigma-s,x,\bullet)\|^2_{L^2_Q}\right)\mathrm{d}s,\ \forall\,\sigma\in[t-\varepsilon,t]\,.$$

By the generalized Gronwall's Lemma I.2.2.1 it follows

$$\sup_{x\in D}\eta^{\varepsilon}_N(\sigma,x)\le C_{N,p,Q,T}\varepsilon^{\frac{p}{2}},\qquad\text{for every }\sigma\in[t-\varepsilon,t]\,.$$

Since for $\sigma\in[t-\varepsilon,t]$, $\|D\xi_N(\sigma,x)\|^p_{\mathcal{H}_{(t-\varepsilon,\sigma)}}=\|D\xi_N(\sigma,x)\|^p_{\mathcal{H}_{(t-\varepsilon,t)}}$ we finally get

$$\sup_{\sigma\in[t-\varepsilon,t]}\sup_{x\in D}\mathbb{E}\|D\xi_N(\sigma,x)\|^p_{\mathcal{H}_{(t-\varepsilon,t)}}\le C_{N,p,Q,T}\varepsilon^{\frac{p}{2}}.$$

□

Theorem I.2.3.2. *Suppose $b>1$ in* (I.1.3.3)*. Let assume that Hypothesis (H1)-(H4) hold and that ξ_0 is a continuous function on D. Then, for every $t\in[0,T]$ and $x\in D$, the image law of the random variable $\xi_N(t,x)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

Proof. In order to prove that $\|D\xi_N(t,x)\|^2_{\mathcal{H}_T}>0\quad\mathbb{P}-a.s.$ we will show that

$$\mathbb{P}(\|D\xi_N(t,x)\|^2_{\mathcal{H}_T}=0)=0,$$

or, better, that

$$\mathbb{P}(\|D\xi_N(t,x)\|^2_{\mathcal{H}_T}<\delta)\to 0\qquad\text{as}\ \ \delta\to 0.\tag{I.2.3.4}$$

Let us fix $\varepsilon>0$ sufficiently small, according to (I.2.2.8), by means of the inequality $(a+b)^2\ge\frac{1}{2}a^2-b^2$, we get

$$\begin{aligned}\|D\xi_N(t,x)\|^2_{\mathcal{H}_T} &= \int_0^T\|D_{r,\bullet}\xi_N(t,x)\|^2_{L^2_Q}\,\mathrm{d}r\ge\int_{t-\varepsilon}^t\|D_{r,\bullet}\xi_N(t,x)\|^2_{L^2_Q}\,\mathrm{d}r\\
&= \int_{t-\varepsilon}^t\left\|g(t-r,x,\bullet)\mathbf{1}_{[0,t]}(r)\sigma(\xi_N(r,\bullet))+\sum_{i=1}^4 I_i(r,\bullet)\right\|^2_{L^2_Q}\mathrm{d}r\\
&\ge\frac{1}{2}\int_{t-\varepsilon}^t\|g(t-r,x,\bullet)\sigma(\xi_N(r,\bullet))\|^2_{L^2_Q}\,\mathrm{d}r-\int_{t-\varepsilon}^t\left\|\sum_{i=1}^4 I_i(r,\bullet)\right\|^2_{L^2_Q}\mathrm{d}r,\end{aligned}$$

where the terms I_i are defined in (I.2.2.11)-(I.2.2.14). Let us set for simplicity

$$I(t,x,\varepsilon)=\int_{t-\varepsilon}^{t}\left\|\sum_{i=1}^{4}I_i(r,\bullet)\right\|_{L^2_Q}^2\mathrm{d}r,\qquad A(x,\varepsilon)=\int_{t-\varepsilon}^{t}\|g(t-r,x,\bullet)\sigma(\xi_N(r,\bullet))\|_{L^2_Q}^2\,\mathrm{d}r.$$

By means of Chebyschev's inequality, for $\delta>0$ sufficiently small, we have

$$\mathbb{P}(\|D\xi_N(t,x)\|_{\mathcal{H}_T}^2<\delta)\le\mathbb{P}(I(t,x,\varepsilon)\ge\frac{1}{2}A(x,\varepsilon)-\delta)\le\frac{\mathbb{E}|I(t,x,\varepsilon)|^{\frac{p}{2}}}{\left(\frac{1}{2}A(x,\varepsilon)-\delta\right)^{\frac{p}{2}}}. \tag{I.2.3.5}$$

Let us find an upper estimate for $\mathbb{E}|I(t,x,\varepsilon)|^{\frac{p}{2}}\le C_p\sum_{i=1}^{4}\mathbb{E}\left|\int_{t-\varepsilon}^{t}\|I_i(r,\bullet)\|_{L^2_Q}^2\,\mathrm{d}r\right|^{\frac{p}{2}}$.

Minkowski's and Hölder's inequalities and (I.1.2.26) imply that

$$\begin{aligned}
&\mathbb{E}\left|\int_{t-\varepsilon}^{t}\|I_1(r,\bullet)\|_{L^2_Q}^2\,\mathrm{d}r\right|^{\frac{p}{2}}\\
&=\mathbb{E}\Bigg[\int_{t-\varepsilon}^{t}\Bigg\|\int_{t-\varepsilon}^{t}\int_D\nabla_y g(t-s,x,y)\cdot v_N(s,y)\\
&\qquad\qquad\Theta_N(\|\xi_N(s,\cdot)\|_{L^p})D_{r,\bullet}\xi_N(s,y)\,\mathrm{d}y\,\mathrm{d}s\Bigg\|_{L^2_Q}^2\,\mathrm{d}r\Bigg]^{\frac{p}{2}}\\
&\le C_N\left(\int_{t-\varepsilon}^{t}\int_D|\nabla_y g(t-s,x,y)|^{\frac{p}{p-1}}\,\mathrm{d}y\,\mathrm{d}s\right)^{p-1}\int_{t-\varepsilon}^{t}\int_D\mathbb{E}\|D\xi_N(s,y)\|_{\mathcal{H}_{(t-\varepsilon,t)}}^p\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}$$

Using Lemma I.2.3.1 with $t-\varepsilon\le s\le t$ and (I.1.2.9), provided $p>4$, we deduce that

$$\mathbb{E}\left|\int_{t-\varepsilon}^{t}\|I_1(r,\bullet)\|_{L^2_Q}^2\,\mathrm{d}r\right|^{\frac{p}{2}}\le C_{N,p,Q,T}\varepsilon^{\frac{p}{2}-2}\varepsilon^{\frac{p}{2}}=C_{N,p,Q,T}\varepsilon^{p-2}.$$

For the term I_2, by means of Fubini Theorem, Hölder's and Minkowski's inequalities and by (I.2.2.16), provided $p>4$, we get

$$\begin{aligned}
&\mathbb{E}\left|\int_{t-\varepsilon}^{t}\|I_2(r,\bullet)\|_{L^2_Q}^2\,\mathrm{d}r\right|^{\frac{p}{2}}\\
&=\mathbb{E}\Bigg[\int_{t-\varepsilon}^{t}\Bigg\|\int_{t-\varepsilon}^{t}\int_D\left(\nabla_y g(t-s,x,y)\cdot\int_D k(y-\alpha)D_{r,\bullet}\xi_N(s,\alpha)\,\mathrm{d}\alpha\right)\\
&\qquad\qquad\Theta_N(\|\xi_N(s,\cdot)\|_{L^p})\xi_N(s,y)\,\mathrm{d}y\,\mathrm{d}s\Bigg\|_{L^2_Q}^2\,\mathrm{d}r\Bigg]^{\frac{p}{2}}\\
&\le C_N\left(\int_{t-\varepsilon}^{t}(t-s)^{\frac{p+2}{2(1-p)}}\,\mathrm{d}s\right)^{p-1}\int_{t-\varepsilon}^{t}\int_D\mathbb{E}\|D\xi_N(s,y)\|_{\mathcal{H}_{(t-\varepsilon,t)}}^p\,\mathrm{d}y\,\mathrm{d}s\\
&\le C_N\varepsilon^{\frac{p}{2}-2}C_{N,p,Q,T}\varepsilon^{\frac{p}{2}}=C_{N,p,Q,T}\varepsilon^{p-2}\qquad\text{by Lemma I.2.3.1.}
\end{aligned}$$

As regards the last term I_3, Minkowski's and Hölder's inequalities and (I.1.6.9) imply that

$$
\begin{aligned}
&\mathbb{E}\left|\int_{t-\varepsilon}^{t} \|I_3(r,\bullet)\|^2_{L^2_Q}\,\mathrm{d}r\right|^{\frac{p}{2}}\\
&= \mathbb{E}\left[\int_{t-\varepsilon}^{t}\left\|p\int_{t-\varepsilon}^{t}\int_D \nabla_y g(t-s,x,y)\cdot \tilde{q}_N(\xi_N(s,\cdot))(y)\|\xi_N(s,\cdot)\|^{1-p}_{L^p}\right.\right.\\
&\qquad\left.\left.\left(\int_D |\xi_N(s,\beta)|^{p-2}\xi_N(s,\beta)D_{r,\bullet}\xi_N(s,\beta)\,\mathrm{d}\beta\right)\mathrm{d}y\,\mathrm{d}s\right\|^2_{L^2_Q}\mathrm{d}r\right]^{\frac{p}{2}}\\
&\le C_{N,p}\left(\int_{t-\varepsilon}^{t}\int_D |\nabla_y g(t-s,x,y)|^{\frac{p}{p-1}}\,\mathrm{d}y\,\mathrm{d}s\right)^{p-1}\\
&\qquad\int_{t-\varepsilon}^{t}\int_D \mathbb{E}\|D\xi_N(s,y)\|^p_{\mathcal{H}_{(t-\varepsilon,t)}}\,\mathrm{d}y\,\mathrm{d}s\\
&\le C_{N,p}\varepsilon^{\frac{p}{2}-2}C_{N,p,Q,T}\varepsilon^{\frac{p}{2}} = C_{N,p,Q,T}\varepsilon^{p-2} \qquad \text{by Lemma I.2.3.1 and (I.1.2.9) if } p>4.
\end{aligned}
$$

As regards the term I_4, proceeding as in the proof of Lemma I.2.3.1 we obtain

$$
\begin{aligned}
&\mathbb{E}\left|\int_{t-\varepsilon}^{t} \|I_4(r,\bullet)\|^2_{L^2_Q}\,\mathrm{d}r\right|^{\frac{p}{2}}\\
&\quad= \mathbb{E}\left[\int_{t-\varepsilon}^{t}\left\|\int_{t-\varepsilon}^{t}\int_D g(t-s,x,y)\sigma'(\xi_N(s,y))D_{r,\bullet}\xi_N(s,y)\,w(\mathrm{d}y,\mathrm{d}s)\right\|^2_{L^2_Q}\mathrm{d}r\right]^{\frac{p}{2}}\\
&\quad\le \|g(t-\cdot,x,\bullet)\|^{p-2}_{\mathcal{H}_{(t-\varepsilon,t)}}\int_{t-\varepsilon}^{t}\sup_{y\in D}\mathbb{E}\|D\xi_N(s,y)\|^p_{\mathcal{H}_{(t-\varepsilon,t)}}\|g(t-s,x,\bullet)\|^2_{L^2_Q}\,\mathrm{d}s\\
&\quad\le \|g(t-\cdot,x,\bullet)\|^p_{\mathcal{H}_{(t-\varepsilon,t)}}\sup_{t-\varepsilon\le s\le t}\sup_{y\in D}\mathbb{E}\|D\xi_N(s,y)\|^p_{\mathcal{H}_{(t-\varepsilon,t)}}.
\end{aligned}
$$

By (I.1.3.4) and the change of variables $s=r-t+\varepsilon$, we get

$$\|g(t-\cdot,x,\bullet)\|^p_{\mathcal{H}_{(t-\varepsilon,t)}} \le C_{p,Q}\varepsilon^{\frac{p}{2}}$$

and by Lemma I.2.3.1

$$\sup_{t-\varepsilon\le s\le t}\sup_{y\in D}\mathbb{E}\|D\xi_N(s,y)\|^p_{\mathcal{H}_{(t-\varepsilon,t)}} \le C_{N,p,Q,T}\varepsilon^{\frac{p}{2}}.$$

So we get

$$\mathbb{E}\left|\int_{t-\varepsilon}^{t}\|I_4(r,\cdot)\|^2_{L^2_Q}\,\mathrm{d}r\right|^{\frac{p}{2}} \le C_{N,p,Q,T}\varepsilon^p.$$

Since we fix an ε sufficiently small, namely $\varepsilon \ll 1$, it holds $\varepsilon^p < \varepsilon^{p-2}$. In conclusion, collecting all the above estimates, we get

$$\mathbb{E}\,|I(t,x,\varepsilon)|^{\frac{p}{2}} \le C_{N,p,Q,T}\varepsilon^{p-2}, \qquad \text{(I.2.3.6)}$$

provided $p > 4$. We now need to find a lower estimate for $A(x,\varepsilon)$. From Hypothesis (H4), we get

$$A(x,\varepsilon) = \int_{t-\varepsilon}^{t} \|g(t-r,x,\bullet)\sigma(\xi_N(r,\bullet))\|^2_{L^2_Q}\, \mathrm{d}r \geq \sigma_0^2 \|g(t-\cdot,x,\bullet)\|^2_{\mathcal{H}_{(t-\varepsilon,t)}}. \tag{I.2.3.7}$$

Proceeding as in (I.1.3.4) we have

$$\|g(t-\cdot,x,\bullet)\|^2_{\mathcal{H}_{(t-\varepsilon,t)}} = \int_{t-\varepsilon}^{t} \|g(t-r,x,\bullet)\|^2_{L^2_Q}\, \mathrm{d}r = \sum_{k\in\mathbb{Z}_0^2} |k|^{-2b}|e_k(x)|^2 \, \frac{1}{2|k|^2}(1-e^{-2|k|^2\varepsilon}).$$

The inequality

$$1-e^{-2|k|^2\varepsilon} \geq \frac{2\varepsilon|k|^2}{1+2\varepsilon|k|^2} \geq \frac{2\varepsilon|k|^2}{1+2T|k|^2}$$

implies that

$$\int_{t-\varepsilon}^{t} \|g(t-r,x,\bullet)\|^2_{L^2_Q}\, \mathrm{d}r \geq \frac{\varepsilon}{(2\pi)^2} \sum_{k\in\mathbb{Z}_0^2} \frac{|k|^{-2b}}{1+2T|k|^2}$$

and the above series is well defined and can be bounded from below by any of its summand, such as the one corresponding to $k = (0,1) \in \mathbb{Z}_0^2$:

$$\int_{t-\varepsilon}^{t} \|g(t-r,x,\bullet)\|^2_{L^2_Q}\, \mathrm{d}r \geq \frac{\varepsilon}{(2\pi)^2(1+2T)} = C_T\, \varepsilon. \tag{I.2.3.8}$$

Thus we obtain

$$A(x,\varepsilon) \geq \sigma_0^2 C_T\, \varepsilon \tag{I.2.3.9}$$

Using estimates (I.2.3.6) and (I.2.3.9) and substituting into (I.2.3.5) we get

$$\mathbb{P}(\|D\xi_N(t,x)\|^2_{\mathcal{H}_T} < \delta) \leq \left(\frac{\sigma_0^2 C_T}{2}\varepsilon - \delta\right)^{-\frac{p}{2}} C_{N,p,Q,T}\, \varepsilon^{p-2}.$$

Thus, if we choose $\varepsilon = \varepsilon(\delta,T)$ sufficiently small in such a way that $\frac{\sigma_0^2 C_T}{2}\varepsilon = 2\delta$ we get

$$\mathbb{P}(\|D\xi_N(t,x)\|^2_{\mathcal{H}_T} < \delta) \leq C_{N,T,Q,p}\delta^{-\frac{p}{2}}\delta^{p-2} = C_{N,T,Q,p}\, \delta^{\frac{p}{2}-2} \to 0 \qquad \text{for } \delta \to 0,$$

since $p > 4$. □

I.2.4 Existence of the density

Now we are ready to prove the main result, Theorem I.2.1.1.

Proof of Theorem I.2.1.1. Let us fix $N \geq 1$ and $p > 4$ and let us define

$$\Omega_N := \left\{\omega \in \Omega : \sup_{t\in[0,T]} \|\xi(t,\cdot,\omega)\|_{L^p} \leq N\right\}. \tag{I.2.4.1}$$

It holds that $\lim_{N\to+\infty} \mathbb{P}(\Omega_N = \Omega) = 1$. In fact we can write

$$\Omega_N = \{\sigma_N = T\}\,,$$

where σ_N is the stopping time defined in (I.1.6.20). So we have that, for $N \to \infty$, $\sup_{N\geq 1} \sigma_N = T$ $\mathbb{P}$-a.s. i.e. $\Omega_N \uparrow \Omega$ $\mathbb{P}$-a.s. Moreover, by the local property of the stochastic integrals we have that $\xi(t,x) \equiv \xi_N(t,x)$ on Ω_N for every $t \in [0,T]$ and $x \in D$. Then it follows that, for every $(t,x) \in [0,T] \times D$, the sequence $(\Omega_N, \xi_N(t,x))$ localizes $\xi(t,x)$ in $\mathbb{D}^{1,p}$. The result then follows by Theorem I.2.3.2: in fact it suffices to show property (B.3.1) on the set $\{t < \sigma_N\}$ for every $N \geq 1$, namely to show (I.2.3.1). □

I.2.5 Notes and Comments: a brief overview of the existing literature concerning analysis in Malliavin sense for solutions to SPDEs

There has been a lot of activity in the last years studying the regularity in the Malliavin sense for solutions to stochastic partial differential equations. Main aim in this direction is to prove the existence (and smoothness) of a density for the law of the random variable given by the solution process at fixed points in time and space. Equations are usually interpreted in Walsh sense and are solved in the space of real valued stochastic processes, considering random field solutions, that is real-valued processes, that are defined for every fixed t and x in the domain. Different kinds of difficulties arise when problems of this type are addressed. To have a better understanding of the existing literature in this field it is useful to focalize the main sources of difficulties one has to deal with. We can summarize them as follows.

1. **The differential operator driving the equation.** Since equations are interpreted in Walsh sense, solutions are random fields and are written as the convolution with the Green function associated to the partial differential operator driving the equation. Most of the existence literature on this subject concerns the heat and wave equations. Suitable estimates on the Green function are fundamental and more the kernel is irregular more the problem becomes difficult to treat, in particular dealing with the stochastic convolution term. Walsh theory cover the case of SPDEs whose Green function is a function. The theory is suitable for solving the heat equation since the Green function associated to the heat operator is very smooth in all dimensions. In [27] Dalang extended the definition of Walsh martingale measure stochastic integral to be able to solve SPDEs whose Green function is a Schwartz distribution. This in particular covers the case of wave equation in dimension greater than two. The fundamental solution keeps tracks also of the boundary conditions associated to the equation.

2. **The spatial dimension of the domain.** If the spatial dimension is equal to one it is possible to take a time-space white noise as random input in the considered equation and obtain a function-valued solution. In higher dimension ($d \geq 2$) solutions to parabolic and hyperbolic equations driven by this kind on noise only exist as random (Schwartz) distributions. This is because the Green function of the heat and wave equations becomes less smooth as the dimension increases. For this reason, in dimension $d \geq 2$ it is common to consider random noises that are smoother than space-time white noise, namely Gaussian noises white in time and colored in space. The choice of the Gaussian random noise driving the equation is fundamental in the Malliavin analysis of the solution process since it provides the underlying Gaussian framework where to work. Obviously computations involving a noise with a spatial covariance are more involved that the space-time white noise case.

3. **The regularity of the non linear terms that appear in the equation.** Commonly, the proofs of the existence of a random field solution and its Malliavin differentiability are proved by means of a fixed point theorem in suitable spaces. If the drift term that appears in the equation is Lipschitz and satisfies a linear growth conditions there are no problems. Problems arise when the non linearities are non Lipschitz. In this case suitable approximation and localization procedures are necessary. Most of the

existing literature concerns the case of drift terms which are Lipschitz and satisfy a linear growth condition. On the other hand, much less attention has been dedicated to SPDEs with non-Lipschitz coefficients. As far as we know papers considering more general non linearities are [74] and [57] where the non linear term satisfies a polynomial growth and [92] where the Burgers equation is considered; in [66] Morien considered a more general type of nonlinearity that in particular includes the Burgers case. Finally in [20] is considered a class of stochastic Cahn-Hilliard equations with locally Lipschitz coefficients.

We recall that, in order to prove the smoothness of the density, stronger assumptions are required: the function has to be infinitely differentiable with bounded derivatives of every order. We notice here that if the non linear term is only locally Lipschitz by a localization procedure is difficult to obtain such a regularity. Papers dealing with this particular case (see [92], [66] and [20]) prove the existence of a density for the image law of the solution, but its smoothness remains an open problem. In fact authors use a localization argument to prove the existence of a density; nevertheless with this type of argument the smoothness of the density can not be proved since the method does not provide the boundedness of the derivatives of every order.

Now that the main technical problems arising in the study of the existence (and smoothness) of a density for the image law of solutions to SPDEs are clear, it becomes easier to give a brief overview of the existing literature on this subject. As pointed out above, most of the literature focus on the problem of existence and regularity of the density for solutions to parabolic and hyperbolic SPDEs with Lipschitz non-linearities. In a one-dimensional domain, the application of Malliavin calculus to the absolute continuity of the solution to the heat equation (with Neumann boundary conditions) perturbed by a space-time white noise, has been taken from Pardoux and Zhang in [74]. [67] considered the same problem with Dirichlet boundary conditions. The smoothness of the density has been studied by Bally and Pardoux in [4]. In the mentioned papers the existence of a density is proved under a Lipschitz and linear growth condition on the drift and diffusion terms. Moreover, a non degeneracy assumption on the diffusion term is needed. To prove the smoothness of the density both the drift and diffusion terms are assumed to be infinitely differentiable with bounded derivatives of every order. Once again we highlight that this kind of assumptions are rather standard and represent the usual assumptions made on the coefficient of the considered SPDE.

For what concerns the case of spatial dimension greater than one, if the spatial domain is equal to $\mathbb{R}^d$, a common choice of random forcing term is given by the Gaussian noise white in time, correlated in space described in Example B.2.2. For instance in [58] (see also the therein references) authors prove existence and smoothness of a density for the heat equation in any spatial dimension $d \geq 2$, with the usual (good) assumptions on the drift and diffusion terms. Working on a bounded domain of $\mathbb{R}^d$, subjected to Dirichlet boundary conditions, in [57] author consider a Gaussian noise as that described in Example B.2.3. Moreover, as recalled above, here the drift term considered is monotone, and growing not faster than a polynomial. As regards the hyperbolic case, a lot of work has been done especially after the papers by Dalang and Frangos [28]. The given extension of Walsh stochastic integration theory provides the necessary tools to study the regularity of the solutions in the Malliavin sense to the wave equation's solutions in dimension $d = 2, 3$; see for instance [65], [58], [79], [80] and the therein references. In these works the considered Gaussian noise driving the equation is white in time, correlated in space and on the drift and diffusion terms are made the (good) usual standard assumptions. For what concerns different type of differential operators, the above mentioned paper [20] deals with the Cahn-Hilliard equation on a bounded domain, driven by a space-time white noise, in dimension $d = 1, 2, 3$.

The smoothness of the density of the projection onto a finite-dimensional subspace of the solution at time $t > 0$ of the two-dimensional Navier-Stokes equations forced by a finite-dimensional Gaussian white noise has been established by Mattingly and Pardoux in [59]. Remaining within the Navier-Stokes equations context, the case $d = 1$ makes no sense since there, the incompressibility condition $\nabla \cdot v = 0$ would imply that v is constant. However, in dimension one, one could consider the Burgers equation which has similar features to the Navier-Stokes equations. In this sense our work can be considered an extension of the results obtained for the Burgers equation in the above cited papers and represent a first step in the study of regularity in Malliavin sense for solutions to stochastic fluid dynamical equations in dimension bigger than one. To conclude, we recall that the vorticity equation we consider can be written as a stochastic parabolic nonlinear equation in a two dimensional spatial domain with a nonlinear term of the form $(k * \xi) \cdot \nabla \xi$. This is of a form different from that studied in other papers about stochastic parabolic nonlinear equations in spatial dimension bigger than one (see [58], [57]).

Part II

Stochastic Navier-Stokes equations: analysis on $\mathbb{R}^2$

Introduction

Main aim of this Part is to study the stochastic equation for the vorticity, driven by a multiplicative Gaussian noise term, on the whole plane $\mathbb{R}^2$. The lack of compactness of the domain throws up a substantial difficulty that led us to use different techniques, than the flat torus case (see Chapter I.1). In this context the equation for the vorticity was rewritten, by means of the Biot-Savart law, as a closed equation for ξ. Existence of the solution was then proved, exploiting some estimates on the Biot-Savart kernel, by a rather classical stopping time argument. On a non compact domain we can not directly handle the closed form for the vorticity equation, where v is given by convolving ξ with the Biot-Savart kernel. Its singularity at the origin prevents us to obtain the needed estimates that allows to the treat the vorticity equation as a closed equation for ξ. The problem has to be approached in a different way: we have to explicitly take into account the Navier-Stokes equations for the velocity.

The problem of the existence and uniqueness of L^2-solutions of the stochastic Navier-Stokes equations (II.0.0.1) has been addressed by many authors. There is also a consistent literature on more regular solutions, but the majority of the work is limited to bounded domains (see e.g. [1, 49] and the therein references). An extension to unbounded domains is not trivial since the direct application of the compactness method, which is central in the proof, fails. Main source of difficulty is the fact that the embedding of the Sobolev space of functions with square integrable gradient into the L^2-space, unlike in the bounded space, is not compact. To address this problem different ideas have been employed. One way is to introduce weighted Sobolev spaces. This is one of the novelties introduced in [19]. The paper represents one of the first result concerning the existence of a global martingale solution of stochastic Navier-Stokes equations in unbounded domains. Spaces with weights are employed also in [18] where authors deal with the stochastic Euler equation. In both the mentioned papers, the results are proved in a rather abstract form (the equation is driven by an Hilbert-valued cylindrical Wiener process with a suitable covariance operator) and then applied in the case of a finite system of independent spatially homogeneous Wiener random fields and a covariance operator of Nemitskii form.

A different approach is used in [63], [64] and [17]. In [64] the authors prove the existence of an L^2-valued continuous solution considering a more general noise than in [19]. Their proof is based on some compactness and tightness criteria in local spaces and in the space L^2 with the weak topology. Differently, in [63] also the vorticity is considered, but the results involve v and ξ in $L^p(\mathbb{R}^d)$ for $p > d$. Inspired by [64], in [17] the authors prove existence and uniqueness of a strong L^2-solution, by means of a modification of the classical Dubinsky compactness theorem that allows to work in unbounded domains. Following the same approach of [17], in [16] authors impose really weak assumptions on the covariance operator of the noise term. In particular, it is not regular enough to allow to use Itô formula in the space of finite energy velocity vectors, which is the basic space in which one looks for existence of solutions.

The original aim of the research was to investigate the regularity of the solution to the vorticity equation on $\mathbb{R}^2$ in the Malliavin sense. The first result we obtain (Chapter II.1) is inspired by [18] and goes in this direction. We prove the existence of a unique strong $L^2 \cap L^q$- valued solution ($q > 2$) to the Navier-Stokes equations in vorticity form, with bounded moments of every order. Mean estimates are fundamental and an existence result which holds only $\mathbb{P}$-a.s. would be too weak: it would not ensure the Malliavin differentiability of the solution process.

As pointed out above, in order to prove the existence of a solution to the equation for the vorticity we can not proceed as in the flat torus case. We proceed as follows. We consider the two-dimensional stochastic Navier-Stokes equations

$$\begin{cases} \mathrm{d}v(t) - [\Delta v(t) + (v(t)\cdot\nabla)v(t) + \nabla p(t)]\,\mathrm{d}t = G(v(t))\,\mathrm{d}W(t) & t \in [0,T] \\ \nabla \cdot v(t) = 0 & t \in [0,T] \\ v(0,x) = v_0(x) & x \in \mathbb{R}^2. \end{cases} \tag{II.0.0.1}$$

Here W is a cylindrical $\mathcal{H}$-Wiener process, with $\mathcal{H}$ a real separable Hilbert space and G a (suitable defined) covariance operator. As usual we project the first equation of (II.0.0.1) onto the space of divergence free vectors fields to get rid of the pressure term. We write the Navier-Stokes equations in the abstract form.

$$\begin{cases} \mathrm{d}v(t) + [Av(t) + B(v(t),v(t))]\,\mathrm{d}t = \mathcal{P}G(v(t))\,\mathrm{d}W(t) & t \in [0,T] \\ v(0,x) = v_0(x) & x \in \mathbb{R}^2, \end{cases} \tag{II.0.0.2}$$

where $\mathcal{P}$ is the projection operator into solenoidal spaces; A and B are appropriate operators corresponding to the Laplacian and the nonlinear term respectively in the Navier-Stokes equations (they will be defined in Section II.0.1.4 where we will recall some of their properties).

We prove that there exists a unique strong (in the probably sense) $H^{1,2} \cap H^{1,q}$-valued solution v to (II.0.0.2). The existing literature on unbounded domains concerns the existence of $L^2 \cap L^q$ solutions to (II.0.0.1). Under stronger assumption on the regularity of the initial datum and the covariance operator we prove a higher space regularity for the solution process. As in [18], in the method we use, an important role is played by the vorticity itself. This latter equation is obtained by taking the curl on both sides of the first equation in (II.0.0.1).

$$\begin{cases} \mathrm{d}\xi(t) + [-\Delta\xi(t) + v(t)\cdot\nabla\xi(t)]\,\mathrm{d}t = \mathrm{curl}(G(v(t))\,\mathrm{d}W(t)) & t \in [0,T] \\ \xi(t) = \nabla^{\perp}\cdot v(t), & t \in [0,T] \\ \xi(0,x) = \xi_0(x) & x \in \mathbb{R}^2. \end{cases} \tag{II.0.0.3}$$

From the existence and uniqueness of a strong $H^{1,2} \cap H^{1,q}$-valued solution v to (II.0.0.2) we infer the existence and uniqueness of a strong $L^2 \cap L^q$-valued solution to (II.0.0.3). The results are proved in a rather abstract setting but in particular cover the case in which the equation is driven by a spatially homogeneous Wiener random field and the covariance operator is of Nemitsky form. This particular case provides the setting where to perform a Malliavin analysis of the solution process. We are working at the moment in this direction.

The second result we obtain (Chapter II.2) is inspired by [16]. In this case we considerably weaken the assumptions on G. In particular the covariance operator appearing in (II.0.0.3) is not regular enough to allow us to use Itô formula in L^q spaces, for $1 < q < \infty$, and an approximation procedure is required. In the study of the approximating problem we use the results of Chapter II.1. In this case we work directly on the equation for the vorticity. This requires a certain regularity on the velocity, which is proved by studying equation (II.0.0.1). Differently from before, in this case the existence and uniqueness result holds only $\mathbb{P}$-almost surely.

The present Part is organized as follows. In Chapter II.1 we prove the existence of a unique solution to (II.0.0.1) (and from that we infer the existence of a unique strong solution to (II.0.0.3)) under quite strong assumptions on the initial datum and the covariance operator G. In Chapter II.2 we deal with the existence and uniqueness problem for (II.0.0.3) under weaker assumptions on G.

Remark II.0.0.1. *As we are working in the intersection of analysis and probability, the terminology concerning the notion of solution can cause some confusion. When we talk about strong and weak solutions we understand them in a probabilistic sense. In the case of strong solutions, the underlying probability space is given in advance. On the other side, the term "weak solution" is used synonymously with the term "martingale solution": the stochastic basis is constructed as part of the solution. In both cases solutions are weak in the sense of PDEs since we test them against smooth functions.*

II.0.1 Mathematical setting and analytic preliminaries

In this Section we recall some basic definitions, we introduce the operators appearing in the abstract formulation of the above stated equations, providing then some results which gathers some useful properties.

II.0.1.1 Notation

Let $(X, |\cdot|_X)$, $(Y, |\cdot|_Y)$ be two real normed spaces. The symbol $\mathcal{L}(X,Y)$ stands for the space of all bounded linear operators from X to Y. If $Y = \mathbb{R}$, then $X^* := \mathcal{L}(X,\mathbb{R})$ is the dual space of X. The symbol ${}_{X^*}\langle \cdot, \cdot \rangle_X$ denotes the standard duality pairing. If both spaces are separable Hilbert, then by $L_{\mathrm{HS}}(X,Y)$ we will denote the Hilbert space of all Hilbert-Schmidt operators from X to Y (see Section A.3). If X is a separable Hilbert space and Y a Banach space, then by $R(X,Y)$ we will denote the separable Banach space of γ-radonifying operators from X to Y (see Section A.4).
Assume that X, Y are Hilbert spaces with scalar products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$ respectively. Let $T : X \supset D(T) \to Y$ be a densely defined linear operator. By T^* we denote the adjoint operator of T. In particular, $D(T^*) \subset Y$, $T^* : D(T^*) \to X$ and

$$\langle Tx, y \rangle_Y = \langle x, T^* y \rangle_X, \qquad x \in D(T), \quad y \in D(T^*). \tag{II.0.1.1}$$

Note that $D(T^*) = Y$ if T is bounded.

II.0.1.2 Functional spaces

Let $q \in [1,\infty)$ and $d = 1, 2$. By $L^q = \left[L^q(\mathbb{R}^2)\right]^d$ we denote the Banach space of Lebesgue measurable $\mathbb{R}^d$-valued p-th power integrable functions on $\mathbb{R}^2$. [1] The norm in L^q is given by

$$\|v\|_{L^q} = \left(\sum_{k=1}^{d} \|v_k\|_{L^q}^q\right)^{\frac{1}{q}} = \left(\sum_{k=1}^{d} \int_{\mathbb{R}^2} |v_k(x)|^q \, \mathrm{d}x\right)^{\frac{1}{q}}, \qquad v \in L^q.$$

By $L^\infty = \left[L^\infty(\mathbb{R}^2)\right]^d$ we denote the Banach space of Lebesgue measurable essentially bounded $\mathbb{R}^2$-valued functions defined on $\mathbb{R}^2$. The norm is given by

$$\|v\|_{L^\infty} = \sum_{k=1}^{d} \|v^k\|_{L^\infty} = \sum_{k=1}^{d} \text{esssup}\{|v^k(x)|, x \in \mathbb{R}^2\}, \qquad v \in L^\infty.$$

If $q = 2$, then L^2 is a Hilbert space with scalar product given by

$$\langle u, v\rangle_{L^2} := \int_{\mathbb{R}^2} u(x) \cdot v(x) \, \mathrm{d}x, \qquad u, v \in L^2.$$

By $\mathbb{L}^q$, $1 \le q \le \infty$ we denote the spaces

$$\mathbb{L}^q = \{u \in L^q : \nabla \cdot u = 0\} \tag{II.0.1.2}$$

with norm inherit from L^q.

Let us denote by $C_0^\infty := \left[C_0^\infty(\mathbb{R}^2)\right]^d$ the space consisting of all mappings $v \in \left[C^\infty(\mathbb{R}^2)\right]^d$ with compact support and, let

$$C_{\text{sol}}^\infty := \{u \in C_0^\infty : \nabla \cdot u = 0\}. \tag{II.0.1.3}$$

Notice that $\mathbb{L}^q =$ the closure of C_{sol}^∞ in L^q.
By $\mathcal{S} := \left[\mathcal{S}(\mathbb{R}^2)\right]^d$ we denote the Schwartz space, that is the space of all infinitely differentiable functions on $\mathbb{R}^2$ for which the semi norms

$$P_{\alpha,\beta}(\varphi) = \sup_{x \in \mathbb{R}^2} |x^\alpha D^\beta \varphi(x)|, \quad \alpha, \beta \in \mathbb{N}$$

are finite. The dual $\mathcal{S}' := \left[\mathcal{S}'(\mathbb{R}^2)\right]^d$ of $\mathcal{S}$ is the space of tempered distributions. By $\langle \cdot, \cdot \rangle$ we denote the usual duality action of $\mathcal{S}'$ on $\mathcal{S}$.
For $s \in \mathbb{R}$ and $1 \le q \le \infty$, set $J^s = (I - \Delta)^{\frac{s}{2}}$. We define the generalized Sobolev spaces as

$$W^{s,q} = \{u \in \mathcal{S}' : \|J^s u\|_{L^q} < \infty\}. \tag{II.0.1.4}$$

We have (see [7]) that J^σ is an isomorphism between $W^{s,q}$ and $W^{s-\sigma,q}$. For $s_1 < s_2$ $W^{s_2,q} \subset W^{s_1,q}$ and the dual space of $W^{s,q}$ is $W^{-s,p}$ with $1 < p \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. We denote by $\langle \cdot, \cdot \rangle$ the $W^{s,q} - W^{-s,p}$ duality bracket:

$$\langle u, v\rangle = \sum_{k=1}^{d} \int_{\mathbb{R}^2} (J^s u_k)(x)(J^{-s} v_k)(x) \, \mathrm{d}x.$$

[1] We shall use the same notation for the case $d = 1$ and $d = 2$. The context shall make clear the case we are considering. Anyway in some ambiguous cases we will specify the dimension d in order to clarify if we are dealing with vector or scalar fields. Moreover, in the definition of divergence-free vector spaces the case $d = 1$ makes no sense. In this case we are implicitly supposing $d = 2$.

Recall that for $s \in \mathbb{N}$, $W^{s,q}$ are the classical Sobolev spaces. In particular, let us focus on the case $s = 1$ needed in the sequel. Let $q \in (1, \infty)$. By $W^{1,q} = \left[W^{1,q}(\mathbb{R}^2)\right]^d$ we denote the Sobolev space of all $u \in \left[L^q(\mathbb{R}^2)\right]^d$ for which there exist weak derivatives $\frac{\partial u}{\partial x_i} \in \left[L^q(\mathbb{R}^2)\right]^d$, $i = 1, 2$. This spaces are endowed with the norm

$$\|u\|^q_{W^{1,q}} = \|u\|^q_{L^q} + \|\nabla u\|^q_{L^q}.$$

If $q = 2$ then $W^{1,2}$ is a Hilbert space with the scalar product given by

$$\langle u, v\rangle_{W^{1,2}} = \langle u, v\rangle_{L^2} + \langle \nabla u, \nabla v\rangle_{L^2}, \qquad u, v \in W^{1,2},$$

where

$$\langle \nabla u, \nabla v\rangle_{L^2} = \sum_{k=1}^{d} \int_{\mathbb{R}^2} \frac{\partial u}{\partial x_k} \cdot \frac{\partial v}{\partial x_k}\, \mathrm{d}x, \qquad u, v \in W^{1,2}.$$

Since we are on the whole space $\mathbb{R}^2$ Poincaré inequality does not holds, thus, in particular we do not have the equivalence of the norms $\|u\|_{W^{1,q}}$ and $\|\nabla u\|_{L^q}$. Nevertheless, we have the following result (see [18, Lemma 3.1]).

Lemma II.0.1.1. *Let $q \in (1, \infty)$. There is a constant C such that $\|\nabla v\|_{L^q} \leq C\|\mathrm{curl}\ v\|_{L^q}$ for every $v \in W^{1,q}$.*

In particular, since $\|\mathrm{curl} v\|_{L^q} \leq c\|\nabla v\|_{L^q}$, for a suitable constant c, we get the equivalence of the norms

$$\|\nabla u\|_{L^q} \sim \|\mathrm{curl} u\|_{L^q}. \tag{II.0.1.5}$$

Next we introduce the generalized Sobolev spaces of divergence free vectors distributions as

$$H^{s,q} = \{u \in W^{s,q} : \nabla \cdot u = 0\} \tag{II.0.1.6}$$

with norm inherit from $W^{s,q}$. The divergence is understood in the distributional sense. Notice that, for $s \in \mathbb{N}$, $H^{s,q} =$ the closure of C^∞_{sol} in $W^{s,q}$.
By $H^{1,2}_{L^2}$ we shall denote the space $H^{1,2}$ endowed with the strong L^2-topology.

We denote by $\mathbb{L}^2_{loc}$ the space $\mathbb{L}^2$ with the topology generated by the seminorms $\|u\|_{\mathbb{L}^2_R} = \left(\int_{|x|<R} |u(x)|^2\, \mathrm{d}x\right)^{\frac{1}{2}}$, $R \in \mathbb{N}$. Similarly, we denote by $L^2(0, T; \mathbb{L}^2_{loc})$ the space $L^2(0, T; \mathbb{L}^2)$ with the topology generated by the seminorms $\|u\|_{L^2(0,T;\mathbb{L}^2_R)} = \left(\int_0^T \int_{|x|<R} |u(x)|^2\, \mathrm{d}x\, \mathrm{d}t\right)^{\frac{1}{2}}$.

We denote by $C([0, T]\,; L^2_W)$ the space of L^2-valued weakly continuous functions with the topology of uniform weak convergence on $[0, T]$; in particular $v_n \to v$ in $C([0, T]\,; L^2_W)$ means

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} |\langle v_n(t) - v(t), h\rangle_{L^2}| = 0 \tag{II.0.1.7}$$

for all $h \in L^2$.

For $0 < \beta < 1$ by $C^\beta([0, T]\,; H^{s,2})$ we denote the Banach space of $H^{s,2}$-valued β-Hölder continuous functions endowed with the following norm

$$\|u\|_{C^\beta([0,T];H^{s,2})} = \sup_{0\leq t\leq T} \|u(t)\|_{H^{s,2}} + \sup_{0\leq s<t\leq T} \frac{\|u(t) - u(s)\|_{H^{s,2}}}{|t - s|^\beta}.$$

Let $q, \alpha > 1$, we denote by $L^\alpha_W(0, T; L^q)$ the space $L^\alpha(0, T; L^q)$ with the weak topologyby and by $L^\infty_W(0, T; L^q)$ the space $L^\infty(0, T; L^q)$ with the weak-* topology.

II.0.1.3 Some embedding theorems

We recall the following Sobolev inequalities that holds in the case $d = 2$.

Theorem II.0.1.2. *i. For every $2 < q < \infty$ the space $W^{1,q}$ is continuously embedded in L^∞, namely there exists a constant C (depending on q) such that:*

$$\|v\|_{L^\infty} \leq C\|v\|_{W^{1,q}}, \tag{II.0.1.8}$$

ii. For every $b > 1$ the space $W^{b,2}$ is continuously embedded in L^∞, namely there exists a constant C (depending on b) such that:

$$\|v\|_{L^\infty} \leq C\|v\|_{W^{b,2}}. \tag{II.0.1.9}$$

iii. For every $q \in [2, \infty)$, $W^{1,2}$ is continuously embedded in L^q, namely there exists a constant C (depending on q) such that:

$$\|v\|_{L^q} \leq C\|v\|_{W^{1,2}}.$$

iv. For every $q \in (2, \infty)$, $W^{1,q}$ is continuously embedded into the space of Hölder continuous functions C^α, $\alpha < 1 - \frac{2}{q}$.

Proof. For statements (i) and (iv) see [9, Theorem 9.12], for statement (ii) see [9, Corollary 9.13] and for statement (iii) see [9, Corollary 9.11]. □

Since $\mathbb{R}^2$ is an unbounded domain, the embedding of $H^{1,2}$ into $\mathbb{L}^2$ is not compact. However, by [44, Lemma 2.5] (see also [17, Lemma C.1]), there exists a separable Hilbert space $\mathbb{U}$ such that

$$\mathbb{U} \subset H^{1,2} \subset \mathbb{L}^2,$$

the embedding i of $\mathbb{U}$ into $H^{1,2}$ being dense and compact. Then we have

$$\mathbb{U} \underset{i}{\subset} H^{1,2} \subset \mathbb{L}^2 \simeq (\mathbb{L}^2)^* \subset H^{-1,2} \underset{i^*}{\subset} \mathbb{U}^* \tag{II.0.1.10}$$

where $(\mathbb{L}^2)^*$ and $H^{-1,2}$ are the dual spaces of $\mathbb{L}^2$ and $H^{1,2}$ respectively, $(\mathbb{L}^2)^*$ being identified with $\mathbb{L}^2$ and i^* is the dual operator to the embedding i. Moreover, i^* is compact as well. The same considerations hold also when we consider the spaces $W^{1,2}$ and L^2. In this case we shall denote by $\mathbb{V}$ the Hilbert space such that $\mathbb{V} \subset W^{1,2} \subset L^2$.

II.0.1.4 Operators

Now we define the operators appearing in the abstract formulation of (II.0.0.2). We refer to [86] and [18] for the details.

Let us consider the projection operator $\mathcal{P}$ which projects vectors into solenoidal vectors. For $u \in L^q$ we set

$$\mathcal{P}u = u - \nabla p, \tag{II.0.1.11}$$

where p is a solution to

$$\Delta p = \nabla \cdot u.$$

The operator $\mathcal{P}$ is a bounded linear projection in L^q and its range is equal to $\mathbb{L}^q$ (see [18, Lemma 3.2]). As usual we introduce the Stokes operator as

$$\begin{cases} A = -\mathcal{P}\Delta = -\Delta\mathcal{P} \\ \mathrm{Dom}(A) = H^{2,q}. \end{cases}$$

It is a linear unbounded operator in $W^{s,q}$ and $H^{s,q}$ ($s \in \mathbb{R}, 1 \leq q < \infty$); it generates a contractive and analytic C_0-semigroup $\{S(t)\}_{t\geq 0}$. Let us notice that, since $\mathcal{P}$ commutes with the laplacian Δ, A is essentially equal to $-\Delta$ and the semigroup $\{S(t)\}_{t\geq 0}$ is essentially the heat semigroup. The following theorem gathers useful analytical properties of A (for a proof see [18, Appendix A]).

Theorem II.0.1.3. *i. For every $u, v \in H^{1,2}$ it holds $\langle Au, v\rangle = \langle \nabla u, \nabla v\rangle_{L^2}$.*

ii. Let $q \in (1, \infty)$ and let q^* be the conjugate to q. Then there is a constant C such that for $v \in \mathrm{Dom}(A)$, $u \in H^{1,q^*}$ one has $|\langle Av, u\rangle| \leq C\|v\|_{H^{1,q}}\|u\|_{H^{1,q^*}}$.

Let us introduce the bilinear form B as follows. For $u, v \in C^\infty_{\mathrm{sol}}$ write

$$B(u, v) = -\mathcal{P}((u \cdot \nabla)v) = -\mathcal{P}(\mathrm{div}(uv_1), \mathrm{div}(uv_2)). \tag{II.0.1.12}$$

For brevity we shall write $B(v) = B(v, v)$. The following theorem gathers the main properties of B we shall need in the following.

Theorem II.0.1.4. *Let $q \in [2, \infty)$, and let B be given by* (II.0.1.12). *Then*

i. the form B is bounded from $\left[H^{1,2}\right]^2 \times \left[H^{1,2}\right]^2$ into $\left[H^{-1,2}\right]^2$ and for every $u, v, z \in \left[H^{1,2}\right]^2$,

$$\langle B(u, v), z\rangle = -\langle B(u, z), v\rangle, \tag{II.0.1.13}$$

ii. for every $u, v \in \left[H^{1,q}\right]^2$, $\langle B(u, v), v\rangle = 0$.

iii. For every $u, v \in \left[H^{2,q}\right]^2$, $\langle \mathrm{curl}\,(B(u, v)), \mathrm{curl} v\ |\mathrm{curl}\ v|^{q-2}\rangle = 0$.

iv. If $q > 2$ then there exists a constant C such that $\|B(u, v)\|_{\mathbb{L}^q} \leq C\|u\|_{[H^{1,q}]^2}\|v\|_{[H^{1,q}]^2}$ for all $u, v \in \left[H^{1,q}\right]^2$ and $\|B(u, v)\|_{\mathbb{L}^2} \leq C\|u\|_{[H^{1,q}]^2}\|v\|_{[H^{1,2}]^2}$ for all $u, v \in \left[H^{1,2} \cap H^{1,q}\right]^2$.

v. For every $q > 2$,

$$\langle |u|^{q-2}u, B(u, u)\rangle = 0, \qquad \forall\ u \in \left[H^{1,2}\right]^2. \tag{II.0.1.14}$$

vi. The operator B maps $\mathbb{L}^2 \times H^{a,2}$ into $\mathbb{L}^2$, for $a > 2$. In particular, there exists a constant C such that

$$\|B(u, v)\|_{\mathbb{L}^2} \leq C\|u\|_{\mathbb{L}^2}\|v\|_{H^{a,2}}.$$

Proof. Statement (i) is a classical result, see for instance [86]. See [18, Theorem 3.2] for a proof of statements (ii)-(iv). Statement (v) is a consequence of the integration by parts formula. As regards statement (vi), from Hölder's inequality and the embedding theorem (II.0.1.9) it follows

$$\|B(u, v)\|_{\mathbb{L}^2} \leq \|u\|_{\mathbb{L}^2}\|\nabla v\|_{\mathbb{L}^\infty} \leq C\|u\|_{\mathbb{L}^2}\|\nabla v\|_{H^{b,2}},$$

for $b > 1$. □

Let us now focus on the non linear term appearing in the equation for the vorticity (II.0.0.3). We define the bilinear operator

$$\langle F(u,\xi),\zeta\rangle := \int_{\mathbb{R}^2} u(x)\cdot\nabla\xi(x)\zeta(x)\,\mathrm{d}x, \qquad \forall u\in [C^\infty_{sol}]^2,\ \xi,\zeta\in C^\infty_0. \tag{II.0.1.15}$$

We summarize the properties of F we shall need in the following.

Lemma II.0.1.5. *i. The operator F is bounded from $\left[H^{1,2}\right]^2\times W^{1,2}$ into $W^{-1,2}$.*

ii. It holds

$$\langle F(u,\xi),\zeta\rangle = -\langle F(u,\zeta),\xi\rangle, \qquad \langle F(u,\xi),\xi\rangle = 0 \qquad \forall u\in\left[H^{1,2}\right]^2, \zeta,\xi\in W^{1,2}. \tag{II.0.1.16}$$

iii. For every $q>2$ we get

$$\langle F(u,\xi),\zeta|\zeta|^{q-2}\rangle = -(q-1)\langle F(u,\zeta),|\zeta|^{q-2}\xi\rangle, \qquad \forall\xi\in W^{1,2},\ \zeta\in L^{2(q-1)}, u\in\mathbb{L}^\infty \tag{II.0.1.17}$$

and

$$\langle F(u,\xi),\xi|\xi|^{q-2}\rangle = 0, \qquad \forall\xi\in W^{1,2},\ u\in\mathbb{L}^\infty \tag{II.0.1.18}$$

iv. F can be extended to a bounded bilinear operator from $[\mathbb{L}^4]^2\times L^4$ to $W^{-1,2}$ and

$$\|F(u,\xi)\|_{W^{-1,2}} \le \|u\|_{\mathbb{L}^4}\|\xi\|_{L^4}. \tag{II.0.1.19}$$

Proof. Statement (iv) follows by the Hölder and Sobolev inequalities:

$$|\langle F(u,\xi),\zeta\rangle| \le \|u\|_{\mathbb{L}^4}\|\nabla\xi\|_{L^2}\|\zeta\|_{L^4}.$$

This, in turn, proves statement (i) since $H^{1,2}$ is dense in L^4. Statements (ii) is a consequence of the integration by parts formula. Statement (iii) is obtained by integrating by parts, where in (II.0.1.17) the proof is done first with smooth functions and then by density is extended on the spaces specified. Notice that the l.h.s. side of (II.0.1.17) is well defined since

$$|\langle F(u,\xi),\zeta|\zeta|^{q-2}\rangle| \le \|u\|_{L^\infty}\|\nabla\xi\|_{L^2}\|\zeta\|_{L^{2(q-1)}} \le \|u\|_{L^\infty}\|\xi\|_{W^{1,2}}\|\zeta\|_{L^{2(q-1)}}.$$

Eventually, (II.0.1.18) is a particular case of (II.0.1.17). □

Chapter II.1

Existence and uniqueness of solutions with regular multiplicative noise

II.1.1 Introduction

The present Chapter is inspired, in particular, by the work of Brzeźniak and Peszat [18] where the authors prove the existence of a martingale solution to the stochastic Euler equation on the whole plane $\mathbb{R}^2$. The main difference of our result is that we can use the regularizing effect of the Laplacian proving a better regularity for the solution. Moreover we prove that pathwise uniqueness holds. Then the solution will be a strong (in the probability sense) solution (see for instance [45]) [1] .

We shall assume the same regularity on the noise term as in [18]. Following the main ideas of this paper we prove the existence of a martingale solution to (II.0.0.1) with the following regularity

$$v \in L^p(\Omega; L^\infty(0,T; H^{1,2} \cap H^{1,q}) \cap L^2(\Omega; L^2(0,T; H^{2,2})), \quad p > 1, q > 2.$$

Since we work on an unbounded domain, the embedding of the Sobolev space of functions with square integrable gradient into the L^2 space is not compact. Compactness is crucial in proving the existence of a solution and so we introduce spaces with weights. The construction of a solution is based on the Faedo-Galerkin approximation, i.e.

$$\begin{cases} \mathrm{d}v^n(t) + \Big[P^{(n)}Av^n(t) + P^{(n)}B_n(v(t), v(t))\Big]\,\mathrm{d}t = \mathcal{P}G_n(v^n(t))\,\mathrm{d}W(t) & t \in [0,T] \\ v^n(0,x) = v_0^n(x) & x \in \mathbb{R}^2 \end{cases} \tag{II.1.1.1}$$

given in Section II.1.4.1. In (II.1.1.1), $P^{(n)}$ represent the orthogonal projection into finite dimensional spaces, B_n, G_n and v_0^n are some regularizations of B, G and v_0 respectively. The solutions v^n to the Galerkin scheme generate a sequence of laws $\{\mathcal{L}(v^n)\}_{n\in\mathbb{N}}$. We prove that this sequence of probability measures is weakly compact by proving its tightness in suitable spaces. Then we construct a martingale solution applying the method used by Da Prato and Zabczyk in [25]. This method is based on the Skorokhod Theorem and the martingale representation Theorem. It is crucial to obtain uniform estimates for $n \in \mathbb{N}$ in the Sobolev

[1] Some results given in what follows are identical to results contained in [18]. In certain cases we choose to report the full proofs.

spaces $H^{1,2}$ and $H^{1,q}$, namely

$$\sup_{n\in\mathbb{N}} \mathbb{E} \sup_{0\le t\le T} \left[\|v^n(t)\|^p_{H^{1,2}} + \|v^n(t)\|^p_{H^{1,q}}\right] < \infty, \tag{II.1.1.2}$$

and in the proof of (II.1.1.2) the equation for the vorticity will play a crucial role. The weak compactness of the family of the laws of $\{v^n, n\in\mathbb{N}\}$ is performed in the weighted space $L^2(0,T;\left[L^2(\mathbb{R}^2; e^{-|x|^2}\,\mathrm{d}x)\right]^2)$. Notice that, in order to obtain a better regularity for the solutions, we use the theory of stochastic integration in Banach spaces. In particular, for suitable initial values, we can show, via a Sobolev embedding, that the solution to (II.0.0.1) is space-Hölder continuous (see Remark II.1.3.3).

Next, under a Lipschitz assumptions on the covariance operator G, we prove the pathwise uniqueness of the solution and from this, in particular, we infer the existence of a unique strong solution to (II.0.0.1). As a consequence we have the existence of a unique strong solution ξ to (II.0.0.3) with the following regularity

$$\xi \in L^p(\Omega; L^\infty(0,T; L^2\cap L^q)) \cap L^2(\Omega; L^2(0,T; W^{1,2})), \quad p>1, q>2.$$

As we shall make clear in Section II.1.6 the existence result, formulated in a rather abstract form, covers the case in which the random forcing term is a $\mathbb{R}$-valued spatially homogeneous Wiener random field. In this case the map G can be of the Nemytski form. There are some good physical motivations to consider the particular case of a spatially homogeneous noise. In many situations experiments show that statistical properties of a turbulent flow are the same at every point of the fluid, except near the boundaries. In mathematical terms, this means that the laws of the processes that we consider (velocity in our case) should be invariant under space translations. Such processes will be called spatially homogeneous.

The present Chapter is organized as follows. In Section II.1.2 we make the assumptions on the noise driving the first equation of (II.0.0.1) and from these we deduce the regularity of the noise driving equation (II.0.0.3). In Section II.1.3 the main results are stated. In Section II.1.4 we give the proof of existence and construction of the martingale solution to (II.0.0.1). In Section II.1.5 the uniqueness result is proved. Finally, in Section II.1.6 the spatially homogeneous noise case is considered and in Section II.1.7 we provide some bibliographical references concerning this specific topic.

II.1.2 Random forcing term

We define the noise forcing term. Given a real separable Hilbert space $\mathcal{H}$, we consider a cylindrical $\mathcal{H}$-Wiener process W defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a right continuous filtration. We can write

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) h_k, \qquad t\in[0,T], \tag{II.1.2.1}$$

where $\{\beta_k\}_{k\in\mathbb{N}}$ is a sequence of standard independent identically distributed Wiener processes defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ and $\{h_k\}_{k\in\mathbb{N}}$ is a complete orthonormal system in $\mathcal{H}$.

We shall make a set of assumptions for the covariance of the noise driving the equation for the velocity (II.0.0.1). Following [18], for the covariance G we make the following assumptions:

(G1) the mapping $G : H^{1,2} \to L_{\mathrm{HS}}(\mathcal{H}; W^{1,2})$ is well defined and there exists $a_1 > 0$ such that

$$\|G(v)\|_{L_{\mathrm{HS}}(\mathcal{H};W^{1,2})} \le a_1(1 + \|v\|_{H^{1,2}}), \qquad \forall v \in H^{1,2}.$$

(G2) Let $q > 2$, the mapping $G : H^{1,q} \to R(\mathcal{H}; W^{1,q})$ is well defined and there exists $a_2 > 0$ such that

$$\|G(v)\|_{R(\mathcal{H};W^{1,q})} \le a_2(1 + \|v\|_{H^{1,q}}), \qquad \forall v \in H^{1,q}.$$

(G3) For all $z \in C^\infty_{sol}$ the real valued function $v \mapsto |G(v)^* z|_{\mathcal{H}}$ is continuous on $H^{1,2}_{L^2}$.

(G4) If assumption **(G1)** holds, then there exist constants $L_1 \ge 0$ and $0 \le L_2 < 1$ such that

$$\|G(v_1) - G(v_2)\|_{L_{\mathrm{HS}}(\mathcal{H};W^{1,2})} \le L_1\|v_1 - v_2\|_{L^2} + L_2\|\nabla v_1 - \nabla v_2\|_{L^2}, \qquad \forall v_1, v_2 \in H^{1,2}.$$

Remark II.1.2.1. *We will prove that the set of measures induced on appropriate spaces by the solutions of the Galerkin equations is tight provided that assumptions **(G1)**-**(G2)** are satisfied. Assumption **(G3)** will be important in passing to the limit as $n \to \infty$ in the Galerkin approximation. Assumption **(G4)** is needed in the proof of uniqueness of the solution. Notice that the existence of a solution does not require assumption **(G4)**: it is sufficient to approximate the covariance operator G in such a way it becomes Lipschitz (see Section II.1.4.1) and then pass to the limit.*

Formally, the noise driving equation (II.0.0.3) is obtained by taking the curl of the noise driving equation (II.0.0.1), namely, bearing in mind (II.1.2.1), it is formally given by

$$\mathrm{curl}(G(v)W(t)) = \sum_{k=1}^{\infty} \beta_k(t)\mathrm{curl}(G(v)h_k), \qquad t \in [0, T]. \tag{II.1.2.2}$$

Notice that, for all $v \in H^{1,2} \cap H^{1,q}$, with $q > 2$ and $k \in \mathbb{N}$, $G(v)h_k \in H^{1,2} \cap H^{1,q}$. Roughly speaking, by taking the curl of this latter quantity we loose one order of derivability, namely $\mathrm{curl}(G(v)h_k) \in L^2 \cap L^q$. Formally, we introduce the operator $\tilde{G}$ in the following way: given $v \in H^{1,2} \cap H^{1,q}$, for all $\psi \in \mathcal{H}$, $\tilde{G}(v)(\psi) := \mathrm{curl}(G(v)\psi)$. Thus we have that the mapping $\tilde{G}$ is well defined from $H^{1,2}$ to $L_{\mathrm{HS}}(\mathcal{H}; L^2)$ and from $H^{1,q}$ to $R(\mathcal{H}; L^q)$. With a little abuse of notation, we shall write

$$\tilde{G} = \mathrm{curl}G, \tag{II.1.2.3}$$

and the random forcing term appearing in equation (II.0.0.3) will be written as $\tilde{G}(v)\mathrm{d}W(t)$ instead of $\mathrm{curl}(G(v)\mathrm{d}W(t))$.

II.1.3 Main results

We shall prove the existence of a martingale solution to problem (II.0.0.1). We give the following notion of solution.

Definition II.1.3.1. *For $q \in [2, \infty)$ let $v_0 \in H^{1,2} \cap H^{1,q}$. Assume that **(G1)** and **(G2)** hold. A martingale solution to the Navier-Stokes problem* (II.0.0.1) *is a triple consisting of a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, an $\{\mathcal{F}_t\}$-adapted cylindrical $\mathcal{H}$-Wiener process $W(t), t \in [0, T]$, and an $\{\mathcal{F}_t\}$-adapted measurable $H^{1,2} \cap H^{1,q}$-valued process v, such that* *i. for every* $p \in [1,\infty)$, $v \in L^p(\Omega; L^\infty(0,T; H^{1,2} \cap H^{1,q})) \cap L^2(\Omega; L^2(0,T; H^{2,2}))$;

ii. for all $z \in C^\infty_{sol}$ *and* $t \in [0,T]$ *one has* $\mathbb{P}$*-a.s.*

$$\langle v(t), z\rangle = \langle v_0, z\rangle + \int_0^t \langle \Delta v(s), z\rangle \, \mathrm{d}s + \int_0^t \sum_{i=1}^2 \langle v_i(s) v(s), \nabla z_i\rangle \, \mathrm{d}s + \langle \int_0^t G(v(s)) \, \mathrm{d}W(s), z\rangle. \tag{II.1.3.1}$$

In the definition of the martingale solution the incompressibility condition is contained in the requirement that v belongs to $H^{1,2} \cap H^{1,q}$. Since $\nabla \cdot v = 0$, the term $v \cdot \nabla v_i$ has been replaced by $\nabla \cdot (v v_i)$. Note that the gradient of the pressure vanishes after projecting both sides of (II.0.0.1) onto the divergence free space and that (ii) is the weak (in the PDEs sense) form of the projected equation. Here is the main result we prove.

Theorem II.1.3.2. *Let* $q \in [2,\infty)$. *Assume that* ***(G1)-(G3)*** *hold. Then for any* $v_0 \in H^{1,2} \cap H^{1,q}$ *there exists a martingale solution to the problem* (II.0.0.1). *Moreover,* $v \in C([0,T]; \mathbb{L}^2)$ $\mathbb{P}$*-a.s..*

Proof of Theorem II.1.3.2 is a variation of proof [18, Theorem 2.1]: there the authors consider the Euler equations on the whole $\mathbb{R}^2$ perturbed by a multiplicative noise term satisfying the same assumptions we made. The main difference with [18] is that here we can use the smoothing properties of the dissipative term to obtain more regularity for the solution process.

Remark II.1.3.3. *If* $q > 2$ *then by the Sobolev embedding theorem (Theorem II.0.1.2 (iv)),* $W^{1,q}$ *is continuously embedded into the space of Hölder continuous functions* C^α, $\alpha < 1 - \frac{2}{q}$. *Thus, as* $H^{1,q}$ *is a subspace of* $W^{1,q}$ *we get the space continuity of the solution to* (II.0.0.1).

We prove a uniqueness result as well.

Theorem II.1.3.4. *Let assumptions of Theorem II.1.3.2 hold. Moreover assume* ***(G4)***. *Then pathwise uniqueness holds for system* (II.0.0.1).

Pathwise uniqueness and existence of martingale solutions imply existence of a strong solution. We recall the definition.

Definition II.1.3.5. *Let* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ *be a fixed filtered probability space, and* $W(t), t \in [0,T]$, *be an* $\{\mathcal{F}_t\}$*-adapted cylindrical* $\mathcal{H}$*-Wiener process. Let* $v_0 \in H^{1,2} \cap H^{1,q}$ *for* $q \in [2,\infty)$. *Assume that* ***(G1)*** *and* ***(G2)*** *hold. A strong solution to the Navier-Stokes problem* (II.0.0.1) *is an* $\{\mathcal{F}_t\}$*-adapted measurable* $H^{1,2} \cap H^{1,q}$*-valued process* v *satisfying conditions (i)-(ii) of Definition II.1.3.1.*

Therefore Theorems II.1.3.2 and II.1.3.4 imply.

Theorem II.1.3.6. *Under the same assumptions of Theorem II.1.3.4 there exists a unique strong solution to* (II.0.0.1).

As a byproduct of the results stated above, we get the existence and uniqueness results concerning the solution to the vorticity equation (II.0.0.3). Similarly to the previous definition we give the following.

Definition II.1.3.7. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a fixed filtered probability space, and $W(t), t \in [0,T]$, be an $\{\mathcal{F}_t\}$-adapted cylindrical $\mathcal{H}$-Wiener process. Let $\xi_0 \in L^2 \cap L^q$ and $v_0 \in \mathbb{L}^2 \cap \mathbb{L}^q$, for $q \in [2,\infty)$. Assume that **(G1)** and **(G2)** hold. A strong solution to the vorticity equation* (II.0.0.3) *is an $\{\mathcal{F}_t\}$-adapted measurable $L^2 \cap L^q$-valued process ξ such that*

i. for every $p \in [1,\infty)$, $\xi \in L^p(\Omega; L^\infty(0,T; L^2 \cap L^q)) \cap L^2(\Omega; L^2(0,T; W^{1,2}))$;

ii. for all $z \in C^\infty_{sol}$ and $t \in [0,T]$ one has $\mathbb{P}$-a.s.

$$\langle \xi(t), z\rangle = \langle \xi_0, z\rangle - \int_0^t \langle \nabla\xi(s), \nabla z\rangle \,\mathrm{d}s + \int_0^t \langle v(s)\xi(s), \nabla z\rangle \,\mathrm{d}s + \langle \int_0^t \tilde{G}(v(s)) \,\mathrm{d}W(s), z\rangle, \tag{II.1.3.2}$$

where v is the solution to (II.0.0.1).

By Theorem II.1.3.6 we deduce the following result.

Corollary II.1.3.8. *Let $q \in [2,\infty)$. Let the same assumptions of Theorem II.1.3.6 hold. Then for any $\xi_0 \in L^2 \cap L^q$, $v_0 \in \mathbb{L}^2 \cap \mathbb{L}^q$ there exists a unique strong $L^2 \cap L^q$-valued solution to system* (II.0.0.3).

II.1.4 Existence of a martingale solution to the Navier-Stokes equations

The present Section concerns the existence of a martingale solution to (II.0.0.2). The proof is a modification of the results of [18] where, we recall, authors consider the two dimensional stochastic Euler equation on the whole plane $\mathbb{R}^2$. In order to prove the existence of a martingale solution they consider a smoothed Faedo-Galerkin scheme of the Euler equation. In particular a diffusion term $\nu\Delta$, $\nu > 0$, is added in order to use its smoothing effect and obtain the desired estimates. In the tightness argument, passing from the finite dimensional approximation to the infinite dimensional non approximated equation, $\nu \to 0$, to recover the Euler equation. The main difference of our result is that we maintain the regularizing effect of the Laplacian also in the limit equation. In this way we prove more regularity for the solution.

II.1.4.1 Smoothed Faedo-Galerkin approximations

The first step to prove the existence of a solution to (II.0.0.2) is to approximate the full equations. We approximate the nonlinear term B and the covariance operator G in such a way they become Lipschitz in appropriate functional spaces. Then we consider a sequence of finite dimensional stochastic differential equations, the Faedo-Galerkin systems. Since all the coefficients are Lipschitz, these SDEs admit a unique solution. The crucial point is to prove the desired estimates uniformly in $n \in \mathbb{N}$.

Let $\{e_k\}_k \subset H^{2,2} \cap H^{2,q}$ be an orthonormal basis of $H^{1,2}$. We assume that it is also a Schauder basis. [2] Let $P^{(n)}$ and P_n be the orthonormal projection of $H^{1,2}$ into the spaces $\mathcal{S}_n := span\{e_1, ..., e_n\}$ and $span\{e_n\} = \mathbb{R}e_n$ respectively. Let $\hat{P}^{(n)} : H^{1,2} \to \mathbb{R}$ be defined by $\hat{P}^{(n)}(v)(e_n) = P_n(v)$, $v \in H^{1,2}$. Let $q > 2$, note that there is a constant C such that

$$\|P^{(n)}v\|_{H^{1,q}} \leq C\|v\|_{H^{1,q}} \quad \text{and} \quad \|P_n v\|_{H^{1,q}} \leq C\|v\|_{H^{1,q}} \tag{II.1.4.1}$$

[2] Recall that (see e.g. [9]), given a Banach space E, a sequence $\{e_k\}_{k\geq 1}$ is said to be a Schauder basis if for every $u \in E$ there exists a unique sequence $\{\alpha_n\}_{n\geq 1}$ in $\mathbb{R}$ such that $u = \sum_{k=1}^\infty \alpha_k e_k$.

for all n and $v \in H^{1,2} \cap H^{1,q}$. Thus $P^{(n)}$ and P_n can be treated as a linear projection on $H^{1,q}$.

From now on, we assume that assumptions **(G1)-(G3)** are fulfilled. Let us start by constructing a suitable approximation of G. Let $\rho \in C_0^\infty(\mathbb{R})$ be a non-negative function with the support in $[0,1]$ and such that $\int_{\mathbb{R}} \rho(x)\,\mathrm{d}x = 1$. Let $\mathbf{1}_n = \mathbf{1}_{[-n,n]}$. Recall that, for all $\psi \in \mathcal{H}$ and for all $v \in H^{1,2} \cap H^{1,q}$, $G(v)\psi \in W^{1,2} \cap W^{1,q}$. For such ψ and v we define

$$[G_n(v)\psi] = n^{-n} P^{(n)} \int_{\mathbb{R}^n} \left[G\left(\sum_{i=1}^n x_i e_i \right) \psi \right] \mathbf{1}_n \left(\left| \sum_{i=1}^n x_i e_i \right|_{H^{1,2}\cap H^{1,q}} \right) \times \rho\left(n(\hat{P}^{(1)} v - x_1) \right) \cdots \rho\left(n(\hat{P}^{(n)} v - x_n) \right) \mathrm{d}x.$$

Notice that $G_n(\cdot)$ is bounded and globally Lipschitz from $H^{1,2}$ into $L_{\mathrm{HS}}(\mathcal{H}; W^{1,2})$ and from $H^{1,q}$ into $R(\mathcal{H}; W^{1,q})$ (with bounds possibly depending on n). Moreover, there exist positive constants $\tilde{a}_1$ and $\tilde{a}_2$ such that, for all $n \in \mathbb{N}$, $v \in H^{1,2} \cap H^{1,q}$ it holds

$$\|G_n(v)\|_{L_{\mathrm{HS}}(\mathcal{H};W^{1,2})} \leq \tilde{a}_1(1 + \|v\|_{H^{1,2}}) \tag{II.1.4.2}$$

and

$$\|G_n(v)\|_{R(\mathcal{H};W^{1,q})} \leq \tilde{a}_2(1 + \|v\|_{H^{1,q}}). \tag{II.1.4.3}$$

Let now consider a suitable approximation of the nonlinear term B. Let $\varphi_n : H^{1,2} \cap H^{1,q} \to H^{1,2} \cap H^{1,q}$ be defined by

$$\varphi_n(u) := \begin{cases} u, & \text{if } \|u\|_{H^{1,2}\cap H^{1,q}} \leq n, \\ n\|u\|^{-1}_{H^{1,2}\cap H^{1,q}} u, & \text{otherwise.} \end{cases}$$

Notice that we can write the function $\varphi_n(u)$ as $\varphi_n(u) = u\Theta_n(\|u\|_{H^{1,2}\cap H^{1,q}})$, where the function $\Theta_n : [0,+\infty) \to [0,1]$ is defined as

$$\Theta_n(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq n \\ \frac{n}{s} & \text{if } s > n. \end{cases} \tag{II.1.4.4}$$

The function φ_n is bounded and globally Lipschitz (see [10, Appendix]). Define $\bar{B}_n(u,v) := B(\varphi_n(u), v)$ and $B_n(v) := B(\varphi_n(v), v)$. It follows from Theorem II.0.1.4 (iv) that B_n is a globally Lipschitz map from $H^{1,2} \cap H^{1,q}$ to $\mathbb{L}^2 \cap \mathbb{L}^q$. Moreover, $P^{(n)} B_n$ is a global Lipschitz continuous function from $\mathcal{S}_n$ into $\mathcal{S}_n$.
Finally, let $v_0^n = P^{(n)} v_0$.

Let us consider the smoothed Faedo-Galerkin approximation scheme in the space $\mathcal{S}_n$,

$$\begin{cases} \mathrm{d}v^n(t) + \left[P^{(n)} A v^n(t) + P^{(n)} B_n(v^n(t), v^n(t)) \right] \mathrm{d}t = \mathcal{P} G_n(v^n(t))\,\mathrm{d}W(t) & t \in [0,T] \\ \nabla \cdot v^n(t) = 0 & t \in [0,T] \\ v^n(0,x) = v_0^n(x) & x \in \mathbb{R}^2. \end{cases} \tag{II.1.4.5}$$

We will prove the following Theorem about *a priori* estimates of the solution v^n to (II.1.4.5).

Theorem II.1.4.1. *Let $v_0 \in H^{1,2} \cap H^{1,q}$. Let W be a cylindrical $\mathcal{H}$-Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$. Then, for every $n \in \mathbb{N}$, there is a unique adapted and continuous $H^{1,2} \cap H^{1,q}$-valued global strong solution v^n to the system* (II.1.4.5)*, that is a process satisfying the following conditions:*

i. $v^n \in L^p(\Omega; L^2(0,T; H^{2,2} \cap H^{2,q}) \cap C([0,T]; H^{1,2} \cap H^{1,q}))$ *for any* $p \in [1,\infty)$*;*

ii. v^n *is a solution to the smoothed Faedo-Galerkin system on* $[0,T]$*, namely for any* $t \in [0,T]$*, it satisfies* $\mathbb{P}$*-a.s.*

$$v^n(t) = v_0^n - \int_0^t \left[P^{(n)} A v^n(s) + P^{(n)} B_n(v^n(s))\right] \mathrm{d}s + \int_0^t \mathcal{P} G_n(v^n(s))\,\mathrm{d}W(s) \quad \text{(II.1.4.6)}$$

Moreover, for any $p \in (1,\infty)$*,*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{t\in[0,T]} \left(\|v^n(t)\|^p_{H^{1,2}} + \|v^n(t)\|^p_{H^{1,q}}\right)\right] < \infty \quad \text{(II.1.4.7)}$$

and

$$\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^T \|\nabla v^n(t)\|^2_{H^{1,2}}\,\mathrm{d}t < \infty. \quad \text{(II.1.4.8)}$$

Since B_n and G_n are globally Lipschitz continuous and $v_0^n \in H^{2,2} \cap H^{2,q}$, the existence and uniqueness of a global mild solution v^n to (II.1.4.5), with the stated regularity, follows from [15, Therorem 4.10]. It follows next from [14, Lemma 4.5] that in fact is a strong solution. Hence Theorem II.1.4.1 will be proved once Lemma II.1.4.2 and Lemma II.1.4.3 formulated below, are verified. The proof of statement (II.1.4.7) is given in [18, Lemma 5.1 and Lemma 5.2]. We recall here the proof (see Lemma II.1.4.2 and Lemma II.1.4.3) for the sake of completeness and since some estimates are required for the proof of (II.1.4.8), given in Lemma II.1.4.2, which does not appear in [18] (recall that there authors consider the Euler equation).

Lemma II.1.4.2. *Let* v^n *be a solution to* (II.1.4.5)*. Then for any* $p \in [1,\infty)$ *there exist finite constants* C_1 *and* C_2 *independent of* n *such that*

$$\mathbb{E} \sup_{t\in[0,T]} \|v^n(t)\|^p_{H^{1,2}} \le C_1 \quad \text{(II.1.4.9)}$$

and

$$\mathbb{E}\int_0^T \|\nabla v^n(t)\|^2_{H^{1,2}}\,\mathrm{d}t \le C_2. \quad \text{(II.1.4.10)}$$

Proof. Let us fix $p \in [2,\infty)$ and $n \in \mathbb{N}$. Set

$$\psi(t) := \mathbb{E} \sup_{0\le s\le t} \|v^n(s)\|^{4p}_{L^2} \qquad \text{and} \qquad \varphi(t) := \mathbb{E} \sup_{0\le s\le t} \|v^n(s)\|^{4p}_{H^{1,2}}$$

For the sake of clearity let us split the proof in different steps.

Step 1. The first step will be to show that there exists a constant C_0 such that, for all $0 \le k \le t \le T$,

$$\psi(t) \le C_0 \left(1 + \psi(k) + \varphi(t)(t-k)\right). \quad \text{(II.1.4.11)}$$

Inequality (II.1.4.11) is obtained by means of Itô formula applied to the function $H(u) = \|u\|^{2p}_{L^2}$, Burkholder-Davies-Gundy's inequality and Theorem II.0.1.4(iii).

Step 2. Let us consider the equation for the vorticity. For every $n \in \mathbb{N}$ set

$$\xi^n(t,x) := \text{curl } v^n(t,x) \qquad \text{and} \qquad \theta(t) = \mathbb{E} \sup_{0\le s\le t} \|\xi^n(s)\|_{L^2}^{4p}.$$

Then, for every $n \in \mathbb{N}$, ξ^n satisfies the following SPDE

$$\frac{\partial \xi^n}{\partial t} = \left[\Delta \xi^n - (v^n \cdot \nabla \xi^n)\Theta_n(\|v^n\|_{H^{1,2}\cap H^{1,q}})\right] \mathrm{d}t + (\text{curl } G_n(v^n))\, \mathrm{d}W(t). \qquad \text{(II.1.4.12)}$$

Let us recall that (see Section II.1.2), by an abuse of notation, we write $(\text{curl } G_n(v^n))\mathrm{d}W(t)$ instead of $\text{curl}(G_n(v^n)\mathrm{d}W(t))$.

For every $n \in \mathbb{N}$, by uniqueness, since (II.1.4.12) has a unique strong L^2 solution (that follows from the fact that (II.1.4.5) admits a unique strong $H^{1,2}$ solution), ξ^n is a strong L^2-valued solution. Hence we can apply again the Itô formula to the function $H(u) = \|u\|_{L^2}^{2p}$. By means of Burkholder-Davies-Gundy's inequality and Theorem II.0.1.4(iii) we obtain the following estimate, valid for all $0 \le k \le t \le T$ and a suitable constant C_1:

$$\theta(t) \le C_1 \left[1 + \theta(k) + (\theta(t) + \varphi(t))\,(t-k)\right]. \qquad \text{(II.1.4.13)}$$

Step 3. Taking into account Lemma II.0.1.1 we can find constants C_2 and C_3 such that

$$C_2\,(\psi(t) + \theta(t)) \le \varphi(t) \le C_3\,(\psi(t) + \theta(t)).$$

So, combining (II.1.4.11) and (II.1.4.13) we get, for suitable constants α_1, α_2, α_3

$$\begin{aligned} \varphi(t) &\le C_3\,(\psi(t) + \theta(t)) \\ &\le C_3\left[C_0\,(1 + \psi(k) + \varphi(t)(t-k)) + C_1(1 + \theta(k) + (\theta(t) + \varphi(t))\,(t-k))\right] \\ &\le \alpha_1 + \alpha_2(1 + \varphi(k)) + \alpha_3(1 + \varphi(t)(t-k)). \end{aligned}$$

Namely, for a suitable constant α it holds

$$\varphi(t) \le \alpha(1 + \varphi(k) + \varphi(t)(t-k)). \qquad \text{(II.1.4.14)}$$

Let $0 = t_1 < \cdots < t_l = T$ be a partition of $[0,T]$ such that for every $i = 1,...,l-1$, $|t_{i+1} - t_i| \le \frac{1}{2}$. Then from (II.1.4.14) it follows

$$\varphi(t_{i+1}) \le \alpha\,(1 + \varphi(t_i) + \varphi(t_{i+1})(t_{i+1} - t_i)) \le \alpha\left(1 + \varphi(t_i) + \frac{\varphi(t_{i+1})}{2}\right)$$

which implies

$$\varphi(t_{i+1}) \le 2\alpha(1 + \varphi(t_i)), \qquad i = 1,...,l-1. \qquad \text{(II.1.4.15)}$$

Iterating inequality (II.1.4.15) we get

$$\mathbb{E} \sup_{0\le t\le T} \|v^n(t)\|_{H^{1,2}}^{4p} = \varphi(T) \le \sum_{j=1}^{l-1} (2\alpha)^j + (2\alpha)^{j-1}\|v_0\|_{H^{1,2}}^{4p}$$

where the estimate does not depend on n. This proves (II.1.4.9).

Step 4. We exploit the regularizing effect of the dissipative term to prove (II.1.4.10).

Let us now prove the estimates stated in Steps 1, 2 and 4.

Proof of Step 1. For $t \in [0,T]$ let us set

$$\Phi(t) = P^{(n)}\Delta v^n(t) - P^{(n)}B_n(v^n(t)) \qquad \text{and} \qquad \mathcal{G}_n(t) = \mathcal{P}G_n(v^n(t)).$$

Let $k \le t$. Since v^n is a strong $\mathbb{L}^2$-valued solution to (II.1.4.6) we can apply Itô formula and infer that

$$\begin{aligned}\|v^n(t)\|_{L^2}^{2p} &= \|v^n(k)\|_{L^2}^{2p} + \int_k^t \langle H'(v^n(s)), \mathcal{G}_n(s)\mathrm{d}W(s)\rangle \\ &\quad + \int_k^t \left[\langle H'(v^n(s)), \Phi(s)\rangle + \frac{1}{2}\mathrm{Tr}\left(H''(v^n(s))\mathcal{G}_n(s)(\mathcal{G}_n(s))^*\right)\right] \mathrm{d}s \\ &=: \|v^n(k)\|_{L^2}^{2p} + \int_k^t \langle H'(v^n(s)), \mathcal{G}_n(s)\mathrm{d}W(s)\rangle + \int_k^t A(s)\,\mathrm{d}s\end{aligned}$$

From Theorem II.0.1.4 (ii) and Theorem II.0.1.3 (i), using the trivial inequality $\|\cdot\|_{L^2} \le \|\cdot\|_{H^{1,2}}$, we obtain

$$\langle H'(v^n(s)), \Phi(s)\rangle = -2p\|v^n(s)\|_{L^2}^{2(p-1)}\|\nabla v^n(s)\|_{L^2}^2 \le 0.$$

Denoting by c_i, $i = 1, ..3$ constants that depends on p and $\tilde{a}_1$, for the second term in the deterministic integral, using (II.1.4.2), we get

$$\begin{aligned}&\frac{1}{2}\mathrm{Tr}\left(H''(v^n(s))\mathcal{G}_n(s)(\mathcal{G}_n(s))^*\right) \\ &\quad = p\|v^n(s)\|_{L^2}^{2(p-2)}\left[\|v^n(s)\|_{L^2}^2\|\mathcal{G}_n(s)\|_{L_{\mathrm{HS}}(\mathcal{H};H^{1,2})}^2 + 2(p-1)\|(\mathcal{G}_n(s))^* v^n(s)\|_{L^2}^2\right] \\ &\quad \le c_1\|v^n(s)\|_{L^2}^{2p-2}\|G_n(v^n(s))\|_{L_{\mathrm{HS}}(\mathcal{H};W^{1,2})}^2 \\ &\quad \le c_2(1 + \|v^n(s)\|_{H^{1,2}}^{2p}),\end{aligned}$$

thus

$$A(s) \le c_3\left(1 + \|v^n(s)\|_{H^{1,2}}^{2p}\right).$$

Therefore, denoting by c_i, $i = 4, ..., 6$ constants that depends on p, T and $\tilde{a}_1$, we obtain

$$\psi(t) \le c_4\left(1 + \psi(k) + c_5\varphi(t)(t-k)\right) + \mathbb{E}\sup_{k\le s\le t}\left|\int_k^s \langle H'(v^n(r)), \mathcal{G}_n(r)\mathrm{d}W(r)\rangle\right|^2.$$

Applying Burkholder-Davies-Gundy's and using the trivial inequality $\|\cdot\|_{L^2} \le \|\cdot\|_{H^{1,2}}$, from (II.1.4.2) we obtain

$$\begin{aligned}&\mathbb{E}\sup_{k\le s\le t}\left|\int_k^s \|v^n(r)\|_{L^2}^{2(p-1)}\langle v^n(r), \mathcal{G}_n(r)\mathrm{d}W(r)\rangle\right|^2 \\ &\qquad \le \mathbb{E}\int_k^t \|v^n(r)\|_{L^2}^{4(p-1)}\|v^n(r)\|_{L^2}^2\|\mathcal{G}_n(r)\|_{L_{\mathrm{HS}}(\mathcal{H};H^{1,2})}^2\,\mathrm{d}r \\ &\qquad \le c_6(1+\varphi(t))(t-k).\end{aligned}$$

Collecting all the above estimates we get (II.1.4.11) for a suitable constant C_0.

Proof of Step 2. For $t \in [0,T]$ set

$$\Psi(t) = P^{(n)}\Delta\xi^n(t) + \operatorname{curl}(P^{(n)}B_n(v(t))) \qquad \text{and} \qquad \tilde{G}_n(t) = \operatorname{curl}(\mathcal{P}G_n(v^n(t))). \tag{II.1.4.16}$$

As observed before, since ξ is a strong L^2-valued solution to equation (II.1.4.12), we can apply the Itô formula to $H(\xi) = \|\xi\|_{L^2}^{2p}$. We get

$$\begin{aligned}\|\xi^n(t)\|_{L^2}^{2p} &= \|\xi^n(k)\|_{L^2}^{2p} + \int_k^t \langle H'(\xi^n(s)), \tilde{G}_n(s)\mathrm{d}W(s)\rangle \\ &\quad + \int_k^t \left[\langle H'(\xi^n(s)), \Psi(s)\rangle + \frac{1}{2}\operatorname{Tr}\left(H''(\xi^n(s))\tilde{G}_n(s)(\tilde{G}_n(s))^*\right)\right] \mathrm{d}s.\end{aligned}$$

Using Theorems II.0.1.3(i) and II.0.1.4(iii) we obtain

$$\langle H'(\xi^n(s)), \Psi(s)\rangle = -2p\|\xi^n(s)\|_{L^2}^{2(p-1)}\|\nabla\xi^n(s)\|_{L^2}^2.$$

Moreover, denoting by c_7 and c_8 constants that depends on p, and p and $\tilde{a}_1$ respectively, from (II.1.4.2) and Young inequality, we get

$$\begin{aligned}&\frac{1}{2}\operatorname{Tr}\left(H''(\xi^n(s))\tilde{G}_n(s)(\tilde{G}_n(s))^*\right) \\ &\quad = p\|\xi^n(s)\|_{L^2}^{2(p-2)}\left[\|\xi^n(s)\|_{L^2}^2\|\tilde{G}_n(s)\|_{L_{\mathrm{HS}}(\mathcal{H};L^2)}^2 + 2(p-1)\|(\tilde{G}(s))^*\xi^n(s)\|_{L^2}^2\right] \\ &\quad \le c_7\|\xi^n(s)\|_{L^2}^{2(p-2)}\|G_n(v^n(s))\|_{L_{\mathrm{HS}}(\mathcal{H};W^{1,2})}^2 \\ &\quad \le c_8\left(1 + \|\xi^n(s)\|_{L^2}^{2p} + \|v^n(s)\|_{H^{1,2}}^{2p}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\|\xi^n(t)\|_{L^2}^{2p} &\le \|\xi^n(k)\|_{L^2}^{2p} - 2p\int_k^t \|\xi^n(s)\|_{L^2}^{2(p-1)}\|\nabla\xi^n(s)\|_{L^2}^2 \,\mathrm{d}s \\ &\quad + c_8\int_k^t \left(1 + \|\xi^n(s)\|_{L^2}^{2p} + \|v^n(s)\|_{H^{1,2}}^{2p}\right) \mathrm{d}s \\ &\quad + 2p\int_k^t \|\xi^n(s)\|_{L^2}^{2(p-1)}\langle\xi^n(s), \tilde{G}_n(s)\mathrm{d}W(s)\rangle.\end{aligned} \tag{II.1.4.17}$$

Applying the Burkholder-Davies-Gundy's inequality, denoting by c_9 constant that depends on p and $\tilde{a}_1$, by (II.1.4.2) and Young inequality we obtain

$$\begin{aligned}&\mathbb{E}\sup_{k\le s\le t}\left|\int_k^s \|\xi^n(s)\|_{L^2}^{2(p-1)}\langle\xi^n(s), \tilde{G}_n(s)\mathrm{d}W(s)\rangle\right|^2 \\ &\qquad \le \mathbb{E}\int_k^t \|\xi^n(s)\|_{L^2}^{4p-2}\|\tilde{G}_n(s)\|_{L_{\mathrm{HS}}(\mathcal{H};L^2)}^2 \,\mathrm{d}s \\ &\qquad \le c_9\mathbb{E}\int_k^t \left(1 + \|\xi^n(s)\|_{L^2}^{4p} + \|v^n(s)\|_{H^{1,2}}^{4p}\right) \mathrm{d}s.\end{aligned}$$

Combining these estimates we get (II.1.4.13) for a suitable constant C_1 depending on p, T and $\tilde{a}_1$.

Proof of Step 4. From (II.1.4.9), bearing in mind (II.0.1.5), it follows in particular

$$\sup_{n\in\mathbb{N}} \mathbb{E} \sup_{t\in[0,T]} \|\xi^n(t)\|^p_{L^2} < \infty. \tag{II.1.4.18}$$

Taking the expectation in (II.1.4.17) (for $k = 0$), from (II.1.4.9) and (II.1.4.18) it follows

$$\mathbb{E}\int_0^T \|\xi^n(s)\|^{2(p-1)}_{L^2} \|\nabla\xi(s)\|^2_{L^2}\, \mathrm{d}s \le C_{p,T,\tilde{a}_1,\|\xi_0\|^{2p}_{L^2}}.$$

For $p = 1$ this gives in particular

$$\mathbb{E}\int_0^T \|\nabla\xi^n(s)\|^2_{L^2}\, \mathrm{d}s < \infty, \qquad \forall n \in \mathbb{N}.$$

Thus (II.1.4.10) immediately follows.

□

Next lemma proves an estimate for solutions, uniform in $n \in \mathbb{N}$, in the space $H^{1,q}$, where $q \in (2,\infty)$. We shall use the inequality

$$\|v\|_{H^{1,q}} \le C\left(\|v\|_{H^{1,2}} + \|\mathrm{curl}\ v\|_{L^q}\right), \tag{II.1.4.19}$$

which is a consequence of Lemma II.0.1.1 and the imbedding $W^{1,2} \hookrightarrow L^q$ (see Theorem II.0.1.2 (iii)).

Lemma II.1.4.3. *Let v^n be a solution to* (II.1.4.5). *Then for any $p \in [1,\infty)$ there is a constant $C < \infty$ independent of n such that*

$$\mathbb{E} \sup_{t\in[0,T]} \|v^n(t)\|^p_{H^{1,q}} \le C.$$

Proof. We proceed similarly to the proof of Lemma II.1.4.2. Let us fix $n \in \mathbb{N}$ and $p \in [1,\infty)$. Define

$$\xi^n(t,x) = \mathrm{curl}\ v^n(t,x) \qquad \text{and} \qquad \theta(t) = \mathbb{E}\sup_{0\le s\le t} \|\xi^n(s)\|^{2p}_{L^q}.$$

We recall (see proof of Lemma II.1.4.2) that ξ^n satisfies (II.1.4.12) for every $n \in \mathbb{N}$. Assume $q > 2$. Note that ξ^n is both strong and mild solution to (II.1.4.12). In particular it follows from [14, Theorem 4.6 and Lemma 4.3] and from [15, Theorem 4.10] that for any $p \in [2,\infty)$,

$$\xi^n \in L^2(\Omega,\mathcal{F},\mathbb{P}; L^2(0,T;H^{2,q})) \cap L^p(\Omega,\mathcal{F},\mathbb{P}; L^\infty(0,T;H^{1,q})). \tag{II.1.4.20}$$

For (II.1.4.20) the uniqueness of a weak solution to (II.1.4.12) and the imbedding $H^{1,q} \hookrightarrow L^\infty$, $q > 2$, are used.

Let $p \in [q,\infty)$ and $0 \le k \le t \le T$. Let Ψ and $\tilde{G}_n$ be given by (II.1.4.16), then by (II.1.4.20), we can apply the Itô formula for $H(\xi) = \|\xi\|^p_{L^q}$. For a full explanation of the details see Chapter A.4 or [18, Section 4], in particular [18, Theorem 4.3. and Remark 4.5.]. We have

$$\begin{aligned}\|\xi^n(t)\|_{L^q}^p \le \|\xi^n(k)\|_{L^q}^p &+ p\int_k^t \|\xi^n(s)\|_{L^q}^{p-q}\langle|\xi^n(s)|^{q-2}\xi^n(s),\Psi(s)\rangle\,\mathrm{d}s\\ &+ p\int_k^t \|\xi^n(s)\|_{L^q}^{p-q}\langle|\xi^n(s)|^{q-2}\xi^n(s),\tilde G_n(s)\mathrm{d}W(s)\rangle\\ &+\frac{p(p-1)}{2}\int_k^t \|\xi^n(s)\|_{L^q}^{p-2}\|\tilde G_n(s)\|_{R(\mathcal{H};L^q)}^2\,\mathrm{d}s.\end{aligned}$$

Using the integration by parts formula and Theorems II.0.1.3(i) and II.0.1.4 (iii), we get

$$\begin{aligned}\int_k^t \|\xi^n(s)\|_{L^q}^{p-q}\langle|\xi^n(s)|^{q-2}\xi(s),\Psi(s)\rangle\,\mathrm{d}s &= \int_k^t \|\xi^n(s)\|_{L^q}^{p-q}\langle|\xi^n(s)|^{q-2}\xi^n(s),\Delta\xi^n(s)\rangle\,\mathrm{d}s\\ &+\int_k^t \|\xi^n(s)\|_{L^q}^{p-q}\langle|\xi^n(s)|^{q-2}\xi(s),\mathrm{curl}(B_n(u^n(s)))\rangle\,\mathrm{d}s\le 0.\end{aligned}$$

Thanks to (II.1.4.3), and using Lemma II.1.4.2 and (II.1.4.19), if we denote by c_1 a constant that depends on $\tilde a_2$, p, T and $\sup_n\|v^n\|_{H^{1,2}}$, we get

$$\begin{aligned}\frac{p(p-1)}{2}\int_k^t \|\xi^n(s)\|_{L^q}^{p-2}\|\tilde G_n(s)\|_{R(\mathcal{H};L^q)}^2\,\mathrm{d}s &\le \frac{\tilde a_2\,p(p-1)}{2}\int_k^t \|\xi^n(s)\|_{L^q}^{p-2}\left(1+\|v^n(s)\|_{H^{1,q})}^2\right)\mathrm{d}s\\ &\le c_1\left(1+\int_k^t \|\xi^n(s)\|_{L^q}^p\,\mathrm{d}s\right).\end{aligned}$$

So we get

$$\begin{aligned}\|\xi^n(t)\|_{L^q}^p &\le \|\xi^n(k)\|_{L^q}^p + p\int_k^t \|\xi^n(s)\|_{L^q}^{p-q}\langle|\xi^n(s)|^{q-2}\xi^n(s),\tilde G_n(s)\mathrm{d}W(s)\rangle\\ &+c_1\left(1+\int_k^t \|\xi^n(s)\|_{L^q}^p\,\mathrm{d}s\right).\end{aligned}$$

The Burkholder-Davies-Gundy's inequality (see Appendix A.4.3 or [18, Theorem 4.2]), (II.1.4.19) and Lemma II.1.4.2 yield

$$\begin{aligned}&\mathbb{E}\sup_{k\le s\le t}\left|p\int_k^s \|\xi^n(s)\|_{L^q}^{p-q}\langle|\xi(s)|^{q-2}\xi^n(s),\tilde G_n(s)\mathrm{d}W(s)\rangle\right|^2\\ &\le p^2\,\mathbb{E}\int_k^t \|\xi^n(s)\|_{L^q}^{2p-2}\|\tilde G_n(s)\|_{R(\mathcal{H};L^q)}^2\,\mathrm{d}s\\ &\le p^2\tilde a_2\,\mathbb{E}\int_k^t \|\xi^n(s)\|_{L^q}^{2p-2}\left(1+\|v^n(s)\|_{H^{1,q}}^{2p}\right)\mathrm{d}s\\ &\le c_2\left(1+\mathbb{E}\int_k^t \|\xi^n(s)\|_{L^q}^{2p}\,\mathrm{d}s\right),\end{aligned}$$

where c_2 is a constant depending on $\tilde a_2$, p, T and $\sup_n\|v^n\|_{H^{1,2}}$. Collecting the above estimates we finally obtain

$$\theta(t)\le C(1+\theta(k)+\theta(t)(t-k)), \qquad \text{(II.1.4.21)}$$

where C depends only on $\tilde{a}_2$, p, T and $\sup_n \|v^n\|_{H^{1,2}}$. Thus, taking a partition $0 = t_1 < \cdots < t_l = T$ such that $|t_{i+1} - t_i| < \frac{1}{2}$, thanks to (II.1.4.19) and Lemma II.1.4.2 we can proceed similarly to the proof of Lemma II.1.4.2 obtaining

$$\mathbb{E} \sup_{0\le t\le T} \|v^n(t)\|_{H^{1,q}}^{2p} \le \mathbb{E} \sup_{0\le t\le T} \left[\|v^n(t)\|_{H^{1,2}}^{2p} + \|\xi^n(t)\|_{L^q}^{2p}\right]$$
$$\le C(1+\theta(T)) \le C\left(1 + \sum_{j=1}^{l-1}(2C)^j + (2C)^{l-1}\|P^{(n)}v_0\|_{H^{1,q}}^{2p}\right).$$

Since $\{e_k\}_k$ is a Schauder basis of $H^{1,q}$, we have

$$\sup_{n\in\mathbb{N}} \|P^{(n)}v_0\|_{H^{1,q}} < \infty$$

and we get the desired result. □

II.1.4.2 Tightness

The construction of a martingale solution to (II.0.0.1) is based on a compactness method. For the sake of completeness let us recall some definitions and necessary tools.

Definition II.1.4.4. *Let S be a complete separable metric space and $\{u^n\}_{n\in\mathbb{N}}$ be a sequence of S-valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that the family of laws $\{\mathcal{L}(u^n)\}_{n\in\mathbb{N}}$ is relatively weakly compact if every sequence of elements of $\{\mathcal{L}(u^n)\}_{n\in\mathbb{N}}$ contains a weakly convergent subsequence. We say that $\{\mathcal{L}(u^n)\}_{n\in\mathbb{N}}$ is tight is for every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset S$ such that $\mathbb{P}(u^n \in K_\varepsilon) \ge 1 - \varepsilon$, for all $n \in \mathbb{N}$.*

The following theorem by Prohorov is the key for a simple characterization of sequences of probability measures that are relatively compact.

Theorem II.1.4.5. ***(Prohorov).*** *Let S be a complete separable metric space and $\{u^n\}_{n\in\mathbb{N}}$ be a sequence of S-valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the following statements are equivalent.*

- *$\{\mathcal{L}(u^n)\}_{n\in\mathbb{N}}$ is relatively compact,*
- *$\{\mathcal{L}(u^n)\}_{n\in\mathbb{N}}$ is tight.*

It is useful to characterize tightness using *compact* embeddings for Banach spaces. We shall need the Dubinsky criterion for compactness (for a proof see [89, Theorem 4.1]).

Lemma II.1.4.6. *Let E_0, E_1 and E be reflexive Banach spaces such that the imbeddings $E_0 \hookrightarrow E \hookrightarrow E_1$ are continuous and the imbedding $E_0 \hookrightarrow E$ is compact. Let $p \in (1, \infty)$ and let Γ be a bounded set in $L^p(0,T;E_0)$ consisting of equicontinuous functions in $C([0,T];E_1)$. Then Γ is relatively compact in $L^p(0,T;E)$ and $C([0,T];E_1)$.*

In the previous Section we have proved that for all $n \in \mathbb{N}$, v^n, is a solution to the smoothed Faedo-Galerkin equations (II.1.4.5). We assume that each v^n is defined on a filtered probably space $(\Omega, \mathcal{F}, \mathbb{P})$ and satisfies (II.1.4.5) driven by a cylindrical $\mathcal{H}$-Wiener process W. Let us denote by $\mathcal{L}(v^n)$ the law of v^n on the space of trajectories $C([0,T];H^{1,2} \cap H^{1,q})$. We aim at proving that this sequence is tight on an appropriate functional space. If we consider an unbounded domain, the embedding of the Sobolev space of functions with square integral gradient into the L^2 space, unlike in the bounded case, is not compact. As said above, compactness is crucial in order to prove the existence of solution and so it is necessary to introduce spaces with weights. Let $\theta \in C^\infty(\mathbb{R}^2)$ be a strictly positive even function equal to $\mathrm{e}^{-|x|}$ for $|x| \geq 1$, and let us denote by L^2_θ the weighted space $\left[L^2(\mathbb{R}^2; \theta(x)\,\mathrm{d}x)\right]^2$. For a proof of the following result see [19, Lemma 3.4(i)].

Lemma II.1.4.7. *The imbedding $W^{1,2} \hookrightarrow L^2_\theta$ is compact.*

Let us set

$$M^n(t) := \int_0^t \mathcal{P}G_n(v^n(s))\,\mathrm{d}W(s), \qquad t \in [0,T],$$

and let $\mathcal{L}(M^n)$ be the law of M^n on $C([0,T]; H^{1,2} \cap H^{1,q})$.

Lemma II.1.4.8. *The family $\mathcal{L}(M^n)$, $n \in \mathbb{N}$ is tight in $C([0,T]; L^2_\theta)$.*

Proof. Let $\varepsilon > 0$. We have to find a relatively compact set $\Gamma \in C([0,T]; L^2_\theta)$ such that $\mathcal{L}(M^n)(\Gamma) = \mathbb{P}(M^n \in \Gamma) > 1 - \varepsilon$ for all $n \in \mathbb{N}$. We shall use the Dubinsky criterion with $E_0 = H^{1,2}$ and $E_1 = E = L^2_\theta$; so it is sufficient to find a set Γ which is bounded in $L^p(0,T; H^{1,2})$ and consists of equicontinuous functions in $C([0,T]; L^2_\theta)$.
Let us fix $\alpha \in (0,1)$ and $p > \frac{1}{\alpha}$. Set

$$J\varphi(t) = \int_0^t (t-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s, \qquad t \in [0,T], \qquad \varphi \in L^p(0,T; H^{1,2}),$$

and

$$Z^n(t) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t (t-s)^{-\alpha}\mathcal{P}G_n(v^n(s))\,\mathrm{d}W(s), \qquad t \in [0,T].$$

By the stochastic Fubini theorem we have the factorization formula (see for instance [22, Chapter 2.2.1] for more details on the factorization method)

$$M^n = J(Z^n).$$

For every $n \in \mathbb{N}$ the process Z^n is well defined in $H^{1,2}$. Moreover,

$$\sup_{n\in\mathbb{N}} \mathbb{E} \int_0^T \|Z^n(t)\|^p_{H^{1,2}}\,\mathrm{d}t < \infty.$$

Infact, from Burkolder-Davies-Gundy's and Hölder's inequalities, for every $n \in \mathbb{N}$, $t \in [0,T]$, we get

$$\begin{aligned}
\mathbb{E}\|Z^n(t)\|^p_{H^{1,2}} &\leq C_\alpha \mathbb{E}\left[\int_0^t (t-s)^{-2\alpha}\|\mathcal{P}G_n(v^n(s))\|^2_{L_{\mathrm{HS}}(\mathcal{H};H^{1,2})}\,\mathrm{d}s\right]^{\frac{p}{2}} \\
&\leq C_\alpha \mathbb{E}\left[\int_0^t (t-s)^{-2\alpha}\|G(v^n(s))\|^2_{L_{\mathrm{HS}}(\mathcal{H};H^{1,2})}\,\mathrm{d}s\right]^{\frac{p}{2}} \\
&\leq C_{\alpha,T} \int_0^t (t-s)^{-\alpha p}\,\mathrm{d}s < \infty,
\end{aligned}$$

thanks to the growth property of G_n and (II.1.4.7). By the Chebyschev's inequality we infer that, for every $n \in \mathbb{N}$ and $r > 0$,

$$\mathbb{P}\left(\int_0^T \|Z^n(t)\|_{H^{1,2}}^p \,\mathrm{d}t > r\right) \leq \frac{\mathbb{E}\int_0^T \|Z^n(t)\|_{H^{1,2}}^p \,\mathrm{d}t}{r} \leq \frac{C_1}{r}.$$

Let R be such that $\frac{C_1}{R} < \varepsilon$, then

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left(\int_0^T \|Z^n(t)\|_{H^{1,2}}^p \,\mathrm{d}t > R\right) \leq \varepsilon.$$

Define the set

$$B = \left\{v \in L^p(0,T;H^{1,2}) : \int_0^T \|v(t)\|_{H^{1,2}}^p \,\mathrm{d}t \leq R_1\right\}.$$

B is bounded in $L^p(0,T;H^{1,2})$. Set $\Gamma = J(B)$. It is known that that J transforms bounded sets in $L^p(0,T;H^{1,2})$ into equicontinuous bounded sets in $C([0,T];H^{1,2})$. From Lemma II.1.4.7 we have then that Γ is bounded in $L^p(0,T;H^{1,2})$ and consists of equicontinuous functions in $C([0,T];L^2_\theta)$. From Lemma II.1.4.6 we infer that Γ is relatively compact in $C([0,T];L^2_\theta)$. Moreover,

$$\mathbb{P}(M^n \in \Gamma) = \mathbb{P}(J(Z^n) \in \Gamma) = \mathbb{P}(Z^n \in B) \geq 1 - \varepsilon.$$

□

A classical result is that $\{M^n(t)\}_n$ are square integrable continuous L^2_θ-martingales with quadratic variation

$$\ll M^n(t) \gg = \int_0^t \left[(j_{H^{1,2};L^2_\theta}\mathcal{P}G_n(v^n(s)))(j_{H^{1,2};L^2_\theta}\mathcal{P}G_n(v^n(s)))^*\right] \mathrm{d}s,$$

where $j_{H^{1,2};L^2_\theta}$ denotes the imbedding of $H^{1,2}$ into L^2_θ. We have the following consequence of Lemma II.1.4.8 and the Métivier-Nakao Theorem (see [60] or [61]).

Corollary II.1.4.9. *The family $\mathcal{L}(\ll M^n(t)) \gg)\}_n$ of the laws of $\{\ll M^n(t) \gg)\}_n$ is tight in $C([0,T];L_1(L^2_\theta, L^2_\theta))$.*

In the proof of Theorem II.1.3.2 we shall need the following result, (for the proof see [18, Lemma 6.4]).

Lemma II.1.4.10. *The family $\{\mathcal{L}(v^n)\}_n$, $n \in \mathbb{N}$ is tight in $L^2(0,T;L^2_\theta)$.*

II.1.4.3 Convergence

We are ready to prove Theorem II.1.3.2. The proof is based on the method used by Da Prato and Zabczyk in [25] and on the Skorokhod Theorem. Let us recall that this last result turns convergence in distribution into pointwise convergence, namely

Theorem II.1.4.11. ***(Skorokhod).*** *Let S be a complete separable metric space and $\{\mu_n\}_{n\in\mathbb{N}}$, μ be distributions on S such that $\lim_{n\to\infty} \mu_n = \mu$ weakly. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and S-valued random variables $\{X_n\}_{n\in\mathbb{N}}$ and X such that*

- *$\mathcal{L}(X^n) = \mu_n$, $n \in \mathbb{N}$ and $\mathcal{L}(X) = \mu$,*

- $\lim_{n\to\infty} X^n = X$ $\mathbb{P}$-*a.s.*.

We shall need the following technical lemma (for the proof see [18, Lemma 6.5]).

Lemma II.1.4.12. *Let $r \in (1,\infty)$, and let v^l, $l \in \mathbb{N}$, be a sequence of processes with trajectories in $L^\infty(0,T;W^{1,r})$, such that, for a fixed $p \in [1,\infty)$,*

$$\sup_{l\in\mathbb{N}} \mathbb{E}\|v^l\|^p_{L^\infty(0,T;W^{1,r})} < \infty.$$

If $\|v^l(t) - v(t)\|_{L^r} \to 0$, $\mathrm{d}t \times \mathbb{P}$-a.s., then the process v has trajectories in $L^\infty(0,T;W^{1,r})$ and $\mathbb{E}\|v\|^p_{L^\infty(0,T;W^{1,r})} < \infty$.

Proof of Theorem II.1.3.2. Let $\tilde{\mathcal{H}}$ be a Hilbert space such that $\mathcal{H} \hookrightarrow \tilde{\mathcal{H}}$ with a Hilbert-Schmidt imbedding. Then W is a process with continuous trajectories on $\tilde{\mathcal{H}}$. Set

$$\mathcal{A} = L^2(0,T;L^2_\theta) \times C([0,T];L^2_\theta) \times C([0,T];L^1(L^2_\theta;L^2_\theta)) \times C([0,T];\tilde{\mathcal{H}}).$$

By Lemmas II.1.4.8 and II.1.4.10 and Corollary II.1.4.9, the family of laws $\{\mathcal{L}(v^n, M^n, \ll M^n \gg, W)\}_n$ of $\{(v^n, M^n, \ll M^n \gg, W)\}_n$ on $\mathcal{A}$ is tight. Hence, by the Prokhorov theorem, it is relatively weakly compact. So, there exists a subsequence $\{n_l\}_{l\in\mathbb{N}}$ such that $\{(v^{n_l}, M^{n_l}, \ll M^{n_l} \gg, W)\}_{n_l}$ converges weakly as $l \to \infty$.

By the Skorokhod imbedding theorem there exists a probability space $\mathcal{Y} = (\tilde\Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_t, \tilde{\mathbb{P}})$, random elements in $\mathcal{A}$, (v, M, m, V) and $\{v^l, M^l, m^l, V^l\}_{l\in\mathbb{N}}$, defined on $\tilde\Omega$, such that

(S1): the laws of $(v^{n_l}, M^{n_l}, \ll M^{n_l} \gg, W)$ and $(v^l, M^l, \ll M^l \gg, V^l)$ are the same,

(S2): $(v^l, M^l, \ll M^l \gg, V^l) \to (v, M, \ll M \gg, V)$, $\tilde{\mathbb{P}}$-a.s. in $\mathcal{A}$.

Note that, for every l, V^l is a cylindrical $\mathcal{H}$-Wiener process. From (S1) it follows, in particular, that v^l is the solution to the appropriate Navier-Stokes equations (II.1.4.5) driven by V^l, with $v^l \in L^p(\Omega; L^2(0,T;H^{2,2} \cap H^{2,q}) \cap C([0,T];H^{1,2} \cap H^{1,q}))$ for any $p \in [1,\infty)$, $q > 2$. Moreover, for any $p \in (1,\infty)$,

$$\sup_{l\in\mathbb{N}} \tilde{\mathbb{E}}\left[\sup_{t\in[0,T]} \left(\|v^l(t)\|^p_{H^{1,2}} + \|v^l(t)\|^p_{H^{1,q}}\right)\right] < \infty \tag{II.1.4.22}$$

and

$$\sup_{l\in\mathbb{N}} \tilde{\mathbb{E}} \int_0^T \|\nabla v^l(t)\|^2_{H^{1,2}}\,\mathrm{d}t < \infty. \tag{II.1.4.23}$$

We shall prove that the limit $(v, M, \ll M \gg, W)$ is the martingale $H^{1,2} \cap H^{1,q}$-valued solution to the Navier-Stokes problem (II.0.0.1). For the sake of simplicity we split the proof in four steps.

Step 1. We prove that v belongs to the space $L^p(\Omega; L^\infty(0,T;H^{1,2} \cap H^{1,q})) \cap L^2(\Omega; L^2(0,T;H^{2,2}))$, $q > 2$, $p \in (1,\infty)$.

From (S2) in particular it follows that

$$\lim_{l\to\infty} \int_0^T \|v^l(t) - v(t)\|^2_{L^2_\theta}\,\mathrm{d}t = 0, \qquad \tilde{\mathbb{P}} - a.s. \tag{II.1.4.24}$$

Let $p \geq 2$. Since $H^{1,2} \subset L^2_\theta$, from (II.1.4.22) and the Fatou's lemma we obtain

$$\tilde{\mathbb{E}}\left[\int_0^T \|v(t)\|^2_{L^2_\theta}\,\mathrm{d}t\right]^p \leq \liminf_{l\to\infty} \tilde{\mathbb{E}}\left[\int_0^T \|v^l(t)\|^2_{L^2_\theta}\,\mathrm{d}t\right]^p < \infty.$$

Thus for any $p \geq 2$ we have

$$\sup_{l\in\mathbb{N}} \tilde{\mathbb{E}}\left[\int_0^T \|v^l(t) - v(t)\|^2_{L^2_\theta}\,\mathrm{d}t\right]^p \leq C_p \sup_{l\in\mathbb{N}} \tilde{\mathbb{E}}\left[\left(\int_0^T \|v^l(t)\|^2_{L^2_\theta}\,\mathrm{d}t\right)^p + \left(\int_0^T \|v(t)\|^2_{L^2_\theta}\,\mathrm{d}t\right)^p\right] < \infty.$$

Hence the sequence $\int_0^T \|v^l(t) - v(t)\|^2_{L^2_\theta}\,\mathrm{d}t$, $l \in \mathbb{N}$, is uniformly integrable. Thus it follows (see e.g. [53, Section 43]),

$$\lim_{l\to\infty} \tilde{\mathbb{E}} \int_0^T \|v^l(t) - v(t)\|^2_{L^2_\theta}\,\mathrm{d}t = 0. \tag{II.1.4.25}$$

At this point, taking a subsequence, we may assume that $v^l(t,x) \to v(t,x)$, $\mathrm{d}t \times \mathrm{d}x \times \tilde{\mathbb{P}}$-a.s. From (II.1.4.22) we know that v^l is bounded in $L^p(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^\infty(0,T;L^r))$, $r = 2, q$. So we may assume that $\|v^l(t) - v(t)\|_{L^r} \to 0$, $\mathrm{d}t \times \tilde{\mathbb{P}}$-a.s. for $r = 2, q$. Lemma II.1.4.12 yields that v has trajectories in $L^\infty(0,T;H^{1,2} \cap H^{1,q})$ and that for every $p \geq 2$ one has

$$\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]} \left(\|v(t)\|^p_{H^{1,2}} + \|v(t)\|^p_{H^{1,q}}\right)\right] < \infty. \tag{II.1.4.26}$$

Moreover, from (II.1.4.22) and (II.1.4.23) it holds

$$\sup_{l\in\mathbb{N}} \tilde{\mathbb{E}} \int_0^T \|v^l(t)\|^2_{H^{2,2}}\,\mathrm{d}t < \infty.$$

Thus we infer that the sequence $\{v^l\}_{l\in\mathbb{N}}$ contains a subsequence, still denoted by $\{v^l\}_{l\in\mathbb{N}}$ convergent weakly in the space $L^2([0,T]\times\tilde{\Omega}; H^{2,2})$. Since from (II.1.4.25) we know that $v^l(t,x) \to v(t,x)$, $\mathrm{d}t\times\mathrm{d}x\times\tilde{\mathbb{P}}$-a.s., we have that the weak limit of v^l is v and in particular we conclude that $v \in L^2([0,T]\times\tilde{\Omega}; H^{2,2})$ and

$$\tilde{\mathbb{E}} \int_0^T \|v(t)\|^2_{H^{2,2}}\,\mathrm{d}t < \infty.$$

Step 2. We prove that M is a square integrable martingale with quadratic variation

$$\ll M \gg (t) = \int_0^t \left[(j_{H^{1,2};L^2_\theta}\mathcal{P}G(v(s)))(j_{H^{1,2};L^2_\theta}\mathcal{P}G(v(s)))^*\right] \mathrm{d}s.$$

From (S1) we know that M^l is a square integrable martingale (w.r.t. $\tilde{\mathbb{P}}$). We can represent it as

$$M^l(t) = \int_0^t \mathcal{P}G_{n_l}(v^l(s))\,\mathrm{d}V^l(s), \qquad t \in [0,T],$$

and its quadratic variation is given by

$$m^l(t) := \ll M^l \gg (t) = \int_0^t \left[\left(j_{H^{1,2};L^2_\theta} \mathcal{P} G_{n_l}(v^l(s)) \right) \left(j_{H^{1,2};L^2_\theta} \mathcal{P} G_{n_l}(v^l(s)) \right)^* \right] \mathrm{d}s.$$

From (S2) we know that $\{M^l\}_l$ converges to M $\tilde{\mathbb{P}}$-a.s., where M is a square integrable L^2_θ-valued martingale with quadratic variation m. Our gaol is to prove that

$$\ll M \gg (t) = \int_0^t \left[\left(j_{H^{1,2};L^2_\theta} \mathcal{P} G(v(s)) \right) \left(j_{H^{1,2};L^2_\theta} \mathcal{P} G(v(s)) \right)^* \right] \mathrm{d}s. \qquad \text{(II.1.4.27)}$$

To do this, for fixed $t \in [0,T]$, we test the processes $M^l(t)$, $l \in \mathbb{N}$, against test functions φ. We consider the corresponding quadratic variation processes and we show the convergence in probability to the desired object. More formally, for every $\varphi \in C_0^\infty$, $t \in [0,T]$, $l \in \mathbb{N}$,

$$M^l_\varphi := \langle M^l(t), \varphi \rangle = \langle \int_0^t \mathcal{P} G_{n_l}(v^l(s)) \, \mathrm{d}V^l(s), \varphi \rangle$$

defines a real square integrable martingale with quadratic variation

$$J_l = \int_0^t \left| \left((j_{H^{1,2};L^2_\theta} \mathcal{P} G_{n_l}(v^l(s)) \right)^* \varphi \right|^2_{\mathcal{H}} \mathrm{d}s, \qquad l \in \mathbb{N}.$$

Set

$$J = \int_0^t \left| \left((j_{H^{1,2};L^2_\theta} \mathcal{P} G(v(s)) \right)^* \varphi \right|^2_{\mathcal{H}} \mathrm{d}s.$$

In order to prove (II.1.4.27) it is sufficient to show that J_l converges to J in probability. Fix $\varepsilon > 0$ and $t \in [0,T]$, then by Chebyschev's inequality we obtain

$$\tilde{\mathbb{P}}(|J - J_l| > \varepsilon) \leq \tilde{\mathbb{P}} \left(|J_l - J| > \varepsilon \text{ and } \sup_{0 \leq s \leq T} \|v^l(s)\|_{L^2} \leq R \right) + \tilde{\mathbb{P}} \left(\sup_{0 \leq s \leq T} \|v^l(s)\|_{L^2} > R \right).$$

Therefore the convergence in probability follows from **(G1)-(G3)**, (II.1.4.24), (II.1.4.26) and the Lebesgue dominated covergence Theorem.

Step 3. We prove that M can be represented as a stochastic integral w.r.t. a suitable cylindrical $\mathcal{H}$-Wiener process.

Thanks to the representation theorem (see [25, Theorem 8.2]) we can find a filtered probability space $(\Omega', \mathcal{F}', \{\mathcal{F}'_t\}_t, \mathbb{P}')$ and a cylindrical $\mathcal{H}$-Wiener process $\bar{W}$, which is defined on the probability space

$$\bar{\mathcal{U}} = (\bar{\Omega} = \tilde{\Omega} \times \Omega', \bar{\mathcal{F}} = \tilde{\mathcal{F}} \times \mathcal{F}', \{\bar{\mathcal{F}}_t\}_t = \{\tilde{\mathcal{F}}_t\}_t \times \{\mathcal{F}'_t\}_t, \bar{\mathbb{P}} = \tilde{\mathbb{P}} \times \mathbb{P}'),$$

such that the process $\mathcal{M}(t, \omega_1, \omega_2) = M(t, \omega_1)$ has the following form

$$\mathcal{M}(t, \omega_1, \omega_2) = \int_0^t \mathcal{P} G(u(s, \omega_1, \omega_2)) \, \mathrm{d}\bar{W}(s, \omega_1, \omega_2),$$

where

$$u(s,\omega_1,\omega_2) = v(s,\omega_1). \tag{II.1.4.28}$$

The process u is adapted to the filtration $\{\bar{\mathcal{F}}_t\}_t$. Moreover, for every $p \in [2,\infty,)$, by (II.1.4.26) it follows

$$\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}\left(\|u(t)\|^p_{H^{1,2}} + \|u(t)\|^p_{H^{1,q}}\right)\right] < \infty.$$

Step 4. It remains to show that the process u satisfies the integral equation (II.1.3.1). Let us fix $t \in [0,T]$ and $\varphi \in C_0^\infty \cap H^{a,2}$, $a > 2$, satisfying $\nabla \cdot \varphi = 0$. From (S2) and Step 3 it follows that, $\bar{\mathbb{P}}$-a.s.,

$$\begin{aligned}\langle \int_0^t \mathcal{P}G_{n_l}(v^l(s))\,\mathrm{d}V^l(s), \varphi\rangle &= \langle M^l(t), \varphi\rangle \to \langle M(t), \varphi\rangle \\ &= \langle \int_0^t \mathcal{P}G(u(s))\,\mathrm{d}\bar{W}(s), \varphi\rangle = \langle \int_0^t G(u(s))\,\mathrm{d}\bar{W}(s), \varphi\rangle.\end{aligned}$$

Clearly $v^l(0) \to u_0$ and

$$\int_0^t \langle v^l(s), \Delta\varphi\rangle\,\mathrm{d}s \to \int_0^t \langle u(s), \Delta\varphi\rangle\,\mathrm{d}s.$$

Finally we are left to prove that,

$$\int_0^t \langle P^{(n_l)}B_{n_l}(v^l(s), v^l(s)), \varphi\rangle\,\mathrm{d}s \to \int_0^t \langle B(u(s), u(s)), \varphi\rangle\,\mathrm{d}s.$$

Let us write (we drop the time parameter for simplicity)

$$\begin{aligned}|\langle P^{(n_l)}B_{n_l}(v^l, v^l), \varphi\rangle - \langle B(u,u), \varphi\rangle| &= |\langle P^{(n_l)}\bar{B}_{n_l}(v^l, \varphi), v^l)\rangle - \langle \bar{B}(u,\varphi), u\rangle| \\ &\le |\langle P^{(n_l)}\bar{B}_{n_l}(v^l - u, \varphi), v^l\rangle| + |\langle P^{(n_l)}\bar{B}_{n_l}(u,\varphi), v^l - u\rangle| \\ &\quad + |\langle P^{(n_l)}\bar{B}_{n_l}(u,\varphi) - \bar{B}(u,\varphi), u\rangle|.\end{aligned}$$

The above dualities are well defined as scalar products in L^2 and, thanks to Theorem II.0.1.4(vi) and (II.1.4.1) we can estimate the three terms as follows.

$$\begin{aligned}|\langle P^{(n_l)}\bar{B}_{n_l}(v^l - u, \varphi), v^l\rangle| &\le \|P^{(n_l)}\bar{B}_{n_l}(v^l - u, \varphi)\|_{L^2}\|v^l\|_{L^2} \\ &\le C\|v^l - u\|_{L^2}\|\varphi\|_{H^{a,2}}\|v^l\|_{L^2} \le C_1\|v^l - u\|_{L^2},\end{aligned}$$

where the last inequality follows from (II.1.4.22).

$$\begin{aligned}|\langle P^{(n_l)}\bar{B}_{n_l}(u,\varphi), v^l - u\rangle| &\le \|P^{(n_l)}\bar{B}_{n_l}(u,\varphi)\|_{L^2}\|v^l - u\|_{L^2} \\ &\le C\|v^l - u\|_{L^2}\|\varphi\|_{H^{a,2}}\|u\|_{L^2} \le C_2\|v^l - u\|_{L^2},\end{aligned}$$

where the last inequality follows from (II.1.4.26) and (II.1.4.28).

$$\begin{aligned}|\langle P^{(n_l)}\bar{B}_{n_l}(u,\varphi) - B(u,\varphi), u\rangle| &\le C\|u\|_{L^2}\|P^{(n_l)}\bar{B}_{n_l}(u,\varphi) - B(u,\varphi)\|_{L^2} \\ &\le C_3\|P^{(n_l)}\bar{B}_{n_l}(u,\varphi) - B(u,\varphi)\|_{L^2}.\end{aligned}$$

Collecting the above estimates we obtain

$$\int_0^t |\langle P^{(n_l)} B_{n_l}(v^l(s), v^l(s)), \varphi\rangle - \langle B(u(s), u(s)), \varphi\rangle| \, \mathrm{d}s$$
$$\leq (C_1 + C_2) \int_0^t \|v^l(s) - u(s)\|_{L^2} \, \mathrm{d}s$$
$$+ C_3 \int_0^t \|P^{(n_l)} B_{n_l}(u(s), \varphi) - B(u(s), \varphi)\|_{L^2} \, \mathrm{d}s.$$

For $l \to \infty$, the second term converges to zero by definition of B_{n_l}. As regards the first term, notice that from (II.1.4.25) and (II.1.4.28), taking a subsequence, we may assume that $v^l(t, x) \to u(t, x)$, $\mathrm{d}t \times \mathrm{d}x \times \bar{\mathbb{P}}$-a.s. Since $\{v^l\}$ is bounded in $L^p(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^\infty(0, T; L^2))$ we may assume that $\|v^l - u\| \to 0$, in $L^\infty(0, T; L^2)$ $\bar{\mathbb{P}}$-a.s. In particular the convergence holds $\bar{\mathbb{P}}$-a.s. in $L^1(0, T; L^2)$ and so the first term converges to zero as $l \to \infty$.

Thus u satisfies the integral equation in Definition II.1.3.1.

Finally, from [17, Lemma 7.2] it follows that $v \in C([0, T]; \mathbb{L}^2)$ $\bar{\mathbb{P}}$-a.s., and the proof is complete.

□

II.1.5 Pathwise uniqueness

We prove that solutions of (II.0.0.1) are pathwise unique. The proof uses the Schmalfuss idea of application of the Itô formula for an appropriate function (see [82]). Let us recall the following.

Definition II.1.5.1. *Suppose that whenever (u, W), $(\Omega, \mathcal{F}, \mathbb{P})$, $\{\mathcal{F}_t\}_t$ and $(\tilde{u}, W)$, $(\Omega, \mathcal{F}, \mathbb{P})$, $\{\tilde{\mathcal{F}}_t\}_t$ are martingale solutions with common noise W (relative to possibly different filtrations) on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with common initial value i.e. $\mathbb{P}(u_0 = \tilde{u}_0) = 1$, the processes u and $\tilde{u}$ are indistinguishable: $\mathbb{P}(u_t = \tilde{u}_t, \ \forall \ 0 \leq t < T) = 1$. We say that pathwise uniqueness holds for equation.*

Theorem II.1.5.2. *Let $q \in [2, \infty)$ and let $\xi_0 \in L^2 \cap L^q$, $v_0 \in \mathbb{L}^2 \cap \mathbb{L}^q$. Assume that **(G1)-(G4)** hold. Then pathwise uniqueness holds for system* (II.0.0.1).

Proof. Let v_1 and v_2 be two martingale solutions to system (II.0.0.1) with $v_1(0) = v_2(0)$. Let $V = v_1 - v_2$. This difference satisfies the equation

$$\begin{cases} \mathrm{d}V(t) + [AV(t) + B(v_1(t), v_1(t)) - B(v_2(t), v_2(t))] \, \mathrm{d}t = [G(v_1(t)) - G(v_2(t))] \, \mathrm{d}W(t) \\ V(0) = 0 \end{cases}$$

and this is equivalent to

$$\begin{cases} \mathrm{d}V(t) + [AV(t) + B(V(t), v_1(t)) + B(v_2(t), V(t))] \, \mathrm{d}t = [G(v_1(t)) - G(v_2(t))] \, \mathrm{d}W(t) \\ V(0) = 0 \end{cases}$$

We shall use the Itô formula for $\mathrm{d}\left(e^{-\int_0^t \psi(s) \, \mathrm{d}s} \|V(t)\|_{L^2}^2\right)$, by choosing ψ as

$$\psi(t) = (a\|\nabla v_1(t)\|_{L^2}^2 + 2L_1^2), \quad t \in [0, T],$$

where L_1 is the Lipschitz constant given in **(G4)** and a is a positive constant given later on. Recall that $v \in H^{1,2}$ and so $\psi \in L^1(0,T)$ $\mathbb{P}$-a.s.. For $t \in [0,T]$, we have

$$\mathrm{d}\left(e^{-\int_0^t \psi(s)\,\mathrm{d}s}\|V(t)\|_{L^2}^2\right) = -\psi(t)e^{-\int_0^t \psi(s)\,\mathrm{d}s}\|V(t)\|_{L^2}^2\mathrm{d}t + e^{-\int_0^t \psi(s)\,\mathrm{d}s}\mathrm{d}\|V(t)\|_{L^2}^2,$$

where the latter differential is given by

$$\begin{aligned}\mathrm{d}\|V(t)\|_{L^2}^2 = {} & 2\left[\langle AV(t),V(t)\rangle + \langle B(V(t),v_1(t)),V(t)\rangle + \langle B(v_2(t),V(t)),V(t)\rangle\right]\mathrm{d}t \\ & + 2\langle [G(v_1(t)) - G(v_2(t))]\,\mathrm{d}W(t), V(t)\rangle \\ & + \|G(v_1(t)) - G(v_2(t))\|_{L_{\mathrm{HS}}(\mathcal{H};W^{1,2})}^2.\end{aligned}$$

For the first term, by Theorem II.0.1.3(i), we get

$$\langle AV(t),V(t)\rangle = -\|\nabla V(t)\|_{L^2}^2.$$

As regards the non linear term, by Theorem II.0.1.4(i) and Gagliardo Nirenberg interpolation's inequality (see e.g. [9, Chapter 8.1]) we get

$$\langle B(V,v_1),V\rangle + \langle B(v_2,V),V\rangle = \langle B(V,v_1),V\rangle \le \|V\|_{L^4}^2\|\nabla v_1\|_{L^2} \le C\|V\|_{L^2}\|\nabla V\|_{L^2}\|\nabla v_1\|_{L^2}.$$

By Young inequality, we can infer that for all $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$2\langle B(V,v_1),V\rangle \le \varepsilon\|\nabla V\|_{L^2}^2 + C_\varepsilon\|\nabla v_1\|_{L^2}^2\|V\|_{L^2}^2.$$

By **(G4)** it follows

$$\|G(v_1) - G(v_2)\|_{L_{\mathrm{HS}}(\mathcal{H};W^{1,2})}^2 \le \left[L_1\|V\|_{L^2} + L_2\|\nabla V\|_{L^2}\right]^2 \le 2L_1^2\|V\|_{L^2}^2 + 2L_2^2\|\nabla V\|_{L^2}^2.$$

So we get

$$\begin{aligned}\mathrm{d}\|V(t)\|_{L^2}^2 \le {} & (-2+\varepsilon+2L_2^2)\|\nabla V(t)\|_{L^2}^2 + \left(C_\varepsilon\|\nabla v_1(t)\|_{L^2}^2 + 2L_1^2\right)\|V(t)\|_{L^2}^2 \\ & + \langle [G(v_1(t)) - G(v_2(t))]\,\mathrm{d}W(t), V(t)\rangle.\end{aligned}$$

Putting $a := C_\varepsilon$, we obtain

$$\begin{aligned}\mathrm{d}\left(e^{-\int_0^t \psi(s)\,\mathrm{d}s}\|V(t)\|_{L^2}^2\right) \le {} & (-2+\varepsilon+2L_2^2)e^{-\int_0^t \psi(s)\,\mathrm{d}s}\|\nabla V(t)\|_{L^2}^2 \\ & + e^{-\int_0^t \psi(s)\,\mathrm{d}s}\langle [G(v_1(t)) - G(v_2(t))]\,\mathrm{d}W(t), V(t)\rangle.\end{aligned}$$

Integrating in both sides we get

$$\begin{aligned} & e^{-\int_0^t \psi(s)\,\mathrm{d}s}\|V(t)\|_{L^2}^2 + (2-\varepsilon-2L_2^2)\int_0^t e^{-\int_0^r \psi(s)\,\mathrm{d}s}\|\nabla V(r)\|_{L^2}^2\,\mathrm{d}r \\ & \quad \le \int_0^t e^{-\int_0^r \psi(s)\,\mathrm{d}s}\langle [G(v_1(r)) - G(v_2(r))]\,\mathrm{d}W(r), V(r)\rangle. \end{aligned} \tag{II.1.5.1}$$

Let us choose $\varepsilon > 0$ such that $2-\varepsilon-2L_2^2 > 0$, then by (II.1.5.1) in particular, we have

$$e^{-\int_0^t \psi(s)\,\mathrm{d}s}\|V(t)\|_{L^2}^2 \le \int_0^t e^{-\int_0^r \psi(s)\,\mathrm{d}s}\langle [G(v_1(r)) - G(v_2(r))]\,\mathrm{d}W(r), V(r)\rangle.$$

Since the r.h.s. is a square integrable martingale, taking the expectation in both members we get

$$\mathbb{E}\left[e^{-\int_0^t \psi(s)\,\mathrm{d}s}\|V(t)\|_{L^2}^2\right] \leq 0, \qquad \forall t \in [0,T]\,,$$

thus in particular, for any $t \in [0,T]$

$$e^{-\int_0^t \psi(s)\,\mathrm{d}s}\|V(t)\|_{L^2}^2 = 0, \qquad \mathbb{P}-a.s..$$

Thus, if we take a sequence $\{t_k\}_{k=1}^{\infty}$, which is dense in $[0,T]$, we have

$$\mathbb{P}\{\|V(t_k)\|_{L^2} = 0 \text{ for all } k \in \mathbb{N}\} = 1.$$

Since each path of the process V belongs to $C([0,T]\,;\mathbb{L}^2)$, we get

$$\mathbb{P}\{v_1(t) = v_2(t) \text{ for all } t \in [0,T]\} = 1.$$

□

II.1.6 The spatially homogeneous noise case

The main existence result, formulated in Theorem II.1.3.6 in a rather abstract form, covers the particular case in which the equation is driven by a $\mathbb{R}$-valued spatially homogeneous Wiener random field and G is a Nemytski operator. In the present Section we provide a formal proof of the result. We point out here that all the results of the present Section are inspired by [18] and proofs are almost the same. We present them here for the sake of completeness.

Let us start by recalling the definition of spatially homogeneous Wiener random field (see [76] and [18]).

Definition II.1.6.1. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space. By a $\mathbb{R}$-valued spatially homogeneous Wiener random field on $[0,T] \times \mathbb{R}^2$ we understand a measurable real valued random field $\mathcal{W}$, on $[0,T] \times \mathbb{R}^2$ such that*

1. *the random vector $(\mathcal{W}(t_1,x_1), \ldots \mathcal{W}(t_n,x_n))$ is Gaussian for an arbitrary finite sequence $(t_1,x_1), \ldots (t_n,x_n) \in [0,T] \times \mathbb{R}^2$;*

2. *for each $x \in \mathbb{R}^2$, $\{\mathcal{W}(t,x)\}_{t\in[0,T]}$ is a real valued Wiener process w.r.t. the filtration $\{\mathcal{F}_t\}_t$;*

3. *for arbitrary $t \in [0,T]$, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathbb{R}^2$ and $h \in \mathbb{R}^2$ the random vectors $(\mathcal{W}(t,x_1 + h), \ldots \mathcal{W}(t,x_n + h))$ and $(\mathcal{W}(t,x_1), \ldots \mathcal{W}(t,x_n))$ have the same distribution.*

If $\mathcal{W}$ is an $\mathbb{R}$-valued spatially homogeneous Wiener random field on $\mathbb{R}^2$ then there exists a uniformly continuous bounded function Γ such that for all $s,t \in [0,T]$ and $x,y \in \mathbb{R}^2$ one has

$$\mathbb{E}\left[\mathcal{W}(t,x)\mathcal{W}(s,y)\right] = (s \wedge t)\;\Gamma(x-y).$$

The function Γ is equal to the Fourier transform of a symmetric positive *finite* measure μ on $\mathbb{R}^2$, namely $\Gamma = \hat{\mu}$. Thus it holds

$$\mathbb{E}\left[\mathcal{W}(t,x)\mathcal{W}(s,y)\right] = (s \wedge t)\;\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i(x-y)\cdot z}\,\mu(\mathrm{d}z).$$

We call Γ and μ respectively the correlation function and the spectral measure of $\mathcal{W}$. The measure μ determines completely the law of $\mathcal{W}$.

Let $\mathcal{W}$ be an $\mathbb{R}$-valued spatially homogeneous Wiener random field on $\mathbb{R}^2$ with a spectral measure μ and let $g : \mathbb{R}^2 \to \mathbb{R}^2$. Consider the following system of equations on $[0,T] \times \mathbb{R}^2$,

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) - \Delta v(t,x) + (v(t,x)\cdot\nabla)v(t,x) + \nabla p(t,x) = g(v(t,x))\dot{\mathcal{W}}(t,x) & (t,x) \in [0,T]\times\mathbb{R}^2 \\ \nabla \cdot v(t,x) = 0 & (t,x) \in [0,T]\times\mathbb{R}^2 \\ v(0,x) = v_0(x) & x \in \mathbb{R}^2 \end{cases} \tag{II.1.6.1}$$

Definition II.1.6.2. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a fixed filtered probability space and let $\mathcal{W}$ be a $\mathbb{R}$-valued spatially homogeneous Wiener random field with spectral measure μ. By a strong solution to* (II.1.6.1) *we mean an $\{\mathcal{F}_t\}$-adapted process v satisfying conditions (i)-(ii) from Definition II.1.3.1. In the integral equation in (ii) we write*

$$\langle \int_0^t G(v(s))\,\mathrm{d}W(s), z\rangle = \int_{\mathbb{R}^2}\int_0^t g(v(s,x))\,\mathcal{W}(\mathrm{d}s,x)\cdot z(x)\,\mathrm{d}x,$$

where $\mathcal{W}(\mathrm{d}s,x)$ means that for fixed x we integrate in Itô sense w.r.t. the real-valued Wiener process $\mathcal{W}(\mathrm{d}s,x)$.

Theorem II.1.6.3. *Let $q \in [2,\infty)$. Let $\mathcal{W}$ be the $\mathbb{R}$-valued spatially homogeneous Wiener random field (driving equation* (II.1.6.1)*), with spectral measure μ. Assume that*

i. $\int_{\mathbb{R}^2}(1+|z|^2)\,\mu(\mathrm{d}z) < \infty$*;*

ii. $g : \mathbb{R}^2 \to \mathbb{R}^2$ *is a measurable function such that*

(a) $|g(x)| \le C|x|$ *and* $|\nabla g(x)| \le C|x|$*, for a positive constant C and for all $x \in \mathbb{R}^2$,*

(b) there exists a positive constant L_h such that

$$|g(x)-g(y)|^2 + |\nabla\,[g(x)-g(y)]\,|^2 \le L_h|x-y|^2$$

for all $x, y \in \mathbb{R}^2$.

Then for any $v_0 \in H^{1,2}\cap H^{1,q}$ there exists a unique strong solution to problem (II.1.6.1) *such that $v(0) = v_0$ a.s.*

The random field $\mathcal{W}$ can be viewed as a cylindrical Wiener W process on the Hilbert space $\mathcal{H}_{\mathcal{W}}$ characterized as follows

$$\mathcal{H}_{\mathcal{W}} = \{\widehat{\varphi\mu} : \varphi \in L^2_{(s)}(\mathbb{R}^2;\mu)\} \tag{II.1.6.2}$$

where $L^2_{(s)}(\mathbb{R}^2;\mu)$ is the closed subspace of $L^2(\mathbb{R}^2;\mu)$ of functions satisfying $\varphi_{(s)} = \varphi$ μ-a.s. Here we use the notation $\varphi_{(s)}(x) = \varphi(-x)$, $x \in \mathbb{R}^2$. $\widehat{\varphi\mu}$ is the Fourier transform of a tempered distribution $\varphi\mu$. Then $\mathcal{H}_{\mathcal{W}}$ endowed with the norm

$$\langle\widehat{\varphi\mu}, \widehat{\psi\mu}\rangle_{\mathcal{H}_{\mathcal{W}}} = \langle\varphi,\psi\rangle_{L^2_{(s)}(\mu)}$$

is the RKHS (reproducing kernel Hilbert space) of $\mathcal{W}$. For a more detailed discussion we remand to Section II.1.7 where some bibliographical references are provided.

Equation (II.1.6.1) can be written as a stochastic evolution equation with G given by

$$(G(v)\psi)(x) = g(v(x))\psi(x), \qquad \psi \in \mathcal{H}_{\mathcal{W}}. \tag{II.1.6.3}$$

Thus we are in the framework of Definition II.1.3.1. Aim of this Section is to show that the operator G satisfies assumptions **(G1)-(G4)** with $\mathcal{H} = \mathcal{H}_{\mathcal{W}}$ and consequently that Theorem II.1.6.3 is a special case of Theorem II.1.3.6. Hence, in order to prove Theorem II.1.6.3 it is sufficient to prove the following.

Lemma II.1.6.4. *Let $\mathcal{W}$ be a $\mathbb{R}$-valued spatially homogeneous Wiener random field with spectral measure μ and let $\mathcal{H}_{\mathcal{W}}$ be its RKHS. Let $q > 2$ and let G be the operator defined in* (II.1.6.3)*. Under assumptions (i)-(ii) of Theorem* (II.1.6.3)*, the operator G satisfies assumptions **(G1)-(G4)**.*

Proof of Lemma II.1.6.4 is based on the following technical result (see [18, Theorem 4.1]).

Theorem II.1.6.5. *Let K be a bounded linear operator acting from a real separable Hilbert space H into $W^{r,q}$, where $r \in [0, \infty)$. Assume that K is given by the formula*

$$(K\psi)(x) = \langle \mathcal{K}(x), \psi \rangle_H \qquad x \in \mathbb{R}^2, \psi \in H, \tag{II.1.6.4}$$

where $\mathcal{K} \in W^{r,q}(\mathbb{R}^2; H)$. Then $K \in R(H; W^{r,q})$ and there is a constant C, independent of K, such that $\|K\|_{R(H;W^{r,q})} \leq C \|\mathcal{K}\|_{W^{r,q}(\mathbb{R}^2;H)}$.

Let us prove Lemma II.1.6.4.

Proof. Let us start by proving that assumptions **(G1)-(G2)** are satisfied. Let us recall that, since $W^{1,2}$ is a separable Hilbert space, then $R(\mathcal{H}_{\mathcal{W}}; W^{1,2}) = L_{\mathrm{HS}}(\mathcal{H}_{\mathcal{W}}; W^{1,2})$ and $\|\cdot\|_{R(\mathcal{H}_{\mathcal{W}};W^{1,2})} = \|\cdot\|_{L_{\mathrm{HS}}(\mathcal{H}_{\mathcal{W}};W^{1,2})}$ (see Appendix A.4); thus let $q \geq 2$.

Take $v \in H^{1,q}$. Let us introduce the operator j as

$$j(\psi) = \widehat{\psi\mu}, \qquad \psi \in L^2_{(s)}(\mu). \tag{II.1.6.5}$$

Clearly $j : L^2_{(s)}(\mu) \to \mathcal{H}_{\mathcal{W}}$ is an isomorphism, thus

$$G(v) \in R(\mathcal{H}_{\mathcal{W}}; W^{1,q}) \iff G(v)j \in R(L^2_{(s)}(\mu); W^{1,q}), \qquad q \geq 2.$$

Let $\psi \in L^2_{(s)}(\mu)$, recalling that G is defined in (II.1.6.3), we have

$$\begin{aligned}(G(v)j)(\psi)(x) &= (G(v))(j\psi)(x) = g(v(x))(j\psi)(x) = g(v(x))\widehat{\psi\mu}(x) \\ &= g(v(x)) \int_{\mathbb{R}^2} \frac{e^{-ix\cdot z}}{2\pi} \psi(z)\, \mu(\mathrm{d}z) \\ &= \int_{\mathbb{R}^2} \left(\frac{e^{-ix\cdot z}}{2\pi} g(v(x)) \right) \psi(z)\, \mu(\mathrm{d}z) \\ &= \langle \mathcal{K}(x), \psi \rangle_{L^2_{(s)}(\mu)},\end{aligned}$$

where

$$\mathcal{K}(x) = \frac{e^{-ix\cdot\star}}{2\pi} g(v(x)).$$

Now we prove that $\mathcal{K} \in W^{1,q}(\mathbb{R}^2; L^2_{(s)}(\mu))$, $q \geq 2$. Recalling that

$$\|\mathcal{K}\|^q_{W^{1,q}(\mathbb{R}^2;L^2_{(s)}(\mu))} = \int_{\mathbb{R}^2} \|\mathcal{K}(x)\|^q_{L^2_{(s)}(\mu)}\,\mathrm{d}x + \int_{\mathbb{R}^2} \|\nabla\mathcal{K}(x)\|^q_{L^2_{(s)}(\mu)}\,\mathrm{d}x,$$

exploiting assumption (ii.a) in Theorem II.1.6.3, we get

$$\begin{aligned}\int_{\mathbb{R}^2} \|\mathcal{K}(x)\|^q_{L^2_{(s)}(\mu)}\,\mathrm{d}x &= \int_{\mathbb{R}^2} \left\|\frac{e^{-ix\cdot\star}}{2\pi} g(v(x))\right\|^q_{L^2_{(s)}(\mu)} \mathrm{d}x \\ &= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\left|\frac{e^{-ix\cdot z}}{2\pi} g(v(x))\right|^2 \mu(\mathrm{d}z)\right)^{\frac{q}{2}} \mathrm{d}x = \frac{1}{(2\pi)^q}\|g(v)\|^q_{L^q}\,(\mu(\mathbb{R}^2))^{\frac{q}{2}} \\ &\leq \frac{C}{(2\pi)^q}\,(\mu(\mathbb{R}^2))^{\frac{q}{2}}\|v\|^q_{L^q}.\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}^2} \|\nabla\mathcal{K}(x)\|^q_{L^2_{(s)}(\mu)}\,\mathrm{d}x &= \int_{\mathbb{R}^2}\left\|\nabla_x\left(\frac{e^{-ix\cdot\star}}{2\pi} g(v(x))\right)\right\|^q_{L^2_{(s)}(\mu)} \mathrm{d}x \\ &= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\left|\nabla_x\left(\frac{e^{-ix\cdot z}}{2\pi} g(v(x))\right)\right|^2 \mu(\mathrm{d}z)\right)^{\frac{q}{2}} \mathrm{d}x \\ &\leq C_q\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\left(|\nabla g(v(x))|^2 + |z|^2\,|g(v(x))|^2\right)\mu(\mathrm{d}z)\right)^{\frac{q}{2}} \mathrm{d}x \\ &\leq C_q\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\left(|v(x)|^2 + |z|^2\,|v(x)|^2\right)\mu(\mathrm{d}z)\right)^{\frac{q}{2}} \mathrm{d}x \\ &= C_q\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}(1+|z|^2)|v(x)|^2\,\mu(\mathrm{d}z)\right)^{\frac{q}{2}} \mathrm{d}x \\ &= C_q\|v\|^q_{L^q}\left(\int_{\mathbb{R}^2}(1+|z|^2)\,\mu(\mathrm{d}z)\right)^{\frac{q}{2}}.\end{aligned}$$

Collecting the above estimates we get

$$\begin{aligned}\|\mathcal{K}\|^q_{W^{1,q}(\mathbb{R}^2;L^2_{(s)}(\mu))} &\leq C_q\|v\|^q_{L^q}\left(\left(\mu(\mathbb{R}^2)\right)^{\frac{q}{2}} + \left(\int_{\mathbb{R}^2}(1+|z|^2)\,\mu(\mathrm{d}z)\right)^{\frac{q}{2}}\right) \\ &\leq C_q\|v\|^q_{L^q}\left(\int_{\mathbb{R}^2}(1+|z|^2)\,\mu(\mathrm{d}z)\right)^{\frac{q}{2}},\end{aligned}$$

where the constant C_q is independent of v and μ. Thus applying Theorem II.1.6.5 for $K = G(v)j$, $r = 1$ and $H = L^2_{(s)}(\mu)$ it follows that assumptions **(G1)-(G2)** are satisfied.

Let us now check that assumption **(G3)** is satisfied. We will show that $G(\cdot)$ is a Lipschitz function from L^2 into $L_{\mathrm{HS}}(\mathcal{H}_{\mathcal{W}}; L^2)$. This guarantees that **(G3)** holds. Let $\{e_k\}_k$ be an orthonormal basis of $L^2_{(s)}(\mu)$. Then

$$
\begin{aligned}
\|G(v)-G(u)\|^2_{L_{\mathrm{HS}}(\mathcal{H}_{\mathcal{W}};L^2)} &= \sum_{k=1}^{\infty}\int_{\mathbb{R}^2} |[g(v(x))-g(u(x))]\,\widehat{e_k\mu}(x)|^2\,\mathrm{d}x \\
&= \frac{1}{4\pi^2}\sum_{k=1}^{\infty}\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2} e^{iy\cdot x}\,[g(v(x))-g(u(x))]\,e_k(y)\mu(\mathrm{d}y)\right|^2\mathrm{d}x \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |e^{iy\cdot x}\,[g(v(x))-g(u(x))]\,|^2\mu(\mathrm{d}y)\,\mathrm{d}x \\
&= \frac{1}{4\pi^2}\,\mu(\mathbb{R}^2)\int_{\mathbb{R}^2} |\,[g(v(x))-g(u(x))]\,|^2\,\mathrm{d}x \le C\|v-u\|^2_{L^2},
\end{aligned}
\tag{II.1.6.6}
$$

where C depends on μ and on the constant L_h appearing in assumption (ii.b) in Theorem II.1.6.3.
It remains to prove **(G4)**. We have

$$
\begin{aligned}
\|G(v)-G(u)\|^2_{L_{\mathrm{HS}}(\mathcal{H}_{\mathcal{W}};H^{1,2})} &= \sum_{k=1}^{\infty}\int_{\mathbb{R}^2} |[g(v(x))-g(u(x))]\,\widehat{e_k\mu}(x)|^2\,\mathrm{d}x \\
&\quad+\sum_{k=1}^{\infty}\int_{\mathbb{R}^2} |\nabla\,[(g(v(x))-g(u(x)))\widehat{e_k\mu}(x)]|^2\,\mathrm{d}x
\end{aligned}
$$

The first term can be estimated as in (II.1.6.6). Regarding the second one, we get

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\int_{\mathbb{R}^2} |\nabla\,[(g(v(x))-g(u(x)))\widehat{e_k\mu}(x)]|^2\,\mathrm{d}x \\
&\qquad= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\sum_{k=1}^{\infty}\left|\int_{\mathbb{R}^2}\nabla\left[(g(v(x))-g(u(x)))e^{ix\cdot y}\right]e_k(y)\mu(\mathrm{d}y)\right|^2\mathrm{d}x \\
&\qquad= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\nabla\left[(g(v(x))-g(u(x)))e^{ix\cdot y}\right]\right|^2\mu(\mathrm{d}y)\,\mathrm{d}x \\
&\qquad\le \frac{1}{2\pi^2}\int_{\mathbb{R}^2}\mu(\mathrm{d}y)\int_{\mathbb{R}^2} |\nabla\,[g(v(x))-g(u(x))]|^2\,\mathrm{d}x \\
&\qquad\qquad+\frac{1}{2\pi^2}\int_{\mathbb{R}^2}|y|^2\mu(\mathrm{d}y)\int_{\mathbb{R}^2}|g(v(x))-g(u(x))|^2\,\mathrm{d}x
\end{aligned}
$$

Thus, exploiting assumption (ii.b) in Theorem II.1.6.3, we obtain

$$
\begin{aligned}
\|G(v)-G(u)\|^2_{L_{\mathrm{HS}}(\mathcal{H}_{\mathcal{W}};H^{1,2})} &\le \frac{1}{2\pi^2}\int_{\mathbb{R}^2}\mu(\mathrm{d}y)\int_{\mathbb{R}^2}|\nabla\,[g(v(x))-g(u(x))]|^2\,\mathrm{d}x \\
&\quad+\frac{1}{4\pi^2}\int_{\mathbb{R}^2}(2|y|^2+1)\mu(\mathrm{d}y)\int_{\mathbb{R}^2}|g(v(x))-g(u(x))|^2\,\mathrm{d}x \\
&\le \frac{1}{2\pi^2}\int_{\mathbb{R}^2}(2|y|^2+1)\mu(\mathrm{d}y)\int_{\mathbb{R}^2}|g(v(x))-g(u(x))|^2+|\nabla\,[g(v(x))-g(u(x))]|^2\,\mathrm{d}x \\
&\le \frac{L_h}{2\pi^2}\left(\int_{\mathbb{R}^2}(2|y|^2+1)\mu(\mathrm{d}y)\right)\,\|u-v\|^2_{L^2}
\end{aligned}
$$

In particular the constant L_h is such that $L_1 = \frac{L_h}{2\pi^2}\int_{\mathbb{R}^2}(2|y|^2+1)\mu(\mathrm{d}y)$.
This concludes the proof. □

As a byproduct of the existence and uniqueness result given in Theorem II.1.6.3, we obtain the existence and uniqueness of a strong solution to the equation for the vorticity

$$\begin{cases} \dfrac{\partial \xi}{\partial t}(t,x) - \Delta\xi(t,x) + v(t,x)\cdot\nabla\xi(t,x) = \mathrm{curl}(g(v(x))\mathcal{W}(t,x)) & (t,x)\in[0,T]\times\mathbb{R}^2 \\ \xi(t,x) = \nabla^\perp\cdot v(t,x) & (t,x)\in[0,T]\times\mathbb{R}^2 \\ \nabla\cdot v(t,x) = 0 & (t,x)\in[0,T]\times\mathbb{R}^2 \\ \xi(0,x) = \xi_0(x) & x\in\mathbb{R}^2. \end{cases} \tag{II.1.6.7}$$

which is formally obtained by taking the *curl* in both sides of the first equation of (II.1.6.1). By solution to (II.1.6.7) we formally mean.

Definition II.1.6.6. *Let* $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_t,\mathbb{P})$ *be a fixed probability space and let* $\mathcal{W}$ *be the* $\mathbb{R}$*-valued spatially homogeneous Wiener random field (with spectral measure* μ*) driving equation* (II.1.6.1)*. By a strong solution to* (II.1.6.7) *we mean an* $\{\mathcal{F}_t\}_t$*-adapted process* ξ *satisfying conditions (i)-(ii) from Definition II.1.3.7. In the integral equation in (ii) we write*

$$\langle \int_0^t \tilde{G}(v(s))\,\mathrm{d}W(s), z\rangle = \int_{\mathbb{R}^2}\int_0^t \mathrm{curl}\,(g(v(s,x))\mathcal{W}(\mathrm{d}s,x))\,z(x)\,\mathrm{d}x.$$

Thus we obtain the following.

Theorem II.1.6.7. *Let* $\mathcal{W}$ *be the* $\mathbb{R}$*-valued spatially homogeneous Wiener random field, driving equation* (II.1.6.1)*, with spectral measure* μ*. Let assumptions of Theorem II.1.6.3 hold. Then there exists a unique strong solution to* (II.1.6.7)*.*

II.1.7 Notes and Comments: the spatially homogeneous noise on $\mathbb{R}^d$ in literature

In recent years, following in particular the papers of Dalang and Frangos [28], Dalang [27] and Peszat and Zabczyk [76] and [77], the spatially homogeneous noise on the whole space $\mathbb{R}^d$ has been used by several researchers. We recall that a process is called spatially homogeneous if the law of the process that we consider is invariant under space translations. The notion of spatially homogeneous noise was introduced for the first time, by Dawson and Salehi in [32] as a model of a random environment for studying population dynamics (using a stochastic heat equation driven by this type of noise). The same class of equations has been considered by Noble [70]. Paper by Dawson and Salehi was a starting point of the next research.

In [76] authors realized how the process introduced in [32] can be interpreted as a Wiener process on a Hilbert space (denoted here by $\mathcal{H}_\mathcal{W}$) continuously embedded into the space of tempered distributions on $\mathbb{R}^d$. They give an explicit characterization of this space, which turns out to be the RKHS (reproducing kernel Hilbert space) of the process. In this way, SPDEs driven by a spatially homogeneous Wiener process taking values in the space of tempered distributions can be treated as evolutions equations, in the contest of the Da Prato-Zabczyk theory (see Section A.3). Results of [76] have been used by Capinski and Peszat [19] for the study of stochastic Navier-Stokes equations. Moreover, Brzeźniak and Peszat in [12] have developed an L^p-theory of stochastic parabolic equations driven by a spatially homogeneous Wiener process. This extension was then used in [18] for the study of two dimensional Euler equation.

On the other hand another, slightly different approach has been used in the works [27], [28], [30], [79] (for instance). Here authors introduce a family of zero-mean Gaussian random variables $W = \{W(\varphi)\}_\varphi$ indexed by elements in the Schwartz space. The random variables are completely characterized by the spatial (smooth) covariance which is given in terms of the spectral measure. Suitable hypothesis on the spectral measure guarantees the spatially homogeneity of the process. Differently from the above cited papers, in these works authors use the theory of stochastic partial differential equations developed in [90] (see Section A.2): in [28] (see also [27]) the authors construct from this Gaussian process a worthy martingale measure. This approach turns out to be useful when one is interested in solutions which are random fields, that is, real-valued processes, that are well defined for every fixed $(t, x) \in [0, T] \times \mathbb{R}^d$. This notion of solution is, for instance, the starting point if one is interested in the analysis of the regularity in the Malliavin sense, that is wants to establish properties of the probability law of the solution. In this context, the isonormal Gaussian process W shall provide the underlying Gaussian setting in which use Malliavin calculus tools. For more details see Appendix B.

Summarizing, starting from the work by Dawson and Salehi, spatially homogeneous noise was studied in two different ways that are suitable to fit two different stochastic integration theories and thus different ways of studying and solving SPDEs. [3] In any case the two different approaches are related. This problem have been addressed by Dalang and Quer-Sardanyons in [30]. There the authors show how it is possible to extend the index set of the isonormal Gaussian process W (as considered in [28]) to an Hilbert space U and how W can be interpreted as a cylindrical U-Wiener process. U is nothing but the RKHS of this

[3] In Section II.1.6 we have interpreted the spatially homogeneous noise as in [76]. We point out that, in view of future work concerning Malliavin analysis of the solution process, interpreting the noise in the sense of [28] is more useful.

process. Then it is proved that the relation between the cylindrical U-Wiener process and the cylindrical $\mathcal{H}_{\mathcal{W}}$-Wiener process as constructed in [76] is given by an isometry between the corresponding RKHS U and $\mathcal{H}_{\mathcal{W}}$. Moreover, exploiting this isometry, the equivalence of Hilbert-space valued and martingale-measure stochastic integrals is proved.

Main aim of the present Section is to provide a brief overview of the existing literature concerning the spatially homogeneous noise. In particular, we recall how the concept of spatially homogeneous noise was introduced in the paper [32] by Dawson and Salehi. Then we provide the interpretation of the noise as an $\mathcal{H}_{\mathcal{W}}$-cylindrical Wiener process given in [76], and the interpretation in terms of a family of Gaussian random variables indexed by elements in the Schwarz space given in [28]. Finally we show the relation between the two ways of interpreting the spatially homogeneous noise, providing the explicit form of the isometry between the two RKHSs.

II.1.7.1 Dawson-Salehi definition

We denote by $\mathcal{F}$ or $\widehat{\cdot}$ the Fourier transform on $\mathcal{S}(\mathbb{R}^2)$, that is

$$\mathcal{F}\varphi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix\cdot y}\varphi(y)\,\mathrm{d}y.$$

Recall that the inverse Fourier transform $\mathcal{F}^{-1}$ (denoted also by $\check{\cdot}$) is given by the formula

$$\mathcal{F}^{-1}\varphi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ix\cdot y}\varphi(y)\,\mathrm{d}y.$$

Given $h \in \mathbb{R}^2$, let us introduce the translation operator $\tau_h : \mathcal{S} \to \mathcal{S}$ as:

$$\tau_h\varphi(\cdot) = \varphi(\cdot + h), \qquad \varphi \in \mathcal{S}.$$

For $\varphi \in \mathcal{S}$ we set $\psi_{(s)}(x) = \psi(-x), x \in \mathbb{R}^2$. Denote by $\mathcal{S}_{(s)}$ the space of all $\varphi \in \mathcal{S}$ such that $\varphi = \varphi_{(s)}$ and by $\mathcal{S}'_{(s)}$ be the space of all $f \in \mathcal{S}'$ such that $\langle f, \varphi\rangle = \langle f, \varphi_{(s)}\rangle$.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$.
In [32] the following definitions are given.

Definition II.1.7.1. *A Wiener process $\mathcal{W}$ defined on Ω and taking values in $\mathcal{S}'$ is a process with continuous trajectories in $\mathcal{S}'$, and such that for each $\varphi \in \mathcal{S}$, $t \to \langle \mathcal{W}(t), \varphi\rangle$ is a one dimensional Wiener process.*

Hence, there exists a bilinear continuous symmetric positive definite form $Q : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ such that

$$\mathbb{E}\left[\langle \mathcal{W}(s), \varphi\rangle\langle \mathcal{W}(t), \psi\rangle\right] = (s \wedge t)Q(\varphi, \psi), \qquad \forall (s, \varphi), (t, \psi) \in [0, T] \times \mathcal{S}. \qquad \text{(II.1.7.1)}$$

Definition II.1.7.2. *$\mathcal{W}$, as introduced in Definition II.1.7.1, is spatially homogeneous if for each fixed $t \geq 0$ the law of $\mathcal{W}(t)$ is invariant with respect to all translations $\tau'_h : \mathcal{S}' \to \mathcal{S}'$. This is equivalent to assume that*

$$\mathbb{P}(\mathcal{W}(t) \in \mathcal{B}(\mathcal{S}')) = \mathbb{P}(\mathcal{W}(t) \in (\tau'_h)^{-1}(\mathcal{B}(\mathcal{S}'))) \qquad h \in \mathbb{R}^2. \qquad \text{(II.1.7.2)}$$

It can be shown that (II.1.7.2) holds iff $Q(\varphi,\psi) = Q(\tau_h\varphi,\tau_h\psi)$ for all $\varphi,\psi \in \mathcal{S}$ and $h \in \mathbb{R}^2$. Moreover, according to [37, Theorem 6], Q is a translation invariant continuous-positive definite real-valued bilinear form on $\mathcal{S}$ iff for all $\varphi,\psi \in \mathcal{S}$

$$Q(\varphi,\psi) = \langle \Gamma, \varphi * \psi_{(s)} \rangle \tag{II.1.7.3}$$

where $\Gamma \in \mathcal{S}'$ (usually called *spatial correlation* of $\mathcal{W}$) is the Fourier transform of a positive-symmetric tempered measure μ on $\mathbb{R}^2$. That means (for more details see e.g. [38, Chapter 2.1]) that, for all $\varphi \in \mathcal{S}$,

$$\langle \Gamma, \varphi \rangle = \int_{\mathbb{R}^2} \widehat{\varphi}(x)\,\mu(\mathrm{d}x),$$

and there exists an integer $m \geq 1$ such that

$$\int_{\mathbb{R}^2} (1+|x|^2)^{-m}\,\mu(\mathrm{d}x) < \infty. \tag{II.1.7.4}$$

The measure μ is called the *spectral measure* of $\mathcal{W}$. The covariance (II.1.7.3) can also be rewritten, using elementary properties of Fourier transform, as

$$Q(\varphi,\psi) = \int_{\mathbb{R}^2} \widehat{\varphi}(x)\ \widehat{\psi}(x)\,\mu(\mathrm{d}x) = \langle \hat{\varphi}, \hat{\psi} \rangle_{L^2_{(s)}(\mu)}, \tag{II.1.7.5}$$

where $L^2_{(s)}(\mu)$ is the subspace of $L^2(\mu)$ consisting of all φ such that $\varphi = \varphi_{(s)}$.
Summing up, a *spatially homogeneous Wiener process* $\mathcal{W}$ *with values in* $\mathcal{S}'$ is a process such that

- for each $\varphi \in \mathcal{S}$, $\{\langle \mathcal{W}(t), \varphi \rangle\}_{t\geq 0}$ is a real-valued Wiener process;
- there exists a $\Gamma \in \mathcal{S}'$ such that for all $\varphi,\psi \in \mathcal{S}$ one has
$$Q(\varphi,\psi) := \mathbb{E}\left[\langle \mathcal{W}(1), \varphi \rangle \langle \mathcal{W}(1), \psi \rangle\right] = \langle \Gamma, \varphi \star \psi_{(s)} \rangle,$$
- Γ is the Fourier transform of a Borel positive and symmetric measure μ on $\mathbb{R}^2$ satisfying
$$\int_{\mathbb{R}^2} (1+|x|^2)^{-m}\,\mu(\mathrm{d}x) < \infty, \quad \text{for a certain } m \geq 1.$$

Let us present some examples of spatially homogeneous Wiener processes.

Example II.1.7.3. ***Space-time white noise.*** *In this case* $Q(\varphi,\psi) = \langle \varphi, \psi \rangle$. *Then* Γ *is equal to the Dirac* δ_0 *function and its spectral density* $\mathrm{d}\mu/\mathrm{d}x$ *is the constant function* $(2\pi)^{-\frac{d}{2}}$.

Example II.1.7.4. ***Wiener random field.*** *As pointed out in [76], if* $\mathcal{W}$ *is a* $\mathbb{R}$*-valued spatially homogeneous Wiener random field on* $[0,T]\times\mathbb{R}^2$ *(see Definition II.1.6.1) then*

$$\langle \mathcal{W}(t), \varphi \rangle = \int_{\mathbb{R}^2} \varphi(x)\mathcal{W}(t,x)\,\mathrm{d}x, \qquad \varphi \in \mathcal{S}, \tag{II.1.7.6}$$

defines a stochastic process in $\mathcal{S}'$ *with the covariance form given by*

$$Q(\varphi,\psi) = \langle \Gamma, \varphi * \psi_{(s)} \rangle = \langle \widehat{\varphi}, \widehat{\psi} \rangle_{L^2_{(s)}(\mu)}, \qquad \varphi,\psi \in \mathcal{S}. \tag{II.1.7.7}$$

Conversely, let $\mathcal{W}$ *be a spatially homogeneous Wiener process on* $\mathbb{R}^2$ *taking values in the space of distributions. Let* $\delta_k \in \mathcal{S}$ *be such that* $\int_{\mathbb{R}^2} \Gamma(x-y)\delta_k(y)\,\mathrm{d}y \to \Gamma(x)$ *uniformly w.r.t.* $x \in \mathbb{R}^2$. $\mathcal{W}(t,x)$ *defined as the* $L^2(\Omega)$*-limit of the series* $\langle \mathcal{W}(t), \tau_x\delta_k \rangle$ *is a spatially homogeneous Wiener random field.*

II.1.7.2 Brzeźniak-Peszat-Zabczyk approach

As explained in [76] p. 191, the process $\mathcal{W}$, as introduced in Definition II.1.7.1 and Definition II.1.7.2, can be regarded as a Hilbert-space valued Wiener process and classical integration theory can be applied (see [25]). Moreover SPDEs driven by a spatially homogeneous Wiener process on $\mathcal{S}'$ can be rewritten as evolution equations driven by this Hilbert-space valued Wiener process. The involved Hilbert space is nothing but the reproducing kernel Hilbert space of the process. Let us recall its definition (for more details see [25, Section 2.2.2.]).

Definition II.1.7.5. *Let μ be a symmetric Gaussian measure on a separable Banach space E. A linear subspace $H \subset E$ equipped with a Hilbert norm $\|\cdot\|_H$ is said to be the reproducing kernel Hilbert space (RKHS) for μ if H is complete, continuously embedded in E and such that for arbitrary $\varphi \in E*$, the law of φ is given by $\mathcal{N}(0, \|\varphi\|_H^2)$.*

The following result holds (see [25, Theorem 2.7]).

Theorem II.1.7.6. *For arbitrary symmetric Gaussian measure μ on a separable Banach space, there exists a unique reproducing kernel Hilbert space $(H, \|\cdot\|_H)$.*

Let now see how to introduce and characterize the RKHS of $\mathcal{W}$. Let us introduce the Hilbert space U as the completion of the set $\mathcal{S} \setminus \mathrm{Ker}Q$ with respect to the norm $\sqrt{Q(\varphi,\varphi)}$. From (II.1.7.5) this is equivalent to understand U as the completion of the Schwartz space $\mathcal{S}$ endowed with the semi inner product

$$\langle \varphi, \psi \rangle_U = \langle \hat{\varphi}, \hat{\psi} \rangle_{L^2_{(s)}(\mu)}, \tag{II.1.7.8}$$

$\varphi, \psi \in \mathcal{S}_{(s)}$ and associated semi-norm $\|\cdot\|_U$. We can give the following explicit characterization of the Hilbert space U (see [30, Remark 2.3]).

$$U = \{g \in \mathcal{S}'_{(s)} : g = \check{\varphi} \text{ where } \varphi \in L^2_{(s)}(\mu)\}, \tag{II.1.7.9}$$

with the inner product

$$\langle g, f \rangle_U = \langle \varphi, \psi \rangle_{L^2_{(s)}(\mu)}, \qquad \text{with } g = \check{\varphi}, f = \check{\psi} \quad \text{and } \varphi, \psi \in L^2_{(s)}(\mu).$$

Let us denote by $\mathcal{H}_\mathcal{W}$ the dual space of U. As proved in [76, Proposition 1.1] $\mathcal{H}_\mathcal{W}$ is the RKHS of the Gaussian law of $\mathcal{W}(1)$. We can give the following characterization of $\mathcal{H}_\mathcal{W}$ (see [76, Proposition 1.2]):

$$\mathcal{H}_\mathcal{W} = \{g \in \mathcal{S}'_{(s)} : g = \widehat{\varphi\mu} \text{ where } \varphi \in L^2_{(s)}(\mu)\}, \tag{II.1.7.10}$$

with the inner product

$$\langle g, f \rangle_{\mathcal{H}_\mathcal{W}} = \langle \varphi, \psi \rangle_{L^2_{(s)}(\mu)}, \qquad \text{with } g = \widehat{\varphi\mu},\ f = \widehat{\psi\mu} \quad \text{and } \varphi, \psi \in L^2_{(s)}(\mu).$$

It can be proved that the map $\mathcal{S} \ni \varphi \to \langle \varphi, \mathcal{W}(t) \rangle$ has a unique continuous extension to $\mathcal{H}_\mathcal{W}$. We denote this extension also by $\mathcal{W}(t)$. $\mathcal{W}$ is a cylindrical $\mathrm{Id}_{\mathcal{H}_\mathcal{W}}$-Wiener process on $\mathcal{H}_\mathcal{W}$. More precisely, let $\tilde{U}$ be a Hilbert space such that there exists a dense Hilbert-Schmidt embedding $J : \mathcal{H}_\mathcal{W} \to \tilde{U}$. Then

$$\mathcal{W}(t) = \sum_{k=1}^{\infty} \beta_k(t) J(e_k) \tag{II.1.7.11}$$

where $\{e_k\}_k$ is a complete orthonormal basis in $\mathcal{H}_\mathcal{W}$ and $\{\beta_k(t)\}_k$ is a sequence of standard independent one dimensional Brownian motions.

Brzeźniak and Peszat in [12, Theorem 2.2] provide an extension to [76, Proposition 1.1]: the cylindrical $\mathrm{Id}_{\mathcal{H}_\mathcal{W}}$-Wiener process on $\mathcal{H}_\mathcal{W}$ takes values in any Banach space B such that the embedding $\mathcal{H}_\mathcal{W} \hookrightarrow B$ is γ-radonyfing.

II.1.7.3 Dalang-Frangos approach

Let us now concentrate on the approach used in the works [27], [28], [30], [79] (see also the therein references). There the noise is understood as an isonormal Gaussian process on a suitable Hilbert space (which is nothing but the RKHS of the process). This approach turns out to be more useful if one is interested in the Malliavin analysis of the solution process: the considered isonormal Gaussian process will be the underlying Gaussian space on which to perform Malliavin calculus (see also Example B.2.2).

On a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we consider a family of mean zero Gaussian random variables $W = \{W(\varphi), \varphi \in C_0^\infty(\mathbb{R}^2 \times [0,T])\}$ with covariance

$$\mathbb{E}\left[W(\varphi)W(\psi)\right] = \int_0^T \langle \widehat{\varphi(t)}, \widehat{\psi(t)} \rangle_{L^2_{(s)}(\mu)} \, \mathrm{d}t, \tag{II.1.7.12}$$

where μ is a symmetric non-negative tempered measure on $\mathbb{R}^2$ and it is called the spectral measure of W.

Starting from the isonormal Gaussian process W it is possible to construct a cylindrical Wiener process on a suitable Hilbert space (in the sense of Section A.3.3). Notice at first that the space U, introduced in Section II.1.7.2 is the RKHS of the Gaussian law of $W(\mathbf{1}_{[0,1]}\varphi)$, $\varphi \in \mathcal{S}$. Now we fix a time interval $[0,T]$ and we set $U_T := L^2([0,T]\,; U)$, equipped with the norm

$$\|g\|^2_{U_T} = \int_0^T \|g(s)\|^2_U \, \mathrm{d}s.$$

W defines a linear isometry from $(C_0^\infty([0,T] \times \mathbb{R}^2, \|\cdot\|_{U_T})$ into $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Since (see [30, Lemma 2.4]) $C_0^\infty([0,T] \times \mathbb{R}^2)$ is dense in U_T for $\|\cdot\|_{U_T}$, $W(\varphi)$ can be extended for all $\varphi \in U_T$ following the standard method for extending an isometry. This establish the following property (see [30, Proposition 2.5]).

Proposition II.1.7.7. *For $t \in [0,T]$ and $\varphi \in U$, set $W_t(\varphi) = W(\mathbf{1}_{[0,t]}(\cdot)\varphi(\bullet))$. Then the process $W = \{W_t(\varphi), t \in [0,T], \varphi \in U\}$ is a standard (i.e. with covariance Id_U) cylindrical Wiener process on U as defined in Definition A.3.8.*

Thanks to this proposition we shall prove that the spatially noise introduced above can be viewed as standard cylindrical Id_U-Wiener process in the sense of Da Prato-Zabczyk theory. As made clear in [30, Remark 3.11], thanks to Proposition A.3.11 it is possible to associate the spatially homogeneous noise, viewed as a cylindrical Wiener process with covariance Id_U in Proposition II.1.7.7, with a cylindrical Id_U-Wiener process as defined in Section A.3.2, on the Hilbert space U.

II.1.7.4 Relation between Brzeźniak-Peszat-Zabczyk and Dalang-Frangos approach

In Sections II.1.7.2 and II.1.7.3 we have briefly explained how a Wiener process taking values in the space $\mathcal{S}'$ and a family of Gaussian random variables indexed by functions in $\mathcal{S}$ can be understood as Hilbert space-valued Wiener processes. The involved Hilbert spaces $\mathcal{H}_{\mathcal{W}}$ and U are nothing but the RKHS of the two processes. The relation between the two different ways of introducing a spatially homogeneous Wiener process is given in terms of an isometry between the two RKHS.

By means of the characterization of the spaces U and $\mathcal{H}_{\mathcal{W}}$, respectively given by (II.1.7.9) and (II.1.7.10) we can write the explicit form of the isometry $I : U \to \mathcal{H}_{\mathcal{W}}$ that links the RKHS of the law of $\langle \mathcal{W}(1), \varphi \rangle$ and $W(\mathbf{1}_{[0,1]}\varphi)$, $\varphi \in \mathcal{S}_{(s)}$, respectively. Since every element $g \in U$ can be written in the form $g = \check{\varphi}$, for $\varphi \in L^2_{(s)}(\mu)$, then for such g, $I(g) \in \mathcal{H}_{\mathcal{W}}$ is defined by $I(g) = \widehat{\varphi\mu}$. The process $\mathcal{W}$, regarded as a cylindrical $\mathrm{Id}_{\mathcal{H}_{\mathcal{W}}}$-Wiener process on $\mathcal{H}_{\mathcal{W}}$ can be written as the serie (II.1.7.11). Now, let $\{u_k\}_k$ be a complete orthonormal basis of the Hilbert space U such that $u_k \in \mathcal{S}_{(s)}$, for all $k \geq 1$. Then we can write the process W, regarded as a cylindrical Id_U-Wiener process on U, as

$$W_t = \sum_{k=1}^{\infty} u_k \beta_k(t).$$

At this point we can assume that e_k and $\beta_k(t)$ that appear in (II.1.7.11) are given by

$$e_k = I(u_k) \qquad \text{and} \qquad \beta_k(t) = W_t(u_k), \tag{II.1.7.13}$$

where I is the isometry defined above.

Summarizing we have

$$\begin{array}{ccccc} L^2_{(s)}(\mu) & \xrightarrow{\mathcal{F}^{-1}} & U & \xrightarrow{I} & \mathcal{H}_{\mathcal{W}} \\ \varphi & \longrightarrow & g = \check{\varphi} & \longrightarrow & I(g) = \widehat{\varphi\mu} \end{array}$$

with

$$\langle \varphi, \psi \rangle_{L^2_{(s)}(\mu)} = \langle \check{\varphi}, \check{\psi} \rangle_U = \langle \widehat{\varphi\mu}, \widehat{\psi\mu} \rangle_{\mathcal{H}_{\mathcal{W}}}.$$

In order to fix ideas the following diagram could be useful (by $\mathcal{L}$ we mean the law of the process).

$$\begin{array}{ccccc} C_0^\infty([0,T] \times \mathbb{R}^2) & \xrightarrow{W} & L^2(\Omega, \mathcal{F}, \mathbb{P}) & \xrightarrow{\mathcal{L}} & U \\ \mathbf{1}_{[0,1]}\varphi & \longrightarrow & W(\mathbf{1}_{[0,1]}\varphi) & \longrightarrow & \mathcal{L}(W(\mathbf{1}_{[0,1]}\varphi)) = \mathcal{N}(0, \|\varphi\|^2_U) \\ & & & & \big\downarrow I \\ \mathcal{S}'(\mathbb{R}^2) & \xrightarrow{\langle \mathcal{W}(1), \cdot \rangle} & L^2(\Omega, \mathcal{F}, \mathbb{P}) & \xrightarrow{\mathcal{L}} & \mathcal{H}_{\mathcal{W}} \\ \varphi & \longrightarrow & \langle \mathcal{W}(1), \varphi \rangle & \longrightarrow & \mathcal{L}(\langle \mathcal{W}(1), \varphi \rangle) = \mathcal{N}(0, \|\varphi\|^2_{\mathcal{H}_{\mathcal{W}}}). \end{array}$$

From the above considerations it is now clear that for $t \in [0,T]$, $\varphi \in \mathcal{S}$ it holds

$$W(\mathbf{1}_{[0,t]}(\cdot)\varphi(\star)) = \langle \mathcal{W}(t), \varphi \rangle,$$

where the equality has to be understood in the following sense: both sides of the equality defines two zero mean Gaussian random variables with equal covariance on isometric spaces.

Chapter II.2

Existence and uniqueness of solutions with not regular multiplicative noise

II.2.1 Introduction

Inspired by [16] we consider the vorticity equations (II.0.0.3) with a multiplicative noise whose covariance is not regular enough to allow to use the Itô formula in L^q spaces, for $1 < q < \infty$; in particular, the covariance of the noise is not a trace class operator in the space of finite energy vorticity and this case has not been considered in previous papers. The aim of this Chapter is to prove the existence of a martingale solution for the vorticity equation (II.0.0.3) in $\mathbb{R}^2$ when $v_0, \xi_0 \in L^2(\mathbb{R}^2)$. Moreover, we prove pathwise uniqueness; this implies existence of a strong solution too. A more regular solution will be found when $v_0, \xi_0 \in L^2(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ for $q > 2$. The results are proved by working directly on the equation for the vorticity (II.0.0.3) and using suitable estimates on v coming from equations (II.0.0.1).

As in [16] we shall introduce an approximation system for both the equation (II.0.0.1) and (II.0.0.3) by regularizing the covariance of the noise. In this way we construct two sequence of approximating processes $\{v_n\}_n$ and $\{\xi_n\}_n$. To obtain the existence of a a martingale solution to (II.0.0.3), we exploit the tightness of the sequence of the laws of $\{\xi_n\}_n$ and $\{v_n\}_n$. Uniform estimates in n are obtained working pathwise. The tightness argument is based on the extension of some compactness criteria proved in [17] and [16].

As far as the contents of the Chapter are concerned, in Section II.2.2 we state the assumptions concerning the random forcing term. In Section II.2.3 we are concerned with the study of the regularity of the velocity solution to equations (II.0.0.1). In Section II.2.4 we prove the existence and uniqueness of a strong solution to the vorticity equations (II.0.0.3).

II.2.2 Random forcing term

We define the noise forcing term driving equation (II.0.0.1). Given a real separable Hilbert space $\mathcal{H}$, we consider a $\mathcal{H}$-cylindrical Wiener process W defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a complete right continuous filtration. We can write

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) h_k, \qquad t \in [0, T], \tag{II.2.2.1}$$

where $\{\beta_k\}_{k\in\mathbb{N}}$ is a sequence of standard independent identically distributed Wiener processes defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ and $\{h_k\}_{k\in\mathbb{N}}$ is a complete orthonormal system in $\mathcal{H}$.

On the covariance operator G appearing in equation (II.0.0.1) we make the following set of assumptions. We consider $q > 2$ and assume that there exists $g \in (0, 1)$ such that

(IG1) The mapping $G : \mathbb{L}^2 \to L_{\mathrm{HS}}(\mathcal{H}; H^{1-g,2})$ is well defined and

$$\sup_{v\in\mathbb{L}^2} \|G(v)\|_{L_{\mathrm{HS}}(\mathcal{H};H^{1-g,2})} =: C_{g,2} < \infty,$$

(IG2) The mapping $G : \mathbb{L}^2 \to R(\mathcal{H}; H^{1-g,q})$ is well defined and

$$\sup_{v\in\mathbb{L}^2} \|G(v)\|_{R(\mathcal{H};H^{1-g,q})} =: C_{g,q} < \infty.$$

(IG3) If assumption **(IG1)** holds, then for any $\varphi \in H^{1-g,2}$ and any $v \in \mathbb{L}^2$ the mapping $v \to G(v)^*\varphi \in \mathcal{H}$ is continuous when in $\mathbb{L}^2$ we consider the Fréchet topology inherited from the space $\mathbb{L}^2_{loc}$ or the weak topology of $\mathbb{L}^2$.

(IG4) For all $z \in C^\infty_{sol}$ the real valued function $v \mapsto \|G(v)^*z\|_{\mathcal{H}}$ is continuous on $H^{1,2}$ endowed with the strong L^2-topology.

(IG5) $G : H^{1,2} \to L_{HS}(\mathcal{H}; \mathbb{L}^2)$ is a Lipschitz continuous map, i.e.

$$\text{there exists } L_g > 0 : \|G(v_1) - G(v_2)\|_{L_{HS}(\mathcal{H};\mathbb{L}^2)} \le L_g\|v_1 - v_2\|_{H^{1,2}},$$

for any $v_1, v_2 \in H^{1,2}$.

Remark II.2.2.1. *i. A map $G : \mathbb{L}^2 \to R(\mathcal{H}; H^{1-g,q})$ is well defined iff the map $J^{1-g}G : \mathbb{L}^2 \to R(\mathcal{H}; \mathbb{L}^q)$ is well defined. Moreover*

$$\|J^{1-g}G(v)\|_{R(\mathcal{H};\mathbb{L}^q)} = \|G(v)\|_{R(\mathcal{H};H^{1-g,q})} < C_{g,q}, \qquad v \in \mathbb{L}^2.$$

ii. From (A.4.4) *and **(IG1)**, for any finite $m \ge 1$ we have*

$$\mathbb{E}\left\|\int_0^t G(v(s))\,\mathrm{d}W(s)\right\|^m_{H^{1-g,2}} \le C_m(C_{g,2})^m t^{\frac{m}{2}}.$$

*iii. If $G(v) \in L_{HS}(\mathcal{H}; H^{1-g,2})$ with the uniform bound of **(IG1)**, then the same holds for the adjoint operator, i.e.*

$$\sup_{v\in\mathbb{L}^2} \|G(v)^*\|_{L_{HS}(H^{1-g,2};\mathcal{H})} = C_{g,2}.$$

The noise driving equation (II.0.0.3) is obtained by taking the curl of the noise driving equation (II.0.0.1). Bearing in mind (II.2.2.1), it is given by

$$\mathrm{curl}(G(v)W(t)) = \sum_{k=1}^\infty \beta_k(t)\mathrm{curl}(G(v)h_k), \qquad t \in [0, T]. \tag{II.2.2.2}$$

Let $q > 2$. Notice that, for all $v \in \mathbb{L}^2$ and $k \in \mathbb{N}$, $G(v)h_k \in H^{1-g,2} \cap H^{1-g,q}$. By taking the curl of this latter quantity we loose one order of differentiability, namely $\mathrm{curl}(G(v)h_k) \in W^{-g,2} \cap W^{-g,q}$. Formally, we introduce the operate $\tilde{G}$ in the following way: given $v \in \mathbb{L}^2$, for all $\psi \in \mathcal{H}$, $\tilde{G}(v)(\psi) := \mathrm{curl}(G(v)\psi)$. Thus we have that the mapping $\tilde{G}$ is well defined from $\mathbb{L}^2$ to $L_{\mathrm{HS}}(\mathcal{H}; W^{-g,2}) \cap R(\mathcal{H}; W^{-g,q})$.

An analogue of Remark II.2.2.1 holds.

Remark II.2.2.2. *Similarly as in Remark II.2.2.1 we have that*

i. a map $\tilde{G} : \mathbb{L}^2 \to R(\mathcal{H}; W^{-g,q})$ *is well defined iff the map* $J^{-g}\tilde{G} : \mathbb{L}^2 \to R(\mathcal{H}; \mathbb{L}^q)$ *is well defined. Moreover*

$$\|J^{-g}\tilde{G}(v)\|_{R(\mathcal{H};\mathbb{L}^q)} = \|\tilde{G}(v)\|_{R(\mathcal{H};W^{-g,q})} < C_{g,q}, \qquad v \in \mathbb{L}^2.$$

ii. From (A.4.4) *and **(IG1)**, for any finite* $m \geq 1$ *we have*

$$\mathbb{E}\left\|\int_0^t \tilde{G}(v(s))\,\mathrm{d}W(s)\right\|^m_{W^{-g,2}} \leq C_m (C_{g,2})^m t^{\frac{m}{2}}. \tag{II.2.2.3}$$

With a little abuse of notation we shall write $\tilde{G}(v)\mathrm{d}W(t)$ instead of $\mathrm{curl}(G(v)\mathrm{d}W(t))$, where $\tilde{G} := \mathrm{curl}G$.

Let us notice that the set of assumptions made on the covariance operator G are rather good to deal with equation (II.0.0.1) in the spaces $\mathbb{L}^2$ or $\mathbb{L}^q$. On the other hand, when we deal with the equation for the vorticity, we are concerned with a covariance operator not regular enough to use the Itô calculus.

For the sake of clarity, among the above assumptions made on G, we rewrite in terms of $\tilde{G}$ those assumptions that we will use in the following. Let $0 < g < 1$ and $q > 2$. Then

(I$\tilde{G}$1) The mapping $\tilde{G} : \mathbb{L}^2 \to L_{HS}(\mathcal{H}; W^{-g,2})$ is well defined and

$$\sup_{v\in\mathbb{L}^2} \|\tilde{G}(v)\|_{L_{HS}(\mathcal{H};W^{-g,2})} =: C_{g,2} < \infty.$$

(I$\tilde{G}$2) The mapping $\tilde{G} : \mathbb{L}^2 \to R(\mathcal{H}; W^{-g,q})$ is well defined and

$$\sup_{v\in\mathbb{L}^2} \|\tilde{G}(v)\|_{R(\mathcal{H};W^{-g,q})} =: C_{g,q} < \infty.$$

(I$\tilde{G}$3) If assumption **(I$\tilde{G}$1)** holds, then for any $\varphi \in W^{-g,2}$ and any $v \in \mathbb{L}^2$ the mapping $v \to \tilde{G}(v)^*\varphi \in \mathcal{H}$ is continuous when in $\mathbb{L}^2$ we consider the Fréchet topology inherited from the space $\mathbb{L}^2_{loc}$ or the weak topology of $\mathbb{L}^2$.

Example II.2.2.3. *Let* $G(v)h_k = c_k\sigma(v)e_k$ *with* $\{e_k\}_k$ *a complete orthonormal system in* $H^{1-g,2}$*,* $c_k \in \mathbb{R}$ *and* $\sigma : \mathbb{L}^2 \to \mathbb{R}$ *such that*

$$\sup_{v\in\mathbb{L}^2} |\sigma(v)| := C^1_\sigma < \infty,$$

$$\exists\ L > 0 : |\sigma(v_1) - \sigma(v_2)| \leq L\|v_1 - v_2\|_{H^{1,2}},\ \forall v_1, v_2 \in H^{1,2}$$

$\sigma(v_1) \to \sigma(v_2)$ if v_1 converges to v_2 in $H^{1,2}$ endowed with the strong $\mathbb{L}^2$ topology,

$\sigma(v_1) \to \sigma(v_2)$ if v_1 converges to v_2 in $\mathbb{L}^2_w$ or $\mathbb{L}^2_{loc}$.

For instance, the above conditions on σ *are fulfilled for* $\sigma(v) = \frac{\langle v,h\rangle^2}{1+\langle v,h\rangle^2}$ *with a given* $h \in \mathbb{L}^2$*. Condition **(IG1)** holds if and only if*

$$\sum_{k=1}^{\infty} c_k^2 < \infty \tag{II.2.2.4}$$

*and **(IG2)** hold if* $e_k \in H^{1-g,q}$ *and*

$$\sum_{k=1}^{\infty} c_k^2 \|e_k\|_{H^{1-g,q}}^2 < \infty. \tag{II.2.2.5}$$

*In order to prove **(IG3)** notice that* $G(v)^* e_k = \sigma(v) c_k h_k$ *for any* k*; therefore, given* $\varphi \in H^{1-g,2}$ *(with* $\varphi = \sum_{k=1}^{\infty} \langle \varphi, e_k \rangle_{H^{1-g,2}} e_k$ *and* $\|\varphi\|_{H^{1-g,2}}^2 = \sum_{k=1}^{\infty} |\langle \varphi, e_k \rangle_{H^{1-g,2}}|^2$*)*

$$\begin{aligned}
\|G(v_1)^*\varphi - G(v_2)^*\varphi\|_{\mathcal{H}}^2 &= \left\| \sum_{k=1}^{\infty} \left[G(v_1)^* \langle \varphi, e_k \rangle_{H^{1-g,2}} e_k - G(v_2)^* \langle \varphi, e_k \rangle_{H^{1-g,2}} e_k \right] \right\|_{\mathcal{H}}^2 \\
&= (\sigma(v_1) - \sigma(v_2))^2 \sum_{k=1}^{\infty} c_k^2 |\langle \varphi, e_k \rangle_{H^{1-g,2}}|^2 \\
&\leq \left(\|\varphi\|_{H^{1-g,2}}^2 \sum_{k=1}^{\infty} c_k^2 \right) (\sigma(v_1) - \sigma(v_2))^2 .
\end{aligned}$$

*In a analogous way we can prove that **(IG4)** holds. Finally, **(IG5)** follows, because*

$$\begin{aligned}
\|G(v_1) - G(v_2)\|_{L_{HS}(\mathcal{H};\mathbb{L}^2)}^2 &= (\sigma(v_1) - \sigma(v_2))^2 \sum_{k=1}^{\infty} c_k^2 \|e_k\|_{\mathbb{L}^2}^2 \\
&\leq (\sigma(v_1) - \sigma(v_2))^2 \sum_{k=1}^{\infty} c_k^2 \|e_k\|_{H^{1-g,2}}^2 \leq \left(\sum_{k=1}^{\infty} c_k^2 \right) L^2 \|v_1 - v_2\|_{H^{1,2}}^2 .
\end{aligned}$$

Notice that in this example we have $curl(G(v)h_k) = c_k \sigma(v) curl\ e_k$*.*

II.2.3 Existence of a unique solution to the Navier-Stokes equations (II.0.0.1)

In order to prove the existence of a solution of (II.0.0.3), as well as the desired regularity, we need a certain regularity on the solution process v of (II.0.0.1). In this Section we remind an existence and uniqueness result concerning system (II.0.0.1) and then, under stronger assumptions on the regularity of the initial datum and the covariance operator of the noise, we prove higher regularity for its solution.

As usual, we project the first equation of (II.0.0.1) onto the space of divergence free vectors. Thus, we get rid of the pressure and we obtain the abstract form of the Navier-Stokes equations

$$\begin{cases} \mathrm{d}v(t) + [Av(t) + B(v(t), v(t))]\,\mathrm{d}t = G(v(t))\,\mathrm{d}W(t), & t \in [0, T] \\ v(0) = v_0, \end{cases} \tag{II.2.3.1}$$

We give the following notion of solution.

Definition II.2.3.1. *A martingale solution to the Navier-Stokes problem* (II.2.3.1) *is a triple consisting of a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$*, an* $\{\mathcal{F}_t\}$*-adapted cylindrical* $\mathcal{H}$*-Wiener process* W *and an* $\{\mathcal{F}_t\}$*-adapted measurable process* v*, such that*

i. $v : [0, T] \times \Omega \to \mathbb{L}^2$ *with* $\mathbb{P}$*-a.e. path*

$$v(\cdot, \omega) \in C([0, T]; \mathbb{L}^2) \cap L^2(0, T; H^{1,2});$$

ii. for all $z \in C^\infty_{sol}$ and $t \in [0,T]$ one has $\mathbb{P}$-a.s.

$$\langle v(t), z\rangle + \int_0^t \langle Av(s), z\rangle \,\mathrm{d}s + \int_0^t \langle B(v(s), v(s)), z\rangle \,\mathrm{d}s = \langle v_0, z\rangle + \langle \int_0^t G(v(s)) \,\mathrm{d}W(s), z\rangle. \tag{II.2.3.2}$$

Under the above assumptions on the covariance operator, the existence of a martingale solution, for square summable initial velocity has already been proved (see [17, Theorem 5.1 and Lemma 7.2]). Moreover Lemma 7.3 in [17] provides the pathwise uniqueness of solutions.

The hypothesis we made on the covariance operator of the noise are stronger than those made in [17]. In particular these latter are implied by our assumptions. Thus the following result holds.

Proposition II.2.3.2. *Assume that $v_0 \in \mathbb{L}^2$. If assumptions **(IG1)** and **(IG3)** are satisfied, then there exists a martingale solution to* (II.2.3.1) *such that*

$$\mathbb{E}\left[\sup_{0\le t\le T} \|v(t)\|^2_{\mathbb{L}^2} + \int_0^T \|\nabla v(t)\|^2_{L^2} \,\mathrm{d}t\right] < \infty. \tag{II.2.3.3}$$

*Moreover, under **(IG5)**, pathwise uniqueness holds.*

In particular, pathwise uniqueness and existence of martingale solutions implies existence of a strong solution (see e.g [45]).
Here we improve the regularity of the solution under stronger assumptions on the regularity of the initial datum and the covariance operator.

Proposition II.2.3.3. *Let $q > 2$ and assume that conditions **(IG1)**, **(IG2)**, **(IG3)** and **(IG5)** hold. If $v_0 \in \mathbb{L}^2 \cap \mathbb{L}^q$, then the unique strong solution v to* (II.2.3.1)*, in addition to* (II.2.3.3)*, satisfies for every $1 \le p < \infty$*

$$\mathbb{E} \sup_{0\le t\le T} \|v(t)\|^p_{\mathbb{L}^q} < C, \tag{II.2.3.4}$$

for a positive constant C, depending on q, T, $\|v_0\|_{\mathbb{L}^q}$ and $C_{g,q}$.

Proof. The proof of existence of solutions requires some Galerkin approximation v^n of v, for which a priori estimates are proved uniformly in n. Then, by a tightness argument one can pass to the limit proving the existence of a solution. Bearing in mind the existence and uniqueness result given by Proposition II.2.3.2, we just compute the needed $\mathbb{L}^q$-estimates in order to get (II.2.3.4).

Let $q \ge 2$ and $p \ge q$. Applying Itô formula to the function $\|\cdot\|^p_{\mathbb{L}^q}$, for all $t \in [0,T]$ we get

$$\begin{aligned}\|v(t)\|^p_{\mathbb{L}^q} \le \|v_0\|^p_{\mathbb{L}^q} &+ p\int_0^t \|v(s)\|^{p-q}_{\mathbb{L}^q}\langle |v(s)|^{q-2}v(s), [-Av(s) - B(v(s), v(s))]\rangle \,\mathrm{d}s \\ &+ p\int_0^t \|v(s)\|^{p-q}_{\mathbb{L}^q}\langle |v(s)|^{q-2}v(s), G(v(s)) \,\mathrm{d}W(s)\rangle \\ &+ \frac{p(q-1)}{2}\int_0^t \|v(s)\|^{p-2}_{\mathbb{L}^q}\|G(v(s))\|^2_{R(\mathcal{H};\mathbb{L}^q)} \,\mathrm{d}s.\end{aligned} \tag{II.2.3.5}$$

Let us estimate separately the various terms appearing in (II.2.3.5). By the integration by parts formula we get

$$
\begin{aligned}
-\langle |v(s)|^{q-2}v(s), Av(s)\rangle = &-\||v(s)|^{\frac{q-2}{2}}\nabla v(s)\|^2_{L^2} \\
&- (q-2)\int |v(s,x)|^{q-4}|\sum_j v_j(s,x)\nabla v_j(s,x)|^2 dx \le 0,
\end{aligned}
$$

and by (II.0.1.14)

$$
\langle |v(s)|^{q-2}v(s), B(v(s),v(s))\rangle = 0.
$$

By the Burkholder-Davis-Gundy inequality we get

$$
\begin{aligned}
\mathbb{E}\sup_{0\le t\le T}&\left| p\int_0^t \|v(s)\|^{p-q}_{\mathbb{L}^q}\langle |v(s)|^{q-2}v(s), G(v(s)\,\mathrm{d}W(s))\rangle\right|^2 \\
&\le C_p\mathbb{E}\int_0^T \|v(s)\|^{2(p-q)}_{\mathbb{L}^q}\|v(s)\|^{2(q-1)}_{\mathbb{L}^q}\|G(v(s))\|^2_{R(\mathcal{H},\mathbb{L}^q)}\,\mathrm{d}s \\
&= C_p\mathbb{E}\int_0^T \|v(s)\|^{2(p-1)}_{\mathbb{L}^q}\|G(v(s))\|^2_{R(\mathcal{H},\mathbb{L}^q)}\,\mathrm{d}s.
\end{aligned}
$$

Therefore, squaring both sides of (II.2.3.5) and then taking the expectation of the sup (in time) norm at first, then using Young inequality we get

$$
\begin{aligned}
\mathbb{E}\sup_{0\le t\le T}\|v(t)\|^{2p}_{\mathbb{L}^q} \le{}& \|v_0\|^{2p}_{\mathbb{L}^q} + \frac{p^2(q-1)^2}{4}\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t \|v(s)\|^{p-2}_{\mathbb{L}^q}\|G(v(s))\|^2_{R(\mathcal{H},\mathbb{L}^q)}\,\mathrm{d}s\right|^2 \\
&+ C_p\mathbb{E}\int_0^T \|v(s)\|^{2(p-1)}_{\mathbb{L}^q}\|G(v(s))\|^2_{R(\mathcal{H},\mathbb{L}^q)}\,\mathrm{d}s \\
\le{}& \|v_0\|^{2p}_{\mathbb{L}^q} + C^1_{p,T}\mathbb{E}\int_0^T \|G(v(s))\|^{2p}_{R(\mathcal{H},\mathbb{L}^q)}\,\mathrm{d}s + C^2_{p,T}\mathbb{E}\int_0^T \|v(s)\|^{2p}_{\mathbb{L}^q}\,\mathrm{d}s \\
\le{}& \|v_0\|^{2p}_{\mathbb{L}^q} + C^1_{p,T}\mathbb{E}\int_0^T \|G(v(s))\|^{2p}_{R(\mathcal{H},\mathbb{L}^q)}\,\mathrm{d}s + C^2_{p,T}\int_0^T \mathbb{E}\sup_{0\le s\le r}\|v(s)\|^{2p}_{\mathbb{L}^q}\,\mathrm{d}r.
\end{aligned}
$$

By Proposition II.2.3.2, $v(t)\in\mathbb{L}^2$ for every $t\in[0,T]$; then, by **(IG1)** and **(IG2)**, we get

$$
\mathbb{E}\int_0^T \|G(v(s))\|^{2p}_{R(\mathcal{H},\mathbb{L}^q)}\,\mathrm{d}s \le T(C_{g,q})^{2p},
$$

thus

$$
\mathbb{E}\sup_{0\le t\le T}\|v(t)\|^{2p}_{\mathbb{L}^q} \le \|v_0\|^{2p}_{\mathbb{L}^q} + C^1_{p,q,T} + C^2_{p,T}\int_0^T \mathbb{E}\sup_{0\le s\le r}\|v(s)\|^{2p}_{\mathbb{L}^q}\,\mathrm{d}r.
$$

Using Gronwall lemma we obtain (II.2.3.4).

This proves the result for $p\ge q$. Therefore it holds also for smaller values, i.e. $1\le p< q$. □

II.2.4 Existence of a unique solution to the vorticity equations (II.0.0.3)

We aim at proving that there exists a martingale solution to (II.0.0.3), in the sense of the following definition.

Definition II.2.4.1. *A martingale solution to equation* (II.0.0.3) *is a triple consisting of a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$*, an* $\{\mathcal{F}_t\}$*-adapted cylindrical Wiener process* W *on* $\mathcal{H}$ *and an* $\{\mathcal{F}_t\}$*-adapted measurable process* ξ *such that* $\xi : [0,T]\times\Omega \to L^2$ *with* $\mathbb{P}$*-a.a. paths*

$$\xi(\cdot,\omega) \in C([0,T]\,; L^2),$$

and such that for all $z \in C^\infty_{sol}$ *and* $t \in [0,T]$

$$\langle \xi(t), z\rangle = \langle \xi_0, z\rangle + \int_0^t \langle \xi(s), \Delta z\rangle \, \mathrm{d}s + \int_0^t \langle v(s)\xi(s), \nabla z\rangle \, \mathrm{d}s + \langle \int_0^t \tilde{G}(v(s))\, \mathrm{d}W(s), z\rangle \quad \text{(II.2.4.1)}$$

$\mathbb{P}$*-a.s., where* v *is the solution to* (II.2.3.1).

The regularity of the paths of this solution and the regularity of v proved in Proposition II.2.3.3 make all the terms in (II.2.4.1) well defined. The well posedness of the stochastic term follows from (II.2.2.3). As regard the well posedness of the non linear term, from (II.0.1.19) and the Gagliardo-Nirenberg inequality we get that

$$|\langle v(s)\xi(s), \nabla z\rangle| \le \|v(s)\|_{\mathbb{L}^4}\|\xi(s)\|_{L^2}\|\nabla z\|_{L^4} \le C\|v(s)\|^{\frac12}_{\mathbb{L}^2}\|\nabla v(s)\|^{\frac12}_{\mathbb{L}^2}\|\xi(s)\|_{L^2}\|\nabla z\|_{L^4}$$

and the r.h.s. is bounded thanks to (II.2.3.3) and the regularity required for ξ.

In order to prove the existence of a martingale solution to problem (II.0.0.3) we cannot use Itô calculus in the spaces $L^2 \cap L^q$, $q \ge 2$, since the covariance of the noise is not regular enough. Following the idea of [16] we introduce an approximation system by regularizing the covariance of the noise: we shall use the Hille-Yosida approximations. In this way we construct a sequence of approximating processes $\{\xi_n\}_n$ and $\{v_n\}_n$. In order to pass to the limit, as $n\to\infty$, we shall exploit the tightness of the sequence of their laws. This is obtained working pathwise with two auxiliary processes β_n and ζ_n with $\xi_n = \beta_n + \zeta_n$.
Thus, we introduce the smoother problems which approximate (II.0.0.1) and (II.0.0.3), then we prove the tightness of the sequence of the laws and finally we show the convergence. In this way we prove the existence of a martingale solution to (II.0.0.3).

II.2.4.1 The approximating equation

Let us introduce the Hille-Yosida approximations

$$R_n = n(nI + A)^{-1}, \qquad n = 1, 2, ...$$

and let us define the approximation sequence

$$G_n = R_n G, \qquad n = 1, 2, ...$$

Every R_n is a contraction operator in $H^{s,q}$ and it converges strongly to the identity operator, i.e. (see [75, Section 1.3])

$$\|R_n\|_{L(H^{s,q},H^{s,q})} \le 1 \quad \text{and} \quad \lim_{n\to\infty} R_n h = h, \quad \forall\, h \in H^{s,q}. \qquad \text{(II.2.4.2)}$$

Moreover, each R_n is a bounded operator from $H^{s,q}$ to $H^{s+t,q}$ for any $t \leq 2$, but the operator norm is not uniformly bounded in n for $t > 0$ (for the details see [16, Section 3.1]). From the above and [5]

$$\|G_n(v)\|_{R(\mathcal{H};H^{1-g,q})} \leq \|G(v)\|_{R(\mathcal{H};H^{1-g,q})}, \qquad \forall n \tag{II.2.4.3}$$

and

$$\lim_{n\to\infty} \|G_n(v) - G(v)\|_{R(\mathcal{H};H^{1-g,q})} = 0. \tag{II.2.4.4}$$

The operator $G_n(v)$ is more regular than $G(v)$. Indeed, assuming **(IG1)** and **(IG2)** (or **(IG3)**), $G_n(v)$ is a γ-radonifying operator in $H^{1,q}$, $q \geq 2$. In fact, for $g \in (0,1)$

$$\begin{aligned}\|G_n(v)\|_{R(\mathcal{H};H^{1,q})} &\leq \|R_n J^g\|_{\mathcal{L}(H^{1,q},H^{1,q})} \|J^{-g}G(v)\|_{R(\mathcal{H};H^{1,q})} \\ &\leq \|R_n\|_{\mathcal{L}(H^{1,q},H^{1+g,q})} \|G(v)\|_{R(\mathcal{H},H^{1-g,q})}.\end{aligned} \tag{II.2.4.5}$$

For every $n \in \mathbb{N}$ we consider the approximating problem

$$\begin{cases} \mathrm{d}v(t) + [Av(t) + B(v(t),v(t))]\,\mathrm{d}t = G_n(v(t))\,\mathrm{d}W(t), & t \in [0,T] \\ v(0) = v_0 \end{cases} \tag{II.2.4.6}$$

By taking the *curl* on both sides of the first equation we obtain the approximating equation for the vorticity:

$$\begin{cases} \mathrm{d}\xi(t) + [A\xi(t) + v(t)\cdot\nabla\xi(t)]\,\mathrm{d}t = \tilde{G}_n(v(t))\,\mathrm{d}W(t), & t \in [0,T] \\ \xi = \nabla^\perp \cdot v \\ \xi(0,x) = \xi_0(x) \end{cases} \tag{II.2.4.7}$$

With the same abuse of notation used above, for every $n \in \mathbb{N}$, we write $\tilde{G}_n(v)\mathrm{d}W(t)$ instead of $\mathrm{curl}(G_n(v)\mathrm{d}W(t))$, where $\tilde{G}_n := \mathrm{curl}G_n$. This is the vorticity equation (II.0.0.3) with a more regular noise.

The next result provides the existence of a unique strong solution to system (II.2.4.7), for any fixed $n \in \mathbb{N}$. We recall that by strong solution to (II.2.4.7) we mean an $\{\mathcal{F}_t\}$-adapted measurable process ξ such that $\xi : [0,T] \times \Omega \to L^2$ with $\mathbb{P}$-a.s. paths $\xi(\cdot,\omega) \in C([0,T];L^2)$, that satisfies (II.2.4.1), where the last term is replaced by $\langle \int_0^t \tilde{G}_n(v(s))\,\mathrm{d}W(s), z\rangle$. Here the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ is given in advance and it is not constructed as a part of the solution. The proof of Proposition II.2.4.2 is based on Theorem II.1.3.2.

Proposition II.2.4.2. *Assume conditions **(IG1)**, **(IG4)** and **(IG5)**. Let $\xi_0 \in L^2$ and $v_0 \in \mathbb{L}^2$. Then, for each $n \in \mathbb{N}$, there exists a unique strong solution ξ_n to* (II.2.4.7). *Moreover,*

$$\xi_n \in L^p(\Omega; L^\infty(0,T;L^2)) \cap L^2(\Omega; L^2(0,T;W^{1,2})), \qquad \forall p > 1$$

and there exists a constant C_n such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \|\xi_n(t)\|^p_{L^2}\right] + \mathbb{E}\left[\int_0^T \|\xi_n(t)\|^2_{W^{1,2}}\,\mathrm{d}t\right] \leq C_n. \tag{II.2.4.8}$$

Proof. Thanks to (II.2.4.5), the operator G_n is regular enough to apply Theorem II.1.3.2. In particular, for $q = 2$ and for any $n \in \mathbb{N}$, we infer the existence of a martingale solution (in the sense of Definition II.1.3.1) to (II.2.4.6), with the stated regularity. Moreover, under assumption **(IG5)**, the solution is pathwise unique (this follows from [17, Lemma 7.3]). Thus (II.2.4.6) admits a unique strong solution. As a consequence we infer that, for any $n \in \mathbb{N}$, there exists a strong solution of the approximating problem (II.2.4.7). This is obtained by taking the *curl* of the solution to equation (II.2.4.6). □

II.2.4.2 Tightness of the law of $\{v_n\}_n$

In this Section we provide the tightness of the sequence of the laws of $\{v_n\}_n$ in proper spaces. The crucial point is to obtain uniform estimates in $n \in \mathbb{N}$.

Proposition II.2.4.3. *Assume* ***(IG1)****,* ***(IG3)*** *and* ***(IG5)****. If $v_0 \in \mathbb{L}^2$, then there exists a unique strong solution to* (II.2.4.6) *for each $n \in \mathbb{N}$.*
Moreover,

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{0\le t\le T} \|v_n(t)\|^2_{\mathbb{L}^2} + \int_0^T \|\nabla v_n(t)\|^2_{L^2}\,\mathrm{d}t\right] < \infty. \tag{II.2.4.9}$$

In particular, for any $\varepsilon > 0$ there exist positive constants α_i, $i = 1, 2, 3$ such that

$$\sup_n \mathbb{P}\left(\|v_n\|_{L^\infty(0,T;\mathbb{L}^2)} > \alpha_1\right) \le \varepsilon, \tag{II.2.4.10}$$

$$\sup_n \mathbb{P}\left(\|v_n\|_{L^2(0,T;H^{1,2})} > \alpha_2\right) \le \varepsilon. \tag{II.2.4.11}$$

$$\sup_n \mathbb{P}\left(\|v_n\|_{L^4(0,T;\mathbb{L}^4)} > \alpha_3\right) \le \varepsilon. \tag{II.2.4.12}$$

Moreover, there exists $\mu > 0$ such that for any $\varepsilon > 0$ there exists a positive constant α_4 such that

$$\sup_n \mathbb{P}\left(\|v_n\|_{C^\mu([0,T];H^{-1,2})} > \alpha_4\right) \le \varepsilon. \tag{II.2.4.13}$$

Proof. The proof of (II.2.4.9) immediately follows from the results of Section II.2.3. Indeed, by (II.2.4.3) we get a uniform estimate on $G_n(v)$. From this we infer the estimates in probability (II.2.4.10) and (II.2.4.11), which in turn imply (II.2.4.12) thanks to the Gagliardo-Nirenberg inequality $\|v_n(s)\|_{\mathbb{L}^4} \le C\|v_n(s)\|^{1/2}_{\mathbb{L}^2}\|\nabla v_n(s)\|^{1/2}_{L^2}$.

Finally, estimate (II.2.4.13) comes from Proposition 3.5 of [16]. Indeed, all the assumptions of that Proposition are fulfilled; in particular the continuous embedding $H^{1-g,2} \subset H^{-g,4}$ implies assumption (G2) of Proposition 3.5 in [16]. □

In the same way, from Proposition II.2.3.3 we get

Proposition II.2.4.4. *Let $q > 2$ and assume* ***(IG1)****,* ***(IG2)****,* ***(IG3)*** *and* ***(IG5)****. Let $v_0 \in \mathbb{L}^2 \cap \mathbb{L}^q$. Let $\{v_n\}$ be the solution to* (II.2.4.6) *as given in Proposition II.2.4.3. Then, in addition to* (II.2.4.9)-(II.2.4.13)*, for any $1 < p < \infty$ it holds,*

$$\sup_{n\in\mathbb{N}} \mathbb{E} \sup_{0\le t\le T} \|v_n(t)\|^p_{\mathbb{L}^q} < \infty. \tag{II.2.4.14}$$

In particular, for any $\varepsilon > 0$ there exists a positive constant α_4, such that

$$\sup_n \mathbb{P}\left(\|v_n\|_{L^\infty(0,T;\mathbb{L}^q)} > \alpha_4\right) \le \varepsilon. \tag{II.2.4.15}$$

II.2.4.3 Tightness of the law of ξ_n

The present Section is devoted to the proof of the tightness of the sequence of the laws of $\{\xi_n\}_n$. Let us start by noticing that estimate (II.2.4.8) is not uniform with respect to n: (II.2.4.5) shows that the γ-radonifying norms of the $G_n(v)$ (and thus of the $\tilde{G}_n(v)$) are not uniformly bounded in n. Therefore, from (II.2.4.8) we cannot obtain the tightness of the sequence of the laws of the ξ_n's. In order to get uniform estimates in n for the sequence $\{\xi_n\}_n$, we follow the idea of [16], splitting our problem in two subproblems in the unknowns ζ_n and β_n with $\xi_n = \zeta_n + \beta_n$.

We define the process ζ_n as the solution of the Ornstein-Uhlenbeck equation

$$\begin{cases} \mathrm{d}\zeta_n(t) + A\zeta_n(t)\ \mathrm{d}t = \tilde{G}_n(v_n(t))\,\mathrm{d}W(t), \qquad t \in [0,T] \\ \zeta_n(0) = 0. \end{cases} \tag{II.2.4.16}$$

Therefore, the process $\beta_n = \xi_n - \zeta_n$ solves

$$\begin{cases} \frac{\mathrm{d}\beta_n}{\mathrm{d}t}(t) + A\beta_n(t) + v_n(t)\cdot\nabla\xi_n(t) = 0, \qquad t \in [0,T] \\ \beta_n(0) = \xi_0. \end{cases} \tag{II.2.4.17}$$

We shall first analyze the Ornstein-Uhlenbeck processes; the solution ζ_n is given by

$$\zeta_n(t) = \int_0^t S(t-s)\tilde{G}_n(v_n(s))\,\mathrm{d}W(s). \tag{II.2.4.18}$$

With a slight modification of the proofs of [16, Lemma 3.2] and [16, Lemma 3.3] respectively, we have the following regularity results. Recall that assumptions **(IG1)**-**(IG2)** reads as **(I$\tilde{G}$1)**-**(I$\tilde{G}$2)** when we deal with the equation for the vorticity.

Lemma II.2.4.5. *Let $q \geq 2$. Assume conditions **(IG1)** and **(IG2)**. Take any $g_0 \in [g,1)$ and put $\varepsilon = g_0 - g \geq 0$. Then, for any integer $m \geq 2$ there exists a constant C independent of n (but depending on m, T, q, g_0 and $\tilde{C}_{g,q}$) such that*

$$\mathbb{E}\|\zeta_n\|^m_{L^m(0,T;W^{\varepsilon,q})} \leq C.$$

In particular, $\zeta_n \in L^m(0,T;W^{\varepsilon,q})$ $\mathbb{P}$-a.s.

Lemma II.2.4.6. *Let $q \geq 2$, assume **(IG1)** and let*

$$0 \leq \beta < \frac{1-g}{2}.$$

Then for any $p \geq 2$ and $\delta \geq 0$ such that

$$\beta + \frac{\delta}{2} + \frac{1}{p} < \frac{1-g}{2}$$

there exists a modification $\tilde{\zeta}_n$ of ζ_n such that

$$\mathbb{E}\|\tilde{\zeta}_n\|^p_{C^\beta([0,T];W^{\delta,q})} \leq \tilde{C} \tag{II.2.4.19}$$

for some constant $\tilde{C}$ independent of n (but depending on T, β, δ, p and q).

As a consequence of Lemma II.2.4.5 and Lemma II.2.4.6 we have that there exist finite constants $K_{m,q}$ and $K_{\beta,\delta,q}$ such that

$$\sup_n \mathbb{E}\|\zeta_n\|^m_{L^m(0,T;W^{\varepsilon,q})} = (K_{m,q})^m$$

and

$$\sup_n \mathbb{E}\|\zeta_n\|^p_{C^\beta([0,T];W^{\delta,q})} = (K_{\beta,\delta,q})^p.$$

Therefore, by Chebychev's inequality, for any $\eta > 0$

$$\sup_n \mathbb{P}\left(\|\zeta_n\|_{L^m(0,T;W^{\varepsilon,q})} > \eta\right) \leq \frac{(K_{m,q})}{\eta} \tag{II.2.4.20}$$

and

$$\sup_n \mathbb{P}\left(\|\zeta_n\|_{C^\beta([0,T];W^{\delta,q})} > \eta\right) \leq \frac{(K_{\beta,\delta,q})}{\eta}. \tag{II.2.4.21}$$

Thanks to these two last inequalities we get uniform estimates in probability for the sequence β_n (see Propositions II.2.4.7 and II.2.4.8), and consequently for $\xi_n = \beta_n + \zeta_n$ (see Proposition II.2.4.9).
Let us now turn to the analysis of equation (II.2.4.17). We shall analyze it pathwise, proving the following result.

Proposition II.2.4.7. *Let $q = 4$ and assume **(IG1)** and **(IG2)**. Let $\xi_0 \in L^2$ and $v_0 \in \mathbb{L}^2$. Then for every $n \in \mathbb{N}$ the paths of the process $\beta_n = \xi_n - \zeta_n$ solving* (II.2.4.17) *are such that*

$$\beta_n \in C([0,T]\,;L^2) \cap L^4(0,T;L^4) \cap L^2(0,T;W^{1,2}) \cap C^{\frac{1}{2}}([0,T]\,;W^{-1,2})$$

$\mathbb{P}$-a.s., and for any $\varepsilon > 0$ there exist constants $C_i = C_i(\varepsilon)$, $i = 1, ...4$ such that

$$\sup_n \mathbb{P}\left(\|\beta_n\|_{L^\infty(0,T;L^2)} > C_1\right) \leq \varepsilon \tag{II.2.4.22}$$

$$\sup_n \mathbb{P}\left(\|\beta_n\|_{L^2(0,T;W^{1,2})} > C_2\right) \leq \varepsilon \tag{II.2.4.23}$$

$$\sup_n \mathbb{P}\left(\|\beta_n\|_{L^4(0,T;L^4)} > C_3\right) \leq \varepsilon \tag{II.2.4.24}$$

$$\sup_n \mathbb{P}\left(\|\beta_n\|_{C^{\frac{1}{2}}([0,T];W^{-1,2})} > C_4\right) \leq \varepsilon. \tag{II.2.4.25}$$

Proof. By definition and merging the regularity of ξ_n and ζ_n we have that $\beta_n \in L^\infty(0,T;L^2)$. Let us prove estimates (II.2.4.22)-(II.2.4.25). We begin with the usual energy estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\beta_n(t)\|^2_{L^2} + \|\nabla\beta_n(t)\|^2_{L^2} = -\langle v_n(t)\cdot\nabla\xi_n(t), \beta_n(t)\rangle. \tag{II.2.4.26}$$

Let us focus on the trilinear term. By means of (II.0.1.16), Gagliardo-Nirenberg and Young's inequalities, we get

$$\begin{aligned}
-\langle v_n(t)\cdot\nabla\xi_n(t),\beta_n(t)\rangle &= \langle v_n(t)\cdot\nabla\beta_n(t),\xi_n(t)\rangle \\
&= \langle v_n(t)\cdot\nabla\beta_n(t),\beta_n(t)\rangle + \langle v_n(t)\cdot\nabla\beta_n(t),\zeta_n(t)\rangle \\
&= \langle v_n(t)\cdot\nabla\beta_n(t),\zeta_n(t)\rangle \\
&\le \|\nabla\beta_n(t)\|_{L^2}\|v_n(t)\|_{\mathbb{L}^4}\|\zeta_n(t)\|_{L^4} \\
&\le C\|\nabla\beta_n(t)\|_{L^2}\|\zeta_n(t)\|_{L^4}\|v_n(t)\|_{\mathbb{L}^2}^{\frac12}\|\nabla v_n(t)\|_{\mathbb{L}^2}^{\frac12} \\
&\le C\|\nabla\beta_n(t)\|_{L^2}\|\zeta_n(t)\|_{L^4}\left(\frac12\|v_n(t)\|_{\mathbb{L}^2}+\frac12\|\nabla v_n(t)\|_{\mathbb{L}^2}\right) \\
&\le C\|\nabla\beta_n(t)\|_{L^2}\|\zeta_n(t)\|_{L^4}\left(\frac12\|v_n(t)\|_{\mathbb{L}^2}+\frac12\|\xi_n(t)\|_{L^2}\right) \\
&\le \frac14\|\nabla\beta_n(t)\|_{L^2}^2 + C\|\zeta_n(t)\|_{L^4}^2\|v_n(t)\|_{\mathbb{L}^2}^2 + \frac14\|\nabla\beta_n(t)\|_{L^2}^2 \\
&\qquad + C\|\zeta_n(t)\|_{L^4}^2\|\beta_n(t)\|_{L^2}^2 + C\|\zeta_n(t)\|_{L^4}^2\|\zeta_n(t)\|_{L^2}^2 \\
&= \frac12\|\nabla\beta_n(t)\|_{L^2}^2 + C_1\|\zeta_n(t)\|_{L^4}^2\left(\|v_n(t)\|_{\mathbb{L}^2}^2+\|\zeta_n(t)\|_{L^2}^2\right) \\
&\qquad + \frac{C_2}{2}\|\zeta_n(t)\|_{L^4}^2\|\beta_n(t)\|_{L^2}^2.
\end{aligned}$$

Let us set

$$\psi_n(t) := 2C_1\|\zeta_n(t)\|_{L^4}^2\left(\|v_n(t)\|_{\mathbb{L}^2}^2+\|\zeta_n(t)\|_{L^2}^2\right). \tag{II.2.4.27}$$

Using Hölder's inequality, by Lemma II.2.4.5 and Proposition II.2.4.3 we have that $\psi_n \in L^1(0,T)$.

Then from (II.2.4.26) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\beta_n(t)\|_{L^2}^2 + \|\nabla\beta_n(t)\|_{L^2}^2 \le \psi_n(t) + C_2\|\zeta_n(t)\|_{L^4}^2\|\beta_n(t)\|_{L^2}^2. \tag{II.2.4.28}$$

Hence from Gronwall's lemma applied to inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\beta_n(t)\|_{L^2}^2 \le \psi_n(t) + C_2\|\zeta_n(t)\|_{L^4}^2\|\beta_n(t)\|_{L^2}^2$$

we infer that

$$\sup_{0\le t\le T}\|\beta_n(t)\|_{L^2}^2 \le \|\xi_0\|_{L^2}^2 e^{C_2\int_0^T\|\zeta_n(r)\|_{L^4}^2\,\mathrm{d}r} + e^{C_2\int_0^T\|\zeta_n(r)\|_{L^4}^2\,\mathrm{d}r}\int_0^T\psi_n(s)\,\mathrm{d}s. \tag{II.2.4.29}$$

Then, integrating in time estimate (II.2.4.28) we infer that for all n

$$\begin{aligned}
\int_0^T\|\nabla\beta_n(t)\|_{L^2}^2\,\mathrm{d}t &\le \|\xi_0\|_{L^2}^2 + \int_0^T\left(\psi_n(t)+C_2\|\zeta_n(t)\|_{L^4}^2\|\beta_n(t)\|_{L^2}^2\right)\mathrm{d}t \\
&\le \|\xi_0\|_{L^2}^2 + C_2\left(\sup_{0\le t\le T}\|\beta_n(t)\|_{L^2}^2\right)\int_0^T\|\zeta_n(t)\|_{L^4}^2\,\mathrm{d}t + \int_0^T\psi_n(t)\,\mathrm{d}t.
\end{aligned} \tag{II.2.4.30}$$

Recalling (II.2.4.10) and (II.2.4.20), we infer that for any $\varepsilon > 0$ there exists a constant C_7 such that

$$\sup_n \mathbb{P}\left(\int_0^T \psi_n(t)\,\mathrm{d}t > C_7\right) \le \varepsilon.$$

Therefore, from (II.2.4.29) and (II.2.4.30) we get that, for any $\varepsilon > 0$ there exist suitable constants $R_1, R_2 > 0$ such that

$$\sup_n \mathbb{P}\left(\|\beta_n\|_{L^\infty(0,T;L^2)} > R_1\right) \le \varepsilon, \qquad \sup_n \mathbb{P}\left(\|\nabla\beta_n\|_{L^2(0,T;L^2)} > R_2\right) \le \varepsilon.$$

From these two last inequalities it is straightforward to see that, for any $\varepsilon > 0$, there exists a suitable constant $R_3 > 0$ such that

$$\sup_n \mathbb{P}\left(\|\beta_n\|_{L^2(0,T;W^{1,2})} > R_3\right) \le \varepsilon.$$

These estimates prove (II.2.4.22) and (II.2.4.23).

Now, as done in Proposition II.2.4.3 we obtain (II.2.4.24) from (II.2.4.22) and (II.2.4.23) by means of Gagliardo-Nirenberg inequality.

Finally, from (II.2.4.17) we infer

$$\left\|\frac{\mathrm{d}\beta_n}{\mathrm{d}t}\right\|_{L^2(0,T;W^{-1,2})} \le \|A\beta_n\|_{L^2(0,T;W^{-1,2})} + \|v_n\cdot\nabla\xi_n\|_{L^2(0,T;W^{-1,2})}.$$

Bearing in mind (II.0.1.19) we get

$$\begin{aligned}\|v_n\cdot\nabla\xi_n\|_{L^2(0,T;W^{-1,2})} &\le \left(\int_0^T \|v_n(t)\|^2_{\mathbb{L}^4}\|\xi_n(t)\|^2_{L^4}\,\mathrm{d}t\right)^{\frac12}\\ &\le \|v_n\|_{L^4(0,T;\mathbb{L}^4)}\|\xi_n\|_{L^4(0,T;L^4)}\\ &\le \|v_n\|^2_{L^4(0,T;\mathbb{L}^4)} + \|\beta_n\|^2_{L^4(0,T;L^4)} + \|\zeta_n\|^2_{L^4(0,T;L^4)}\end{aligned}$$

Thus,

$$\left\|\frac{\mathrm{d}\beta_n}{\mathrm{d}t}\right\|_{L^2(0,T;W^{-1,2})} \le \|\beta_n\|_{L^2(0,T;W^{1,2})} + \|v_n\|^2_{L^4(0,T;\mathbb{L}^4)} + \|\beta_n\|^2_{L^4(0,T;L^4)} + \|\zeta_n\|^2_{L^4(0,T;L^4)}.$$

From (II.2.4.23), (II.2.4.12), (II.2.4.23) and (II.2.4.20) we find that for any $\varepsilon > 0$, there exists a suitable constant $R_4 > 0$ such that

$$\sup_n \mathbb{P}\left(\left\|\frac{\mathrm{d}\beta_n}{\mathrm{d}t}\right\|_{L^2(0,T;W^{-1,2})} > R_4\right) \le \varepsilon.$$

Now we recall the Sobolev's embedding Theorem $H^{1,2}(0,T) = \{u \in L^2(0,T) : u' \in L^2(0,T)\} \subset C^{\frac12}([0,T])$. Hence, there exists a constant $R_5 > 0$ such that

$$\sup_n \mathbb{P}\left(\|\beta_n\|_{C^{\frac12}([0,T];W^{-1,2})} > R_5\right) \le \varepsilon,$$

which proves (II.2.4.25).

We conclude proving the continuity in time. From the previous estimates we have that $\frac{\mathrm{d}\beta_n}{\mathrm{d}t} \in L^2(0,T;W^{-1,2})$ and $\beta_n \in L^2(0,T;W^{1,2})$. Therefore (see [87, Theorem III.1.2]) we get $\beta_n \in C([0,T];L^2)$. □

Proposition II.2.4.8. *Assume that conditions of Proposition II.2.4.7 hold. In addition assume that condition **(IG2)** holds also for a $q > 2$. Let $\xi_0 \in L^2 \cap L^q$ and $v_0 \in \mathbb{L}^2 \cap \mathbb{L}^q$. Then for every $n \in \mathbb{N}$ the paths of the process $\beta_n = \xi_n - \zeta_n$ solving* (II.2.4.17) *are such that*

$$\beta_n \in C([0,T]\,;L^2) \cap L^\infty(0,T;L^q) \cap L^4(0,T;L^4) \cap L^2(0,T;W^{1,2}) \cap C^{\frac{1}{2}}([0,T]\,;W^{-1,2})$$

$\mathbb{P}$*-a.s., and, in addition to* (II.2.4.22)-(II.2.4.25)*, for any $\varepsilon > 0$ there exist a constant C_5, such that*

$$\sup_n \mathbb{P}\left(\|\beta_n\|_{L^\infty(0,T;L^q)} > C_5\right) \leq \varepsilon. \tag{II.2.4.31}$$

Proof. Let us estimate the L^q-norm for $q > 2$. Let $x \in \mathbb{R}^2$ and $t \in [0,T]$. We get

$$\frac{\partial}{\partial t}|\beta_n(t,x)|^q = q|\beta_n(t,x)|^{q-2}\beta_n(t,x)\left(\Delta\beta_n(t,x) - v_n(t,x)\cdot\nabla\xi_n(t,x)\right).$$

Integrating on $\mathbb{R}^2$, by means of the integration by parts formula we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\beta_n(t)\|_{L^q}^q &= q\langle|\beta_n(t)|^{q-2}\beta_n(t),\Delta\beta_n(t)\rangle - q\langle|\beta_n(t)|^{q-2}\beta_n(t)v_n(t),\nabla\xi_n(t)\rangle \\ &= -q(q-1)\||\beta_n(t)|^{\frac{q-2}{2}}\nabla\beta_n(t)\|_{L^2}^2 - q\langle|\beta_n(t)|^{q-2}\beta_n(t)v_n(t),\nabla\xi_n(t)\rangle.\end{aligned} \tag{II.2.4.32}$$

Let us estimate the nonlinear term. Thanks to (II.0.1.17)-(II.0.1.18) we get

$$\begin{aligned}&-q\langle|\beta_n(t)|^{q-2}\beta_n(t)v_n(t),\nabla\xi_n(t)\rangle \\ &\qquad= -q\langle|\beta_n(t)|^{q-2}\beta_n(t)v_n(t),\nabla\beta_n(t)\rangle - q\langle|\beta_n(t)|^{q-2}\beta_n(t)v_n(t),\nabla\zeta_n(t)\rangle \\ &\qquad= q(q-1)\langle|\beta_n(t)|^{q-2}\zeta_n(t)v_n(t),\nabla\beta_n(t)\rangle\end{aligned}$$

By means of Young's inequality and (II.0.1.8), recalling (II.0.1.5) we get

$$\begin{aligned}&|\langle|\beta_n(t)|^{\frac{q-2}{2}}\zeta_n(t)v_n(t),\nabla\beta_n(t)\rangle| \\ &\quad\leq \||\beta_n(t)|^{q-2}\nabla\beta_n(t)\|_{L^2}\|\zeta_n(t)\|_{L^2}\|v_n(t)\|_{L^\infty} \\ &\quad\leq \frac{1}{2}\||\beta_n(t)|^{q-2}\nabla\beta_n(t)\|_{L^2}^2 + \frac{C}{2}\|\zeta_n(t)\|_{L^2}^2\|v_n(t)\|_{H^{1,q}}^2 \\ &\quad= \frac{1}{2}\||\beta_n(t)|^{q-2}\nabla\beta_n(t)\|_{L^2}^2 + \frac{C}{2}\|\zeta_n(t)\|_{L^2}^2\left(\|v_n(t)\|_{\mathbb{L}^q}^2 + \|\xi_n(t)\|_{L^q}^2\right) \\ &\quad\leq \frac{1}{2}\||\beta_n(t)|^{q-2}\nabla\beta_n(t)\|_{L^2}^2 + C\|\zeta_n(t)\|_{L^2}^2\left(\|v_n(t)\|_{\mathbb{L}^q}^2 + \|\beta_n(t)\|_{L^q}^2 + \|\zeta_n(t)\|_{L^q}^2\right) \\ &\quad\leq \frac{1}{2}\||\beta_n(t)|^{q-2}\nabla\beta_n(t)\|_{L^2}^2 + C_1\|\beta_n(t)\|_{L^q}^q + C_2\|\zeta_n(t)\|_{L^2}^{\frac{2q}{q-2}} \\ &\qquad\qquad + \|\zeta_n(t)\|_{L^2}^2\left(\|v_n(t)\|_{\mathbb{L}^q}^2 + \|\zeta_n(t)\|_{L^q}^2\right)\end{aligned}$$

Let us set

$$\varphi_n(t) = C_2\|\zeta_n(t)\|_{L^2}^{\frac{2q}{q-2}} + \|\zeta_n(t)\|_{L^2}^2\left(\|v_n(t)\|_{\mathbb{L}^q}^2 + \|\zeta_n(t)\|_{L^q}^2\right),$$

then from (II.2.4.32) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\beta_n(t)\|_{L^q}^q + \frac{q(q-1)}{2}\||\beta_n(t)|^{q-2}\nabla\beta_n(t)\|_{L^2}^2 \leq q(q-1)\left(\varphi_n(t) + C_1\|\beta_n(t)\|_{L^q}^q\right). \tag{II.2.4.33}$$

Using Hölder's inequality, by Lemma II.2.4.5 and Proposition II.2.4.4, we have that $\varphi_n \in L^1(0,T)$ uniformly in n. Hence from Gronwall's lemma applied to inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\beta_n(t)\|^q_{L^q} \le q(q-1)\left(\varphi_n(t) + C_1\|\beta_n(t)\|^q_{L^q}\right),$$

we infer that for all n

$$\sup_{0\le t\le T}\|\beta_n(t)\|^q_{L^q} \le \|\xi_0\|^q_{L^q} e^{q(q-1)C_1T} + q(q-1)\int_0^T \varphi_n(s)\,\mathrm{d}s. \tag{II.2.4.34}$$

Recalling (II.2.4.15) and (II.2.4.20), we infer that for any $\varepsilon > 0$ there exists a constant C_0 such that

$$\sup_n \mathbb{P}\left(\int_0^T \varphi_n(t)\,\mathrm{d}t > C_0\right) \le \varepsilon.$$

Therefore, from (II.2.4.34) we get that, for any $\varepsilon > 0$ there exist suitable constant $R_4 > 0$ such that

$$\sup_n \mathbb{P}\left(\|\beta_n\|_{L^\infty(0,T;L^q)} > R_4\right) \le \varepsilon.$$

This proves (II.2.4.24). □

In order to pass to the limit we shall now apply a tightness argument. Merging the estimates (II.2.4.20)-(II.2.4.21) for ζ_n and those for β_n in Proposition II.2.4.7 we get the estimates of $\xi_n = \zeta_n + \beta_n$. These estimates in probability are uniform with respect to n.

Proposition II.2.4.9. *i) Let $q = 4$ and assume conditions **(IG1)**, **(IG2)**, **(IG4)** and **(IG5)**. Let $\xi_0 \in L^2$ and $v_0 \in \mathbb{L}^2$. Let ξ_n be the solution to* (II.2.4.7) *as given in Proposition II.2.4.2.*
Then there exist $\gamma, \delta > 0$ such that for any $\varepsilon > 0$ there exist positive constants η_i, $i = 1, ..., 4$ such that

$$\sup_n \mathbb{P}\left(\|\xi_n\|_{L^\infty(0,T;L^2)} > \eta_1\right) \le \varepsilon$$

$$\sup_n \mathbb{P}\left(\|\xi_n\|_{L^4(0,T;L^4)} > \eta_2\right) \le \varepsilon$$

$$\sup_n \mathbb{P}\left(\|\xi_n\|_{L^2(0,T;W^{\delta,2})} > \eta_3\right) \le \varepsilon$$

$$\sup_n \mathbb{P}\left(\|\xi_n\|_{C^\gamma([0,T];W^{-1,2})} > \eta_4\right) \le \varepsilon.$$

*ii) If in addition we assume that condition **(IG2)** holds also for a $q > 2$ and if $\xi_0 \in L^2 \cap L^q$, $v_0 \in \mathbb{L}^2 \cap \mathbb{L}^q$, then for any $\varepsilon > 0$ there exist positive constants η_5, such that*

$$\sup_n \mathbb{P}\left(\|\xi_n\|_{L^\infty(0,T;L^q)} > \eta_5\right) \le \varepsilon.$$

Let us notice that $\gamma = \min(\beta, \frac{1}{2})$, with β and γ fulfilling hypothesis in Lemma II.2.4.6; thus $0 < \gamma < \frac{1}{2}$ and $0 < \delta < 1$.

II.2.4.4 Convergence and existence of a unique strong solution

In order to pass to the limit we exploit a tightness argument. This requires some technical results. If we proceed as in [17, Lemma 3.3.] and [16, Lemma 5.3], we get the following compactness result.

Lemma II.2.4.10. *Let $\alpha, q > 1$ and define*

$$Z = L^{\alpha}_{w}(0,T;L^q) \cap C([0,T];\mathbb{V}') \cap L^2(0,T;L^2_{loc}) \cap C([0,T];L^2_w).$$

Let $\mathcal{T}$ be the supremum of the corresponding topologies. Then a set $K \subset Z$ is $\mathcal{T}$-relatively compact if the following conditions hold:

i. $\sup_{f\in K} \|f\|_{L^{\alpha}(0,T;L^q)} < \infty$

ii. $\exists\ \gamma > 0 : \sup_{f\in K} \|f\|_{C^{\gamma}([0,T];W^{-1,2})} < \infty$

iii. $\exists\ \delta > 0 : \sup_{f\in K} \|f\|_{L^2(0,T;W^{\delta,2})} < \infty$

iv. $\sup_{f\in K} \|f\|_{L^{\infty}(0,T;L^2)} < \infty$

From this Lemma we also get the following tightness criterion.

Lemma II.2.4.11. *We are given parameters $\gamma > 0$, $\delta > 0$, $\alpha, q > 1$ and a sequence $\{f_n\}_{n\in\mathbb{N}}$ of adapted processes in $C([0,T];\mathbb{V}')$.*
Assume that for any $\varepsilon > 0$ there exist positive constants $R_i = R_i(\varepsilon)$ $(i = 1, ..., 4)$ such that

$$\sup_n \mathbb{P}\left(\|f_n\|_{L^{\alpha}(0,T;L^q)} > R_1\right) \le \varepsilon$$
$$\sup_n \mathbb{P}\left(\|f_n\|_{C^{\gamma}([0,T];W^{-1,2})} > R_2\right) \le \varepsilon$$
$$\sup_n \mathbb{P}\left(\|f_n\|_{L^2(0,T;W^{\delta,2})} > R_3\right) \le \varepsilon$$
$$\sup_n \mathbb{P}\left(\|f_n\|_{L^{\infty}(0,T;L^2)} > R_4\right) \le \varepsilon$$

Let μ_n be the law of f_n on $Z = L^{\alpha}_{w}(0,T;L^q) \cap C([0,T];\mathbb{V}') \cap L^2(0,T;L^2_{loc}) \cap C([0,T];L^2_w)$. Then the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is tight in Z.

Remark II.2.4.12. *Lemma II.2.4.11 holds true also for the case of divergence free vector field spaces.*

The results of Sections II.2.4.2 and II.2.4.3, Lemma II.2.4.11 and Remark II.2.4.12 provide the tightness to pass to the limit. Proceeding similarly as in the proof of [16, Theorem 3.6.] we get the following result.

Theorem II.2.4.13. *i) Let $q = 4$ and assume conditions **(IG1)**, **(IG2)**, **(IG3)** and **(IG4)**. Let $\xi_0 \in L^2$ and $v_0 \in \mathbb{L}^2$. Then there exists a martingale solution $((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), \tilde{W}, \tilde{\xi})$ (in the sense of Definition II.2.4.1) to* (II.0.0.3). *In addition $\tilde{\xi} \in L^4(0,T;L^4)$ $\mathbb{P}$-a.s..*

*ii) If, in addition, we assume that condition **(IG2)** holds also for a $q > 2$, and $\xi_0 \in L^2 \cap L^q$, $v_0 \in \mathbb{L}^2 \cap \mathbb{L}^q$, then also $\tilde{\xi} \in L^{\infty}(0,T;L^q)$ $\mathbb{P}$-a.s..*

Proof. Let us prove (i). One proceeds as in [16]. We fix $0 < \gamma < \frac{1}{2}$ and $0 < \delta < 1$ appearing in Proposition II.2.4.9 and define the spaces

$$Z = L^4_w(0,T;L^4) \cap C([0,T]\,;\mathbb{V}') \cap L^2(0,T;L^2_{loc}) \cap C([0,T]\,;L^2_w),$$

$$\mathbb{Z} = L^4_w(0,T;\mathbb{L}^4) \cap C([0,T]\,;\mathbb{U}') \cap L^2(0,T;\mathbb{L}^2_{loc}) \cap C([0,T]\,;\mathbb{L}^2_w),$$

with the topology $\mathcal{T}$ and τ respectively, given by the supremum of the corresponding topologies. According to Lemma II.2.4.11 (with $\alpha = 4$, $q = 4$), Proposition II.2.4.9(i) provides that the sequence of laws of the processes ξ_n is tight in Z. Moreover, according to Lemma II.2.4.11 (with $\alpha = q = 4$) and Remark II.2.4.12, Propositions II.2.4.3 provide that the sequence of laws of the processes v_n is tight in $\mathbb{Z}$. So the pair (ξ_n, v_n) is tight in $Z \times \mathbb{Z}$.
By the Jakubowski's generalization of the Skorokhod Theorem in non metric spaces (see [17], [47] and [46]) there exist subsequences $\{\xi_{n_k}\}_{k=1}^{\infty}$ and $\{v_{n_k}\}_{k=1}^{\infty}$, a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, Z-valued Borel measurable variables $\tilde{\xi}$ and $\{\tilde{\xi}_k\}_{k=1}^{\infty}$, $\mathbb{Z}$-valued Borel measurable variables $\tilde{v}$ and $\{\tilde{v}_k\}_{k=1}^{\infty}$ such that

- the laws of ξ_{n_k} and $\tilde{\xi}_k$ are the same and $\tilde{\xi}_k$ converges to $\tilde{\xi}$ $\tilde{\mathbb{P}}$-a.s. with the topology $\mathcal{T}$
- the laws of v_{n_k} and $\tilde{v}_k$ are the same and $\tilde{v}_k$ converges to $\tilde{v}$ with the topology τ.

Since each $\tilde{\xi}_k$ has the same law as ξ_{n_k}, it is a martingale solution to (II.2.4.7); therefore each process

$$\tilde{M}_k(t) = \tilde{\xi}_k(t) - \tilde{\xi}(0) + \int_0^t A\tilde{\xi}_k(s)\,\mathrm{d}s + \int_0^t \tilde{v}_k(s)\cdot\nabla\tilde{\xi}_k(s)\,\mathrm{d}s$$

is a martingale with quadratic variation

$$\ll \tilde{M}_k \gg (t) = \int_0^t \tilde{G}_k(\tilde{v}_k(s))\tilde{G}_k(\tilde{v}_k(s))^*\,\mathrm{d}s.$$

Proceeding as in [16] we can prove that

$$\langle \tilde{M}_k(t) - \tilde{M}(t), \varphi\rangle \to 0 \qquad \tilde{\mathbb{P}} - a.s.$$

for any $\varphi \in H^{s,2}$, with $s > 2$, with compact support, and every $t \in [0,T]$, where

$$\tilde{M}(t) = \tilde{\xi}(t) - \tilde{\xi}(0) + \int_0^t A\tilde{\xi}(s)\,\mathrm{d}s + \int_0^t \tilde{v}(s)\cdot\nabla\tilde{\xi}(s)\,\mathrm{d}s.$$

In particular, the convergence of the non linear term

$$\langle \int_0^t \tilde{v}_k(s)\cdot\nabla\tilde{\xi}_k(s)\,\mathrm{d}s, \varphi\rangle \to \langle \int_0^t \tilde{v}(s)\cdot\nabla\tilde{\xi}(s)\,\mathrm{d}s, \varphi\rangle$$

is obtained with a slightly modification of the proof of [17, Lemma B1], exploiting the convergence of $\tilde{v}_k$ in $C([0,T]\,;\mathbb{L}^2_{loc})$ and of $\tilde{\xi}_k$ in $C([0,T]\,;L^2_{loc})$.
For the convergence of the quadratic variation process

$$\int_0^t \langle \tilde{G}_k(\tilde{v}_k(s))^*\varphi_1, \tilde{G}_k(\tilde{v}_k(s))^*\varphi_2\rangle_{\mathcal{H}}\,\mathrm{d}s \to \int_0^t \langle \tilde{G}(\tilde{v}(s))^*\varphi_1, \tilde{G}(\tilde{v}(s))^*\varphi_2\rangle_{\mathcal{H}}\,\mathrm{d}s,$$

for any $\varphi_1, \varphi_2 \in H^{-g}$, we proceed exactly as in [16, Theorem 3.6].
Similar convergence results show that the limit is a martingale. Therefore, we conclude appealing to the usual martingale representation Theorem: there exists a cylindrical $\mathcal{H}$-Wiener process $\tilde{w}$ such that

$$\langle \tilde{M}(t), \varphi\rangle = \langle \varphi, \int_0^t \tilde{G}(\tilde{v}(s)\,\mathrm{d}\tilde{w}(s)\rangle = \int_0^t \langle G(\tilde{v}(s))^*\varphi,\, \mathrm{d}\tilde{w}(s)\rangle.$$

Therefore, $\tilde{\xi}$ is a martingale solution to (II.0.0.3) and $\tilde{\xi} \in L^4(0,T;L^4)$ $\mathbb{P}$-a.s..
From Proposition II.2.4.9(ii) and Proposition II.2.4.4 immediately follows $\tilde{\xi} \in L^\infty(0,T;L^q)$ and thus statement (ii) follows. □

From the pathwise uniqueness for v, stated in Proposition II.2.3.2, we infer pathwise uniqueness for ξ. In particular, pathwise uniqueness and existence of martingale solutions implies existence of strong a solution.

Corollary II.2.4.14. *Assume that the same assumptions as Theorem II.2.4.13(i) hold, moreover assume **(IG5)**. Then there exists a unique strong solution of* (II.0.0.3).

As a byproduct result of Theorem II.2.4.13 we gain more regularity for the solution v to equation (II.2.3.1).

Corollary II.2.4.15. *Under the same assumptions of Theorem II.2.4.13(i) the solution process v of* (II.2.3.1) *has $\mathbb{P}$-a.s. paths in $C([0,T];H^{1,2})$. Moreover, under the same assumptions of Theorem II.2.4.13(ii), v has also $\mathbb{P}$-a.s. paths in $L^\infty(0,T;H^{1,q})$.*

Appendices

Appendix A

Stochastic Integration

A.1 Introduction

The theory of stochastic partial differential equations is developed on the one hand, from the work of J.B. Walsh, and on the other hand, through work on stochastic evolution equations in Hilbert spaces, for which a fundamental reference is the book of Da Prato and Zabczyk [25]. The latter theory has been lately extended to cover more general classes of Banach space valued processes. These two approaches led to the development of two distinct schools of study for stochastic partial differential equations (SPDEs), based on different theories of stochastic integration.

The Walsh theory emphasizes integration with respect to worthy martingale measures. The developed theory is used in [90] to give rigorous meaning to SPDEs, primarily parabolic equations driven by space-time white noise, though Walsh also considered the wave equation in one spatial dimension, and various linear equations in higher dimensions. Solutions are random fields, that is real-valued processes, that are defined for every fixed t and x in the domain. These are written as the convolution with the Green function associated to the partial differential operator driving the equation. Walsh theory covers the case of SPDEs whose Green function is a function. In [27] Dalang extended the definition of Walsh martingale measure stochastic integral to be able to solve SPDEs whose Green function is a Schwartz distribution. This in particular covers the case of the wave equation in dimension greater than two.
Even when the integrands is a distribution the value of the stochastic integral process is a real-valued martingale. Dalang's extension theory is not needed dealing with parabolic equations (the case we consider in the thesis) since the Green function associated to the Laplacian is very smooth in all dimensions. Nevertheless, in the parabolic case too, as the dimension increases, the Green function becomes less regular. Space-time white noise does not have enough regularity to make the stochastic convolution well defined as a real-valued process. One way to achieve this is to consider random noises that are smoother than white noise, namely a Gaussian noise that is white in time but has a smooth spatial covariance. The formulation of solution in the Walsh sense turns out to be useful if one is interested in the study of the sample path regularity properties. For instance, establish properties of the probability law of the solution, for every fixed t and x.

The theory of integration with respect to Hilbert-space-valued processes centers around solutions in Hilbert spaces of functions: equations are converted into abstract Cauchy problems, that can be seen as evolution equations in a suitable Hilbert space, driven by a given operator. From the point of view of SPDEs the limitation to the Hilbert space framework is rather restricting, and in the last years the theory of stochastic integration in a more general class of Banach spaces has been developed. For general Banach spaces there are difficulties in defining a meaning stochastic Itô integral. However a theory of stochastic integration has been developed for M-type 2 Banach spaces.

Main aim of this Appendix is to give an idea of these two approaches (we consider stochastic integration in Banach spaces as an extension of the Hilbert space-valued theory) and each theory is presented rather succinctly. We present both the stochastic integration theories since in the thesis both are needed. In particular, in Part I (the flat torus case) Walsh stochastic theory provides the underlying context where to study both the existence of a solution and its regularity in the Malliavin sense. As regards the case on the whole space, some technical problems related to the fact that the domain is unbounded, lead us to use the Hilbert and Banach space integration theory in order to prove the existence and uniqueness of solutions with the desired regularity.
It is well known that in certain cases, the Hilbert-space valued integral is equivalent to a martingale-measure stochastic integral. In [30] authors proved that in the case of a spatially homogeneous noise white in time with a correlation in space, the Walsh stochastic integral is equivalent to an infinite dimensional stochastic integral as in [25]. The (spatially homogeneous) noise we consider in Part I has similar features to the noise considered in the above mentioned paper (is in a sense its "discrete counterpart"). Thus in the present Appendix we shall also provide the tools needed in Sction I.1.3.4 where we focus on the particular case of a spatially homogeneous noise on the flat torus and show how, in that context, the Hilbert space-valued integral and the martingale measure stochastic integral turns out to be equivalent.

For the first approach we remand to [90] for a complete and detailed discussion; see also [26] and [27] for the Dalang's extension of Walsh theory. For the Hilbert space-valued approach our main references are [25], [78] and [30]. It should be mentioned that the general theory of integration with respect to Hilbert-space-valued processes and its generalizations was well-developed several years before [90] and before reference [25] appeared: see, for instance, the book of Métivier and Pellaumail [62]. See [30] and [48] for a comparison of the two stochastic integration theories. Finally, see [69], [34] and [6] for the part concerning stochastic integration in Banach spaces.

The present Appendix is organized as follows. In Section A.2 we present Walsh integration theory ([90]): we recall the concept of worthy martingale measure, we define the class of predictable process and briefly show how to define the stochastic integral of such processes w.r.t. worthy martingale measures. In Section A.3 we present the stochastic integration theory in Hilbert spaces. In particular, in Section A.3.2 we sketch the construction of the infinite dimensional stochastic integral in the setup of Da Prato and Zabczyk ([25]). In Section A.3.3 we recall the notion of cylindrical Wiener process and the stochastic integral with respect to such processes. Then Section A.3.4 gives the relationship between a Hilbert-space-valued Wiener process and a cylindrical Wiener process, in the case where the covariance operator has finite and infinite trace. Moreover it is showed how the integrals of Section A.3.3 can be interpreted in the infinite-dimensional context. In particular the results of this latter Section shall be used in Section I.1.3.4 to show the equivalence between Walsh stochastic integral and Hilbert-valued stochastic integral in the case of the spatially homogeneous noise we consider on the flat torus. Finally, in Section A.4 we briefly present the extension of the previous theory to the Banach-spaces case.

A.2 Walsh stochastic integration theory

The theory of stochastic integration developed by Walsh is based on the concept of (worthy) martingale measure. Like the classical Itô integral, the stochastic integral of a process X w.r.t. a worthy martingale measure is defined when X is in the completion of a suitable space of elementary functions. In this Section we recall the definition of martingale measure and worthy martingale measure. Then we briefly explain the Walsh construction of the stochastic integral.

A.2.1 Worthy martingale measures

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that $(L, \mathcal{L})$ is a Lusin space, i.e. a measurable space homeomorphic to a Borel subset of the line (this space includes all Euclidean spaces and, more generally, all Polish spaces). Suppose $\mathcal{A} \subset \mathcal{L}$ is an algebra. By $\mathcal{B}$ we shall denote the Borel sets on $[0, T]$.

Definition A.2.1. *Let $M: \mathcal{A} \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ We say that M is a σ-finite L^2-valued measure if the measure induced by M on the space $L^2(\Omega, \mathcal{F}, \mathbb{P})$, given by $\|M(A)\|^2_{L^2(\Omega)} = \mathbb{E}\left[M(A)^2\right]$, for any $A \in \mathcal{A}$, is a σ-finite additive measure.*

Definition A.2.2. *Let $\{\mathcal{F}_t\}_t$ be a right continuous filtration. A process $\{w_t(A), \{\mathcal{F}_t\}_t, t \in [0, T], A \in \mathcal{A}\}$ is a martingale measure if*

1. $w_0(A) = 0$,
2. *if $t > 0$, w_t is a σ-finite L^2-valued measure,*
3. $\{w_t(A), \{\mathcal{F}_t\}_t, t \in [0, T]\}$ *is a martingale.*

It is not possible to construct a stochastic integral with respect to all martingale measures. We need a technical condition called "worthiness" on the martingale measure. This requires a little background.

Definition A.2.3. *Let w be a martingale measure. The covariance functional of w is*

$$\bar{R}_t(A, B) = \langle w_\cdot(A), w_\cdot(B)\rangle_t, \qquad \forall t \in [0, T], \quad A, B \in \mathcal{A}. \tag{A.2.1}$$

Given $A, B \in \mathcal{A}$, notice that $\bar{R}_t$ is symmetric in A and B and biadditive: for fixed A, $\bar{R}_t(A, \cdot)$ and $\bar{R}_t(\cdot, A)$ are additive set functions.
Now we define a random set function as follows. For all $t \geq s \geq 0$ and $A, B \in \mathcal{L}$ define

$$R(A \times B \times (s, t]) = \bar{R}_t(A, B) - \bar{R}_s(A, B).$$

If $A_i \times B_i \times (s_i, t_i]$, $(1 \leq i \leq n)$ are disjoint, then we can define

$$R\left(\bigcup_{i=1}^n A_i \times B_i \times (s_i, t_i]\right) := \sum_{i=1}^n R\left(A_i \times B_i \times (s_i, t_i]\right).$$

This extends the definition of R to rectangles (see [90, Chapter 2]).
It turns out that, in general, one cannot go beyond this, namely is not always possible to extend R to a measure on $\mathcal{L} \times \mathcal{L} \times \mathcal{B}$. This will be make it impossible to define a completely general theory of stochastic integration in this setting. However all work fine if w is "worthy". To introduce this concept the following definition is needed.

Definition A.2.4. *A signed measure $K(\mathrm{d}x, \mathrm{d}y, \mathrm{d}s)$ on $\mathcal{L} \times \mathcal{L} \times \mathcal{B}$ is positive defined if for each bounded measurable function f for which the integral makes sense,*

$$\int_{L\times L\times\mathbb{R}^+} f(x,s)f(y,s)\, K(\mathrm{d}x, \mathrm{d}y, \mathrm{d}s) \geq 0.$$

For such a positive definite signed measure K, define

$$(f,g)_K = \int_{L\times L\times\mathbb{R}^+} f(x,s)g(y,s)\, K(\mathrm{d}x, \mathrm{d}y, \mathrm{d}s).$$

We are lead to the following definition.

Definition A.2.5. *A martingale measure w is worthy if there exists a random σ-finite measure $K(A \times B \times C, \omega)$, where $A, B \in \mathcal{L}$, $C \in \mathcal{B}$ and $\omega \in \Omega$, such that*

1. *$A \times B \mapsto K(A \times B \times C, \omega)$ is positive defined and symmetric,*
2. *$\{K(A \times B \times (0,t]), t \in [0,T]\}$ is a predictable process, for any $A, B \in \mathcal{L}$,*
3. *for all compact sets $A, B \in \mathcal{L}$ and $t > 0$, $\mathbb{E}\left[K(A \times B \times (0,t])\right] < \infty$,*
4. *for all $A, B \in \mathcal{L}$ and $t > 0$, $|R(A \times B \times (0,t])| \leq K(A \times B \times (0,t])$ a.s..*

If and when such a K exists, then we call K the dominating measure of w.

Remark A.2.6. *If w is a worthy martingale measure, then R can be extended to a measure on $\mathcal{L} \times \mathcal{L} \times \mathcal{B}$. By R_w and K_w we denote its covariation and dominating measure, respectively.*

A.2.2 Integration with respect to a worthy martingale measure

The construction of the stochastic integral w.r.t. the worthy martingale measure w follows Itô's construction in a different setting. In the classical case, one constructs the stochastic integral as a process rather than as a random variable, i.e. one constructs $\{\int_0^t f\, \mathrm{d}B_s, t \geq 0\}$ simultaneously for all t; one can then say that the integral is a martingale. The analogue of "martingale" in this setting is a "martingale measure". Accordingly, the stochastic integral will be defined as a martingale measure. As usual, the integral is defined first for elementary functions, then for simple functions, and then for all functions in a certain class by a functional completion argument.
Let us start constructing the stochastic integral for elementary and simple functions.

Definition A.2.7. *A function $f : L \times [0,T] \times \Omega \to \mathbb{R}$ is elementary if it is of the form*

$$f(x,s,\omega) = X(\omega)\mathbf{1}_{(a,b]}(s)\mathbf{1}_A(x), \tag{A.2.2}$$

where $0 \leq a < t$, X is bounded and $\mathcal{F}_a$-measurable, and $A \in \mathcal{L}$. Finite linear combinations of elementary functions are called simple functions. We denote the class of simple functions by $\mathcal{Z}$.

The right class of integrands are functions f that are "predictable". That is, they are measurable w.r.t. the predictable σ-algebra $\mathcal{P}$ that is defined next.

Definition A.2.8. *The predictable σ-field $\mathcal{P}$ on $\Omega \times L \times [0,T]$ is the σ-field generated by $\mathcal{Z}$. A function is predictable if it is $\mathcal{P}$-measurable.*

We define a norm $\|\cdot\|_+$ on the predictable functions by

$$\|f\|_+ = \mathbb{E}\left[(|f|,|f|)_K\right]^{\frac{1}{2}}.$$

Let $\mathcal{P}_+$ be the class of all predictable f for which $\|f\|_+ < \infty$. It can be proved (see [90, Proposition 2.3]) that $(\mathcal{P}_+, \|\cdot\|_+)$ is a Banach space and $\mathcal{Z}$ is dense in $\mathcal{P}_+$.

We have all the ingredients to sketch the construction of the stochastic integral.
If w is a worthy martingale measure and f is an elementary function of the form (A.2.2), we define the stochastic integral process of f as

$$f \cdot w_t(B) = X(\omega)(w_{t\wedge b}(A\cap B) - w_{t\wedge a}(A \cap B)). \tag{A.2.3}$$

The following result holds

Proposition A.2.9. *If w is a worthy martingale measure and f is an elementary function, then $f\cdot w$ is a worthy martingale measure. If R_w and K_w respectively define the covariance and dominating measure of w, then the covariance and dominating measure $R_{f\cdot w}$ and $K_{f\cdot w}$ of $f\cdot w$ are given by*

$$R_{f\cdot w}(\mathrm{d}x,\mathrm{d}y,\mathrm{d}s) = f(x,s)f(y,s)R_w(\mathrm{d}x,\mathrm{d}y,\mathrm{d}s), \tag{A.2.4}$$

$$K_{f\cdot w}(\mathrm{d}x,\mathrm{d}y,\mathrm{d}s) = |f(x,s)f(y,s)|K_w(\mathrm{d}x,\mathrm{d}y,\mathrm{d}s). \tag{A.2.5}$$

Moreover,

$$\mathbb{E}\left[(f\cdot w_t(B))^2\right] \le \|f\|_+^2, \qquad \forall\ B \in \mathcal{L}, t \le T. \tag{A.2.6}$$

Proof. See [90, Lemma 2.4]. □

We can define $f \cdot w$ for $f \in \mathcal{Z}$ by linearity. Then it can be proved that (A.2.4)-(A.2.6) hold for $f \in \mathcal{Z}$.

Now let us see how to extend the definition of stochastic integral to a larger class of functions. Suppose that $f \in \mathcal{P}_+$. Since $\mathcal{Z}$ is dense in $\mathcal{P}_+$, there exists $f_n \in \mathcal{Z}$ such that $\|f - f_n\|_+ \to 0$. By (A.2.6), if $A \in \mathcal{L}$ and $t \le T$,

$$\mathbb{E}\left[(f_m\cdot w_t(A) - f_n \cdot w_t(A))^2\right] \le \|f_m - f_n\|_+^2 \to 0$$

as $m,n \to \infty$. It follows that $(f_n \cdot w_t(A))$ is a Cauchy sequence in $L^2(\Omega,\mathcal{F},\mathbb{P})$, so that it converges in L^2 to a martingale which we shall call $f\cdot w_t(A)$. The limit is independent of the chosen sequence $\{f_n\}_n$.
Thus we obtain the following.

Theorem A.2.10. *Let w be a worthy martingale measure. Then for all $f \in \mathcal{P}_+$, $f\cdot w$ is a worthy martingale measure that satisfies* (A.2.4) *and* (A.2.5). *Moreover, for all $t \in (0,T]$, $A, B \in \mathcal{L}$,*

$$\langle f\cdot w(A), g\cdot w(B)\rangle_t = \iiint_{A\times B\times[0,t]} f(x,s)f(y,s)\, R_w(\mathrm{d}x,\mathrm{d}y,\mathrm{d}s);$$

$$\mathbb{E}\left[(f\cdot w_t(B))^2\right] \le \|f\|_+^2. \tag{A.2.7}$$

Proof. See [90, Theorem 2.5]. □

Notation: Now that the stochastic integral is defined as a martingale measure, we define the usual stochastic integrals by

$$\iint_{A\times[0,t]} f\,\mathrm{d}w = f \cdot w_t(A).$$

We shall use also the notation

$$\int_A \int_0^t f(x,s)\,w(\mathrm{d}x,\mathrm{d}s).$$

To conclude, we recall that in this context the Burkholder-Davis-Gundy inequality reads as follows.

Theorem A.2.11. *For all $p \geq 2$ there exists $c_p \in (0,\infty)$ such that for all predictable f and all $t > 0$,*

$$\mathbb{E}\left[|f \cdot w_t(B)|^p\right] \leq c_p \mathbb{E}\left[\left(\iiint_{L\times L\times(0,T]} |f(x,t)f(y,t)|\, K_w(\mathrm{d}x,\mathrm{d}y,\mathrm{d}t)\right)^{\frac{p}{2}}\right].$$

A.3 Stochastic integration in Hilbert spaces

In this Section we briefly present the theory of stochastic integration developed by Da Prato and Zabczyk in [25]. The stochastic integral presented in [25] is defined w.r.t. a class of Hilbert-space valued processes, namely Q-Wiener processes. For the following part we refer to [30], for more details see the therein references. Let us point out that we shall use the terminology used in [30] which is different from that used in [25].
We shall recall also the notion of cylindrical Wiener process and the stochastic integral with respect to such process; we explain the relationship between a Hilbert-space-valued Wiener process and a cylindrical Wiener process, in the case where the covariance operator has finite and infinite trace. To conclude, we show how the integral w.r.t. a cylindrical Wiener process can be interpreted in the infinite-dimensional context.

A.3.1 Notation and analytic preliminaries

In order to introduce the general concept of Hilbert-space-valued processes, we begin by recalling some facts concerning nuclear and Hilbert-Schmidt operators on Hilbert spaces.

By $(H, \langle\cdot,\cdot\rangle)$ and $(V, \langle\cdot,\cdot\rangle)$ we shall denote separable Hilbert spaces; by E and F Banach spaces.
As usual, with $\mathcal{L}(E,F)$ we denote the vector space of all linear bounded operators from E into F. If we consider an Hilbert space H with $\mathcal{L}(H)$ we denote the space $\mathcal{L}(H,H)$. An important class of elements of $\mathcal{L}(H)$ is the class of all linear operators Q from H into H that are symmetric and non-negative defined, i.e. they satisfy

$$\langle Qx, y\rangle = \langle x, Qy\rangle \quad \text{and} \quad \langle Qx, x\rangle \geq 0 \quad \forall x, y \in H.$$

Definition A.3.1. *An element $T \in \mathcal{L}(E, F)$ is said to be a nuclear operator if there exist two sequences $\{a_k\}_k \subset F$ and $\{\phi_k\}_k \subset E^*$ such that*

$$T(x) = \sum_{k=1}^{\infty} a_k \phi_k(x), \quad \forall x \in E, \qquad \text{and} \qquad \sum_{k=1}^{\infty} ||a_k||_F ||\phi_k||_{E^*} < +\infty. \tag{A.3.1}$$

The space of all nuclear operators from E into F is denoted by $L_1(E, F)$. It turns to be a Banach space when endowed with the norm

$$\|T\|_1 = \inf \left\{ \sum_{k=1}^{\infty} ||a_k||_F ||\phi_k||_{E^*} : T(x) = \sum_{k=1}^{\infty} a_k \phi_k(x), x \in E \right\}.$$

Definition A.3.2. *Let V be a separable Hilbert space and let $\{v_k\}_k$ be a complete ortonormal basis in V. For $T \in L_1(V) := L_1(V, V)$, the trace of T is defined as*

$$\text{Tr } T = \sum_{k=1}^{\infty} \langle T(v_k), v_k \rangle_V. \tag{A.3.2}$$

It can be proved that if $T \in L_1(V)$, then Tr T is a well defined real number and its value does not depend on the choice of the orthonormal basis (see[25] Proposition C.1). Moreover, ([25] Proposition C.3) a non-negative definite operator $T \in \mathcal{L}(V)$ is nuclear if and only if, for an orthonormal basis $\{v_k\}_j$ on V, $\sum_{k=1}^{\infty} \langle T(v_k), v_k \rangle_V < +\infty$.

Definition A.3.3. *Let V and H be two separable Hilbert spaces and $\{v_k\}_k$ a complete orthonormal basis of V. A bounded linear operator $T : V \to H$ is said to be Hilbert-Schmidt if*

$$\sum_{k=1}^{\infty} \|T(v_k)\|_H^2 < +\infty$$

(and it turns out that this property is independent of the choice of the basis in V). We will denote the set of all Hilbert-Schmidt operators from V into H by $L_{HS}(V, H)$. The norm in this space is defined by

$$\|T\|_{L_{HS}} = \left(\sum_{k=1}^{\infty} \|T(v_k)\|_H^2 \right)^{\frac{1}{2}}, \tag{A.3.3}$$

and defines a Hilbert space with inner product $\langle S, T \rangle_{L_{HS}} = \sum_{k=1}^{\infty} \langle S(v_k), T(v_k) \rangle_H$.

Definition A.3.4. *Let $T \in \mathcal{L}(V, H)$ and Ker $T := \{x \in V : T(x) = 0\}$. The pseudo-inverse of the operator T is defined by*

$$T^{-1} := \left(T|_{(Ker T)^{\perp}} \right)^{-1} : \; T(V) \to (Ker \; T)^{\perp}.$$

Notice that T is one-to-one on (Ker $T)^{\perp}$ and T^{-1} is linear and bijective.
Let us conclude this Section by stating some notation we shall use in what follows.
Let Q be a linear, symmetric, non negative defined and bounded operator on the Hilbert space V. Set $V_0 = Q^{\frac{1}{2}}(V)$ and denote by L^0_{HS} the space $L_{\text{HS}}(V_0, H)$. By $\{e_k\}_k$ we shall denote an orthonormal basis of V that consists of eigenfunctions of Q with corresponding eigenvalues

μ_k, $k \in \mathbb{N}_0$. Let $\hat{e}_k = Q^{\frac{1}{2}} e_k$, $\{\hat{e}_k\}_k$ is an orthonormal basis of V_0. We denote by V_Q the Hilbert space V endowed with the inner product

$$\langle h, g\rangle_{V_Q} = \langle Qh, g\rangle_V, \qquad h, g \in V. \tag{A.3.4}$$

Let $\tilde{e}_k = Q^{-\frac{1}{2}}(e_k) = \mu_k^{-\frac{1}{2}} e_k$, for $k \in \mathbb{N}_0$ with $\mu_k \neq 0$. $\{\tilde{e}_k\}_k$ is a complete orthonormal basis of the space V_Q. By $\{\beta_k\}_k$ we denote a sequence of standard independent one-dimensional Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

A.3.2 Da Prato-Zabczyk stochastic integration theory

In the definition of Q-Wiener processes and in the construction of the Hilbert-space valued stochastic integral, we shall consider a linear, symmetric (self-adjoint) non-negative defined and bounded operator Q on an Hilbert space V. First we shall consider Q such that $\mathrm{Tr}Q < \infty$ and then we shall extend the obtained results to the case $\mathrm{Tr}Q = \infty$.

The finite trace class case

Let us fix a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$ and let us consider a linear, symmetric (self-adjoint) non-negative defined and bounded operator Q on V such that $\mathrm{Tr}Q < \infty$.

Definition A.3.5. *A V-valued stochastic process $\{\mathcal{W}_t, t \geq 0\}$ is called a Q-Wiener process if (1) $\mathcal{W}_0 = 0$, (2) $\mathcal{W}$ has continuous trajectories, (3) $\mathcal{W}$ has independent increments, and (4) the law of $\mathcal{W}_t - \mathcal{W}_s$ is Gaussian with mean zero and covariance operator $(t-s)Q$, for all $0 \leq s \leq t$.*

In [25, Proposition 4.1, Proposition 4.3 and Theorem 4.4] it is proved that the V-valued process

$$\mathcal{W}_t = \sum_{k=1}^{\infty} \sqrt{\mu_k} \beta_k(t) e_k \tag{A.3.5}$$

(the series converges in $L^2(\Omega; C\left(0,T\right]; V))$), defines a Q-Wiener process on V. We denote by $\{\mathcal{F}_t\}_t$ the (completed) filtration generated by $\mathcal{W}$.

Let us fix $T > 0$ and let X be a Hilbert space. Let us denote by $\mathcal{N}^2_{\mathcal{W}}(0,T;X)$ the Hilbert space of all $\{\mathcal{F}_t\}_t$-predictable processes such that

$$\|\Phi\|_{\mathcal{N}^2_{\mathcal{W}}(0,T;X)} := \left[\mathbb{E}\left(\int_0^T \|\Phi(s)\|^2_X \, \mathrm{d}s\right)\right]^{\frac{1}{2}}$$

is finite. In [25] authors define the stochastic integral w.r.t. $\mathcal{W}$ of any element in $\mathcal{N}^2(0,t;L^0_{\mathrm{HS}})$, namely they define the H-valued stochastic integral $\int_0^t \Phi(s)\,\mathrm{d}\mathcal{W}_s$, for $t \in [0,T]$.
We shall give only an idea of the construction of the integral. For a detailed analysis see [25, Chapter 4].
The integral is defined at first for simple processes, defined as follows.

Definition A.3.6. *An $\mathcal{L}(V,H)$-valued process $\Phi(t)$, $t \in [0,T]$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$ is said to be simple if there exists a partition $0 = t_0 < \ldots < t_n = T$, $n \in \mathbb{N}$, such that $\Phi(t) = \sum_{k=0}^{n-1} \Phi_k \mathbf{1}_{(t_k,t_{k+1}]}(t)$, for $t \in [0,T]$, where for all $0 \leq k \leq n-1$, Φ_k are $\mathcal{L}(V,H)$-valued $\mathcal{F}_{t_k}$-mesurable w.r.t. the strong Borel σ-algebra on $\mathcal{L}(V,H)$, and Φ_k takes only a finite number of values in $\mathcal{L}(V,H)$.*

For such processes, the stochastic integral takes value in H and is defined by the formula

$$I_t^{\mathcal{W}}(\Phi) := \sum_{k=0}^{n-1} \Phi_k \left(\mathcal{W}_{t_{k+1}\wedge t} - \mathcal{W}_{t_k \wedge t} \right), \qquad t \in [0,T].$$

Then, it is proved that the map $\Phi \mapsto I_\cdot^{\mathcal{W}}(\Phi)$ is an isometry between the set of elementary processes, equipped with the norm $\|\cdot\|_{\mathcal{N}^2_{\mathcal{W}}(0,T;L^0_{\mathrm{HS}})}$, and the space of square-integrable H-valued $(\mathcal{F}_t)$-martingales $X = \{X_t, t \in [0,T]\}$ endowed with the norm $(\mathbb{E}\left[\|X_T\|_H^2\right])^{\frac{1}{2}}$ (see Proposition 4.5 in [25]). This isometry property for simple processes reads as

$$\mathbb{E}\left[\|I_T^{\mathcal{W}}(\Phi)\|_H^2\right] = \|\Phi\|^2_{\mathcal{N}^2_{\mathcal{W}}(0,T;L^0_{\mathrm{HS}})}. \tag{A.3.6}$$

Once the isometry property (A.3.6) is established, a completion argument is used to extend the above definition to all elements in $\mathcal{N}^2_{\mathcal{W}}(0,T;L^0_{\mathrm{HS}})$ and the isometry property is preserved for such processes. More precisely, it can be proved (see Proposition 4.7 in [25]) that elementary processes form a dense set in $\mathcal{N}^2_{\mathcal{W}}(0,T;L^0_{\mathrm{HS}})$. Thanks to the isometric transformation (A.3.6) and using a density argument the definition of the integral can be immediately extended to all elements of $\mathcal{N}^2_{\mathcal{W}}(0,T;L^0_{\mathrm{HS}})$. We denote the value of the extension of $I_t^{\mathcal{W}}$ at $\Phi \in \mathcal{N}^2_{\mathcal{W}}(0,T;L^0_{\mathrm{HS}})$ by $\int_0^t \Phi(s)\, \mathrm{d}\mathcal{W}_s$.

The infinite trace class case

The above definition of H-valued stochastic integral required the assumption $\mathrm{Tr}Q < \infty$. However it can be extended in order to consider also operators Q such that $\mathrm{Tr}Q = \infty$. We shall recall here the needed main ingredients; we refer to [30, Section 3.5] (for more details see the therein references).

Let us consider a symmetric non-negative definite and bounded operator on V such that $\mathrm{Tr}Q = \infty$. It is always possible to find a Hilbert space V_1 and a bounded linear injective operator $J : (V, \|\cdot\|_V) \to (V_1, \|\cdot\|_{V_1})$ such that the restriction $J_0 = J_{|V_0} : (V_0, \|\cdot\|_{V_0}) \to (V_1, \|\cdot\|_{V_1})$ is Hilbert-Schmidt. Let us define $Q_1 = J_0 J_0^* : V_1 = \mathrm{Im}(J_0) \to V_1$. Then Q_1 is a symmetric (self-adjoint), non-negative definite and $\mathrm{Tr}Q_1 < \infty$. Then

$$\mathcal{W}_t := \sum_{k=1}^{\infty} \beta_k(t) J_0(\hat{e}_k), \qquad t \in [0,T] \tag{A.3.7}$$

is a Q_1-Wiener process in V_1. Moreover, it can be proved that

$$J_0 : V_0 \to Q_1^{\frac{1}{2}}(V_1) \quad \text{is an isometry} \tag{A.3.8}$$

(for the detailed proofs of this statements see [25, Proposition 4.11] and [78, Proposition 2.5.2]). The Q_1-Wiener process $\{\mathcal{W}_t, t \geq 0\}$ is usually called *cylindrical Q-Wiener process.*
Now, let $\{\mathcal{W}_t, t \geq 0\}$ be as in (A.3.7). A predictable stochastic process $\{\Phi(t), t \in [0,T]\}$ will be integrable w.r.t. $\mathcal{W}$ if it takes values in $L_{\mathrm{HS}}(Q^{\frac{1}{2}}(V_1), H)$ and

$$\mathbb{E}\left[\int_0^T \|\Phi(t)\|^2_{L_{\mathrm{HS}}(Q^{\frac{1}{2}}(V_1),H)}\, \mathrm{d}t\right] < \infty.$$

By (A.3.8) we have

$$\Phi \in L^0_{\mathrm{HS}} = L_{\mathrm{HS}}(V_0, H) \iff \Phi \circ J_0^{-1} \in L_{\mathrm{HS}}(Q_1^{\frac{1}{2}}(V_1), H).$$

Definition A.3.7. *For every square integrable predictable process Φ with values in L^0_{HS} such that*

$$\mathbb{E}\left[\int_0^T \|\Phi(t)\|^2_{L^0_{HS}}\,\mathrm{d}t\right] < \infty,$$

the H-valued stochastic integral $\Phi \cdot \mathcal{W}$ is defined by

$$\int_0^T \Phi(s)\,\mathrm{d}\mathcal{W}_s := \int_0^T \Phi(s) \circ J_0^{-1}\,\mathrm{d}\mathcal{W}_s.$$

Notice that the class of integrable processes w.r.t. $\mathcal{W}$ does not depend on the choice of V_1.

A.3.3 Stochastic integration w.r.t. cylindrical Wiener processes

In this section, we recall the notion of cylindrical Wiener process and the stochastic integral with respect to such processes.

Definition A.3.8. *Let Q be a symmetric (self-adjoint) and non-negative definite bounded linear operator on V. A family of random variables $W = \{W_t(h), t \geq 0, h \in V\}$ is a cylindrical Wiener process on V if the following two conditions are fulfilled.*

- *for any $h \in V$, $\{W_t(h), t \geq 0\}$ defines a Brownian motion with variance $t\langle Qh, h\rangle_V$;*
- *for all $s, t \in \mathbb{R}^+$, and $h, g \in V$,*

$$\mathbb{E}(W_s(h)W_t(g)) = (s \wedge t)\langle Qh, g\rangle_V.$$

If $Q = Id_V$ is the identity operator in V, then W will be called a standard cylindrical Wiener process. We will refer to Q as the covariance of W.

Let $\{\mathcal{F}_t\}_t$ be the σ-field generated by the random variables $\{W_s(h), h \in V, 0 \leq s \leq t\}$ and the $\mathbb{P}$-null sets. We define the predictable σ-field as the σ-field in $[0,T] \times \Omega$ generated by the sets $\{(s,t] \times A, A \in \mathcal{F}_s, 0 \leq s < t \leq T\}$.
We can define the stochastic integral of any predictable square-integrable process with values in V_Q (see (A.3.4)). Recall that by $\{\tilde{e}_k\}_k$ we denote a complete orthonormal basis of the Hilbert space V_Q. For any predictable process $g \in L^2(\Omega \times [0,T]; V_Q)$ it can be proved that the series

$$g \cdot W = \sum_k \int_0^T \langle g_s, \tilde{e}_k\rangle_{V_Q}\,\mathrm{d}W_s(\tilde{e}_k) \tag{A.3.9}$$

is convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and the sum does not depend on the chosen orthonormal system. Each summand in the series is a classical Itô integral with respect to a standard Brownian motion, and the resulting stochastic integral is a real-valued random variable. The independence of the terms in the series leads to the isometry property

$$\mathbb{E}\left[(g \cdot W)^2\right] = \mathbb{E}\left[\int_0^T \|g_s\|^2_{V_Q}\,\mathrm{d}s\right].$$

Notice that it is possible to define this integral in an alternative way: one can start by defining the stochastic integral in (A.3.9) for a class of simple predictable V_Q-valued processes, and then use the isometry property to extend the integral to elements of $L^2(\Omega \times [0,T]; V_Q)$ by checking that these simple processes are dense in this set.

A.3.4 Interpreting the stochastic integration theory w.r.t. a cylindrical Wiener process in the Da Prato-Zabczyk setting

Let us explain the relationship between Q-Wiener processes and cylindrical Wiener processes with covariance operator Q (considering both the cases Tr $Q < \infty$ and Tr $Q = \infty$) and relate the corresponding notion of integrals in the particular case in which the involved Hilbert space, where the Da Prato-Zabczyk integral takes value, is $H = \mathbb{R}$.

Relation between Q-Wiener processes and cylindrical Wiener processes with covariance Q (the case Tr$Q < \infty$)

If $\{\mathcal{W}_t, t \geq 0\}$ is a Q-Wiener process on V, there is a natural way to associate to it a cylindrical Wiener process in the sense of Definition A.3.8: for any $h \in V$, $t \in [0, T]$ we set

$$W_t(h) := \langle \mathcal{W}_t, h \rangle_V.$$

One checks that $\{W_t(h), t \in [0, T], h \in V\}$ is a cylindrical Wiener process on V with covariance operator Q. Since $\{\tilde{e}_k\}_k$ is a complete orthonormal basis of V_Q and so $\{W_t(\tilde{e}_k)\}_k$ defines a sequence of standard one-dimensional Brownian motions, one can write

$$\mathcal{W}_t = \sum_{k=1}^{\infty} \sqrt{\mu_k}\, W_t(\tilde{e}_k) e_k.$$

However, it is not true in general that any cylindrical Wiener process is associated to a Q-Wiener process on a Hilbert space. Indeed the following result holds (see [30, Theorem 3.2]).

Theorem A.3.9. *Let V be a separable Hilbert space and W a cylindrical Wiener process on V with covariance Q. Then, the following three conditions are equivalent:*

1. *W is associated to a V-valued Q-Wiener process $\mathcal{W}$, in the sense that $\langle \mathcal{W}_t, h \rangle_V = W_t(h)$, for all $h \in V$.*
2. *for any $t \geq 0$, $h \to W_t(h)$ defines a Hilbert-Schmidt operator from V into $L^2(\Omega, \mathcal{F}, \mathbb{P})$*
3. *Tr$Q < \infty$.*

As a consequence, if dim$V = +\infty$ and if W is a standard cylindrical Wiener process on V, that is $Q = Id_V$, then there is non Q-Wiener process $\mathcal{W}$ associated to W. However, as we shall explain next, it will possible to find a Hilbert-space-valued Wiener process with values in a larger Hilbert space V_1 which will correspond to W in a certain sense.

Equivalence of integrals in the case $H = \mathbb{R}$

Now let us see how the stochastic integral w.r.t. W constructed in Subsection A.3.3 is equal to an integral w.r.t. $\mathcal{W}$ constructed in [25] when the Hilbert space in which the integral takes values is $H = \mathbb{R}$.
Let us consider a cylindrical Wiener process W on V with covariance Q such that Tr$Q < \infty$. By Theorem A.3.9, W is associated to a V-valued Q-Wiener process $\mathcal{W}$. Let g be a predictable

process in $L^2(\Omega \times [0,T]\,; V_Q)$ as introduced in Subsection A.3.3. For any $s \in [0,T]$ we define the operator

$$\Phi_s^g : V \to \mathbb{R} \tag{A.3.10}$$

$$\Phi_s^g(\eta) := \langle g_s, \eta \rangle_V, \qquad \eta \in V. \tag{A.3.11}$$

The following result holds (see [30, Proposition 3.5]).

Proposition A.3.10. $\Phi^g = \{\Phi_s^g, s \in [0,T]\}$ *defines a predictable process with values in* L^0_{HS} *such that*

$$\int_0^T \|\Phi_s^g\|^2_{L^0_{HS}} \, \mathrm{d}s \equiv \int_0^T \|g_s\|^2_{V_Q} \, \mathrm{d}s.$$

Therefore the stochastic integral of Φ^g *w.r.t.* $\mathcal{W}$ *can be defined as in Subsection A.3.3, namely it holds*

$$\int_0^T \Phi_s^g \, \mathrm{d}\mathcal{W}_s = \int_0^T g_s \, \mathrm{d}W_s.$$

Relation between cylindrical Q-Wiener processes and cylindrical Wiener processes with covariance Q (the case Tr$Q = \infty$) and equivalence of integrals in the case $H = \mathbb{R}$

Let us consider the case Tr $Q = \infty$. Let $\{W_t, t \in [0,T]\}$ be a cylindrical Wiener process with covariance Q on the Hilbert space V and let $g \in L^2(\Omega \times [0,T]\,; V_Q)$ be a predictable process such that $g \cdot W$ is well defined as in Subsection A.3.3. Then we can consider the cylindrical Q-Wiener process $\{\mathcal{W}_t, t \in [0,T]\}$ defined by

$$\mathcal{W}_t = \sum_{k=1}^{\infty} W_t(\tilde{e}_k) J_0(\hat{e}_k).$$

This process takes values in some Hilbert space V_1.

For $g \in L^2(\Omega \times [0,T]\,; V_Q)$ we define, as in (A.3.10), the operator

$$\Phi_s^g(\eta) := \langle g_s, \eta \rangle_V, \qquad \eta \in V,$$

which takes values in $H = \mathbb{R}$. Then the following result holds (see [30, Proposition 3.10]).

Proposition A.3.11. *The process* $\{\Phi_s^g, s \in [0,T]\}$ *defines a predictable process with values in* $L_{HS}(V_0; \mathbb{R})$*, such that*

$$\mathbb{E}\left[\int_0^T \|\Phi_s^g\|^2_{L_{HS}} \, \mathrm{d}s\right] = \mathbb{E}\left[\int_0^T \|g_s\|^2_{V_Q} \, \mathrm{d}s\right],$$

and

$$\int_0^T \Phi_s^g \, \mathrm{d}\mathcal{W}_s = \int_0^T g_s \, \mathrm{d}W_s.$$

Proposition A.3.11 allows us to associate a cylindrical Wiener process W with covariance Q, with a cylindrical Q-Wiener process, as described in Subsection A.3.2, and to relate the associated stochastic integrals.

A.4 Stochastic integration in M-type 2 Banach spaces

In this Section we present basic facts concerning stochastic integration in M-type 2 Banach spaces. We briefly outline the construction of the Itô integral in this context. Then we recall some technical results as the Burkholder-Davis-Gundy inequality and the Itô formula. For this last result we shall focus in particular on the case of L^q-spaces. We mainly refer to [10] and [12] where the basic facts useful for our needs are well exposed in a succinctly and clear way. For a more detailed and complete discussion of the topic see the therein references.

Let us fix some notations. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$ be a fixed filtered probability space. By $(V, \langle \cdot, \cdot \rangle)$ we shall denote again a real separable Hilbert space and by $\{e_k\}_k$ we denote a fixed orthonormal basis of V. Let $(E, |\cdot|)$ be a Banach space.
We denote by γ the standard Gaussian cylindrical distribution on V.

A.4.1 γ-radonifying operators

Definition A.4.1. *A bounded linear operator $K : V \to E$ is called γ-radonifying iff the image $K(\gamma) := \gamma \circ K^{-1}$ of γ under K is σ-additive on the algebra of cylindrical sets in E. We set*

$$R(V, E) := \{K : V \to E : K \in \mathcal{L}(V, E) \text{ and } K \text{ is } \gamma\text{-radonifying}\}.$$

The algebra of cylindrical sets in E generates the Borel σ-algebra $\mathcal{B}(E)$ on E (see [50]). Thus $K(\gamma)$ extends to a Borel measure on $\mathcal{B}(E)$ which we denote by γ_K. In particular, γ_K is a Gaussian measure on $\mathcal{B}(E)$, i.e., for each $f \in E^*$, the image measure $f(\gamma_K)$ is a Gaussian measure on $\mathcal{B}(\mathbb{R})$ (see e.g. [25, Chapter 2.2]). For $K \in R(V, E)$ we put

$$\|K\|^2_{R(V,E)} := \int_E |x|^2_E \, \mathrm{d}\gamma_K(x). \tag{A.4.1}$$

As γ_k is Gaussian, then by the Fernique-Landau-Shepp Theorem (see [50]), $\|K\|_{R(V,E)}$ is finite. Moreover, see (see [69]), $R(V, E)$ is a separable Banach space endowed with the norm (A.4.1). It is a well known fact that, if E is a Hilbert space, then $K : V \to E$ is γ-radonifying means that K is Hilbert-Schmidt. In this case it holds $\|K\|_{L_{HS}(V;E)} = \|K\|_{R(V;E)}$.

We have the following characterization of γ-radonifying operators when $E = L^q$, see [88, Proposition 13.7] and [13, Theorem 2.3].

Proposition A.4.2. *Let $1 \leq q < \infty$ and $\{h_j\}_{k=1}^{\infty}$ a complete orthonormal system in V. For an operator $K \in \mathcal{L}(V; L^q)$ the following two conditions are equivalent:*

- *$K \in R(V, L^q)$;*
- *$\left(\sum_{k=1}^{\infty} |Kh_k|^2\right)^{\frac{1}{2}} \in L^q$.*

Moreover, the norms $\|K\|_{R(V;L^q)}$ and $\| \left(\sum_{k=1}^{\infty} |Kh_k|^2\right)^{\frac{1}{2}} \|_{L^q}$ are equivalent.

A.4.2 Construction of the stochastic integral

Let $\mathcal{W}$ be a cylindrical Q-Wiener process on V in the sense of Section A.3.2, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$. Let us fix $T > 0$ and let Y be a Banach space. Let us denote by $\mathcal{M}^p_{\mathcal{W}}(0, T; Y)$ the Banach space of all $\{\mathcal{F}_t\}_t$-predictable Y-valued processes Φ such that

$$\|\Phi\|_{\mathcal{M}^p_{\mathcal{W}}(0,T;Y)} := \left(\mathbb{E} \int_0^T \|\Phi(t)\|^p_Y \, \mathrm{d}t\right)^{\frac{1}{p}}$$

is finite. The stochastic integral $I_t^{\mathcal{W}}(\Phi)$ is defined at first for simple processes Φ in $\mathcal{M}^2_{\mathcal{W}}(0,T;R(V,E))$. In general, $I^{\mathcal{W}}$ can not be extended continuously to the whole $\mathcal{M}^2_{\mathcal{W}}(0,T;R(V,E))$. This requires some additional assumptions on the Banach space E, namely we have to consider M-type 2 Banach spaces.

Definition A.4.3. *A Banach space E is called M-type 2 iff there is a constant $C>0$ such that for any finite E-valued martingale $\{M_k\}$ the following inequality holds*

$$\sup_k \mathbb{E}\|M_k\|_E^2 \le C(E)\sum_k \mathbb{E}\left[|M_k - M_{k-1}|^2\right]. \tag{A.4.2}$$

For the proof of the following theorem below see [69] or [15].

Theorem A.4.4. *Assume that E is a M-type 2 Banach space. Then the class of simple processes in $\mathcal{M}^2_{\mathcal{W}}(0,T;R(V,E))$ is a dense subspace of $\mathcal{M}^2_{\mathcal{W}}(0,T;R(V,E))$, and for every $t\in[0,T]$ there exists a unique extension of $I_t^{\mathcal{W}}$ to a linear bounded operator acting from $\mathcal{M}^2_{\mathcal{W}}(0,T;R(V,E))$ into $L^2(\Omega,\mathcal{F},\{\mathcal{F}_t\}_t,\mathbb{P};E)$. Moreover, there exists $C>0$ such that for any $T>0$ and $\Phi\in\mathcal{M}^2_{\mathcal{W}}(0,T;R(V,E))$ one has*

$$\mathbb{E}\|I_T^{\mathcal{W}}(\Phi)\|_E^2 \le C\mathbb{E}\int_0^T \|\Phi(t)\|^2_{R(V,E)}\,\mathrm{d}t. \tag{A.4.3}$$

We denote value of the extension of $I_t^{\mathcal{W}}$ at $\Phi\in\mathcal{M}^2_{\mathcal{W}}(0,T;R(V,E))$ by $\int_0^t \Phi(s)\,\mathrm{d}\mathcal{W}_s$.

Remark A.4.5. *Any Hilbert space is a M-type 2 Banach space. In such a case we have the equality in* (A.4.2) *with $C(E)=1$. The spaces L^q and $W^{r,q}$, $q>2$, $r\ge 1$ are examples of M-type 2 Banach spaces which are not Hilbert spaces.*

A.4.3 Burkholder-Davis-Gundy inequality and Itô formula

A Burkholder-Davis-Gundy type inequality holds in M-type 2 Banach spaces (see [34, Theorem 2.4, Theorem 3.3].

Theorem A.4.6. *Assume that E is a M-type 2 Banach space. Then for every $\Phi\in\mathcal{M}^2_{\mathcal{W}}(0,T;R(V,E))$, $\int_0^t\Phi(s)\,\mathrm{d}\mathcal{W}(s)$, $t\in[0,T]$ is an E-valued square integrable martingale with continuous modification and zero mean. Moreover, for every $p\in[2,\infty)$ there is a constant C independent of T and Φ such that*

$$\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t \Phi(s)\mathrm{d}\mathcal{W}(s)\right|_E^p \le C\mathbb{E}\left[\int_0^T \|\Phi(s)\|^2_{R(V,E)}\,\mathrm{d}s\right]^{\frac{p}{2}}. \tag{A.4.4}$$

Next we state the Itô lemma. We do this at first for an Itô process with values in an abstract M-type 2 Banach space E and for a Fréchet differentiable mapping $\Psi:[0,T]\times E\to\mathbb{R}$. Then we derive Itô's lemma for an L^q-valued process, and $\Psi=|\cdot|^p_{L^q}$, $p\ge q$. Before formulating the theorem we need to introduce some notation. For any Banach space Y and any bounded linear map $L:E\times E\to Y$ we define

$$\mathrm{tr}_K L = \int_E L(x,x)\,\mathrm{d}\gamma_K(x).$$

Denote by $\bar{\mathcal{L}}(E,Y)$ the space of all bounded bilinear operators acting from E into Y. By the Fernique-Landau-Shepp theorem tr_K is a bounded linear operator from $\bar{\mathcal{L}}(E,Y)$ into Y. Moreover,

$$|\mathrm{tr}_K L|_Y \le \|L\|_{\bar{\mathcal{L}}(E,Y)} \|K\|^2_{R(V,E)}, \quad \forall\, K \in R(V,E),\ L \in \bar{\mathcal{L}}(E,Y). \tag{A.4.5}$$

Let us dente by $C^2_b(E)$ the class of all Fréchet differentiable functions $\Psi : E \to \mathbb{R}$ with bounded derivatives. For the proof of the following Theorem see [69].

Theorem A.4.7. *Assume that E is a M-type 2 Banach space. Let $\Psi \in C^2_b(E)$. Let $b \in \mathcal{M}^1_{\mathcal{W}}(0,T;E)$ and $\sigma \in \mathcal{M}^2_{\mathcal{W}}(0,T;R(V,E))$. Let*

$$X(t) = X(0) + \int_0^t b(s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}\mathcal{W}(s), \qquad t \in [0,T]\,.$$

Then, for all $t \in [0,T]$,

$$\Psi(X(t)) = \Psi(X(0)) + \int_0^t \Psi'(X(s))b(s)\,\mathrm{d}s + \int_0^t \Psi(X(s))\sigma(s)\,\mathrm{d}\mathcal{W}(s) + \frac{1}{2}\int_0^t tr_{\sigma(s)}\Psi''(X(s))\,\mathrm{d}s.$$

Let us now formulate the result in the particular case in which $\Psi = |\cdot|^p_{L^q}$. Let $u \in L^q(D)$, where $D \subseteq \mathbb{R}^d$; notice that $|u|^{q-2}u \in L^{q^*} = (L^q)^*$. Below, $\langle\cdot,\cdot\rangle$ denotes the duality form on $L^{q*} \times L^q$.

Theorem A.4.8. *Let $q \in [2,\infty)$, and $p \ge q$. Assume that*

$$X(t) = X(0) + \int_0^t b(s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}\mathcal{W}(s), \qquad t \in [0,T]\,,$$

with $b \in \mathcal{M}^p_{\mathcal{W}}(0,T;L^q)$ and $\sigma \in \mathcal{M}^p_{\mathcal{W}}(0,T;R(V,L^q))$. Then for all $t \in [0,T]$

$$\begin{aligned}\|X(t)\|^p_{L^q} &= \|X(0)\|^p_{L^q} + p\int_0^t \|X(s)\|^{p-q}_{L^q}\langle |X(s)|^{q-2}X(s), b(s)\rangle\,\mathrm{d}s\\ &\quad + p\int_0^t \|X(s)\|^{p-q}_{L^q}\langle |X(s)|^{q-2}X(s), \sigma(s)\mathrm{d}\mathcal{W}(s)\rangle\\ &\quad + \frac{1}{2}\int_0^t tr_{\sigma(s)}\Psi''(X(s))\,\mathrm{d}s.\end{aligned}$$

Notice that $V \ni \psi \to \langle |X(s)|^{q-2}X(s), \sigma(s)\psi\rangle$ belongs to $R(V,\mathbb{R})$, so the Itô integral above is well defined.

Remark A.4.9. *Notice that for $u, v_1, v_2 \in L^q$ we have*

$$\begin{aligned}\Psi''(u)(v_1,v_2) &= p(q-1)\|u\|^{p-q}_{L^q}\int_D |u(x)|^{q-2}v_1(x)v_2(x)\,\mathrm{d}x\\ &\quad + p(p-q)|u|^{p-2q}_{L^q}\int_D |u(x)|^{q-2}u(x)v_1(x)\,\mathrm{d}x \times \int_D |u(x)|^{q-2}u(x)v_2(x)\,\mathrm{d}x.\end{aligned} \tag{A.4.6}$$

Hence combining (A.4.5) *with* (A.4.6)*, we obtain*

$$tr_\sigma\Psi''(u) \le p(p-1)\|u\|^{p-2}_{L^q}\|\sigma\|^2_{R(V;L^q)}.$$

Appendix B

Malliavin calculus and the problem of the existence of a density

B.1 Introduction

Malliavin calculus was conceived in the 70's and in the following years a huge amount of research has been done in this field. Nowadays several monographs on this subject are available and we mainly refer to Nualart [72]. In his initial papers Malliavin used the absolute continuity criterion in order to prove that under Hörmander's condition, the law of the diffusion process has a smooth density and in this way he gave a probabilistic proof of Hörmander's theorem.

One of the main application to Malliavin calculus is to give sufficient conditions in order that the law of a random variable has a smooth density with respect to the Lebesgue measure. In particular, there has been a lot of activity in the last years studying the regularity in the Malliavin sense for solutions to stochastic partial differential equations. One is interested in looking for the existence (and smoothness) of a density for the law of the random variable given by the solution process at fixed points in time and space. This property is important in the analysis of hitting probabilities (see [29, 31]) and concentration inequalities (see [71]).

In the present Chapter, we briefly introduce the Gaussian framework where to perform Malliavin calculus and recall the definitions of Malliavin derivative and divergence operator. Then we briefly recall the Bouleau-Hirsh criterium which provides sufficient conditions for existence of densities of finite measures on $\mathbb{R}$ and therefore for densities of probability laws. To conclude we recall some useful rules of calculus for the derivative and divergence operators. All the results are stated without proof. We remand to [72] for a complete exposition of what briefly recalled here. See also [81] and [3].

B.2 Isonormal gaussian processes, derivative and divergence operators

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Consider a real separable Hilbert space H endowed with inner product $\langle \cdot, \cdot \rangle_H$.

Definition B.2.1. *We say that a stochastic process $W = \{W(h), h \in H\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a isonormal gaussian process (or a gaussian process on H) if W is a centered Gaussian family of random variables, namely a closed subset of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ whose elements zero-mean*

Gaussian random variables, such that

$$\mathbb{E}(W(h)W(g)) = \langle h, g\rangle_H, \qquad \text{for all } h, g \in H.$$

This map provides a linear isometry of H onto a closed subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ that we denoted by $\mathcal{O}$. The elements $W(h)$ of $\mathcal{O}$, are zero mean Gaussian random variables with variance $\|h\|_H^2$. Note that by Kolmogorov's Theorem, given the Hilbert space H we can always construct a probability space and a Gaussian process $\{W(h), h \in H\}$ verifying the above conditions.
In the sequel we will replace $\mathcal{F}$ by the filtration generated by W. The fact that we work with the σ-algebra generated by the isonormal Gaussian process W is central: all the objects involved in Malliavin calculus are "functionals " of that process and so we work in the universe generated by W.
In the study of SPDEs the (Gaussian) noise driving the equation can be introduced as an isonormal Gaussian process on a proper Hilbert space. This process provides the underlying Gaussian context where to study the equation and, in particular, perform its Malliavin analysis. A well known example is given by the time-space white noise; another important class of examples are based on noises white in time and weighted in space. We have already encountered these kind of noises troughout the thesis. Let us recall them here, in the context of isonormal Gaussian processes.

Example B.2.2. ***Gaussian noise white in time correlated in space*** *In Section II.1.7.3 we introduced, on the complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *the Gaussian family* $W = \{W(\varphi), \varphi \in C_0^\infty(\mathbb{R}^2 \times [0,T])\}$ *and we showed how it is possible to associate the Hilbert space* $U_T := L^2(0,T;U)$ *to* W. $W = \{W(h), h \in U_T\}$ *defines an isonormal Gaussian process; its elements are zero-mean Gaussian random variables with covariance*

$$\mathbb{E}\left[W(\varphi)W(\psi)\right] = \int_0^T \langle \widehat{\varphi(t)}, \widehat{\psi(t)}\rangle_{L^2_{(s)}(\mu)}\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^2} \left(\varphi(t) * \psi_{(s)}(t)\right)(x)\,\Gamma(\mathrm{d}x)\,\mathrm{d}t, \qquad \text{(B.2.1)}$$

where we recall μ *is the spectral measure of* W *and* $\Gamma = \hat{\mu}$ *the correlation function.*
In particular, for every $A \in \mathcal{B}_b(\mathbb{R}^2)$, $W(\mathbf{1}_{[0,T]}\mathbf{1}_A)$, *defined by an approximation procedure, is called Gaussian noise white in time and correlated in space. Notice that, for* $\Gamma = \delta_0$, *we obtain the famous time-space white noise.*

Example B.2.3. ***Gaussian noise white in time weighted in space*** *In Section I.1.3.1 we considered, on a complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *the isonormal Gaussian process* $W = \{W(h) : h \in \mathcal{H}_T\}$, *where* $\mathcal{H}_T = L^2(0,T;L^2_Q)$. *Its elements are zero-mean Gaussian random variables with covariance*

$$\mathbb{E}\left[W(h)W(g)\right] = \int_0^T \langle h(s), g(s)\rangle_{L^2_Q}\,\mathrm{d}s.$$

For every $A \in \mathcal{B}(D)$, $W(\mathbf{1}_{[0,T]}\mathbf{1}_A)$, *defines a Gaussian noise white in time and weighted in space. Notice that, for* $Q = Id$, *we obtain the time-space white noise.*

Now that the general Gaussian framework is clear, let us introduce the Malliavin derivative operator. Let $(\Omega, \mathcal{F}, \mathbb{P})$ and W be respectively the compete probability space and the isonormal Gaussian process introduced above. To start with, let us denote by $\mathcal{C}$ the set of smooth cylindrical random variables of the following form

$$F = f(W(h_1), ..., W(h_d)), \qquad \text{(B.2.2)}$$

where $d \geq 1$, $h_i \in H$ and $f \in C_p^\infty(\mathbb{R}^d)$, which is the set of all infinitely continuously differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}$ such that f and all its derivatives have polynomial growth. We define the Malliavin derivative of $F \in \mathcal{C}$ as the H-valued random variable given by

$$DF = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(W(h_1), ..., W(h_d))h_i.$$

We introduce the Sobolev norm of F as

$$\|F\|_{1,p} = \left[(\mathbb{E}|F|^p) + \mathbb{E}(\|DF\|_H^p)\right]^{\frac{1}{p}}. \tag{B.2.3}$$

It is possible to prove that the derivative operator D is closable and take the extension of D in the standard way. We can now define in the obvious way DF for any F in the closure of $\mathcal{C}$ with respect to this norm and denote the domain of D in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$. That means that $\mathbb{D}^{1,p}$ is the closure of $\mathcal{C}$ w.r.t. the norm (B.2.3). Notice that, by definition, $\mathbb{D}^{1,q} \subset \mathbb{D}^{1,p}$ for $p \leq q$. By convention, $\mathbb{D}^{0,p} = L^p(\Omega)$. The above procedure can be iterated to define the operator D^k, $k \in \mathbb{N}$, for more details see [72, Section 1.2]. We can localize the domain of the operator D as follows: we will dente by $\mathbb{D}^{1,p}_{loc}$, $p \geq 1$, the set of random variables F such that there exists a sequence $\{(\Omega_N, F_N), N \geq 1\} \subset \mathcal{F} \times \mathbb{D}^{1,p}$ such that $\Omega_N \uparrow \Omega$, $\mathbb{P}$-a.s. and $F = F_N$ $\mathbb{P}$-a.s. on Ω_N. We say that (Ω_N, F_N) localizes F in $\mathbb{D}^{1,p}$ and DF is defined without ambiguity by $DF = DF_N$ on Ω_N, $N \geq 1$.

Remark B.2.4. *Let us consider the Hilbert space H of the particular form $H = L^2(0, T; \mathcal{U})$, with $\mathcal{U}$ an Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{U}}$. Consider the isonormal Gaussian process W on H. Notice that, respectively for $\mathcal{U} = U$ and $\mathcal{U} = L^2_Q$ we obtain the isonormal Gaussian processes of examples B.2.2 and B.2.3.*
Let $X \in \mathbb{D}^{1,p}$, for a certain $p \geq 1$; for $h \in H$ set $D_h X = \langle DX, h\rangle_H$. Since $H = L^2(0, T; \mathcal{U})$, for $r \in [0, T]$, $DX(r)$ defines an element in $\mathcal{U}$, which will be denoted by $D_{r,\bullet}X$. Then, clearly, for any $h \in H$,

$$D_h X = \int_0^T \langle D_{r,\bullet}X, h(r)\rangle_{\mathcal{U}} \, dr.$$

We shall write $D_{r,\varphi}X = \langle D_{r,\bullet}X, \varphi\rangle_{\mathcal{U}}$, $r \in [0, T]$, $\varphi \in \mathcal{U}$.

We now introduce the divergence operator, defined as the adjoint of the Malliavin derivative. Let us recall that the Malliavin derivative is an unbounded operator from $L^2(\Omega)$ into $L^2(\Omega; H)$ and its domain $\mathbb{D}^{1,2}$ is dense in $L^2(\Omega)$. Then, by a standard procedure (see [91]) one can define the adjoint of D, that we shall denote by δ. The domain of the adjoint, denoted by Domδ, is the set of random variables $u \in L^2(\Omega; H)$ such that for any $F \in \mathbb{D}^{1,2}$,

$$|\mathbb{E}\left[\langle DF, u\rangle_H\right]| \leq c\|F\|_{L^2(\Omega)},$$

where c is a constant depending on u. If $u \in$ Domδ, then δu is the element of $L^2(\Omega)$ characterized by the identity

$$\mathbb{E}\left[F\delta(u)\right] = \mathbb{E}\left[\langle DF, u\rangle_H\right], \tag{B.2.4}$$

for all $F \in \mathbb{D}^{1,2}$. Equation (B.2.4) expresses the duality between D and δ and it is called the integration by parts formula.

B.3 The Bouleau-Hirsch criterium for the existence of a density and some calculus results

Let us present here the Bouleau-Hirsch criterium for the existence of the density of random vectors defined on a Gaussian probability space. We recall the results formulated for a one-dimensional random variable (for the case of d-dimensional random vector see [72, Theorem 2.1.2]). Let us recall that there are other criteria regarding existence of densities for random vectors; we use the Bouleau-Hirsch criterium since its requirements are weaker. Moreover under stronger requirements there exists some criteria concerning the smoothness of the density (see [72, Chapter 2]).

Proposition B.3.1. ***Bouleau-Hirsch criterium*** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a complete probability space and let* F *be a random variable of the space* $\mathbb{D}^{1,1}_{loc}$*. Suppose that*

$$\|DF\|_H > 0 \tag{B.3.1}$$

$\mathbb{P}$*-a.s., then the law of* F *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}$*.*

To conclude, let us recall some basic useful rules of calculus for the derivative and divergence operators defined so far. The first result is a chain rule.

Proposition B.3.2. *Let* $\varphi : \mathbb{R}^d \to \mathbb{R}$ *be a continuously differentiable function with bounded partial derivatives. Let* $F = (F_1, ..., F_d)$ *be a random vector whose components belong to* $\mathbb{D}^{1,p}$ *for some* $p \geq 1$*. Then* $\varphi(F) \in \mathbb{D}^{1,p}$ *and*

$$D(\varphi(F)) = \sum_{i=1}^{d} \partial_i \varphi(F) DF_i.$$

The following result is an essential approximation procedure to study the Malliavin differentiability of solutions to SPDEs.

Proposition B.3.3. *Let* $p \in (1, \infty)$ *and let* $\{F_n, n \geq 1\}$ *be a sequence of random variables in* $\mathbb{D}^{1,p}$ *converging to* F *in* $L^p(\Omega)$ *and such that*

$$\sup_n \mathbb{E}\left[\|DF_n\|_H^p\right] < \infty.$$

Then F *belongs to* $\mathbb{D}^{1,p}$ *and the sequence of derivatives* $\{DF_n, n \geq 1\}$ *converges weakly in* $\mathbb{D}^{1,p}$ *to* DF*.*

To conclude we recall the commutativity relation between Malliavin derivative and divergence operator. Recall that for any $u \in \mathbb{D}^{1,2}$ and $h \in H$ we set $D_h u = \langle Du, h\rangle_H$.

Proposition B.3.4. *Suppose that* $h \in H$*,* $u \in \mathbb{D}^{1,2}$ *and* $D_h u \in Dom\delta$*. Then*

$$D_h(\delta(u)) = \langle u, h\rangle_H + \delta(D_h u). \tag{B.3.2}$$

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