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Sara Torelli

Fujita decompositions and infinitesimal invariants on fibred surfaces

Supervisor:

Gian Pietro Pirola

A passo svelto.

Introduction

The theory of fibred surfaces has applications in different fields of research (e.g. the classification of surfaces, the study of moduli of curves), where they represent an important tool of work. The thesis is devoted to the study of some infinitesimal conditions in relation to some relative objects globally defined by the fibration. The aim is to capture some geometric properties of fibred surfaces.

A fibred surface is the data (S, f) of a smooth complex surface S and a fibration $f : S \to B$ over a smooth complex curve B, i.e. a proper surjective morphism with connected fibres. The general fibre is a smooth compact curve F of constant geometric genus $g = g(F)$.

We consider the direct image sheaf $f_*\omega_{S/B}$ of the relative dualizing sheaf $\omega_{S/B} = \omega_S \otimes f^*\omega_B^\vee$, as a relative object globally defined by f. This is a vector bundle of rank g and has general fibre equal to the space $H^0(\omega_F)$ of holomorphic forms on the fibre of f (see e.g. [BHPVdV04]), which relates the canonical sheaves ω_S, ω_B and ω_F of the surface S, the base B and the general fibre F.

The infinitesimal information is given by Griffiths' theory on variation of the Hodge structure (see e.g. [Gri71]). The invariants are pointwise constructed in the kernel of the differential of the period map $\mathcal{P} : B^0 \to (\mathcal{D}/\text{monodromy})$ on the classifying space $\mathcal{D}$ of weight one Hodge structures modulo monodromy, namely the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties, associated to the geometric variation induced by f. The core of the thesis (Chapter 2) has the purpose to analyse a certain kind of cohomological objects, the *Massey-products*, linked to the Griffiths infinitesimal invariants, and follows a formalism similar to that of Deligne (see [Del71]). The last chapter (Chapter 3) focuses on the differential of a Prym map, which turns out to be related to the problem of existence of non-isotrivial fibred surfaces in the case of maximal relative irregularity.

The key point of our study of infinitesimal invariants over $f_*\omega_{S/B}$ is provided by two splittings known as *first* and *second Fujita decompositions*, respectively.

Theorem ([Fuj78a],[Fuj78b],[CD])**.** *Let $f : S \to B$ be a fibration over a smooth projective curve B. Then there exist two splittings on $f_*\omega_{S/B}$ given respectively by the unitary flat vector bundles $\mathcal{O}_B^{\oplus q_f}$ and $\mathcal{U}$ as a direct sum of vector bundles*

$$f_*\omega_{S/B} = \mathcal{O}_B^{\oplus q_f} \oplus \mathcal{E} = \mathcal{U} \oplus \mathcal{A},$$

where $\mathcal{O}_B^{\oplus q_f}$ is the trivial bundle of rank q_f the relative irregularity of f, $\mathcal{E}$ is nef and such that $h^1(B, \omega_B(\mathcal{E})) = 0$ and $\mathcal{A}$ is ample.

The investigation of the infinitesimal conditions is narrowed down to the unitary flat summand $\mathcal{U}$ involved in the second decomposition. This is possible according to the description provided in [CD] in terms of geometric variation of the Hodge structure.

On the locus $j : B^0 \hookrightarrow B$ of the smooth fibres of f the unitary flat bundle $\mathcal{U}_{|B^0}$ is the subbundle of the kernel of the Gauss Manin connection ∇ of $\mathcal{H}_{\mathbb{C}} = j^* R^1 f_* \mathbb{C} \otimes \mathcal{O}_{B^0}$, which is also in the image $\mathcal{F}^1 \hookrightarrow \mathcal{H}_{\mathbb{C}}$ of the Hodge filtration, i.e. $\mathcal{U}_{|B^0} = \ker \nabla \cap \mathcal{F}^1$. The unitary structure is given by the hermitian metric defined by the intersection form, which is positively defined on it. In case of semistable fibrations over smooth projective curves, the extension of $\mathcal{U}_{|B^0}$ to a unitary flat bundle over B is unique, as a consequence of the behaviour under local monodromies. The above description says that the unitary flat bundle $\mathcal{U}$ is completely determined by the VHS and it fixes the relation with the period map $\mathcal{P} : B^0 \to \mathcal{D}/\text{monodromy}$.

The first result we present is a description of the local system $\mathbb{U}$ underlying the unitary flat summand $\mathcal{U}$ of the second Fujita decomposition of semistable fibred surfaces. We start from the description of $\mathcal{U}$ mentioned above moving deeper into the geometry of the surface. Note that in our study it is crucial to distinguish the behavior of the local system from the one of the associated bundle. We consider the subsheaf $\Omega^1_{S,d} \subset \Omega^1_S$ of closed holomorphic 1-forms on the surface S. The direct image $f_*\Omega^1_{S,d}$ is a sheaf whose sections are given on every open subset $A \subset B$ by families of holomorphic forms on the fibres of f which provide closed holomorphic 1-forms on the tubular subset $f^{-1}(A)$ of S.

Lemma (Lifting/Splitting lemma 2.2.5/2.2.8, [PT17])**.** *Let $f : S \to B$ be a semistable fibration over a smooth projective curve B. Then there is a splitting on the short exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & \omega_B \arrow[r] & f_*\Omega^1_{S,d} \arrow[r] & \mathbb{U} \arrow[r] \arrow[l, bend right, "\eta'"'] & 0, \end{tikzcd}$$

which is defined by comparing the holomorphic de-Rham sequence on B with the push-forward of that on S truncated at Ω^1_S.

The lemma states that the unitary local system $\mathbb{U}$ is the image of the sheaf $f_*\Omega^1_{S,d}$ through a map $f_*\Omega^1_{S,d} \to R^1 f_*\mathbb{C}$, which is defined by the connecting morphism of the push-forward functor.

On the one hand, the lemma is preparatory to the main results. On the other hand, it has its own meaning since it can be reviewed as a fixed-part theorem type (see Corollary 2.2.10).

The geometry of the unitary flat bundle is completely contained in its monodromy group, which is defined as the image of the monodromy representation. The first piece of information one wants to understand is whether the monodromy is finite or not. Note that flat bundles with finite monodromy are trivial after some finite étale base changes. By the Splitting lemma 2.2.8, in the concrete case of $\mathbb{U}$ this means that, up to a finite base change, there is no obstruction on holomorphic forms on the fibres that locally define a closed form on a tubular set to glue to a global holomorphic form on the surface. The existence of fibred surfaces whose associated unitary summand has infinite monodromy has been recently proved in [CD], where the authors provide an infinite sequence of examples with arbitrary big genus.

In the thesis we provide a criterion for the finiteness of the monodromy group of $\mathcal{U}$, which is obtained by a local study of infinitesimal objects, the *Massey product*.

We consider the description of $\mathcal{U}$ in terms of VHS and we look at the differential of the period map over a point $b \in B^0$

$$d\mathcal{P}_b : T_b B^0 \to \mathrm{Hom}(H^0(\omega_{F_b}), H^1(\mathcal{O}_{F_b})), \qquad w \mapsto \cup\xi_b : H^0(\omega_{F_b}) \to H^1(\mathcal{O}_{F_b}),$$

which sends a vector $w \in T_b B^0$ to the homomorphism defined by the cup product $\cup$ with the Kodaira Spencer class $\xi_b \in H^1(T_{F_b})$ of F_b over w. On semistable fibred surfaces, the vector space $\mathrm{Ker} \cup\xi_b$ is the fibre over each point $b \in B^0$ of a vector bundle $\mathcal{K}_\partial$ defined as the kernel of the connecting morphism

$$\partial : f_*\Omega^1_{S/B} \longrightarrow R^1 f_*\mathcal{O}_S \otimes \omega_B$$

induced by the push forward on the short exact sequence of the relative differentials (see sequence (1.4)). By the Splitting lemma 2.2.8, there is an injection

$$\mathcal{U} = \mathbb{U} \otimes \mathcal{O}_B \hookrightarrow \mathcal{K}_\partial,$$

which allows the study over $\mathbb{U}$ (thought as identified with its fibre over b) of conditions defined on $\mathrm{Ker} \cup\xi_b$.

Massey products are defined as follows. The Kodaira Spencer class $\xi_b \in H^1(T_{F_b})$ of F_b corresponds to the extension

$$\xi_b : \qquad 0 \longrightarrow \mathcal{O}_{F_b} \longrightarrow \Omega^1_{S|F_b} \longrightarrow \Omega^1_{F_b} \longrightarrow 0$$

defined by restriction to $F_b \hookrightarrow S$ of Sequence (1.4). The connecting morphism in the long exact sequence in cohomology is exactly $\cup\xi_b : H^0(\omega_{F_b}) \to H^1(\mathcal{O}_{F_b})$ and describes the kernel as

$$\ker \cup\xi_b = \mathrm{Im}\{H^0(\Omega^1_{S|F_b}) \to H^0(\omega_{F_b})\},$$

namely as the vector space of holomorphic $1-$forms on F_b that are liftable to infinitesimal deformations on S. The Adjoint map (see e.g. [CP95])

$$\wedge_{\xi_b} : \ \bigwedge^2 H^0(\Omega^1_{S|F_b}) \longrightarrow H^0(\bigwedge^2 \Omega^1_{S|F_b}) \simeq H^0(\omega_{F_b})$$

maps a pair of liftings on $H^0(\Omega^1_{S|F_b})$ of two fixed independent elements (α_1, α_2) in $\ker \cup\xi_b$ to $H^0(\omega_{F_b})$. Since each lifting is determined up to an element in $H^0(\mathcal{O}_{F_b}) \simeq \mathbb{C}$, the images define a class

$$\mathfrak{m}_{\xi_b}(\alpha_1, \alpha_2) \in H^0(\omega_{F_b}) / < \alpha_1, \alpha_2 >_{\mathbb{C}} .$$

This is called *Massey product of the pair* (α_1, α_2) along ξ_b and it vanishes if the image of the map lies in the complex vector space generated by the pair.

We define a subspace $W \subset H^0(A, \mathcal{K}_\partial)$ to be *Massey-trivial* if each pair of local sections has vanishing Massey-products on the general fibre over A and we study this condition on subspaces of $H^0(A, \mathbb{U})$. The criterion is provided by the following result and is valid in case of surfaces fibred over a smooth projective curve (not necessarily semistable).

Theorem (Theorem 2.3.1,[PT17]). *Let $f : S \to B$ be a fibration over a smooth projective curve B of genus $g(F) \geq 2$ and $\mathcal{U}$ be the unitary bundle of the second Fujita decomposition of f. Let $\mathcal{M} \subset \mathcal{U}$ be a flat subbundle of $\mathcal{U}$ generated by a Massey-trivial subspace. Then $\mathcal{M}$ has finite monodromy.*

For the sake of clarity, the flat bundle generated by a subspace of sections over an open contractile set $A \subset B$ is obtained by the action of the monodromy group over the fixed subspace (see Section 1.3). We note that this defines the smallest flat bundle containing the fixed subspace, which is unitary and gives a splitting on $\mathcal{U}$ via the unitary structure.

A more accurate piece of information is given by an explicit description of the local system $\mathbb{U}$ that reflects interesting properties of the fibred surface, e.g. existence of other fibrations, curves inside it and so on. Our main result provides such geometric information.

Theorem (Theorem 2.3.2,[PT17]). *Let $f : S \to B$ be a semistable fibration over a smooth projective curve B of genus $g(F) \geq 2$ and $\mathcal{M} \subset \mathcal{U}$ be a unitary flat subbundle generated by a maximal dimensional Massey-trivial subspace. Then the monodromy group G_{M} of $\mathcal{M}$ is in one to one correspondence with a subgroup of the bijections $\mathcal{S}_{\mathscr{K}}$ on a finite set $\mathscr{K}$ of morphisms $k_g : F \to \Sigma$ from the general fiber F to a smooth compact curve Σ of genus $g(\Sigma) \geq 2$. Moreover, after a finite étale base change $u_{M} : B_{M} \to B$ trivializing the monodromy, the pullback bundle of $\mathcal{M}$ becomes the trivial bundle $V \otimes \mathcal{O}_{B_M}$ of fibre $V = \sum_{g \in G_M} k_g^* H^0(\omega_\Sigma) \subset H^0(\omega_F)$.*

As it can be seen in the proof, the description of the local system reflects a special property of the fibred surface, namely the existence of an additional fibration on S over Σ after a finite étale base change trivializing the monodromy. In the framework of variation of the Hodge structure this means that after a finite base change there is a fixed part defined by the Jacobian of Σ. The main tool in the proof is provided by the Lifting Lemma 2.2.5 together with a Castelnuovo and de Franchis theorem generalized to the case of surfaces fibred over a complex (not compact in general) smooth curve (proved in [GST17]). A central point is the content of a classical de Franchis theorem, which states that the set of morphisms between smooth compact curves of genus greater that 2 is finite (see e.g. [Mar88]).

The first and second Fujita decompositions provide the chain of inclusions

$$\mathcal{O}_B^{q_f} \subset \mathcal{U} \subset f_* \omega_{S/B} \tag{1}$$

and consequently it is natural to capture numerical information on the fibred surface by looking to bounds that involve their ranks. Note that the ranks of $\mathcal{O}_B^{q_f}$ and $f_*\omega_{S/B}$ are the relative irregularity q_f of the fibred surface and the geometric genus g of the general fibre of f, respectively.

We give a result in this direction, which is a numerical bound involving the Clifford index c_f of the general fibre.

Theorem (Theorem 2.4.6,[GST17]). *Let $f : S \to B$ be a non-isotrivial fibration of genus g, flat unitary rank u_f and Clifford index c_f. Then*

$$u_f \leq g - c_f.$$

We note that the bound is a natural generalization of the one given in [BGAN16],

$$q_f \leq g - c_f,$$

which links the trivial bundle $\mathcal{O}_B^{q_f}$ and $f_*\omega_{S/B}$, according to the inclusion provided by the first Fujita decomposition. The proof of Theorem 2.4.6 follows the line of [BGAN16] but needs the Lifting Lemma 2.2.5 and the generalization of the Castelnuovo de Franchis theorem mentioned above in order to localize the used techniques.

The previous chain of inclusions can be refined as

$$\mathcal{O}_B^{q_f} \subset_1 \mathcal{U} \subset_2 \mathcal{K}_\partial \subset_3 f_*\omega_{S/B}, \tag{2}$$

where inclusion

1 is provided by the compatibility of the first and the second Fujita decompositions;

2 is the content of a splitting lemma (lemma 2.2.8);

3 is given by $f_*\Omega^1_{S/B} \hookrightarrow f_*\omega_{S/B}$.

But also

$$\mathbb{U} \subset_4 f_*\Omega^1_{S,d}, \qquad \mathcal{K}_\partial \subset_5 f_*\Omega^1_S, \tag{3}$$

where inclusion

4 follows from a splitting (lemma 2.2.8);

5 is given by a splitting (lemma 2.2.1).

A challenging problem should be understanding relations between them. First of all numerically, by providing bounds that involve their ranks. Then trying to construct more ad hoc examples (some are already discussed in the applications of the theorem).

We end the introduction to the first part of the thesis with some applications of the results.

The first application is a criterion for the semiampleness of $f_*\omega_{S/B}$. In the original interest the study of the bundle $f_*\omega_{S/B}$ was stimulated by the problem of understanding positivity properties. In the paper [Fuj78a], Fujita proved that $f_*\omega_{S/B}$ is semipositive (i.e. nef) and provided the first decomposition, disproving ampleness. Then in [Fuj78b], he sketched the proof of the second decomposition with the purpose of investigating semiampleness but he did not solve the problem. The answer appeared in some recent works ([CD14],[CD16],[CD]), due to Catanese and Dettweiler, where they construct examples of unitary flat summands with non finite momodromy group that are not semiample according to a characterization theorem (see [CD, Theorem 2.5]). We use the characterization mentioned above to apply Theorem 2.3.1 to the study of semiampleness of $f_*\omega_{S/B}$.

Corollary 1 (Corollary 2.3.10,[PT17]). *Let $f : S \to B$ be a projective semistable fibration of genus $g(F) \geq 2$. Assume that $\mathcal{U}$ is Massey-trivial generated, then $f_*\omega_{S/B}$ is semiample.*

The criterion allows to obtain information on moduli of curves. Indeed the assumptions are satisfied by fibrations of hyperelliptic curves, which means that one can not expect to find unitary flat bundles with non finite monodromy group by moving inside the hyperelliptic locus (see Section 2.3). The result has been proved in a different way in [LZ17].

Another important application is related with Hodge theory and provided by a formula proved in [CP95], which reviews the condition of Massey-triviality in terms of the Griffiths infinitesimal invariant.

Corollary 2 (Corollary 2.3.15,,[PT17])**.** *Let $f : S \to B$ be a fibration of genus $g(F) \geq 2$. Assume that the monodromy of $\mathcal{U}$ is not finite, then the Griffiths infinitesimal invariant on the canonical normal function $\nu : B^0 \to \mathcal{P}(f)$ is not zero at the general point $b \in B^0$, where $\mathcal{P}(f)$ denotes the family of primitive intermediate Jacobians of weight $2g-3$. In particular, ν is not a torsion section.*

In particular, the corollary applies to the examples provided in [CD16] and moreover, it is consistent with the Volumetric theorem provided in [PZ03].

The bound proved in Theorem 2.4.6 can be analysed in the contest of the problem of Xiao, where the purpose is to estimate the relative irregularity of fibred surfaces. The state of art (see Section 2.3) leads to the conjecture that the relative irregularity of any non-isotrivial fibred surface of genus $g \geq 2$ satisfies $q_f \leq \left\lceil \frac{g+1}{2} \right\rceil$ (see [BGAN16]).

We note that the bound in [BGAN16] implies the conjecture in the (general) case of maximal Clifford index $c_f = \left\lfloor \frac{g-1}{2} \right\rfloor$ and also on the rank u_f of $\mathcal{U}$, in case of finite monodromy group.

The last chapter contains a generalization of the infinitesimal study as in [Kan04] and is a project born during the school of Pragmatic. The purpose is to investigate the existence of non-isotrivial fibred surfaces with maximal relative irregularity, namely those with $g_f = g-1$ (see Beauville [Deb82, Appendix]). We note that in this case the first and second Fujita decompositions collapse into

$$f_*\omega_{S/B} = \mathcal{O}_B^{\oplus g-1} \oplus \mathcal{L},$$

where $\mathcal{L}$ is an ample line bundle. The trivial summand $\mathcal{O}_B^{\oplus g-1}$ corresponds to the fixed part in the variation of the Hodge structure defined by the Albanese variety of the general fibre. The Albanese map fixes the link between the data (S, f) and a local Prym map

$$\phi : \mathcal{H} \longrightarrow \mathcal{A}_{g-1}, \qquad p \longmapsto [P(\pi_p)],$$

defined between a local parametrizing space $\mathcal{H}$ of degree-d coverings $\pi_p : F \to E$ over elliptic curves E and a moduli space $\mathcal{A}_{g-1}$ of abelian varieties. More precisely the map sends a covering π_p (constructed by the data p) to a Prym variety $P(\pi_p)$ (generalized to the case of degree d morphisms).

Let $\mathrm{alb}(f) : \mathrm{Alb}(S) \to \mathrm{Alb}(B)$ be the map induced by f between the Albanese varieties $\mathrm{Alb}(S)$ and $\mathrm{Alb}(B)$ of S and B, respectively. Consider the abelian variety $K_f = \mathrm{Ker}(\mathrm{Alb}(f))_0$ of dimension $q_f = g-1$ defined as the connected component containing the zero of the kernel of the above map. The injection $F_b \hookrightarrow S$ of the fibre F_b of f over $b \in B^0$ defines a surjective morphism $J(F_b) \simeq \mathrm{Alb}(F_b) \twoheadrightarrow K_f$

together with a sequence

$$0 \longrightarrow K_f^{\vee} \longrightarrow Alb(F_b)^{\vee} \xrightarrow{\psi} E_b \longrightarrow 0 \tag{4}$$

whose cokernel E_b is an elliptic curve. The composition of $\mathrm{Alb}(F_b) : F_b \to \mathrm{Alb}(F_b)$ with ψ constructs a covering $\pi(b) : F_b \to E_b$ with generalized Prym variety $P(\pi_p) = \mathcal{K}_f^{\vee}$. The fibres vary but the Prym remains fixed by construction. In other words, the existence of a non-isotrivial fibred surface of maximal relative irregularity is directly linked to a positive dimensional fibre of the differential of the local Prym map as defined above.

We generalize a formula of [Kan04] for the differential of the local Prym map (Theorem 3.2.3) and then we use it to estimate the dimension of the kernel of the differential of the local Prym map (Theorem 3.3.3).

A geometric approach via the canonical embedding $F_b \hookrightarrow \mathbb{P}(\omega_{F_b})^{\vee}$ reviews the above theorem in terms of intersection of quadrics. Let q^- of $\mathbb{P}H^0(\omega_F)^{\vee}$ be the intersection point of all hyperplanes defined by elements in $H^0(\omega_F)^-$, where $H^0(\omega_F)^-$ is the kernel of the trace map providing the splitting $H^0(\omega_F) = H^0(\omega_F)^- \oplus \pi^* H^0(\omega_E)$.

Theorem (Theorem 3.3.4)**.** *Let F be a non-hyperelliptic curve. Then*

$$q^- \notin \bigcap_{F \subset Q} Q \Longrightarrow \dim(\mathrm{Ker}(d_{\mathrm{o}}P)) = 1,$$

where Q ranges in the set of quadrics of $\mathbb{P}H^0(\omega_F)^{\vee}$ containing F (identified with its canonical model).

We highlight the link with the modified Xiao conjecture (see [BGAN16]). The existence of a non-isotrivial fibration of maximal irregularity corresponds to a point in $\mathcal{H}$ where the kernel of the local Prym map has a dimension at least 2 (elliptic curve is not allowed to move under automorphisms in our definition). The bound $q_f \leq g - c_f$ given in [BGAN16] implies the conjecture in the (general) case of maximal Clifford index. In case of $\mathrm{Cliff}(f) = 1$, the general fibre is trigonal or isomorphic to a plane quintic curve. The problem is solved for fibrations with general fibre isomorphic to a plane quintics in [FNP17].

Summarizing, the thesis is organized as follows.

Chapter 1. It contains general preliminaries on fibrations over curves and related topics such as relative objects, local systems, infinitesimal deformation theory, Hodge theory, Massey products on fibred surfaces and supporting divisors. In this chapter we fix our framework: notations, definitions and classical results.

Chapter 2. It contains the original results of the thesis concerning the study of infinitesimal invariants (i.e. Massey products) in relation to the Fujita decompositions. It is divided in four sections. In section 2.1 we introduce the reader to Fujita decompositions. In Section 2.2 we state the Lifting/Splitting lemma 2.2.5/2.2.8, preparatory results and applications. Section 2.3 is devoted to Monodromy results, Theorem 2.3.1 and Theorem 2.3.2, and related preparatory results and applications. The results of this section will appear in [PT17]. Section 2.4 is dedicated to Theorem 2.4.6, which is a result achieved in collaboration with Victor González-Alonso and Lidia Stoppino (see [GST17]).

Chapter 3. It contains a local study of the differential of a generalized Prym map, in collaboration with Filippo Favale (see [FT17]). This project has been suggested during the school of Pragmatic.

Contents

Chapter 1

Preliminaries on fibrations over curves and related topics

The chapter is devoted to fix some preliminaries and notations on fibrations over curves and related topics. The main references are the books of [Voi02],[Voi03], [Gri70], [PS08] and also [BPVdV84].

1.1 Fibrations over curves and semistable reduction

In the section we introduce fibrations over smooth complex curves and we fix our notations.

We consider a *(smooth) complex curve* B as a (smooth) reduced and irreducible connected variety of dimension 1, analytic or algebraic depending on the contest. More precisely, we will assume that B is an analytic variety (that is, locally defined as the zero locus of holomorphic functions) when we will study *local properties*, while algebraic and defined over the complex field $\mathbb{C}$ (that is, globally defined as the zero locus af polynomials over $\mathbb{C}$, e.g. affine, quasi-projective, projective) when we will look at *global properties*. Moreover, we will use the same conventions for *a complex variety* X (smooth or not) of higher dimension and unless otherwise specified, we will also assume that X is smooth, that is a complex manifold.

Definition 1.1.1. A *fibration* $f : X \to B$ *of a smooth complex variety* X *over a smooth complex curve* B is a proper surjective morphism of complex varieties with connected fibres and it is *smooth* when the morphism is submersive (that is, the differential of f is surjective).

As a convention, we will sometimes omit to write "smooth complex varieties" assuming this framework as fixed.

By the regular value theorem, the fibre X_b over $b \in B$ is a complex variety, smooth over every point b in the locus B^0 where the fibration is a submersive morphism, namely where the differential of f is surjective.

A *singular point* $x \in X$ is a critical point of f, that is a point where the differential of f vanishes. The locus Z of the singular points of f is called *singular locus of* f and it is a proper subvariety (analytic or algebraic) of X.

A *branch point* $b \in B$ of f is a critical value of f, that is the image of a singular point of f. The locus B_0 of the branch points f is called *branch locus of* f, while its complement $B^0 = B \setminus B_0$ is called the *smooth locus of* f. By the Remmert mapping theorem (see [BPVdV84, Theorem 8.4]), B_0 is a subvariety of B (again analytic or algebraic depening on the assumptions on B). Moreover, under the assumption that f is defined between smooth varieties, both Z and B_0 cannot be the full space X and B, respectively, which means that B_0 is at most a discrete subset of B (Zariski closed, when B is algebraic) and then the general fibre X_g of f is smooth. The fibres over B_0 are called *singular* since they have singularities. Clearly, B_0 is empty when a fibration $f : X \to B$ is smooth and in this case all the fibres are smooth.

By Ehresmann theorem (see [Voi02, Theorem 9.3]), a smooth fibration is topologically locally trivial, which means that there is a topological isomorphism $(i_g, f_{|U}) : X_U \simeq X_g \times U$ between the restriction X_U of X to U and the product $X_g \times U$, where U is an open contractible coordinate subset of B, the general fibre X_g is identified with a fixed one as a topological variety and i_g maps the fibres to X_g. The property always holds on the restriction $f^0 : X^0 = f^{-1}(B^0) \to B^0$ of f to the smooth locus B^0, which is smooth by definition.

We recall some definitions on smooth fibrations.

Definition 1.1.2 ([Voi03], Definition 4.14)**.** A smooth fibration $f : X \to B$ over B is

(•) *projective* if it admits a holomorphic embedding $i : X \hookrightarrow B \times \mathbb{P}^r$ such that $f = pr_1 \circ i$ provides an integral Kaehler form $\omega \in H^2(X, \mathbb{Z})$ defined by $\omega = (pr_2 \circ i)^* c_1(\mathcal{O}_{\mathbb{P}^r})$, which restricts to a Kaehler class $\omega_{|X_b} \in H^2(X_b, \mathbb{Z})$;

(••) *Kaehler* if X is Kaehler and the Kaehler form $\omega \in H^2(X, \mathbb{R})$ restricts to a Kaehler class $\omega_{|X_y} \in H^2(X_y, \mathbb{R})$ over each $b \in B$.

A fibration $f : X \to B$ can be also thought as a *family of complex varieties* X_b parametrized by $b \in B$. In this sense, the smooth variety X is called *ambient* or *total space* of f, while B is called *base* or *parametrizing space*.

Definition 1.1.3 ([Voi02, Definition 7.8])**.** A *polarised smooth complex variety* is a pair (Y, ω), where Y is a compact complex smooth variety and ω is an integral Kaehler class on Y.

A projective fibration as above naturally defines *an algebraic family of polarized algebraic (projective) varieties* as in [Gri70], while a Kaehler fibration a *family of smooth compact Kaehler varieties.*

As in [GA13, Definition 3.1.4], we give the following definitions.

Definition 1.1.4. A fibration $f : X \to B$ of smooth complex varieties over B is

(T) *trivial* if S it is isomorphic to the product $X_g \times B$ and f is the projection over B;

(IT) *isotrivial* if all the smooth fibres are isomorphic;

(LT) *locally-trivial* if it is smooth isotrivial but not necessarily trivial (that is, a fibre bundle).

A very simple type of singularity is locally described as the intersection of coordinate hyperplanes. A common strategy is in fact provided by reducing the study to those fibrations with at most this kind of singularities. As in [Gri70], we give the following.

Definition 1.1.5. A fibration $f : X \to B$ of smooth complex varieties over B is

(ss) *semistable* if all the singular fibres are reduced and normal crossing divisors.

A *normal crossing divisor* is a divisor $X_{nc} = \sum X_i$ given as a finite sum of irreducible smooth varieties X_i (called *components*), which intersect transversally. They are locally described as intersections of coordinate hyperplanes, namely they look like

$$t = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$$

with respect to the natural choice of local coordinates given by $f : t = 0$ locally describing B and $x_1, \dots x_k$ as part of a local coordinate system on X.

Definition 1.1.6 ([Gri71, Pag. 102])**.** A *degeneration* $f : X \to \Delta$ is a fibration of smooth complex varieties over a complex disk $\Delta \subset \mathbb{C}$ with only one singular fibre F_0 over 0 and it is *semistable* if F_0 is reduced and normal crossing.

Examples are provided by restriction of semistable fibrations around singular fibres.

Let $f : X \to B$ be a fibration over a smooth projective curve B. Combining Hironaka and Mumford theorems (see [KKMSD73], pag. 98 and pag. 53, respectively, and also [Gri71], pag. 102-103) we can associate to f a semistable fibration $f' : X' \to B'$, which we will call *semistable reduction of f*. We present the *semistable reduction* as in [CD].

Theorem 1.1.7 (**[CD, Semistable reduction]**)**.** *Let $f : X \to B$ be a fibration over a smooth projective curve B. Then there exists a finite Galois covering $u : B' \to B$ and a semistable fibration $f' : X' \to B'$ such that the following diagram commutes*

$$\begin{tikzcd} X' \arrow[r, dashed, "\phi"] \arrow[dr, "f'"] & X \times_B B' \arrow[r, "\phi_u"] \arrow[d, "f_u"] & X \arrow[d, "f"] \\ & B' \arrow[r, "u"] & B \end{tikzcd} , \tag{1.1}$$

where $X \times_B B'$ is the fibre product and $\phi : X' \dashrightarrow X_u$ is a resolution of singularities.

A *smooth compactification* ([Gri70]) of a fibration $f : X \to B$ over a quasi-projective curve B is a fibration $\overline{f} : \overline{X} \to \overline{B}$ over a smooth projective curve $\overline{B}$, which fits into the diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"] & \overline{X} \arrow[d, "\overline{f}"] \\ B \arrow[r, hook] & \overline{B} \end{tikzcd} , \tag{1.2}$$

where $\overline{X}$ and $\overline{B}$ are smooth complete projective varieties containing X and B respectively as Zarisky open subset and $\overline{X} - X$ and $\overline{B} - B$ are normal crossing divisors.

According to the semistable reduction theorem as stated above, smooth compactifications exist and we can reduce to them to the study of certain properties of fibrations.

A different point of view is given by fixing a smooth complex variety X and looking at the existence of fibrations on it. In this point of view the variety X is called *fibred space over B by f* if there exists a fibration $f : X \to B$ over B.

1.2 "Relative" sheaves and cohomology on fibrations over curves

Euristically, the existence of fibrations over a smooth complex variety X allows to study the geometry of the fibred variety $f : X \to B$ through the one of the base B and the one of the fibres X_b, which are of lower dimension. This approach leads to the definition of some *relative objects* attached to a fibration, e.g. sheaves and their cohomological spaces. In this section we introduce three kind of sheaves naturally attached to a fibration which will be used and related in the following chapters: (•) the *Leray sheaves*, (••) the *sheaf of relative differentials* and (•••) the *relative dualizing sheaf* .

(•) Let $f : X \to B$ be a smooth fibration over a smooth complex curve B and let $\mathcal{F}$ be a sheaf on X. The $q-$th derived sheaf $R^q f_*\mathcal{F}$ of the direct image sheaf $f_*\mathcal{F}$ is by definition the sheaf attached to the presheaf $U \mapsto H^q(U, f_*\mathcal{F})$, for each U open subset in B. The *Leray spectral sequence of $\mathcal{F}$* is canonically defined by the following result (see [Voi03, Chapter 4]).

Theorem 1.2.1 (Leray). *Let $\mathcal{F}$ be a sheaf on X. Then there exists a canonical filtration L on $H^q(X, \mathcal{F})$ and a spectral sequence $E_r^{p,q} \Rightarrow H^{p+q}(X, \mathcal{F})$ that is canonically starting from E_2 and satisfies*

$$E_2^{p,q} = H^p(B, R^q f_*\mathcal{F}), \quad E_\infty^{p,q} = Gr_L^p H^{p+q}(X, \mathcal{F}). \tag{1.3}$$

As the Leray spectral sequence started canonically from the $q-$th derived sheaf $R^q f_*\mathcal{F}$ on B, it is also called *Leray sheaf of $\mathcal{F}$*.

Remark 1.2.2. The beginning of the spectral sequence leands to the exact sequence

$$0 \longrightarrow H^1(B, f_*\mathcal{F}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^0(B, R^1 f_*\mathcal{F}) \longrightarrow H^2(B, f_*\mathcal{F}). \tag{1.4}$$

Examples are provided by locally constant sheaves over a ring R. Let $\mathcal{F}$ be a constant sheaf on X over a ring R (e.g. $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$), that is a sheaf of locally constant functions with values on R. Then the sheaf $R^q f_*\mathcal{F}$ is a local system (a sheaf locally isomorphic to a constant sheaf). The Leray spectral sequence on $R^q f_* R$ shows how to compute the cohomology $H^{p+q}(X, R)$ starting from the cohomology of B with values in $R^q f_* R$ but this one is not enough to compute the cohomology $H^{p+q}(X, R)$ unless the spectral sequence degenerates at E_2. As a smooth fibration is locally trivial, the space $H^q(U, R)$ is simply the cohomology $H^q(X_b, R)$ of a fibre of $b \in U$, where U is a local trivialization. This is not true in general on arbitrary open subsets where the cohomology depends also on the homotopy of the total space. Then the property to degenerate at E_2 is satisfied when there is no obstruction to extend the cohomology of the general fibre to those of the total space.

Theorem 1.2.3 (Deligne degeneracy theorem [Del71]**).** *Let $f : X \to B$ be a smooth projective fibration of complex varieties over a smooth curve B. Then the Leray spectral sequence of f with rational coefficients (that is $\mathcal{F} = \mathbb{Q}$) degenerates at E_2.*

Remark 1.2.4. A spectral sequence degenerates at E_2 if the associated differential d_2 vanishes identically. The proof of the theorem above follows by looking at each term of the primitive cohomology (see section 1.5.8) and proving by induction that the differentials on each term of the primitive cohomology on the spectral sequence vanish. The argument can be repeated in the same way on Kaehler fibrations and more in general every time we consider a a local system $\mathcal{F}$ of a ring which has the *Hard Lefshetz property* as in [PS08, Definition 1.31].

Remark 1.2.5. The construction of the spectral sequence does not require that B is compact and allows to study branches as elements in homotopy.

(••) Let Y be a smooth complex variety of dimension l. Let $\mathcal{O}_Y$ be the structure sheaf, T_Y the (holomorphic) tangent sheaf of Y and Ω^1_Y the sheaf of holomorphic 1-forms, its dual (i.e. $\Omega^1_Y = T_Y^\vee$).

Let $f : X \to B$ be a fibration over a smooth complex curve B. Let m be the dimension of X and $n = m - 1$ be the dimension of the fibres of f. Note that $\Omega^1_B \simeq \omega_B$, where ω_B is the canonical sheaf of B.

Definition 1.2.6. The *sheaf of relative differentials* $\Omega^1_{X/B}$ is defined as the cokernel in the short exact sequence

$$0 \longrightarrow f^*\omega_B \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/B} \longrightarrow 0, \tag{1.5}$$

given by the differential morphism $Df : T_X \to f^*T_B$ of sheaves induced by the exterior differential d of f.

Remark 1.2.7. The sequence above is exact on the left because the first homomorphism is injective on a dense open set and $f^*\omega_B$ is locally free.

The sheaf of relative differentials $\Omega^1_{X/B}$ is not locally free in general (e.g. the property fails over a non-reduced fibre of f). Let $i : X_b \hookrightarrow X$ be the natural injection of a smooth fibre X_b in the ambient space X. Then

$$\Omega^1_{X/B|X_b} \simeq \Omega^1_{X_b}. \tag{1.6}$$

Let Ω^k_Y be the sheaf of holomorphic k-forms on Y. (i.e. $\Omega^k_Y = \wedge^k\Omega^1_Y = \wedge^k T_Y^\vee$). For $k = l$, as Y is smooth, the sheaf $\omega_l = \wedge^l\Omega^1_Y$ is the canonical sheaf of Y and it is a line bundle $\mathcal{O}(K_Y)$ on Y, where K_Y is a canonical divisor. For $k \geq l$ the sheaves vanish.

Sequence 1.4 also defines the *complex of relative differentials* $(\Omega^\bullet_{X/B}, \mathrm{d})$ where

$$\Omega^p_{X/B} = \bigwedge^p \Omega^1_{X/B}. \tag{1.7}$$

Remark 1.2.8. When $f : X \to B$ is a smooth fibration over a smooth curve B, the complex above is exactly the *Relative de-Rham complex* of f as in [PS08].

(•••) Let $f : X \to B$ be a fibration over a smooth complex curve B, ω_X and ω_B be the canonical sheaves of X and B respectively.

Definition 1.2.9. The line bundle $\omega_{X/B} = \omega_X \otimes f^*\omega_B^\vee$ on X is called the *relative dualizing sheaf*.

The direct image $f_*\omega_{X/B}$ is a coherent sheaf with interesting properties which we recall in the next proposition (see e.g. [Fuj78a]).

Proposition 1.2.10. *Let $f : X \to B$ be a fibration of complex varieties over a smooth curve B. Then $f_*\omega_{X/B}$ is a locally free sheaf of $\mathcal{O}_B$-modules (that is, a vector bundle) and the rank* $\operatorname{rank} f_*\omega_{X/B}$ *is the geometric genus $p_g(X_g)$ of the general fibre X_g.*

As proved for example in [CD, Proposition 2.9], the direct image of the relative dualizing sheaf is not stable under semistable reduction but the behaviour is well described by a short exact sequence.

Proposition 1.2.11 ([CD, Proposition 2.9])**.** *Let $f : X \to B$ be a fibration over a projective curve B. Let $u : B' \to B$ be a finite morphism of curves given by the semistable reduction theorem (1.1.7) and $f' : X' \to B'$ be the semistable reduced fibration of f. Then there is an injection of sheaves $f'_*\omega_{X'/B'} \hookrightarrow u^*f_*\omega_{X/B}$, which gives an exact sequence of sheaves*

$$0 \longrightarrow f'_*\omega_{X'/B'} \longrightarrow u^*f_*\omega_{X/B} \longrightarrow \mathcal{G} \longrightarrow 0, \tag{1.8}$$

such that the cokernel $\mathcal{G}$ is sky-scraper supported on the branch lucus B_0 of f.

Remark 1.2.12. The previous is true also in more general cases (e.g. in [Tan94] it is proved for finite base changes on fibred surfaces).

The direct image of the relative dualizing sheaf turns out to have interesting positivity properties.

We recall some notions of positivity of vector bundles over a curve following [Bar00] and [CD]. Let Y be a smooth projective variety, D be a divisor on Y and $\mathcal{L} = \mathcal{O}(D)$ a line bundle over Y.

Definition 1.2.13. A line bundle $\mathcal{L} = \mathcal{O}(D)$ is

(Ne) *nef* if $D \cdot C \geq 0$, for every curve C in Y, where $D \cdot C$ is the intersection of D with C;

(sNe) *strictly nef* if $D \cdot C > 0$, for every curve C in Y, where $D \cdot C$ is the intersection of D with C;

(vA) *very ample* if there is an embedding $j : Y \to \mathbb{P}^n$ for some positive integer n such that $\mathcal{L} = j^*\mathcal{O}(1)$;

(A) *ample* if $\mathcal{L}^{\otimes r}$ is very ample, for some integer $r \geq 0$. Equivalently, if for every subvariety W in Y of dimension k the intersection of D^k with W is positive, that is $W \cdot D^k > 0$.

Let Y be a smooth projective variety and $\mathcal{F}$ be a locally free sheaf on Y, that is a vector bundle. Consider the projective bundle $\mathbb{P} = \mathbb{P}_Y(\mathcal{F})$ and its tautological line bundle $\mathcal{O}_{\mathbb{P}}(L_{\mathcal{F}})$, that is $L_{\mathcal{F}}$ is a divisor on $\mathbb{P}$ such that $p_*\mathcal{O}_{\mathbb{P}}(L_{\mathcal{F}}) = \mathcal{F}$, where $p : \mathbb{P} \to Y$ is the natural projection.

Definition 1.2.14. A locally free sheaf $\mathcal{F}$ over a smooth projective variety Y is *nef, strictly nef, ample* if its tautological line bundle is nef, strictly nef, ample, respectively.

Remark 1.2.15. The properties introduced are numerical.

Theorem 1.2.16 ([CD, Proposition 2.3]). *Let $\mathcal{F}$ be a vector bundle over a smooth projective curve B. Then $\mathcal{F}$ is nef if and only every quotient bundle Q of $\mathcal{F}$ has degree* $\deg Q \geq 0$ *(* numerically positive*); it is ample if and only if every quotient bundle Q of $\mathcal{F}$ has degree* $\deg Q > 0$.

Theorem 1.2.17 (**[Fuj78a, Theorem 2.7]**). *Let $f : X \to B$ be a fibration over a projective curve B. Then $f_*\omega_{X/B}$ is nef.*

The sheaf $f_*\omega_{X/B}$ is not ample in general. This follows from the existence of a decomposition on $f_*\omega_{X/B}$ as the direct sum of a trivial bundle and a locally free sheaf, known as *first Fujita decomposition*, which we will state formally in section 2.1. Already in the case of fibred surfaces, where the trivial summand has rank the relative irregularity, it appears whenever the fibration is not isotrivial. A non numerical property of positivity is provided by the semiampleness.

Definition 1.2.18. A line bundle $\mathcal{L}$ over a smooth projective variety M is *semiample* if it admits a positive integer that is globally generated. A locally free sheaf $\mathcal{F}$ is *semiaple* if its tautological line bundle $\mathcal{O}_{\mathbb{P}}(L_{\mathcal{F}})$ is a semiample line bundle.

We will discuss some details related with the *second Fujita decomposition* in section 2.3.

1.3 Local systems, flat bundles and monodromy representations

In this section we introduce definitions and terminology on local systems, flat bundles and monodromy representation and we recall the correspondence between them. We will work over a smooth irreducible curve B even if everything we will say holds in a more general setting. We follow the references given at the top of the chapter, but also [Kob87].

Definition 1.3.1. Let B be a smooth irreducible curve, $\pi_1(B, b)$ be the fundamental group of B with base point b.

(LS) A *Local system* of $\mathbb{C}$-vector spaces over B is a sheaf $\mathbb{V}$ of $\mathbb{C}$-vector spaces which is locally isomorphic to the constant sheaf of stalk a $\mathbb{C}$-vector space V.

(FB) A *Flat vector bundle* over B of fiber a $\mathbb{C}$-vector space V is a pair $(\mathcal{V}, \nabla)$ given by a locally free sheaf $\mathcal{V}$ of $\mathcal{O}_B$-modules and stalk V and a flat connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_B$ (i.e. such that the curvature $\Theta = \nabla^2$ is identically zero).

(MS) A *Monodromy representation* of B over a $\mathbb{C}$-vector space V is a representation of the fundamental group $\pi_1(B, b)$, that is a homomorphism

$$\rho_{\mathrm{V}} : \pi_1(B, b) \to \mathrm{Aut}(V),$$

and the image $\mathrm{Im}\,\rho_{\mathrm{V}}$ is called the monodromy group.

Morphisms in the respective category are the natural expected: maps of sheaves of $\mathbb{C}-$vector spaces on local systems, maps of vector bundles preserving the connection on flat vector bundles and maps of representations on monodromy representations. In the sequel we assume that V has finite dimension.

Proposition 1.3.2. *There are 1 to 1 correspondences between local systems, flat vector bundles and monodromy representations modulo isomorphisms in the respective categories. More precisely, it holds*

$$\left\{ \begin{array}{c} \textit{Local systems } \mathbb{V} \textit{ over } B \\ \textit{of } \mathbb{C}-\textit{vector spaces} \end{array} \right\}_{/iso} \rightleftarrows \left\{ \begin{array}{c} \textit{Flat vector bundles} \\ (\mathcal{V}, \nabla) \end{array} \right\}_{/iso} \tag{1.9}$$

$$\left\{ \begin{array}{c} \textit{Local systems } \mathbb{V} \textit{ over } B \\ \textit{of } \mathbb{C}-\textit{vector spaces} \end{array} \right\}_{/iso} \rightleftarrows \left\{ \begin{array}{c} \textit{Monodromy representations} \\ \rho_V : \pi_1(B,b) \to \mathrm{Aut}(V) \end{array} \right\}_{/con} \tag{1.10}$$

where "iso" denotes the action given by isomorphisms in the correspondent category, while "con" the action given by conjugation.

We shortly recall the constructions of the stated correspondences.

Correspondence 1.9. Correspondence $\mathbb{V} \to (\mathcal{V}, \nabla)$ is constructed by taking $\mathcal{V}$ the vector bundle $\mathcal{V} := \mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_B$ and ∇ the flat connection defined by $\ker \nabla \simeq \mathbb{V}$; the inverse $(\mathcal{V}, \nabla) \to \mathbb{V}$ is given setting $\mathbb{V}$ to be the sheaf $\ker \nabla$, called *the local system of flat sections* of $\mathcal{V}$.

Correspondence 1.10. Correspondence $\mathbb{V} \to \rho_{\mathbb{V}}$ is constructed fixing a point $b \in B$, considering the isomorphism $\alpha : \mathbb{V}_b \simeq V$ and defining $\rho_V(\gamma) = \alpha \circ \gamma^* \alpha^{-1}$, where $\gamma^* : \mathbb{V}_b \simeq \mathbb{V}_b$ is the isomorphism induced by $\gamma \in \pi_1(B,b)$. Conversely, $\rho_{\mathbb{V}} \to \mathbb{V}$ is given by looking at the action of $\pi_1(B,b)$ on $\widetilde{B} \times V$, where $\widetilde{B}$ is the universal covering of B, induced by $\rho_{\mathbb{V}}$.

Remark 1.3.3. There is a natural isomorphism $\Gamma(A, \mathbb{V}) \to V$ over any contractible subset A of B since $\mathbb{V}$ is trivial over A.

The space of global sections instead is smaller in general and depends on the monodromy action as stated in the following

Lemma 1.3.4 ([Voi03, Lemma 4.17])**.** *Let $\mathbb{V}$ be a local system over B. Then the space $\Gamma(B, \mathbb{V})$ of global sections on $\mathbb{V}$ can be identified with the space of invariants*

$$\mathbb{V}^{inv} = \{ v \in \mathbb{V} \mid \rho_V(\gamma)(v) = v, \, \forall \gamma \in \pi_1(B,b) \}$$

ans we also use the notation $H^0(B, \mathbb{V})^{inv}$.

We recall some properties about the local systems and we give a sketch of the proof of those that will be use in the proof of some results of the thesis.

Proposition 1.3.5. *Let $u : B' \to B$ be a morphism of curves and $\mathbb{V}$ be a local system over B. Then $u^{-1}\mathbb{V}$ is a local system over B'. Moreover, the associated monodromy representation factors through $u_* : \pi_1(B', b') \to \pi_1(B,b)$, where $b' \in u^{-1}(b)$.*

Proposition 1.3.6. *Let $\mathbb{V}_1$ and $\mathbb{V}_2$ be two local subsystems of the local system $\mathbb{V}$. If they both have finite monodromy, then the local subsystem $\mathbb{V}_1 + \mathbb{V}_2$ of $\mathbb{V}$ has finite monodromy.*

Proof. Let $\rho_{\mathbb{V}}$ be the monodromy representation of $\mathbb{V}$, H the kernel of $\rho_{\mathbb{V}}$ and H_i the kernel of the sub-representations induced by ρ on $\mathbb{V}_i$, for $i = 1, 2$. Then $H_{12} := H_1 \cap H_2$ is the kernel of the sub-representation of $\mathbb{V}_1 + \mathbb{V}_2$. We prove that H_{12} has finite index in $\pi_1(B,b)$. By assumption, $\mathbb{V}_1$ and $\mathbb{V}_2$

have both finite monodromy, which means that $\pi_1(B,b)/H_i$ for $i = 1, 2$ are finite. Consider the chain of normal extensions $H_{12} \lhd H_1 \lhd H$. Then $H_1 \lhd H$ has finite index by assumption while $H_{12} \lhd H_1$ has finite index since there is a natural injective morphism $H_1/H_{12} \hookrightarrow \pi_1(B,b)/H_2$ and $\pi_1(B,b)/H_2$ is finite by assumption. Thus $H_{12} \lhd H$ has finite index. □

Let $\mathbb{V}$ be a local system over a smooth curve B of stalk V and let $\rho_{\mathrm{V}} : \pi_1(B,b) \to \mathrm{Aut}(V)$ be its monodromy representation. We will always denote by $H_{\mathrm{V}} = \ker \rho_{\mathrm{V}}$ the kernel of ρ_{V} and $G_{\mathrm{V}} = \pi_1(B,b)/H_{\mathrm{V}}$ the quotient group, which is isomorphic to the monodromy group $\mathrm{Im}\,\rho_V$. We want to attach a local subsystem of $\mathbb{V}$ to a vector subspace $W \subset V$. Given a vector subspace W of V we define

$$G_{\mathrm{V}} \cdot W := \sum_{g \in G_{\mathrm{V}}} g \cdot W,$$

where $g \cdot W := \rho_{\mathrm{V}}(g)(W)$ (we will also use the notation gW). We remark that $G_{\mathrm{V}} \cdot W$ is smallest subspace of V containing W and invariant under the action ρ_{V}. Thus it defines the smallest sub-representation of V containing W.

Definition 1.3.7. Let $\mathbb{V}$ be a local system over a smooth curve B of stalk the $\mathbb{C}$-vector space V and W be a vector subspace of V. The local system $\widehat{\mathbb{W}}$ *generated by* W is the local sub-system of $\mathbb{V}$ of stalk $\widehat{W} = G_{\mathrm{V}} \cdot W$.

As usual, we denote with $\rho_{\widehat{W}}$ the monodromy representation of $\widehat{W}$, with $H_{\widehat{W}}$ the kernel and with $G_{\widehat{W}}$ the quotient. We also denote with H_{W} the subgroup of H_{V} which fix W. We remark that $H_{\widehat{W}}$ is the normalization of H_{W}.

Let $\mathbb{V}$ be a local system of stalk V over a smooth curve B and let $A \hookrightarrow B$ be an open contractible subset of B and we use the canonical isomorphism $V \simeq \Gamma(A, \mathbb{V})$ between the stalk V and the sections over A. In the followings we will prove some properties of generated local systems.

Proposition 1.3.8. *Let W_1 and W_2 be two subspaces of $\Gamma(A, \mathbb{V})$ such that $W_1 \subset W_2$. If the local system $\widehat{\mathbb{W}}_2$ generated by W_2 has finite monodromy, then the local system $\widehat{\mathbb{W}}_1$ generated by W_1 has finite monodromy.*

Proof. Let $H = \ker \rho$ be the kernel of the unitary representation of $\mathbb{U}$ and $H_i = \ker \rho_i$ be the kernel of the sub-representations ρ_i defining $\widehat{\mathbb{W}}_i$, for $i = 1, 2$. Then we have an inclusion $H_2 \lhd H_1$ of subgroups which gives a surjection

$$G_2 := \pi_1(B,b)/H_2 \longrightarrow G_1 := \pi_1(B,b)/H_1 \longrightarrow 0 \tag{1.11}$$

on the quotients groups respectively isomorphic to the monodromy groups of $\widehat{\mathbb{W}}_2$ and $\widehat{\mathbb{W}}_1$. Thus whenever the monodromy of $\widehat{\mathbb{W}}_2$ is finite, the monodromy of $\widehat{\mathbb{W}}_1$ is finite. □

Proposition 1.3.9. *Let $\mathbb{V}$ be a local system over a curve B, let $W \subset \Gamma(A, \mathbb{V})$ be a vector subspace and let $\widehat{\mathbb{W}}$ be the local subsystem of $\mathbb{V}$ generated by W. Then a finite morphism of curves $u : B' \to B$ induces a isomorphism of local systems over B'*

$$u^{-1}\widehat{\mathbb{W}} \longleftrightarrow \textstyle\sum_{g_i \in I_u} \widehat{\mathbb{W}}_{g_i}, \tag{1.12}$$

where $\widehat{\mathbb{W}}_{g_i}$ *is the local subsystem generated by* $u^*g_i \cdot W$ *via* ρ_W^{-1}, *for* g_i *varying in a set* $I_u \subset \pi_1(B,b)$ *of generators of the quotient given by* $u_* : \pi_1(B',b') \to \pi_1(B,b)$.

Proof. Consider the local system $\widehat{\mathbb{W}}$ generated by W, which is by definition the local system on B of stalk $\pi_1(B,b)\cdot W$ and monodromy representation ρ_{W}. The inverse image $u^{-1}\widehat{\mathbb{W}}$ is a local system of the same stalk (i.e. $\pi_1(B,b)\cdot W$) and monodromy representation ρ_{W}^{-1} given by the action of $\pi_1(B',b')$ via the composition $\rho \circ u_*$, where $u(b') = b$ and $u_* : \pi_1(B',b') \to \pi_1(B,b)$ is the natural homomorphism induced by u. Consider the local system $\widehat{\mathbb{W}}_g$, which is a local system on B' of stalk generated by u^*gW (i.e. $\pi_1(B',b')\cdot u^*gW$). Then, since the monodromy action of $g \in \pi_1(B,b)$ sends W to gW, it is clear that the sum over a set of generators of the cokernel of u_* reconstructs exactly $u^{-1}\widehat{\mathbb{W}}$. □

The correspondences given in proposition 1.3.2 generalize when some suitable metric structures are introduced: a *unitary local system* $(\mathbb{V}, h)$, with h an hermitian structure on $\mathbb{V}$; a *unitary flat vector bundle* $(\mathcal{V}, \nabla, h)$, with h an hermitian metric compatible with the holomorphic connection ∇; a *unitary monodromy representation* $(\rho_V, (V,h))$, with (V,h) a hermitian vector space and h preserved under the monodromy action. Under this assumption, there is a fundamental structure theorem due to Narasimhan and Seshadri [NS65], which links unitary bundles to stable bundles. We recall that a holomorphic vector bundle on a complete smooth curve B is stable if the slope (i.e. the number given by the degree over the rank of a vector bundle) decreases on subbundles.

Theorem 1.3.10 ([NS65])**.** *Let* B *be a smooth complete irreducible curve of genus* $g(B) \geq 2$. *Then a holomorphic vector bundle* $\mathcal{V}$ *on* B *of degree zero is stable if and only if it is induced by a unitary irreducible representation of the fundamental group of* B.

1.4 Infinitesimal deformation theory on fibrations over curves

We recall the notion of first order deformation of a smooth projective variety and then we restrict to the special case of fibrations over curves.

Definition 1.4.1. Let X_0 be a smooth projective variety of dimension $m = n+1$. A *first-order deformation* of X_0 is an extension

$$\xi : \qquad 0 \longrightarrow \mathcal{O}_{X_0} \longrightarrow \mathcal{F} \longrightarrow \Omega^1_{X_0} \longrightarrow 0 \tag{1.13}$$

defined by the element $\xi \in \mathrm{Ext}^1(\Omega^1_{X_0}, \mathcal{O}^1_{X_0}) \simeq H^1(X_0, \Theta_{X_0})$, which is called *Kodaira-Spencer class.*

We also use the notation $f_\epsilon : \mathcal{X} \to \Delta$, where $\Delta = \mathrm{Spec}(k[\epsilon]/(\epsilon^2))$ is the spectrum over the ring of dual numbers. The reason, unformally, is that first order deformations are studied to understand the local structure of a moduli space $\mathcal{M}$ around $[X_0]$, when a moduli space $\mathcal{M}$ exists. As we have fixed a complex projective variety X_0, we can think to some kind of moduli space $\mathcal{M}$ of isomorphism classes of projective varieties defined over the field $\mathbb{C}$ of complex numbers. In this setting, the cocycle $\xi \in H^1(X_0, \Theta_{X_0})$ is constructed, using a local trivialization $\theta_\alpha : X_{|U_\alpha} \simeq U_\alpha \times \mathrm{Spec}(k[\epsilon]/(\epsilon^2))$, which is the identity on the central fibre $U_\alpha = U_\alpha \times \mathrm{Spec}\mathbb{C}$, as the cocycle $\theta_{\alpha\beta} = \theta_\beta\theta_\alpha^{-1} \in \Gamma(U_{\alpha\beta}, \Theta_{X_0})$. This defines a map $T_{\mathcal{M},[X_0]} \to H^1(X_0, \Theta_{X_0})$ and an estension

$$\xi : \qquad 0 \longrightarrow \mathcal{O}_{X_0} \longrightarrow \Omega^1_{\mathcal{X}} \otimes \mathcal{O}_{X_0} \longrightarrow \Omega^1_{X_0} \longrightarrow 0 \tag{1.14}$$

with extension class $\xi = [\{\theta_{\alpha\beta}\}_{\alpha\beta}] \in H^1(X_0, \Theta_{X_0})$. More precisely, the differential at 0 of the functorial morphism $\Delta \to \mathcal{M}$ defines a map $KS : T_{\Delta,0} \to H^1(X_0, \Theta_{X_0})$ called *Kodaira Spencer map* and the image of $\xi = KS(v)$ of $v \in T_{\Delta,0}$ is the *Kodaira Spencer class* at v (see [Voi02] for details).

A special case of first order deformations is provided by $1-$dimensional families smooth projective varieties, namely by fibrations $f : X \to B$ over a smooth complex curve B.

Let $f : X \to B$ be a fibration over a smooth curve B. Let $b \in B$ be a regular value of f and X_b the fibre over $b \in B$, which is a smooth projective variety. The fibration f induces a first order deformation

$$\xi_b : \qquad 0 \longrightarrow \mathcal{N}^\vee_{X/B} \simeq T^\vee_{B,b} \otimes \mathcal{O}_{X_b} \longrightarrow \Omega^1_{X|X_b} \longrightarrow \Omega^1_{X_b} \longrightarrow 0 \tag{1.15}$$

of X_b with Kodaira-Spencer class $\xi_b \in \mathrm{Ext}^1(\Omega^1_{X_b}, \mathcal{O}^1_{X_b} \otimes T^\vee_{B,b}) \simeq H^1(X_b, T_{X_b}) \otimes T^\vee_{B,b}$. The extension is obtained by restriction through $i_b : X_b \hookrightarrow X$ to X_b of the sequence 1.4

$$0 \longrightarrow f^*\omega_B \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/B} \longrightarrow 0,$$

as $\Omega^1_{X/B_{|X_b}} = \Omega^1_{X_b}$ over each b regular value. The connecting homomorphism δ on the associated long exact sequence in cohomology

$$0 \longrightarrow H^0(\mathcal{O}_{X_b}) \longrightarrow H^0(\Omega^1_{X|X_b}) \longrightarrow H^0(\omega_{X_b}) \xrightarrow{\delta = \cup\xi_b} H^1(\mathcal{O}_{X_b}), \tag{1.16}$$

is given by the cut product $\cup\xi_b : H^0(\omega_{X_b}) \to H^1(\mathcal{O}_{X_b})$ with $\xi_b \in H^1(T_{X_b})$ and given by the connecting homomorphism

$$\partial : f_*\Omega^1_{X/B} \longrightarrow R^1 f_*\mathcal{O}_X \otimes \omega_B \tag{1.17}$$

of the pushforward sequence

$$0 \longrightarrow f_*f^*\omega_B \simeq \omega_B \longrightarrow f_*\Omega^1_X \longrightarrow f_*\Omega^1_{X/B} \xrightarrow{\partial} (R^1 f_*\mathcal{O}_X) \otimes \omega_B. \tag{1.18}$$

The sheaf $\mathcal{K}_\partial = \ker \partial \subset R^1 f_*\mathcal{O}_X \otimes \omega_B$ is the subsheaf whose sections over suitable subsets $A \subset B$ are holomorphic $1-$forms on the fibres with are litfable to holomorphic $1-$forms of the ambient space X over $f^{-1}(A)$, as

$$H^0(A, \mathcal{K}_\partial) = H^0(f^{-1}(A), f^*\mathcal{K}_\partial). \tag{1.19}$$

It is not torsion free in general, but the fibre over a general value $b \in B$ (not a torsion point) is

$$\mathcal{K}_\partial \otimes \mathbb{C}(b) = \mathcal{K}_\partial^{\vee\vee} \otimes \mathbb{C}(b) = \ker \cup\xi_b$$

and thus it is the vector space

$$\ker \cup \xi_b = \mathrm{Im}\{H^0(\Omega^1_{X|X_b}) \to H^0(\omega_{X_b})\}$$

of holomorphic 1−forms on X_b which are liftable to infinitesimal deformations on X. Clearly over each open contractible A in B the equality 1.19 holds and there is no obstruction to lift infinitesimal extensions to holomorphic forms in the ambient space, as A is Stein in a curve B. But for open subsets that are not contractible, the consideration above is not automatic and fits into a vanishing problem depending on the geometry of the ambient space.

1.5 Geometric variation of the Hodge structure

In this section we introduce terminology on (polarized) variations of the Hodge structure and recall some classical well known facts. We are interested in the case of (polarized) variations of the Hodge structure " which come from the geometry ", that is those naturally defined by smooth fibrations over curves. For more details we refer to [Gri70], [Voi02], [Voi03] and also [PS08].

Definition 1.5.1. A *Hodge structure (HS) of weight k over the integers* is the data $(V_{\mathbb{Z}}, {V^{p,q}}_{p,q\geq 0, p+q=k})$ (shortly, $(V_{\mathbb{Z}}, V^{p,q})$) of

(HS1) a finitely generated free abelian group (a lattice) $V_{\mathbb{Z}}$;

(HS2) a direct sum decomposition

$$V_{\mathbb{C}} = \oplus_{p+q=k} V^{p,q}, \quad V^{p,q} = \overline{V^{q,p}} \tag{1.20}$$

on the complexification $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C}$ of $V_{\mathbb{Z}}$, called *Hodge decomposition.*

A *polarized Hodge structure of weight k* is the data $(V_{\mathbb{Z}}, V^{p,q}, Q)$ of a Hodge structure of weight k $(V_{\mathbb{Z}}, V^{p,q})$ as above and

(HS3) a $(-1)^k$−symmetric bilinear form $Q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$, whose $\mathbb{C}$−linear extension to $V_{\mathbb{C}}$ satisfies the *Hodge-Riemann bilinear relations*

(HR1) $Q(V^{p,q}, V^{p',q'}) = 0$ unless $p' = k - p$ and $q' = k - q$;

(HR2) $i^{p-q} Q(u, \bar{u}) > 0$, for $u \in V^{p,q}$, $u \neq 0$.

The bilinear form Q is called *(integral) polarization* of the Hodge structure.

Equivalently, a Hodge structure is defined by a filtration $F^p V_{\mathbb{C}} = \oplus_{p' \geq p} V^{p',q}$ on $V_{\mathbb{C}}$, called *Hodge filtration*, which satisfies $V_{\mathbb{C}} = F^p V_{\mathbb{C}} \oplus \overline{F^{k-p+1} V_{\mathbb{C}}}$.

Remark 1.5.2. By the Hodge-Riemann relation (HR1), Q defines a perfect pairing between $V^{p,q}$ and $V^{q,p}$; by relation (HR2), Q defines a positive definite hermitian form $h(u, v) = Q(Cu, \bar{v})$ on $V^{p,q}$, where $C = i^{p-q}$ is called the *Weil operator.* In terms of the Hodge filtration, $F^p V_{\mathbb{C}}$ and $F^{k-p-1} V_{\mathbb{C}}$ are orthogonal complements via the metric h.

Remark 1.5.3. Hodge structures as in Definition 1.5.1 are *integral HS*, as they are defined starting from a lattice $V_{\mathbb{Z}}$. More in general, one can define $\mathbb{Q}$-Hodge structures, or $\mathbb{R}$-Hodge structures, respectively, by starting from $\mathbb{Q}-$modules or $\mathbb{R}-$modules, respectively. This is usual whenever one can find polarizations as above but which takes values on $\mathbb{Q}$ or $\mathbb{R}$ instead of $\mathbb{Z}$.

We introduce variations of the Hodge structure over a smooth connected curve B (not necessarily compact).

Definition 1.5.4. A *Variation of the Hodge structure (VHS) of weight k over B* is the data $(\mathbb{V}, \{\mathcal{F}^p\}_{k\geq p\geq 0})$ (shortly $(\mathbb{V}, \mathcal{F}^p)$) of

(VHS1) a local system $\mathbb{V}$ of free $\mathbb{Z}$-modules of finite rank k (lattices of rank k);

(VHS2) a finite decreasing filtration $\{\mathcal{F}^p\}_{k\geq p\geq 0}$ of holomorphic sub-bundles $\mathcal{F}^p \subset \mathcal{V} = \mathbb{V} \otimes_{\mathbb{Z}} \mathcal{O}_B$ called *Hodge filtration* such that

(VHS2a) it satisfies the *infinitesimal period relation*

$$\nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_B, \tag{1.21}$$

where ∇ is the flat connection of $\mathcal{V}$ defined by $\mathbb{V}$;

(VHS2b) it defines pointwise a HS of weight k over $\mathbb{V}_{\mathbb{Z},s}$ as a Hodge filtration

$$\mathcal{F}^p(s) \subset \mathbb{V}_s \simeq \mathbb{V}_{\mathbb{Z},s} \otimes_{\mathbb{Z}} \mathbb{C}, \tag{1.22}$$

where $\mathcal{F}^p(s)$ is the fibre of $\mathcal{F}^p$ over $s \in B$.

A *Polarized variation of the Hodge structure (PVHS) of weight k over B* is the data $(\mathbb{V}, \mathcal{F}^p, Q)$ of a VHS $(\mathbb{V}, \mathcal{F}^p)$ as above and

(VHS3) a morphism of VHS $Q : \mathbb{V} \times \mathbb{V} \to \mathbb{Z}(-k)_s$, which is called *polarization*, defining pointwise a polarization on the HS of weight k, where $\mathbb{Z}(-k) = (2\pi i)^{-k}\mathbb{Z}$.

The quotients bundles $\mathcal{V}^{p,q} = \mathcal{F}^p/\mathcal{F}^{p+1}$, $p+q = k$ are called *Hodge bundles* and there is a C^∞ (not holomorphic) decomposition

$$\mathcal{V} = \oplus_{p+q=k} \mathcal{V}^{p,q}, \quad \mathcal{V}^{p,q} = \overline{\mathcal{V}^{p,q}} \tag{1.23}$$

which punctually gives the HS of weight k.

Remark 1.5.5. As well as for Hodge structures (see Remark 1.5.3), (polarized) variations of the Hodge structures can be defined over $\mathbb{Q}$ or $\mathbb{R}$, by taking local systems over this fields that punctually define rational or real Hodge structures, respectively.

Remark 1.5.6. In section 1.3 we introduced the notion of local systems of $\mathbb{C}-$vector spaces and we saw the correspondence with holomorphic flat bundles. Variations of the HS as above actually fixed restrictions to the structure of a local system.

The monodromy group of the VHS is the monodromy group of the local system $\mathbb{V}_{\mathbb{Z}}$. Let

$$\rho_{\mathbb{V}} : \pi_1(B, b) \to \mathrm{Aut}_{\mathbb{Z}}(\mathbb{V}_b)$$

be the monodromy representation of the local system $\mathbb{V}$. From the fact that the form Q is locally constant, the image of $\rho_{\mathbb{V}}$ is included in the monodromy group $G_{\mathbb{Z}} := \mathrm{Aut}(\mathbb{V}_b, Q)$.

Definition 1.5.7. An *Infinitesimal variation of the Hodge structure of weight* k (shortly *IVHS*) is the data $(V_{\mathbb{Z}}, V^{p,q}, Q, T, \delta)$ of a polarized HS $(V_{\mathbb{Z}}, V^{p,q}, Q)$ of weight k together with a vector space T and a linear map

$$\delta : T \to \oplus_{1\geq p\geq n}\mathrm{Hom}(V^{p,k-p}, V^{p-1,k-p+1}) \tag{1.24}$$

satisfying

IVHS1 $\delta_{p-1}(v_1)\delta_p(v_2) = \delta_{p-1}(v_2)\delta_p(v_1)$, for $v_1, v_2 \in T$;

IVHS2 $Q(\delta(v)v_1, v_2) + Q(v_1, \delta(v)v_2) = 0$, for $v_1, v_2 \in V$, $v \in T$.

It is clear that VHS pointwise defines IVHS.

Now we restrict our attention on Hodge structures and corresponding variations that " come from the geometry".

Example 1.5.8. (**Geometric variations of the Hogde structure**) Let $f : X \to B$ be a smooth Kaehler fibration over a smooth complex curve B. Let X_b be the fibre of X over $b \in B$ of dimension n. As a compact Kaehler manifold, X_b carries a strong Hodge decomposition

$$H^k(X_b, \mathbb{C}) = \oplus_{p+q=k}H^{p,q}(X_b), \quad H^{p,q}(X_b) = \overline{H^{q,p}(X_b)} \tag{1.25}$$

which defines for each $0 \leq k \leq n$ a Hodge structure $(H_{\mathbb{Z}} = H^k(X_b, \mathbb{Z})/\mathrm{torsion}, F^pH_{\mathbb{C}} = \oplus_{p'\geq p}H^{p',q})$ of weight k called *geometric Hodge structure.* Moreover,

$$H^{p,q}(X_b) = H^q(X_b, \Omega^p_{X_b}). \tag{1.26}$$

Consider the local system $\mathbb{H}_{\mathbb{Z}} = R^kf_*\mathbb{Z}$ of lattices of rank k over B, the local system $\mathbb{H}_{\mathbb{C}} = R^kf_*\mathbb{C}$ of $\mathbb{C}-$vector spaces and the holomorphic flat bundle $(\mathcal{H}, \nabla)$ defined by the local system $R^kf_*\mathbb{C}$,

$$\mathcal{H} = R^kf_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_B, \quad \nabla = 1 \otimes \mathrm{d}_B : \mathcal{H} \mapsto \mathcal{H} \otimes \omega_B. \tag{1.27}$$

The Hodge filtration $\mathcal{F}^p$ is pointwise defined by the Hodge filtration of the fibres

$$\mathcal{H} \otimes \kappa(b) \simeq R^kf_*\mathbb{C}_b \simeq H^k(X_b, \mathbb{C}), \quad \mathcal{F}^p \otimes \kappa(b) \simeq \oplus_{p'\geq p}H^q(X_b, \Omega^{p'}_{X_b}). \tag{1.28}$$

The data $(\mathbb{H}_{\mathbb{Z}}, \mathcal{F}^p)$ is a VHS, which is called *geometric VHS* of f, since it comes from the geometry of the fibration f. More precisely, as a consequence of local triviality of f then $R^kf_*\mathbb{Z}$ is a local system and the filtration pointwise defined glues into a filtration of holomorphic subbundles of the bundle $\mathcal{H} = R^kf_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_X$.

The geometric VHS introduced above is not polarized in general. Let $f : X \to Y$ be a smooth projective fibration of smooth projective varieties and $\omega \in H^2(X, \mathbb{Z})$ be the integral Kaehler form the defined by the holomorphic embedding $i : X \hookrightarrow B \times \mathbb{P}^r$ as $\omega = (pr_2 \circ i)^*c_1(\mathcal{O}_{\mathbb{P}^r})$ and whose restriction to each fibre X_b gives a Kaehler class $\omega_{|X_b} \in H^2(X_b,\mathbb{Z})$. The cohomology class $\omega \in H^2(X,\mathbb{Z})$ induces the *relative Lefschetz operator* L, which is a morphism of local systems between $R^k f_*\mathbb{Q}$ and $R^{k+2} f_*\mathbb{Q}$

$$L := \cup\omega : R^k f_*\mathbb{Q} \to R^{k+2} f_*\mathbb{Q} \tag{1.29}$$

punctually defined by the cup product with $\omega_{|X_b} \in H^2(X_b,\mathbb{Z})$ in this sense: on X_b the cup-product with the Kaehler class $\omega_{|X_b} \in H^2(X_b,\mathbb{Z})$ defines the *Lefschetz operator* L_b on the rational cohomology $H^k(X_b,\mathbb{Q})$,

$$L_b = \cup\omega_{|X_b} : H^k(X_b,\mathbb{Q}) \to H^{k+2}(X_b,\mathbb{Q}), \tag{1.30}$$

and the punctual construction fits into a morphism of local system since ω is $d-$closed. Moreover, each Lefschetz operator L_b defines the *primitive cohomology* on the fibre X_b

$$H^s(X_b,\mathbb{Q})_{\mathrm{prim}} = \ker\{L_b^{n-s+1} : H^s(X_b,\mathbb{Q}) \to H^{2n+s+2}(X_b,\mathbb{Q})\}, \tag{1.31}$$

the *Lefschetz deomposition*

$$H^k(X_b,\mathbb{Q}) = \bigoplus_{2h<k} L_b^h H^{k-2h}(X_b,\mathbb{Q})_{\mathrm{prim}} \tag{1.32}$$

and the *Lefschetz isomorphism*

$$L_b^{n-k} : H^k(X_b,\mathbb{Q}) \to H^{2n-k}(X_b,\mathbb{Q}) \tag{1.33}$$

which fit into the *relative* ones: the *relative primitive cohomology* $R^m f_*\mathbb{Q}_{\mathrm{prim}}$, the *relative Lefschtz decomposition*

$$R^k f_*\mathbb{Q} = \bigoplus_{2h<k} L^h R^{k-2h} f_*\mathbb{Q}_{\mathrm{prim}} \tag{1.34}$$

and the *relative Lefschetz isomorphism*

$$L^{n-k} : R^k f_*\mathbb{Q} \to R^{2n-k} f_*\mathbb{Q}. \tag{1.35}$$

The sheaf $R^m f_*\mathbb{Q}_{\mathrm{prim}}$ is locally constant and carries a VHS induced by the geometric VHS on $R^k f_*\mathbb{C}$ since the Lefschetz relative decomposition is pointwise compatible with the Hodge decomposition. The intersection form $Q : R^m f_*\mathbb{Q}_{\mathrm{prim}} \times R^m f_*\mathbb{Q}_{\mathrm{prim}} \to \mathbb{Q}$ given by the integer Kaehler class on X as before, that is $Q = (-1)^{k(k-1)/2}(L^{n-k}u, v)$, for $u, v \in R^m f_*\mathbb{Q}_{\mathrm{prim}}$, is non degenerate on the primitive cohomology and induces a polarization on the VHS. More precisely, on the fibre X_b, the intersection form $Q_b = (L_b^{n-k}u_b, v_b) = (-1)^{k(k-1)/2}\int_{X_b} u_b \wedge v_b \wedge \omega_{|X_b}^{n-k}$ gives a polarization on the HS over the primitive cohomology on X_b and the construction is locally constant in $b \in B$.

Remark 1.5.9. The same construction holds more in general on Kaehler manifolds releasing the assumption of projectivity, by using $\omega \in H^2(X,\mathbb{R})$ such that $\omega_{|X_b} \in H^2(X_b,\mathbb{R})$ is a Keahler class on the fibres.

1.5.1 Monodromy and the classical invariant cycles theorems

In the previous section we saw how fibrations over curves define a geometric VHS, up to restrict to the locus of regular values. As a local system, the geometric VHS of weight k defines a monodromy action over its stalk, which is the k cohomology group of the general fibre of the fibration. In what follows we will recall some well known results relating the cohomology of the ambient space with that of the general fibre of the fibration.

Let $f : X \to B$ be a smooth projective fibration of complex varieties over a smooth curve B (that is proper submersive projective morphism). Consider the local system $R^k f_*\mathbb{Q}$ associated to the geometric VHS $(\mathbb{H}_\mathbb{Z}, \mathcal{F}^p)$ of f (as in 1.5.8). As a local system of stalk isomorphic to $H^k(X_b, \mathbb{Q})$, it has a monodromy action $\rho : \pi_1(B, b) \to Aut(H^k(X_b, \mathbb{Q}))$. We consider the space $H^k(X_b, \mathbb{Q})^{\text{inv}}$ of cohomology classes in the fibre X_b invariant under the monodromy action. An important remark is the following interpretation of $H^0(B, R^k f_*\mathbb{Q})$.

Remark 1.5.10. The spaces $H^0(B, R^k f_*\mathbb{Q})$ of global sections of the local system $R^k f_*\mathbb{Q}$ and $H^k(X_b, \mathbb{Q})^{\text{inv}}$ of the cohomology on the fibre invariant under the monodromy action coincide, that is

$$H^k(X_b, \mathbb{Q})^{\text{inv}} \simeq H^0(B, R^k f_*\mathbb{Q}). \tag{1.36}$$

Theorem 1.5.11 (Global invariant cycle). *Let $f : X \to B$ be a smooth projective fibration over a curve B. Then for any k the restriction map*

$$H^k(X, \mathbb{Q}) \to H^k(X_b, \mathbb{Q})^{inv} \tag{1.37}$$

is surjective.

Sketch of the proof. Consider the the morphism

$$H^k(X, \mathbb{Q}) \to H^0(B, R^k f_*\mathbb{Q}) \tag{1.38}$$

given by 1.37 up to the identification 1.36. This can be interpreted in terms of the morphism

$$H^k(X, \mathbb{Q}) \to E_\infty^{0,k} \subset E_2^{0,k} = H^0(B, R^k f_*\mathbb{Q}) \tag{1.39}$$

induced by the Leray spectral sequence defined by $E_2^{0,k} = R^k f_*\mathbb{Q}$. The morphism is surjective when the equality $E_2^{0,k} = E_\infty^{0,k}$ holds, which is the case when the Leray spectral sequence of f degenerates at E_2 (Deligne degeneracy theorem, see Theorem 1.2.3).

Remark 1.5.12. The Deligne degeneracy theorem still work on the local system over the fields $\mathbb{R}$ or $\mathbb{C}$, providing a weaker version of the Global invariant theorem. Looking at the cohomology over the complex field $\mathbb{C}$, the Invariant cycle theorem states that the complex forms $H^k(X_b, \mathbb{C})$ invariants under the monodromy action associated to $R^k f_*\mathbb{C}$ are given by restriction of the complex forms $H^k(X, \mathbb{C})$ on the ambient space, i.e.

$$H^k(X, \mathbb{C}) \to H^k(X_b, \mathbb{C})^{\text{inv}} \simeq H^0(B, R^k f_*\mathbb{C}) \tag{1.40}$$

The problem to relate the behaviour of the cohomology of the fibres with that of the ambient space under monodromy action generalizes to the case on fibrations with singular fibres (degenerations), where one must take care of local monodromies. The analogue of the theorem above, proved by Deligne in [Del71], holds under the assumption of semistable degenerations.

Let $f : X \to B$ be a fibration of smooth complex vaierties over the curve B, let B_0 be the locus of singular values and the complement $B^0 = B \setminus B_0$ the locus of regular values. Assume that $j : B^0 \hookrightarrow B$ is quasi projective subset, the restriction $f^0 : X^0 = f^{-1}(B^0) \to B^0$ of f to B^0 is a smooth projective fibration. Then f is a smooth compactification of f^0. As before, we consider the local system $R^k f^0_* \mathbb{Q}$ of stalk isomorphic to $H^k(X_b, \mathbb{Q})$ and its monodromy action $\rho : \pi_1(B^0, b) \to Aut(H^k(X_b, \mathbb{Q}))$. The space $H^k(X_b, \mathbb{Q})^{\text{inv}}$ of cohomology classes in the fibre X_b invariant under the monodromy action is naturally isomorphic to $H^0(B^0, R^k f^0_* \mathbb{Q})$, i.e

$$H^k(X_b, \mathbb{Q})^{\text{inv}} \simeq H^0(B^0, R^k f^0_* \mathbb{Q}). \tag{1.41}$$

Again the Larey spectral sequence of f^0 degenerates at E_2 and applying the Global invariant cycle theorem 1.5.11 on it we get the description of the invariant elements in terms of X^0. The following is a stronger result due to Deligne.

Theorem 1.5.13 (Invariant cycle theorem). *Let $f : X \to B$ be a fibration over the curve B as above (the singular locus $j : B^0 \hookrightarrow B$ is quasi projective , the restriction of f to B^0 is projective). Then the map*

$$H^k(X, \mathbb{Q}) \to H^k(X_b, \mathbb{Q})^{inv} \tag{1.42}$$

is surjective.

As observed before, the map $H^k(X^0, \mathbb{Q}) \to H^k(X_b, \mathbb{Q})^{\text{inv}}$ is surjective by theorem 1.5.11. The key point is to compare the image of the above map with that of the map $H^k(X, \mathbb{Q}) \to H^k(X_b, \mathbb{Q})$ and they turns out to be equal. This last step does not depend only on the topology and requires some results in mixed hodge theory that we will not discuss here.

The theorem above is also called *Theorem of the fixed part* and the motivation to the name is the following.

Corollary 1.5.14. *Let $f : X \to B$ a fibration of smooth complex varieties as before. Then for all $b \in B$ the space of invariants $H^k(X_b, \mathbb{Q})^{inv}$ under the monodromy action defines a rational Hodge structure substructure of $H^k(X_b, \mathbb{Q})$.*

Remark 1.5.15. We notice that the theorem above holds under the assumption of semistable degenerations so that it is usual to reduce to this case by using the semistable reduction (Theorem 1.1.7).

In the dual formulation we can we look at the problem in homology. Let $\gamma \in H_k(X_b, \mathbb{Q})$ a homology class invariant under the monodromy action of $\pi_1(B^0, b)$ on the homology of the fibres. There is a cycle $\mathcal{L}(\gamma) \in H_{k+n}(X^0, \mathbb{Q})$ such that the intersection

$$\mathcal{L}(\gamma) \cdot X_b = \gamma \tag{1.43}$$

and it is called *locus of* γ. Moving $b \in B^0$ we can repeat the previous argument and it is expected that the cycle whose intersection with $X_{b'}$ gives $\gamma_{b'}$ is the same of γ, that is $\mathcal{L}(\gamma) \in H_{k+n}(X^0, \mathbb{Q})$ is such that

$$\mathcal{L}(\gamma) \cdot X_{b'} = \gamma_{b'} \tag{1.44}$$

for every $b' \in B^0$, moving γ along the family to $\gamma_{b'}$. This is the homological version of the Deligne degeneracy theorem, or better of the Global invariant cycle over X^0. The question posed and answered positively under the above assumptions by the Global invariant cycle theorem 1.5.13 is if the cycle is in $H_{n+d}(X, \mathbb{Q})$. In other words the theorem says that the cycle $\mathcal{L}(\gamma)$ remains fixed and it is called *locus of the invariant cycles*.

There are two kinds of "loops" generating the fundamental group of B^0. The ones which comes from B and the branchs of f. Restricting the attention on the behaviour of cycles around a branch point $b \in B_0$ we obtain a local formulation of the problem. The associated monodromy action is called *local monodromy*.

Let $b_0 \in B_0$ be a singular value of f and Δ be a coordinate complex disk centered in b_0, which does not contain other critical values of f. Let $f_\Delta : S_\Delta = f^{-1}(\Delta) \to \Delta$ be the fibration given by restriction and with only one degeneration on $t = 0$. Fixed $t \neq 0$, the monodromy action of $\pi_1(\Delta \setminus \{0\}, t)$ on the cohomology $H^k(X_t)$ of the fibre X_t around 0 defines a homomorphism

$$T_0 : H^k(X_t, \mathbb{Q}) \to H^k(X_t, \mathbb{Q}). \tag{1.45}$$

This is called *local monodromy operator* or *Picard-Lefschetz transformation* around b_0 and by definition describes the *local monodromy around* b_0. Let $r : X_t \to X_{b_0}$ be the map obtained by composition of the inclusion $X_t \hookrightarrow X_\Delta$ with the retraction $S_\Delta \to X_0$ to the singular fibre X_0. The *local invariant cycle problem* asks if there are obstructions to obtain the cohomology of the smooth fibre X_t from the cohomology of the singular fibre X_0.

Definition 1.5.16. Let $f : X \to B$ a fibration with a degeneration over b_0. We say that f satisfies the *Local invariant cycle property around* b_0 if the sequence

$$H^k(X_{b_0}, \mathbb{Q}) \stackrel{r_0^*}{\to} H^k(X_t, \mathbb{Q}) \stackrel{T_0 - I}{\to} H^k(X_t, \mathbb{Q}), \tag{1.46}$$

where r_0 is defined by composition of the injection $X_t \hookrightarrow X_U$ ($X_U = f^{-1}(U)$, U contractible open set around b_0) with a retraction to X_{b_0}, is exact. In this case, the space $H^k(F_t)^{\mathrm{inv}0} := \ker(T_0 - I)$ of invariants under the local monodromy $T_0 : H^k(X_t, \mathbb{Q}) \to H^k(X_t, \mathbb{Q})$ is given by the cohomology of the singular fibre X_{b_0}.

Theorem 1.5.17 (Monodromy theorem). *Let $T : H^k(X_t, \mathbb{Q}) \to H^k(X_t, \mathbb{Q})$ be a local monodromy operator as define before. Then T is quasi-unipotent with index of unipotency at most k, i.e.*

$$(T^l - I)^{k+1} = 0. \tag{1.47}$$

Moreover, when T is a local monodromy operator around a branch over a semistable degeneration, *then T is unipotent, i.e*

$$(T-I)^{k+1}=0. \tag{1.48}$$

The Local invariant cycle property allows to understand when the sheaf $R^k f_*\mathbb{Q}$ is determined by the local system $j^*R^k f_*\mathbb{Q}$, since the pullback j^* of $j : B^0 \hookrightarrow B$ is just the restriction and we have $j^*R^k f_*\mathbb{Q} \simeq R^k f^0_*\mathbb{Q}$. Equivalently, it happens when the morphism

$$\alpha : R^k f_*\mathbb{Q} \to j_*j^*R^k f_*\mathbb{Q}, \tag{1.49}$$

defined by composition of the pullback and the pushforward is surjective.

Theorem 1.5.18. *Let $f : X \to B$ be a fibration satisfying the local invariant cycle property around each singular point. Then the morphism 1.49 is surjective.*

The problem to determine *completely* $R^k f_*\mathbb{Q}$ from $j^*R^k f_*\mathbb{Q}$ is then translated into the study of the injectivity of the morphism above.

Lemma 1.5.19. *Let $f : X \to B$ a fibration as before and let n be the dimension of the smooth fibres of f. Assume that f satisfies the local invariant cycle property around all the singular values. Then the morphism 1.49 with $k = n$ is an isomorphism.*

The lemma is obtained keeping together the contents of Lemma C.13, pag. 434 of [PS08] and Theorem 5.3.4, pag. 266 of [CEZGT14].

Remark 1.5.20. The sheaf $R^n f_*\mathbb{Q}$ is not always a local system even if completely determined by $j^*R^n f_*\mathbb{Q}$. Lemma 1.5.19 just provides a critirion of unicity of extension of sheaves but not inside the category of local systems.

1.5.2 Infinitesimal variations of the Hodge structure and first order deformations

We relate infinitesimal variations of weight 1 Hodge structures to first order deformations in the case of smooth fibrations over curves.

Let $f : X \to B$ be a smooth fibration over a smooth curve B. As recalled in section 1.4, the fibration f defines by restriction $i_b : X_b \hookrightarrow X$ a first order deformation of the fibre X_b over $b \in B$

$$\xi_b : \qquad 0 \longrightarrow \mathcal{O}_{X_b} \otimes T^\vee_{B,b} \longrightarrow \Omega^1_{X|X_b} \longrightarrow \Omega^1_{X_b} \longrightarrow 0 \tag{1.50}$$

whose extension class is the Kodaira Spencer class $\xi_b \in H^1(T_{X_b}) \otimes T^\vee_{B,b}$. Moreover, the connecting morphism is given by the cup product with ξ_b

$$\cup\, \xi_b : H^0(\Omega^1_{X_b}) \to H^1(\mathcal{O}_{X_b}) \tag{1.51}$$

after a fixed choice of generator $w \in T_{B,b}$.

We consider the IVHS induced by the geometric VHS $H = (\mathbb{H}_\mathbb{Z} = R^1 f_*\mathbb{Z}, \mathcal{F}^1)$ of weight 1, which is defined by and differential

$$d\mathcal{P}_b : T_{B,b} \to \mathrm{Hom}(H^0(\Omega^1_{X_b}), H^1(\mathcal{O}_{X_b})) \tag{1.52}$$

of the period map corresponding to the VHS at $b \in B$. By Griffiths theorem (see [Voi02, Theorem 10.21]), it is computed by the Kodaira Spencer map by composition

$$H^1(T_{X_b}) \to \mathrm{Hom}(H^0(\Omega^1_{X_b}), H^1(\mathcal{O}_{X_b})) \tag{1.53}$$

of the cup product with the Kodaira-Spencer class and the interior product $T_{X_b} \otimes \Omega^1_{X_b} \to \Omega^0_{X_b} \simeq \mathcal{O}_{X_b}$ induced in cohomology, where we have fixed a generator $w \in T_{B,b}$ as above. This fixes the relation between first order deformations and IVHS.

Just to complete the picture, let $(\mathcal{H} = R^1 f_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_B, \nabla)$ be the holomorphic flat bundle defined by the VHS. On weight 1 variations of the Hodge structure, we only have these two Hodge bundles $\mathcal{F}^1 = \mathcal{H}^{1,0}$ and $\mathcal{H}/\mathcal{F}^1 \simeq \mathcal{H}^{0,1}$, which are related by the exact sequence

$$\overline{\nabla}^{1,0}: \qquad \mathcal{F}^1 \xrightarrow{\nabla} \mathcal{H} \otimes \Omega^1_B \longrightarrow \mathcal{H}/\mathcal{F}^1 \otimes \Omega^1_B \tag{1.54}$$

defined by the Griffiths trasversality property. The differential of the period map $d\mathcal{P}_b$ is computed by adjuction by the morphism $\overline{\nabla}^{1,0}_b$.

We conclude by comparing the kernels of the IVHS (or equivalently of the first order deformation) $\ker \cup\xi_b$ and the kernel $\ker \nabla$ of VHS and the following shows that they should differ.

Lemma 1.5.21. *Let $f : X \to B$ be a smooth fibration over a smooth compact curve B. Let ∇ the Gauss Manin connection of the geometric polarized VHS of weight 1 of f. Then*

$$\ker \nabla_b \subset \ker \cup\xi_b. \tag{1.55}$$

In other words, the previous lemma states that the request on periods to remain locally constant (that is flatness via ∇) is much stronger than the infinitesimal one.

1.6 Fibred surfaces and Massey-Products

We restrict our attention to the case of fibred surfaces.

A *fibred surface* is the data (S, f) of a complex surface (a smooth complex variety of dimension $m = 2$) and a fibration $f : S \to B$ over a smooth complex curve B. In this case the general fibre F of f is a smooth compact curve (as it is a smooth complex subvariety of S of dimension $n = m - 1 = 1$) and with geometric genus $p_g(F) = g(F)$.

The singular locus Z of f is a subvariety of S (analytic or algebraic) and to be more precise $Z = Z_d + Z_0$, where Z_d is a codimension one subvariety of S (a divisor) and Z_0 is supported in codimension 2 (that is, in a set of isolated points). Let $D = f^{-1}(B_0)$ be of singular fibres of f. As in [BPVdV84], in the framework of fibred surfaces we will use the following.

Definition 1.6.1. A projective fibred surface $f : S \to B$ is *semistable* if is a semistable fibration as in Definition 1.1.5 and satisfies the extra condition:

(•) S is *relatively minimal*, that is the fibres do not contain any $(-1)-$curve (i.e. a smooth connected rational curve with self-intersection -1).

We fix the relation between the sheaves $\omega_{S/B}$ and $\Omega^1_{S/B}$ (see e.g. [GA13] or [GA16]).

Dualizing the Sequence 1.4, we obtain the exact sequence

$$0 \longrightarrow T_{S/B} \longrightarrow T_S \longrightarrow f^*T_B \longrightarrow \mathcal{N} \longrightarrow 0, \tag{1.56}$$

where

$$T_{S/B} = \mathrm{Hom}_{\mathcal{O}_S}(\Omega^1_{S/B}, \mathcal{O}_S), \quad \mathcal{N} = \mathrm{Ext}^1_{\mathcal{O}_S}(\Omega^1_{S/B}, \mathcal{O}_S).$$

Then $\mathcal{N}$ is the *normal sheaf of* f and it is supported on the set of singular points of the fibres of f; $T_{S/B}$ is the *relative tangent sheaf*, it is dual to $\Omega^1_{S/B}$ and it is an invertible sheaf since it is the second syzygy of the $\mathcal{O}_S$-module $\mathcal{N}$. Moreover, its inverse sheaf depends on the non-reduced fibres of f as computed in the following formula due to Serrano (see [Ser92]).

Lemma 1.6.2 (**[Ser92, Lemma1.1]**). *Let $\{\nu_i E_i\}$ be the set of all components of the singular fibres E_i of f of multeplicity ν_i. Then*

$$T^{\vee}_{S/B} \simeq \omega_{S/B}\left(-\sum_i (\nu_i - 1)E_i\right). \tag{1.57}$$

In particular when all the fibres of f are reduced, then $T^{\vee}_{S/B} \simeq \omega_{S/B}$.

We re-write as follows the result [GA13, Lemma 3.2.6].

Proposition 1.6.3. *Let $f : S \to B$ be a fibration of a surface S over a smooth curve B. Then the torsion of $\Omega^1_{S/B}$ is supported on non-reduced components of the fibres. In other words, $\Omega^1_{S/B}$ is locally free when the locus Z of the singularities of f is supported on isolated points.*

Proposition 1.6.4. *Let $f : S \to B$ be a fibration over a smooth projective curve B. Then $f_*\omega_{S/B}$ is a locally free sheaf of $\mathcal{O}_B$-modules (namely, a vector bundle) of* rank $f_*\omega_{S/B} = g(F)$. *Moreover, $f_*\Omega^1_{S/B}$ is a sheaf of the same rank and there is an injection of sheaves*

$$(f_*\Omega^1_{S/B})^{\vee\vee} \overset{\nu'}{\hookrightarrow} f_*\omega_{S/B} \tag{1.58}$$

defined by the exact sequence

$$0 \longrightarrow f^*\omega_B(Z_d)_{|Z_d} \longrightarrow \Omega^1_{S/B} \xrightarrow{\nu} \omega_{S/B} \longrightarrow \omega_{S/B}{}_{|Z_0} \longrightarrow 0, \tag{1.59}$$

where $Z = Z_d + Z_0$ is the singular locus of f, with Z_d a divisor and Z_0 supported on isolated points, and $(f_*\Omega^1_{S/B})^{\vee\vee}$ is the double dual sheaf which is locally free. In particular, when all the fibres of f are reduced the morphism ν is injective and we get

$$(f_*\Omega^1_{S/B}) \overset{\nu'}{\hookrightarrow} \omega_{S/B} \tag{1.60}$$

In other words, the sheaves $f_*\omega_{S/B}$ and $f_*\Omega^1_{S/B}$ are isomorphic vector bundles over B^0, that is outside the singular locus B_0 and the relation between them depends on the subvariety Z of the singular fibres, where the rank of $\Omega^1_{S/B}$ can jump.

1.6.1 The adjoint images and (or) Massey-prodcuts on fibred surfaces

We recall the construction of the *adjoint images* and (or) *Massey-products.* Details in the topic are given e.g. in [CP95], [GA16], [PZ03], [NPZ04], [Riz08].

Let $f : S \to B$ be a fibration of a smooth surface S over a smooth curve B. Let F be the general fibre of f and assume $g(F) \geq 2$. As in 1.4, the fibration f induces on the general fibre F a first order deformation, namely an extension

$$\xi : \qquad 0 \longrightarrow \mathcal{O}_F \otimes T^{\vee}_{\Delta_\epsilon,0} \longrightarrow \Omega^1_{S|F} \longrightarrow \omega_F \longrightarrow 0 \tag{1.61}$$

corresponding to the Kodaira-Spencer class $\xi \in \mathrm{Ext}^1_{\mathcal{O}_F}(\omega_F, \mathcal{O}_F \otimes T^{\vee}_{\Delta_\epsilon,0}) \simeq H^1(T_F) \otimes T^{\vee}_{\Delta_\epsilon,0}$ which is defined by restriction to F of the exact sequence

$$0 \longrightarrow f^*\Omega^1_B \longrightarrow \Omega^1_B \longrightarrow \Omega^1_{S/B} \longrightarrow 0.$$

In this case, the sheaf Ω^1_F of holomorphic 1−forms on the fibres equals the canonical sheaf ω_F, as the fibres of f are curves. Up to fix a generator σ of $T^{\vee}_{\Delta_\epsilon,0}$, we get an induced isomorphism $\sigma : \mathcal{O}_F \otimes T^{\vee}_{\Delta_\epsilon,0} \simeq \mathcal{O}_F$, which we will call σ itself with a little abuse of notation, and we can look at $\xi \in H^1(T_F) \otimes T^{\vee}_{\Delta_\epsilon,0}$ as the Kodaira-Spencer class $\xi \in H^1(T_F)$ defining the short exact sequence

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \Omega^1_{S|F} \longrightarrow \omega_F \longrightarrow 0. \tag{1.62}$$

The connecting homomorphism δ on the associated long exact sequence in cohomology

$$0 \longrightarrow H^0(\mathcal{O}_F) \longrightarrow H^0(\Omega^1_{S|F}) \longrightarrow H^0(\omega_F) \xrightarrow{\delta=\cup\xi} H^1(\mathcal{O}_F), \tag{1.63}$$

is given by the cut product $\cup\xi : H^0(\omega_F) \to H^1(\mathcal{O}_F)$ with $\xi \in H^1(T_F)$. Let $K_\xi = \ker(\cup\xi)$ and assume $\dim K_\xi \geq 2$. We consider the map

$$\wedge_\xi : \ \bigwedge^2 H^0(\Omega^1_{S|F}) \longrightarrow H^0(\bigwedge^2 \Omega^1_{S|F}) \simeq H^0(\omega_F) \tag{1.64}$$

defined by the composition of the wedge product with the isomorphism $\bigwedge^2 H^0(\Omega^1_{S|F}) \simeq \omega_F$ induced by sequence 1.62. On any pair (s_1, s_2) of linearly independent elements of K_ξ, we can choose a pair of liftings $(\tilde{s_1}, \tilde{s_2})$ in $H^0(\Omega^1_{S|F})$ and take the image $\tilde{s_1} \wedge_\xi \tilde{s_2} \in H^0(\omega_F)$ of the map $\wedge_\xi$, where we have set $\wedge_\xi(\tilde{s_1} \wedge \tilde{s_2}) = \tilde{s_1} \wedge_\xi \tilde{s_2}$. Such image depends on the choice of both liftings but it turns out to be well defined modulo the $\mathbb{C}$-vector space $< s_1, s_2 >_{\mathbb{C}}$ generated by (s_1, s_2), since each lifting must differ from the previous one for an element in $H^0(\mathcal{O}_F) \simeq \mathbb{C}$ according to 1.63.

Definition 1.6.5. Let (s_1, s_2) be a pair of linearly independent elements of K_ξ and $(\tilde{s}_1, \tilde{s}_2)$ be a pair of liftings in $H^0(\Omega^1_{S|F})$. The image

$$\tilde{s}_1 \wedge_\xi \tilde{s}_2 \in H^0(\omega_F)$$

is called *adjoint image* of the pair of liftings $(\tilde{s}_1, \tilde{s}_2)$. The equivalent class

$$\mathfrak{m}_\xi(s_1, s_2) := [(\tilde{s}_1 \wedge_\xi \tilde{s}_2)] \in H^0(\omega_F)/ < s_1, s_2 >_{\mathbb{C}} \tag{1.65}$$

is called *Massey product of* (s_1, s_2) along ξ.

A natural definition of vanishing in the setting of Massey-products is the following.

Definition 1.6.6. We say that a pair $(s_1, s_2) \in K_\xi$ is *Massey-trivial* or equivalently has *vanishing Massey-products* if the equivalent class $\mathfrak{m}_\xi(s_1, s_2)$ vanishes. In other words, if the adjoint image $\tilde{s}_1 \wedge_\xi \tilde{s}_2$ lies in $< s_1, s_2 >_{\mathbb{C}}$, for a (and thus each) pair $(\tilde{s}_1, \tilde{s}_2)$ of liftings of (s_1, s_2) in $H^0(\Omega^1_{S|F})$.

As observed in 1.4, $\ker \cup \xi_b = \operatorname{Im}\{H^0(\Omega^1_{X|X_b}) \to H^0(\omega_{X_b})\}$ is the space of holomorphic $1-$forms on X_b which are liftable to infinitesimal deformations on X. Massey products study the relation on pairs of forms and their vanishing adds conditions on the liftings, as they must lie in a very small subspace of $H^0(\omega_F)$. In the following subsections we will see how this extra condition applies to the study of liftings on forms on the ambient space and also to supports on deformations.

We now put Massey-products in families. Consider the push forward sequence

$$0 \longrightarrow f_* f^* \omega_B \simeq \omega_B \longrightarrow f_*\Omega^1_S \longrightarrow f_*\Omega^1_{S/B} \xrightarrow{\partial} (R^1 f_* \mathcal{O}_S) \otimes \omega_B \tag{1.66}$$

which is an exact sequence given by 1.62. Let $\mathcal{K}_\partial = \ker \partial$ be the subsheaf of $f_*\Omega^1_{S/B}$ defined as the kernel of the morphism ∂. Assume the locus Z of the singularities of f is supported on isolated points. Then $f_*\Omega^1_{S/B}$ is locally free (see 1.6.3) and therefore the same holds on the subsheaf $\mathcal{K}_\partial$ (a subsheaf of a torsion free sheaf is torsion free and thus locally free if defined over a curve).

The connecting morphism $\partial : f_*\Omega^1_{S/B} \to (R^1 f_* \mathcal{O}_S) \otimes \omega_B$ at a regular value $b \in B$ is the cup product with the Kodaira spencer class $\xi_b \in H^1(F_b, T_{F_b}) \otimes T^\vee_{B,b}$ induced by f over $F_b \hookrightarrow S$,

$$\cup \, \xi_b : H^0(F_b, \omega_{F_b}) \to H^1(F_b, \mathcal{O}_{F_b}) \otimes T^\vee_{B,b} \tag{1.67}$$

and therefore the fibre at a regular value $b \in B$ of $\mathcal{K}_\partial$ is simply the kernel $K_{\xi_b} = \ker \cup \xi_b$ (see details in [GA16]). Let (s_1, s_2) be a pair of sections of $\mathcal{K}_\partial$ over an open subset A of B^0. We can punctually repeat the construction of Massey products on $(s_1(b), s_2(b))$ in $\mathcal{K}_{\xi_b}$ using Definition 1.6.5 of Massey-products on the fibre of f over b. Taking care of the choice of representatives, we get a section

$$\mathfrak{m}_\sigma(s_1, s_2) \in \Gamma(A, f_*\Omega^1_{S/B}), \tag{1.68}$$

which is well defined modulo the $\mathcal{O}_B(A)-$submodule $< s_1, s_2 >_{\mathcal{O}_B(A)}$ of $\Gamma(A, f_*\Omega^1_{S/B})$ generated by s_1, s_2. Observe that there is no evident reason because the constructed section must lie in $\mathcal{K}_\partial$. Indeed, there are cases where this does not happen (see e.g. [Pir92])

Definition 1.6.7. The section $\mathfrak{m}_\sigma(s_1, s_2) \in \Gamma(A, f_*\Omega^1_{S/B})$ defined modulo the $\mathcal{O}_B(A)-$submodule $< s_1, s_2 >_{\mathcal{O}_B(A)}$ of $\Gamma(A, f_*\Omega^1_{S/B})$ is called *local family of Massey-products over* A or *Massey-product section* over A.

Remark 1.6.8. Let A be a connected open subset of B. Then it is equivalent that a pair (s_1, s_2) of sections in $\Gamma(A, \mathcal{K}_\partial)$ is Massey-trivial (Definition 1.6.7) and that the restriction of s_1 and s_2 to the general point b is Massey-trivial.

1.6.2 Vanishing Massey Products and Supporting divisors

We focus on supports of infinitesimal deformations of curves and we recall the relation between *supporting divisors* of first-order deformations of curves and the vanishing of Massey-products. Details of the theory in [CP95] [GA16], [Gin08]. [BGAN16].

Let $f : S \to B$ be a fibration of a surface S over a curve B. Let F be a general fibre of f and $\xi \in H^1(F, T_F)$ the Kodaira-Spencer class of the first-order infinitesimal deformation defined by f, which corresponds to the exact sequence 1.61. Let $D = \sum n_i p_i$ be an effective divisor on F of degree $d = \deg D = \sum n_i$ and set $r = r(D) = h^0(F, \mathcal{O}_F(D))$ the dimension of the complete linear series of the line bundle $\mathcal{O}_F(D)$ defined by D. Consider the exact sequence

$$0 \longrightarrow T_F \longrightarrow T_F(D) \longrightarrow T_F(D)_{|D} \longrightarrow 0 \tag{1.69}$$

obtained by twisting the exact sequence

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{O}_F(D) \longrightarrow \mathcal{O}_F(D)_{|D} \longrightarrow 0 \tag{1.70}$$

with the tangent sheaf T_F. Let $\partial_D : H^0(F, T_F(D)_{|D}) \to H^1(F, T_F)$ be the connecting morphism of the long exact sequence

$$0 \longrightarrow H^0(T_F) \longrightarrow H^0(T_F(D)) \longrightarrow H^0(T_F(D)_{|D}) \xrightarrow{\partial_D} H^1(T_F), \tag{1.71}$$

in cohomology. Then the image $\operatorname{Im} \partial_D$ is simply the kernel of $H^1(F, T_F) \to H^1(F, T_F(D))$.

Definition 1.6.9. We say that ξ is *supported on* an effective divisor D if and only if

$$\xi \in \ker\left(H^1\left(F, T_F\right) \longrightarrow H^1\left(F, T_F\left(D\right)\right)\right). \tag{1.72}$$

Furthermore, ξ is *minimally* supported on D if it is not supported on any strictly effective subdivisor $D' < D$.

Equivalently, ξ is supported on D if and only if the subsheaf $\omega_F(-D)$ lifts to $\Omega^1_{S|F}$, that is the inclusion $\omega_F(-D) \hookrightarrow \omega_F$ factors through $\Omega^1_{S|F} \to \omega_F$ and the digram

$$\begin{tikzcd}
\xi_D & 0 \arrow[r] & \mathcal{N}^\vee \arrow[r] \arrow[d, equal] & \mathcal{F}_D \arrow[r] \arrow[d, hook] & \omega_F(-D) \arrow[r] \arrow[d, hook] \arrow[l, bend right] & 0 \\
\xi & 0 \arrow[r] & \mathcal{N}^\vee \arrow[r] & \Omega^1_{S|F} \arrow[r] & \omega_F \arrow[r] & 0,
\end{tikzcd} \tag{1.73}$$

is split.

Remark 1.6.10. The equivalence of the definitions above follows from the fact that the map $H^1(F, T_F) \longrightarrow H^1(F, T_F(D))$ can be identified with the pullback

$$\mathrm{Ext}^1_{\mathcal{O}_F}(\omega_F, \mathcal{O}_F) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_F}(\omega_F(-D), \mathcal{O}_F), \qquad \xi \longmapsto \xi_D$$

This equivalence does not always attach on fibrations over curves, where the above pointwise setting represents only the general case (see [GA16] for details).

We now fix the relation between Massey products and the existence of interesting supporting divisors. Note that supporting divisors alwasys exist. Indeed, for D ample enough, $H^1(C, T_C(D)) = 0$ and such D supports any deformation ξ. This happens e.g. when $\deg D > 2g(C) - 2$. However, these divisors are uninteresting, as they don't contain information on the specific deformation. The existence of "small" supporting divisors is provided by the a construction given by the *adjoint theorem* proved in [CP95] together with a numerical sufficient condition of existence obtained in [GA16].

Theorem 1.6.11 ([CP95] Theorem 1.1.8)**.** *Let $W \subset K_\xi$ be a $2-$dimensional subspace and let D_W be the base locus of the corresponding linear series $|D_W| \subset |\omega_F|$. Then W has vanishing Massey-products if and only if D_W supports the Kodair-Spencer class ξ.*

Proposition 1.6.12 ([GA16] Corollary 3.1)**.** *If $\dim K_\xi > \frac{g(F)+1}{2}$, then there is a two-dimensional subspace $W \subseteq K_\xi \subset H^0(F, \omega_F)$ whose base divisor D has vanishing Massey-products.*

We now explain the interest in "small" supporting divisor, as are those provided by the above theorems.

Definition 1.6.13. The *rank* of ξ is defined as

$$\operatorname{rank} \xi = \operatorname{rank} \partial_\xi = \operatorname{rank}(\cup \xi).$$

The rank of ξ is a numerical invariant and it is related to supporting divisors by the following result proved in [BGAN16].

Definition 1.6.14. Let D be a divisor on a smooth curve B. The *Cliffird index* of D is defined as

$$\mathrm{Cliff}(D) = \deg D - 2r(D).$$

Theorem 1.6.15 (Lemma 2.3 and Thm 2.4 in [BGAN16])**.** *Suppose ξ is supported on D. Then $H^0(F, \omega_F(-D)) \subseteq \ker \partial_\xi$. In particular,*

$$\operatorname{rank} \xi \leq \deg D - r(D). \tag{1.74}$$

If moreover ξ is minimally supported on D, then

$$\operatorname{rank} \xi \geq \deg D - 2r(D) = \mathrm{Cliff}(D). \tag{1.75}$$

As a consequence, divisors "ample enough" are uninteresting in order to apply Theorem 1.6.15, since the inequalities provided by the theorem are trivial.

We introduce supporting divisors on fibrations, restricting to the simplest case of smooth fibrations over a constractible curve, which is enough to deal with local families with no singular fibres. Let $f : S \to A$ be a smooth fibration of a complex surface S over an open contractible set $A \subset \mathbb{C}$ (e.g. a complex disk, which we can think as an open cohordinate subset of a smooth curve B, base of a fibration whose f is the restriction). Let F_b be the fibre over $b \in A$ amd $\xi_b \in H^1(F_b, T_{F_b})$ the Kodaira-Spencer class of the first-order deformation of F_b induced by f (well-defined up to non-zero scalar multiplication).

Definition 1.6.16. We say that an effective divisor $\mathcal{D} \subset S$ supports f if the restrictions $D_b = \mathcal{D}_{|F_b}$ support ξ_b for general $b \in B$.

Remark 1.6.17. Note that Definition 1.6.16 considers general fibres, hence we can always assume that a divisor $\mathcal{D}$ supporting a family f contains no fibre.

1.6.3 The Griffiths' infinitesimal invariant of the canonical normal function and Massey-products

We recall the relation between the Griffiths infinitesimal invariant of the canonical normal function with Massey-products, which is fixed by a formula provided in [CP95].

Let $f : S \to B$ be a smooh fibration of a complex surface S over a complex curve B. Let F be the general fibre of f. The Jacobian $J(F)$ of F

$$J(F) = H^1(F, \mathbb{C})/(H^1(F, \mathbb{Z}) + H^0(\omega_F)) \simeq H^0(\omega_F)^*/H_1(F, \mathbb{Z}) \tag{1.76}$$

is a principally polarized abelian variety of dimension $g = g(F)$. Consider the weight-$(2g-3)$ geometric HS$(H_{\mathbb{Z}} = H^{2g-3}(J(F), \mathbb{Z}), F^pH^{2g-3}(J(F), \mathbb{C}))$ of $J(F)$, where

$$H^{p,q} = \bigwedge^p H^{1,0} \otimes \bigwedge H^{0,1}, \quad H^{1,0} \simeq H^0(\omega_F). \tag{1.77}$$

The $(g-1)-$Griffiths intermediate Jacobian of $J(F)$ is defined by the weight-$(2g-3)$ geometric HS of $J(F)$ as

$$J^{g-1}(J(F)) = H^{2g-3}(J(F), \mathbb{C})/(F^{g-1}H^{2g-3}(J(F), \mathbb{C}) \oplus H_{\mathbb{Z}}^{2g-3}) \simeq F^2H^3(J(F), \mathbb{C})^*/H_3(J(C), \mathbb{Z}), \tag{1.78}$$

where the isomorphism is given by the Poincaré duality. Let $[\Theta] \in H^2(J(F), \mathbb{Z})$ be the principal polarization on $J(F)$ given by the Θ-divisor. The associated Lefschetz operator induces a decomposition

$$H^3(J(F), \mathbb{C}) = P^3(J(F), \mathbb{C}) \oplus [\Theta] \cdot H^1(J(C), \mathbb{C}), \tag{1.79}$$

where $P^3(J(F), \mathbb{C})$ is the primitive cohomology. This defines the intermediate primitive cohomology

$$P(F) = F^2P^3(J(F), \mathbb{C})^*/H_3(J(C), \mathbb{Z})_{prim}, \tag{1.80}$$

where $H_3(J(F), \mathbb{Z})_{prim}$ is the image of $H_3(J(F), \mathbb{Z})$ in $F^2P^3(J(F), \mathbb{C})^*$.

Let $Z^{g-1}(J(F))_{\mathrm{hom}}$ be the group of one dimensional algebraic cycles homologically equivalent to zero in $J^{g-1}(F)$. The Abel Jacobi map

$$\phi\,:\,Z^{g-1}(J(F))_{\mathrm{hom}} \to J^{g-1}(F) \tag{1.81}$$

is the map given by integration of cycles, depending on a base point $b \in B$. The composition of the map above with the projection $J^{g-1}(F) \to P(F)$ is independent on the choice of the base point.

The *Ceresa cycle* of the general fibre F is defined as the one cycle $[F - F^-] \in Z^{g-1}(J(F))_{\mathrm{hom}}$ given by the image of $F - F^-$ in $J(F)$ via the Abel Jacobi map.

We consider the Jacobian fibration $\mathcal{J}(f) : \mathcal{J} \to B^0$ induced by f, which has fibre over $b \in B^0$ the Jacobian $J(F_b)$ of the fibre F_b of f. This is a fibration of $g-$dimensional polarized abelian varieties over the smooth curve B^0 and has a geometric variation of the Hodge structure of weight $2g-3$, which defines a fibration $\mathcal{J}^{g-1}(f) : \mathcal{J}^{g-1} \to B^0$ with fibres the intermediate Jacobians $J^{g-1}(F)$. The construction is compatible with the Lefschetz decomposition and defines a fibration $\mathcal{P}(f) : \mathcal{P} \to B^0$ with fibres the primitive intermediate Jacobian as before.

The *canonical normal function* is the section $\nu : B^0 \to \mathcal{P}$ associating to each $b \in B^0$ the image of the Ceresa cycle $[F_b - F_b^-] \in Z^{g-1}(J(F_b))_{\mathrm{hom}}$ via the higher Abel-Jacobi map.

The *Griffiths infinitesimal invariant* $\delta(\nu)$ (see e.g. [CP95] for the explicit definition) induces pointwise over a point $b \in B^0$ a linear map

$$\ker(\gamma) \to \mathbb{C},$$

where $\gamma : T_{B,b} \otimes P^{2,1}J(F_b) \to P^{1,2}J(F_b)$ is naturally defined by the IVHS on the primitive cohomological groups $P^{1,2}J(F_b)$ and $P^{1,2}J(F_b)$ in $H^3(J(F_b), \mathbb{C})$. The formula depends on the description above, on the polarization $Q(-,-) = \frac{i}{2}\int_{F_b} - \wedge -$ of the HS $(H_{\mathbb{Z}} = H^1(F_b, \mathbb{Z}), H^{1,0} = H^0(\omega_{F_b}))$, given by the intersection form, and on the Kodaira-Spencer class $\xi_b \in H^1(T_{F_b})$ of the fibre F_b.

Lemma 1.6.18. *([CP95]) Let $\omega_1, \omega_2, \sigma \in H^0(\omega_{F_b})$ be such that $\xi_b \cdot \omega_1 = \xi_b \cdot \omega_2 = 0$ and $Q(\omega_1, \bar{\sigma}) = Q(\omega_2, \bar{\sigma}) = 0$. Then $\omega_1 \wedge \omega_2 \wedge \bar{\sigma}$ lies, up to a natural isomorphism, in $\ker \gamma$ and we have*

$$\delta(\nu)(\xi_b \otimes \omega_1 \wedge \omega_2 \wedge \bar{\sigma}) = -2Q(\mathfrak{m}_{\xi_b}(\omega_1, \omega_2), \bar{\sigma}). \tag{1.82}$$

Chapter 2

Fujita decompositions on fibred surfaces

In this chapter we present the main results of the thesis, concerning the unitary flat summand involved in the second Fujita decomposition of a fibred surface.

2.1 Introduction to Fujita decompositions and relation with Hodge theory

We introduce the *first and second Fujita decompositions* and their relation with the geometric variation of the Hodge structure. We will present the topic in the more general setting of fibrations over curves with fibres of arbitrary dimension even if the results are specific to the case of fibred surfaces. We follow essentially [CD].

Let $f : X \to B$ be a fibration over a smooth projective curve B of a compact Kaehler manifold X of dimension $m = n + 1 \geq 2$. Let $f_*\omega_{X/B} = f_*(\omega_X \otimes f^*\omega_B^\vee)$ be the direct image of the relative dualizing sheaf, which is a vector bundle of rank $p_g(X_g)$, where $p_g(X_g)$ is the geometric genus of the general fibre X_g of f (see chapter 1, or directly [Fuj78a]). In [Fuj78a] and [Fuj78b] Fujita studied some properties of $f_*\omega_{X/B}$ and provided two splittings known as *First and Second Fujita decompositions*, given by unitary flat vector bundles. The first one, completely proved in [Fuj78a], while the second one, sketched in [Fuj78b] and completely proved later in [CD14] (see also [CD] and [CD16]).

Theorem 2.1.1 (first and second Fujita decompositions)**.** *Let $f : X \to B$ be a fibration over a smooth projective curve B. Then there exist two splitings on $f_*\omega_{X/B}$ given respectively by the unitary flat vector bundles $\mathcal{O}_B^{\oplus h}$ and $\mathcal{U}$ as a direct sum of vector bundles*

$$f_*\omega_{X/B} = \mathcal{O}_B^{\oplus h} \oplus \mathcal{E} = \mathcal{U} \oplus \mathcal{A}, \tag{2.1}$$

where $\mathcal{O}_B^{\oplus h}$ is the trivial bundle of rank $h = h^1(B, f_*\omega_X)$, $\mathcal{E}$ is nef and such that $h^1(B, \omega_B(\mathcal{E})) = 0$ and $\mathcal{A}$ is ample. These are called respectively first and second Fujita decompositions *of $f_*\omega_{X/B}$ (or shortly of f).*

Remark 2.1.2. There is a natural injection $\mathcal{O}_B^{\oplus h} \hookrightarrow \mathcal{U}$, which is compatible with the unitary flat structure and gives a decomposition $\mathcal{U} = \mathcal{O}_B^{\oplus h} \oplus \check{\mathcal{U}}$ of unitary flat bundles. Starting from Fujita decompositions, we get another decomposition

$$f_*\omega_{X/B} = \mathcal{O}_B^{\oplus q_f} \oplus \check{\mathcal{U}} \oplus \mathcal{A}, \tag{2.2}$$

with $h^1(\omega_B(\check{\mathcal{U}})) = 0$. We note that the condition $h^1(B, \omega_B(\check{\mathcal{U}})) = 0$ says that the unitary flat bundle $\check{U}$ has no glogal sections, i.e. $h^0(B, \check{\mathcal{U}}) = 0$ (indeed, $h^0(B, \check{\mathcal{U}}) \simeq h^0(B, \check{\mathcal{U}}^\vee) = h^1(B, \omega_B(\check{\mathcal{U}}))$).

Remark 2.1.3. When $f : S \to B$ is a fibred surface, then $h = q_f$, where $q_f = h^0(S, \Omega^1_S) - h^0(B, \omega_B)$ is the relative irregularity of f (see for instance [GA13, Corollary 3.2.10]).

We focus on the unitary flat bundle $\mathcal{U}$ of the second Fujita decomposition of f. As a unitary flat bundle (Proposition 1.3.2, together with compatibility with the unitary structure), it admits

(•) an underlying unitary local system $\mathbb{U}$ of stalk U such that $\mathcal{U} = \mathbb{U} \otimes \mathcal{O}_B$, where U is a vector space isomorphic to a fibre of $\mathcal{U}$ (modulo isomorphisms of sheaves);

(••) a unitary monodromy representation $\rho_U : \pi_1(B, b) \to \mathrm{Aut}(U, h)$ (modulo conjugacy classes).

We denote with H_U the kernel of ρ_U and with G_U the quotient $\pi_1(B, b)/H_U$. We recall that G_U is naturally isomorphic to the monodromy group $\mathrm{Im}\,\rho_U$ of $\mathcal{U}$ and we identify them. Moreover, one has the underlying compatible decompositions as the direct sums $\mathbb{U} = \check{\mathbb{U}} \oplus \mathbb{C}^{\oplus h}$ and $\rho_U = \rho_{\check{U}} \oplus \rho_{\mathbb{C}^{\oplus h}}$ of local systems and monodromy representations, respectively.

We now explain the relation with the geometric variation of the Hodge structure (shortly, VHS) of f, following essentially [CD].

Let$f : X \to B$ be a fibration of a Kaehler manifold X over a smooth projective curve B. Consider the restriction $f^0 : X^0 = f^{-1}(B^0) \to B^0$ of f to the locus of regular values B^0. This is smooth and defines a geometric VHS of weight n on the local system $\mathbb{H} = R^n f^0_* \mathbb{Z}$ over B^0. Let $\mathcal{F}^n\mathcal{H} \hookrightarrow \mathcal{H} = \mathbb{H} \otimes_{\mathbb{Z}} \mathcal{O}_{B^0}$ be the Hodge filtration, which is given exactly by the Hodge bundle $\mathcal{H}^{n,0}$. Under the natural isomorphism

$$f_*\omega_{X/B_{|B^0}} \simeq \mathcal{F}^n\mathcal{H}, \tag{2.3}$$

the restriction $f_*\omega_{X/B_{|B^0}}$ is a holomorphic sub-bundle of $\mathcal{H}$. We consider the unitary flat bundle $\mathcal{U}$ provided by the second Fujita decomposition of f over B. The restriction $\mathcal{U}_{|B^0}$ of $\mathcal{U}$ over the locus B^0 is also a holomorphic sub-bundle $\mathcal{U}_{|B^0} \subset \mathcal{H}^0$.

Proposition 2.1.4. *The holomorphic sub-bundle $\mathcal{U}_{|B^0} \subset \mathcal{H}^0$ of $\mathcal{U}$ is unitary flat with respect to the flat structure on $\mathcal{H}$ given by the Gauss-Manin connection ∇^0 and the metric structure is given by the intersection form on $f_*\omega_{X/B_{|B_0}}$. Moreover, the injection $\mathcal{U}_{|B^0} \hookrightarrow f_*\omega_{X/B_{|B_0}}$ gives the splitting of the second Fujita decomposition on B^0.*

Sketch of the proof (see [CD, Proof of Theorem 3.3] for details). Let us first assume that $f : S \to B$ is a smooth fibration over B. The geometric VHS over the local system $\mathbb{H} = R^n f_* \mathbb{Z}$ is polarized by the intersection form on the fibres, which induces an hermitian structure on the flat bundle $\mathcal{H} = R^n f_* \mathbb{C} \otimes \mathcal{O}_B$ compatible with the holomorphic flat connection (see 1.5). We consider the Hodge bundle $\mathcal{H}^{n,0}$ and the anti-holomorphic bundle $\overline{\mathcal{H}^{n,0}}$ given by conjugation. The direct sum $\mathcal{H}^{n,0} \oplus \overline{\mathcal{H}^{n,0}}$ defines a sub-bundle of $\mathcal{H}$, even if not holomorphic subbundle. This allows to identify $\overline{\mathcal{H}^{n,0}}$ with the dual bundle $\mathcal{H}^{n,0^\vee} \simeq R^n f^0_* \mathcal{O}_{B^0}$, which is semi-negative (see [Gri70]) and consequently to conclude that $\mathcal{H}^{n,0}$ is nef (see [Fuj78a, Theorem 2.7] for a different proof).

Assume that $\mathcal{H}^{n,0}$ is not ample. As $\mathcal{H}^{n,0}$ is nef , by Proposition 1.2.16 (see also [CD, Proposition 2.3]) the bundle must have a degree-zero quotient $\mathcal{Q}$. Then its dual $\mathcal{Q}^\vee$ is a degree-zero sub-bundle of $\mathcal{H}^{n,0^\vee}$, which is also semi-negative (by the curvature formula for sub-bundles) and thus flat. Then $\mathcal{Q}$ is flat, injects in $\mathcal{H}^{n,0}$ and is unitary since the intersection form is strictly positive on $\mathcal{H}^{n,0}$. The above argument applied to the bigger quotient provides $\mathcal{U}$ together with the splitting $\mathcal{U} \hookrightarrow f_*\omega_{S/B}$. This concludes the argument in the case of smooth fibrations.

Assume now that $B^0 \subset B$ is a proper quasi-projective subspace. Then a similar argument as before holds (see [CD]) and provides the description for $\mathcal{U}_{|B^0} \hookrightarrow f_*\omega_{S/B_{|B^0}}$. More precisely, $f_*\omega_{S/B}$ is nef and we can assume that it is not ample (otherwise there is nothing to prove). As above we get a degree-zero quotient $\mathcal{U}$ of maximal rank. The restriction $\mathcal{U}_{|B^0}$ over the locus B^0 is the unitary flat bundle as defined in case of smooth fibrations.

The last point is to understand whenever $\mathcal{U}_{|B^0}$ extends unique to a unitary flat subbundle $\widehat{\mathcal{U}_{|B^0}}$ of $f_*\omega_{X/B}$. The answer depends on the behaviour of local monodromies around the singularities of f, which must be trivial. In other words, this is a problem of local invariant cycles (subsection 1.5.1). Keeping together results on the growth of the hermitian structure around the singularities and on unipotency of local monodromies, in [CD] the authors proved that the extension of $\mathcal{U}_{|B^0}$ over B exists unique on semistable fibrations. The argument follows by applying " big " theorems to get a meromorphic extension of the hermitian structure with unitary local monodromies. The local monodromies on semistable fibrations are unipotent (see 1.5.1), so that they are both unitary and unipotent and thus trivial. In other words, we get a local system completely described by the VHS as the sheaf of locally flat sections of $f_*\omega_{X/B_{|B^0}}$ with respect to the Gauss-Manin connection $\nabla_{\mathcal{H}^0}$. This is isomorphic to the degree zero quotient $\mathcal{U}$ of $f_*\omega_{X/B}$ providing the unitary flat bundle, the isomorphism is given by the injection on B^0 since it extends to an injection on B and gives the desired splitting.

Let now assume that the fibration $f : X \to B$ is not semistable. By applying the semistable-reduction theorem (see Theorem 1.1.7 or directly [CD, Semistable reduction]), we get a base change $u : B' \to B$ given by a ramified finite morphism of curves and a resolution of the fiber product

$$\begin{tikzcd} X' := \widetilde{X \times_B B'} \arrow[r, "\varphi"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ B' \arrow[r, "u"] & B \end{tikzcd} \quad , \tag{2.4}$$

producing a semistable fibration $f' : X' \to B'$ from a smooth compact surface X' to a smooth projective curve B', called semistable-reducted fibration of f. Let $\mathcal{U}'$ be the unitary bundle given by the second Fujita decomposition of f'. A description of $\mathcal{U}$ in this case follows by comparing the unitary bundle $\mathcal{U}$ of f with the unitary bundle $\mathcal{U}'$ of its semistable reduction.

Lemma 2.1.5. *There exists a short exact sequence*

$$0 \longrightarrow \mathcal{K}_U \longrightarrow \mathcal{U}' \longrightarrow u^*\mathcal{U} \longrightarrow 0, \tag{2.5}$$

which is split. Moreover, $\mathcal{K}_U$ is unitary flat and the splitting is compatible with the underlying local systems.

Proof. Let $\mathcal{U}'$ be the unitary bundle of the semistable reducted fibration f' of f and $\mathcal{U}$ be the unitary bundle of f. By looking at the short exact sequence (see 1.2.11 or directly [CD, Proposition 2.9])

$$0 \longrightarrow f'_*\omega_{X'/B'} \longrightarrow u^*f_*\omega_{X/B} \longrightarrow \mathcal{G} \longrightarrow 0, \tag{2.6}$$

where $\mathcal{G}$ is a skyscraper sheaf supported on points over the singular fibers of f. Comparing the second Fujita decompositions of f and f' we get

$$0 \longrightarrow \mathcal{A}' \oplus \mathcal{U}' \xrightarrow{i'} u^*\mathcal{A} \oplus u^*\mathcal{U} \longrightarrow \mathcal{G} \longrightarrow 0, \tag{2.7}$$

which induces by projection a morphism $\mathcal{U}' \to u^*\mathcal{U}$. The map from an ample bundle to a unitary flat bundle must be the null map (this is consequence of Proposition 1.2.16) and then using a characterization theorem of unitary flat bundle over curves of genus greater that 2 (see 1.3.10 or directly [NS65]), the morphism above must be surjective of vector bundles. The cases of genus 0 and 1 are trivial. We refer to [CD14], [CD] and [CD16] for details. $\square$

Remark 2.1.6. As a consequence of the proof of the previous theorem, we also have the exact sequence

$$0 \longrightarrow \mathcal{A}' \oplus \mathcal{K}_{\mathrm{U}} \longrightarrow u^*\mathcal{A} \longrightarrow \mathcal{G} \longrightarrow 0, \tag{2.8}$$

which gives information on the behavior under semistable reduction of the ample summand in the decomposition.

2.2 A local lifting property

In this section we study the relation between the unitary flat bundle $\mathcal{U}$ of the second Fujita decomposition of a fibred surface and the sheaf of the closed holomorphic 1-forms on the surface. The main result is Lemma 2.2.5, called the Lifting lemma (or a more refined version Lemma 2.2.8). This turns out to have interesting applications, which we will discuss in this section, but more to be a key point in the proof of some other original results presented in the next sections. The result appeared in [PT17].

2.2.1 Preparatory results

Let $f : S \to B$ be a fibred surface over a smooth curve B. We consider

($\circ$) the sheaf $\mathcal{K}_\partial$ as defined in Section 1.4, whose sections are defined by holomorphic forms on the fibres that glue to holomorphic 1−forms on S;

($\circ\circ$) the sheaf $f_*\Omega^1_{S,d}$, where $\Omega^1_{S,d}$ is the sheaf of closed holomorphic 1−forms on S, whose sections are defined by holomorphic forms on the fibres that glue to closed holomorphic 1−forms on S.

A first relation is clearly given by the injections

$$f_*\Omega^1_{S,d} \subset \mathcal{K}_\partial \subset f_*\Omega^1_{S/B}. \tag{2.9}$$

In what follows, we prove some facts and we define a local system $\mathbb{D}$ related to both of them.

($\bullet$) **A splitting lemma for the sheaf $\mathcal{K}_\partial$.** We recall that $\mathcal{K}_\partial$ is the kernel of the connecting morphism 1.16

$$\partial : f_*\Omega^1_{S/B} \longrightarrow R^1 f_*\mathcal{O}_S \otimes \omega_B \tag{2.10}$$

in the pushforward sequence 1.18. Consider the exact sequence

$$\zeta : \qquad 0 \longrightarrow \omega_B \longrightarrow f_*\Omega^1_S \longrightarrow \mathcal{K}_\partial \longrightarrow 0 \tag{2.11}$$

of locally free sheaves of $\mathcal{O}_B$−modules naturally defined 1.18 and let $\zeta \in \operatorname{Ext}^1_{\mathcal{O}_B}(\mathcal{K}_\partial, \omega_B)$ be the extension class.

Lemma 2.2.1 (Splitting on $\mathcal{K}_\partial$)**.** *Let $f : S \to B$ be a semistable fibration over a smooth projective curve B. Then the short exact sequence 2.11*

$$\zeta : \qquad \begin{tikzcd} 0 \arrow[r] & \omega_B \arrow[r] & f_*\Omega^1_S \arrow[r] & \mathcal{K}_\partial \arrow[l, bend right, "\eta"'] \arrow[r] & 0, \end{tikzcd} \tag{2.12}$$

is split, namely $f_*\Omega^1_S \simeq \omega_B \oplus \mathcal{K}_\partial$.

Proof. Let $\zeta \in \operatorname{Ext}^1(K_\partial, \omega_B)$ be the extension class of the short exact sequence 2.12. Through the chain of isomorphisms

$$\operatorname{Ext}^1(K_\partial, \omega_B) \simeq \operatorname{Ext}^1(\mathcal{O}_B, K_\partial^\vee \otimes \omega_B) \simeq H^1(K_\partial^\vee \otimes \omega_B)$$

we look at ζ as an element in $H^1(K_\partial^\vee \otimes \omega_B)$. By definition, the short exact sequence 2.12 is split if ζ is zero.

Consider the long exact sequence induced in cohomology

$$0 \longrightarrow H^0(\omega_B) \longrightarrow H^0(f_*\Omega^1_S) \longrightarrow H^0(\mathcal{K}_\partial) \xrightarrow{\delta} H^1(\omega_B) \longrightarrow H^1(f_*\Omega^1_S) \longrightarrow \tag{2.13}$$

Consider the map $\delta^\vee : H^1(\omega_B)^\vee \to H^0(\mathcal{K}_\partial)^\vee$ dual to δ. Through the isomorphisms $H^1(\omega_B)^\vee \simeq H^0(\mathcal{O}_B)$ and $H^0(\mathcal{K}_\partial)^\vee \simeq H^1(\omega_B \otimes \mathcal{K}^\vee)$, we can identify the map above with

$$\delta^\vee : H^0(\mathcal{O}_B) \to H^1(\omega_B \otimes \mathcal{K}^\vee) \tag{2.14}$$

which naturally sends $1 \mapsto \zeta$. In what follows, we prove that $\delta^\vee$ is the null map. To do this we consider again the map $\delta : H^0(\mathcal{K}_\partial) \to H^1(\omega_B)$ and we prove that it is the null map, which is equivalent to prove that $\delta^\vee$ is the null map. By the long exact sequence 2.13, this is actually equivalent to prove the map $H^1(\omega_B) \to H^1(f_*\Omega^1_S)$ is an injection.

We consider the map $H^1(B, \omega_B) \to H^1(B, f_*\Omega^1_S)$ induced by the long exact sequence in cohomology and we prove that it is an injection. First, observe that the pullback map $H^1(B, \omega_B) \to H^1(S, \Omega^1_S)$ is an injection, as it sends the class of a point b on B (which corresponds to a Kähler form) to the class of the fibre F in S, which is non zero. Then, also the map $H^1(B, \omega_B) \to H^1(B, f_*\Omega^1_S)$ is non zero, since it must factorize through the Leray spectral sequence (see 1.4)

$$\begin{tikzcd} & & H^1(B,\omega_B) \arrow[dl, hook'] \arrow[d, hook] & \\ 0 \arrow[r] & H^1(B, f_*\Omega^1_S) \arrow[r] & H^1(S,\Omega^1_S) \arrow[r] & H^0(B, R^1 f_*\Omega^1_S). \end{tikzcd} \tag{2.15}$$

□

Remark 2.2.2. The splitting in the lemma above is far away to be unique.

(••) **Relative holomorphic de-Rham: a usefull short exact sequence.**

Let $f : S \to B$ be a fibration over a smooth curve B. We construct a useful short exact sequence, comparing the holomorphic de-Rahm sequences of the surface S and the base B.

Let us consider the holomorphic de-Rham sequence on S

$$0 \longrightarrow \mathbb{C}_S \longrightarrow \mathcal{O}_S \xrightarrow{\mathrm{d}} \Omega^1_{S,d} \longrightarrow 0, \tag{2.16}$$

where $\Omega^1_{S,d}$ denotes the sheaf of d-closed holomorphic 1-forms on S. Then, we get the exact sequence

$$0 \longrightarrow f_*\mathbb{C}_S \longrightarrow f_*\mathcal{O}_S \xrightarrow{\mathrm{d}} f_*\Omega^1_{S,d} \longrightarrow R^1 f_*\mathbb{C}_S \longrightarrow R^1 f_*\mathcal{O}_S. \tag{2.17}$$

We compare it with the holomorphic de-Rham sequence on B

$$0 \longrightarrow \mathbb{C}_B \longrightarrow \mathcal{O}_B \xrightarrow{\mathrm{d}} \omega_B \longrightarrow 0, \tag{2.18}$$

using the natural morphisms $\mathbb{C}_B \to f_*\mathbb{C}_S$ and $\mathcal{O}_B \to f_*\mathcal{O}_S$ induced by f, which are both isomorphisms in this case. We obtain a diagram

$$\begin{tikzcd} 0 \arrow[r] & f_*\mathbb{C}_S \arrow[r] & f_*\mathcal{O}_S \arrow[r, "\mathrm{d}"] & f_*\Omega^1_{S,d} \arrow[r] & R^1 f_*\mathbb{C}_S \arrow[r] & R^1 f_*\mathcal{O}_S \\ 0 \arrow[r] & \mathbb{C}_B \arrow[r] \arrow[u, "\parallel"] & \mathcal{O}_B \arrow[r, "\mathrm{d}"] \arrow[u, "\parallel"] & \omega_B \arrow[rr] \arrow[u, hook] & & 0, \end{tikzcd} \tag{2.19}$$

which induces an injective morhpism $\omega_B \hookrightarrow f_*\Omega^1_{S,d}$ together with a short exact sequence

$$0 \longrightarrow \omega_B \longrightarrow f_*\Omega^1_{S,d} \longrightarrow \widehat{D} \longrightarrow 0, \tag{2.20}$$

where $\widehat{D}$ denotes the image of the morphism $f_*\Omega^1_{S,d} \to R^1 f_*\mathbb{C}_S$. We will call the sequence above *Relative holomorphic de-Rahm sequence.*

We analyse some properties of the subsheaf $\widehat{D} \hookrightarrow R^1 f_*\mathbb{C}$, which turns out to be (possibly strictly) connected with $\mathbb{U}$.

Lemma 2.2.3. *There is an injection of sheaves $i_{\widehat{D}} : \widehat{D} \hookrightarrow \mathcal{K}_\partial$.*

Proof. Consider the natural injection of sheaves $\Omega^1_{S,d} \overset{i_d}{\hookrightarrow} \Omega^1_S$ and compare the Relative holomorphic de-Rahm sequence (2.20)

$$0 \longrightarrow \omega_B \longrightarrow f_*\Omega^1_{S,d} \longrightarrow \widehat{D} \longrightarrow 0$$

with sequence (1.66)

$$0 \longrightarrow f_*f^*\omega_B \simeq \omega_B \longrightarrow f_*\Omega^1_S \longrightarrow f_*\Omega^1_{S/B} \xrightarrow{\partial} (R^1 f_*\mathcal{O}_S) \otimes \omega_B,$$

using the induced morphism on the direct image sheaves. We get a diagram

$$\begin{tikzcd}
0 \arrow[r] & \omega_B \arrow[r] \arrow[d, "\parallel"] & f_*\Omega^1_{S,d} \arrow[r] \arrow[d, hook, "i_{\mathrm{d}}"] & \widehat{D} \arrow[r] \arrow[d, hook, "i_{\widehat{D}}"] & 0 \arrow[d] \\
0 \arrow[r] & f_*f^*\omega_B \arrow[r] & f_*\Omega^1_S \arrow[r] & f_*\Omega^1_{S/B} \arrow[r, "\partial"] & R^1 f_*\mathcal{O}_S \otimes \omega_B,
\end{tikzcd} \tag{2.21}$$

defining the injection $i_{\widehat{D}} : \widehat{D} \hookrightarrow \mathcal{K}_\partial$ as claimed.

□

Lemma 2.2.4. *Let $f : S \to B$ be a semistable fibration over a smooth projective curve B. Then $\widehat{D}$ is a local system over B and in this case we denote it with $\mathbb{D}$. Moreover, the stalk D of $\mathbb{D}$ is isomorphic over the general fiber F to the maximal subspace of $H^{1,0}(F) \subset H^1(F, \mathbb{C})$ that defines a local system over B.*

Proof. Let B_0 be the branch locus of f and $j : B^0 = B \setminus B_0 \hookrightarrow B$ be the natural injection. Consider the injection $k_{\widehat{D}} : \widehat{D} \hookrightarrow R^1 f_*\mathbb{C}_S$ defined by Diagram 2.19 and the morphism $\alpha : R^1 f_*\mathbb{C} \to j_*j^*R^1 f_*\mathbb{C}$ defined in (1.49), which we recall is given by restriction. By Lemma 1.5.19, α is an isomorphism whenever f satisfies the local invariant cycle property around each branch point (e.g. when f is semistable). In this case, the local system $j^*R^1 f_*\mathbb{C}$ determines completely the sheaf $R^1 f_*\mathbb{C}_S$ even if it is not a local system.

We consider the injective morphism $\alpha_{\widehat{D}} : \widehat{D} \to j_*j^*R^1 f_*\mathbb{C}_S$ given by the composition

$$\begin{tikzcd}
\widehat{D} \arrow[r, "k_{\widehat{D}}"] \arrow[dr, "\alpha_{\widehat{D}}"'] & R^1 f_*\mathbb{C}_S \arrow[d, "\alpha"] \\
& j_*j^*R^1 f_*\mathbb{C}_S.
\end{tikzcd} \tag{2.22}$$

and we identify $\widehat{D}$ with its image under the morphism above.

We recall that the stalk of the local system $j^*R^1f_*\mathbb{C}$ is isomorphic to the first cohomology group $H^1(F,\mathbb{C})$ of the general fibre F of f. Then the subsheaf $j^*\widehat{D} \subset j^*R^1f_*\mathbb{C}$ must be a local subsystem of $j^*R^1f_*\mathbb{C}$ and the stalk over $b \in B^0$ must be isomorphic to a vector subspace $D_b \subset H^{1,0}(F_b)$. Indeed, we note that D_b is obtained by evaluation over $b \in B^0$ of the stalk of the image sheaf provided by the map $f_*\Omega^1_{S,d} \to R^1f_*\mathbb{C}$ and by the exactness it is the kernel of the map $R^1f_*\mathbb{C} \to R^1f_*\mathcal{O}_S$. On the general value $b \in B$ the fibre of the vector bundle $R^1f_*\mathcal{O}_S$ is $H^{0,1}(F_b)$ and the map above defines the projection $H^1(F_b,\mathbb{C}) \to H^{0,1}(F_b)$.

We claim that the direct image $j_*j^*\widehat{D}$ is a local system over B and $j_*j^*\widehat{D} \simeq \widehat{D}$. To prove this we look at the local monodromies around the singular values. We recall that these are unipotent on semistable degenerations (see Theorem 1.5.17). On the other hand the intersection form, which gives a polarization on the weight-1 hodge structure on $H^1(F,\mathbb{Z})$, provides pointwise an hermitian structure on $H^1(F,\mathbb{C})$ that is positive definite over $H^{1,0}(F_b)$. The local system $j^*\mathbb{D}$ is unitary with this metric structure. Consequently, local monodromies are both unipotent and unitary on D_b and thus trivial. In other words, $j^*\widehat{D}$ extends to a local system on B.

We prove that $j^*\mathbb{D}$ defines the maximal local subsystem of $j^*R^1f_*\mathbb{C}$ with stalk D isomorphic to a subspace of $H^{1,0}(F)$ on the general fiber F. Indeed, if $\mathbb{D}'$ is a local system with stalk contained in $H^{1,0}(F)$ on the general fibre F, then the map $\mathbb{D}' \to R^1f_*\mathcal{O}_S$ is zero and therefore $\mathbb{D}' \subset \mathbb{D}$.

□

2.2.2 The lifting Lemma: the local system $\mathbb{U}$ and the sheaf $f_*\Omega^1_{S,d}$

We state and prove two lemmas, the *lifting and splitting lemmas.*

Lemma 2.2.5 (Lifting lemma)**.** *Let $f : S \to B$ be semistable fibration over a smooth projective curve B and $\mathbb{U}$ be the local system underlying the unitary bundle $\mathcal{U}$ in the second Fujita decomposition of f. Then, there is a short exact sequence of sheaves*

$$0 \longrightarrow \omega_B \longrightarrow f_*\Omega^1_{S,d} \longrightarrow \mathbb{U} \longrightarrow 0. \tag{2.23}$$

In particular, the sequence above remains exact

$$0 \longrightarrow H^0(A,\omega_B) \longrightarrow H^0(f^{-1}(A),\Omega^1_{S,d}) \longrightarrow H^0(A,\mathbb{U}) \longrightarrow 0 \tag{2.24}$$

over any proper open subset A of B.

Proof. We consider the Relative holomorphic de-Rahm sequence (sequence 2.20). The proof follows immediately by taking sequence 2.23 equal to 2.20 and noting that by definition the local system $\mathbb{D}$ is naturally isomorphic to $\mathbb{U}$. The last part of the statement follows taking the long exact sequence in cohomology and noting that it remains exact on any proper open subset A of B, that is $H^1(A,\omega_B)$ is zero, since A is Stein. □

Remark 2.2.6. The analogue description of the trivial bundle $\mathcal{O}^{\oplus q_f}$ of rank the relative irregularity $\oplus q_f$ of f in terms of $H^0(S,\Omega^1_S)/H^0(B,\omega_B)$ has been given in [GA13].

Remark 2.2.7. The local liftings of $\mathbb{U}$ provided by the Lifting lemma are all closed holomorphic forms. Indeed, two liftings differ from the pullback of a holomorphic form on the curve B, which is automatically closed.

In [CLZ16] the authors proved a lifting of $\mathcal{U}$ (and thus also on $\mathbb{U}$) to the sheaf Ω^1_S (Corollary 7.2). This suggested us to investigate deeper the Lifting lemma, as the local system $\mathbb{U}$ is quite different from the vector bundle $\mathcal{U}$.

Lemma 2.2.8 (Splitting Lemma). *Let $f : S \to B$ be a semistable fibration over a smooth projective curve B. Then the short exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & \omega_B \arrow[r] & f_*\Omega^1_{S,d} \arrow[r] & \mathbb{U} \arrow[r] \arrow[l, bend right, "\eta'"'] & 0 \end{tikzcd} \tag{2.25}$$

is split.

Proof. Let $\eta \in H^0(\mathcal{K}_\partial^\vee \otimes f_*\Omega^1_S)$ be the section that splits sequence 2.12

$$\begin{tikzcd} \zeta : & 0 \arrow[r] & \omega_B \arrow[r] & f_*\Omega^1_S \arrow[r] & \mathcal{K}_\partial \arrow[r] \arrow[l, bend right, "\eta"'] & 0, \end{tikzcd}$$

as proved in lemma 2.2.1. We prove that η induces also a splitting on sequence 2.25. Consider the diagram (see 2.21)

$$\begin{tikzcd} 0 \arrow[r] & \omega_B \arrow[r] \arrow[d, "\|"] & f_*\Omega^1_{S,d} \arrow[r] \arrow[d, hook, "i_{\mathrm{d}}"] & \mathbb{U} \arrow[r] \arrow[d, hook, "i_{\mathbb{U}}"] & 0 \\ 0 \arrow[r] & \omega_B \arrow[r] & f_*\Omega^1_S \arrow[r] & \mathcal{K}_\partial \arrow[r] & 0, \end{tikzcd} \tag{2.26}$$

Then the proof follows immediately since the kernel of the two sequences is the same. The morphism $\eta' : \mathbb{U} \to f_*\Omega^1_S$ given by composition of η with the injection $i_{\mathbb{U}} : \mathbb{U} \hookrightarrow \mathcal{K}_\partial$ is in fact in the image $i_d : f_*\Omega^1_{S,d} \hookrightarrow f_*\Omega^1_S$ and this gives the desired splitting. □

Remark 2.2.9. The splitting above is long far away to be unique, as for that given by 2.2.1.

2.2.3 Local liftings on $\mathbb{U}$ and the theorem of the fixed part

We relate the unitary flat bundle $\mathcal{U}$ with the theorem of the fixed part (Theorem 1.5.13).

Let $f : S \to B$ be a projective fibration over a smooth projective curve B. As seen in Remark 1.5.10, the space of invariants under the monodromy action are described as

$$H^1(X_b, \mathbb{Q})^{\mathrm{inv}} \simeq H^0(B, R^1 f_*\mathbb{Q})$$

and also

$$H^1(X_b, \mathbb{C})^{\mathrm{inv}} \simeq H^0(B, R^1 f_*\mathbb{C}).$$

By construction, Sequence 2.23 in the Splitting lemma (2.2.8) is provided by the relative holomorphic de-Rham sequence (2.20) and we get an injection $\mathbb{U} \hookrightarrow R^1 f_* \mathbb{C}$ of sheaves, which defines the map

$$H^0(B, \mathbb{U}) \to H^0(B, R^1 f_* \mathbb{C}) = H^0(B, R^1 f_* \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{C}). \tag{2.27}$$

Moreover, we also have

$$H^0(B, \mathbb{U}) = H^0(B, \mathcal{U})^{\text{inv}} = H^0(B^0, \mathcal{U})^{\text{inv}}, \tag{2.28}$$

where first equality is a property of the local systems (see Lemma 1.3.4) while the second requires the assumption of semistability (see 2.1).

As an application of the Splitting lemma (2.2.8), we state an analogous of the theorem of the fixed part (Theorem 1.5.13) specific for the case of $\mathbb{U}$.

Corollary 2.2.10 (Fixed part on $\mathcal{U}$). *Let $f : S \to B$ be a semistable projective fibration of a surface S over the smooth projective curve B (the singular locus $j : B^0 \hookrightarrow B$ is quasi projective, the restriction of f to B^0 is projective). Then the map*

$$H^{1,0}(S) \simeq H^0(B, f_* \Omega^1_{S,d}) \to H^0(B^0, \mathcal{U})^{inv} \tag{2.29}$$

is surjective. Moreover, the map $H^1(S, \mathbb{C}) \to H^1(F, \mathbb{C})$ in the theorem of the fixed part as above fits into the commutative diagram

$$\begin{tikzcd} H^{1,0}(S) \arrow[r] \arrow[d, hook] & H^0(B^0, \mathcal{U})^{inv} \arrow[d, hook] \\ H^1(S, \mathbb{C}) \arrow[r] & H^{1,0}(F) \end{tikzcd} \quad . \tag{2.30}$$

Proof. According to Proposition 2.1.4, $H^0(B^0, \mathcal{U})^{\text{inv}} = H^0(B, \mathbb{U})$. The first assertion is nothing more than the Splitting lemma (2.2.8). The second one follows from the first one, by using equality 2.28 and the fact that by the Splitting lemma the type of the forms must be preserved locally. ☐

Remark 2.2.11. The above corollary does not take care of the monodromy of $\mathbb{U}$, which could be finite or not.

Remark 2.2.12. In the statement above, it is not possible to replace the complex field $\mathbb{C}$ with the rational field $\mathbb{Q}$, as the fibre of $\mathbb{U}$ is a vector subspace of $H^{1,0}(F)$, which is not invariant under the complex conjugation (it is not defined over the real field $\mathbb{R}$).

We consider the local system $\mathbb{U} \oplus \overline{\mathbb{U}}$ undelrying the complex vector bundle $\mathcal{U} \oplus \overline{\mathcal{U}}$, which is by definition invariant under conjugation. In the case of finite monodromy, it is possible to relate the Lifting Lemma (2.2.5) with the theorem of the fixed part over the rational field $\mathbb{Q}$.

Corollary 2.2.13. *Ler $f : S \to B$ be a semistable projective fibration over a smooth projective curve B. Assume that $\mathcal{U}$ has finite monodromy. Then the local system $\mathbb{U} + \overline{\mathbb{U}}$ is defined over the rationals, meaning that $\mathbb{U} \oplus \overline{\mathbb{U}} = \mathbb{U}_{\mathbb{Q}} \otimes \mathbb{C}$ where $\mathbb{U}_{\mathbb{Q}}$ is a local subsystem of $R^1 f^0_* \mathbb{Q}$.*

Proof. As the monodromy is finite, we have a finite étale base change $u : B' \to B$ trivializing the monodromy and after applying it, we get $u^*(\mathbb{U} \oplus \overline{\mathbb{U}}) \simeq H^0(B^0, R^1 f^0_* \mathbb{C})$. This follows from the fact that

$$R^1 f^0_* \mathbb{C} \otimes \mathcal{O}_{B^0} = (f_* \omega_{S/B} \oplus R^1 f_* \mathcal{O}_S)_{|B^0} = (f_* \omega_{S/B} \oplus f_* \omega^\vee_{S/B})_{|B^0}.$$

Then

$$R^1 f^0_* \mathbb{C} = (\mathbb{U} \oplus \overline{\mathbb{U}} \oplus \mathbb{V})_{|B^0}$$

and $H^0(B^0, \mathbb{V}) = 0$. Consequently, $H^0(B^0, R^1 f^0 \mathbb{C}) = H^0(B^0, \mathbb{U} \oplus \overline{\mathbb{U}})$ and by the theorem of the fixed part combined with the Splitting lemma the decomposition is compatible with the Hodge structure. By Corollary 1.5.14, this defines a constant variation of the Hodge structure and thus a Hodge structure on the fibre F together with a monodromy representation defined over $\mathbb{Q}$. Since the fibre does not change after étale base changes we get the proof on the fibre of $\mathbb{U}$, as it is invariant under local monodromies. □

2.2.4 The unitary flat bundle and the kernel of the IVHS

We relate the IVHS and the unitary flat bundle $\mathcal{U}$, using the description provided by Lemma 2.2.5 and 2.2.8. We recall that in Lemma 1.5.21 we saw the relation between the Gauss Manin connection and the kernel of the cup product with the Kodaira-Spencer class of the infinitesimal deformation induced by the fibration f on the general fibre F.

Let $f : S \to B$ be a projective fibration of $g(F)$. It is well known (see for instance [GA13]) the following result.

Proposition 2.2.14. *Let $\xi \in H^1(T_F)$ be the Kodaira-Spencer class of F. Then*

$$\dim \ker \cup\xi \geq q_f \tag{2.31}$$

In terms of the rank of locally free sheves one can state

Proposition 2.2.15. *Let $f : S \to B$ be a semistable fibration over a smooth projective curve B. Then*

$$\operatorname{rank} \mathcal{K}_\partial \geq \operatorname{rank} \mathcal{O}_B^{q_f} \tag{2.32}$$

As an application of the Splitting lemma, we get the following result, which is consistent with Lemma 1.5.21.

Corollary 2.2.16. *Let $f : S \to B$ be a semistable fibration over a smooth projective curve B. Then*

$$\operatorname{rank} \mathcal{K}_\partial \geq \operatorname{rank} \mathbb{U} = \operatorname{rank} \mathcal{U} \tag{2.33}$$

More precisely,

$$\operatorname{rank} \mathcal{K}_\partial \geq q_f + \operatorname{rank} \check{\mathbb{U}}, \tag{2.34}$$

where $\check{\mathbb{U}}$ is the local system underlying the unitary flat sub-bundle $\check{\mathcal{U}}$ of $\mathcal{U}$ of maximal rank such that $H^1(\check{\mathcal{U}}) = 0$.

The second part of the statement is clear according to Remark 2.1.2.

Remark 2.2.17. In terms of the general fibre F of f, the corollary above says that the Kodaira-Spencer class $\xi \in H^1(F, T_F)$ satisfies

$$\dim \cup\xi \geq q_f \oplus \operatorname{rank} \check{\mathbb{U}}. \tag{2.35}$$

2.2.5 Local liftings with vanishing Massey-products and a generalized Castelnuovo de Franchis theorem

In this section we introduce Massey-trivial subspaces $W \subset \Gamma(A, \mathbb{U})$ of flat local sections of the unitary bundle $\mathcal{U}$ of a fibred surface and with general fibre F of geometric genus $g(F) \geq 2$. Then we relate them to the existence (up to base changes) of morphisms from the surface into a smooth compact curve Σ of genus greater than 2. The construction is given by a generalization of the Castelnuovo de Franchis' theorem (see e.g. [Bea96, Proposition X.6]) for fibred surfaces provided in [GST17].

In ($\bullet$) we state the adapted version of the Castelnuovo de-Franchis theorem for fibred surfaces; in ($\bullet\bullet$) we introduce Massey-trivial subspaces on the sheaf $\mathcal{K}_\partial$ and in ($\bullet\bullet\bullet$) we focus on Massey-trivial subspaces on $\mathbb{U}$ and fix the relation with the Castelnuovo de-Franchis theorem.

($\bullet$) **Castelnuovo de-Franchis for fibred surfaces.** The classical theorem of Castelnuovo de-Franchis (see e.g. [Bea96, Proposition X.6]) states that a pair of holomorphic forms on S with wedge zero defines a fibration on S.

In our case S is already fibred, that is there is a fibration $f : S \to B$ over a smooth complex curve B (not necessarly compact). The holomorphic forms given by pullback from the base B satisfy the assumption of the theorem but are uninteresting since they reproduce the same fibration. In other words, the interesting holomorphic forms on a fibred surface S in order to produce other fibrations on S are those that come from the fibres. More precisely, the holomorphic forms on the fibres which lift to closed holomorphic forms on the surface and have wedge zero produce other fibrations on S.

In [GST17], it is proved a tubular version of the classical Castelnuovo-de Franchis theorem.

Theorem 2.2.18 ([GST17])**.** *Let $f : S \to \Delta$ be a family of smooth projective curves over a disk. Let $\omega_1, \dots, \omega_k \in H^0\left(S, \Omega^1_S\right)$ $(k \geq 2)$ be closed holomorphic 1-forms such that $\omega_i \wedge \omega_j = 0$ for every i, j, and whose restrictions to a general fibre F are linearly independent. Then there exist a projective curve Σ and a morphism $\phi : S \to \Sigma$ such that $\omega_i \in \phi^* H^0(\Sigma, \omega_\Sigma)$ for every i (possibly after shrinking Δ).*

Proof. The argument follows the one of Beauville [Bea96, Proposition X.6]. Since all the 1-forms $\omega_1, \dots, \omega_k$ are pointwise proportional, there are meromorphic functions $g_2, \dots, g_k$ on S such that $\omega_i = g_i \omega_1$. Differentiating these equalities and using that $d\omega_i = 0$, we obtain $0 = dg_i \wedge \omega_1$ for $i = 2, \dots, k$. In particular, there is another meromorphic function g_1 such that $\omega_1 = g_1 dg_2$, and also $0 = dg_1 \wedge dg_2$. This means that the meromorphic differentials $dg_2, \dots, dg_k$ are pointwise proportional to ω_1 (wherever they are defined) and also to dg_1, hence $dg_i \wedge dg_j = 0$ for any i, j.

These functions define a meromorphic map $\psi : S \dashrightarrow \mathbb{P}^k$ as

$$\phi(p) = (1 : g_1(p) : g_2(p) : \dots : g_k(p)) \,.$$

Let $\epsilon : \hat{S} \to S$ be a resolution of the indeterminacy locus of ψ, and let $\hat{\psi} : \hat{S} \to \mathbb{P}^k$ be the corresponding holomorphic map. Note that this resolution might not exist on the original surface due to the possible existence of infinitely many indeterminacy points. However, over a smaller disk there are only finitely many such points, and each of them can be resolved after finitely many blow-ups. If $(x_1, \ldots, x_k)$ are affine coordinates on $\{X_0 \neq 0\} \subset \mathbb{P}^k$, then by construction $\epsilon^*\omega_1 = \hat{\psi}^*(x_1 dx_2)$ and $\epsilon^*\omega_i = \hat{\psi}^*(x_1 x_i dx_2)$ for any $i \geq 2$. Furthermore $\hat{\psi}^*(dx_i \wedge dx_j) = dg_i \wedge dg_j = 0$, which implies that the image of $\hat{\psi}$ is locally an analytic curve $\hat{\Sigma} \subset \mathbb{P}^k$. Since $\hat{S}$ is smooth, $\hat{\psi}$ factors throught the normalization $\nu : \Sigma \to \hat{\Sigma}$, giving a holomorphic map $\hat{\phi} : \hat{S} \to \Sigma$.

The meromorphic 1-forms on Σ defined as $\alpha_1 = \nu^*(x_1 dx_2)$ and $\alpha_i = \nu^*(x_1 x_i dx_2)$ (for $i \geq 2$) verify that $\hat{\phi}^*\alpha_i = \epsilon^*\omega_i$ are holomorphic. A straightforward computation in local coordinates shows that

$$\operatorname{div}\left(\hat{\phi}^*\alpha_i\right) = \hat{\phi}^* \operatorname{div}(\alpha_i) + \sum (n_j - 1) E_j,$$

where the E_j are the irreducible components of the fibres of $\hat{\phi}$, and n_j is the corresponding multiplicity. Since the $\hat{\phi}^*\alpha_i$ are holomorphic, these divisors are effective, but any contribution of a pole of α_i to E_j would be smaller or equal than $-n_j$. Therefore the α_i are holomorphic 1-forms on Σ.

To conclude, let $F \subset \hat{S}$ be a general fibre of $\epsilon \circ f$, and let $\pi : F \to \Sigma$ be the map induced by $\hat{\phi}$. Since the $\pi^*\alpha_i = (\epsilon^*\omega_i)_{|F}$ are linearly independent by hypothesis, in particular they are not zero and π is hence surjective. This implies that Σ is indeed a compact curve of genus $g(\Sigma) \geq k$.

Note that, *a fortiori*, the resolution of indeterminacy ϵ is not necessary, since every ϵ-exceptional divisor is contracted by $\hat{\phi}$ because $g(\Sigma) \geq k \geq 2$. □

The theorem can be generalized to higher dimensional vector spaces and non-contractible bases.

Definition 2.2.19. A subspace V of $H^0(S, \Omega^1_{S,d})$ is *isotropic* if the $\wedge-$map restricts to the null map on $\bigwedge^2 V$. Moreover, it is *maximal* if it is not properly contained in any bigger isotropic space.

Then the theorem is the following.

Theorem 2.2.20 (Castelnuovo-de Franchis for fibred surfaces)**.** *Let $f : S \to B$ be a fibred surface over a smooth curve B and let $V \subset H^0(S, \Omega^1_{S,d})$ be a maximal isotropic subspace of dimension $r \geq 2$ such that the restriction $V \to H^0(\omega_F)$ to a general fibre F is injective. Then there is a non-constant morphism $\varphi : S \to \Sigma$ from the surface S to a smooth compact curve Σ of genus $g(\Sigma) \geq 2$ such that $\varphi^* H^0(\omega_\Sigma) = V$.*

Remark 2.2.21. The theorem is proved using the same argument of Theorem 2.2.18, where the base B is a complex disk. This because the key point is that closed forms give rise to integrable foliations and then every check is local.

(••) Vanishing Massey-products and and liftings on sections of $\mathcal{K}_\partial$ and $\mathbb{U}$.

We generalize the property of vanishing of Massey-products from pairs of sections to subspaces of sections in $\mathcal{K}_\partial$ (see Definition 1.6.7) and we study the relation with suitable local liftings in $f_*\Omega^1_S$.

Assume $\mathcal{K}_\partial$ has rank greater than 2. Let A be an open subset of B and $W \subset \Gamma(A, \mathcal{K}_\partial)$ be a subspace of sections over A such that $\dim_{\mathbb{C}} W \geq 2$. The vanishing property is the following.

Definition 2.2.22. A subspace $W \subset \Gamma(A, \mathcal{K}_\partial)$ is *Massey-trivial* if each pair of sections on W is Massey-trivial (Definition 1.6.7).

By the Splitting lemma (see 2.12) on $\mathcal{K}_\partial$, we get a choice, even if long far away to be unique, to lifts local sections and then compute the Massey-product pointwise.

As seen in Proposition 1.6.4, when the fibres of f are all reduced, there is an injective morphism

$$(f_*\Omega^1_{S/B}) \xhookrightarrow{\nu'} f_*\omega_{S/B} \tag{2.36}$$

which naturally induces $\nu : \mathcal{K}_\partial \hookrightarrow f_*\omega_{S/B}$ by restriction. Then one can think at a local family of Massey-products as a section of $f_*\omega_{S/B}$ through the injection ν'.

Up to a choice of a pair of liftings of $s_1, s_2 \in \Gamma(A, f_*\omega_{S/B})$ in $\Gamma(A, f_*\Omega^1_S)$ the section $\mathfrak{m}_\sigma(s_1, s_2) \in \Gamma(A, f_*\omega_{S/B})$ is computed by the map

$$\bigwedge^2 \Gamma(A, f_*\Omega^1_S) \otimes \Gamma(A, T_B) \to \Gamma(A, f_*\omega_{S/B}), \tag{2.37}$$

where the last isomorphism is given by the projection formula. This is indeed a local version of the adjoint map (see Section 1.6 or [CP95]). The construction glues to a sheaf map under the assumption of isolated singularities, where $\mathcal{K}_\partial$ is locally free. Let us now introduce the morphism

$$\phi : \bigwedge^2 f_*\Omega^1_S \otimes T_B \longrightarrow f_*\omega_{S/B}, \tag{2.38}$$

defined by the morphism $\bigwedge^2 f_*\Omega^1_S \to f_* \bigwedge^2 \Omega^1_S$ twisted by T_B. By the projection formula, we have $f_* \bigwedge^2 \Omega^1_S \otimes T_B \simeq f_*\omega_{S/B}$. By looking at the short exact sequence

$$0 \longrightarrow \omega_B \otimes \mathcal{K}_\partial \longrightarrow \bigwedge^2 f_*\Omega^1_S \longrightarrow \bigwedge^2 \mathcal{K}_\partial \longrightarrow 0 \tag{2.39}$$

induced by ζ, which is split by lemma 2.12, we get an injection $\bigwedge^2 \mathcal{K}_\partial \hookrightarrow \bigwedge^2 f_*\Omega^1_S$ that factorizes through the morphism 2.38. Thus we obtain

$$\phi : \bigwedge^2 \mathcal{K}_\partial \otimes T_B \longrightarrow f_*\omega_{S/B}. \tag{2.40}$$

where we denoted the restriction with ϕ, with a little abuse of notation. Moreover, a direct computation shows that the image of $\omega_B \otimes \mathcal{K}_\partial$ via the morphism ϕ is contained $\mathcal{K}_\partial \hookrightarrow f_*\omega_{S/B}$.

The Massey-product $\mathfrak{m}_\sigma(s_1, s_2) \in \Gamma(A, f_*\omega_{S/B})$ of the pair of sections s_1, s_2 of $\mathcal{K}_\partial$ over a subset A of B is computed by 2.40 modulo the the $\mathcal{O}(A)-$submodule $< s_1, s_2 >$ of $\Gamma(A, f_*\omega_{S/B})$.

The splitting given in 2.12 lifts W to $f_*\Omega^1_S$ and then we can apply the morphism 2.40 using the liftings above to compute the Massey-products of each pair of W. We get sections of $f_*\omega_{S/B}$ that lie in $< s_1, s_2 >_{\mathcal{O}_B(A)}$, by definition of Massey-trivial pairs. We prove that one can choose suitable liftings on $f_*\Omega^1_S$ with wedge zero. We note that these can be different from those given by the splitting fixed in 2.12.

Proposition 2.2.23. *Let A be an open contractible set of B and $W \subset \Gamma(A, \mathcal{K}_\partial)$ be a Massey-trivial subspace of sections of $\mathcal{K}_\partial$ on A. Assume that the evaluation map $W \otimes \mathcal{O}_A \to \mathcal{K}_{\partial|A}$ defines an injective map of vector bundles. Then there exists a unique $\widetilde{W} \subset H^0(A, f_*\Omega^1_S)$ which lifts W to $f_*\Omega^1_S$ and such* *that the map 2.38 vanishes identically over $\widetilde{W}$, namely*

$$\tilde{\phi} : \bigwedge^2 \widetilde{W} \otimes T_A \longrightarrow f_* \omega_{S/B|A}, \tag{2.41}$$

is the null map.

Proof. Let $\tau \in T_A$ be a trivialization of T_A and set β its dual (that is, $\tau \cdot \beta = 1$). By composition of $W \otimes \mathcal{O}_A \to \mathcal{K}_{\partial|A}$ with the splitting $\mathcal{K}_\partial \to f_*\Omega^1_S$, we obtain a lifting map $\rho : W \to \Gamma(A, f_*\Omega^1_S)$. Let $V = \rho(W)$ be its image. Let $\{s_1, s_2, \ldots, s_n\}$ be a basis of W and set $v_i = \rho(s_i)$. Since the Massey products are zero on any pairs of sections of W, we have

$$\tilde{\phi}(v_1 \wedge v_i, \sigma) = f_i s_1 + g_i s_i$$

and

$$\tilde{\phi}(v_1 \wedge \sum_2^n v_i, \sigma) = f_0 s_1 + g_0(\sum_2^n s_i),$$

where the f_i and the g_i are holomorphic functions on A. Assume $n = 2$ and set $\tilde{v}_1 = v_1 - g_2\beta$ and $\tilde{v}_2 = v_2 - f_2\beta$. Then $\tilde{\phi}(\tilde{v}_1 \wedge \tilde{v}_2, \sigma) = 0$ and therefore $\tilde{v}_1 \wedge \tilde{v}_2 = 0 \in \Gamma(A, f_*\omega_S)$. We remark that the unicity of the liftings $\tilde{v}_1$ and $\tilde{v}_2$ follows at once.

Assume by induction on $n > 2$ that the proposition holds for $k < n$. Consider the space W' generated by $\{s_1 \ldots s_{n-1}\}$ where we find liftings $\tilde{v}_i$, for $i = 1, \ldots n-1$, such that $\tilde{v}_i \wedge \tilde{v}_j = 0$. Moreover,

$$\tilde{\phi}(\tilde{v}_1 \wedge v_n, \sigma) = e s_1 + f s_n$$

and

$$\tilde{\phi}(\tilde{v}_1 \wedge (\sum_2^{n-1} \tilde{v}_i + v_n), \sigma) = \tilde{\phi}(\tilde{v}_1 \wedge v_n, \sigma) = l s_1 + m(\sum_2^n s_i).$$

We get $es_1 + fs_n = ls_1 + m(\sum_2^n s_i)$ and $e = l$ and $f = 0 = m$, since the sections are independent. Set $\tilde{v}_n = v_n - e\beta$. Then $\tilde{\phi}(\tilde{v}_1 \wedge \tilde{v}_n, \sigma) = 0$ and thus also $\tilde{v}_1 \wedge \tilde{v}_n = 0$. In the same way we obtain $\tilde{v}_i \wedge \tilde{v}_n = 0$. Unicity of the lifting follows immediately.

□

(•••) Local families of vanishing Massey-products on $\mathbb{U}$ and the Castelnuovo de-Franchis theorem. We want to relate Massey-trivial subspaces W of flat local sections of $\mathcal{K}_\partial$ with the Castelnuovo de-Franchis theorem above.

Remark 2.2.24. Let $i : F \hookrightarrow S$ be the general fibre of S. Assume that a pair (ω_1, ω_2) of holomprphic forms on F, that is sections on $H^0(\omega_F)$ lifts via pullback $i^* : H^0(F, \omega_F) \to H^0(S, \Omega^1_S)$ to a pair $(\widetilde{\omega_1}.\widetilde{\omega_2})$ on $H^0(S, \Omega^1_S)$ of closed holomorphic forms on S with $\wedge$ zero. Then the adjoint image (see Definition 1.6.5) of the pair $(\widetilde{\omega_1}_{|F}, \widetilde{\omega_2}_{|F})$ on $H^0(F, \Omega^1_{S|F})$ given by restriction $H^0(S, \Omega^1_S) \to H^0(F, \Omega^1_{S|F*})$ is zero (and thus has vanishing Massey-products).

Repeating the previous argument, holomorphic forms on S with wedge zero restricts to vanishing Massey products on the general fibre F. But the map $H^0(\Omega^1_S) \to H^1(\Omega^1_{S|F})$ is not surjective, that is infinitesimal extensions are not all restrictions of holomorphic forms on S.

Starting from a pair (ω_1, ω_2) in K_ξ of holomorphic forms on the general fibre F with vanishing Massey-products, we study local families of vanishing Massey-products to find conditions that produces fibrations on S.

Remark 2.2.25. Families of vanishing Massey products over a complex disk (thought as local families on a fibration over a smooth curve B) lift to $\Gamma(S, \Omega^1_S)$, which is the space of holomorphic forms on the fibres that attach to a holomorphic form on S. These are not closed in general since the base is a complex disk, which is not compact, and thus we are not able to apply the previous theorem.

In other words, the remark above says that it is not enough to look at sections on $\mathcal{K}_\partial$ since the purpose is to find liftings with the additional property to be closed. The Lifting lemma 2.2.5 suggests to look at the existence of Massey-trivial subspaces W of flat local sections of $\mathcal{U} \hookrightarrow \mathcal{K}_\partial$.

Let $W \subset \Gamma(A, \mathbb{U})$ be a subspace of sections over an open subset A. In section 2.2, we showed that there is an injection $\mathbb{U} \hookrightarrow \mathcal{K}_\partial$ and moreover each lifting of $\mathbb{U}$ to $f_*\Omega^1_S$ lies in $f_*\Omega_{S,d}$ (lemma 2.2.8). We get the following.

Proposition 2.2.26. *Let A be an open set of B and $W \subset \Gamma(A, \mathbb{U})$ be a Massey-trivial subspace of sections of $\mathbb{U}$ on A. Then there exists a unique $\widetilde{W} \subset H^0(A, f_*\Omega^1_{S,d})$ which lifts W to $f_*\Omega^1_{S,d}$ and such that $\bigwedge^2 \widetilde{W} \to \Gamma(A, f_*\omega_S)$ is the zero map.*

Proof. The proof follows immediately by proposition 2.2.23. It is enough to observe that we can choose liftings in $f_*\Omega^1_{S,d}$ by lemma 2.2.8 and then all the other admissible splittings are still sections of $f_*\Omega^1_{S,d}$ since they must differs from the first ones by sections of ω_B. Moreover, the evaluation map $W \otimes \mathcal{O}_A \to \mathcal{K}_{\partial|A}$ is automatically injective map of vector bundles since $\Gamma(A, \mathbb{U})$ is the space of the flat sections of $\mathcal{U}$. □

The above proposition shows that Massey-trivial subspaces W of sections on $\mathbb{U}$ correspond to isotropic subspaces on $f_*\Omega_{S,d}$. Let H be the kernel of the monodromy representation of $\mathbb{U}$, which is a normal subgroup of $\pi_1(B, b)$. Set H_{W} be the subgroup of H which acts trivially on W, that is

$$H_{\mathrm{W}} = \{g \in H \mid g \cdot w = w, \forall w \in W\}. \tag{2.42}$$

As an application of the Castelnuovo-de Franchis for fibred surfaces 2.2.20 we obtain the following.

Theorem 2.2.27. *Let $f : S \to B$ be a semistable fibration of genus $g(F) \geq 2$ and $W \subset \Gamma(A, \mathbb{U})$ be a maximal Massey-trivial subspace of sections over A. Then for any subgroup K of H_W, the fibred surface $f_K : S_K \to B_K$ defined by the étale base change $u_K : B_K \to B$ classified by K has an irrational pencil $h_K : S_K \to \Sigma$ onto a smooth compact curve Σ such that $W \simeq h_K^* H^0(\omega_\Sigma)$.*

Proof. Let $u_{\mathrm{W}} : B_{\mathrm{W}} \to B$ be the étale covering classified by the subgroup H_{W} of H. By construction, the pull back of W extends to a subspace $\widehat{W}$ of global sections $\Gamma(B_{\mathrm{W}}, \mathbb{U}_{\mathrm{W}})$, where $\mathbb{U}_{\mathrm{W}}$ is the unitary bundle of the fibration $f_{\mathrm{W}} : S_{\mathrm{W}} \to B_{\mathrm{W}}$ defined by the base change. The proof for H_{W} follows by applying proposition 2.2.26 to $\widehat{W}$ and theorem 2.2.20. Then by using the same argument, we get the proof for each étale covering $u_{\mathrm{K}} : B_{\mathrm{K}} \to B$ given by a subgroup K of H_{W}, in a natural way.

□

Remark 2.2.28. Releasing the assumption of maximality, we only get an inclusion $\widetilde{W} \subset h^*H^0(\omega_\Sigma)$.

A subspace $W \subset \Gamma(A, \mathbb{U})$ of local sections over a contractible open subset A of B is not necessarily invariant under the monodromy action of $\mathbb{U}$. This motivates the following definition.

Definition 2.2.29. Let $\mathbb{M}$ be a local subsystem of $\mathbb{U}$ of stalk M. We say that $\mathbb{M}$ is *Massey-trivial* if the stalk M is isomorphic to a Massey-trivial subspace of $\Gamma(A, \mathbb{U})$ of sections over an open contractible subset A of B. Moreover, we say that $\mathbb{M}$ is *Massey-trivial generated* if the stalk M is generated by a Massey-trivial subspace of $\Gamma(A, \mathbb{U})$.

Remark 2.2.30. The request of Massey-triviality over the general point b of A is strong. By using a standard argument of analytic continuation, the above property on subspaces $W \subset \Gamma(A, \mathbb{U})$ of local flat sections on $\mathbb{U}$ is stable under the monodromy action.

2.3 Monodromy of the unitary flat bundle

The main results of this section are a criterion for the finiteness of the monodromy group on unitary flat sub-bundle of the second Fujita decomposition of a fibred surface and a description in terms of morphisms of curves under the assumpion of semistability. We state and prove the theorems and then we apply them in the framework of fibrations (see also [PT17]).

2.3.1 The main theorems: a finiteness monodromy criterion and morphisms of curves

Let $f : S \to B$ be a fibration over a smooth projective curve B with general fibre F of genus $g(F) \geq 2$ and $\mathcal{U}$ be the unitary bundle in the second Fujita decomposition of f. Let $b \in B$ be a regular value and F_b be the (smooth) fibre over b. Let $\mathbb{U}$ be the underlying local system (i.e $\mathcal{U} = \mathbb{U} \otimes \mathcal{O}_B$), r_U the rank of $\mathcal{U}$, $\rho : \pi_1(B, b) \to U(r_U, \mathbb{C})$ the unitary representation of $\mathbb{U}$, $H = \ker \rho$ the kernel and $G = \pi_1(B, b)/H$. We recall that G is naturally isomorphic to the monodromy group $\operatorname{Im} \rho$ of $\mathcal{U}$ and we identify them.

Consider $W \subset \Gamma(A, \mathbb{U})$ a Massey-trivial subspace of sections on a contractible open subset A of B around b and $\mathbb{M}$ the local sub-system of $\mathbb{U}$ generated by W. By definition, $\mathbb{M}$ is the local system with stalk $M = G \cdot W$ and defines a unitary flat subbundle $\mathcal{M}$ of rank $r_M = \dim M$ of $\mathcal{U}$ together with a unitary sub-representation $\rho_M : \pi_1(B, b) \to U(r_M, \mathbb{C})$ of ρ. We denote with H_{M} the kernel and $G_{\mathrm{M}} = \pi_1(B, b)/H_{\mathrm{M}}$ the quotient, again isomorphic to the monodromy group $\operatorname{Im} \rho_{\mathrm{M}}$ (see section 1.3 for details on notations).

Theorem 2.3.1. *Let $f : S \to B$ be a fibration over a smooth projective curve B of genus $g(F) \geq 2$ and $\mathcal{U}$ be the unitary bundle in the second Fujita decomposition of f. Let $\mathcal{M} \subset \mathcal{U}$ be a flat subbundle of $\mathcal{U}$ generated by a Massey-trivial subspace. Then $\mathcal{M}$ has finite monodromy.*

Theorem 2.3.2. *Let $f : S \to B$ be a semistable fibration over a smooth projective curve B of genus $g(F) \geq 2$ and $\mathcal{M} \subset \mathcal{U}$ be a unitary flat subbundle generated by a maximal dimensional Massey-trivial subspace. Then the monodromy group G_M of $\mathcal{M}$ is in one to one correspondence with the group of bijections $\mathcal{S}_{\mathscr{K}}$ of a finite set $\mathscr{K}$ of morphisms $k_g : F \to \Sigma$ from the general fiber F to a smooth compact curve Σ of genus $g(\Sigma) \geq 2$. Moreover, after a finite étale base change $u_M : B_M \to B$*

trivializing the monodromy, the pullback bundle of $\mathcal{M}$ *becomes the trivial bundle* $V \otimes \mathcal{O}_{B_M}$ *of fibre* $V = \sum_{g \in G_M} k_g^* H^0(\omega_\Sigma) \subset H^0(\omega_F)$.

Proof of theorem 2.3.2.

The proof is developed in three steps.

(1) The construction of the set $\mathscr{K}$ of morphisms of curves;

(2) The proof of the finitness of the monodromy group G;

(3) The geometric description of $\mathcal{M}$.

(1) **The construction of the set** $\mathscr{K}$ **of morphisms of curves.** Let $u_{\mathrm{M}} : B_{\mathrm{M}} \to B$ be the Galois covering map classified by the normal group H_{M} of $\pi_1(B, b)$ and given exactly by the action of the monodromy group $G_{\mathrm{M}} = \pi_1(B, b)/H_{\mathrm{M}}$ of $\mathcal{M}$. By construction, $u_{\mathrm{M}} : B_{\mathrm{M}} \to B$ trivializes the monodromy of $\mathcal{M}$ (that is, $u_{\mathrm{M}}^{-1}\mathbb{M}$ is a trivial local system on B_{M}) and we consider the étale base change

$$\begin{tikzcd} S_{\mathrm{M}} := S \times_B B_{\mathrm{M}} \arrow[r, "\varphi_{\mathrm{M}}"] \arrow[d, "f_{\mathrm{M}}"] & S \arrow[d, "f"] \\ B_{\mathrm{M}} \arrow[r, "u_{\mathrm{M}}"] & B \end{tikzcd} , \tag{2.43}$$

where S_{M} is a smooth surface (not a priori compact) given by the fibred product $S \times_B B_{\mathrm{M}}$ and $\varphi_{\mathrm{M}} : S_{\mathrm{M}} \to S$ is an étale Galois covering. The action of $G_{\mathrm{M}} \times S_{\mathrm{M}} \to S_{\mathrm{M}}$ sends a point (p, b') to the point $g(p, b') := (p, gb')$, for $g \in G_{\mathrm{M}}$, where $b' \mapsto gb'$ is the automorphism of B_{M} defined by the action of G_{M} on B_{M}. Note that $g : S_{\mathrm{M}} \to S_{\mathrm{M}}$ is an automorphism of S_{M} compatible with the fibration f_{M}.

Let $W \subset \Gamma(A, \mathbb{U})$ be a maximal Massey-trivial subspace of sections over A around b and generating $\mathbb{M}$. From now on we identify W with its isomorphic subspace of $H^0(\omega_{F_b})$ (Remark 1.3.3). It is easy to check that H_{M} can be described as

$$H_{\mathrm{M}} = \{g \in \pi_1(B, b) |\, gg'w = g'w,\ \forall w \in W,\ \forall g' \in G_{\mathrm{M}}\}. \tag{2.44}$$

Applying Proposition 2.2.27 to the subgroup $K = H_{\mathrm{M}}$ of H_{W}, we get a non-constant map $h : S_{\mathrm{M}} \to \Sigma$ on S_{M} over a smooth compact curve Σ such that $u_{\mathrm{M}}^* W \simeq h^* H^0(\omega_\Sigma)$. We start from the map above to construct a family

$$\mathscr{H} := \{h_g : S_{\mathrm{M}} \to \Sigma \,|\, g \in G_{\mathrm{M}}\}$$

of non-constant morphisms and a family

$$\mathscr{K} := \{k_g : F_b \to \Sigma \,|\, g \in G_{\mathrm{M}}\}$$

of morphisms of curves from the smooth fiber F_b over b as follows.

Let b_0 be a preimage of a point $b \in A$ via u_{M} and F be the fibre of f_{M} over b_0 (isomorphic to the fibre of f over b). For any $g \in G_{\mathrm{M}}$, we consider the automorphism $g : S_{\mathrm{M}} \to S_{\mathrm{M}}$ and we define h_g and

k_g by composition

$$\begin{tikzcd} F \arrow[r, hook, "i"] \arrow[drr, "k_g"'] & S_{\mathrm{M}} \arrow[r, "g"] \arrow[dr, "h_g"] & S_{\mathrm{M}} \arrow[d, "h"] \\ & & \Sigma \end{tikzcd} \tag{2.45}$$

Consider the actions on $\mathscr{H}$ and $\mathscr{K}$

$$G_{\mathrm{M}} \times \mathscr{H} \longrightarrow \mathscr{H}, \qquad (g_1, h_{g_2}) \longmapsto g_1 \cdot h_{g_2} := h_{g_1 g_2} \tag{2.46}$$

$$G_{\mathrm{M}} \times \mathscr{K} \longrightarrow \mathscr{K}, \qquad (g_1, k_{g_2}) \longmapsto g_1 \cdot k_{g_2} := (g_1 \cdot h_{g_2}) \circ i = k_{g_1 g_2} \tag{2.47}$$

defined in the natural way by the action of G_{M} on S_{M}.

Remark 2.3.3. By construction $\mathscr{K}$ is the orbit of the elements k that corresponds to the neutral element $e \in G_{\mathrm{M}}$. In general $\mathscr{K}$ is not in one to one correspondence with G_{M} since two elements should define the same map. In fact when there is $g \in G_{\mathrm{M}}$ that fixes not only W but every element of W under the monodromy action, then $k_g = k$. In other words, the action is not automatically effective.

We consider the group $\mathcal{S}_{\mathscr{K}}$ of the bijections over $\mathscr{K}$ and we prove that it is in one to one correspondence with G_{M}.

More precisely, we consider the homomorphism

$$\Psi_{\mathrm{M}} : G_{\mathrm{M}} \longrightarrow \mathrm{Aut}(\mathscr{K}) = \mathcal{S}_{\mathscr{K}}, \qquad g_1 \longmapsto g_1 \cdot : \mathscr{K} \to \mathscr{K} \tag{2.48}$$

defined by the action above and we prove that it is injective. The proof requires the formula proved in the following.

Lemma 2.3.4. *Let $e \in G_M$ be the neutral element and $\alpha \in H^0(\omega_\Sigma)$. Then for each $g \in G_M$,*

$$k_g^*(\alpha) = g^{-1} k_e^*(\alpha), \tag{2.49}$$

where g^{-1} acts over $k_e^*(\alpha) \in W$ via the monodromy action ρ_M defining $\mathbb{M}$.

Proof. Let b_0 be the preimage of b via u_{M} and A_0 an open contractible subset of B_{M} such that $u_{\mathrm{M}}(A_0) \subset A$. Let $\widetilde{W} \in \Gamma(A_0, f_{\mathrm{M}*}\Omega^1_{S_{\mathrm{M}},d})$ be the unique lifting of W provided by proposition 2.2.26. By construction, we can lift in a natural way the monodromy action from W to $\widetilde{W}$. Then for $\eta \in \widetilde{W}$, we have $g\eta = \eta$, for each $g \in H_{\mathrm{M}}$. This means that $\widetilde{W}$ extends to a subspace of global forms $H^0(S_{\mathrm{M}}, \Omega^1_{S_{\mathrm{M}},d})$ and we identify them. Let $\eta = h_e^*\alpha \in \widetilde{W}$ and let $w = \eta_{|F}$. Then $k_g^*(\alpha) = (g^* h_e^* \alpha)_{|F} = (g^*\eta)_{|F} = \eta_{|F_{g^{-1}b}}$ and $g^{-1}k_e^*(\alpha) = g^{-1}(\eta_{|F}) = g^{-1}w = \eta_{|F_{g^{-1}b}}$ are equal. □

Proposition 2.3.5. *The map Ψ_M is injective.*

Proof. Let e be the neutral element in G_{M} and $g_1 \in G_{\mathrm{M}}$, $g_1 \neq e$. We prove that $\Psi_{\mathrm{M}}(g_1) \neq \Psi_{\mathrm{M}}(e)$, i.e. that there exists $g_2 \in G_{\mathrm{M}}$ such that $g_1 \cdot k_{g_2} \neq e \cdot k_{g_2}$. By looking at the pullback functor $\mathscr{M}(F, \Sigma) \to \mathrm{Hom}(H^{1,0}(\Sigma), H^{1,0}(F))$ from the set of morphisms between F and Σ to the group of homomorphisms on the spaces of holomorphic one forms, we prove that $k^*_{g_1 g_2} \neq k^*_{g_2}$, which is enough to get the thesis.

Since $g_1 \neq e$, then $g_1 \notin H_{\mathrm{M}}$. According to Description 2.44 of H_{M}, there exists $w \in W \subset H^0(\omega_F)$ and $g_2 \in G_{\mathrm{M}}$ such that $g_1 g_2 w \neq g_2 w$. Let $\alpha \in H^0(\omega_\Sigma)$ such that $k_e^* \alpha = w$. Applying formula 2.49 with $g = g_1 g_2$, we obtain $g_1 g_2 w = (g_1 g_2)^{-1} k_e^*(\alpha) = k_{g_1 g_2}^* w$, while the same formula applied with $g = g_2$ gives $g_2 w = g_2^{-1} k_e^*(\alpha) = k_{g_2}^*(\alpha)$. By assumption, $g_1 g_2 w \neq g_2 w$ and thus $k_{g_1 g_2}^* \neq k_{g_2}^*$.

□

(2) **The proof of the finiteness of the monodromy group** G. The proof follows from a classical de-Franchis' theorem (see e.g. [Mar88]) applied to the set of morphisms $\mathscr{M}(F, \Sigma)$. Let C and C' be smooth compact curves and $\mathscr{M}(C, C')$ the set of non-constant morphisms between them.

Theorem 2.3.6 (de Franchis)**.** *Let C, C' be smooth compact curves of genus ≥ 2. Then the set $\mathscr{M}(C, C')$ is finite.*

Moreover, we recall the following.

Proposition 2.3.7. *Let C, C' be smooth projective curves of genus ≥ 2. Then the map*

$$\mathscr{M}(C, C') \to \mathrm{Hom}(H^{1,0}(C), H^{1,0}(C')) \tag{2.50}$$

given by the pullback functor is injective.

Proof. We give a sketch the proof. Let $\phi_i : C \to C'$, for $i = 1, 2$, be two morphims between C and C'. Let $\phi_i^{1,0} : H^{1,0}(C') \to H^{1,0}(C)$ and $\phi_i^{0,1} : H^{0,1}(C') \to H^{0,1}(C)$ be the linear maps defined by the pullback. Assume that $\phi_1^{1,0} = \phi_2^{1,0}$. By conjugation, we get $\phi_1^{0,1} = \phi_2^{0,1}$ and using the Hodge decomposition also that the maps $\phi_{i\mathbb{C}}^* : H^1(C', \mathbb{C}) \to H^{1,0}(C, \mathbb{C})$ and $\phi_{i\mathbb{Z}}^* : H^1(C', \mathbb{Z}) \to H^{1,0}(C, \mathbb{Z})$ must be equal.

Consequently, the maps between the Jacobians are the same. The result follows by the standard proof of de Franchis given by Martens in [Mar88] (see also [AP90]). □

Applying de-Franchis's theorem to the set of morphisms $\mathscr{M}(F, \Sigma)$, we have that this is a finite set. Then also $\mathscr{K}$ and consequently $\mathcal{S}_{\mathscr{K}}$ is finite. By Lemma 2.3.5, $\Psi_{\mathrm{M}}(G_{\mathrm{M}})$ is in one to one correspondence with G_{M}, thus the monodromy G_{M} is finite group and u_{M} is a finite covering.

(3) **The geometric description of** $\mathcal{M}$. Using the family $\mathscr{K}$ of morphisms h_g parametrized by G_{M}, we get $k_g^* H^0(\omega_\Sigma) = i^* h_g^* H^0(\omega_\Sigma) = gW$, for each $g \in G_{\mathrm{M}}$ and then the stalk of $\mathbb{M}$ is described by $\sum_{a \in G} k_g^* H^0(\omega_\Sigma) \subset H^0(\omega_F)$

Proof of theorem 2.3.1 Let $f : S \to B$ be a fibration over a smooth projective curve B and assume that f is not semistable (otherwise we can lead back to theorem 2.3.2). Let $\mathcal{U}$ be the unitary bundle in the second Fujita decomposition of f, $W \subset \Gamma(A, \mathbb{U})$ be a Massey-trivial subspace and $\mathbb{M}$ be the generated local subsystem of $\mathbb{U}$. We want to prove that $\mathbb{M}$ has finite monodromy.

Following [CD], we apply the semistable-reduction theorem to reduce to the semistable case. Then the proof follows using theorem 2.3.2 together with some basic facts concerning the behaviour of the monodromy on local systems.

More precisely, by applying the semistable-reduction theorem (see [CD, Theorem 2.7]), we get a base change $u : B' \to B$ given by a ramified finite morphism of curves and a resolution on the fiber product

$$
\begin{tikzcd}
S' := \widetilde{S \times_B B'} \arrow[r, "\varphi"] \arrow[d, "f'"] & S \arrow[d, "f"] \\
B' \arrow[r, "u"] & B
\end{tikzcd}
\quad , \tag{2.51}
$$

producing a semistable fibration $f' : S' \to B'$ from a smooth compact surface S' to a smooth compact curve B'. The base change induces the short exact sequence 2.5

$$
0 \longrightarrow \mathcal{K}_{\mathrm{U}} \longrightarrow \mathcal{U}' \longrightarrow u^*\mathcal{U} \longrightarrow 0, \tag{2.52}
$$

compatible with the underlying structure of local systems. Thus in particular $u^{-1}\mathbb{M}$ is a local system contained in $\mathcal{U}'$ which by compatibility is a local subsystem of the local system $\mathbb{U}'$ underlying $\mathcal{U}$. We prove that $u^{-1}\mathbb{M}$ has finite monodromy, which is enough to prove that $\mathbb{M}$ has finite monodromy (by proposition 1.3.9). Let $\widehat{\mathbb{W}}_{g_i}$ be the local system generated by u^*g_iW via $\operatorname{Im}\rho_W^{-1}$, for $g_i \in I_u$ and $I_u \subset \pi_1(B, b)$ be a set of generators of the quotient given by $u_* : \pi_1(B', b') \to \pi_1(B, b)$. We apply theorem 2.3.2 to the local system generated by the maximal Massey-trivial subspace of $\mathbb{U}$ containing u^*g_iW, which remains Massey-trivial by pullback, and we get that this one has finite monodromy. By proposition 1.3.8, maximality is not obstructive to the finiteness of the monodromy group. Thus we can conclude that $\widehat{\mathbb{W}}_{g_i}$ has finite monodromy. Finally, propositions 1.3.9 and 1.3.6 show that $u^{-1}\mathbb{M}$ has finite monodromy.

We end the section by fixing a basic case satisfying the theorem and we relate it with variation of the Hodge structure.

Let $f : S \to B$ be a fibration over a smooth curve B. It follows from the statement that the condition studied produces a finite number of morphisms of curves from the general fibre F over a fixed curve Σ (up to a finite base change given by the semistable reduction). This means that the variation of the Hodge structure of weight one, which corresponds to a family of abelian varieties $\mathrm{Pic}^0(F_b)$ over the smooth locus B^0, is " roughly speaking " in the image of a certain Hurwitz space inside the moduli space of curves.

Example 2.3.8. Quasi isotrivial case. Let Σ be a smooth compact curve of genus $g(\Sigma)$ and $k : F \to \Sigma$ a ramified covering of a smooth compact curve F of genus $g(F) \geq 2$ over Σ. Moving the branches we can construct a family of curves $f : S \to B$ over a base B. The pullback $k^*H^{1,0}(\Sigma)$ is fixed on the space $H^{1,0}(F)$ of the general fibre F. This defines a trivial sub-HS and thus a variation under the monodromy action given by the base B.

The example above describes the base case of Massey trivial bundles. Indeed, let $f : S \to B$ be a fibration as before and assume that B is projective. Then by construction $k^*H^{1,0}(\Sigma)$ is a Massey-trivial subspace and satisfies the assumption of Theorem 2.3.2.

2.3.2 Monodromy of unitary flat bundle and semiampleness

We relate the monodromy of unitary flat bundles with the study of positivity (or semipositivity) on vector bundles over a curve. As recalled in the preliminaries, in [Fuj78a] Fujita proved that $f_*\omega_{S/B}$ is nef (numerically effective) but not ample in general, as a consequence of the first Fujita decomposition. These properties are numerical (see theorem 1.2.16). In terms of sections, an equivalent definition of *ample line bundle* $\mathcal{L}$ is that it admits a multiple $\mathcal{L}^{\otimes k}$, for some positive integer $k \geq 1$, which is very ample and thus gives an embedding into the projective space. From this point of view, a natural weaker notion to be studied is the *semiapleness*.

We recall (see Definition 1.2.18) that a line bundle $\mathcal{L}$ over a smooth projective variety M is *semiample* if it admits a positive integer that is globally generated and a locally free sheaf $\mathcal{F}$ is *semimaple* if its tautological line bundle $\mathcal{O}_{\mathbb{P}}(L_{\mathcal{F}})$ is a semiample.

By definition, semiample line bundles defines fibration or finite morphisms over M so that it contains geometric even if not numerical information.

Fujita posed the question to understand semiampleness on the sheaf $f_*\omega_{X/B}$ of a fibration $f : X \to B$ of smooth complex varieties over a complete curve B but the property fails in general already in fibred surfaces, as proved in [CD].

The property of semiapleness is not a numerical and thus it is interesting to find conditions which imply it. As a consequence of the second Fujita decomposition $f_*\omega_{X/B} = \mathcal{U} \oplus \mathcal{A}$, semiampleness depends only on the unitary flat bundle $\mathcal{U}$. Semiampleness of unitary flat bundles are caracterized by the following criterion.

Proposition 2.3.9 ([CD, Theorem 2.5])**.** *A unitary flat bundle $\mathcal{V}$ over a smooth projective curve B is semiample if and only if it has finite monodromy.*

As a corollary of theorem 2.2.30, together with the property above, we can state a criterion for the semiampleness of $f_*\omega_{S/B}$ on fibrations $f : S \to B$ of genus $g(F) \geq 2$.

Corollary 2.3.10. *Let $f : S \to B$ be a projective semistable fibration of genus $g(F) \geq 2$ and $\mathcal{U}$ be the unitary bundle in the second Fujita decomposition of f. If $\mathcal{U}$ is Massey-trivial generated, then $f_*\omega_{S/B}$ is semiample.*

Remark 2.3.11. The property to be Massey-trivial generated is a local condition on a vector bundle and actually it depends on a infinitesimal one which has to hold on an open contractible (not only over a isolated point of course). Thus the criterion shows how a local condition implies a global one.

Example 2.3.12. Hyperelliptic fibrations of genus $g \geq 2$. In [LZ17], the authors studied hyperelliptic fibrations and in particular they proved that the monodromy of the unitary flat summand of the second Fujita decomposition is finite. We give a more precise description on the unitary flat bundle and, as an application of the above criterion we get a different proof of the finiteness of the monodromy group.

Let $f : S \to B$ be a fibration over a smooth projective curve B and let F be the general fibre of f. A fibration as above is hyperelliptic of genus $g(F)$ if the general fibre F of f is a hyperelliptic curve of genus $g(F)$, that is the general fibre F admits a covering of degree 2 $\pi : F \to \mathbb{P}^1$over the projective space $\mathbb{P}^1$ and the covering induces an involution $\sigma : F \to F$ over F, the hyperelliptic involution.

Proposition 2.3.13. *Let $f : S \to B$ be an hyperelliptic fibration of over a smooth projective curve B and assume $g(F) \geq 2$. Then the unitary flat bundle $\mathcal{U}$ in the second Fujita decomposition of f is Massey-trivial generated.*

Proof. We prove that $\mathcal{U}$ is Massey-trivial generated. Let F be the general fibre of f and $\xi \in H^1(T_F)$ the extension class attached to F. Consider $s_1, s_2 \in U \subset H^0(\omega_F)$ two independent vectors in the fibre U of $\mathcal{U}$. Observe that since f is hyperelliptic, each $s \in U$ lies in K_ξ and we can compute the Massey product of the pair (s_1, s_2). By the formula 1.65, $m_\xi(s_1, s_2)$ is antisymmetric in s_1, s_2. Applying the hyperelliptic involution, which acts on $H^0(\omega_F)$ by pullback $\sigma^* : H^0(\omega_F) \to H^0(\omega_F)$ as the -1 multiplication map, we get $\sigma^* m_\xi(s_1, s_2) = -m_\xi(s_1, s_2)$. On the other hand, $\sigma^* m_\xi(s_1, s_2) = m_\xi(-s_1, -s_2) = m_\xi(s_1, s_2)$ and thus by antisymmetry it must be zero. □

Applying the criterion 2.3.10 of semiapleness for the unitary flat bundles we can state the following.

Corollary 2.3.14. *Let $f : S \to B$ be an hyperelliptic fibration over a smooth projective curve B and assume $g(F) \geq 2$. Then $f_*\omega_{S/B}$ is semimaple.*

2.3.3 Monodromy of the unitary flat bundle and the canonical normal function

In section 1.6.3 we recalled the relation between the Griffiths infinitesimal invariant of the canonical normal function and Massey-products (see Lemma 1.6.18), given by a formula proved in [CP95]. We apply it in relation to the unitary flat summand $\mathcal{U}$ of the second Fujita decomposition of a fibration $f : S \to B$ of genus $g(F) \geq 2$. The formula together with Theorem 2.3.2 provides a non-vanishing criterion for the Griffiths infinitesimal invariant of the canonical normal function in terms of the monodromy of $\mathcal{U}$. This is the content of Corollary 2.3.15.

Let $f : S \to B$ be a fibration over a smooth projective curve B and with general fiber F of genus $g(F) \geq 2$. Consider the inclusion $j : B^0 \hookrightarrow B$ of f to the smooth locus B^0 and the restriction $f^0 : S^0 \to B^0$. Let $\mathcal{P}(f) : \mathcal{P} \to B^0$ be the fibration of primitive intermediate Jacobians associated to the Hodge structure $H^{2g-3}(J(F), \mathbb{C})$ of the Jacobian $J(F)$ of the general fibre F of f. Let $\nu : B^0 \to \mathcal{P}$ be the canonical normal function, which is the section associating to each $b \in B^0$ the image of the Ceresa cycle $[F_b - F_b^-] \in Z^{g-1}(J(F_b))_{\mathrm{hom}}$ in $\mathcal{P}$ through the higher Abel-Jacobi map composed with the projection over $\mathcal{P}$.

Let $\delta(\nu)$ be Griffiths infinitesimal invariant of the canonical normal function ν, pointwise identified with a linear map

$$\ker(\gamma) \to \mathbb{C},$$

where $\gamma : T_{B,b} \otimes P^{2,1}J(F_b) \to P^{1,2}J(F_b)$ is naturally defined by the IVHS on the primitive cohomological groups $P^{1,2}J(F_b)$ and $P^{1,2}J(F_b)$ in $H^3(J(F_b), \mathbb{C})$ (see Section 1.6.3).

As an application of Theorem 2.3.1, we get the following.

Corollary 2.3.15. *Let $f : S \to B$ be a fibration of genus $g(F) \geq 2$ and $\mathcal{U}$ be the unitary summand in the second Fujita decomposition of f. If the monodromy of $\mathcal{U}$ is not finite, then the Griffiths infinitesimal invariant on the canonical normal function $\nu : B^0 \to \mathcal{P}$ is not zero at the general point $b \in B^0$. In particular, ν is not a torsion section.*

Proof. We apply the formula 1.82 to sections of $j^*\mathcal{U} \subset j^*\mathcal{K}_\partial$. Since the monodromy of $\mathcal{U}$ is not finite, then by Theorem 2.3.2 is not Massey-trivial generated and we can find a pair $(\omega_1, \omega_2) \subset H^0(\omega_{F_b})$ of independent element such that $\mathfrak{m}_{\xi_b}(\omega_1, \omega_2) \neq 0$, where ξ_b is the Kodaira spencer class of F_b. Applying the formula 1.82 to $\omega_1, \omega_2 \in H^0(\omega_{F_b})$ (which are such that $\xi_b \cdot \omega_1 = \xi_b \cdot \omega_2 = 0$) and $\sigma = \mathfrak{m}_{\xi_b}(\omega_1, \omega_2) \in H^0(\omega_{F_b})$ we get

$$\delta(\nu)(\xi_b \otimes \omega_1 \wedge \omega_2 \wedge \bar{\sigma}) = -2Q(\mathfrak{m}_{\xi_b}(\omega_1, \omega_2), \bar{\mathfrak{m}}_{\xi_b}(\omega_1, \omega_2)) < 0 \tag{2.53}$$

This concludes the proof, since the fact that the normal function is non-torsion when the Griffiths infinitesimal invariant in not zero has been proven in [Gri83], [Gre89],[Voi88]. □

In particular, the previous result applies to the examples provided in [CD14], in [CD] and also in [CD16], concerning the construction of fibrations where the monodromy of $\mathcal{U}$ is not finite. More precisely, one can state the following.

Corollary 2.3.16. *Let $f : S \to B$ be a fibration as those constructed in [CD14], [CD] and [CD16] with $\mathcal{U}$ of not finite monodromy. Then the canonical normal function is not torsion.*

2.4 The rank of the unitary flat bundle

The main result of this section is an upper bound on the rank of the unitary flat bundle of the second Fujita decomposition of fibred surface. Let $f : S \to B$ be a fibration of a surface over a smooth projective curve B and $\mathcal{U}$ be the unitary summand in the second Fujita decomposition of f. We assume that the general fibre F has genus $g = g(F)$ greater than 2. We prove a bound that involves the genus g of the general fibre F and the Clifford index c_f of the fibration f (see also [GST17]).

2.4.1 Preparation to the results

The proof of the theorem needs some preparatory results concerning supporting divisors (see Section 1.6.2). The following is a simplification of some results provided in [GA16, Sec. 2], which is enough for the setting of this work.

Lemma 2.4.1 ([GST17])**.** *Let $f : S \to B$ be a smooth fibration over a complex disk B (i.e. it has no singular fibres) and $\mathcal{D} \subset S$ is a divisor supporting f such that $\mathcal{D} \cdot F < 2g - 2$ for any fibre F. Then the inclusion $\omega_{S/B}(-\mathcal{D}) \hookrightarrow \omega_{S/B}$ factors uniquely as*

$$\omega_{S/B}(-\mathcal{D}) \overset{\iota}{\hookrightarrow} \Omega^1_S \twoheadrightarrow \omega_{S/B}.$$

Proof. First of all let us note that for any fibre F, $\deg\left(\omega_{S/B}(-\mathcal{D})_{|F}\right) = 2g - 2 - \mathcal{D} \cdot F > 0$. Hence $f_*\left(\omega_{S/B}(-\mathcal{D})^\vee\right) = 0$ and

$$\operatorname{Hom}\left(\omega_{S/B}(-\mathcal{D}), f^*\omega_B\right) = H^0\left(B, f_*\left(\omega_{S/B}(-\mathcal{D})\right)^\vee \otimes \omega_B\right) = 0.$$

The (left-exact) functor $\operatorname{Hom}\left(\omega_{S/B}(-\mathcal{D}), -\right)$ applied to the exact sequence

$$\xi : \quad 0 \longrightarrow f^*\omega_B \longrightarrow \Omega^1_S \longrightarrow \omega_{S/B} \longrightarrow 0.$$

gives

$$0 \longrightarrow \mathrm{Hom}\left(\omega_{S/B}\left(-\mathcal{D}\right), \Omega^1_S\right) \longrightarrow \mathrm{Hom}\left(\omega_{S/B}\left(-\mathcal{D}\right), \omega_{S/B}\right) \xrightarrow{\mu} \mathrm{Ext}^1\left(\omega_{S/B}\left(-\mathcal{D}\right), f^*\omega_B\right).$$

The image of the morphism of sheaves $\phi : \omega_{S/B}\left(-\mathcal{D}\right) \to \omega_{S/B}$ by the map μ is $\phi^*\xi$, the short exact sequence obtained from ξ by pull-back in the last term and completing the diagram. Thus the natural inclusion ι factors through Ω^1_S if and only if $\mu\left(\iota\right) = 0$.

By [GA16, Lemma 2.4], $\mathrm{Ext}^1\left(\omega_{S/B}\left(-\mathcal{D}\right), f^*\omega_B\right) \cong H^0\left(B, \mathcal{E}\right)$, where $\mathcal{E} = \mathcal{E}\mathrm{xt}^1_f\left(\omega_{S/B}\left(-\mathcal{D}\right), \mathcal{O}_S\right) \otimes \omega_B$ is a vector bundle whose fibre over a point $b \in B$ is

$$\mathrm{Ext}^1\left(\omega_{F_b}\left(-\mathcal{D}_{|F_b}\right), T^\vee_{B,b}\right) \cong H^1\left(F_b, T_{F_b}\left(\mathcal{D}_{|F_b}\right)\right) \otimes T^\vee_{B,b}$$

(cf. [GA16, Lemma 2.3, Proposition 2.1 and Appendix]). Moreover, under this isomorphism $\mu\left(\iota\right)$ corresponds to a section σ of $\mathcal{E}$, whose value $\sigma\left(b\right)$ at any $b \in B$ is the class of the restriction to F_b of the pull-back $\iota^*\xi$. Since this pull-back and the restriction clearly conmute, $\sigma\left(b\right)$ coincides with the image of $\xi_b \in H^1\left(F_b, T_{F_b}\right)$ in $H^1\left(F_b, T_{F_b}\left(\mathcal{D}_{|F_b}\right)\right)$, which is generically zero (hence identically zero, $\mathcal{E}$ being torsion-free) because the family f is supported on $\mathcal{D}$. This shows that $\mu\left(\iota\right) = 0$ and completes the proof. □

Remark 2.4.2. When $\mathcal{D}$ supports f, then $f_*\omega_{S/B}\left(-\mathcal{D}\right) \subseteq \mathcal{K}_\partial$.

Proposition 2.4.3. *[GST17] Suppose* $\mathrm{rank}\, \mathcal{K}_\partial > \frac{g+1}{2}$. *Then there is an open disk $V \subseteq B$ and a divisor $\mathcal{D} \subset f^{-1}\left(V\right)$* minimally *supporting the restriction of f such that $h^0\left(F_b, \omega_{F_b}\left(-\mathcal{D}_{|F_b}\right)\right) \geq 2$ and $H^0\left(F_b, \omega_{F_b}\left(-\mathcal{D}_{|F_b}\right)\right) \subseteq \mathcal{K}_{\partial b}$ for any fibre F_b with $b \in V$.*

Proof. The proof is analogous to that of Theorem 3.3 and Corollary 3.2 in [GA16]. Roughly speaking, applying Proposition 1.6.12 pointwise we can locally (over any disk $V \subseteq B$) find a rank-2 vector subbundle $\mathcal{W} \subseteq \mathcal{K}_\partial$ such that the divisorial base locus $\mathcal{D}$ of the relative evaluation map

$$f^*\mathcal{W} \longrightarrow f^*f_*\omega_{S/B} \longrightarrow \omega_{S/B}$$

supports f. Up to shrinking V we may assume that $\mathcal{D}$ consists of disjoint sections of f over V. If $\mathcal{D}$ is not minimal supporting f, we can remove some of the components (or reduce their multiplicities) until obtaining a minimal one. □

2.4.2 Bound for the rank of the unitary flat bundle

Let $f : S \to B$ be a fibration of a surface over a smooth projective curve B and $\mathcal{U}$ be the unitary summand in the second Fujita decomposition of f. We assume that the general fibre F has genus $g = g(F)$ greater than 2. We prove a bound over the rank u_f of $\mathcal{U}$ that involves the genus g of the general fibre F and the Clifford index c_f of the fibration f

Definition 2.4.4. The *Clifford index* of a smooth projective curve C is defined as

$$\mathrm{Cliff}\left(C\right) = \min\left\{\mathrm{Cliff}\left(D\right) = \deg D - 2r\left(D\right) \mid h^0\left(C, \mathcal{O}_C\left(D\right)\right), h^1\left(C, \mathcal{O}_C\left(D\right)\right) \geq 2\right\}.$$

It satisfies $0 \leq \mathrm{Cliff}\,(C) \leq \left\lfloor \frac{g-1}{2} \right\rfloor$, and the second is an equality for general $C \in \mathcal{M}_g$.

Definition 2.4.5. Let $f : S \to B$ be a fibration over a smooth projective curve B. The *Clifford index* c_f of f is the maximal Clifford index of the smooth fibres.

We note that the locus B^0 is a Zariski-open subset and c_f is attained over a Zariski-open subset of B, which means that there are at most finitely many smooth fibres with smaller clifford index.

The bound on the rank of the unitary summand in the second Fujita decomposition is provided by the following.

Theorem 2.4.6. *Let $f : S \to B$ be a non-isotrivial fibration of genus g, flat unitary rank u_f and Clifford index c_f. Then*

$$u_f \leq g - c_f. \tag{2.54}$$

proof of Theorem 2.4.6.

We assume that the fibration is semistable by applying the semistable reduction. Indeed, as shown for instance in [Tan94, proof of Xiao's Theorem], even if the Hodge bundle is not stable under finite base changes, the rank of the flat unitary summand does not decrease. More precisely, if $u : B' \to B$ is a finite map, S' is a desingularization of the fibre product $S \times_B B'$ and $f' : S' \to B'$ is the induced fibration, then $f'_*\omega_{S'/B'}$ is a subsheaf of the pullback $u^* f'_*\omega_{S'/B'}$ and the quotient is a skyscraper sheaf supported on the u-branch points $b \in B$ such that $f^{-1}(b)$ is not semistable. Hence $u_{f'} = \mathrm{rank}\,\mathcal{U}' \geq \mathrm{rank}\,\pi^*\mathcal{U} = u_f$. In other words, the analogous of [CD, Proposition 2.9] (see 1.2.11) holds for finite morphisms and one can look at the exact sequence 2.7, repeating the argument of Lemma 2.1.5.

Consider now an open disk $V \subseteq B$ such that:

- V contains only regular values and the corresponding smooth fibres have Clifford index c_f,
- the connecting homomorphism $\partial : f_*\Omega^1_{S/B} \to R^1 f_*\mathcal{O}_S \otimes \omega_B$ has constant rank on V.

From now on, let $f : S \to B$ denote only the restriction to $f^{-1}(V) \to V$. Thus f is a smooth fibration, $\Omega^1_{S/B} \cong \omega_{S/B}$, and $\mathcal{K}_\partial$ is a vector bundle whose fibre K_b over any $b \in B$ is exactly $\ker\left(\cup \xi_b : H^0\left(F_b, \omega_{F_b}\right) \to H^1\left(F_b, \mathcal{O}_{F_b}\right)\right)$.

Suppose that $u_f > g - c_f$. Since $c_f \leq \left\lfloor \frac{g-1}{2} \right\rfloor$, in particular $u_f > \frac{g+1}{2}$. Therefore, by Proposition 2.4.3 there is (up to shrinking B) a divisor $\mathcal{D} \subset S$ *minimally* supporting f such that

$$h^1\left(F_b, \mathcal{O}_{F_b}\left(\mathcal{D}_{|F_b}\right)\right) = h^0\left(F_b, \omega_{F_b}\left(-\mathcal{D}_{|F_b}\right)\right) \geq 2$$

for general $b \in B$.

As in the proof of [BGAN16, Theorem 1.2], we consider now two cases:

Case 1: The divisor $\mathcal{D}$ is relatively rigid, that is $h^0\left(F_b, \mathcal{O}_{F_b}\left(\mathcal{D}_{|F_b}\right)\right) = 1$ for a general $b \in B$. Then Theorem 1.6.15, together with Riemann-Roch, implies that

$$H^0\left(F_b, \omega_{F_b}\left(-\mathcal{D}_{|F_b}\right)\right) = K_b,$$

and hence $\mathcal{K}_\partial = f_* \omega_{S/B}(-\mathcal{D})$.

Let now $\omega_{S/B}(-\mathcal{D}) \hookrightarrow \Omega^1_S$ be the maps provided by Lemma 2.4.1, which gives a splitting $\mathcal{K}_\partial \hookrightarrow f_* \Omega^1_S$. Thus every section of $\mathcal{K}_\partial$ corresponds to a 1-form on S, and in particular every $w \in \ker \cup \xi_b \subseteq H^0(F_b, \omega_{F_b})$ can be extended to S. Moreover, all these extensions are sections of the same line bundle $\omega_{S/B}(-\mathcal{D})$, and therefore any two such extensions wedge to 0 (this means in particular that any adjoint image is 0).

Let now $\eta_1, \ldots, \eta_{u_f}$ be a basis of flat sections of $\mathcal{U} \subseteq \mathcal{K}_\partial$, i.e. a basis of $H^0(V, \mathbb{U})$, and let $\omega_1, \ldots, \omega_{u_f} \in H^0\left(S, \Omega^1_S\right)$ be the extensions provided by the splitting chosen above.

We will combine Lemmas 2.2.5 and 2.4.1 to lift sections of $\mathbb{U} \subset \mathcal{K}_\partial$ to closed differential forms on S. A priori, they can be different but any two possible ι differ by sections of ω_B, which is exactly the kernel in Lemma 2.2.5. Thus any ι as in Lemma 2.4.1 maps flat sections of $\mathcal{U} \subseteq \mathcal{K}_\partial$ to closed differential forms.

The forms are closed and, by the previous discussion, any two of them wedge to zero. We are thus in the situation of Theorem 2.2.18, which gives a new fibration $\phi : S \to C$ over a compact curve of genus $g(C) = u_f$. Let now F_b be any fibre of f, and $\pi : F_b \to C$ be the restriction of ϕ to the smooth fibre F_b. Applying Riemann-Hurwitz we obtain

$$2g - 2 \geq \deg \pi \, (2u_f - 2).$$

At the beginning of the proof we obtained that $u_f > \frac{g+1}{2}$, so that $2u_f - 2 > g - 1$, and thus

$$2(g-1) > \deg \pi \, (g-1).$$

It follows that $\deg \pi = 1$, so every smooth fibre is isomorphic to C and hence f is isotrivial.

Case 2: The divisor $\mathcal{D}$ moves on any smooth fibre, i.e. $h^0\left(F_b, \mathcal{O}_{F_b}\left(\mathcal{D}_{|F_b}\right)\right) \geq 2$ for every regular value $b \in B$. We use Theorem 1.6.15 to obtain

$$\operatorname{rank} \xi_b \geq \operatorname{Cliff}\left(\mathcal{D}_{|F_b}\right) = c_f. \tag{2.55}$$

But $\mathcal{U}_b \subseteq \ker \partial_{\xi_b} = K_{\xi_b}$, so that $\operatorname{rank} \xi_b = g - \dim K_{\xi_b} \leq g - u_f$, and the inequality (2.55) implies that

$$g - u_f \geq c_f,$$

contradicting our very first hypothesis.

2.4.3 The rank of the unitary flat bundle and the Xiao Conjecture

The bound provided in Theorem 2.4.6 is a natural generalization of that in [BGAN16],

$$q_f \leq g - c_f,$$

involving the relative irregularity q_f of the fibration. This is natural since the relative irregularity q_f is the rank of the trivial bundle $\mathcal{O}_B^{q_f}$ involved in the first Fujita decomposition of f and the two decompositions provide a chain of inclusions

$$\mathcal{O}_B^{q_f} \subset \mathcal{U} \subset f_*\omega_{S/B}. \tag{2.56}$$

The relations between the relative irregularity q_f and the genus g have been analyzed intensively since the first results and conjecture of Xiao in the '80's [Xia87b].

We summarize the main known results on the relative irregularity q_f and then we analyse the case of the rank u_f of the unitary flat summand in the second Fujita decomposition.

1. (Beauville [Deb82, Appendix]) $q_f \leq g$ and equality holds if and only if f is trivial, i.e. S is birational to $B \times F$ and f corresponds to the first projection.

2. (Serrano [Ser96]) If f is isotrivial (i.e. its smooth fibres are all mutually isomorphic) but not trivial, then $q_f \leq \frac{g+1}{2}$.

3. (Xiao [Xia87b], [Xia87a]) If f is non-isotrivial and $B \cong \mathbb{P}^1$, then $q_f = q \leq \frac{g+1}{2}$. For general non-isotrivial fibrations the bound $q_f \leq \frac{5g+1}{6}$ holds.

4. (Cai [Cai98]) If f is non-isotrivial and the general fibre is either hyperelliptic or bielliptic, the same bound $q_f \leq \frac{g+1}{2}$ holds.

5. Pirola in [Pir92] constructs for non-isotrivial fibrations a higher Abel-Jacobi map whose vanishing implies $q_f \leq \frac{g+1}{2}$.

6. (Barja-González-Naranjo [BGAN16]) If f is non-isotrivial, then $q_f \leq g - c_f$, where c_f is the Clifford index of the general fibre of f.

Xiao's original conjecture [Xia87b] says that the first bound of (3) holds for any non-trivial fibration has been disproved by Pirola in [Pir92] (more counterexamples have been found later by Albano and Pirola in [AP16]) and was modified in [BGAN16] as follows.

Conjecture 1 (Modified Xiao's conjecture for the relative irregularity)**.** For any non-isotrivial fibred surface $f\colon S \to B$ of genus $g \geq 2$ it holds

$$q_f \leq \left\lceil \frac{g+1}{2} \right\rceil \tag{2.57}$$

The results of [BGAN16] imply the conjecture in the (general) case of maximal Clifford index $c_f = \left\lfloor \frac{g-1}{2} \right\rfloor$. All counterexamples to the original conjecture found by Albano and Pirola satisfy equality for the modified one.

Now we analyse the behaviour of u_f in relation with the previous results.

1. Clearly $u_f \leq \operatorname{rank} f_*\omega_{S/B} = g$ and equality holds if and only if f is locally trivial (i.e. f is a holomorphic fibre bundle): this follows immediately observing that $u_f = g$ if and only if $\chi_f = \deg f_*\omega_{S/B} = 0$.

2. In the case of isotrivial but non-trivial fibrations, there is no better bound for u_f, analogous to Serrano's bound for q_f. Indeed, it is easy to construct such fibred surfaces with $u_f = g$ as appropriate quotients of products of curves. We do not know whether isotrivial non-locally trivial fibrations satisfy some bound or u_f can be arbitrarily close to g.

3. If $B \cong \mathbb{P}^1$, then clearly the flat unitary bundle is trivial, so $u_f = q_f = q \leq \frac{g+1}{2}$ for the non-isotrivial case. Moreover, as observed in [CLZ16, Lemma 3.3.1], Xiao's argument can be extended to bound the rank of the whole unitary bundle, hence $u_f \leq \frac{5g+1}{6}$ for f non-isotrivial (the authors only consider semistable fibrations, but Xiao's proof works identically for any fibration).

4. Lu and Zuo in [LZ14, Theorem 4.7] prove that for a non-isotrivial hyperelliptic fibration there is a finite base change such that the flat bundle of the new Hodge bundle is trivial, and moreover it is known that u_f is non-decreasing by base change (as explained in the proof of Theorem 2.4.6), and so one can apply the bound for the relative irregularity proved by Cai and get the bound $u_f \leq \frac{g+1}{2}$.

5. Pirola's construction is specific to the case of the trivial part of $f_*\omega_{S/B}$, it does not apply directly to the flat unitary bundle.

In case (4) the bound follows by trivializing the unitary flat part by base change. However, the examples of Catanese and Dettweiler precisely show that this is not always possible: indeed they prove that the flat bundle can be trivialized via a base change if and only if the image of the monodromy map associated to the unitary bundle is finite (see also [Bar00]). Catanese and Dettweiler provide examples where the monodromy image is not finite (see [CD16],[CD]).

After [CD] the modified Xiao conjecture implies this one.

Conjecture 2. For any non-isotrivial fibred surface $f: S \to B$ of genus $g \geq 2$ such that the flat unitary summand has finite monodromy, it holds

$$u_f \leq \left\lceil \frac{g+1}{2} \right\rceil . \tag{2.58}$$

Indeed, we go back to the original conjecture by applying the a finite base change trivializing the monodromy.

The same conjecture is false if we release the assumption of finitness of the monodromy group, as recently observed by Lu in [Lu17], where he computes the rank of examples constructed along the line of Catanese and Dettweiler in [CD].

We end the section giving more details about some examples mentioned before.

Example 2.4.7 (cf. [AP16])**.** We first consider some non-isotrivial fibrations $f : S \to B$ constructed by Albano and Pirola in [AP16] with invariants $(q_f, g) = (4, 6), (6, 10)$. We will now show that these fibrations satisfy $q_f = u_f$, and hence the equality (2.58).

The general fibres F_b of f are simultaneously double covers of a fixed curve D and étale cyclic covers (of prime order p) of hyperelliptic curves E_b with simple Jacobian variety. Moreover, the Jacobian of F_b is isogenous to $J(D) \times J(D) \times J(E_b)$, and the simplicity of $J(E_b)$ implies that the relative irregularity of f is $q_f = 2g(D)$ and does not grow under base change.

The curves E_b's form an hyperelliptic fibration $h : T \to B$ and (possibly after finite base change) there is a compatible étale cyclic cover $S \to T$ of order p. Thus the Hodge bundle of h is a direct summand of the Hodge bundle of f, and is complementary to the trivial bundle $\mathcal{O}_B^{\oplus q_f} = \mathcal{O}_B^{\oplus 2g(D)}$ that corresponds to the constant bundles $J(D) \times J(D)$ of the Jacobians (up to isogeny). Thus the non-trivial flat unitary bundle $\mathcal{U}'$ of $f_*\omega_{S/B}$ coincides with the flat unitary bundle of E_h.

Suppose that $u_f > q_f$, that is, $\operatorname{rank}\mathcal{U}' > 0$. Since h is an hyperelliptic fibration, by [LZ14, Theorem Theorem 4.7] the monodromy representation of $\pi_1(B)$ corresponding to $\mathcal{U}'$ has finite non-trivial image. But this would provide a finite covering $\pi : \widetilde{B} \to B$ such that $\pi^*\mathcal{U}'$ is a trivial bundle of the Hodge bundle of base-change of h, contradicting the simplicity of the Jacobian of the general fibres of h.

Example 2.4.8 (cf. [CD])**.** In [CD], Catanese and Dettweiler construct a non-isotrivial fibration $f : S \to B$ with $g = 6$ and $u_f = 4$, hence satisfying the equality in (2.58). We will now show that that the Clifford index of this fibration is $c_f = 1 < \left\lfloor \frac{g-1}{2} \right\rfloor$, hence it is not extremal for Theorem 2.4.6.

Indeed, the smooth fibres of f are cyclic coverings $\pi : C \to \mathbb{P}^1$ of order 7 branched on 4 points. Moreover, a crucial step in Catanese-Dettweiler's construction is that $H^0(C, \omega_C)$ has two 2-dimensional subspaces V_1, V_2 invariant by the action of $\mathbb{Z}/7\mathbb{Z}$. Each of these invariant subspaces corresponds to an invariant g^1_{10}, which has to be of the form $|V_i| = \pi^* |\mathcal{O}_{\mathbb{P}^1}(1)| + D_i$, for (different) effective divisors D_1, D_2 of degree 3. But since both linear series consist of canonical divisors, D_1 and D_2 are linearly equivalent, generating a g^1_3. Thus the smooth fibres of f are trigonal, hence of Clifford index 1.

Chapter 3

Coverings over elliptic curves and the Prym map

In this chapter we analyse local families of coverings over elliptic curves, studying the differential of a local Prym map. The results are in collaboration with Filippo Favale (see [FT17]).

3.1 Generalized Prym variety and local Prym map

In this section we introduce the notion of *generalized Prym variety* of a degree d covering $\pi : F \to E$ of a smooth curve F of genus $g(F) \geq 2$ over an elliptic curve E. The construction of a local parametrizing space, meaning a parametrizing space constructed over a contractible base which does not care of any monodromy action, allows to define a *local* version of the Prym map in this contest, which is enough in order to make computations on the differential.

Let $\pi : F \to E$ be a degree d covering of a smooth compact curve F of genus $g(F) \geq 2$ over an elliptic cuve E, that is a smooth compact curve of genus $g = 1$. Let $\mathrm{Alb}(C)$ be the albanese variety of the curve C, $C = F, E$ and recall that $\mathrm{Alb}(C) \simeq J(C)$ on curves. Moreover, as E is a complex torus we also have $E \simeq J(E)$. The albanese map $\mathrm{alb}(C) : C \to Alb(C)$ induces the commutative diagram

$$\begin{tikzcd} F \arrow[r, "\mathrm{alb}(F)"] \arrow[d, "\pi"] & \mathrm{Alb}(F) \arrow[d, "\mathrm{alb}(\pi)"] \\ E \arrow[r, "\mathrm{alb}(E)"] & \mathrm{Alb}(E) \end{tikzcd} \quad , \tag{3.1}$$

Definition 3.1.1. Let $\pi : F \to E$ be a covering of a smooth compact curve F over an elliptic curve E. The *generalized Prym variety* $P(\pi)$ associated to $\pi : F \to E$ (or simply *Prym variety*) is the connected component of $\mathrm{Ker}(\mathrm{Alb}(\pi))$ that contains the 0, i.e.

$$P(\pi) = \mathrm{Ker}(\mathrm{Alb}(\pi))_0. \tag{3.2}$$

Remark 3.1.2. The generalized Prym variety $P(\pi)$ is an abelian variety of dimension $g - 1$ with the polarization Θ_P given by $\Theta_{JF}|_P$ via the embedding

$$P(\pi) \hookrightarrow JF.$$

The map $\pi : F \to E$ induces a map $tr_\pi : H^0(\omega_F) \to H^0(\omega_E)$ called the *trace* of π (see Appendix A of [Kan04] for the definition). The trace satisfies

$$tr_\pi \circ \pi^* = \mathrm{Deg}(\pi)\mathrm{Id}_{H^0(\omega_E)}.$$

If we define

$$H^0(\omega_F)^- = \mathrm{Ker}(tr_\pi) \tag{3.3}$$

we have a canonical splitting

$$H^0(\omega_F) = \pi^* H^0(\omega_E) \oplus H^0(\omega_F)^- \tag{3.4}$$

and we can identify the quotient $H^0(\omega_F)/\pi^* H^0(\omega_E)$ with $H^0(\omega_F)^-$. In particular, the tangent bundle of $P(\pi)$ can be described as

$$TP(\pi) = \left(\frac{H^0(\omega_F)}{\pi^* H^0(\omega_E)}\right)^\vee \otimes \mathcal{O}_{P(\pi)} = (H^0(\omega_F)^-)^\vee \otimes \mathcal{O}_{P(\pi)}. \tag{3.5}$$

We introduce *Local families of coverings over elliptic curves* and their parametrizing spaces, the *Local Hurwitz spaces.*

Let F be a smooth curve of genus $g \geq 2$ and $\pi : F \to E$ a degree d covering of F over an elliptic curve E. Denote with

$$R = \sum_{j=1}^{n} (n_j - 1) a_j$$

the ramification divisor defined by the ramification points $a_j \in F$ of ramification index n_j and let b_j be the branch point corresponding to the ramification point a_j, that is $\pi(a_j) = b_j$.

Remark 3.1.3. We assume to index with each ramification point a branch point. In other words, we allow to have only one ramification point over each branch point and thus n_j is the degree of π when restricted to a suitable neighborhood of a_j. The general case in treated as a limit.

Fix α a generator of $H^0(\omega_E)$. Choose a suitable set $\{\Delta_j\}$ of coordinate neighborhoods centered in the points b_j and call w_j the corresponding coordinate on E. This is not needed at the moment but observe that we can assume that $\alpha|_{\Delta_j} = dw_j$. We can chose a collection of pairwise disjoint coordinate neighborhoods (U_j, z_j) centered in a_j in such a way that $w_j = \pi|_{U_j}(z_j) = z_j^{n_j}$.

Denote by $\mathcal{H}_E$ the polydisc $\Pi_{j=1}^n \Delta_j$ and consider the coordinates $t = (t_j)_{j=1}^n$ defined by the relation

$$t_j(P_1, \cdots, P_n) = w_j(P_j).$$

We follow the construction as in [Kan04, Section 4.1], generalizing the case of simple ramifications to arbitrary ones.

Definition 3.1.4. A *Local family of coverings deforming π parametrized by $\mathcal{H}_E$* is a family

$$(\Psi, f) : \mathcal{F} \to E \times \mathcal{H}_E$$

of d-sheeted branched coverings deforming π parametrized by $\mathcal{H}_E$ which satisfies the condition:

$$w_j = \Psi|_{U_j}(z_j, t) = z_j^{n_j} + t_j. \tag{3.6}$$

We will usually forget to write *local*, assuming that the base is always a polydisk $\mathcal{H}_E$.

Let $\pi : F \to E$ be a d-sheeted covering of a smooth compact curve F of genus $g(F) \geq 2$ over an elliptic curve E. A local family of coverings xparametrized by $\mathcal{H}_E$ is a *(Local) deformation of π parametrized by $\mathcal{H}_E$* if (up to restrict $\mathcal{H}_E$) $\pi : F \to E$ is the central fibre of the family.

Remark 3.1.5. Assumption (3.6) forces the ramification orders to remain costant and allows different branch points to move indipendently.

In other words, each family is simply a deformation of the the central fiber $\pi : F \to E$ which corresponds to each $b' \in \mathcal{H}_E$ a covering $\pi_{b'} : F_{b'} \to E$ preserving the ramification index.

Remark 3.1.6. In the previous definition, we assumed that the elliptic curve is fixed and thus the families of coverings are obtained just by moving the branches.

The tangent space to $\mathcal{H}_E$ in $b = (b_1, \dots, b_n) \in \mathcal{H}_E$ is

$$T_b\mathcal{H}_E \simeq \bigoplus_{j=1}^{n} T_{b_j}E \simeq \bigoplus_{j=1}^{n} \mathbb{C}\frac{\partial}{\partial t_j},$$

where the tangent vectors on the right are evaluated in 0.

We now take into account the deformation of the elliptic curve. Following [ACG11], if one chooses $c \in E$ not among the b_j and considers a small coordinate neighborhood (N, v) of c (eventually shrinking Δ_j in such a way that for all j they are disjoint from N), one can consider the associated Schiffer variation $\mathcal{E} \to N$ of E with coordinate s. Observe that we can assume $\alpha|_N = dv$. Taking into account also the movement of the branch points one has a family $f : \mathcal{F} \to \mathcal{H}_E \times N$ of curves of genus g that fits into the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "p"] \arrow[d, "f"'] & \mathcal{E} \arrow[d] \\ \mathcal{H}_E \times N \arrow[r] & N \end{tikzcd} \tag{3.7}$$

For a choice $(b', s') \in \mathcal{H} = \mathcal{H}_E \times N$ we have an elliptic curve $E_{s'}$, the fiber of the map $\mathcal{E} \to N$ over s', a curve $F_{(b',s')}$ of genus g and a covering

$$\pi_{(b',s')} = p|_{F_{(b',s')}} : F_{(b',s')} \to E_{s'}.$$

Definition 3.1.7. A *Local family of coverings with central fiber π parametrized by $\mathcal{H}$* (or simply *local family of coverings*) is a morphism as above.

The tangent space to $\mathcal{H}$ in (b,s) is

$$T_{(b,s)}\mathcal{H} \simeq \left(\bigoplus_{j=1}^{n} \mathbb{C}\frac{\partial}{\partial t_j}\right) \oplus \mathbb{C}\frac{\partial}{\partial s}.$$

and, clearly, containts $T_b\mathcal{H}_E$ in a natural way. We stress that unless otherwise stated, we will always refer to the families of coverings constructed in this sections.

We have constructed a parametrizing space $\mathcal{H}$ of degree d coverings from curves of genus g greater than 2 over elliptic curves as the branches and the ramification type are enough to reconstruct the covering). Thus for each (b,s) we can construct a generalized Prym variety of the covering constructed with the data (b,s). As by construction, the polarization-type remains constant, we get a map with image a certain the moduli space of abelian variety.

Definition 3.1.8. The *Local Prym map* is defined as the map

$$\begin{aligned} \mathcal{H} &\xrightarrow{\Phi} \mathcal{A}_{g-1} \\ (b,s) &\longmapsto [P(\pi_{(b,s)})] \end{aligned} \tag{3.8}$$

where $\mathcal{A}_{g-1}$ is the moduli space of polarized abelian varieties with polarization-type given by 3.1.2.

Remark 3.1.9. In the same way, one can construct a local Prym map Φ_E on local family of coverings over a fixed elliptic curve E.

To avoid technical subtleties around singular points of $\mathcal{A}_{g-1}$, we will consider the period map $P : \mathcal{H} \to \mathbb{D}$ (or $P_E : \mathcal{H}_E \to \mathbb{D}$) instead of the Prym map Φ (respectively Φ_E), where $\mathbb{D}$ is a suitable period domain for $\mathcal{A}_{g-1}$. The interested reader is referred to [Kan04, Section 3] for technical details.

3.2 A direct formula for the differential of the Prym map

In this section we will prove an explicit formula for the codifferential of the period map in terms of the residue at the ramification points of some forms. The framework is similar to the one in [Kan04] with the main difference being that we don't restrict ourselves to the case of simple ramification. First of all we introduce some notations.

Fix an elliptic curve E and let $\pi : F \to E$ be a covering of E with F of genus g. Consider

$$(\Psi, f) : \mathcal{F} \to E \times \mathcal{H}_E,$$

the local family of coverings with fixed base E, central fiber π and parameter space $\mathcal{H}_E$ constructed in Section 3.1. By construction, it induces a family $f : \mathcal{F} \to \mathcal{H}_E$ with central fiber $\mathcal{F}_0 = F$. If we consider a minimal versal deformation $f' : \mathcal{F}' \to M$ of F then the previous family is induced by f' by means of a pullback. More precisely, there exists a holomorphic map $h_E : \mathcal{H}_E \to M$ such that

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "f"'] & \mathcal{F}' \arrow[d, "f'"] \\ \mathcal{H}_E \arrow[r, "h_E"'] & M \end{tikzcd} \tag{3.9}$$

is commutative, the tangent spaces satisfy

$$T_0 M \simeq H^1(T_F) \simeq H^0(\omega_F^{\otimes 2})^\vee$$

and the evaluation of dh_E in a tangent vector v in $T_0\mathcal{H}_E$ gives exactly the Kodaira-Spencer map KS_E associated to $\mathcal{F} \to \mathcal{H}_E$ and evaluated in v.

The following provides a formula for the codifferential of the Kodaira Spencer map in terms of the residue of differential forms.

Proposition 3.2.1. *Under the identifications introduced above, we have that the codifferential of the Kodaira Spencer map*

$$dh_E^\vee : T_0^\vee M \to T_0^\vee \mathcal{H}_E$$

can be written as

$$dh_E^\vee(\varphi) = \sum_{j=1}^n \gamma_j dt_j \qquad \textit{where} \qquad \gamma_j = 2\pi i \operatorname{Res}_{a_j}\left(\frac{\varphi}{\pi^*\alpha}\right) \tag{3.10}$$

and $\varphi \in T_0^\vee M = H^0(\omega_F^{\otimes 2})$.

Proof. For every $\varphi \in H^0(\omega_F^{\otimes 2})$ we have that $dh_E^\vee(\varphi)$ is identified, as cotangent vector on M in 0, by the complex numbers γ_j such that

$$dh_E^\vee(\varphi) = \sum_{j=1}^n \gamma_j dt_j.$$

By construction, we can obtain these numbers simply by pairing $dh_E^\vee(\varphi)$ against $\frac{\partial}{\partial t_j}$:

$$\gamma_j = dh_E^\vee(\varphi)\left(\frac{\partial}{\partial t_j}\right) = \varphi\left(dh_E\left(\frac{\partial}{\partial t_j}\right)\right) = \varphi\left(KS_E\left(\frac{\partial}{\partial t_j}\right)\right).$$

In order to develop the computation we may proceed using a description of KS_E in terms of the Čech cohomology (see [Hor73]). To do it consider the exact sequence

$$0 \longrightarrow T_F \xrightarrow{d\pi} \pi^* T_E \xrightarrow{\psi} \mathcal{R} \longrightarrow 0 \tag{3.11}$$

and let δ be the coboundary map $H^0(\mathcal{R}) \to H^1(T_F)$. Then KS_E factors as $\delta \circ \tau = KS_E$ where $\tau : T_b H \to H^0(\mathcal{R})$ is the *characteristic map* of the family (see [Hor73] for the definition and the proof of this fact). Hence we can unfold the calculation using these exact sequences.

If one restricts the exact sequence (3.11) on U_j (or some sufficiently small subset of this coordinate neighborhood), it can be identified with

$$0 \longrightarrow \mathcal{O}_{U_j}\frac{\partial}{\partial z_j} \xrightarrow{d\pi} \mathcal{O}_{U_j}\frac{\partial}{\partial w_j} \xrightarrow{\psi} \mathcal{R}|_{U_j} \longrightarrow 0. \tag{3.12}$$

The first map sends $\frac{\partial}{\partial z_j}$ to $n_j z_j^{n_j-1}\frac{\partial}{\partial w_j}$ while the second one is simply the restriction to the ramification locus. Let $\mathcal{U} = \{U_0, U_1, \ldots, U_n\}$ where U_j for $j = 1, \ldots, n$ are the neighborhoods defined above and $U_0 = F \setminus \{a_j\}$. Let, as usual, $U_{\alpha,\beta}$, be a shorthand for $U_\alpha \cap U_\beta$ with $\alpha < \beta$. If $\eta = [\eta_j] \in H^0(\mathcal{U}, \mathcal{R})$ with $\eta_0 = 0$ and $\eta_j = p_j(z_j)\frac{\partial}{\partial w_j}$ we have

$$\delta(\eta) = [\lambda_{\alpha,\beta}] \qquad \text{with} \qquad \lambda_{0,j} = \frac{p_j(z_j)}{n_j z^{n_j-1}}\frac{\partial}{\partial z_j}$$

for $j > 0$ and $\lambda_{\alpha,\beta} = 0$ if $\alpha, \beta > 0$. Following [Hor73] and using Equation (3.6) we have

$$\tau\left(\frac{\partial}{\partial t_j}\right) = [\tau_k^{(j)}] \qquad \text{with} \qquad \tau_k^{(j)} = \begin{cases} 0 & k \neq j \\ \frac{\partial}{\partial w_j} & k = j. \end{cases} \tag{3.13}$$

Hence we have

$$KS_E\left(\frac{\partial}{\partial t_j}\right) = \delta\left(\tau\left(\frac{\partial}{\partial t_j}\right)\right) = [\chi_{\alpha,\beta}^{(j)}] \qquad \text{with} \qquad \chi_{\alpha,\beta}^{(j)} = \begin{cases} \frac{1}{n_j z^{n_j-1}}\frac{\partial}{\partial z_j} & (\alpha,\beta) = (0,j) \\ 0 & \text{otherwise.} \end{cases}$$

If $\varphi \in H^0(\omega_F^{\otimes 2})$ we can represent it as Čech-cocycle as $[\phi_j]$ where

$$\phi_0 = \phi|_{U_0} \qquad \text{and} \qquad \phi_j = q_j(z_j)dz_j^2$$

are the local expressions of φ in coordinates around a_j. The numbers we are interested in are simply the ones obtained by considering the perfect pairing

$$H^0(\omega_F^{\otimes 2}) \otimes H^1(T_F) \longrightarrow H^1(\omega_F) \xrightarrow{\simeq} \mathbb{C} \tag{3.14}$$

applied to $KS_E\left(\frac{\partial}{\partial t_j}\right)$ and φ. Using Čech cohomology, the image in $H^1(\omega_F)$ of our product is given by the Čech class $[\epsilon_{\alpha,\beta}^{(j)}]$ with

$$\epsilon_{\alpha,\beta}^{(j)} = \begin{cases} \frac{q_j(z_j)}{n_j z^{n_j-1}}dz_j & (\alpha,\beta) = (0,j) \\ 0 & \text{otherwise.} \end{cases}$$

What remains to be proven is the analogous to the calculation of [Kan04] for the case of simple ramification: roughly, one can adapt the techniques of [ACG11, pag. 14-15] to develop the last isomorphism of (3.14) in order to finally get

$$\gamma_j = 2\pi i \operatorname{Res}_0 \frac{q_j(z_j)dz_j^2}{n_j z_j^{n_j-1}dz_j} = 2\pi i \operatorname{Res}_{a_j} \frac{\varphi}{\pi^*\alpha}.$$

□

Consider now the family $p : \mathcal{F} \to \mathcal{E}$ with central fiber $\pi : F \to E$ and parameter space $\mathcal{H} = \mathcal{H}_E \times N$ as defined in Section 3.1. As before, we have an induced deformation $f : \mathcal{F} \to \mathcal{H}$ of F, its associated Kodaira-Spencer map KS and, when a minimal versal deformation $f' : \mathcal{F}' \to M$ of F is chosen, an holomorphic map $h : \mathcal{H} \to M$ such that

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "f"'] & \mathcal{F}' \arrow[d, "f'"] \\ \mathcal{H} \arrow[r, "h"'] & M \end{tikzcd} \tag{3.15}$$

is commutative. Again, as $T_0M \simeq H^1(T_F)$, we can identify dh with KS. We will denote by $x_1, \dots, x_d$ the points of the fiber of π over the point c which, by construction, are all different.

Proposition 3.2.2. *Under the identifications introduced above, we have that the codifferential of the Kodaira Spencer map*

$$dh^\vee : T_0^\vee M \to T_0^\vee \mathcal{H}$$

can be written for any $\varphi \in H^0(\omega_F^{\otimes 2}) = T_0^\vee M$ *as* $dh^\vee(\varphi) = \sum_{j=1}^n \gamma_j dt_j + \gamma ds$ *where*

$$\gamma_j = 2\pi i \operatorname{Res}_{a_j}\left(\frac{\varphi}{\pi^*\alpha}\right) \quad \text{and} \quad \gamma = 2\pi i \sum_{k=1}^{d} \frac{\varphi}{\pi^*\alpha}(x_k). \tag{3.16}$$

Proof. As before, by duality,

$$dh^\vee(\varphi) = \varphi \circ dh = \varphi \circ KS.$$

It is then clear that the formula for γ_j follows directly from Proposition 3.2.1. The one that gives γ, as it involves calculations done far from the ramification points, doesn't depend on the type of the ramifications. Hence, the one given in [Kan04] when π as only simple ramification is still valid. □

Recall that we have a decomposition of $H^0(\omega_F)$ given by $H^0(\omega_F)^- \oplus \pi^* H^0(\omega_E)$ where the first space is the vector space of 1-forms on F with trivial trace. This induces a decomposition on $\operatorname{Sym}^2(H^0(\omega_F))$. Unless otherwise specified, consider $\operatorname{Sym}^2(H^0(\omega_F)^-)$ as a subspace of $\operatorname{Sym}^2(H^0(\omega_F))$ in the natural way. Let $m : \operatorname{Sym}^2(H^0(\omega_F)^2) \to H^0(\omega_F^{\otimes 2})$ be the multiplication map. Denote by $P : \mathcal{H} \to \mathbb{D}$ the period map associated to the Prym map $\Phi : \mathcal{H} \to \mathcal{A}_{g-1}$ where $\mathbb{D}$ is a suitable period domain. We are ready to prove the following, which gives a direct formula for the period map.

Theorem 3.2.3. *With the notation introduced in this section, for any* $\varphi \in \operatorname{Sym}^2(H^0(\omega_F)^-)$ *we have*

$$d_o P^\vee(\varphi) = \sum_{j=1}^{n} \operatorname{Res}_{a_j}\left(\frac{m(\varphi)}{\pi^*\alpha}\right) dt_j + \left(\sum_{k=1}^{d} \frac{m(\varphi)}{\pi^*\alpha^2}(x_k)\right) ds. \tag{3.17}$$

Proof. Theorem 3.21 of [Kan04] expresses the codifferential of the period map calculated in $\varphi \in$

$\mathrm{Sym}^2(H^0(\omega_F)^-)$ and paired with $\frac{\partial}{\partial t_j}$ as

$$\varphi\left(KS\left(\frac{\partial}{\partial t_j}\right)\right)$$

without any restriction on the ramification type. In particular, this formula, together with Proposition 3.2.2 ends the proof of the Theorem. □

Remark 3.2.4. As a consequence of the last Theorem we can conclude that, if we fix E, the codifferential $d_{\mathrm{o}}P_{\mathrm{E}}^{\vee} : \mathrm{Sym}^2(H^0(\omega_F)^-) \to T_0^{\vee}\mathcal{H}_E$ factors as

$$\begin{tikzcd} H^0(\omega_F^{\otimes 2}) \arrow[d, "dh_E^{\vee}"'] & \mathrm{Sym}^2\left(H^0(\omega_F)\right) \arrow[l, "dT^{\vee}"'] \\ T_0^{\vee}\mathcal{H}_E & \mathrm{Sym}^2(H^0(\omega_F)^-) \arrow[u, hook, "\sigma"'] \arrow[l, "d_{\mathrm{o}}P_{\mathrm{E}}^{\vee}"] \end{tikzcd} \tag{3.18}$$

where T is the Torelli map (so that $m = dT^{\vee}$) and σ is the lifting of the projection of $\mathrm{Sym}^2(H^0(\omega_F)) \to \mathrm{Sym}^2(H^0(\omega_F)^-)$ induced by the decomposition $H^0(\omega_F) = H^0(\omega_F)^- \oplus \pi^*H^0(\omega_E)$. The commutativity of the diagram is a consequence of Proposition 3.2.1 as, for any $\varphi \in H^0(\omega_F)\,\hat{\otimes}\,\pi^*H^0(\omega_E)$, we have that $\varphi/\pi^*\alpha$ is holomorphic and hence has residue zero everywhere.

3.3 The canonical embedding and the kernel of the Prym map

In this section we will use the technical result of the previous section in order to prove that $\dim(\mathrm{Ker}(d_{\mathrm{o}}P_{\mathrm{E}})) = 1$ for arbitrary ramification types and a geometric criterion to determine whether $\dim(\mathrm{Ker}(d_{\mathrm{o}}P)) = 1$ or not. First we fix some notation and facts about the canonical curves that we are going to use extensively in the following.

As F has genus $g \geq 3$ and is not hyperelliptic, we may identify it with its canonical model in $\mathbb{P} = \mathbb{P}H^0(\omega_F)^{\vee}$. This is a non-degenerate curve of degree $2g-2$, which is also projectively normal by a classical result of Max Noether (see, for example, [ACGH85]). One of the consequences of this fact is that the multiplication map $m_k : \mathrm{Sym}^k H^0(\omega_F) \to H^0(\omega_F^{\otimes k})$ is surjective. As before we will denote m_2 simply by m. We will use frequently the natural identifications $H^0(\mathcal{O}_{\mathbb{P}}(d)) = \mathrm{Sym}^d H^0(\omega_F)$ which enable us to identify $\mathbb{P}(\mathrm{Ker}(m_d))$ with the space of hypersurfaces of degree d in $\mathbb{P}H^0(\omega_F)^{\vee}$ that contain F. By abuse of notation we will simply say that an element in $\mathrm{Sym}^d H^0(\omega_F)$ is an hypersurface of degree d if no confusion arises. In particular, if I_F is the ideal sheaf of F in $\mathbb{P}H^0(\omega_F)^{\vee}$, then $\mathrm{Ker}(m) = H^0(I_F(2))$ gives the set of all quadrics in $\mathbb{P}H^0(\omega_F)^{\vee}$ containing the curve F, and has dimension $\frac{(g-2)(g-3)}{2}$.

Recall that the decomposition

$$H^0(\omega_F) = H^0(\omega_F)^- \oplus \pi^*H^0(\omega_E)$$

where the first space is the space of forms with zero trace.

Since elements in $H^0(\omega_F)$ are linear equations on $\mathbb{P}H^0(\omega_F)^\vee$, all the hyperplanes defined by elements in $H^0(\omega_F)^-$ intersect in a single point q^- of $\mathbb{P}$ which is a point really important in what will follows. We have also a particular hyperplane, the one defined by the subspace $\pi^* H^0(\omega_E)$ which will be denoted by H^-. More precisely,

$$q^- = \mathbb{P}((H^0(\omega_F)^-)^\perp) \qquad \text{and} \qquad H^- = \mathbb{P}((\pi^* H^0(\omega_E))^\perp)$$

As before, we will fix a generator α of $H^0(\omega_E)$ so that

$$\mathrm{Sym}^2(H^0(\omega_F)) = \mathrm{Sym}^2(H^0(\omega_F)^-) \oplus \left(\pi^*\alpha \,\hat{\otimes}\, H^0(\omega_F)\right). \tag{3.19}$$

Given a quadric Q in $\mathbb{P}$ we will denote by $G_Q \in \mathrm{Sym}^2(H^0(\omega_F))$ one of its equations and by $G_Q^- \in \mathrm{Sym}^2(H^0(\omega_F)^-)$ and $\omega_Q \in H^0(\omega_F)$ the only elements such that

$$G_Q = G_Q^- + \pi^*\alpha \,\hat{\otimes}\, \omega_Q$$

under the decomposition (3.19). Finally, given a quadric Q, we will denote by Q^- the cone given by the equation G_Q^-, i.e. the quadric such that $G_{Q^-} = G_{Q^-}^- = G_Q^-$.

We have the following result.

Lemma 3.3.1. *There is a natural inclusion of $H^0(I_F(2))$ in* $\mathrm{Ker}(d_o P_{\mathrm{E}}^\vee)$.

Proof. Recall that, fixed a family of coverings with base E and central fiber $\pi : F \to E$, by fixing a minimal versal deformation $\mathcal{F}' \to M$ of F, we can construct $h_E : \mathcal{H} \to M$ like in diagram (3.9). As observed in Remark 3.2.4 we have a commutative diagram

$$\begin{tikzcd}
 & & 0 & & \\
0 \arrow[r] & \mathrm{Ker}\, d_o P_{\mathrm{E}}^\vee \arrow[r, hook, "j"] & \mathrm{Sym}^2(H^0(\omega_F)^-) \arrow[u] \arrow[r, "d_0 P_E^\vee"] & T_0^\vee \mathcal{H}_E & \\
0 \arrow[r] & H^0(I_F(2)) \arrow[u, hook, "\gamma"] \arrow[r, hook, "\iota"] & \mathrm{Sym}^2(H^0(\omega_F)) \arrow[u, two heads, "pr"] \arrow[r, two heads, "m"] & H^0(\omega_F^{\otimes 2}) \arrow[u, "dh_E^\vee"'] \arrow[r] & 0 \\
 & & H^0(\omega_F) \,\hat{\otimes}\, \pi^* H^0(\omega_E) \arrow[u, hook] & & \\
 & & 0 \arrow[u] & &
\end{tikzcd} \tag{3.20}$$

It is easy to see that the image of $pr \circ \iota$ lives in $\mathrm{Ker}(d_o P_{\mathrm{E}}^\vee)$ so we have a well defined map $\gamma : H^0(I_F(2)) \to \mathrm{Ker}(d_o P_{\mathrm{E}}^\vee)$. We want to prove that this map is indeed injective. This follows from the geometry of the problem. Indeed, if a quadric Q contains F, i.e. if the quadric has equation

$$G_Q = G_Q^- + \pi^*\alpha \,\hat{\otimes}\, \omega_Q \in H^0(I_F(2)),$$

and if $\gamma(G_Q) = 0$ then we have that the quadric has equation $\pi^*\alpha \,\hat{\otimes}\, \omega_Q$. But this is impossible because such a quadric the union of two planes (one of which is H^-) and the canonical curve is non-degenerate. Hence γ is injective. $\square$

Theorem 3.3.2. *Let $\pi : F \to E$ be a covering with F non-hyperelliptic, consider a local family of coverings with base E and parameter space $\mathcal{H}_E$ constructed in Section 3.1. Let P_E be the period mapping associated to the Prym map Φ_E. Then* $\dim(\mathrm{Ker}(d_o P_{\mathrm{E}})) = 1$.

Proof. First of all, observe that for dimensional reasons, one has $\dim(\mathrm{Ker}(d_o P_{\mathrm{E}})) = 1$ if and only if

$$\dim(\mathrm{Ker}(d_o P_{\mathrm{E}}^{\vee})) = \frac{g(g-1)}{2} - n + 1.$$

From the splitting $H^0(\omega_F) = H^0(\omega_F)^- \oplus \pi^* H^0(\omega_E)$ we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^0(\omega_F) \arrow[r, "\hat{\otimes}\,\pi^*\alpha"] & \mathrm{Sym}^2(H^0(\omega_F)) \arrow[r, "pr"] \arrow[dr, "\Psi"'] & \mathrm{Sym}^2(H^0(\omega_F)^-) \arrow[r] \arrow[d, "d_0 P_E^{\vee}"] & 0 \\ & & & T_0\mathcal{H}_E & \end{tikzcd}$$

with Ψ defined by extending the formula in Theorem 3.2.3 to $\mathrm{Sym}^2(H^0(\omega_F))$. This can be done because, as previously observed (see Remark 3.2.4), $d_o P_{\mathrm{E}}^{\vee}(H^0(\omega_F) \,\hat{\otimes}\, \pi^*\alpha) = \{0\}$. In particular, we have the relation

$$\dim(\mathrm{Ker}(d_o P_{\mathrm{E}}^{\vee})) = \dim(\mathrm{Ker}(\Psi)) - \dim(\mathrm{Ker}(pr)) = \dim(\mathrm{Ker}(\Psi)) - g. \tag{3.21}$$

By definition, Ψ factors through the multiplication map m as $\Psi = \bar{\Psi} \circ m$. The map $\bar{\Psi}$ is well defined as, by Lemma 3.3.1, $\mathrm{Ker}(m) \subset \mathrm{Ker}(\Psi)$.

$$\begin{tikzcd} 0 \arrow[r] & H^0(\omega_F) \arrow[r, "\hat{\otimes}\,\pi^*\alpha"] & \mathrm{Sym}^2(H^0(\omega_F)) \arrow[r, "pr"] \arrow[d, "m"] \arrow[dr, "\Psi"] & \mathrm{Sym}^2(H^0(\omega_F)^-) \arrow[r] \arrow[d, "d_0 P_E^{\vee}"] & 0 \\ & & H^0(\omega_F^{\otimes 2}) \arrow[r, "\bar{\Psi}"] & T_0\mathcal{H}_E & \end{tikzcd}$$

Being m surjective (as F is non-hyperelliptic) we obtain the further relation

$$\dim(\mathrm{Ker}(\Psi)) = \dim(\mathrm{Ker}(\bar{\Psi})) + \dim(\mathrm{Ker}(m)) = \dim(\mathrm{Ker}(\bar{\Psi})) + \frac{(g-2)(g-3)}{2}. \tag{3.22}$$

As the divisor associated to $\pi^*\alpha$ is exactly R, the ramification divisor, we have that $\omega_F = \mathcal{O}_F(R)$ and there is an exact sequence

$$0 \longrightarrow \omega_F \xrightarrow{\cdot\pi^*\alpha} \omega_F^{\otimes 2} \longrightarrow \omega_F^{\otimes 2}|_R \longrightarrow 0 \tag{3.23}$$

which yields, denoting with V the quotient $H^0(\omega_F^{\otimes 2})/(H^0(\omega_F)\cdot\pi^*\alpha)$, the exact sequences

$$0 \longrightarrow H^0(\omega_F) \xrightarrow{\cdot\pi^*\alpha} H^0(\omega_F^{\otimes 2}) \xrightarrow{\quad\epsilon\quad} V \longrightarrow 0 \tag{3.24}$$

$$0 \longrightarrow V \xrightarrow{\quad\zeta\quad} H^0(\omega_F^{\otimes 2}|_R) \longrightarrow H^1(\omega_F) \longrightarrow 0$$

Let $\eta \in \mathrm{Ker}(\epsilon)$. We want to prove that $\bar\Psi(\eta) = 0$. This is easily proven: write η as $\omega\cdot\pi^*\alpha$ and observe that

$$\bar\Psi(\eta) = (\bar\Psi \circ m)(\omega \mathbin{\hat\otimes} \pi^*\alpha) = (d_{\mathrm{o}} P_{\mathrm{E}}^\vee \circ pr)(\omega \mathbin{\hat\otimes} \pi^*\alpha) = 0$$

because $\omega \mathbin{\hat\otimes} \pi^*\alpha \in \mathrm{Ker}(pr)$. In particular, $\mathrm{Ker}(\epsilon) \subset \mathrm{Ker}(m)$ and we can define a map $\lambda : V \to T_0^\vee\mathcal{H}_E$ such that $\bar\Psi = \lambda\circ\epsilon$. Moreover

$$\dim(\mathrm{Ker}(\bar\Psi)) = \dim(\mathrm{Ker}(\lambda)) + g. \tag{3.25}$$

Using the second exact sequence in 3.24 we can also define a map $\mu : H^0(\omega_F^{\otimes 2}|_R) \to T_0\mathcal{H}_E$ such that $\mu\circ\zeta = \lambda$.

$$\begin{tikzcd}
 & H^0(I_F(2)) \arrow[d, hook] & & & \\
H^0(\omega_F) \arrow[r, hook, "\hat\otimes\,\pi^*\alpha"] \arrow[dr, hook, "\cdot\pi^*\alpha"'] & \mathrm{Sym}^2(H^0(\omega_F)) \arrow[rr, two heads, "pr"] \arrow[d, "m"] \arrow[drr, "\Psi"] & & \mathrm{Sym}^2(H^0(\omega_F)^-) \arrow[d, "d_0 P_E^\vee"] & \\
 & H^0(\omega_F^{\otimes 2}) \arrow[rr, "\bar\Psi"] \arrow[dr, "\epsilon"'] \arrow[drr] & & T_0^\vee\mathcal{H}_E & \\
 & & V \arrow[r, hook, "\zeta"'] \arrow[ur, "\lambda"] & H^0(\omega_F^{\otimes 2}|_R) \arrow[u, "\mu"'] \arrow[r, two heads] & H^1(\omega_F)
\end{tikzcd} \tag{3.26}$$

Note that we have several ways to define μ. Since $\omega_F^{\otimes 2}|_R = \omega_F(R)|_R$ the global sections of $\omega_F^{\otimes 2}|_R$ are just collections of meromorphic tails on the points of ramification, i.e. elements

$$\left\{\sum_{j=1}^{n_k-1}\beta_{jk}\frac{dz_k}{z_k^j}\right\}_{a_k\in R}$$

where n_k is the ramification index of the point a_k. In particular, we can define μ as the map which gives the residue in the corresponding point of the meromorphic tail. This ensures that the diagram is commutative. In addition, μ is surjective (this because the image of a collection of meromorphic tails $\{s_k\}$, one for each point of ramification, with $\beta_{1m} = \delta_{km}$, generates the image), and as a consequence, $\zeta|_{\mathrm{Ker}(\lambda)}$ is an isomorphism between $\mathrm{Ker}(\lambda)$ and $\mathrm{Ker}(\mu)$. Hence, as wanted. □

Now we will prove the first main theorem:

Theorem 3.3.3. *Let $\pi : F \to E$ be a covering with F non-hyperelliptic, consider the local family of coverings with parameter space $\mathcal{H}$ constructed in Section 3.1. Let P be the period mapping associated* *to the Prym map $\Phi : \mathcal{H} \to \mathcal{A}_{g-1}$. Using the same notations of Theorem 3.2.3 we have*

$$\dim(\mathrm{Ker}(d_{\mathrm{o}}P))) = 1 \qquad \Longleftrightarrow \qquad \exists \beta \in \mathrm{Ker}(d_{\mathrm{o}}P_{\mathrm{E}}^{\vee}) \quad | \quad \sum_{k=1}^{d} \frac{m(\beta)}{\pi^*\alpha^2}(x_k) \neq 0.$$

Proof. First of all consider the diagrams

$$\begin{tikzcd} T_0\mathcal{H} \arrow[r, "d_0P"] & T_{P(o)}\mathbb{D} \\ T_0\mathcal{H}_E \arrow[u, hook] \arrow[ur, "d_0P_E"'] & \end{tikzcd} \qquad \begin{tikzcd} T_0^{\vee}\mathcal{H} \arrow[d] & T_{P(o)}^{\vee}\mathbb{D} \arrow[l, "d_0P^{\vee}"'] \arrow[dl, "d_0P_E^{\vee}"] \\ T_0^{\vee}\mathcal{H}_E & \end{tikzcd}$$

and observe that one always has

$$\mathrm{Ker}(d_{\mathrm{o}}P_{\mathrm{E}}) \subseteq \mathrm{Ker}(d_{\mathrm{o}}P) \qquad \mathrm{Ker}(d_{\mathrm{o}}P^{\vee}) \subseteq \mathrm{Ker}(d_{\mathrm{o}}P_{\mathrm{E}}^{\vee}).$$

Moreover, the codimensions are at most 1. If one considers the exact sequences

$$0 \longrightarrow \mathrm{Ker}(d_{\mathrm{o}}P_{\mathrm{E}}) \longrightarrow T_0\mathcal{H}_E \longrightarrow T_{P(o)}\mathbb{D} \longrightarrow \mathrm{Ker}(d_{\mathrm{o}}P_{\mathrm{E}}^{\vee})^{\vee} \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Ker}(d_{\mathrm{o}}P) \longrightarrow T_0\mathcal{H} \longrightarrow T_{P(o)}\mathbb{D} \longrightarrow \mathrm{Ker}(d_{\mathrm{o}}P^{\vee})^{\vee} \longrightarrow 0$$

it is clear that $\mathrm{Ker}(d_{\mathrm{o}}P_{\mathrm{E}}) = \mathrm{Ker}(d_{\mathrm{o}}P)$ if and only if $\mathrm{Ker}(d_{\mathrm{o}}P^{\vee}) \subsetneq \mathrm{Ker}(d_{\mathrm{o}}P_{\mathrm{E}}^{\vee})$. Hence we have

$$\dim(\mathrm{Ker}(d_{\mathrm{o}}P))) = 1 \Longleftrightarrow \mathrm{Ker}(d_{\mathrm{o}}P^{\vee}) \subsetneq \mathrm{Ker}(d_{\mathrm{o}}P_{\mathrm{E}}^{\vee}).$$

This is true if and only there exists an element $\beta \in \mathrm{Ker}(d_{\mathrm{o}}P_{\mathrm{E}}^{\vee})$ on which $d_{\mathrm{o}}P^{\vee}$ doesn't vanish. This can only be possible if $d_{\mathrm{o}}P^{\vee}(\beta)$ is not zero on $\frac{\partial}{\partial s}$, where s is the parameter taking into account the moduli of the elliptic curve. By using Theorem 3.2.3 we have

$$d_{\mathrm{o}}P^{\vee}(\beta) = \sum_{k=1}^{d} \frac{m(\beta)}{\pi^*\alpha^2}(x_k)$$

and this concludes the proof. □

This result improves the one in [Kan04] where it is proved only for simple ramification. In the same work is proved that, for simple ramification, having the sum in Theorem 3.3.3 different from zero for some $\beta \in \mathrm{Ker}(d_{\mathrm{o}}P_{\mathrm{E}}^{\vee})$ is equivalent to ask that the intersection of the quadrics that contain the canonical model of F doesn't contain the point q^- defined before. Unfortunately, in the case of arbitrary ramification, we are not able to prove this equivalence but only one implication.

Theorem 3.3.4. *With the same hypotesis of Theorem 3.3.3, if we identify F with its canonical model in $\mathbb{P}H^0(\omega_F)^{\vee}$, then we have*

$$q^- \notin \bigcap_{F \subset Q} Q \Longrightarrow \dim(\mathrm{Ker}(d_{\mathrm{o}}P)) = 1, \tag{3.27}$$

where Q ranges in the set of quadrics of $\mathbb{P}H^0(\omega_F)^{\vee}$ containing F.

The proof of the theorem uses some arguments developed in [Kan04] that we have summarized in the following Lemma.

Lemma 3.3.5. *Let Q be a quadric of $\mathbb{P}H^0(\omega_F)^\vee$ containing F and denote by $G_Q = G_Q^- + \pi^*\alpha \hat{\otimes} \omega_Q$ one of its equations. Then*

$$\sum_{k=1}^{d} \frac{m(G_Q^-)}{\pi^*\alpha^2}(x_k) = 0 \Longleftrightarrow G_Q(q^-) = 0 \Longleftrightarrow q^- \in Q.$$

Proof. The last statement is clear by definition so we really need to prove only the first one. First of all observe that we can choose the coordinate s in such a way that α is locally given by ds. Then, as $G_Q^- = G_Q - \pi^*\alpha \hat{\otimes} \omega_Q$ and $Q \in H^0(I_F(2)) = \mathrm{Ker}(m)$ by hypotesis, one has

$$\sum_{k=1}^{d} \frac{m(G_Q^-)}{\pi^*\alpha^2}(x_k) = -\sum_{k=1}^{d} \frac{m(\pi^*\alpha \hat{\otimes} \omega_Q)}{\pi^*\alpha^2}(x_k) = -\frac{Tr_\pi(\omega_Q)}{\alpha}(c).$$

But $Tr_\pi(\omega_Q)$ is an element of $H^0(\omega_E)$ so it is equal to $r \cdot \alpha$ for some r. Thus we have

$$\sum_{k=1}^{d} \frac{m(G_Q^-)}{\pi^*\alpha^2}(x_k) = -r$$

which is zero if and only if ω_Q has trace 0, i.e. if and only if $\omega_Q \in H^0(\omega_F)^-$. This happens if and only if $(\pi^*\alpha)^{\otimes 2}$ doesn't appear in the equation of Q, i.e. if and only if $q^- \in Q$. □

Using Lemma 3.3.1 and Lemma 3.3.5 the proof of Theorem 3.3.4 is straightforward.

Proof of Theorem 3.3.4. Assume that

$$q^- \notin \bigcap_{F \subset Q} Q.$$

Then, there exists a quadric which cointains F but doesn't contain q^-. Denote by G_Q its equation. By Lemma 3.3.1 we know that $\beta = \gamma(G_Q) = G_Q^- \in \mathrm{Ker}(d_\mathrm{o} P_\mathrm{E}^\vee)$ and by Lemma 3.3.5 we have that

$$\sum_{k=1}^{d} \frac{m(\beta)}{\pi^*\alpha^2}(x_k) \neq 0.$$

Hence, using Theorem 3.3.3 we have the thesis. □

Remark 3.3.6. In [Kan04], with different methods, it is proved that $H^0(I_F(2)) = \mathrm{Ker}(d_\mathrm{o} P_\mathrm{E}^\vee)$ if the ramification is simple. This fact is exactly what allows to prove the converse implication of Theorem 3.3.4.

Remark 3.3.7. Notice that $H^0(I_F(2)) = \mathrm{Ker}(d_\mathrm{o} P_\mathrm{E}^\vee)$ if and only if all the ramification indices are equal to 2. Indeed, denote by R_{red} the reduced divisor whose support equals the support of the ramification divisor. Let $\bar{R}$ be $R - R_{red}$. From Riemann-Hurwitz we have

$$2g - 2 = \deg(R) = \deg(R_{red}) + \deg(\bar{R}) = n + \deg(\bar{R}).$$

Hence, one has

$$\dim \operatorname{Ker} d_0 P_E^\vee = h^0(I_F(2)) + \deg(\bar{R}).$$

As $\bar{R} \geq 0$ and is trivial if and only if all the ramification indices are equal to 2 the claim follows. In particular, the converse implication of (3.27) in Theorem 3.3.4 holds for coverings whose ramification indices are all equal to 2.

We conclude this section by proving the existence of an exact sequence which should help to measure, in a more intrinsic way, how much $H^0(I_F(2))$ and $\operatorname{Ker}(d_{\mathrm{o}} P_{\mathrm{E}}^\vee)$ differ.

Proposition 3.3.8. *Under the hypotesis of Theorem 3.3.4 there is an exact sequence*

$$0 \longrightarrow H^0(I_F(2)) \xhookrightarrow{\gamma} \operatorname{Ker}(d_{\mathrm{o}} P_{\mathrm{E}}^\vee) \longrightarrow \frac{\operatorname{Ker}(dh^\vee)}{H^0(\omega_F)\,\hat{\otimes}\,\pi^* H^0(\omega_E)} \longrightarrow 0. \tag{3.28}$$

Proof. Starting from diagram (3.20) it is easy to see that the composition of the inclusion of $H^0(\omega_F)\,\hat{\otimes}\,\pi^* H^0(\omega_E)$ with m has image in $H^0(\omega_F^{\otimes 2})$ but also in the kernel of $dh^\vee$. Hence there is a map

$$\epsilon : H^0(\omega_F)\,\hat{\otimes}\,\pi^* H^0(\omega_E) \to \operatorname{Ker}(dh^\vee),$$

which is easily proven to be injective as we have done with γ. We can also complete the diagram on the right by adding two (trivial) vertical arrows. The complete diagram looks like this

$$\begin{tikzcd}
 & & 0 & & & \\
0 \arrow[r] & \operatorname{Ker} d_{\mathrm{o}} P_{\mathrm{E}}^\vee \arrow[r, hook, "j"] & \operatorname{Sym}^2(H^0(\omega_F)^-) \arrow[u] \arrow[r, "d_0 P_E^\vee"] & T_0^\vee \mathcal{H}_E \arrow[r] & (\operatorname{Ker} d_{\mathrm{o}} P_{\mathrm{E}})^\vee \arrow[r] & 0 \\
0 \arrow[r] & H^0(I_F(2)) \arrow[u, hook, "\gamma"] \arrow[r, hook, "\iota"] & \operatorname{Sym}^2(H^0(\omega_F)) \arrow[u, two heads, "pr"] \arrow[r, two heads, "m"] & H^0(\omega_F^{\otimes 2}) \arrow[u, "dh^\vee"'] \arrow[r] & 0 \arrow[u] & \\
 & 0 \arrow[u] \arrow[r] & H^0(\omega_F)\,\hat{\otimes}\,\pi^* H^0(\omega_E) \arrow[u, hook] \arrow[r, hook, "\epsilon"] & \operatorname{Ker} dh^\vee \arrow[u] \arrow[r] & \operatorname{Coker} \epsilon \arrow[u] \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd} \tag{3.29}$$

By using the snake lemma on the central columns one obtain the wanted sequence. $\square$

3.4 Coverings over elliptic curves and fibred surfaces with maximal relative irregularity

3.4.1 The link construction

We explain the relation between the world of non isotrivial fibrations with maximal relative irregularity and the one of the families of coverings as introduced in the previous sections. The *Link contruction* is the following. Let $f : S \to B$ be a non isotrivial fibration of a smooth surface over a smooth compact curve B and let $g = g(F)$ be the genus of the general fibre. The fibration f induces through the albalnese functor $\mathrm{Alb}(\cdot)$ a map $\mathrm{alb}(f) : \mathrm{Alb}(S) \to \mathrm{Alb}(B) = J(B)$ which is surjective and has $\dim(\mathrm{Ker}(\mathrm{Alb}(f))) = q_f$. Let $K_f = \mathrm{Ker}(\mathrm{Alb}(f))_0$ be the connected component containing 0.Fix F_b be a fibre over a regular value $b \in B^0$. The inclusion $F_b \hookrightarrow S$ induces a map $JF_b \simeq \mathrm{Alb}(F_b \to \mathrm{Alb}(S)$ whose image is, up to translation, exactly K_f. Dualizing we have a map

$$K_f^\vee \hookrightarrow JF_b^\vee = JF_b$$

Note that $K_f^\vee$ doesn't depend on b whereas F_b strongly depends on it. In particular we have proved that the Jacobian of every smooth fiber of a non isotrivial fibration contains a fixed abelian variety of dimension q_f. Assume $q_f = g - 1$. According to Beauville [Deb82, Appendix]), each fibration satisfies $0 \leq q_f \leq g$ and the equality $q_f = g$ holds if and only if the fibration is trivial. Thus the case of $q_f = g - 1$ is the extremal for non isotrivial fibrations and we say that f has *maximal relative irregularity.* As K_f has dimension $\dim(K_f^\vee) = q_f = g - 1$, the quotient $JF_b/K_f^\vee$ will be an abelian variety of dimension $g - q_f = 1$, that is an elliptic curve E_b.

$$\begin{tikzcd}
\Phi(\pi_b) = K_f^\vee \arrow[r, hook] & JF_b \arrow[r, two heads, "p_b"] & E_b \arrow[d, equal] \\
& F_b \arrow[u] \arrow[ur, "\pi_b"] \arrow[r, "\pi_b"'] & E_b
\end{tikzcd}$$

The diagram defines a family of coverings $\pi_b : F_b \to E_b$ as introduced in the previous sections and we can take the Prym varieties $\Phi(\pi_b)$. By construction, the abelian variety $K_f^\vee$ is the connected component through the origin of the kernel of the norm map associated to the ramified covering $\pi_b : F_b \to E_b$, that is Prym variety $\Phi(\pi_b)$, and thus

$$K_f^\vee = \Phi(\pi_b)$$

provides the *Link construction.*

3.4.2 Fibred surfaces with maximal relative irregularity and the Xiao Conjecture

By the *Link construction* given in the previous section, a counterexample to the modified conjecture of Xiao, under the additional assumption $q_f = g - 1$, would give a family of coverings of elliptic curves with constant Prym variety and a fibre of positive dimension in the differential of local Prym map. In Subsection 2.4.3, we summarized the state of art on the *Xiao Conjecture* (modified of not). According to it, non isotrivial fibrations with maximal relative irregularity can exist only if $g \leq 7$. On the other hand, the bound $q_f \leq g - c_f$ proves the (modified) Xiao's conjecture for fibrations of general Clifford index. Thus, the link construction must be applied on fibrations with $\mathrm{Cliff}(f) = 1$, that is on non isotrivial fibrations with general fibre F which is trigonal or isomorphic to a plane quintic ($\mathrm{Cliff}(F) = 1$), where the geometric approach gives less informations.

Remark 3.4.1. In our framework, as we have to take care of the automorphisms of the elliptic curves, then $\dim(\mathrm{Ker}(d_o P))) = 1$ implies that there are no non isotrivial fibrations. Thus the obstruction to the construction of a non isotrivial fibration is $\dim(\mathrm{Ker}(d_o P))) = 1$.

Remark 3.4.2. The probelm has recently been solved for fibrations with general fibre isomorphic to a plane quintics by Favale, Naranjo and Pirola in [FNP17]. More precisely, in [FNP17, Corollary1.2] they proved that the modified Xiao's conjecture holds for fibrations with generic fibre a quintic plane curve.

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Per passare ai singoli che spero si autoidentificheranno, perdonandomi se non cito ognuno persona per persona. Un ringraziamento solo non basta verso le persone ”piccole, medio, grandi” con le quali ho vissuto questi tre anni, partendo da coloro con cui ho battuto la testa contro un pezzetto di matematica che non mi tornava o anche solo di vita, per arrivare a coloro con i quali ho collaborato e/o sub-collaborato. La regia dice che ho la fama di essere un po’ ”faticosa” da gestire e quindi ”grazie!” a queste persone con molto zen che, tra up and down, non mi hanno negato il loro tempo!

Concludo ringraziando la ”mia famiglia”, secondo la mia personale definizione del termine. A voi che sapete di farne parte e che quindi non vi stupirete molto per la scelta delle mie parole, voglio dire di non rilassarvi troppo: troverò sicuramente il modo di tenervi impegnati ancora! Le appendici smelense ve le dico (forse) per posta..

Grazie a tutti di cuore,

Sara