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Smoothness Estimates for Semi-regular Subdivision via Wavelet Tight Frames

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Acknowledgments

*In geometric order
An insulated border [...]
Growing up, it all seems so one-sided [...]
Detached and subdivided*

RUSH
Subdivisions

Maybe starting a Ph.D. thesis in mathematics citing a progressive rock song about the pressure that society applies to each individual for fitting into a prescribed lifestyle model is not really appropriate. Those lines however stuck into my head since I started working with subdivision schemes and it is undeniable that they could be used to describe such objects! Moreover my Ph.D. programme started and ended in the office 2112 which is the name of another success of the Canadian band Rush. As a human being, I want to look at these coincidences as hints I did the correct choice sailing for this adventure, even if sometimes I felt I was not capable enough for it. Indeed, *“human lives are composed [...] like music. Guided by his sense of beauty, an individual transforms a fortuitous occurrence [...] into a motif [...]. Without realizing it, the individual composes his life according to the laws of beauty even in times of greatest distress”* (M. KUNDERA, *The Unbearable Lightness of Being*). Now that the adventure is approaching its end, it is correct and useful to look back, to analyse what was good and what could have been done differently, maybe better, but most important to say thank you to all the people that, in either big or small way, influenced this journey.

Looking at what I already wrote, it seems natural to start thanking Federico and Jacopo, even if we do not see each other so often, for having me introduced to Rush's music and Chiara, even if this is not her top contribution to these years, for being the reason why I read Kundera's book.

In these years I met a lot of wonderful people that enriched me under various aspects, while a other wonderful people I already knew kept having an active part in my life. I was thinking about mentioning each and every name but, after listing all of them, I decided not to write an enormous and sterile list of names. Some readers will then forgive me if their names are not explicitly written here: I promise they will get my thanks personally.

I start thanking all the professors I had the pleasure to know, for everything they taught me, directly or indirectly. I owe them a lot for all the things I learnt, even if one of them is that I will probably never be as good as a mathematician as them. A special thank goes to Prof. Vignati, for the role he played both during my master degree and the choice to embark on a Ph.D., and to my supervisors, Prof. Charina and Prof.

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Introduction

There are two main protagonists of this thesis: subdivision schemes and wavelet frames. On the one hand, subdivision schemes are iterative methods for generating curves and surfaces, widely used in CAGD and animation (see e.g. [10, 52, 57, 63]). On the other hand we have wavelets and wavelet frames, from the seminal works of Daubechies, Meyer, Ron, Shen and many others [16, 20, 24, 50, 55, 56], which are families of functions that provide methods for useful decompositions of the elements of application related function spaces such as $L^2(\mathbb{R}^d)$. These two topics are deeply related since both methods are entwined with the concept of refinable functions, i.e. functions $[\varphi_k]_{k \in \mathbb{Z}}$ satisfying a refinement equation of the type

$$[\varphi_k]_{k \in \mathbb{Z}} = \mathbf{P}^T [\varphi_k(2 \cdot)]_{k \in \mathbb{Z}} \quad (0.1)$$

for some bi-infinite real-valued matrix \mathbf{P} . In particular, subdivision schemes generate refinable functions which build the foundation for wavelet frame constructions.

One of the major open issue in the realm of subdivision nowadays is to understand how to construct schemes that produce \mathcal{C}^2 surfaces in settings with arbitrary topology. In particular, the crucial case is when the initial mesh used for the subdivision process features one or more extraordinary vertices, i.e. vertices with a number of incoming edges different from 6 for triangular meshes or 4 for quadrilateral meshes. Since the subdivision schemes useful for applications are local, the standard approach would be to try to tune locally a known regular scheme around an extraordinary vertex, in such a way that the resulting surface is globally \mathcal{C}^2 . Not having a clue on how to do this tuning process properly, the consequential step is to go deeper into the smoothness analysis for more insight. In this direction, results were obtained exploiting spectral analysis, e.g. [52, 54, 62, 63], however without achieving a full understanding of the smoothness phenomena. The initial idea behind this work was to follow the same direction through a different path, i.e. applying another method for the analysis of the global subdivision smoothness. In the regular (shift-invariant) setting, other methods have been used successfully for this task, such as Fourier techniques, the Joint Spectral Radius approach [6, 25] and wavelet analysis [20, 50]. The first two of them do not naturally extend to the case of extraordinary vertices since they rely on the shift-invariance of the mesh. We choose instead to focus on the latter, the wavelet analysis. To test the availability of this path, we focused on the easiest non-regular setting, as done in [29], i.e. we considered univariate schemes over a semi-regular initial mesh given by

$$\mathbf{t}_0 = -h_\ell \mathbb{N} \cup \{0\} \cup h_r \mathbb{N}, \quad h_\ell, h_r \in (0, \infty). \quad (0.2)$$

In the regular case, wavelet analysis rely on the characterization of Besov spaces $\mathcal{B}_{p,q}^r(\mathbb{R})$ provided by Lemarié and Meyer [47] in their follow-up on the results by Frazier and Jawerth [35].

Theorem 0.1 ([50], Section 6.10). *Let $s > 0$ and $1 \leq p, q \leq \infty$. Assume*

$$\{\phi_k = \phi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi_{j,k} = 2^{(j-1)/2} \psi_{1,0}(2^{j-1} \cdot -k) : k \in \mathbb{Z}, j \in \mathbb{N}\} \subset \mathcal{C}^s(\mathbb{R})$$

is a compactly supported orthogonal wavelet system with v vanishing moments.

Then, for $r \in (0, \min(s, v))$,

$$\mathcal{B}_{p,q}^r(\mathbb{R}) = \left\{ \sum_{k \in \mathbb{Z}} a_k \phi_k + \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} b_{j,k} \psi_{j,k} : \{a_k\}_{k \in \mathbb{Z}} \in \ell^p(\mathbb{Z}), \left\{ 2^{j(r+\frac{1}{2}-\frac{1}{p})} \|\{b_{j,k}\}_{k \in \mathbb{Z}}\|_{\ell_p} \right\}_{j \in \mathbb{N}} \in \ell^q(\mathbb{Z}) \right\}.$$

To be able to apply Theorem 0.1, i.e. to extract the smoothness of a given function f from the decay of its coefficients

$$\{a_k = \langle f, \phi_k \rangle : k \in \mathbb{Z}\} \quad \text{and} \quad \{b_{j,k} = \langle f, \psi_{j,k} \rangle : j \in \mathbb{N}, k \in \mathbb{Z}\}, \quad (0.3)$$

one must first compute these inner products. In the context of subdivision, the analytic expressions neither of the analysed function f nor of the functions $\phi_k, \psi_{j,k}$ are usually known. However, in the regular setting, the desired inner products can be computed explicitly (or numerically) using results of [44]. Unfortunately, in the semi-regular case the exact same strategy does not work. The questions naturally arising at this point are the following. Can we construct function systems that mimic the properties of orthogonal wavelets in the semi-regular setting? Can we characterize at least $\mathcal{B}_{\infty,\infty}^r(\mathbb{R})$ similarly to Theorem 0.1? And will we be able to compute the inner products in (0.3)? In this work we are able to answer all these questions affirmatively.

The natural choice to generalize wavelets are wavelet tight frames. These families of functions are similar to wavelets since they are also based on refinable functions $\Phi = [\phi_k]_{k \in \mathbb{Z}}$ and retain the underlying multi-resolution structure, i.e. there exists a sequence of bi-infinite matrices $\{\mathbf{Q}_j\}_{j \in \mathbb{N}}$ such that

$$\Psi_j = 2^{j/2} \mathbf{Q}_j^T \Phi(2^j \cdot), \quad j \in \mathbb{N}. \quad (0.4)$$

On the other hand wavelet tight frames are more flexible than wavelets, not requiring either orthogonality or linear independence. The property which is kept is the perfect reconstruction property for $L^2(\mathbb{R})$, i.e.

$$f = \sum_{k \in \mathbb{Z}} \langle f, \phi_k \rangle \phi_k + \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}, \quad f \in L^2(\mathbb{R}). \quad (0.5)$$

The choice of tight frames over more general systems, such as dual frames (see e.g. [23, 30, 31, 37, 38, 42]), is due to the easier applicability of the former which use the same functions in (0.5) both for the computation of the coefficients and for the reconstruction of f .

In the stationary regular setting, i.e. when the subdivision rules are shift-invariant and do not change between the iterations, the so-called *Unitary Extension Principle* (UEP) [55, 56] and *Oblique Extension Principle* (OEP) from [16, 24] are used for constructions of wavelet tight frames with one or more vanishing moments. UEP and OEP are based on Fourier techniques and on factorizations of trigonometric polynomials. A generalization of the UEP procedure for nonstationary regular schemes, when the subdivision rules can change from one iteration to the other, was presented in [40]. A general setting that also covers the semi-regular case is the one proposed in [14, 15] where matrix formulations of the UEP and OEP is given and examples of wavelet tight frames based on non-uniform B-spline schemes are presented.

We construct wavelet tight frames with n vanishing moments from the semi-regular Dubuc-Deslauriers $2n$ -point subdivision schemes and, to meet this goal, we also present the convergence analysis of such semi-regular schemes. The family of Dubuc-Deslauriers $2n$ -point schemes was introduced in [27] in the regular case and extended to meshes of the type (0.2) in [62]. Our convergence analysis of this family uses the local eigenvalue analysis [52, 63]. Our construction of the corresponding wavelet tight frames on the regular part of the mesh uses the UEP and is based on the well known (see e.g [51]) link between Dubuc-Deslauriers and Daubechies refinable functions [20]. The interpolation and polynomial generation (up to degree $2n - 1$) properties of the corresponding subdivision schemes ensure n vanishing moments for the framelets. On the irregular part of the mesh, in a neighborhood of $\mathbf{t}_0(0)$, we apply the matrix factorization technique from [14, 15]. Similarly to [8], instead of factorizing a certain global positive semi-definite matrix, we used the regular framelets to reduce the process to the factorization of a finite positive semi-definite matrix involving an appropriate approximation of the inverse Gramian matrix, which guarantees n vanishing moments of the framelets. However, the existence of the underlying refinable functions is ensured only for certain values of h_r/h_ℓ .

The advantage of our construction is in its simplicity. Indeed, with our UEP based construction we obtain regular framelets with n vanishing moments without endeavouring into more tedious computations required in general by the OEP. Furthermore, compared to the B-spline based wavelet tight frames in [16] (the only other semi-regular wavelet tight frame in the literature) whose filters have size (number of non-zero coefficients) $3n - 1$, the corresponding filters obtained from the Dubuc-Deslauriers $2n$ -point framelets are of size $2n + 1$. The disadvantage occurs on the irregular part of the mesh, where our filters have possibly larger supports.

Using our wavelet tight frames we are able to estimate the regularity of other semi-regular subdivision schemes. Our method relies on a new characterization of Hölder-Zygmund spaces, $\mathcal{B}_{\infty,\infty}^r(\mathbb{R})$, $r > 0$. It generalizes successful wavelet frame methods [9, 10, 11, 20, 43, 50, 53, 58] from the regular to the semi-regular and even to the irregular setting. In comparison to the method in [21], our approach yields numerical estimates for the optimal Hölder-Zygmund regularity of a refinable function given just the refinement equation without requiring any ad hoc norm estimates for the corresponding subdivision scheme. Our numerical estimates turn out to be optimal in all considered cases and require fewer computational steps than the standard linear regression method.

We provide a generalization of Theorem 0.1 for function systems

$$\mathcal{F} = \{\phi_k : k \in \mathbb{Z}\} \cup \{\psi_{j,k} : j \in \mathbb{N}, k \in \mathbb{Z}\} \subset L^2(\mathbb{R})$$

satisfying properties (3.2)-(3.7). These requirements express conditions about the localized properties of the framelets $\psi_{j,k}$. They also guarantee a quasi-uniform behaviour of the system over \mathbb{R} , even if no shift-invariance is required as for orthogonal wavelets. Our main result states the following:

Theorem 0.2. *Let $s > 0$ and $v \in \mathbb{N}$. Assume $\mathcal{F} \subset \mathcal{C}^s(\mathbb{R})$ satisfies assumptions (3.2)-(3.7) with v vanishing moments. Then, for $r \in (0, \min(s, v))$,*

$$\mathcal{B}_{\infty, \infty}^r(\mathbb{R}) = \left\{ \sum_{k \in \mathbb{Z}} a_k \phi_k + \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} b_{j,k} \psi_{j,k} : \{a_k\}_{k \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}), \left\{ 2^{j(r+\frac{1}{2})} \|\{b_{j,k}\}_{k \in \mathbb{Z}}\|_{\ell^\infty} \right\}_{j \in \mathbb{N}} \in \ell^\infty(\mathbb{Z}) \right\}.$$

The setting described by assumptions (3.2)-(3.7) includes some cases not addressed in the results of Frazier and Jawerth [35] or of Cordero and Gröchenig in [18]. The results of [35] require that the elements of \mathcal{F} in the decomposition of $\mathcal{B}_{p,q}^r(\mathbb{R})$ are linked to dyadic intervals. The results in [18] impose the so-called localization property which implies that the system \mathcal{F} is semi-orthogonal (in particular, non-redundant).

On the other hand, one could view assumptions (3.2)-(3.7) to be somehow restrictive, since they were designed to fit wavelet tight frames \mathcal{F} constructed using results of [15], such as the one constructed here from the Dubuc-Deslauriers $2n$ -point schemes. For function families \mathcal{F} satisfying (0.4), assumptions (3.3)-(3.7) reflect properties of the matrices $\{\mathbf{Q}_j\}_{j \in \mathbb{N}}$: (3.3) controls the support of the columns of \mathbf{Q}_j , (3.4) controls the slantedness of \mathbf{Q}_j and (3.6)-(3.7) are linked to eigenproperties of \mathbf{Q}_j .

Nevertheless, the spirit of assumptions (3.3)-(3.7) merges with the spirit of atoms and molecules in [35] and compactly supported orthogonal wavelet systems, for which (3.3)-(3.7) are also satisfied. These similarities are also visible in the structure of the proofs of Propositions 3.5 and 3.6.

For the sake of completeness, we point out that our setting includes some of the wavelet frames considered in [41] for which a characterization of the spaces $\mathcal{B}_{2,2}^r(\mathbb{R})$, $r \in \mathbb{R}$, is given. However, those frames are shift-invariant, i.e. (0.4) holds with block 2-slanted $\{\mathbf{Q}_j\}_{j \in \mathbb{N}}$. The approach in [41] applies Fourier techniques that are not feasible in our case, due to the lack of shift-invariance. The lack of shift-invariance makes also the techniques in [3] inapplicable in our case.

To exploit Theorem 0.2 for the smoothness estimates we provide the tools to compute the frame coefficients of the expansion given by the perfect reconstruction property (0.5) for the refinable functions arising from semi-regular subdivision. The main ingredient is the computation of the cross-Gramian matrix between the refinable functions of the wavelet tight frame considered for the analysis and the refinable functions given by the semi-regular scheme one wants to analyse. To do so we exploit the result in [44] (proven in the regular case) to adjust the idea sketched for B-spline in [48] in the case of generic semi-regular schemes. Furthermore, we show how to compute moments of semi-regular refinable functions, thus, extending results in [19] to the semi-regular setting.

These algorithms are crucial for both the construction of the semi-regular wavelet tight frames and the application of Theorem 0.2 to the smoothness estimates of semi-regular subdivision.

The organization of the thesis mirrors the steps followed during the three year period of the doctoral programme. In Chapter 1, we introduce subdivision schemes, highlighting, in Section 1.1, the similarities and differences between the regular and the semi-regular case. Then in Section 1.2 we extend the methods in [19, 44, 48] and compute moments (Section 1.2.1) and (cross-)Gramian matrices (Section 1.2.2) in the semi-regular setting. These methods are fundamental for both the construction and the application of our wavelet tight frames.

Chapter 2 is devoted to the construction of suitable semi-regular wavelet tight frames, see also [60]. We start with the known tools for the construction of wavelet tight frames in the regular case, namely the Unitary and Oblique Extension Principles and their extension in matrix form to a general formulation that includes the semi-regular setting (Section 2.1). A first detailed example based on cubic B-spline is provided in Section 2.1.1. It shows the difficulties arising in the semi-regular case when using the OEP. Next, in Section 2.2, we introduce the family of Dubuc-Deslauriers $2n$ -point interpolatory schemes on which our frame construction is based, providing their convergence analysis. Consequently we define the corresponding scaling functions. Our wavelet tight frame construction is presented in Section 2.2.1, followed by examples for $n = 1, 2$ in Section 2.2.2, which illustrate our theoretical results.

At last, in Chapter 3 we discuss the extension of Theorem 0.1 and its application to the approximation of the smoothness of semi-regular subdivision [7]. In Section 3.1.1, the proof of Theorem 0.2 is split into two cases: Theorem 3.4 treats the case $r \in (0, \infty) \setminus \mathbb{N}$ and, in Section 3.1.2, Theorem 3.9 provides the proof of Theorem 0.2 for $r \in \mathbb{N}$. We would like to emphasize that the results in Sections 3.1.1 and 3.1.2 hold in regular, semi-regular and irregular cases. The proofs in Sections 3.1.1 and 3.1.2 are reminiscent of the continuous wavelet transform techniques in [20, 50] and references therein. In Section 3.2, we illustrate our results with several examples. There the underlying semi-regular setting becomes crucial. In particular, we use wavelet tight frames constructed in Section 2.2, to approximate the Hölder-Zygmund regularity of semi-regular subdivision schemes based on B-splines, the family of Dubuc-Deslauriers subdivision schemes and interpolatory radial basis functions based subdivision. Semi-regular B-spline and Dubuc-Deslauriers schemes were introduced, e.g in [21, 62, 63]. The construction of semi-regular RBFs based schemes is our generalization of [45, 46] to the semi-regular case. The numerical computations have been done in MATLAB 2018a on a Windows 10 (x64) laptop (CPU: Intel Core i7-7700HQ 2.80 GHz, RAM: 16 GB).

Notation

Vectors and Matrices

We use bold letters and numbers to indicate numerical vectors and matrices, whereas lowercase letters denote vectors and capital letters matrices. Depending on the context, $\mathbf{1}$ can indicate both a vector or a matrix whose components are equal to 1. All vectors are column vectors and their size, as well as the sizes of matrices, are specified when not clear from the context. Moreover, sometimes all the finite vectors and matrices are extended to bi-infinite vectors and matrices padded with zeros. For clarity the element at position $(0, 0)$ is framed in a box.

Vectors and matrices are interpreted as functions of their indices, i.e. $\mathbf{v}(-3)$ refers to the component of \mathbf{v} at position -3 . We use the MatLab notation for matrices, i.e. $\mathbf{M}(-3 : 3, :)$ indicates the submatrix of \mathbf{M} consisting of all the rows with indices between -3 and 3 .

The support of a vector \mathbf{v} is defined by

$$\text{supp}(\mathbf{v}) = [a, b] \cap \mathbb{Z},$$

where

$$\begin{aligned} a &= \max\{ k \in \mathbb{Z} : \mathbf{v}(m) = 0, \quad \forall m < k \}, \\ b &= \min\{ k \in \mathbb{Z} : \mathbf{v}(m) = 0, \quad \forall m > k \}. \end{aligned}$$

We use the notation $[\varphi_k]_{k \in \mathbb{Z}}$ to indicate the vector of basic limit functions of a convergent subdivision scheme and $\Phi_0 = [\phi_{0,k}]_{k \in \mathbb{Z}}$ the vector of scaling functions obtained from $[\varphi_k]_{k \in \mathbb{Z}}$ after a proper renormalization. These vectors are functions from \mathbb{R} to $\ell(\mathbb{Z})$ and all the operations involving them, such as integration or differentiation, are done componentwise. Similarly, for two sets of indices or real numbers \mathcal{A} and \mathcal{B} we define

$$\mathcal{A} + \mathcal{B} = \{ a + b : a \in \mathcal{A}, b \in \mathcal{B} \}.$$

Function and Sequence Spaces

We use the standard notation for the function spaces $\mathcal{C}^s(\mathbb{R})$, $s \in \mathbb{N}_0$, the Hölder spaces

$$\mathcal{C}^s(\mathbb{R}) = \left\{ f \in \mathcal{C}^\ell(\mathbb{R}) : \sup_{x, h \in \mathbb{R}} \frac{|f^{(\ell)}(x+h) - f^{(\ell)}(x)|}{|h|^\alpha} < \infty \right\},$$

for $s = \ell + \alpha$, $\ell \in \mathbb{N}_0$, $\alpha \in (0, 1)$, with $f^{(\ell)}$ denoting the ℓ -th derivative of f , the *Zygmund class*

$$\Lambda(\mathbb{R}) = \left\{ f : \mathbb{R} \rightarrow \mathbb{R} : \sup_{x, h \in \mathbb{R}} \frac{|f(x+h) - 2f(x) + f(x-h)|}{|h|} < \infty \right\},$$

the Lebesgue spaces $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, and for sequence spaces $\ell^p(\mathbb{Z})$, $1 \leq p \leq \infty$.

Besov spaces $\mathcal{B}_{p,q}^r(\mathbb{R})$, e.g in [50], are defined, for $1 \leq p, q \leq \infty$, $r \in (0, \infty)$, by

$$\mathcal{B}_{p,q}^r(\mathbb{R}) := \left\{ f \in L^p(\mathbb{R}) : \|f\|_{\mathcal{B}_{p,q}^r} = \|\{2^{jr} \omega_p^{[r]+1}(f, 2^{-j})\}_{j \in \mathbb{N}}\|_{\ell^q} < \infty \right\},$$

with the p -th modulus of continuity of order $n \in \mathbb{N}$

$$\omega_p^n(f, x) = \sup_{|h| \leq x} \|\Delta_h^n(f, \cdot)\|_{L^p}$$

and the difference operator of order $n \in \mathbb{N}$ and step $h > 0$

$$\Delta_h^n(f, x) = \sum_{\ell=0}^n \binom{n}{\ell} (-1)^{n-\ell} f(x + \ell h).$$

The special case $p = q = \infty$ reduces to

$$\mathcal{B}_{\infty,\infty}^r(\mathbb{R}) = \begin{cases} \mathcal{C}^r(\mathbb{R}) \cap L^\infty(\mathbb{R}), & \text{if } r \in (0, \infty) \setminus \mathbb{N}, \\ \{f \in \mathcal{C}^{r-1}(\mathbb{R}) \cap L^\infty(\mathbb{R}) : f^{(r-1)} \in \Lambda(\mathbb{R})\}, & \text{if } r \in \mathbb{N}. \end{cases}$$

The corresponding sequence spaces $\ell_{p,q}^r$, $r \in (0, \infty)$, are defined, for $1 \leq p \leq \infty$ and $1 \leq q < \infty$, by

$$\ell_{p,q}^r = \left\{ (a, b) \in \mathbb{Z} \times (\mathbb{N} \times \mathbb{Z}) : \|(a, b)\|_{\ell_{p,q}^r} = \left(\|a\|_{\ell_p}^q + \sum_{j=1}^{\infty} 2^{j(r+\frac{1}{2}-\frac{1}{p})q} \|\{b_{j,k}\}_{k \in \mathbb{Z}}\|_{\ell_p}^q \right)^{1/q} \right\}$$

and, for $1 \leq p \leq \infty$ and $q = \infty$, by

$$\ell_{p,\infty}^r = \left\{ (a, b) \in \mathbb{Z} \times (\mathbb{N} \times \mathbb{Z}) : \|(a, b)\|_{\ell_{p,\infty}^r} = \max \left\{ \|a\|_{\ell_p}, \sup_{j \in \mathbb{N}} 2^{j(r+\frac{1}{2}-\frac{1}{p})} \|\{b_{j,k}\}_{k \in \mathbb{Z}}\|_{\ell_p} \right\} \right\}.$$

1 Subdivision Schemes

Subdivision schemes are iterative refining processes that starting from a coarse set of initial data (control points) produce functions, curves or surfaces. The local nature of subdivision schemes and the simplicity of their implementation made them a standard tool in Computer Aided Geometric Design (CAGD), computer graphics and animation. In this work, we consider only univariate, binary subdivision, i.e. one-dimensional subdivision processes that double the number of control points at each iteration. We focus on the construction of the so-called *basic limit functions* of such subdivision schemes. The importance of these limit functions is twofold: on one hand they characterize the behaviour of the scheme, so that one can analyze a scheme via its basic limit functions, while, on the other hand, they are excellent candidates for the construction of wavelet tight frames in Chapter 2. In return, wavelet tight frames are the tools we will use for the analysis of subdivision in Chapter 3.

We start this chapter with a short excursus into subdivision. The selection and order of facts about subdivision and subdivision properties aims to show the similarities and the differences between regular and semi-regular settings and at presenting the basic tools needed for handling the topics treated in this thesis. For the proofs of well known results, the reader is referred to [5, 10, 52, 63].

1.1 Regular vs. Semi-Regular

Both regular and semi-regular subdivision processes we are interested in fall in the following category.

Definition 1.1. Let \mathbf{P} be a bi-infinite matrix with compactly supported columns. It defines a *stationary subdivision operator*

$$\begin{aligned} P : \ell(\mathbb{Z})^2 &\longrightarrow \ell(\mathbb{Z})^2 \\ (\mathbf{t}, \mathbf{f}) &\longmapsto (\mathbf{u}, \mathbf{g}) \end{aligned}$$

by

$$\begin{aligned} \mathbf{u}(2k) &= \mathbf{t}(k) \\ \mathbf{u}(2k+1) &= \frac{\mathbf{t}(k) + \mathbf{t}(k+1)}{2} \quad , \quad k \in \mathbb{Z} \end{aligned} \tag{1.1}$$

and

$$\mathbf{g} = \mathbf{P} \mathbf{f}. \tag{1.2}$$

Starting with an initial mesh \mathbf{t}_0 of the form

$$\mathbf{t}_0(k) = \begin{cases} h_\ell k, & \text{if } k < 0, \\ 0, & \text{if } k = 0, \\ h_r k, & \text{if } k > 0, \end{cases} \quad (1.3)$$

for some $h_\ell, h_r \in (0, \infty)$, and a vector of initial data $\mathbf{f}_0 \in \ell(\mathbb{Z})$, the iterative application of the subdivision operator P to the couple $(\mathbf{t}_0, \mathbf{f}_0)$, i.e.

$$(\mathbf{t}_j, \mathbf{f}_j) = P(\mathbf{t}_{j-1}, \mathbf{f}_{j-1}) = P^j(\mathbf{t}_0, \mathbf{f}_0), \quad j \in \mathbb{N},$$

is called *subdivision scheme*. The matrix \mathbf{P} is called *subdivision matrix*. If $h_\ell = h_r$, then the mesh \mathbf{t}_0 is called *regular*, otherwise it is called *semi-regular*.

Remark 1.2. The assumption on the compact support of the columns of the subdivision matrix \mathbf{P} is natural for applications. Moreover, this assumption ensures the locality of the subdivision process, in the sense that each element of \mathbf{f} only influences a finite number of values of \mathbf{g} . \diamond

Every subdivision scheme satisfying Definition 1.1 over a regular initial mesh \mathbf{t}_0 is called regular. On the other hand, we call a scheme semi-regular if, after a finite number of subdivision steps, it can be described locally by schemes satisfying Definition 1.1. In general, semi-regular schemes can be non-stationary, i.e. the subdivision matrix \mathbf{P} depends on j , and be defined over more general initial meshes \mathbf{t}_0 , but, as observed in [62], it suffices to consider the schemes satisfying Definition 1.1 over semi-regular meshes \mathbf{t}_0 to get the full picture.

Remark 1.3. In contrast to a regular mesh, a semi-regular mesh is non-shift-invariant. Indeed, for a regular mesh \mathbf{t}_0 there exists $h > 0$ such that, for every $k \in \mathbb{Z}$, $\mathbf{t}_0 - hk = \mathbf{t}_0(\cdot - k)$. This is not the case for any semi-regular mesh \mathbf{t}_0 with $h_\ell \neq h_r$. \diamond

Once we have the sequence of vector-tuples $\{(\mathbf{t}_j, \mathbf{f}_j)\}_{j \in \mathbb{N}}$, we consider the sequence of piecewise linear functions $\{F_j\}_{j \in \mathbb{N}}$, where F_j interpolate the values \mathbf{f}_j over the mesh \mathbf{t}_j , i.e., for $x \in [\mathbf{t}_j(k), \mathbf{t}_j(k+1)] = 2^{-j}h[k, k+1]$, $k \in \mathbb{Z}$,

$$F_j(x) = \frac{(x - \mathbf{t}_j(k))(\mathbf{f}_j(k+1) - \mathbf{f}_j(k))}{(\mathbf{t}_j(k+1) - \mathbf{t}_j(k))} + \mathbf{f}_j(k). \quad (1.4)$$

The convergence of the subdivision process then is defined as follows.

Definition 1.4. A subdivision scheme satisfying Definition 1.1 is said to be *convergent* if and only if, for every initial data $\mathbf{f}_0 \in \ell^\infty(\mathbb{Z})$, there exists a *limit function* $F \in \mathcal{C}^0(\mathbb{R})$ such that

$$\lim_{j \rightarrow \infty} \|F - F_j\|_\infty = 0$$

and $F \not\equiv 0$ for at least one set of initial data $\mathbf{f}_0 \in \ell^\infty(\mathbb{Z}) \setminus \{\mathbf{0}\}$. Moreover, if the limit function F belongs to $\mathcal{C}^r(\mathbb{R})$, $r > 0$, for any initial data, the scheme is said to be \mathcal{C}^r and r is referred as the *smoothness* of the scheme.

A very basic example of convergent subdivision scheme is the following.

Example 1.5 (Linear B-spline, part I). Consider the subdivision matrix

$$\mathbf{P} = \begin{bmatrix} & & 1/2 & & & \\ & \ddots & & 1 & & \\ & & 1/2 & 1/2 & & \\ & & & \boxed{1} & & \\ & & & 1/2 & 1/2 & \\ & & & & 1 & \ddots \\ & & & & 1/2 & \end{bmatrix}$$

and an initial regular or semi-regular mesh \mathbf{t}_0 . Then, for $\mathbf{f}_0 \in \ell^\infty(\mathbb{Z})$, from (1.2) and (1.4), we have, for every $j \in \mathbb{N}$, $k \in \mathbb{Z}$,

$$F_j(\mathbf{t}_j(2k)) = \mathbf{f}_j(2k) = \mathbf{f}_{j-1}(k) = F_{j-1}(\mathbf{t}_{j-1}(k)) = F_{j-1}(\mathbf{t}_j(2k))$$

and

$$\begin{aligned} F_j(\mathbf{t}_j(2k+1)) &= \mathbf{f}_j(2k+1) = \frac{\mathbf{f}_{j-1}(k) + \mathbf{f}_{j-1}(k+1)}{2} \\ &= \frac{F_{j-1}(\mathbf{t}_{j-1}(k)) + F_{j-1}(\mathbf{t}_{j-1}(k+1))}{2} = F_{j-1}(\mathbf{t}_j(2k+1)). \end{aligned}$$

Since F_j and F_{j-1} are piecewise linear interpolants on the meshes \mathbf{t}_j and \mathbf{t}_{j-1} , respectively, and they coincide on the finer mesh, we have $F_j = F_{j-1}$. Thus, $F_j \equiv F \in \mathcal{C}^0(\mathbb{R})$ and the scheme converges. The resulting limit function F belongs to $\mathcal{C}^\infty(\mathbb{R})$ when $\mathbf{f}_0 \equiv c \in \mathbb{R}$. For general $\mathbf{f}_0 \in \ell^\infty(\mathbb{R})$, however, F is a bounded piecewise linear function over equispaced knots which is known to be only Lipschitz-continuous, thus, the corresponding scheme is $\mathcal{C}^{1-\epsilon}$, $\epsilon > 0$. This is the simplest example of a convergent subdivision scheme called *linear B-spline scheme* since it produces piecewise linear functions. \triangle

Convergence is one of the fundamental properties of subdivision. Without convergence subdivision schemes are very less appealing for visual applications, such as CAGD. Moreover, since this work aims to study the smoothness of convergent subdivision, in what follows we will often omit the word “convergent” by taking it for granted. When talking about graphic design a good trade off between high smoothness and fast implementation is the goal to achieve. The typical aim for the smoothness in applications is \mathcal{C}^2 smoothness. In graphics, for example, the human eye struggles to distinguish \mathcal{C}^2 curves and surfaces from the more regular ones. With this link to graphics in mind, it is natural to refer to j as the resolution level, since all the functions F_j approximate the smooth

function F better and better on finer meshes as j grows. This fact is also the reason why they are called subdivision schemes.

Equations (1.1) and (1.2) imply that the schemes we consider are linear. Since the action of a subdivision operator in Definition 1.1 on the mesh is only a dyadic cut, the subdivision process is characterized by the subdivision matrix \mathbf{P} .

Subdivision Matrix

In the regular case, a subdivision matrix \mathbf{P} is a 2-slanted matrix, i.e. there exists a compactly supported vector \mathbf{p} such that

$$\mathbf{P}(k, m) = \mathbf{p}(k - 2m), \quad k, m \in \mathbb{Z}. \quad (1.5)$$

Definition 1.6. The vector \mathbf{p} in (1.5) is called *mask* of the scheme defined by \mathbf{P} .

Thus, the subdivision step in (1.2) in the regular case coincides with the convolution

$$\mathbf{g}(k) = \sum_{m \in \mathbb{Z}} \mathbf{p}(k - 2m) \mathbf{f}(m), \quad k \in \mathbb{Z}. \quad (1.6)$$

In the case of a semi-regular scheme instead there exist two compactly supported vectors \mathbf{p}_ℓ and \mathbf{p}_r such that

$$\begin{aligned} \text{(i)} \quad & k_\ell(\mathbf{P}) < k_r(\mathbf{P}) - 1, \quad \text{where} \\ & k_\ell(\mathbf{P}) = \max(\text{supp}(\mathbf{p}_\ell)) \quad \text{and} \quad k_r(\mathbf{P}) = \min(\text{supp}(\mathbf{p}_r)); \\ \text{(ii)} \quad & \text{for every } m \in \mathbb{Z}, \end{aligned} \quad (1.7)$$

$$\mathbf{P}(m, k) = \begin{cases} \mathbf{p}_\ell(m + 2(k_\ell(\mathbf{P}) - k)) & \text{for } k \leq k_\ell(\mathbf{P}), \\ \mathbf{p}_r(m + 2(k_r(\mathbf{P}) - k)) & \text{for } k \geq k_r(\mathbf{P}). \end{cases}$$

The vectors \mathbf{p}_ℓ and \mathbf{p}_r are referred, respectively, as *left and right regular masks* of the scheme defined by \mathbf{P} . The submatrix of \mathbf{P} given by

$$\mathbf{P}_{irr} := \mathbf{P}(:, k_\ell(\mathbf{P}) + 1 : k_r(\mathbf{P}) - 1), \quad (1.8)$$

is called *irregular part* of \mathbf{P} .

The first difference between the two settings is the structure of the subdivision matrix \mathbf{P} . The semi-regular case allows for a finite number of different consequent columns. The first and the last of such columns, which can be different, are repeated in a 2-slanted fashion as in the regular case. The regular case is a special case of the semi-regular one, where \mathbf{P} is 2-slanted and $k_\ell(\mathbf{P}), k_r(\mathbf{P})$ are chosen such that

$$\text{supp}(\mathbf{p}) = \{ -k_r(\mathbf{P}), \dots, -k_\ell(\mathbf{P}) \}.$$

Since in Example 1.5 we have a 2-slanted subdivision matrix \mathbf{P} that works both in the regular and semi-regular setting, it is natural to ask if this higher generality is necessary. In general, the transition from the regular to the semi-regular setting is not so smooth. Indeed, if F is the limit function of a regular convergent subdivision scheme with the subdivision matrix \mathbf{P} over the regular initial mesh \mathbb{Z} which is obtained from the initial values \mathbf{f}_0 , then attaching the values $\mathbf{P}^j \mathbf{f}_0$, $j \in \mathbb{N}$, to the refined meshes \mathbf{t}_j , obtained from a semi-regular initial mesh \mathbf{t}_0 , we obtain the limit function G that satisfies

$$G(x) = \begin{cases} F(x/h_\ell), & \text{if } x < 0, \\ F(x/h_r), & \text{if } x \geq 0. \end{cases}$$

Hence, if $F \in \mathcal{C}^n(\mathbb{R})$, $n \in \mathbb{N}$, then, for every $k \in \mathbb{N}$, $k \leq n$,

$$G^{(k)}(x) = \begin{cases} h_\ell^{-k} F^{(k)}(x/h_\ell), & \text{if } x < 0, \\ h_r^{-k} F^{(k)}(x/h_r), & \text{if } x > 0, \end{cases} \quad (1.9)$$

and $G^{(k)}$ is unlikely to be continuous at 0 for $h_\ell \neq h_r$. To prevent this loss of smoothness, we must allow modifications to the columns of the subdivision matrix that influence the limit functions at 0.

Usually, dealing with \mathbf{P} can be difficult since it is a bi-infinite matrix. Luckily, some properties can be studied on a finite section of \mathbf{P} .

Definition 1.7. The finite square section of the subdivision matrix \mathbf{P} defined by

$$\mathring{\mathbf{P}} = \mathbf{P}(k_\ell(\mathbf{P}) : k_r(\mathbf{P}), k_\ell(\mathbf{P}) : k_r(\mathbf{P})), \quad (1.10)$$

is called *invariant neighborhood matrix* of the scheme.

Remark 1.8. The dimension of the invariant neighbourhood matrix is $k_r(\mathbf{P}) - k_\ell(\mathbf{P}) + 1$ which in the regular case coincides with $|\text{supp}(\mathbf{p})|$. Moreover, in the regular case, $\mathring{\mathbf{P}}$ does not depend on the indexing of the mask \mathbf{p} . Indeed, the diagonal of $\mathring{\mathbf{P}}$ is always the flipped mask \mathbf{p} and the off-diagonal elements are uniquely determined by the 2-slantedness of \mathbf{P} . \diamond

Via the invariant neighbourhood matrix $\mathring{\mathbf{P}}$ we can analyse the right-spectrum of the subdivision matrix \mathbf{P} .

Proposition 1.9 ([63]). *Consider a subdivision matrix \mathbf{P} and its corresponding invariant neighborhood matrix $\mathring{\mathbf{P}}$. Then there is a one-to-one correspondence between the right-eigenvectors of \mathbf{P} and $\mathring{\mathbf{P}}$. In addition, \mathbf{P} and $\mathring{\mathbf{P}}$ have the same right-eigenvalues.*

Proof. On one hand, due to (1.7), for every $k, m \in \mathbb{Z}$ such that $m > k_r(\mathbf{P})$ and $k \leq k_r(\mathbf{P})$ or $m < k_\ell(\mathbf{P})$ and $k \geq k_\ell(\mathbf{P})$,

$$\mathbf{P}(k, m) = 0. \quad (1.11)$$

Thus, for every bi-infinite vector \mathbf{v} ,

$$(\mathbf{P} \mathbf{v})(k_\ell(\mathbf{P}) : k_r(\mathbf{P})) = \mathring{\mathbf{P}} \mathbf{v}(k_\ell(\mathbf{P}) : k_r(\mathbf{P})).$$

In particular, this holds for every right-eigenvector \mathbf{v} of \mathbf{P} with respect to a right-eigenvalue $\lambda \in \mathbb{C}$, which leads to

$$\mathring{\mathbf{P}} \mathbf{v}(k_\ell(\mathbf{P}) : k_r(\mathbf{P})) = \lambda \mathbf{v}(k_\ell(\mathbf{P}) : k_r(\mathbf{P})).$$

On the other hand, with the same argument as in (1.11), $\mathring{\mathbf{P}}$ contains all the non-zero diagonal elements of \mathbf{P} . Thus, if \mathbf{v} is a right-eigenvector of $\mathring{\mathbf{P}}$ with respect to an eigenvalue $\lambda \in \mathbb{C}$, we have that, for

$$\mathbf{v}^{(1)} = \begin{bmatrix} \frac{1}{\lambda} \mathbf{P}(k_\ell(\mathbf{P}) - 1, k_\ell(\mathbf{P}) : k_r(\mathbf{P})) \mathbf{v} \\ \mathbf{v} \\ \frac{1}{\lambda} \mathbf{P}(k_r(\mathbf{P}) + 1, k_\ell(\mathbf{P}) : k_r(\mathbf{P})) \mathbf{v} \end{bmatrix},$$

$$\mathbf{P}(k_\ell(\mathbf{P}) - 1 : k_r(\mathbf{P}) + 1, k_\ell(\mathbf{P}) - 1 : k_r(\mathbf{P}) + 1) \mathbf{v}^{(1)} = \lambda \mathbf{v}^{(1)}.$$

Thus, we extended uniquely \mathbf{v} to $\mathbf{v}^{(1)}$ to a right-eigenvector of $\mathbf{P}(k_\ell(\mathbf{P}) - 1 : k_r(\mathbf{P}) + 1, k_\ell(\mathbf{P}) - 1 : k_r(\mathbf{P}) + 1)$. This procedure can be repeated indefinitely yielding a bi-infinite eigenvector of \mathbf{P} and, thus, the claim. \square

Remark 1.10. Since $\mathring{\mathbf{P}}$ is a finite matrix, its left and right-eigenvalues are the same. This is not true for bi-infinite matrices such as \mathbf{P} . Proposition 1.9 gives information only about the right-eigenvalues of \mathbf{P} . \diamond

From the subdivision matrix \mathbf{P} of a scheme one can easily read a well known necessary condition for convergence.

Theorem 1.11 ([5]). *Consider a convergent subdivision scheme with subdivision matrix \mathbf{P} and let $\{\lambda_i\}_{i=0}^I$, $I \in \mathbb{N}$, be the right-eigenvalues of \mathbf{P} with*

$$|\lambda_0| \geq |\lambda_1| \geq \dots \geq |\lambda_I|.$$

Then,

$$(i) \quad 1 = \lambda_0 > |\lambda_1|;$$

$$(ii) \quad \mathbf{P} \mathbf{1} = \mathbf{1}.$$

In particular, in the regular case, a consequence of (ii) are the so-called *sum rules*, i.e.

$$\sum_{k \in \mathbb{Z}} \mathbf{p}(2k) = \sum_{k \in \mathbb{Z}} \mathbf{p}(2k + 1) = 1. \quad (1.12)$$

In the semi-regular case, (1.7) implies that (1.12) holds for both regular masks \mathbf{p}_ℓ and \mathbf{p}_r . This gives a constraint on the support of the columns of \mathbf{P}_{irr} which must be a subset

of the set

$$\{ k_\ell(\mathbf{P}) - |\text{supp}(\mathbf{p}_\ell)| + 3, \dots, k_r(\mathbf{P}) + |\text{supp}(\mathbf{p}_r)| - 3 \}.$$

Example 1.12 (Linear B-spline, part II). From the subdivision matrix \mathbf{P} in Example 1.5, it is easy to check that the corresponding mask and invariant neighbourhood matrix are respectively

$$\mathbf{p}(k) = \begin{cases} 1, & \text{if } k = 0, \\ 1/2, & \text{if } k \in \{-1, 1\}, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \mathring{\mathbf{P}} = \begin{bmatrix} 1/2 & 1/2 & \\ & 1 & \\ & 1/2 & 1/2 \end{bmatrix}. \quad (1.13)$$

The eigenvalues of $\mathring{\mathbf{P}}$ are 1 with multiplicity one and 1/2 with multiplicity two, with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_{1/2,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_{1/2,2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

By Proposition 1.9, \mathbf{P} has the same right-eigenvalues of $\mathring{\mathbf{P}}$ and the corresponding right-eigenvectors of \mathbf{P} can be obtained by extending \mathbf{v}_1 , $\mathbf{v}_{1/2,1}$ and $\mathbf{v}_{1/2,2}$. In particular, the right-eigenvector of \mathbf{P} corresponding to the eigenvalue 1 is the bi-infinite vector of all ones $\mathbf{1}$. To extend $\mathbf{v}_{1/2,1}$ to an eigenvector of \mathbf{P} , following Proposition 1.9, we want to find

$$\mathbf{v}^{(1)} = \begin{bmatrix} a \\ \mathbf{v}_{1/2,1} \\ b \end{bmatrix}$$

such that

$$\begin{bmatrix} 0 & 1 & & & \\ & 1/2 & 1/2 & & \\ & & 1 & & \\ & & 1/2 & 1/2 & \\ & & & 1 & 0 \end{bmatrix} \mathbf{v}^{(1)} = \frac{1}{2} \mathbf{v}^{(1)}.$$

This leads to $a = 2$ and $b = 0$. If we repeat this process indefinitely we get

$$\mathbf{v}^{(\infty)}(k) = \begin{cases} -k, & \text{for } k < 0, \\ 0, & \text{for } k \geq 0. \end{cases}$$

The extension of $\mathbf{v}_{1/2,2}$ is done in the same way. \triangle

Basic Limit Functions

As already pointed out, by Definition 1.4, convergent subdivision schemes produce different limit functions based on the initial set of data. Among those functions there is a function family of special interest.

Definition 1.13. Consider a convergent subdivision scheme. Its limit functions $[\varphi_k]_{k \in \mathbb{Z}} \subset \mathcal{C}^0(\mathbb{R})$, where φ_k is obtained via the subdivision process from the initial data

$$\mathbf{e}^{(k)}(m) = \delta_{km}, \quad m \in \mathbb{Z},$$

are called *basic limit functions*

In the regular case, the global 2-slanted structure of the subdivision matrix \mathbf{P} leads to the following result.

Proposition 1.14 ([5]). *Consider a convergent regular subdivision scheme over the mesh $\mathbf{t}_0 = h\mathbb{Z}$, $h > 0$. Let $\mathbf{f}_0, \mathbf{g}_0 \in \ell(\mathbb{Z})$ such that, for some $k^* \in \mathbb{Z}$, $\mathbf{g}_0(k) = \mathbf{f}_0(k - k^*)$, $k \in \mathbb{Z}$. Then $G = F(\cdot - hk^*)$, where F and G denote the limit function obtained applying the subdivision scheme to the initial data \mathbf{f}_0 and \mathbf{g}_0 , respectively.*

Proposition 1.14 is a shift-invariance property of regular subdivision, in the sense that any shift of the initial data results in the shift of the limit functions. In particular, it means that the basic limit functions are actually the shifts of a single function, i.e.

$$\varphi_k(x) = \varphi_0(x - hk), \quad x \in \mathbb{R}, \quad k \in \mathbb{Z},$$

where h is the stepsize of the underlying regular mesh.

In the semi-regular case, instead, we only have that

$$\varphi_k(x) = \begin{cases} \varphi_{k_\ell(\mathbf{P})}(x + h_\ell(k_\ell(\mathbf{P}) - k)) & \text{for } k \leq k_\ell(\mathbf{P}) \\ \varphi_{k_r(\mathbf{P})}(x + h_r(k_r(\mathbf{P}) - k)) & \text{for } k \geq k_r(\mathbf{P}) \end{cases}, \quad x \in \mathbb{R}, \quad (1.14)$$

while nothing of the kind can be said about the functions φ_k for $k_\ell(\mathbf{P}) < k < k_r(\mathbf{P})$, leaving us with $k_r(\mathbf{P}) - k_\ell(\mathbf{P}) + 1$ different basic limit functions. This lack of shift-invariance around \mathbf{t}_0 is one of the most significant difference between the regular and the semi-regular settings.

Remark 1.15. A global scaling of the initial mesh amounts to a uniform scale of all the basic limit functions, i.e. if we change the initial mesh from \mathbf{t}_0 to $h\mathbf{t}_0$, for some $h > 0$, the basic limit functions will change from φ_k to $\varphi_k(\cdot/h)$, for every $k \in \mathbb{Z}$. \diamond

The importance of the basic limit functions, from the subdivision point of view, resides in the following fundamental result.

Theorem 1.16 ([5]). *Consider a convergent subdivision scheme with basic limit functions $[\varphi_k]_{k \in \mathbb{Z}}$. Then, for every initial data $\mathbf{f}_0 \in \ell^\infty(\mathbb{Z})$, the corresponding limit function F satisfies*

$$F(x) = \sum_{k \in \mathbb{Z}} \mathbf{f}_0(k) \varphi_k(x), \quad x \in \mathbb{R}. \quad (1.15)$$

In particular, the scheme is \mathcal{C}^r , $r > 0$, if and only if $[\varphi_k]_{k \in \mathbb{Z}} \subset \mathcal{C}^r(\mathbb{R})$.

The basic limit functions of a scheme have the ability to describe all the limit functions that the scheme is able to produce. Thus, the study of a scheme can be reduced to the study of its basic limit functions.

From the point of view of the basic limit functions, the necessary condition for the convergence of a scheme stated in Theorem 1.11, is translated into the following result, which states that the basic limit functions of a convergent scheme form a partition of unity. In particular, a convergent scheme is able to produce constant polynomials.

Theorem 1.17 ([10]). *Consider a convergent subdivision scheme with basic limit functions $\varphi_k \in \mathcal{C}^0(\mathbb{R})$. Then, for every $x \in \mathbb{R}$,*

$$\mathbf{1}^T [\varphi_k(x)]_{k \in \mathbb{Z}} = \sum_{k \in \mathbb{Z}} \varphi_k(x) = 1. \quad (1.16)$$

Definition 1.18. A subdivision scheme on the initial mesh \mathbf{t}_0 , with basic limit functions $[\varphi_k]_{k \in \mathbb{Z}}$, is said to *generate polynomials of degree $n \in \mathbb{N}_0$* , if for every polynomial π of degree at most n there exists a bi-infinite vector \mathbf{c} such that

$$\pi(x) = \sum_{k \in \mathbb{Z}} \mathbf{c}(k) \varphi_k(x) = \mathbf{c}^T [\varphi_k(x)]_{k \in \mathbb{Z}}, \quad x \in \mathbb{R}.$$

If, for every $k \in \mathbb{Z}$, $\mathbf{c}(k) = \pi(\mathbf{t}_0(k))$, then the scheme is said to *reproduce polynomials of degree n* .

Refinement Equation

Another fundamental aspect about the basic limit functions of a convergent scheme, is that they satisfy a very useful equation called *refinement equation*, which is one of the key ingredient for our frame construction later on. This is a relation that links the basic limit functions to their dilated versions.

Theorem 1.19 ([63]). *Consider a convergent subdivision scheme with subdivision matrix \mathbf{P} and basic limit function $[\varphi_k]_{k \in \mathbb{Z}}$. Then*

$$[\varphi_k(x)]_{k \in \mathbb{Z}} = \mathbf{P}^T [\varphi_k(2x)]_{k \in \mathbb{Z}}, \quad x \in \mathbb{R}. \quad (1.17)$$

In particular, in the regular case, over the initial mesh $\mathbf{t}_0 = h\mathbb{Z}$, $h > 0$,

$$\varphi_0(x) = \sum_{k \in \mathbb{Z}} \mathbf{p}(k) \varphi_0(2x - hk), \quad x \in \mathbb{R}, \quad (1.18)$$

where \mathbf{p} is the mask of the scheme.

Example 1.20 (Linear B-spline, part III). Example 1.5 shows that the linear B-spline scheme produces piecewise linear functions interpolating the initial data $\mathbf{f} \in \ell(\mathbb{Z})$ over $\mathbf{t}_0 = h\mathbb{Z}$, $h > 0$. Thus, the basic limit function of the scheme is

$$\varphi_0(x) = (1 - |x|/h) \chi_{[-h, h]}(x). \quad (1.19)$$

It is easy to see that (1.15) and (1.18) hold. In particular, see Figure 1.1,

$$\varphi_0(x) = \frac{1}{2} \varphi_0(2x + h) + \varphi_0(2x) + \frac{1}{2} \varphi_0(2x - h), \quad x \in \mathbb{R}. \quad (1.20)$$

If we consider the semi-regular initial mesh \mathbf{t}_0 as in (1.3), with $h_\ell, h_r > 0$, then the

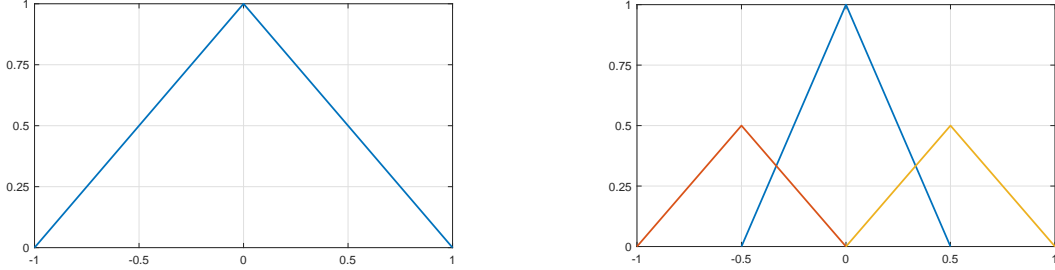


Figure 1.1: The basic limit function φ_0 of the linear B-spline scheme on the mesh $\mathbf{t}_0 = \mathbb{Z}$ on the left and its decomposition as in the refinement equation (1.20), with $h = 1$.

corresponding basic limit functions are

$$\varphi_k(x) = \begin{cases} (1 - |x - h_\ell k|/h_\ell) \chi_{h_\ell[k-1, k+1]}(x), & \text{if } k < 0, \\ (1 + x/h_\ell) \chi_{[-h_\ell, 0]}(x) + (1 - x/h_r) \chi_{[0, h_r]}(x), & \text{if } k = 0, \\ (1 - |x - h_r k|/h_r) \chi_{h_r[k-1, k+1]}(x), & \text{if } k > 0, \end{cases} \quad x \in \mathbb{R}. \quad (1.21)$$

Thus, Proposition 1.14 no longer holds and we have three different basic limit functions instead of one (see e.g. Figure 1.2 with $h_\ell = 1$ and $h_r = 2$). Here the refinement equation holds in its matrix form (1.17) but not as in (1.18). Again the basic limit functions form a partition of unity (1.16) and the scheme reproduces polynomials of degree 1. Since the subdivision matrix \mathbf{P} is the same as in the regular case, see Example 1.5, all the eigenproperties of \mathbf{P} are kept. Moreover, we observe that at 0 the basic limit functions are still $\mathcal{C}^{1-\epsilon}$, $\epsilon > 0$. \triangle

Now that we introduced all the necessary tools, we would like to take a step back and prove Theorem 1.11 to point out a specific fact about the invariant neighbourhood matrix and the structure of semi-regular subdivision matrices.

Proof of Theorem 1.11. For the sake of simplicity we only consider the regular case. Let $\text{supp}(\mathbf{p}) = \{a, \dots, b\}$, and, for $k \in \mathbb{Z}$, $\{\xi_j^{(k)}\}_{j \in \mathbb{N}}$ the piecewise linear functions that interpolate $\mathbf{P}^j \mathbf{e}^{(k)}$, $\mathbf{e}^{(k)}(m) = \delta_{km}$, over the mesh $\mathbf{t}_j = 2^{-j} \mathbf{t}_0$ approximating φ_k . We consider the invariant neighborhood matrix $\mathring{\mathbf{P}}$. By Proposition 1.9, studying the right-spectrum of \mathbf{P} and the spectrum of $\mathring{\mathbf{P}}$ is equivalent. Due to Definition 1.7,

$$\left[\xi_j^{(k)}(\mathbf{t}_j(m)) \right]_{m=-b, \dots, -a} = \mathring{\mathbf{P}}^j \mathbf{e}^{(k)}(-b : -a), \quad j \in \mathbb{N}, k \in \mathbb{Z}. \quad (1.22)$$

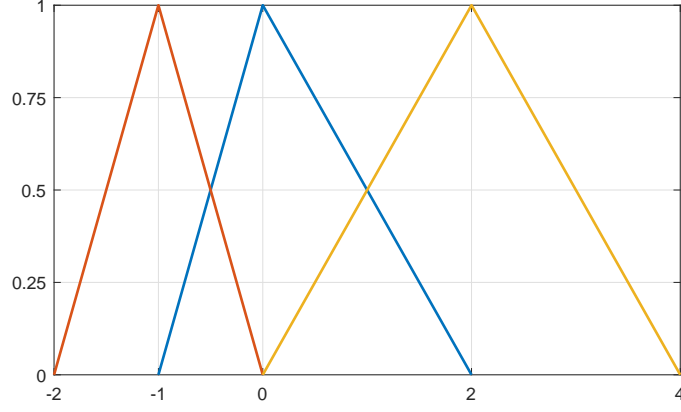


Figure 1.2: The basic limit functions φ_{-1} , φ_0 and φ_1 of the linear B-spline scheme on the semi-regular mesh with $h_\ell = 1$ and $h_r = 2$.

Since the scheme is convergent, the left-hand side of (1.22) converges, for $j \rightarrow \infty$, to $\varphi_k(0)\mathbf{1}$, for every $k \in \mathbb{Z}$. Now, if we suppose $|\lambda_0| < 1$, (1.22) implies $\varphi_0(-hk) = \varphi_k(0) = 0$ for every $k \in \mathbb{Z}$. But in this case, from the refinement equation (1.18) we also get, for $x = hm/2$, $m \in \mathbb{Z}$,

$$\varphi_0(hm/2) = \sum_{k \in \mathbb{Z}} \mathbf{p}(k) \varphi_0((m-k)h) = 0.$$

Repeating this argument it is easy to check that $\varphi_0(x) = 0$ for every $x \in \{2^{-j}hk\}_{j \in \mathbb{N}, k \in \mathbb{Z}}$ which is a dense set in \mathbb{R} . At this point, the continuity of φ_0 implies $\varphi_0 \equiv 0$ which is against the hypothesis of convergence. On the other hand, if we suppose $|\lambda_0| > 1$, with eigenvector $\mathbf{v}_0 \in \mathbb{R}^{b-a+1} \setminus \{\mathbf{0}\}$, we have

$$\begin{aligned} \infty &= \lim_{j \rightarrow \infty} \left\| \mathring{\mathbf{P}}^j \mathbf{v} \right\|_\infty = \lim_{j \rightarrow \infty} \left\| \mathring{\mathbf{P}}^j \left(\sum_{k=1}^{b-a+1} \mathbf{v}(k) \mathbf{e}^{(k-b-1)} \right) \right\|_\infty \\ &= \left\| \sum_{k=1}^{b-a+1} \mathbf{v}(k) \varphi_{k-b-1}(0) \mathbf{1} \right\|_\infty < \infty, \end{aligned}$$

which is a contradiction. The only case left then is $|\lambda_0| = 1$. Supposing $\lambda_0 \neq 1$ implies that the limit for $j \rightarrow \infty$ of λ_0^j does not exist which means that, for $\mathbf{v}_0 \in \mathbb{R}^{b-a+1} \setminus \{\mathbf{0}\}$ an eigenvector associated to λ_0 , also the limit of $\mathring{\mathbf{P}}^j \mathbf{v} = \lambda_0^j \mathbf{v}$ does not exist and this is again in contrast with

$$\lim_{j \rightarrow \infty} \mathring{\mathbf{P}}^j \mathbf{v} = \sum_{k=1}^{b-a+1} \mathbf{v}(k) \varphi_{k-b-1}(0) \mathbf{1}. \quad (1.23)$$

This leaves us with $\lambda_0 = 1$ and the same argument as (1.23) shows that it can only have algebraic multiplicity one with associated eigenspace generated by $\mathbf{1}$. In particular,

since $\mathring{\mathbf{P}}$ contains all the non-zero elements of the rows $-b, \dots, -a$ of \mathbf{P} (1.11), $\mathring{\mathbf{P}} \mathbf{1} = \mathbf{1}$ implies (ii). In particular, from (1.5),

$$1 = (\mathbf{P} \mathbf{1})(m) = \sum_{k \in \mathbb{Z}} \mathbf{p}(m - 2k) = \sum_{k \equiv m \pmod{2}} \mathbf{p}(k), \quad m \in \mathbb{Z}.$$

□

From (1.22) and its limit for $j \rightarrow \infty$ we see how $\mathring{\mathbf{P}}$ encodes the information about the basic limit functions around 0. Indeed, the only basic limit functions that can not vanish at 0 are the one with index k between $1 - b$ and $-a - 1$, since the supports of φ_{-a} and φ_{-b} start and end at 0 respectively. As for \mathbf{P}_{irr} , due to (1.8) and (1.10), its columns are exactly the ones in common with $\mathring{\mathbf{P}}$ which refer to the basic limit functions having 0 inside their support.

We now focus on the regular case, where we exploit the refinement equation to evaluate the basic limit function φ_0 .

Proposition 1.21 ([5]). *Let φ_0 be the basic limit function of a regular convergent subdivision scheme over the initial mesh $\mathbf{t}_0 = h\mathbb{Z}$, $h > 0$, with the subdivision matrix \mathbf{P} and the mask \mathbf{p} , $\text{supp}(\mathbf{p}) = \{a, \dots, b\}$. Then,*

$$[\varphi_0(-hk)]_{k \in \mathbb{Z}} = \mathbf{P}^T [\varphi_0(-hk)]_{k \in \mathbb{Z}}.$$

In particular, $\varphi_0(-hk) = 0$ for $k \leq -b$ or $k \geq -a$, and

$$[\varphi_0(-hk)]_{k=-b, \dots, -a} = \mathring{\mathbf{P}}^T [\varphi_0(-hk)]_{k=-b, \dots, -a} \quad \text{with} \quad \sum_{k=a}^b \varphi_0(hk) = 1. \quad (1.24)$$

Moreover, for every $j \in \mathbb{N}$, $m \in \mathbb{Z}$,

$$[\varphi_0(2^{-j}h(m - 2^j k))]_{k \in \mathbb{Z}} = (\mathbf{P}^T)^j [\varphi_0(h(m - k))]_{k \in \mathbb{Z}}. \quad (1.25)$$

We can get more insight about φ_0 looking at the refinement equation (1.18) again from a different angle. The structure of (1.18) suggests to pass to the Fourier side.

Theorem 1.22 ([10]). *Let $\mathbf{p} \in \ell(\mathbb{Z})$ be a compactly supported mask. If there exists $\varphi_0 \in L^1(\mathbb{R})$ satisfying the refinement equation associated to \mathbf{p} (1.18), for some $h > 0$, then*

$$\hat{\varphi}_0(\omega) = p(h\omega/2) \hat{\varphi}_0(\omega/2), \quad \omega \in \mathbb{R},$$

where $\hat{\varphi}_0$ is the Fourier transform of φ_0 , i.e.

$$\hat{\varphi}_0(\omega) = \int_{\mathbb{R}} \varphi_0(x) e^{-2\pi i x \omega} dx, \quad \omega \in \mathbb{R},$$

and $p(\omega)$ is the trigonometric polynomial

$$p(\omega) := \frac{1}{2} \sum_{k \in \mathbb{Z}} \mathbf{p}(k) e^{-2\pi i k \omega}. \quad (1.26)$$

In particular, if \mathbf{p} is the mask of a convergent subdivision scheme, we have

$$p(\omega) = \frac{1 + e^{-2\pi i \omega}}{2} p^{[1]}(\omega), \quad (1.27)$$

where $p^{[1]}(\omega)$ is a trigonometric polynomial with $p^{[1]}(0) = 1$.

Definition 1.23. Consider a regular subdivision scheme with compactly supported mask \mathbf{p} . The trigonometric polynomial $p(\omega)$ associated with \mathbf{p} in (1.26) is called *symbol* of the scheme.

Remark 1.24. A consequence of (1.5) and Proposition 1.14 is that, given a mask \mathbf{p} , all the subdivision schemes associated to shifts of \mathbf{p} are equivalent. This implies that the symbol $p(\omega)$ of a scheme is unique up to a factor $e^{-2\pi i k \omega}$, $k \in \mathbb{Z}$. \diamond

Every manipulation requiring the symbol of a scheme can only be exploited in the regular case, due to the shift-invariance of the basic limit functions. In the semi-regular case, the lack of shift-invariance does not allow for a meaningful extension of the concept of symbol.

Example 1.25 (Linear B-spline, part IV). From (1.26) one has

$$p(\omega) = \frac{1}{4} e^{-2\pi i \omega} + \frac{1}{2} + \frac{1}{4} e^{2\pi i \omega} = \frac{1 + e^{-2\pi i \omega}}{2} \frac{1 + e^{2\pi i \omega}}{2} = \frac{1 + e^{-2\pi i \omega}}{2} p^{[1]}(\omega)$$

as in Theorem 1.22, but also

$$p(\omega) = \left(\frac{e^{\pi i \omega} + e^{-\pi i \omega}}{2} \right)^2 = (\cos(\pi \omega))^2, \quad \omega \in \mathbb{R}.$$

In particular, the symbol of the linear B-spline scheme has a double zero at $\omega = 1/2$. Moreover, from the analytic expression of φ_0 (1.19), it is easy to check

$$\sum_{k \in \mathbb{Z}} \varphi_0(x - hk) = 1 \quad \text{and} \quad \sum_{k \in \mathbb{Z}} hk \varphi_0(x - hk) = x, \quad x \in \mathbb{R},$$

which means, by the linearity of the subdivision process, that the linear B-spline scheme reproduces polynomials of degree 1. \triangle

Remark 1.26. The multiplicity of the zero at $\omega = 1/2$ of the symbol $p(\omega)$ and the degree of the polynomial generation of the associated scheme are intrinsically linked. Indeed, a regular scheme generates polynomials of degree $n \in \mathbb{N}_0$ if and only if

$$p(\omega) = \left(\frac{1 + e^{-2\pi i \omega}}{2} \right)^{n+1} p^{[n]}(\omega),$$

where $p^{[n]}(\omega)$ is a trigonometric polynomial such that $p^{[n]}(0) = 1$ (see e.g. [10]). \diamond

Via symbol manipulations one can also produce new subdivision schemes from the known ones.

Proposition 1.27 ([10]). *Consider two convergent regular subdivision schemes over the same regular mesh $\mathbf{t}_0 = h\mathbb{Z}$, $h > 0$. If $\varphi_0 \in \mathcal{C}^m(\mathbb{R})$, $m > 0$, and $\zeta_0 \in \mathcal{C}^n(\mathbb{R})$, $n > 0$, are the basic limit functions of the two schemes associated to the symbols $p(\omega)$ and $z(\omega)$ with compactly supported masks \mathbf{p} and \mathbf{z} , respectively, then the symbol*

$$g(\omega) = p(\omega) z(\omega), \quad \omega \in \mathbb{R}, \quad (1.28)$$

defines a convergent subdivision scheme over the same initial mesh \mathbf{t}_0 with basic limit function given by

$$\gamma_0(x) = \int_{\mathbb{R}} \varphi_0(y) \zeta_0(x - y) dy, \quad x \in \mathbb{R}.$$

Moreover, $\gamma_0 \in \mathcal{C}^{\max(m,n)}(\mathbb{R})$.

Remark 1.28. In terms of the masks, equation (1.28) for $g(\omega) = \sum_{k \in \mathbb{Z}} \mathbf{g}(k) e^{-2\pi i k \omega}$ is a convolution

$$\mathbf{g}(k) = \sum_{m \in \mathbb{Z}} \mathbf{p}(m) \mathbf{z}(k - m), \quad k \in \mathbb{Z}.$$

\diamond

Remark 1.29. Due to the smoothing property of the convolution, Proposition 1.27 works even if one of the functions is a non-continuous solution of a refinement equation, e.g. the indicator function $\chi_{[0,h]} \notin \mathcal{C}^0(\mathbb{R})$, $h > 0$, which satisfies

$$\chi_{[0,h]}(x) = \chi_{[0,h/2]}(x) + \chi_{[h/2,h]}(x) = \chi_{[0,h]}(2x) + \chi_{[0,h]}(2x - h), \quad x \in \mathbb{R},$$

and, on the Fourier side,

$$\widehat{\chi}_{[0,h]}(\omega) = p(h\omega/2) \widehat{\chi}_{[0,h]}(\omega/2) \quad \text{with} \quad p(\omega) = \frac{1 + e^{-2\pi i \omega}}{2}, \quad \omega \in \mathbb{R}.$$

\diamond

Remark 1.30. The basic limit function γ_0 in Proposition 1.27 has a wider support than each of φ_0 or ζ_0 . Indeed, if

$$\text{supp}(\mathbf{p}) = \{a, \dots, b\} \quad \text{and} \quad \text{supp}(\mathbf{z}) = \{c, \dots, d\},$$

then $\text{supp}(\mathbf{g}) = \{a + c, \dots, b + d\}$. By Theorem 1.16,

$$|\text{supp}(\gamma_0)| = h(|\text{supp}(\mathbf{g})| - 1) = h(b + d - a - c) = |\text{supp}(\varphi_0)| + |\text{supp}(\zeta_0)|.$$

\diamond

We illustrate the result of Proposition 1.27 on the following example.

Example 1.31 (Cubic B-spline, part I). We choose the symbols in Proposition 1.27 to be both the one associated to the linear B-spline scheme in Examples 1.5, 1.12, 1.20 and 1.25. Then, by Proposition 1.27, we obtain the product symbol

$$\begin{aligned}
 p(\omega) &= \left(\frac{1}{4}e^{-2\pi i\omega} + \frac{1}{2} + \frac{1}{4}e^{2\pi i\omega} \right)^2 \\
 &= \left(\frac{1}{16}e^{-4\pi i\omega} + \frac{1}{4}e^{-2\pi i\omega} + \frac{3}{8} + \frac{1}{4}e^{2\pi i\omega} + \frac{1}{16}e^{4\pi i\omega} \right) \\
 &= \left(\frac{1 + e^{-2\pi i\omega}}{2} \right)^4 e^{4\pi i\omega} = (\cos(\pi\omega))^4, \quad \omega \in \mathbb{R}.
 \end{aligned}$$

Thus, we have

$$\mathbf{P} = \begin{bmatrix} & 1/8 & & & \\ & 1/2 & & & \\ \ddots & 3/4 & 1/8 & & \\ & 1/2 & 1/2 & & \\ 1/8 & \boxed{3/4} & 1/8 & & \\ & 1/2 & 1/2 & & \\ & 1/8 & 3/4 & \ddots & \\ & & 1/2 & & \\ & & 1/8 & & \end{bmatrix} \quad \text{and} \quad \mathring{\mathbf{P}} = \begin{bmatrix} 1/8 & 3/4 & 1/8 & & \\ & 1/2 & 1/2 & & \\ & 1/8 & 3/4 & 1/8 & \\ & & 1/2 & 1/2 & \\ & & 1/8 & 3/4 & 1/8 \end{bmatrix}. \quad (1.29)$$

The eigenvalues of $\mathring{\mathbf{P}}$ are 1, 1/2, 1/4, 1/8 and 1/8 with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_{1/2} = \begin{bmatrix} -1 \\ -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_{1/4} = \begin{bmatrix} 1 \\ 2/11 \\ -1/11 \\ 2/11 \\ 1 \end{bmatrix}, \quad \mathbf{v}_{1/8,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_{1/8,2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

These vectors can be extended uniquely to eigenvectors of \mathbf{P} as in Proposition 1.9, e.g. for $\mathbf{v}_{1/2}$,

$$\mathbf{v}_{1/2}^{(1)} = \begin{bmatrix} a \\ \mathbf{v}_{1/2} \\ b \end{bmatrix}$$

such that

$$\begin{bmatrix} 0 & 1/2 & 1/2 & & & \\ & 1/8 & 3/4 & 1/8 & & \\ & & 1/2 & 1/2 & & \\ & & 1/8 & 3/4 & 1/8 & \\ & & & 1/2 & 1/2 & \\ & & & 1/8 & 3/4 & 1/8 \\ & & & & 1/2 & 1/2 & 0 \end{bmatrix} \mathbf{v}_{1/2}^{(1)} = \frac{1}{2} \mathbf{v}_{1/2}^{(1)},$$

which leads to $a = -3/2$ and $b = 3/2$. We can then repeat the process to add a further component on the top and on the bottom of $\mathbf{v}_{1/2}^{(1)}$, considering the square submatrix of \mathbf{P} obtained by $\mathring{\mathbf{P}}$ this time adding two rows and two columns on each sides.

The basic limit function φ_0 (Figure 1.3 for the initial mesh \mathbb{Z}), is the convolution of the hat function (1.19) with itself and is a piecewise cubic polynomial. Moreover, the scheme is $\mathcal{C}^{3-\epsilon}(\mathbb{R})$, $\epsilon > 0$, and generates cubic polynomials. \triangle

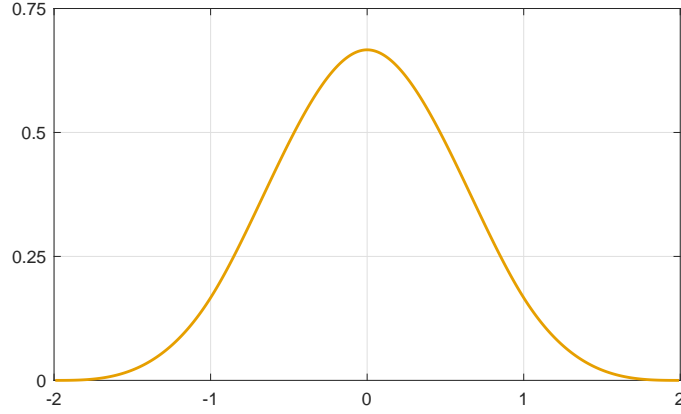


Figure 1.3: The basic limit function φ_0 of the regular cubic B-spline scheme over the initial mesh $\mathbf{t}_0 = \mathbb{Z}$.

Before we proceed with some computational aspect about subdivision, it is worthwhile to fully review the cubic B-spline scheme of Example 1.31 in a semi-regular setting.

Example 1.32 (Cubic B-spline, part II). Consider the cubic B-spline scheme, Example 1.31. The regular basic limit function in Figure 1.3 is a piecewise cubic polynomial with \mathcal{C}^2 junctions and it is increasing (decreasing) at $x = -1$ ($x = 1$). Thus, if we use the regular subdivision matrix (1.29) over a semi-regular mesh with $h_\ell \neq h_r$, due to (1.9), we end up with φ_{-1} and φ_1 to be not \mathcal{C}^1 . However, given a semi-regular mesh \mathbf{t}_0 one can compute via knot insertion, e.g. with the Oslo algorithm [17], the matrix that describes the cubic B-splines on \mathbf{t}_0 as a linear combination of the cubic B-splines on \mathbf{t}_1 . This is indeed the semi-regular subdivision matrix for the cubic B-spline scheme. For example,

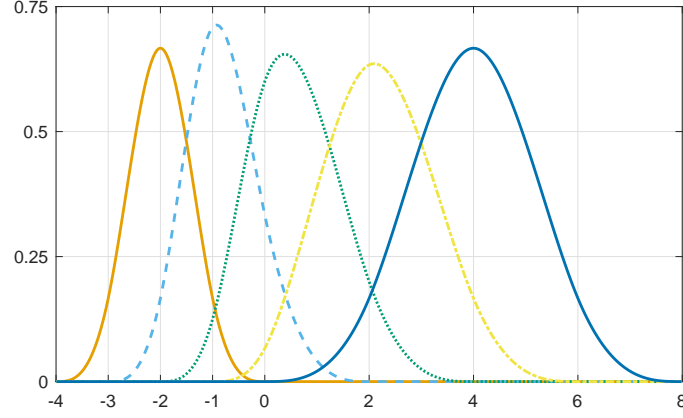


Figure 1.4: The basic limit functions $\varphi_{-2}, \dots, \varphi_2$ of the semi-regular cubic B-spline scheme over the initial semi-regular mesh \mathbf{t}_0 with $h_\ell = 1$ and $h_r = 2$.

1.2 Computing Moments and Inner Products

In this section, we present algorithms for the computation of two fundamental quantities for the construction and for the application of wavelet tight frames. These two quantities are the moments and the (cross-)Gramian matrices.

Definition 1.33. Let $f : \mathbb{R} \rightarrow \mathbb{R}$. The quantity

$$\int_{\mathbb{R}} x^\alpha f(x) dx, \quad \alpha \in \mathbb{N}_0 \quad (1.30)$$

is called the $(\alpha + 1)$ -th moment of f .

Definition 1.34. Consider two families of functions $\mathcal{F} = [f_k]_{k \in \mathbb{Z}}$ and $\mathcal{G} = [g_k]_{k \in \mathbb{Z}}$. The matrix

$$\mathbf{G} := \left[\int_{\mathbb{R}} f_k(x) g_m(x) dx \right]_{k, m \in \mathbb{Z}} \quad (1.31)$$

is called the *cross-Gramian matrix* between \mathcal{F} and \mathcal{G} . If the two families coincide, \mathbf{G} is called the *Gramian matrix* of \mathcal{F} .

In this section, the families \mathcal{F} and \mathcal{G} of basic limit functions of subdivision schemes are built from both regular and semi-regular subdivision. We start with the computation of the moments, using the work of Dahmen and Micchelli [19] for the regular case and a simple generalization of their argument to deal with the semi-regular one. Then we proceed by computing inner products between refinable functions. For the regular case we use an idea of Kunoth [44], while for the semi-regular case we exploit a method suggested by Lounsbery [48], proving its viability.

1.2.1 Moments of Limit Functions

We start with the simplest case. Let φ_0 be the basic limit function of a convergent regular subdivision scheme over the initial mesh $\mathbf{t}_0 = h\mathbb{Z}$, $h > 0$. In particular it is

continuous, compactly supported and it satisfies the refinement equation (1.18) with respect to a compactly supported mask $\mathbf{p} \in \ell(\mathbb{Z})$ that satisfies the sum rule (1.12).

Proposition 1.35 ([19]). *For every $\alpha \in \mathbb{N}$, we have*

$$\int_{\mathbb{R}} x^{\alpha} \varphi_0(x) dx = \frac{1}{2(2^{\alpha} - 1)} \sum_{0 \leq \beta < \alpha} c_{\alpha}(\beta) \int_{\mathbb{R}} x^{\beta} \varphi_0(x) dx,$$

where

$$c_{\alpha}(\beta) := \binom{\alpha}{\beta} \sum_{k \in \mathbb{Z}} \mathbf{p}(k) (hk)^{\alpha-\beta}, \quad 0 \leq \beta < \alpha.$$

Proof. Using the refinement equation (1.18) and the fact that the sequence \mathbf{p} is compactly supported we have

$$\int_{\mathbb{R}} x^{\alpha} \varphi_0(x) dx = \sum_{k \in \mathbb{Z}} \mathbf{p}(k) \int_{\mathbb{R}} x^{\alpha} \varphi_0(2x - hk) dx.$$

After the substitution $y = 2x - hk$, using the binomial formula we get

$$\begin{aligned} \int_{\mathbb{R}} x^{\alpha} \varphi_0(x) dx &= \frac{1}{2^{1+\alpha}} \sum_{k \in \mathbb{Z}} \mathbf{p}(k) \int_{\mathbb{R}} (y + hk)^{\alpha} \varphi_0(y) dy \\ &= \frac{1}{2^{1+\alpha}} \sum_{k \in \mathbb{Z}} \mathbf{p}(k) \int_{\mathbb{R}} \varphi_0(y) \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} y^{\beta} (hk)^{\alpha-\beta} dy \end{aligned}$$

Now, changing the order of the sums and the integral and recalling the definition of $c_{\alpha}(\beta)$, we arrive at

$$\int_{\mathbb{R}} x^{\alpha} \varphi_0(x) dx = \frac{1}{2^{1+\alpha}} \sum_{0 \leq \beta \leq \alpha} c_{\alpha}(\beta) \int_{\mathbb{R}} y^{\beta} \varphi_0(y) dy$$

Since, by (1.12),

$$c_{\alpha}(\alpha) = \sum_{k \in \mathbb{Z}} \mathbf{p}(k) = 2,$$

we can bring all the term with $\beta = \alpha$ on the left-hand side and get

$$\left(1 - \frac{1}{2^{\alpha}}\right) \int_{\mathbb{R}} x^{\alpha} \varphi_0(x) dx = \frac{1}{2^{1+\alpha}} \sum_{0 \leq \beta < \alpha} c_{\alpha}(\beta) \int_{\mathbb{R}} x^{\beta} \varphi_0(x) dx.$$

Thus the claim follows. □

Remark 1.36. From the moments of φ_0 , one can easily compute the moments of its shifts.

Indeed, for every $\alpha \in \mathbb{N}_0$, $y \in \mathbb{R}$,

$$\int_{\mathbb{R}} x^\alpha \varphi_0(x - y) dx = \int_{\mathbb{R}} (x + y)^\alpha \varphi_0(x) dx = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} y^{\alpha - \beta} \int_{\mathbb{R}} x^\beta \varphi_0(x) dx.$$

In particular we have

$$\int_{\mathbb{R}} \varphi_k(x) dx = \int_{\mathbb{R}} \varphi_0(x) dx$$

and

$$\int_{\mathbb{R}} x^\alpha \varphi_k(x) dx = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} (hk)^{\alpha - \beta} \int_{\mathbb{R}} x^\beta \varphi_0(x) dx, \quad k \in \mathbb{Z}.$$

Moreover, for $\lambda > 0$,

$$\int_{\mathbb{R}} x^\alpha \varphi_k(\lambda x) dx = \frac{1}{\lambda^{1 + \alpha}} \int_{\mathbb{R}} x^\alpha \varphi_k(x) dx \quad (1.32)$$

◇

Proposition 1.35, together with Remark 1.36, tells us that, if we know the value of the integral of φ_0 , all the moments of all the basic limit functions can be computed in a recursive fashion. However, as we already observed in Remark 1.15, the refinement equation (1.18) alone does not give any information about the integral of φ_0 . With the following result we compute the integral of all compactly supported limit functions that a regular scheme can generate. As a consequence, we have that the integral of φ_0 is very easy to compute and depends only on the stepsize h of the initial mesh \mathbf{t}_0 .

Proposition 1.37. *Consider a regular convergent subdivision scheme over the initial mesh $\mathbf{t}_0 = h\mathbb{Z}$, $h > 0$, with mask $\mathbf{p} \in \ell(\mathbb{Z})$, subdivision matrix \mathbf{P} and basic limit function φ_0 . If $f \in \mathcal{C}^0(\mathbb{R})$ is the limit function obtained by the subdivision scheme starting with the compactly supported data $\mathbf{f}_0 \in \ell^\infty(\mathbb{Z})$, then*

$$\int_{\mathbb{R}} f(x) dx = h \sum_{k \in \mathbb{Z}} \mathbf{f}_0(k). \quad (1.33)$$

In particular,

$$\int_{\mathbb{R}} \varphi_0(x) dx = h. \quad (1.34)$$

Proof. Let $\{f_j\}_{j \in \mathbb{N}}$ be the piecewise linear functions interpolating the data $\mathbf{f}_j = \mathbf{P}^j \mathbf{f}_0$ over the mesh $\mathbf{t}_j = 2^{-j} \mathbf{t}_0 = 2^{-j} h\mathbb{Z}$. Due to the convergence of the scheme we have $\lim_{j \rightarrow \infty} \|f - f_j\|_\infty = 0$. Now, for every $j \in \mathbb{N}$, being f_j piecewise linear over the mesh \mathbf{t}_j , we

have

$$\begin{aligned}
 \int_{\mathbb{R}} f_j(x) dx &= \sum_{k \in \mathbb{Z}} \frac{f_j(\mathbf{t}_j(k)) + f_j(\mathbf{t}_j(k+1))}{2} (\mathbf{t}_j(k+1) - \mathbf{t}_j(k)) \\
 &= \frac{h}{2^{1-j}} \sum_{k \in \mathbb{Z}} \mathbf{f}_j(k) + \mathbf{f}_j(k+1) = \frac{h}{2^{-j}} \sum_{k \in \mathbb{Z}} \mathbf{f}_j(k) \\
 &= \frac{h}{2^{-j}} \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \mathbf{p}(k-2m) \mathbf{f}_{j-1}(m) = \frac{h}{2^{1-j}} \sum_{m \in \mathbb{Z}} \mathbf{f}_{j-1}(m) \\
 &= h \sum_{k \in \mathbb{Z}} \mathbf{f}_0(k) < \infty,
 \end{aligned}$$

where we used (1.6) and (1.12). Thus, by uniform convergence, we have

$$\int_{\mathbb{R}} f(x) dx = \lim_{j \rightarrow \infty} \int_{\mathbb{R}} f_j(x) dx = h \sum_{k \in \mathbb{Z}} \mathbf{f}_0(k).$$

In particular, since φ_0 is obtained by the initial data $\mathbf{e}^{(0)} = [\delta_{0k}]_{k \in \mathbb{Z}}$, we get (1.34). \square

Remark 1.38. If one is able to compute every moment of every basic limit function φ_k , $k \in \mathbb{Z}$, then one can compute all the inner products between every polynomial π and every limit function f . Indeed, if

$$\pi(x) = \sum_{\alpha=0}^N \pi_{\alpha} x^{\alpha} \quad \text{and} \quad f(x) = \sum_{k \in \mathbb{Z}} \mathbf{f}_0(k) \varphi_k(x),$$

for some $\mathbf{f} \in \ell(\mathbb{Z})$, then

$$\langle \pi, f \rangle = \int_{\mathbb{R}} \sum_{\alpha=0}^N \pi_{\alpha} x^{\alpha} \sum_{k \in \mathbb{Z}} \mathbf{f}_0(k) \varphi_k(x) dx = \sum_{k \in \mathbb{Z}} \mathbf{f}_0(k) \sum_{\alpha=0}^N \pi_{\alpha} \int_{\mathbb{R}} x^{\alpha} \varphi_k(x) dx.$$

\diamond

Remark 1.39. Proposition 1.35 can be generalized in a straightforward way to higher dimensional refinable functions with general diagonal dilation, i.e. if $\varphi_0 \in \mathcal{C}^0(\mathbb{R}^d)$, $d \in \mathbb{N}$, with

$$\varphi_0(x) = \sum_{k \in \mathbb{Z}^d} \mathbf{p}(k) \varphi_0(\mathbf{A}x - \mathbf{H}k), \quad x \in \mathbb{R}^d$$

for some $\mathbf{A}, \mathbf{H} \in GL_d(\mathbb{R})$, with \mathbf{A} diagonal, $\min_{1 \leq m \leq d} (\mathbf{A}(m, m)) > 1$, and some multi-vector \mathbf{p} with

$$\sum_{k \in \mathbb{Z}^d} \mathbf{p}(k) = \det(\mathbf{A}),$$

we have that, for every multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d \setminus \{\mathbf{0}\}$,

$$\int_{\mathbb{R}^d} x^\alpha \varphi_0(x) dx = \frac{1}{\det(\mathbf{A}) (\text{diag}(\mathbf{A})^\alpha - 1)} \sum_{0 \leq \beta < \alpha} c_\alpha(\beta) \int_{\mathbb{R}^d} x^\beta \varphi_0(x) dx$$

where

$$c_\alpha(\beta) := \binom{\alpha}{\beta} \sum_{k \in \mathbb{Z}^d} \mathbf{p}(k) (\mathbf{H}k)^{\alpha-\beta}, \quad 0 \leq \beta < \alpha,$$

$$x^\alpha = \prod_{m=1}^d x_m^{\alpha_m}, \quad |\alpha| = \sum_{m=1}^d \alpha_m,$$

and $0 \leq \beta < \alpha$ meaning that, componentwise, all the components of β are not greater than the components of α , with at least one component strictly lesser. Moreover, Remark 1.36 still holds. Proposition 1.37 is more tricky to extend to higher dimensions. With $d = 2$, if we consider a diagonal dilation matrix $\mathbf{A} \in GL_2(\mathbb{R})$, with $\max_{m=1,2} \mathbf{A}(m, m) > 1$, an initial set of regular knots is of the form

$$\mathbf{t}_0 = \mathbf{H} \mathbb{Z}^2 \quad \text{with} \quad \mathbf{H} = \begin{bmatrix} h_1 & h_2 \cos(\theta_1) \\ 0 & h_2 \cos(\theta_1) \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & \sin(\theta_2) \\ -\sin(\theta_2) & \cos(\theta_2) \end{bmatrix},$$

where θ_1 is the angle between the two main axes, θ_2 the angle describing the rotation of the system and $h_1, h_2 > 0$ the sizes of the intervals between two consecutive knots on each main axis. Indeed, we have that, for every $j \in \mathbb{N}$, the knots of $\mathbf{t}_j = \mathbf{A}^{-j} \mathbf{t}_0$ belong to \mathbf{t}_{j+1} , and, as in Remark 1.3, for every $k \in \mathbb{Z}^2$, $\mathbf{t}_0 - \mathbf{H}k = \mathbf{t}_0(\cdot - k)$. In contrast with the univariate case, with these knots we can have different quadrilateral and triangular meshes. This creates different sequences of approximants to the limit function and in general one should prove analogous of (1.33) for each of these choices. In the bivariate case, however, is not hard to prove that the integral of a limit function f obtained starting from the compactly supported data $\mathbf{f}_0 \in \ell(\mathbb{Z}^2)$ satisfies

$$\int_{\mathbb{R}^2} f(x) dx = h_1 h_2 \sin(\theta_1) \sum_{k \in \mathbb{Z}^2} \mathbf{f}_0(k),$$

independently from the considered mesh. \diamond

When dealing with semi-regular schemes, due to (1.14), we realize that we already know all the moments of most of the basic limit functions. Indeed, if we consider a semi-regular scheme over the initial mesh \mathbf{t}_0 in (1.3), $h_\ell, h_r > 0$, with the subdivision matrix \mathbf{P} , the left and right regular masks $\mathbf{p}_\ell, \mathbf{p}_r$, and $k_\ell(\mathbf{P}), k_r(\mathbf{P})$ in (1.7), then the basic limit functions φ_k , $k \leq k_\ell(\mathbf{P})$ are also basic limit functions of the regular scheme defined by the mask \mathbf{p}_ℓ over the mesh $h_\ell \mathbb{Z}$, and similarly for φ_k , $k \geq k_r(\mathbf{P})$ and \mathbf{p}_r . Thus, we only need to compute the moments of φ_k , $k_\ell(\mathbf{P}) < k < k_r(\mathbf{P})$ and to do so we can exploit the knowledge of the moments of the other regular basic limit functions of the same subdivision scheme.

Proposition 1.40. *For every $\alpha \geq 0$,*

$$\mathbf{P}^T \left[\int_{\mathbb{R}} x^\alpha \varphi_k(x) dx \right]_{k \in \mathbb{Z}} = 2^{1+\alpha} \left[\int_{\mathbb{R}} x^\alpha \varphi_k(x) dx \right]_{k \in \mathbb{Z}}. \quad (1.35)$$

In particular, for every $\alpha \in \mathbb{N}_0$,

$$\mathbf{A}_\alpha \left[\int_{\mathbb{R}} x^\alpha \varphi_k(x) dx \right]_{k_\ell(\mathbf{P}) < k < k_r(\mathbf{P})} = \mathbf{b}_\alpha, \quad (1.36)$$

where

$$\mathbf{b}_\alpha = \left(\sum_{k \leq k_\ell(\mathbf{P})} + \sum_{k \geq k_r(\mathbf{P})} \right) \int_{\mathbb{R}} x^\alpha \varphi_k(x) dx \mathbf{P}_{irr}(k, :)^T$$

and the matrix

$$\mathbf{A}_\alpha = 2^{1+\alpha} \mathbf{I} - \mathbf{P}_{irr}(k_\ell(\mathbf{P}) + 1 : k_r(\mathbf{P}) - 1, :)^T$$

is invertible.

Proof. Since the columns of \mathbf{P} are compactly supported, we can multiply the refinement equation (1.17) by x^α and integrate componentwise, thus obtaining

$$\left[\int_{\mathbb{R}} x^\alpha \varphi_k(x) dx \right]_{k \in \mathbb{Z}} = \mathbf{P}^T \left[\int_{\mathbb{R}} x^\alpha \varphi_k(2x) dx \right]_{k \in \mathbb{Z}}, \quad x \in \mathbb{R}.$$

Then the substitution $y = 2x$ on the right-hand side leads to (1.35). Moreover, due to (1.7), if we select on both sides of (1.35) the rows with indices between $k_\ell(\mathbf{P})$ and $k_r(\mathbf{P})$ we obtain

$$2^{1+\alpha} \left[\int_{\mathbb{R}} x^\alpha \varphi_k(x) dx \right]_{k_\ell(\mathbf{P}) < k < k_r(\mathbf{P})} = \mathbf{P}_{irr}^T \left[\int_{\mathbb{R}} x^\alpha \varphi_k(x) dx \right]_{k \in \mathbb{Z}}.$$

Isolating the contribution of the $(\alpha + 1)$ -th moments of φ_k , $k_\ell(\mathbf{P}) < k < k_r(\mathbf{P})$, on the left-hand side we get to the linear system in (1.36). Now, we observe that $\mathbf{P}_{irr}(k_\ell(\mathbf{P}) + 1 : k_r(\mathbf{P}) - 1, :)$ is the submatrix of $\mathring{\mathbf{P}}$ obtained by eliminating the first and the last rows and columns of $\mathring{\mathbf{P}}$. Since the first and last columns of $\mathring{\mathbf{P}}$ are two of its eigenvectors of the form

$$\mathring{\mathbf{P}}(:, 1) = \begin{bmatrix} \mathring{\mathbf{P}}(1, 1) \\ \mathbf{0} \end{bmatrix} \quad \text{and}$$

$$\mathring{\mathbf{P}}(:, k_r(\mathbf{P}) - k_\ell(\mathbf{P}) + 1) = \begin{bmatrix} \mathbf{0} \\ \mathring{\mathbf{P}}(k_r(\mathbf{P}) - k_\ell(\mathbf{P}) + 1, k_r(\mathbf{P}) - k_\ell(\mathbf{P}) + 1) \end{bmatrix},$$

the spectrum of $\mathbf{P}_{irr}(k_\ell(\mathbf{P}) + 1 : k_r(\mathbf{P}) - 1, :)$ is contained in the spectrum of $\mathring{\mathbf{P}}$ and in particular the dominant eigenvalue of $\mathbf{P}_{irr}(k_\ell(\mathbf{P}) + 1 : k_r(\mathbf{P}) - 1, :)$ does not exceed 1 (Theorem 1.11 and Proposition 1.9). This is sufficient to guarantee the invertibility of the matrix \mathbf{A}_α , for every $\alpha \in \mathbb{N}_0$. \square

Remark 1.41. Even if the right-spectrum of a semi-regular subdivision matrix \mathbf{P} is discrete (Proposition 1.9), its left-spectrum contains at least the interval $[2, \infty)$. It also contains 1 due to Proposition 1.21. \diamond

1.2.2 Gramian and Cross-Gramian Matrices

Let us start considering a simple example of two regular convergent subdivision schemes over the same regular mesh $\mathbf{t}_0 = h\mathbb{Z}$, $h > 0$, with the basic limit functions $\{\varphi_k\}_{k \in \mathbb{Z}}$, $\{\zeta_k\}_{k \in \mathbb{Z}}$, compactly supported masks \mathbf{p} , \mathbf{z} , subdivision matrices \mathbf{P} , \mathbf{Z} and symbols $p(\omega)$, $z(\omega)$, respectively. The main tools for computing the cross-Gramian matrix

$$\mathbf{G}(k, m) = \int_{\mathbb{R}} \varphi_k(x) \zeta_m(x) dx, \quad k, m \in \mathbb{Z}, \quad (1.37)$$

are given in Propositions 1.27 and 1.21.

Proposition 1.42. *The cross-Gramian matrix \mathbf{G} in (1.37) is a band-limited Toeplitz matrix. Moreover, $\mathbf{G}(k, 0) = \gamma_0(hk)$, $k \in \mathbb{Z}$, where*

$$\hat{\gamma}_0(\omega) = \overline{p(h\omega/2)} z(h\omega/2) \hat{\gamma}_0(\omega/2), \quad \omega \in \mathbb{R}. \quad (1.38)$$

Proof. We start by observing that, for every $k, m \in \mathbb{Z}$,

$$\begin{aligned} \mathbf{G}(k, m) &= \int_{\mathbb{R}} \varphi_k(x) \zeta_m(x) dx \\ &= \int_{\mathbb{R}} \varphi_0(x - hk) \zeta_0(x - hm) dx \\ &= \int_{\mathbb{R}} \varphi_0(y - h(k - m)) \zeta_0(y) dy = \mathbf{G}(k - m, 0). \end{aligned}$$

Thus, \mathbf{G} is a Toeplitz matrix. Now, consider $\eta_0 = \varphi_0(-\cdot)$. It is easy to see that η_0 satisfies the refinement equation on the Fourier side (1.26) with respect to the symbol $\bar{p}(\omega)$. Thus, by Proposition 1.27, we get

$$\mathbf{G}(k, 0) = \int_{\mathbb{R}} \varphi_0(x - hk) \zeta_0(x) dx = \int_{\mathbb{R}} \eta_0(hk - x) \zeta_0(x) dx = \gamma_0(hk), \quad k \in \mathbb{Z},$$

with γ_0 satisfying (1.38). \square

Proposition 1.42 shifts the problem of computing the entries of \mathbf{G} to the problem of evaluating a certain basic limit function γ_0 at the knots of the initial mesh $\mathbf{t}_0 = h\mathbb{Z}$, $h > 0$, and we already know how to do it by Proposition 1.21.

Remark 1.43 ([51]). When $\zeta_0 = \varphi_0$, γ_0 is the so-called *autocorrelation function* of φ_0 . In this case, we have that $g(\omega) = |p(\omega)|^2$, a real symmetric non-negative trigonometric polynomial and so the mask \mathbf{g} is symmetric. Moreover, $\gamma_0 = \|\varphi_0\|_2^2$. If φ_k 's are

orthonormal, then

$$\gamma_0(0) = 1 \quad \text{and} \quad \gamma(hk) = \int_{\mathbb{R}} \varphi_0(x) \varphi_0(x - hk) dx = 0, \quad k \in \mathbb{Z} \setminus \{0\},$$

and so the scheme associated to γ_0 is an interpolatory scheme. The converse is also true. \diamond

Suppose now that the schemes at hand are semi-regular over the same semi-regular mesh \mathbf{t}_0 in (1.3) with $h_\ell, h_r > 0$. We already know most of the entries of the corresponding cross-Gramian matrix \mathbf{G} from the regular case, since, far from 0 we are still working with regular subdivision. The situation we have to deal with is the following:

$$\mathbf{G} = \left[\begin{array}{c} \text{green area} \\ \vdots \\ \text{blue area} \\ \vdots \\ \text{green area} \end{array} \right] \left\{ \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right\} n(\mathbf{P})$$

$\underbrace{\hspace{10em}}_{n(\mathbf{Z})}$

where the green (light) area represents the values that we already know from the regular case and the blue (dark) area represents the unknown entries of \mathbf{G} , where

$$n(\mathbf{P}) = k_r(\mathbf{P}) - k_\ell(\mathbf{P}) - 1 \quad \text{and} \quad n(\mathbf{Z}) = k_r(\mathbf{Z}) - k_\ell(\mathbf{Z}) - 1,$$

are the numbers of the irregular basic limit functions of the two schemes respectively. The size of the blue area of course depends on the size of the supports of the irregular basic limit functions of the two schemes. Since they also have compact support, see Proposition 1.9, the number of unknown entries of \mathbf{G} is bounded by

$$(n(\mathbf{P}) + n(\mathbf{Z}) + |\text{supp}(\mathbf{p}_\ell)| + |\text{supp}(\mathbf{p}_r)| + |\text{supp}(\mathbf{z}_\ell)| + |\text{supp}(\mathbf{z}_r)| - 11)^2.$$

As we will see in a moment the number of unknowns is not relevant. It is crucial that

there are finitely many of them. The other ingredient of the method in Proposition 1.44 is the following relation, already observed in [48]. Using the refinement equation (1.17) for both schemes we obtain

$$\begin{aligned}
 \mathbf{G} &= \int_{\mathbb{R}} [\varphi_k(x)]_{k \in \mathbb{Z}} [\zeta_m(x)]_{m \in \mathbb{Z}}^T dx \\
 &= \int_{\mathbb{R}} \mathbf{P}^T [\varphi_k(2x)]_{k \in \mathbb{Z}} [\zeta_m(2x)]_{m \in \mathbb{Z}}^T \mathbf{Z} dx \\
 &= \frac{1}{2} \mathbf{P}^T \mathbf{G} \mathbf{Z}.
 \end{aligned} \tag{1.39}$$

This relation yields a linear system of equations for the unknown entries of \mathbf{G} . We prove that the knowledge of the regular entries of \mathbf{G} determines the unknown entries uniquely.

Proposition 1.44. *The finite system of linear equations obtained from (1.39) is uniquely solvable for the irregular entries of \mathbf{G} .*

Proof. Seeking a contradiction, suppose there is another bi-infinite matrix \mathbf{F} such that \mathbf{F} differs from \mathbf{G} only on the irregular part and

$$\mathbf{F} = \frac{1}{2} \mathbf{P}^T \mathbf{F} \mathbf{Z}.$$

We then consider the matrix $\Delta = \mathbf{G} - \mathbf{F} \neq \mathbf{0}$. By the linearity of the matrix multiplication

$$\Delta = \mathbf{G} - \mathbf{F} = \frac{1}{2} \mathbf{P}^T (\mathbf{G} - \mathbf{F}) \mathbf{Z} = \frac{1}{2} \mathbf{P}^T \Delta \mathbf{Z}. \tag{1.40}$$

Let $n_1 < \min(k_\ell(\mathbf{P}), k_\ell(\mathbf{Z})) < \max(k_r(\mathbf{P}), k_r(\mathbf{Z})) < n_2$ such that $\tilde{\mathbf{G}} = \mathbf{G}(n_1 : n_2, n_1 : n_2)$ contains all the irregular entries of \mathbf{G} . If we consider analogously $\tilde{\Delta}$, $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Z}}$, we have that $\tilde{\Delta}$ contains all the non-zero elements of Δ and, since, as in Proposition 1.9, $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Z}}$ contain all the non-zero elements of the corresponding rows of \mathbf{P} and \mathbf{Z} , respectively. We get an equivalent finite version of (1.40), namely,

$$\tilde{\Delta} = \frac{1}{2} \tilde{\mathbf{P}}^T \tilde{\Delta} \tilde{\mathbf{Z}}. \tag{1.41}$$

Now, if $\tilde{\Delta} \neq \mathbf{0}$, there exist $k, m \in \mathbb{N}$ such that $\tilde{\Delta}(k, m) \neq 0$. Consider the vectors $\mathbf{e}^{(k)}$ and $\mathbf{e}^{(m)}$ of the canonical basis. From (1.41), we have

$$0 \neq \tilde{\Delta}(k, m) = (\mathbf{e}^{(k)})^T \tilde{\Delta} \mathbf{e}^{(m)} = \frac{1}{2^j} (\mathbf{e}^{(k)})^T (\tilde{\mathbf{P}}^j)^T \tilde{\Delta} \tilde{\mathbf{Z}}^j \mathbf{e}^{(m)}.$$

Now, $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Z}}$ have $\mathring{\mathbf{P}}$ and $\mathring{\mathbf{Z}}$, respectively, as submatrices on their diagonals, see Proposition 1.9 and Proposition 1.9. Thus, $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Z}}$ share the spectral properties of $\mathring{\mathbf{P}}$ and $\mathring{\mathbf{Z}}$,

respectively. In particular $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Z}}$ both have dominant eigenvalue 1 with multiplicity one. So there exist $0 < C(\mathbf{P}), C(\mathbf{Z}) < \infty$ such that

$$\begin{cases} \|(\mathbf{e}^{(k)})^T (\tilde{\mathbf{P}}^T)^j\| \leq C(\mathbf{P}) \\ \|(\tilde{\mathbf{Z}})^j \mathbf{e}^{(m)}\| \leq C(\mathbf{Z}) \end{cases}, \quad j \in \mathbb{N}.$$

This means that

$$\begin{aligned} 0 < |(\mathbf{e}^{(k)})^T \tilde{\Delta} \mathbf{e}^{(m)}| &\leq \left| \frac{1}{2^j} (\mathbf{e}^{(k)})^T (\tilde{\mathbf{P}}^j)^T \tilde{\Delta} (\tilde{\mathbf{Z}})^j \mathbf{e}^{(m)} \right| \\ &\leq \frac{1}{2^j} C(\mathbf{P}) C(\mathbf{Z}) \|\tilde{\Delta}\| \xrightarrow{j \rightarrow +\infty} 0, \end{aligned}$$

which leads to a contradiction. □

2 Wavelet Tight Frames

The concept of a frame was introduced by Duffin and Schaeffer in the fifties [28] as a generalization of the idea of a basis of an inner product vector space. In the last twenty years, frames, in particular wavelet frames, found a place in a wide spectrum of applications such as audio, image and surface compression, edge detection, inpainting, approximation of PDE solutions. Frames are flexible and efficient in implementations. The idea behind a wavelet frame is an efficient representation of the functions of an inner product vector space that splits the functions in a way convenient for their analysis and other manipulations.

Definition 2.1. Let \mathcal{V} be an inner product vector space with the norm $\|\cdot\|$ and $\mathcal{F} = \{\psi_j\}_{j \in I} \subset \mathcal{V}$ with the index set I at most countable. \mathcal{F} is a *frame* for \mathcal{V} if and only if $\overline{\text{span}(\mathcal{F})}^{\|\cdot\|} = \mathcal{V}$ and there exists $0 < A \leq B < \infty$ such that

$$A \|f\|^2 \leq \sum_{j \in I} |\langle f, \psi_j \rangle|^2 \leq B \|f\|^2, \quad f \in \mathcal{V}.$$

If $A = B$ the frame is said to be *tight*.

Remark 2.2. If a frame \mathcal{F} is tight, then w.l.o.g. $A = B = 1$. Indeed, \mathcal{F}/\sqrt{A} is still a tight frame for which the Parseval's equality holds

$$\sum_{j \in I} \left| \left\langle f, \frac{\psi_j}{\sqrt{A}} \right\rangle \right|^2 = \|f\|^2, \quad f \in \mathcal{V}. \quad (2.1)$$

Moreover, in this case we have the so called *perfect reconstruction property*, i.e.

$$f = \sum_{j \in I} \left\langle f, \frac{\psi_j}{\sqrt{A}} \right\rangle \frac{\psi_j}{\sqrt{A}}, \quad f \in \mathcal{V}, \quad (2.2)$$

where the equality is intended in the norm $\|\cdot\|$. ◇

Of course, orthonormal bases are frames, and in particular tight frames, but the notion of a frame allows for weaker assumptions on the elements of \mathcal{F} which for example can be linearly dependent, see Example 2.3.

Example 2.3. Let $\mathcal{V} = \mathbb{R}^2$ and consider the set

$$\mathcal{F} = \left\{ \mathbf{f}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{f}_2 = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}, \mathbf{f}_3 = \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix} \right\}.$$

Then, for every $x = [x_1, x_2] \in \mathbb{R}^2$,

$$\begin{aligned} |\langle x, \mathbf{f}_1 \rangle|^2 + |\langle x, \mathbf{f}_2 \rangle|^2 + |\langle x, \mathbf{f}_3 \rangle|^2 &= \\ &= x_1^2 + \left(-\frac{x_1}{2} + \frac{\sqrt{3}}{2}x_2 \right)^2 + \left(-\frac{x_1}{2} - \frac{\sqrt{3}}{2}x_2 \right)^2 \\ &= \frac{3}{2}x_1^2 + \frac{3}{2}x_2^2 = \frac{3}{2}\|x\|^2. \end{aligned}$$

Thus, \mathcal{F} is a tight frame for \mathbb{R}^2 . \triangle

The linear dependence of the frame elements introduces redundancy which, even if not always nice in theory, is very useful in applications. It adds robustness to noise when the information about a function is encoded via its frame coefficients, i.e. the values $\{\langle f, \psi_j \rangle\}_{j \in I}$. Since in applications most of the functions/signals are compactly supported and bounded, it is very convenient to see them as elements of $L^2(\mathbf{R}^d)$, $d \in \mathbb{N}$, which is a Hilbert space and, in particular, an inner product vector space. We again focus on the univariate case $d = 1$. On one hand, the concept of a frame is very general. On the other hand, we would like to have some exploitable structure both for the construction of frames and for their application. The most convenient way to this goal is via the so-called multi-resolution analysis. We use here the general definition introduced by Chui, He and Stöckler in [14, 15].

Definition 2.4. A family $\{\mathcal{V}_j\}_{j \in \mathbb{N}_0}$ of closed subspaces of $L^2(\mathbb{R})$ is said to be a *multi-resolution analysis* for $L^2(\mathbb{R})$ if the following conditions are satisfied.

- (i) $\mathcal{V}_{j-1} \subset \mathcal{V}_j$, for all $j \in \mathbb{N}$; (increasing subspaces)
- (ii) $\overline{\bigcup_{j \in \mathbb{N}_0} \mathcal{V}_j}^{\|\cdot\|_{L^2}} = L^2(\mathbb{R})$; (completeness in $L^2(\mathbb{R})$)
- (iii) there exist families of functions $\Phi_0 = [\phi_k]_{k \in \mathbb{Z}}$ and $\Phi_j = [\phi_{j,k}]_{k \in \mathbb{Z}}$, $j \in \mathbb{N}$, such that

$$\overline{\text{span}(\Phi_j)}^{L^2} = \mathcal{V}_j, \quad j \in \mathbb{N}_0,$$

and, for every $j \in \mathbb{N}$, $\Phi_{j-1} = \mathbf{P}_{j-1}^T \Phi_j$, for some matrix \mathbf{P}_{j-1} ;

- (iv) there exists a vector \mathbf{c}_0 such that $\mathbf{c}_0^T \Phi_0 \equiv 1$.

The index j is called *resolution level*.

Definition 2.5. Consider a multi-resolution analysis in Definition 2.4. If there exists a family of matrices $\{\mathbf{Q}_j\}_{j \in \mathbb{N}}$ such that the set $\mathcal{F} = \Phi_0 \cup \{\Psi_j = \mathbf{Q}_j^T \Phi_j\}_{j \in \mathbb{N}}$ is a tight frame for $L^2(\mathbb{R})$, then \mathcal{F} is called *wavelet tight frame*. The elements of Φ_0 are called *scaling functions* and the elements of Ψ_j *framelets* (*wavelets* if \mathcal{F} is orthonormal). Furthermore,

\mathcal{F} is said to be \mathcal{C}^s , $s > 0$, if all its elements belong to $\mathcal{C}^s(\mathbb{R})$ and to have $v \in \mathbb{N}$ *vanishing moments* if, for every $j \in \mathbb{N}, k \in \mathbb{Z}$,

$$\int_{\mathbb{R}} x^\alpha \psi_{j,k}(x) dx = 0, \quad \alpha \in \{0, \dots, v-1\}. \quad (2.3)$$

The smoothness and the number of vanishing moments of \mathcal{F} will be crucial in Chapter 3. Roughly speaking, if a function $f \in L^2(\mathbb{R})$ is regular, then locally it can be approximated by a high degree polynomial. Thus, if the wavelet tight frame has a high number of vanishing moments, the frame coefficients of f with respect to the framelets will be negligible. This fact means, on one hand, that one can approximate signals thresholding the frame coefficients and, on the other hand, that from the decay of the frame coefficients of f one can extract the information about the smoothness of f .

The simplest example of such a function system is the so called *Haar system* [36], which is orthonormal. The Haar system in its simplicity is not so desirable since it lacks both smoothness and vanishing moments. For other interesting wavelet tight frames with nicer properties one had to wait until the seminal works of Daubechies, Meyer, Mallat and others at the end of the eighties (see e.g. [20, 49]) and what came afterwards.

Example 2.6 (Haar system). Consider a set of non-trivial intervals $\{I_k = [a_k, a_{k+1})\}_{k \in \mathbb{Z}}$ with disjointed interiors and that form a partition of \mathbb{R} . For every interval, we consider the function $\phi_k(x) = \chi_{I_k}(x)/\sqrt{|I_k|}$. Of course,

$$\sum_{k \in \mathbb{Z}} \sqrt{|I_k|} \phi_k(x) \equiv 1, \quad x \in \mathbb{R}.$$

We then consider the dyadic cuts of the intervals $\{I_k\}_{k \in \mathbb{Z}}$ namely $\{I_{1,k}\}_{k \in \mathbb{Z}}$, closed on the left and open on the right, such that

$$I_k = I_{1,2k} \cup I_{1,2k+1}, \quad |I_{1,2k} \cap I_{1,2k+1}| = 0 \quad \text{and} \quad |I_{1,2k}| = |I_{1,2k+1}| = \frac{|I_k|}{2},$$

and the corresponding indicator functions $\phi_{1,k} = \chi_{I_{1,k}}(x)/\sqrt{|I_{1,k}|}$. Iterating this process, we obtain $\{\Phi_j\}_{j \in \mathbb{N}_0}$ such that

$$\Phi_{j-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \ddots & 1 & & & \\ & 1 & & & \\ & & \boxed{1} & & \\ & & 1 & & \\ & & & 1 & \ddots \\ & & & 1 & \end{bmatrix}^T \Phi_j, \quad j \in \mathbb{N}.$$

Moreover, the closed subspaces $\{\mathcal{V}_j = \overline{\text{span}(\Phi)}^{\|\cdot\|_{L^2}}\}_{j \in \mathbb{N}_0}$ form a multi-resolution analysis for $L^2(\mathbb{R})$. Indeed for every $k \in \mathbb{Z}$, $\bigcup_{j \in \mathbb{N}_0} \mathcal{V}_j$ contains all the simple functions defined on

all dyadic subintervals of I_k which are dense in $L^2(I_k)$. Let

$$\Psi_j = \frac{1}{\sqrt{2}} \begin{bmatrix} \ddots & & 1 & & \\ & & -1 & & \\ & & \boxed{1} & & \\ & & -1 & & \\ & & & 1 & \ddots \\ & & & -1 & \ddots \end{bmatrix}^T \Phi_j, \quad j \in \mathbb{N}.$$

It is easy to see that $\mathcal{F} = \Phi_0 \cup \{\Psi_j\}_{j \in \mathbb{N}}$ is an orthonormal set. Moreover, since for every $j \in \mathbb{N}$,

$$\mathcal{V}_j = \overline{\text{span}(\Phi_{j-1} \cup \Psi_j)}^{\|\cdot\|_{L^2}} = \overline{\text{span}(\Phi_0 \cup \{\Psi_m\}_{m=1}^j)}^{\|\cdot\|_{L^2}} \xrightarrow{j \rightarrow \infty} L^2(\mathbb{R}),$$

the orthonormality of \mathcal{F} , implies that \mathcal{F} is a wavelet tight frame for $L^2(\mathbb{R})$. In particular, the elements of \mathcal{F} are not continuous and \mathcal{F} has only one vanishing moment. In particular, we observe that, when $I_k = [k, k+1)$, $k \in \mathbb{Z}$, we obtain a shift-invariant system with $\phi_k = \phi_0(\cdot - k)$, where

$$\hat{\phi}_0(\omega) = \frac{1 + e^{-2\pi i \omega}}{2} \hat{\phi}_0(\omega/2), \quad \omega \in \mathbb{R}.$$

△

Whenever, for every $j \in \mathbb{N}_0$, $\mathbf{P}_j \equiv \mathbf{P}_0$, $\mathbf{Q}_j \equiv \mathbf{Q}_1$ and $\Phi_j = 2^{j/2} \Phi_0(2^j \cdot)$, in Definition 2.4 and 2.5, we have that

$$\Phi_0 = 2^{j/2} (\mathbf{P}_0^j)^T \Phi_0(2^j \cdot) \quad \text{and} \quad \Psi_j = 2^{j/2} \mathbf{Q}_1 \Phi_0(2^j \cdot) = 2^{(j-1)/2} \Psi_1(2^{j-1} \cdot). \quad (2.4)$$

At this point the first of these two equations should remind us of the refinement equation (1.17). Indeed, a good candidate for the set of scaling functions is the set of basic limit functions of a subdivision scheme, after a proper renormalization.

Remark 2.7. The role of the framelets Ψ_j is to describe the features needed to pass from resolution level $j-1$ to resolution level j . Indeed, for $f \in L^2(\mathbb{R})$, due to (2.2) we have that the sequence

$$f_0 = \langle f, \Phi_0 \rangle^T \Phi_0 \in \mathcal{V}_0$$

$$f_j = f_{j-1} + \langle f, \Psi_j \rangle^T \Psi_j \in \mathcal{V}_j, \quad j \in \mathbb{N},$$

converges to f in $\|\cdot\|_{L^2}$. Moreover, due to Definition 2.4 (iii) and (iv), it is always possible to ensure that the framelets have at least one vanishing moment. Thus, Definition 2.5 is consistent. ◇

In the regular case, where the underlying structure is shift-invariant, we can use

powerful tools from the literature, namely the *Unitary Extension Principle* [55, 56] and the *Oblique Extension Principle* [16, 24], to construct wavelet tight frames, see Section 2.1. These constructions, in our case, are based on the symbols of convergent subdivision schemes. In the semi-regular case, due to the lack of shift invariance, those tools are not available in the same form. Indeed, we leave the Fourier domain and use the general framework provided in [14, 15], where the construction method works directly on the matrices involved. After reviewing these methods, we present a first example of semi-regular wavelet tight frame with two vanishing moments based on the cubic B-spline scheme, see Section 2.1.1. This example illustrates the difficulties of the OEP type construction. In Section 2.2, we propose a family of semi-regular wavelet tight frames based on the Dubuc-Deslauriers interpolatory schemes whose construction overcomes those difficulties and is UEP-based.

2.1 The Unitary and Oblique Extension Principles: from Symbols to Matrices

We start with the regular case. Consider a convergent regular subdivision scheme over the initial mesh $\mathbf{t}_0 = h\mathbb{Z}$, $h > 0$, with the basic limit functions $[\varphi_k = \varphi_0(\cdot - hk)]_{k \in \mathbb{Z}}$, the compactly supported mask \mathbf{p} , the symbol $p(\omega)$ and the subdivision matrix \mathbf{P} . To transform our basic limit functions into the scaling functions we need to renormalize them in the following way. We define

$$\Phi_0 = [\phi_k]_{k \in \mathbb{Z}} = \mathbf{D}^{-1/2} [\varphi_k]_{k \in \mathbb{Z}}, \quad \mathbf{D}(k, m) = \begin{cases} \int_{\mathbb{R}} \varphi_k(x) dx, & k = m, \\ 0, & \text{otherwise.} \end{cases} \quad (2.5)$$

Moreover, from (2.4) using (1.17) we get

$$\begin{aligned} 2^{1/2} \mathbf{P}_0^T \Phi_0(2\cdot) &= \Phi_0 = \mathbf{D}^{-1/2} [\varphi_k]_{k \in \mathbb{Z}} = \mathbf{D}^{-1/2} \mathbf{P}^T [\varphi_k(2\cdot)]_{k \in \mathbb{Z}} \\ &= \mathbf{D}^{-1/2} \mathbf{P}^T \mathbf{D}^{1/2} [\varphi_k(2\cdot)]_{k \in \mathbb{Z}} = \mathbf{D}^{-1/2} \mathbf{P}^T \mathbf{D}^{1/2} \Phi_0(2\cdot). \end{aligned} \quad (2.6)$$

Thus, $\mathbf{P}_0 = 2^{-1/2} \mathbf{D}^{1/2} \mathbf{P} \mathbf{D}^{-1/2}$. In particular, in the regular case, from Section 1.2.1, we know that $\mathbf{D}(k, k) \equiv h$, so $\mathbf{P}_0 = 2^{-1/2} \mathbf{P}$. Next we would like to find a matrix \mathbf{Q}_1 such that

$$\mathcal{F} = \Phi_0 \cup \{ \Psi_j = 2^{j/2} \mathbf{Q}_1^T \Phi_0(2^j \cdot) \}_{j \in \mathbb{N}} \quad (2.7)$$

is a wavelet tight frame for $L^2(\mathbb{R})$. It would be also preferable that the shift-invariant structure is kept also with respect to the framelets. In particular, we would like that

$$\psi_{1,k}(x) = \psi_{1,m} \left(x - h \frac{k-m}{M} \right), \quad \text{for } k \equiv m \pmod{M},$$

for some $M \in \mathbb{N}$. This is reflected on the \mathbf{Q}_1 in the following way:

$$\mathbf{Q}_1(k, n + m - 1) = 2^{-1/2} \mathbf{q}_m(k - 2j), \quad k, n \in \mathbb{Z}, \quad m = 1, \dots, M$$

for some vectors \mathbf{q}_m , $m = 1, \dots, M$. Asking the $\psi_{j,k}$ to be compactly supported then means that the vectors $\{\mathbf{q}_m\}_{m=1}^M$ must be compactly supported. The following result characterizes all the wavelet tight frames that have this form.

Theorem 2.8 (Oblique Extension Principle (OEP), [16, 24]). *The set \mathcal{F} in (2.7) is a wavelet tight frame with $v \in \mathbb{N}$ vanishing moments if and only if there exists*

$$s(\omega) = \frac{s_n(\omega)}{s_d(\omega)}, \quad \omega \in \mathbb{R},$$

with $s_n(\omega), s_d(\omega)$ trigonometric polynomials, continuous at 0 with $s(0) = 1$ such that

$$\left\{ \begin{array}{l} s(2\omega) p(\omega) \overline{p(\omega)} + \sum_{m=1}^M q_m(\omega) \overline{q_m(\omega)} = s(\omega), \\ s(2\omega) p(\omega) \overline{p(\omega - 1/2)} + \sum_{m=1}^M q_m(\omega) \overline{q_m(\omega - 1/2)} = 0, \end{array} \right. \quad a.e., \quad (2.8)$$

for

$$q_m(\omega) = \frac{1}{2} \sum_{k \in \mathbb{Z}} \mathbf{q}_m(k) e^{-2\pi i k \omega}, \quad m = 1, \dots, M,$$

and

$$q_m(\omega) = \left(\frac{1 - e^{-2\pi i \omega}}{2} \right)^v q_m^{[v]}(\omega), \quad m = 1, \dots, M,$$

for some trigonometric polynomials $\{q_m^{[v]}(\omega)\}_{m=1, \dots, M}$,

The Unitary Extension Principle (UEP) [55, 56] is a particular case of the OEP when $s(\omega) \equiv 1$. In general, this restriction yields wavelet tight frames with at most one vanishing moment, except for special cases. On the other hand, the UEP is much easier to handle. For instance, this simple and interesting result holds.

Theorem 2.9. *Let $p(\omega)$ and $z(\omega)$ be the symbols of convergent regular subdivision schemes satisfying (2.8) with $s(\omega) \equiv 1$, trigonometric polynomials $\{q_m(\omega)\}_{m=1, \dots, M_p}$, $\{b_m(\omega)\}_{m=1, \dots, M_z}$, and $v_p, v_z \in \mathbb{N}$, respectively. Then the symbol $p(\omega)z(\omega)$ satisfies (2.8) with $s \equiv 1$ together with the trigonometric polynomials*

$$\{ p(\omega) b_m(\omega) \}_{m=1, \dots, M_z} \cup \{ z(\omega) q_m(\omega) \}_{m=1, \dots, M_p} \cup \{ q_{m_1}(\omega) b_{m_2}(\omega) \}_{m_1=1, \dots, M_p, m_2=1, \dots, M_z}$$

and $v = \min\{v_a, v_p\}$.

Proof. The proof simply follows by multiplying the systems arising from (2.8) for $p(\omega)$ and for $z(\omega)$, term by term. \square

Remark 2.10. The strategy of Theorem 2.9 in general is not valid for the OEP case, when $s(\omega) \neq 1$. A counterexample is presented in Remark 2.22. \diamond

Theorem 2.9 can be used for example to obtain explicit algebraic expressions for framelets with 1 vanishing moment for the family of B-spline schemes, since the B-spline symbols are powers of $(1 + e^{-2\pi i\omega})/2$, which is the symbol associated to the regular Haar system, see Example 2.6. This is an alternative, more straightforward way for obtaining the framelets in [8] in the regular case.

Example 2.11 (Cubic B-spline, part III). As we saw in Example 1.31, the symbol associated to the cubic B-spline scheme is

$$p(\omega) = \left(\frac{1 + e^{-2\pi i\omega}}{2} \right)^4, \quad \omega \in \mathbb{R},$$

up to a unitary factor, see Remark 1.24. In particular, $p(\omega)$ is the fourth power of the Haar symbol. It is easy to see that the Haar symbol also satisfies (2.8) with $s(\omega) \equiv 1$, $v = 1$ and one trigonometric polynomial $q_1(\omega) = (1 - e^{-2\pi i\omega})/2$. We can use Theorem 2.9 three times ending up essentially with the following four trigonometric polynomials,

$$q_m = \sqrt{\binom{4}{m}} \left(\frac{1 - e^{-2\pi i\omega}}{2} \right)^m \left(\frac{1 + e^{-2\pi i\omega}}{2} \right)^{4-m}, \quad m = 1, \dots, 4.$$

In the end, the corresponding vectors are

$$\mathbf{q}_1 = \begin{bmatrix} -1/4 \\ -1/2 \\ 0 \\ 1/2 \\ 1/4 \end{bmatrix}, \quad \mathbf{q}_2 = \sqrt{6} \begin{bmatrix} 1/8 \\ 0 \\ -1/4 \\ 0 \\ 1/8 \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} -1/4 \\ 1/2 \\ 0 \\ -1/2 \\ 1/4 \end{bmatrix}, \quad \mathbf{q}_4 = \begin{bmatrix} 1/8 \\ -1/2 \\ 3/4 \\ -1/2 \\ 1/8 \end{bmatrix}.$$

\triangle

Before diving into the semi-regular case, let us take a look at the UEP conditions from a different angle. First of all, we can interpret a symbol $p(\omega)$ as the scalar product of two particular vectors

$$p(\omega) = \frac{1}{2} \sum_{k \in \mathbb{Z}} \mathbf{p}(\omega) e^{-2\pi i k \omega} = \frac{1}{2} [e^{-2\pi i k \omega}]_{k \in \mathbb{Z}} \mathbf{p}. \quad (2.9)$$

The same holds for all the $\{q_m(\omega)\}_{m=1,\dots,M}$. Thus, we write

$$\left\{ \begin{array}{l} 2 = p(\omega)\overline{p(\omega)} + \sum_{m=1}^M q_m(\omega)\overline{q_m(\omega)} \\ = \frac{1}{2} [e^{-2\pi i k \omega}]_{k \in \mathbb{Z}}^T \left(\mathbf{p}\mathbf{p}^T + [\mathbf{q}_1, \dots, \mathbf{q}_M] [\mathbf{q}_1, \dots, \mathbf{q}_M]^T \right) [e^{2\pi i k \omega}]_{k \in \mathbb{Z}}, \\ 0 = p(\omega)\overline{p(\omega - 1/2)} + \sum_{m=1}^M q_m(\omega)\overline{q_m(\omega - 1/2)} \\ = \frac{1}{2} [e^{-2\pi i k \omega}]_{k \in \mathbb{Z}}^T \left(\mathbf{p}\mathbf{p}^T + [\mathbf{q}_1, \dots, \mathbf{q}_M] [\mathbf{q}_1, \dots, \mathbf{q}_M]^T \right) [(-1)^k e^{2\pi i k \omega}]_{k \in \mathbb{Z}}, \end{array} \right. \quad \text{a.e.,}$$

where we multiplied by 2 both equations in (2.8) and set $s(\omega) \equiv 1$. Now we observe that $\mathbf{p}\mathbf{p}^T/2$ is one of the elements of the rank-1 expansion of $\mathbf{P}_0\mathbf{P}_0^T$, and the same holds for $[\mathbf{q}_1, \dots, \mathbf{q}_M][\mathbf{q}_1, \dots, \mathbf{q}_M]^T/2$ and $\mathbf{Q}_1\mathbf{Q}_1^T$. In particular, if

$$\mathbf{B}_0 := \mathbf{p}\mathbf{p}^T + [\mathbf{q}_1, \dots, \mathbf{q}_M] [\mathbf{q}_1, \dots, \mathbf{q}_M]^T, \quad (2.10)$$

we have that

$$\sum_{k \in \mathbb{Z}} \mathbf{B}_0(\cdot - 2k, \cdot - 2k) = \mathbf{P}_0\mathbf{P}_0^T + \mathbf{Q}_1\mathbf{Q}_1^T. \quad (2.11)$$

Moreover, from the first UEP condition we have that the sum of each of the diagonals of \mathbf{B}_0 gives one of the coefficients of the polynomial 2, i.e.

$$\sum_{k \in \mathbb{Z}} \mathbf{B}_0(k, k) = 2 \quad \text{and} \quad \sum_{k \in \mathbb{Z}} \mathbf{B}_0(k, k + m) = 0, \quad m \in \mathbb{Z} \setminus \{0\}. \quad (2.12)$$

The second UEP condition implies that on each diagonal of \mathbf{B}_0 the sum of the elements at odd positions is the same as the sum of the elements at even positions, i.e.,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \mathbf{B}_0(2k, 2k) &= \sum_{k \in \mathbb{Z}} \mathbf{B}_0(2k + 1, 2k + 1), \\ \sum_{k \in \mathbb{Z}} \mathbf{B}_0(2k, 2k + m) &= \sum_{k \in \mathbb{Z}} \mathbf{B}_0(2k + 1, 2k + 1 + m), \quad m \in \mathbb{Z} \setminus \{0\}. \end{aligned} \quad (2.13)$$

Due to (2.12), the first identity in (2.13) must be equal to 1, while the second identity (2.13) must be always 0. Thus, (2.10) becomes

$$\mathbf{I} = \mathbf{P}_0\mathbf{P}_0^T + \mathbf{Q}_1\mathbf{Q}_1^T. \quad (2.14)$$

Thus, in the regular case, finding solutions to the UEP conditions is equivalent to find a block 2-slanted symmetric factorization of the matrix $\mathbf{I} - \mathbf{P}_0\mathbf{P}_0^T$.

Remark 2.12. Due to (2.14), a necessary condition for the construction of a wavelet tight frame in the regular case is that $\mathbf{I} - \mathbf{P}_0 \mathbf{P}_0^T$ is a positive semi-definite matrix. \diamond

We can go deeper from (2.14). If we multiply both sides in (2.14) by $\Phi_j(x)^T$ from the left and by $\Phi_j(y)$ from the right and use (2.4), we obtain

$$\begin{aligned} \Phi_j(x)^T \Phi_j(y) &= \Phi_j(x)^T \mathbf{P}_0 \mathbf{P}_0^T \Phi_j(y) + \Phi_j(x)^T \mathbf{Q}_1 \mathbf{Q}_1^T \Phi_j(y) \\ &= \Phi_{j-1}(x)^T \Phi_{j-1}(y) + \Psi_j(x)^T \Psi_j(y), \quad x, y \in \mathbb{R}. \end{aligned} \quad (2.15)$$

The quantity $\Phi_j(x)^T \Phi_j(y)$ is a kernel which defines a projection from $L^2(\mathbb{R})$ onto the element of the associated multi-resolution analysis \mathcal{V}_j , i.e

$$\begin{aligned} K_j : L^2(\mathbb{R}) &\longrightarrow \mathcal{V}_j \\ f &\longmapsto \int_{\mathbb{R}} f(x) \Phi_j(x)^T \Phi_j(\cdot) dx. \end{aligned} \quad (2.16)$$

This point of view does not depend on symbols or masks, but only on the family of functions and matrix refinement relation between them. Indeed, under suitable assumptions, this approach can be extended to include the semi-regular case and even the irregular one. The more general OEP conditions can also be incorporated. This is the essence of the works [14, 15] that we are going to briefly review, focusing on the key points needed for our further construction.

In the general case, we consider a family of vectors of refinable functions $\{\Phi_j\}_{j \in \mathbb{N}_0}$ that defines a multi-resolution analysis in Definition 2.4. To proceed we need this family to satisfy the following assumptions. Assumption 1 requires uniform behaviour for the functions defining the multi-resolution analysis, even if the setting is not shift-invariant.

Assumption 1. For every $j \in \mathbb{N}_0$,

(a) Φ_j is a *Riesz basis* for \mathcal{V}_j , i.e. there exist $0 < A_j \leq B_j < \infty$ such that

$$A_j \|\mathbf{f}\|_{\ell^2}^2 \leq \|\Phi_j^T \mathbf{f}\|_{L^2}^2 \leq B_j \|\mathbf{f}\|_{\ell^2}^2, \quad \forall \mathbf{f} \in \ell^2(\mathbb{Z}).$$

(b) Φ_j is *uniformly bounded*, i.e. $\sup_{k \in \mathbb{Z}} \|\phi_{j,k}\|_{L^\infty} < \infty$.

(c) Φ_j is *strictly local*, i.e. $\sup_{k \in \mathbb{Z}} |\text{supp}(\phi_{j,k})| < \infty$ and there exists $m_j < \infty$ such that, for every index set $\mathcal{I} \subset \mathbb{Z}$ with $|\mathcal{I}| > m_j$,

$$\bigcap_{k \in \mathcal{I}} \text{supp}(\phi_{j,k}) = \emptyset.$$

(d) $\lim_{j \rightarrow \infty} \sup_{k \in \mathbb{Z}} |\text{supp}(\phi_{j,k})| = 0$.

Remark 2.13. Conditions (b) and (c) of Assumption 1, imply that, for every $j \in \mathbb{N}_0$, the family Φ_j is a Bessel sequence, i.e., there exists a constant $C_{B,j} > 0$ such that

$$\sum_{k \in \mathbb{Z}} |\langle f, \phi_{j,k} \rangle|^2 \leq C_{B,j} \|f\|_{L^2}^2, \quad f \in L^2(\mathbb{R}).$$

Indeed, for every $f \in L^2(\mathbb{R})$,

$$\sum_{k \in \mathbb{Z}} |\langle f, \phi_{j,k} \rangle|^2 \leq \left(\sup_{k \in \mathbb{Z}} \|\phi_{j,k}\|_{L^\infty} \right)^2 \sum_{k \in \mathbb{Z}} \int_{\text{supp}(\phi_{j,k})} |f(x)|^2 dx,$$

where $\sup_{k \in \mathbb{Z}} \|\phi_{j,k}\|_{L^\infty}$ is bounded due to (b) and the sum is bounded by $C(m_j) \|f\|_{L^2}^2$, for some $C(m_j) > 0$, due to (c). \diamond

The second assumption is a condition on the Gramian matrices

$$\mathbf{G}_j := \int_{\mathbb{R}} \Phi_j(x) \Phi_j(x)^T dx, \quad j \in \mathbb{N}_0.$$

Assumption 2. There exists $v \in \mathbb{N}$ such that, for every $j \in \mathbb{N}_0$,

$$\int_{\mathbb{R}} x^\alpha \Phi_j(x) \mathbf{G}_j^{-1} \Phi_j(y) dy = y^\alpha, \quad y \in \mathbb{R}, \alpha \in \{0, \dots, v-1\}.$$

Remark 2.14. Assumption 2 is linked to the degree of the polynomial space that the functions Φ_j are able to span. Indeed, a necessary condition for Assumption 2 is that there exist vectors $\{\mathbf{c}_{j,\alpha}\}_{\alpha=0}^v$ such that

$$\mathbf{c}_{j,\alpha}^T \Phi_j(x) = x^\alpha, \quad x \in \mathbb{R}.$$

In particular, if we denote by $\mathbf{m}_{j,\alpha}$ the vector of the $(\alpha+1)$ -th moment of Φ_j , then

$$\mathbf{G}_j^{-1} \mathbf{m}_{j,\alpha} = \mathbf{c}_{j,\alpha}. \quad (2.17)$$

\diamond

The last assumption is related, roughly speaking, to the primitives of the functions Φ_j and, together with Assumption 2, is the one closely related to the vanishing moments of the resulting wavelet tight frame.

Assumption 3. There exists $w \in \mathbb{N}$ such that there exists a family of scaling functions $\{\Phi_j^{[-w]}\}_{j \in \mathbb{N}_0}$ that generates a multi-resolution analysis in Definition 2.4 where $\Phi_j^{[-w]}$ satisfies Assumption 1 and the following conditions, for every $j \in \mathbb{N}_0$.

- (a) for every $k \in \mathbb{Z}$, $\Phi_j^{[-w]} \subset \mathcal{H}^w(\mathbb{R})$, where \mathcal{H}^w denotes the Sobolev spaces of square integrable functions with w square integrable weak derivatives.

(b) $f \in \mathcal{V}_j$ has w vanishing moments if and only if there exists $\mathbf{f} \in \ell^2(\mathbb{Z})$ such that

$$f(x) = \mathbf{f}^T \frac{d^w}{dx^w} \Phi_j^{[-w]}(x), \quad x \in \mathbb{R}.$$

Moreover, \mathbf{f} decays exponentially if f does.

As we will see in the following constructions, Assumptions 1-3 are easily satisfied in real applications.

We are now able to state the fundamental result of Chui, He, Stöckler which characterizes wavelet tight frames in a generic setting.

Theorem 2.15 ([15]). *Under Assumption 1, a set of bi-infinite matrices $\{\mathbf{Q}_j\}_{j \in \mathbb{N}}$ defines a wavelet tight frame in Definition 2.5 if and only if there exists a set of bi-infinite symmetric semi-positive definite matrices $\{\mathbf{S}_j\}_{j \in \mathbb{N}}$ such that:*

(i) *for every $j \in \mathbb{N}_0$, there exists $C_j > 0$ such that*

$$\int_{\mathbb{R}^2} f(x) \Phi_j(x)^T \mathbf{S}_j \Phi_j(y) f(y) dx dy \leq C_j \|f\|_{L^2}^2, \quad f \in L^2(\mathbb{R});$$

(ii) *for every $f \in L^2(\mathbb{R})$,*

$$\int_{\mathbb{R}^2} f(x) \Phi_j(x)^T \mathbf{S}_j \Phi_j(y) f(y) dx dy \xrightarrow{j \rightarrow \infty} \|f\|_{L^2}^2;$$

(iii) *for every $j \in \mathbb{N}$,*

$$\mathbf{S}_j - \mathbf{P}_{j-1} \mathbf{S}_{j-1} \mathbf{P}_{j-1}^T = \mathbf{Q}_j \mathbf{Q}_j^T.$$

Moreover, if Assumptions 2 and 3 are satisfied with $v = w \in \mathbb{N}$, then the corresponding wavelet tight frame has v vanishing moments if and only if, for every $j \in \mathbb{N}_0$, the matrix \mathbf{S}_j satisfies

$$(a) \quad \|\mathbf{S}_j\|_{\ell^2 \rightarrow \ell^2} < \infty;$$

$$(b) \quad \exists C > 0, r > 2v + 1:$$

$$|\Phi_j(x)^T \mathbf{S}_j \Phi_j(y)| \leq \frac{C}{(1 + |x - y|)^r}, \quad x, y \in \mathbb{R};$$

$$(c) \quad \int_{\mathbb{R}} x^\alpha \Phi_j(x)^T (\mathbf{G}_j^{-1} - \mathbf{S}_j) \Phi_j(y) dx = 0, \quad a.e., \alpha \in \{0, \dots, v-1\};$$

$$(d) \quad \int_{\mathbb{R}} y^\alpha \int_{-\infty}^x \frac{(x-t)^{v-1}}{(v-1)!} \Phi_j(t)^T (\mathbf{G}_j^{-1} - \mathbf{S}_j) \Phi_j(y) dt dy = 0 \quad a.e., \alpha \in \{0, \dots, v-1\}.$$

Remark 2.16. If one desires a wavelet tight frame with compactly supported elements, due to Theorem 2.15 (iii), the matrices $\{\mathbf{Q}_j\}_{j \in \mathbb{N}}$ must have compactly supported columns, which means \mathbf{S}_j must be bandlimited, for every $j \in \mathbb{N}$. \diamond

Remark 2.17. Condition (d) in Theorem 2.15 is implied by condition (c) if one can exchange the order of integration. This, however, is not always the case, see e.g. Example 2.11. On the other hand, for our construction in the semi-regular, which is based on a local modification of \mathbf{S}_j , (d) will be implied by (c) trivially. \diamond

Remark 2.18. In the regular case, when $s(\omega)$ in (2.8) is a symmetric polynomial, the resulting matrices \mathbf{S}_j are equal to the Toeplitz matrix obtained from the coefficients of $s(\omega)$. \diamond

Theorem 2.15 basically gives us the roadmap for the construction of a wavelet tight frame:

1st: choose a suitable multi-resolution analysis satisfying Assumptions 1, 2 and 3;

2nd: find a suitable set of matrices $\{\mathbf{S}_j\}_{j \in \mathbb{Z}}$ satisfying Theorem 2.15;

3rd: take the square root of the matrices $\mathbf{S}_j - \mathbf{P}_{j-1}\mathbf{S}_{j-1}\mathbf{P}_{j-1}^T$.

We already have good candidates for the 1st step - the renormalized basic limit functions of subdivision schemes. We still need to check the properties required by Assumptions 1, 2 and 3, which we will do case by case when needed. For the second step, Theorem 2.15 itself already gives us a clue on how to choose the matrices \mathbf{S}_j . Indeed, if we are able to choose \mathbf{S}_j such that

$$\mathbf{S}_j \mathbf{m}_{j,\alpha} = \mathbf{c}_{j,\alpha}, \quad \alpha \in \{0, \dots, v-1\},$$

then Remark 2.14 guarantees that hypothesis (c) of Theorem 2.15 holds. Moreover, working in the semi-regular setting is advantageous since $\mathbf{S}_j \equiv \mathbf{S}_0$, $j \in \mathbb{N}$, and additionally we restrict the search to bandlimited matrices. In general, it is quite hard to factorize a semi-positive definite bi-infinite matrix, even if it is bandlimited and consists of positive semi-definite blocks (2.11). The factorization requires the solution of several quadratic systems. Thus, it is better to exploit the properties of the considered system case by case.

With a general strategy at hand, we focus on the regular and semi-regular cases. In both cases we start from a convergent subdivision scheme with continuous basic limit functions $[\varphi_k]_{k \in \mathbb{Z}}$, which satisfy the refinement equation (1.17) with respect to a subdivision matrix \mathbf{P} . First of all we still need to justify the renormalization (2.5). In the regular UEP case, we saw that it is always possible and it works just fine, translating the condition $s(\omega) \equiv 1$ in (2.8) on the polynomial side to the condition $\mathbf{S}_0 = \mathbf{I}$ in (2.11) on the matrix side. No doubt then that this is the natural renormalization one wants for defining the scaling function. However, a question arises: is this renormalization always possible? The answer to this question is yes if and only if we can prove that for the considered semi-regular convergent scheme we have

$$\int_{\mathbb{R}} \varphi_k(x) dx > 0, \quad k \in \mathbb{Z}. \quad (2.18)$$

This unfortunately is not the case (a counterexample can be found in Section 2.2.2, $n = 2$). On the other hand, cases where the condition (2.18) fails are of no interest for us because of Assumption 3. Indeed, if we are able to construct a wavelet tight frame with one vanishing moment, since the frame elements are linear combination of the scaling functions, Assumption 3 implies that there exists a convergent subdivision scheme with basic limit functions whose derivatives are linear combinations of the basic limit functions of the scheme considered to construct the scaling functions. As observed in [22], this is true if and only if there exists a monotone sequence $\mathbf{b} \in \ell(\mathbf{Z})$ such that

$$\mathbf{b}^T \Delta = \frac{1}{2} \mathbf{b}^T \Delta \mathbf{P} \quad \text{and} \quad \mathbf{b}(k+1) - \mathbf{b}(k) = C \int_{\mathbb{R}} \varphi_k(x) dx, \quad k \in \mathbb{Z}, \quad (2.19)$$

for some constant $C \neq 0$, where

$$\Delta(k, m) = \begin{cases} (-1)^{k+m+1}, & \text{if } k-1 \leq m \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

This requires that all the integrals of φ_k must be w.l.o.g. strictly positive. Thus, if a wavelet tight frame can be constructed via Theorem 2.15 starting from a convergent subdivision scheme, the integrals of its basic limit functions must be positive in order to perform the renormalization (2.5). Due to Proposition 1.40, we are able to compute those integrals.

Assumption 1, conditions (b), (c) and (d) are rather easy to check both in the regular and the semi-regular settings. The Riesz basis condition (a) is trickier to prove in general, but for the regular setting we have the following sufficient condition ensuring (b).

Proposition 2.19. *Let \mathbf{G}_0 be the Gramian matrix of $\Phi_0 = [\phi_{0,k} = \phi_{0,0}(\cdot - hk)]_{k \in \mathbb{Z}}$, $h > 0$. If there exist $0 < A_0 \leq B_0 < \infty$ such that*

$$A_0 \leq g(\omega) = \sum_{k \in \mathbb{Z}} \mathbf{G}_0(0, k) e^{-2\pi i h k \omega} \leq B_0, \quad \omega \in \mathbb{R}, \quad (2.20)$$

then Φ_0 is a Riesz basis for \mathcal{V}_0 with bounds A_0 and B_0 .

Proof. Let $\mathbf{f} \in \ell^2(\mathbb{Z})$ and consider $f = \Phi_0^T \mathbf{f}$. Since the Fourier transform is a linear isometry on $L^2(\mathbb{R})$, we have

$$\begin{aligned} \|f\|_{L^2}^2 &= \int_{\mathbb{R}} |\hat{f}(\omega)|^2 d\omega = \int_{\mathbb{R}} \left| \sum_{k \in \mathbb{Z}} \mathbf{f}(k) \hat{\phi}_{0,k}(\omega) \right|^2 d\omega \\ &= \int_{\mathbb{R}} \left| \hat{\phi}_{0,0}(\omega) \sum_{k \in \mathbb{Z}} \mathbf{f}(k) e^{-2\pi i h k \omega} \right|^2 d\omega. \end{aligned}$$

Since the function $F(\omega) = \sum_{k \in \mathbb{Z}} \mathbf{f}(k) e^{-2\pi i k \omega}$ is periodic with period $1/h$, we can split \mathbb{R} into $\bigcup_{k \in \mathbb{Z}} [k, k+1)/h$ and obtain

$$\begin{aligned} \|f\|_{L^2}^2 &= \sum_{k \in \mathbb{Z}} \int_{k/h}^{(k+1)/h} |\hat{\phi}_{0,0}(\omega)|^2 |F(\omega)|^2 d\omega \\ &= \int_0^{1/h} |F(\omega)|^2 \sum_{k \in \mathbb{Z}} \left| \hat{\phi}_{0,0} \left(\omega - \frac{k}{h} \right) \right|^2 d\omega. \end{aligned} \quad (2.21)$$

Now, focusing on the internal sum, we observe that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \left| \hat{\phi}_{0,0} \left(\omega - \frac{k}{h} \right) \right|^2 &= \sum_{k \in \mathbb{Z}} \hat{\phi}_{0,0} \left(\omega - \frac{k}{h} \right) \overline{\hat{\phi}_{0,0} \left(\omega - \frac{k}{h} \right)} \\ &= \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \phi_{0,0}(x) e^{-2\pi i x \left(\omega - \frac{k}{h} \right)} dx \int_{\mathbb{R}} \phi_{0,0}(y) e^{2\pi i y \left(\omega - \frac{k}{h} \right)} dy \end{aligned}$$

We can then rearrange the order of integration and substitute $x = z + y$ to get

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \left| \hat{\phi}_{0,0} \left(\omega - \frac{k}{h} \right) \right|^2 &= \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} e^{-2\pi i z \left(\omega - \frac{k}{h} \right)} \int_{\mathbb{R}} \phi_{0,0}(z+y) \phi_{0,0}(y) dy dz \\ &= \sum_{k \in \mathbb{Z}} \hat{\gamma}_0 \left(\omega - \frac{k}{h} \right), \end{aligned}$$

where

$$\gamma_0(x) = \int_{\mathbb{R}} \phi_{0,0}(x+y) \phi_{0,0}(y) dy, \quad x \in \mathbb{R}.$$

Finally, due to the Poisson summation formula (see e.g. [64]),

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \left| \hat{\phi}_{0,0} \left(\omega - \frac{k}{h} \right) \right|^2 &= h \sum_{k \in \mathbb{Z}} \gamma_0(hk) e^{2\pi i h k \omega} \\ &= h \sum_{k \in \mathbb{Z}} \mathbf{G}_0(0, k) e^{-2\pi i h k \omega} = h g(\omega). \end{aligned}$$

Thus, from (2.21), by hypothesis,

$$hA_0 \int_0^{1/h} |F(\omega)|^2 d\omega \leq \|f\|_{L^2}^2 \leq hB_0 \int_0^{1/h} |F(\omega)|^2 d\omega.$$

To conclude the proof, we observe that

$$\begin{aligned} \int_0^{1/h} |F(\omega)|^2 d\omega &= \sum_{k,m \in \mathbb{Z}} \mathbf{f}(k) \mathbf{f}(m) \int_0^{1/h} e^{-2\pi i h \omega (k-m)} d\omega \\ &= \frac{1}{h} \|\mathbf{f}\|_{\ell^2}^2. \end{aligned}$$

□

Remark 2.20. Proposition 2.19 is equivalent to a property of the so-called *bracket product* $[\hat{\phi}_{0,0}, \hat{\phi}_{0,0}]$, see [39] and references therein. ◇

In the regular case then, we only need to compute the Gramian matrix of Φ_0 using the results in Section 1.2.2 and then compute the bounds for the trigonometric polynomial in (2.20). For the other levels $j \in \mathbb{N}$, $\Phi_j = 2^{j/2} \Phi_0(2^j \cdot)$ and we get for free the Riesz property, since

$$\begin{aligned} \|\Phi_j^T \mathbf{f}\|_{L^2}^2 &= \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |\phi_{j,k}(x) \mathbf{f}(k)|^2 dx \\ &= 2^j \int_{\mathbb{R}} \left| \sum_{k \in \mathbb{Z}} \phi_{0,k}(2^j x) \mathbf{f}(k) \right|^2 dx = \|\Phi_0^T \mathbf{f}\|_{L^2}^2. \end{aligned} \tag{2.22}$$

In the semi-regular setting, we use Proposition 2.19 to check that the regular schemes on the left and on the right of $\mathbf{t}_0(0)$ satisfy Assumption 1. Additionally we need to guarantee that the overall resulting semi-regular scheme satisfies Assumption 1. In particular, the linear independence of the basic limit functions is enough to guarantee the Riesz basis property for the whole family of scaling functions.

Proposition 2.21. *Let Φ_0 be the vector of the scaling functions of a semi-regular multi-resolution analysis with the corresponding left and right regular schemes which induce multi-resolution analysis satisfying Assumption 1. If Φ_0 is linearly independent, then it is a Riesz basis for \mathcal{V}_0 .*

Proof. Let $B_\ell, B_r > 0$ be the Riesz upper bounds for the left and right regular multi-resolution analysis, respectively. Then, for every $\mathbf{f} \in \ell^2(\mathbb{Z})$,

$$\begin{aligned} \|\Phi_0^T \mathbf{f}\|_{L^2}^2 &= \left\| \left(\sum_{k \leq k_l(\mathbf{P})} + \sum_{k_\ell(\mathbf{P}) < k < k_r(\mathbf{P})} + \sum_{k \geq k_r(\mathbf{P})} \right) \mathbf{f}(k) \phi_{0,k} \right\|_{L^2}^2 \\ &\leq \left(B_\ell + \max_{k_\ell(\mathbf{P}) < k < k_r(\mathbf{P})} \|\phi_{0,k}\|_{L^2}^2 + B_r \right) \|\mathbf{f}\|_{\ell^2}^2. \end{aligned}$$

The lower estimate follows directly from the linear independence of Φ_0 . □

We proceed further with the cubic B-spline scheme in Examples 1.31, 1.32 and 2.11. Using Theorem 2.15, we are searching for a wavelet tight frame with two vanishing moments.

2.1.1 A First Example from Cubic B-spline

Regular Case

We consider the initial regular mesh $\mathbf{t}_0 = h\mathbb{Z}$, $h > 0$. In Example 1.31, the regular cubic B-spline scheme produces piecewise cubic basic limit functions $\varphi_k = \varphi_0(\cdot - hk) \in \mathcal{C}^{3-\epsilon}(\mathbb{R})$, $\epsilon > 0$, that satisfy a refinement equation (1.17) with respect to the subdivision matrix in (1.29). By Proposition 1.37, the integrals of φ_k are equal to h for all $k \in \mathbb{Z}$. Thus, the renormalized scaling functions are $\phi_k = \varphi_k/\sqrt{h}$, $k \in \mathbb{Z}$. To compute the Gramian matrix of the scaling functions in Φ_0 , we compute the Gramian matrix of the basic limit functions via Proposition 1.42 and then rescale. Indeed,

$$\mathbf{G}_0 = \int_{\mathbb{R}} \Phi_0(x) \Phi_0(x)^T dx = \frac{1}{h} \int_{\mathbb{R}} [\varphi_k(x)]_{k \in \mathbb{Z}} [\varphi_k(x)]_{k \in \mathbb{Z}}^T dx = \frac{1}{h} \mathbf{G}.$$

The matrix \mathbf{G} is a banded Toeplitz matrix and it is defined by the vector \mathbf{g} obtained as a solution of

$$\begin{bmatrix} 1/16 & 7/16 & 7/16 & 1/16 & & & \\ 1/128 & 7/32 & 35/64 & 7/32 & 1/128 & & \\ & 1/16 & 7/16 & 7/16 & 1/16 & & \\ & 1/128 & 7/32 & 35/64 & 7/32 & 1/128 & \\ & & 1/16 & 7/16 & 7/16 & 1/16 & \\ & & 1/128 & 7/32 & 35/64 & 7/32 & 1/128 \\ & & & 1/16 & 7/16 & 7/16 & 1/16 \end{bmatrix}^T \mathbf{g} = \mathbf{g}, \quad (2.23)$$

with the sum of the elements of \mathbf{g} equal to h . The entries of the matrix in (2.23) are obtained from the coefficients of the square of the symbol in Example 1.31. The solution we seek has sum of the elements equal to h , which leads to the unique vector

$$\mathbf{g} = h \begin{bmatrix} 1/5040 & 1/42 & 397/1680 & 151/315 & 397/1680 & 1/42 & 1/5040 \end{bmatrix}^T.$$

Thus, to prove the Riesz basis property, due to Proposition 2.19, we have to find bounds for the function

$$g(\omega) = \frac{151}{315} + \frac{397}{840} \cos(2\pi\omega) + \frac{1}{21} \cos(4\pi\omega) + \frac{1}{2520} \cos(6\pi\omega), \quad \omega \in \mathbb{R}.$$

The function g is periodic of period 1 and it satisfies

$$\frac{17}{315} = g\left(\frac{1}{2}\right) \leq g(\omega) \leq g(0) = 1.$$

Thus, the scaling functions Φ_0 form a Riesz basis for the closure of their span \mathcal{V}_0 . Then, by (2.22), our system Φ_0 satisfies Assumption 1 (a). Conditions (b), (c) and (d) of Assumption 1 are trivially satisfied.

The partition of unity property (1.16) implies that, for every $j \in \mathbb{N}$,

$$\Phi_j(x)^T \frac{\sqrt{h}}{2^{j/2}} \mathbf{1} = \Phi_0(2^j x)^T \sqrt{h} \mathbf{1} = [\varphi_k(2^j x)]_{k \in \mathbb{Z}}^T \mathbf{1} = 1, \quad x \in \mathbb{R},$$

and we have

$$\mathbf{c}_{j,0} = \frac{\sqrt{h}}{2^{j/2}} \mathbf{1} \quad \text{and} \quad \mathbf{m}_{j,0} = \int_{\mathbb{R}} \Phi_j(x) dx = 2^{j/2} \int_{\mathbb{R}} \Phi_0(2^j x) dx = \frac{\sqrt{h}}{2^{j/2}} \mathbf{1}$$

satisfying (2.17). This is coherent with the tight frame construction (with one vanishing moment) in Example 1.32, where $\mathbf{S}_j \equiv \mathbf{I}$. To get one more vanishing moment, however, we need to check Assumptions 2 and 3 for $v = w = 2$ and then choose a suitable \mathbf{S}_0 to apply Theorem 2.15.

Assumption 3 follows directly from (2.19) and it is a well known property of B-splines (see e.g. [26]). To check Assumption 2 with $v = 2$, we observe that, from Example 1.31, the right-eigenspace of $\dot{\mathbf{P}}$ related to the eigenvalue $1/2$ can be extended uniquely, as in Proposition 1.9, to the corresponding right-eigenspace of \mathbf{P} with respect to the same eigenvalue. The resulting right-eigenspace of \mathbf{P} is formed by the samples of all the polynomials of the form λx , $\lambda \in \mathbb{R} \setminus \{0\}$. In particular, $[hk]_{k \in \mathbb{Z}}$ is a right-eigenvector of \mathbf{P} and we have, for every $j \in \mathbb{N}_0$,

$$\Phi_j(x)^T \frac{\sqrt{h}}{2^{\frac{3}{2}j}} [k]_{k \in \mathbb{Z}} = \Phi_0(2^j x)^T \frac{\sqrt{h}}{2^j} [k]_{k \in \mathbb{Z}} = [\varphi_k(2^j x)]_{k \in \mathbb{Z}}^T \frac{1}{2^j} [k]_{k \in \mathbb{Z}} = x, \quad x \in \mathbb{R}.$$

Thus, Assumption 2 is satisfied with $v = 2$ and, moreover, we get

$$\mathbf{c}_{j,1} = \frac{\sqrt{h}}{2^{\frac{3}{2}j}} [k]_{k \in \mathbb{Z}} \quad \text{and} \quad \mathbf{m}_{j,1} = \mathbf{G}_j \mathbf{c}_{j,1}, \quad j \in \mathbb{N}_0.$$

In particular, we get $\mathbf{m}_{j,1} = \mathbf{c}_{j,1}$, $j \in \mathbb{N}_0$. Unfortunately, even if the choice of $\mathbf{S}_0 = \mathbf{I}$ fits into hypothesis (a), (b) and (c) of Theorem 2.15, it fails to meet (d) for $\alpha = 1$. Roughly speaking \mathbf{S}_0 is not a good enough approximation of \mathbf{G}_0^{-1} . Indeed, as observed in [13] in the regular case, to get v vanishing moments from the OEP (2.8) one has to choose s such that

$$s(\omega) - g(\omega)^{-1} = \mathcal{O}(\omega^{2v}), \quad \omega \rightarrow 0,$$

with $g(\omega)$ as in (2.20). One way to do that is to choose $s(\omega)$ to be the Taylor polynomial of degree $2v - 1$ of $g^{-1}(\omega)$ at $\omega = 0$. Then we can easily construct the matrices \mathbf{S}_j in Remark 2.18. Since $0 < g(\omega) \leq 1$ is real, we can exploit the Taylor expansion of the

function $1/(1-x)$ at 0 to get

$$g(\omega)^{-1} = \frac{1}{1 - (1 - g(\omega))} = \sum_{j \in \mathbb{N}_0} (1 - g(\omega))^j, \quad \omega \rightarrow 0$$

where the factor $(1 - g(\omega))^j$ adds a polynomial term of lowest degree j , since $g(0) = 1$. Thus, we can easily choose $s(\omega)$ such that

$$\sum_{j=0}^{2v-1} (1 - g(\omega))^j = s(\omega) + \mathcal{O}(\omega^{2v}), \quad \omega \rightarrow 0.$$

In our case,

$$\begin{aligned} \sum_{j=0}^3 (1 - g(\omega))^j &= 1 + \frac{(2\pi\omega)^2}{3} + \mathcal{O}(\omega^4) \\ &= \frac{5}{3} - \frac{2}{3} \left(1 - \frac{(2\pi\omega)^2}{2} \right) + \mathcal{O}(\omega^4) \\ &= \frac{5}{3} - \frac{2}{3} \cos(2\pi\omega) + \mathcal{O}(\omega^4) \\ &= -\frac{1}{3} e^{-2\pi i\omega} + \frac{5}{3} - \frac{1}{3} e^{2\pi i\omega} + \mathcal{O}(\omega^4), \quad \omega \rightarrow 0, \end{aligned}$$

so we can choose

$$\mathbf{S}_j = \begin{bmatrix} \ddots & -1/3 & & & \\ & 5/3 & -1/3 & & \\ & -1/3 & \boxed{5/3} & -1/3 & \\ & & -1/3 & 5/3 & \\ & & & -1/3 & \ddots \end{bmatrix}.$$

Remark 2.22. If we apply the same procedure for the linear B-spline scheme, see Examples 1.5, 1.12, 1.20 and 1.25, to get two vanishing moments we obtain

$$s_1(\omega) = -\frac{1}{6} e^{-2\pi i\omega} + \frac{4}{3} - \frac{1}{6} e^{2\pi i\omega}, \quad \omega \in \mathbb{R}.$$

If Theorem 2.9 could be applied to the OEP, then we should have

$$s_1(\omega)^2 = s(\omega) + \mathcal{O}(\omega^4), \quad \omega \rightarrow 0,$$

but

$$\begin{aligned}
 s_1(\omega)^2 - s(\omega) &= -\frac{1}{36}e^{-4\pi i\omega} - \frac{1}{9}e^{-2\pi i\omega} + \frac{1}{6} - \frac{1}{9}e^{2\pi i\omega} - \frac{1}{36}e^{4\pi i\omega} \\
 &= -\frac{1}{9} - \frac{2}{9}(2\pi\omega)^2 + \mathcal{O}(\omega^4), \quad \omega \rightarrow 0.
 \end{aligned}$$

◇

Hence, we proceed differently. Having $\mathbf{P}_j = \mathbf{P}/\sqrt{2}$ as in (2.6), with \mathbf{P} in Example 1.31, to get the matrices \mathbf{Q}_j , we need to factorize

$$\mathbf{R}_j = \mathbf{S}_j - \mathbf{P}_{j-1} \mathbf{S}_{j-1} \mathbf{P}_{j-1}^T = \mathbf{S}_0 - \frac{1}{2} \mathbf{P} \mathbf{S}_0 \mathbf{P}.$$

The resulting matrix is the following

$$\mathbf{R}_j = \begin{bmatrix}
 1/384 & & & & & & & \\
 1/96 & & & & & & & \\
 7/384 & 1/96 & 1/384 & & & & & \\
 1/48 & 1/24 & 1/96 & & & & & \\
 -7/128 & 1/48 & 7/384 & 1/96 & 1/384 & & & \\
 \ddots & -59/96 & -1/8 & 1/48 & 1/24 & 1/96 & & \\
 79/64 & -59/96 & -7/128 & 1/48 & 7/384 & 1/96 & 1/384 & \\
 -59/96 & 4/3 & -59/96 & -1/8 & 1/48 & 1/24 & 1/96 & \\
 -7/128 & -59/96 & 79/64 & -59/96 & -7/128 & 1/48 & 7/384 & \\
 1/48 & -1/8 & -59/96 & 4/3 & -59/96 & -1/8 & 1/48 & \\
 7/384 & 1/48 & -7/128 & -59/96 & 79/64 & -59/96 & -7/128 & \\
 1/96 & 1/24 & 1/48 & -1/8 & -59/96 & 4/3 & -59/96 & \\
 1/384 & 1/96 & 7/384 & 1/48 & -7/128 & -59/96 & 79/64 & \\
 & & & & & & & \ddots \\
 & & & 1/96 & 1/24 & 1/48 & -1/8 & -59/96 \\
 & & & 1/384 & 1/96 & 7/384 & 1/48 & -7/128 \\
 & & & & & 1/96 & 1/24 & 1/48 \\
 & & & & & 1/384 & 1/96 & 7/384 \\
 & & & & & & & 1/96 \\
 & & & & & & & 1/384
 \end{bmatrix}.$$

To get the frame elements, we factorize $\mathbf{R}_j = \mathbf{Q}_j \mathbf{Q}_j^T$. Since we want to keep the shift-invariant structure and to have compactly supported framelets, we are searching first for a blocking decomposition of \mathbf{R}_j as in (2.11), i.e. we want to find a finite positive semi-definite block \mathbf{B} such that

$$\mathbf{R}_j = \sum_{k \in \mathbb{Z}} \mathbf{B}(\cdot - 2k, \cdot - 2k),$$

and then factorize \mathbf{B} to obtain the vectors \mathbf{q}_m . From the structure of \mathbf{R}_j we deduce that the smallest \mathbf{B} possible must belong to $\mathbb{R}^{7 \times 7}$. Since the Fejér-Riesz Theorem states

that the OEP conditions (2.8) can always be satisfied with two polynomials q_1, q_2 , then there exists a rank-2 \mathbf{B} that does what we seek. This leads to a quadratic system in the unknown entries of \mathbf{q}_1 and \mathbf{q}_2 . This system is already very complex for MatLab. Nevertheless, one of the possible solutions is shown in Figure 2.1.

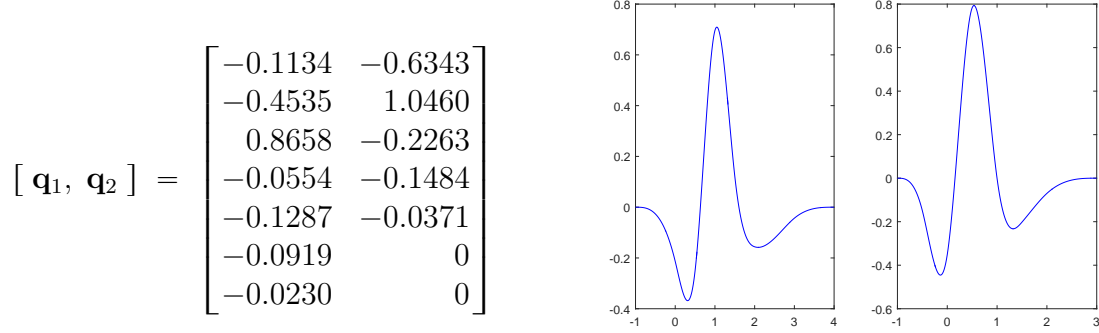


Figure 2.1: Two possible generators for the cubic B-spline regular wavelet tight frame over the initial mesh \mathbb{Z} .

If we take a look back to the structure of \mathbf{R}_j we can try to infer more dependency to simplify the quadratic system. First we observe that the upper right and lower left 2×2 corners in red (for the interpretation of the references to color, the reader is referred to the pdf version of this manuscript) should belong to only one block $\mathbf{B}(\cdot - 2k, \cdot - 2k)$. Moreover, the numbers in blue, in magenta and cyan are sum of the entries of two, three and four consecutive blocks, respectively. Since we cannot have \mathbf{q}_m with only one element (the corresponding framelet would be a multiple of a scaling function which has no vanishing moments), we can guess

$$\mathbf{R}_j = \sum_{k \in \mathbb{Z}} \mathbf{B}_7(\cdot - 2k, \cdot - 2k) + \mathbf{B}_5(\cdot - 2k, \cdot - 2k) + \mathbf{B}_3(\cdot - 2k, \cdot - 2k),$$

with $\mathbf{B}_n \in \mathbb{R}^{n \times n}$ being rank-1 blocks. Moreover, due to the bi-symmetry of \mathbf{R}_j we suppose

$$\mathbf{B}_7 = \mathbf{q}_1 \mathbf{q}_1^T, \quad \mathbf{B}_5 = \mathbf{q}_2 \mathbf{q}_2^T, \quad \mathbf{B}_3 = \mathbf{q}_3 \mathbf{q}_3^T,$$

with symmetric $\{\mathbf{q}_m\}_{m=1,2,3}$. This simplifies enormously the resulting quadratic system and leads to a unique solution with symmetric generators, Figure 2.2. The price for our assumptions is that only one generator has a small support.

Semi-regular Case

Consider now the initial semi-regular mesh \mathbf{t}_0 in (1.3) with $h_\ell = 1$ and $h_r = 2$. Similarly to Example 1.32, we get a subdivision matrix \mathbf{P} that differs from the regular one over three columns, w.l.o.g. $k_\ell(\mathbf{P}) = -2$ and $k_r(\mathbf{P}) = 2$. Thus, we have three irregular basic limit functions. To construct the wavelet tight frame, first of all we need to renormalize the basic limit functions and the subdivision matrix as in (2.5) and (2.6). To do so we

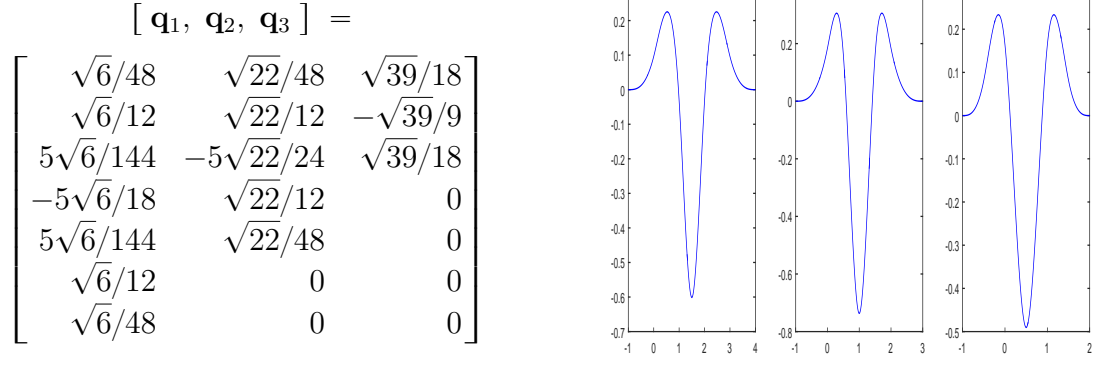


Figure 2.2: Three possible symmetric generators for the cubic B-spline regular wavelet tight frame over the initial mesh \mathbb{Z} .

compute the integrals of the basic limit functions $\{\varphi_k\}_{k \in \mathbb{Z}}$. From the regular case, we already know that

$$\int_{\mathbb{R}} \varphi_k(x) dx = \begin{cases} 1, & \text{if } k \leq k_\ell(\mathbf{P}), \\ 2, & \text{if } k \geq k_r(\mathbf{P}). \end{cases}$$

To compute the missing integrals we rely on Proposition 1.40. Thus,

$$\begin{bmatrix} 1/8 \\ 1/2 \\ 25/32 & 3/32 \\ 5/8 & 3/8 \\ 5/24 & 29/40 & 1/15 \\ & 3/5 & 2/5 \\ & 3/20 & 29/40 \\ & & 1/2 \\ & & 1/8 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ \int_{\mathbb{R}} \varphi_{-1}(x) dx \\ \int_{\mathbb{R}} \varphi_0(x) dx \\ \int_{\mathbb{R}} \varphi_1(x) dx \\ 2 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} \int_{\mathbb{R}} \varphi_{-1}(x) dx \\ \int_{\mathbb{R}} \varphi_0(x) dx \\ \int_{\mathbb{R}} \varphi_1(x) dx \end{bmatrix},$$

which leads to

$$\int_{\mathbb{R}} \varphi_{-1}(x) dx = \frac{5}{4}, \quad \int_{\mathbb{R}} \varphi_0(x) dx = \frac{3}{2}, \quad \int_{\mathbb{R}} \varphi_1(x) dx = \frac{7}{4}.$$

ular, see Example 1.32,

$$\mathbf{c}_{j,1} = 2^{-\frac{3}{2}j} \mathbf{D}^{1/2} \begin{bmatrix} \vdots \\ -2 \\ -1 \\ 1/3 \\ 2 \\ 4 \\ \vdots \end{bmatrix} = 2^{-\frac{3}{2}j} \begin{cases} k, & k \leq -2, \\ -\frac{\sqrt{5}}{2}, & k = -1, \\ \frac{\sqrt{6}}{6}, & k = 0, \\ \sqrt{7}, & k = 1, \\ 2\sqrt{2}k, & k \geq 2, \end{cases}, \quad j \in \mathbb{N}_0.$$

To compute $\mathbf{m}_{j,1}$ we use again Proposition 1.40. From the regular case, together with (1.32), we have

$$\int_{\mathbb{R}} \varphi_k(x) dx = \begin{cases} k, & \text{if } k \leq k_\ell(\mathbf{P}) = -2, \\ 4k, & \text{if } k \geq k_r(\mathbf{P}) = 2, \end{cases}$$

thus,

$$\begin{bmatrix} 1/8 \\ 1/2 \\ 25/32 & 3/32 \\ 5/8 & 3/8 \\ 5/24 & 29/40 & 1/15 \\ & 3/5 & 2/5 \\ & 3/20 & 29/40 \\ & & 1/2 \\ & & 1/8 \end{bmatrix}^T \begin{bmatrix} -4 \\ -3 \\ -2 \\ \int_{\mathbb{R}} x \varphi_{-1}(x) dx \\ \int_{\mathbb{R}} x \varphi_0(x) dx \\ \int_{\mathbb{R}} \varphi_1(x) dx \\ 8 \\ 12 \\ 16 \end{bmatrix} = 4 \begin{bmatrix} \int_{\mathbb{R}} x \varphi_{-1}(x) dx \\ \int_{\mathbb{R}} x \varphi_0(x) dx \\ \int_{\mathbb{R}} x \varphi_1(x) dx \end{bmatrix},$$

which leads to

$$\int_{\mathbb{R}} x \varphi_{-1}(x) dx = -1, \quad \int_{\mathbb{R}} x \varphi_0(x) dx = \frac{9}{10}, \quad \int_{\mathbb{R}} x \varphi_1(x) dx = \frac{77}{20}.$$

Therefore,

$$\mathbf{m}_{j,1}(k) = 2^{-\frac{3}{2}j} \begin{cases} k, & k \leq -2, \\ -\frac{2\sqrt{5}}{5}, & k = -1, \\ \frac{3\sqrt{6}}{10}, & k = 0, \\ \frac{11\sqrt{7}}{10}, & k = 1, \\ 2\sqrt{2}k, & k \geq 2. \end{cases}$$

If, similarly to the regular case, we suppose \mathbf{S}_{irr} to be tridiagonal, then the condition (2.24) yields a unique solution

$$\mathbf{S}_{irr} = \begin{bmatrix} 29/18 & -\sqrt{5}/9 & & & & \\ -\sqrt{5}/9 & 14/9 & -\sqrt{30}/18 & & & \\ & -\sqrt{30}/18 & \boxed{27/16} & -59\sqrt{42}/1008 & & \\ & & -59\sqrt{42}/1008 & 2683/1512 & -20\sqrt{14}/189 & \\ & & & -20\sqrt{14}/189 & 46/27 & \end{bmatrix},$$

which leads to a positive semi-definite $\mathbf{S}_j - \mathbf{P}_{j-1}\mathbf{S}_{j-1}\mathbf{P}_{j-1}^T$. Indeed, the changes in \mathbf{P}_j and \mathbf{S}_j affect the columns of \mathbf{R}_j with indices -6 to 6 , Figure 2.3. If we subtract the regular blocks we found before, e.g. in Figure 2.2, from both sides until we eliminate all the entries which did not change from the regular to the semi-regular case, we end up with the 13×13 block \mathbf{R}_{irr} in Figure 2.4. \mathbf{R}_{irr} is positive semi-definite and can be factorized as $\mathbf{Q}_{irr}\mathbf{Q}_{irr}^T$ in different ways. For example, we can use the eigenvalue decomposition, Figure 2.5. This way we obtain nine frame elements around $\mathbf{t}_0(0)$ which all have support in $[-3, 6]$. This choice however spoils completely the blocking structure still visible in \mathbf{R}_{irr} . Another possible choice to preserve this block structure is to use a Cholesky-like factorization. This is what have been done in Figure 2.6. Here the Cholesky algorithm has been applied alternatively from the left and from the right side, not to promote either of the directions approaching $\mathbf{t}_0(0)$. This approach results in a much sparser \mathbf{Q}_{irr} and in framelets with much shorter support. In Figure 2.6, one can observe also that the framelets obtained this way have a more uniform behaviour and are less oscillating than the ones in Figure 2.5. The same process can be done starting from the regular framelets in 2.1, with similar results. This example shows that already in a fairly simple case the matrices involved are messy and difficult to handle. Studying the whole process with respect to the initial mesh parameters h_ℓ, h_r then becomes a nightmare, apart from very special cases. In the next section, we will construct a family of wavelet tight frames for which a lot of the steps presented here are much simpler. The trade-off is a toll with respect to the possible choices of h_ℓ and h_r .

$$\mathbf{Q}_{irr} = \begin{bmatrix} -0.0068 & -0.0411 & 0.0395 & -0.0580 & 0.0799 & -0.1228 & 0.2294 & 0.2075 & 0.0956 \\ 0.0504 & 0.3056 & -0.2687 & 0.3645 & -0.4198 & 0.3503 & -0.2792 & -0.0481 & 0.0509 \\ -0.1399 & -0.7702 & 0.5243 & -0.4412 & 0.2929 & 0.0171 & -0.2003 & -0.1212 & 0.0191 \\ 0.2198 & 0.9527 & -0.2021 & -0.2285 & 0.4223 & -0.2789 & -0.0090 & -0.1354 & -0.0074 \\ -0.2441 & -0.6365 & -0.6252 & 0.5174 & -0.0646 & -0.2979 & 0.1377 & -0.1150 & -0.0277 \\ 0.4285 & 0.1531 & 0.8435 & 0.1903 & -0.4615 & -0.0239 & 0.2088 & -0.0622 & -0.0461 \\ -0.8627 & 0.1755 & -0.2397 & -0.5319 & -0.1743 & 0.3067 & 0.1541 & 0.0248 & -0.0566 \\ 0.9804 & -0.3061 & -0.3794 & -0.0930 & 0.2726 & 0.2852 & 0.0060 & 0.1007 & -0.0559 \\ -0.5258 & 0.1894 & 0.3966 & 0.6212 & 0.4387 & 0.0001 & -0.1479 & 0.1318 & -0.0409 \\ 0.0748 & -0.0327 & -0.1107 & -0.3454 & -0.4655 & -0.4447 & -0.2495 & 0.1216 & -0.0199 \\ 0.0231 & -0.0088 & -0.0209 & -0.0370 & -0.0243 & 0.0274 & 0.0584 & -0.0697 & 0.0506 \\ 0.0035 & -0.0005 & 0.0055 & 0.0395 & 0.0737 & 0.1108 & 0.0966 & -0.0801 & 0.0445 \\ 0.0009 & -0.0001 & 0.0014 & 0.0099 & 0.0184 & 0.0277 & 0.0242 & -0.0200 & 0.0111 \end{bmatrix}$$

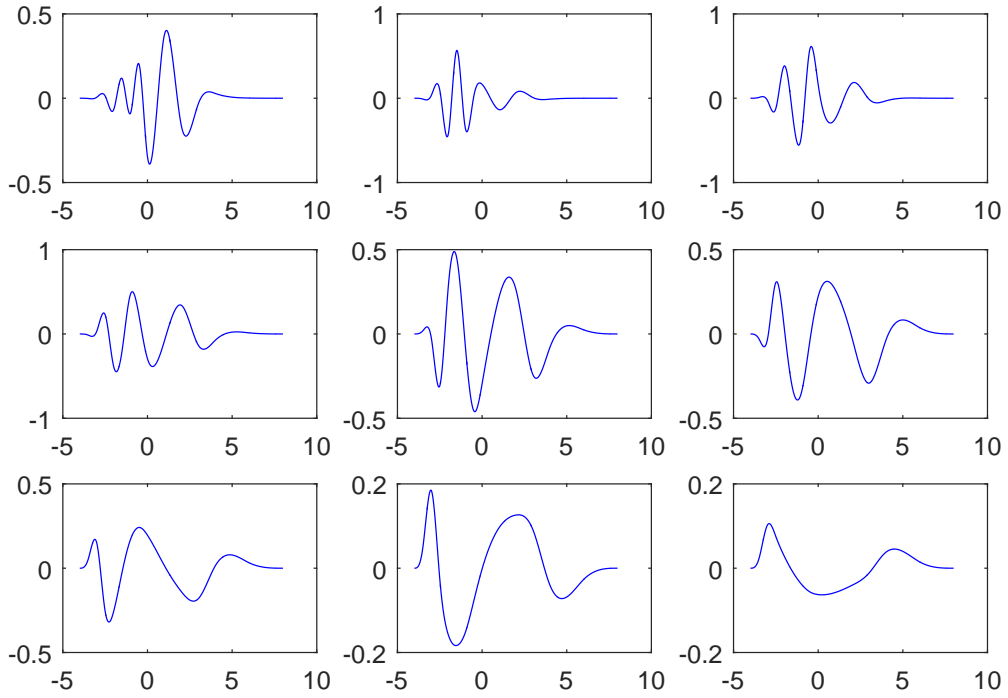


Figure 2.5: The result of the eigenvalue decomposition of \mathbf{R}_{irr} and the corresponding framelets.

$$\mathbf{Q}_{irr} = \begin{bmatrix} 0.3646 & & & & & & & & & \\ -0.5222 & 0.6405 & & & & & & & & \\ 0.0849 & -0.8650 & 0.6841 & & & & & & & \\ 0.0048 & 0.0147 & -0.9067 & 0.7080 & & & & & & \\ 0.0262 & 0.0810 & -0.0030 & -0.8272 & 0.7401 & & 0.0975 & 0.0195 & & \\ 0.0266 & 0.0823 & 0.1150 & -0.0740 & -0.8693 & 0.6320 & 0.1703 & 0.0874 & & \\ 0.0097 & 0.0301 & 0.0569 & 0.0498 & -0.0983 & -1.0817 & 0.2167 & 0.1817 & 0.0317 & \\ & & 0.0161 & 0.0829 & 0.2662 & 0.4673 & -1.0085 & 0.1436 & 0.2057 & \\ & & 0.0043 & 0.0222 & & & 0.5521 & -0.8709 & 0.1581 & \\ & & & & & & & 0.3971 & -0.6873 & \\ & & & & & & & 0.0993 & 0.0687 & \\ & & & & & & & & 0.1925 & \\ & & & & & & & & & 0.0481 \end{bmatrix}$$

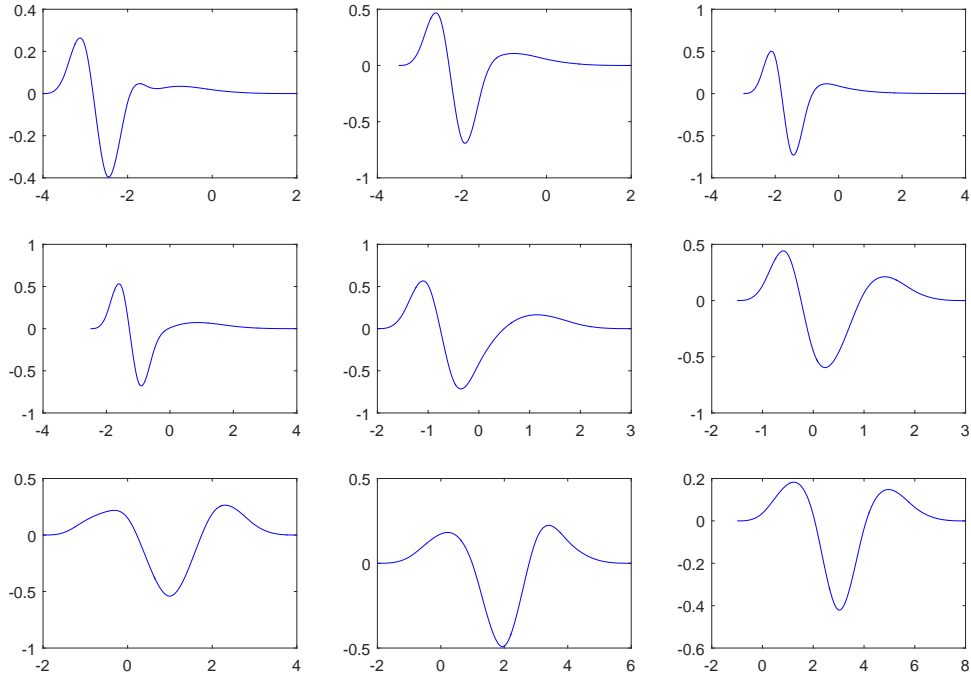


Figure 2.6: The result of the alternating application of the Cholesky factorization of \mathbf{R}_{irr} and the corresponding framelets.

2.2 Semi-regular Dubuc-Deslauriers Wavelet Tight Frames

In this section, we present the work published in [60]. The aim is to provide an easy strategy to construct a semi-regular family of wavelet tight frames with a high number of vanishing moments. The starting point for this construction is the family of Dubuc-Deslauriers subdivision schemes, introduced in [27] in the regular case and extended to the semi-regular setting in [62]. These schemes are interpolatory, i.e. their basic limit functions $\{\varphi_k\}_{k \in \mathbb{Z}}$ satisfy

$$\varphi_k(\mathbf{t}_0(m)) = \delta_{km}, \quad k, m \in \mathbb{Z}, \quad (2.25)$$

where \mathbf{t}_0 is the initial mesh in (1.3). Moreover, the Dubuc-Deslauriers family depends on a parameter $n \in \mathbb{N}$, that describes the polynomial generation of each scheme. Indeed, the Dubuc-Deslauriers schemes can be constructed as solutions of the following interpolation problems, which already incorporates the semi-regular setting.

Definition 2.23. Let $n \in \mathbb{N}$ and \mathbf{t}_0 a semi-regular initial mesh in (1.3). The subdivision matrix \mathbf{P} defining the Dubuc-Deslauriers $2n$ -point scheme over \mathbf{t}_0 is the unique bi-infinite matrix satisfying the following conditions:

1. $\mathbf{P}(2k, k) = 1$, $k \in \mathbb{Z}$,
2. the entries $\bar{\mathbf{P}}_k = [\mathbf{P}(2k+1, m) : m = k-n+1, \dots, k+n]$, $k \in \mathbb{Z}$, satisfy

$$\bar{\mathbf{P}}_k \begin{bmatrix} 1 & \mathbf{t}_0(k-n+1) & \cdots & \mathbf{t}_0(k-n+1)^{2n-1} \\ 1 & \mathbf{t}_0(k-n+2) & \cdots & \mathbf{t}_0(k-n+2)^{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \mathbf{t}_0(k+n) & \cdots & \mathbf{t}_0(k+n)^{2n-1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\mathbf{t}_0(2k+1)}{2} & \cdots & \left(\frac{\mathbf{t}_0(2k+1)}{2}\right)^{2n-1} \end{bmatrix}.$$

3. all other entries of \mathbf{P} are equal to zero.

Remark 2.24. Definition 2.23 is well posed since the linear systems in part 2. of Definition 2.23 are uniquely solvable due to \mathbf{t}_0 being monotonically increasing. Moreover, for every $k \in \mathbb{Z}$,

$$\text{supp}(\mathbf{P}(:, k)) = \{2k - 2n + 1, \dots, 2k + 2n - 1\}$$

which means that the columns of \mathbf{P} have support of length $4n - 1$ and that the eventual basic limit functions have

$$\text{supp}(\varphi_k) = [\mathbf{t}_0(2k - 2n + 1), \mathbf{t}_0(2k + 2n - 1)]. \quad (2.26)$$

◇

Here we present a proof, for any mesh \mathbf{t}_0 , with arbitrary $h_\ell, h_r \in (0, \infty)$, of the convergence of these schemes in the semi-regular setting.

Proposition 2.25. Let $n \in \mathbb{N}$ and \mathbf{P} be the subdivision matrix constructed in 1.-3. over the semi-regular mesh \mathbf{t}_0 . Then

- (i) 1 is a simple eigenvalue of \mathbf{P} associated to the right eigenvector $\mathbf{1}$ and all other eigenvalues of \mathbf{P} are less than 1 in absolute value;
- (ii) the subdivision scheme with the subdivision matrix \mathbf{P} converges.

Proof. Part (i): Let $n \in \mathbb{N}$ and $\mathcal{I} = \{1 - 2n, \dots, 2n - 1\}$. By Proposition 1.9, all non-zero eigenvalues of \mathbf{P} are uniquely determined by the eigenvalues of its finite section, the square matrix $\mathring{\mathbf{P}} = (\mathbf{P}(m, k))_{m, k \in \mathcal{I}}$. By construction, due to 2., $\mathring{\mathbf{P}}\mathbf{1} = \mathbf{1}$. We show next, that $\lambda \in \mathbb{C} \setminus \{0, 1\}$ with $\mathring{\mathbf{P}}\mathbf{v} = \lambda\mathbf{v}$, $\mathbf{v} \in \mathbb{C}^{4n-3} \setminus \{\mathbf{0}\}$, must satisfy $|\lambda| < 1$. The proof is by contradiction, we assume that $|\lambda| \geq 1$. Note first that $\mathring{\mathbf{P}}(0, 0) = 1$ and by step 3. of the construction above, we get $\mathbf{v}(0) = \lambda \mathbf{v}(0)$, thus, $\mathbf{v}(0) = 0$. Step 1. of the construction forces $\mathbf{v}(k) = \lambda \mathbf{v}(2k)$ for $k, 2k \in \mathcal{I}$. To determine the odd entries of \mathbf{v} , let $m \in \mathcal{I}$ be odd and consider the polynomial interpolation problem with the pairwise distinct knots $\mathbf{t}_0(k)$ and values $\mathbf{v}(k)$ for $j \in \left\{ \frac{m+1}{2} - n, \dots, n + \frac{m+1}{2} \right\}$. This interpolation problem possesses a unique solution, possibly complex-valued, interpolation polynomial $\pi \in \Pi_{2n-1}$. Therefore, by the interpolation property of \mathbf{P} , we have

$$\lambda \pi(\mathbf{t}_0(m)) = \lambda \mathbf{v}(m) = (\mathring{\mathbf{P}} \mathbf{v})(m) = \pi\left(\frac{\mathbf{t}_0(m)}{2}\right).$$

Iterating we obtain

$$\lim_{r \rightarrow \infty} |\lambda|^r |\pi(\mathbf{t}_0(m))| = \lim_{r \rightarrow \infty} \left| \pi\left(\frac{\mathbf{t}_0(m)}{2^r}\right) \right| = |\pi(0)| = |\mathbf{v}(0)| = 0, \quad (2.27)$$

which leads to a contradiction: for $|\lambda| = 1$, we have $\mathbf{v}(m) = \pi(\mathbf{t}_0(m)) = 0$ or, for $|\lambda| > 1$, the identity (2.27) is violated. It is left to show that $\lambda = 1$ is simple. The proof is by contradiction. w.l.o.g. we assume that 1 has an algebraic multiplicity 2. Note that $\delta^T \mathring{\mathbf{P}} = \delta^T$, where $\delta(0) = 1$ and its other entries are equal to zero and define $\mathbf{A} = \mathring{\mathbf{P}} - \mathbf{1}\delta^T$. Then \mathbf{A} has a simple eigenvalue 1 with $\mathbf{A}\mathbf{v} = \mathbf{v}$, $\mathbf{v} \neq \mathbf{0}$. By construction $\mathbf{v}(0) = 0$, thus $\mathbf{A}\mathbf{v} = \mathring{\mathbf{P}}\mathbf{v}$. Following a similar argument as above we arrive at the contradiction $\mathbf{v} = \mathbf{0}$. Part (ii): To prove the convergence of the scheme, due to [27, 62], it suffices to prove the continuity of the basic limit functions at 0, i.e.

$$|\mathbf{f}_j(0) - \mathbf{f}_j(1)| \xrightarrow{j \rightarrow \infty} 0, \quad \text{and} \quad |\mathbf{f}_j(0) - \mathbf{f}_j(-1)| \xrightarrow{j \rightarrow \infty} 0 \quad (2.28)$$

for $\mathbf{f}_j = \mathbf{P}^j \mathbf{e}^{(k)}$, $k \in \mathbb{Z}$. Equivalently, we show that

$$|\mathbf{f}_j(0) - \mathbf{f}_j(1)| = |[0, \dots, 0, \boxed{1}, -1, 0, \dots, 0] \mathring{\mathbf{P}}^j \mathbf{e}^{(k)}| \rightarrow 0$$

and, similarly, for the other difference in (2.28). The claim follows then by part (i), together with steps 1. and 3. of the construction, which imply that $\mathring{\mathbf{P}}^j \mathbf{e}^{(k)} \rightarrow \mathbf{e}^{(k)}(0) \mathbf{1}$. \square

Due to 1. these schemes preserves at each level all the data computed at the previous

levels. In particular, for every initial data $\mathbf{f}_0 \in \ell(\mathbf{Z})$,

$$\mathbf{f}_j(2k) = (\mathbf{P}\mathbf{f}_{j-1})(k), \quad j \in \mathbb{N}, \quad k \in \mathbb{Z},$$

and this is sufficient to guarantee (2.25). The odd entries of each finer level instead are computed using linear combinations of the $2n$ neighbouring points. Moreover, condition 2. of Definition 2.23 implies that the constructed schemes reproduces polynomials of degree $2n - 1$, see Definition 1.18.

Remark 2.26. A subdivision scheme over the initial mesh \mathbf{t}_0 that reproduces polynomials of degree $v \in \mathbb{N}$ has a subdivision matrix \mathbf{P} that satisfies, for every polynomial $\pi \in \Pi_v$ of degree v ,

$$\pi(\mathbf{t}_j) = \mathbf{P} \pi(\mathbf{t}_{j-1}), \quad k \in \mathbb{Z}.$$

◇

The convergence and the interpolation property imply that the functions $\{\varphi_k : k \in \mathbb{Z}\}$ are linearly independent and, thus, the representation

$$x^\alpha = \sum_{k \in \mathbb{Z}} \mathbf{t}_0(k)^\alpha \varphi_k(x), \quad x \in \mathbb{R}, \quad \alpha \in \{0, \dots, 2n - 1\}, \quad (2.29)$$

is unique. If we suppose that the integrals of $\{\varphi_k\}_{k \in \mathbb{Z}}$ are positive, the corresponding scaling functions $\Phi_j = [\phi_{j,k} : k \in \mathbb{Z}]$ in (2.5) inherit the polynomial reproduction property in (2.29) and we have

$$x^\alpha = \Phi_j^T(x) \mathbf{c}_{j,\alpha}, \quad \mathbf{c}_{j,\alpha} = 2^{-\frac{3}{2}j} \mathbf{D}^{1/2} \mathbf{t}_0^\alpha, \quad \alpha \in \{0, \dots, 2n - 1\}. \quad (2.30)$$

Furthermore, in the semi-regular case, we have $k_\ell(\mathbf{P}) = 1 - 2n$ and $k_r(\mathbf{P}) = 2n - 1$, which means the presence of $4n - 3$ irregular basic limit functions corresponding to the indices

$$\mathcal{I}_{irr} = \{2 - 2n, \dots, 2n - 2\}. \quad (2.31)$$

We denote with Φ_ℓ and Φ_r the vectors of scaling functions over the regular meshes $\mathbf{t}_\ell = h_\ell \mathbb{Z}$ and $\mathbf{t}_r = h_r \mathbb{Z}$ respectively, and thus

$$\mathbf{I}_\ell \Phi = \mathbf{I}_\ell \Phi_\ell \quad \text{and} \quad \mathbf{I}_r \Phi = \mathbf{I}_r \Phi_r, \quad (2.32)$$

where

$$\mathbf{I}_\ell(m, k) = \begin{cases} 1, & \text{if } m = k < 2 - 2n, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathbf{I}_r(m, k) = \begin{cases} 1, & \text{if } m = k > 2n - 2 \\ 0, & \text{otherwise.} \end{cases}$$

The choice of the Dubuc-Deslauriers schemes is due to the fact that they are one of the special cases in which the frames constructed via the UEP conditions, (2.8) with $s(\omega) \equiv 1$, achieve naturally more than one vanishing moment, and this can be exploited to simplify the construction in the semi-regular case.

2.2.1 Wavelet Tight Frames Construction

We present the construction of the wavelet tight frames based on the Dubuc-Deslauriers $2n$ -point schemes as we did for the cubic B-spline scheme in Section 2.1.1, starting with the regular case, where we will exploit Theorem 2.9, and then passing to the semi-regular case with an ad hoc construction. Unfortunately, as it will be shown in Section 2.2.2, the construction is not possible for every choice of $h_\ell, h_r > 0$, but the larger n is the smaller the interval which the ratio h_ℓ/h_r can belong, see Section 2.2.2, case $n > 2$.

Regular Case

The construction in the regular case is based on another family of schemes leading to the Daubechies $2n$ -tap wavelet systems (see e.g. [20]), which form orthonormal basis for $L^2(\mathbb{R})$. In particular, for $n \in \mathbb{N}$, the Daubechies $2n$ -tap schemes are characterized by the unique symbol $d(\omega)$ with coefficients \mathbf{d} supported on $\{1 - 2n, \dots, 0\}$ which satisfies (2.8) with $s(\omega) \equiv 1$ and

$$q_d(\omega) = q_1(\omega) = e^{-i2\pi(2n-1)\omega} \overline{d(\omega - 1/2)} \quad \text{and} \quad v = n, \quad n \in \mathbb{N}. \quad (2.33)$$

Daubechies wavelets are closely connected to the Dubuc-Deslauriers subdivision schemes. Indeed, see [51], the symbol $p(\omega)$ of the Dubuc-Deslauriers $2n$ -point scheme satisfies

$$p(\omega) = d(\omega) \overline{d(\omega)}, \quad \omega \in \mathbb{R}. \quad (2.34)$$

The identity (2.34), together with Theorem 2.9, leads to our construction of Dubuc-Deslauriers wavelet tight frames for the regular case.

Proposition 2.27. *Let $n \in \mathbb{N}$ and $d(\omega)$ and $p(\omega)$ be the symbols of the Daubechies $2n$ -tap scheme and the Dubuc-Deslauriers $2n$ -point scheme, respectively. Then*

$$\begin{aligned} q_1(\omega) &= \sqrt{2} e^{i2\pi(2n-1)\omega} d(\omega) \overline{d(\omega - 1/2)} \\ q_2(\omega) &= d(\omega - 1/2) \overline{d(\omega - 1/2)}, \end{aligned} \quad \omega \in \mathbb{R},$$

define a wavelet tight frame with n vanishing moments for $p(\omega)$ in Theorem 2.8.

Proof. Note that the convergence of the subdivision associated to $d(\omega)$ implies the convergence of the subdivision associated to $\overline{d(\omega)}$. Thus, applying Theorem 2.9 to $d(\omega)$ and $\overline{d(\omega)}$, due to (2.34), we obtain a wavelet tight frame for $p(\omega) = d(\omega) \overline{d(\omega)}$ with n vanishing moments with the polynomials

$$d(\omega) \overline{q_d(\omega)}, \quad \overline{d(\omega)} q_d(\omega), \quad \text{and} \quad q_d(\omega) \overline{q_d(\omega)}.$$

To reduce the number of frame generators we take a closer look to the structure of these

framelets. Indeed, the UEP identities, (2.8) with $s(\omega) \equiv 1$, and (2.33) yield

$$\left\{ \begin{array}{l} 1 = \left(d(\omega) \overline{d(\omega)} + q_d(\omega) \overline{q_d(\omega)} \right) \left(\overline{d(\omega)} d(\omega) + \overline{q_d(\omega)} q_d(\omega) \right) \\ \quad = \left(d(\omega) \overline{d(\omega)} \right) \overline{\left(d(\omega) \overline{d(\omega)} \right)} + \sqrt{2} d(\omega) \overline{q_d(\omega)} \overline{\left(\sqrt{2} d(\omega) \overline{q_d(\omega)} \right)} + \left(q_d(\omega) \overline{q_d(\omega)} \right) \overline{\left(q_d(\omega) \overline{q_d(\omega)} \right)}, \\ 0 = \left(d(\omega) \overline{d(\omega - 1/2)} + q_d(\omega) \overline{q_d(\omega - 1/2)} \right) \left(\overline{d(\omega)} d(\omega - 1/2) + \overline{q_d(\omega)} q_d(\omega - 1/2) \right) \\ \quad = \left(d(\omega) \overline{d(\omega)} \right) \overline{\left(d(\omega - 1/2) \overline{d(\omega - 1/2)} \right)} + \left(q_d(\omega) \overline{q_d(\omega)} \right) \overline{\left(q_d(\omega - 1/2) \overline{q_d(\omega - 1/2)} \right)} \\ \quad \quad + d(\omega) \overline{q_d(\omega)} \overline{\left(d(\omega - 1/2) \overline{q_d(\omega - 1/2)} \right)} + \overline{d(\omega)} q_d(\omega) d(\omega - 1/2) \overline{q_d(\omega - 1/2)}. \end{array} \right. \quad (2.35)$$

From (2.33), we have

$$q_d(\omega) d(\omega - 1/2) = \overline{q_d(\omega)} \overline{d(\omega - 1/2)}, \quad \omega \in \mathbb{R}.$$

Moreover, the periodicity of the symbols implies

$$\overline{d(\omega)} \overline{q_d(\omega - 1/2)} = d(\omega) q_d(\omega - 1/2), \quad \omega \in \mathbb{R}.$$

Next, we rewrite the last term of the second identity in (2.35)

$$\overline{d(\omega)} q_d(\omega) d(\omega - 1/2) \overline{q_d(\omega - 1/2)} = d(\omega) \overline{q_d(\omega)} \overline{\left(d(\omega - 1/2) \overline{q_d(\omega - 1/2)} \right)}, \quad \omega \in \mathbb{R},$$

obtaining $q_1(\omega) = \sqrt{2}d(\omega)\overline{q_d(\omega)}$ and $q_2(\omega) = q_d(\omega)\overline{q_d(\omega)}$. The claim follows by (2.33). \square

Remark 2.28. This construction leads to the same result as in [12, Section 3.1.2], but in a more straightforward way. \diamond

An important consequence of Proposition 2.27 is that from the matrix point of view $\mathbf{S}_j \equiv \mathbf{I}$ gives n vanishing moments in Theorem 2.15. Similarly to the construction in Section 2.1.1 in the semi-regular case, we exploit the regular wavelet tight frame and the fact that $\mathbf{S}_j \equiv \mathbf{I}$ to isolate the irregular part of the matrix in Theorem 2.15 (iii).

Semi-regular Case

Let $n \in \mathbb{N}$, we consider the semi-regular Dubuc-Deslauriers $2n$ -point scheme over the semi-regular initial mesh \mathbf{t}_0 in (1.3) with $h_\ell, h_r > 0$. To apply Theorem 2.15, we need to check first Assumptions 1, 2 and 3. Assumption 1 follows from Proposition 2.21 due to the linear independence of the scaling functions guaranteed by the interpolation property (2.25). Assumption 2 is a direct consequence of the polynomial reproduction (2.30) and the third one is satisfied, as long as the integrals of the basic limit functions are positive, due to (2.19) and [22].

The next step, for the construction of the wavelet tight frames is the choice of the matrices $\{\mathbf{S}_j\}_{j \in \mathbb{N}}$, $j \in \mathbb{N}$. A natural choice, at least to get one vanishing moment, should

be $\mathbf{S}_j \equiv \mathbf{I}$. After all, we always have $\mathbf{c}_{j,0} = \mathbf{m}_{j,0}$, $j \in \mathbb{N}_0$. Unfortunately, already for the 4-point scheme: the matrix $\mathbf{I} - \mathbf{P}_j \mathbf{P}_j^T$ is not positive semi-definite. The idea is to change the matrix \mathbf{S}_j locally nearby the irregular scaling functions, i.e. we have the following situation

$$\mathbf{S}_j = \begin{bmatrix} \ddots & & & & & & & & & & \\ & 1 & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \mathbf{S}_{irr} & \ddots & \ddots & & & \\ & & & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & & & 1 & & & & \\ & & & & & & & 1 & & & \\ & & & & & & & & \ddots & & \end{bmatrix}$$

with $\mathbf{S}_{irr} \in \mathbb{R}^{(4n-3) \times (4n-3)}$. Since in the regular case we obtain n vanishing moments, \mathbf{S}_{irr} must be chosen to maintain this property. To achieve this goal, since \mathbf{S}_j must satisfy (2.17), we need to know the vectors $\mathbf{c}_{j,\alpha}$ and $\mathbf{m}_{j,\alpha}$, for every $\alpha \in \{0, \dots, 2n-1\}$. This is not difficult thanks to the polynomial reproduction property (2.30) and Proposition 1.40. Since outside \mathbf{S}_{irr} , the matrices \mathbf{S}_j are diagonal, there is no interaction between regular and irregular entries. The idea, roughly speaking, is to choose \mathbf{S}_{irr} such that it is the minimal block that achieves (2.24) to get the desired number of vanishing moments.

Algorithm 1:

1. Define the $(4n-3) \times n$ matrix

$$\mathbf{C} = \begin{bmatrix} [\mathbf{c}_{j,0}(k)]_{k \in \mathcal{I}_{irr}} & \dots & [\mathbf{c}_{j,n-1}(k)]_{k \in \mathcal{I}_{irr}} \end{bmatrix};$$

2. Compute the QR factorization $\mathbf{C} = \mathbf{O}\mathbf{U}$ with orthogonal $\mathbf{O} \in \mathbb{R}^{(4n-3) \times (4n-3)}$ and upper triangular $\mathbf{U} \in \mathbb{R}^{(4n-3) \times n}$;
3. Define $\mathbf{S}_j \equiv \mathbf{I}$ and, if $h_\ell \neq h_r$, modify

$$\mathbf{S}_{irr} := [\mathbf{S}_j(k, m)]_{k, m \in \mathcal{I}_{irr}} = \tilde{\mathbf{O}} \tilde{\mathbf{O}}^T, \quad (2.36)$$

where

$$\tilde{\mathbf{O}} = \begin{bmatrix} [\mathbf{O}(k, 1)]_{k \in \mathcal{I}_{irr}} & \dots & [\mathbf{O}(k, n)]_{k \in \mathcal{I}_{irr}} \end{bmatrix}.$$

We notice that Algorithm 1, when $h_\ell \neq h_r$, generates a matrix \mathbf{S}_{irr} which is not full rank and this is a significant downside for some applications, e.g. signal compression. Nonetheless, for analysis of subdivision smoothness we only need the decomposition part of the corresponding wavelet tight frame algorithm and the matrices \mathbf{S}_j constructed via Algorithm 1 define a family of kernels $\Phi_j(x)^T \mathbf{S}_j \Phi_j(y)$ with the desired approximation properties.

Proposition 2.29. *Let \mathbf{S}_j be defined by Algorithm 1. Then, for all $f \in L^2(\mathbb{R})$,*

$$(i) \quad \exists C_j > 0 : \int_{\mathbb{R}^2} f(x) \Phi_j(x)^T \mathbf{S}_j \Phi_j(y) f(y) dx dy \leq C_j \|f\|_{L^2}^2,$$

$$(ii) \quad \int_{\mathbb{R}^2} f(x) \Phi_j(x)^T \mathbf{S}_j \Phi_j(y) f(y) dx dy \longrightarrow \|f\|_{L^2}^2, \quad j \rightarrow \infty.$$

Proof. Part (i): Since $\Phi_j = 2^{j/2} \Phi_0(2^j \cdot)$, we only need to prove the claim for $j = 0$. Recall that the regular elements in Φ_0 are the elements of Φ_ℓ and Φ_r in (2.32). For such regular families of scaling functions Theorem 2.15 with $\mathbf{S} = \mathbf{I}$ implies the existence of $C_\ell > 0$ and $C_r > 0$, respectively, such that, for all $f \in L^2(\mathbb{R})$,

$$\begin{aligned} \max \left\{ \int_{\mathbb{R}^2} f(x) \Phi_\ell(x)^T \Phi_\ell(y) f(y) dx dy, \quad \int_{\mathbb{R}^2} f(x) \Phi_r(x)^T \Phi_r(y) f(y) dx dy \right\} &\leq \\ &\leq \max\{C_\ell, C_r\} \|f\|_{L^2}^2. \end{aligned}$$

Decompose the bi-infinite identity $\mathbf{I} = \mathbf{I}_\ell + \mathbf{I}_{irr} + \mathbf{I}_r$ with

$$\mathbf{I}_\ell(j, j) = \begin{cases} 1, & j < 1 - 2n, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathbf{I}_r(j, j) = \begin{cases} 1, & j > 2n - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for all $f \in L^2(\mathbb{R})$, by (2.36) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^2} f(x) \Phi_0(x)^T \mathbf{S}_0 \Phi_0(y) f(y) dx dy &= \\ &= \int_{\mathbb{R}^2} f(x) \Phi(x)_0^T (\mathbf{I}_\ell + \mathbf{I}_{irr} \mathbf{S}_0 \mathbf{I}_{irr} + \mathbf{I}_r) \Phi(y)_0 f(y) dx dy \\ &\leq \max\{C_\ell, C_r\} \|f\|_{L^2}^2 + \sum_{j=2-2n}^{2n-2} \sum_{k=2-2n}^{2n-2} |\mathbf{S}_0(j, k)| \left| \int_{\mathbb{R}} f(x) \phi_{0,j}(x) dx \right| \left| \int_{\mathbb{R}} \phi_{0,k}(y) f(y) dy \right| \\ &\leq \left(\max\{C_\ell, C_r\} + (4n - 3) \|\mathbf{S}_{irr}\|_\infty \max_{k \in \mathcal{I}_{irr}} \|\phi_{0,k}\|_{L^2}^2 \right) \|f\|_{L^2}^2. \end{aligned}$$

Part (ii): Using again $\mathbf{I} = \mathbf{I}_\ell + \mathbf{I}_{irr} + \mathbf{I}_r$ and (2.26), for every $f \in L^2(\mathbb{R})$, we get

$$\begin{aligned} \left\| f - \int_{\mathbb{R}} f(x) \Phi_j(x)^T \mathbf{S}_j \Phi_j(\cdot) dx \right\|_{L^2}^2 &\leq \left\| f \chi_{(-\infty, 0)} - 2^j \int_{-\infty}^0 f(x) \Phi_\ell(2^j x)^T \Phi_\ell(2^j \cdot) dx \right\|_{L^2}^2 \\ &\quad + \left\| 2^j \int_{\mathbb{R}} f(x) \Phi(2^j x)^T \mathbf{I}_{irr} \mathbf{S}_0 \mathbf{I}_{irr} \Phi(2^j \cdot) dx \right\|_{L^2}^2 \\ &\quad + \left\| f \chi_{(0, \infty)} - 2^j \int_0^\infty f(x) \Phi_r(2^j x)^T \Phi_r(2^j \cdot) dx \right\|_{L^2}^2 \\ &=: \gamma_\ell + \gamma_{irr} + \gamma_r. \end{aligned}$$

The indices of the non-zero elements of $\mathbf{I}_{irr} \mathbf{S}_0 \mathbf{I}_{irr}$ belong to the set $\mathcal{I}_{irr} \times \mathcal{I}_{irr}$ in (2.31), thus,

$$\bigcup_{k \in \mathcal{I}_{irr}} \text{supp}(\phi_{0,k}) =: [a, b], \quad -\infty < a \leq b < \infty.$$

The continuity of Φ_0 and the Cauchy-Schwarz inequality yield

$$\gamma_{irr} := 2^{2j} \int_{\frac{[a,b]}{2^j}} \left| \int_{\frac{[a,b]}{2^j}} f(x) \Phi_0(2^j x)^T \mathbf{I}_{irr} \mathbf{S} \mathbf{I}_{irr} \Phi_0(2^j y) dx \right|^2 dy \leq C \|f\|_{L^2(\frac{[a,b]}{2^j})}^2$$

with the constant $C = (b-a)^2 \|\Phi^T \mathbf{I}_{irr} \mathbf{S}_0 \mathbf{I}_{irr} \Phi\|_{L^\infty}^2$. Thus, γ_{irr} goes to zero as j goes to ∞ . Moreover, since $f \chi_{(-\infty, 0)}$ and $f \chi_{(0, \infty)}$ belong to $L^2(\mathbb{R})$, by the argument from the regular case, both γ_ℓ and γ_r go to zero as j goes to ∞ . \square

Examples in Section 2.2.2 and numerical evidence for $n = 3, \dots, 8$ with different $h_\ell, h_r > 0$ (defining the mesh \mathbf{t}_0) lead to the following conjecture.

Conjecture 2.30. *Let \mathbf{S}_j be defined by Algorithm 1, \mathbf{P}_j and \mathbf{D} as in (2.6) for the Dubuc-Deslauriers $2n$ -point scheme and \mathbf{p} , \mathbf{q}_1 and \mathbf{q}_2 as in Proposition 2.27.*

(i) For

$$\mathbf{R}_j = \mathbf{S}_j - \mathbf{P}_{j-1} \mathbf{S}_{j-1} \mathbf{P}_{j-1}^T, \quad (2.37)$$

and \mathbf{B}_k , $k \notin \mathcal{I}_{irr}$ with entries

$$\mathbf{B}_k(t, u) = \mathbf{p} \mathbf{p}^T(t - 2k, u - 2k) + \sum_{m=1}^2 \mathbf{q}_m \mathbf{q}_m^T(t - 2k, u - 2k), \quad t, u \in \mathbb{Z}, \quad (2.38)$$

the matrix

$$\mathbf{R}_{irr} = \mathbf{R}_j - \frac{1}{2} \sum_{k \notin \mathcal{I}_{irr}} \mathbf{B}_k \quad (2.39)$$

is positive semi-definite.

(ii) For all $\alpha \in \{0, \dots, n-1\}$, \mathbf{S}_j satisfy (2.24).

Remark 2.31. Note that the requirement that \mathbf{R}_{irr} is positive semi-definite is stronger than the positive semi-definiteness of \mathbf{R}_j . Moreover, we get explicit

$$\mathbf{Q}_j = \begin{bmatrix} \dots & \overline{\mathbf{Q}}_{\min(\mathcal{I}_{irr})-1} & \mathbf{Q}_{irr} & \overline{\mathbf{Q}}_{\max(\mathcal{I}_{irr})+1} & \dots \end{bmatrix}$$

with $\mathbf{R}_{irr} = \mathbf{Q}_{irr} \mathbf{Q}_{irr}^T$ and $\mathbf{B}_k = \overline{\mathbf{Q}}_k \overline{\mathbf{Q}}_k^T$ for $k \notin \mathcal{I}_{irr}$. \diamond

Apart from the examples in Section 2.2.2 and numerical evidence for $n = 3, \dots, 8$, there are other facts that support Conjecture 2.30. First of all, in the regular case, the construction in Algorithm 1 reduces to the standard UEP construction. Indeed, even if we modify \mathbf{S}_{irr} as in Algorithm 1 step 3, since in the regular case $\mathbf{m}_{j,\alpha} = \mathbf{c}_{j,\alpha}$, $\alpha \in \{0, \dots, n-1\}$, it is easy to check that Conjecture (i) and (ii) are satisfied. Secondly, in the semi-regular case, (2.24) reduces to an identity for certain finite matrices. For $\alpha = 0, \dots, n-1$, define

$$\mathbf{M}_j = \begin{bmatrix} [\mathbf{m}_{j,0}(k)]_{k \in \mathcal{I}_{irr}} & \dots & [\mathbf{m}_{j,n-1}(k)]_{k \in \mathcal{I}_{irr}} \end{bmatrix}$$

and

$$\mathbf{C}_j = \begin{bmatrix} [\mathbf{c}_{j,0}(k)]_{k \in \mathcal{I}_{irr}} & \dots & [\mathbf{c}_{j,n-1}(k)]_{k \in \mathcal{I}_{irr}} \end{bmatrix}.$$

Then the irregular part of (2.24) becomes $\mathbf{S}_{irr} \mathbf{M}_j = \mathbf{C}_j$, which implies

$$\mathbf{M}_j^T \mathbf{C}_j = \mathbf{M}_j^T \mathbf{S}_{irr} \mathbf{M}_j.$$

Thus, for (2.24) to hold the matrix $\mathbf{M}_j^T \mathbf{C}_j$ must be symmetric. Indeed, for every $\alpha, \beta \in \{0, \dots, n-1\}$, by (2.30), we get

$$\begin{aligned} 0 &= x^\alpha \Phi_j^T \mathbf{c}_{j,\beta} - \mathbf{c}_{j,\alpha}^T \Phi_j x^\beta \\ &= [x^\alpha \phi_{j,k}(x)]_{k \in \mathcal{I}_{irr}}^T [\mathbf{c}_{j,\beta}(k)]_{k \in \mathcal{I}_{irr}} - [\mathbf{c}_{j,\alpha}(k)]_{k \in \mathcal{I}_{irr}}^T [x^\beta \phi_{j,k}(x)]_{k \in \mathcal{I}_{irr}} \\ &\quad + \sum_{k \notin \mathcal{I}_{irr}} (\mathbf{c}_{j,\beta}(k) x^\alpha \phi_{j,k}(x) - \mathbf{c}_{j,\alpha}(k) x^\beta \phi_{j,k}(x)), \quad x \in \mathbb{R}. \end{aligned}$$

Integrating both sides of the above identity and using the fact that $\mathbf{m}_{j,\alpha}(k) = \mathbf{c}_{j,\alpha}(k)$ for $\alpha \in \{0, \dots, n-1\}$, $k \notin \mathcal{I}_{irr}$, we obtain

$$\begin{aligned} (\mathbf{M}_j^T \mathbf{C}_j)(\alpha, \beta) - (\mathbf{M}_j^T \mathbf{C}_j)(\beta, \alpha) &= \\ &= [\mathbf{m}_{j,\alpha}(k)]_{k \in \mathcal{I}_{irr}}^T [\mathbf{c}_{j,\beta}(k)]_{k \in \mathcal{I}_{irr}} - [\mathbf{c}_{j,\alpha}(k)]_{k \in \mathcal{I}_{irr}}^T [\mathbf{m}_{j,\beta}(k)]_{k \in \mathcal{I}_{irr}} \\ &= 0, \quad \alpha, \beta \in \{0, \dots, n-1\}. \end{aligned}$$

We strongly believe that part (ii) of Conjecture 2.30 is due to some special, intriguing property of the Dubuc-Deslauriers schemes.

Remark 2.32. If one chooses $\tilde{n} < n$ columns of \mathbf{O} in (2.36), then the corresponding matrix \mathbf{S} would generate a wavelet tight frame with \tilde{n} vanishing moments. Since it is not possible to get more than n vanishing moments in the regular case, the choice $\tilde{n} = n$ is optimal in the semi-regular case. \diamond

2.2.2 Examples

We present two simple examples illustrating the construction in Section 2.2 for $n = 1$ and $n = 2$, respectively. The small bandwidth of the corresponding subdivision matrices \mathbf{P} allows for exact computations in terms of the mesh parameter $h_\ell, h_r > 0$. Without loss of generality, after a suitable renormalization, we consider the semi-regular mesh \mathbf{t}_0 with $h_\ell = 1$ and $h_r = h, h > 0$. In the case $n = 1$, the Dubuc-Deslauriers 2-point scheme corresponds to the linear B-spline scheme, Examples 1.5, 1.12, 1.20 and 1.25. The case $n = 2$, is more interesting and involved due to the high complexity of the entries of the corresponding matrices. For these two examples we are able to prove both parts of Conjecture 2.30.

For the interested reader, the irregular filters \mathbf{Q}_{irr} for $n = 2, 3, 4, 5$ and for several values of $h_r > 0$ are available in [59].

Case $n = 1$: linear B-spline scheme

In the regular case, i.e. $h_\ell = h_r = 1$, the linear B-spline scheme is defined by the mask

$$[\mathbf{p}(k) : k = -1, 0, 1] = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}^T.$$

By Proposition 2.27, with $[\mathbf{d}(k) : k = -1, 0] = \begin{bmatrix} 1 & 1 \end{bmatrix}$, we get

$$[\mathbf{q}_1(k) : k = -1, 0, 1] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T,$$

$$[\mathbf{q}_2(k) : k = -1, 0, 1] = \frac{1}{2} \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}^T.$$

In the semi-regular case, the subdivision matrix \mathbf{P} does not depend on h and is the 2-slanted matrix with columns determined by \mathbf{p} . The corresponding basic limit functions are the ones defined in (1.21), Example 1.20. Thus, the entries of \mathbf{D} in (2.6) are well defined for every $h > 0$ and, the first moments of the scaling functions satisfy

$$\mathbf{m}_{j,0} = \mathbf{D}^{1/2} \mathbf{1} = \begin{bmatrix} \dots & 1 & 1 & \sqrt{\frac{1+h}{2}} & \sqrt{h} & \sqrt{h} & \dots \end{bmatrix}^T = \mathbf{c}_{j,0}.$$

yields

$$[\mathbf{q}_1(k) : k = -3, \dots, 3] = \frac{\sqrt{2}}{16} [\sqrt{3} - 2 \quad 0 \quad 6 - \sqrt{3} \quad 0 \quad -6 - \sqrt{3} \quad 0 \quad \sqrt{3} + 2]^T,$$

$$[\mathbf{q}_2(k) : k = -3, \dots, 3] = \frac{1}{16} [1 \quad 0 \quad -9 \quad 16 \quad -9 \quad 0 \quad 1]^T.$$

Applying Proposition 1.37 in the regular part of the mesh, we obtain $\mathbf{m}_{0,0}(k) = 1$, $k < -2$, and $\mathbf{m}_{0,0}(k) = \sqrt{h}$, $k > 2$, and, by Proposition 1.40, we get

$$\mathbf{m}_{0,0}(-2) = \sqrt{\frac{1}{120} \left(h - \frac{3}{2}\right)^2 + \frac{479}{480}}, \quad \mathbf{m}_{0,0}(-1) = \sqrt{\frac{(7-2h)(h+2)}{15}},$$

$$\mathbf{m}_{0,0}(0) = \sqrt{\frac{(h+1)^3}{8h}}, \quad \mathbf{m}_{0,0}(1) = \sqrt{\frac{(7h-2)(2h+1)}{15h}}$$

$$\text{and } \mathbf{m}_{0,0}(2) = \sqrt{\frac{122}{120h} \left(h - \frac{3}{244}\right)^2 + \frac{479}{58560h}}.$$

The expressions for $\mathbf{m}_{0,0}(-1)$ and $\mathbf{m}_{0,0}(1)$ imply that \mathbf{D} in (2.6) is positive definite if and only if $h \in \left(\frac{2}{7}, \frac{7}{2}\right)$. Thus, the frame construction in Section 2.2 is not valid for other h . Moreover, for the second moment we have

$$\mathbf{m}_{0,1}(k) = k, \quad k < -2, \quad \text{and} \quad \mathbf{m}_{0,1}(k) = k\sqrt{h}, \quad k > 2,$$

and in the irregular part

$$[\mathbf{m}_{0,1}(k)]_{k=-2, \dots, 2} =$$

$$\begin{aligned} & \text{diag}([\mathbf{m}_{0,0}(k)]_{k=-2, \dots, 2})^{-1} \left[\frac{h^3 - 3h^2 + 7h - 1205}{600}; -\frac{(h+2)(4h^2 - 14h + 35)}{75}; \dots \right. \\ & \left. \dots \frac{(h+1)(h-1)(31h^2 + 40h + 31)}{600h}; \frac{(2h+1)(35h^2 - 14h + 4)}{75h}; \frac{1205h^3 - 7h^2 + 3h - 1}{600h} \right]. \end{aligned}$$

Next, we construct \mathbf{S}_{irr} to check the validity of Conjecture 2.30. The entries of \mathbf{S}_{irr} depend in an intricate way on the parameter h , thus, we work with $\tilde{\mathbf{S}}_{irr}$ instead, where, for

$$\alpha = \frac{5(h+1)}{2}, \quad \beta = \frac{37(h^2-1)}{12}, \quad \gamma = \frac{5(h+1)^3(431h^2 + 938h + 431)}{288},$$

we have

$$\mathbf{S}_{irr} = \frac{1}{\alpha\gamma} \text{diag}([\mathbf{m}_{0,0}(k)]_{k=-2,\dots,2}) \tilde{\mathbf{S}}_{irr} \text{diag}([\mathbf{m}_{0,0}(k)]_{k=-2,\dots,2}).$$

with

$$\begin{aligned} \tilde{\mathbf{S}}_{irr} &= (\alpha\beta^2 + \gamma) [(\mathbf{1}\mathbf{1}^T)(m, k)]_{-2 \leq m, k \leq 2} + \alpha^3 [(\mathbf{t}_0 \mathbf{t}_0^T)(m, k)]_{-2 \leq m, k \leq 2} - \alpha^2 \beta [(\mathbf{1}\mathbf{t}_0^T + \mathbf{t}_0 \mathbf{1}^T)(m, k)]_{-2 \leq m, k \leq 2} \\ &= \frac{25(h+1)^3}{48} \begin{bmatrix} 60h^2 + 88h + 32 & 60h^2 + 51h + 9 & 60h^2 + 14h - 14 & 23h^2 - 9h - 14 & -14h^2 - 32h - 14 \\ 60h^2 + 51h + 9 & 60h^2 + 14h + 16 & 60h^2 - 23h + 23 & 23h^2 - 16h + 23 & -14h^2 - 9h + 23 \\ 60h^2 + 14h - 14 & 60h^2 - 23h + 23 & 60h^2 - 60h + 60 & 23h^2 - 23h + 60 & -14h^2 + 14h + 60 \\ 23h^2 - 9h - 14 & 23h^2 - 16h + 23 & 23h^2 - 23h + 60 & 16h^2 + 14h + 60 & 9h^2 + 51h + 60 \\ -14h^2 - 32h - 14 & -14h^2 - 9h + 23 & -14h^2 + 14h + 60 & 9h^2 + 51h + 60 & 32h^2 + 88h + 60 \end{bmatrix}. \end{aligned}$$

Note that part (ii) of Conjecture 2.30 is equivalent to the system

$$\begin{cases} \tilde{\mathbf{S}}_{irr} \text{diag}([\mathbf{m}_{0,0}(k)]_{k=-2,\dots,2}) [\mathbf{m}_{0,0}(k)]_{k=-2,\dots,2} = \alpha \gamma [\mathbf{1}(k)]_{k=-2,\dots,2} \\ \tilde{\mathbf{S}}_{irr} \text{diag}([\mathbf{m}_{0,0}(k)]_{k=-2,\dots,2}) [\mathbf{m}_{0,1}(k)]_{k=-2,\dots,2} = \alpha \gamma [\mathbf{t}_0(k)]_{k=-2,\dots,2} \end{cases}$$

of polynomial equations, whose validity we checked with the help of MATLAB symbolic tool. Due to $\alpha, \gamma > 0$ for $h \in \left(\frac{2}{7}, \frac{7}{2}\right)$, part (i) of Conjecture 2.30 is equivalent to checking that

$$\tilde{\mathbf{R}}_{irr} = \alpha \gamma h \text{diag}(\mathbf{m}_{0,0})^{-1} \mathbf{R}_{irr} \text{diag}(\mathbf{m}_{0,0})^{-1}$$

is positive semi-definite. The renormalization leads to $\tilde{\mathbf{R}}_{irr}$ with polynomial entries and allows for symbolic manipulations. Indeed, this way, the generalized Sylvester criterion, confirms that $\tilde{\mathbf{R}}_{irr}$ is positive semi-definite for $h \in \left(\frac{2}{7}, \frac{7}{2}\right)$. In Figure 2.7 one can see the framelets corresponding to a possible factorization of \mathbf{R} with $h = 2$.

Remark 2.33. The value $2/7 \approx 0.2857$ resembles the corresponding critical value in [33, 34] computed for the irregular knot insertion for the 4-point scheme. Below this critical value the scheme loses smoothness. This fact makes the restriction on the range of the stepsize h less surprising in this case. \diamond

Case $n > 2$

Unfortunately for $n = 3$ it is already too difficult to compute the moments with respect to the mesh parameter h , as for $n = 1, 2$. However we approximated, with accuracy 10^{-6} , the critical value h_{crit} which defines the interval $(h_{crit}^{-1}, h_{crit})$ available for h to construct a wavelet tight frame from the Dubuc-Deslauriers $2n$ -point scheme.

n	h_{crit}
3	2.622482436768618
4	2.359070119036101
5	2.234640490839382
6	2.164064886283900
7	2.119382199894698
8	2.089127660655424

From this table, we guess that h_{crit} goes to 2 for n going to ∞ , which means that seeking for more vanishing moments this way requires to pay a toll on the flexibility of the initial mesh.

Different $n \in \{1, \dots, 8\}$ and $h \in [1, h_{crit} - 10^6)$ have been considered. In the following table are listed the minimum eigenvalues of \mathbf{R}_{irr} for some choices of n and h , computed with the command `min(eig(double(.)))` of MATLAB.

(n,h)	4/3	5/3	2
3	$-2.6741e - 16$	$-2.0526e - 16$	$-2.0990e - 16$
4	$-3.3300e - 16$	$-3.8006e - 16$	$-3.1009e - 16$
5	$-8.4996e - 13$	$-8.4977e - 13$	$-8.4999e - 13$
6	$-5.3231e - 13$	$-5.3221e - 13$	$-5.3187e - 13$
7	$-2.0486e - 12$	$-2.0435e - 12$	$-2.0390e - 12$
8	$-2.3581e - 12$	$-2.3571e - 12$	$-2.3566e - 12$

For $n = 3$, we still have the explicit algebraic expression of the Daubechies symbol and, thus, of the regular columns of \mathbf{Q}_j . Moreover, for fixed $h > 0$, we can compute algebraically the matrix \mathbf{S}_{irr} , which satisfies (2.24), and, thus, we have the precise explicit expression for \mathbf{R}_{irr} , which can be proved to be positive semi-definite by the Sylvester criterion. For $n = 4$, we lose the algebraic expression of the Daubechies symbol, thus, the regular columns of \mathbf{Q}_j are just approximations of the ones given by Proposition 2.27. Thus, we can only have an approximate expression of \mathbf{R}_{irr} , even if \mathbf{S}_{irr} can be still computed exactly. For $n \geq 5$, also the computation of \mathbf{S}_{irr} is approximate because the symbolic expressions involved are too long and complicated to be managed by a standard computer. These are the reasons why there is a drop in precision in the above table for $n \geq 5$. After a proper thresholding, the \mathbf{Q}_{irr} obtained from \mathbf{R}_{irr} by singular value decomposition has proved to be performing good for applications in Chapter 3.

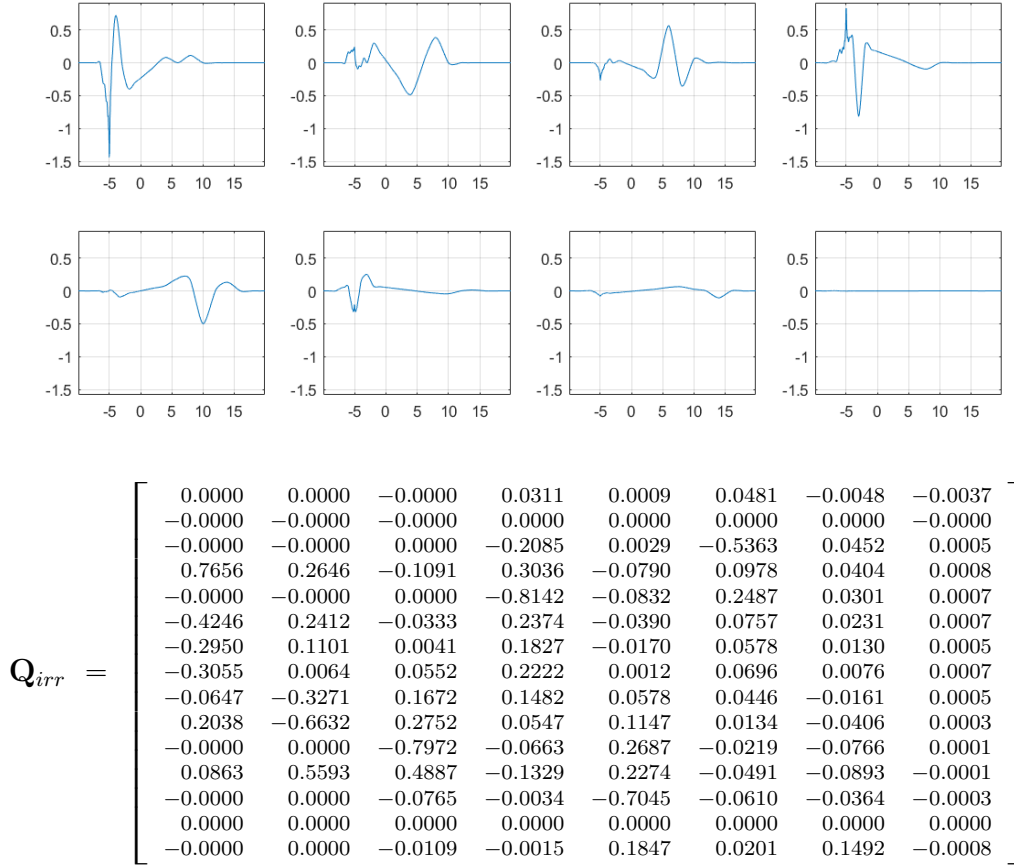


Figure 2.7: Irregular framelets for the Dubuc-Deslauriers 4-point scheme with $h = 2$ corresponding, from left to right, to the columns of a possible factorization of $\mathbf{R}_{irr} = \mathbf{Q}_{irr} \mathbf{Q}_{irr}^T$.

3 Regularity Analysis via Wavelet Tight Frames

This last chapter is devoted to the results in [7]. The goal is to develop theoretic tools (based on the wavelet tight frames constructed in Chapter 2) for the analysis of the smoothness of semi-regular subdivision schemes.

Since their appearance, wavelet systems became of crucial importance in several applications. One of them, which is of special interest to us, is the wavelet based characterization of the Besov spaces $\mathcal{B}_{p,q}^r(\mathbb{R})$, $1 \leq p, q \leq \infty$, $r > 0$, and, in particular, of the Hölder-Zygmund spaces, for $p = q = \infty$.

Theorem 3.1 ([50], Section 6.10). *Let $s \in (0, \infty)$ and $1 \leq p, q \leq \infty$. Assume*

$$\{\phi_k = \phi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi_{j,k} = 2^{(j-1)/2} \psi_{1,0}(2^{j-1} \cdot -k) : k \in \mathbb{Z}, j \in \mathbb{N}\} \subset \mathcal{C}^s(\mathbb{R})$$

is a compactly supported orthogonal wavelet system with v vanishing moments. Then, for $r \in (0, \min(s, v))$,

$$\mathcal{B}_{p,q}^r(\mathbb{R}) = \left\{ \sum_{k \in \mathbb{Z}} a_k \phi_k + \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} b_{j,k} \psi_{j,k} : \{a_k\}_{k \in \mathbb{Z}} \in \ell_p(\mathbb{Z}), \left\{ 2^{j(r + \frac{1}{2} - \frac{1}{p})} \|\{b_{j,k}\}_{k \in \mathbb{Z}}\|_{\ell_p} \right\}_{j \in \mathbb{N}} \in \ell_q(\mathbb{Z}) \right\}.$$

Due to the perfect reconstruction property (2.2), Theorem 3.1 basically asserts that if we have the information about the decay of the coefficients $\{a_k = \langle f, \phi_k \rangle : k \in \mathbb{Z}\}$ and $\{b_{j,k} = \langle f, \psi_{j,k} \rangle : j \in \mathbb{N}, k \in \mathbb{Z}\}$ of a function $f \in L^2(\mathbb{R})$, we can determine the smoothness of f . To do that however one must first compute those inner products. In the regular case we are able to do that by Proposition 1.42. However, orthogonal wavelet systems are not suitable for the semi-regular setting and we can not use Proposition 1.44 in this case. Proposition 1.44 is applicable to wavelet tight frames constructed in Chapter 2. A question then arises naturally: can we extend Theorem 3.1 to the semi-regular wavelet tight frames? The answer is yes and we prove it in what follows. We actually do more, providing a generalization of Theorem 3.1, in the case $p = q = \infty$, for the wider class of function systems

$$\mathcal{F} = \{\phi_k : k \in \mathbb{Z}\} \cup \{\psi_{j,k} : j \in \mathbb{N}, k \in \mathbb{Z}\} \quad (3.1)$$

with the following properties

(I) \mathcal{F} forms a (Parseval/normalized) tight frame for $L^2(\mathbb{R})$, i.e.

$$f = \sum_{k \in \mathbb{Z}} \langle f, \phi_k \rangle \phi_k + \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}, \quad f \in L^2(\mathbb{R}); \quad (3.2)$$

(II) there exists a constant $C_{supp} > 0$ such that

$$\begin{aligned} \sup_{k \in \mathbb{Z}} \{|\text{supp}(\phi_k)|\} &\leq C_{supp}, \\ \sup_{k \in \mathbb{Z}} \{|\text{supp}(\psi_{j,k})|\} &\leq C_{supp} 2^{-j}, \quad j \in \mathbb{N}; \end{aligned} \quad (3.3)$$

(III) there exists a constant $C_\Gamma > 0$ such that for every bounded interval $K \subset \mathbb{R}$ the sets

$$\Gamma_0(K) = \{k \in \mathbb{Z} : \text{supp}(\phi_k) \cap K \neq \emptyset\},$$

$$\Gamma_j(K) = \{k \in \mathbb{Z} : \text{supp}(\psi_{j,k}) \cap K \neq \emptyset\}, \quad j \in \mathbb{N},$$

satisfy

$$|\Gamma_j(K)| \leq C_\Gamma (2^j |K| + 1), \quad j \geq 0; \quad (3.4)$$

(IV) \mathcal{F} has $v \in \mathbb{N}$ vanishing moments, i.e.

$$\int_{\mathbb{R}} x^n \psi_{j,k}(x) dx = 0, \quad n \in \{0, \dots, v-1\}, \quad j \in \mathbb{N}, \quad k \in \mathbb{Z}, \quad (3.5)$$

and there exists a sequence of points $\{x_{j,k} : j \in \mathbb{N}, k \in \mathbb{Z}\}$ such that, for every $0 \leq r \leq v$, there exists a constant $C_{vm,r} > 0$ such that

$$\sup_{k \in \mathbb{Z}} \int_{\mathbb{R}} |x|^r |\psi_{j,k}(x + x_{j,k})| dx \leq C_{vm,r} 2^{-j(r+\frac{1}{2})}; \quad (3.6)$$

(V) $\mathcal{F} \subset \mathcal{C}^s(\mathbb{R})$, $s > 0$, and for every $0 \leq r \leq s$ there exists a constant $C_{sm,r} > 0$ such that

$$\begin{aligned} \sup_{k \in \mathbb{Z}} \{\|\phi_k\|_{C^r}\} &\leq C_{sm,r}, \\ \sup_{k \in \mathbb{Z}} \{\|\psi_{j,k}\|_{C^r}\} &\leq C_{sm,r} 2^{j(r+\frac{1}{2})}, \quad j \in \mathbb{N}. \end{aligned} \quad (3.7)$$

Remark 3.2. Note that (3.6) is implied by conditions (II) and (V) for $0 \leq r \leq s$. Indeed,

choosing $x_{j,k}$ to be the midpoint of $\text{supp}(\psi_{j,k})$, $j \in \mathbb{N}$, $k \in \mathbb{Z}$, we get

$$\begin{aligned} \int_{\mathbb{R}} |x|^r |\psi_{j,k}(x + x_{j,k})| dx &\leq C_{sm,0} \left(\frac{|\text{supp}(\psi_{j,k})|}{2} \right)^r 2^{j/2} |\text{supp}(\psi_{j,k})| \\ &\leq \frac{C_{supp}^{r+1} C_{sm,0}}{2^r} 2^{-j(r+\frac{1}{2})}. \end{aligned}$$

We state the assumptions (II) and (V) separately to emphasize their duality, which becomes even more evident in the statements of Propositions 3.5 and 3.6 in Section 3.1.1. Indeed, Theorems 3.1 and 3.3 require $0 < r < \min(s, v)$, where the value of s or v affects only one of the inclusions in either Proposition 3.5 or in Proposition 3.6. Moreover, stating (IV) and (V) separately, we can easily generalize our results to the case of dual frames with the analysis frame satisfying (II) and (IV) and the synthesis frame satisfying (III) and (V). \diamond

Properties (II)-(V) on one hand leave freedom to the tight frame to behave differently at different places along \mathbb{R} , while on the other hand they still require a sort of uniform behaviour of the frame elements with respect to the length of the supports, the density of the elements along \mathbb{R} levelwise, the vanishing moments and the smoothness. These restrictions, however, are not such a big deal, since for most wavelet tight frames used in applications, including the ones constructed in Chapter 2, they are easily satisfied. Indeed, (II) and (III) follow from Assumption 1 (c) and (d) together with (2.7), since the columns of \mathbf{Q}_j have uniformly bounded support. The structure of \mathbf{Q}_j and (2.7) are also responsible for (IV) and (V), once the considered system with v vanishing moments has been constructed from a \mathcal{C}^s subdivision scheme. Our main result, Theorem 3.3, reads as follows.

Theorem 3.3. *Let $s > 0$ and $v \in \mathbb{N}$. Assume $\mathcal{F} \subset \mathcal{C}^s(\mathbb{R})$ satisfies assumptions (I)-(V) with v vanishing moments. Then, for $r \in (0, \min(s, v))$,*

$$\mathcal{B}_{\infty,\infty}^r(\mathbb{R}) = \left\{ \sum_{k \in \mathbb{Z}} a_k \phi_k + \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} b_{j,k} \psi_{j,k} : \{a_k\}_{k \in \mathbb{Z}} \in \ell_\infty(\mathbb{Z}), \left\{ 2^{j(r+\frac{1}{2})} \|\{b_{j,k}\}_{k \in \mathbb{Z}}\|_{\ell_\infty} \right\}_{j \in \mathbb{N}} \in \ell_\infty(\mathbb{Z}) \right\}.$$

This chapter is organized as follows. In Section 3.1 we give the proof of Theorem 3.3 dividing it into two parts: Theorem 3.4, in Section 3.1.1, gives the proof of Theorem 3.3 in the case $r \in (0, \infty) \setminus \mathbb{N}$ and Theorem 3.9, in Section 3.1.2, provides the proof for $r \in \mathbb{N}$. We would like to emphasize that the results in Sections 3.1.1 and 3.1.2 are true in regular, semi-regular and even irregular cases. Theorem 3.3 implies the norm equivalence between Besov spaces $\mathcal{B}_{\infty,\infty}^r(\mathbb{R})$ and the sequence spaces $\ell_{\infty,\infty}^r$, $r \in (0, \infty)$, see Remark 3.10. The proofs in Sections 3.1.1 and 3.1.2 are reminiscent of the continuous wavelet transform techniques in [20, 50] and references therein. In Section 3.2, we illustrate our results with several examples. In particular, we use the wavelet tight frames constructed in Chapter 2, to approximate the Hölder-Zygmund regularity of semi-regular subdivision schemes based on B-splines, the family of Dubuc-Deslauriers subdivision schemes and

interpolatory radial basis functions (RBFs) based subdivision. The construction of semi-regular RBFs based schemes is a generalization of [45, 46] to the semi-regular case. We would like to point out that such semi-regular schemes could be used for blending curve pieces with different properties.

3.1 Characterization of Hölder-Zygmund Spaces $\mathcal{B}_{\infty,\infty}^r(\mathbb{R})$ via Tight Frames

3.1.1 The Case $r \in (0, \infty) \setminus \mathbb{N}$

In this section, in Theorem 3.4 we characterize the Hölder spaces $\mathcal{B}_{\infty,\infty}^r(\mathbb{R}) = \mathcal{C}^r(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for $r \in (0, \infty) \setminus \mathbb{N}$ in terms of the function system \mathcal{F} in (3.1). The proof of Theorem 3.4 follows after Propositions 3.5 and 3.6 that stress the duality between conditions (IV) and (V). Proposition 3.5, provides the inclusion " \supseteq " under assumptions (III), (V) and $r \in (0, \min(s, 1))$. Whereas Proposition 3.6 yields the other inclusion " \subseteq " under assumptions (I), (II), (IV) and $r \in (0, 1)$. The proof of Theorem 3.4 then extends the argument of Propositions 3.5 and 3.6 to the case $r > 1$, $r \notin \mathbb{N}$. Our results show that the continuous wavelet transform techniques from [20, 50] and references therein are almost directly applicable in the irregular setting.

Theorem 3.4. *Let $s \in (0, \infty)$ and $v \in \mathbb{N}$. Assume \mathcal{F} satisfies (I)-(V) with v vanishing moments. Then, for $r \in (0, \min(s, v)) \setminus \mathbb{N}$,*

$$\mathcal{B}_{\infty,\infty}^r(\mathbb{R}) = \left\{ \sum_{k \in \mathbb{Z}} a_k \phi_k + \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} b_{j,k} \psi_{j,k} : (a, b) \in \ell_{\infty,\infty}^r \text{ with } a = \{a_k\}_{k \in \mathbb{Z}}, b = \{b_{j,k}\}_{j \in \mathbb{N}, k \in \mathbb{Z}} \right\}.$$

We start by proving the following result.

Proposition 3.5. *Let $s > 0$. Assume \mathcal{F} satisfies (III) and (V). Then, for $r \in (0, \min(s, 1))$,*

$$\mathcal{B}_{\infty,\infty}^r(\mathbb{R}) \supseteq \left\{ \sum_{k \in \mathbb{Z}} a_k \phi_k + \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} b_{j,k} \psi_{j,k} : (a, b) \in \ell_{\infty,\infty}^r \text{ with } a = \{a_k\}_{k \in \mathbb{Z}}, b = \{b_{j,k}\}_{j \in \mathbb{N}, k \in \mathbb{Z}} \right\}.$$

Proof. We consider $f(x) = f_0(x) + g(x)$, $x \in \mathbb{R}$, where

$$f_0(x) = \sum_{k \in \mathbb{Z}} a_k \phi_k(x) \quad \text{and} \quad g(x) = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} b_{j,k} \psi_{j,k}(x), \quad (3.8)$$

with finite

$$C_a := \sup_{k \in \mathbb{Z}} \{|a_k|\} \quad \text{and} \quad C_b := \sup_{j \in \mathbb{N}} 2^{j(r+\frac{1}{2})} \sup_{k \in \mathbb{Z}} \{|b_{j,k}|\}. \quad (3.9)$$

Since on every open bounded interval in \mathbb{R} the sum defining f_0 is finite due to (III), we

have $f_0 \in \mathcal{C}^s(\mathbb{R}) \subseteq \mathcal{C}^r(\mathbb{R})$ due to $r < s$. Moreover, by (III) and (V), we obtain

$$\|f_0\|_{L^\infty} \leq C_a C_\Gamma C_{sm,0} < \infty. \quad (3.10)$$

Analogously, since $r > 0$, we have

$$\|g\|_{L^\infty} \leq C_b C_\Gamma C_{sm,0} \sum_{j \in \mathbb{N}} 2^{-jr} < \infty, \quad (3.11)$$

thus, $f \in L^\infty(\mathbb{R})$. Let $x, h \in \mathbb{R}$. By (3.9), we get

$$\begin{aligned} |g(x+h) - g(x)| &\leq \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} |b_{j,k}| |\psi_{j,k}(x+h) - \psi_{j,k}(x)| \\ &\leq C_b |h|^r \sum_{j \in \mathbb{N}} \frac{2^{-j(r+\frac{1}{2})}}{|h|^r} \sum_{k \in \mathbb{Z}} |\psi_{j,k}(x+h) - \psi_{j,k}(x)|. \end{aligned}$$

Since there exists $J \in \mathbb{Z}$ such that

$$2^{-J} < |h| \leq 2^{-J+1},$$

we have

$$\begin{aligned} |g(x+h) - g(x)| &\leq C_b |h|^r \sum_{j \in \mathbb{N}} 2^{(J-j)r-j/2} \sum_{k \in \mathbb{Z}} |\psi_{j,k}(x+h) - \psi_{j,k}(x)| \\ &= C_b |h|^r (A + B), \end{aligned}$$

where

$$\begin{aligned} A &= \sum_{j=1}^{J-1} 2^{(J-j)r-j/2} \sum_{k \in \mathbb{Z}} |\psi_{j,k}(x+h) - \psi_{j,k}(x)| \quad \text{and} \\ B &= \sum_{j=J}^{\infty} 2^{(J-j)r-j/2} \sum_{k \in \mathbb{Z}} |\psi_{j,k}(x+h) - \psi_{j,k}(x)|. \end{aligned} \quad (3.12)$$

If $J \leq 1$, $A = 0$. Otherwise, for every $\epsilon > 0$ with $r < r + \epsilon < \min(s, 1)$, due to (V) we have

$$|\psi_{j,k}(x+h) - \psi_{j,k}(x)| \leq C_{sm,r+\epsilon} 2^{j(r+\epsilon+\frac{1}{2})} |h|^{r+\epsilon} \leq C_{sm,r+\epsilon} 2^{(j-J)(r+\epsilon)} 2^{j/2} 2^{r+\epsilon},$$

and, by (III), the sum in A over k has at most

$$|\Gamma_j(x+h)| + |\Gamma_j(x)| \leq 2 C_\Gamma$$

non-zero elements. Thus,

$$A \leq 2^{r+\epsilon+1} C_\Gamma C_{sm,r+\epsilon} \sum_{j=1}^{J-1} (2^{-\epsilon})^{(J-j)} \leq 4 C_\Gamma C_{sm,r+\epsilon} \sum_{j=1}^{J-1} (2^{-\epsilon})^j. \quad (3.13)$$

Therefore, since $\epsilon > 0$, A is bounded. To conclude the proof, we observe that, by (V),

$$B \leq 2 C_\Gamma C_{sm,0} \sum_{j=J}^{\infty} (2^{-r})^{j-J} = 2 C_\Gamma C_{sm,0} \frac{1}{1-2^{-r}}.$$

Thus $|g(x+h) - g(x)|/|h|^r$ is uniformly bounded in x and h , which leads to $g \in \mathcal{B}_{\infty,\infty}^r(\mathbb{R})$ and $f \in \mathcal{B}_{\infty,\infty}^r(\mathbb{R})$ with $\|f\|_{\mathcal{B}_{\infty,\infty}^r} \leq C \|(a,b)\|_{\ell_{\infty,\infty}^r}$ for some constant $C > 0$. \square

Next, we give a proof of Proposition 3.6.

Proposition 3.6. *Assume $\mathcal{F} \subset \mathcal{C}^0(\mathbb{R})$ with uniformly bounded $\{\phi_k : k \in \mathbb{Z}\}$ satisfies (I), (II) and (IV) with 1 vanishing moment. Then, for $r \in (0, 1)$,*

$$\mathcal{B}_{\infty,\infty}^r(\mathbb{R}) \subseteq \left\{ \sum_{k \in \mathbb{Z}} a_k \phi_k + \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} b_{j,k} \psi_{j,k} : (a,b) \in \ell_{\infty,\infty}^r \text{ with } a = \{a_k\}_{k \in \mathbb{Z}}, b = \{b_{j,k}\}_{j \in \mathbb{N}, k \in \mathbb{Z}} \right\}.$$

Proof. Consider $f \in \mathcal{B}_{\infty,\infty}^r(\mathbb{R}) \cap L^2(\mathbb{R})$. We choose a representative of f in (3.2) with coefficients $a_k = \langle f, \phi_k \rangle$ and $b_{j,k} = \langle f, \psi_{j,k} \rangle$. On one hand, due to (II) and the uniform boundedness of Φ , there exists $C_\phi > 0$ such that

$$|a_k| \leq C_\phi \|f\|_\infty, \quad k \in \mathbb{Z}. \quad (3.14)$$

On the other hand, with $x_{j,k}$ as in (IV), we can exploit the vanishing moment of the tight frame and the regularity of f to get

$$\begin{aligned} |b_{j,k}| &= \left| \int_{\mathbb{R}} f(x) \psi_{j,k}(x) dx \right| = \left| \int_{\mathbb{R}} (f(x) - f(x_{j,k})) \psi_{j,k}(x) dx \right| \\ &\leq \|f\|_{\mathcal{B}_{\infty,\infty}^r} \int_{\mathbb{R}} |x - x_{j,k}|^r |\psi_{j,k}(x)| dx \\ &= \|f\|_{\mathcal{B}_{\infty,\infty}^r} \int_{\mathbb{R}} |x|^r |\psi_{j,k}(x + x_{j,k})| dx \leq 2^{-j(r+\frac{1}{2})} C_{vm,r} \|f\|_{\mathcal{B}_{\infty,\infty}^r} \end{aligned} \quad (3.15)$$

For a general $f \in \mathcal{B}_{\infty,\infty}^r(\mathbb{R})$ the claim follows by a density argument. Thus, there exists a constant $C > 0$ such that $\|f\|_{\mathcal{B}_{\infty,\infty}^r} \geq C \|(a,b)\|_{\ell_{\infty,\infty}^r}$. \square

Remark 3.7. In Proposition 3.6, there is no need for the tight frame to be more than continuous: only the vanishing moment matters. The same phenomenon happens for the inclusion \subseteq in Theorem 3.4 - the number of vanishing moments being the key ingredient for its proof. On the other hand, the regularity of the wavelet tight frame \mathcal{F} plays the

key role both in Proposition 3.5 and in the proof of the inclusion \supseteq in Theorem 3.4. This explains the duality between assumptions (IV) and (V). \diamond

We are now ready to complete the proof of Theorem 3.4.

Proof of Theorem 3.4. For the case $r \in (0, 1)$ see Propositions 3.5 and 3.6. Let $r = n + \alpha$, with $n \in \mathbb{N}$ and $\alpha \in (0, 1)$.

1st step, proof of “ \supseteq ”: similarly to Proposition 3.5, we define constants C_a and C_b as in (3.9) and make use of the estimates in (3.10) and (3.11) to conclude that $f \in L^\infty(\mathbb{R})$. The next step is to show the existence of the n -th derivative $g^{(n)}$ of g in (3.8). This follows by uniform convergence since, for every $x \in \mathbb{R}$ and $0 \leq \ell \leq n < r$, by (V), we have

$$\left| \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} b_{j,k} \psi_{j,k}^{(\ell)}(x) \right| \leq \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} |b_{j,k}| |\psi_{j,k}^{(\ell)}(x)| \leq C_b C_\Gamma C_{sm,\ell} \sum_{j \in \mathbb{N}} 2^{-j(r-\ell)} < \infty,$$

The same argument as in Proposition 3.5 leads to $g^{(n)} \in \mathcal{C}^\alpha(\mathbb{R})$ and, thus, $f \in \mathcal{C}^r(\mathbb{R})$.

2nd step, proof of “ \subseteq ” resembles [43]: similarly to Proposition 3.6, we consider $f \in \mathcal{B}_{\infty,\infty}^r(\mathbb{R}) \cap L^2(\mathbb{R})$ and the uniform bound for $|\langle f, \phi_k \rangle|$ is obtained as in (3.14). Exploiting the first n vanishing moments of the tight frame we have

$$\begin{aligned} |\langle f, \psi_{j,k} \rangle| &= \left| \int_{\mathbb{R}} f(x) \psi_{j,k}(x) dx \right| \\ &= \left| \int_{\mathbb{R}} \left(f(x) - f(x_{j,k}) - \sum_{\ell=1}^{n-1} \frac{f^{(\ell)}(x_{j,k})}{\ell!} (x - x_{j,k})^\ell \right) \psi_{j,k}(x) dx \right|, \end{aligned}$$

where the $x_{j,k}$ are as in (IV). Using the property of the Taylor expansion of f centered in $x_{j,k}$ with the Lagrange remainder term, we have that, for every $x \in \mathbb{R}$, there exists a measurable $\xi(x) \in \mathbb{R}$, with $|\xi(x) - x_{j,k}| \leq |x - x_{j,k}|$, such that

$$|\langle f, \psi_{j,k} \rangle| \leq \left| \int_{\mathbb{R}} \frac{f^{(n)}(\xi(x))}{n!} (x - x_{j,k})^n \psi_{j,k}(x) dx \right|.$$

Now we can exploit $n + 1$ vanishing moments, the Hölder regularity α of $f^{(n)}$ and (IV)

to get

$$\begin{aligned}
 |\langle f, \psi_{j,k} \rangle| &\leq \frac{1}{n!} \left| \int_{\mathbb{R}} (f^{(n)}(\xi(x)) - f^{(n)}(x_{j,k})) (x - x_{j,k})^n \psi_{j,k}(x) dx \right| \\
 &\leq \frac{\|f^{(n)}\|_{\mathcal{B}_{\infty,\infty}^\alpha}}{n!} \int_{\mathbb{R}} |\xi(x) - x_{j,k}|^\alpha |x - x_{j,k}|^n |\psi_{j,k}(x)| dx \\
 &\leq \frac{\|f^{(n)}\|_{\mathcal{B}_{\infty,\infty}^\alpha}}{n!} \int_{\mathbb{R}} |x - x_{j,k}|^r |\psi_{j,k}(x)| dx \\
 &\leq \frac{\|f^{(n)}\|_{\mathcal{B}_{\infty,\infty}^\alpha}}{n!} \int_{\mathbb{R}} |y|^r |\psi_{j,k}(y + x_{j,k})| dy \leq 2^{-j(r+\frac{1}{2})} \frac{C_{vm,r} \|f^{(n)}\|_{\mathcal{B}_{\infty,\infty}^\alpha}}{n!}.
 \end{aligned} \tag{3.16}$$

Thus, the claim follows. \square

Remark 3.8. If $s \in \mathbb{N}$ and $v \geq s$ in Theorem 3.4, then the wavelet tight frame \mathcal{F} does not need to belong to $\mathcal{C}^s(\mathbb{R})$. It suffices to have $\mathcal{F} \subset \mathcal{C}^{s-1}(\mathbb{R})$ with the $(s-1)$ -st derivatives of its elements being Lipschitz-continuous. \diamond

3.1.2 The Case $r \in \mathbb{N}$

It is well known [50] that the Hölder spaces with integer Hölder exponents cannot be characterized via either a wavelet or a wavelet tight frame system \mathcal{F} . Indeed, if $r = 1$, the estimate (3.13) does not follow from (3.9). Thus, similarly to Theorem 3.1, the natural spaces in this context are the Hölder-Zygmund spaces $\mathcal{B}_{\infty,\infty}^r(\mathbb{R})$ for $r \in \mathbb{N}$. This section is devoted to the proof of the wavelet tight frame characterization of such spaces, see Theorem 3.9. The results of Theorems 3.4 and 3.9 yield Theorem 3.3.

Theorem 3.9. *Let $s > 0$ and $v \in \mathbb{N}$. Assume $\mathcal{F} \subset \mathcal{C}^s(\mathbb{R})$ satisfies (I)-(V) with v vanishing moments. Then, for $0 < r < \min(s, v)$, $r \in \mathbb{N}$,*

$$\mathcal{B}_{\infty,\infty}^r(\mathbb{R}) = \left\{ \sum_{k \in \mathbb{Z}} a_k \phi_k + \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} b_{j,k} \psi_{j,k} : (a, b) \in \ell_{\infty,\infty}^r \text{ with } a = \{a_k\}_{k \in \mathbb{Z}}, b = \{b_{j,k}\}_{j \in \mathbb{N}, k \in \mathbb{Z}} \right\}.$$

The significant case is when $r = 1$, since for all other integers one usually argues similarly to the proof of Theorem 3.4. In the case $r = 1$, for the inclusion \supseteq we use the argument similar to the one in Proposition 3.5. On the other hand, for the inclusion \subseteq , we cannot exploit the vanishing moments as done in (3.15). To circumvent this problem, inspired by [58], we consider an auxiliary orthogonal wavelet system which satisfies the assumptions of Theorem 3.1. This way we get a convenient expansion for $f \in \mathcal{B}_{\infty,\infty}^r(\mathbb{R})$ and make use of the wavelet characterization of $\mathcal{B}_{\infty,\infty}^\alpha(\mathbb{R})$ for $\alpha \in (r, s) \setminus \mathbb{N}$ in Theorem 3.4.

Proof. We only prove the claim for $r = 1 < \min(s, v)$. In this case $\mathcal{B}_{\infty,\infty}^r(\mathbb{R}) = \Lambda(\mathbb{R}) \cap L^\infty(\mathbb{R})$. The general case follows using an argument similar to the one in the proof of Theorem 3.4.

1st step, proof of “ \supseteq ”: similarly to Proposition 3.5, we define constants C_a and C_b as in (3.9) and make use of the estimates in (3.10) and (3.11) to conclude that $f \in L^\infty(\mathbb{R})$. Let $x \in \mathbb{R}$. It suffices to consider $h > 0$. Then, for $r = 1$, we obtain

$$\begin{aligned} & |g(x+h) - 2g(x) + g(x-h)| \leq \\ & \leq \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} |b_{j,k}| |\psi_{j,k}(x+h) - 2\psi_{j,k}(x) + \psi_{j,k}(x-h)| \\ & \leq C_b h \sum_{j \in \mathbb{N}} \frac{2^{-j\frac{3}{2}}}{h} \sum_{k \in \mathbb{Z}} |\psi_{j,k}(x+h) - 2\psi_{j,k}(x) + \psi_{j,k}(x-h)|. \end{aligned}$$

Since there exists $J \in \mathbb{Z}$ such that

$$2^{-J} < h \leq 2^{-J+1},$$

we have

$$\begin{aligned} & |g(x+h) - 2g(x) + g(x-h)| \leq \\ & \leq C_b h \sum_{j \in \mathbb{N}} 2^{J-j\frac{3}{2}} \sum_{k \in \mathbb{Z}} |\psi_{j,k}(x+h) - 2\psi_{j,k}(x) + \psi_{j,k}(x-h)|. \end{aligned}$$

To estimate $|g(x+h) - 2g(x) + g(x-h)|$, we consider

$$A = \sum_{j=1}^{J-1} 2^{J-j\frac{3}{2}} \sum_{k \in \mathbb{Z}} |\psi_{j,k}(x+h) - 2\psi_{j,k}(x) + \psi_{j,k}(x-h)|$$

and

$$B = \sum_{j=J}^{\infty} 2^{J-j\frac{3}{2}} \sum_{k \in \mathbb{Z}} |\psi_{j,k}(x+h) - 2\psi_{j,k}(x) + \psi_{j,k}(x-h)|.$$

If $J \leq 1$, $A = 0$. Otherwise, since the tight frame \mathcal{F} belongs to $\mathcal{C}^s(\mathbb{R})$, $s > 1$, we use the mean value theorem twice for every framelet and find $\xi_{j,k}(x) \in [x, x+h]$ and $\eta_{j,k}(x) \in [x-h, x]$ such that

$$\begin{aligned} & |\psi_{j,k}(x+h) - 2\psi_{j,k}(x) + \psi_{j,k}(x-h)| = \\ & = |\psi_{j,k}(x+h) - \psi_{j,k}(x) - (\psi_{j,k}(x) - \psi_{j,k}(x-h))| \\ & = h |\psi'_{j,k}(\xi_{j,k}(x)) - \psi'_{j,k}(\eta_{j,k}(x))|. \end{aligned}$$

Now, for $\epsilon > 0$ with $r = 1 < 1 + \epsilon < s$, using (V) we get

$$\begin{aligned} |\psi_{j,k}(x+h) - 2\psi_{j,k}(x) + \psi_{j,k}(x-h)| &= \\ &= C_{sm,1+\epsilon} 2^{j(\epsilon+\frac{3}{2})} h |\xi_{j,k}(x) - \eta_{j,k}(x)|^\epsilon \\ &\leq C_{sm,1+\epsilon} 2^{j(\epsilon+\frac{3}{2})+\epsilon} h^{1+\epsilon} \leq C_{sm,1+\epsilon} 2^{(j-J)(1+\epsilon)+\frac{j}{2}+1+2\epsilon}. \end{aligned}$$

Moreover, the sum in A over k has at most

$$|\Gamma_j(x+h)| + |\Gamma_j(x-h)| + |\Gamma_j(x)| \leq 3C_\Gamma$$

non-zero summands and, thus, we get

$$A \leq C_\Gamma C_{sm,1+\epsilon} 2^{1+2\epsilon} 3 \sum_{j=1}^{J-1} (2^{-\epsilon})^{J-j}.$$

Since $\epsilon > 0$, A is bounded. To conclude the proof, we observe that

$$B \leq C_\Gamma C_{sm,0} 3 \sum_{j=J}^{\infty} 2^{J-j} = 6C_\Gamma C_{sm,0}.$$

Thus, $g \in \mathcal{B}_{\infty,\infty}^1(\mathbb{R})$ and, therefore, $f \in \mathcal{B}_{\infty,\infty}^1(\mathbb{R})$ with $\|f\|_{\mathcal{B}_{\infty,\infty}^1} \leq C\|(a,b)\|_{\ell_{\infty,\infty}^1}$ for some constant $C > 0$.

2nd step, proof of “ \subseteq ”: similarly to Proposition 3.6 we only consider $f \in \Lambda(\mathbb{R}) \cap L^2(\mathbb{R})$. The uniform bound for $|\langle f, \phi_k \rangle|$ is obtained similarly to (3.14). To obtain the bound for $|\langle f, \psi_{j,k} \rangle|$, we let $\tilde{\Phi} \cup \{\tilde{\Psi}_\ell\}_{\ell \in \mathbb{N}} \subseteq \mathcal{C}^s(\mathbb{R})$ be an auxiliary compactly supported orthogonal wavelet system with v vanishing moments (e.g. Daubechies $2n$ -tap wavelets [20] with large enough $n \in \mathbb{N}$). $\tilde{\Phi} \cup \{\tilde{\Psi}_\ell\}_{\ell \in \mathbb{N}}$ satisfies the assumptions of Theorem 3.1 and fulfills (I)-(V), with appropriate $\tilde{\Gamma}_j$ and $\tilde{C}_{supp} > 0$, $\tilde{C}_\Gamma > 0$, $\tilde{C}_{vm,r} > 0$ and $\tilde{C}_{sm,r} > 0$. Then, from (3.2), we have $f(x) = \tilde{f}_0(x) + \tilde{g}(x)$, $x \in \mathbb{R}$, where

$$\tilde{f}_0(x) = \sum_{m \in \mathbb{Z}} \langle f, \tilde{\phi}_m \rangle \tilde{\phi}_m(x) \quad \text{and} \quad \tilde{g}(x) = \sum_{\ell \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \langle f, \tilde{\psi}_{\ell,m} \rangle \tilde{\psi}_{\ell,m}(x).$$

Thus, for every $j \in \mathbb{N}$ and $k \in \mathbb{Z}$, we get

$$|\langle f, \psi_{j,k} \rangle| \leq |\langle \tilde{f}_0, \psi_{j,k} \rangle| + \sum_{\ell \in \mathbb{N}} \sum_{m \in \mathbb{Z}} |\langle f, \tilde{\psi}_{\ell,m} \rangle| |\langle \tilde{\psi}_{\ell,m}, \psi_{j,k} \rangle|.$$

Let $\alpha \in (1, s) \setminus \mathbb{N}$. Since both tight frames belong to $\mathcal{C}^s(\mathbb{R}) \subseteq \mathcal{C}^\alpha(\mathbb{R})$, the function \tilde{f}_0 , which is locally the finite sum of \mathcal{C}^α -functions, belongs to $\mathcal{C}^\alpha(\mathbb{R})$, and, by Theorem 3.4,

there exists $C_1 > 0$, such that

$$\sup_{k \in \mathbb{Z}} |\langle \tilde{f}_0, \psi_{j,k} \rangle| \leq C_1 2^{-j(\alpha + \frac{1}{2})} \leq C_1 2^{-j\frac{3}{2}}, \quad j \in \mathbb{N}.$$

Moreover, by Theorem 3.1 and due to $f \in \Lambda(\mathbb{R})$, there exists $C_2 > 0$ such that

$$\sup_{m \in \mathbb{Z}} |\langle f, \tilde{\psi}_{\ell,m} \rangle| \leq C_2 2^{-\ell\frac{3}{2}}, \quad \ell \in \mathbb{N}.$$

Thus,

$$\begin{aligned} |\langle f, \psi_{j,k} \rangle| &\leq C_1 2^{-j\frac{3}{2}} + C_2 \sum_{\ell \in \mathbb{N}} 2^{-\ell\frac{3}{2}} \sum_{m \in \mathbb{Z}} |\langle \tilde{\psi}_{\ell,m}, \psi_{j,k} \rangle| \\ &= C_1 2^{-j\frac{3}{2}} + C_2 \left(\sum_{\ell=1}^{j-1} 2^{-\ell\frac{3}{2}} \sum_{m \in \mathbb{Z}} |\langle \tilde{\psi}_{\ell,m}, \psi_{j,k} \rangle| + \sum_{\ell=j}^{\infty} 2^{-\ell\frac{3}{2}} \sum_{m \in \mathbb{Z}} |\langle \tilde{\psi}_{\ell,m}, \psi_{j,k} \rangle| \right) \\ &= C_1 2^{-j\frac{3}{2}} + C_2 (A + B). \end{aligned} \quad (3.17)$$

The sums in (3.17) over m have at most

$$|\tilde{\Gamma}_{\ell}(\text{supp}(\psi_{j,k}))| \leq \tilde{C}_{\Gamma} (2^{\ell} |\text{supp}(\psi_{j,k})| + 1) \leq \tilde{C}_{\Gamma} (C_{\text{supp}} 2^{\ell-j} + 1)$$

non-zero summands. When $\ell < j$, by assumption (V) for $\tilde{\psi}_{\ell,m}$ and (3.16) with $f = \tilde{\psi}_{\ell,m}$, due to Theorem 3.4, we have

$$|\langle \tilde{\psi}_{\ell,m}, \psi_{j,k} \rangle| \leq C_{vm,\alpha} 2^{-j(\alpha + \frac{1}{2})} \|\tilde{\psi}_{\ell,m}\|_{C^{\alpha}} \leq C_{vm,\alpha} \tilde{C}_{sm,\alpha} 2^{(\ell-j)(\alpha + \frac{1}{2})},$$

uniformly in m and k . Thus, substituting $\ell' = j - \ell$, we obtain

$$\begin{aligned} A &\leq C_{vm,\alpha} \tilde{C}_{\Gamma} \tilde{C}_{sm,\alpha} \sum_{\ell=1}^{j-1} 2^{-\ell\frac{3}{2}} 2^{(\ell-j)(\alpha + \frac{1}{2})} (C_{\text{supp}} 2^{\ell-j} + 1) \\ &= C_{vm,\alpha} \tilde{C}_{\Gamma} \tilde{C}_{sm,\alpha} 2^{-j\frac{3}{2}} \sum_{\ell'=1}^{j-1} 2^{-\ell'(\alpha-1)} (C_{\text{supp}} 2^{-\ell'} + 1) \\ &\leq C_3 2^{-j\frac{3}{2}}, \end{aligned} \quad (3.18)$$

for some $C_3 > 0$, due to the fact that $\alpha > 1$.

On the other hand, when $\ell \geq j$, using (II) and (V), we get

$$|\langle \tilde{\psi}_{\ell,m}, \psi_{j,k} \rangle| \leq C_{sm,0} \tilde{C}_{sm,0} 2^{(j+\ell)/2} \min(C_{\text{supp}} 2^{-j}, \tilde{C}_{\text{supp}} 2^{-\ell}) = C_4 2^{(j-\ell)/2},$$

uniformly in m and k . Thus, after the substitution $\ell' = \ell - j$, we obtain

$$\begin{aligned} B &\leq C_4 \tilde{C}_\Gamma \sum_{\ell=j}^{\infty} 2^{-\ell \frac{3}{2}} (C_{supp} 2^{\ell-j} + 1) 2^{(j-\ell)/2} \\ &= C_4 \tilde{C}_\Gamma 2^{-j \frac{3}{2}} \sum_{m=0}^{\infty} 2^{-2m} (C_{supp} 2^m + 1) \leq C_5 2^{-j \frac{3}{2}} \end{aligned} \quad (3.19)$$

for some constant $C_5 > 0$. Combining (3.17), (3.18) and (3.19) we finally get

$$\sup_{k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle| \leq (C_1 + C_2 C_3 + C_2 C_5) 2^{-j \frac{3}{2}}, \quad j \in \mathbb{N}.$$

Thus, the claim follows, i.e., there exists a constant $C > 0$ such that $\|f\|_{\mathcal{B}_{\infty,\infty}^1} \geq C \|(a, b)\|_{\ell_{\infty,\infty}^1}$. \square

Remark 3.10. The norm equivalence between the Besov norm $\|\cdot\|_{\mathcal{B}_{\infty,\infty}^r}$ and $\|\cdot\|_{\ell_{\infty,\infty}^r}$, $r \in (0, \infty)$, is a consequence of Theorem 3.3 and the Open Mapping Theorem. \diamond

3.2 Approximation of the Optimal Hölder-Zygmund Exponent

In this section, we show how to apply Theorem 3.3 for estimating the Hölder-Zygmund regularity of a semi-regular subdivision scheme from the decay of the frame coefficients of its basic limit functions with respect to a given tight frame \mathcal{F} satisfying (I)-(V) for some $s > 0$ and $v \in \mathbb{N}$. In Section 3.2.1, in a general irregular setting, we discuss how to obtain such regularity estimates using the result of Theorem 3.3. In Section 3.2.2, we describe how to compute the frame coefficients in the semi-regular case using Proposition 1.44. Lastly, in Section 3.2.3, we illustrate our results with examples of semi-regular schemes such as B-spline, Dubuc-Deslauriers subdivision and interpolatory schemes based on radial basis functions. The latter example in the regular setting reduces to the construction in [45].

3.2.1 Two Methods for the Estimation of the Optimal Hölder-Zygmund Exponent

Definition 3.11. Let $f \in L^\infty(\mathbb{R})$. We call *optimal Hölder-Zygmund (smoothness) exponent of f* the real number

$$r(f) = \sup\{r > 0 : f \in \mathcal{B}_{\infty,\infty}^r(\mathbb{R})\}.$$

Assume that $r(f) \in (0, \min(s, v))$ and that we are given

$$\gamma_j = \sup_{k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|, \quad j \in \mathbb{N}.$$

By Theorem 3.3, for every $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that, for every $j \in \mathbb{N}$,

$$\gamma_j \leq C_\epsilon 2^{-j(r(f) - \epsilon + \frac{1}{2})}, \quad \text{i.e.} \quad j \left(r(f) - \epsilon + \frac{1}{2} \right) - \log_2(C_\epsilon) \leq -\log_2(\gamma_j). \quad (3.20)$$

From (3.20) we infer that searching for $r(f)$ is equivalent to searching for the largest slope of a line lying under the set of points $\{(j, -\log_2(\gamma_j))\}_{j \in \mathbb{N}}$. With this interpretation in mind, the natural approach (see e.g. [20]) to approximate $r(f)$ is to compute the real-valued sequence $\{r_n(f)\}_{n \in \mathbb{N}}$, where $r_n(f) - 1/2$ is the slope of the regression line for the points $\{(j, -\log_2(\gamma_j))\}_{j=1}^{n+1}$. This method is robust, i.e. for larger n the contributions of the levels $j \geq n$ become less significant, thus, the difference between $r_n(f)$ and $r_{n+1}(f)$ is small and we are able to estimate the overall distribution of $\{(j, -\log_2(\gamma_j))\}_{j \in \mathbb{N}}$. However, examples in Subsection 3.2.3 illustrate that the convergence of $\{r_n(f)\}_{n \in \mathbb{N}}$ towards $r(f)$ is very slow. One of the main reasons for such a behavior is the value of the unknown C_ϵ , which can be significant, e.g. when $f \notin \mathcal{B}_{\infty, \infty}^{r(f)}(\mathbb{R})$.

An alternative approach for estimating the Hölder-Zygmund exponent is given by the following Proposition.

Proposition 3.12. *Let $r(f)$ be the optimal Hölder-Zygmund exponent of $f \in \mathcal{C}^0(\mathbb{R})$. If*

$$0 < r^*(f) = \lim_{n \rightarrow \infty} \log_2 \left(\frac{\gamma_n}{\gamma_{n+1}} \right) - \frac{1}{2} < \min(s, v),$$

then $r^(f) = r(f)$.*

Proof. We first prove that $r^*(f) \leq r(f)$ and then, by contradiction, that $r^*(f) = r(f)$. Let $\epsilon > 0$. We consider the series

$$S(r^*(f) - \epsilon) = \sum_{j=1}^{\infty} 2^{j(r^*(f) - \epsilon + \frac{1}{2})} \gamma_j. \quad (3.21)$$

By the assumption, we obtain

$$\lim_{n \rightarrow \infty} \frac{2^{(n+1)(r^*(f) - \epsilon + \frac{1}{2})} \gamma_{n+1}}{2^{n(r^*(f) - \epsilon + \frac{1}{2})} \gamma_n} = 2^{r^*(f) - \epsilon + \frac{1}{2}} \lim_{n \rightarrow \infty} \frac{\gamma_{n+1}}{\gamma_n} = 2^{-\epsilon}. \quad (3.22)$$

Thus, by the ratio test, the series $S(r^*(f) - \epsilon)$ in (3.21) converges for every $\epsilon > 0$. Consequently, the non-negative summands of $S(r^*(f) - \epsilon)$ are uniformly bounded, i.e. there exists $C_\epsilon > 0$ such that

$$\gamma_j \leq C_\epsilon 2^{-j(r^*(f) - \epsilon + \frac{1}{2})}, \quad j \in \mathbb{N}.$$

Therefore, by Definition 3.11 and by (3.20), $r^*(f) \leq r(f)$.

On the other hand, by (3.20), if $r^*(f) < r(f)$, then there exists $\delta > 0$ and a constant $C_\delta > 0$ such that

$$\gamma_j \leq C_\delta 2^{-j(r^*(f) + \delta + \frac{1}{2})}, \quad j \in \mathbb{N}.$$

Therefore, similarly to (3.22), we obtain that the series $S(r^*(f) + \delta/2)$ diverges and at the same time is bounded from above by

$$S(r^*(f) + \delta/2) = \sum_{j=1}^{\infty} 2^{j(r^*(f) + \frac{\delta}{2} + \frac{1}{2})} \gamma_j \leq C_\delta \sum_{j=1}^{\infty} 2^{-j\delta/2}.$$

Thus, due to this contradiction, $r^*(f) = r(f)$. \square

The advantage of the approach in Proposition 3.12 is that it eliminates the effect of the constant C_ϵ in (3.20). Even though the existence of $r^*(f)$ is not guaranteed and the elements of the sequence

$$r_n^*(f) = \log_2 \left(\frac{\gamma_n}{\gamma_{n+1}} \right) - \frac{1}{2}, \quad n \in \mathbb{N},$$

can oscillate wildly, our numerical experiments in Subsection 3.2.3 provide examples which illustrate the cases when $\{r_n^*(f)\}_{n \in \mathbb{N}}$ converges to $r(f)$ rapidly. The convergence in these examples is much faster than that of the linear regression method.

Remark 3.13. The series in (3.21) with $\epsilon = 0$ becomes

$$S(r) = \left\| \left\{ 2^{j(r + \frac{1}{2})} \left\| \{ \langle f, \psi_{j,k} \rangle \}_{k \in \mathbb{Z}} \right\|_{\ell^\infty} \right\}_{j \in \mathbb{N}} \right\|_{\ell^1}.$$

This norm appears in the characterization of $\mathcal{B}_{\infty,1}^r(\mathbb{R})$ in Theorem 3.1 and corresponds to $(a, b) \in \ell_{\infty,1}^r$, $r \in (0, \infty)$. Even if the case $p = \infty$ and $q = 1$ is not covered by Theorem 3.3, this observation is consistent with $\mathcal{B}_{p,q_1}^r(\mathbb{R}) \subseteq \mathcal{B}_{p,q_2}^r(\mathbb{R})$ for $q_1 \leq q_2$. \diamond

3.2.2 Computation of Frame Coefficients

Conditions (I)-(V) do not require the semi-regularity of the mesh and all the above results hold even in the irregular case. To use the presented results in practice, however, we need an efficient method for computing the frame coefficients

$$\{a_k = \langle f, \phi_k \rangle\}_{k \in \mathbb{N}} \quad \text{and} \quad \{b_{j,k} = \langle f, \psi_{j,k} \rangle\}_{j \in \mathbb{N}, k \in \mathbb{Z}}.$$

If \mathcal{F} as in (2.7) is a wavelet tight frame obtained from a convergent subdivision scheme with subdivision matrix \mathbf{P} and basic limit functions $\{\varphi_k\}_{k \in \mathbb{Z}}$ as in Chapter 2 and the function f we want to analyse is a limit function of another convergent semi-regular subdivision scheme with subdivision matrix \mathbf{Z} and basic limit functions $\{\zeta_k\}_{k \in \mathbb{Z}}$, we can do it in a rather simple way exploiting Proposition 1.44 and the following result.

Proposition 3.14. *For every $j \in \mathbb{N}$, $i, k \in \mathbb{Z}$, we have $\langle \zeta_k, \psi_{j,m} \rangle = \mathbf{C}_j(k, m)$, where*

$$\mathbf{C}_j = 2^{-j/2} (\mathbf{Z}^j)^T \mathbf{G} \mathbf{D}^{-1/2} \mathbf{Q}_j,$$

with the cross-Gramian $\mathbf{G} = [\langle \zeta_k, \varphi_m \rangle]_{k,m \in \mathbb{Z}}$, \mathbf{D} as in (2.6) and \mathbf{Q}_j are the matrices defining the wavelet tight frame in (2.7).

Proof. Applying (2.7), the substitution $y = 2^j x$, (1.17) and (2.5), we get

$$\begin{aligned} \int_{\mathbb{R}} [\zeta_k(x)]_{k \in \mathbb{Z}} \Psi_j(x)^T dx &= 2^{j/2} \int_{\mathbb{R}} [\zeta_k(x)]_{k \in \mathbb{Z}} \Phi_0(2^j x)^T dx \mathbf{Q}_j \\ &= 2^{-j/2} \int_{\mathbb{R}} [\zeta_k(2^{-j} y)]_{k \in \mathbb{Z}} \Phi_0(y)^T dy \mathbf{Q}_j \\ &= 2^{-j/2} (\mathbf{Z}^j)^T \int_{\mathbb{R}} [\zeta_k(y)]_{k \in \mathbb{Z}} \Phi_0(y)^T dy \mathbf{Q}_j \\ &= 2^{-j/2} (\mathbf{Z}^j)^T \int_{\mathbb{R}} [\zeta_k(y)]_{k \in \mathbb{Z}} [\varphi_m(y)]_{m \in \mathbb{Z}}^T dy \mathbf{D}^{-1/2} \mathbf{Q}_j \\ &= 2^{-j/2} (\mathbf{Z}^j)^T \mathbf{G} \mathbf{D}^{-1/2} \mathbf{Q}_j. \end{aligned}$$

□

In practice then, given a wavelet frame, we only need to compute the cross-Gramian matrix \mathbf{G} and then we can obtain the frame coefficients of $\{\zeta_k\}_{k \in \mathbb{Z}}$ just by matrix multiplication. The only thing one has to be careful about is to cut properly the matrices at each step, since the products in Proposition 3.14 are between bi-infinite matrices.

3.2.3 Numerical Estimates

For simplicity of presentation, in this subsection we choose $h_\ell = 1$ and $h_r = 2$ for the initial semi-regular mesh \mathbf{t}_0 (1.3). The wavelet tight frames used for our numerical experiments are the ones constructed in Section 2.2 from the Dubuc-Deslauriers $2n$ -point subdivision schemes. We present an application of the methods in Section 3.2.1 to four cases: the quadratic B-spline scheme, the Dubuc-Deslauriers 4-point scheme and the semi-regular version of two interpolatory schemes based on radial basis functions. The optimal Hölder-Zygmund exponents of semi-regular B-splines schemes and Dubuc-Deslauriers 4-point scheme are known. These examples are used as a benchmark to test our theoretical results.

Example 3.15. The quadratic B-spline scheme generates basic limit functions which are piecewise polynomials of degree two, supported between four consecutive knots of \mathbf{t}_0 . The corresponding subdivision matrix \mathbf{Z} is constructed to satisfy these conditions. There are $k_r(\mathbf{Z}) - k_\ell(\mathbf{Z}) - 1 = 2$ irregular functions whose supports contain the point

$\mathbf{t}(0) = 0$ and it is well known that these functions are $\mathcal{C}^{2-\epsilon}(\mathbb{R})$, $\epsilon > 0$, thus their optimal exponent is equal to 2.

In Figure 3.1 we give the estimates of the optimal Hölder-Zygmund exponents by both the linear regression method and by the method in Proposition 3.12. For the analysis we used the semi-regular tight wavelet frame constructed from the limits of the semi-regular Dubuc-Deslauriers 6-point subdivision scheme. This toy example already illustrates that the method proposed in Proposition 3.12 reaches the optimal exponent in few steps, while the linear regression method converges much slower. \triangle

Example 3.16. The scheme considered here is the one obtained in Definition 2.23 with $n = 2$. In this case, there are 5 irregular basic limit functions depicted in Figure 3.2. Due to results in [21], it is well known that the optimal exponent of all these irregular functions is equal to 2. Again, the method in Proposition 3.12 remarkably outperforms the linear regression method. \triangle

Radial basis functions based interpolatory schemes

Using techniques similar to the ones in Definition 2.23, we extend the subdivision schemes [45, 46] based on radial basis functions to the semi-regular setting. Let $L \in \mathbb{N}$. We require that the subdivision matrix \mathbf{Z} satisfies $\mathbf{Z}(2i, k) = \delta_{i,k}$ for $i, k \in \mathbb{Z}$. To determine the other entries of the 2-slanted matrix \mathbf{Z} whose columns are centered at $\mathbf{Z}(2k, k)$, $k \in \mathbb{Z}$ and have support length at most $4L - 1$, we proceed as follows. We first choose a radial basis function $g(x) = g(|x|)$, $x \in \mathbb{R}$, which is conditionally positive definite of order $\eta \in \mathbb{N}$, i.e., for every set of pairwise distinct points $\{x_i\}_{i=1}^N \subset \mathbb{R}$ and coefficients $\{c_i\}_{i=1}^N \subset \mathbb{R}$, $N \in \mathbb{N}$, there exists a polynomial π of degree at most $\eta - 1$ such that

$$\sum_{i=1}^N c_i \pi(x_i) = 0$$

and the function g satisfies

$$\sum_{i=1}^N \sum_{k=1}^N c_i c_k g(x_i - x_k) \geq 0.$$

The next step is to choose the order $m \in \{\eta, \dots, 2L\}$ of polynomial reproduction and, for every set of $2L$ consecutive points $\mathbf{t}_0(k - L + 1), \dots, \mathbf{t}_0(k + L)$, $k \in \mathbb{Z}$, of the mesh \mathbf{t}_0 in (1.3), solve the linear system of equations

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} \quad (3.23)$$

with

$$\{ \mathbf{A}(i, j) = g(\mathbf{t}_0(k - L + i) - \mathbf{t}_0(k - L + j)) \}_{i,j=1,\dots,2L},$$

$$\begin{aligned} \{ \mathbf{B}(i, j) &= \mathbf{t}_0(k - L + i)^{j-1} \}_{i=1, \dots, 2L, j=1, \dots, m}, \\ \{ \mathbf{r}(i) &= g(x_k - \mathbf{t}_0(k - L + i)) \}_{i=1, \dots, 2L}, \quad \{ \mathbf{s}(j) = x_k^{j-1} \}_{j=1, \dots, m} \quad \text{and} \\ x_k &= (\mathbf{t}_0(k) + \mathbf{t}_0(k + 1)) / 2. \end{aligned}$$

Lastly, the vector \mathbf{u} contains the entries of the $(2k + 1)$ -th row of \mathbf{Z} associated to the columns $k - L + 1$ to $k + L$. For the interested reader, a MATLAB function for the generation of semi-regular RBFs-based interpolatory schemes is available at [61].

Remark 3.17. (i) Determining the rows of \mathbf{Z} by solving the linear systems (3.23) for $k \in \mathbb{Z}$ guarantees the polynomial reproduction of degree at most $m - 1$. Indeed, the condition $\mathbf{B}^T \mathbf{u} = \mathbf{s}$ forces \mathbf{Z} to map samples over \mathbf{t}_0 of a polynomial of degree at most $m - 1$ onto sample over the finer knots $\mathbf{t}_0/2$ of the same polynomial.

(ii) If $m = 2L$, the system of equations $\mathbf{B}^T \mathbf{u} = \mathbf{s}$ coincides with the one defining the Dubuc-Deslauriers $2L$ -point scheme in Definition 2.23. In this case, the system $\mathbf{B}^T \mathbf{u} = \mathbf{s}$ has a unique solution, which makes the presence of \mathbf{A} , i.e. of the radial basis function g obsolete.

(iii) In general, if $m < 2L$, the structure of the irregular basic limit functions around $\mathbf{t}_0(0)$ reflects the transition (blending) between the two (one on the left and one on the right of $\mathbf{t}_0(0)$) subdivision schemes of different regularity, see Example 3.18. This depends on the properties of the chosen underlying radial basis function g . For example, the blending produces no visible effect if g is homogeneous, i.e. $g(\lambda x) = |\lambda|g(x)$, $\lambda \in \mathbb{R}$. In this case, for $\lambda > 0$, the linear system of equations

$$\begin{bmatrix} \lambda \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}/\lambda \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \lambda \mathbf{L}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \lambda \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} \quad (3.24)$$

with the identity matrix \mathbf{I} and $\mathbf{L} = \text{diag}([\lambda^{j-1}]_{j=1, \dots, m})$ is equivalent to the system in (3.23) for the mesh $\lambda \mathbf{t}_0$. The structure of the linear system in (3.24) implies that \mathbf{u} is the same as the one determined by (3.23).

(iv) The argument in (iii) with $\mathbf{L} = \mathbf{I}$ shows that the subdivision matrix obtained this way does not depend on the normalization of the radial basis function g , i.e. all functions λg , $\lambda > 0$, lead to the same subdivision scheme. \diamond

Example 3.18. We consider the radial basis function introduced by M. Buhmann in [4]

$$g(x) = \begin{cases} 12x^4 \log |x| - 21x^4 + 32|x|^3 - 12x^2 + 1, & \text{if } |x| < 1, \\ 0, & \text{otherwise,} \end{cases} \quad (3.25)$$

and choose $L = 2$ and $m = 1$. The resulting irregular functions $\zeta_{-2}, \dots, \zeta_2$ are shown in Figure 3.3. The structure of ζ_0 illustrates the blending effect (described in Remark 3.17 part (iii)) of two different subdivision schemes meeting at $\mathbf{t}_0(0)$. Figure 3.3 also presents the estimates of the optimal Hölder-Zygmund exponents of $\zeta_{-2}, \dots, \zeta_2$. These

exponents are determined using the tight wavelet frame based on the Dubuc-Deslauriers 4-point subdivision scheme (see Example 3.16). We again observe the phenomenon that the method in Proposition 3.12 converges faster than the linear regression. \triangle

Example 3.19. Another radial basis function that we consider is the polyharmonic function $g(x) = |x|$, $x \in \mathbb{R}$. The corresponding irregular part of the interpolatory subdivision matrix \mathbf{Z} is determined for $L = 2$ and $m = 3$, see Figure 3.4. Note that the regular part of the subdivision matrix \mathbf{Z} (see the first and the last columns corresponding to the regular parts of the mesh) coincides with the subdivision matrix of the regular Dubuc-Deslauriers 4-point scheme. Due to the observation in Remark 3.17 part (iii), the absence of the blending effect is due to our choice of a homogeneous function g . We would like to emphasise that the resulting subdivision scheme around $\mathbf{t}_0(0)$ is not the semi-regular Dubuc-Deslauriers 4-point scheme, compare with Figure 3.2. Indeed, the polynomial reproduction around $\mathbf{t}_0(0)$ is of one degree lower. We also lose regularity (the Dubuc-Deslauriers 4-point scheme is $C^{2-\epsilon}$, $\epsilon > 0$) but overall the irregular limit functions on Figure 3.4 have a more uniform behavior than those in Figure 3.2. We again observe that the method in Proposition 3.12 yields better estimates for the optimal Hölder-Zygmund exponent, see tables on Figure 3.4. \triangle

To conclude, we tested this method for a large number of other semi-regular families of subdivision schemes (i.e. B-splines, Dubuc-Deslauriers and interpolatory schemes based on (inverse) multi-quadrics, gaussians, Wendland's functions, Wu's functions, Buhmann's functions, polyharmonic functions and Euclid's hat functions [32]) obtaining similar results.

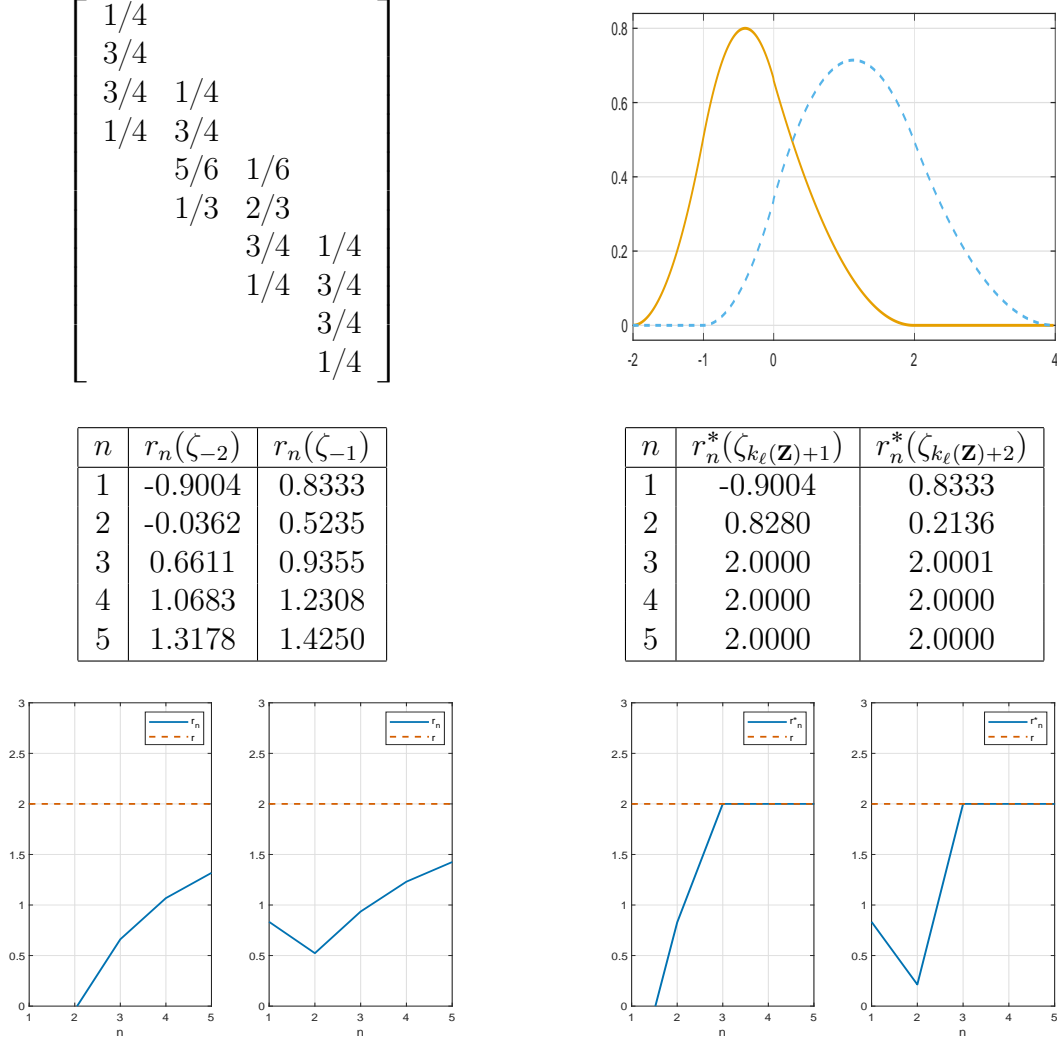
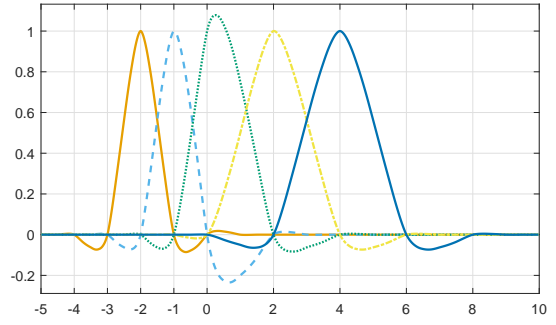
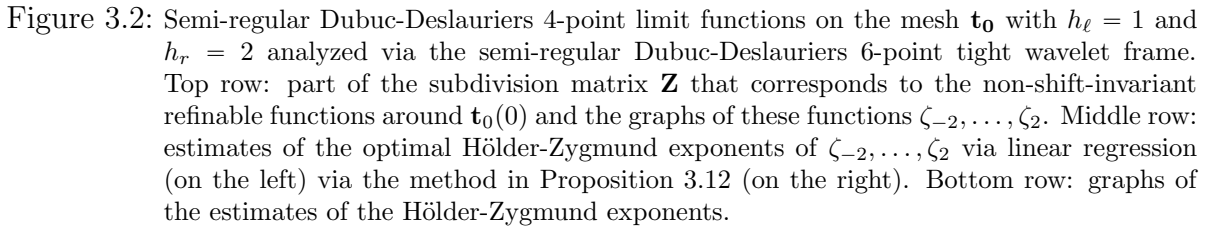


Figure 3.1: Semi-regular quadratic B-spline functions on the mesh \mathbf{t}_0 with $h_\ell = 1$ and $h_r = 2$ analyzed with the semi-regular Dubuc-Deslauriers 6-point tight wavelet frame. Top row: part of the subdivision matrix \mathbf{Z} that corresponds to the non-shift-invariant refinable functions around $\mathbf{t}_0(0)$ and the graphs of these functions ζ_{-2} and ζ_{-1} . Middle row: estimates of the optimal Hölder-Zygmund exponents of ζ_{-2} and ζ_{-1} via linear regression (on the left) via the method in Proposition 3.12 (on the right). Bottom row: graphs of the estimates of the Hölder-Zygmund exponents.



n	$r_n^*(\zeta_{-2})$	$r_n^*(\zeta_{-1})$	$r_n^*(\zeta_0)$	$r_n^*(\zeta_1)$	$r_n^*(\zeta_2)$
1	-0.0589	-0.6276	-0.4561	-0.6674	1.4706
2	3.0835	1.5448	1.7165	2.4105	2.2007
3	2.0585	2.0261	2.1006	2.7262	3.0086
4	1.8721	1.9392	1.9742	2.2284	2.4769
5	1.9764	1.9883	2.0064	2.0489	2.1474
6	1.9792	1.9897	1.9984	2.0131	2.0013
7	1.9920	1.9960	2.0004	2.0096	1.9997
8	1.9954	1.9977	1.9999	2.0041	2.0001
9	1.9979	1.9989	2.0000	2.0022	2.0000
10	1.9989	1.9995	2.0000	2.0011	2.0000
11	1.9995	1.9997	2.0000	2.0005	2.0000
12	1.9997	1.9999	2.0000	2.0003	2.0000
13	1.9999	1.9999	2.0000	2.0001	2.0000
14	1.9999	2.0000	2.0000	2.0001	2.0000
15	2.0000	2.0000	2.0000	2.0000	2.0000
16	2.0000	2.0000	2.0000	2.0000	2.0000



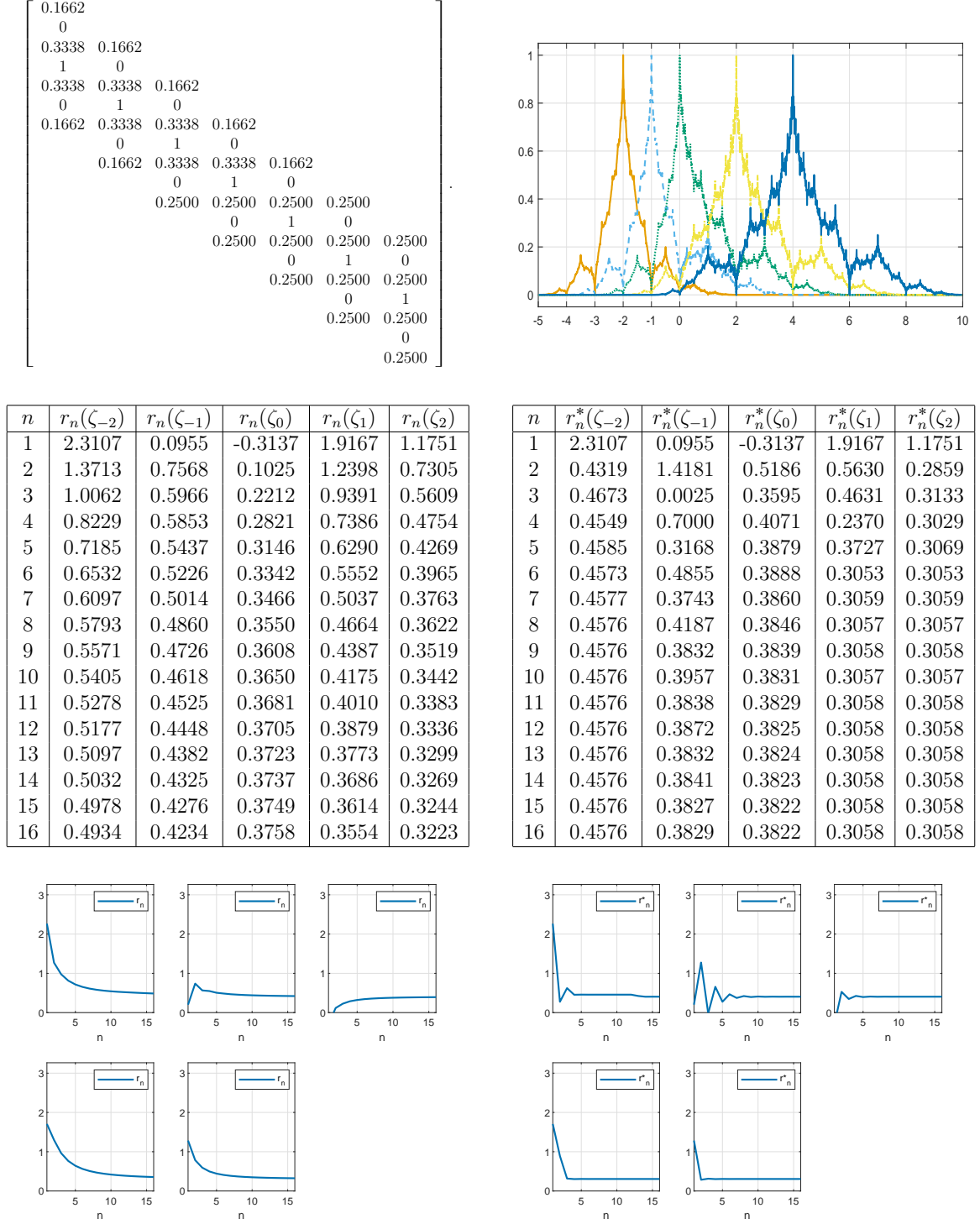
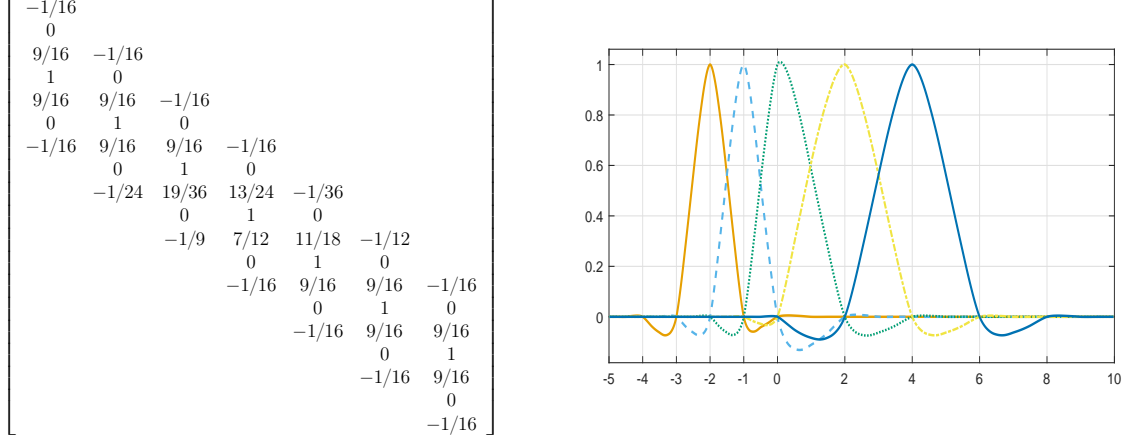


Figure 3.3: Semi-regular interpolatory subdivision scheme based on g in (3.25), $L = 2$, $m = 1$, on the mesh \mathbf{t}_0 with $h_\ell = 1$ and $h_r = 2$ analyzed via the semi-regular Dubuc-Deslauriers 4-point tight wavelet frame. Top row: part of the subdivision matrix \mathbf{Z} that corresponds to the non-shift-invariant refinable functions around $\mathbf{t}_0(0)$ and the graphs of these functions $\zeta_{-2}, \dots, \zeta_2$. Middle row: estimates of the optimal Hölder-Zygmund exponents of $\zeta_{-2}, \dots, \zeta_2$ via linear regression (on the left) via the method in Proposition 3.12 (on the right). Bottom row: graphs of the estimates of the Hölder-Zygmund exponents.



n	$r_n(\zeta_{-2})$	$r_n(\zeta_{-1})$	$r_n(\zeta_0)$	$r_n(\zeta_1)$	$r_n(\zeta_2)$	n	$r_n^*(\zeta_{-2})$	$r_n^*(\zeta_{-1})$	$r_n^*(\zeta_0)$	$r_n^*(\zeta_1)$	$r_n^*(\zeta_2)$
1	-0.0163	-0.5174	-0.3984	-0.5637	1.4525	1	-0.0163	-0.5174	-0.3984	-0.5637	1.4525
2	1.8110	0.6301	0.2509	0.5646	1.5324	2	3.6383	1.7777	0.9002	1.6929	1.6124
3	2.3259	0.9516	0.7115	1.0643	1.6848	3	2.9182	1.3190	1.5699	1.8541	2.0135
4	2.1635	1.1421	1.0006	1.3048	1.7340	4	0.9991	1.5830	1.6965	1.7672	1.7787
5	2.0346	1.2779	1.1895	1.4400	1.7552	5	1.5859	1.7110	1.7445	1.7700	1.7838
6	1.9538	1.3766	1.3178	1.5230	1.7640	6	1.7105	1.7467	1.7568	1.7642	1.7665
7	1.9029	1.4495	1.4081	1.5777	1.7677	7	1.7468	1.7580	1.7612	1.7638	1.7649
8	1.8695	1.5043	1.4740	1.6155	1.7692	8	1.7580	1.7615	1.7625	1.7633	1.7636
9	1.8466	1.5463	1.5232	1.6429	1.7696	9	1.7615	1.7626	1.7630	1.7632	1.7633
10	1.8303	1.5791	1.5610	1.6633	1.7695	10	1.7626	1.7630	1.7631	1.7632	1.7632
11	1.8183	1.6050	1.5906	1.6789	1.7693	11	1.7630	1.7631	1.7632	1.7632	1.7632
12	1.8092	1.6260	1.6141	1.6911	1.7689	12	1.7631	1.7632	1.7632	1.7632	1.7632
13	1.8022	1.6430	1.6332	1.7008	1.7685	13	1.7632	1.7632	1.7632	1.7632	1.7632
14	1.7967	1.6571	1.6488	1.7087	1.7681	14	1.7632	1.7632	1.7632	1.7632	1.7632
15	1.7922	1.6689	1.6618	1.7152	1.7677	15	1.7632	1.7632	1.7632	1.7632	1.7632
16	1.7886	1.6788	1.6727	1.7206	1.7674	16	1.7632	1.7632	1.7632	1.7632	1.7632

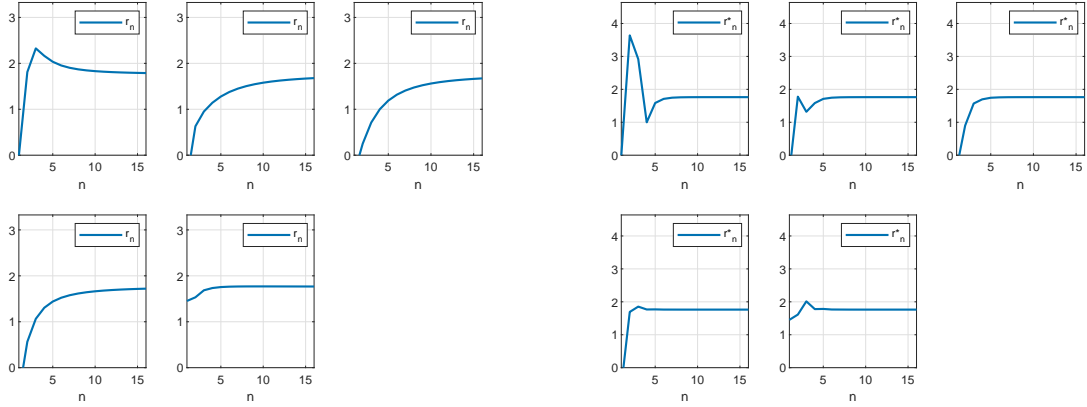


Figure 3.4: Semi-regular interpolatory scheme based on the polyharmonic function $g(x) = |x|$, $L = 2$ and $m = 3$ on the mesh \mathbf{t}_0 with $h_\ell = 1$, $h_r = 2$ analyzed with the semi-regular Dubuc-Deslaurliers 6-point scheme. Top row: part of the subdivision matrix \mathbf{Z} that corresponds to the non-shift-invariant refinable functions around $\mathbf{t}_0(0)$ and the graphs of these functions $\zeta_{-2}, \dots, \zeta_2$. Middle row: estimates of the optimal Hölder-Zygmund exponents of $\zeta_{-2}, \dots, \zeta_2$ via linear regression (on the left) via the method in Proposition 3.12 (on the right). Bottom row: graphs of the estimates of the Hölder-Zygmund exponents.

Conclusions

We presented a new characterization of the Hölder-Zygmund spaces via irregular families of tight frames, in particular the one developed from semi-regular subdivision schemes. We provided the tools for the computation of the moments and the (Cross-)Gramian matrices involving basic limit functions of such semi-regular schemes. This opened up the possibility for estimating the regularity of any semi-regular scheme via the decay of the frame coefficients of its basic limit functions. For the analysis we constructed wavelet tight frames associated with the Dubuc-Deslauriers family of semi-regular interpolatory subdivision schemes. We presented their convergence analysis, to the choice of a suitable approximation of the corresponding Gramian matrix. This UEP construction was developed to overcome the difficulties arising in the OEP semi-regular setting and its simplicity may also have a strength in other practical applications.

The results presented here can be generalized in a straightforward way to tensor products of semi-regular schemes in the bivariate case. This could be the first step towards the analysis of the bivariate subdivision in presence of extraordinary vertices.

Bibliography

- [1] C. BECCARI, G. CASCIOLA, AND L. ROMANI, *Polynomial-based non-uniform interpolatory subdivision with features control*, J. Comput. Appl. Math., 235 (2011), pp. 4754–4769.
- [2] C. V. BECCARI, G. CASCIOLA, AND L. ROMANI, *Non-uniform interpolatory curve subdivision with edge parameters built upon compactly supported fundamental splines*, BIT, 51 (2011), pp. 781–808.
- [3] L. BORUP, R. GRIBONVAL, AND M. NIELSEN, *Tight wavelet frames in Lebesgue and Sobolev spaces*, J. Funct. Spaces Appl., 2 (2004), pp. 227–252.
- [4] M. D. BUHMANN, *Radial basis functions*, in Acta numerica, 2000, vol. 9 of Acta Numer., Cambridge Univ. Press, Cambridge, 2000, pp. 1–38.
- [5] A. S. CAVARETTA, W. DAHMEN, AND C. A. MICCHELLI, *Stationary subdivision*, Mem. Amer. Math. Soc., 93 (1991).
- [6] M. CHARINA, C. CONTI, N. GUGLIELMI, AND V. PROTASOV, *Regularity of non-stationary subdivision: a matrix approach*, Numer. Math., 135 (2017), pp. 639–678.
- [7] M. CHARINA, C. CONTI, L. ROMANI, J. STÖCKLER, AND A. VISCARDI, *Optimal Hölder-Zygmund exponent of semi-regular refinable functions*, J. Approx. Theory, (submitted). <https://arxiv.org/abs/1807.10909>.
- [8] M. CHARINA AND J. STÖCKLER, *Tight wavelet frames for irregular multiresolution analysis*, Appl. Comput. Harmon. Anal., 25 (2008), pp. 98–113.
- [9] O. CHRISTENSEN, *An introduction to frames and Riesz bases*, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, [Cham], second ed., 2016.
- [10] C. CHUI AND J. DE VILLIERS, *Wavelet subdivision methods*, CRC Press, Boca Raton, FL, 2011.
- [11] C. K. CHUI, *An introduction to wavelets*, vol. 1 of Wavelet Analysis and its Applications, Academic Press, Inc., Boston, MA, 1992.
- [12] C. K. CHUI AND W. HE, *Compactly supported tight frames associated with refinable functions*, Appl. Comput. Harmon. Anal., 8 (2000), pp. 293–319.

- [13] C. K. CHUI, W. HE, AND J. STÖCKLER, *Compactly supported tight and sibling frames with maximum vanishing moments*, Appl. Comput. Harmon. Anal., 13 (2002), pp. 224–262.
- [14] ———, *Nonstationary tight wavelet frames. I. Bounded intervals*, Appl. Comput. Harmon. Anal., 17 (2004), pp. 141–197.
- [15] ———, *Nonstationary tight wavelet frames. II. Unbounded intervals*, Appl. Comput. Harmon. Anal., 18 (2005), pp. 25–66.
- [16] C. K. CHUI, W. HE, J. STÖCKLER, AND Q. SUN, *Compactly supported tight affine frames with integer dilations and maximum vanishing moments*, Adv. Comput. Math., 18 (2003), pp. 159–187.
- [17] E. COHEN, T. LYCHE, AND R. RIESENFELD, *Discrete B-splines and subdivision techniques in computer-aided geometric design and computer graphics*, Computer graphics and image processing, 14 (1980), pp. 87–111.
- [18] E. CORDERO AND K. GRÖCHENIG, *Localization of frames. II*, Appl. Comput. Harmon. Anal., 17 (2004), pp. 29–47.
- [19] W. DAHMEN AND C. A. MICCHELLI, *Using the refinement equation for evaluating integrals of wavelets*, SIAM J. Numer. Anal., 30 (1993), pp. 507–537.
- [20] I. DAUBECHIES, *Ten lectures on wavelets*, vol. 61 of CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.
- [21] I. DAUBECHIES, I. GUSKOV, AND W. SWELDENS, *Regularity of irregular subdivision*, Constr. Approx., 15 (1999), pp. 381–426.
- [22] I. DAUBECHIES, I. GUSKOV, AND W. SWELDENS, *Commutation for irregular subdivision*, Constr. Approx., 17 (2001), pp. 479–514.
- [23] I. DAUBECHIES AND B. HAN, *Pairs of dual wavelet frames from any two refinable functions*, Constr. Approx., 20 (2004), pp. 325–352.
- [24] I. DAUBECHIES, B. HAN, A. RON, AND Z. SHEN, *Framelets: MRA-based constructions of wavelet frames*, Appl. Comput. Harmon. Anal., 14 (2003), pp. 1–46.
- [25] I. DAUBECHIES AND J. C. LAGARIAS, *Sets of matrices all infinite products of which converge*, Linear Algebra Appl., 161 (1992), pp. 227–263.
- [26] C. DE BOOR, *A practical guide to splines*, vol. 27 of Applied Mathematical Sciences, Springer-Verlag, New York, revised ed., 2001.
- [27] G. DESLAURIERS AND S. DUBUC, *Symmetric iterative interpolation processes*, Constr. Approx., 5 (1989), pp. 49–68.

- [28] R. J. DUFFIN AND A. C. SCHAEFFER, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., 72 (1952), pp. 341–366.
- [29] N. DYN, J. A. GREGORY, AND D. LEVIN, *Piecewise uniform subdivision schemes*, in Mathematical methods for curves and surfaces (Ulvik, 1994), Vanderbilt Univ. Press, Nashville, TN, 1995, pp. 111–119.
- [30] M. EHLE, *The multiresolution structure of pairs of dual wavelet frames for a pair of Sobolev spaces*, Jaen J. Approx., 2 (2010), pp. 193–214.
- [31] M. EHLE AND B. HAN, *Wavelet bi-frames with few generators from multivariate refinable functions*, Appl. Comput. Harmon. Anal., 25 (2008), pp. 407–414.
- [32] G. E. FASSHAUER, *Meshfree approximation methods with MATLAB*, vol. 6 of Interdisciplinary Mathematical Sciences, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007. With 1 CD-ROM (Windows, Macintosh and UNIX).
- [33] M. S. FLOATER, *The loss of Hölder regularity of four-point interpolatory subdivision on irregularly spaced points*, Jaen J. Approx., 3 (2011), pp. 261–270.
- [34] —, *A piecewise polynomial approach to analyzing interpolatory subdivision*, J. Approx. Theory, 163 (2011), pp. 1547–1563.
- [35] M. FRAZIER AND B. JAWERTH, *Decomposition of Besov spaces*, Indiana Univ. Math. J., 34 (1985), pp. 777–799.
- [36] A. HAAR, *Zur Theorie der orthogonalen Funktionensysteme*, Math. Ann., 69 (1910), pp. 331–371.
- [37] B. HAN, *On dual wavelet tight frames*, Appl. Comput. Harmon. Anal., 4 (1997), pp. 380–413.
- [38] —, *Recent developments on dual wavelet frames*, in Representations, wavelets, and frames, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2008, pp. 103–130.
- [39] B. HAN AND R.-Q. JIA, *Characterization of Riesz bases of wavelets generated from multiresolution analysis*, Appl. Comput. Harmon. Anal., 23 (2007), pp. 321–345.
- [40] B. HAN AND Z. SHEN, *Compactly supported symmetric C^∞ wavelets with spectral approximation order*, SIAM J. Math. Anal., 40 (2008), pp. 905–938.
- [41] —, *Characterization of Sobolev spaces of arbitrary smoothness using nonstationary tight wavelet frames*, Israel J. Math., 172 (2009), pp. 371–398.
- [42] —, *Dual wavelet frames and Riesz bases in Sobolev spaces*, Constr. Approx., 29 (2009), pp. 369–406.

- [43] M. HOLSCHNEIDER AND P. TCHAMITCHIAN, *Régularité locale de la fonction “non-différentiable” de Riemann*, in Les ondelettes en 1989 (Orsay, 1989), vol. 1438 of Lecture Notes in Math., Springer, Berlin, 1990, pp. 102–124, 209–210.
- [44] A. KUNOTH, *On the fast evaluation of integrals of refinable functions*, in Wavelets, images, and surface fitting (Chamonix-Mont-Blanc, 1993), A K Peters, Wellesley, MA, 1994, pp. 327–334.
- [45] B.-G. LEE, Y. J. LEE, AND J. YOON, *Stationary binary subdivision schemes using radial basis function interpolation*, Adv. Comput. Math., 25 (2006), pp. 57–72.
- [46] Y. J. LEE AND J. YOON, *Analysis of stationary subdivision schemes for curve design based on radial basis function interpolation*, Appl. Math. Comput., 215 (2010), pp. 3851–3859.
- [47] P. G. LEMARIÉ AND Y. MEYER, *Ondelettes et bases hilbertiennes*, Rev. Mat. Iberoamericana, 2 (1986), pp. 1–18.
- [48] J. M. LOUNSBERY, *Multiresolution analysis for surfaces of arbitrary topological type*, ProQuest LLC, Ann Arbor, MI, 1994. Thesis (Ph.D.)—University of Washington.
- [49] S. MALLAT, *A wavelet tour of signal processing*, Elsevier/Academic Press, Amsterdam, third ed., 2009.
- [50] Y. MEYER, *Wavelets and operators*, vol. 37 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1992. Translated from the 1990 French original by D. H. Salinger.
- [51] C. A. MICCHELLI, *Interpolatory subdivision schemes and wavelets*, J. Approx. Theory, 86 (1996), pp. 41–71.
- [52] J. PETERS AND U. REIF, *Subdivision surfaces*, vol. 3 of Geometry and Computing, Springer-Verlag, Berlin, 2008.
- [53] S. PILIPOVIĆ, D. S. RAKIĆ, AND J. VINDAS, *New classes of weighted Hölder-Zygmund spaces and the wavelet transform*, J. Funct. Spaces Appl., (2012).
- [54] U. REIF, *A unified approach to subdivision algorithms near extraordinary vertices*, Comput. Aided Geom. Design, 12 (1995), pp. 153–174.
- [55] A. RON AND Z. SHEN, *Affine systems in $L_2(\mathbf{R}^d)$. II. Dual systems*, J. Fourier Anal. Appl., 3 (1997), pp. 617–637.
- [56] —, *Affine systems in $L_2(\mathbf{R}^d)$: the analysis of the analysis operator*, J. Funct. Anal., 148 (1997), pp. 408–447.
- [57] M. SABIN, *Analysis and design of univariate subdivision schemes*, vol. 6 of Geometry and Computing, Springer-Verlag, Berlin, 2010.

- [58] L. F. VILLEMOES, *Wavelet analysis of refinement equations*, SIAM J. Math. Anal., 25 (1994), pp. 1433–1460.
- [59] A. VISCARDI, *Irregular filters for semi-regular Dubuc-Deslauriers wavelet tight frames*. Mendeley Data, 2018. <http://dx.doi.org/10.17632/f5rc39k9m4.1>.
- [60] A. VISCARDI, *Semi-regular Dubuc-Deslauriers wavelet tight frames*, J. Comput. Appl. Math., (2018), p. accepted. <http://dx.doi.org/10.1016/j.cam.2018.07.049>.
- [61] ———, *Semi-regular interpolatory RBF-based subdivision schemes*. Mendeley Data, 2018. <http://dx.doi.org/10.17632/7fpccm8y9j.1>.
- [62] J. WARREN, *Binary subdivision schemes for functions over irregular knot sequences*, in Mathematical methods for curves and surfaces (Ulvik, 1994), Vanderbilt Univ. Press, Nashville, TN, 1995, pp. 543–562.
- [63] J. WARREN AND H. WEIMER, *Subdivision Methods for Geometric Design: A Constructive Approach*, Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 1st ed., 2001.
- [64] A. ZYGMUND, *Trigonometric series. Vol. I, II*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, third ed., 2002.

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