Non Abelian Asymptotics of Szegö Kernels on Compact Quantized Manifolds

Doctoral Thesis

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Abstract

Let M be a complex projective manifold with a positive line bundle A on it. The unit circle bundle X, inside the dual of A, is a contact and CR manifold by positivity of A. Let $H(X) \subset L^2(X)$ be the associated Hardy space, and Π be the corresponding orthogonal projector, which is known in the literature as the Szegö projector. Assume given a Hamiltonian and holomorphic action of a compact Lie group G on M, and suppose that the action lifts to a contact CR action on X. There is a naturally induced unitary representation of G on $H(X)$, which therefore splits equivariantly into a direct sum of isotypical components, indexed by the irreducible representations of G. Assuming that the moment map is nowhere vanishing, each isotypical summand is finite-dimensional, and therefore the kernel of the corresponding orthogonal projector ('equivariant Szegö kernel') is a C^∞ function on $X \times X$. One is led to study the local asymptotics of the equivariant projectors pertaining to the irreducible representations in a given ray in weight space. In this thesis we consider the case of special unitary group $SU(2)$ and unitary group $U(2)$.

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Contents

Chapter 1

Introduction

1.1 Geometric setting and goal

Let M be a connected compact complex d-dimensional projective manifold, with a complex structure J (for a more detailed discussion and for definitions see the preliminaries in the next chapter). Let $(A,\, h)$ be a positive line bundle on M, where h is the Hermitian structure, with connection ∇ compatible with the metric such that $\operatorname{curv}\nabla = -2i\,\omega$, where ω is a Kähler form on M. We shall denote by g the corresponding Riemannian structure on M, given by

$$g_m(\mathbf{v},\, \mathbf{w}) := \omega_m\,(\mathbf{v},\, J_m(\mathbf{w})) \quad (m \in M,\ \mathbf{v},\, \mathbf{w} \in T_mM)\,. \tag{1.1}$$

Furthermore M has a natural choice of a volume form: $\mathrm{dV}_M := (1/d!)\,\omega^d$.

If $A^\vee \supset X \xrightarrow{\pi} M$ is the unit circle bundle in the dual of A, then ∇ naturally corresponds to a connection 1-form α on X, such that $\mathrm{d}\alpha = 2\,\pi^*(\omega)$. Let us notice that (X, α) is a contact manifold and inherits the volume form $\mathrm{dV}_X := (2\pi)^{-1}\,\alpha\wedge\pi^*(\mathrm{dV}_M)$. Furthermore α determines an invariant splitting of the tangent bundle of X as

$$TX := \mathcal{V}(X/M) \oplus \mathcal{H}(X/M)\,, \tag{1.2}$$

where $\mathcal{V}(X/M) := \ker(\mathrm{d}\pi)$ is the *vertical* tangent bundle, and $\mathcal{H}(X/M) := \ker(\alpha)$ is the *horizontal* one. Given $V \in \mathfrak{X}(M)$ (the Lie algebra of smooth vector fields on M), we shall denote by $V^\sharp \in \mathfrak{X}(X)$ its horizontal lift to X. If the vector field $\partial_\theta \in \mathfrak{X}(X)$ is the generator of the structure S^1-action, then ∂_θ spans the vertical tangent bundle.

The holomorphic structure on M, pulled-back to the horizontal tangent bundle, endows X with a CR structure. Explicitly, the complex structure J on M naturally lifts to a vector bundle endomorphism of TX, also denoted by J, such that $J(\partial_\theta) = 0$ and

$$J(\mathbf{v}^\sharp) = J(\mathbf{v})^\sharp \qquad (\mathbf{v} \in \mathfrak{X}(M))\,. \tag{1.3}$$

By the positivity of (A, h), X is the boundary of a strictly pseudo-convex domain. The corresponding Hardy space $H(X) \subset L^2(X)$ encapsulates the holomorphic structure of A and its tensor powers. The corresponding orthogonal projector $\Pi : L^2(X) \to H(X)$ is called the Szegö projector. For more details concerning the Szegö Kernel and its description as a Fourier Integral Operator see section 3.3.1 in the next chapter.

Suppose that, in addition, one is given a compact Lie group G and a holomorphic Hamiltonian action $\mu : G \times M \to M$ with moment map $\Phi_G : M \to \mathfrak{g}^\vee$; then G acts on M via Riemannian isometries (see the next chapter for a more detailed description of the representation theory of compact Lie groups and for Hamiltonian group actions). In particular, for each $\xi \in \mathfrak{g}$, let us denote with $\xi_M \in \mathfrak{X}(M)$ the vector field associated to ξ. We shall assume that the action μ lifts to an action $\tilde{\mu} : G \times X \to X$ preserving the contact and CR structures. The vector field ξ_X associated to $\xi \in \mathfrak{g}$ is related to ξ_M via the following formula

$$\xi_X(x) = \xi_M^\sharp(m) - \langle \Phi_G(m), \, \xi \rangle \, \partial_\theta \,. \tag{1.4}$$

Then pull-back of functions, given by $g \cdot s := \tilde{\mu}^*_{g^{-1}}(s)$, is a unitary representation of G on $L^2(X)$ leaving $H(X) \subset L^2(X)$ invariant. This yields a unitary representation $\hat{\mu} : \; G \to U(H(X))$ which commutes with the standard S^1 action.

By the Theorem and Peter and Weyl, all the irreducible representations of G are finite dimensional, and any unitary representation of G splits equivariantly as a direct sum of irreducible representations of G.

We shall focus in particular on the two cases $G = SU(2)$ and $G = U(2)$. If $G = U(2)$, the irreducible representations of G are indexed by the couples $\boldsymbol{\nu} = (\nu_1, \, \nu_2)$ of integers satisfying $\nu_1 > \nu_2$. There is an equivariant unitary isomorphism

$$H(X) \cong \bigoplus_{\nu_1 > \nu_2} H(X)_{\boldsymbol{\nu}}\,,$$

where $H(X)_{\boldsymbol{\nu}} \subset H(X)$ is the $\boldsymbol{\nu}$-isotypical component. Correspondingly,

$$\Pi = \sum_{\nu_1 > \nu_2} \Pi_{\boldsymbol{\nu}} \,,$$

where $\Pi_{\boldsymbol{\nu}} : L^2(X) \to H(X)_{\boldsymbol{\nu}}$ is the orthogonal projector. In the same spirit, the action of $SU(2)$ gives rise to the decomposition

$$H(X) \cong \bigoplus_{\nu > 0} H(X)_{\boldsymbol{\nu}}$$

where here $\boldsymbol{\nu} = (\nu, 0)$ and the corresponding equivariant Szegö kernels will be denoted by $\Pi_{\boldsymbol{\nu}}$.

In general, $H(X)_{\boldsymbol{\nu}}$ may well be infinite dimensional; however, if $\mathbf{0} \notin \Phi_G(M)$ then $\dim(H(X)_{\boldsymbol{\nu}}) < +\infty$, see [Pao12]. In this case $\Pi_{\boldsymbol{\nu}}$ is a smoothing operator, with a distributional kernel

$$\Pi_{\boldsymbol{\nu}}(\,\cdot\,,\,\cdot\,) \in \mathcal{C}^{\infty}(X \times X)\,.$$

In particular,

$$\dim\, H(X)_{\boldsymbol{\nu}} = \int_X \Pi_{\boldsymbol{\nu}}(x,\, x)\, \mathrm{dV}_X(x)\,.$$

The goal of this thesis is to study the concentration behavior of $\Pi_{k\boldsymbol{\nu}}$ when $k \to +\infty$, in the specific cases where $G = SU(2)$ and $G = U(2)$.

A few words about the organization of this thesis. In chapter §3 we review some definitions and theorems about harmonic analysis, moment maps and Szegö projector.

In chapter §1 we described the main results of this thesis. In section 1.2 (based on [GP18a]) we study the case $G = SU(2)$. We obtain an asymptotic expansion in descending powers of k for both the free and locally free action on X. First we provide some quantitative information about the rate of decay of $\Pi_{k\boldsymbol{\nu}}(x,\, y)$ when x approaches to $G\cdot y$. An explicit expression of the leading order term is found for the asymptotic of $\Pi_{k\boldsymbol{\nu}}(x,\, x)$ for a locally free action at x and for a near-diagonal rescaled displacement when the the stabilizer of x is contained in the center of G. In section 1.3 (based on [GP18b]) we prove the results for $U(2)$. It turns out that a very special role is played by a locus $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, which is the inverse image via the moment map Φ_G of the cone through the co-adjoint orbit of $\boldsymbol{\nu}$. Under some reasonable hypotheses,

$M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ is an hyper-surface and it divides M in two connected components: the "outside" and the "inside". We are able to give a precise result about the rate of decay of the asymptotics of $\Pi_{k\boldsymbol{\nu}}(x,\,y)$ when one approaches to $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ from the outside. Nevertheless, we prove that the equivariant Szegö kernel has a rapidly decreasing asymptotic on compact subset not lying in $\mathcal{Z}_{\boldsymbol{\nu}} := \{(x,\,y)\in X^G_{\mathcal{O}_{\boldsymbol{\nu}}}\times X^G_{\mathcal{O}_{\boldsymbol{\nu}}}\,:\,y\in G\cdot x\}$, where $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ denotes the pull-back of $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ on X. If the action of G on $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ is free, we find an explicit expression for the asymptotic along the diagonal, an asymptotic expansion for near-diagonal rescaled displacements and a lower bound for the dimension of the isotypes $H_{k\boldsymbol{\nu}}(X)$. Similar results are given also for the locally free case.

The proofs are collected in the last chapter. The case of $G = SU(2)$ bears a close resemblance to the case of S^1, in sharp contrast with the case of $G = U(2)$; this reflects the fact that the cone over the coadjoint orbit is open and dense in the case of $SU(2)$ and S^1, but has codimension 1 in the case of $U(2)$. The first main step in the proofs is to show that the equivariant Szegö kernel has the same asymptotic as a compact supported oscillatory integral. The leading term of the asymptotic expansion is found by the use of *stationary phase Lemma*. In the case $G = U(2)$, the proof concerning the rapidly decreasing asymptotic of $\Pi_{k\boldsymbol{\nu}}(x,\,y)$ on compact subset in the complement of $\mathcal{Z}_{\boldsymbol{\nu}}$ has a different approach: it is grounded on results of Guillemin and Sternberg ([GS82c]) regarding functorial properties of some distributions related with the equivariant Szegö kernel.

1.2 The case $G = SU(2)$

In this section we denote by G the group $SU(2)$, The Lie algebra $\mathfrak{g}$ of $SU(2)$ consists of skew-Hermitian matrices with trace zero. There is a natural G-invariant Euclidean scalar product on $\mathfrak{g}$, given by $\langle \xi_1,\,\xi_2\rangle = \mathrm{trace}(\xi_1\bar{\xi}_2^t)$. By means of $\langle\cdot,\,\cdot\rangle$, we can equivariantly identify $\mathfrak{g}$ and $\mathfrak{g}^\vee$. The first step will be to provide quantitative information about the rate of decay of $\Pi_{k\boldsymbol{\nu}}(x,\,y)$ when x approaches to $G\cdot y$.

Theorem 1.2.1. *Assume that* $\mathbf{0}$ *does not lie in the image of the moment map* $\Phi_G(M)$. *Let* $C,\,\epsilon>0$. *Then, uniformly for*

$$\mathrm{dist}_X(x,\,G\cdot y)\geq C\,k^{\epsilon-1/2}\,,$$

we have $\Pi_{k\boldsymbol{\nu}}(x,\,y)=O(k^{-\infty})$ *as* $k\to+\infty$.

Before stating our next Theorem, we need to introduce some terminology. Let us suppose that $\mathbf{0} \notin \Phi_G(M)$. Thus there exists one and only one $h_m T \in G/T$ such that

$$h_m \Phi_G(m) h_m^{-1} = i \begin{pmatrix} \lambda(m) & 0 \\ 0 & -\lambda(m) \end{pmatrix} ,$$

where $\lambda(m)$ is the unique positive eigenvalue of $\Phi_G(m)$ and clearly $\lambda : M \to (0, +\infty)$ is smooth.

Remark 1. The positive eigenvalue $\lambda(m)$ has a symplectic interpretation, being closely related to the moment map for the action restricted to a suitable torus $T_m \leq G$. Let us set

$$\beta := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} ;$$

thus β is the infinitesimal generator of the standard torus T, and therefore $\mathrm{Ad}_{h_m}(\beta)$ is the infinitesimal generator of the torus $T_m := C_{h_m}(T)$ (here $C_g(h) := g\, h\, g^{-1}$, for all $g,\, h \in G$). Then for any $m \in M$ we have

$$2\lambda(m) = \langle h_m^{-1} \Phi_G(m) h_m,\, \beta \rangle = \langle \Phi_G(m),\, \mathrm{Ad}_{h_m}(\beta) \rangle .$$

Let us denote by G_x the stabilizer of $x \in X$. By equivariance of Φ_G, G_x stabilizes $\Phi_G(m)$. Hence, $G_x \subset h_{m_x}\, T\, h_{m_x}^{-1}$. In particular, if the lifted action $\tilde{\mu} : G \times X \to X$ is locally free at x, then G_x is finite and Abelian. There exists $e^{i\vartheta_j} \in S^1$, $j = 1, \ldots, N_x$, such that

$$G_x = \left\{ h_{m_x} \begin{pmatrix} e^{i\,\vartheta_j} & 0 \\ 0 & e^{-i\,\vartheta_j} \end{pmatrix} h_{m_x}^{-1} : \quad j = 1, \ldots, N_x \right\} .$$

Let $Z := \{\pm I_2\}$ be the center of G, and set $Z_x := G_x \cap Z$. We shall see that each $g \in G_x$ contributes to the rescaled asymptotics of $\Pi_{k\boldsymbol{\nu}}$ near x, and that the nature of the contribution is quite different depending on whether $g \in Z_x$ or $g \in G \setminus G_x$.

Definition 1.2.1. If $l \in \mathbb{Z}$, let us define $f_l : T \to \mathbb{C}$ by letting

$$f_l : e^{i\vartheta}\, I_2 \in T \mapsto e^{i\,l\,\vartheta} \in \mathbb{C}^* .$$

If $h \in G_x \setminus Z_x$, then $h \neq h^{-1}$. Hence $G_x \setminus Z_x$ has even cardinality $b_x = 2\, a_x$, and perhaps after renumbering its elements can be arranged in pairs $(g_j,\, g_j^{-1})$, $j = 1, \ldots a_x$. For every $j = 1, \ldots, a_x$. For every $j = 1, \ldots, a_x$, let us set

$$t_j := h_{m_x}^{-1}\, g_j\, h_{m_x} = \begin{pmatrix} e^{i\vartheta_j} & 0 \\ 0 & e^{-i\vartheta_j} \end{pmatrix} . \tag{1.5}$$

Definition 1.2.2. If $z \in \mathbb{C}$, let us set

$$A(z) := i \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} \in \mathfrak{g}\,.$$

The the $\mathbb{R}$-linear map

$$\eta_j\,:\,z \in \mathbb{C} \mapsto \left(\mathrm{Ad}_{t_j^{-1}} - \mathrm{id}_{\mathfrak{g}}\right)(A(z)) \in \mathfrak{g}$$

is injective. Therefore, since $\tilde{\mu}$ is locally free at x, there is a positive 2×2 matrix $C(x;\,j)$ such that

$$\left\|\mathrm{Ad}_{h_{m_x}}\left(\eta_j(z)\right)_X(x)\right\|^2 = \frac{1}{2} \cdot Z^t\,C(x;\,j)\,Z \quad (z \in \mathbb{C})$$

where $Z := (a,\,b)^t \in \mathbb{R}^2$ if $z = a + i\,b$. Let us define

$$B(x;\,j) := C(x;\,j) + 4\,i\,\sin(2\vartheta_j) \cdot \lambda\,(m_x)\,I_2\,.$$

Finally, let us denote by V_3 the area of the unit sphere $S^3 \subset \mathbb{R}^4$, and set

$$D_{G/T} := 2\pi/V_3$$

We shall prove the following.

Theorem 1.2.2. *Le us assume that $\mathbf{0} \notin \Phi_G(M)$, $\tilde{\mu}$ is locally free at x and that $G_x \setminus Z_x = \{g_1,\,g_1^{-1},\ldots,g_{a_x},\,g_{a_x}^{-1}\}$. Then as $k \rightarrow +\infty$ there is an asymptotic expansion*

$$\Pi_{k\boldsymbol{\nu}}(x,\,x) \sim \Pi_{k\boldsymbol{\nu}}(x,\,x)_{Z_x} + \Pi_{k\boldsymbol{\nu}}(x,\,x)_{G_x \setminus Z_x}\,,$$

where

$$\Pi_{k\boldsymbol{\nu}}(x,\,x)_{Z_x} \sim \quad \frac{1}{2\,\lambda(m_x)} \cdot \left(\frac{\nu\,k}{2\,\pi\,\lambda(m_x)}\right)^d \cdot \sum_{g \in Z_x} f_{1-k\cdot\nu}(g) \cdot \left[1 + \sum_{j=1}^{+\infty} k^{-j/2}\,B_{g\,j}(x)\right]\,,$$

and

$$\Pi_{k\boldsymbol{\nu}}(x,\,x)_{G_x \setminus Z_x} \sim \quad 4\,\pi \cdot D_{G/T} \cdot \left(\frac{\nu\,k}{2\,\pi\,\lambda(m_x)}\right)^d \cdot \left[\sum_{j=1}^{a_x} \Re\left(\frac{i\,\sin(\vartheta_j) \cdot e^{-i\,k\,\nu\cdot\vartheta_j}}{\sqrt{\det\left(B(x;\,j)\right)}}\right) + \sum_{l \geq 1} k^{-l/2}\,P_{j\,l}(m_x)\right]\,,$$

for appropriate smooth functions $B_{g\,j},\ P_{j\,l} : M \rightarrow \mathbb{R}$.

We shall consider the asymptotic expansion for near-diagonal rescaled displacement. Let us analyze the less general case where $\tilde{\mu}$ is locally free at x and G_x is contained in Z.

Let us choose a set of Heisenberg local coordinates centered at $x \in X$ for which we refer to [SZ02]. We shall set $x+(\theta,\, \mathbf{v})$ for $v = (\theta,\, \mathbf{v}) \in (-\pi,\, \pi)\times\mathbb{R}^{2d}$. When $\theta = 0$, we shall write $x + \mathbf{v}$ for $x + (0,\, \mathbf{v})$.

Definition 1.2.3. If $m \in M$ and $\mathbf{v}_1,\, \mathbf{v}_2 \in T_m M$, following [SZ02] let us set

$$\psi_2\left(\mathbf{v}_1,\, \mathbf{v}_2\right) := -i\omega_m(\mathbf{v}_1,\, \mathbf{v}_2) - \frac{1}{2}\left\|\mathbf{v}_1 - \mathbf{v}_2\right\|^2 .$$

Let us set $\mathbf{v}^{(g)} := \mathrm{d}_{m_x}\mu_g(\mathbf{v})$ and define

$$u_0(\nu,\, x) := \frac{\nu}{2\lambda(m_x)} .$$

Given this definition we can formulate our result about the scaling asymptotics for $\Pi_{k\boldsymbol{\nu}}(x,\, x)$.

Theorem 1.2.3. *Let us assume that $\mathbf{0} \notin \Phi_G(M)$ and that $\tilde{\mu}$ is locally free on X. Let G_x be the stabilizer subgroup of x, and suppose that $G_x \leq Z$. Suppose $C > 0$, $\epsilon \in (0,\, 1/6)$, and if $x \in X$ let us set $m := \pi(x)$. Then, uniformly in $\mathbf{v}_1,\, \mathbf{v}_2 \in \mathfrak{g}_M(m_x)^{\perp_h}$ satisfying $\|\mathbf{v}_j\| \leq C\, k^{\epsilon}$, belonging to subspace transverse to the orbit through x and with $\theta_1 - \theta_2 = 0$; we have for $k \to +\infty$ an asymptotic expansion*

$$\begin{aligned}
&\Pi_{k\boldsymbol{\nu}}\left(x + \frac{1}{\sqrt{k}}\, v_1,\, x + \frac{1}{\sqrt{k}}\, v_2\right)\\
&\quad\sim \frac{1}{2\lambda(m_x)}\left(\frac{k\nu}{2\pi\,\lambda(m_x)}\right)^{d} \cdot \sum_{g\in G_x} f_{1-k}(g)\cdot e^{u_0\,\psi_2(\mathbf{v}_1^{(g)},\, \mathbf{v}_2)}\\
&\quad\cdot\left[1 + \sum_{j\geq 1} k^{-j/2} a_j(\nu,\, m,\, \mathbf{v}_1,\, \mathbf{v}_2)\right],
\end{aligned}$$

where $a_j(\nu,\, m,\, \cdot,\, \cdot)$ is a polynomial of degree $\leq 3j$ and parity $(-1)^j$.

We can apply Theorems 1.2.3 and 1.2.2 to estimate the dimension of $H(X)_{\boldsymbol{\nu}}$ when $k \to +\infty$. Let us make this explicit in the case where $\tilde{\mu}$ is generically free.

Corollary 1.2.1. *Le us assume that* $\mathbf{0} \notin \Phi_G(M)$ *and that* $\tilde{\mu}$ *is free on* X*, we have*

$$\lim_{k\to+\infty}\left[\left(\frac{2\pi}{k\nu}\right)^d \cdot \dim\left(H_{k\boldsymbol{\nu}}(X)\right)\right] = \frac{1}{2}\int_M \lambda(m_x)^{-d-1}\,\mathrm{dV}_M(m)\,.$$

Proof. For $k = 1, 2, \ldots$ let us define $f_k \in \mathcal{C}^\infty(M)$ by setting $f_k(m) := k^{-d}\,\Pi_{k\boldsymbol{\nu}}(x, x)$ if $m = \pi(x)$. Given Theorem 1.2.2, $f_k \leq C$ for some constant $C > 0$ and $f_k \to (\nu/2\pi)^d\,\Phi^{-(d+1)}/2$ for $k \to +\infty$. By the dominate convergence theorem we can conclude. □

1.3 The case $G = U(2)$

First, let us set some notations. In this section we will denote by G the group $U(2)$, its Lie algebra $\mathfrak{g}$ is the space of skew-Hermitian matrices. There is a natural G-invariant Euclidean scalar product on $\mathfrak{g}$, given by $\langle \xi_1, \xi_2\rangle = \mathrm{trace}(\xi_1\bar{\xi}_2^t)$. By means of $\langle\cdot, \cdot\rangle$, we can equivariantly identify $\mathfrak{g}$ and $\mathfrak{g}^\vee$. If $\boldsymbol{\nu} \in \mathbb{Z}^2$, $\nu_1 > \nu_2$, let us set

$$D_{\boldsymbol{\nu}} := \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}\,.$$

Let us introduce the following loci:

1. $\mathcal{O}_{\boldsymbol{\nu}}$ is the (co)adjoint orbit of $i\,D_{\boldsymbol{\nu}}$;
2. $\mathcal{C}(\mathcal{O}_{\boldsymbol{\nu}}) := \mathbb{R}_+ \cdot \mathcal{O}_{\boldsymbol{\nu}}$ is the cone over $\mathcal{O}_{\boldsymbol{\nu}}$;
3. in M and X, respectively, we have the inverse images

$$M^G_{\mathcal{O}_{\boldsymbol{\nu}}} := \Phi_G^{-1}\left(\mathcal{C}(\mathcal{O}_{\boldsymbol{\nu}})\right), \quad X^G_{\mathcal{O}_{\boldsymbol{\nu}}} := \pi^{-1}\left(M^G_{\mathcal{O}_{\boldsymbol{\nu}}}\right)\,.$$

Finally, let us define the following smooth functions

$$m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}} \mapsto h_m T \in G/T, \quad m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}} \mapsto \lambda_{\boldsymbol{\nu}}(m) \in (0, +\infty)$$

by the equality

$$\Phi_G(m) = i\lambda_{\boldsymbol{\nu}}(m) h_m D_{\boldsymbol{\nu}} h_m^{-1}\,. \tag{1.6}$$

Our first result is the following.

Theorem 1.3.1. *Assume that $\mathbf{0} \notin \Phi_G(M)$, and Φ_G is transverse to $\mathcal{C}(\mathcal{O}_{\boldsymbol{\nu}})$. Let us define $G \times G$-invariant subset of $X \times X$*

$$\mathcal{Z}_{\boldsymbol{\nu}} := \left\{(x, y) \in X^G_{\mathcal{O}_{\boldsymbol{\nu}}} \times X^G_{\mathcal{O}_{\boldsymbol{\nu}}} \, : \, y \in G \cdot x\right\}.$$

Then, uniformly on compact subset of $(X \times X) \setminus \mathcal{Z}_{\boldsymbol{\nu}}$, we have

$$\Pi_{k\boldsymbol{\nu}}(x, y) = O(k^{-\infty}).$$

Corollary 1.3.1. *Uniformly on compact subset of $X \setminus X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, we have*

$$\Pi_{k\boldsymbol{\nu}}(x, x) = O(k^{-\infty}) \quad \textit{for} \quad k \to +\infty.$$

The hypothesis of Theorem 1.3.1 imply that $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ is a compact and smooth real hypersurface of M. Our next aim is to elucidate the geometry of $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, we need to introduce some further loci related to the action.

Definition 1.3.1. Let

$$M^G_{\boldsymbol{\nu}} := \Phi_G^{-1}\left(i\mathbb{R}_+ \cdot D_{\boldsymbol{\nu}}\right), \quad X^G_{\boldsymbol{\nu}} := \pi^{-1}\left(M^G_{\boldsymbol{\nu}}\right).$$

Remark 2. Under the assumption of Theorem 1.3.1, $M^G_{\boldsymbol{\nu}}$ is a compact submanifold of M, of real codimension 3. Clearly, $M^G_{\mathcal{O}_{\boldsymbol{\nu}}} = G \cdot M^G_{\boldsymbol{\nu}}$, i.e. the G-saturation of $M^G_{\boldsymbol{\nu}}$, by the equivariance of Φ_G.

There is a natural map $\mathrm{diag} : \mathfrak{g}^\vee \to \mathfrak{t}^\vee$ given by

$$\mathrm{diag} : i\begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix} \mapsto i\begin{pmatrix} a \\ b \end{pmatrix}$$

where $a, b \in \mathbb{R}$ and $z \in \mathbb{C}$. The action of T on M induced by restriction of μ is also Hamiltonian, with moment map given by

$$\Phi_T := \mathrm{diag} \circ \Phi_G : M \to \mathfrak{t}.$$

Let us introduce the loci

$$M^T_{\boldsymbol{\nu}} := \Phi_T^{-1}(\mathbb{R}_+ \cdot i\boldsymbol{\nu}), \quad X^T_{\boldsymbol{\nu}} := \pi^{-1}(M^T_{\boldsymbol{\nu}}).$$

Let us assume that $\mathbf{0} \notin \Phi_T(M)$ and that Φ_T is transverse to $\mathbb{R}_+ \cdot i\boldsymbol{\nu}$; then $M^T_{\boldsymbol{\nu}}$ is a compact smooth real hypersurface of M. Since $M^G_{\boldsymbol{\nu}} \subset M^T_{\boldsymbol{\nu}}$, we have $M^G_{\mathcal{O}_{\boldsymbol{\nu}}} \subset G \cdot M^T_{\boldsymbol{\nu}}$.

In §4.5.1 we shall construct a vector field $\Upsilon = \Upsilon_{\mu,\boldsymbol{\nu}}$ tangent to M along $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, naturally associated to the action and the weight, which is nowhere vanishing and everywhere normal to $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$. Explicitly, $\Upsilon = \Upsilon_{\mu,\boldsymbol{\nu}}$ along $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ is

$$\Upsilon(m) := J_m\left(\boldsymbol{\rho}(m)_M(m)\right) \qquad \left(m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}}\right),$$

where

$$\boldsymbol{\rho}(m) := i\,h_m D_{\boldsymbol{\nu}_\perp} h_m^{-1}$$

and $\boldsymbol{\nu}_\perp := (-\nu_2,\, \nu_1)^t$.

Theorem 1.3.2. *Let us assume that:*

1. Φ_G *and* Φ_T *are both transverse to* $\mathbb{R}_+ \cdot i D_{\boldsymbol{\nu}}$*;*
2. $\mathbf{0} \notin \Phi_T(M)$ *(hence also* $\mathbf{0} \notin \Phi_G(M)$*);*
3. $M^G_{\boldsymbol{\nu}} \neq \emptyset$ *(equivalently* $M^G_{\mathcal{O}_{\boldsymbol{\nu}}} \neq \emptyset$*);*
4. $\nu_1 + \nu_2 \neq 0$.

Then

1. $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ *is a connected and orientable smooth hypersurface in* M*, and separates* M *in two connected components: the "outside"* $A := M \setminus G \cdot M^T_{\boldsymbol{\nu}}$ *and the "inside"* $B := G \cdot M^T_{\boldsymbol{\nu}} \setminus M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$*;*
2. *the normal bundle to* $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ *in* M *is the real line sub-bundle of* $TM|_{M^G_{\mathcal{O}_{\boldsymbol{\nu}}}}$ *spanned by* Υ*;*
3. Υ *is "outer" oriented if* $\nu_1+\nu_2 > 0$ *and "inner" oriented if* $\nu_1+\nu_2 < 0$*;*
4. $M^G_{\mathcal{O}_{\boldsymbol{\nu}}} \cap M^T_{\boldsymbol{\nu}} = M^G_{\boldsymbol{\nu}}$*, and the two hypersurcafes meet tangentially along* $M^G_{\boldsymbol{\nu}}$*.*

Let us clarify the meaning of the partition $M = A \,\dot\cup\, M^G_{\mathcal{O}_{\boldsymbol{\nu}}} \,\dot\cup\, B$. Clearly, $G \cdot M^T_{\boldsymbol{\nu}} = \bar{B}$, $A = \left(G \cdot M^T_{\boldsymbol{\nu}}\right)^c$. For any $m \in M$, let $\mathcal{O}_{\Phi(m)} := \Phi_G(G \cdot m)$ be the coadjoint orbit of $\Phi_G(m)$. By the Schur-Horn Theorem, see [Hor54], $\mathrm{diag}\left(\mathcal{O}_{\Phi(m)}\right)$ is the segment J_m joining $i\,(\lambda_1,\, \lambda_2)^t$ with $i\,(\lambda_2,\, \lambda_1)^t$. Then:

1. $m \in A$ if and only if the orthogonal projection of the orbit $\mathcal{O}_{\Phi(m)}$ in $\mathfrak{t}$, $\mathrm{diag}\left(\mathcal{O}_{\Phi(m)}\right)$, is disjoint from $i\mathbb{R}_+ \cdot \boldsymbol{\nu}$;

2. $m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ if and only if $\mathrm{diag}(\mathcal{O}_{\Phi(m)}) \cap (i\mathbb{R}_+ \cdot \boldsymbol{\nu})$ is an endpoint of J_m;

3. $m \in B$ if and only if $\mathrm{diag}(\mathcal{O}_{\Phi(m)}) \cap (i\mathbb{R}_+ \cdot \boldsymbol{\nu})$ is an interior point of J_m.

The next step will be to provide some more precise quantitative information on the rate of decay of $\Pi_{k\boldsymbol{\nu}}(\cdot, \cdot)$ on the complement of $\mathcal{Z}_{\boldsymbol{\nu}}$. Namely, we shall show that $\Pi_{k\boldsymbol{\nu}}(x, y)$ is still rapidly decreasing when either $y \to G \cdot x$ at a sufficiently slow rate, or when at least one of x and y belongs to the "outer" component A, and converges to $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ sufficiently slowly.

Let us consider on X the Riemannian structure which is uniquely determined by the following conditions:

1. (1.2) is an orthogonal direct sum;

2. $\pi : X \to M$ is a Riemannian submersion;

3. the S^1-orbits have unit length.

The corresponding density is dV_X. Let $\mathrm{dist}_X : X \times X \to [0, +\infty)$ denote the associated distance function.

Theorem 1.3.3. *In the situation of Theorem 1.3.1, assume in addition that G acts freely on $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$. For any fixed $C, \epsilon > 0$, we have $\Pi_{k\boldsymbol{\nu}}(x, y) = O(k^{-\infty})$ uniformly for*

$$\max\{\mathrm{dist}_X(x, G \cdot y), \mathrm{dist}_X(x, G \cdot X^T_{\boldsymbol{\nu}})\} \geq C\, k^{\epsilon - 1/2}.$$

Remark 3. Theorem 1.3.3 is built on the general theory of Guillemin and Sternberg; we have established that the $U(2)$-equivariant asymtptotics of the Szegö kernels are rapidly decreasing for points not lying in $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$. However, the precise structure of this rapid decrease along the boundary is not clear at the moment, since the usual scaling techniques based on [SZ02] do not seem to cope with "inner" directions.

In Theorem 1.3.4 below we shall consider the diagonal asymptotics behaviour of $\Pi_{k\boldsymbol{\nu}}$ along $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ assuming the action is free on it. Before giving the statement, some further notation is needed.

Definition 1.3.2. Given $\boldsymbol{\nu} \in \mathbb{Z}^2$, we shall denote by $\boldsymbol{\nu}_M \in \mathfrak{X}(M)$ and $\boldsymbol{\nu}_X \in \mathfrak{X}(X)$ the vector fields induced by $iD_{\boldsymbol{\nu}}$; similarly, for every $gT \in G/T$, $\mathrm{Ad}_g(\boldsymbol{\nu})_M$ and $\mathrm{Ad}_g(\boldsymbol{\nu})_X$ will be the vector field induced by $\mathrm{Ad}_g(iD_{\boldsymbol{\nu}})$ respectively on M and X.

Definition 1.3.3. Let $\|\cdot\|_m : T_mM \to \mathbb{R}$ and $\|\cdot\|_x : T_xX \to \mathbb{R}$ be the norm functions. If $\boldsymbol{\nu}_1,\, \boldsymbol{\nu}_2 \in \mathbb{Z}^2$, $\nu_1 > \nu_2$, let us set $\boldsymbol{\nu}_\perp := (-\nu_2,\, \nu_1)$. Let us define the smooth function $\mathcal{D}_{\boldsymbol{\nu}} :\, M^G_{\mathcal{O}_{\boldsymbol{\nu}}} \to (0,\, +\infty)$ by posing

$$\mathcal{D}_\nu(m) := \frac{\|\boldsymbol{\nu}\|}{\|\mathrm{Ad}_{h_m}(\boldsymbol{\nu}_\perp)_M(m)\|_m}\,.$$

For every $m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ and $gT \in G/T$ we have $\langle \Phi_G(m),\, \mathrm{Ad}_g(iD_{\boldsymbol{\nu}_\perp})\rangle = 0$. Let us set $\pi(x) = m_x$. Hence, by (1.4), the function $\mathcal{D}_{\boldsymbol{\nu}}$ is well-defined since

$$\|\mathrm{Ad}_{h_m}(\boldsymbol{\nu}_\perp)_M(m_x)\|_{m_x} = \left\|\mathrm{Ad}_{h_{m_x}}(\boldsymbol{\nu}_\perp)_X(x)\right\|_x > 0\,.$$

Let us record one more piece of notation. If V_3 is the area of the unit sphere $S^3 \subset \mathbb{R}^4$, let us set

$$D_{G/T} := 2\pi\, V_3^{-1}\,.$$

Theorem 1.3.4. *Under the same hypothesis as in Theorem 1.3.2, let us assume in addition that G atcs freely on $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$. Then uniformly in $x \in X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ we have for $k \to +\infty$ an asymptotics expansion of the form*

$$\Pi_{k\boldsymbol{\nu}}(x,\, x) \sim \frac{D_{G/T}}{\sqrt{2}} \frac{1}{\|\Phi_G(m_x)\|^{d+1/2}} \left(\frac{k\,\|\boldsymbol{\nu}\|}{\pi}\right)^{d-1/2} \cdot \mathcal{D}_{\boldsymbol{\nu}}(m_x)$$
$$\left[1 + \sum_{j\geq 1} k^{-j/4} a_j(\boldsymbol{\nu},\, m_x)\right]\,.$$

Let us also discuss how the expansion generalizes to the case where the action on $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ is only locally free. Let $x \in X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, then the stabilizer G_x is a discrete, hence finite, subgroup of G. By the equivariance of Φ_G, we have that g commutes with $\Phi_G(m_x)$ for each $g \in G_{m_x}$. Thus, let h_{m_x} as in (1.6), all the matrices that commute with $\Phi_G(m_x)$ have the form

$$h_{m_x} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} h^{-1}_{m_x}\,,$$

for some $e^{i\alpha},\, e^{i\beta} \in S^1$. Since $G_x \subseteq G_{m_x}$, there exists a well-defined finite subgroup R_x of T such that $G_x = h_m\, R_x\, h_m^{-1}$.

Let Z the subgroup of scalar matrices, $Z \cong S^1$, and denote with $Z_x := Z \cap G_x$. Since Z_x is finite subgroup of S^1, it is a cyclic group, i.e. there exists $b_x \in \mathbb{N}$ such that

$$\beta_x = e^{\frac{2\pi}{b_x}\, i}\, I_2$$

is the generator of Z_x, $|Z_x| = b_x$. Let us define the function

$$\Gamma_k(x) = \sum_{g \in Z_x} g^k = \sum_{j=0}^{b_x - 1} (\beta_x^k)^j \,.$$

Furthermore $G_x \setminus Z_x$ has even cardinality $2a_x$, and perhaps after renumbering its elements can be arranged in pairs $(g_j,\, g_j^{-1})$, $j = 1, \ldots, a_x$. . Thus for each $j = 1, 2, \ldots, a_x$ the corresponding $g_j \in G_x \setminus Z_x$ has the form

$$g_j = h_m \begin{pmatrix} e^{i\alpha_j} & 0 \\ 0 & e^{i\beta_j} \end{pmatrix} h_m^{-1} \,,$$

where $\alpha_j \neq \beta_j$. In §4.8 we shall define a complex-valued smooth function $\Theta_j(x,\, \boldsymbol{\nu})$, depending on α_j, β_j, $\boldsymbol{\nu}$ and x; for ease of exposition we postpone its definition and the proof of the following Theorem 1.3.5 in section §4.8 .

Theorem 1.3.5. *Assume that the same hypothesis as in Theorem 1.3.2 hold. Then uniformly in $x \in X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ we have for $k \to +\infty$ that the leading term of the asymptotics expansion of $\Pi_{k\boldsymbol{\nu}}(x,\, x)$ is*

$$\frac{D_{G/T}}{\|\Phi_G(m_x)\|^{d+1/2}} \cdot \left(\frac{k\,\|\boldsymbol{\nu}\|}{\pi} \right)^{d-1/2} \left\{ \frac{1}{\sqrt{2}}\, \mathcal{D}_{\boldsymbol{\nu}}(m_x)\, \Gamma_{1-(\nu_1+\nu_2)k}(x) \right.$$
$$\left. + \, 2\sqrt{2\, \|\Phi_G(m_x)\|\, \|\boldsymbol{\nu}\|}\, (\nu_1 - \nu_2) \cdot \sum_{j=0}^{a_x} \Re \left[\frac{\left(e^{i\alpha_j} - e^{i\beta_j} \right) e^{-i\,k\,(\nu_1 \alpha_j + \nu_2 \beta_j)}}{\sqrt{\Theta_j(x,\, \boldsymbol{\nu})}} \right] \right\} .$$

From now on, to simplify our exposition, we shall make the stronger assumption that $\tilde{\mu}$ is actually free along $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$.

We can refine the previous asymptotic expansion at a fixed diagonal point $(x,\, x) \in X^G_{\mathcal{O}_{\boldsymbol{\nu}}} \times X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ to an asymptotic expansion for near-diagonal rescaled displacement; however, for the sake of simplicity we shall restrict the directions of the displacements.

Definition 1.3.4. If $m \in M$, let $\mathfrak{g}_M(m) \subset T_m M$ be the image of the linear evaluation map $\mathrm{val}_m : \mathfrak{g} \to T_m M$, $\xi \mapsto \xi_M(m)$; also, let $\mathfrak{g}_M(m)^{\perp_\omega} \subseteq T_m M$ be its symplectic orthocomplement with respect to ω_m, and let $\mathfrak{g}_M(m)^{\perp_g} \subseteq T_m M$ be its Riemannian orthocomplement with respect to g_m. Hence,

$$\mathfrak{g}_M(m)^{\perp_h} := \mathfrak{g}_M(m)^{\perp_\omega} \cap \mathfrak{g}_M(m)^{\perp_g} \subseteq T_m M$$

is the Hermitian orthocomplement of the complex subspace generated by $\mathfrak{g}_M(m)$ with respect to $h_m := g_m - i\omega_m$.

Definition 1.3.5. If $\mathbf{v}_1,\, \mathbf{v}_2 \in T_m M$, following [SZ02] let us set

$$\psi_2(\mathbf{v}_1,\, \mathbf{v}_2) := -i\, \omega_m(\mathbf{v}_1,\, \mathbf{v}_2) - \frac{1}{2}\, \|\mathbf{v}_1 - \mathbf{v}_2\|_m^2 \, . \tag{1.7}$$

Here $\|\mathbf{v}\|_m := g_m(\mathbf{v},\, \mathbf{v})^{1/2}$. The same invariant can be introduced in any Hermitian vector space. Given the choice of a system of Heisenberg local coordinates centred at $x \in X$ ([SZ02]), there is built-in unitary isomorphism $T_m M \cong \mathbb{C}^d$; with this implicit, (1.7) will be used with $\mathbf{v}_j \in \mathbb{C}^d$.

The choice of Heisenberg local coordinates centred at $x \in X$ gives a meaning to the expression $x + (\theta, \mathbf{v})$ for $(\theta, \mathbf{v}) \in (\pi, \pi) \times \mathbb{R}^{2d}$ with $\|\mathbf{v}\|$ of sufficiently small norm. When $\theta = 0$, we shall write $x + \mathbf{v}$.

Theorem 1.3.6. *Let us assume the same hypothesis as in Theorem 1.3.4. Suppose $C > 0$, $\epsilon \in (0, 1/6)$, and if $x \in X$ let us set $m_x := \pi(x)$. Then, uniformly in $x \in X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ and $\mathbf{v}_1,\, \mathbf{v}_2 \in \mathfrak{g}_M(m_x)^{\perp_h}$ satisfying $\mathbf{v}_j \leq C\, k^{\epsilon}$, we have for $k \to +\infty$ an asymptotics expansion*

$$\Pi_{k\boldsymbol{\nu}}\left(x + \frac{1}{\sqrt{k}}\mathbf{v}_1,\, x + \frac{1}{\sqrt{k}}\mathbf{v}_2\right) \sim \frac{D_{G/T}}{\sqrt{2}}\, \frac{e^{\psi_2(\mathbf{v}_1,\, \mathbf{v}_2)/\lambda_{\boldsymbol{\nu}}(m_x)}}{\|\Phi_G(m_x)\|^{d+1/2}} \left(\frac{k\,\|\boldsymbol{\nu}\|}{\pi}\right)^{d-1/2} \cdot \mathcal{D}_{\boldsymbol{\nu}}(m_x)\left[1 + \sum_{j\geq 1} k^{-j/4} a_j(\nu,\, m_x,\, \mathbf{v}_1,\, \mathbf{v}_2)\right]$$

where $a_j(\nu,\, m,\, \cdot,\, \cdot)$ is a polynomial of degree $\leq \lceil 3j/2 \rceil$.

Furthermore, we shall provide an integral formula of independent interest for the asymptotics of $\Pi_{k\boldsymbol{\nu}}(x',\, x')$ when $x' \to X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ at a "fast" pace from the "outside" (that is, $x' \in A$ in the notation of Theorem 1.3.2). While the latter formula is a bit too technical to be described in this introduction, by global integration it leads to a lower bound on $\dim H(X)_{\boldsymbol{\nu}}$ which can be stated in a compact form. With the notation of Theorem 1.3.2, we have

$$\dim H(X)_{\boldsymbol{\nu}} = \dim_{in} H(X)_{\boldsymbol{\nu}} + \dim_{out} H(X)_{\boldsymbol{\nu}}\, ,$$

where

$$\dim_{out} H(X)_{\boldsymbol{\nu}} := \int_A \Pi_{\boldsymbol{\nu}}(x,\, x)\, \mathrm{dV}_X(x)\, ,$$

and similarly for $\dim_{in} H(X)_{\boldsymbol{\nu}}$, with A replaced by B. Hence an asymptotic estimate for $\dim_{out} H(X)_{\boldsymbol{\nu}}$ when $k \to +\infty$ implies an asymptotic lower bound for $\dim H(X)_{k\boldsymbol{\nu}}$. In Theorem 1.3.7 below, we shall show that $\dim_{out} H(X)_{k\boldsymbol{\nu}}$ is given by an asymptotic expansion of descending fractional powers of k, the leading power being k^{d-1}.

Theorem 1.3.7. *Under the assumption of Theorem 1.3.4,* $\dim_{out} H(X)_{\boldsymbol{\nu}}$ *is given by an asymptotic expansion in descending powers of* $k^{1/4}$ *as* $k \to +\infty$*, with leading order term*

$$\frac{1}{4}\, D_{G/T} \left(\frac{k\,\|\boldsymbol{\nu}\|}{\pi}\right)^{d-1} \int_{M^G_{\mathcal{O}_{\boldsymbol{\nu}}}} \frac{1}{\|\Phi_G(m)\|^d} \cdot \mathcal{D}_{\boldsymbol{\nu}}(m)\, \mathrm{dV}_{M^G_{\mathcal{O}_{\boldsymbol{\nu}}}}(m)\,.$$

1.4 Recalls on prior literature

The asymptotic expansions of the Szegö kernel and its variants were studied in a lot of papers and they are grounded on a well-known foundational result due to Boutet de Monvel and Sjöstrand [BdMS76], who proved that Π is a Fourier integral operator with complex phase. Beginning with the papers of Zelditch on a theorem of Tian, [Zel98], the FIO construction for these kernel functions has proved extremely useful in the study of the behavior, as the power tends to infinity, of the space of sections of powers of a positive line bundle over an algebraic variety.

More explicitly, let (A, h) be an Hermitian line bundle over a compact d-dimensional Kähler manifold (M, ω). Let $(s_0^k, \dots, s_{d_k}^k)$ be any orthonormal basis of $H^0(M, A^{\otimes k})$ (where $d_k + 1$ is the dimension of $H^0(M, A^{\otimes k})$), with respect to the inner product

$$\langle s_1,\, s_2\rangle_{h_k} = \int_M h_k(s_1(m),\, s_2(m))\, \mathrm{dV}_M(m).$$

Catlin and Zelditch, indipendently, sharpened a theorem conjectured by Yau, solved by Tian, proving that there exists a complete asymptotic expansion:

$$\sum_{j=0}^{d_k} \left\| s_j^k(m) \right\|^2_{h_k} = a_0\, k^d + a_1(m)\, k^{d-1} + a_2(m)\, k^{d-2} + \dots \tag{1.8}$$

for certain smooth coefficients $a_j(m)$ with $a_0 = \pi^{-d}$. More recently, Z. Lu [Lu00] proved that the lower terms of the asymptotic expansion (1.8) is a polynomial of the curvature and its covariant derivatives at m of the metric

g, associated to the form ω; and he computed the first three coefficients of this expansion (see also [Loi04] and [Loi05] for the computations of the coefficients a_j's).

The expansion (1.8) is sometimes called the TYZ (Tian-Yau-Zelditch) expansion (also Catlin in [Cat99] published the same result as Zelditch in the same year). Since the CR structure of the circle bundle X is S^1-invariant (see the next chapter for definitions and details), there is a naturally induced unitary representation of S^1 on $H(X)$; therefore, $H(X)$ splits unitarily and equivariantly as a direct sum of isotypical components $H_k(X)$, $k \in \mathbb{Z}$. We have $H_k(X) = 0$ if $k < 0$, and for $k \geq 0$ there is a natural unitary isomorphism between $H_k(X)$ and $H^0\left(M, A^{\otimes k}\right)$. Thus the expansion (1.8) coincides with the asymptotics of the kernel of the projector $\Pi_k : L^2(X) \to H_k(X)$ along the diagonal.

In the situation described above the standard S^1 action on X is trivial on M and thus it is Hamiltonian with respect to 2ω with constant moment map. Thus we can generalize the situation described in the previous paragraph as follow. Suppose that we have another Hamiltonian holomorphic action μ of S^1 on M with moment map $\Phi : M \to \mathbb{R}$; suppose furthermore that it can be linearized to a holomorphic action $\tilde{\mu}$ on A. Then S^1 acts on X as a group of contactomorphisms under the naturally induced action. Since the lifted action $\tilde{\mu}$ preserves the contact form α and it is a lifting of the holomorphic action μ, it leaves $H(X)$ invariant; therefore it determines a unitary action of S^1 on $H(X)$. Thus $H(X)$ equivariantly and unitarily decomposes into the Hilbert direct sum of its isotypes $H_k^\mu(X)$. In general, however, the isotypical components in point don't correspond to subspaces of holomorphic sections of some higher tensor power of the polarizing line bundle.

Nonetheless, if $\Phi > 0$ then $H_k^\mu(X)$ is finite-dimensional for any $k \in \mathbb{Z}$, and is the null space if $k < 0$; in particular, the orthogonal projector $\Pi_k^\mu(X) : L^2(X) \to H_k^\mu(X)$ is a smoothing operator. Uniformly on $x \in X$, there is an asymptotic expansion of the form

$$\Pi_k^\mu(x,x) \sim \left(\frac{k}{\pi}\right)^d \cdot \left[\Phi(m)^{-(d+1)} + \sum_{j\geq 1} k^{-j} a_j^\mu(m_x)\right],$$

and the first term a_1^μ is computed in [Pao15]; in particular if $\Phi = 1$, one recovers Lu's subprincipal term [Lu00].

In the n-dimensional toric case, this theme has been studied in [Pao12], [Pao15] and [Cam16]. One has a decomposition of the Hardy space into isotypes $H_{k\boldsymbol{\nu}}(X)$, $\boldsymbol{\nu} \in \mathbb{Z}^n$, and one is led to investigate the behavior of the kernel of the projector $\Pi_{k\boldsymbol{\nu}} : L^2(X) \to H_{k\boldsymbol{\nu}}(X)$ as k goes to infinity. Suppose that $\mathbf{0} \notin \Phi(M)$; then

1. If $\Phi(m) \notin \mathbb{R}_+ \cdot \boldsymbol{\nu}$, then $\Pi_{k\boldsymbol{\nu}} = O(k^{-\infty})$.

2. Assume that Φ is transversal to $\mathbb{R}_+ \cdot \boldsymbol{\nu}$. Then for every $m = \pi(x) \in \Phi^{-1}(\mathbb{R}_+ \cdot \boldsymbol{\nu})$ (where π is the projection $\pi : A \to M$) as $k \to \infty$ we have

$$\Pi_{k\boldsymbol{\nu}}(x,x) \sim \frac{1}{(\sqrt{2}\,\pi)^{n-1}} \left(\|\boldsymbol{\nu}\| \cdot \frac{k}{\pi} \right)^{d+(1-n)/2} \cdot \sum_{g \in T_m} \chi_{\boldsymbol{\nu}}(g)^k \cdot \frac{1}{\mathcal{D}(m)} \left(\frac{1}{\|\Phi(m)\|} \right)^{d+1+(1-n)/2} \cdot \left(1 + \sum_{l \geq 1} B_l(m_x)\, k^{-l} \right);$$

where T_m is the stabilizer of m, $\chi_{\boldsymbol{\nu}}$ is the character pertaining to $\boldsymbol{\nu}$, $\mathcal{D} : M \to \mathbb{R}$ is a distortion function and B_l's are smooth functions on $\Phi^{-1}(\mathbb{R}_+ \cdot \boldsymbol{\nu})$ (we refer to [Pao12] for definitions).

In general, if the standard circle action is replaced by a geometrically induced representation of a generic compact Lie group G, then, under appropriate hypothesis, the isotypes are finite dimensional and one can ask whether there are some extension of the above asymptotic expansions. In this thesis, we turn to non-Abelian actions, and consider specifically the cases of $G = SU(2)$ and $G = U(2)$.

The analysis of Szegö kernels have other applications in complex and symplectic geometry. By studying the off-diagonal of Π_k, Shiffman and Zelditch obtain an analytic proof of the Kodaira embedding theorem (see [SZ02]). Explicitly, when $x \in X$ tends to the orbit through y at a suitable fast pace, there is an asymptotic expansion which captures an exponential decrease of $\Pi_k(x, y)$ away from $(\pi \times \pi)^{-1}(\Delta_M)$, in a family of shrinking neighborhoods of the latter (Δ_M is the diagonal in $M \times M$). To express this, following [BSZ00] and [SZ02] let us define $\psi_2 : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}$ by

$$\psi_2(\mathbf{v}_1, \mathbf{v}_2) := -\imath\, \omega(\mathbf{v}_1, \mathbf{v}_2) - \frac{1}{2}\, \|\mathbf{v}_1 - \mathbf{v}_2\|^2,$$

in the so-called Heisenberg local coordinates (henceforth, HLC's) centered at

$x \in X$, where $v_l = (\theta, \mathbf{v}_l) \in \mathbb{R} \otimes \mathbb{C}^d \cong T_x X$. We have

$$\Pi_k \left(x + \left(\theta_1, \frac{\mathbf{v}_1}{\sqrt{k}} \right), x + \left(\theta_2, \frac{\mathbf{v}_2}{\sqrt{k}} \right) \right) \sim \left(\frac{k}{\pi} \right)^d \cdot e^{\imath\, k\, (\theta_1 - \theta_2) + \psi_2(\mathbf{v}_1, \mathbf{v}_2)} \cdot \left[1 + \sum_{j \geq 1} k^{-j/2} \cdot R_j(m_x; \mathbf{v}_1, \mathbf{v}_2) \right]. \tag{1.9}$$

Here $R_j(m_x; \cdot, \cdot)$ is a polynomial of degree $\leq 3\, j$. For fixed $C > 0$ and $\epsilon \in (0, 1/6)$, the asymptotic expansion (1.9) holds uniformly for $\|\mathbf{v}_j\| \leq C\, k^{\epsilon}$.

For a general torus action we have, under some suitable hypothesis and uniformly in $m_x \in \Phi^{-1}(\mathbb{R}_+ \cdot \boldsymbol{\nu})$, the following asymptotic expansion as $k \to +\infty$

$$\begin{aligned} \Pi_{k\boldsymbol{\nu}} \left(x + \frac{\mathbf{v}_1}{\sqrt{k}}, x + \frac{\mathbf{v}_2}{\sqrt{k}} \right) \sim & \\ \frac{1}{(\sqrt{2}\pi)^{n-1}} \left(\|\boldsymbol{\nu}\| \cdot \frac{k}{\pi} \right)^{d + (1-n)/2} & \frac{1}{\mathcal{D}(m)} \left(\frac{1}{\|\Phi(m)\|} \right)^{(d+1) + (1-n)/2} \\ \cdot \left(\sum_{t \in T_m} \chi_{\boldsymbol{\nu}}(t)^k \cdot e^{H_m(\mathrm{d}_m \tilde{\mu}_{t^{-1}}(\mathbf{v}_1), \mathbf{v}_2)} \right) & \\ \cdot \left[1 + \sum_{j \geq 1} k^{-j/2} \cdot R_j(m_x; \mathbf{v}_1, \mathbf{v}_2) \right], & \end{aligned} \tag{1.10}$$

where $\mathbf{v}_l \in N_m \subseteq T_m M$ (here N is the normal bundle of $\Phi^{-1}(\mathbb{R}_+ \cdot \boldsymbol{\nu})$ in M) and $\|\mathbf{v}_l\| \leq C\, k^{1/9}$. The smooth functions R_j's are polynomial in the $\mathbf{v}_l$'s and $H : TM \oplus TM \to \mathbb{C}$ is a smooth function (see [Pao12] for the precise definition). We provided similar results for the case of $G = SU(2)$ and $G = U(2)$, see Theorem 1.2.3 and Theorem 1.3.6. While we have restricted the exposition to the complex projective setting, the results of these thesis admit natural generalizations to the almost Kähler context, following the theory of generalized Szegö kernels in [SZ02].

In the paper [SZ99] Shiffman and Zelditch also use the Szegö kernel to show that the zeros of a 'random section' of $H^0\left(M, A^{\otimes k}\right)$ become uniformly distributed as $k \to +\infty$; it is also a key ingredient in the investigations of balanced metrics in Donaldsons terminology [Don01] (see also [AL04]). For variants of and alternative approaches to the general theme of asymptotic expansions see for instance [BU00], [Cha16], [MM07] and [MZ05].

In closing, let us emphasize that while the present analysis and results belong to the general framework of geometric quantization, since the unitary representation on $H(X)$ is a 'quantization' of the Hamiltonin action on M, they do not fit into the traditional framework of Berezin-Toeplitz quantization, since we are not working here within a fixed isotype for the structure S^1-action. Rather, a more appropriate heuristic framework is the one discussed in [GS82c] (and of course [BdMG81]). In this respect, let us note that, at least heuristically, the isotypical components in point should relate to the Riemann Roch numbers of certain symplectic reductions, of which the local asymtptotic expansions above can be seen as an estimate and geometric reinterpretation.

Chapter 2

Examples

2.1 Examples about $G = SU(2)$

Let us describe some examples where the geometric hypothesis of the previous discussion are verified.

Example 1

Let A be the hyperplane line bundle on $M = \mathbb{P}^1$, then the circle bundle $X \subset A^\vee$ may be identified with $S^3 \subset \mathbb{C}^2$, and the projection $\pi : X \to M$ with the Hopf map. There is a natural action of G on X given by left translation on itself. This action is free and descend to an Hamiltonian action on M given by

$$\mu_A([Z]) := [AZ], \qquad \big(Z = (z_0,\, z_1) \in \mathbb{C}^2 \setminus \{0\}\big) \ .$$

The moment map is

$$\Psi([Z]) := \frac{1}{\|Z\|^2} \begin{pmatrix} \frac{1}{2}(|z_0|^2 - |z|_1^2) & z_0\bar{z}_1 \\ \bar{z}_0 z_1 & \frac{1}{2}(|z_1|^2 - |z|_0^2) \end{pmatrix} ,$$

which is everywhere non-vanishing. Then $\lambda([z_0 : z_1]) = 1/2$ for any $[z_0 : z_1] \in \mathbb{P}^1$, and the contact action $\tilde{\mu}$ on S^3 is free, since it may be identified with action of $SU(2)$ on itself by left translations. Furthermore, $H_{k\boldsymbol{\nu}}(X) = H_{k\boldsymbol{\nu}-1}(X)$, where the right hand side is the $(k \cdot \nu - 1)$-th isotype for the S^1-action. With $\nu = 1$ the leading order term of the expansion of Theorem 1.2.2 is $(k/\pi)^d \cdot e^{\psi_2(\mathbf{v}_1, \mathbf{v}_2)}$, in agreement with the standard off-diagonal scaling asymptotics for Szegö kernels on $\mathbb{P}^1$, [SZ02].

Example 2

Let us consider the diagonal action of G on $\mathbb{P}^1 \times \mathbb{P}^1$,

$$\mu_A([Z],[W]) = ([AZ],[AW])\,.$$

For $r = 1, 2, \ldots$, consider the symplectic structure $\Omega_r := \omega_{FS} \boxtimes (r\,\omega_{FS})$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Then μ is Hamiltonian with respect to $2\,\Omega_r$, with the moment map

$$\Phi_r : ([Z],\,[W]) \mapsto \Psi([Z]) + r\,\Psi([W])\,.$$

If $r \geq 2$, then Φ_r is nowhere vanishing.

On the other hand, Ω_r is the normalized curvature of the positive line bundle $A_r := \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(r)$. The unit circle bundle X_r associated to A_r is the image of $S^3 \times S^3$ under the map

$$(Z,\,W) \in S^3 \times S^3 \subset \mathbb{C}^2 \times \mathbb{C}^2 \mapsto Z \otimes W^{\otimes r} \in \mathbb{C}^{2(r+1)}\,,$$

and the contact lift of μ is given by

$$\tilde{\mu}_A(Z \otimes W^{\otimes r}) = (AZ) \otimes (AW)^{\otimes r}\,.$$

Let us consider the stabilizer subgroup of $Z \otimes W^{\otimes r}$. We have

$$\tilde{\mu}_A(Z \otimes W^{\otimes r}) = Z \otimes W^{\otimes r} \Leftrightarrow AZ = \lambda_1\,Z, \quad AW = \lambda_2\,W$$

for certain $\lambda_1,\,\lambda_2 \in S^1$ with $\lambda_1 \cdot \lambda_2^r = 1$.

If Z and W are linearly dependent, then $\lambda_1 = \lambda_2$ and $\lambda_1^{r+1} = 1$. The stabilizer subgroup of $Z \otimes W^{\otimes r}$ is therefore cyclic of order $r+1$. Otherwise, $(Z,\,W)$ is an eigenbasis of A and $\lambda_2 = \lambda_1^{-1}$, $\lambda_1^{r-1} = 1$. Hence, assuming that $Z \wedge W \neq 0$, the stabilizer subgroup of $Z \otimes W^{\otimes r}$ is cyclic of order $r-1$ when $(Z,\,W)$ is an orthonormal basis of $\mathbb{C}^2$, and otherwise it is trivial when r is even and $\{\pm I_2\}$ when r is odd. Thus $\tilde{\mu}$ is locally free for $r \geq 2$. Furthermore, the action is generically free when r is even, and the stabilizer is generically of order two when r is odd.

Let us now consider how $V_{k\boldsymbol{\nu}}$ appears in

$$H(X_r) = \bigoplus_{l=0}^{+\infty} H_l(X_r), \quad H_l(X_r) \cong H^0(\mathbb{P}^1 \times \mathbb{P}^1,\, A_r^{\otimes l})\,.$$

Since $A_r^{\otimes l} = \mathcal{O}_{\mathbb{P}^1}(l) \boxtimes \mathcal{O}_{\mathbb{P}^1}(l\,r)$, by the Künneth formula we have

$$H^0(\mathbb{P}^1 \times \mathbb{P}^1,\, A_r^{\otimes l}) \cong H^0\left(\mathbb{P}^1,\, \mathcal{O}_{\mathbb{P}^1}(l)\right) \otimes H^0\left(\mathbb{P}^1,\, \mathcal{O}_{\mathbb{P}^1}(l\,r)\right)$$
$$\cong V_{(l+1,0)} \otimes V_{(lr+1,0)}\,.$$

Thus the character of $H^0(\mathbb{P}^1 \times \mathbb{P}^1, A_r^{\otimes l})$ as a G-representation is $\chi_{l+1} \cdot \chi_{l\,r+1}$. We see by a few computations that

$$\begin{aligned}
&(\chi_{l+1} \cdot \chi_{l\,r+1}) \left(\begin{pmatrix} e^{i\,\vartheta} & 0 \\ 0 & e^{-i\,\vartheta} \end{pmatrix}\right) \\
&\quad = \left(e^{i\,l\,\theta} + e^{i\,(l-2)\,\theta} + \cdots + e^{-i\,l\,\theta}\right) \cdot \frac{e^{i\,(l\,r+1)\,\theta} - e^{-i\,(l\,r+1)\,\theta}}{e^{i\,\theta} - e^{-i\,\theta}} \\
&\quad = \sum_{j=0}^{l} \frac{e^{i\,(l+l\,r+1-2j)\,\theta} - e^{-i\,(l+l\,r+1-2j)\,\theta}}{e^{i\,\theta} - e^{-i\,\theta}}\,.
\end{aligned}$$

Therefore,

$$H^0\left(\mathbb{P}^1 \times \mathbb{P}^1,\, A_r^{\otimes l}\right) \cong \bigoplus_{j=0}^{l} V_{(l+l\,r+1-2j,\,0)}\,.$$

We conclude that $V_{k\boldsymbol{\nu}}$ appears at most once in each $H_l(X_r)$; it does appear once, in fact, if and only if $k\,\nu$ and $l\,(r+1)+1$ have the same parity, and

$$\frac{k\,\nu - 1}{r-1} \geq l \geq \frac{k\,\nu - 1}{r+1}\,. \tag{2.1}$$

Suppose, for example, that $k\,\nu$ and $r+1$ are both even. Then $l\,(r+1)+1$ is odd for any choice of l and we conclude that $H_{k\boldsymbol{\nu}}(X)$ vanishes. Notice that at the general $x \in X_r$ we have $G_x = \{\pm I_2\}$, and $\sum_{g \in G_x} f_{1-k\,\nu}(g) = 0$. If, on the other hand, $r+1$ is even and $k\nu$ is odd, then there is a copy of $V_{k\boldsymbol{\nu}}$ in $H_l(X_r)$ for every integer l satisfying (2.1). Hence the number of copies of $V_{k\boldsymbol{\nu}}$ in $H(X_r)$ is $\sim 2k\nu/(r^2-1)$, so that the dimension of $H_{k\boldsymbol{\nu}}(X)$ is $\sim 2(k\,\nu)^2/(r^2-1)$. For the general $x \in X_r$, we have in this case $\sum_{g \in G_x} f_{1-k\,\nu}(g) = 2$.

When $r+1$ is odd, on the other hand, the generic stabilizer is trivial. For the general $x \in X_r$, therefore, $\sum_{g \in G_x} f_{1-k\,\nu}(g) = 1$ irrespective of $k\,\boldsymbol{\nu}$. If $k\boldsymbol{\nu}$ is even (respectively, odd) then there is a copy of $V_{k\boldsymbol{\nu}}$ in $H_l(X_r)$ if and only if l is odd (respectively, even) and satisfies (2.1). Thus the number of copies of $V_{k\boldsymbol{\nu}}$ in $H(X_r)$ is $\sim k\nu/(r^2-1)$, so that the dimension of $H_{k\boldsymbol{\nu}}(\mathrm{X})$ is $\sim (k\nu)^2 = (r^2-1)$.

2.2 Example about $G = U(2)$

Let A be the hyperplane line bundle on $M = \mathbb{P}^3$; then the unit circle bundle $X \subseteq A^\vee \setminus \{\mathbf{0}\}$ may be identified with $S^7 \subset \mathbb{C}^4 \setminus \{\mathbf{0}\}$, and the projection $\pi : X \to \mathbb{P}^3$ with the Hopf map.

Consider the unitary representation of $U(2)$ on $\mathbb{C}^4 \equiv \mathbb{C}^2 \oplus \mathbb{C}^2$ given by

$$A \cdot (Z, W) = (AZ, AW); \tag{2.2}$$

here $Z = (z_1, z_2)^t$, $W = (w_1, w_2)^t \in \mathbb{C}^2$. This linear action yields by restriction a contact action $\mu : G \times S^7 \to S^7$, and descends to an holomorphic action $\mu : G \times \mathbb{P}^3 \to \mathbb{P}^3$. If ω_{FS} is the Fubini-Study form on $\mathbb{P}^3$, then μ is Hamiltonian with respect to $2\,\omega_{FS}$. The moment map is

$$\Phi_G : [Z : W] \in \mathbb{P}^3 \mapsto \frac{i}{\|Z\|^2 + \|W\|^2}\,[z_i\,\bar{z}_j + w_i\,\bar{w}_j] \in \mathfrak{g}\,.$$

Furthermore, $\tilde{\mu}$ is the contact lift of μ.

From this, one can draw the following conclusions:

Lemma 2.2.1. *Under the previous assumptions, we have:*

1. $-i\,\Phi_G([Z : W])$ *is a convex linear combination of the orthogonal projections onto the subspaces of* $\mathbb{C}^2$ *spanned by* Z *and* W*, respectively;*

2. $-i\,\Phi_G([Z : W])$ *has rank* 2 *if and only if* Z *and* W *are linearly independent, rank* 1 *otherwise;*

3. $\Phi_G(M) = i\,K$*, where* K *denotes the set of all positive semi-definite Hermitian matrices of trace* 1*;*

4. *the determinant of* $-i\,\Phi_G([Z : W])$ *is*

$$\det(-i\,\Phi_G([Z : W])) = \frac{|Z \wedge W|^2}{(\|Z\|^2 + \|W\|^2)^2}\,,$$

where $Z \wedge W = z_1 w_2 - z_2 w_1 \in \mathbb{C}$*;*

5. *the eigenvalues of* $-i\,\Phi_G([Z : W])$ *are both real and given by*

$$\lambda_{1,2}([Z : W]) = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4\,|Z \wedge W|^2}{(\|Z\|^2 + \|W\|^2)^2}}\right)\,.$$

Let us fix $\boldsymbol{\nu} \in \mathbb{Z}^2$ with $\nu_1 > \nu_2 \geq 0$. Let, as above, $\mathcal{O}_{\boldsymbol{\nu}} \subseteq \mathfrak{g}$ denote the coadjoint orbit od $i D_{\boldsymbol{\nu}}$. With $M = \mathbb{P}^3$, the locus $M^G_{\mathcal{O}_{\boldsymbol{\nu}}} = \Phi_G^{-1}(\mathbb{R}_+ \cdot \mathcal{O}_{\boldsymbol{\nu}})$ is given by the condition

$$\nu_2 \, \lambda_1([Z:W]) - \nu_1 \, \lambda_2([Z:W]) = 0 \, .$$

In view of Lemma 2.2.1, this implies:

Corollary 2.2.1. *Under the previous hypothesis,*

$$M^G_{\mathcal{O}_{\boldsymbol{\nu}}} = \left\{ [Z:W] \in \mathbb{P}^3 : \, \frac{|Z \wedge W|}{\|Z\|^2 + \|W\|^2} = \frac{\sqrt{\nu_1 \nu_2}}{\nu_1 + \nu_2} \right\} .$$

Let us now consider transversality. By Lemma 4.5.1 below (see also the discussion in §2 of [Pao12]), Φ_G is transverse to the ray $\mathbb{R}_+ \cdot i D_{\boldsymbol{\nu}}$ in $\mathfrak{g}$ if and only if $\tilde{\mu}$ is locally free along $X^G_{\boldsymbol{\nu}}$ (that is, each $x \in X^G_{\boldsymbol{\nu}}$ has discrete stabilizer).

On the other hand, by (2.2) $\tilde{\mu}$ is locally free at $(Z, W) \in S^7$ if and only if $Z \wedge W \neq 0$, and this is equivalent to $\Phi_G([Z:W])$ having rank 2; this means that $-i \, \Phi_G([Z:W])$ has two positive eigenvalues. Thus we obtain the following.

Corollary 2.2.2. *The following conditions are equivalent:*

1. *Φ_G is transverse to $\mathbb{R}_+ \cdot i D_{\boldsymbol{\nu}}$ and $\Phi_G^{-1}(\mathbb{R}_+ \cdot i D_{\boldsymbol{\nu}}) \neq \emptyset$;*

1. *Φ_G is transverse to $\mathcal{O}_{\boldsymbol{\nu}}$, and $\Phi_G^{-1}(\mathbb{R}_+ \cdot \mathcal{O}_{\boldsymbol{\nu}}) \neq \emptyset$;*

3. *$\nu_1, \nu_2 > 0$.*

Let us now consider the restricted Hamiltonian action of T. Identifying $\mathfrak{t}$ with $i \, \mathbb{R}^2$, $\Phi_T : M \to \mathfrak{t}$ may be written:

$$\Phi_T : \mathbb{P}^3 \ni [Z:W] \mapsto \frac{i}{\|Z\|^2 + \|W\|^2} \begin{pmatrix} |z_1|^2 + |w_1|^2 \\ |z_2|^2 + |w_2|^2 \end{pmatrix} \in \mathfrak{t} \, . \tag{2.3}$$

Thus we obtain

Lemma 2.2.2. *Assume that $\nu_1 > \nu_2 \geq 0$; then:*

1. *the image of Φ_T in $\mathfrak{t} \cong i\mathbb{R}^2$ is*

$$\Phi_T(M) = i \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \, x + y = 1, \; x, \, y \geq 0 \right\} ;$$

2. *the locus* $M^T_{\boldsymbol{\nu}}$ *is given by*

$$M^T_{\boldsymbol{\nu}} = \{[Z : W] \in \mathbb{P}^3 \ : \ \nu_2(|z_1|^2 + |w_1|^2) = \nu_1(|z_2|^2 + |w_2|^2)\} \, ;$$

3. Φ_T *is transverse to* $\mathbb{R}_+ \cdot iD_{\boldsymbol{\nu}}$ *and* $M^G_{\boldsymbol{\nu}} \neq \emptyset$ *if and only if* $\nu_1, \, \nu_2 > 0$.

Proof. The first two statements follow immediately from (2.3). As to the third, let us recall again that Φ_T is transverse to $\mathbb{R}_+ \cdot iD_{\boldsymbol{\nu}}$ if and only if the action of T on $X^T_{\boldsymbol{\nu}} \subseteq S^7$ is locally free, [Pao12]. On the other hand, T acts locally freely at $(Z, W) \in S^7$ if and only if Z and W are neither both scalar multiples of $\mathbf{e}_1$, nor both scalar multiples of $\mathbf{e}_2$, where $(\mathbf{e}_1, \mathbf{e}_2)$ is the standard basis of $\mathbb{C}^2$. By 2), there are no points (Z, W) of this form in $X^T_{\boldsymbol{\nu}}$ if and only if $\nu_2 > 0$. □

Hence if $\nu_1, \, \nu_2 > 0$, then Φ_G and Φ_T are transverse to $\mathbb{R}_+ \cdot \boldsymbol{\nu}$, and $M^G_{\boldsymbol{\nu}} \neq \emptyset$, $M^T_{\boldsymbol{\nu}} \neq \emptyset$. For instance,

$$\left[\sqrt{\frac{\nu_1}{\nu_1 + \nu_2}} \boldsymbol{e}_1 \ : \ \sqrt{\frac{\nu_1}{\nu_1 + \nu_2}} \boldsymbol{e}_2 \right] \in M^G_{\boldsymbol{\nu}} \cap M^T_{\boldsymbol{\nu}} \, .$$

More generally, we have the following.

Lemma 2.2.3. *For any* $\boldsymbol{\nu}$, $M^G_{\boldsymbol{\nu}} \cap M^T_{\boldsymbol{\nu}} = \Phi_G^{-1}\{i(\nu_1 + \nu_2)^{-1} D_{\boldsymbol{\nu}}\} \, .$

Proof. By Lemma 2.2.1, $[Z : W] \in M^G_{\boldsymbol{\nu}}$ if and only if $-i\Phi_G([Z : W])$ is similar to $D_{\boldsymbol{\nu}/(\nu_1+\nu_2)}$; on the other hand, by Lemma 2.2.2, $[Z : W] \in M^G_{\boldsymbol{\nu}}$ if and only if for some $z \in \mathbb{C}$

$$-i\Phi_G([Z : W]) = \begin{pmatrix} \nu_1/(\nu_1 + \nu_2) & z \\ \bar{z} & \nu_1/(\nu_1 + \nu_2) \end{pmatrix} \, .$$

Equating determinants, we conclude that $z = 0$. This concludes the proof. □

Let $\mathfrak{g}_i \subset \mathfrak{g}$ be the affine hyperplane of the skew-Hermitian matrices of trace i; we may interpret Φ_G as a smooth map $\Phi'_G : \mathbb{P}^3 \to \mathfrak{g}_i$.

Lemma 2.2.4. *If* $\nu_1 > \nu_2 > 0$, *then* $i(\nu_1 + \nu_2)^{-1} D_{\boldsymbol{\nu}} \in \mathfrak{g}_i$ *is a regular value of* Φ'_G.

Proof. Clearly, the latter matrix is a regular value of Φ'_G if and only if Φ_G is transverse to the ray $\mathbb{R}_+ \cdot iD_{\boldsymbol{\nu}}$; thus the statement follows from Corollary 2.2.2. □

By Lemmata 2.2.3 and 2.2.4, we obtain

Corollary 2.2.3. *Suppose $\nu_1 > \nu_2 > 0$. Then, with $M = \mathbb{P}^3$:*

1. $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ *and* $M^T_{\boldsymbol{\nu}}$ *are smooth compact (real) hypersurfaces in* M*;*
2. $M^G_{\mathcal{O}_{\boldsymbol{\nu}}} \cap M^T_{\boldsymbol{\nu}}$ *is a smooth submanifold of* M *of real codimension* 3*.*

Let us now describe the saturation $G \cdot M^T_{\boldsymbol{\nu}}$.

Lemma 2.2.5. *Under the previous assumptions*

$$G \cdot M^T_{\boldsymbol{\nu}} = \left\{ [Z : W] \in \mathbb{P}^3 \, : \, \frac{\|Z \wedge W\|}{\|Z\|^2 + \|W\|^2} \leq \frac{\sqrt{\nu_1 \nu_2}}{\nu_1 + \nu_2} \right\} .$$

Proof. Consider $[Z : W] \in \mathbb{P}^3$ with $(Z, W) \in S^7$. By definition, $[Z : W] \in G \cdot M^T_{\boldsymbol{\nu}}$ if and only if there exists $A \in G$ such that $[AZ : AW] \in M^T_{\boldsymbol{\nu}}$; we may actually require without loss that $A \in SU(2)$. Let us write

$$A = \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \in SU(2), \quad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix};$$

then $[AZ : AW] \in M^T_{\boldsymbol{\nu}}$ if and only if (with some computations)

$$\begin{aligned} 0 &= \nu_2(|a\, z_1 - \bar{c}\, z_2|^2 + |a\, w_1 - \bar{c}\, w_2|^2) - \nu_1(|c\, z_1 + \bar{a}\, z_2|^2 + |c\, w_1 + \bar{a}\, w_2|^2) \\ &= \nu_2 \left\| \begin{pmatrix} z_1 & z_2 \\ w_1 & w_2 \end{pmatrix} \begin{pmatrix} a \\ -\bar{c} \end{pmatrix} \right\|^2 - \nu_1 \left\| \begin{pmatrix} z_1 & z_2 \\ w_1 & w_2 \end{pmatrix} \begin{pmatrix} c \\ \bar{a} \end{pmatrix} \right\|^2 . \end{aligned}$$

In other words, $[Z : W] \in G \cdot M^T_{\boldsymbol{\nu}}$ if and only if there exits an orthonormal basis $\mathcal{B} = (V_1, V_2)$ of $\mathbb{C}^2$ such that

$$\nu_2 \left\| \begin{pmatrix} z_1 & z_2 \\ w_1 & w_2 \end{pmatrix} V_1 \right\|^2 = \nu_1 \left\| \begin{pmatrix} z_1 & z_2 \\ w_1 & w_2 \end{pmatrix} V_2 \right\|^2 . \tag{2.4}$$

Now, for any $V \in S^7$ we have

$$\left\| \begin{pmatrix} z_1 & z_2 \\ w_1 & w_2 \end{pmatrix} V \right\|^2 = V^t \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix} \begin{pmatrix} \bar{z}_1 & \bar{z}_2 \\ \bar{w}_1 & \bar{w}_2 \end{pmatrix} \bar{V} = V^t \, \frac{1}{i} \, \Phi_G([Z : W]) \, \bar{V} .$$

If $\lambda_1(Z, W) \geq \lambda_2(Z, W) \geq 0$ are the eigenvalues of $-i\Phi_G([Z : W])$ (see Lemma 2.2.1), we then obtain for any $V \in S^7$

$$\lambda_1(Z, W) \geq \left\| \begin{pmatrix} z_1 & z_2 \\ w_1 & w_2 \end{pmatrix} V \right\|^2 \geq \lambda_2(Z, W) , \tag{2.5}$$

with left (respectively, right) equality holding if and only if V is an eigenvector of $-i\Phi_G([Z:W])$ relative to $\lambda_1(Z,W)$ (respectively $\lambda_2(Z,W)$). We conclude from (2.4) and (2.5) that if $(Z,W)\in G\cdot X^T_{\boldsymbol{\nu}}$ then the following inequalities holds:

$$\nu_1\,\lambda_1(Z,W)\geq\nu_2\,\lambda_2(Z,W),\quad \nu_2\,\lambda_1(Z,W)\geq\nu_1\,\lambda_2(Z,W)\,. \tag{2.6}$$

While the former is trivial, since $\nu_1>\nu_2>0$ and $\lambda_1(Z,W)\geq\lambda_2(Z,W)\geq 0$, the latter is equivalent to the other

$$\frac{\sqrt{\nu_1\,\nu_2}}{\nu_1+\nu_2}\geq\|Z\wedge W\|\,. \tag{2.7}$$

Suppose, conversely, that (2.7) holds. Then (2.6) also holds. Let (W_1,W_2) be an orthonormal basis of eigenvectors of $-i\Phi_G([Z:W])$ with respect to the eigenvalues $\lambda_1(Z,W)$ and $\lambda_2(Z,W)$, respectively. Evaluating the two sides of (2.4) with $V_1'=W_1$ and $V_2'=W_2$ in place of (V_1,V_2). We obtain

$$\nu_2\left\|\begin{pmatrix}z_1 & z_2\\ w_1 & w_2\end{pmatrix}V_1'\right\|^2=\nu_2\,\lambda_1(Z,W)\geq\nu_1\,\lambda_2(Z,W)=\nu_1\left\|\begin{pmatrix}z_1 & z_2\\ w_1 & w_2\end{pmatrix}V_2'\right\|^2.$$

Using instead $V_1''=W_2$ and $V_2''=W_1$ in place of (V_1,V_2), we obtain

$$\nu_2\left\|\begin{pmatrix}z_1 & z_2\\ w_1 & w_2\end{pmatrix}V_1''\right\|^2=\nu_2\,\lambda_2(Z,W)\leq\nu_1\,\lambda_1(Z,W)=\nu_1\left\|\begin{pmatrix}z_1 & z_2\\ w_1 & w_2\end{pmatrix}V_2''\right\|^2.$$

Since $G=U(2)$ is connected, and acts transitively on the family of all orthonormal basis of $\mathbb{C}^2$, we conclude by continuity that there exists an orthonormal basis (V_1,V_2) on which (2.4) is satisfied. □

In view of Corollary 2.2.1, we deduce

Corollary 2.2.4. *$M^G_{\mathcal{O}_{\boldsymbol{\nu}}}=\partial(G\cdot M^T_{\boldsymbol{\nu}})$.*

The boundary $\partial(G\cdot M^T_{\boldsymbol{\nu}})$ consists of those $[Z:W]\in\mathbb{P}^3$ such that $-i\,\Phi_G([Z:W])$ is similar to $(\nu_1+\nu_2)^{-1}D_{\boldsymbol{\nu}}$, while the interior $(G\cdot M^T_{\boldsymbol{\nu}})^0$ consists of those $[Z:W]\in\mathbb{P}^3$ such that $-i\,\Phi_G([Z:W])$ is similar to a matrix of the form

$$\frac{1}{\nu_1+\nu_2}\begin{pmatrix}\nu_1 & z\\ \bar z & \nu_2\end{pmatrix},$$

for some complex number $z\neq 0$.

Finally, the locus $X' \subseteq X = S^7$ of those (Z, W) at which $\tilde{\mu}$ is not locally free is defined by the condition $Z \wedge W = 0$, and therefore it is contained in $(G{\cdot}M^T_{\boldsymbol{\nu}})^0$. It is the unit circle bundle over a non-singular quadric hypersurface in $\mathbb{P}^3$. The stabilizer subgroup of $(Z, W) \in S^7$ is trivial if $Z \wedge W \neq 0$, and it is isomorphic to S^1 otherwise.

For any fixed $\boldsymbol{\nu} = (\nu_1, \nu_2) \in \mathbb{Z}^2$ with $\nu_1 > \nu_2$, let consider how $V_{k\boldsymbol{\nu}}$ appears in the isotypical decomposition of $H(X)$ under $\hat{\mu}$. The Hopf map $\pi : X = S^7 \to \mathbb{P}^3$ is the quotient map for the standard action $r : S^1 \times S^7 \to S^7 \subset \mathbb{C}^4$, given by complex scalar multiplication. The corresponding unitary representation of S^1 on $H(X)$ yields an isotypical decomposition $H(X) = \bigoplus_{l \in \mathbb{Z}} H_l(X)$, where for $l \in \mathbb{N}$ we set

$$H_l(X) := \left\{ f \in H(X) \, : \, f(e^{i\theta}\, x) = e^{i\, l\, \theta}\, f(x) \; \forall x = (Z, W) \in X, \, e^{i\theta} \in S^1 \right\} .$$

As is well-known, there are natural $U(2)$-equivariant unitary isomorphisms

$$\begin{aligned} H_l(X) &\cong H^0(\mathbb{P}^3, \, \mathcal{O}_{\mathbb{P}^3}(l)) \cong \mathrm{Sym}^l(\mathbb{C}^2 \oplus \mathbb{C}^2) \\ &= \bigoplus_{h=0}^{l} \mathrm{Sym}^h(\mathbb{C}^2) \otimes \mathrm{Sym}^{l-h}(\mathbb{C}^2) \, . \end{aligned}$$

On the other hand, a character computation yields the following.

Lemma 2.2.6. *For $p \geq q$.*

$$\mathrm{Sym}^p(\mathbb{C}^2) \otimes \mathrm{Sym}^q(\mathbb{C}^2) \cong \bigoplus_{a=0}^{q} (\det)^{\otimes a} \otimes \mathrm{Sym}^{p+q-2a}(\mathbb{C}^2) \, .$$

as $U(2)$-representations.

Proof of Lemma 2.2.6. The character of $\mathrm{Sym}^p(\mathbb{C}^2)$ is $\chi_{(p+1,0)}$. Since the character of a tensor product of representations is the product of the respective characters, the character of $\mathrm{Sym}^p(\mathbb{C}^2) \otimes \mathrm{Sym}^q(\mathbb{C}^2)$ is $\chi' := \chi_{(p+1,0)} \cdot \chi_{(q+1,0)}$. Let us evaluate χ on a diagonal matrix $D_{\boldsymbol{z}}$ with diagonal $\boldsymbol{z} = (z_1, z_2)$. We obtain

$$\begin{aligned}\chi'(D_{\boldsymbol{z}}) &= \frac{z_1^{p+1}-z_2^{p+1}}{z_1-z_2}\cdot(z_1^q+z_1^{q-1}z_2+\cdots+z_1z_2^{q-1}+z_2^q) \qquad (2.8)\\ &= \frac{1}{z_1-z_2}\cdot\left(\sum_{j=0}^{q} z_1^{p+1+q-j}z_2^j-\sum_{j=0}^{q} z_1^j z_2^{p+1+q-j}\right)\\ &= \sum_{j=0}^{q}\frac{1}{z_1-z_2}\cdot(z_1^{p+1+q-j}z_2^j-z_1^jz_2^{p+1+q-j})\\ &= \sum_{j=0}^{q}\chi_{p+1+q-j,\,j}(D_{\boldsymbol{z}})\,.\end{aligned}$$

Now, a character is uniquely determined by its restriction to T, and on the other hand the character of a direct sum is the sum of the characters; therefore, in view of (3.4), we conclude from (2.8) that

$$\mathrm{Sym}^p(\mathbb{C}^2)\otimes\mathrm{Sym}^q(\mathbb{C}^2)\cong\bigoplus_{j=0}^{q}V_{(p+1+q-j,\,j)}=\bigoplus_{j=0}^{q}\det{}^{\otimes j}\,\mathrm{Sym}^{p+q-2j}(\mathbb{C}^2)\,.$$

□

Therefore

$$H_l(X)\cong\bigoplus_{h=0}^{l}H_{l,\,h}(X), \qquad (2.9)$$

where we set

$$H_{l,\,h}(X):=\bigoplus_{a=0}^{\min(h,\,l-h)}(\det)^{\otimes a}\otimes\mathrm{Sym}^{l-2a}(\mathbb{C}^2)\,. \qquad (2.10)$$

In order for the a-th summand in (2.9) to be isomorphic to $V_{k\boldsymbol{\nu}}$, we need to have $a=k\nu_2$ and $l-2a=k(\nu_1-\nu_2)-1$; hence in this special case $H(X)_{k\boldsymbol{\nu}}\subseteq H_l(X)$ with $l=k(\nu_1+\nu_2)-1$. Let us estimate the multiplicity of $H(X)_{k\boldsymbol{\nu}}$ in $H_l(X)$. In order for the a-th summand with $a=k\nu_2$ to appear in $H_{lh}(X)$ in (2.10) for some $h\le k(\nu_1+\nu_2)-1$ we need to have

$$\begin{aligned}a=k\nu_2 &\le \min(h,\,k(\nu_1+\nu_2)-1-h)\\ &\Rightarrow k\nu_2\le h,\quad k\nu_2\le k(\nu_1+\nu_2)-1-h\\ &\Rightarrow k\nu_2\le h\le k\nu_1-1\,.\end{aligned}$$

Hence there are $k(\nu_1 - \nu_2) - 1$ values of h for which $H_{l,h}(X)$ contains one copy of $V_{k\boldsymbol{\nu}}$. The dimension of $H(X)_{k\boldsymbol{\nu}}$ is thus

$$(k(\nu_1 - \nu_2) - 1)\, k(\nu_1 - \nu_2) \sim k^2(\nu_1 - \nu_2)^2 + O(k)\,.$$

Chapter 3

Preliminaries

3.1 Quantized manifolds

Recall that a d-dimensional complex projective manifold is a complex submanifolds of the complex projective space $\mathbb{P}^n(\mathbb{C})$. A positive complex line bundle on M is a triple (A, h, ∇) where

1. A is a complex line bundle,
2. h is an Hermitian metric on L (that is a smooth field of Hermitian inner products $\langle\,,\,\cdot\rangle_h$ in the fibers of A);
3. ∇ is a connection on M compatible with h, that is

   $$\mathrm{d}\,\langle s_1,\, s_2\rangle_h = \langle \nabla s_1,\, s_2\rangle_h + \langle s_1,\, \nabla s_2\rangle_h$$

   for each smooth section s_1, s_2, such that the curvature of ∇ (up to a factor $2/i$) is a positive form ω.

As a consequence M is a Kähler manifold with Kähler form ω.

From the bundle (A, h) we pass to its dual $(A^\vee, h^\vee)$ and we consider the circle bundle

$$X = \{a \in A \,:\, h^\vee(a, a) = 1\},$$

which is the boundary of the following strictly pseudoconvex domain (by the positivity of (A, h))

$$D = \{a \in A \,:\, h^\vee(a, a) \leq 1\}.$$

Furthermore X is a contact manifold, that is there exists a 1-form α on X such that $\alpha \wedge (\mathrm{d}\alpha)^n \neq 0$ (see Example 3.5.11, p. 130 in [MS17]).

3.2 Compact Lie groups and actions

A Lie group $(G, \cdot_G)$ is a group and a manifold such that the following maps $\cdot_G : G \times G \to G$ and $^{-1} : G \to G$ are smooths. Let us review some basics facts concerning Lie groups and its actions on symplectic manifolds. (We will write $g\,h$ instead of $g \cdot_G h$).

3.2.1 Volume elements

We will denote by $\mathrm{dV}_G(g)$ the Haar measure on a compact Lie group G, which is the unique measure on G invariant under both left and right translations, normalized by the condition

$$\int_G \mathrm{dV}_G(g) = 1\,.$$

For later use, we compute explicitly the volume element for $SU(2)/T$. There exists a diffeomorphism γ from $SU(2)$ into the unit sphere $S^3 \subset \mathbb{C}^2$,

$$g := \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \xrightarrow{\gamma} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in S^3\,, \tag{3.1}$$

where α and β are complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$. Furthermore, γ intertwines the right action of $T \cong S^1$ on $SU(2)$ with the standard circle action on S^3. Therefore, the projection $SU(2) \to SU(2)/T$ may be identified with the Hopf map $S^3 \to \mathbb{P}^1 \cong S^2$. It follows that the Haar measure on $SU(2)/T$ is a positive multiple of the pull-back of the standard measure on S^2.

In particular, when $\alpha \cdot \beta \neq 0$, we can set the following local coordinates on $SU(2)$:

$$\alpha = e^{i\theta_\alpha} \cos\theta\,, \quad \beta = e^{i\theta_\beta} \sin\theta\,. \tag{3.2}$$

Thus, the Haar volume element on $SU(2)/T$ is

$$\omega_{FS} = \frac{1}{2\pi} \sin(2\,\theta)\, d\theta\, d\delta\,.$$

3.2.2 Harmonic analysis on compact Lie groups

In this section we will resume some basic theorems in harmonic analysis of compact Lie groups G, in particular for $G = U(2)$ and $G = SU(2)$. We refer to the books [Var89] and [BtD95].

The central notion is that of the *character* of a finite dimensional unitary representation. Let us denote with κ the representation, its character χ_κ is the function on G defined by

$$\chi_\kappa(g) = \mathrm{trace}\,(\kappa(g)), \quad g \in G.$$

Let us recall that two representations $\kappa : G \to GL(V)$ and $\kappa' : G \to GL(V')$, respectively on the vector spaces V and V', are called equivalent if there is a linear isomorphism $T : V \to V'$ such that $\kappa'(g) = T\,\kappa(g)\,T^{-1}$ for all $g \in G$. The character of κ depends only on the equivalence class $[\kappa]$. The set of equivalence classes of irreducible representations of G is written $\hat{G}$ and is called *unitary dual* of G.

Furthermore the characters are *class functions*, i.e. are invariant under the action of the group by conjugation. We will denote by $L^2(G)^c$ the space of square integrable class functions.

In addition to the characters we can associate with a finite dimensional unitary representation κ its *matrix elements*, namely, the functions

$$\mathcal{M}^{(\kappa)}_{a,b} : \, g \to (\kappa(g)\, a,\, b) \quad (a,\, b \in V)\,.$$

Theorem 3.2.1 (Completeness Theorem of Peter-Weyl)**.** *The irreducible unitary representations of* G *are all finite-dimensional and they separate the points of* G*. The irreducible characters form an orthonormal basis of* $L^2(G)^c$*, and* $L^2(G)$ *is the orthogonal direct sum of matrix elements.*

We now want to show some explicit expressions for the characters of $U(2)$ and we will deduce from these the ones of $SU(2)$. One is able to write explicitly a formula for the restriction of χ_κ to a maximal compact abelian subgroup of $U(2)$, the torus T. Let dV_T be the Haar measure on T and define the function $\Delta : \, T \to \mathbb{C}$ by setting

$$\Delta(t) := t_1 - t_2, \quad t := (t_1,\, t_2) \in T\,;$$

here we identify T with $S^1 \times S^1$ in a natural manner. Furthermore, for any $f \in \mathcal{C}^\infty(U(2))$ let us define $F_f : \, T \to \mathbb{C}$ by setting

$$F_f(t) := \int_{U(2)/T} f(gtg^{-1})\, \mathrm{dV}_{U(2)/T}(gT)\,.$$

If f is a class function $F_f(t) = f(t)$ for any $t \in T$.

Theorem 3.2.2 (Weyl's integration formula)**.** *A Borel function f on $U(2)$ lies in $L^1(U(2))$ if and only if F_f lies in $L^1(T,\, \Delta\bar{\Delta}\, \mathrm{dV}_T)$; in this case*

$$\int_{U(2)} f \ \mathrm{dV}_{U(2)} = \frac{1}{2}\int_T F_f\, \Delta\bar{\Delta}\ \mathrm{dV}_T\,.$$

In particular, an invariant function f on $U(2)$ lies in $L^1(U(2))$ if and only if its restriction to T lies in $L^1(T,\, \Delta\bar{\Delta}\, \mathrm{dV}_T)$, and then

$$\int_{U(2)} f\, \mathrm{dV}_{U(2)} = \frac{1}{2}\int_T f\big|_T\, \Delta\bar{\Delta}\, \mathrm{dV}_T\,.$$

We will use Weyl's integration formula in the proof of Theorems described in the introduction and it is also the main tool in the proof of the following theorem, see [Var89].

Theorem 3.2.3 (Weyl's character formula)**.** *The irreducible characters of $U(2)$ are in one-one correspondence with the decreasing couples of integers. For each $\boldsymbol{\nu} = (\nu_1,\, \nu_2)$, with $\nu_1 > \nu_2$, the character $\chi_{\boldsymbol{\nu}}$ is given on T by*

$$\chi_{\boldsymbol{\nu}}(t) := \frac{t_1^{\nu_1} t_2^{\nu_2} - t_1^{\nu_2} t_2^{\nu_1}}{t_1 - t_2}\,. \tag{3.3}$$

Namely, $\boldsymbol{\nu}$ corresponds to the irreducible representation

$$\det{}^{\nu_2} \otimes \mathrm{Sym}^{\nu_1-\nu_2-1}(\mathbb{C}^2)\,, \tag{3.4}$$

where $\mathrm{Sym}^{\nu_1-\nu_2-1}$ is the space of homogeneous polynomials in two complex variables of degree $\nu_1 - \nu_2 - 1$. Thus, the representations of $U(2)$ labeled by $\boldsymbol{\nu}$ has dimension $k\,(\nu_1 - \nu_2)$.

The above theorems for $U(2)$ remain valid for $SU(2)$ with the sole modification that the character is $\chi_{(\nu,0)}$, $\nu > 0$. Explicitly, the Weyl character formula for $SU(2)$ is

$$\chi_{k\boldsymbol{\nu}}(e^{i\vartheta}) = \frac{e^{ik\,\nu\,\vartheta} - e^{-ik\,\nu\,\vartheta}}{e^{i\vartheta} - e^{-i\vartheta}} = e^{i(k\nu-1)\vartheta} + e^{i(k\nu-3)\vartheta} + \cdots + e^{-i(k\nu-1)\vartheta}\,. \tag{3.5}$$

Similarly every finite dimensional irreducible representation of $SU(2)$ is isomorphic to $\mathrm{Sym}^{\nu-1}(\mathbb{C}^2)$, for some $\nu > 0$. Hence, the representations of $SU(2)$ labeled by $\boldsymbol{\nu} = (\nu,\, 0)$ has dimension $k\,\nu$.

3.2.3 Hamiltonian group actions

Let M be a manifold, G a compact connected Lie group and $\mathfrak{g}$ its Lie algebra. Let us suppose that the smooth map $\mu : G \times M \to M$ defines an action of G on M. For each point $m \in M$, the mapping

$$\begin{aligned} f_m \ : & G \to M \\ & g \mapsto \mu_g(m) := g \cdot m \end{aligned}$$

is a smooth map and its differential

$$\begin{aligned} \mathrm{d}_m f \ : & \mathfrak{g} \to T_m M \\ & \xi \mapsto \xi_M(m) \end{aligned}$$

is a Lie algebra homeomorphism. The vector field ξ_M is called *infinitesimal vector field* associated to ξ. If we have defined on M a complex line bundle on A (with projection $p : A \to M$) then we say that μ can be lifted to a group action $\tilde{\mu} : G \times A \to A$ on A if there exists a group action $\tilde{\mu}$ of G on A such that the induced restricted action on the fiber of A is linear.

If (M, ω) is symplectic, we suppose that G acts on M via symplectomorphism $\mu_g : M \to M$, then for every $\xi \in \mathfrak{g}$ the infinitesimal vector field ξ_M is symplectic. This means that $\iota(\xi_M)\,\omega$ is closed for every ξ; if the 1-form $\iota(\xi_M)\,\omega$ is exact then we say that the action is *weakly Hamiltonian*. More explicitly there exists a function $\Phi^\xi \in \mathcal{C}^\infty(M)$ such that $\mathrm{d}\Phi^\xi = \iota(\xi_M)\,\omega$.

The group G acts on itself by conjugation $C : G \times G \to G$. The differential of C_g defines an action $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ of G on its Lie algebra. The dual Ad^* of Ad is called the co-adjoint action. The action is called *Hamiltonian* if the map $\xi \to \Phi^\xi$ can be chosen to be G-equivariant, which means

$$\Phi^\xi \circ \mu_g = \Phi^{\mathrm{Ad}_g \xi},$$

for every $\xi \in \mathfrak{g}$ and $g \in G$. This last condition is equivalent to require that the map $\xi \to \Phi^\xi$ is a Lie algebra homomorphism with respect to the Poisson structure $\{\,,\,\}$ on $\mathcal{C}^\infty(M)$ (see Lemma 5.2.1, p. 203 of [MS17]).

Now, assume that the action μ is Hamiltonian. Then a moment map $\Phi : M \to \mathfrak{g}^\vee$ is a G-equivariant smooth map such that $\Phi^\xi(m) = \langle \Phi(m), \xi \rangle$, where $\langle \cdot, \cdot \rangle$ is the pairing between $\mathfrak{g}$ and its dual. Equivalently,

$$\mathrm{d}\Phi^\xi_m(\mathbf{v}) = \omega_m(\mathbf{v}, \xi_M(m))$$

for each $m \in M$, $\mathbf{v} \in T_m M$ and $\xi \in \mathfrak{g}$.

3.3 The Szegö projector

Let $W \subseteq \mathbb{C}^n$ be a compact strictly pseudo-convex domain with smooth boundary. Let r be a smooth function on W with $r > 0$ on $\operatorname{Int} W$, $r = 0$ on ∂W and $\mathrm{d}r \neq 0$ near ∂W. Let $\iota : \partial W \to W$ be the inclusion map; the one-form $\alpha := \iota^* \mathrm{Im}\, \bar{\partial} r$ is a contact form on ∂W. Thus, $\nu := \alpha \wedge (\mathrm{d}\alpha)^{n-1}$ defines a volume form on ∂W. Let $L^2(\partial W, \nu)$ be the space of square integrable function over ∂W with respect to the measure ν. The *Hardy Space* is the closure in $L^2(\partial W, \nu)$ of the space of smooth function on ∂W which can be extended to holomorphic function on W. The orthogonal projector $\Pi : L^2(\partial W, \nu) \to H^2(\partial W)$ is called the *Szegö projector*.

Let M be a connected complex d-dimensional projective manifold with a $(A,\, h)$ a positive line bundle on M, as described in chapter 1.1. If $A^\vee \supset X \xrightarrow{\pi} M$ is the unit circle bundle in the dual of A, then X is the boundary of a strictly pseudo-convex domain and the structures defined in the previous paragraph carry over this case. Here the domain W is the disc bundle,

$$W := \{(m, \nu) \in A^\vee \,:\, h(\nu, \nu) \leq 1\}$$

and the defining function r of the previous paragraph is $r(m, \nu) = 1 - \|\nu\|_m^2$, where $\|\cdot\|$ is the norm in the metric induced by h.

3.3.1 The Szegö kernel

Let us recall some basic facts concerning the Szegö kernel from [BdMS76], [Zel98]. The Szegö kernel is a Fourier integral operator of complex type, in particular there exists a symbol $s \in S^d(X \times X \times \mathbb{R}^+)$ of the type

$$s(x,\, y,\, u) \sim \sum_{k=0}^{+\infty} u^{d-k}\, s_k(x,\, y)\,,$$

to be so that

$$\Pi(x,\, y) = \int_0^{+\infty} e^{i\, u \psi(x,y)}\, s(x,\, y,\, u)\, \mathrm{d}u\,,$$

where the phase $\psi \in \mathcal{C}^\infty(X \times X)$ is determined up to a function which vanishes to infinite order along the diagonal. Explicitly, ψ is determined by the following properties:

1. $\psi(x,\, x) = \frac{1}{i}\, r(x)$, where r is the defining function of X;

2. $\mathrm{d}''_x\psi$ and $\mathrm{d}'_y\psi$ vanish to infinite order along the diagonal;

3. $\psi(x,y) = -\overline{\psi(y,x)}$.

Example 3.3.1. Consider the unit ball in $\mathbb{C}^{n+1}$. The above formula has the form

$$\Pi(x,\, y) = \frac{1}{(1-\langle x,y\rangle)^{n+1}} = \int_0^\infty e^{i\,u\psi(x,y)}\,u^n\,\mathrm{d}u$$

where $\psi(x,y) = 1-\langle x,y\rangle$.

In [SZ02], Shiffman and Zelditch provide some results about the scaling asymptotics of Szegö kernel and their implications in symplectic geometry. Their proofs rely on the notion of Heisenberg local coordinates. A set of Heisenberg local coordinates on X centred at x determines a linear isometry $T_xX \cong \mathbb{R}\oplus\mathbb{C}^d$, furthermore, they are horizontal at x with respect to the connection 1-form.

3.3.2 Equivariant Szegö projector

Suppose we have an unitary action of a compact Lie group G on the Hardy space $H(X)$. We have a natural decomposition, given by

$$H(X) = \bigoplus_{\boldsymbol{\nu}\in\hat{G}} H_{\boldsymbol{\nu}}(X).$$

Let us focus on the case $G = U(2)$. Let $\Pi_{\boldsymbol{\nu}}$ the projector onto the isotypical component $H_{\boldsymbol{\nu}}(X)$. We will denote its kernel by $\Pi_{\boldsymbol{\nu}}(\cdot,\cdot)$. For any $x,\, y\in X$, we have an explicit formula for the equivariant projector (see [GS82c]):

$$\Pi_{\boldsymbol{\nu}}(x,\, y) = d_{\boldsymbol{\nu}}\int_G \overline{\chi_{\boldsymbol{\nu}}(g)}\,\Pi(\tilde{\mu}_{g^{-1}}(x),\, y)\;\mathrm{dV}_G(g)\,, \tag{3.6}$$

where $d_{\boldsymbol{\nu}}$ is the dimension of the representation labeled by $\boldsymbol{\nu}$. In order to prove the main results we need to write (3.6) more explicitly. In view of the Weyl's integration formula the expression (3.6) can be rewritten

$$\Pi_{\boldsymbol{\nu}}(x,\, y) = \frac{d_{\boldsymbol{\nu}}}{2}\int_T \overline{\chi_{\boldsymbol{\nu}}(t)}\,\overline{\Delta(t)}\Delta(t)\,F(t;\, x,\, y)\;\mathrm{dV}_T(t)\,, \tag{3.7}$$

where

$$F(t;\, x,\, y) := \int_{G/T}\Pi(\tilde{\mu}_{gt^{-1}g^{-1}}(x),\, y)\;\mathrm{dV}_{G/T}(gT)\,. \tag{3.8}$$

Thus, inserting the Weyl's character formula (3.3) for $U(2)$ we obtain, instead of (3.7),

$$\Pi_{\boldsymbol{\nu}}(x,\, y) = \frac{d_{\boldsymbol{\nu}}}{2} \int_T \left(t_1^{-\nu_1} t_2^{-\nu_2} - t_1^{-\nu_2} t_2^{-\nu_1}\right) \Delta(t)\, F(t;\, x,\, y)\; \mathrm{dV}_T(t)\,.$$

The previous integral splits in two summands that, after a change of variables, add up to

$$\Pi_{\boldsymbol{\nu}}(x,\, y) = d_{\boldsymbol{\nu}} \int_T t^{-\boldsymbol{\nu}} \Delta(t)\, F(t;\, x,\, y)\; \mathrm{dV}_T(t)\,, \tag{3.9}$$

where $t^{-\boldsymbol{\nu}} = t_1^{-\nu_1} t^{-\nu_2}$ and $d_{\boldsymbol{\nu}}$ is explicitly given by $d_{\boldsymbol{\nu}} = \nu_1 - \nu_2$. For $G = SU(2)$ the computations are similar. In the proof of Theorems in the next chapters we will gloss over these computations starting directly with (3.9).

Chapter 4

Proofs

4.1 Proof of Theorem 1.2.1

Proof. We have

$$\Pi_{k\boldsymbol{\nu}}(x,\,y) = k\nu \int_G \overline{\chi_{k\boldsymbol{\nu}}(g)}\,\Pi(\tilde{\mu}_{g^{-1}}(x),\,y)\;\mathrm{dV}_G(g)\,.$$

Let $\delta > 0$ and let $\rho : \mathbb{R} \to (0,\,+\infty)$ be a bump function with $\rho \equiv 1$ on $(-\delta,\,\delta)$ and $\rho \equiv 0$ on $(-2\delta,\,2\delta)^c$. We have

$$\Pi_{k\boldsymbol{\nu}}(x,\,y) = \Pi_{k\boldsymbol{\nu}}(x,\,y)' + \Pi_{k\boldsymbol{\nu}}(x,\,y)''\,,$$

where

$$\begin{aligned}&\Pi_{k\boldsymbol{\nu}}(x,\,y)' \\ &\quad = k\nu \int_G \rho\left(\mathrm{dist}_X(\tilde{\mu}_{g^{-1}}(x),\,y)\right)\,\overline{\chi_{k\boldsymbol{\nu}}(g)}\,\Pi(\tilde{\mu}_{g^{-1}}(x),\,y)\;\mathrm{dV}_G(g)\,,\end{aligned}$$

and

$$\begin{aligned}&\Pi_{k\boldsymbol{\nu}}(x,\,y)'' \\ &\quad = k\nu \int_G \left(1 - \rho\left(\mathrm{dist}_X(\tilde{\mu}_{g^{-1}}(x),\,y)\right)\right)\,\overline{\chi_{k\boldsymbol{\nu}}(g)}\,\Pi(\tilde{\mu}_{g^{-1}}(x),\,y)\;\mathrm{dV}_G(g)\,.\end{aligned}$$

Since the function

$$g \mapsto \left(1 - \rho\left(\mathrm{dist}_X(\tilde{\mu}_{g^{-1}}(x),\,y)\right)\right)\,\Pi(\tilde{\mu}_{g^{-1}}(x),\,y)$$

is smooth on G, we obtain that $\Pi_{k\boldsymbol{\nu}}(x,\, y)'' = O(k^{-\infty})$.

We need to exploit the explicit description of Π as an FIO developed in [BdMS76]. Namely, up to a smoothing contribution, we have

$$\begin{aligned}
&\Pi_{k\boldsymbol{\nu}}(x,\, y)\\
&\quad\sim k\nu \int_0^{+\infty} \mathrm{d}u \int_G \mathrm{dV}_G(g) \left[\rho_x(g)\, \overline{\chi_{k\boldsymbol{\nu}}(g)}\, e^{iu\psi\left(\tilde{\mu}_{g^{-1}}(x),\, y\right)}\, s(\tilde{\mu}_{g^{-1}}(x),\, y,\, u)\right]\\
&\quad\sim k^2\nu \int_0^{+\infty} \mathrm{d}u \int_G \mathrm{dV}_G(g) \left[\rho_x(g)\, \overline{\chi_{k\boldsymbol{\nu}}(g)}\, e^{iku\psi\left(\tilde{\mu}_{g^{-1}}(x),\, y\right)}\, s(\tilde{\mu}_{g^{-1}}(x),\, y,\, ku)\right]
\end{aligned}$$

where we have set

$$\rho_x(g) := \rho\left(\mathrm{dist}_X(\tilde{\mu}_{g^{-1}}(x),\, y)\right)\,.$$

Now, let $D \gg 0$ and $\rho_1 : \mathbb{R} \to [0,\, +\infty)$ such that $\rho_1 \equiv 1$ on $[0,\, D]$ and $\rho \equiv 0$ on $[2D,\, +\infty)$. We have

$$\Pi_{k\boldsymbol{\nu}}(x\, y) \sim \Pi_{k\boldsymbol{\nu}}(x,\, y)_1 + \Pi_{k\boldsymbol{\nu}}(x,\, y)_2\,,$$

where

$$\begin{aligned}
\Pi_{k\boldsymbol{\nu}}(x,\, y)_1 := k^2\nu \int_0^{+\infty} \mathrm{d}u \int_G \mathrm{dV}_G(g)&\\
\left[\rho_x(g)\, \rho_1(u)\, \overline{\chi_{k\boldsymbol{\nu}}(g)}\, e^{iku\psi\left(\tilde{\mu}_{g^{-1}}(x),\, y\right)}\, s(\tilde{\mu}_{g^{-1}}(x),\, y,\, ku)\right]&,
\end{aligned}$$

and

$$\begin{aligned}
\Pi_{k\boldsymbol{\nu}}(x,\, y)_2 := k^2\nu \int_0^{+\infty} \mathrm{d}u \int_G \mathrm{dV}_G(g)&\\
\left[\rho_x(g)\, (1-\rho_1(u))\, \overline{\chi_{k\boldsymbol{\nu}}(g)}\, e^{iku\psi\left(\tilde{\mu}_{g^{-1}}(x),\, y\right)}\, s(\tilde{\mu}_{g^{-1}}(x),\, y,\, ku)\right]&.
\end{aligned}$$

Lemma 4.1.1. *$\Pi_{k\boldsymbol{\nu}}(x,\, y)_2 = O(k^{-\infty})$ as $k \to +\infty$.*

Proof of Lemma 4.1.1. Let $\{G', G''\}$ be an open cover of G where

$$\begin{aligned}
G' &:= \{g \in G \,:\, \mathrm{dist}_G\left(g, \{\pm I_2\}\right) < 2\delta\},\\
G'' &:= \{g \in G \,:\, \mathrm{dist}_G\left(g, \{\pm I_2\}\right) > \delta\}.
\end{aligned}$$

We may consider a subordinate partition of unity $\beta' + \beta'' = 1$ on G. Let us set

$$\varrho' := \varrho \cdot \beta', \qquad \varrho'' := \varrho \cdot \beta''\,.$$

Then $\varrho = \varrho' + \varrho''$, where ϱ' is supported in a small neighborhood of $\{\pm I_2\}$, and ϱ'' is supported away from $\{\pm I_2\}$.

Accordingly, we have

$$\Pi_{k\boldsymbol{\nu}}(x,y)_2 = \Pi_{k\boldsymbol{\nu}}(x,y)'_2 + \Pi_{k\boldsymbol{\nu}}(x,y)''_2\,,$$

where in the former (respectively, latter) summand $\varrho(g)$ has been replaced by $\varrho'(g)$ (respectively, $\varrho''(g)$). Let us study the two summands in the previous expression separately.

Lemma 4.1.2. *$\Pi_{k\boldsymbol{\nu}}(x,y)'_2 = O(k^{-\infty})$ as $k \to +\infty$.*

Proof of Lemma 4.1.2. Let us assume to fix ideas that $k\boldsymbol{\nu} = 2l+1$ is odd. Then $V_{k\boldsymbol{\nu}}$ may be identified with the vector space $\mathbb{C}^{(2l)}[z_1, z_2]$ of complex homogeneous polynomials of degree $2l$ in two variables. A natural basis of the latter is given by the monomials $P_\mu(z_1,z_2) := z_1^{l-\mu}\, z_2^{l+\mu}$, where $\mu \in \{-l,\ldots,0,\ldots l\}$. We shall accordingly denote the matrix elements of the representation $V_{k\boldsymbol{\nu}}$ by

$$\mathcal{M}^{(k\boldsymbol{\nu})}_{a,b} : G \to \mathbb{C}\,, \quad (a, b \in \{-l,\ldots,0,\ldots,l\})\,.$$

We have

$$\chi_{k\boldsymbol{\nu}}(g) = \sum_{a=-l}^{l} \mathcal{M}^{(k\boldsymbol{\nu})}_{a,\,a}(g)\,.$$

Thus, we get

$$\Pi_{k\boldsymbol{\nu}}(x,\,y)_2 \sim \sum_{a=-l}^{l} \Pi_{k\boldsymbol{\nu}}(x,\,y)_{2,\,a}\,,$$

where

$$\begin{aligned}\Pi_{k\boldsymbol{\nu}}(x,\,y)_{2,\,a} \quad &:= k^2\nu \int_0^{+\infty} \mathrm{d}u \int_G \mathrm{dV}_G(g) \\ &\Big[\rho_x(g)\,(1-\rho_1(u))\,\overline{\mathcal{M}^{k\boldsymbol{\nu}}_{a,a}(g)}\, e^{iku\psi\left(\tilde{\mu}_{g^{-1}}(x),\,y\right)}\, s(\tilde{\mu}_{g^{-1}}(x),\,y,\,ku)\Big]\,.\end{aligned}$$

On the support of ϱ', either $g \sim I_2$ or $g \sim -I_2$, and hence we can write explicatively

$$g = \begin{pmatrix} A(g)\,e^{i\theta_G(g)} & -\overline{\gamma(g)} \\ \gamma(g) & A(g)\,e^{-i\theta_G(g)} \end{pmatrix}\,, \tag{4.1}$$

where $A(g) > 0$, and either $\theta_G(g) \approx 0$ or $\theta_G(g) \approx \pi$. Furthermore, A, γ and θ_G are smooth functions of $g \in G$ on a neighborhood of the support of ϱ'.

We can write (see [App14], §2.7, formula (2.6.40)), with g as in (4.1), the following expression for the matrix elements

$$\mathcal{M}^{(k\boldsymbol{\nu})}_{a,\,a}(g) = e^{-2ia\theta_G(g)} \cdot P_{l,a}(A(g)^2)\,, \tag{4.2}$$

where $P_{l,a}$ may be expressed in terms of suitable Jacobi polynomials, and is itself a real polynomial. Since the left hand side of (4.2) is an entry of a unitary matrix, we have at any rate $|P_{l,a}(A(g)^2)| \leq 1$.

Hence, we have

$$\begin{aligned}\Pi_{k\boldsymbol{\nu}}(x,\,y)_{2,\,a} := k^2\nu \int_G \mathrm{dV}_G(g) \int_{D/2}^{+\infty} \mathrm{d}u \big[e^{ik\Psi_{a/k}(x,y;g,\,u)} \\ \varrho_x(g) \cdot \varrho'_2(u) \cdot P_{l,a}(A(g)^2)\; s(\tilde{\mu}_{g^{-1}}(x),\,y,\,ku)\big]\end{aligned}$$

where we have set

$$\Psi_{a/k}(x, y; g,\,u) := u \cdot \psi\,(\tilde{\mu}_{g^{-1}}(x),\,y) + \frac{2a}{k} \cdot \theta_G(g)\,.$$

Since

$$-\frac{\nu}{2} \leq \frac{a}{k} \leq \frac{\nu}{2}\,,$$

the family of phases $\Psi_{a/k}$ is finite.

Let us view β, the infinitesimal generator of the standard torus T, as a left-invariant vector field on G; the corresponding 1-parameter group of diffeomorphisms is $\varphi_\tau(g) := ge^{\tau\,\beta}$. Therefore, if L_β is the same vector field viewed as a differential operator on G, then $L_\beta(\theta_G) = 1$ on the support of ϱ'. Also, L_β is a skew-hermitian operator on $L^2(G)$, since φ_τ induces a 1-parameter group of unitary automorphisms of $L^2(G)$; hence, $L_\beta^t = -\bar{L}_\beta$. Furthermore, the function $g \mapsto A(g)^2$ is smooth, real and φ_τ-invariant; therefore, $L_\beta(A(g)^2) = \bar{L}_\beta(A(g)^2) = 0$.

On the other hand, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{\mu}_{\varphi_\tau(g)^{-1}}(x)\Big|_{\tau=0} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{\mu}_{e^{-\tau\,\beta}}(\tilde{\mu}_{g^{-1}}(x))\Big|_{\tau=0} \\ &= -\beta_X(\tilde{\mu}_{g^{-1}}(x)) \\ &= -\beta_M(\tilde{\mu}_{g^{-1}}(m_x))^\sharp + \langle \Phi_G(\mu_{g^{-1}}(m_x)), \beta\rangle\,\partial_\theta\,.\end{aligned}$$

For $\varrho(g) \neq 0$ we have $\mathrm{dist}_X(\tilde{\mu}_{g^{-1}}(x), y) \leq 2\delta$; therefore,

$$\langle \Phi_G(\mu_{g^{-1}}(m_x)), \beta \rangle = \langle \Phi_G(m_y), \beta \rangle + O(\delta). \tag{4.3}$$

If δ is sufficiently small, (4.3) is non-zero, since we are assuming that $\Phi_G(m_y)$ is not anti-diagonal; assuming to fix ideas that $\Phi_G(m_y)$ is diagonal, then the right hand side of (4.3) is $\langle \Phi_G(m_y), \beta \rangle = 2\,\lambda(m_x) + O(\delta)$.

In addition, by the discussion in [SZ02], where $\varrho(g) \neq 0$

$$\mathrm{d}_{(\tilde{\mu}_{g^{-1}}(x), y)}\psi = \left(\alpha_{\tilde{\mu}_{g^{-1}}(x)}, -\alpha_y\right) + O(\delta)\,.$$

Therefore,

$$L_\beta(\Psi_{a/k}(x; u, g)) = 2\left[u \cdot \lambda(m_x) + \frac{a}{k}\right] + O(\delta)\,.$$

For $u \gg 0$, we conclude that $L_\beta\left(\Psi_{a/k}(u, g)\right) \geq C' \cdot u + 1$ for some $C' > 0$, which can be chosen uniformly for all $a \in \{-l, \ldots, 0, \ldots l\}$; by iteratively integrating by parts by the transpose operator $L_\beta^t = -\bar{L}_\beta$, we conclude that $\Pi_{k\boldsymbol{\nu}}(x, y)'_{2,a}$ is rapidly decreasing as $k \to +\infty$ uniformly for $a \in \{-l\, \ldots, l\}$. Since this holds uniformly for each of the $k\boldsymbol{\nu}$ summands, the statement of Lemma is established in the case where $k\boldsymbol{\nu}$ is odd.

The case where $k\boldsymbol{\nu}$ is even is only slightly different - one takes l to be half-integer (see Theorem 11.7.1 of [RT10]). □

Lemma 4.1.3. *$\Pi_{k\boldsymbol{\nu}}(x, y)''_2 = O(k^{-\infty})$ as $k \to +\infty$.*

Proof of Lemma 4.1.3. The proof is similar to the one of previous Lemma, except that we shall use eigenvalues rather than matrix elements, so we'll be somewhat sketchy.

If $g \in G \setminus \{\pm I_2\}$, then there is a unique $\vartheta_G(g) = \cos^{-1}(\mathrm{trace}(g)/2) \in (0,\, \pi)$ such that the eigenvalues of g are $e^{\pm i\vartheta_G(g)}$. The map $g \in G \setminus \pm I_2 \mapsto \vartheta_G(g) \in (0,\, \pi)$ is $\mathcal{C}^\infty$. On the same domain, the character of $V_{k\boldsymbol{\nu}}$ is thus given by

$$\chi_{k\,\boldsymbol{\nu}}(g) = \sum_{j=0}^{k\,\nu-1} e^{(k\nu-1-2j)\,\vartheta_G(g)}\,.$$

We shall now write

$$\Pi_{k\boldsymbol{\nu}}(x, y)''_2 \sim \sum_{j=0}^{k\,\nu-1} \Pi_{k\boldsymbol{\nu}}(x, y)''_{2,j}\,,$$

where $\Pi_{k\boldsymbol{\nu}}(x,y)''_{2,j}$ has the same integrand as $\Pi_{k\boldsymbol{\nu}}(x,y)''_2$ with $\chi_{k\,\boldsymbol{\nu}}(g)$ replaced by $e^{(k\nu-1-2j)\,\vartheta_G(g)}$. Thus we can see $\Pi_{k\boldsymbol{\nu}}(x,\,y)''_{2,j}$ as a sum of oscillatory integrals with phases

$$\Psi_b(x,y;u,g) = u\,\psi\left(\tilde{\mu}_{g^{-1}}(x),y\right) - (\nu+b)\cdot\vartheta_G(g)\,,$$

where $b\in\mathbb{R}$ and again the phases Ψ_b vary in a bounded family. Notice that, away from $\{\pm I_2\}$, $\vartheta_G : G\to\mathbb{C}$ is a smooth function. We can argue as in the proof of the previous Lemma. □

□

Thus, we have

$$\Pi_{k\boldsymbol{\nu}}(x,\,y)\sim\Pi_{k\boldsymbol{\nu}}(x,\,y)_1\,.$$

Let us shrink further the support of the integrand. Let $\rho_2:\mathbb{R}\to[0,\,+\infty)$ be a smooth function such that $\rho_2\equiv 0$ on $(0,\,1/(2D)]$ and $\rho_2\equiv 1$ on $[1/D,\,+\infty)$. We have

$$\Pi_{k\boldsymbol{\nu}}(x,\,y)\sim\Pi_{k\boldsymbol{\nu}}(x,\,y)_{11}+\Pi_{k\boldsymbol{\nu}}(x,\,y)_{12}\,,$$

where

$$\begin{aligned}\Pi_{k\boldsymbol{\nu}}(x,\,y)_{11} :=& k^2\nu\int_0^{+\infty}\mathrm{d}u\int_G\mathrm{dV}_G(g)\\ &\left[\rho_x(g)\,\rho_2(u)\,\rho_1(u)\,\overline{\chi_{k\boldsymbol{\nu}}(g)}\,e^{iku\psi\left(\tilde{\mu}_{g^{-1}}(x),\,y\right)}\,s(\tilde{\mu}_{g^{-1}}(x),\,y,\,ku)\right],\end{aligned}$$

and

$$\begin{aligned}\Pi_{k\boldsymbol{\nu}}(x,\,y)_{12} :=& k^2\nu\int_0^{+\infty}\mathrm{d}u\int_G\mathrm{dV}_G(g)\\ &\left[\rho_x(g)\,(1-\rho_2(u))\,\rho_1(u)\,\overline{\chi_{k\boldsymbol{\nu}}(g)}\,e^{iku\psi\left(\tilde{\mu}_{g^{-1}}(x),\,y\right)}\,s(\tilde{\mu}_{g^{-1}}(x),\,y,\,ku)\right].\end{aligned}$$

Lemma 4.1.4. *$\Pi_{k\boldsymbol{\nu}}(x\,x)_{12}=O(k^{-\infty})$ as $k\to+\infty$.*

Proof. By the use of the Weyl integration formula we have

$$\begin{aligned}\Pi_{k\boldsymbol{\nu}}(x,\,y)_{12} =& \frac{k^2\nu}{2\pi}\int_0^{+\infty}\mathrm{d}u\int_{G/T}\mathrm{dV}_{G/T}(gT)\int_{-\pi}^{\pi}\mathrm{d}\vartheta\\ &\Bigg[\rho_x(ge^{i\boldsymbol{\vartheta}}g^{-1})\,(1-\rho_2(u))\,\rho_1(u)\,(e^{i\vartheta}-e^{-i\vartheta})\,e^{-ik\nu\vartheta}\\ &\quad e^{iku\psi\left(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\,x\right)}\,s(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\,x,\,ku)\Bigg].\end{aligned}$$

Let us set

$$\Psi(u,\, gT,\, \vartheta) = u\psi\left(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\, y\right) - \nu\vartheta\,.$$

If $D \gg 0$ and $0 < u < 1/D$ we have

$$|\partial_\vartheta \Psi(u,\, gT,\, \vartheta)| \geq \frac{\nu}{2}\,.$$

The statement follows integrating by parts in $\mathrm{d}\vartheta$. □

In view of the previous lemmas, we can write

$$\Pi_{k\boldsymbol{\nu}}(x,\, y) \sim \frac{k^2\nu}{2\pi}\int_{1/D}^{D}\mathrm{d}u\int_{G/T}\mathrm{dV}_{G/T}(gT)\int_{-\pi}^{\pi}\mathrm{d}\vartheta \tag{4.4}$$

$$\left[\rho_x(ge^{i\boldsymbol{\vartheta}}g^{-1})\,\rho_1(u)\,(e^{i\vartheta}-e^{-i\vartheta})\,e^{ik\Psi_{x,\,y}(u,\,gT,\,\vartheta)}\,s(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\, y,\, ku)\right].$$

where

$$\Psi_{x,y}(u,\, gT,\, \vartheta) = u\,\psi\left(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\, y\right) - \nu\,\vartheta \tag{4.5}$$

Let $\Im(z)$ denote the imaginary part of $z \in \mathbb{C}$. In view of Corollary 1.3 of [BdMS76], there exists a fixed constant C_1, depending only on X, such that

$$\Im\left(\psi(x,\, y)\right) \geq C_1 \mathrm{dist}_X(x,\, y)^2\,. \tag{4.6}$$

Thus, in the range fixed in the hypothesis, we have

$$\mathrm{dist}_X\left(\tilde{\mu}_{ge^{i\boldsymbol{\vartheta}}g^{-1}}(x),\, y\right) \geq C\,k^{\epsilon-1/2}$$

for every $gT \in G/T$ and $e^{i\vartheta} \in T$. In view of (4.5) and (4.6),

$$|\partial_u \Psi_{x,y}(u,\, gT,\, \vartheta)| = |\partial_u\psi\left(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\, y\right)|$$
$$\geq \Im\left(\psi\left(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\, y\right)\right) \geq C_1\mathrm{dist}_X\left(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\, y\right)^2 \geq C_1\,C^2\,k^{2\epsilon-1}\,.$$

Eventually, by the identity

$$-\frac{i}{k}\,\partial_u\psi\left(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\, x\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}u}e^{ik\Psi_{x,\,y}} = e^{ik\Psi_{x,\,y}}\,,$$

we can iteratively integrate by parts in $\mathrm{d}u$ in (4.4) and the claim follows. □

4.2 Proof of Theorem 1.2.2

Proof. Let $\varrho : G \to [0, +\infty)$ be a smooth bump function supported in a small neighborhood of the origin, and identically equal to 1 on a smaller neighborhood. Let us set $\varrho_j(g) := \varrho(g\, g_j)$. We then have

$$\begin{aligned} \Pi_{k\boldsymbol{\nu}}(x,\,x) &= k\nu \cdot \int_G \mathrm{dV}_G(g) \left[\overline{\chi_{k\boldsymbol{\nu}}(g)}\,\Pi\left(\tilde{\mu}_{g^{-1}}(x),\,x\right)\right] \\ &\sim \sum_{j=0}^{N} k\nu \cdot \int_G \mathrm{dV}_G(g) \left[\varrho_j(g)\cdot \overline{\chi_{k\boldsymbol{\nu}}(g)}\,\Pi\left(\tilde{\mu}_{g^{-1}}(x),\,x\right)\right]\,. \end{aligned} \tag{4.7}$$

Let us write $\Pi_{k\boldsymbol{\nu}}(x,\,x)_j$ for the j-th summand on the last line of (4.7).

Let $Z_x := G_x \cap Z$, where $Z = \{\pm I_2\}$ is the center of G. We shall distinguish two cases, depending on whether $g_j \in Z_x$ or not. Let us write

$$\Pi_{k\boldsymbol{\nu}}(x,\,x) \sim \Pi_{k\boldsymbol{\nu}}(x,\,x)_{Z_x} + \Pi_{k\boldsymbol{\nu}}(x,\,x)_{G_x\setminus Z_x}\,,$$

where

$$\Pi_{k\boldsymbol{\nu}}(x,\,x)_{Z_x} := \sum_{g_j\in Z_x} \Pi_{k\boldsymbol{\nu}}(x,\,x)_j, \quad \Pi_{k\boldsymbol{\nu}}(x,\,x)_{G_x\setminus Z_x} := \sum_{g_j\notin Z_x} \Pi_{k\boldsymbol{\nu}}(x,\,x)_j\,.$$

Let us study $\Pi_{k\boldsymbol{\nu}}(x,\,x)_{G_x\setminus Z_x}$. We shall consider the summand in (4.7) with $g_j \notin Z_x$. Then $g_j \neq g_j^{-1}$ and thus $G_x \setminus Z_x$ has even cardinality $b_x = 2\,a_x$, and perhaps after renumbering its elements can be arranged in pairs $(g_j,\,g_j^{-1})$, $j = 1,\ldots,a_x$.

Let $p : G/T \times T \to G$, $(gT,\,t) \mapsto g\,t\,g^{-1}$; since g_j is a regular element of G, it is a regular value of p. For every $j = 1,\ldots,a_x$, let us set

$$t_j := h_{m_x}^{-1}\, g_j\, h_{m_x} = \begin{pmatrix} e^{i\,\vartheta_j} & 0 \\ 0 & e^{-i\,\vartheta_j}\end{pmatrix}\,.$$

We have

$$p^{-1}(g_j) = \left\{(h_{m_x}T,\,t_j),\,(k_{m_x}T,\,t_j^{-1})\right\}\,, \quad k_{m_x} := h_{m_x}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\,. \tag{4.8}$$

Definition 4.2.1. If $z \in \mathbb{C}$, let us set

$$A(z) := i\,\begin{pmatrix} 0 & z \\ \bar{z} & 0\end{pmatrix} \in \mathfrak{g}\,.$$

Then the $\mathbb{R}$-linear map

$$\eta_j \,:\, z \in \mathbb{C} \mapsto \left(\mathrm{Ad}_{t_j^{-1}} - \mathrm{id}_{\mathfrak{g}}\right)(A(z)) \in \mathfrak{g}$$

is injective. Therefore, since $\tilde{\mu}$ is locally free at x, there is a positive define 2×2 matrix $C(x;\, j)$ such that

$$\left\|\mathrm{Ad}_{h_{m_x}}(\eta_j(z))_X(x)\right\|_x^2 = \frac{1}{2} \cdot Z^t\, C(x;\, j)\, Z \quad (z \in \mathbb{C})$$

where $Z := (a,\, b)^t \in \mathbb{R}^2$ if $z = a + i\, b$. Let us define

$$B(x;\, j) := C(x;\, j) + 4\, i\, \sin(2\vartheta_j) \cdot \lambda\,(m_x)\, I_2\,.$$

We shall prove the following.

Proposition 4.2.1. *Assume that $G_x \setminus Z_x = \{g_1,\, g_1^{-1}, \ldots, g_{a_x},\, g_{a_x}^{-1}\}$. Then as $k \to +\infty$ there is an asymptotic expansion*

$$\Pi_{k\boldsymbol{\nu}}(x,\, x)_{G_x\setminus Z_x} \sim \quad 4\,\pi \cdot D_{G/T} \cdot \left(\frac{\nu\, k}{2\,\pi\,\lambda(m_x)}\right)^d \cdot \left[\sum_{j=1}^{a_x} \Re\left(\frac{i\, \sin(\vartheta_j) \cdot e^{-i\,k\,\nu\cdot\vartheta_j}}{\sqrt{\det\left(B(x;\, j)\right)}}\right) + \sum_{l\geq 1} k^{-l/2}\, P_{j\,l}(m_x)\right].$$

Proof of Proposition 4.2.1. Let $E \,:\, \mathfrak{g} \to G$ be the exponential map. We shall write the general $t \in T$ in exponential form as

$$\begin{pmatrix} e^{i\,\vartheta} & 0 \\ 0 & e^{-i\,\vartheta} \end{pmatrix} = E\,(i\,\vartheta\, B)\,, \quad B := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By the Weyl integration formula and character formulae, we have

$$\Pi_{k\boldsymbol{\nu}}(x,\, x)_j = \frac{k\,\nu}{2\,\pi} \cdot \int_{G/T} \mathrm{dV}_{G/T}(gT) \int_{-\pi}^{\pi} \mathrm{d}\vartheta \left[\rho_j\left(g\, e^{i\vartheta\, B} g^{-1}\right) \cdot e^{-ik\nu\cdot\vartheta}\, \Pi(\tilde{\mu}_{ge^{-i\vartheta\, B}g^{-1}}(x),\, x)(e^{i\vartheta} - e^{-i\vartheta})\right]. \tag{4.9}$$

The pulled-back cut-off $(gT,\, e^{i\vartheta}) \mapsto \varrho_j(g\, e^{i\vartheta\, B}\, g^{-1})$ is supported in a small open neighborhood of the pair (4.8). Therefore, we can further split (4.9) as

$$\Pi_{k\boldsymbol{\nu}}(x,\, x)_j = \Pi_{k\boldsymbol{\nu}}(x,\, x)_{j1} + \Pi_{k\boldsymbol{\nu}}(x,\, x)_{j2}\,,$$

where in $\Pi_{k\boldsymbol{\nu}}(x,\,x)_{j1}$ (respectively, $\Pi_{k\boldsymbol{\nu}}(x,\,x)_{j2}$) integration is over a small neighborhood of $(h_{mx}T,\,t_j)$ (respectively, $(k_{mx}T,\,t_j^{-1})$).

Let us consider each $\Pi_{k\boldsymbol{\nu}}(x,\,x)_{jl}$ separately, starting with $l=1$.

Let us introduce local coordinates on G/T and on T. First, for $z \in D(0,\,\delta) \subset \mathbb{C}$ for some suitably small $\delta > 0$, we set

$$h(z) := h_{m_x}\,E\left(A(z)\right)\,,$$

where $A(z)$ is as in Definition 1.2.2; then the assignment $z \in D(0,\,\delta) \mapsto h(z)T \in G/T$ is a system of local coordinates on G/T centered at $h_{m_x}T$. The Haar measure on G/T, expressed in the z coordinates, is $\mathcal{V}_{G/T}(z)\;\mathrm{d}V_{\mathbb{C}}(z)$, for an appropriate smooth function $\mathcal{V}_{G/T}$. The proof of the following Lemma will be omitted.

Lemma 4.2.1. *Let us set $D_{G/T} = \mathcal{V}_{G/T}(0)$. Then $D_{G/T} = 2\pi/V_3$, where V_3 is the surface of S^3.*

Next,

$$\theta \in (-\delta,\,\delta) \mapsto t_j\,E\left(i\theta\,B\right) \in T$$

is a system of local coordinates on T centered at t_j. Furthermore, since $(\tilde{\mu}_{ge^{-i\vartheta\,B}g^{-1}}(x),\,x)$ is in a small neighborhood of the diagonal in $X \times X$, we may replace Π by its representation as an FIO. After performing the rescaling $u \mapsto ku$, we obtain

$$\begin{aligned}\Pi_{k\boldsymbol{\nu}}(x,\,x)_{j1} \sim\; & \frac{k^2\,\nu}{2\,\pi}\cdot e^{-ik\nu\cdot\vartheta_j}\cdot\int_{D(0,\,\delta)}\mathrm{dV}_{\mathbb{C}}(z)\int_{-\delta}^{\delta}\mathrm{d}\theta\int_0^{+\infty}\mathrm{d}u \\ & \left[e^{-ik\left[u\,\psi\left(\tilde{\mu}_{h(z)E(-i\vartheta\,B)t_j^{-1}h(z)^{-1}}(x),\,x\right)-\nu\cdot\theta\right]}\left(e^{i(\vartheta_j+\theta)}-e^{-i(\vartheta_j+\theta)}\right)\right. \\ & \left. s_{j1}\left(\tilde{\mu}_{h(z)E(-i\vartheta\,B)t_j^{-1}h(z)^{-1}}(x),\,x,\,k\,u\right)\,\mathcal{V}_{G/T}(z)\right]\,.\end{aligned} \tag{4.10}$$

Here, $\mathrm{dV}_{\mathbb{C}}(z)$ is the Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$, and s_{j1} denotes the usual amplitude of the representation of Π as an FIO, with the above cut-offs incorporated. In addition, by the same argument used before, only a rapidly decreasing contribution is lost if the integrand is multiplied by a bump function in u, compactly supported in $(1/D,\,D)$, and $\equiv 1$ in $(2/D,\,D/2)$ for some $D \gg 0$ (also implicitly incorporated in the amplitude).

In order to proceed, we need to express the phase more explicitly. We have

$$\begin{aligned}
&h(z)\, E(-i\vartheta\, B)\, t_j^{-1}\, h(z)^{-1} \\
&\quad = C_{h_{m_x}}\left(E(A(z))\, E(-i\vartheta\, B)\, E\left(-\operatorname{Ad}_{t_j^{-1}}(A(z))\right)\right) g_j^{-1} \\
&\quad = E\left(-\operatorname{Ad}_{h_{m_x}}(\gamma(z,\,\theta))\right) g_j^{-1}\,,
\end{aligned}$$

where (by the use of Baker-Campbell-Hausdorff)

$$\gamma(z,\theta) = \gamma_1(z,\theta) + \gamma_2(z,\theta) + R_3(z,\,\theta)\,,$$

with

$$\begin{aligned}
\gamma_1(z,\,\theta) &:= i\theta\, B + \left(\operatorname{Ad}_{t_j^{-1}} - \operatorname{id}_{\mathfrak{g}}\right)(A(z))\;, \\
\gamma_2(z,\,\theta) &:= -\frac{i}{2}[\theta\, B,\; A(z) + \operatorname{Ad}_{t_j^{-1}}(A(z))] + \frac{1}{2}[A(z),\; \operatorname{Ad}_{t_j^{-1}}(A(z))]\,.
\end{aligned}$$

while R_j denotes a generic C^∞ function vanishing to j-th order at the origin. By Corollary 2.2 of [Pao12], we obtain in HLC's

$$\tilde{\mu}_{h(z)E(-i\vartheta\, B)t_j^{-1}h(z)^{-1}}(x) = \tilde{\mu}_{E\left(-\operatorname{Ad}_{h_{m_x}}(\gamma(z,\theta))\right)}(x) = x + (\Theta(z,\theta),\, V(\theta,z))\;,$$

where

$$\begin{aligned}
\Theta(z,\theta) &:= \left\langle \Phi_G(m_x),\, \operatorname{Ad}_{h_{m_x}}(\gamma(z,\theta))\right\rangle + R_3(z,\theta) \\
V(\theta,z) &:= \operatorname{Ad}_{h_{m_x}}(\gamma(z,\theta))_M\,(m) + R_2(z,\theta)\,.
\end{aligned}$$

By the discussion §3 of [SZ02] (see especially (65)) we conclude that

$$\begin{aligned}
&u\psi\left(\tilde{\mu}_{h(z)E(-i\vartheta\, B)t_j^{-1}h(z)^{-1}}(x), x\right) - \nu\,\theta \\
&\quad = u\psi\left(x + (\Theta(z,\theta),\, V(\theta,z))\,,\, x\right) - \nu\,\theta \\
&\quad = i u\cdot\left[1 - e^{i\Theta(z,\theta)}\right] + \frac{i\,u}{2}\cdot\|V(\theta,z)\|^2 + u\, R_3(z,\theta) \\
&\quad = u\,\Theta(z,\theta) + \frac{i\,u}{2}\cdot\left[\Theta(z,\theta)^2 + \|V(\theta,z)\|^2\right] + u\, R_3(z,\,\theta)\,.
\end{aligned}$$

Let us choose $C > 0$, $\epsilon \in (0, 1/6)$. Since p is a local diffeomorphism at $(h_{m_x}T,\, t_j)$ and $\tilde{\mu}$ is locally free at x, the contribution to the asymptotics of (4.10) of the locus where $\|(z,\theta)\| \geq C\,k^{\epsilon-1/6}$ is $O(k^{-\infty})$. Adopting the rescaling $z \mapsto z/\sqrt{k}$, $\theta \mapsto \theta/\sqrt{k}$ we can rewrite (4.10) in the following form:

$$\Pi_{k\boldsymbol{\nu}}(x,\,x)_{j1} \sim \frac{k^{1/2}\,\nu}{2\,\pi}\cdot e^{-ik\nu\cdot\vartheta_j}\int_{D(0,\,\delta)}\mathrm{dV}_{\mathbb{C}}(z)\int_{-\delta}^{\delta}\mathrm{d}\theta\int_0^{+\infty}\mathrm{d}u \left[e^{ik\Psi_k(x;\,u,\,\theta,\,z)}\,\mathcal{A}_k(x;\,u,\,\theta,\,z)\right]; \tag{4.11}$$

here we have set

$$\begin{aligned} i\,k\,\Psi_k(x;\,u,\,\theta,\,z) :=& \,i\,\sqrt{k}\,\left[u\cdot\left\langle\Phi_G(m_x),\,\mathrm{Ad}_{h_{m_x}}\left(\gamma_1(z,\theta)\right)\right\rangle - \theta\,\nu\right] \\ & + i\,u\cdot\left\langle\Phi_G(m_x),\,\mathrm{Ad}_{h_{m_x}}\left(\gamma_2(z,\theta)\right)\right\rangle \\ & - \frac{u}{2}\,\left\|\mathrm{Ad}_{h_{m_x}}\left(\gamma_1(z,\theta)\right)_X(x)\right\|^2 + k\,R_3\left(\frac{z}{\sqrt{k}},\,\frac{\theta}{\sqrt{k}}\right), \end{aligned}$$

$$\begin{aligned} \mathcal{A}_k(x;\,u,\,\theta,\,z) :=& s_{j1}\left(\tilde{\mu}_{h(z/\sqrt{k})E(-i\vartheta/\sqrt{k}\,B)t_j^{-1}h(z/\sqrt{k})^{-1}}(x),\,x,\,k\,u\right) \\ & \cdot\varrho(k^{-\epsilon}(z,\theta))\cdot\left(e^{i(\vartheta_j+\theta/\sqrt{k})} - e^{-i(\vartheta_j+\theta/\sqrt{k})}\right)\cdot\mathcal{V}_{G/T}\left(\frac{z}{\sqrt{k}}\right), \end{aligned}$$

with ϱ an appropriate bump function. Integration in (z,θ) in (4.11) is over a ball of radius $O(k^{-\infty})$ centered at the origin.

The second summand in $\gamma_1(z,\theta)$ in $\gamma_2(z)$ is anti-diagonal. Therefore,

$$\begin{aligned} \left\langle\Phi_G(m_x),\,\mathrm{Ad}_{h_{m_x}}\left(\gamma_1(z,\theta)\right)\right\rangle &= \left\langle\mathrm{Ad}_{h_{m_x}^{-1}}\left(\Phi_G(m_x)\right),\,\gamma_1(z,\theta)\right\rangle \\ &= \langle i\lambda(m_x)\,B,\,i\theta\,B\rangle = 2\,\lambda(m_x)\,\theta\,. \end{aligned}$$

Let us define

$$I_k(x,\,z) := \int_{-\infty}^{+\infty}\mathrm{d}\theta\int_0^{+\infty}\mathrm{d}u\left[e^{i\sqrt{k}\,\Psi_k(u,\theta)}\,\mathcal{A}_k(x;\,u,\,\theta,\,z)\right], \tag{4.12}$$

where now

$$\begin{aligned} \Psi_k(u,\theta) &:= \theta\,[2\,\lambda(m_x)\cdot u - \nu] \\ \mathcal{A}_k(x;\,u,\,\theta,\,z) &:= e^{\mathcal{E}_k(u,\,\theta,\,z)}\cdot A_k(x;\,u,\theta,z), \end{aligned}$$

with

$$\mathcal{E}(u,\,\theta,\,z) := i\,u\cdot\left\langle \Phi_G(m_x),\,\mathrm{Ad}_{h_{m_x}}\left(\gamma_2(z,\theta)\right)\right\rangle - \frac{u}{2}\cdot\left\|\mathrm{Ad}_{h_{m_x}}\left(\gamma_1(z,\,\theta)_X(x)\right)\right\|_x^2 + k\,R_3\left(\frac{z}{\sqrt{k}},\,\frac{\theta}{\sqrt{k}}\right). \tag{4.13}$$

Since $\tilde{\mu}$ is locally free at x, on the support of the integrand

$$\Re\left(\mathcal{E}_k(z,\theta)\right) \leq -D'\cdot\left(|z|^2+|\theta|^2\right)$$

for some positive constant $D>0$.

Then we can rewrite (4.11) in the following form:

$$\Pi_{k\boldsymbol{\nu}}(x,\,x)_{j1} \sim \frac{k^{1/2}\,\nu}{2\pi}\cdot e^{-ik\nu\cdot\vartheta_j}\cdot\int_{\mathbb{C}}\mathrm{dV}_{\mathbb{C}}(z)\left[I_k(x,\,z)\right]. \tag{4.14}$$

The following is straightforward.

Lemma 4.2.2. Ψ_k *has a unique critical point, which is non-degenerate and given by* $(u_0,\theta_0)=(\nu/(2\,\lambda(m_x)),0)$*; we have* $\Psi_k(u_0,\theta_0)=0$*. The Hessian matrix has determinant* $-4\lambda(m_x)^2$ *and vanishing signature.*

We can apply the Stationary Phase Lemma to determine the asymptotic expansion of (4.12). By a few computations we get

$$\gamma_1(z,0) = i\begin{pmatrix} 0 & \left(e^{-2i\,\vartheta_j}-1\right)\cdot z \\ \left(e^{2i\,\vartheta_j}-1\right)\cdot\bar{z} & 0\end{pmatrix},$$
$$\gamma_2(z,0) = -i\cdot|z|^2\cdot\sin(2\,\vartheta_j)\,B\,.$$

If $z=a+ib$ with $a,\,b\in\mathbb{R}$, let $Z=(a,\,b)^t\in\mathbb{R}^2$ be the corresponding vector; thus $|z|=\|Z\|$. Then

$$\left\|\mathrm{Ad}_{h_{m_x}}\left(\gamma_1(z,0)\right)_X(x)\right\|_x^2 = \frac{1}{2}\cdot Z^t\,C^t(x,\,j)\,Z$$

where $C(x,\,j)$ is as in Definition 1.2.2. From (4.13) we conclude

$$\mathcal{E}_k(u_0,0,z) = -\frac{u_0}{2}\,Z^t\,B(x,\,j)\,Z + k\,R_3\left(\frac{z}{\sqrt{k}}\right),$$

where $B(x,\,j)$ is also as in Definition 1.2.2.

On the other hand, we have (noting that $\sin(\vartheta_j) \neq 0$ as $g_j \notin Z_x$)

$$e^{i(\vartheta_j+\theta/\sqrt{k})} - e^{-i(\vartheta_j+\theta/\sqrt{k})} = 2\,i\,\sin(\vartheta_j) \cdot \left[1 + \sum_{j\geq 1} k^{-l/2} a_{jl}(\theta)\right] .$$

Applying the Stationary Phase Lemma, we obtain that as $k \to +\infty$

$$I_k(x,\,z) \sim \quad D_{G/T} \left(\frac{k\,u_0}{\pi}\right)^d \cdot \frac{\pi}{\sqrt{k}} \cdot \frac{2\,i\,\sin(\vartheta_j)}{\lambda(m_x)} \cdot e^{-\frac{u_0}{2} Z^t\,A(x,j)\,Z} \cdot \left[1 + \sum_{l\geq 1} k^{-l/2}\,R_{jl}(m_x,\,Z)\right] , \tag{4.15}$$

where $R_{jl}(m_x,\,Z)$ is polynomial in Z of degree $\leq 3l$. Inserting (4.15) in (4.14), we obtain

$$\Pi_{k\boldsymbol{\nu}}(x,\,x)_{j1} \sim \quad 4\pi \cdot D_{G/T} \cdot \frac{\sin(\vartheta_j) \cdot e^{-ik\nu\cdot\vartheta_j}}{\sqrt{\det B(x,\,j)}} \cdot \left(\frac{\nu\,k}{2\,\pi \cdot \lambda(m_x)}\right)^d \cdot \left[1 + \sum_{l\geq 1} k^{-l/2}\,P_{jl}(m_x)\right] . \tag{4.16}$$

Let us remark that $g_j \neq g_j^{-1}$ since $g_j \neq \pm I_2$; summing the contributions (4.16) corresponding to g_j and $g_{j'} = g_j^{-1}$, we obtain

$$\Pi_{k\boldsymbol{\nu}}(x,\,x)_{j1} + \Pi_{k\boldsymbol{\nu}}(x,\,x)_{j'1} = \quad 8\pi \cdot \Re\left(\frac{i\,\sin(\vartheta_j) \cdot e^{-i\,k\nu\vartheta_j}}{\sqrt{\det\left(A(x,\,j)\right)}}\right) \cdot \left(\frac{\nu\,k}{2\pi \cdot \lambda(m_x)}\right)^d \cdot \left[D_{G/T} + \sum_{l\geq 1} k^{-l/2}\,P_{jl}(m_x)\right] .$$

To deal with $\Pi_{k\boldsymbol{\nu}}(x,\,x)_{j2}$, in view of (4.8) we need only go over the previous computations replacing h_{m_x} with k_{m_x}, and t_j with t_j^{-1}. In the analogue of (4.12), in place of the phase Ψ_x, we obtain

$$\Psi'_x(u,\,\theta) = -\theta \cdot \left[2\,\lambda(m_x) \cdot u + \nu\right] ,$$

so that $\partial_\theta \Psi'_x(u,\,\theta) = -[2\,\lambda(m_x) \cdot u + \nu] \leq -\nu$ on the domain of integration. Thus $\Pi_{k\boldsymbol{\nu}}(x,\,x)_{j2} = O(k^{-\infty})$.

The proof of Proposition 4.2.1 is complete. □

□

4.3 Proof of Theorem 1.2.3

Proof of Theorem 1.2.3. We have

$$\Pi_{k\boldsymbol{\nu}}(x_{1k}, x_{2k}) = k\nu \int_G \mathrm{dV}_G(g) \left[\overline{\chi_{k\,\boldsymbol{\nu}}(g)}\, \Pi\left(\tilde{\mu}_{g^{-1}}(x_{1\,k}),\, x_{2\,k}\right)\right] .$$

Only a rapidly decreasing contribution to the asymptotic is lost, if integration is restricted to a small neighborhood of G_x. Thus we may multiply the integrand by a cut-off function $\varrho \in \mathcal{C}^\infty(G)$ supported in a small neighborhood of G_x and $\equiv 1$ in a slightly smaller neighborhood, without altering the asymptotics. We may assume that ϱ is invariant under conjugation; thus we shall write $\varrho = \varrho(t)$, or $\varrho(\vartheta)$ working in coordinates.

By the Weyl integration formula and character formula,

$$\begin{aligned}\Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k}) \sim k\nu &\int_T \mathrm{dV}_T(t) \int_{G/T} \mathrm{dV}_{G/T}(gT) \\ &\left[t^{-k\nu}(t - t^{-1})\, \varrho(t)\, \Pi(\tilde{\mu}_{gt^{-1}g^{-1}}(x_{1k},\, x_{2k}))\right] \\ = \frac{k\nu}{2\pi} &\int_{-\pi/2}^{3\pi/2} \mathrm{d}\vartheta \int_{G/T} \mathrm{dV}_{G/T}(gT) \\ &\left[e^{-i\,k\nu\vartheta}(e^{i\vartheta} - e^{-i\vartheta})\, \varrho(\vartheta)\, \Pi(\tilde{\mu}_{ge^{-i\vartheta}\,B_{g^{-1}}}(x_{1k}),\, x_{2k})\right] .\end{aligned}$$

On the support of ϱ, $(\tilde{\mu}_{ge^{-i\vartheta}\,B_{g^{-1}}}(x_{1k}),\, x_{2k})$ lies in a small neighborhood of the diagonal; hence we may replace Π by its representation as an FIO, without changing the asymptotics. Therefore,

$$\begin{aligned}\Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k}) \sim &\frac{k^2\nu}{2\pi} \int_0^{+\infty} \mathrm{d}u \int_{-\pi/2}^{3\pi/2} \mathrm{d}\vartheta \int_{G/T} \mathrm{dV}_{G/T}(gT) \Big[\rho(\vartheta)\,(e^{i\vartheta} - e^{-i\vartheta}) \\ &e^{ik\left[u\,\psi\left(\tilde{\mu}_{ge^{-i\vartheta}\,B_{g^{-1}}}(x_{1k}),\, x_{2k}\right) - \nu\vartheta\right]}\, s(\tilde{\mu}_{ge^{-i\vartheta}\,B_{g^{-1}}}(x_{1k}),\, x_{2k},\, ku)\Big] ,\end{aligned}$$

where we have performing the rescaling $u \mapsto k\,u$.

The same arguments of Lemmas 4.1.1 and 4.1.4 apply here with minor modifications. In particular, we have

Lemma 4.3.1. *Let $D \gg 0$ and let $\varrho_1 \in \mathcal{C}_c(\mathbb{R})$ be ≥ 0, supported in $(1/D,\, D)$, and $\equiv 1$ on $(2/D,\, D/2)$. Then only a rapidly decreasing contribution to the asymptotics is lost, if the integrand is multiplied by $\varrho_1(u)$.*

We obtain

$$\Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k}) \sim \frac{k^2\nu}{2\pi}\int_0^{+\infty}\mathrm{d}u\int_{-\pi/2}^{3\pi/2}\mathrm{d}\vartheta\int_{G/T}\mathrm{dV}_{G/T}(gT)\Big[\varrho_1(u)\,(e^{i\vartheta}-e^{-i\vartheta}) \tag{4.17}$$
$$\rho(\vartheta)\,e^{ik\left[u\,\psi\left(\tilde{\mu}_{ge^{-i\vartheta}\,B}{}_{g^{-1}}(x_{1k}),\,x_{2k}\right)-\nu\vartheta\right]}\,s(\tilde{\mu}_{ge^{-i\vartheta}\,B}{}_{g^{-1}}(x_{1k}),\,x_{2k},\,ku)\Big],$$

We can write $\varrho(\vartheta)=\varrho_0(\vartheta)+\varrho_\pi(\vartheta)$, where $\varrho_0(\vartheta)$ is supported in a small neighborhood of 0, and identically equal to 1 in a smaller neighborhood; on the other hand, $\varrho_\pi(\vartheta)$ is supported in a small neighborhood of π, and in fact it vanishes identically if $-I_2\notin G_x$, while it is identically equal to 1 on a smaller neighborhood if $I_2\in G_x$.

Inserting the latter identity in (4.17), we shall accordingly write

$$\Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k}) \sim \Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k})_0 + \Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k})_\pi\,,$$

and examine the two summands separately.

First, let us consider the asymptotics of $\Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k})_0$. We shall prove the following.

Proposition 4.3.1. *Under the assumptions of Theorem 1.2.3, as $k\to+\infty$ we have*

$$\Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k})_0 \sim \frac{1}{2\,\lambda(m_x)}\cdot\left(\frac{\nu\,k}{2\,\pi\,\lambda(m_x)}\right)^d\cdot e^{u_0\,\psi_2(\mathbf{v}_1,\,\mathbf{v}_2)}$$
$$\cdot\left[1+\sum_{j\geq 1}^{+\infty}k^{-j/2}A_j(x,\,\mathbf{v}_1,\,\mathbf{v}_2)\right],$$

where $A_j(x,\,\cdot,\,\cdot)$ is a polynomial of degree $\leq 3j$ and parity $(-1)^j$.

Proof of Proposition 4.3.1. Here $\varrho(\vartheta)$ has been replaced by $\varrho_0(\vartheta)$, therefore integration in $\mathrm{d}\vartheta$ is restricted to $(-2\delta,\,2\delta)$. Let us fix $C_1>0$, $\epsilon_1\in(0,\,1/6)$. Iteratively integrating by parts in $\mathrm{d}u$, we conclude that the locus where $|\vartheta|>C_1k^{\epsilon_1-1/2}$ contributes negligibly to the asymptotics of $\Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k})_0$. More precisely, we have the following.

Lemma 4.3.2. *Suppose that $\varrho_2\in\mathcal{C}_c(\mathbb{R})$ is ≥ 0, supported in $(-2,\,2)$, and $\equiv 1$ on $(-1,\,1)$. Then the asymptotics of $\Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k})_0$ are unchanged, if the integrand is multiplied by $\varrho_2(k^{1/2-\epsilon_1}\vartheta)$.*

Applying the rescaling $\vartheta \mapsto \vartheta/\sqrt{k}$, we recover

$$\begin{aligned}
&\Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k})_0 \\
&\sim \frac{k^{3/2}\,\nu}{2\pi} \int_0^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \int_{G/T} \mathrm{dV}_{G/T}(gT) \Big[\rho_2(k^{-\epsilon_1}\,\vartheta)\,\varrho_1(u)\,e^{ik\Psi_k} \\
&\quad (e^{i\vartheta/\sqrt{k}} - e^{-i\vartheta/\sqrt{k}})\, s(\tilde{\mu}_{ge^{-i\vartheta\,B/\sqrt{k}}g^{-1}}(x_{1k}),\, x_{2k},\, ku)\Big]\,,
\end{aligned}$$

where

$$\Psi_k(u,\, v_1,\, v_2,\, \vartheta, gT) := \psi\left(\tilde{\mu}_{ge^{-i\vartheta\,B/\sqrt{k}}g^{-1}}(x_{1k}),\, x_{2k}\right) - \frac{\vartheta}{\sqrt{k}}\,\nu\,.$$

Integration in $\mathrm{d}\vartheta$ is effectively over an interval of length $4\,k^{\epsilon_1}$ centered at the origin.

The next step is to make Ψ_k more explicit. By Corollary 2.2 of [Pao12], with $m_x = \pi(x)$ we have

$$\begin{aligned}
&\tilde{\mu}_{ge^{-i\vartheta\,B/\sqrt{k}}g^{-1}}(x_{1k}) = \tilde{\mu}_{e^{-\vartheta\ \mathrm{Ad}_g(\beta)/\sqrt{k}}}(x_{1k}) \\
&\quad x + \left(\Theta_k(v_1,\, \vartheta,\, gT),\, \frac{1}{\sqrt{k}} V(v_1,\, \vartheta,\, gT) + R_2\left(\frac{1}{\sqrt{k}}\vartheta,\, \frac{1}{\sqrt{k}}\mathbf{v}_1\right)\right)\,,
\end{aligned}$$

where (for appropriate R_3 and R_2)

$$\begin{aligned}
\Theta_k(v_1,\, \vartheta,\, gT) :=& \frac{1}{\sqrt{k}}\left[\theta_1 + \vartheta \cdot \langle \Phi_G(m_x),\, \mathrm{Ad}_g(\beta)\rangle\right] \\
&+ \frac{1}{k}\vartheta \cdot \omega_{m_x}\left(\mathrm{Ad}_g(\beta)_M(m),\, \mathbf{v}_1\right) + R_3\left(\frac{1}{\sqrt{k}}\vartheta,\, \frac{1}{\sqrt{k}}\mathbf{v}_1\right)
\end{aligned}$$

and

$$V(\mathbf{v}_1,\, \vartheta,\, gT) := \mathbf{v}_1 - \vartheta\ \mathrm{Ad}_g(\beta)_M(m_x)\,.$$

Let us set

$$\tilde{\Theta}_k(v_1,\, v_2,\, \vartheta,\, gT) := \frac{1}{\sqrt{k}}A + \frac{1}{k}B + R_3\left(\frac{1}{\sqrt{k}}\vartheta,\, \frac{1}{\sqrt{k}}\mathbf{v}_1\right)$$

where

$$A = A(v_1,\, v_2,\, \vartheta,\, gT) := \vartheta \cdot \langle \Phi_G(m_x),\, \mathrm{Ad}_g(\beta)\rangle\,,$$

and

$$B = B(v_1,\, \vartheta,\, gT) := \vartheta \cdot \omega_{m_x}\left(\mathrm{Ad}_g(\beta)_M(m),\, \mathbf{v}_1\right).$$

Then, in view of the discussion of §3 of [SZ02] (see especially (65)), we conclude that

$$\Psi_k(u,\, v_1,\, v_2,\, \vartheta,\, gT) = i\, u \left[1 - e^{\tilde{\Theta}_k}\right] - \frac{\vartheta}{\sqrt{k}}\, \nu - i\, \frac{u}{k}\, \psi_2\,(V,\, \mathbf{v}_2) + R_3\left(\frac{1}{\sqrt{k}}(\vartheta,\, v_1,\, v_2)\right).$$

By a few computations, this leads to the following.

Lemma 4.3.3. *We have*

$$\Psi_k(u,\, v_1,\, v_2,\, \vartheta,\, gT) = \frac{1}{\sqrt{k}}\, \mathcal{G}_{\theta_1,\,\theta_2}(u,\, \vartheta,\, gT) + \frac{1}{k}\mathcal{D}(u,\, v_1,\, v_2,\, \vartheta,\, gT) + R_3\left(\frac{1}{\sqrt{k}}(\vartheta,\, v_1,\, v_2)\right).$$

where

$$\mathcal{G}_{\theta_1,\,\theta_2}(u,\, \vartheta,\, gT) = u\, A - \vartheta\, \nu$$

and

$$\mathcal{D}(u,\, v_1,\, v_2,\, \vartheta,\, gT) = u\, \left[B + i\left(\frac{1}{2}A^2 - \psi_2(V,\, \mathbf{v}_2)\right)\right].$$

Thus, we conclude that

$$\Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k})_0 \sim \frac{k^{3/2}\,\nu}{2\pi} \int_0^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \int_{G/T} \mathrm{dV}_{G/T}(gT) \Big[e^{i\sqrt{k}\mathcal{G}_{\theta_1,\,\theta_2}(u,\vartheta,gT)}\, e^{iu\,B + u\,\left[\psi_2(V,\,\mathbf{v}_2) - \frac{1}{2}\,A^2\right]}\, \rho_2(k^{-\epsilon_1}\,\vartheta)\, \varrho_1(u)\,(e^{i\vartheta/\sqrt{k}} - e^{-i\vartheta/\sqrt{k}})\, s(\tilde{\mu}_{ge^{-i\vartheta\, B/\sqrt{k}}g^{-1}}(x_{1k}),\, x_{2k},\, ku)\Big]. \tag{4.18}$$

In order to make (4.18) yet more explicit, let us make recourse to the coordinates $(\theta,\, \delta)$ on G/T discussed in section §3.2.1. Thus we shall make the replacement

$$\int_{G/T} \mathrm{dV}_{G/T}(gT) \mapsto \frac{1}{2\pi} \int_0^{\pi/2} \mathrm{d}\theta \int_{-\pi}^{\pi} \mathrm{d}\delta \sin(2\theta)\,.$$

Furthermore, let $h_m T \in G/T$ be as in the introduction, we shall operate the change of variable $gT \mapsto h_m\, gT$ in G/T, and write g as in section §3.2.1 with $\alpha = \cos(\theta)\, e^{i\delta}$ and $\beta = \sin(\theta)$. Then

$$\begin{aligned}\mathcal{G}_{\theta_1,\,\theta_2}(u,\,\vartheta,\,h_m gT) &= u\cdot\left[\vartheta\cdot\left\langle i\,g^{-1}\begin{pmatrix}\lambda(m_x) & 0\\ 0 & -\lambda(m_x)\end{pmatrix}g,\,\beta\right\rangle\right] - \vartheta\,\nu \\ &= u\cdot[2\vartheta\cdot\cos(2\theta)\cdot\lambda(m_x)] - \vartheta\,\nu\,.\end{aligned} \tag{4.19}$$

Let $\mathcal{G}'_{\theta_1,\,\theta_2}(u,\,\vartheta,\,\theta)$ denote the expression on the last line of (4.19). We can rewrite (4.18) in the following form

$$\begin{aligned}\Pi_{k\boldsymbol{\nu}}(x_{1k},\,x_{2k})_0 \sim \frac{k^{3/2}\,\nu}{4\pi^2}\int_0^{+\infty}\mathrm{d}u\int_{-\infty}^{+\infty}\mathrm{d}\vartheta\int_0^{\pi/2}\mathrm{d}\theta\int_{-\pi}^{\pi}\mathrm{d}\delta \\ \Big[e^{i\sqrt{k}\,\mathcal{G}_{\theta_1,\,\theta_2}(u,\vartheta,gT)}\,e^{iu\,B+u\left[\psi_2(V,\mathbf{v}_2)-\frac{1}{2}\,A^2\right]}\,\rho_2(k^{-\epsilon_1}\,\vartheta) \\ \varrho_1(u)\,(e^{i\vartheta/\sqrt{k}} - e^{-i\vartheta/\sqrt{k}})\,s(\tilde{\mu}_{ge^{-i\vartheta\,B/\sqrt{k}}g^{-1}}(x_{1k}),\,x_{2k},\,ku)\,\sin(2\theta)\Big]\,,\end{aligned} \tag{4.20}$$

where (with abuse of notation) $g = g(\theta,\,\delta)$ and $A = A(\theta,\,\delta)$, $B = B(\theta,\,\delta)$ with the obvious change of variables.

By the change of integration variable $t = \cos(2\theta)$, we can further reformulate (4.20) as follows. With some abuse of notation, let us write $gT = g(t,\,\delta)T$ and set

$$\begin{aligned}\Gamma_{\theta_1,\,\theta_2}(u,\,\vartheta,\,t) &:= \mathcal{G}'_{\theta_1,\,\theta_2}(u,\,\vartheta,\,\theta) \\ &= u\cdot[2\vartheta\cdot t\cdot\lambda(m_x)] - \vartheta\,\nu\,.\end{aligned}$$

Then

$$\begin{aligned}\Pi_{k\boldsymbol{\nu}}(x_{1k},\,x_{2k})_0 \sim \frac{1}{2}\cdot\frac{k^{3/2}\,\nu}{(2\pi)^2}\int_0^{+\infty}\mathrm{d}u\int_{-\infty}^{+\infty}\mathrm{d}\vartheta\int_{-1}^{1}\mathrm{d}t\int_{-\pi}^{\pi}\mathrm{d}\delta \\ \Big[e^{i\sqrt{k}\,\Gamma_{\theta_1,\,\theta_2}(u,\vartheta,t)}\,e^{iu\,B_t+u\left[\psi_2(V,\mathbf{v}_2)-\frac{1}{2}\,A_t^2\right]}\,\rho_2(k^{-\epsilon_1}\,\vartheta) \\ \varrho_1(u)\,\left(e^{i\vartheta/\sqrt{k}} - e^{-i\vartheta/\sqrt{k}}\right)\,s(\tilde{\mu}_{ge^{-i\vartheta\,B/\sqrt{k}}g^{-1}}(x_{1k}),\,x_{2k},\,ku)\,\sin(2\theta)\Big]\,;\end{aligned} \tag{4.21}$$

we have denoted by A_t, B_t the functions

$$A_t(v_1,\,v_2,\,\vartheta,\,\delta) = A_t(v_1,\,v_2,\,\vartheta,\,g(t,\,\delta)T),\;\; B_t(v_1,\,\vartheta,\,\delta) = B_t(v_1,\,\vartheta,\,g(t,\,\delta)T),$$

and similarly for V_t.

Let us remark that

$$e^{i\vartheta/\sqrt{k}} - e^{-i\vartheta/\sqrt{k}} = \frac{2\,i}{\sqrt{k}}\,\vartheta\,\sum_{j=0}^{+\infty}\frac{(-1)^j}{(2j+1)!}\,\frac{\vartheta^{2j}}{k^j}$$
$$= \frac{2\,i}{\sqrt{k}}\,\vartheta\,\left[1 + R_2\left(\frac{\vartheta}{\sqrt{k}}\right)\right] .$$

Furthermore, working in HLC's, Taylor expansion yields an asymptotic expansion

$$s(\tilde{\mu}_{ge^{-i\vartheta\,B/\sqrt{k}}g^{-1}}(x_{1k}),\, x_{2k},\, ku) \sim \left(\frac{k\,u}{\pi}\right)^d \left[1 + R_1\left(\frac{\vartheta}{\sqrt{k}}\right)\right] .$$

It follows that (4.21) is given by an asymptotic expansion in descending half-integer powers of k; with a few computations, one sees that the dominant term is to be extracted from

$$I_{v_1,\,v_2}(k) := \frac{\nu}{8\pi}\cdot\left(\frac{k}{\pi}\right)^{d+1}\int_0^{+\infty}\mathrm{d}u\int_{-\infty}^{+\infty}\mathrm{d}\vartheta\int_{-1}^{1}\mathrm{d}t\int_{-\pi}^{\pi}\mathrm{d}\delta \tag{4.22}$$
$$\Big[e^{i\sqrt{k}\,\Gamma_{\theta_1,\,\theta_2}(u,\vartheta,t)}\cdot(2i\,\vartheta\,u)$$
$$e^{iu\,B_t+u\left[\psi_2(V,\mathbf{v}_2)-\frac{1}{2}\,A_t^2\right]}\,\varrho_1(u)\,\rho_2(k^{-\epsilon_1}\,\vartheta)\,u^{d-1}\Big] ;$$

the latter may in turn be rewritten

$$I_{v_1,\,v_2}(k) = \frac{\nu}{8\pi}\cdot\left(\frac{k}{\pi}\right)^{d+1}\frac{1}{\sqrt{k}\cdot\lambda(m_x)}$$
$$\cdot\int_0^{+\infty}\mathrm{d}u\int_{-\infty}^{+\infty}\mathrm{d}\vartheta\int_{-1}^{1}\mathrm{d}t\int_{-\pi}^{\pi}\mathrm{d}\delta\Big[\partial_t\left(e^{i\sqrt{k}\,\Gamma_{\theta_1,\,\theta_2}(u,\vartheta,t)}\right)$$
$$e^{iu\,B_t+u\left[\psi_2(V,\mathbf{v}_2)-\frac{1}{2}\,A_t^2\right]}\,\varrho_1(u)\,\rho_2(k^{-\epsilon_1}\,\vartheta)\,u^{d-1}\Big] .$$

Integrating by parts in dt, we obtain

$$I_{v_1,\,v_2}(k) = \frac{\nu}{8\pi}\cdot\left(\frac{k}{\pi}\right)^{d+1}\frac{1}{\sqrt{k}\cdot\lambda(m_x)}\cdot\left[J'_{v_1,v_2}(k) - J''_{v_1,v_2}(k) - J'''_{v_1,v_2}(k)\right]$$

where

$$J'_{v_1,v_2}(k) := \int_0^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \int_{-\pi}^{\pi} \mathrm{d}\delta \Big[e^{i\sqrt{k}\,\Gamma_{\theta_1,\,\theta_2}(u,\,\vartheta,\,1)}$$

$$e^{iu\,B_1 + u\left[\psi_2(V,\,\mathbf{v}_2) - \frac{1}{2}\,A_1^2\right]}\,\varrho_1(u)\,\rho_2(k^{-\epsilon_1}\,\vartheta)\,u^{d-1}\Big],$$

$$J''_{v_1,v_2}(k) := \int_0^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \int_{-\pi}^{\pi} \mathrm{d}\delta \Big[e^{i\sqrt{k}\,\Gamma_{\theta_1,\,\theta_2}(u,\,\vartheta,\,-1)}$$

$$e^{iu\,B_{-1} + u\left[\psi_2(V,\,\mathbf{v}_2) - \frac{1}{2}\,A_{-1}^2\right]}\,\varrho_1(u)\,\rho_2(k^{-\epsilon_1}\,\vartheta)\,u^{d-1}\Big],$$

and

$$J'''_{v_1,v_2}(k) := \int_{-1}^{1} \mathrm{d}t \int_0^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \int_{-\pi}^{\pi} \mathrm{d}\delta \Big[e^{i\sqrt{k}\,\Gamma_{\theta_1,\,\theta_2}(u,\,\vartheta,\,t)}$$

$$\cdot\,\partial_t \left(e^{iu\,B_t + u\left[\psi_2(V,\,\mathbf{v}_2) - \frac{1}{2}\,A_t^2\right]} \right)\,\varrho_1(u)\,\rho_2(k^{-\epsilon_1}\,\vartheta)\,u^{d-1}\Big].$$

Let us estimate the three summands separately.

Lemma 4.3.4. *As $k \to +\infty$, there is an asymptotic expansion of the form*

$$J'_{v_1,v_2}(k) \sim \frac{2\,\pi^2}{\lambda(m_x)\,\sqrt{k}} \cdot e^{u_0\,\psi_2(\mathbf{v}_1,\,\mathbf{v}_2)} \cdot \left(\frac{\nu}{2\,\lambda(m_x)}\right)^{d-1}$$

$$\cdot \left[1 + \sum_{l=1}^{+\infty} k^{-l/2}\,a_l(m_x;\,v_1,v_2) \right],$$

where $a_l(m_x;\,\cdot,\,\cdot)$ is a polynomial of degree $\leq 3j$, whose coefficient are smooth function on M.

Proof of Lemma 4.3.4. Let us view $J'_{v_1,v_2}(k)$ as an oscillatory integral in the parameter $\sqrt{k}$, with real phase $\Gamma_{\theta_1,\,\theta_2}(1;\,u,\,\vartheta)$

$$\begin{aligned}\Gamma_{\theta_1,\,\theta_2}(1;\,u,\,\vartheta) &= u\cdot[2\,\lambda(m_x)\cdot\vartheta] - \vartheta\,\nu \\ &= u\cdot\left[\left\langle\Phi_G(m_x),\,\mathrm{Ad}_{h_{m_x}}(\beta)\right\rangle\cdot\vartheta\right] - \vartheta\,\nu\,,\end{aligned}$$

and amplitude

$$e^{iu\,B_1 + u\left[\psi_2(V,\,\mathbf{v}_2) - \frac{1}{2}\,A_1^2\right]}\,\varrho_1(u)\,\rho_2(k^{-\epsilon_1}\,\vartheta)\,u^{d-1}.$$

Explicitly, the exponent is

$$\mathcal{E}(u,\,\vartheta,\,v_1,\,v_2) := i\,u\,B_1 + u\,\left[\psi_2\,(V,\,\mathbf{v}_2) - \frac{1}{2}\,A_1^2\right]$$
$$= u\Big[-i\,\omega_m(\mathbf{v}_1,\,\mathbf{v}_2) + i\,\vartheta\,\omega_{m_x}\,\big(\mathrm{Ad}_{h_{m_x}}(\beta)_M(m_x), \mathbf{v}_1 + \mathbf{v}_2\big)$$
$$- \frac{1}{2}\left\|(v_1 - v_2) - \vartheta\;\mathrm{Ad}_{h_{m_x}}(\beta)_X(x)\right\|^2\Big]\,.$$

Under the hypothesis of the Theorem, therefore, $\Re(\mathcal{E}_1) \le -C'\,\vartheta^2 + C''$ for some constants $C',\,C'' > 0$.

Furthermore, the phase has a unique critical point, given by $(u_0,\,\vartheta_0) = (\nu/2\,\lambda(m_x),\,0)$, and Hessian matrix of the phase $\Gamma_{\theta_1,\,\theta_2}(1,\,\cdot,\,\cdot)$ has determinant $-4\,\lambda(m_x)^2$ and its signature is zero. Thus the critical point is non-degenerate, and the critical value is $\Gamma_{\theta_1,\,\theta_2}(1,\,u_0,\,0) = 0$. At the critical point, the exponent in the amplitude is

$$\mathcal{E}_1(u_0,\,0,\,\mathbf{v}_1,\,\mathbf{v}_2) = u_0 \cdot \psi_2(\mathbf{v}_1,\,\mathbf{v}_2)\,.$$

When applying the Stationary Phase Lemma, at the l-th step we need to let the differential operator $k^{-l/2}R_\Gamma^l$ act on the amplitude, where

$$R_\Gamma := \frac{i}{4\cdot\lambda(m_x)}\cdot\frac{\partial^2}{\partial u\,\partial\vartheta}\,,$$

and evaluate the result at the critical point. One sees inductively that

$$k^{-l/2}\,R_\Gamma^l\left(e^{i\,\mathcal{E}_1(u,\vartheta,\mathbf{v}_1,\mathbf{v}_2)}\right) = H_l\,e^{i\,\mathcal{E}_1(u,\vartheta,\mathbf{v}_1,\mathbf{v}_2)}\,,$$

where H_l is a polynomial of degree $\le 3l$ in $(\vartheta, \mathbf{v}_1, \mathbf{v}_2)$ and parity $(-1)^l$. The claimed asymptotic expansion follows by applying the Stationary Phase Lemma. □

Lemma 4.3.5. *As $k \to +\infty$, we have $J''_{v_1,v_2}(k) = O(k^{-\infty})$.*

Proof of Lemma 4.3.5. Let us view $J''_{v_1,v_2}(k)$ as an oscillatory integral in $\sqrt{k}$, with the phase $\Gamma_{\theta_1,\,\theta_2}(-1,\,u,\,\vartheta)$. We have

$$\partial_\vartheta\Gamma_{\theta_1,\,\theta_2}(-1,\,u,\,\vartheta) = -2\,u\cdot\lambda(m_x) - \nu \le -\nu\,.$$

The claim follows by integration by parts in ϑ. □

Lemma 4.3.6. $\mathcal{J}'''_{v_1,v_2}(k;t,\delta)_2 = O(k^{-1})$ *as* $k \to +\infty$.

Proof of Lemma 4.3.6. Let us choose $\epsilon' \in (0,\, \nu/(8\, D \cdot \lambda(m_x)))$, and consider the open cover $\mathcal{U} := \{[-1,\, 2\epsilon'), (\epsilon',\, 1]\}$. Let $\gamma_1(t) + \gamma_2(t) = 1$ be a smooth partition of unity on $[-1,\, 1]$ subordinate to $\mathcal{U}$. Thus

$$\mathcal{J}'''_{v_1,v_2}(k) = \mathcal{J}'''_{v_1,v_2}(k)_1 + \mathcal{J}'''_{v_1,v_2}(k)_2,$$

where $\mathcal{J}'''_{v_1,v_2}(k)_j$ is defined as $\mathcal{J}'''_{v_1,v_2}(k)$, except that the integrand has been further multiplied by $\gamma_j(t)$. Explicitly, let us write

$$\mathcal{J}'''_{v_1,v_2}(k)_j = \int_{-\pi}^{\pi} \mathrm{d}\delta \int_{-1}^{1} \mathrm{d}t \left[\mathcal{J}'''_{v_1,v_2}(k;t,\delta)_j\right]$$

where

$$\mathcal{J}'''_{v_1,v_2}(k;t,\delta)_j = \int_0^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \Bigg[e^{i\sqrt{k}\Gamma_{\theta_1,\,\theta_2}(t,\,u,\,\vartheta)} \cdot \gamma_j(t) \cdot \partial_t \left(e^{i\,u\,B_t + u\left[\psi_2(V_1,\,\mathbf{v}_2) - \frac{1}{2}A_t^2\right]} \right) \cdot \varrho_1(u) \cdot \varrho_2(k^{-\epsilon_1}\vartheta) \cdot u^{d-1} \Bigg].$$

Let us view the integral $\mathcal{J}'''_{v_1,v_2}(k;t,\delta)_j$ as an oscillatory integral with phase $\Gamma_{\theta_1,\,\theta_2}(t,\,u,\,\vartheta)$.

On the support of γ_1, we have $u \le 2D$ and $t \le \epsilon'$; therefore,

$$\partial_\vartheta \Gamma_{\theta_1,\,\theta_2}(t,\,u,\,\vartheta) = 2u \cdot t \cdot \lambda(m_x) - \nu \le 4\,D \cdot \epsilon' \cdot \lambda(m_x) - \nu \le -\frac{\nu}{2}\,.$$

Therefore, integration by parts in ϑ implies that $\mathcal{J}'''_{v_1,v_2}(k;t,\delta)_j = O(k^{-\infty})$, uniformly on the support of γ_1. It follows that $\mathcal{J}'''_{v_1,v_2}(k)_1 = O(k^{-\infty})$.

On the support of γ_2, on the other hand, $\Gamma_{\theta_1,\,\theta_2}(t,\,\cdot,\,\cdot)$ has the non-degenerate critical point

$$(u(t),\, \vartheta(t)) = \left(\frac{\nu}{2\,t\,\lambda(m_x)},\, 0\right),$$

with Hessian matrix

$$\mathrm{Hess}(\Gamma_{\theta_1,\,\theta_2}(t,\,\cdot,\,\cdot)) = \begin{pmatrix} 0 & 2\,t \cdot \lambda(m_x) \\ 2\,t \cdot \lambda(m_x) & 0 \end{pmatrix}.$$

On the other hand, the integrand is divisible by ϑ, hence it vanishes at the critical point. Furthermore, the integrand is of class L^1 as a function of the parameter t, since the exponent getting differentiated is a smooth function of t and $\sqrt{1-t^2}$.

Hence $\mathcal{J}'''_{v_1,\,v_2}(k,\,t,\,\delta)_2$ admits an asymptotic expansion in descending half-integer powers of k, with leading power k^{-1}, and coefficients of class L^1 as functions of t, since the exponent getting differentiated is a smooth function of t and $\sqrt{1-t^2}$.

Hence $\mathcal{J}'''_{v_1,v_2}(k;t,\delta)_2$ admits an asymptotic expansion in descending half-integer powers of k, with leading power k^{-1}, and coefficients of class L^1 as functions of t. The general term of the expansion will be a scalar multiple of

$$k^{-(l+1)/2} \cdot R^l_{\Gamma_t}\left(\partial_t\left(\mathcal{E}_t\left(u,\vartheta,\mathbf{v}_1,\mathbf{v}_2\right)\right) \cdot e^{\mathcal{E}_t(u,\vartheta,\mathbf{v}_1,\mathbf{v}_2)}\right), \tag{4.23}$$

where

$$\mathcal{E}_t\left(u,\vartheta,\mathbf{v}_1,\mathbf{v}_2\right) := i\,u\,B_t + u\left[\psi_2\left(V_t,\mathbf{v}_2\right) - \frac{1}{2}A_t^2\right].$$

Given integers $a,b \geq 0$, let us denote by $H_{a,b}(\vartheta;\mathbf{v}_1;\mathbf{v}_2)$ a generic polynomial in $(\vartheta;\mathbf{v}_1,\mathbf{v}_2)$, which is separately homogeneous of degree a in ϑ, and of degree b in $(\mathbf{v}_1,\mathbf{v}_2)$, and by $H_a(\vartheta;\mathbf{v}_1,\mathbf{v}_2)$ a generic polynomial in $(\vartheta;\mathbf{v}_1,\mathbf{v}_2)$ homogeneous of degree a (but perhaps not polyhomogeneous); both $H_{a,b}$ and H_a are allowed to vary from line to line, and their coefficients depend smoothly on u. Thus $\mathcal{E}_t = u \cdot H_2 = u\cdot(H_{2,0}+H_{1,1}+H_{0,2})$, $\partial_t\mathcal{E}_t = u\cdot(H_{2,0}+H_{1,1})$ (here the polynomials do not depend on u). Hence we can split (4.23) as

$$\begin{aligned}k^{-(l+1)/2} \cdot \Big[&R^l_{\Gamma_t}\left(\rho(u)\cdot H_{2,0}\cdot e^{u\cdot(H_{2,0}+H_{1,1}+H_{0,2})}\right)\\ &+ R^l_{\Gamma_t}\left(\rho(u)\cdot H_{1,1}\cdot e^{u\cdot(H_{2,0}+H_{1,1}+H_{0,2})}\right)\Big].\end{aligned}$$

The proof of Lemma is completed by the following two claims, which can be proved inductively from the cases $l = 0, 1$:

Claim 1. For $l = 0, 1, 2, \ldots$

$$R^l_{\Gamma_t}\left(\rho(u)\cdot H_{1,1}\cdot e^{\mathcal{E}_t}\right)$$

is a sum of term of the form

$$\left[H_{0,1}\cdot H_{p_l} + H_{1,1}\cdot H_{q_l}\right]\cdot e^{\mathcal{E}_t},$$

where $p_l + 1 \leq 3l$, $(-1)^{p_l+1} = (-1)^l$, and $q_l \leq 3l$, $(-1)^{q_l} = (-1)^l$.

Claim 2. For $l = 0, 1, 2, \dots$

$$R^l_{\Gamma_t}\left(\rho(u) \cdot H_{2,0} \cdot e^{\mathcal{E}_t}\right)$$

is a sum of term of the form $H_{a,b} \cdot e^{\mathcal{E}_t}$, where $b \leq 3l$, $(-1)^{a+b} = (-1)^l$.

□

Since at the critical point $\vartheta = 0$, the summands with a factor of the form $H_{a,b}$ with $a \geq 1$ all vanish at the critical point. It follows that the asymptotic expansion for (4.22) is as in the statement of Proposition 4.3.1. This completes the proof. □

Let us next consider the asymptotics of $\Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k})_\pi$. We shall prove the following analogue of the latter Proposition.

Proposition 4.3.2. *Under the assumption of Theorem 1.2.3, as $k \to +\infty$ we have*

$$\begin{aligned}\Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k})_\pi \sim & \frac{1}{2\,\lambda(m_x)} \cdot \left(\frac{\nu\, k}{2\,\pi\,\lambda(m_x)}\right)^d \cdot e^{u_0\,\psi_2(\mathbf{v}_2,\,\mathbf{v}_2)} \\ & \cdot \left[1 + \sum_{j\geq 1}^{+\infty} k^{-j/2}\, A_j(x;\, \mathbf{v}_1,\, \mathbf{v}_2)\right],\end{aligned}$$

where $A_j(x,\,\cdot,\,\cdot)$ is a polynomial of degree $\leq 3j$ and parity $(-1)^j$.

Proof of Proposition 4.3.2. Since the proof is a slight modification of the one for Proposition 4.3.1, we shall be sketchy. In the integrand, $\varrho(\vartheta)$ has now been replaced by $\varrho_\pi(\vartheta)$, therefore integration in $\mathrm{d}\vartheta$ is restricted to $(\pi - 2\,\delta, \pi + 2\,\delta)$. Since ϱ_π vanishes identically unless $-I_2 \in G_x$, we shall assume that the latter condition holds. Let us set $\mathbf{v}_1' := \mathrm{d}_{m_x}\mu_{-I_2}(\mathbf{v}_1)$ and

$$\begin{aligned}x_{1k} :=& \tilde{\mu}_{-I_2}(x_{1k}) \\ =& x + \left(\frac{1}{\sqrt{k}}\,\theta_1 + R_3\left(\frac{1}{\sqrt{k}}\,\theta_1\right),\, \frac{1}{\sqrt{k}}\,\mathbf{v}_1' + R_2(\mathbf{v}_1)\right).\end{aligned}$$

We may assume without loss that $\varrho_\pi(\vartheta) = \varrho_0(\vartheta - \pi)$. Let us perform the change of variable $\vartheta \mapsto \pi + \vartheta$, so that integration in $\mathrm{d}\vartheta$ is over $(-2\,\delta,\, 2\,\delta)$.

We have

$$\begin{aligned}
&\Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k})_\pi \\
&\sim \frac{k^{3/2}\,\nu}{2\pi}\int_0^{+\infty}\mathrm{d}u\int_{-\infty}^{+\infty}\mathrm{d}\vartheta\int_{G/T}\mathrm{dV}_{G/T}(gT)\Big[\rho_2(k^{-\epsilon_1}\,\vartheta)\,\varrho_1(u)\,e^{ik\Gamma_k} \\
&\qquad \left(e^{i\left(\pi+\vartheta/\sqrt{k}\right)}-e^{-i\left(\pi+\vartheta/\sqrt{k}\right)}\right)\,s(\tilde{\mu}_{ge^{-i\vartheta\,B/\sqrt{k}}g^{-1}}(x'_{1k}),\,x_{2k},\,ku)\Big]\,,
\end{aligned}$$

where

$$\Gamma_k(u,\,v_1,\,v_2,\,\vartheta,gT) := u\,\psi\left(\tilde{\mu}_{ge^{-i\vartheta\,B/\sqrt{k}}g^{-1}}(x'_{1k}),\,x_{2k}\right)-\frac{\vartheta}{\sqrt{k}}\,\nu-\pi\,\nu \tag{4.24}$$

$$= \Psi_k(u,\,v'_1,\,v_2,\,\vartheta,gT)-\pi\,\nu\,. \tag{4.25}$$

Let us write $\Psi'_k=\Psi_k(u,\,v'_1,\,v_2,\,\vartheta,gT)$. Thus we may rewrite the latter integral in the following manner

$$\begin{aligned}
&\Pi_{k\boldsymbol{\nu}}(x_{1k},\, x_{2k})_\pi \\
&\sim e^{i\,\pi(1-k\nu)}\frac{k^{3/2}\,\nu}{2\pi}\int_0^{+\infty}\mathrm{d}u\int_{-\infty}^{+\infty}\mathrm{d}\vartheta\int_{G/T}\mathrm{dV}_{G/T}(gT)\Big[\rho_2(k^{-\epsilon_1}\,\vartheta)\,e^{ik\Psi'_k} \\
&\quad\cdot\varrho_1(u)\,\left(e^{i\left(\pi+\vartheta/\sqrt{k}\right)}-e^{-i\left(\pi+\vartheta/\sqrt{k}\right)}\right)\,s(\tilde{\mu}_{ge^{-i\vartheta\,B/\sqrt{k}}g^{-1}}(x'_{1k}),\,x_{2k},\,ku)\Big]\,, \\
&\sim e^{i\,\pi(1-k\nu)}\cdot\Pi_{k\boldsymbol{\nu}}(x'_{1k},\,x_{2k})_0\,.
\end{aligned}$$

The statement of Proposition follows from Proposition 4.3.1. □

Thus the proof is concluded. □

4.4 Proof of Theorem 1.3.1

Before we delve into the proof let us recall some relevant concepts and results from [GS82c]. We shall use throughout the identification $T^*G\cong G\times\mathfrak{g}^\vee$ induced by right translations. If R and S are manifolds and $\Lambda\subset T^*R\times T^*S$ is a Lagrangian submanifold, the corresponding canonical relation is

$$\Lambda' := \{((r,v),\,(s,-\gamma))\,:\,(r,v),\,(s,\gamma)\in\Lambda\}\,.$$

Definition 4.4.1. For every $f \in \mathcal{C}(\mathcal{O}_{\boldsymbol{\nu}})$, let $G_f \leq G$ be the stabilizer subgroup of f, and let $\mathfrak{g}_f$ be its Lie algebra. Let H_f be the closed connected subgroup with Lie subalgebra $\mathfrak{h}_f := \{\xi \in \mathfrak{g}_f \,:\, \langle f, \xi \rangle = 0\}$. The locus

$$\Lambda_L := \{(g, rf) \in G \times \mathfrak{g}^\vee \,:\, f \in \mathcal{O}_{\boldsymbol{\nu}},\, r > 0,\, g \in H_f\}$$

is a Lagrangian submanifold of T^*G.

Definition 4.4.2. For every weight $\boldsymbol{\nu}$, Let us denote by $L = L_{\boldsymbol{\nu}} := (k\boldsymbol{\nu})_{k=0}^{+\infty}$ the *ladder sequence* of weights generated by $\boldsymbol{\nu}$, and set

$$\chi_L := \sum_{k=1}^{+\infty} d_{k\boldsymbol{\nu}}\, \chi_{k\boldsymbol{\nu}} \in \mathcal{D}'(G).$$

Then we have the following

Theorem 4.4.1 (Theorem 6.3 of [GS82c]). *χ_L is a Lagrangian distribution on G, and its associated conic Lagrangian submanifold of $T^*G \cong G \times \mathfrak{g}^\vee$ is Λ_L.*

Consider the Hilbert space direct sum

$$H(X)_L := \bigoplus_{k=1}^{+\infty} H(X)_{k\boldsymbol{\nu}}\,,$$

and let $\Pi_L : L^2(X) \to H(X)_L$ denote the corresponding orthogonal projector, $\Pi_L(\cdot,\, \cdot) \in \mathcal{D}'(X \times X)$ its Schwartz kernel. then

$$\Pi_L(x,\, y) := \int_G \overline{\chi_L(g)}\, \Pi(\tilde{\mu}_{g^{-1}}(x),\, y)\, \mathrm{d}V_G(g)\,.$$

We shall express Π_L in functorial notation using functorial behaviour of wave fronts under pull-back and push-forward (see for instance §1.3 of [Dui96]) to draw conclusions on the singularities of Π_L.

To this end, let us consider the map

$$f\,:\, G \times X \times X \to X \times X, \quad (g,\, x,\, y) \mapsto (\tilde{\mu}_{g^{-1}}(x),\, y)$$

and the distribution $\hat{\Pi} := f^*(\Pi) \in \mathcal{D}'(G \times X \times X)$. Let

$$\Sigma := \{(x,\, r\alpha) \,:\, x \in X,\, r > 0\} \subset T^*X \setminus (0)$$

denote the closed symplectic cone sprayed by the connection 1-form; by [BdMS76], the wave front of Π satisfies

$$\mathrm{WF}'(\Pi) = \mathrm{diag}(\Sigma) \subset \Sigma \times \Sigma\,.$$

It follows that $\mathrm{WF}'(\hat{\Pi}) \subseteq f^*(\mathrm{diag}(\Sigma))$. This implies the following.

Lemma 4.4.1. *In terms of the identification $T^*G \cong G \times \mathfrak{g}^\vee$ induced by the right translations, the canonical relation of $\hat{\Pi}$ is*

$$\mathrm{WF}'(\hat{\Pi}) = \Big\{ ((g,\, r\,\Phi_G(m_x)),\, (x,\, r\,\alpha_x),\, (y,\, r\,\alpha_y)) \in T^*G \times T^*X \times T^*X \ : \\ g \in G, x \in X,\, r > 0,\, y = \tilde{\mu}_{g^{-1}}(x) \Big\}$$

where $m_x := \pi(x)$.

Now let us give the functorial reformulation of the kernel of the ladder projector. Consider the diagonal map

$$\Delta : G \times X \times X \to G \times G \times X \times X, \quad (g,\, x,\, y) \mapsto (g,\, g,\, x,\, y)\,,$$

and the projection

$$p :\, G \times X \times X \to X \times X, \quad (g,\, x,\, y) \mapsto (x,\, y)\,.$$

Lemma 4.4.2. *The Schwarts kernel $\Pi_L \in \mathcal{D}'(X \times X)$ is given by*

$$\Pi_L = p_* \left(\Delta^* (\overline{\chi_L} \boxtimes \hat{\Pi}) \right)\,.$$

Let $\sigma :\, T^*G \to T^*G$ be given by $(g,\, f) \mapsto (g,\, -f)$. Then

$$\begin{aligned} &\mathrm{WF}\left(\overline{\chi_L} \boxtimes \hat{\Pi} \right) \\ &\quad \subseteq \left(\sigma(\Lambda_L) \times (0) \right) \cup \left(\sigma(\Lambda_L) \times \mathrm{WF}(\hat{\Pi}) \right) \cup \left((0) \times \mathrm{WF}(\hat{\Pi}) \right) \\ &\quad \subset T^*G \times (T^*G \times T^*X \times T^*X)\,. \end{aligned}$$

Therefore, the pull back $\Delta^*(\overline{\chi_L} \boxtimes \hat{\Pi})$ is well-defined, and

$$\begin{aligned} &\mathrm{WF}\left(\Delta^* \left(\overline{\chi_L} \boxtimes \hat{\Pi} \right) \right) \subseteq \mathrm{d}\Delta^* \left(\overline{\chi_L} \boxtimes \hat{\Pi} \right) \\ &\quad \subseteq \left(\sigma(\Lambda_L) \times (0) \right) \cup \mathrm{d}\Delta^* \left(\sigma(\Lambda_L) \times \mathrm{WF}(\hat{\Pi}) \right) \cup \mathrm{WF}\left(\hat{\Pi} \right) \\ &\quad \subset T^*G \times T^*X \times T^*X\,. \end{aligned}$$

Explicitly, we have

$$\mathrm{d}\Delta^*\left(\sigma(\Lambda_L)\times \mathrm{WF}(\hat{\Pi})\right) = \Big\{\, ((g,\, -f + r\,\Phi_G(m_x)),\, (x,\, r\,\alpha_x),\, (y,\, -r\,\alpha_y)) \,: \\ f \in \mathcal{C}(\mathcal{O})\, g \in H_f, x \in X,\, r>0,\, y = \tilde{\mu}_{g^{-1}}(x)\Big\}\,.$$

Using that Φ_G is nowhere vanishing, we can now apply Proposition 1.3.4 of [Dui96] to conclude the following.

Corollary 4.4.1. *The wave front* $\mathrm{WF}(\Pi_L) \subseteq (T^*X\setminus(0))\times(T^*X\setminus(0))$ *of the distributional kernel* Π_L *satisfies*

$$\mathrm{WF}(\Pi_L) = \{((x,\, r\,\alpha_x),\, (y,\, -r\,\alpha_y)) \,:\, \Phi_G(m_x)\in\mathcal{C}(\mathcal{O}),\, y \in H_f\cdot x\}\,.$$

where $H_f \cdot x$ *is the* H_f*-orbit of* x.

Corollary 4.4.2. *Let* $\mathrm{SS}(\Pi_L)\subseteq X\times X$ *be the singular support of the distributional kernel* Π_L*. Then* $\mathrm{SS}(\Pi_L)\subseteq \mathcal{Z}_{\boldsymbol{\nu}}$.

4.4.1 The proof

Proof of Theorem 1.3.1. For every $\boldsymbol{\nu} = (\nu_1,\, \nu_2)\in\mathbb{Z}^2$ with $\nu_1>\nu_2$, let $P_{\boldsymbol{\nu}} : L^2(X)\to L^2(X)_{\boldsymbol{\nu}}$ be the orthogonal projector. Clearly

$$\Pi_{k\boldsymbol{\nu}} = P_{k\boldsymbol{\nu}}\circ \Pi_L\,. \tag{4.26}$$

In terms of the Schwartz kernels, (4.26) can be reformulated as follows:

$$\Pi_{k\boldsymbol{\nu}}(x,\, y) = d_{k\boldsymbol{\nu}}\int_G \mathrm{dV}_G(g)\left[\overline{\chi_{k\boldsymbol{\nu}}(g)}\,\Pi_L(\tilde{\mu}_{g^{-1}}(x)\,,\, y)\right]\,. \tag{4.27}$$

Using the Weyl integration, character and dimension formulae, (4.27) can in turn be rewritten as follows (see chapter §3):

$$\Pi_{k\boldsymbol{\nu}}(x,\, y) = \frac{k\,(\nu_1-\nu_2)}{(2\,\pi)^2}\int_{(-\pi,\,\pi)^2}\mathrm{d}\boldsymbol{\vartheta}\,\left[e^{i\,k\langle\boldsymbol{\nu},\,\boldsymbol{\vartheta}\rangle}\,\left(e^{i\,\vartheta_1}-e^{i\,\vartheta_2}\right)\,F_L(x,\, y;\, e^{i\,\boldsymbol{\vartheta}})\right]\,. \tag{4.28}$$

where for $t\in T$ we set

$$F_L(x,\, y;\, e^{i\,\boldsymbol{\vartheta}}) := \int_{G/T}\mathrm{dV}_{G/T}(gT)\,[\Pi_L(\tilde{\mu}_{gt^{-1}g^{-1}}(x)\,,\, y)]\,. \tag{4.29}$$

Now suppose $K \Subset (X \times X) \setminus \mathcal{Z}_{\boldsymbol{\nu}}$. We may assume without loss that K is $G \times G$-invariant. There exist $G \times G$-invariant open subsets A, $B \subset X \times X$ such that

$$K \subset A \Subset (X \times X) \setminus \mathcal{Z}_{\boldsymbol{\nu}}, \quad \mathcal{Z}_{\boldsymbol{\nu}} \subset B \Subset (X \times X) \setminus K, \quad X \times X = A \cup B\,.$$

Hence A is a $G \times G$-invariant open neighbourhood of K in $X \times X$, and the restriction of Π_L to A is $\mathcal{C}^\infty$.

Therefore, we get a $\mathcal{C}^\infty$ function

$$R\,:\,T \times G/T \times A \to \mathbb{C}, \quad (t,\,gT,\,(x,\,y)) \mapsto \Pi_L\left(\tilde{\mu}_{gt^{-1}g^{-1}}(x),\,y\right)\,.$$

With F_L as in (4.29), we obtain a $\mathcal{C}^\infty$ function on $T \times A$ by setting

$$\beta:\,(t,\,(x,\,y)) \mapsto \Delta(t)\,F_L(x,\,y;\,t)\,.$$

Let us denote with $\mathcal{F}_T$ the Fourier transform with respect to $t \in T$ of a function on $T \times A$, viewed as a function on $\mathbb{Z}^2 \times A$; then (4.28) may be rewritten

$$\Pi_{k\boldsymbol{\nu}}(x,\,y) = \frac{k}{2}(\nu_1 - \nu_2) \cdot \mathcal{F}_T(\beta)(k\,\boldsymbol{\nu};\,x,\,y)\,. \tag{4.30}$$

The statement of Theorem 1.3.1 follows from (4.30) and the previous considerations. □

4.5 Proof of Theorem 1.3.2

We shall assume in this section that the assumptions of Theorem 1.3.2 hold.

4.5.1 Preliminaries

Before attacking the proof, it is in order to list some useful preliminaries (see also the discussion of §2 of [Pao12]).

For any $m \in M$, let $\mathrm{val}_m\,:\,\mathfrak{g} \to T_mM$ be the evaluation map $\xi \mapsto \xi_M(m)$, and similarly for any $x \in X$ let $\mathrm{val}_x\,:\,\mathfrak{g} \to T_xX$ be the evaluation map $\xi \mapsto \xi_X(x)$.

Ray transversality and locally free actions

Since $\tilde{\mu}$ preserves the connection 1-form, the induced action of G on T^*X leaves the symplectic cone Σ invariant. The restricted action is of course Hamiltonian, and its moment map $\tilde{\Phi}_G : \Sigma \to \mathfrak{g}$ is the restriction to Σ of the cotangent Hamiltonian map on T^*X.

If $m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, then by equivariance Φ_G is transverse to $\mathbb{R}_+ \cdot \Phi_G(m)$. Hence,

$$\mathrm{d}_m\Phi_G(T_mM) + \mathrm{span}\,(\Phi_G(m)) = \mathfrak{g}\,. \tag{4.31}$$

Suppose $x \in \pi^{-1}(m) \subset X$ and $r > 0$, and consider $\sigma = (x, r\alpha_x) \in \Sigma$. Then it follows from (4.31) that

$$\mathrm{d}_\sigma\tilde{\Phi}_G(T_\sigma\Sigma) = \mathrm{d}_m\Phi_G(T_mM) + \mathrm{span}\,(\Phi_G(m)) = \mathfrak{g}\,.$$

Thus $\tilde{\Phi}_G$ is submersive at any $(x, r\alpha_x)$ with $x \in X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$. If we let $\Sigma^G_{\mathcal{O}_{\boldsymbol{\nu}}} \cong X^G_{\mathcal{O}_{\boldsymbol{\nu}}} \times \mathbb{R}_+$ denote the inverse image of $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ in Σ, we conclude therefore that G acts locally free on $\Sigma^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, and this clearly implies that it acts locally freely on $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$.

The previous implications may obviously be reversed, and we obtain the following.

Lemma 4.5.1. *The following conditions are equivalent*

1. *Φ_G is transverse to $\mathbb{R}_+ \cdot i\boldsymbol{\nu}$;*
2. *$\tilde{\mu}$ is locally free on $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$;*
3. *for any $x \in X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, val_x is injective;*
4. *for any $m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, val_m is injective on $\Phi_G(m)^{\perp_{\mathfrak{g}}}$.*

The vector field $\Upsilon_{\mu,\boldsymbol{\nu}}$

Let us consider the normal vector field $\Upsilon = \Upsilon_{\mu,\boldsymbol{\nu}}$ to $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ appearing in the statement of Theorem 1.3.2.

By definition, $m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ if and only if $\Phi_G(m)$ is similar to $i\lambda_{\boldsymbol{\nu}}(m)D_{\boldsymbol{\nu}}$, for some $\lambda_{\boldsymbol{\nu}}(m) > 0$. Equating norms and traces, we obtain

$$\lambda_{\boldsymbol{\nu}}(m) = \frac{\|\Phi_G(m)\|}{\|\boldsymbol{\nu}\|} = -i\,\frac{\mathrm{trace}\,(\Phi_G(m))}{\nu_1 + \nu_2} \qquad (m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}})\,.$$

Since $\nu_1 > \nu_2$, there exists a unique coset $h_m T \in G/T$ such that

$$\Phi_G(m) = i\, \lambda_{\boldsymbol{\nu}}(m)\, h_m\, D_{\boldsymbol{\nu}}\, h_m^{-1}\,. \tag{4.32}$$

Let us set $\boldsymbol{\nu}_\perp := (-\nu_2,\, \nu_1)^t$, and define $\boldsymbol{\rho} = \boldsymbol{\rho}_{\boldsymbol{\nu}} \,:\, M^G_{\mathcal{O}_{\boldsymbol{\nu}}} \to \mathfrak{g}$ by setting

$$\boldsymbol{\rho}(m) := i\, h_m\, D_{\boldsymbol{\nu}_\perp}\, h_m^{-1} \qquad (m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}})\,. \tag{4.33}$$

Then $\boldsymbol{\rho}(m)_M \in \mathfrak{X}(M)$ is the vector field on M induced by $\boldsymbol{\rho}(m) \in \mathfrak{g}$; its evaluation at $m' \in M$ is $\boldsymbol{\rho}(m)_M(m')$ (and similarly for X).

Definition 4.5.1. The vector field $\Upsilon = \Upsilon_{\mu,\boldsymbol{\nu}}$ along $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ is

$$\Upsilon(m) := J_m\left(\boldsymbol{\rho}(m)_M(m)\right) \qquad (m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}})\,.$$

With abuse of notation, recalling (1.3) we shall also denote by Υ the vector field along $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ given by

$$\Upsilon(x) := J_x\left(\boldsymbol{\rho}(m_x)_X(x)\right), \qquad m_x := \pi(x)\,.$$

Notice that

$$\langle \Phi_G(m),\, \boldsymbol{\rho}(m)\rangle = \lambda_{\boldsymbol{\nu}}(m)\, \langle \boldsymbol{\nu},\, \boldsymbol{\nu}_\perp\rangle = 0 \qquad (m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}})\,.$$

Therefore, in view of (1.4) for any $x \in \pi^{-1}(m)$ we have

$$\boldsymbol{\rho}(m)_X(x) = \boldsymbol{\rho}(m)^\sharp_M(x)\,.$$

A spectral characterization of $G \cdot M^G_{\boldsymbol{\nu}}$

Suppose that $-i\,\Phi_G(m)$ has eigenvalues $\lambda_1(m) \geq \lambda_2(m)$. Then $m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ if and only if $\lambda_1(m)\,\nu_2 - \lambda_2(m)\,\nu_1 = 0$. We shall give a similar spectral characterization of $G \cdot M^T_{\boldsymbol{\nu}}$. Notice that if $\lambda_1(m) = \lambda_2(m)$, then $\Phi_G(m)$ is a multiple of the identity, hence certainly $m \notin M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$. Thus we may as well assume that $\lambda_1(m) > \lambda_2(m)$.

Proposition 4.5.1. *Suppose $m \in M$, and let the eigenvalues of $-i\,\Phi_G(m)$ be $\lambda_1(m) > \lambda_2(m)$. Then $m \in G \cdot M^T_{\boldsymbol{\nu}}$ if and only if*

$$t(m,\, \boldsymbol{\nu}) := \frac{\lambda_1(m)\,\nu_2 - \lambda_2(m)\,\nu_1}{(\nu_1 + \nu_2)(\lambda_1(m) - \lambda_2(m))} \in [0,\, 1/2)\,. \tag{4.34}$$

Proof of Proposition 4.5.1. Let us set $\boldsymbol{\lambda}(m) := (\lambda_1(m), \lambda_2(m))$, and let $D_{\boldsymbol{\lambda}}$ be the corresponding diagonal matrix. By definition, $m \in G \cdot M^T_{\boldsymbol{\nu}}$ if and only if there exists $g \in SU(2) \leq G$ such that $\mathrm{diag}(g\, D_{\boldsymbol{\lambda}}\, g^{-1}) \in \mathbb{R}_+ \cdot \boldsymbol{\nu}$. This is equivalent to the condition that there exists $u, v \in \mathbb{C}$ such that

$$\begin{pmatrix} u & -\bar{w} \\ w & \bar{u} \end{pmatrix} D_{\boldsymbol{\lambda}} \begin{pmatrix} \bar{u} & \bar{w} \\ -w & u \end{pmatrix} = c \begin{pmatrix} \nu_1 & a \\ \bar{a} & \nu_2 \end{pmatrix}, \tag{4.35}$$

for some $c > 0$ and $a \in \mathbb{C}$. If we set $t = |w|^2$, we conclude that $m \in G \cdot M^T_{\boldsymbol{\nu}}$ if and only if there exists $t \in [0, 1]$ such that

$$\boldsymbol{\lambda}_t(m) := \begin{pmatrix} (1-t)\lambda_1(m) + t\,\lambda_2(m) \\ (1-t)\lambda_2(m) + t\,\lambda_1(m) \end{pmatrix} \in \mathbb{R}_+ \cdot \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}.$$

The condition $\boldsymbol{\lambda}_t(m) \wedge \boldsymbol{\nu} = 0$ translates into the equality $t = t(m, \boldsymbol{\nu})$. Hence we need to have $t(m, \boldsymbol{\nu}) \in [0, 1]$. Given this, $\boldsymbol{\lambda}_t(m)$ is a *positive* multiple of $\boldsymbol{\nu}$ if and only if

$$(1 - t(m, \boldsymbol{\nu}))\lambda_1(m) + t(m, \boldsymbol{\nu})\lambda_2(m) > t(m, \boldsymbol{\nu})\lambda_1(m) + (1 - t(m, \boldsymbol{\nu}))\lambda_2(m),$$

and this is equivalent to $t(m, \boldsymbol{\nu}) < 1/2$.

Conversely, suppose that $t(m, \boldsymbol{\nu}) \in [0, 1/2)$, and define

$$g := \begin{pmatrix} \sqrt{1 - t(m, \boldsymbol{\nu})} & -\sqrt{t(m, \boldsymbol{\nu})} \\ \sqrt{t(m, \boldsymbol{\nu})} & \sqrt{1 - t(m, \boldsymbol{\nu})} \end{pmatrix}.$$

□

4.5.2 The proof

Proof of Theorem 1.3.2. As Φ_G is equivariant, it is transverse to $\mathbb{R}_+ \cdot iD_{\boldsymbol{\nu}}$ if and only if it is transverse to $\mathbb{R}_+ \cdot \mathcal{O}_{\boldsymbol{\nu}}$. Given that $\nu_1 > \nu_2$, $\mathcal{O}_{\boldsymbol{\nu}}$ is 2-dimensional (and diffeomorphic to S^2); therefore, $\mathbb{R}_+ \cdot \mathcal{O}_{\boldsymbol{\nu}}$ has codimension 1 in $\mathfrak{g}$. Similarly, $\mathbb{R}_+ \cdot iD_{\boldsymbol{\nu}}$ has codimension 1 in $\mathfrak{t}^\vee$. Given $\mathbf{0} \notin \Phi_G(M)$, we conclude the following.

Step 4.5.1. $M^G_{\boldsymbol{\nu}}$, $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ and $M^T_{\boldsymbol{\nu}}$ are compact and smooth (real) submanifolds of M. $M^G_{\boldsymbol{\nu}}$ has codimension 3, and $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ and $M^T_{\boldsymbol{\nu}}$ are hypersurfaces.

The Weyl chambers in $\mathfrak{t}$ are the half-planes

$$\mathfrak{t}_+ := \{\boldsymbol{\mu} \,:\, \mu_1 > \mu_2\}, \quad \mathfrak{t}_- := \{\boldsymbol{\mu} \,:\, \mu_1 < \mu_2\}$$

and clearly with our identifications $iD_{\boldsymbol{\nu}} \leftrightarrow \boldsymbol{\nu} \in \mathfrak{t}_+$. Since $\Phi_G(m) \cap \mathfrak{t}_+$ is a convex polytope ([GS82a], [GS84], [Kir84a]), $\Phi_G(M) \cap \mathbb{R}_+ \cdot iD_{\boldsymbol{\nu}}$ is a closed segment I. Furthermore, for any $a \in I$, the inverse image $\Phi_G^{-1}(a) \subseteq M$ is also connected ([Kir84b], [Ler95]). Thus we obtain the following conclusion.

Step 4.5.2. $M^G_{\boldsymbol{\nu}}$, $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ and $M^T_{\boldsymbol{\nu}}$ are connected.

Proof of Step 4.5.2. The previous considerations imply easily that $M^G_{\boldsymbol{\nu}}$ is connected. Given this, since $M^G_{\mathcal{O}_{\boldsymbol{\nu}}} = G \cdot M^G_{\boldsymbol{\nu}}$ the connectedness of G implies the one of $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$. Let us consider $M^T_{\boldsymbol{\nu}}$. Since $\Phi_T(M)$ is a convex polytope ([GS82a], [Ati82]), $\Phi_T(M) \cap \mathbb{R}_+ \cdot iD_{\boldsymbol{\nu}}$ is also a connected segment I'. The statement follows since the fibers of Φ_T are connected again by [Kir84b] and [Ler95]. □

For any $m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, let us set

$$M^G_{\Phi_G(m)} := \Phi_G^{-1}(\mathbb{R}_+ \cdot \Phi_G(m))\,.$$

Since Φ_G is transverse to $\mathbb{R}_+ \cdot \boldsymbol{\nu}$, by equivariance it is also transverse to $\mathbb{R}_+ \cdot \Phi_G(m)$; hence $M^G_{\Phi_G(m)}$ is also a connected real submanifold of M, of real codimension 3 and contained in $M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$.

Let us consider the normal bundle $N(M^G_{\Phi_G(m)})$ to $M^G_{\Phi_G(m)} \subset M$. For any $\xi \in \mathfrak{g}$, let $\xi^\perp \subset \mathfrak{g}$ be the orthocomplement to ξ. Under the equivariant identification $\mathfrak{g} \cong \mathfrak{g}^\vee$, $\xi^\perp$ corresponds to ξ^0.

For any subset $L \subseteq \mathfrak{g}$, let $L^{\perp_\mathfrak{g}}$ denote the orthocomplement of L (that is, of the linear span of L) under the pairing $\langle \cdot, \cdot \rangle_\mathfrak{g}$.

Lemma 4.5.2. *For any $m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, we have*

$$N_m\left(M^G_{\Phi_G(m)}\right) = J_m \circ \mathrm{val}_m\left(\Phi_G(m)^{\perp_\mathfrak{g}}\right)\,.$$

Similarly, for any $m \in M^T_{\boldsymbol{\nu}}$, we have

$$N_m\left(M^T_{\boldsymbol{\nu}}\right) = J_m \circ \mathrm{val}_m\left((i\boldsymbol{\nu})^{\perp_\mathfrak{t}}\right) = J_m \circ \mathrm{val}_m(i\boldsymbol{\nu}_\perp)$$

Proof of Lemma 4.5.2. If $v \in T_m M^G_{\Phi_G(m)}$, then $\mathrm{d}_m\Phi_G(v) = a\Phi_G(m)$ for some $a \in \mathbb{R}$. Given $\eta \in \Phi_G(m)^{\perp_{\mathfrak{g}}}$, and with ρ as in (1.1), we have

$$\begin{aligned}\rho_m\left(J_m\left(\eta_M(m)\right), v\right) &= \omega_m\left(\eta_M(m), v\right) = \mathrm{d}_m\Phi^\eta(v)\\ &= \langle \mathrm{d}_m\Phi(v), \eta\rangle_{\mathfrak{g}} = a\langle \Phi_G(m), \eta\rangle_{\mathfrak{g}} = 0\,.\end{aligned}$$

Therefore $J_m \circ \mathrm{val}_m\left(\Phi_G(m)^{\perp_{\mathfrak{g}}}\right) \subseteq N_m(M^G_{\Phi_G(m)})$. Since we have that both $\Phi_G^{\perp_{\mathfrak{g}}}$ and $N_m(M^G_{\Phi_G(m)})$ are 3-dimensional, it suffices to recall that by Lemma 4.5.1 val_m is injective when restricted to $\Phi_G(m)^{\perp_{\mathfrak{g}}}$.

The proof of the second statement is similar. □

For any vector subspace $L \subseteq \mathfrak{g}$, let us set $L_M(m) := \mathrm{val}_m(L) \subseteq T_m M$ $(m \in M)$. For any $m \in M^G_{\mathcal{O}_\nu}$, given that $M^G_{\mathcal{O}_\nu}$ is the G-saturation of $M^G_{\Phi_G(m)}$, we have

$$T_m M^G_{\mathcal{O}_\nu} = T_m M^G_{\Phi_G(m)} + \mathfrak{g}_M(m)\,. \tag{4.36}$$

Therefore, passing to ρ_m-orthocomplements

$$N_m(M^G_{\mathcal{O}_\nu}) = N_m\left(M^G_{\Phi_G(m)}\right) \cap \mathfrak{g}_M(m)^{\perp_{\rho_m}}\,.$$

We conclude from relation (4.36) and Lemma 4.5.2 that the normal bundle $N_m(M^G_{\mathcal{O}_\nu})$ is the set of all vectors $J_m(\eta_M(m)) \in T_m M$, with $\eta \in \Phi_G(m)^{\perp_{\mathfrak{g}}}$, such that $\rho_m\left(J_m(\eta_M(m)), \xi_M\right) = 0$ for every $\xi \in \mathfrak{g}$. From this remark we can draw the following conclusion.

Step 4.5.3. Let $\Upsilon = \Upsilon_{\mu,\nu}$ be as in §4.5.1. Then for any $m \in M^G_{\mathcal{O}_\nu}$ we have

$$N_m(M^G_{\mathcal{O}_\nu}) = \mathrm{span}\left(\Upsilon(m)\right)\,.$$

In particular, $M^G_{\mathcal{O}_\nu}$ is orientable.

Proof of Step 4.5.3. By the above,

$$\begin{aligned} &N_m(M^G_{\mathcal{O}_\nu})\\ &= \left\{J_m(\eta_M(m))\,;\, \eta \in \Phi_G(m)^{\perp_{\mathfrak{g}}} \;\wedge\; \rho_m\left(J_m(\eta_M(m)), \xi_M(m)\right) = 0,\ \forall \xi \in \mathfrak{g}\right\}\\ &= \left\{J_m(\eta_M(m))\,;\, \eta \in \Phi_G(m)^{\perp_{\mathfrak{g}}} \;\wedge\; \omega_m\left(\eta_M(m), \xi_M(m)\right) = 0,\ \forall \xi \in \mathfrak{g}\right\}\\ &= \left\{J_m(\eta_M(m))\,;\, \eta \in \Phi_G(m)^{\perp_{\mathfrak{g}}} \;\wedge\; \eta_M(m) \in \ker(\mathrm{d}_m\Phi_G),\ \forall \xi \in \mathfrak{g}\right\}\\ &= \left\{J_m(\eta_M(m))\,;\, \eta \in \Phi_G(m)^{\perp_{\mathfrak{g}}} \;\wedge\; [\eta, \Phi_G(m)] = 0,\ \forall \xi \in \mathfrak{g}\right\}\,.\end{aligned}$$

The latter equality holds because, by the equivariance of Φ_G, we have

$$\mathrm{d}_m\Phi_G\left(\eta_M(m)\right) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi_G\left(\mu_{e^{t\eta}}(m)\right)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Ad}_{e^{t\eta}}\Phi_G\left(m\right)\Big|_{t=0}$$
$$= \left[\eta,\, \Phi_G(m)\right].$$

There exists a unique $h_mT \in G/T$ such that $\Phi_G(m) = i\lambda_{\boldsymbol{\nu}}(m)h_m D_{\boldsymbol{\nu}} h_m^{-1}$. It is then clear that $\langle\Phi_G(m),\, \eta\rangle_{\mathfrak{g}} = 0$ and $[\eta,\, \Phi_G(m)] = 0$ if and only if

$$\eta \in \mathrm{span}\left(i h_m D_{\boldsymbol{\nu}_\perp} h_m^{-1}\right) = \mathrm{span}\left(\boldsymbol{\rho}(m)\right),$$

where $\boldsymbol{\rho}(m)$ is as in (4.33). This completes the proof on Step 4.5.3. □

Step 4.5.4. $M^G_{\mathcal{O}_{\boldsymbol{\nu}}} \cap M^T_{\boldsymbol{\nu}} = M^G_{\boldsymbol{\nu}}$.

Proof of Step 4.5.4. Obviously, $M^G_{\mathcal{O}_{\boldsymbol{\nu}}} \cap M^T_{\boldsymbol{\nu}} \supseteq M^G_{\boldsymbol{\nu}}$. Conversely, suppose $m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}} \cap M^T_{\boldsymbol{\nu}}$. Then on the one hand $\Phi_G(m)$ is similar to a positive multiple of $iD_{\boldsymbol{\nu}}$; for a unique $h_mT \in G/T$,

$$\Phi_G(m) = i\lambda_{\boldsymbol{\nu}}(m)h_m D_{\boldsymbol{\nu}} h_m^{-1},$$

where we can assume without loss that $h_m \in SU(2)$. On the other hand $\mathrm{diag}\left(\Phi_G(m)\right)$ is a positive multiple of $i\boldsymbol{\nu}$. Hence the diagonal of $h_m D_{\boldsymbol{\nu}} h_m^{-1}$ is a positive multiple of $\boldsymbol{\nu}$. Let us write h_m as in (4.35), and argue as in the proof of Proposition 4.5.1; using that $\nu_1^2 \neq \nu_2^2$, one concludes readily that h_m is diagonal. Hence $h_m D_{\boldsymbol{\nu}} h_m^{-1} = D_{\boldsymbol{\nu}}$, and so $\Phi_G(m) \in \mathbb{R}_+ \cdot i\boldsymbol{\nu}$. Thus $m \in M^G_{\boldsymbol{\nu}}$. □

Step 4.5.5. For any $m \in M^G_{\boldsymbol{\nu}}$, $T_m M^G_{\mathcal{O}_{\boldsymbol{\nu}}} = T_m M^T_{\boldsymbol{\nu}}$.

Proof of Step 4.5.5. If $m \in M^G_{\boldsymbol{\nu}}$, then h_m is the identity matrix in (4.32) and (4.33); therefore, $\Upsilon(m) = J_m((iD_{\boldsymbol{\nu}_\perp})_M(m))$. Hence, we obtain that $N_m(M^G_{\mathcal{O}_{\boldsymbol{\nu}}}) = \mathrm{span}\left(J_m((iD_{\boldsymbol{\nu}_\perp})(m))\right)$. The claim follows from this remark and Lemma 4.5.2. □

Step 4.5.6. For any $M^G_{\mathcal{O}_{\boldsymbol{\nu}}} = \partial\left(G \cdot M^T_{\boldsymbol{\nu}}\right)$.

Proof of Step 4.5.6. Suppose $m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$. Thus $\Phi_G(m) = i\lambda(m)h_m D_{\boldsymbol{\nu}} h_m^{-1}$ for a unique $h_mT \in G/T$. Let us choose $\delta > 0$ arbitrarily small, and let $M(m,\, \delta) \subseteq M$ be the open ball centred at m and radius δ in the Riemannian distance on M. Since Φ_G is transverse to $\mathbb{R}_+ \cdot i\boldsymbol{\nu}$, there exists $\epsilon_1 > 0$ such that the following holds. For every $\epsilon \in (-\epsilon_1,\, \epsilon_1)$ there exists $m' \in M(m,\, \delta)$ with

$$\Phi_G(m') = i\lambda(m')h_m D_{\boldsymbol{\nu}+\epsilon\boldsymbol{\nu}_\perp} h_m^{-1} \tag{4.37}$$

for some $\lambda(m') > 0$ (see §2 of [Pao12]). This implies that the eigenvalues of $-i\Phi_G(m')$ are

$$\lambda_1(m') := \lambda(m')(\nu_1 - \epsilon\,\nu_2), \quad \lambda_2(m') := \lambda(m')(\nu_2 + \epsilon\,\nu_1)\,.$$

Therefore, the invariant defined in (4.34) takes the following value at m':

$$t(m',\, \boldsymbol{\nu}) = -\frac{\epsilon}{\nu_1 + \nu_2} \cdot \frac{\nu_1^2 + \nu_2^2}{(\nu_1 - \nu_2) - \epsilon(\nu_1 + \nu_2)}\,.$$

Therefore, if $\epsilon\,(\nu_1 + \nu_2) > 0$ (and ϵ is sufficiently small) then $m' \notin G \cdot M^T_{\boldsymbol{\nu}}$ by Proposition 4.5.1. This implies $M^G_{\mathcal{O}_{\boldsymbol{\nu}}} \subseteq \partial(G \cdot M^T_{\boldsymbol{\nu}})$.

To prove the reverse inclusion, assume that $m \in G \cdot M^T_{\boldsymbol{\nu}} \setminus M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$. Then $t(m,\, \boldsymbol{\nu}) \in [0, 1/2)$ by 4.5.1. Furthermore, $t(m,\, \boldsymbol{\nu}) \neq 0$, since otherwise $m \in M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$. Hence $t(m,\, \boldsymbol{\nu}) \in (0, 1/2)$; by continuity, then, $t(m',\, \boldsymbol{\nu}) \in (0, 1/2)$ for every m' in a sufficiently small open neighbourhood of m. Hence Proposition 4.5.1 implies that $G \cdot M^T_{\boldsymbol{\nu}} \setminus M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ contains an open neighborhood of m in M. Thus $G \cdot M^T_{\boldsymbol{\nu}} \setminus M^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ is open, and in particular $m \notin \partial(G \cdot M^T_{\boldsymbol{\nu}})$. □

Step 4.5.7. Υ is outer oriented if $\nu_1 + \nu_2 > 0$ and inner oriented if $\nu_1 + \nu_2 < 0$.

Proof of Step 4.5.7. Let denote by $\mathcal{B}_{\boldsymbol{\nu}}$ the collection of all $B \in \mathfrak{g}$ such that $\mathrm{diag}(gBg^{-1}) \in \mathbb{R}_+ \cdot i\boldsymbol{\nu}$ for some $g \in G$. Thus $\mathcal{B}_{\boldsymbol{\nu}}$ is a conic and invariant closed subset of $\mathfrak{g} \setminus \{0\}$; in addition, $m \in G \cdot M^T_{\boldsymbol{\nu}}$ if and only if $\Phi_G(m) \in \mathcal{B}_{\boldsymbol{\nu}}$.

If $\lambda_1(B) \geq \lambda_2(B)$ are the eigenvalues of $-iB$, the Proposition 4.5.1 implies that $B \in \mathcal{B}_{\boldsymbol{\nu}}$ if and only if $\lambda_1(B) > \lambda_2(B)$ and

$$t(B,\, \boldsymbol{\nu}) := \frac{\lambda_1(B)\nu_2 - \lambda_2(B)\nu_1}{(\nu_1 + \nu_2)(\lambda_1(B) - \lambda_2(B))} \in [0,\, 1/2)\,.$$

In particular, if $t(B,\, \boldsymbol{\nu}) \in (0,\, 1/2)$ then B belongs to the interior of $\mathcal{B}_{\boldsymbol{\nu}}$.

Suppose $m \in M^G_{\boldsymbol{\nu}}$ and consider the path

$$\gamma_1 : \, (-\epsilon,\, \epsilon) \ni \tau \mapsto \Phi_G(m + \tau\Upsilon(m)) \in \mathfrak{g}\,,$$

defined for sufficiently small $\epsilon > 0$; the expression $m + \tau\Upsilon(m) \in M$ is meant in an adapted coordinate system on M centred at m. Then

$$\gamma_1(0) = \Phi_G(m) = i\lambda_{\boldsymbol{\nu}}(m)\, D_{\boldsymbol{\nu}}\,,$$
$$\dot{\gamma}_1(0) = \omega_m(\,\cdot\,,\, \Upsilon(m)) = \rho_m(\,\cdot\,,\, (iD_{\boldsymbol{\nu}_\perp})_M(m))\,.$$

Let us consider a smooth positive function $y : (-\epsilon, \epsilon) \to \mathbb{R}$, to be determined but subject to the condition $y(0) = \lambda_{\boldsymbol{\nu}}(m)$. Let us define a second path of the form

$$\gamma_2(\tau) := i\, y(\tau)\, \mathrm{Ad}_{e^{\tau\,\xi}}\left(i D_{\boldsymbol{\nu}+a\,\tau\,\boldsymbol{\nu}_\perp}\right),$$

where $a > 0$ is a constant also to be determined.

Then

$$\begin{aligned}
&\gamma_1(0) = \gamma_2(0)\,,\\
&\dot\gamma_2(0) = i\left[\dot y(0)\, D_{\boldsymbol{\nu}} + \lambda_{\boldsymbol{\nu}}(m)\,[\boldsymbol{\xi},\, \boldsymbol{\nu}] + a\,\lambda_{\boldsymbol{\nu}}(m)\, D_{\boldsymbol{\nu}_\perp}\right].
\end{aligned}$$

Clearly, we can choose $a > 0$ uniquely so that

$$a\,\lambda_{\boldsymbol{\nu}}(m)\,\|\boldsymbol{\nu}\|^2 = \rho_m\left((iD_{\boldsymbol{\nu}_\perp})_M(m),\, (iD_{\boldsymbol{\nu}_\perp})_M(m)\right),$$

so that $\langle\dot\gamma_2(0),\, \boldsymbol{\nu}_\perp\rangle = \langle\dot\gamma_1(0),\, \boldsymbol{\nu}_\perp\rangle$. Having fixed a, we can then choose $\dot y(0)$ uniquely so that

$$\dot y(0)\,\|\boldsymbol{\nu}\|^2 = \rho_m\left((iD_{\boldsymbol{\nu}})_M(m),\, (iD_{\boldsymbol{\nu}_\perp})_M(m)\right),$$

so that we also have $\langle\dot\gamma_2(0),\, \boldsymbol{\nu}\rangle = \langle\dot\gamma_1(0),\, \boldsymbol{\nu}\rangle$. Finally, if we set

$$\boldsymbol{v}_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \boldsymbol{v}_2 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

we can choose $\boldsymbol{\xi} \in \mathrm{span}_{\mathbb{R}}\{\boldsymbol{v}_1,\, \boldsymbol{v}_2\}$ uniquely so that

$$\lambda_{\boldsymbol{\nu}}(m)\langle[\boldsymbol{\xi},\, i\boldsymbol{\nu}],\, \boldsymbol{v}_j\rangle = \rho_m\left((\boldsymbol{v}_j)_M(m),\, (iD_{\boldsymbol{\nu}_\perp})_M(m)\right),$$

so that in addition $\langle\dot\gamma_2(0),\, \boldsymbol{v}_j\rangle = \langle\dot\gamma_1(0),\, \boldsymbol{v}_j\rangle$ for $j = 1,\, 2$. With this choices, γ_1 and γ_2 agree to first order at 0.

Let us remark that when τ is sufficiently small $\gamma_2(\tau)$ has eigenvalues

$$\lambda_1(\gamma_2(\tau)) = y(\tau)(\nu_1 - a\tau\nu_2) > \lambda_2(\gamma_2(\tau)) = y(\tau)(\nu_2 + a\tau\nu_1)\,.$$

Hence

$$t(B,\, \boldsymbol{\nu}) := -\frac{a\,\tau}{\nu_1 + \nu_2} \cdot \frac{\nu_1^2 + \nu_2^2}{\nu_1 - \nu_2 + a\tau(\nu_1 + \nu_2)}\,.$$

Thus if $\nu_1 + \nu_2 > 0$ then $\gamma_2(\tau) \notin \mathcal{B}_{\boldsymbol{\nu}}$ when $\tau \in (0,\, \epsilon)$; since γ_1 and γ_2 agree to second order at 0, we also have $\Phi_G(m + \tau\Upsilon(m)) \notin \mathcal{B}_{\boldsymbol{\nu}}$ when $\tau \sim 0^+$.

The argument when $\nu_1 + \nu_2 < 0$ is similar. □

The proof of Theorem 1.3.2 is complete. □

4.6 Proof of Theorem 1.3.3

4.6.1 An *a priori* polynomial bound

Let us record the following rough *a priori* polynomial bound.

Lemma 4.6.1. *There is a constant $C_{\boldsymbol{\nu}} > 0$ such that for any $x \in X$ one has*

$$|\Pi_{k\boldsymbol{\nu}}(x,\, x)| \leq C_{\boldsymbol{\nu}}\, k^{d+1}$$

for $k \gg 0$.

Proof of Lemma 4.6.1. Let $r:\ S^1 \times X \to X$ be the standard structure action on the unit circle bundle X. Thus there is a natural decomposition

$$H(X) = \bigoplus_{l=0}^{+\infty} H(X)_l$$

into isotypes for the S^1-action.

Since $\tilde{\mu}$ commutes with the structure action S^1 on X, we have

$$H(X)_{k\boldsymbol{\nu}} = \bigoplus_{l=0}^{+\infty} H(X)_{k\boldsymbol{\nu}} \cap H(X)_l\,.$$

On the other hand, by the theory of [GS82b] we have $H(X)_{k\boldsymbol{\nu}} \cap H(X)_l \neq (0)$ only if the highest weight vector $\boldsymbol{r}(k\boldsymbol{\nu})$ of the representation indexed by $k\boldsymbol{\nu}$ satisfies

$$\boldsymbol{r}(k\boldsymbol{\nu}) = (k\nu_1 - 1,\, k\nu_2) = k\boldsymbol{\nu} + (-1,\, 0) \in l\,\Phi_G(M) \subset \mathfrak{g}\,.$$

Let us define

$$a_G := \min \|\Phi_G\|\,, \quad A_G := \max \|\Phi_G\|\ .$$

Thus $A_G \geq a_G > 0$. Therefore, we need to have

$$l\, a_G \leq \|\boldsymbol{r}(k\boldsymbol{\nu})\| \leq k\,\|\boldsymbol{\nu}\| + 1 \Rightarrow l \leq L_1(k) := \left\lceil \frac{\|\boldsymbol{\nu}\|}{a_G}\, k + \frac{1}{a_G} \right\rceil\,.$$

Similarly,

$$k\,\|\boldsymbol{\nu}\| - 1 \leq \|\boldsymbol{r}(k\boldsymbol{\nu})\| \leq l\, A_G \Rightarrow L_2(k) := \left\lfloor \frac{\|\boldsymbol{\nu}\|}{A_G}\, k - \frac{1}{A_G} \right\rfloor \leq l\,.$$

On the other hand, in view of the asymptotic expansion of $\Pi_k(x,\,x)$ from the article [Zel98] we also have $\Pi_l(x,\,x) \le 2\,(l/\pi)^d$ for $l \gg 0$. We conclude that

$$\Pi_{k\boldsymbol{\nu}}(x,\,x) \le \sum_{l=L_1(k)}^{L_2(k)} \Pi_l(x,\,x) \le \frac{2}{\pi^d} \sum_{l=L_1(k)}^{L_2(k)} l^d \le C_{\boldsymbol{\nu}}\, k^{d+1}$$

for some $C_{\boldsymbol{\nu}} > 0$. □

4.6.2 The proof

We shall use the following shorthand notation. If $x \in X$, $g \in G$, $t \in T$, let us set

$$x(g,t) := \tilde{\mu}_{gt^{-1}g^{-1}}(x)\,;$$

similarly, if $m \in M$

$$m(g,\,t) := \mu_{gt^{-1}g^{-1}}(m)\,.$$

If $t = e^{i\boldsymbol{\vartheta}} := (e^{i\vartheta_1},\,e^{i\vartheta_2})$, we shall write $x(g,\,t) = x(g,\,\boldsymbol{\vartheta})$, $m(g,\,t) = m(g,\,\boldsymbol{\vartheta})$. Since $\tilde{\mu}$ is a lifting of μ, if $m = \pi(x)$ then

$$m(g,\,\boldsymbol{\vartheta}) = \pi\left(x(g,\,\boldsymbol{\vartheta})\right)\,.$$

Proof of Theorem 1.3.3. We replace $\boldsymbol{\nu}$ by $k\boldsymbol{\nu}$ in (3.9), and use the angular coordinates on T, we obtain

$$\Pi_{k\boldsymbol{\nu}}(x,\,y) = \frac{k(\nu_1-\nu_2)}{(2\pi)^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-ik\langle\boldsymbol{\nu},\,\boldsymbol{\vartheta}\rangle}\Delta\left(e^{i\boldsymbol{\vartheta}}\right) F\left(e^{i\boldsymbol{\vartheta}};\,x,\,y\right) \mathrm{d}\boldsymbol{\vartheta}\,, \tag{4.38}$$

here $e^{i\boldsymbol{\vartheta}} = \left(e^{i\vartheta_1},\,e^{i\vartheta_2}\right)$.

For $\delta > 0$, let us define

$$V_\delta := \{(x,\,y) \in X\,:\,\mathrm{dist}_X(x,\,G\cdot y) \ge \delta\}\,.$$

Proposition 4.6.1. *For any $\delta > 0$, we have $\Pi_{k\boldsymbol{\nu}}(x,\,y) = O(k^{-\infty})$ uniformly on V_δ.*

Proof of Proposition 4.6.1. The singular support of Π is the diagonal in $X \times X$. Therefore,

$$\beta:\,((x,\,y),\,gT,\,t) \in V_\delta \times G/T \times T \mapsto \Pi(x(g,\,t),y) \in \mathbb{C}$$

is $\mathcal{C}^\infty$. The same then holds of $((x,\,y),\,t) \in V_\delta \times T \mapsto \Delta(t)\,F(t;\,x,\,y)$. Hence its Fourier transform (4.38) is rapidly decreasing for $k \to +\infty$. □

We are thus reduced to assuming that $\mathrm{dist}_X(x,\, G\cdot y) < \delta$ for some fixed $\delta > 0$. Let $\varrho \in \mathcal{C}^\infty_0(\mathbb{R})$ be $\equiv 1$ on $[-1\,,1]$ and $\equiv 0$ on $\mathbb{R}\setminus(-2,\,2)$. We can write

$$\Pi_{\boldsymbol{\nu}}(x,\,y) = \Pi_{\boldsymbol{\nu}}(x,\,y)_1 + \Pi_{\boldsymbol{\nu}}(x,\,y)_2\,,$$

where the two summands on the right are defined as in (4.38), but with the integrand of $F(t;\,x,\,y)$ multiplied respectively by $\varrho\,(\delta^{-1}\,\mathrm{dist}_X\,(x(g,\,\boldsymbol{\vartheta}),\,y))$ and $1-\varrho\,(\delta^{-1}\,\mathrm{dist}_X\,(x(g,\,\boldsymbol{\vartheta}),\,y))$.

Lemma 4.6.2. $\Pi_{k\boldsymbol{\nu}}(x,\,y)_2 = O(k^{-\infty})$ *for* $k\to+\infty$.

Proof of Lemma 4.6.2. On the support of the integrand in $\Pi_{k\boldsymbol{\nu}}(x,\,y)_2$ we have $\mathrm{dist}_X(x(g,\,t),\,y)\geq\delta$. We can then apply with minor changes the argument in the proof of Proposition 4.6.1. □

On the support of the integrand in $\Pi_{k\boldsymbol{\nu}}(x,\,y)_1$, $\mathrm{dist}_X\,(x(g,\,t),\,y)\leq 2\delta$; therefore, perhaps after discarding a smoothing term contributing negligibly to the asymptotics, we can apply the explicit description of Π as an FIO. With some passages, we obtain in place of (4.38):

$$\begin{aligned}\Pi_{k\boldsymbol{\nu}}(x,\,y) &\sim \Pi_{k\boldsymbol{\nu}}(x,\,y)_1 \\ &\sim \frac{k^2(\nu_1-\nu_2)}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{G/T}\int_0^{+\infty} e^{ik\Psi_{x,\,y}}\mathcal{A}_{x,\,y}\,\mathrm{d}u\;\mathrm{dV}_{G/T}(gT)\;\mathrm{d}\boldsymbol{\vartheta}\,;\end{aligned}\tag{4.39}$$

we have applied the rescaling $u\mapsto k\,u$ to the parameter in (4.38), and set

$$\Psi_{x,\,y} = \Psi_{x,\,y}(u,\,\boldsymbol{\vartheta},\,gT) := u\psi(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\,y) - \langle\boldsymbol{\nu},\,\boldsymbol{\vartheta}\rangle\,,\tag{4.40}$$

$$\mathcal{A}_{x,\,y} = \mathcal{A}_{x,\,y}(u,\,\boldsymbol{\vartheta},\,gT) := \Delta\left(e^{i\boldsymbol{\vartheta}}\right)s'(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\,y,\,ku)\,,$$

with

$$s'(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\,y,\,ku) := s(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\,y,\,ku)\,\varrho\left(\delta^{-1}\,\mathrm{dist}_X\,(x(g,\,\boldsymbol{\vartheta}),\,y)\right)\,.$$

Lemma 4.6.3. *Only a rapidly decreasing contribution to the asymptotics is lost, if in* (4.39) *integration in* du *is restricted to an interval of the form* $(1/D,\,D)$ *for some* $D\gg 0$.

Proof of Lemma 4.6.3. Suppose that $x,\,y\in X$, $(g_0T,\,e^{i\boldsymbol{\vartheta}_0})\in(G/T)\times T$ and

$$\mathrm{dist}_X(x(g_0,\,\boldsymbol{\vartheta}_0),\,y) < \delta\,.$$

In view of [SZ02], in any system of local coordinates we have

$$\mathrm{d}_{(x(g_0,\,\boldsymbol{\vartheta}_0),\,y)}\psi = (\alpha_{x(g_0,\,\boldsymbol{\vartheta}_0)},\, -\alpha_y) + O(\delta)\,. \tag{4.41}$$

Let $\mathrm{d}^{(\boldsymbol{\vartheta})}$ denote the differential with respect to the variable $\boldsymbol{\vartheta}$. If $i\boldsymbol{\eta} \in \mathfrak{t}$, we obtain with $m_x := \pi(x)$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau} x(g_0,\, \boldsymbol{\vartheta}_0 + \tau\boldsymbol{\eta})\Big|_{\tau=0} &= -\operatorname{Ad}_{g_0}(i\boldsymbol{\eta})_X(x(g_0,\, \boldsymbol{\vartheta}_0)) \\ &= -\operatorname{Ad}_{g_0}(i\boldsymbol{\eta})_M(x(g_0,\, \boldsymbol{\vartheta}_0))^\sharp + \langle \Phi_G\left(m_x(g_0,\, \boldsymbol{\vartheta}_0)\right),\, \operatorname{Ad}_{g_0}(i\boldsymbol{\eta})\rangle\, \partial_\theta\,.\end{aligned} \tag{4.42}$$

On the other hand, as Φ_G is G-equivariant we get

$$\begin{aligned}\langle \Phi_G\left(m_x(g_0,\, \boldsymbol{\vartheta}_0)\right),\, \operatorname{Ad}_{g_0}(i\boldsymbol{\eta})\rangle &= \left\langle \operatorname{Ad}_{g_0^{-1}}\left(\Phi_G\left(m_x(g_0,\, \boldsymbol{\vartheta}_0)\right)\right),\, i\boldsymbol{\eta}\right\rangle \\ = \left\langle \Phi_G\left(\tilde{\mu}_{g_0^{-1}}\left(m_x(g_0,\, \boldsymbol{\vartheta}_0)\right)\right),\, i\boldsymbol{\eta}\right\rangle &= \left\langle \Phi_T\left(\tilde{\mu}_{g_0^{-1}}\left(m_x(g_0,\, \boldsymbol{\vartheta}_0)\right)\right),\, i\boldsymbol{\eta}\right\rangle\,.\end{aligned} \tag{4.43}$$

Now, (4.41), (4.42) and (4.43) imply

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\psi\left(x(g_0,\, \boldsymbol{\vartheta}_0 + \tau\boldsymbol{\eta}),\, y\right)\Big|_{\tau=0} \\ &= -\,\mathrm{d}_{(x(g_0,\,\boldsymbol{\vartheta}_0),\,y)}\psi\Big(\operatorname{Ad}_{g_0}(i\boldsymbol{\eta})_X(x(g_0,\, \boldsymbol{\vartheta}_0)),\, 0\Big) \\ &= -\alpha_{x(g_0,\boldsymbol{\vartheta}_0)}\Big(\operatorname{Ad}_{g_0}(i\boldsymbol{\eta})_X(x(g_0,\, \boldsymbol{\vartheta}_0))\Big) + \langle O(\delta),\, \boldsymbol{\eta}\rangle \\ &= \left\langle \operatorname{Ad}_{g_0^{-1}}\left(\frac{1}{i}\,\Phi_T\left(m_x(g_0,\, \boldsymbol{\vartheta}_0)\right)\right) + O(\delta),\, \boldsymbol{\eta}\right\rangle\,.\end{aligned}$$

Recalling (4.40), we obtain

$$\mathrm{d}^{(\boldsymbol{\vartheta})}_{(u,\,g_0T,\boldsymbol{\vartheta}_0)}\Psi_{x,\,y} = \frac{u}{i}\,\Phi_T\left(\mu_{g_0^{-1}}(m_x)\right) - \boldsymbol{\nu} + O(\delta)\,. \tag{4.44}$$

By assumption, $\mathbf{0} \notin \Phi_T(M)$. Let us set

$$a_T := \min \|\Phi_T\|\,, \quad A_T := \max \|\Phi_T\|\ .$$

Then $A_T \geq a_T > 0$, and (4.44) implies

$$\left\|\mathrm{d}^{(\boldsymbol{\vartheta})}_{(u,\,g_0T,\boldsymbol{\vartheta}_0)}\Psi_{x,\,y}\right\| \geq \max\{u\, a_T - \|\boldsymbol{\nu}\| + O(\delta),\, \|\boldsymbol{\nu}\| - u\, A_T + O(\delta)\}\,.$$

Thus if $D \gg 0$ and $u \geq D$ we have

$$\left\| \mathrm{d}^{(\boldsymbol{\vartheta})}_{(u,\, g_0 T, \boldsymbol{\vartheta}_0)} \Psi_{x,\, y} \right\| \geq \frac{a_T}{2}\, u + 1\,, \tag{4.45}$$

while for $0 < u < 1/D$

$$\left\| \mathrm{d}^{(\boldsymbol{\vartheta})}_{(u,\, g_0 T, \boldsymbol{\vartheta}_0)} \Psi_{x,\, y} \right\| \geq \frac{\|\boldsymbol{\nu}\|}{2}\,. \tag{4.46}$$

The Lemma then follows from (4.45) and (4.46) by a standard iterated integration by parts in $\boldsymbol{\vartheta}$ (in view of the compactness of T). $\square$

Suppose that $\varrho \in \mathcal{C}^\infty_0((0,\, +\infty))$ is $\equiv 1$ on $(1/D,\, D)$ and is supported on $(1/(2D),\, 2D)$. By Lemma 4.6.3, the asymptotics of (4.39) is unaltered, if the integrand is multiplied by $\varrho(u)$. Thus we obtain

$$\Pi_{k\boldsymbol{\nu}}(x,\, y) \sim \frac{k^2(\nu_1 - \nu_2)}{(2\pi)^2} \int_{(-\pi,\pi)^2} \int_{G/T} \int_{1/2D}^{2D} e^{ik\Psi_{x,\, y}} \mathcal{A}'_{x,\, y}\, \mathrm{d}u\ \mathrm{dV}_{G/T}(gT)\ \mathrm{d}\boldsymbol{\vartheta}\,, \tag{4.47}$$

where we have set

$$\mathcal{A}'_{x,\, y}(u,\, \boldsymbol{\vartheta},\, gT) := \varrho(u)\, \mathcal{A}_{x,\, y}(u,\, \boldsymbol{\vartheta},\, gT)\,,$$

and the integration in $\mathrm{d}u$ is now over a compact interval.

Let $\Im(z)$ denote the imaginary part of $z \in \mathbb{C}$. In view of Corollary 1.3 of [BdMS76], there exists a fixed constant D', depending only on X, such that

$$\Im\left(\psi(x',\, x'')\right) \geq D' \,\mathrm{dist}_X(x',\, x'')^2 \qquad (x',\, x'' \in X)\,. \tag{4.48}$$

Proposition 4.6.2. *Uniformly for*

$$\mathrm{dist}_X(x,\, G \cdot y) \geq C\, k^{\epsilon - 1/2} \tag{4.49}$$

we have $\Pi_{k\boldsymbol{\nu}}(x,\, y) = O(k^{-\infty})$*.*

Proof of Proposition 4.6.2. In the range (4.49), we have

$$\mathrm{dist}_X(x(g,\, \boldsymbol{\vartheta}),\, y) \geq C\, k^{\epsilon - 1/2}$$

for every $gT \in G/T$ and $e^{i\boldsymbol{\vartheta}} \in T$. In view of (4.40) and (4.48),

$$|\partial_u \Psi_{x,y}(u,\, \boldsymbol{\vartheta},\, gT)| = |\psi(x(g,\, \boldsymbol{\vartheta}),\, y)| \geq \Im(\psi(x(g,\, \boldsymbol{\vartheta}),\, y)) \tag{4.50}$$

$$\geq D\, \mathrm{dist}_X(x(g,\, \boldsymbol{\vartheta}),\, y)^2 \geq D\, C^2\, k^{2\epsilon-1}\,. \tag{4.51}$$

Let us use the identity

$$-\frac{i}{k}\psi(x(g,\, \boldsymbol{\vartheta}),\, y)^{-1}\frac{\mathrm{d}}{\mathrm{d}u}e^{ik\Psi_{x,\,y}} = e^{ik\Psi_{x,\,y}} \tag{4.52}$$

to iteratively integrate by parts in du in (4.47); then by (4.50) at each step we introduce a factor $O(k^{-2\epsilon})$. The claim follows. □

To complete the proof of Theorem 1.3.3, we need to establish the following.

Proposition 4.6.3. *Uniformly for*

$$\mathrm{dist}_X(x,\, G\cdot X^T_{\boldsymbol{\nu}}) \geq C\, k^{\epsilon-1/2} \tag{4.53}$$

we have $\Pi_{k\boldsymbol{\nu}}(x,\, y) = O(k^{-\infty})$.

Remark 4. Let dist_M denote the distance function on M; if $m = \pi(x)$, then $\mathrm{dist}_X(x,\, G\cdot X^T_{\boldsymbol{\nu}}) = \mathrm{dist}_M(m,\, G\cdot M^T_{\boldsymbol{\nu}})$.

Proof of Proposition 4.6.3. Since G acts on M as a group of Riemannian isometries, (4.53) means that for any $g \in G$ we have

$$C\, k^{\epsilon-1/2} \leq \mathrm{dist}_M(m,\, \mu_g(M^T_{\boldsymbol{\nu}})) = \mathrm{dist}_M(\mu_{g^{-1}}(m),\, M^T_{\boldsymbol{\nu}})\,.$$

On the other hand, as $-i\Phi_T$ is transverse to $\mathbb{R}_+ \cdot \boldsymbol{\nu}$, by the discussion in §2.1.3 of [Pao17] there is a constant $b_{\boldsymbol{\nu}} > 0$ such that every $u \in [1/(2D),\, 2D]$ we have

$$\|-iu\, \Phi_T(\mu_{g^{-1}}(m)) - \boldsymbol{\nu}\| \geq b_{\boldsymbol{\nu}}\, C\, k^{\epsilon-1/2}\,. \tag{4.54}$$

Let us consider (4.47) with $x = y$:

$$\Pi_{k\boldsymbol{\nu}}(x,\, x) \sim \tag{4.55}$$

$$\frac{k^2(\nu_1-\nu_2)}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{G/T}\int_{1/2D}^{2D} e^{ik\Psi_{x,\,x}}\mathcal{A}'_{x,\,x}\, \mathrm{d}u\; \mathrm{dV}_{G/T}(gT)\; \mathrm{d}\boldsymbol{\vartheta}\,,$$

Let us choose $\epsilon' \in (0,\, \epsilon)$ and multiply the integrand in (4.55) by the identity

$$\varrho\left(k^{1/2-\epsilon'}\,\mathrm{dist}_X(x(g,\,\boldsymbol{\vartheta}),\,x)\right) + \left[1 - \varrho\left(k^{1/2-\epsilon'}\,\mathrm{dist}_X(x(g,\,\boldsymbol{\vartheta}),\,x)\right)\right] = 1\,.$$

Here ϱ is as in the discussion preceding Lemma 4.6.2. We obtain a further splitting

$$\Pi_{k\boldsymbol{\nu}}(x,\,x) \sim \Pi_{k\boldsymbol{\nu}}(x,\,x)_a + \Pi_{k\boldsymbol{\nu}}(x,\,x)_b\,, \tag{4.56}$$

where $\Pi_{k\boldsymbol{\nu}}(x,\,x)_a$ is given by (4.55) with $\mathcal{A}'_{x,\,x}$ replaced by

$$\mathcal{B}'_{x,\,x} := \varrho\left(k^{1/2-\epsilon'}\,\mathrm{dist}_X(x(g,\,\boldsymbol{\vartheta}),\,x)\right)\,\mathcal{A}'_{x,\,x} \tag{4.57}$$

similarly $\Pi_{k\boldsymbol{\nu}}(x,\,x)_b$ is given by (4.55) with $\mathcal{A}'_{x,\,x}$ replaced by

$$\mathcal{B}''_{x,\,x} := \left[1 - \varrho\left(k^{1/2-\epsilon'}\,\mathrm{dist}_X(x(g,\,\boldsymbol{\vartheta}),\,x)\right)\right]\,\mathcal{A}'_{x,\,x}\,.$$

Lemma 4.6.4. $\Pi_{k\boldsymbol{\nu}}(x,\,x)_b = O(k^{-\infty})$ *as* $k \to +\infty$*.*

Proof of Lemma 4.6.4. On the support of $\mathcal{B}''_{x,\,x}$, we have

$$\mathrm{dist}_X(x(g,\,\boldsymbol{\vartheta}),\,x) \ge k^{\epsilon'-1/2}\,.$$

Thus we may again appeal to (4.52) and iteratively integrate by parts in $\mathrm{d}u$, introducing at each step a factor $O(k^{-1}k^{1-2\epsilon'}) = O(k^{-2\epsilon'})$. □

Thus the proof of the Theorem will be complete once we establish the following.

Lemma 4.6.5. $\Pi_{k\boldsymbol{\nu}}(x,\,x)_a = O(k^{-\infty})$ *as* $k \to +\infty$*.*

Before attacking the proof of Lemma 4.6.5, let us prove the following.

Lemma 4.6.6. *If* (4.53) *holds, then for any* $u \in [1/(2D),\,2D]$ *and* $k \gg 0$

$$\left\|\mathrm{d}^{\boldsymbol{\vartheta}}_{(u,\,gT,\,\boldsymbol{\vartheta})}\Psi_{x,\,x}\right\| \ge \frac{b_{\boldsymbol{\nu}}}{2}\,C\,k^{\epsilon-1/2}$$

on the support of $\mathcal{B}'_{x,\,x}$*.*

Proof of Lemma 4.6.6. On the support of $\mathcal{B}'_{x,\,x}$, we have

$$\mathrm{dist}_X(x(g,\,\boldsymbol{\vartheta}),\,x) \leq 2\,k^{\epsilon'-1/2}\,.$$

Thus, instead of (4.41) we have

$$\mathrm{d}_{(x(g,\,\boldsymbol{\vartheta}),\,x)}\psi = (\alpha_{x(g,\,\boldsymbol{\vartheta})},\,-\alpha_x) + O\left(k^{\epsilon'-1/2}\right)$$

Therefore, in place of (4.44), on the support of $\mathcal{B}'_{x,\,x}$ we have

$$\mathrm{d}^{(\boldsymbol{\vartheta})}_{(u,\,gT,\,\boldsymbol{\vartheta})}\Psi_{x,\,x} = \frac{u}{i}\,\Phi_T(\mu_{g^{-1}}(m_x)) - \boldsymbol{\nu} + +O\left(k^{\epsilon'-1/2}\right)$$

Thus in view of (4.54) the claim follows from $0 < \epsilon' < \epsilon$. □

Given Lemma 4.6.6, we can prove Lemma 4.6.5 essentially by iteratively integrating by parts in $\mathrm{d}\boldsymbol{\vartheta}$.

Proof of Lemma 4.6.5. Since $\tilde{\mu}$ is free on $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, it is also free on a small tubular neighbourhood X' of $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ in X. Without loss, we may restrict our analysis to X in view of Theorem 1.3.3. On the support of $\mathcal{B}'_{x,\,x}$, therefore, $e^{i\boldsymbol{\vartheta}} \in T$ varies in a small neighbourhood of I_2. Let $f : T \to [0,\,+\infty)$ be a bump function compactly supported in a small neighbourhood $U \subset T$ of I_2 (identified with $(1,\,1)$), and identically $\equiv 1$ near I_2. Then we obtain

$$\Pi_{k\boldsymbol{\nu}}(x,\,x)_a \sim \frac{k^2(\nu_1-\nu_2)}{(2\pi)^2}\int_U\int_{G/T}\int_{1/2D}^{2D} e^{ik\Psi_{x,\,x}}\,f(t)\,\mathcal{B}'_{x,\,x}\,\mathrm{d}u\,\mathrm{dV}_{G/T}(gT)\,\mathrm{d}\boldsymbol{\vartheta}\,,$$

Let us introduce the differential operator

$$P = \sum_{h=1}^{2}\frac{\partial_{\vartheta_h}\Psi_{x,\,x}}{(\partial_{\vartheta_1}\Psi_{x,\,x})^2 + (\partial_{\vartheta_2}\Psi_{x,\,x})^2}\frac{\partial}{\partial\vartheta_h}\,,$$

so that

$$\frac{1}{i\,k}P\left(e^{i\,k\,\Psi_{x,\,x}}\right) = e^{i\,k\,\Psi_{x,\,x}}\,.$$

Thus,

$$\begin{aligned}
&\int_U e^{ik\Psi_{x,\,x}}\, f(t)\, \mathcal{B}'_{x,\,x}\ \mathrm{d}\boldsymbol{\vartheta} \\
&\quad = \frac{1}{i\,k}\sum_{h=1}^{2}\int_U \frac{\partial_{\vartheta_h}\Psi_{x,\,x}}{(\partial_{\vartheta_1}\Psi_{x,\,x})^2+(\partial_{\vartheta_2}\Psi_{x,\,x})^2}\frac{\partial}{\partial\vartheta_h}\left[e^{ik\Psi_{x,\,x}}\right]\ f(e^{i\,\boldsymbol{\vartheta}})\,\mathcal{B}'_{x,\,x}\ \mathrm{d}\boldsymbol{\vartheta} \\
&\quad = \frac{i}{k}\sum_{h=1}^{2}\int_U e^{ik\Psi_{x,\,x}}\,\frac{\partial}{\partial\vartheta_h}\left[\frac{\partial_{\vartheta_h}\Psi_{x,\,x}}{(\partial_{\vartheta_1}\Psi_{x,\,x})^2+(\partial_{\vartheta_2}\Psi_{x,\,x})^2}\, f(e^{i\,\boldsymbol{\vartheta}})\,\mathcal{B}'_{x,\,x}\right]\ \mathrm{d}\boldsymbol{\vartheta} \\
&\quad = \frac{i}{k}\int_U e^{ik\Psi_{x,\,x}}\, P^t\left[f(e^{i\,\boldsymbol{\vartheta}})\,\mathcal{B}'_{x,\,x}\right]\ \mathrm{d}\boldsymbol{\vartheta}\,,
\end{aligned}$$

where

$$P^t(\gamma) := \sum_{h=1}^{2}\frac{\partial}{\partial\vartheta_h}\left[\frac{\partial_{\vartheta_h}\Psi_{x,\,x}}{(\partial_{\vartheta_1}\Psi_{x,\,x})^2+(\partial_{\vartheta_2}\Psi_{x,\,x})^2}\,\gamma\right]. \tag{4.58}$$

Iterating, for any $r \in \mathbb{N}$ we have

$$\int_U e^{ik\Psi_{x,\,x}}\, f(e^{i\,\boldsymbol{\vartheta}})\,\mathcal{B}'_{x,\,x}\ \mathrm{d}\boldsymbol{\vartheta} = \frac{i^r}{k^r}\int_U e^{ik\Psi_{x,\,x}}\,(P^t)^r\left[f(e^{i\,\boldsymbol{\vartheta}})\,\mathcal{B}'_{x,\,x}\right]\ \mathrm{d}\boldsymbol{\vartheta}\,. \tag{4.59}$$

Let us consider the function

$$\mathcal{D}:\ \boldsymbol{\vartheta}\mapsto \mathrm{dist}_X(x(g,\,\boldsymbol{\vartheta})) = \mathrm{dist}_X(\tilde{\mu}_{e^{-i\boldsymbol{\vartheta}}}\circ\tilde{\mu}_{g^{-1}}(x),\,\tilde{\mu}_{g^{-1}}(x))\,.$$

We have the following.

Lemma 4.6.7. *For $\boldsymbol{\vartheta}\sim\mathbf{0}$, we have*

$$\mathrm{dist}_X(x(g,\,\boldsymbol{\vartheta}),\,x) = F_1(gT,\,\boldsymbol{\vartheta}) + F_2(gT,\,\boldsymbol{\vartheta}) + \dots\,,$$

where $F_j(gT,\,\boldsymbol{\vartheta})$ is homogeneous of degree j in $\boldsymbol{\vartheta}$, and $\mathcal{C}^\infty$ for $\boldsymbol{\vartheta}\neq\mathbf{0}$. In addition, $F_1(gT,\,\boldsymbol{\vartheta}) = \|\mathrm{Ad}_g(\boldsymbol{\vartheta})_X(x)\| = \|\boldsymbol{\vartheta}_X(\tilde{\mu}_{g^{-1}}(x))\|$.

For any $c\in\mathbb{N}$ let $\mathcal{D}^{(c)}$ denote a generic iterated derivative of the form

$$\frac{\partial^c\mathcal{D}}{\partial\vartheta_{i_1}\cdots\partial\vartheta_{i_c}}\,;$$

clearly $\mathcal{D}^{(c)}$ is not uniquely determined by c. By Lemma 4.6.7, as $k\to+\infty$

$$\mathcal{D}^{(c)}\left(\frac{\boldsymbol{\vartheta}}{\sqrt{k}}\right) = O\left(k^{(c-1)(1/2-\epsilon')}\right)$$

where $\varrho(k^{1/2-\epsilon'}\mathcal{D}) \neq 1$. For any multi-index $\boldsymbol{C} = (c_1, \cdots, c_s)$ let us denote by $\mathcal{D}^{(\boldsymbol{C})}$ a generic product of the form $\mathcal{D}^{(c_1)} \cdots \mathcal{D}^{(c_s)}$; then

$$\mathcal{D}^{(\boldsymbol{C})}\left(\frac{\boldsymbol{\vartheta}}{\sqrt{k}}\right) = O\left(k^{(1/2-\epsilon')\sum_j (c_j-1)}\right).$$

Lemma 4.6.8. *For any $r \in \mathbb{N}$, $(P^t)^r(f(t)\mathcal{B}'_{x,x})$ is a linear combination of summands of the form*

$$\varrho^{(b)}\left(k^{1/2-\epsilon'}\mathcal{D}_k(\boldsymbol{\vartheta})\right) \frac{P_{a_1}(\Psi_{x,x},\, \partial\Psi_{x,x})}{[(\partial_{\vartheta_1}\Psi_{x,x})^2 + (\partial_{\vartheta_2}\Psi_{x,x})^2]^{a_2}}\, k^{b(1/2-\epsilon')}\, \mathcal{D}^{(\boldsymbol{C})}, \tag{4.60}$$

times omitted factors bounded in k depending on f and its derivatives, where

1. *P_{a_1} denotes a generic differential polynomial in $\Psi_{x,x}$ homogeneous of degree a_1 in the first derivatives $\partial\Psi_{x,x}$;*
2. *if $a := 2a_2 - a_1$, then a, b, $\boldsymbol{C}$ are subjected to the bound*

$$a + b + \sum_{j=1}^{r}(c_j - 1) \leq 2r \tag{4.61}$$

   (the sum is over $c_j > 0$);

3. *$\boldsymbol{C}$ is not zero if and only if $b > 0$.*

Here $\varrho^{(b)}$ is the b-th derivative of the one-variable real function ϱ.

Proof of Lemma 4.6.8. Let us set $F := f(e^{i\boldsymbol{\vartheta}})\mathcal{B}'_{x,x}$. For $r = 1$, we have

$$\frac{\partial}{\partial\vartheta_h}\left[\frac{\partial_{\vartheta_h}\Psi_{x,x}}{(\partial_{\vartheta_1}\Psi_{x,x})^2 + (\partial_{\vartheta_2}\Psi_{x,x})^2}\, F\right] \tag{4.62}$$
$$= \frac{\partial_{\vartheta_h}\Psi_{x,x}}{(\partial_{\vartheta_1}\Psi_{x,x})^2 + (\partial_{\vartheta_2}\Psi_{x,x})^2}\, \frac{\partial F}{\partial\vartheta_h} + F\,\frac{\partial}{\partial\vartheta_h}\left[\frac{\partial_{\vartheta_h}\Psi_{x,x}}{(\partial_{\vartheta_1}\Psi_{x,x})^2 + (\partial_{\vartheta_2}\Psi_{x,x})^2}\right].$$

We have

$$\frac{\partial_{\theta_h}\Psi_{x,x}}{(\partial_{\theta_1}\Psi_{x,x})^2 + (\partial_{\theta_2}\Psi_{x,x})^2}\, \frac{\partial F}{\partial\vartheta_h}$$
$$= \frac{\partial_{\theta_h}\Psi_{x,x}}{(\partial_{\theta_1}\Psi_{x,x})^2 + (\partial_{\theta_2}\Psi_{x,x})^2}\, \left[\frac{\partial f}{\partial\vartheta_h}\, \mathcal{B}'_{x,x} + \frac{\partial\mathcal{B}'_{x,x}}{\partial\vartheta_h}\, f\right].$$

Thus, in view of (4.57), the first summand on the right hand side of (4.62) splits as a linear combination of terms as in the statement, with $a_1 = a_2 = 1$, b and $\boldsymbol{C}$ both zero, or $a_1 = a_2 = 1$, $b = 1$, $\boldsymbol{C} = (1)$. Hence $a+b+\sum_j (c_j - 1) = 2$ in either case. On the other hand, the second summand on the right hand side of (4.62) satisfies

$$F \, \frac{\partial}{\partial \vartheta_h} \left[\frac{\partial_{\vartheta_h} \Psi_{x,x}}{\left(\partial_{\vartheta_1} \Psi_{x,x}\right)^2 + \left(\partial_{\vartheta_2} \Psi_{x,x}\right)^2} \right] = \frac{F}{\left[\left(\partial_{\vartheta_1} \Psi_{x,x}\right)^2 + \left(\partial_{\vartheta_2} \Psi_{x,x}\right)^2\right]^2}$$

$$\cdot \left\{ \partial^2_{\vartheta_h, \vartheta_h} \Psi_{x,x} \left[\left(\partial_{\vartheta_1} \Psi_{x,x}\right)^2 + \left(\partial_{\vartheta_2} \Psi_{x,x}\right)^2 \right] - 2 \partial_{\vartheta_h} \Psi_{x,x} \sum_{a=1}^{2} \partial_{\vartheta_a} \Psi_{x,x} \, \partial^2_{\vartheta_a \, \vartheta_h} \Psi_{x,x} \right\} .$$

This is of the stated type with $a_1 = a_2 = 2$, b and $\boldsymbol{C}$ both zero. Hence $a = 4 - 2 = 2$.

Passing to the inductive step, let us consider (4.58) with γ given by (4.60), and assume that (4.61) is satisfied. Let us write $\varrho^{(l)}$ for the factor in front in (4.60). We obtain a linear combination of expressions of the form

$$\frac{\partial}{\partial \vartheta_h} \left[\varrho^{(b)} \, \frac{P_{a_1+1}(\Psi_{x,x}, \, \partial \Psi_{x,x})}{[(\partial_{\vartheta_1} \Psi_{x,x})^2 + (\partial_{\vartheta_2} \Psi_{x,x})^2]^{a_2+1}} \, k^{b(1/2-\epsilon')} \, \mathcal{D}^{(\boldsymbol{C})} \, k^{b(1/2-\epsilon')} \, \mathcal{D}^{(\boldsymbol{C})} \right] . \tag{4.63}$$

It is now clear that (4.63) splits as a linear combination of summands of the following forms:

$$\varrho^{(b)} \frac{P_{a'}(\Psi_{x,x}, \, \partial \Psi_{x,x})}{[(\partial_{\vartheta_1} \Psi_{x,x})^2 + (\partial_{\vartheta_2} \Psi_{x,x})^2]^{a_2+1}} \, k^{b(1/2-\epsilon')} \, \mathcal{D}^{(\boldsymbol{C})} \, k^{b(1/2-\epsilon')} \, \mathcal{D}^{(\boldsymbol{C})} \, ,$$

with $a' \in \{a_1, \, a_1 + 1, \, a_1 + 2\}$;

$$\varrho^{(b)} \frac{P_{a_1+2}(\Psi_{x,x}, \, \partial \Psi_{x,x})}{[(\partial_{\vartheta_1} \Psi_{x,x})^2 + (\partial_{\vartheta_2} \Psi_{x,x})^2]^{a_2+2}} \, k^{b(1/2-\epsilon')} \, \mathcal{D}^{(\boldsymbol{C})} \, k^{b(1/2-\epsilon')} \, \mathcal{D}^{(\boldsymbol{C})} \, ;$$

$$\varrho^{(b+1)} \frac{P_{a_1+1}(\Psi_{x,x}, \, \partial \Psi_{x,x})}{[(\partial_{\vartheta_1} \Psi_{x,x})^2 + (\partial_{\vartheta_2} \Psi_{x,x})^2]^{a_2+1}} \, k^{(b+1)(1/2-\epsilon')} \, \mathcal{D}^{(\boldsymbol{C})} \, k^{b(1/2-\epsilon')} \, \mathcal{D}^{(\boldsymbol{C}')} \, ,$$

where $\boldsymbol{C}'$ is of the form $\boldsymbol{C}' = (1, \, \boldsymbol{C})$;

$$\varrho^{(b)} \frac{P_{a_1+1}(\Psi_{x,x}, \, \partial \Psi_{x,x})}{[(\partial_{\vartheta_1} \Psi_{x,x})^2 + (\partial_{\vartheta_2} \Psi_{x,x})^2]^{a_2+1}} \, k^{b(1/2-\epsilon')} \, \mathcal{D}^{(\boldsymbol{C}')} \, ,$$

where $\boldsymbol{C}'$ is obtained from $\boldsymbol{C}$ (if the latter is not zero) by replacing one of the c_j's by $c_j + 1$, and leaving all the others unchanged.

In all these cases we obtain a term of the form (4.60), satisfying (4.61) with r replaced by $r + 1$. This completes the proof of Lemma 4.6.8. $\square$

As $0 < \epsilon' < \epsilon$, the general summand (4.61) is

$$O\left(k^{a(1/2-\epsilon)+[b+\sum_j(c_j-1)](1/2-1\epsilon')}\right) = O\left(k^{[a+b+\sum_j(c_j-1)](1/2-1\epsilon')}\right)$$
$$= O\left(k^{2r(1/2-1\epsilon')}\right) = O\left(k^{r(1-2\epsilon')}\right) .$$

Making use of the latter estimate in (4.59), we obtain the following

Corollary 4.6.1. *For any $r \in \mathbb{N}$,*

$$\int_{U_j} e^{i\,k\,\Psi_{x,\,x}}\, f(t)\, \mathcal{B}'_{x,\,x}\, \mathrm{d}\boldsymbol{\vartheta} = O\left(k^{-2r\epsilon'}\right) .$$

The proof of Lemma 4.6.5 is thus complete. $\square$

Given (4.56), Proposition 4.6.3 follows from Lemmata 4.6.4 and 4.6.5. $\square$

Thus the statement of Theorem 1.3.3 holds true when $x = y$. The general case follows from this and the Schwartz inequality

$$|\Pi_{\boldsymbol{\nu}}(x,\, y)| \le \sqrt{\Pi_{\boldsymbol{\nu}}(x,\, x)}\,\sqrt{\Pi_{\boldsymbol{\nu}}(y,\, y)}\,;$$

in fact both factors on the right hand side have at most polynomial growth in k by Lemma 4.6.1, and if say (4.53) holds, then the first one is rapidly decreasing. The proof of Theorem 1.3.3 is complete. $\square$

4.7 Proof of Theorem 1.3.4, 1.3.6 and 1.3.7

4.7.1 Preliminaries on local rescaled asymptotics

In the proof of Theorem 1.3.4, 1.3.6 and 1.3.7, we are interested in the asymptotics of $\Pi_{k\boldsymbol{\nu}}(x',\, x'')$ when $(x',\, x'')$ approaches the diagonal of $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ in $X \times X$ along appropriate directions and at a suitable pace.

In Theorems 1.3.4 and 1.3.7, we consider $x' = x''$ in a shrinking "one-sided" neighbourhood of $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$. In Theorem 1.3.6, we shall assume that

$(x',\, x'')$ approaches the diagonal in $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ along "horizontal" directions orthogonal to the orbits. We shall treat the former case in detail, and then briefly discuss the necessary changes for the latter. Suppose $x \in X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ and let $m = \pi(x)$. Let us choose a system of HLC centered at x, and let us consider the collection of points

$$x_{\tau,\,k} := x + \frac{\tau}{\sqrt{k}} \Upsilon_{\boldsymbol{\nu}}(m)\,,$$

where $k = 1, 2, \ldots$, and $|\tau| \leq C\, k^{\epsilon}$ for some fixed $C > 0$ and $\epsilon \in (0, 1/6)$. The sign of τ is chosen so that $\tau\, \Upsilon_{\boldsymbol{\nu}}(m)$ is either zero or outer oriented. Thus $\tau\,(\nu_1 + \nu_2) \geq 0$. We shall provide an integral expression for the asymptotics of $\Pi_{k\boldsymbol{\nu}}(x_{\tau,\,k},\, x_{\tau,\,k})$ when $k \to +\infty$. Applying as before the Weyl integration and character formulae, inserting the micro-local description of Π as an FIO and making use of the rescaling $u \mapsto k\,u$, $\boldsymbol{\vartheta} \mapsto \boldsymbol{\vartheta}/\sqrt{k}$, we obtain that, as $k \to +\infty$,

$$\begin{aligned} &\Pi_{k\boldsymbol{\nu}}(x_{\tau,\,k},\, x_{\tau,\,k}) \sim \qquad (4.64)\\ &\frac{k(\nu_1 - \nu_2)}{(2\pi)^2} \int_{G/T} \mathrm{dV}_{G/T}(gT) \int_{-\infty}^{+\infty} \mathrm{d}\vartheta_1 \int_{-\infty}^{+\infty} \mathrm{d}\vartheta_2 \int_0^{+\infty} \mathrm{d}u \\ &\Bigg[e^{ik\left[u\,\psi\left(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}/\sqrt{k}}g^{-1}}(x_{\tau,\,k}),\, x_{\tau,\,k}\right) - \langle \boldsymbol{\vartheta}, \boldsymbol{\nu}\rangle/\sqrt{k}\right]} \\ &\Delta\left(e^{i\boldsymbol{\vartheta}/\sqrt{k}}\right) s\left(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}/\sqrt{k}}g^{-1}}\left(x_{\tau,\,k}\right),\, x_{\tau,\,k},\; ku\right)\Bigg]. \end{aligned}$$

Integration in $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2)$ is over a ball centered at the origin and radius $O(k^{\epsilon})$ in $\mathbb{R}^2$. A cut-off function of the form $\varrho(k^{\epsilon}\,\boldsymbol{\vartheta})$ is implicitly incorporated into the amplitude.

In order to express the previous phase more explicitly, we need the following.

Definition 4.7.1. Let us define $\boldsymbol{\rho} = \boldsymbol{\rho}_m : G/T \to \mathfrak{t} \cong \mathbb{R}^2$, $gT \mapsto \boldsymbol{\rho}_{gT}$, by requiring

$$\langle \boldsymbol{\rho}_{gT},\, \boldsymbol{\vartheta}\rangle = \omega_m\left(\mathrm{Ad}_g\,(i D_{\boldsymbol{\vartheta}})_M\,(m),\, \Upsilon_{\boldsymbol{\nu}}(m)\right) \quad (\boldsymbol{\vartheta} \in \mathbb{R}^2)\,.$$

Next, let the symmetric and positive definite matrix $E(gT) = E_x(gT)$ be defined by the equality

$$\boldsymbol{\vartheta}^t\, E(gT)\, \boldsymbol{\vartheta} = \left\|\mathrm{Ad}_g\,(i\, D_{\boldsymbol{\vartheta}})_X\,(x)\right\|_x^2 \quad (\boldsymbol{\vartheta} \in \mathbb{R}^2)\,.$$

Furthermore, let us define $\tilde{\Psi}(u,\, gT,\, \tau) = \tilde{\Psi}_m(u,\, gT,\, \tau) \in \mathfrak{t}$ by setting

$$\tilde{\Psi}(u,\, gT,\, \boldsymbol{\vartheta}) := u \,\mathrm{diag}\left(\mathrm{Ad}_{g^{-1}}\left(\Phi'_G(m)\right)\right) - \boldsymbol{\nu}, \quad \Phi'_G(m) := -i\, \Phi_G(m)\,.$$

Finally, let us pose

$$\Psi(u,\, gT,\, \boldsymbol{\vartheta}) := \left\langle \tilde{\Psi}(u,\, gT),\, \boldsymbol{\vartheta} \right\rangle\,.$$

The following Proposition is proved by a rather lengthy computation, along the lines of those in the proof of Theorem 1.3.3 and in [Pao12].

Proposition 4.7.1.

$$\begin{aligned} &ik\left[u\,\psi\left(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}/\sqrt{k}}g^{-1}}\left(x_{\tau,k}\right),\, x_{\tau,k}\right) - \langle \boldsymbol{\vartheta},\, \boldsymbol{\nu}\rangle/\sqrt{k}\right] \\ &\quad = i\sqrt{k}\,\Psi(u,\, gT,\, \boldsymbol{\vartheta}) - \frac{u}{2}\boldsymbol{\vartheta}^t\, E(gT)\,\boldsymbol{\vartheta} + 2i\, u\, \tau\, \langle \boldsymbol{\rho}_{gT},\, \boldsymbol{\vartheta}\rangle + k\, R_3\left(\frac{\tau}{\sqrt{k}},\, \frac{\boldsymbol{\vartheta}}{\sqrt{k}}\right)\,. \end{aligned}$$

Corollary 4.7.1. (4.64) *may be rewritten as follows:*

$$\begin{aligned} &\Pi_{k\boldsymbol{\nu}}(x_{\tau,k},\, x_{\tau,k}) \sim \\ &\quad \frac{k(\nu_1-\nu_2)}{(2\pi)^2}\int_{G/T} \mathrm{dV}_{G/T}(gT) \int_{-\infty}^{+\infty} \mathrm{d}\vartheta_1 \int_{-\infty}^{+\infty} \mathrm{d}\vartheta_2 \int_0^{+\infty} \mathrm{d}u \\ &\quad \left[e^{i\sqrt{k}\,\Psi(u,\,gT,\,\boldsymbol{\vartheta})}\, \mathcal{A}_{k,\,\boldsymbol{\nu}}(u,\, gT,\, \tau,\, \boldsymbol{\vartheta})\right]\,. \end{aligned} \tag{4.65}$$

where (leaving implicit the dependence on x)

$$\begin{aligned} \mathcal{A}_{k,\,\boldsymbol{\nu}}(u,\, gT,\, \tau,\, \boldsymbol{\vartheta}) &:= e^{-\frac{u}{2}\boldsymbol{\vartheta}^t\, E(gT)\,\boldsymbol{\vartheta} + 2i\,u\,\tau\,\langle \boldsymbol{\rho}_{gT},\,\boldsymbol{\vartheta}\rangle + k\,R_3\left(\frac{\tau}{\sqrt{k}},\,\frac{\boldsymbol{\vartheta}}{\sqrt{k}}\right)}\, \Delta\left(e^{i\boldsymbol{\vartheta}/\sqrt{k}}\right) \\ &\quad \cdot s\left(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}/\sqrt{k}}g^{-1}}\left(x_{\tau,k}\right),\, x_{\tau,k},\, ku\right)\,. \end{aligned} \tag{4.66}$$

Let $h_mT \in G/T$ be the unique coset such that $h_m^{-1}\Phi_G(m)h_m$ is diagonal. Then only a rapidly decreasing contribution to the asymptotics is lost in (4.65), if integration in $\mathrm{dV}_{G/T}$ is localized in a small neighbourhood of h_mT. In the following, a C^∞ bump function on G/T, supported in a small neighbourhood of h_mT and identically equal to 1 near h_mT, will be implicitly incorporated into the amplitude (4.66).

For some choice of $h_m \in h_m T$ and $\delta > 0$ sufficiently small, let us consider the real-analytic map

$$h : w \in B(0;\,\delta) \subset \mathbb{C} \mapsto h(w) := h_m \exp\left(i \begin{pmatrix} 0 & w \\ \bar{w} & 0 \end{pmatrix} \right) \in G\,.$$

By composition with the projection $\pi' : G \to G/T$, we obtain a real-analytic coordinate chart on G/T centred at $h_m T \in G/T$, given by $w \in B(0;\,\delta) \mapsto h(w)T \in G/T$. The Haar volume form on G/T has the form $\mathcal{V}_{G/T}(w)\,\mathrm{dV}_{\mathbb{C}}(w)$, where $\mathrm{dV}_{\mathbb{C}}(w)$ is the Lebesgue measure on $\mathbb{C}$, and $\mathcal{V}_{G/T}$ is a uniquely determined C^∞ positive function on $B(0;\,\delta)$. We record the following statements, whose proofs we shall omit for the sake of brevity.

Lemma 4.7.1. *$\mathcal{V}_{G/T}$ is rotationally invariant, that is,*

$$\mathcal{V}_{G/T}(w) = \mathcal{V}_{G/T}(e^{i\theta} w)\,,$$

for all $w \in B(0;\,\delta)$ and $e^{i\theta} \in S^1$. In particular, $\mathcal{V}_{G/T}$ is given by a convergent power series in $r^2 = |w|^2$ on $B(0;\delta)$.

Thus we shall write

$$\mathcal{V}_{G/T}(w) = \mathcal{V}_{G/T}(r) = D_{G/T} \cdot \mathcal{S}_{G/T}(r)\,,$$

where $D_{G/T}$ is a constant, and $\mathcal{S}_{G/T}(r) = 1 + \sum_j s_j r^{2j}$.

Lemma 4.7.2. *Let V_3 be the total area of the unit sphere $S^3 \subset \mathbb{C}^2$. Then*

$$D_{G/T} = 2\pi\, V_3^{-1}\,.$$

Furthermore, let us introduce the real-analytic function

$$\kappa = \kappa_m \,:\, w \in B(0;\,\delta) \mapsto \mathrm{diag}\left(\mathrm{Ad}_{h(w)^{-1}}\left(\Phi'_G(m)\right)\right) \in \mathbb{R}^2\,.$$

Then we also have the following.

Lemma 4.7.3. *κ is rotationally invariant, and is given by a convergent power series of the following form*

$$\kappa(w) = \lambda_{\boldsymbol{\nu}}(m)[\boldsymbol{\nu} - r^2(\nu_1 - \nu_2) S_\kappa(r)\,\mathbf{b}]\,, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\,,$$

where $r = |w|$ and $S_\kappa(r)$ is a real-analytic function of r, of the form

$$S_\kappa(r) = 1 + \sum_{j \geq 1} b_j\, r^{2j}\,.$$

If $w = r\, e^{i\theta}$ in polar coordinates, we shall write accordingly $\mathcal{V}_{G/T} = \mathcal{V}_{G/T}(r)$ and $\kappa = \kappa(r)$.

Recalling the previous notations, let us set

$$\tilde{\Psi}_w(u) := u\,\kappa(r) - \boldsymbol{\nu}\,, \quad \Psi_w(u,\,\boldsymbol{\vartheta}) := \left\langle \tilde{\Psi}_w(u),\, \boldsymbol{\vartheta} \right\rangle\,.$$

We obtain the following integral formula (dependence on x on the right hand sides is left implicit).

Proposition 4.7.2. *As $k \to +\infty$ we have*

$$\Pi_{k\boldsymbol{\nu}}(x_{\tau,\,k},\, x_{\tau,\,k}) \quad \sim D_{G/T}\, \frac{k(\nu_1 - \nu_2)}{(2\pi)^2} \int_{-\pi}^{+\pi} \mathrm{d}\theta \int_0^{+\infty} \mathrm{d}r\, \left[I_k(\tau,\, r,\, \theta)\right]\,, \quad (4.67)$$

where

$$I_k(\tau,\, r,\, \theta) = I_k(\tau,\, w) := \int_{-\infty}^{+\infty} \mathrm{d}\vartheta_1 \int_{-\infty}^{+\infty} \mathrm{d}\vartheta_2 \int_0^{+\infty} \mathrm{d}u \quad (4.68)$$
$$\left[e^{i\sqrt{k}\,\Psi_w(u,\,\boldsymbol{\vartheta})}\, \mathcal{A}_{k,\,\boldsymbol{\nu}}(u,\, h(r\, e^{i\theta})\, T,\, \tau,\, \boldsymbol{\vartheta}) \mathcal{S}_{G/T}(r)\, r\right]\,.$$

Our next goal is to produce an asymptotic expansion for $I_k(\tau,\, r,\, \theta)$.

Definition 4.7.2. Let us set

$$\mathbf{n}_1(r) := \frac{\kappa(r)}{\|\kappa(r)\|}\,,$$

and let $\mathbf{n}_2(r)$ be uniquely determined for $|r| < \delta$ so that $\mathcal{B}_r := (\mathbf{n}_1(r),\, \mathbf{n}_2(r))$ is a positively oriented orthonormal basis of $\mathbb{R}^2$. We shall write the change of basis matrix in the form

$$M^{\mathcal{B}_r}_{\mathcal{C}_2}\,(id_{\mathbb{R}^2}) = \begin{pmatrix} C(r) & -S(r) \\ S(r) & C(r) \end{pmatrix}\,, \quad (4.69)$$

where $\mathcal{C}_2$ is the canonical basis of $\mathbb{R}^2$, and denote the change of coordinates by $\boldsymbol{\vartheta} = \zeta_1\, \mathbf{n}_1(w) + \zeta_2\, \mathbf{n}_2(w)$.

A straightforward computation then yields the following.

Corollary 4.7.2. *With $w = r\,e^{i\theta} \in B(0;\,\delta)$ and $I_k(\tau;\,w)$ as in (4.68), we have*

$$I_k(\tau,\,w) = \int_{-\infty}^{+\infty} \mathrm{d}\zeta_2 \left[e^{-i\sqrt{k}\langle \boldsymbol{\nu},\,\mathbf{n}_2(w)\rangle \zeta_2} J_k(\tau,\,w;\zeta_2)\,\mathcal{S}_{G/T}(r)\,r\right]\,, \tag{4.70}$$

where

$$J_k(\tau,\,w;\zeta_2) := \int_{-\infty}^{+\infty} \mathrm{d}\zeta_1 \int_0^{+\infty} \mathrm{d}u \left[e^{i\sqrt{k}\,\Upsilon_r(u,\,\zeta_1)} \mathcal{A}_{k,\,\boldsymbol{\nu}}\,(u,\,h(w)T,\,\tau,\,\boldsymbol{\vartheta}(\boldsymbol{\zeta}))\right]\,, \tag{4.71}$$

and

$$\Upsilon_r(u,\,\zeta_1) := [u\,\|\kappa(r)\| - \langle \boldsymbol{\nu},\,\mathbf{n}_1(r)\rangle]\,\zeta_1\,.$$

Let us view J_k as an oscillatory integral with phase Υ_r.

Lemma 4.7.4. *Υ_r has the unique critical point*

$$P_r = (u(r),\,0) := \left(\frac{\langle \boldsymbol{\nu},\,\mathbf{n}_1(r)\rangle}{\|\kappa(r)\|},\,0\right)\,.$$

Furthermore, $\Upsilon_r(P_r) = 0$, and the Hessian matrix is

$$H(\Upsilon_r)_{P_r} = \begin{pmatrix} 0 & \|\kappa(r)\| \\ \|\kappa(r)\| & 0 \end{pmatrix}\,.$$

Hence its signature is zero and the critical point is non-degenerate.

Recalling that $s_0(x,\,x) = \pi^{-d}$, the amplitude in (4.71) may be written in the following form

$$\begin{aligned} &\mathcal{A}_{k,\boldsymbol{\nu}}(u,\,h(w)T,\,\tau,\,\boldsymbol{\vartheta}(\boldsymbol{\zeta})) \\ &\quad\sim e^{-\frac{u}{2}\boldsymbol{\vartheta}(\boldsymbol{\zeta})^t\,E(w)\,\boldsymbol{\vartheta}(\boldsymbol{\zeta}) + 2\,i\,u\,\tau\,\left\langle \boldsymbol{\rho}_{h(w)T},\,\boldsymbol{\vartheta}(\boldsymbol{\zeta})\right\rangle} \left[e^{\frac{i}{\sqrt{k}}\boldsymbol{\vartheta}_1(\boldsymbol{\zeta})} - e^{\frac{i}{\sqrt{k}}\boldsymbol{\vartheta}_2(\boldsymbol{\zeta})}\right] \\ &\quad\cdot \left(\frac{ku}{\pi}\right)^d \left[1 + \sum_{j\geq 1} a_j(u,\,w;\,\tau,\,\boldsymbol{\vartheta}(\boldsymbol{\zeta}))k^{-j/2}\right]\,; \end{aligned} \tag{4.72}$$

in (4.72) we have set $E(w) := \tilde{E}(h(w)T)$, and in view of the exponent $k\,R_3(\tau/\sqrt{k},\,\boldsymbol{\vartheta}/\sqrt{k})$ appearing in (4.66), $a_j(u,\,w;\,\cdot,\,\cdot)$ is an appropriate polynomial in $(\tau,\,\boldsymbol{\vartheta})$ of degree $\leq 3j$.

Given Lemma 4.7.4, we may evaluate J_k in (4.71) by the Stationary Phase Lemma, and obtain an asymptotic expansion in descending powers of $k^{1/2}$. The latter expansion may be inserted in (4.70), and integrated term by term, thus leading to an asymptotic expansion for I_k. The leading order term of either expansion is determined by the contribution of the leading order term in the asymptotic expansion for the amplitude in (4.72), which is given by the following:

$$J'_k(\tau,\, w;\, \zeta_2) = \left(\frac{k}{\pi}\right)^d \int_{-\infty}^{+\infty} \mathrm{d}\zeta_1 \int_0^{+\infty} \mathrm{d}u \left[e^{i\sqrt{k}\,\Upsilon_w(u,\,\zeta_1)}\, u^d \left(e^{\frac{i}{\sqrt{k}}\boldsymbol{\vartheta}_1(\boldsymbol{\zeta})} - e^{\frac{i}{\sqrt{k}}\boldsymbol{\vartheta}_2(\boldsymbol{\zeta})} \right) \right.$$
$$\left. \cdot\, e^{-\frac{u}{2}\boldsymbol{\vartheta}(\boldsymbol{\zeta})^t\, E(w)\, \boldsymbol{\vartheta}(\boldsymbol{\zeta}) + 2\, i\, u\, \tau\, \left\langle \boldsymbol{\rho}_{h(w)T},\, \boldsymbol{\vartheta}(\boldsymbol{\zeta}) \right\rangle} \right].$$

Definition 4.7.3. Suppose $w = r\, e^{i\theta} \in B(0,\, \delta)$ and let $C(r)$ and $S(r)$ be as in (4.69). Let us set

$$\begin{aligned} \mathfrak{a}(w) :=& u(r) \begin{pmatrix} -S(r) & C(r) \end{pmatrix} E(w) \begin{pmatrix} -S(r) \\ C(r) \end{pmatrix} \\ =& u(r)\, \left\| \mathrm{Ad}_{h(w)}(\mathbf{n}_2(r))_X(x) \right\|_x^2 \end{aligned}$$

and

$$\begin{aligned} \mathfrak{r}(w) :=& 2\, u(r)\, \langle \boldsymbol{\rho}_{h(w)T},\, \mathbf{n}_2(r) \rangle \\ =& 2\, u(r)\, \omega_m \left(\mathrm{Ad}_{h(w)}(\mathbf{n}_2(r))_M(m),\, \Upsilon_{\boldsymbol{\nu}}(m) \right). \end{aligned}$$

Given the previous considerations, an application of the Stationary Phase Lemma yields the following.

Definition 4.7.4. With $|r| < \delta$, let us set $\mathfrak{b}(r) := \langle \boldsymbol{\nu},\, \mathbf{n}_2(r) \rangle$, and

$$D_l(r) := \frac{i^l}{l!\, \|\kappa(r)\|} \left[C(r)^l + (-1)^{l-1} S(r)^l \right].$$

The definition of $\mathfrak{b}(r)$ implies:

$$\mathfrak{b}(r) = -\frac{(\nu_1 - \nu_2)(\nu_1 + \nu_2)}{\|\boldsymbol{\nu}\|}\, r^2\, S_1(r),$$

where S_1 is a real-analytic function of the form $S_1(r) = 1 + \sum_{j \geq 1} c_j r^{2j}$.

Proposition 4.7.3. *Suppose $x \in X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$. Then as $k \to +\infty$ we have*

$$\Pi_{k\boldsymbol{\nu}}(x_{\tau,k},\, x_{\tau,k}) \sim D_{G/T}\,\frac{k\,(\nu_1-\nu_2)}{(2\pi)^2}\int_{-\pi}^{\pi}\mathrm{d}\theta\int_0^{+\infty}\mathrm{d}r\,\left[I_k(\tau,\,r,\,\theta)\right]\,, \tag{4.73}$$

where $I_k(\tau;\,r;\,\theta)$ is given by an asymptotic expansion in descending powers of $k^{1/2}$, the leading power being k^{d-1}. As a function of τ, aside from a phase factor, the coefficient of $k^{d-(1+j)/2}$ is a polynomial of degree $\le 3j$. Up to non-dominant terms we may replace $I_k(\tau,\,w)$ by

$$I_k(\tau,\,w)' = -\left(\frac{k}{\pi}\right)^d\left(\frac{2\pi}{\sqrt{k}}\right)\mathcal{S}_{G/T}(r)\,r\cdot u(w)^d \cdot\sum_{l\ge 1}\frac{D_l(r)}{k^{l/2}}\int_{-\infty}^{+\infty}d\zeta_2\left[e^{-i\sqrt{k}\,\zeta_2\,\mathfrak{f}(\tau,w)}\zeta_2^l\cdot e^{-\frac{1}{2}\mathfrak{a}(w)\zeta_2^2}\right]\,, \tag{4.74}$$

where for $k = 1,\,2,\ldots,$ we have set

$$\mathfrak{f}_k(\tau,\,w) := \mathfrak{b}(r) - \frac{\tau}{k^{1/2}}\mathfrak{r}(w)\,. \tag{4.75}$$

The Gaussian integrals in (4.74) may be estimated recalling that

$$\int_{-\infty}^{+\infty}x^l\,e^{-i\xi\,x-\frac{1}{2}\lambda x^2}\,\mathrm{d}x = \sqrt{2\pi}\frac{(-i)^l}{\lambda^{l+1/2}}\,P_l(\xi)\,e^{-\frac{1}{2\lambda}\xi^2}\,, \tag{4.76}$$

where $P_l(\xi) = \xi^l + \sum_{j\ge 1}p_{l_j}\,\xi^{l-2j}$ is a monic *polynomial* in ξ, of degree l and parity $(-1)^l$ (thus the previous sum is *finite*). Applying (4.76) with

$$\xi = k^{1/2}\,\mathfrak{f}_k(w,\,\tau), \quad \lambda = \mathfrak{a}(w)$$

we obtain the following conclusion.

Proposition 4.7.4. *Let us set*

$$F_l(\tau,\,w) := \frac{\sqrt{2\pi}}{l!}\left[\frac{C(r)^l + (-1)^{l-1}S(r)^l}{\|\kappa(r)\|}\right]\frac{P_l\left(\sqrt{k}\,\mathfrak{f}_k(\tau,\,w)\right)}{k^{l/2}\,\mathfrak{a}(w)^{l+1/2}}\,. \tag{4.77}$$

Up to lower order terms, we can replace I'_k in (4.74) by

$$I_k(\tau,\,w)'' := -\left(\frac{k}{\pi}\right)^d\left(\frac{2\pi}{\sqrt{k}}\right)\mathcal{S}_{G/T}(r)\cdot r\cdot u(w)^d \cdot e^{-\frac{1}{2}\,k\,\frac{\mathfrak{f}_k(\tau,\,w)^2}{\mathfrak{a}(w)}}\sum_{l\ge 1}F_l(\tau,\,w)\,. \tag{4.78}$$

Thus the leading order asymptotics of $\Pi_{k\boldsymbol{\nu}}(x_{\tau,k},\,x_{\tau,k})$ are obtained by replacing $I_k(\tau,\,r,\,\theta)$ in (4.73) by $I_k(\tau,\,w)''$.

4.7.2 Proof of Theorem 1.3.4

We shall set $\tau = 0$ in 4.7.3 and obtain an asymptotic estimate for $\Pi_{k\boldsymbol{\nu}}(x,\,x)$ when $x \in X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ and $k \to +\infty$.

Proof of Theorem 1.3.4. It follows from the definitions that

$$\frac{\mathfrak{f}_k(0,\,w)^2}{\mathfrak{a}(w)} = \frac{\mathfrak{b}(r)^2}{\mathfrak{a}(w)} = \lambda_{\boldsymbol{\nu}}(m)\,D(\boldsymbol{\nu})\,r^4\,\mathcal{S}(r,\,\theta)\,,$$

where $\mathcal{S}(r,\,\theta) = 1 + \sum_{j\geq 1} r^j\,d_j(\theta)$, and

$$D(\boldsymbol{\nu}) := \frac{(\nu_1-\nu_2)^2(\nu_1+\nu_2)^2}{\|\mathrm{Ad}_{h_m}(\boldsymbol{\nu}_\perp)_M(m)\|^2_m}\,.$$

Similarly,

$$\begin{aligned}\frac{P_l\left(\sqrt{k}\,\mathfrak{f}_k(0,\,w)\right)}{k^{l/2}\,\mathfrak{a}(w)^{l+1/2}} &= \frac{P_l\left(\sqrt{k}\,\mathfrak{b}(r)\right)}{k^{l/2}\,\mathfrak{a}(w)^{l+1/2}}\\ &= \frac{1}{\mathfrak{a}(w)^{l+1/2}}\left[\mathfrak{b}(r)^l + \sum_{j\geq 1}^{\lfloor l/2\rfloor} p_{l_j}\,k^{-j}\,\mathfrak{b}(r)^{l-2j}\right]\\ &= \sum_{j=0}^{\lfloor l/2\rfloor}\frac{1}{k^j}\,r^{2l-4j}\mathcal{S}_{l_j}(r,\,\theta)\,,\end{aligned}$$

where $\mathcal{S}_{l_j}(r,\,\theta)$ is a convergent power series in r. The resulting series may be integrated term by term. The l-th summand in (4.78) then gives rise to a convergent series of summands of the form

$$B_{\boldsymbol{\nu},l,j}(m,\,\theta)\,\frac{1}{k^j}\int_0^{+\infty}\tilde{r}^{2l-4j+a}e^{-\frac{1}{2}k\,\lambda_{\boldsymbol{\nu}}(m)\,D(\boldsymbol{\nu})\cdot\tilde{r}^4}\,\tilde{r}\,\mathrm{d}\tilde{r} = O\left(\frac{1}{k^{\frac{l+1}{2}+\frac{a}{4}}}\right)$$

with $j \leq \lfloor l/2\rfloor$ and $a = 0,\,1,\,2,\,\ldots$.

The previous discussion shows that $\Pi_{k\boldsymbol{\nu}}(x,\,x)$ is given by an asymptotic expansion in descending powers of $k^{1/4}$, and that the leading order term occurs for $l = 1$ and $a = 0$.

By equation (4.76), $P_l(\xi) = \xi$; by Lemma 4.7.3, $\|\kappa(r)\| = \lambda_{\boldsymbol{\nu}}(m)\,\|\boldsymbol{\nu}\| \cdot \mathcal{S}'_\kappa(r)$, where $\mathcal{S}'_\kappa(r)$ is a convergent power series in r^2 with $\mathcal{S}'_\kappa(0) = 1$.

In view of (4.77), we obtain

$$F_l(0,\, w) = -\sqrt{2\pi} \cdot \frac{(\nu_1-\nu_2)(\nu_1+\nu_2)^2}{\|\mathrm{Ad}_{h_m}(\boldsymbol{\nu}_\perp)_M(m)\|^3}\,\lambda_{\boldsymbol{\nu}}(m)^{1/2}\,r^2\,\mathcal{S}_{F_1}(r,\,\theta),$$

where $\mathcal{S}_{F_1}$ is real-analytic function and $\mathcal{S}''(0,\,\theta) = 1$.

Hence the leading order term of the asymptotic expansion of $\Pi_{k\boldsymbol{\nu}}(x,\,x)$ is given by

$$D_{G/T}\,\frac{k\,(\nu_1-\nu_2)}{(2\,\pi)^2}\int_{-\pi}^{\pi}\mathrm{d}\theta\int_0^{+\infty}\mathrm{d}r\,\left[L_k(r,\,\theta)\right]\,, \tag{4.79}$$

where

$$\begin{aligned} L_k(r) :=& 2^{3/2}\,\frac{k^{d-1/2}}{\pi^{d-3/2}}\lambda_{\boldsymbol{\nu}}(m)^{-(d-1/2)} \\ &\cdot \left[\frac{(\nu_1-\nu_2)(\nu_1+\nu_2)^2}{\|\mathrm{Ad}_{h_m}(\boldsymbol{\nu}_\perp)_M(m)\|^3}\right]\,e^{-\frac{1}{2}\,k\,\lambda_{\boldsymbol{\nu}}(m)\,D(\boldsymbol{\nu})\,r^4\,\mathcal{S}(r,\theta)}\,r^3\,\tilde{\mathcal{S}}(r,\,\theta)\,, \end{aligned} \tag{4.80}$$

where again $\tilde{\mathcal{S}}$ is real-analytic and e $\tilde{\mathcal{S}}(0,\,\theta) \equiv 1$.

We need to integrate in $\mathrm{d}r$ the product of the last two factors in (4.80). Let us perform the coordinate change $s = \sqrt{k}\,r^2\,\mathcal{S}(r,\,\theta)^{1/2}$, and argue as above. To leading order, we are reduced to computing

$$\frac{1}{2k}\int_0^{+\infty}\mathrm{d}s\left[e^{-\frac{1}{2}\lambda_{\boldsymbol{\nu}}(m)\,D(\boldsymbol{\nu})\,s^2}\,s\right] = \frac{1}{2k}\cdot\frac{1}{\lambda_{\boldsymbol{\nu}}(m)\,D(\boldsymbol{\nu})}\,.$$

Inserting this in (4.79), we conclude that the leading order term in the asymptotic expansion of $\Pi_{k\boldsymbol{\nu}}(x,\,x)$ is

$$\frac{D_{G/T}}{\sqrt{2}}\,\frac{1}{\|\Phi(m)\|^{d+1/2}}\left(\frac{k\,\|\boldsymbol{\nu}\|}{\pi}\right)^{d-1/2}\cdot\frac{\|\boldsymbol{\nu}\|}{\|\mathrm{Ad}_{h_m}(\boldsymbol{\nu}_\perp)_M(m)\|}\,.$$

The proof of Theorem 1.3.4 is complete. □

4.7.3 Proof of Theorem 1.3.6

The proof is a modification of the one of Theorem 1.3.4, so the discussion will be sketchy. We shall set

$$x_{j,\,k} := x + \frac{1}{\sqrt{k}}\,\mathbf{v}_j, \quad j = 1,\,2,$$

Definition 4.7.5. With the previous notation, let us set

$$\begin{aligned}&\Gamma(\boldsymbol{\vartheta},\, gT,\, \mathbf{v}_j)\\ &\quad := -\frac{1}{2}\left[\langle \mathrm{diag}\left(\mathrm{Ad}_{g^{-1}}(\Phi'_G(m))\right),\, \boldsymbol{\vartheta}\rangle^2 + \|\mathbf{v}_1 - \mathbf{v}_2 + \mathrm{Ad}_g(i\, D_{\boldsymbol{\vartheta}})_M(m)\|_m^2\right]\\ &\qquad + i\left[-\omega_m(\mathbf{v}_1,\, \mathbf{v}_2) + \omega_m(\mathrm{Ad}_g(i\, D_{\boldsymbol{\vartheta}})_M(m),\, \mathbf{v}_1 + \mathbf{v}_2)\right].\end{aligned}$$

Then, the same computations leading to Proposition 4.7.1 yield the following.

Proposition 4.7.5.

$$\begin{aligned}&i k\left[u\, \psi\left(\tilde{\mu}_{g e^{-i\boldsymbol{\vartheta}/\sqrt{k}} g^{-1}}\left(x_{1,k}\right),\, x_{2,k}\right) - \langle \boldsymbol{\vartheta},\, \boldsymbol{\nu}\rangle/\sqrt{k}\right]\\ &\quad = i\sqrt{k}\, \Psi(u,\, gT,\, \boldsymbol{\vartheta}) + u\, \Gamma(\boldsymbol{\vartheta},\, gT,\, \mathbf{v}_j) + k\, R_3\left(\frac{\mathbf{v}_j}{\sqrt{k}},\, \frac{\boldsymbol{\vartheta}}{\sqrt{k}}\right).\end{aligned}$$

Remark 5. Assuming $\mathbf{v}_1,\, \mathbf{v}_2 \in \mathfrak{g}_M(m_x)^{\perp_h}$ and recalling Definition 4.7.1 we have

$$\Gamma(\boldsymbol{\vartheta},\, gT,\, \mathbf{v}_j) = \psi_2(\mathbf{v}_1,\, \mathbf{v}_2) - \frac{1}{2}\boldsymbol{\vartheta}^t E(gT)\boldsymbol{\vartheta}.$$

In place of Corollary 4.7.1, we then obtain the following:

$$\begin{aligned}&\Pi_{k\boldsymbol{\nu}}(x_{1,k},\, x_{2,k}) \sim\\ &\quad \frac{k(\nu_1 - \nu_2)}{(2\pi)^2}\int_{G/T} \mathrm{dV}_{G/T}(gT)\int_{-\infty}^{+\infty} \mathrm{d}\vartheta_1 \int_{-\infty}^{+\infty} \mathrm{d}\vartheta_2 \int_0^{+\infty} \mathrm{d}u\\ &\quad \left[e^{i\sqrt{k}\,\Psi(u,gT,\boldsymbol{\vartheta})}\, \mathcal{A}'_{k,\boldsymbol{\nu}}(u,\, gT,\, \boldsymbol{\vartheta},\, \mathbf{v}_j)\right].\end{aligned}$$

with the amplitude

$$\begin{aligned}\mathcal{A}'_{k,\boldsymbol{\nu}}(u,\, gT,\, \boldsymbol{\vartheta},\, \mathbf{v}_j) :=& e^{u\psi_2(\mathbf{v}_1,\mathbf{v}_2) - \frac{u}{2}\boldsymbol{\vartheta}^t\, E(gT)\,\boldsymbol{\vartheta} + k\, R_3\left(\frac{\mathbf{v}_j}{\sqrt{k}}\, \frac{\boldsymbol{\vartheta}}{\sqrt{k}}\right)}\, \Delta\left(e^{i\boldsymbol{\vartheta}/\sqrt{k}}\right)\\ &\cdot s\left(\tilde{\mu}_{g e^{-i\boldsymbol{\vartheta}/\sqrt{k}} g^{-1}}\left(x_{1,k}\right),\, x_{2,k},\; ku\right).\end{aligned}$$

Similarly in place of (4.72) we now have the following expansion:

$$\begin{aligned}&\mathcal{A}'_{k,\boldsymbol{\nu}}(u,\, gT,\, \boldsymbol{\vartheta},\, \mathbf{v}_j)\\ &\quad \sim e^{u\psi_2(\mathbf{v}_1,\mathbf{v}_2) - \frac{u}{2}\boldsymbol{\vartheta}^t\, E(gT)\,\boldsymbol{\vartheta} + k\, R_3\left(\frac{\mathbf{v}_j}{\sqrt{k}}\, \frac{\boldsymbol{\vartheta}}{\sqrt{k}}\right)}\left[e^{\frac{i}{\sqrt{k}}\boldsymbol{\vartheta}_1(\boldsymbol{\zeta})} - e^{\frac{i}{\sqrt{k}}\boldsymbol{\vartheta}_2(\boldsymbol{\zeta})}\right]\\ &\quad \cdot \left(\frac{ku}{\pi}\right)^d \cdot \left[1 + \sum_{j\geq 1} a_j(u,\, w;\, \mathbf{v}_1,\, \mathbf{v}_2,\, \boldsymbol{\vartheta}(\boldsymbol{\zeta}))k^{-j/2}\right];\end{aligned}$$

where a_j is, as a function of $\mathbf{v}_1$ and $\mathbf{v}_2$, a polynomial of degree $\leq 3j$.

With these changes, Theorem 1.3.6 can be proved by applying the arguments in the proof of Theorem 1.3.4 with minor modifications.

4.7.4 Proof of Theorem 1.3.7

Proof. Let $A' \subset X$ be a one-sided "outer" tubular neighbourhood of $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, that is, the intersection of A with a tubular neighbourhood of $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ in X.

By Theorem 1.3.1, we have

$$\begin{aligned} &\dim_{out} H(X)_{k\boldsymbol{\nu}} \\ &\quad = \int_A \Pi_{k\boldsymbol{\nu}}(x,\, x)\, \mathrm{dV}_X(x) \sim \int_{A'} \Pi_{k\boldsymbol{\nu}}(x,\, x)\, \mathrm{dV}_X(x)\,. \end{aligned}$$

Let us denote by $m_x = \pi(x)$ the sign of $\nu_1 + \nu_2$. Then, locally along $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, for some sufficiently small $\delta > 0$ we can parametrize A' by a diffeomorphism

$$\Gamma\,:\, X^G_{\mathcal{O}_{\boldsymbol{\nu}}} \times [0,\, \delta) \to A', \quad (x,\, \tau) \mapsto x + \tau(\boldsymbol{\nu})\Upsilon_{\boldsymbol{\nu}}(m_x),$$

where $m_x = \pi(x)$. The latter expression is meant in terms of a collection of smoothly varying systems of Heisenberg local coordinates centred at $x \in X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, locally defined along $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$ (to be precise, one ought to work locally on $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, introduce an appropriate open cover of $X^G_{\mathcal{O}_{\boldsymbol{\nu}}}$, and a subordinate partition of unity; however for the sake of exposition we shall omit details on this).

We shall set $x_\tau := \Gamma(x,\, \tau)$, and write

$$\Gamma^*(\mathrm{dV}_X) = \mathcal{V}_X(x,\, \tau)\, \mathrm{dV}_{X^G_{\mathcal{O}_{\boldsymbol{\nu}}}}(x)\, \mathrm{d}\tau,$$

where $\mathcal{V}_X\,:\, X^G_{\mathcal{O}_{\boldsymbol{\nu}}} \times [0,\, \delta) \to (0,\, +\infty)$ is $\mathcal{C}^\infty$ and $\mathcal{V}_X(x,\, 0) = \|\Upsilon_{\boldsymbol{\nu}}(m_x)\|$.

Hence we obtain

$$\begin{aligned} &\dim_{out} H(X)_{k\boldsymbol{\nu}} \\ &\quad \sim \int_{X^G_{\mathcal{O}_{\boldsymbol{\nu}}}} \mathrm{dV}_{X^G_{\mathcal{O}_{\boldsymbol{\nu}}}}(x) \int_0^\delta \mathrm{d}\tau [\mathcal{V}_X(x,\, \tau)\, \Pi_{k\boldsymbol{\nu}}(x_\tau,\, x_\tau)] \end{aligned} \tag{4.81}$$

By Theorem 1.3.3, only a rapidly decreasing contribution to (4.81) is lost, if integration in (4.81) is restricted to the locus where $\tau \leq C\, k^{\epsilon - 1/2}$. Thus the asymptotics of $\dim_{out} H(X)_{k\boldsymbol{\nu}}$ are unchanged, if the integrand is multiplied by a rescaled cut-off function $\varrho(k^{1/2-\epsilon}\tau)$, where ϱ is identically one sufficiently near the origin in $\mathbb{R}$, and vanishes outside a slightly larger neighbourhood.

With the rescaling $\tau \mapsto \tau/\sqrt{k}$, we obtain

$$\dim_{out} H(X)_{k\boldsymbol{\nu}} \sim \frac{1}{\sqrt{k}} \int_{X^G_{\mathcal{O}_{\boldsymbol{\nu}}}} \mathrm{dV}_{X^G_{\mathcal{O}_{\boldsymbol{\nu}}}}(x) \left[\mathcal{H}_k(x)\right] ,$$

where with $x_{\tau,k} := \Gamma(x,\, k^{-1/2}\tau)$ we have set

$$\mathcal{H}_k(x) := \int_0^{+\infty} \mathrm{d}\tau \left[\varrho(k^{-\epsilon}\tau)\, \mathcal{V}_X\left(x,\, \frac{\tau}{\sqrt{k}}\right) \Pi_{k\boldsymbol{\nu}}(x_{\tau,k},\, x_{\tau,k})\right] . \tag{4.82}$$

Integration in $\mathrm{d}\tau$ is now over an expanding interval of the form $[0,\, C'\, k^{\epsilon})$.

Let us consider the asymptotics of 4.82. Having in mind (4.78), and inserting the Taylor expansion of $\mathcal{V}_X$, we are led to considering double integrals of the form

$$\frac{1}{k^{(l+j)/2}} \int_0^{+\infty} \mathrm{d}\tau \int_0^{+\infty} \mathrm{d}r \tag{4.83}$$
$$\left[r\, C(r)^l\, \tau^j\, \mathcal{S}'(r)\, \frac{P_l\left(\sqrt{k}\, \mathfrak{f}_k(\tau,\, w)\right)}{\mathfrak{a}(w)^{l+1/2}} \cdot e^{-\frac{1}{2}\,k\,\frac{\mathfrak{f}_k(\tau,\,w)^2}{\mathfrak{a}(w)}} \right]$$

with $l \geq 1$ and $j \geq 0$, and their analogues with $S(r)$ in place of $C(r)$; $\mathcal{S}'$ is some real-analytic function (dependence on θ and x is implicit).

In view of (4.75), we have

$$\frac{\mathfrak{f}_k(\sigma(\boldsymbol{\nu})\,\tau,\, w)}{\sqrt{\mathfrak{a}(w)}} = -\sigma(\boldsymbol{\nu}) \left[\frac{(\nu_1-\nu_2)|\nu_1+\nu_2|}{\|\boldsymbol{\nu}\|\, \sqrt{\mathfrak{a}(0)}}\, r^2\, S_1(r) + \frac{\tau}{k^{1/2}}\, \frac{\mathfrak{r}(0)}{\sqrt{\mathfrak{a}(0)}}\, S_2(r,\, \theta) \right] ,$$

where again $S_2(0,\, \theta) = 1$. Therefore, with the change of variables

$$s := k^{1/4}\, r\, \sqrt{S_1(r)}, \quad \tilde{\tau} := \tau\, S_2(r,\, \theta)$$

we obtain

$$\frac{\mathfrak{f}_k(\sigma(\boldsymbol{\nu})\,\tau,\, w)}{\sqrt{\mathfrak{a}(w)}} = -\frac{\sigma(\boldsymbol{\nu})}{\sqrt{k}} \left[\frac{(\nu_1-\nu_2)|\nu_1+\nu_2|}{\|\boldsymbol{\nu}\|\, \sqrt{\mathfrak{a}(0)}}\, s^2 + \frac{\mathfrak{r}(0)}{\sqrt{\mathfrak{a}(0)}}\, \tilde{\tau} \right] .$$

Therefore, we also have

$$\mathfrak{f}_k(\sigma(\boldsymbol{\nu})\,\tau,\,w) = -\frac{\sigma(\boldsymbol{\nu})}{\sqrt{k}}\left[\frac{(\nu_1-\nu_2)|\nu_1+\nu_2|}{\|\boldsymbol{\nu}\|}\,s^2 + \mathfrak{r}(0)\,\tilde{\tau}\right]\cdot\left[1+R_1\left(\frac{s}{\sqrt[4]{k}}\right)\right].$$

With the substitution $a = s^2$, (4.83) may be rewritten as a linear combination of summands of the form

$$\frac{1}{k^{(l+j+1)/2}}\int_0^{+\infty}\mathrm{d}\tau\int_0^{+\infty}\mathrm{d}a$$
$$\left[C\left(\frac{\sqrt{a}}{\sqrt[4]{k}}\right)^l (A_1\,a+B_1\,\tau)^b\,\tau^j\cdot\left[1+R_1\left(\frac{\sqrt{a}}{\sqrt[4]{k}}\right)\right]\cdot e^{-\frac{1}{2}\,(A_1\,a+B_1\,\tau)^2}\right]$$
$$= O\left(\frac{1}{k^{(l+j+1)/2}}\right).$$

Hence the leading contribution occurs for $l=1$, $j=0$, and dropping the term $R_1(k^{-1/4}\sqrt{a})$. The conclusion of Theorem 1.3.7 then follows by a fairly simple computation. □

4.8 Proof of Theorem 1.3.5

Arguing as in the proof of Theorem 1.3.4, we can restrict the integration in dV_G over a small neighbourhood of each $g_j \in G_x$. Let us denote with ϱ_j the bump function vanishing outside a small neighborhood of g_j in G. We have,

$$\Pi_{k\boldsymbol{\nu}}(x,\,x) \tag{4.84}$$
$$\sim \quad \sum_{j=1}^{|G_x|}\frac{k^2(\nu_1-\nu_2)}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{G/T}\int_0^{+\infty} e^{ik\Psi_{x,\,x}}\mathcal{A}_j\;\mathrm{d}u\;\mathrm{dV}_{G/T}(gT)\;\mathrm{d}\boldsymbol{\vartheta}\,;$$

where

$$\begin{aligned}\Psi_{x,\,x} = \Psi_{x,\,x}(u,\,\boldsymbol{\vartheta},\,gT) &:= u\,\psi(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\,x) - \langle\boldsymbol{\nu},\,\boldsymbol{\vartheta}\rangle\,,\\ \mathcal{A}_j = \mathcal{A}_j(u,\,\boldsymbol{\vartheta},\,gT) &:= \Delta\left(e^{i\boldsymbol{\vartheta}}\right)s'(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\,x,\,ku)\,,\end{aligned} \tag{4.85}$$

with

$$s'(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\,x,\,ku) := s(\tilde{\mu}_{ge^{-i\boldsymbol{\vartheta}}g^{-1}}(x),\,x,\,ku)\,\varrho_j\left(ge^{-i\boldsymbol{\vartheta}}g^{-1}\right).$$

Suppose that $\varrho \in \mathcal{C}_0^\infty((0, +\infty))$ is $\equiv 1$ on $(1/D, D)$ and is supported on $(1/(2D), 2D)$. Lemma 4.6.3 holds with minor modifications. Thus the asymptotics of (4.84) is unaltered, if the integrand is multiplied by $\varrho(u)$. We obtain

$$\Pi_{k\boldsymbol{\nu}}(x, y) \sim \sum_{j=1}^{|G_x|} \frac{k^2(\nu_1 - \nu_2)}{(2\pi)^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{G/T}\int_{1/2D}^{2D} e^{ik\Psi_{x,y}} \mathcal{A}_j' \, \mathrm{d}u \, \mathrm{dV}_{G/T}(gT) \, \mathrm{d}\boldsymbol{\vartheta}\,, \tag{4.86}$$

where we have set

$$\mathcal{A}_j'(u, \boldsymbol{\vartheta}, gT) := \varrho(u)\, \mathcal{A}_j(u, \boldsymbol{\vartheta}, gT)\,,$$

and the integration in du is now over a compact interval. Let us write $\Pi_{k\boldsymbol{\nu}}(x, x)_j$ for the j-th summand in (4.86).

We shall distinguish two cases, depending on whether $g_j \in Z_x$ or not. Let us write

$$\Pi_{k\boldsymbol{\nu}}(x, x) = \Pi_{k\boldsymbol{\nu}}(x, x)_{Z_x} + \Pi_{k\boldsymbol{\nu}}(x, x)_{G_x \setminus Z_x}\,,$$

where

$$\Pi_{k\boldsymbol{\nu}}(x, x)_{Z_x} := \sum_{g_j \in Z_x} \Pi_{k\boldsymbol{\nu}}(x, x)_j, \quad \Pi_{k\boldsymbol{\nu}}(x, x)_{G_x \setminus Z_x} := \sum_{g_j \notin Z_x} \Pi_{k\boldsymbol{\nu}}(x, x)_j\,.$$

First, let us consider the summand in (4.86) with g_j lying in the centre of the group. Thus, in this case the integral is over a compact neighbourhood of $e^{i\,\beta_j} I_2$. Let us perform the following change $\vartheta_i \mapsto \vartheta_i + \beta_j$, for $i = 1, 2$. We have

$$\Psi_{x,x}(u, \boldsymbol{\vartheta} + \boldsymbol{\beta}_j, gT) = \Psi_{x,x}(u, \boldsymbol{\vartheta}, gT) - (\nu_1 + \nu_2)\,\beta_j\,,$$

where $\boldsymbol{\beta}_j^t = (\beta_j, \beta_j)$. Thus, the j-th summand in (4.86) becomes

$$\Pi_{k\boldsymbol{\nu}}(x, x)_j \sim e^{i\beta_j\,[1-(\nu_1+\nu_2)k]} \frac{k^2(\nu_1 - \nu_2)}{(2\pi)^2} \cdot \int_{-\pi}^{\pi} \mathrm{d}\vartheta_1 \int_{-\pi}^{\pi} \mathrm{d}\vartheta_2 \int_{G/T} \mathrm{dV}_{G/T}(gT) \int_0^{+\infty} \mathrm{d}u \left[e^{ik\Psi_{x,x}(u,\boldsymbol{\vartheta},gT)} \mathcal{A}_0'(u, \boldsymbol{\vartheta}, gT) \right], \tag{4.87}$$

where, for $j = 0$ we have set $g_0 = I_2$. Hence the integral (4.87) is the same appearing in (4.64) with $\tau = 0$. Thus we can argue as in the proof of Theorem 1.3.4.

Let us now consider the case $g_j \notin Z$. By the discussion preceding the statement of Theorem 1.3.5 in §1.3, there exist $e^{i\alpha_j}$, $e^{i\beta_j} \in S^1$ and two unique cosets h_{m_x}, $k_{m_x} \in G/T$ such that

$$g_j = h_{m_x}\, t_{j1}\, h_{m_x}^{-1} = k_{m_x}\, t_{j2}\, k_{m_x}^{-1}$$

where

$$k_m = h_m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_{j1} = \begin{pmatrix} e^{i\alpha_j} & 0 \\ 0 & e^{i\beta_j} \end{pmatrix} \quad \text{and} \quad t_{j2} = \begin{pmatrix} e^{i\beta_j} & 0 \\ 0 & e^{i\alpha_j} \end{pmatrix}.$$

The pulled-back cut-off $(gT,\, e^{i\boldsymbol{\vartheta}}) \mapsto \varrho_j(g\, e^{\boldsymbol{\vartheta}}\, g^{-1})$ is supported in two small open neighbourhoods of $(h_{m_x}T,\, t_{j1})$ and $(k_{m_x}T,\, t_{j2})$. Therefore, we can further split $\Pi_{k\boldsymbol{\nu}}(x,\, x)_j$, $g_j \in G_x \setminus Z_x$,

$$\Pi_{k\boldsymbol{\nu}}(x,\, x)_j = \Pi_{k\boldsymbol{\nu}}(x,\, x)_{j1} + \Pi_{k\boldsymbol{\nu}}(x,\, x)_{j2}\,,$$

where in $\Pi_{k\boldsymbol{\nu}}(x,\, x)_{j1}$ (respectively $\Pi_{k\boldsymbol{\nu}}(x,\, x)_{j2}$) integration is over a small neighbourhood of $(h_{m_x}T,\, t_{j1})$ (respectively $(k_{m_x}T,\, t_{j2})$).

Let us focus on $\Pi_{k\boldsymbol{\nu}}(x,\, x)_{j1}$. By the previous remark and by composition with the projection $\pi' : G \to G/T$, we obtain a real-analytic coordinate chart $h(z)$ on G/T centred at $h_{m_x}T \in G/T$, and $t(\boldsymbol{\vartheta}_j)$ on T centred at $(e^{i\,\alpha_j},\, e^{i\,\beta_j})$; explicitly given by the following.

Definition 4.8.1. Let us denote by

$$h(z) := h_{m_x}\, e^{A(z)}, \quad A(z) := i \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix},$$

and

$$t(\boldsymbol{\vartheta}_j) := \begin{pmatrix} e^{i(\alpha_j + \vartheta_1)} & 0 \\ 0 & e^{i(\beta_j + \vartheta_2)} \end{pmatrix},$$

where $\alpha_j \neq \beta_j$.

The Haar volume form on G/T has the form $\mathcal{V}_{G/T}(w)\, \mathrm{dV}_{\mathbb{C}}(w)$, where $\mathrm{dV}_{\mathbb{C}}(w)$ is the Lebesgue measure on $\mathbb{C}$, and $\mathcal{V}_{G/T}$ is a uniquely determined C^∞ positive function on $B(0;\, \delta)$. In view of Lemma 4.7.2, we obtain

$$\mathcal{V}_{G/T}(w) = D_{G/T} \cdot \mathcal{S}_{G/T}(r)\,,$$

where $D_{G/T}$ is a constant, and $\mathcal{S}_{G/T}(r) = 1 + \sum_j s_j r^{2j}$.

Thus, in this case, the j-th summand in (4.86) becomes

$$\Pi_{k\boldsymbol{\nu}}(x,\,x)_{j1} \sim \quad e^{-i\,(\alpha_j\,\nu_1+\beta_j\,\nu_2)\,k}\cdot\frac{k^2(\nu_1-\nu_2)}{(2\pi)^2} \cdot\int_{-\pi}^{\pi}\mathrm{d}\vartheta_1\int_{-\pi}^{\pi}\mathrm{d}\vartheta_2\int_{\mathbb{C}}\mathrm{dV}_{\mathbb{C}}(z)\int_0^{+\infty}\mathrm{d}u \left[e^{ik\Psi_{x,\,x}(u,\,\boldsymbol{\vartheta},\,gT(z))}\mathcal{A}'_j(u,\,\boldsymbol{\vartheta},\,gT(z))\,\mathcal{V}_{G/T}(z)\right]. \tag{4.88}$$

where

$$\Psi_{x,x}(u,\,\boldsymbol{\vartheta},\,gT(z)) := u\,\psi(\tilde{\mu}_{h(z)t(\boldsymbol{\vartheta}_j)^{-1}h(z)^{-1}}(x),\,x)-\langle\boldsymbol{\nu},\,\boldsymbol{\vartheta}\rangle\,,$$
$$\mathcal{A}'_j(u,\,\boldsymbol{\vartheta},\,gT(z)) := \Delta\left(t(\boldsymbol{\vartheta}_j)\right)s'(\tilde{\mu}_{h(z)t(\boldsymbol{\vartheta}_j)^{-1}h(z)^{-1}}(x),\,x,\,ku)\,\varrho(u)\,.$$

As usual let us express $\tilde{\mu}_{h(z)t(\boldsymbol{\vartheta}_j)^{-1}h(z)^{-1}}(x)$ in Heisenberg local coordinates centred in x. We have

$$h(z)t(\boldsymbol{\vartheta}_j)^{-1}h(z)^{-1} = C_{h_m}\left[e^{A(z)}e^{-i\boldsymbol{\vartheta}}\,C_{t_{j1}^{-1}}\left[e^{-A(z)}\right]\right]\,g_j^{-1}\,,$$

where C_g denotes the conjugation by $g\in G$. Thus, by the Baker-Campbell-Hausdorff formula, we get

$$e^{A(z)}\,e^{-i\boldsymbol{\vartheta}}\,C_{t_{j1}^{-1}}\left[e^{-A(z)}\right] = e^{A(z)}\,e^{-i\boldsymbol{\vartheta}}\,e^{-\,\mathrm{Ad}_{t_{j1}^{-1}}A(z)} = e^{-\gamma_j(z,\boldsymbol{\vartheta})}\,,$$

where we have set

$$\gamma_j(z,\boldsymbol{\vartheta}) := \gamma_{j1}(z,\,\boldsymbol{\vartheta})+\gamma_{j2}(z,\,\boldsymbol{\vartheta})+R_3\left(z,\,\boldsymbol{\vartheta}\right)\,,$$

with

$$\gamma_{j1}(z,\boldsymbol{\vartheta}) := -A(z)+i\boldsymbol{\vartheta}+\mathrm{Ad}_{t_{j1}^{-1}}A(z)\,,$$

$$\gamma_{j2}(\boldsymbol{\vartheta},\,z) := \frac{1}{2}\left[A(z),\,\mathrm{Ad}_{t_{j1}^{-1}}A(z)\right]-\frac{i}{2}\left[\boldsymbol{\vartheta},\,A(z)+\mathrm{Ad}_{t_{j1}^{-1}}A(z)\right] \tag{4.89}$$

and $R_3\left(z,\,\boldsymbol{\vartheta}\right)$ denote a smooth function vanishing at 3-rd order at the origin in $\mathbb{C}\times\mathbb{R}^2$. Thus, by Corollary 2.2 of [Pao12], we obtain

$$\tilde{\mu}_{h(z)t(\boldsymbol{\vartheta}_j)^{-1}h(z)^{-1}}(x) = x+\left(\Theta(z,\,\boldsymbol{\vartheta}),\,V(z,\,\boldsymbol{\vartheta})\right),$$

where

$$\Theta(z,\,\boldsymbol{\vartheta}) := \langle \mathrm{Ad}_{h_{m_x}^{-1}}(\Phi(m_x)), \gamma_j(z,\boldsymbol{\vartheta})\rangle + R_3(z,\boldsymbol{\vartheta})\,,$$
$$V(z,\,\boldsymbol{\vartheta}) := -\left(\mathrm{Ad}_{h_{m_x}}\gamma_j(z,\boldsymbol{\vartheta})\right)_M(m_x) + R_2(z,\boldsymbol{\vartheta})\,.$$

In view of the discussion in chapter §3 of [SZ02], after some computations, we can rewritten the phase ψ as follow

$$u\,\psi(\tilde{\mu}_{h(z)t(\boldsymbol{\vartheta}_j)^{-1}h(z)^{-1}}(x),\,x) = u\,\Theta(z,\,\boldsymbol{\vartheta}) + \frac{i\,u}{2}\cdot\left[\Theta(z,\,\boldsymbol{\vartheta})^2 + \|V(z,\,\boldsymbol{\vartheta})\|^2\right] + u\,R_3(z,\,\boldsymbol{\vartheta})\,.$$

Let us notice that $(\mathrm{Ad}_{t_{j1}^{-1}} - I_2)\,A(z)$ and the second summand in (4.89) vanish on the diagonal. Thus the phase of the compact supported oscillatory integral $\Pi_{k\boldsymbol{\nu}}(x,\,x)_{j1}$ is

$$\Psi_{x,\,x}(u,\,\boldsymbol{\vartheta},\,gT) = u\left\langle \mathrm{Ad}_{h(w)^{-1}}(\Phi(m))\,,\,\boldsymbol{\vartheta}\right\rangle - \langle\boldsymbol{\nu},\,\boldsymbol{\vartheta}\rangle \quad (4.90)$$
$$\frac{i\,u}{2}\left\|\mathrm{Ad}_{h_{m_x}}(\gamma_j(z,\,\boldsymbol{\vartheta}))_X(x)\right\|_x^2 + u\left\langle \mathrm{Ad}_{h(w)^{-1}}(\Phi(m))\,,\,\frac{1}{2}\left[A(z),\,\mathrm{Ad}_{t_{j1}^{-1}}A(z)\right]\right\rangle$$
$$+\,R_3(z,\,\boldsymbol{\vartheta})\,.$$

The following is straightforward.

Lemma 4.8.1. *The phase* (4.90) *can be written in the following form*

$$\Psi_{x,\,x}(u,\,\boldsymbol{\vartheta}_j,\,z) = \left(\frac{u}{u_0} - 1\right)\langle\boldsymbol{\nu},\,\boldsymbol{\vartheta}\rangle + u\,|z|^2\,\sin(\beta_j - \alpha_j)(\nu_1 - \nu_2) + \frac{i\,u}{2}\,\mathbf{w}^t G_x\mathbf{w} + R_3(z,\,\boldsymbol{\vartheta})\,, \quad (4.91)$$

where $u_0 := \|\Phi(m)\|\,/\,\|\boldsymbol{\nu}\|$ *and* G_x *is a positive-defined symmetric matrix defined by*

$$\mathbf{w} := (z,\,\boldsymbol{\vartheta}) \mapsto \left\|\mathrm{Ad}_{h_{m_x}}\left((\mathrm{Ad}_{t_{j1}^{-1}} - I_2)\,A(z) - i\,\boldsymbol{\vartheta}\right)_X(x)\right\|_x^2\,.$$

The phase (4.91) *has a unique non degenerate critical point,*

$$P_0 = (u_0,\,\mathbf{0},\,\mathbf{0})\;,$$

and the Hessian at the critical point is

$$H_{P_0}(\Psi_{x,x}) = i \begin{pmatrix} 0 & -i\,\mathbf{b}_{\boldsymbol{\nu}}^t \\ -i\,\mathbf{b}_{\boldsymbol{\nu}} & H' \end{pmatrix}$$

where $\mathbf{b}_{\boldsymbol{\nu}}^t = (0,\, 0;\, \boldsymbol{\nu}/u_0)$ *and*

$$i\,H' := i\,u_0\,G_x + 2\,u_0\,\sin(\beta_j - \alpha_j)(\nu_1 - \nu_2)\begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0}^t & 0 \end{pmatrix}.$$

We are now in position to apply the Stationary Phase Lemma in (4.88). By Lemma 4.8.1 we have

$$\det\left(\frac{H_{P_0}(\Psi_j)}{2\,\pi\,i}\right) = -\frac{1}{(2\pi)^5}\det\left(\begin{pmatrix} 0 & \mathbf{b}_{\boldsymbol{\nu}}^t \\ \mathbf{b}_{\boldsymbol{\nu}} & H' \end{pmatrix}\right) = \frac{1}{(2\pi)^5}\,\mathbf{b}_{\boldsymbol{\nu}}^t\,(H')^{-1}\mathbf{b}_{\boldsymbol{\nu}}\cdot\det(H'), \tag{4.92}$$

where, for the last equality, we have used Lemma 2.1.3 of [Pao12]. For ease of notation let us set $\Theta_j(\boldsymbol{\nu},\, x) := \mathbf{b}_{\boldsymbol{\nu}}^t\,(H')^{-1}\mathbf{b}_{\boldsymbol{\nu}}\cdot\det(H')$.

Since H' and $(H')^{-1}$ are complex matrices with positive defined real part, we have

$$\Re\left(\mathbf{b}_{\boldsymbol{\nu}}^t\,(H')^{-1}\mathbf{b}_{\boldsymbol{\nu}}\right) > 0 \quad \text{and} \quad \Re\left(\det(H')\right) > 0\,.$$

Thus, for both factors appearing in (4.92) we can choose a well-defined square root with argument in $(-\pi/4,\, \pi/4)$, in such a way that

$$\Re\left(\sqrt{\det\left(\frac{H_{P_0}(\Psi_j)}{2\,\pi\,i}\right)}\right) > 0\,.$$

Thus, we can compute the leading term of $\Pi_{k\boldsymbol{\nu}}(x,\, x)_{j1}$, $g_j \in G_x \setminus Z_x$ by the Stationary Phase Lemma, it is

$$\sqrt{2}\left(\frac{\|\boldsymbol{\nu}\|}{\|\Phi_G(m_x)\|}\right)^d \cdot D_{G/T}\cdot\left(\frac{k}{\pi}\right)^{d-1/2}\cdot\frac{(\nu_1-\nu_2)\left(e^{i\alpha}-e^{i\beta}\right)}{\sqrt{\Theta_j(\boldsymbol{\nu},\, x)}}\cdot e^{-i\,k\,(\nu_1\alpha+\nu_2\beta)}\,. \tag{4.93}$$

Notice that $\Pi_{k\boldsymbol{\nu}}(x,\, x)_{j2}$ has the same expression as (4.88) with h_{m_x} replaced by k_{m_x} and t_{j_1} replaced by t_{j_2}. Thus it is an oscillatory integral with phase

$$\begin{aligned}\Psi'_{x,x}(u,\, \boldsymbol{\vartheta}_j,\, gT) &= \frac{i\,u}{2}\left\|\mathrm{Ad}_{k_{m_x}}(\gamma(z,\, \boldsymbol{\vartheta}))_X(x)\right\|_x^2 - \langle\boldsymbol{\nu},\, \boldsymbol{\vartheta}\rangle \\ &\quad + u\left\langle \mathrm{Ad}_{k(w)^{-1}}\left(\Phi(m)\right),\, i\boldsymbol{\vartheta} + \frac{1}{2}\left[A(z),\, \mathrm{Ad}_{t_{j2}^{-1}}A(z)\right]\right\rangle + R_3(z,\, \boldsymbol{\vartheta})\,.\end{aligned}$$

In a similar way as before, one can prove that $\Psi'_{x,x}$ has not critical point, thus $\Pi_{k\boldsymbol{\nu}}(x, x)_{j2}$ has rapidly decreasing asymptotic expansion.

Eventually, let us remark that $g_j \neq g_j^{-1}$ since $g_j \notin Z_x$; summing the contribution corresponding to $g_{j'} := g_j^{-1}$, we obtain the desired asymptotic expansion.

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