Università degli studi di Milano Bicocca
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Joint Ph.D. Program in Mathematics

Two Fractional Stochastic Problems:

Semi-Linear Heat Equation

and

Singular Volterra Equation

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Contents

Abstract

This thesis deals with two fractional stochastic equations: the fractional stochastic heat equation (a *partial stochastic differential equation* or SPDE) and a fractional stochastic Volterra equation (a *stochastic differential equation* or SDE).

In the first part, we deal with the fractional stochastic heat equation

$$\partial_t u = -(-\Delta)^{\frac{\alpha}{2}} u + f(u)\,\dot{W}$$

in the spatial domain $\mathbb{R}$, driven by space-time white noise W. We prove the existence, uniquess and regularity of the solution for $\alpha \in (1, 2]$ through classical methods of stochastic integration. Our first contribution is to prove these results under optimal assumptions on the initial datum, which was previously known only in the non-fractional case $\alpha = 2$. Our second main contribution is to study the behavior of the solution for $t \to 0$, by proving a quantitative comparison result between the solution of the stochastic problem and the solution of the corresponding deterministic problem. As a by-product, we derive a new proof of the strict positivity of the solution, in the linear case $f(u) = u$.

In the second part, we deal with the stochastic Volterra integral equation

$$u_t = \xi + \int_0^t f(u_s)\,(t-s)^{H-\frac{1}{2}}\,\mathrm{d}W_s,$$

which can be written in differential form $D^\alpha u = f(u)\,\dot{W}$ for $\alpha = H + \frac{1}{2}$. We perform a robust analysis of this equation, proving existence, uniqueness and regularity of the solution for $H \in (\frac{1}{4}, \frac{1}{2})$ in the framework of *rough paths theory*. We also prove finer estimates on the solution, which lead to a finite-increment reformulation of the equation. To develop our analysis, we enrich the usual rough paths theory with an extension of independent interest, namely the integration with respect to singular kernels, in the spirit of Hairer's theory of *Regularity Structures*, but in a rather elementary way.

Introduction

This thesis is divided in two parts: we consider two distinct stochastic differential problems and we analyze them with two different approaches and methods.

In the first part, we deal with a stochastic partial differential equation (SPDE) in one space dimension, called *fractional stochastic heat equation* (FSHE), written formally as

$$\frac{\partial u}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} u = f(u)\,\dot{W} \tag{FSHE}$$

where f is a Lipschitz function, $\dot{W}$ is the "density of a white noise" on $\mathbb{R}_+ \times \mathbb{R}$, and $(-\Delta)^{\frac{\alpha}{2}}$ is the *fractional Laplacian* of order α, for $\alpha \in (1,2]$. This kind of stochastic differential equation has been widely studied in literature; for the case $\alpha = 2$, (FSHE) coincides with the classical stochastic heat equation, which has been intensively studied; for $\alpha \in (1,2)$, the existence, uniqueness and regularity of a solution has been proven only in recent years.

In particular the case $\alpha = 2$ was initially motivated by the *parabolic Anderson model* (in which $f(u) = u$) (see [Carmona, Molchanov 94]). A study of the stochastic heat equation can be found in [Bertini, Cancrini 95], and, more recently, in [Dalang et al. 09], [Conus et al. 10], [Conus et al. 14], and [Chen, Dalang 15 A]. The Hölder continuity of the solution was already studied in [Walsh 86], in the case of bounded inital data, and in [Shiga 94] and [Pospisil, Tribe 07] in the case in which the initial data is a continuous function with tails that grow at most exponentially. In [Dalang et al. 07],[Dalang et al. 09], the authors proved the Hölder continuity of the solution of (FSHE) with vanishing initial conditions and in [Chen, Dalang 15 A], the authors extended the above results proving the regularity of the solution for $\alpha = 2$ under the weakest possible condition on the initial data.

The case $\alpha \in (1,2)$ is a particular case of the fractional stochastic equations studied in [Debbi, Dozzi 05], [Debbi 06]. They proved the existence and uniqueness of a solution with the assumption that the initial condition is a bounded function. In [Chen, Dalang 15 A] and [Chen, Kim 14] the authors extended the results by enlarging the space of the possible initial data to locally uniformly bounded measures (see (1.2)) below.

Our first contribution is to present a general proof of the existence, uniqueness and regularity of a solution of (FSHE) for $\alpha \in (1,2]$ under optimal assumptions on the initial datum (see Theorems 1.3 and 1.4). In particular, we prove that, under optimal assumptions, the solution u of (FSHE) is a locally Hölder continuous function with exponents $(\frac{\alpha-1}{2\alpha})^-$ in time and $(\frac{\alpha-1}{2})^-$ in space.

For $\alpha = 2$ this was done in [Chen, Dalang 15 A], while for $\alpha \in (1,2)$ this is a new result. We follow a standard approach, based on Picard's iteration scheme. However, instead of the classical Gronwall's inequality, we use Gronwall-type inequalities that involve space-time convolutions. We obtain sharper results for $\alpha \in (1,2)$ thanks to sharper estimates on the convolutions of the square of the *fractional heat kernel* g (see Propostion 1.5), that is the solution of the following

$$\begin{cases} \dfrac{\partial g}{\partial t} = \Delta^{\frac{\alpha}{2}}\, g \\ g_0(x) = \delta_0(x), \end{cases}$$

where δ represent the Dirac delta measure.

Our second contribution concerns the behavior of the solution $u(t,x)$ as $t \downarrow 0$, which depends of course on the initial datum $u(0,\cdot) = \mu_0(\cdot)$. We prove that $u(t,x)$ is close to the solution $I_0(t,x) = (\mu_0 \star g_t)(x)$ of the corresponding deterministic problem in a rather strong sense: the ratio $u(t,x)/I_0(t,x))$ converges to 1 as $t \downarrow 0$ uniformly for x in compact sets. Actually this ratio, that we call *normalized solution*, can be extended to a Hölder function on $[0,\infty) \times \mathbb{R}$ (including $t = 0$) with explicit Hölder exponents (see Theorem 2.2).

Finally, we present an alternative proof of the *strict positivity* of the solution of the *linear* (FSHE) (that is when $f(u) = cu$) started with non-negative initial conditions. This result was proven first in [Mueller 91] for the case $\alpha = 2$, through a strong comparison principle but under strong conditions on the initial datum; see [Moreno 14] and [Gubinelli, Perkowski 17] for alternative approaches. In [Chen, Kim 14], authors proved the strict positivity to the general (*non-linear*) (FSHE), extending Mueller's comparison principle for the case $\alpha \in (1,2)$. Here we stick to the linear case, but we present an alternative approach, which is based on the proof of the continuity of the *normalized solution* (see Theorem 2.19).

In the second part of this thesis, we consider the one-dimensional *rough fractional differential equation* (RFDE), written as

$$D^\alpha u = f(u)\,\dot{W}, \tag{RFDE}$$

where f is a Lipschitz function, $\dot{W}$ is the density of a white noise on $\mathbb{R}_+$ (i.e. the derivative of Brownian motion) and D^α denotes the fractional differential operator of order α, such that the integral formulation of (RFDE) is written as

$$u_t = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t f(u_s)\,(t-s)^{\alpha-1}\,\mathrm{d}W_s. \tag{I}$$

Equation (I) represent the so-called *Volterra integral equation* with kernel given by $p_s = s^{\alpha-1}$. This kind of equation has been studied in literature and solved by different techniques. The first studies were carried in [Berger, Mizel 80a]-[Berger, Mizel 80b], where (I) is solved by using the classical theory of Itô stochastic integration. Then, many authors studied stochastic Volterra equations in different settings and generality, for instance [Protter 85], [Pardoux, Protter 90], [Chocran et al. 95], [Zhang 10]. In recent years, stochastic Volterra equations attracted attention in financial modelling, because stochastic Volterra equations with singular kernels constitute very suitable models for the rough behaviour of volatility in financial markets; this was first observed in [Gatheral et al. 18]. Very recently, in [Bayer et al. 17], the authors used a new and powerful tool to analyze rough volatility models, that is the theory of regularity structures ([Hairer 14]). Meanwhile [Prömel, Trabs 18] develop a pathwise approach and a solution theory for Volterra equations, using the theory of paracontrolled distributions.

Like [Bayer et al. 17]-[Prömel, Trabs 18] (which appeared as a preprint simultaneously to the writing of this thesis), here we present a robist pathwise analysis of (I). We prove the existence, uniqueness and regularity of a solution of (I) (see Theorem 4.6) in the framework of the *rough path theory*, which is more elementary than regularity structures or paracontrolled distributions. To solve (I) with the theory of rough paths, developed by Lyons ([Lyons 98]) and extended by [Gubinelli 04], we need some generalizations of independent interest: as in [Hairer 14], but with more elementary techniques, we define the integration of singular kernel.

It is convenient to set $\alpha = H + \frac{1}{2}$. The solution turns out to be a *controlled path* of the function

$$X_t = \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \int_0^t (t-s)^{H-\frac{1}{2}} \,\mathrm{d}W_s, \tag{RL-fBM}$$

that is the so-called *Riemann-Liouville fractional Brownian motion* with Hurst parameter $H \in (0, \frac{1}{2})$, close to the usual fractional Brownian motion. We are able to give finer estimate for the solution (for the linear case in which $f(u) = u$), see Theorem 5.2, providing an equivalent characterization of the equation based on increments.

Part I

Fractional Stochastic Heat Equation

Introduction to Part I

In the first part of this thesis, we consider the *fractional stochastic heat equation* (FSHE) in the spatial domain $\mathbb{R}$.

In Chapter 1, we investigate the existence, the uniqueness and the regularity of the solution of (FSHE), taking inspiration from [Chen, Dalang 15 B] and [Chen, Kim 14], and extending their results to a wider class of initial data.

In Chapter 2, we focus on the *normalized solution*, that is the ratio of the solution of (FSHE) and the fractional heat kernel, that is the solution of the (deterministic) fractional heat equation. We study the properties of the normalized solution, in particular its continuity, from which we derive an alternative proof of the strict positivity of the solution of (FSHE) (first proved in [Mueller 91] for $\alpha = 2$ and in [Chen, Kim 14] for $\alpha \in (1, 2]$), when f is linear.

Chapter 1

Fractional Stochastic Heat Equation

Introduction

We consider the Fractional Stochastic Heat Equation (FSHE), which is formally written as

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta^{\frac{\alpha}{2}} u(t,x) + f(u(t,x))\,\dot{W}(t,x) & \text{for } t>0,\ x\in\mathbb{R}, \\ u(0,\cdot) = \mu_0, \end{cases} \tag{1.1}$$

where $\alpha \in (1,2]$ is fixed, $f:\mathbb{R}\to\mathbb{R}$ is a function, μ_0 is the initial datum, which may be a function or a measure on $\mathbb{R}$, $\dot{W}(t,x)$ is "the density of a white noise" $W(\mathrm{d}t,\mathrm{d}x)$ on $\mathbb{R}_+\times\mathbb{R}$ and $\Delta^{\frac{\alpha}{2}}$ denotes the fractional *Laplacian operator*. Actually, it would be more correct to write $-(-\Delta)^{\frac{\alpha}{2}}$ instead of $\Delta^{\frac{\alpha}{2}}$, since $-\Delta$ is positive definite; however, for convenience, we will stick to the notation used in (1.1).

In the case of $\alpha = 2$, (1.1) is the usual stochastic heat equation. Indeed, $\Delta^{\frac{\alpha}{2}}$ becomes the Laplacian (since we are in the one-dimensional case, the Laplacian is just the second derivative in space). The *stochastic heat equation* has been intensively studied (i.e. [Mueller 91], [Bertini, Cancrini 95], [Khoshnevisan 09], [Hairer, Pardoux 15]). When f is a linear function, that is $f(x) = \beta x$ for some $\beta\in\mathbb{R}$, the problem (1.1) is known as the *parabolic Anderson model*, which has been studied in depth since [Carmona, Molchanov 94] and permits to model random motions in random media.

When $\alpha\in(1,2)$, the operator $\Delta^{\frac{\alpha}{2}}$ can be defined as the generator of a symmetric α-stable Lévy process (recall that for $\alpha = 2$, the Laplacian can be defined as the generator of a Brownian Motion). We stick to the case $\alpha\in(1,2]$ since, according to Theorem 11 in [Dalang 99], even the simplest form of (1.1), with $f\equiv 1$, does not have a solution if $\alpha\le 1$.

This equation has been studied in literature also in recent years: in particular, we took inspiration from [Chen, Dalang 15 B] and [Chen, Kim 14]. (In both articles, the authors actually consider a more general fractional operator of order $\alpha\in(1,2]$ with skewness δ (where $|\delta|\le\min(\alpha,2-\alpha)$); here we stick for simplicity to the symmetric case $\delta = 0$). They proved existence, uniqueness and regularity of a solution to (1.1) for the class $\mathcal{M}(\alpha)$ of initial data, where

$$\mathcal{M}(\alpha) = \Big\{\mu_0 \text{ Borel measures on } \mathbb{R}, \quad s.t. \quad \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} \frac{1}{1+|x-y|^{\alpha+1}}\,\mu_0(\mathrm{d}y) < \infty\Big\} \qquad \text{for } \alpha\in(1,2) \tag{1.2}$$

and

$$\mathcal{M}(2) = \Big\{\mu_0 \text{ Borel measures on } \mathbb{R}, \quad s.t. \quad \int_{\mathbb{R}} e^{-ay^2}\mu_0(\mathrm{d}y) < \infty \quad \text{for all } a>0\Big\}. \tag{1.3}$$

Note that $\mathcal{M}(2)$ is the widest class of initial data for which the usual deterministic heat equation (i.e. with $f(u)\dot{W} = 0$) has a solution defined for all times, so it is an optimal choice.

Our first contribution, described in this chapter, is to extend the above results by allowing an optimal class of initial data also for the case $\alpha \in (1,2)$. Indeed, condition (1.2) is much stronger than the condition of the case $\alpha = 2$, due to the presence of sup. We are able to remove the sup thanks to sharp estimates of the convolutions of the sqare of the "fractional heat kernels", i.e. the fundamental solution of the deterministic fractional heat equation.

Description of the chapter. This first chapter follows the following scheme:

- In Section 1.1, we define the problem (1.1) more precisely and give the results of existence, uniqueness and regularity of the solution; we re-write the problem in integral formulation, using the theory of stochastic integration.
- In the remaining Sections, we give the proofs of existence and uniquess (Section 1.3) and regularity (Section 1.4). Before that, in Section 1.2 we prove a basic but crucial estimate on the convolution of the square of the square of the fractional heat kernel.
- The main properties of the fractional heat kernel are given in Appendix A and the most technical proofs are deferred to Appendix B.

1.1. Mild solution, assumptions and main results

Fix a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and a white noise W on $\mathbb{R}_+ \times \mathbb{R}$ defined on Ω. All the stochastic processes we will deal with, will be defined on $(\Omega, \mathcal{A}, \mathbb{P})$. We also fix a complete filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$, which is compatible with the white noise W. All the notions about measurability, adaptedness and progressive measurability are done with respect to $\mathcal{F}$. For our goals, we could actually work with the augmented filtration generated by the white noise, that is $\mathcal{F}^W = (\mathcal{F}^W_t)_{t\geq 0}$, where

$$\mathcal{F}^W_t = \sigma\Big(\{W_s(A) \mid s \in [0,t],\, A \in \mathcal{B}^*(\mathbb{R})\} \cup \mathcal{N}\Big),$$

where $\mathcal{B}^*(\mathbb{R})$ are the Borel sets with finite Lebesgue measure, and $\mathcal{N}$ is the collection of the $\mathbb{P}$-null sets. Working with a more general filtration can be useful, e.g., to allow for random initial data.

1.1.1. Mild solution and main results.

Consider the stochastic differential problem (1.1) with a measure as initial datum, that is formally

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta^{\frac{\alpha}{2}} u(t,x) + f(u(t,x))\, \dot{W}(t,x) & \text{for } t > 0,\, x \in \mathbb{R}, \\ u(0,\cdot) = \mu_0(\cdot), \end{cases} \tag{1.4}$$

We now give a rigorous meaning to this equation.

Definition 1.1. A *mild solution* of (1.4) is a progressively measurable stochastic process $(u(t,x))_{(t,x)\in(0,\infty)\times\mathbb{R}}$ such that, for every $(t,x) \in (0,\infty)\times\mathbb{R}$,

$$\int_0^t \int_{\mathbb{R}} \mathbb{E}(|f(u(s,y))|^2) g^2_{t-s}(x-y)\, \mathrm{d}s\, \mathrm{d}y < \infty, \tag{1.5}$$

and

$$u(t,x) = \int_{\mathbb{R}} g_t(x-y)\, \mu_0(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}} f(u(s,y)) g_{t-s}(x-y)\, W(\mathrm{d}s, \mathrm{d}y) \quad \mathbb{P}\text{-a.s.}. \tag{1.6}$$

A mild solution consists of two terms: the first one is a standard "deterministic" integral, the second one is a stochastic integral with respect to W. Condition (1.5) and the progressive misurability of u ensure that the stochastic integral in the right hand side of (1.6) is well defined, see [Walsh 86].

For the deterministic term we will use the following notation:

$$I_0(t,x) = \int_{\mathbb{R}} g_t(x-y)\,\mu_0(\mathrm{d}y), \tag{1.7}$$

that is the solution of the deterministic differential problem:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta^{\frac{\alpha}{2}}\, u(t,x) & \text{for } t>0, x\in\mathbb{R} \\ u(0,\cdot) = \mu_0(\cdot). \end{cases} \tag{1.8}$$

We are going to study:

1. the existence of a mild solution;
2. the uniqueness;
3. the regularity.

We make the following hypothesis on f and μ_0.

Hypothesis on f. The function $f:\mathbb{R}\to\mathbb{R}$ is globally Lipschitz and it has at most linear growth: there exist constants $L, K>0$ such that

$$|f(x)-f(y)| \le L|x-y| \quad \text{for all } x,y\in\mathbb{R}, \tag{1.9}$$

$$|f(x)| \le K(1+|x|) \quad \text{for all } x\in\mathbb{R}. \tag{1.10}$$

Remark 1.2. Condition (1.9) implies condition (1.10). Indeed, suppose that f satisfies (1.9), then, for all $x,y\in\mathbb{R}$,

$$|f(x)| \le |f(x)-f(y)| + |f(y)| \le L|x-y| + |f(y)|.$$

We can choose $y=0$ and, if we define $K=\max\{L,|f(0)|\}$, then we get (1.10). However, it is customary to separate the two conditions, since they will play different roles in the proofs of the theorems of existence, uniqueness and regularity of the solution. Moreover, (1.9) can in principle be weakened to a *local* Lipschitz condition – something we will not consider – in which case it no longer implies (1.10).

Hypothesis on μ_0. The initial datum μ_0 is a deterministic positive Borel measure on $\mathbb{R}$, where we recall that a Borel measure is a measure defined on the sigma-algebra of Borel sets.

We require that

$$\int_{\mathbb{R}} g_t(x-y)\mu_0(\mathrm{d}y) < \infty \quad \text{for all } t>0,\, x\in\mathbb{R}, \tag{1.11}$$

where $g_t(x)$ denotes the functional heat kernel, i.e. the solution of (1.8) with $\mu_0=\delta_0$. See Appendix A for a quick remainder of its main properties. In particular, by properties (3) and (8) of Proposition A.2, condition (1.11) is equivalent to (1.2) with $\sup_{x\in\mathbb{R}}$ removed.

For the case $\alpha = 2$, where g is the classic heat kernel, condition (1.11) is equivalent to the condition in [Chen, Dalang 15 B] (see (1.3)):

$$\int_{\mathbb{R}} e^{-ay^2} \mu_0(\mathrm{d}y) < \infty \quad \text{for all } a > 0.$$

For the case $\alpha \in (1, 2)$, condition (1.11) is weaker than condition used in [Chen, Dalang 15 B] (see (1.2)). Indeed, condition (1.2) implies that the admissible measures have to uniformly bounded on compact sets: $\sup_{x\in\mathbb{R}} \mu_0([x, x + 1]) < \infty$. Instead, condition (1.11), which we underline is the weakest possible condition on μ_0 to have the possibility to define the deterministic solution I_0 in (1.7), allows the initial datum to be a measure with a polynomial growth, e.g. $\mu_0([x, x + 1]) \sim |x|^\gamma$ as $|x| \to \infty$, with $\gamma < \alpha$.

Theorem 1.3 (Existence and uniqueness)**.** If f satisfies (1.9) and (1.10), and μ_0 satisfies (1.11), then the fractional stochastic heat equation (1.4) has an unique (up to modification) mild solution such that, for all $(t, x) \in (0, \infty) \times \mathbb{R}$,

$$\int_0^t \int_{\mathbb{R}} \mathbb{E}(|u(s, y)|^2) g^2_{t-s}(x - y) \,\mathrm{d}s \,\mathrm{d}y < \infty. \tag{1.12}$$

For every $p \in [2, \infty)$,

$$\|u(t, x)\|_p^2 \leq \begin{cases} C_p(t)(1 + |I_0(t, x)|^2) & \text{if } f(0) \neq 0 \\ C_p(t)\,|I_0(t, x)|^2 & \text{if } f(0) = 0, \end{cases} \tag{1.13}$$

for all $(t, x) \in (0, \infty) \times \mathbb{R}$, where

$$C_p(t) := \sum_{k=0}^{\infty} (\tilde{c}_p 4K^2)^k \, t^{\frac{k(\alpha-1)}{\alpha}} \, \frac{\Gamma\left(\frac{\alpha-1}{\alpha}\right)^k}{\Gamma\left(\frac{(k+1)(\alpha-1)}{\alpha}\right)} \tag{1.14}$$

and $\tilde{c}_p$ is the constant that appears in Corollary B.20 and depends only on p.

Theorem 1.4 (Regularity)**.** The mild solution of (1.4) has a locally $(\frac{\alpha-1}{2\alpha}^-, \frac{\alpha-1}{2}^-)$-Hölder continuous modification in $(0, \infty) \times \mathbb{R}$.

Sketch of the proof. We divided the proofs of existence and uniqueness (Theorem 1.3) and regularity (Theorem 1.4).

The strategy of the proof of the theorem of existence and uniqueness (see Theorem 1.3) is similar to the usual one adopted for SDEs (*stochastic differential equations*), in which the two main tools are Gronwall's inequality and Picard's iteration scheme. One can proceed with the standard Gronwall's lemma (for example, see [Walsh 86] or [Khoshnevisan 09]); however, here we use the properties of the convolutions of g, which play a similar role. They turn out to be more specific for SPDEs, since they involve space-time convolutions, not only time ones and they lead to sharper results. This kind of inequalities are of Gronwall-type: we present them and their proofs in Appendix B. This approach is similar to the one used in [Chen, Dalang 15 A] and [Chen, Dalang 15 B]. The novel ingredient, which allows us to prove Theorems 1.3 and

1.4 under the weakest possible assumption on the initial datum (see (1.11)) is the following estimate on the square of the fractional heat kernel, whose proof can be found in Section 1.2.

Proposition 1.5. Let $\alpha \in (1,2]$. For all $0 < s < t < \infty$ and $x \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} \mathrm{d}y\, [I_0(s,y)]^2 g^2_{t-s}(x-y) \leq c\, \frac{t^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(t-s)^{\frac{1}{\alpha}}}\, [I_0(t,x)]^2,$$

where $c(\alpha)$ is a positive constant which depends only on α.

To prove the regularity of a mild solution (see Theorem 1.11), we need some useful estimates on its p-norms. The starting points are the Generalized Kolmogorov Continuity Theorem (see Theorem B.17), the Burkholder-Davis-Gundy (BDG) inequality and the regularity of the Gaussian integral. All the most technical results can be found in Section B.3 of Appendix B.

1.2. Proof of Proposition 1.5

In this section we contain the proof of Proposition 1.5, which is our key fundamental result to prove the existence, uniqueness and regularity of a solution for the case $\alpha \in (1,2)$.

Proof of Proposition 1.5. It is enough to prove that, for all $z_1, z_2 \in \mathbb{R}$,

$$\int_{\mathbb{R}} \mathrm{d}y\, g_s(y-z_1)\, g_s(y-z_2)\, g^2_{t-s}(x-y) \leq c\, \frac{t^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(t-s)^{\frac{1}{\alpha}}}\, g_t(x-z_1)\, g_t(x-z_2). \tag{1.15}$$

We consider two case:

1. $|x-z_1| \leq 2m\, t^{\frac{1}{\alpha}}$ or $|x-z_2| \leq 2m\, t^{\frac{1}{\alpha}}$;

2. $|x-z_1| \geq 2m\, t^{\frac{1}{\alpha}}$ and $|x-z_2| \geq 2m\, t^{\frac{1}{\alpha}}$.

where m is the same of the Lemma A.6. In the first case, we can suppose that $|x-z_1| \leq 2m\, t^{\frac{1}{\alpha}}$ and we can write

$$\begin{aligned}
\int_{\mathbb{R}} \mathrm{d}y\, g_s(y-z_1)\, g_s(y-z_2)\, g^2_{1-s}(x-y) &\leq \|g\|^2_\infty\, \frac{1}{s^{\frac{1}{\alpha}}(t-s)^{\frac{1}{\alpha}}} \int_{\mathbb{R}} \mathrm{d}y\, g_s(y-z_2)\, g_{t-s}(x-y) \\
&= \|g\|^2_\infty\, \frac{1}{s^{\frac{1}{\alpha}}(t-s)^{\frac{1}{\alpha}}}\, g_t(x-z_2) \\
&\leq \frac{\|g\|^2_\infty}{g(2m)}\, \frac{t^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(t-s)^{\frac{1}{\alpha}}}\, g_t(x-z_1)\, g_t(x-z_2),
\end{aligned}$$

having used (A.5) and the semigroup property of g and the fact that $z \mapsto g(z)$ is a symmetric function which decreases for $z > 0$ and then $1 \leq g_t(x-z_1)\frac{t^{\frac{1}{\alpha}}}{g(2m)}$. If we suppose that $|x-z_2| \leq 2m\, t^{\frac{1}{\alpha}}$, we just switch the role of z_1 and z_2 and then (1.15) is proved in case (1).

In the case (2), we have both $|x-z_1| \geq 2m$ and $|x-z_2| \geq 2m$. We can suppose that $z_1 < z_2$

and we divide the integral in three parts:

$$
\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}y\, g_s(y-z_1)\, g_s(y-z_2)\, g^2_{t-s}(x-y) = \\
&= \Big[\int_{y>\frac{x+z_2}{2}} \mathrm{d}y\, g_s(y-z_1)\, g_s(y-z_2)\, g^2_{t-s}(x-y) + \int_{y<\frac{x+z_1}{2}} \mathrm{d}y\, g_s(y-z_1)\, g_s(y-z_2)\, g^2_{t-s}(x-y) + \\
&\quad + \int_{\frac{x+z_1}{2}\le y\le \frac{x+z_2}{2}} \mathrm{d}y\, g_s(y-z_1)\, g_s(y-z_2)\, g^2_{t-s}(x-y)\Big].
\end{aligned}
$$

Let us first fix x such that $x > z_2$. When $y > \frac{x+z_2}{2}$, then also $y > \frac{x+z_1}{2}$ (recall that $z_2 > z_1$) and then we have

$$
\begin{aligned}
y - z_2 &> \frac{x-z_2}{2} \quad \text{which implies that} \quad g_s(y-z_2) \le g_s\Big(\frac{x-z_2}{2}\Big) \\
y - z_1 &> \frac{x-z_1}{2} \quad \text{which implies that} \quad g_s(y-z_1) \le g_s\Big(\frac{x-z_1}{2}\Big).
\end{aligned}
$$

Now, using Lemma A.6, we have

$$
g_s(y-z_2) \le g_t\Big(\frac{x-z_2}{2}\Big) \quad \text{and} \quad g_s(y-z_1) \le g_t\Big(\frac{x-z_1}{2}\Big).
$$

Then

$$
\begin{aligned}
\int_{y>\frac{x+z_2}{2}} \mathrm{d}y\, g_s(y-z_1)\, g_s(y-z_2)\, g^2_{t-s}(x-y) &\le g_t\Big(\frac{x-z_1}{2}\Big)\, g_t\Big(\frac{x-z_2}{2}\Big) \int_{\mathbb{R}} g^2_{t-s}(x-y) \\
&\le c\, \frac{1}{(t-s)^{\frac{1}{\alpha}}} g_t(x-z_1)\, g_t(x-z_2).
\end{aligned}
$$

When $y < \frac{x+z_1}{2}$, which implies that $y < \frac{x+z_2}{2}$, we use the fact that

$$
\begin{aligned}
x - y &> \frac{x-z_1}{2} > 0 \quad \text{which implies that} \quad g_{t-s}(x-y) \le g_{t-s}\Big(\frac{x-z_1}{2}\Big) \\
x - y &> \frac{x-z_2}{2} > 0 \quad \text{which implies that} \quad g_{t-s}(x-y) \le g_{t-s}\Big(\frac{x-z_2}{2}\Big).
\end{aligned}
$$

Then,

$$
\begin{aligned}
g^2_{t-s}(x-y) = g_{t-s}(x-y)\, g_{t-s}(x-y) &\le g_{t-s}\Big(\frac{x-z_1}{2}\Big)\, g_{t-s}\Big(\frac{x-z_2}{2}\Big) \\
&\le c\, g_t(x-z_1)\, g_t(x-z_2),
\end{aligned}
$$

using Lemma A.6 as above. We get

$$
\begin{aligned}
\int_{y<\frac{x+z_1}{2}} \mathrm{d}y\, g_s(y-z_1)\, g_s(y-z_2)\, g^2_{1-s}(x-y) &\le c\, \frac{\|g\|_\infty}{s^{\frac{1}{\alpha}}} g_t(x-z_1)\, g_t(x-z_2) \int_{\mathbb{R}} \mathrm{d}y\, g_s(y-z_1) \\
&= c\, \frac{\|g\|_\infty}{s^{\frac{1}{\alpha}}} g_t(x-z_1)\, g_t(x-z_2)
\end{aligned}
$$

The last part of the integral is for $\frac{x+z_1}{2} \le y \le \frac{x+z_2}{2}$. In this case, we get:

$$
\begin{aligned}
x - y &\ge \frac{x-z_2}{2} > 0 \quad \text{and then} \quad g_{t-s}(x-y) \le c\, g_t(x-z_2) \\
y - z_1 &\ge \frac{x-z_1}{2} > 0 \quad \text{and then} \quad g_s(y-z_1) \le c\, g_t(x-z_1).
\end{aligned}
$$

Finally,

$$\begin{aligned}
&\int_{\frac{x+z_1}{2}\le y\le\frac{x+z_2}{2}} \mathrm{d}y\, g_s(y-z_1)\, g_s(y-z_2)\, g^2_{1-s}(x-y)\\
&\le c\, g_t(x-z_1)\, g_t(x-z_2) \int_{\mathbb{R}} g_s(y-z_1)\, g_{t-s}(x-y)\\
&\le c\,\frac{\|g\|_\infty}{t^{\frac{1}{\alpha}}}\, g_t(x-z_1)\, g_t(x-z_2).
\end{aligned}$$

By summing up the estimate in the case (1) and these three estimates in the case (2), relation (1.15) is proved for for every $x > z_2$. However, it is easy to extend the result also in the case $x < z_1$ or $z_1 < x < z_2$. Indeed, in the prove of (1) nothing changes, while in the prove of (2) we have to use always Lemma A.6 and find convenient relations between $x-y, y-z_1, y-z_2, x-z_1, x-z_2$, which gives convenient relations between g, recalling that $z \mapsto g(z)$ is increasing when $z < 0$, and descreasing when $z > 0$. □

1.3. Proof of Existence and Uniqueness

In this section, we are going to prove the existence and uniqueness, stated in Theorem 1.3. Let us recap the statement.

Theorem 1.6 (Existence and uniqueness)**.** If μ_0 is a positive Borel measure on $\mathbb{R}$ that satisfies (1.11) and $f : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function (that is satisfies (1.9) and (1.10)), then the fractional stochastic heat equation (1.1) has an unique (up to modification) mild solution such that, for all $(t, x) \in (0, \infty) \times \mathbb{R}$,

$$\int_0^t \int_{\mathbb{R}} \mathbb{E}(|u(s, y)|^2) g^2_{t-s}(x-y)\, \mathrm{d}s\, \mathrm{d}y < \infty. \tag{1.16}$$

For every $p \in [2, \infty)$,

$$\|u(t, x)\|_p < \infty, \tag{1.17}$$

and, for every even integer $p \in [2, \infty)$,

$$\|u(t, x)\|_p^2 \le \begin{cases} C_p(t)(1 + |I_0(t, x)|^2) & \text{if } f(0) \ne 0\\ C_p(t)\, |I_0(t, x)|^2 & \text{if } f(0) = 0\end{cases} \tag{1.18}$$

where $t \mapsto C_p(t)$ is an increasing function and depends only on p (and on α and f, which are fixed). We can define C_p as

$$C_p(t) := \sum_{k=0}^{\infty} (c\,\tilde{c}_p\, K^2)^k\, t^{\frac{k(\alpha-1)}{\alpha}}\, \frac{\Gamma\left(\frac{\alpha-1}{\alpha}\right)^k}{\Gamma\left(\frac{(k+1)(\alpha-1)}{\alpha}\right)}, \tag{1.19}$$

where $\tilde{c}_p$ is the constant that appears in Corollary B.20 and depends only on p, c is a linear combination of $\|g\|_\infty$ and $c(\alpha)$ (see (A.8)), and K depends on the function f (see (1.10)).

Remark 1.7. Note that, if (1.18) holds for some $p \geq 2$, then (1.16) is automatically satisfied. Indeed, by $\|\cdot\|_2 \leq \|\cdot\|_p$,

$$\int_0^t \int_{\mathbb{R}} \mathbb{E}(|u(s,y)|^2) g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y \leq \int_0^t \int_{\mathbb{R}} C_p(s)(1+|I_0(s,y)|^2) g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y$$
$$\leq C_p(t) t^{\frac{\alpha-1}{\alpha}} (1+|I_0(t,x)|^2) < \infty,$$

by Corollary B.5.

Moreover, for any (possibly non integer) $p \in [2,\infty)$, denoting by $p_- := 2\lfloor \frac{p}{2} \rfloor$ and $p_+ := 2\lceil \frac{p}{2} \rceil$ the even integers closer to p, we have

$$\|u(t,x)\|_{p^-} \leq \|u(t,x)\|_p \leq \|u(t,x)\|_{p^+},$$

and then (1.18) implies also (1.17).

Remark 1.8. It is important to distinguish the case $f(0) = 0$. Indeed, this holds in the special linear case in which $f(u) = u$. Relation (1.18) will be used in Chapter 2 to prove the strict positivity for the linear case.

We divide the proof in two parts: in the first one we prove the uniquess, and, in the secon part, we prove the existence.

Proof of Theorem 1.3, Uniqueness. Suppose u and v are mild solutions of (1.1), which satisfy (1.16). We have to prove that u and v are modifications of one another. For all $(t,x) \in (0,\infty) \times \mathbb{R}$, define

$$\begin{aligned} d(t,x) &:= u(t,x) - v(t,x) \\ &= \int_0^t \int_{\mathbb{R}} \Big[f(u(s,y)) - f(v(s,y)) \Big] g_{t-s}(x-y)\, W(\mathrm{d}s,\mathrm{d}y) \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

thanks to the definition (1.6) of a mild solution. We will show that $\mathbb{P}(d(t,x) = 0) = 1$, by proving that $\mathbb{E}(d(t,x)^2) = 0$. By the Ito isometry and the Lipschitz condition (1.9) on f,

$$\begin{aligned} \mathbb{E}\big(|d(t,x)|^2\big) &\leq L^2 \int_0^t \int_{\mathbb{R}} \mathbb{E}\big(\big|u(s,y) - v(s,y)\big|^2\big) g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y \\ &= L^2 \int_0^t \int_{\mathbb{R}} \mathbb{E}\big(|d(s,y)|^2\big) g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y. \end{aligned}$$

Moreover, applying the triangle inequality, we have

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \mathbb{E}\big(|d(s,y)|^2\big) g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y &\leq 2\Big(\int_0^t \int_{\mathbb{R}} \mathbb{E}\big(|u(s,y)|^2\big) g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y + \\ &\quad + \int_0^t \int_{\mathbb{R}} \mathbb{E}\big(|v(s,y)|^2\big) g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y \Big) \\ &=: I_1(t,x) < \infty, \end{aligned}$$

thanks to (1.16).

Let us define $\varphi_n(t,x) := \mathbb{E}(|d(t,x)|^2)$ for every $n \in \mathbb{N}$ (even though, of course, it does not depend on n): it is a sequence of non-negative measurable functions defined on $(0,\infty) \times \mathbb{R}$ that satisfy (B.4) with $A = 0$ and $B = L^2$. (The measurability of φ_n follows by Fubini's theorem, because u and v, and hence d, being progressively measurable, are jointly measurable functions of (ω, t, x).) We can apply Lemma B.2 and get

$$\mathbb{E}(|d(t,x)|^2) \leq L^{2n}\, c^n t^{\frac{(n-1)(\alpha-1)}{\alpha}} \frac{\Gamma\left(\frac{\alpha-1}{\alpha}\right)^n}{\Gamma\left(\frac{n(\alpha-1)}{\alpha}\right)} I_1(t,x)$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we have $\mathbb{E}(|d(t,x)|^2) = 0$, and then $\mathbb{P}(u(t,x) = v(t,x)) = 1$, for all $(t,x) \in (0,\infty) \times \mathbb{R}$. □

The next proof of the existence part is based on Theorem 2.4 of [Chen, Dalang 15 A].

Proof of Theorem 1.3, Existence. We proceed with the standard Picard iteration scheme, showing that the solution can be written as the limit in $L^p(\Omega)$ of a Cauchy sequence, for every $p \geq 2$ (which means $p \in [2,\infty)$). For all $(t,x) \in (0,\infty) \times \mathbb{R}$ define

$$v_0(t,x) := I_0(t,x) = \int_{\mathbb{R}} g_t(x-y)\, \mu_0(\mathrm{d}y), \tag{1.20}$$

$$v_{n+1}(t,x) := I_0(t,x) + \int_0^t \int_{\mathbb{R}} f(v_n(s,y)) g_{t-s}(x-y)\, W(\mathrm{d}s,\mathrm{d}y), \tag{1.21}$$

for all $n \in \mathbb{N}_0$.

We divide the proof in four steps: in the first one we prove that the sequence $(v_n)_{n\in\mathbb{N}}$ is well defined; in the second one we find useful estimates on the p-norms; in the third step we show that, for every $(t,x) \in (0,\infty) \times \mathbb{R}$, $(v_n(t,x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$ and so it converges to some $u(t,x)$ in $L^p(\Omega)$; finally we prove that the process $(u(t,x))_{(t,x)\in(0,\infty)\times\mathbb{R}}$ is a mild solution of (1.1) satisfying (1.18).

Step 1. (v_n) are "well-defined", that is the stochastic integral which appears in (1.21) is well-defined and v_{n+1} is progressively measurable, for all $n \in \mathbb{N}$.

Let us start with the case $n = 0$: since I_0 is a non-random continuous function over $(0,\infty)\times\mathbb{R}$ (see Proposition A.2), clearly the following hold:

- I_0 is adapted;
- $I_0(t,x) < \infty$ by (1.11) and I_0 is $L^2(\Omega)$-continuous.

Thanks to Proposition B.15, with $\varphi = v_0 = I_0$, these properties imply that the process

$$\left(\int_0^t \int_{\mathbb{R}} f(I_0(s,y)) g_{t-s}(x-y)\, W(\mathrm{d}s,\mathrm{d}y)\right)_{(t,x)\in(0,\infty)\times\mathbb{R}},$$

which appears in the definition of v_1, is well-defined, adapted, continuous in L^2 and we have

$$\int_0^t \int_{\mathbb{R}} \mathbb{E}(|f(I_0(s,y))|^2) g_{t-s}^2(x-y)\, \mathrm{d}s\, \mathrm{d}y \leq \tilde{C}_1(t)(1_{f(0)\neq 0} + |I_0(t,x)|^2) < \infty,$$

by Proposition B.15, where, in this case, we can calculate directly $\tilde{C}_1(t) = 2K^2 c(\alpha) t^{\frac{\alpha-1}{\alpha}}$, thanks to the growth condition (1.10) on f and Corollary B.5 (notice that when $f(0) = 0$, $|f(I_0)|^2 \leq K^2 |I_0|^2$). Then, even the process

$$v_1 = \left(I_0(t,x) + \int_0^t \int_{\mathbb{R}} f(I_0(s,y)) g_{t-s}(x-y)\, W(\mathrm{d}s,\mathrm{d}y)\right)_{(t,x)\in(0,\infty)\times\mathbb{R}}$$

is well-defined, adapted, continuous in L^2 and, by the triangle inequality and the Ito isometry,

$$\mathbb{E}(|v_1(t,x)|^2) \leq C_1(t)(1_{f(0)\neq 0} + |I_0(t,x)|^2)$$

where $C_1 : (0,\infty) \to \mathbb{R}$ is a non-decreasing map.

Let us proceed by induction: let $n \geq 1$ and suppose that

$$v_n = \Big(I_0(t,x) + \int_0^t \int_{\mathbb{R}} f(v_{n-1}(s,y))g_{t-s}(x-y)\, W(\mathrm{d}s,\mathrm{d}y)\Big)_{(t,x)\in(0,\infty)\times\mathbb{R}} \tag{1.22}$$

is a well-defined stochastic process and the following properties hold:

1. $(v_n(t,x))_{(t,x)\in(0,\infty)\times\mathbb{R}}$ is adapted;
2. $(t,x) \mapsto v_n(t,x)$ is $L^2(\Omega)$-continuous;
3. for all $(t,x) \in (0,\infty)\times\mathbb{R}$, $\mathbb{E}(|v_n(t,x)|^2) \leq C_n(t)(1 + |I_0(t,x)|^2)$ for some non-decreasing function $C_n : (0,\infty) \to \mathbb{R}$.

By Proposition B.15, we have that the process

$$(f(v_n)\dot{W}) \star g := \Big(\int_0^t \int_{\mathbb{R}} f(v_n(s,y))g_{t-s}(x-y)\, W(\mathrm{d}s,\mathrm{d}y)\Big)_{(t,x)\in(0,\infty)\times\mathbb{R}} \tag{1.23}$$

is well-defined and

1. $(f(v_n)\dot{W}) \star g$ is adapted;
2. $(f(v_n)\dot{W}) \star g$ is $L^2(\Omega)$-continuous;
3. for all $(t,x) \in (0,\infty)\times\mathbb{R}$, $\mathbb{E}(|(f(v_n)\dot{W}) \star g|^2) \leq \tilde{C}_n(t)(1_{f(0)\neq 0} + |I_0(t,x)|^2)$ for some non-decreasing function $\tilde{C}_n : (0,\infty) \to \mathbb{R}$.

But, since $I_0(t,x)$ is a continuous and deterministic function, (1) and (2) holds even for $v_{n+1} = I_0 + (f(v_n)\dot{W}) \star g$. Moreover,

$$\mathbb{E}(|v_{n+1}(t,x)|^2) \leq 2|I_0(t,x)|^2 + 2\tilde{C}_n(t)(1_{f(0)\neq 0} + |I_0(t,x)|^2) \leq C_{n+1}(t)(1_{f(0)\neq 0} + |I_0(t,x)|^2),$$

where C_{n+1} is a non-decreasing function over $(0,\infty)$. We have just proved that the properties (1), (2) and (3) holds for any $n \in \mathbb{N}_0$, which means that the sequence (v_n) is well-defined.

Step 2. Once we know that the Picard iteration scheme is well defined, we would like to have some estimates on the p-norms of (v_n) (that is to have a control on the C_n's which will be useful in order to prove the property (1.17) satisfied by a mild solution. Fix an even integer $p \geq 2:$ for every $(t,x) \in (0,\infty)\times\mathbb{R}$, by (1.21) and the triangle inequality,

$$\|v_{n+1}(t,x)\|_p^2 \leq 2|I_0(t,x)|^2 + 2\Big\|\int_0^t \int_{\mathbb{R}} f(v_n(s,y))g_{t-s}(x-y)\, W(\mathrm{d}s,\mathrm{d}y)\Big\|_p^2.$$

Now, since p is even, we can apply Corollary B.20 on the second term, getting

$$\begin{aligned}\|v_{n+1}(t,x)\|_p^2 &\leq 2|I_0(t,x)|^2 + 2\tilde{c}_p \int_0^t \int_{\mathbb{R}} \|f(v_n(s,y))\|_p^2\, g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y \\ &\leq 2|I_0(t,x)|^2 + \tilde{c}_p 4K^2 \int_0^t (1_{f(0)\neq 0} + \|v_n(s,y)\|_p^2) g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y\end{aligned}$$

by the growth condition (1.10) on f. If we define $\varphi_n(t,x) := \|v_{n+1}(t,x)\|_p^2$, then $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of non-negative and measurable functions that satisfy (B.10) with $A = 2$ and $B = \tilde{c}_p 4K^2$ when $f(0) \neq 0$ and $(\varphi_n)_{n\in\mathbb{N}}$ satisfies (B.7) with the same A and B. Then, by Lemma B.7 (when $f(0) \neq 0$) and Lemma B.6 (when $f(0) = 0$), we can write

$$\|v_{n+1}(t,x)\|_p^2 \leq 2\,|I_0(t,x)|^2 + (1_{f(0)\neq 0} + 2|I_0(t,x)|^2) \sum_{k=1}^{n+1} (c\,\tilde{c}_p 4K^2)^k\, t^{\frac{k(\alpha-1)}{\alpha}} \frac{\Gamma\left(\frac{\alpha-1}{\alpha}\right)^{k+1}}{\Gamma\left(\frac{(k+1)(\alpha-1)}{\alpha}\right)}. \tag{1.24}$$

Step 3. Fix an arbitrary even $p \geq 2$; we are going to prove that, for every $(t,x) \in (0,\infty)\times\mathbb{R}$, $\{v_n(t,x)\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in $L^p(\Omega)$.

For all $(t,x) \in (0,\infty)\times\mathbb{R}$ define

$$\begin{aligned} d_{n+1}(t,x) &= v_{n+1}(t,x) - v_n(t,x) \\ &= \int_0^t \int_{\mathbb{R}} \left[f(v_n(s,y)) - f(v_{n-1}(s,y))\right] g_{t-s}(x-y)\, W(\mathrm{d}s, \mathrm{d}y), \end{aligned}$$

for $n \in \mathbb{N}$, and

$$d_1(t,x) = v_1(t,x) - I_0(t,x) = \int_0^t \int_{\mathbb{R}} f(I_0(s,y)) g_{t-s}(x-y)\, W(\mathrm{d}s, \mathrm{d}y).$$

We notice that, for all $(t,x) \in (0,\infty)\times\mathbb{R}$ and for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \|d_{n+1}(t,x)\|_p^2 &\leq \tilde{c}_p \int_0^t \int_{\mathbb{R}} \|f(v_n(s,y)) - f(v_{n-1}(s,y))\|_p^2 g_{r-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y \\ &\leq \tilde{c}_p L^2 \int_0^t \int_{\mathbb{R}} \|d_n(s,y)\|_p^2 g_{r-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y, \end{aligned}$$

having used Corollary B.20 and the Lipschitz condition (1.9) on f. Similarly, for $n = 0$, using the growth condition (1.10),

$$\begin{aligned} \|d_1(t,x)\|_p^2 &\leq \tilde{c}_p \int_0^t \int_{\mathbb{R}} \|f(I_0(s,y))\|_p^2 g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y \\ &\leq \tilde{c}_p\, 2K^2 \int_0^t \int_{\mathbb{R}} (1_{f(0)\neq 0} + |I_0(s,y)|^2) g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y \\ &\leq \tilde{c}_p\, 2K^2\, c_\alpha t^{\frac{\alpha-1}{\alpha}}\, (1_{f(0)\neq 0} + |I_0(t,x)|^2), \end{aligned} \tag{1.25}$$

by Corollary B.5.

Denoting by $\varphi_n(t,x) = \|d_n(t,x)\|_p^2$, then $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of non-negative and measurable functions defined on $(0,\infty)\times\mathbb{R}$ that satisfy the condition (B.4) of Lemma B.2 with $A = 0$ and $B = \tilde{c}_p L^2$. Then

$$\|d_{n+1}(t,x)\|_p^2 \leq (c\,\tilde{c}_p L^2)^n \frac{\Gamma\left(\frac{\alpha-1}{\alpha}\right)^n}{\Gamma\left(\frac{n(\alpha-1)}{\alpha}\right)}\, t^{\frac{(n-1)(\alpha-1)}{\alpha}} I_1(t,x)$$

where, by (B.3) and (1.25),

$$
\begin{aligned}
I_1(t,x) &= \int_0^t \int_{\mathbb{R}} \|d_1(s,y)\|_p^2 g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y \\
&\le c_\alpha\,\tilde{c}_p\,2K^2 t^{\frac{\alpha-1}{\alpha}} \int_0^t \int_{\mathbb{R}} (1_{f(0)\neq 0} + |I_0(s,y)|^2) g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y \\
&\le c_\alpha\,\tilde{c}_p\,2K^2\,t^{\frac{2(\alpha-1)}{\alpha}}\,(1_{f(0)\neq 0} + |I_0(t,x)|^2),
\end{aligned}
$$

(in the same way we have done in (1.25)) and this is finite for every fixed (t,x). Then,

$$
\|d_{n+1}(t,x)\|_p^2 \le c^n\,(\tilde{c}_p)^{n+1} L^{2n} K^2 \frac{\Gamma\left(\frac{\alpha-1}{\alpha}\right)^n}{\Gamma\left(\frac{n(\alpha-1)}{\alpha}\right)}\, t^{\frac{(n+1)(\alpha-1)}{\alpha}}\,(1+|I_0(t,x)|^2).
$$

In particular, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} \|v_{n+1}(t,x) - v_n(t,x)\|_p &= \|d_1(t,x)\|_p + \sum_{n=1}^{\infty} \|d_{n+1}(t,x)\|_p \\
&\le \sqrt{2c_\alpha \tilde{c}_p} K\, t^{\frac{\alpha-1}{2\alpha}} \sqrt{1+|I_0(t,x)|^2} + t^{\frac{\alpha-1}{2\alpha}}\,\sqrt{(1+|I_0(t,x)|^2)} \times \\
&\qquad \times \sum_{n=1}^{\infty} c^n\,(\tilde{c}_p)^{\frac{n+1}{2}} L^n K \frac{\Gamma\left(\frac{\alpha-1}{\alpha}\right)^{\frac{n}{2}}}{\sqrt{\Gamma(\frac{n(\alpha-1)}{\alpha})}} t^{\frac{(n-1)(\alpha-1)}{2\alpha}} < \infty.
\end{aligned} \tag{1.26}
$$

Hence, for every $(t,x) \in (0,\infty)\times\mathbb{R}$, and for every even $p \ge 2$, $\{v_n(t,x)\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in $L^p(\Omega)$ and so it converges to some random variable in $L^p(\Omega)$. We define the process $\big(u(t,x)\big)_{(t,x)\in(0,\infty)\times\mathbb{R}}$ such that, for every $(t,x) \in (0,\infty)\times\mathbb{R}$,

$$
u(t,x) = \lim_{n\to\infty} v_n(t,x) \quad \text{in } L^p(\Omega). \tag{1.27}
$$

We stress that, for any fixed (t,x), we have defined $u(t,x)$ *by choosing arbitrarily a random variable* in the L^p equivalence class of the limit of $v_n(t,x)$, which is uniquely determined only for a.e. ω. As a consequence, for the moment we have no information on the path properties of $u(t,x)$ and, in particular, we have no guarantee that the process $(u(t,x))_{(t,x)\in(0,\infty)\times\mathbb{R}}$ is progressively measurable. We are now going to show that this is the case, provided we choose a suitable modification.

Fix $(t,x) \in (0,\infty)\times\mathbb{R}$, then, for every even $p \ge 2$, passing to the limit in (1.24),

$$
\begin{aligned}
\|u(t,x)\|_p^2 &\le 2|I_0(t,x)|^2 + (1_{f(0)\neq 0} + 2|I_0(t,x)|^2) \sum_{k=1}^{\infty} (c\tilde{c}_p 4K^2)^k t^{\frac{k(\alpha-1)}{\alpha}} \frac{\Gamma\left(\frac{\alpha-1}{\alpha}\right)^{k+1}}{\Gamma(\frac{(k+1)(\alpha-1)}{\alpha})} \\
&\le C_p(t)(1_{f(0)\neq 0} + |I_0(t,x)|^2),
\end{aligned} \tag{1.28}
$$

where C_p is defined by (1.19). This proves that (1.18) holds, as well as (1.16) and (1.17) (see Remark 1.7).

Moreover, for every $(t,x) \in (0,\infty)\times\mathbb{R}$, $u(t,x)$ is $\mathcal{F}_t$-measurable, since each $v_n(t,x)$ is; in other words, the process $(u(t,x))_{(t,x)\in(0,\infty)\times\mathbb{R}}$ is adapted.

Since the convergence holds for every even $p \geq 2$, we can say that it actually holds for every $p \geq 2$.

Step 4. We shall verify that $(u(t,x))_{(t,x)\in(0,\infty)\times\mathbb{R}}$ is a mild solution of (1.1). By Step 2, we know that

- u is adapted;
- for all $(t,x) \in (0,\infty)\times\mathbb{R}$, $\mathbb{E}(|u(t,x)|^2) \leq C_2(t)(1+|I_0(t,x)|^2)$.

If we prove that u is $L^2(\Omega)$-continuous, then Proposition B.15 will say that

$$(f(u)\dot{W}) \star g := \Big(\int_0^t \int_{\mathbb{R}} f(u(s,y))g_{t-s}(x-y)\, W(\mathrm{d}s,\mathrm{d}y)\Big)_{(t,x)\in(0,\infty)\times\mathbb{R}}$$

is a well defined stochastic process (this is the minimal requirement on u to be a mild solution).

To prove that u is $L^2(\Omega)$-continuous, we show that $v_n(t,x) \to u(t,x)$ in $L^2(\Omega)$ *uniformly* with respect to (t,x), on any compact set of $(0,\infty)\times\mathbb{R}$. Let $M>0$ and define $K_M = [\frac{1}{M}, M]\times[-M,M]$; then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sup_{(t,x)\in K_M} \|v_{n+1}(t,x)-v_n(t,x)\|_2 \\
&\leq \underbrace{\sup_{(t,x)\in K_M} \Big[t^{\frac{\alpha-1}{\alpha}}(1_{f(0)\neq 0}+|I_0(t,x)|^2)\Big]^{\frac{1}{2}}}_{S_M} \sum_{n=0}^{\infty} 2c^n L^n K \frac{\Gamma\big(\frac{\alpha-1}{\alpha}\big)^{\frac{n}{2}}}{\sqrt{\Gamma\big(\frac{n(\alpha-1)}{\alpha}\big)}} M^{\frac{\alpha-1}{2\alpha}}
\end{aligned}$$

(see (1.26)). Since I_0 is continuous over K_M, and K_M is a compact set, $S_M < \infty$ for any $M>0$. This implies that $v_n \to u$ uniformly over K_M : indeed, for any $(t,x)\in K_M$,

$$\|u(t,x)-v_n(t,x)\|_2 \leq \lim_{k\to\infty} \|v_k(t,x)-v_n(t,x)\|_2 \leq \sum_{i=n}^{\infty} \sup_{(s,y)\in K_M} \|v_{i+1}(s,y)-v_i(s,y)\|_2,$$

which is the tail of a convergence series and proves that

$$\sup_{(t,x)\in K_M} \|u(t,x)-v_n(t,x)\|_2 \xrightarrow[n\to\infty]{} 0.$$

Then u is $L^2(\Omega)$-continuous over K_M for any $M>0$, which leads to say that u is $L^2(\Omega)$-continuous over $(0,\infty)\times\mathbb{R}$.

It remains to prove that u satisfies (1.6) for all $(t,x)\in(0,\infty)\times\mathbb{R}$. Passing to the limit as $n\to\infty$ in (1.21), for every $(t,x)\in(0,\infty)\times\mathbb{R}$, we get

$$u(t,x) = \int_{\mathbb{R}} g_t(x-y)\,\mu_0(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}} f(u(s,y))g_{t-s}(x-y)\, W(\mathrm{d}s,\mathrm{d}y)$$

in $L^2(\Omega)$. Indeed, for every $(t,x)\in(0,\infty)\times\mathbb{R}$,

$$\int_0^t \int_{\mathbb{R}} f(v_n(s,y))g_{t-s}(x-y)\, W(\mathrm{d}s,\mathrm{d}y) \xrightarrow[n\to\infty]{} \int_0^t \int_{\mathbb{R}} f(u(s,y))g_{t-s}(x-y)\, W(\mathrm{d}s,\mathrm{d}y)$$

in $L^2(\Omega)$, since, by the Ito isometry and the Lipschitz condition (1.9) on f, we can write

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^t\int_{\mathbb{R}}\big(f(u(s,y))-f(v_n(s,y))\big)g_{t-s}(x-y)\,W(\mathrm{d}s,\mathrm{d}y)\Big|^2\Big]\\
&=\int_0^t\int_{\mathbb{R}}\mathbb{E}(|f(u(s,y))-f(v_n(s,y))|^2)g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y\\
&\le L^2\int_0^t\int_{\mathbb{R}}\mathbb{E}(|u(s,y)-v_n(s,y)|^2)g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y \xrightarrow[n\to\infty]{} 0
\end{aligned}$$

by dominated convergence. In fact, for all $(s,y)\in(0,\infty)\times\mathbb{R}$,

$$\|u(s,y)-v_n(s,y)\|_2^2 \xrightarrow[n\to\infty]{} 0$$

by (1.27), and

$$\begin{aligned}
\mathbb{E}(|u(s,y)-v_n(s,y)|^2)g^2_{t-s}(x-y) &\le 2\Big[\mathbb{E}(|u(s,y)|^2)+\mathbb{E}(|v_n(s,y)|^2)\Big]g^2_{t-s}(x-y)\\
&\le 4C_2(s)(1+|I_0(s,y)|^2)g^2_{t-s}(x-y),
\end{aligned}$$

(see (1.24) and (1.28)) and $(s,y)\mapsto 4C_2(s)(1+|I_0(s,y)|^2)g^2_{t-s}(x-y)$ is in $L^1([0,t]\times\mathbb{R})$, since we have

$$\int_0^t\int_{\mathbb{R}}4C_2(s)(1+|I_0(s,y)|^2)g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y<\infty$$

by Corollary B.5. ☐

1.3.1. Continuity with respect to the initial datum. In Section 1.3 we proved the existence of a solution to the differential stochastic problem (FSHE-α). Clearly this solution depends on the initial datum μ_0. We now prove that this dependence is continuos in the sense of the weak convergence of measures.

We start by proving this Lemma, that gives an estimate for the difference in the L^2-norm of solutions with different initial data.

Lemma 1.9 (Distance between solutions)**.** Let $\mu^{(1)}$ and $\mu^{(2)}$ be Borel measures that satisfy condition (1.11). Let $u^{(1)}$ be the solution to (FSHE-α) with $\mu^{(1)}$ as initial condition and let $u^{(2)}$ be the solution to (FSHE-α) with $\mu^{(2)}$ as initial condition. Then

$$\mathbb{E}[|u^{(1)}(t,x)-u^{(2)}(t,x)|^2]\le C(t,\alpha)\,(I_0^{(1)}(t,x)-I_0^{(2)}(t,x))^2, \tag{1.29}$$

where

$$I_0^{(i)}(t,x)=\int_{\mathbb{R}}g_t(x-y)\,\mu^{(i)}(\mathrm{d}y)\qquad \text{for } i=1,2,$$

and $C>0$ is a real constant which depends only on time t and α and can be chosen as

$$C(t,\alpha)=1+\sum_{k\in\mathbb{N}}c^{k-1}\,L^k\,t^{\frac{k(\alpha-1)}{\alpha}}\,\frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^{k+1}}{\Gamma\Big(\frac{(k+1)(\alpha-1)}{\alpha}\Big)}.$$

PROOF. Since $u^{(1)}$ and $u^{(2)}$ are solutions to (FSHE-α) with initial data given by $\mu^{(1)}$ and $\mu^{(2)}$ respectively, then, thanks to the integral formulation (1.6), for every $(t,x) \in (0,\infty) \times \mathbb{R}$,for $i = 1,2$, we can write

$$u^{(i)}(t,x) = \int_{\mathbb{R}} g_t(x-y)\,\mu^{(i)}(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}} f(u^{(i)}(s,y))\,g_{t-s}(x-y)\,W(\mathrm{d}s,\mathrm{d}y).$$

Then, for every $(t,x) \in (0,\infty) \times \mathbb{R}$,

$$\begin{aligned} u^{(1)}(t,x) - u^{(2)}(t,x) &= \Big[\int_{\mathbb{R}} g_t(x-y)\,\mu^{(1)}(\mathrm{d}y) - \int_{\mathbb{R}} g_t(x-y)\,\mu^{(2)}(\mathrm{d}y)\Big] \\ &\quad + \int_0^t \int_{\mathbb{R}} \big(f(u^{(1)}(s,y)) - f(u^{(2)}(s,y))\big)\,g_{t-s}(x-y)\,W(\mathrm{d}s,\mathrm{d}y). \end{aligned}$$

For the second term, we can write

$$\begin{aligned} &\mathbb{E}\Big[\Big|\int_0^t \int_{\mathbb{R}} \big(f(u^{(1)}(s,y)) - f(u^{(2)}(s,y)\big)\,g_{t-s}(x-y)\,W(\mathrm{d}s,\mathrm{d}y)\Big|^2\Big] \\ &= \int_0^t \int_{\mathbb{R}} \mathbb{E}\Big[\big|f(u^{(1)}(s,y)) - f(u^{(2)}(s,y)\big|^2\Big]\,g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y \\ &\leq L\int_0^t \int_{\mathbb{R}} \mathbb{E}\Big[\big|u^{(1)}(s,y) - u^{(2)}(s,y)\big|^2\Big]\,g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y. \end{aligned}$$

Hence, we can write

$$\begin{aligned} &\mathbb{E}\Big[\big|u^{(1)}(t,x) - u^{(2)}(t,x)\big|^2\Big] \\ &\leq \Big(I_0^{(1)}(t,x) - I_0^{(2)}(t,x)\Big)^2 + L\int_0^t \int_{\mathbb{R}} \mathbb{E}\Big[\big|u^{(1)}(s,y) - u^{(2)}(s,y)\big|^2\Big]\,g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y. \end{aligned}$$

We can apply Lemma B.6: in this case $A = 1$, $B = L$, $\varphi_n(t,x) = \mathbb{E}[|u^{(1)}(t,x) - u^{(2)}(t,x)|^2]$ for every (t,x) and for every n (of couse it does not depend on n) and we can use as $I_0(t,x)$ the map

$$(t,x) \mapsto \int_{\mathbb{R}} g_t(x-y)(\mu - \nu)(\mathrm{d}y)$$

indeed, by triange inequality, it satisfies Proposition 1.5. Then,

$$\mathbb{E}\Big[\big|u^{(1)}(t,x) - u^{(2)}(t,x)\big|^2\Big] \leq \Big(I_0^{(1)}(t,x) - I_0^{(2)}(t,x)\Big)^2 \left(1 + \sum_{k\in\mathbb{N}} c^{k-1}\,L^k\,t^{\frac{k(\alpha-1)}{\alpha}}\,\frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^{k+1}}{\Gamma\Big(\frac{(k+1)(\alpha-1)}{\alpha}\Big)}\right),$$

that is (1.29). □

We recall that, given a sequence of measures $(\mu_n)_{n\in\mathbb{N}}$, we say that they converge weakly to the measure μ, and we write

$$\mu_n \xrightarrow[n\to\infty]{\text{weakly}} \mu$$

if, for all bounded and continuos functions $h : \mathbb{R} \to \mathbb{R}$, we have

$$\int_{\mathbb{R}} h(y)\, \mu_n(\mathrm{d}y) \longrightarrow_{n\to\infty} \int_{\mathbb{R}} h(y)\, \mu(\mathrm{d}y).$$

We have the following.

Proposition 1.10. Let $(\mu_n)_{n\in\mathbb{N}}$ and μ be Borel measures that satisfy (1.11). For every $n \in \mathbb{N}$, let $(u_n(t,x))_{(t,x)\in[0,\infty)\times\mathbb{R}}$ be the solution to (FSHE-α) with initial datum given by μ_n and let $(u(t,x))_{(t,x)\in[0,\infty)\times\mathbb{R}}$ be the solution to (FSHE-α) with μ as initial datum.

If $\mu_n \xrightarrow[n\to\infty]{\text{weakly}} \mu$, then

$$\lim_{n\to\infty} \mathbb{E}[|u(t,x) - u_n(t,x)|^2] = 0 \qquad \text{for every } (t,x) \in (0,\infty)\times\mathbb{R}. \tag{1.30}$$

Proof. Thanks to Lemma 1.9, for every $n \in \mathbb{N}$ and for every $(t,x) \in (0,\infty)\times\mathbb{R}$,

$$\mathbb{E}[|u(t,x) - u_n(t,x)|^2] \le C(t,\alpha) \Big[\int_{\mathbb{R}} g_t(x-y)\, \mu(\mathrm{d}y) - \int_{\mathbb{R}} g_t(x-y)\, \mu_n(\mathrm{d}y) \Big]^2.$$

But we know that μ_n converge weakly to μ, and then

$$\lim_{n\to\infty} \Big[\int_{\mathbb{R}} g_t(x-y)\, \mu(\mathrm{d}y) - \int_{\mathbb{R}} g_t(x-y)\, \mu_n(\mathrm{d}y) \Big] = 0,$$

since, for every fixed $t > 0$, the map $y \mapsto g_t(x-y)$ is bounded and continuous. This implies (1.30). $\square$

We would like to point out the the convergence in (1.30) is actually uniform for $(t,x) \in [\varepsilon,\infty)\times\mathbb{R}$.

1.4. Proof of the Regularity

Once we know a solution exists, we would like to know its regularity: in this section we will prove that, away from $t = 0$, the solution of (1.1) *has a continuous modification*. In the sequel, we will implicitly fix such a modification. Thanks to this fact, we can exchange "a.s." and "for all (t,x)" in the definition of mild solution. Namely, a mild solution of (1.1) is a continuous adapted stochastic process, $(u(t,x))_{(t,x)\in(0,\infty)\times\mathbb{R}}$, such that, $\mathbb{P}$-a.s.,

$$u(t,x) = \int_{\mathbb{R}} g_t(x-y)\, \mu_0(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}} f(u(s,y)) g_{t-s}(x-y)\, W(\mathrm{d}s,\mathrm{d}y)$$

for all $(t,x) \in (0,\infty)\times\mathbb{R}$. We have the following:

Theorem 1.11 (Regularity)**.** Suppose that f is a globally Lipschitz function and μ_0 is a Borel measure such that $g_t \star \mu_0(x) < \infty$ for all $(t,x) \in (0,\infty)\times\mathbb{R}$. Then, the solution of (1.1) has a locally $\Big((\frac{\alpha-1}{2\alpha})^-, (\frac{\alpha-1}{2})^-\Big)$-Hölder continuous modification in $(0,\infty)\times\mathbb{R}$.

The first step is the following result, which says that the solution $u(t,x)$ at time t of the SHE on the interval $[0,t]$, with initial value μ_0 at time 0, coincides with the solution at time t of the SHE on the interval $[\epsilon,t]$, with initial value $u(\epsilon,\cdot)$ at time ϵ.

Proposition 1.12. For all $0<\varepsilon<t<\infty$ and $x\in\mathbb{R}$,

$$u(t,x)=\int_{\mathbb{R}} g_{t-\varepsilon}(x-y)u(\varepsilon,y)\,\mathrm{d}y+\int_\varepsilon^t\int_{\mathbb{R}} f(u(s,y))g_{t-s}(x-y)\,W(\mathrm{d}s,\mathrm{d}y) \tag{1.31}$$

$\mathbb{P}$-a.s..

Proof. Fix $\varepsilon>0$ and $(t,x)\in(\varepsilon,\infty)\times\mathbb{R}$. We know that

$$u(\varepsilon,y)=\int_{\mathbb{R}} g_\varepsilon(y-z)\,\mu_0(\mathrm{d}z)+\int_0^\varepsilon\int_{\mathbb{R}} f(u(s,z))g_{\varepsilon-s}(y-z)\,W(\mathrm{d}s,\mathrm{d}z),$$

by Definition 1.1. So we can write

$$\begin{aligned}\int_{\mathbb{R}} g_{t-\varepsilon}(x-y)u(\varepsilon,y)\,\mathrm{d}y&=\int_{\mathbb{R}} g_{t-\varepsilon}(x-y)\Big[\int_{\mathbb{R}} g_\varepsilon(y-z)\,\mu_0(\mathrm{d}z)\Big]\,\mathrm{d}y+\\&+\int_{\mathbb{R}} g_{t-\varepsilon}(x-y)\Big[\int_0^\varepsilon\int_{\mathbb{R}} f(u(s,z))g_{\varepsilon-s}(y-z)\,W(\mathrm{d}s,\mathrm{d}z)\Big]\,\mathrm{d}y.\end{aligned} \tag{1.32}$$

Consider the first term: thanks to Fubini it can be rewritten as

$$\int_{\mathbb{R}}\Big[\int_{\mathbb{R}} g_{t-\varepsilon}(x-y)g_\varepsilon(y-z)\,\mathrm{d}y\Big]\,\mu_0(\mathrm{d}z)=\int_{\mathbb{R}} g_t(x-z)\,\mu_0(\mathrm{d}z),$$

by the semigroup property of the fractional heat kernel. Now, consider the second term in (1.32): we claim that also in this case we can switch the order of integration:

$$\begin{aligned}&\int_{\mathbb{R}} g_{t-\varepsilon}(x-y)\Big[\int_0^\varepsilon\int_{\mathbb{R}} f(u(s,z))g_{\varepsilon-s}(y-z)\,W(\mathrm{d}s,\mathrm{d}z)\Big]\,\mathrm{d}y\\&\overset{(\star)}{=}\int_0^\varepsilon\int_{\mathbb{R}} f(u(s,z))\Big[\int_{\mathbb{R}} g_{t-\varepsilon}(x-y)g_{\varepsilon-s}(y-z)\,\mathrm{d}y\Big]\,W(\mathrm{d}s,\mathrm{d}z)\\&=\int_0^\varepsilon\int_{\mathbb{R}} f(u(s,z))g_{t-s}(x-z)\,W(\mathrm{d}s,\mathrm{d}z).\end{aligned}$$

This implies that

$$\begin{aligned}&\int_{\mathbb{R}} g_{t-\varepsilon}(x-y)u(\varepsilon,y)\,\mathrm{d}y+\int_\varepsilon^t\int_{\mathbb{R}} f(u(s,y))g_{t-s}(x-y)\,W(\mathrm{d}s,\mathrm{d}y)\\&=\int_{\mathbb{R}} g_t(x-z)\,\mu_0(\mathrm{d}z)+\int_0^\varepsilon\int_{\mathbb{R}} f(u(s,z))g_{t-s}(x-z)\,W(\mathrm{d}s,\mathrm{d}z)\\&\qquad+\int_\varepsilon^t\int_{\mathbb{R}} f(u(s,y))g_{t-s}(x-y)\,W(\mathrm{d}s,\mathrm{d}y)\\&=\int_{\mathbb{R}} g_t(x-z)\,\mu_0(\mathrm{d}z)+\int_0^t\int_{\mathbb{R}} f(u(s,y))g_{t-s}(x-y)\,W(\mathrm{d}s,\mathrm{d}y)\\&=u(t,x)\end{aligned}$$

in $L^2(\Omega)$, and then (1.31) will be proved.

It remains to justify $(\star)$: we can switch the integrals thanks to the stochastic Fubini's theorem. We have to show that the hypothesis holds: it is clear that $\big(f(u(s,z))g_{\varepsilon-s}(y-z)g_{t-\varepsilon}(x-y)\big)_{(s,z,y)\in[0,\varepsilon)\times\mathbb{R}^2}$ is progressively measurable; we must prove that

$$\int_{\mathbb{R}}\Big[\int_0^\varepsilon\int_{\mathbb{R}}\mathbb{E}(|f(u(s,z))|^2)\,g^2_{\varepsilon-s}(y-z)\mathrm{d}s\,\mathrm{d}z\Big]^{\frac{1}{2}}g_{t-\varepsilon}(x-y)\mathrm{d}y<\infty. \tag{1.33}$$

We can write

$$\begin{aligned}
&\int_0^\varepsilon\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}z\,g^2_{\varepsilon-s}(y-z)\mathbb{E}(|f(u(s,z))|^2)\\
&\le 2K^2\int_0^\varepsilon\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}z\,g^2_{\varepsilon-s}(y-z)\Big(1+\mathbb{E}(|u(s,z)|^2)\Big)\\
&\le 2K^2\frac{\alpha}{\alpha-1}\,\varepsilon^{\frac{\alpha-1}{\alpha}}+2K^2\int_0^\varepsilon\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}z\,g^2_{\varepsilon-s}(y-z)\,C_2(s)\Big(1+2|I_0(s,z)|^2\Big)\\
&\le D_1(\varepsilon)+D_2(\varepsilon)|I_0(\varepsilon,y)|^2,
\end{aligned}$$

by Proposition B.5, where $D_1(\varepsilon)$ and $D_2(\varepsilon)$ are positive constant which depend only on ε (and K), and C_2 is defined by (1.19). Then, since $\sqrt{a+b}\le\sqrt{a}+\sqrt{b}$ for any $a,b\ge 0$,

$$\begin{aligned}
&\int_{\mathbb{R}}\Big[\int_0^\varepsilon\int_{\mathbb{R}}\mathbb{E}(|f(u(s,z))|^2)\,g^2_{\varepsilon-s}(y-z)\mathrm{d}s\,\mathrm{d}z\Big]^{\frac{1}{2}}g_{t-\varepsilon}(x-y)\mathrm{d}y\\
&\le\sqrt{D_1(\varepsilon)}+\sqrt{D_2(\varepsilon)}\int_{\mathbb{R}}I_0(\varepsilon,y)g_{t-\varepsilon}(x-y)\,\mathrm{d}y.
\end{aligned}$$

Hence, to prove (1.33), we just have to show that

$$\int_{\mathbb{R}}I_0(\varepsilon,y)\,g_{t-\varepsilon}(x-y)\mathrm{d}y<\infty. \tag{1.34}$$

This holds since

$$\begin{aligned}
\int_{\mathbb{R}}I_0(\varepsilon,y)\,g_{t-\varepsilon}(x-y)\mathrm{d}y&=\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}g_\varepsilon(y-z)\,\mu_0(\mathrm{d}z)\Big]\,g_{t-\varepsilon}(x-y)\mathrm{d}y\\
&=\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}g_\varepsilon(y-z)g_{t-\varepsilon}(x-y)\,\mathrm{d}y\Big]\,\mu_0(\mathrm{d}z)\\
&=\int_{\mathbb{R}}g_t(x-z)\,\mu_0(\mathrm{d}z)=I_0(t,x),
\end{aligned}$$

thanks to (classic) Fubini and the semigroup property of the fractional heat kernel. It is finite for every fixed (t,x) by (1.11), and then we have finished. □

Proof of Theorem 1.11. Fix $\varepsilon>0$; by Proposition 1.12, for all $(t,x)\in(\varepsilon,\infty)\times\mathbb{R}$, the mild solution of (1.1) can be written as

$$u(t,x)=A_\varepsilon(t,x)+B_\varepsilon(t,x),\quad\text{in }\mathbb{P}\text{-a.s.},$$

where

$$\begin{aligned}
A_\varepsilon(t,x)&=\int_{\mathbb{R}}g_{t-\varepsilon}(x-y)u(\varepsilon,y)\,\mathrm{d}y\\
B_\varepsilon(t,x)&=\int_\varepsilon^t\int_{\mathbb{R}}f(u(s,y))g_{t-s}(x-y)\,W(\mathrm{d}s,\mathrm{d}y).
\end{aligned}$$

First let us prove that A_ε is in $C^\infty((\varepsilon,\infty)\times\mathbb{R})$. We note that A_ε is a path-by-path integral, and, for every $\omega\in\Omega$, $A_\varepsilon(t,x)(\omega)$ is a deterministic space convolution between the fractional heat kernel and the map $y\mapsto u(\varepsilon,y)(\omega)$. We can write

$$A_\varepsilon(t,x)(\omega)=I_0(\mu_\varepsilon^+(\omega);t-\varepsilon,x)-I_0(\mu_\varepsilon^-(\omega);t-\varepsilon,x), \tag{1.35}$$

where $I_0(\nu;r,z)=\int_{\mathbb{R}} g_r(z-y)\,\nu(\mathrm{d}y)$, according to definition of I_0 (here we show the dependence on the measure for convenience), and we define

$$\mu_\varepsilon^\pm(\omega)(\mathrm{d}y)=\big[u(\varepsilon,y)(\omega)\big]^\pm\,\mathrm{d}y.$$

We claim that $\mu_\varepsilon^+(\omega),\mu_\varepsilon^-(\omega)$ are positive Borel measures on $\mathbb{R}$ which satisfy (1.11) for almost every ω. In fact, for every $(t,x)\in(0,\infty)\times\mathbb{R}$, $I_0(\mu_\varepsilon^\pm(\omega);t,x)<\infty$, since $\|u(\varepsilon,y)\|_2^2\le C_2(\varepsilon)\,(1+|I_0(\varepsilon,y)^2)$. Hence, since $I_0\in C^\infty((0,\infty)\times\mathbb{R})$, and recalling (1.35), one has that, for almost every ω, $A_\varepsilon(t,x)(\omega)$ is in $C^1((\varepsilon,\infty)\times\mathbb{R})$. In particular, it is locally Hölder of any exponent in $(0,1)$.

Now we have to show the continuity of the stochastic process given by the stochastic integral, $B_\varepsilon(t,x)=\int_\varepsilon^t\int_{\mathbb{R}} f(u(s,y))g_{t-s}(x-y)W(\mathrm{d}s,\mathrm{d}y)$ for $(t,x)\in(\varepsilon,\infty)\times[-M,M]$. The strategy of the proof is to use the (generalized) Kolmogorov continuity theorem (see Theorem B.17): let us fix $T>0$, $M>0$ and show that, for all even $p\ge 2$,

$$E\big(|B_\varepsilon(t,x)-B_\varepsilon(t',x')|^p\big)\le A\Big(|t-t'|^{p\frac{\alpha-1}{2\alpha}}+|x-x'|^{p\frac{\alpha-1}{2}}\Big), \tag{1.36}$$

for all $(t,x),(t',x')\in(\varepsilon,T]\times[-M,M]$, where $A=A(p,\varepsilon,T,M)\in\mathbb{R}_+$ is a constant which depends only on p,ε,T and M.

Suppose that we proved (1.36): we can apply Theorem B.17 with $\alpha_1=\frac{p(\alpha-1)}{2\alpha}$ and $\alpha_2=\frac{p(\alpha-1)}{2}$, where we can choose p arbitrarily large. This implies that $(B_\varepsilon(t,x))_{(t,x)\in(\varepsilon,T]\times[-M,M]}$ has a $\big((\frac{\alpha-1}{2\alpha})^-,(\frac{\alpha-1}{2})^-\big)$-Hölder continuous modification in $(\varepsilon,T]\times[-M,M]$, and the theorem will be proved.

First of all, we can write

$$\|B_\varepsilon(t,x)-B_\varepsilon(t',x')\|_p^2=\left\|\int_\varepsilon^t\int_{\mathbb{R}} f(u(s,y))\,(g_{t-s}(x-y)-g_{t'-s}(x'-y))W(\mathrm{d}s,\mathrm{d}y)\right\|_p^2,$$

with the convention that $g_r(z)=0$ if $r\le 0$. By applying Corollary B.20, for every p even, we get for $0<t'<t$

$$\begin{aligned}
\|B_\varepsilon(t,x)-B_\varepsilon(t',x')\|_p^2&\le\tilde c_p\int_\varepsilon^t\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\,\|f(u(s,y))\|_p^2\,(g_{t-s}(x-y)-g_{t'-s}(x'-y))^2\\
&\le K^2\tilde c_p\int_\varepsilon^t\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\,(1+\|u(s,y)\|_p^2)\,(g_{t-s}(x-y)-g_{t'-s}(x'-y))^2\\
\le K^2\tilde c_p(1+C_p(T))&\int_\varepsilon^t\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\,(g_{t-s}(x-y)-g_{t'-s}(x'-y))^2+\\
&+K^2\tilde c_p\,C_p(T)\int_\varepsilon^t\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\,I_0^2(s,y)(g_{t-s}(x-y)-g_{t'-s}(x'-y))^2\\
=:I+II,&
\end{aligned}$$

having used condition (1.10) of f and (1.13) to bound $\|u\|_p^2$. In order to bound I and II, we will use the following theorems, whose proofs use the fundamental result in Proposition 1.5 and are deferred in Section B.3 in Appendix B (see Theorems B.21 and B.22).

Theorem 1.13. For all $x, x' \in \mathbb{R}$ and t, t' with $0 < t' < t$, we have

$$\int_{t'}^{t} \int_{\mathbb{R}} g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y \le K_1\,|t-t'|^{\frac{\alpha-1}{\alpha}} \tag{1.37}$$

$$\int_{0}^{t} \int_{\mathbb{R}} \big(g_{t-s}(x-y) - g_{t-s}(x'-y)\big)^2\,\mathrm{d}s\,\mathrm{d}y \le K_2\,|x-x'|^{\alpha-1}, \tag{1.38}$$

$$\int_{0}^{t'} \int_{\mathbb{R}} \big(g_{t-s}(x-y) - g_{t'-s}(x-y)\big)^2\,\mathrm{d}s\,\mathrm{d}y \le K_3\,|t-t'|^{\frac{\alpha-1}{\alpha}} \tag{1.39}$$

Theorem 1.14. For all $x, x' \in \mathbb{R}$, with $x' < x$, and t, t' with $0 < t' < t$, we have

$$\int_{t'}^{t} \int_{\mathbb{R}} I_0^2(s,y)\,g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y \le \tilde{K}_1\,I_0^2(t,x)\,|t-t'|^{\frac{\alpha-1}{\alpha}} \tag{1.40}$$

$$\begin{aligned} &\int_{0}^{t} \int_{\mathbb{R}} I_0^2(s,y)\,\big(g_{t-s}(x-y) - g_{t-s}(x'-y)\big)^2\,\mathrm{d}s\,\mathrm{d}y \\ &\qquad\qquad \le \tilde{K}_2\left(1 + \frac{1}{\sqrt{t}}\,1_{\alpha=2}\right)\left(\max_{w\in[x',x]} I_0^2(t,w)\right)|x-x'|^{\alpha-1}, \end{aligned} \tag{1.41}$$

$$\begin{aligned} &\int_{0}^{t'} \int_{\mathbb{R}} I_0^2(s,y)\,\big(g_{t-s}(x-y) - g_{t'-s}(x-y)\big)^2\,\mathrm{d}s\,\mathrm{d}y \\ &\qquad\qquad \le \tilde{K}_3\left(1 + \frac{1}{\sqrt{t}}\,1_{\alpha=2}\right)\left(\max_{c\in[t',t]} I_0^2(c,x)\right)|t-t'|^{\frac{\alpha-1}{\alpha}} \end{aligned} \tag{1.42}$$

Thanks to Theorem 1.13, I can be bounded by:

$$K^2 \tilde{c}_p(1 + C_p(T))\,(K_1 + K_2 + K_3)\left[|t-t'|^{\frac{\alpha-1}{\alpha}} + |x-x'|^{\alpha-1}\right],$$

and, in the meanwhile, thanks to Theorem 1.14, II can be bounded by:

$$K^2 \tilde{c}_p\,C_p(T)\,(\tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3)\left(1 + \frac{1}{\sqrt{\varepsilon}}\right)\left(\max_{c\in[\varepsilon,T],w\in[-M,M]} I_0^2(c,w)\right)\left[|t-t'|^{\frac{\alpha-1}{\alpha}} + |x-x'|^{\alpha-1}\right]$$

Since I_0^2 is a continuous function, in the compact $[\varepsilon, T]\times[-M, M]$ its maximum is a real positive constant which depends only on ε, T, M. Hence we have just proved (1.36) and we are done.

We point out that from Theorems 1.13 and 1.14, that is from Proposition 1.5 which is widely used to prove them, we are able to write the Hölder constant of the solution. $\square$

Chapter 2

Normalized Solution and Strict Positivity

Introduction

In this chapter we prove the strict positivity of the solution to the linear fractional stochastic heat equation with measure-valued initial data, that is formally written as

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta^{\frac{\alpha}{2}}\, u(t,x) + \beta\, u(t,x)\, \dot{W}(t,x) \quad \text{for } t>0,\ x\in\mathbb{R}, \\ u(0,\cdot) = \mu_0, \end{cases} \tag{2.1}$$

(hence, with $f(u) = \beta\, u$ for some $\beta \in \mathbb{R}$; cfr. (1.1) in Chapter 1).

For the case $\alpha = 2$, that is the "classical" *stochastic heat equation*, there are many references in literature, starting by the well-known theorem proved by Mueller ([Mueller 91]) of a strong comparison principle, and then, due to the links between the SHE and the *KPZ models* (see [Quastel 11] for a review), the strict positivity of the solution was studied and proved in [Moreno 14] and [Gubinelli, Perkowski 17] with alternative techniques based on concentration of measure arguments and paracontrolled pathwise arguments, respectively. The problem of strict positivity, for a different choice of noise, was also studied in [Tessitore, Zabczyk 98] and [Wang 18].

In [Chen, Kim 14], the authors proved the strict positivity of the solution to the nonlinear *fractional* stochastic heat equation through a comparison principle, extending Mueller's comparison principle on the stochastic heat equation ([Mueller 91]) to allow more general initial data.

We are going to prove the strict positivity with an alternative proof, by showing the continuity of the *normalized solution* $\hat{u}$, defined as the ratio of the solution u of the stochastic fractional heat equation (1.1) with the same initial datum, which is $I_0(t,x) = \int_{\mathbb{R}} g_t(x-y)\,\mu_0(\mathrm{d}y)$ (see (1.7)). This result is interesting in itself and becomes the real aim of this chapter. Indeed, this yields information on the behavior of the solution to (2.1) when time goes to zero. We point out that in the recent preprint [Han, Kim 19], in a multi-dimensional setting, the authors obtained Hölder regularity and boundary behavior using a suitable notion of normalized solution, defined there as the ratio of the solution u and its distance from the boundary.

Our results also permit to prove the continuity of the *four-parameter* fundamental solution (see Section 2.4.2), that is a continuity also with respect to the initial time. This feature is essential to define a long-range version of the continuum directed polymer in random environments ([Alberts, Khanin, Quastel 14a], [Alberts, Khanin, Quastel 14b], [Caravenna, Sun, Zygouras 16], [Caravenna, Sun, Zygouras 17]).

Description of the Chapter.

- In Section 2.1, we define the *normalized solution* and we present our main results.

- In Sections 2.2 and 2.3, we prove the Hölder continuity of the normalized solution in the case $\alpha \in (1,2)$ and $\alpha = 2$, respectively. For the case $\alpha = 2$, we just prove the continuity of the *fundamental normalized solution*, which actually is enough to prove the strict positivity for the linear case.
- In Section 2.4, we define the *Stochastic Fundamental Solution*, which is the solution to the fractional stochastic heat equation with the delta measure as initial datum. We present a "four parameter" fundamental solution and we analyze its specific properties. In Section 2.4.3, we prove the continuity of the normalized fundamental solution.
- In the last Section 2.5 we prove the strict positivity of the fundamental solution, which implies the strict positivity of the solution of the fractional stochastic heat equation in the linear case.
- In Section 2.6, we deferred some technical proofs.

2.1. Normalized Solution and Main Results

Let $(u(t,x))_{(t,x)\in(0,\infty)\times\mathbb{R}}$ be the solution of

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta^{\frac{\alpha}{2}}\, u(t,x) + f(u(t,x))\,\dot{W}(t,x) \quad \text{for } t>0,\ x\in\mathbb{R}, \\ u(0,\cdot) = \mu_0, \end{cases} \tag{FSHE-α}$$

We know that u is well defined, unique and Hölder continuos and we can write

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}} g_t(x-y)\,\mu_0(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}} f(u(s,y)) g_{t-s}(x-y)\, W(\mathrm{d}s,\mathrm{d}y) \\ &= \mu_0 \star g(t,x) + \big(f(\varphi)\,\dot{W}\big) \star g(t,x), \end{aligned} \tag{2.2}$$

where we have used the notation (B.16) for the stochastic integral, while the first term denotes the classic convolution and is another way to write the solution $I_0(t,x)$ of the deterministic problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta^{\frac{\alpha}{2}}\, u(t,x) \quad \text{for } t>0,\ x\in\mathbb{R}, \\ u(0,\cdot) = \mu_0, \end{cases} \tag{2.3}$$

Roughly speaking, for times close to 0, we expect that $u(t,x)$ behaves like $I_0(t,x)$, and then $u(t,x)/I_0(t,x)$ should be close to 1. This is a motivation to introduce the following definition.

Definition 2.1. Let us define the *Normalized Solution* of (FSHE-α) as the process $(\hat{u}(t,x))_{(t,x)\in(0,\infty)\times\mathbb{R}}$ such that

$$\hat{u}(t,x) = \begin{cases} \dfrac{u(t,x)}{I_0(t,x)} & \text{if } t>0; \\ 1 & \text{if } t=0. \end{cases} \tag{2.4}$$

We know will prove the following theorem.

THEOREM 2.2. Let u be the solution of (1.1) where f is a globally Lipschitz function such that $f(0) = 0$ and let μ_0 satisfy (1.11), that is μ_0 is a Borel measure such that $I_0(t,x)$ is well defined for every $(t,x) \in (0,\infty) \times \mathbb{R}$. Then the process $(\hat{u}(t,x))_{(t,x)\in[0,\infty)\times\mathbb{R}}$, defined in (2.4) has a locally (γ,δ)-Hölder continuous modification, where

1. if $\alpha \in (1,2)$, we can choose any $\gamma < \frac{\alpha-1}{2\alpha}$ and any $\delta < \frac{\alpha-1}{2}$;
2. if $\alpha = 2$, we can choose any $\gamma < \frac{1}{6}$ and any $\delta < \frac{1}{2}$.

REMARK 2.3. In the case $\alpha \in (1,2)$, we can write that both u and $\hat{u}$ have a locally $\left(\left(\frac{\alpha-1}{2\alpha}\right)^-, \left(\frac{\alpha-1}{2}\right)^-\right)$-Hölder continuous modification. The case $\alpha = 2$ is worse, since the coefficient of Hölder continuity in time of the normalized solution $\hat{u}$ is lower that the one of the solution u. This is not an artifact of the proof: it can be shown that the exponent $(\frac{1}{6})^-$ in time is optimal for $\alpha = 2$.

The idea to prove Theorem 2.2 is to use (a generalized version of) the Kolmogorov continuity theorem (see Theorem B.17). The key step is the proof of the following result:

THEOREM 2.4. Fix $T, M > 0$. For all p large enough, for all $0 \le t' \le t < T$ and $x, x' \in [-M, M]$,

$$\|\hat{u}(t,x) - \hat{u}(t',x')\|_p^2 \le \begin{cases} C\left[|t-t'|^{\frac{\alpha-1}{\alpha}} + |x-x'|^{\alpha-1}\right] & \text{if } \alpha \in (1,2) \\ C\left[|t-t'|^{\frac{1}{3}} + |x-x'|\right] & \text{if } \alpha = 2 \end{cases} \tag{2.5}$$

where C is a positive constant which depends only on p, T, M.

In order to prove Theorem 2.4, which leads directly to Theorem 2.2 thanks to the Kolmogorov continuity theorem (see Theorem B.17), we need to prove the following Theorem, whose proof is deferred to Section 2.2 and Section 2.3, since we divide the cases $\alpha \in (1,2)$ and $\alpha = 2$. For $\alpha = 2$ we prove the following theorem only in the case $I_0(s,y) = g_s(y)$, which correspond to the case $\mu_0 = \delta_0$, that is the Dirac delta measure is the initial datum of the problem (FSHE-α).

THEOREM 2.5. For all $0 < t' < t < \infty$ and $x, x' \in \mathbb{R}$, we have

$$\int_{t'}^{t} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \frac{g_{t-s}^2(x-y)}{I_0^2(t,x)} \le C_1 |t-t'|^{\frac{\alpha-1}{\alpha}}$$

$$\int_0^{t'} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \left(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t'-s}(x-y)}{I_0(t',x)}\right)^2 \le C_2 \begin{cases} |t-t'|^{\frac{\alpha-1}{\alpha}} & \text{if } \alpha \in (1,2) \\ |t-t'|^{\frac{1}{3}} & \text{if } \alpha = 2 \end{cases};$$

$$\int_0^{t} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \left(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t-s}(x'-y)}{I_0(t,x')}\right)^2 \le C_3 |x-x'|^{\alpha-1}.$$

PROOF OF THEOREM 2.4. If $t = t' = 0$, then $\hat{u}(t,x) = \hat{u}(t',x')$ and we have nothing to prove.

If $t' = 0 < t$, then we can write

$$\begin{aligned}\hat{u}(t,x) - 1 &= \frac{1}{I_0(t,x)} \Big(u(t,x) - I_0(t,x) \Big) \\ &= \frac{1}{I_0(t,x)} \int_0^t \int_{\mathbb{R}} f(u(s,y))\, g_{t-s}(x-y)\, W(\mathrm{d}s, \mathrm{d}y).\end{aligned}$$

Then,

$$\begin{aligned}\|\hat{u}(t,x) - 1\|_p^2 =& \frac{1}{I_0^2(t,x)} \left\| \int_0^t \int_{\mathbb{R}} f(u(s,y))\, g_{t-s}(x-y)\, W(\mathrm{d}s, \mathrm{d}y) \right\|_p^2 \\ &\le C \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y)\, \frac{g_{t-s}^2(x-y)}{I_0^2(t,x)},\end{aligned}$$

as we have done to get (2.6). Thanks to the first relation of Theorem 2.5, we have

$$\|\hat{u}(t,x) - 1\|_p^2 \le C\, t^{\frac{\alpha-1}{\alpha}}$$

and we have done.
So, from now on, we consider the case in which $t, t' > 0$. We first get a general estimate: if $0 < t' < t < T$, $x, x' \in \mathbb{R}$ and p be is an integer, then, thanks to Theorem 1.3, we know that

$$\begin{aligned}&\|\hat{u}(t,x) - \hat{u}(t',x')\|_p^2 = \left\| \int_0^t \int_{\mathbb{R}} f(u(s,y)) \Big(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t'-s}(x'-y)}{I_0(t',x')} \Big)^2 W(\mathrm{d}s, \mathrm{d}y) \right\|_p^2 \\ &\le 2K^2\, C(T)\, \tilde{c}_p \int_0^t \int_{\mathbb{R}} \|f(u(s,y))\|_p^2 \Big(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t'-s}(x'-y)}{I_0(t',x')} \Big)^2 \mathrm{d}s\, \mathrm{d}y,\end{aligned}$$

having used also Corollary B.20.

Now, since we suppose that $f(0) = 0$, then we know that $\|f(u(s,y))\|_p^2 \le 2K^2\, C_p(s)\, |I_0(s,y)|^2$ and we can write

$$\|\hat{u}(t,x) - \hat{u}(t',x')\|_p^2 \le 2K^2\, C(T)\, \tilde{c}_p \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \Big(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t'-s}(x'-y)}{I_0(t',x')} \Big)^2. \tag{2.6}$$

We can write

$$\begin{aligned}\|\hat{u}(t,x) - \hat{u}(t',x')\|_p &= \|\hat{u}(t,x) - \hat{u}(t,x') + \hat{u}(t,x') - \hat{u}(t',x')\|_p \\ &\le \|\hat{u}(t,x) - \hat{u}(t',x')\|_p + \|\hat{u}(t,x') - \hat{u}(t',x')\|_p,\end{aligned}$$

so it is enough to prove (2.5) by considering the case $t \neq t'$ and $x = x'$ and the case $t = t'$ and $x \neq x'$.

CASE $x = x'$. We have

$$\begin{aligned}&\|\hat{u}(t,x) - \hat{u}(t',x)\|_p^2 \le C \int_0^t \int_{\mathbb{R}} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \Big(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t'-s}(x-y)}{I_0(t',x)} \Big)^2 \\ &= C \int_0^{t'} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \int_{\mathbb{R}} I_0^2(s,y) \Big(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t'-s}(x-y)}{I_0(t',x)} \Big)^2 + C \int_{t'}^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \frac{g_{t-s}^2(x-y)}{I_0^2(t,x)}\end{aligned}$$

thanks to (2.6) and the fact that we consider $g_r(z) := 0$ when $r \le 0$. By using the second relation of Theorem 2.5, we get

$$\|\hat{u}(t,x) - \hat{u}(t',x)\|_p^2 \le \begin{cases} C\, |t - t'|^{\frac{\alpha-1}{\alpha}} & \text{if } \alpha \in (1,2) \\ C\, |t - t'|^{\frac{1}{3}} & \text{if } \alpha = 2 \end{cases}$$

Case $t = t'$. Thanks to (2.6), we have

$$\begin{aligned}\|\hat{u}(t,x') - \hat{u}(t,x')\|_p &\le \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \Big(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t-s}(x'-y)}{I_0(t,x')}\Big)^2 \\ &\le C\,|x-x'|^{\alpha-1},\end{aligned}$$

having used the third relation of Theorem 2.5. □

2.2. Proof of the Hölder continuity for $\alpha \in (1,2)$

This is the fundamental theorem which permits to prove Theorem 2.2 when $\alpha \in (1,2)$.

Theorem 2.6. For all $0 < t' < t < \infty$ and $x, x' \in \mathbb{R}$, we have

$$\int_{t'}^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y)\, \frac{g_{t-s}^2(x-y)}{I_0^2(t,x)} \le C_1\, |t-t'|^{\frac{\alpha-1}{\alpha}} \tag{2.7}$$

$$\int_0^{t'} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \left(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t'-s}(x-y)}{I_0(t',x)}\right)^2 \le C_2\, |t-t'|^{\frac{\alpha-1}{\alpha}}; \tag{2.8}$$

$$\int_0^{t} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \left(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t-s}(x'-y)}{I_0(t,x')}\right)^2 \le C_3\, |x-x'|^{\alpha-1}. \tag{2.9}$$

We first prove a simple lemma that will be used in the proof of Theorem 2.2.

Lemma 2.7. For all $0 < s < t < \infty$ and $x, y \in \mathbb{R}$, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}c}\frac{g_{c-s}(x-y)}{I_0(c,x)}\right| \le \frac{2}{\alpha}\,\frac{1}{c-s}\,\frac{g_{c-s}(x-y)}{I_0(c,x)} \tag{2.10}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}w}\frac{g_{t-s}(w-y)}{I_0(t,w)}\right| \le \frac{2}{(t-s)^{\frac{1}{\alpha}}}\,\frac{g_{t-s}(w-y)}{I_0(t,w)} \tag{2.11}$$

Proof. First of all, since we can change the order of integral and derivative, relations (A.12) and (A.13) show that the following hold:

$$\left|\frac{\mathrm{d}}{\mathrm{d}c} I_0(c,z)\right| \le \frac{1}{\alpha}\,\frac{1}{c}\, I_0(c,z) \tag{2.12}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}w} I_0(t,w)\right| \le \frac{1}{t^{\frac{1}{\alpha}}}\, I_0(t,w) \tag{2.13}$$

Then we can write

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}c}\frac{g_{c-s}(x-y)}{I_0(c,x)}\right| &= \frac{1}{I_0^2(c,x)}\left|\frac{\mathrm{d}}{\mathrm{d}c} g_{c-s}(x-y)\, I_0(c,x) - g_{c-s}(x-y)\,\frac{\mathrm{d}}{\mathrm{d}c} I_0(c,x)\right| \\ &\le \frac{1}{I_0^2(c,x)}\left|\frac{1}{\alpha}\,\frac{1}{c-s}\, g_{c-s}(x-y)\, I_0(c,x) + g_{c-s}(x-y)\,\frac{1}{\alpha}\frac{1}{c} I_0(c,x)\right| \\ &\le \frac{2}{\alpha}\,\frac{1}{c-s}\,\frac{g_{c-s}(x-y)}{I_0(c,x)},\end{aligned}$$

having used (A.12) and (2.12). To prove (2.11), we proceed similarly and we write

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}w}\frac{g_{t-s}(w-y)}{I_0(t,w)}\right| &= \frac{1}{I_0^2(t,w)}\left|\frac{\mathrm{d}}{\mathrm{d}w}g_{t-s}(w-y)\,I_0(t,w) - g_{t-s}(w-y)\,\frac{\mathrm{d}}{\mathrm{d}w}I_0(t,w)\right| \\ &\le \frac{1}{I_0^2(t,w)}\left|\frac{1}{(t-s)^{\frac{1}{\alpha}}}\,g_{t-s}(w-y)\,I_0(t,w) + g_{t-s}(w-y)\,\frac{1}{t^{\frac{1}{\alpha}}}\,I_0(t,w)\right| \\ &\le \frac{2}{(t-s)^{\frac{1}{\alpha}}}\,\frac{g_{t-s}(w-y)}{I_0(t,w)}\end{aligned}$$

□

Now we can pass to prove Theorem 2.6.

PROOF OF THEOREM 2.6. The proof of (2.7) is straightforward: thanks to Proposition 1.5, we have

$$\begin{aligned}\int_{t'}^{t}\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\,I_0^2(s,y)\,\frac{g_{t-s}^2(x-y)}{I_0^2(t,x)} &\le c(\alpha)\int_{t'}^{t}\mathrm{d}s\,\frac{t^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(t-s)^{\frac{1}{\alpha}}} \\ &\le c(\alpha)\,(t-t')^{\frac{\alpha-1}{\alpha}}\,\mathrm{BETA}\Big(\frac{\alpha-1}{\alpha},\frac{\alpha-1}{\alpha}\Big) \\ &=: C_1\,|t-t'|^{\frac{\alpha-1}{\alpha}},\end{aligned}$$

having used also Lemma A.11.

Before proving (2.8) and (2.9), we can write

$$\begin{aligned}&\int_0^{t'}\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\,I_0^2(s,y)\left(\frac{g_{t-s}(x-y)}{I_0(t,x)}-\frac{g_{t'-s}(x'-y)}{I_0(t',x')}\right)^2 \\ &\le 2\int_0^{t'}\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\,I_0^2(s,y)\,\frac{g_{t-s}^2(x-y)}{I_0^2(t,x)}+2\int_0^{t'}\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\,I_0^2(s,y)\,\frac{g_{t'-s}^2(x'-y)}{I_0^2(t',x')} \\ &\le 2c(\alpha)\int_0^{t'}\mathrm{d}s\,\frac{t^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(t-s)^{\frac{1}{\alpha}}}+2c(\alpha)\int_0^{t'}\mathrm{d}s\,\frac{(t')^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(t'-s)^{\frac{1}{\alpha}}}\le 4c(\alpha)\int_0^{t'}\mathrm{d}s\,\frac{(t')^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(t'-s)^{\frac{1}{\alpha}}} \\ &\le 4c(\alpha)\,\mathrm{BETA}\Big(\frac{\alpha-1}{\alpha},\frac{\alpha-1}{\alpha}\Big)\left(t'\right)^{\frac{\alpha-1}{\alpha}},\end{aligned}$$

having used Proposition 1.5 and Lemma A.11. In particular, we have

$$\int_0^{t'}\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\,I_0^2(s,y)\left(\frac{g_{t-s}(x-y)}{I_0(t,x)}-\frac{g_{t'-s}(x-y)}{I_0(t',x)}\right)^2\le C\,|t-t'|^{\frac{\alpha-1}{\alpha}}\quad\text{if } t'\le 2|t-t'| \tag{2.14}$$

and

$$\int_0^{t}\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\,I_0^2(s,y)\left(\frac{g_{t-s}(x-y)}{I_0(t,x)}-\frac{g_{t-s}(x'-y)}{I_0(t,x')}\right)^2\le C\,|x-x'|^{\alpha-1}\quad\text{if } t\le 2|x-x'|^{\alpha} \tag{2.15}$$

Hence it is sufficient to prove (2.8) in the case $t'>2|t-t'|$ and we can prove (2.9) in the case $t>2|x-x'|^{\alpha}$; let us fix t',t,x,x' which satisfy these relations.

Now we show (2.8); we split the integral:

$$\begin{aligned}
\diamondsuit_{0,t'} &:= \int_0^{t'} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \left(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t'-s}(x-y)}{I_0(t',x)} \right)^2 \\
&= \int_0^{t'-|t-t'|} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \left(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t'-s}(x-y)}{I_0(t',x)} \right)^2 + \\
&\qquad + \int_{t'-|t-t'|}^{t'} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \left(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t'-s}(x-y)}{I_0(t',x)} \right)^2 \\
&=: \diamondsuit_{0,t'-|t-t'|} + \diamondsuit_{t'-|t-t'|,t'}.
\end{aligned}$$

Estimate of $\diamondsuit_{t'-|t-t'|,t'}$. For the second integral, we have

$$\begin{aligned}
&\diamondsuit_{t'-|t-t'|,t'} \\
&\le 2 \int_{t'-|t-t'|}^{t'} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \frac{g_{t-s}^2(x-y)}{I_0^2(t,x)} + 2 \int_{t'-|t-t'|}^{t'} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \frac{g_{t'-s}^2(x-y)}{I_0^2(t',x)} \\
&\le 2c(\alpha) \int_{t'-|t-t'|}^{t'} \mathrm{d}s \frac{t^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(t-s)^{\frac{1}{\alpha}}} + 2c(\alpha) \int_{t'-|t-t'|}^{t'} \mathrm{d}s \frac{(t')^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(t'-s)^{\frac{1}{\alpha}}} \\
&\le 4c(\alpha)\, \mathrm{Beta}\Big(\frac{\alpha-1}{\alpha}, \frac{\alpha-1}{\alpha}\Big) |t-t'|^{\frac{\alpha-1}{\alpha}},
\end{aligned}$$

as we have done to prove (2.7).

Estimate of $\diamondsuit_{0,t'-|t-t'|}$. To prove (2.8), we now have just to prove that

$$\diamondsuit_{0,t'-|t-t'|} \le C\, |t-t'|^{\frac{\alpha-1}{\alpha}}.$$

We write

$$\begin{aligned}
\diamondsuit_{0,t'-|t-t'|} &= \int_0^{t'-|t-t'|} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \left(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t'-s}(x-y)}{I_0(t',x)} \right)^2 \\
&= \int_0^{t'-|t-t'|} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \left(\int_{t'}^{t} \frac{\mathrm{d}}{\mathrm{d}c} \frac{g_{c-s}(x-y)}{I_0(c,x)}\, \mathrm{d}c \right)^2 \\
&\le (t-t') \int_0^{t'-|t-t'|} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \int_{t'}^{t} \left(\frac{\mathrm{d}}{\mathrm{d}c} \frac{g_{c-s}(x-y)}{I_0(c,x)} \right)^2 \mathrm{d}c,
\end{aligned}$$

having used Jensen's inequality. By using the estimate (2.10) for the derivative, we get

$$\begin{aligned}
\diamondsuit_{0,t'-|t-t'|} &\le \frac{4}{\alpha^2}(t-t') \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c \frac{1}{(c-s)^2} \int_{\mathbb{R}} \mathrm{d}y \frac{I_0^2(s,y) g_{c-s}^2(x-y)}{I_0^2(c,x)} \\
&\le \frac{4c(\alpha)}{\alpha^2}(t-t') \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c \frac{1}{(c-s)^2} \frac{c^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(c-s)^{\frac{1}{\alpha}}},
\end{aligned}$$

thanks to Proposition 1.5.

Since $t' > 2(t-t')$, we have $t' - (t-t') > t'/2$ and then we can split the integral over s:

$$
\begin{aligned}
&\diamondsuit_{0,t'-|t-t'|}\\
&\le \frac{4c(\alpha)}{\alpha^2}\,(t-t')\left[\int_0^{\frac{t'}{2}} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{1}{(c-s)^2}\, \frac{c^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(c-s)^{\frac{1}{\alpha}}} + \int_{\frac{t'}{2}}^{t'-(t-t')} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{1}{(c-s)^2}\, \frac{c^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(c-s)^{\frac{1}{\alpha}}}\right]\\
&\le \frac{4c(\alpha)}{\alpha^2}\,(t-t')\left[\frac{t^{\frac{1}{\alpha}}}{(t'/2)^{2+\frac{1}{\alpha}}} \int_0^{\frac{t'}{2}} \mathrm{d}s\, \frac{1}{s^{\frac{1}{\alpha}}} \int_{t'}^{t} \mathrm{d}c + \frac{t^{\frac{1}{\alpha}}}{(t'/2)^{\frac{1}{\alpha}}} \int_{\frac{t'}{2}}^{t'-(t-t')} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{1}{(c-s)^{2+\frac{1}{\alpha}}}\right]\\
&\le \frac{2^{2+\frac{2}{\alpha}} c(\alpha)}{\alpha(\alpha-1)}\,, \frac{t^{\frac{1}{\alpha}}}{(t')^{2+\frac{1}{\alpha}}}\,(t-t')^2\,(t')^{1-\frac{1}{\alpha}} + \frac{4c(\alpha)(2^{\frac{1}{\alpha}}+1)}{\alpha+1}\,(t-t')\,\frac{t^{\frac{1}{\alpha}}}{(t')^{\frac{1}{\alpha}}}\,(t-t')^{-\frac{1}{\alpha}};
\end{aligned}
$$

indeed,

$$
\begin{aligned}
\int_{\frac{t'}{2}}^{t'-(t-t')} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{1}{(c-s)^{2+\frac{1}{\alpha}}} &= \int_{\frac{t'}{2}}^{t'-(t-t')} \mathrm{d}s\, \frac{1}{(-1-\frac{1}{\alpha})}\left[(t-s)^{-1-\frac{1}{\alpha}} - (t'-s)^{-1-\frac{1}{\alpha}}\right]\\
&= \frac{\alpha}{(1+\frac{1}{\alpha})}\left[-(t-s)^{-\frac{1}{\alpha}} + (t'-s)^{-\frac{1}{\alpha}}\right]_{s=\frac{t'}{2}}^{s=t'-(t-t')}\\
&\le \frac{\alpha^2}{1+\alpha}\left[(t-\frac{t'}{2})^{-\frac{1}{\alpha}} + (t-t')^{-\frac{1}{\alpha}}\right] \le \frac{\alpha^2}{1+\alpha}\,(2^{-\frac{1}{\alpha}}+1)\,(t-t')^{-\frac{1}{\alpha}},
\end{aligned}
$$

since $t - t'/2 > 2(t-t')$. Hence

$$
\diamondsuit_{0,t'-|t-t'|} \le (\text{const.})\,\frac{t^{\frac{1}{\alpha}}}{(t')^{\frac{1}{\alpha}}}\left[\frac{(t-t')^2}{(t')^{1+\frac{1}{\alpha}}} + (t-t')^{1-\frac{1}{\alpha}}\right] \le (\text{const.})\,(t-t')^{\frac{\alpha-1}{\alpha}},
$$

reminding that $t' > t - t'$.

Now we prove the last relation, (2.9). First of all, we remind that we can consider only the case $t > 2\,|x-x'|^\alpha$, since the other case can be proved simply (see (2.15)). We write

$$
\begin{aligned}
\diamondsuit_{0,t} :&= \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y I_0(s,y)^2 \left(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t-s}(x'-y)}{I_0(t,x')}\right)^2\\
&= \int_0^{t-|x-x'|^\alpha} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y I_0(s,y)^2 \left(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t-s}(x'-y)}{I_0(t,x')}\right)^2 +\\
&\qquad + \int_{t-|x-x'|^\alpha}^{t} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y I_0(s,y)^2 \left(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t-s}(x'-y)}{I_0(t,x')}\right)^2\\
&=: \diamondsuit_{0,t-|x-x'|^\alpha} + \diamondsuit_{t-|x-x'|^\alpha,t}.
\end{aligned}
$$

Estimate of $\diamondsuit_{t-|x-x'|^\alpha,t}$. For $\diamondsuit_{t-|x-x'|^\alpha,t}$, we write

$$
\begin{aligned}
&\diamondsuit_{t-|x-y|^\alpha,t}\\
&\le 2\int_{t-|x-x'|^\alpha}^{t} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y)\, \frac{g_{t-s}^2(x-y)}{I_0^2(t,x)} + 2\int_{t-|x-y|^\alpha}^{t} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y)\, \frac{g_{t-s}^2(x'-y)}{I_0^2(t,x')}\\
&\le 4c(\alpha) \int_{t-|x-y|^\alpha}^{t} \mathrm{d}s \frac{t^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(t-s)^{\frac{1}{\alpha}}}\\
&\le 4c(\alpha)\,\mathrm{Beta}\Big(\frac{\alpha-1}{\alpha}, \frac{\alpha-1}{\alpha}\Big)\,\Big(t-(t-|x-y|^\alpha)\Big)^{\frac{\alpha-1}{\alpha}} = (\text{const.})\,|x-y|^{\alpha-1}.
\end{aligned}
$$

having used Proposition 1.5, and Lemma A.11.

Estimate of $\diamondsuit_{0,t-|x-x'|^\alpha}$**.** We write

$$\begin{aligned}
\diamondsuit_{0,t-|x-x'|^\alpha} &= \int_0^{t-|x-x'|^\alpha} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \left(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t-s}(x'-y)}{I_0(t,x')} \right)^2 \\
&= \int_0^{t-|x-x'|^\alpha} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \left(\int_{x'}^{x} \frac{\mathrm{d}}{\mathrm{d}w} \frac{g_{t-r}(w-y)}{I_0(t,w)} \mathrm{d}w \right)^2 \\
&\le |x-x'| \int_0^{t-|x-x'|^\alpha} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \int_x^y \mathrm{d}w \left(\frac{\mathrm{d}}{\mathrm{d}w} \frac{g_{t-r}(w-y)}{I_0(t,w)} \right)^2,
\end{aligned}$$

having used Jensen's inequality. Then, by using (2.11), which gives an estimate on the derivative with respect to w, we can write

$$\begin{aligned}
\diamondsuit_{0,t-|x-x'|^\alpha} &\le 4\,|x-x'| \int_0^{t-|x-x'|^\alpha} \mathrm{d}s \frac{1}{(t-s)^{\frac{2}{\alpha}}} \int_{\mathbb{R}} \mathrm{d}y \int_{x'}^{x} \mathrm{d}w \frac{I_0^2(s,y) g_{t-s}^2(w-y)}{I_0^2(t,w)} \\
&= 4\,|x-x'| \int_0^{t-|x-x'|^\alpha} \mathrm{d}s \frac{1}{(t-s)^{\frac{2}{\alpha}}} \int_{x'}^{x} \mathrm{d}w \int_{\mathbb{R}} \mathrm{d}y \frac{I_0^2(s,y) g_{t-s}^2(w-y)}{I_0^2(t,w)}.
\end{aligned}$$

By using Proposition 1.5, we write

$$\diamondsuit_{0,t-|x-x'|^\alpha} \le 4c(\alpha)\,|x-x'|^2 \int_0^{t-|x-x'|^\alpha} \mathrm{d}s \frac{t^{\frac{1}{\alpha}}}{(t-s)^{\frac{3}{\alpha}} s^{\frac{1}{\alpha}}}.$$

Since we are in the case $t > 2|x-x'|^\alpha$, then $\frac{t}{2} < t - |x-x'|^\alpha$, and we divide the integral by convenience:

$$\begin{aligned}
\diamondsuit_{0,t-|x-x'|^\alpha} &\le 4c(\alpha)\,|x-x'|^2 \left[\int_0^{\frac{t}{2}} \mathrm{d}s \frac{t^{\frac{1}{\alpha}}}{(t-s)^{\frac{3}{\alpha}} s^{\frac{1}{\alpha}}} + \int_{\frac{t}{2}}^{t-|x-x'|^\alpha} \mathrm{d}s \frac{t^{\frac{1}{\alpha}}}{(t-s)^{\frac{3}{\alpha}} s^{\frac{1}{\alpha}}} \right] \\
&\le 4c(\alpha)\,|x-x'|^2 \left[\frac{t^{\frac{1}{\alpha}}}{(\frac{t}{2})^{\frac{3}{\alpha}}} \int_0^{\frac{t}{2}} \mathrm{d}s \frac{1}{r^{\frac{1}{\alpha}}} + \frac{t^{\frac{1}{\alpha}}}{(\frac{t}{2})^{\frac{1}{\alpha}}} \int_{\frac{t}{2}}^{t-|x-x'|^\alpha} \mathrm{d}s \frac{1}{(t-s)^{\frac{3}{\alpha}}} \right] \\
&\le 4c(\alpha)\,|x-x'|^2 \left[\frac{2^{\frac{3}{\alpha}}\alpha}{\alpha-1} t^{\frac{\alpha-3}{\alpha}} + \frac{2^{\frac{1}{\alpha}}\alpha}{3-\alpha} |x-x'|^{\alpha-3} \right] \\
&\le 2^3 c(\alpha) \frac{\alpha}{\alpha-1}\,|x-x'|^{\alpha-1},
\end{aligned}$$

and we have finished. □

2.3. Proof of the Hölder continuity for $\alpha = 2$

In the case $\alpha = 2$, the proof of the Hölder continuity is similar to the proof of the case $\alpha \in (1,2)$, and can be done in the same way exept to

$$\diamondsuit_{0,t-|t-t'|} = \int_0^{t'-|t-t'|} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0(s,y)^2 \left(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t'-s}(x-y)}{I_0(t',x)} \right)^2$$

$$\diamondsuit_{0,t-|x-x'|^2} = \int_0^{t-|x-x'|^2} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0(s,y)^2 \left(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t-s}(x'-y)}{I_0(t,x')} \right)^2$$

where we can assume that $t' > 2\,|t-t'|$ and $t > 2\,|x-x'|$, otherwise we can apply (2.14) and (2.15) and get the expected results of Theorem 2.5.

The problem is in these integrals, $\diamondsuit_{0,t-|t-t'|}$ and $\diamondsuit_{0,t-|x-x'|^2}$, since here we use an approximation with derivatives, which have different behavior from the case $\alpha \in (1,2)$, as can be seen by comparing Lemma 2.7 above and Lemma 2.9 below. Indeed, in the case $\alpha = 2$, we have an exponential function whose derivatives have polynomial factors.

We recall that here we report the proof only in the case in which the initial measure is the Dirac delta measure, and then $I_0(s,y) = g_s(y)$.

In order to prove Theorem 2.5 for $\alpha = 2$, we need to show the following:

Theorem 2.8. Let us fix $0 < t' < t < \infty$ and $x, x' \in \mathbb{R}$ with $t' > 2\,|t-t'|$ and $t > 2\,|x-x'|$. Then we have

$$\diamondsuit_{0,t-|t-t'|} := \int_0^{t'-|t-t'|} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, g_s^2(y) \left(\frac{g_{t-s}(x-y)}{g_t(x)} - \frac{g_{t'-s}(x-y)}{g_{t'}(x)}\right)^2 \le C\,|t-t'|^{\frac{1}{3}} \quad (2.16)$$

$$\diamondsuit_{0,t-|x-x'|^2} := \int_0^{t-|x-x'|^2} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, g_s^2(y) \left(\frac{g_{t-s}(x-y)}{g_t(x)} - \frac{g_{t-s}(x'-y)}{g_t(x')}\right)^2 \le C\,|x-x'|. \quad (2.17)$$

We first need the following lemma (cfr. Lemma 2.7).

Lemma 2.9. For all $0 < s < t < \infty$ and $x, y \in \mathbb{R}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}c} \frac{g_{c-s}(x-y)}{g_c(x)} = \frac{g_{c-s}(x-y)}{g_c(x)} \frac{1}{2}\left[\frac{(x-y)^2}{2(c-s)^2} - \frac{x^2}{2c^2} - \frac{s}{c(c-s)}\right] \quad (2.18)$$

$$= \frac{g_{c-s}(x-y)}{g_c(x)} \frac{1}{4}\left[\frac{(y-\frac{s}{c}x)^2}{(c-s)^2} - \frac{2x(y-\frac{s}{c}x)}{c(c-s)} - \frac{2s}{c(c-s)}\right] \quad (2.19)$$

$$\frac{\mathrm{d}}{\mathrm{d}w} \frac{g_{t-s}(w-y)}{g_t(w)} = \frac{1}{2(t-s)} \frac{g_{t-s}(w-y)}{g_t(w)} \left(y - \frac{s}{t}w\right). \quad (2.20)$$

Proof. The proof just follows from Lemma A.8 in Appendix A.

Thanks to (A.14), we write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}c} \frac{g_{c-s}(x-y)}{g_c(x)} &= \frac{g_{c-s}(x-y)}{g_c(x)} \frac{1}{2}\left[\frac{1}{c-s}\left(\frac{(x-y)^2}{2(c-s)} - 1\right) - \frac{1}{c}\left(\frac{x^2}{2c} - 1\right)\right] \\ &= \frac{g_{c-s}(x-y)}{g_c(x)} \frac{1}{2}\left[\frac{(x-y)^2}{2(c-s)^2} - \frac{x^2}{2c^2} - \frac{s}{c(c-s)}\right],\end{aligned}$$

that is (2.18). Moreover, by computation, one get

$$\frac{(x-y)^2}{(c-s)^2} - \frac{x^2}{c^2} = \frac{(y-\frac{s}{c}x)^2}{(c-s)^2} - \frac{2x(y-\frac{s}{c}x)}{c(c-s)},$$

and then relation (2.19) follows.

Thanks to (A.15), we can write

$$\frac{\mathrm{d}}{\mathrm{d}w}\frac{g_{t-s}(w-y)}{g_t(w)} = -\frac{(w-y)}{2(t-s)}\frac{g_{t-s}(w-y)}{g_t(w)} + \frac{w}{2t}\frac{g_{t-s}(w-y)}{g_t(w)} = \frac{g_{t-s}(w-y)}{g_t(w)}\left[\frac{-wt+yt+wt-ws}{2t(t-s)}\right]$$
$$= \frac{1}{2(t-s)}\frac{g_{t-s}(w-y)}{g_t(w)}\left(y-\frac{s}{t}w\right),$$

that is (2.20). □

Now we can prove Theorem 2.8.

Proof of Theorem 2.8. We divide the proof of the two estimates (2.16) and (2.17).

Estimate (2.16) for $\diamondsuit_{0,t-|t-t'|}$. We can always write

$$\begin{aligned}\diamondsuit_{0,t-|t-t'|} &= \int_0^{t-|t-t'|}\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\,g_s^2(y)\left(\frac{g_{t-s}(x-y)}{g_t(x)}-\frac{g_{t'-s}(x-y)}{g_{t'}(x)}\right)^2\\
&= \int_0^{t-|t-t'|}\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\,g_s^2(y)\left(\int_{t'}^{t}\mathrm{d}c\,\frac{\mathrm{d}}{\mathrm{d}c}\frac{g_{c-s}(x-y)}{g_c(x)}\right)^2\\
&\le |t-t'|\int_0^{t-|t-t'|}\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\,g_s^2(y)\int_{t'}^{t}\mathrm{d}c\left(\frac{\mathrm{d}}{\mathrm{d}c}\frac{g_{c-s}(x-y)}{g_c(x)}\right)^2,\end{aligned}$$

having used Jensen inequality.

By using (2.19) for the derivative, we get

$$\begin{aligned}\diamondsuit_{0,t-|t-t'|} &\le \frac{1}{4}|t-t'|\int_0^{t-|t-t'|}\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\,g_s^2(y)\int_{t'}^{t}\mathrm{d}c\,\frac{g_{c-s}^2(x-y)}{g_c^2(x)}\left(\frac{(y-\frac{s}{c}x)^2}{(c-s)^2}-\frac{2x(y-\frac{s}{c}x)}{c(c-s)}-\frac{2s}{c(c-s)}\right)^2\\
&\le \diamondsuit^{(1)}_{0,t'-|t-t'|}+\diamondsuit^{(2)}_{0,t'-|t-t'|}+\diamondsuit^{(3)}_{0,t'-|t-t'|},\end{aligned}$$

where

$$\begin{aligned}\diamondsuit^{(1)}_{0,t'-|t-t'|} &= 3|t-t'|\int_0^{t'-|t-t'|}\mathrm{d}s\int_{t'}^{t}\mathrm{d}c\int_{\mathbb{R}}\mathrm{d}y\frac{g_s^2(y)g_{c-s}^2(x-y)}{g_c^2(x)}\frac{(y-\frac{s}{c}x)^4}{(c-s)^4}\\
\diamondsuit^{(2)}_{0,t'-|t-t'|} &= 3|t-t'|\int_0^{t'-|t-t'|}\mathrm{d}s\int_{t'}^{t}\mathrm{d}c\int_{\mathbb{R}}\mathrm{d}y\frac{g_s^2(y)g_{c-s}^2(x-y)}{g_c^2(x)}\frac{4x^2(y-\frac{s}{c}x)^2}{c^2(c-s)^2}\\
\diamondsuit^{(3)}_{0,t'-|t-t'|} &= 3|t-t'|\int_0^{t'-|t-t'|}\mathrm{d}s\int_{t'}^{t}\mathrm{d}c\int_{\mathbb{R}}\mathrm{d}y\frac{g_s^2(y)g_{c-s}^2(x-y)}{g_c^2(x)}\frac{4s^2}{c^2(c-s)^2},\end{aligned}$$

having used the fact that $(a+b+c)^2 \le 3a^2+3b^2+3c^2$. Thanks to (A.7) and Lemma A.9, we recall that

$$\begin{aligned}\frac{g_s^2(y)g_{c-s}^2(x-y)}{g_c^2(x)} &= \frac{1}{\sqrt{8\pi}}\sqrt{\frac{c}{s(c-s)}}\,\frac{g_{\frac{s}{2}}(y)g_{\frac{c-s}{2}}(x-y)}{g_{\frac{c}{2}}(x)}\\
&= \frac{1}{\sqrt{8\pi}}\sqrt{\frac{c}{s(c-s)}}\,g_{\frac{s}{2c}(c-s)}\left(y-\frac{s}{c}x\right).\end{aligned}$$

Hence, with the change of variable

$$z := \frac{y - \frac{s}{c}x}{\sqrt{\frac{s}{2c}(c-s)}},$$

we can write

$$\begin{aligned}
\diamondsuit^{(1)}_{0,t'-|t-t'|} &= \frac{3}{\sqrt{8\pi}}\,|t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c \sqrt{\frac{c}{s(c-s)}} \int_{\mathbb{R}} \mathrm{d}z\, g(z)\, \frac{z^4\, s^2}{4c^2(c-s)^2}\\
\diamondsuit^{(2)}_{0,t'-|t-t'|} &= \frac{3}{\sqrt{8\pi}}\,|t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c \sqrt{\frac{c}{s(c-s)}} \int_{\mathbb{R}} \mathrm{d}z\, g(z)\, \frac{2\,x^2\, z^2\, s}{c^3\,(c-s)}\\
\diamondsuit^{(3)}_{0,t'-|t-t'|} &= \frac{3}{\sqrt{8\pi}}\,|t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c \sqrt{\frac{c}{s(c-s)}} \int_{\mathbb{R}} \mathrm{d}z\, g(z)\, \frac{s^2}{c^2(c-s)^2}
\end{aligned}$$

We now proceed by estimating the three integrals.

Estimate for $\diamondsuit^{(1)}_{0,t'-|t-t'|}$. We write

$$\diamondsuit^{(1)}_{0,t'-|t-t'|} \le \frac{3}{4\sqrt{8\pi}}\,|t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c \sqrt{\frac{c}{s(c-s)}}\, \frac{s^2}{c^2(c-s)^2}\, C(4),$$

where

$$C(4) := \int_{\mathbb{R}} \mathrm{d}z\, g(z)\, z^4 < \infty, \tag{2.21}$$

since we recall that $g(z) = \frac{1}{\sqrt{4\pi}}\, e^{-\frac{z^2}{4}}$ has an exponential decay. Then

$$\begin{aligned}
\diamondsuit^{(1)}_{0,t'-|t-t'|} &\le \frac{3}{4\sqrt{8\pi}}\, C(4)\,|t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{s^{\frac{3}{2}}}{c^{\frac{3}{2}}\,(c-s)^{\frac{5}{2}}}\\
&\le \frac{3}{4\sqrt{8\pi}}\, C(4)\,|t-t'|^2 \int_0^{t'-|t-t'|} \mathrm{d}s\, \frac{1}{(t'-s)^{\frac{5}{2}}}\\
&= \frac{1}{2\sqrt{8\pi}}\, C(4)\,|t-t'|^2 \Big[|t-t'|^{-\frac{3}{2}} - (t')^{-\frac{3}{2}}\Big] \le \frac{1}{2\sqrt{8\pi}}\, C(4)\,|t-t'|^{\frac{1}{2}},
\end{aligned} \tag{2.22}$$

since $s < c$ and $(c-s) > (t'-s)$ for any $s \in (0, t'-|t-t'|)$ and $c \in (t', t)$ and we recall that $t' > 2\,(t-t')$.

Estimate for $\diamondsuit^{(2)}_{0,t'-|t-t'|}$. We have

$$\diamondsuit^{(2)}_{0,t'-|t-t'|} = \frac{3}{\sqrt{8\pi}}\, C(2)\, x^2\,|t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c \sqrt{\frac{c}{s(c-s)}}\, \frac{2s}{c^3\,(c-s)}$$

where

$$C(2) := \int_{\mathbb{R}} \mathrm{d}z\, g(z)\, z^2 < \infty.$$

We recall that, since (2.14) holds in general, then, if $t' \le 2(t-t')^{\frac{2}{3}}$, then (2.16) follows. Hence, we can suppose $t' > 2\,(t-t')^{\frac{2}{3}}$, and then $t' > 8\frac{(t.t')^2}{(t')^2}$. We will find an estimate for the integral over $[0,t']$ with a different split: we will prove that

$$\diamondsuit^{(2)}_{0,t'-\frac{|t-t'|^2}{(t')^2}} := \frac{3}{\sqrt{8\pi}}\,C(2)\,x^2\,|t-t'| \int_0^{t'-\frac{|t-t'|^2}{(t')^2}} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c \sqrt{\frac{c}{s(c-s)}}\,\frac{2s}{c^3\,(c-s)} \le C\,|t-t'|^{\frac{1}{3}} \tag{2.23}$$

and

$$\diamondsuit_{t'-\frac{|t-t'|^2}{(t')^2},t'} := \int_{t'-\frac{|t-t'|^2}{(t')^2}}^{t'} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, g_s^2(y) \left(\frac{g_{t-s}(x-y)}{g_t(x)} - \frac{g_{t'-s}(x-y)}{g_{t'}(x)}\right)^2 \le C\,|t-t'|^{\frac{1}{3}}, \tag{2.24}$$

which combined together leads to the proof of (2.16) and then to Theorem 2.5 for $\alpha = 2$.

For the second integral, we can use the same argument used for the proof of the estimate for $\diamondsuit_{t'-|t-t'|,t'}$ in Theorem 2.6: we can write

$$\begin{aligned}\diamondsuit_{t'-\frac{|t-t'|^2}{(t')^2},t'} &:= \int_{t'-\frac{|t-t'|^2}{(t')^2}}^{t'} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, g_s^2(y) \left(\frac{g_{t-s}(x-y)}{g_t(x)} - \frac{g_{t'-s}(x-y)}{g_{t'}(x)}\right)^2 \\ &\le 4\,c(2)\,\mathrm{B{\small ETA}}\Big(\frac{1}{2},\frac{1}{2}\Big)\,\frac{|t-t'|}{t'} \le 4\pi\,c(2)\,|t-t'|^{\frac{1}{3}},\end{aligned}$$

by using triangle inequality, Proposition 1.5 and Lemma A.11, and recalling that we assume $t' > 2|t-t'|^{\frac{2}{3}}$.

Now it remains to prove (2.23). We can write

$$\diamondsuit^{(2)}_{0,t'-\frac{|t-t'|^2}{(t')^2}} \le \frac{3}{\sqrt{2\pi}}\,C(2)\,x^2\,|t-t'| \int_0^{t'-\frac{|t-t'|^2}{(t')^2}} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{s^{\frac{1}{2}}}{c^{\frac{5}{2}}\,(c-s)^{\frac{3}{2}}}.$$

Now, since $s < t'$ (and then $s^{\frac{1}{2}} < (t')^{\frac{1}{2}}$) and $c > t'$ (and then $c^{\frac{5}{2}}(c-s)^{\frac{3}{2}} > (t')^{\frac{5}{2}}(t'-s)^{\frac{3}{2}}$), we can write

$$\begin{aligned}\diamondsuit_{0,t'-\frac{|t-t'|^2}{(t')^2}} &\le \frac{3}{\sqrt{2\pi}}\,C(2)\,x^2\,\frac{|t-t'|^2}{(t')^2} \int_0^{t'-\frac{|t-t'|^2}{(t')^2}} \mathrm{d}s\, \frac{1}{(t'-s)^{\frac{3}{2}}} \\ &\le \frac{3}{\sqrt{2\pi}}\,C(2)\,x^2\,\frac{|t-t'|}{(t')} \le \frac{3}{\sqrt{2\pi}}\,C(2)\,x^2\,|t-t'|^{\frac{1}{3}},\end{aligned}$$

since $t' > 2|t-t'|^{\frac{2}{3}}$, and we have done also for this part.

Estimate for $\diamondsuit^{(3)}_{0,t'-|t-t'|}$. We have

$$\begin{aligned}\diamondsuit^{(3)}_{0,t'-|t-t'|} &= \frac{12}{\sqrt{8\pi}}\,|t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c \sqrt{\frac{c}{s(c-s)}}\,\frac{s^2}{c^2(c-s)^2} \\ &= \frac{12}{\sqrt{8\pi}}\,|t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{s^{\frac{3}{2}}}{c^{\frac{3}{2}}\,(c-s)^{\frac{5}{2}}} \le \frac{4}{\sqrt{2\pi}}\,|t-t'|^{\frac{1}{2}},\end{aligned}$$

as we have done for $\diamondsuit^{(1)}_{0,t'-|t-t'|}$ (see (2.22)).

Estimate (2.17) **for** $\diamondsuit_{0,t-|x-x'|^2}$**.** We can always write

$$\begin{aligned}\diamondsuit_{0,t-|x-x'|^2} &= \int_0^{t-|x-x'|^2} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \left(\frac{g_{t-s}(x-y)}{I_0(t,x)} - \frac{g_{t-s}(x'-y)}{I_0(t,x')} \right)^2 \\ &= \int_0^{t-|x-x'|^2} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \left(\int_{x'}^{x} \mathrm{d}w\, \frac{\mathrm{d}}{\mathrm{d}w} \frac{g_{t-s}(w-y)}{I_0(t,w)} \right)^2 \\ &\le |x-x'| \int_0^{t-|x-x'|^2} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \int_{x'}^{x} \mathrm{d}w \left(\frac{\mathrm{d}}{\mathrm{d}w} \frac{g_{t-s}(w-y)}{I_0(t,w)} \right)^2,\end{aligned}$$

having used Jensen inequality.

Recalling that

$$g_t^2(x) = \frac{1}{\sqrt{8\pi t}}\, g_{\frac{t}{2}}(x),$$

(see (A.7)), and by changing the order of integration, we have

$$\begin{aligned}&|x-x'| \int_0^{t-|x-x'|^2} \mathrm{d}s \int_{x'}^{x} \mathrm{d}w \int_{\mathbb{R}} \mathrm{d}y\, g_s^2(y) \left(\frac{\mathrm{d}}{\mathrm{d}w} \frac{g_{t-s}(w-y)}{g_t(w)} \right)^2 \\ &= \frac{1}{4} |x-x'| \int_0^{t-|x-x'|^2} \mathrm{d}s\, \frac{1}{(t-s)^2} \int_{x'}^{x} \mathrm{d}w \int_{\mathbb{R}} \mathrm{d}y\, g_s^2(y) \frac{g_{t-s}^2(w-y)}{g_t^2(w)} \left(y - \frac{s}{t} w \right)^2,\end{aligned}$$

having used relation (2.20). Now, from (A.7), we can write

$$\begin{aligned}g_s^2(y) \frac{g_{t-s}^2(w-y)}{g_t^2(w)} &= \frac{1}{\sqrt{8\pi}} \sqrt{\frac{t}{s(t-s)}}\, \frac{g_{\frac{s}{2}}(y) g_{\frac{t-s}{2}}(w-y)}{g_{\frac{t}{2}}(w)} \\ &= \frac{1}{\sqrt{8\pi}} \sqrt{\frac{t}{s(t-s)}}\, g_{\frac{s}{2t}(t-s)}\left(y - \frac{s}{t} w \right),\end{aligned}$$

having used also Lemma A.9. We recall the fact that the following quantity

$$\int_{\mathbb{R}} \mathrm{d}y\, g_{\frac{s}{2t}(t-s)}\left(y - \frac{s}{t} w \right) \left(y - \frac{s}{t} w \right)^2$$

represent the second moment of a Gaussian random variable with mean zero and standard variation given by $\frac{s}{2t}(t-s)$.

Then, we write

$$
\begin{aligned}
&\diamondsuit_{0,t-|x-x'|^2}\\
&\le \frac{1}{8\sqrt{2\pi}}\,|x-x'|\int_0^{t-|x-x'|^2}\mathrm{d}s\,\frac{1}{(t-s)^2}\sqrt{\frac{t}{s(t-s)}}\int_{x'}^{x}\mathrm{d}w\int_{\mathbb{R}}\mathrm{d}y\,g_{\frac{s}{2t}(t-s)}\Big(y-\frac{s}{t}w\Big)\Big(y-\frac{s}{t}w\Big)^2\\
&=\frac{1}{8\sqrt{2\pi}}\,|x-x'|\int_0^{t-|x-x'|^2}\mathrm{d}s\,\frac{1}{(t-s)^2}\sqrt{\frac{t}{s(t-s)}}\int_{x'}^{x}\mathrm{d}w\,\mathbb{E}\Big[\mathcal{N}(0,\frac{s}{2t}(t-s))^2\Big]\\
&=\frac{1}{16\sqrt{2\pi}}\,|x-x'|^2\int_0^{t-|x-x'|^2}\mathrm{d}s\,\frac{1}{(t-s)^2}\sqrt{\frac{t}{s(t-s)}\frac{s}{t}(t-s)}\\
&=\frac{1}{16\sqrt{2\pi}}\,|x-x'|^2\int_0^{t-|x-x'|^2}\mathrm{d}s\,\sqrt{\frac{s}{t}}\,\frac{1}{(t-s)^{\frac{3}{2}}}\\
&\le\frac{1}{16\sqrt{2\pi}}\,|x-x'|^2\int_0^{t-|x-x'|^2}\mathrm{d}s\,\frac{1}{(t-s)^{\frac{3}{2}}}\\
&=\frac{1}{8\sqrt{2\pi}}\,|x-x'|^2\Big[|x-x'|^{-1}-t^{-\frac{1}{2}}\Big]\\
&\le\frac{3}{8\sqrt{2\pi}}\,|x-x'|,
\end{aligned}
$$

since $t > 2\,|x-x'|^2$. □

2.4. The Stochastic Fundamental Solution

2.4.1. Two-parameter fundamental solution. Let us fix $\beta\in\mathbb{R}$ and consider the linear SHE with zero initial time and delta initial datum:

$$
\begin{cases}
\dfrac{\partial u}{\partial t}(t,x)=\Delta^{\frac{\alpha}{2}}u(t,x)+\beta\,u(t,x)\,\dot{W}(t,x) & \text{for } t>0,\ x\in\mathbb{R}\\
u(0,\cdot)=\delta_0(\cdot),
\end{cases}
\tag{2.25}
$$

where δ_0 is the Dirac delta in 0, which is the positive Borel measure on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} h(z)\,\delta_0(\mathrm{d}z)=h(0),\quad \text{for all } h\in C_b(\mathbb{R}),$$

where $C_b(\mathbb{R})$ denotes the set of the continuous and bounded functions. Clearly the measure δ_0 satisfies the hypothesis (1.11) of Theorem 1.3 and Theorem 1.4. Hence, all the results of existence, uniqueness and regularity of a solution can be applied to this particular case.

We can give the following definition.

Definition 2.10. The *two-parameter fundamental solution of the linear SHE*, also called *stochastic fundamental solution*, is the unique (up to indistinguishability) continuous and adapted process, denoted by $(U(t,x))_{(t,x)\in(0,\infty)\times\mathbb{R}}$, which is a mild solution of (2.25). It is locally Hölder continuous of any order $\beta_1<\frac{\alpha-1}{2\alpha}$ in time and $\beta_2<\frac{\alpha-1}{2}$ in space.

REMARK 2.11. By standard arguments any two mild solutions of (2.25) with continuous trajectories are indinstinguishable. Indeed, by the up-to-modification uniqueness, we have

$$\mathbb{P}\big(u(t,x) = v(t,x)\big) = 1 \quad \text{for all } t > 0, x \in \mathbb{R}.$$

Since the countable intersection of amost sure events is also almost sure, this implies that

$$\mathbb{P}\big(u(t,x) = v(t,x) \quad \text{for all } t \in \mathbb{Q}^+,\ x \in \mathbb{Q}\big) = 1.$$

But, for almost every $\omega \in \Omega$, both $(t,x) \mapsto u(t,x)(\omega)$ and $(t,x) \mapsto v(t,x)(\omega)$ are continuous: if they agree on all the rational points, they must be the same function on $(0,\infty) \times \mathbb{R}$. In other words,

$$\mathbb{P}\big(u(t,x) = v(t,x) \quad \text{for all } t \in (0,\infty) \times \mathbb{R}\big) = 1.$$

2.4.2. FOUR-PARAMETER FUNDAMENTAL SOLUTION. It is useful to define a stochastic fundamental solution which depends on four parameters.

Fix $(s,y) \in [0,\infty) \times \mathbb{R}$ and consider now the linear fractional stochastic heat equation with initial time s and initial datum δ_y,

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta^{\frac{\alpha}{2}} u(t,x) + \beta\, u(t,x)\, \dot{W}(t,x) & \text{for } t > s,\ x \in \mathbb{R} \\ u(s,\cdot) = \delta_0(\cdot - y). \end{cases} \tag{2.26}$$

A mild solution of (2.26) is a progressively measurable process, which will be denoted by $U^{s,y} = (U^{s,y}(t,x))_{(t,x)\in(s,\infty)\times\mathbb{R}}$, such that, for every $(t,x) \in (s,\infty) \times \mathbb{R}$,

$$U^{s,y}(t,x) = \int_{\mathbb{R}} g_{t-s}(x-y) + \int_s^t \int_{\mathbb{R}} U^{s,y}(r,z) g_{t-r}(x-z)\, W(\mathrm{d}r, \mathrm{d}z) \tag{2.27}$$

$\mathbb{P}$-a.s..

It is easy to show that Theorems 1.3 and 1.4, combined with the fact that white noise has a good behaviour with translations, ensure the existence, uniqueness and regularity of a mild solution of (2.26).

DEFINITION 2.12. The *(four-parameter) fundamental solution of* (1.1) is a stochastic process,

$$(U(s,y;t,x))_{(s,y;t,x)\in[0,\infty)^2_< \times \mathbb{R}^2}$$

such that, for every $(s,y) \in [0,\infty) \times \mathbb{R}$, $(U(s,y;t,x;\beta))_{(t,x)\in(s,\infty)\times\mathbb{R}}$ is a mild solution of (2.26) with continuous trajectories.

We note that $U(t,x) = U(0,0;t,x)$ $\mathbb{P}$-a.s., for all $(t,x) \in (0,\infty) \times \mathbb{R}$.

REMARK 2.13. As things stand, the regularity of the four-parameter fundamental solution is far from obvious. Indeed, for any *fixed* $(s,y) \in [0,\infty) \times \mathbb{R}$, the map

$$(t,x) \mapsto U(s,y;t,x)(\omega)$$

is continuous on $(s,\infty) \times \mathbb{R}$, for all $\omega \in \Omega$. However, this does not imply the joint continuity in *all four* variables. We are going to prove that joint continuity does hold.

Now we can give the definition of the *four-parameter normalized fundamental solution* (cfr. Definition 2.1).

DEFINITION 2.14. We define the *normalized fundamental solution* as the four-parameter process

$$(\hat{U}(s,y;t,x))_{(s,y;t,x)\in[0,\infty)^2_{\leq}\times\mathbb{R}^2}$$

such that, for all $(s,y;t,x)\in[0,\infty)^2_{\leq}\times\mathbb{R}^2$,

$$\hat{U}(s,y;t,x) = \begin{cases} \dfrac{U(s,y;t,x)}{g_{t-s}(x-y)} & \text{if } t>s \\ 1 & \text{if } t=s. \end{cases} \tag{2.28}$$

We denote by $\hat{U}(t,x) := \hat{U}(0,0;t,x)$, for all $t\geq 0$ and $x\in\mathbb{R}$.

We now state some basic properties for the processes U. When necessary, we will write the dependence on β by the writing "$U(s,y;t,x;\beta)$" and "$\hat{U}(s,y;t,x;\beta)$", where $\beta\in\mathbb{R}$ is the one that appears in (2.25).

PROPOSITION 2.15 (BASIC PROPERTIES)**.** For every $\beta>0$, the following properties hold.

- Stationarity: for all $(t_0,x_0)\in[0,\infty)\times\mathbb{R}$,

$$\Big(U(s,y;t,x;\beta)\Big)_{(s,y;t,x)\in[0,\infty)^2_{<}\times\mathbb{R}^2} \overset{(d)}{=} \Big(U(s+t_0,y+x_0;t+t_0,x+x_0;\beta)\Big)_{(s,y;t,x)\in[-t_0,\infty)^2_{<}\times\mathbb{R}^2}.$$

- Diffusive scaling: for every $r>0$,

$$\Big(U(r^2s,ry;r^2t,rx;\beta)\Big)_{(s,y;t,x)\in[0,\infty)^2_{<}\times\mathbb{R}^2} \overset{(d)}{=} \Big(\frac{1}{r}U(s,y;t,x;\beta\sqrt{r})\Big)_{(s,y;t,x)\in[0,\infty)^2_{<}\times\mathbb{R}^2}.$$

- Independence: for any finite disjoint intervals $\{[s_i,t_i)\}_{i=1}^n$ and for all $x_i,y_i\in\mathbb{R}$, the random variables $\{U(s_i,y_i;t_i,x_i;\beta)\}_{i=1}^n$ are mutually independent.
- Semigroup property: For all $x,y\in\mathbb{R}$ and $0\leq s<r<t$,

$$U(s,y;t,x;\beta) = \int_{\mathbb{R}} U(s,y;r,z;\beta)U(r,z;t,x;\beta)\,\mathrm{d}z, \quad \mathbb{P}\text{-a.s.}. \tag{2.29}$$

- Non-negativity: for all $s\geq 0$, $y\in\mathbb{R}$,

$$\mathbb{P}\Big(U(s,y;t,x;\beta)\geq 0 \quad \text{for all } t>s,\ x\in\mathbb{R}\Big) = 1. \tag{2.30}$$

The key property of the normalized fundamental solution is its continuity *in all four variables*, on the domain $0\leq s\leq t<\infty$ and $x,y\in\mathbb{R}$. This permits us to prove the two missing key properties of the four-parameter fundamental solution: the continuity and the strict positivity of U. The first one follows clearly from the corrisponding statement for $\hat{U}$; the second one is proved in the next section.

Theorem 2.16. The process $(\hat{U}(s,y;t,x))_{(s,y;t,x)\in[0,\infty)^2_{\leq}\times\mathbb{R}^2}$ has a locally $(\gamma^-,\delta^-;\gamma^-,\delta^-)$-Hölder continuous modification;

1. if $\alpha\in(1,2)$, $\gamma=\frac{\alpha-1}{2\alpha}$ and $\delta=\frac{\alpha-1}{2}$;
2. if $\alpha=2$, $\gamma=\frac{1}{6}$ and $\delta=\frac{1}{2}$.

2.4.3. Continuity of the normalized fundamental solution.

Proof of Theorem 2.16. Fix $T,M>0$ and let $(s',y';t',x'),(s,y;t,x)\in[0,T]_{\leq}\times[-M,M]$. In order to prove the continuity of the process, we use the generalized Kolmogorov continuity theorem (Theorem B.17): it suffices to show that

$$\begin{aligned}&\mathbb{E}(|\hat{U}(s,y;t,x)-\hat{U}(s',y';t',x')|^p)\leq\\&C(T,M,p)\Big(|s-s'|^{p\gamma}+|y-y'|^{p\delta}+|t-t'|^{p\gamma}+|x-x'|^{p\delta}),\end{aligned}\tag{2.31}$$

where

- if $\alpha\in(1,2)$, then $\gamma=\frac{\alpha-1}{2\alpha}$ and $\delta=\frac{\alpha-1}{2}$;
- if $\alpha=2$, then $\gamma=\frac{1}{6}$ and $\delta=\frac{1}{2}$.

The constant $C(T,M,p)>0$ depends only on T,M,p (and whose value may change from line to line throughout the proof) and for all p large enough (we need to let $p\to\infty$ in order to get the optimal exponents).

By symmetry, we can suppose that $s'\leq s$; applying the triangle inequality,

$$\begin{aligned}&\|\hat{U}(s,y;t,x)-\hat{U}(s',y';t',x')\|_p\leq\\&\|\hat{U}(s,y;t,x)-\hat{U}(s',y;t,x)\|_p+\|\hat{U}(s',y;t,x)-\hat{U}(s',y';t,x)\|_p+\\&\|\hat{U}(s',y';t,x)-\hat{U}(s',y';t',x)\|_p+\|\hat{U}(s',y';t',x)-\hat{U}(s',y';t',x')\|_p.\end{aligned}\tag{2.32}$$

By the stationarity property of $\hat{U}$ (which follows from Proposition 2.15) and by Theorem 2.2, we can write

$$\begin{aligned}\|\hat{U}(s,y;t,x)-\hat{U}(s',y;t,x)\|_p&=\|\hat{U}(t-s,x-y)-\hat{U}(t-s',x-y)\|_p\\&\leq C(T,M,p)\,|s-s'|^{\gamma},\\\|\hat{U}(s',y;t,x)-\hat{U}(s',y';t,x)\|_p&=\|\hat{U}(t-s',x-y)-\hat{U}(t-s',x-y')\|_p\\&\leq C(T,M,p)\,|y-y'|^{\delta},\\\|\hat{U}(s',y';t,x)-\hat{U}(s',y';t',x)\|_p&=\|\hat{U}(t-s',x-y')-\hat{U}(t'-s',x-y')\|_p\\&\leq C(T,M,p)\,|t-t'|^{\gamma},\\\|\hat{U}(s',y';t',x)-\hat{U}(s',y';t',x')\|_p&=\|\hat{U}(t'-s',x-y')-\hat{U}(t'-s',x'-y')\|_p\\&\leq C(T,M,p)\,|x-x'|^{\delta}.\end{aligned}$$

Then (2.31) holds true for every p big enough. By the Kolmogorov continuity theorem (see Theorem B.17), $\hat{U}$ has a continuous modification, which indeed is locally $(\gamma^-, \delta^-; \gamma^-, \delta^-)$-Holder continuous. $\square$

Theorem 2.17 (Regularity). $(U(s,y;t,x))_{(s,y;t,x)\in[0,\infty)^2_<\times\mathbb{R}^2}$ has a continuous modification, *jointly* in all four variables.

Proof. We can write $U(s,y;t,x) = g_{t-s}(x-y)\,\hat{U}(s,y;t,x)$; the statement follows from Theorem 2.16 and from the regularity of the fractional heat kernel. $\square$

From now on, every time we write U, we implicitly consider the continuous modification of the four-parameter process.

Corollary 2.18. We have

$$\mathbb{P}\Big(U(s,y;t,x) = \int_{\mathbb{R}} U(s,y;r,z)\,U(r,z;t,x)\,\mathrm{d}z \quad \text{for all} \quad 0 \le s < r < t\ , x, y \in \mathbb{R}\Big) = 1, \tag{2.33}$$

and

$$\mathbb{P}\Big(U(s,y;t,x) \ge 0 \quad \text{for all } (s,y;t,x) \in [0,\infty)^2_< \times \mathbb{R}^2\Big) = 1. \tag{2.34}$$

Proof. In Proposition 2.15 we proved the *semigroup property* (2.29) (see also Proposition 2.22), that is, for all $(s,y;t,x) \in [0,\infty)^2_< \times \mathbb{R}^2$, and for all $r \in (s,t)$,

$$U(s,y;t,x) = \int_{\mathbb{R}} U(s,y;r,z)\,U(r,z;t,x)\,\mathrm{d}z \quad \mathbb{P}\text{-a.s.}. \tag{2.35}$$

This implies that

$$\begin{aligned}\mathbb{P}\Big(U(s,y;t,x) = &\int_{\mathbb{R}} U(s,y;r,z)\,U(r,z;t,x)\,\mathrm{d}z \\ &\text{for all} \quad (s,y;t,x) \in \big([0,\infty)^2_< \times \mathbb{R}^2\big) \cap \mathbb{Q}^4 \text{ and } r \in \mathbb{Q}\Big) = 1.\end{aligned} \tag{2.36}$$

If we prove that the following map is continuous:

$$(s,y;t,x;r) \mapsto \int_{\mathbb{R}} U(s,y;r,z)(\omega)\,U(r,z;t,x)(\omega)\,\mathrm{d}z \tag{2.37}$$

on the domain $s < r < t$ and $x, y \in \mathbb{R}$, then (2.33) follows, since both sides of (2.35) are continuous, and by (2.36) they must be equal almost surely (see Remark 2.11 for more details).

With the same argument on the continuity, we can pass from (2.30) of Proposition 2.15 to (2.34).

$\square$

2.5. Strict Positivity in the Linear Case

We now focus our attention on the *linear (fractional) stochastic heat equation* with general initial datum, that is

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta^{\frac{\alpha}{2}} u(t,x) + \beta\, u(t,x)\, \dot W & \text{for } x \in \mathbb{R}, t > 0, \\ u(0,\cdot) = \mu_0(\cdot), \end{cases} \tag{2.38}$$

for some $\beta \in \mathbb{R}$ and $\alpha \in (1,2)$. If the initial datum μ_0 satisfies (1.11), that is $\int_{\mathbb{R}} g_t(x-y)\,\mu_0(\mathrm{d}y) < \infty$ for all $(t,x) \in (0,\infty) \times \mathbb{R}$, we can apply Theorems 1.3 and 1.4, getting all the results of existence, uniqueness and regularity of the mild solution. Moreover, we are in the case of $f(0) = 0$ and then we can apply Theorem 2.2. The approach we use to prove the *strict* positivity relies on the continuity of the normalized fundamental solution and on the semigroup property. Roughly speaking, we have proved that $\hat U$ is continuous: since $\hat U(s,\cdot;s,\cdot) = 1$, this implies that $\hat U(s,\cdot;t,\cdot)$ must be strictly positive for close times, i.e. $t - s > 0$ small enough. Thanks to the semigroup property, we can bootstrap this argument to prove strict positivity for all times.

Theorem 2.19 (Strict Positivity for the Fundamental Solution).

$$\mathbb{P}(\hat U(s,y;t,x) > 0 \quad \text{for every } (s,y;t,x) \in [0,\infty)^2_{\le} \times \mathbb{R}^2) = 1, \tag{2.39}$$

and

$$\mathbb{P}(U(s,y;t,x) > 0 \quad \text{for every } (s,y;t,x) \in [0,\infty)^2_{<} \times \mathbb{R}^2) = 1. \tag{2.40}$$

Proof. Note that it suffices to prove (2.39). Indeed, recalling that $U(s,y;t,x) = g_{t-s}(x-y)\hat U(s,y;t,x)$, (2.40) follows from (2.39) and the strict positivity of the map $(s,y;t,x) \mapsto g_{t-s}(x-y)$ over $[0,\infty)^2_< \times \mathbb{R}^2$.

In order to prove (2.39) it is enough to show that

$$\mathbb{P}(\hat U(s,y;t,x) > 0 \quad \text{for every } (s,y;t,x) \in [0,T]^2_{\le} \times [-M,M]^2) = 1 \tag{2.41}$$

for every $T > 0$ and $M > 0$. Indeed, let us suppose we proved (2.41): define

$$A_{T,M} = \left\{ \hat U(s,y;t,x) > 0 \quad \text{for every } (s,y;t,x) \in [0,T]^2_{\le} \times [-M,M]^2 \right\}, \text{ and}$$

$$A = \bigcap_{T\in\mathbb{N}} \bigcap_{M\in\mathbb{N}} A_{T,M}.$$

Then we have $\mathbb{P}(A) = 1$, since it is a countable intersection of almost sure events; moreover $A = \left\{ \hat U(s,y;t,x) > 0 \quad \text{for every } (s,y;t,x) \in [0,\infty)^2_{\le} \times \mathbb{R}^2 \right\}$, so we get (2.39).

Let us prove (2.41): fix $T > 0$, $M > 0$ and denote by $\hat\Omega$, a measurable set in $\mathcal{A}$, such that, for every $\omega \in \hat\Omega$, $\hat U(\cdot,\cdot,\cdot,\cdot)(\omega)$ is a continuous function, and both the semigroup property (2.50) and the non-negativity (2.56) hold true. Thanks to Theorem 2.16 and Corollary 2.18, we have $\mathbb{P}(\hat\Omega) = 1$.

Let fix $\omega \in \hat\Omega$. Then, the function

$$\begin{aligned} &[0,T]^2_{\le} \times [-M,M]^2 \to \mathbb{R} \\ &(s,y;t,x) \mapsto \hat U(s,y;t,x)(\omega) \end{aligned}$$

is uniformly continuous: for every $\eta \in (0,1)$, there exists $\varepsilon = \varepsilon(\eta, \omega, T, M) > 0$ such that, if $|(s,y;t,x) - (\tilde{s},\tilde{y};\tilde{t},\tilde{x})| \le \varepsilon$, then

$$|\hat{U}(s,y;t,x)(\omega) - \hat{U}(\tilde{s},\tilde{y};\tilde{t},\tilde{x})(\omega)| < \eta, \tag{2.42}$$

where we denote by $|\cdot|$ the Euclidean norm. For all $a \in (0,T]$, we define

$$D(a,T,M) = \{(s,y;t,x) \in [0,T]^2 \times [-M,M]^2 \mid 0 \le t-s \le a\}.$$

Clearly $D(T,T,M) = [0,T]^2_{\le} \times [-M,M]^2$; our aim is to show that

$$\hat{U}(s,y;t,x)(\omega) > 0 \quad \text{for every } (s,y;t,x) \in D(T,T,M). \tag{2.43}$$

By induction, we will prove that, for all $N = 1, \dots, \left\lceil \frac{T}{\varepsilon} \right\rceil$,

$$\hat{U}(s,y;t,x)(\omega) > 0 \quad \text{for every } (s,y;t,x) \in D(N\varepsilon,T,M). \tag{2.44}$$

For $N = 1$, we use the uniform continuity: for all $(s,y;t,x) \in [0,T]^2_{\le} \times [-M,M]^2$, such that $t - s \le \varepsilon$, since $\hat{U}(s,y;s,x) = 1$, by (2.42)

$$|\hat{U}(s,y;t,x)(\omega) - 1| < \eta \Longrightarrow \hat{U}(s,y;t,x)(\omega) > 1 - \eta > 0.$$

Let $N \ge 2$ and suppose we proved that

$$\hat{U}(s,y;t,x)(\omega) > 0 \quad \text{for every } (s,y;t,x) \in D\big((N-1)\varepsilon,T,M\big). \tag{2.45}$$

We are now going to switch for a moment to U, which enjoys the semigroup property (2.50): for every $(s,t) \in [0,T]^2_{\le}$, such that $(N-1)\varepsilon < t-s \le N\varepsilon$, and for every $x, y \in [-M,M]$,

$$\begin{aligned} U(s,y;t,x)(\omega) &= \int_{\mathbb{R}} U\big(s,y;s+(N-1)\varepsilon,z\big)(\omega)\; U\big(s+(N-1)\varepsilon,z;t,x\big)(\omega)\,\mathrm{d}z \\ &\ge \int_{|z|\le M} U\big(s,y;s+(N-1)\varepsilon,z\big)(\omega)\; U\big(s+(N-1)\varepsilon,z;t,x\big)(\omega)\,\mathrm{d}z, \end{aligned}$$

where the last inequality is ensured by the non-negativity (2.56). We want to prove that the last integral is strictly positive: we are going to show that the integrand is strictly positive. The function defined by

$$\begin{aligned} &[-M,M] \to \mathbb{R} \\ &z \mapsto U\big(s,y;s+(N-1)\varepsilon,z\big)(\omega) \end{aligned}$$

is strictly positive by the induction hypothesis (2.45), in fact, for every $|z| \le M$, $\big(s,y;s+(N-1)\varepsilon,z\big) \in D\big((N-1)\varepsilon,T,M\big)$, and $U = g\,\hat{U}$ where $g > 0$. Analogously, the function defined by

$$\begin{aligned} &[-M,M] \to \mathbb{R} \\ &z \mapsto U\big(s+(N-1)\varepsilon,z;t,x\big)(\omega) \end{aligned}$$

is strictly posivite by the case $N = 1$, since $t - s - (N-1)\varepsilon \le \varepsilon$.

We have just proved that $U(s,y;t,x)(\omega) > 0$ for $(N-1)\varepsilon < t-s \le N\varepsilon$ and the same clearly holds for $\hat{U}$. Joining this fact with the induction hypothesis (2.45),

$$\hat{U}(s,y;t,x)(\omega) > 0 \quad \text{for every } (s,y;t,x) \in D(N\varepsilon,T,M),$$

which is what we had to prove.

$\square$

Now we have to prove the strict positivity of any mild solution u of (2.38) with general measure as initial datum (but it always has to satisfy (1.11)).

Lemma 2.20. Let $(u(t,x))_{(t,x)\in(0,\infty)\times\mathbb{R}}$ be the solution to the linear fractional stochastic heat equation (2.38) with μ_0 as initial datum, where μ_0 is a Borel measure that satisfies (1.11). Then, for every $(t,x)\in(0,\infty)\times\mathbb{R}$,

$$u(t,x)=\int_{\mathbb{R}}\mu_0(\mathrm{d}y)\,U(0,y;t,x),\tag{2.46}$$

and, for any $s\in(0,t)$,

$$u(t,x)=\int_{\mathbb{R}}\mathrm{d}y\,u(s,y)\,U(s,y;t,x).\tag{2.47}$$

Proof. The proof is similar to the proof of Proposition 2.22 and here we just rewrite the fundamental steps. We fix $s>0$ and we define the process v such that, for every $t>s$ and $x\in\mathbb{R}$,

$$v(t,x)=\int_{\mathbb{R}}\mathrm{d}y\,u(s,y)\,U(s,y;t,x).$$

It is well defined, moreover, using the definition of mild solution for U, we have

$$\begin{aligned}v(t,x)&:=\int_{\mathbb{R}}\mathrm{d}y\,u(s,y)\,U(s,y;t,x)\\&=\int_{\mathbb{R}}\mathrm{d}y\,u(s,y)\left[g_{t-s}(x-y)+\int_s^t\int_{\mathbb{R}}U(s,y;r,z)\,g_{t-r}(x-z)\,W(\mathrm{d}r,\mathrm{d}z)\right]\\&=\int_{\mathbb{R}}\mathrm{d}y\,u(s,y)\,g_{t-s}(x-y)+\int_s^t\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\mathrm{d}y\,u(s,y)\,U(s,y;r,z)\right)g_{t-r}(x-z)\,W(\mathrm{d}r,\mathrm{d}z)\\&=\int_{\mathbb{R}}\mathrm{d}y\,u(s,y)\,g_{t-s}(x-y)+\int_s^t\int_{\mathbb{R}}v(s,y)\,g_{t-r}(x-z)\,W(\mathrm{d}r,\mathrm{d}z),\end{aligned}$$

recalling Proposition 1.12. Then v is a modification of u and we have proved (2.47). In order to prove (2.46), one can running through the proof above and replace "$\mathrm{d}y\,u(s,y)$" with "$\mu_0(\mathrm{d}y)$". □

The strict positivity of a solution u of a linear stochastic heat equation follows directly from (2.46) and from the strict positivity of the four-parameter fundamental solution (see Theorem 2.19).

Theorem 2.21. Let $(u(t,x))_{(t,x)\in(0,\infty)\times\mathbb{R}}$ be the solution to a linear fractional stochastic heat equation (2.38) with initial datum given by a Borel positive measure μ_0, not identically null.

Then

$$\mathbb{P}\big(u(t,x)>0\qquad\text{for every }(t,x)\in(0,\infty)\times\mathbb{R}\big)=1.\tag{2.48}$$

Proof. With the same argument used in the proof of Theorem 2.19, it is enough to prove (2.48) by proving that

$$\mathbb{P}\big(u(t,x)>0\big)=1,\qquad\text{for any }(t,x)\in(0,T]\times[-M,M].\tag{2.49}$$

Thanks to Lemma 2.20, we can write

$$u(t,x) = \int_{\mathbb{R}} \mu_0(\mathrm{d}y)\, U(s,y;t,x),$$

for any $s \in (0,t)$. Then we have done, since we know that μ_0 is a positive Borel measure not identically null by assumptions and U is a continuous strictly positive function (Theorem 2.19). □

2.6. Technical Proofs

2.6.1. Proof of Proposition 2.15. In this section we are going to prove the basic properties of the four-parameter fundametal solution U. We split Proposition 2.15 and prove each statement independently. The stationarity, the diffusive scaling and the independence properties are peculiar to the stochastic framework and due to the definition of a white noise (in particular its covariance function). Then, we are going to prove the *semigroup property* and the *non-negativity*, which requires some further specifications.

Semigroup property. Relation (2.50) below is usually called *Chapman–Kolmogorov equation*.

Proposition 2.22 (Semigroup property)**.** For all $x, y \in \mathbb{R}$ and $0 \le s < r < t$,

$$U(s,y;t,x) = \int_{\mathbb{R}} U(s,y;r,z)U(r,z;t,x)\,\mathrm{d}z, \quad \mathbb{P}\text{-a.s.}. \tag{2.50}$$

Proof. Let us define the four-parameter process V, given by

$$V(s,y;t,x) := \int_{\mathbb{R}} U(s,y;r,z)U(r,z;t,x)\,\mathrm{d}z, \tag{2.51}$$

where $r \in (s,t)$ is fixed. Fix $(s,y;t,x) \in [0,\infty)^2_< \times \mathbb{R}^2$ and $r \in (s,t)$. First of all we should prove that $V(s,y;t,x)$ is well defined, i.e.

$$\mathbb{P}\Big((U(s,y;r,z)U(r,z;t,x))_{z\in\mathbb{R}} \in L^1(\mathbb{R}) \Big) = 1. \tag{2.52}$$

The measurability of the function

$$z \mapsto U(s,y;r,z)\, U(r,z;t,x)(\omega)$$

follows from the property of mild solution. Moreover,

$$\begin{aligned}
&\mathbb{E}\Big(\int_{\mathbb{R}} \big|U(s,y;r,z)U(r,z;t,x)\big|\,\mathrm{d}z \Big) \le \int_{\mathbb{R}} \|U(s,y;r,z)U(r,z;t,x)\|_2\,\mathrm{d}z \\
&= \int_{\mathbb{R}} \|U(s,y;r,z)\|_2 \|U(r,z;t,x)\|_2\,\mathrm{d}z \\
&= \sqrt{C_2(r-s)\, C_2(t-r)}\, g_{t-s}(x-y),
\end{aligned} \tag{2.53}$$

by Cauchy-Schwarz, the independence property, estimate (1.13) in the linear case and the semigroup property of the heat kernel. Relation (2.52) follows.

Now our aim is to prove that V is a modification of U, by the uniqueness of the solution. We know that

$$U(r,z;t,x) = g_{t-r}(x-z) + \int_r^t \int_{\mathbb{R}} U(r,z;t_1,x_1) g_{t-t_1}(x-x_1)\, W(\mathrm{d}t_1,\mathrm{d}x_1).$$

Let us substitute the formula above in (2.51):

$$\begin{aligned} &V(s,y;t,x)\\ &= \int_{\mathbb{R}} U(s,y;r,z)\Big[g_{t-r}(x-z) + \int_r^t \int_{\mathbb{R}} U(r,z;t_1,x_1) g_{t-t_1}(x-x_1)\, W(\mathrm{d}t_1,\mathrm{d}x_1)\Big]\,\mathrm{d}z\\ &= \int_{\mathbb{R}} U(s,y;r,z)\, g_{t-r}(x-z)\,\mathrm{d}z + \int_{\mathbb{R}} U(s,y;r,z)\Big[\int_r^t \int_{\mathbb{R}} U(r,z;t_1,x_1) g_{t-t_1}(x-x_1)\, W(\mathrm{d}t_1,\mathrm{d}x_1)\Big]\,\mathrm{d}z. \end{aligned}$$

We claim that

$$\begin{aligned} &\int_{\mathbb{R}} U(s,y;r,z;\beta)\Big[\int_r^t \int_{\mathbb{R}} U(r,z;t_1,x_1;\beta) g_{t-t_1}(x-x_1)\, W(\mathrm{d}t_1,\mathrm{d}x_1)\Big]\,\mathrm{d}z =\\ &= \int_r^t \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} U(s,y;r,z;\beta) U(r,z;t_1,x_1;\beta)\,\mathrm{d}z\Big]\, g_{t-t_1}(x-x_1)\, W(\mathrm{d}t_1,\mathrm{d}x_1). \end{aligned} \tag{2.54}$$

This implies that

$$V(s,y;t,x) = \int_{\mathbb{R}} U(s,y;r,z) g_{t-r}(x-z)\,\mathrm{d}z + \int_r^t \int_{\mathbb{R}} V(s,y;t_1,x_1) g_{t-t_1}(x-x_1)\, W(\mathrm{d}t_1,\mathrm{d}x_1),$$

$\mathbb{P}$-a.s. and then V is a modification of U.

It remains to prove (2.54): it holds thanks to the stochastic Fubini's theorem. In fact, we can apply it since

$$\begin{aligned} &\int_{\mathbb{R}} \Big[\int_r^t \int_{\mathbb{R}} \mathbb{E}(|U(s,y;r,z)U(r,z;t_1,x_1)|^2)\, g_{t-t_1}^2(x-x_1)\,\mathrm{d}t_1\,\mathrm{d}x_1\Big]^{\frac{1}{2}}\mathrm{d}z\\ &= \int_{\mathbb{R}} \Big[\int_r^t \int_{\mathbb{R}} \mathbb{E}(|U(s,y;r,z)|^2)\mathbb{E}(|U(r,z;t_1,x_1)|^2)\, g_{t-t_1}^2(x-x_1)\,\mathrm{d}t_1\,\mathrm{d}x_1\Big]^{\frac{1}{2}}\mathrm{d}z\\ &\le C(t)\int_{\mathbb{R}} g_{r-s}(z-y)\Big[\int_r^t \int_{\mathbb{R}} g_{t_1-r}^2(x_1-z)\, g_{t-t_1}^2(x-x_1)\,\mathrm{d}t_1\,\mathrm{d}x_1\Big]^{\frac{1}{2}}\mathrm{d}z, \end{aligned}$$

and by Proposition 1.5 and Lemma A.11, this is less than

$$C(t)\left[\mathrm{B}\text{ETA}\Big(\frac{\alpha-1}{\alpha},\frac{\alpha-1}{\alpha}\Big)(t-r)^{\frac{\alpha-1}{\alpha}}\right]^{\frac{1}{2}}\int_{\mathbb{R}} g_{r-s}(z-y) g_{t-r}(x-z)\,\mathrm{d}z \le \tilde{C}(t)\, g_{t-s}(x-y) < \infty,$$

where $\tilde{C}(t)$ is a constant which depens only on t (and α). For the first equality, we have used the independence property. □

NON-NEGATIVITY. The non-negativity, as stated in Proposition 2.24 below, can be proven through the *comparison principle*; a proof of this, in the case $\alpha = 2$ and with the assumptions of continuity on initial data, can be found in [Mueller 91] and in [Bertini, Cancrini 95] there is a generalization for measures. The proof for the general case with $\alpha \in (1,2]$ is in [Chen, Kim 14] (Theorem 1.1), which we report here:

Theorem 2.23 (Weak comparison Principle)**.** Let $\mu^{(1)}$ and $\mu^{(2)}$ be positive Borel measure on $\mathbb{R}$ such that $\mu^{(1)} \leq \mu^{(2)}$ as measures and, for all $T > 0$,

- for $\alpha \in (1,2)$

$$\sup_{y\in\mathbb{R}} \int_{\mathbb{R}} \frac{1}{1+|y-z|^{1+\alpha}}\, \mu^{(i)}(\mathrm{d}z) < \infty, \qquad \text{for } i = 1,2.$$

- for $\alpha = 2$,

$$\int_{\mathbb{R}} e^{-az^2}\, \mu^{(i)}(\mathrm{d}z) < \infty \quad \text{for all } a > 0.$$

Suppose that $u^{(1)}$ and $u^{(2)}$ are the mild solutions of the following

$$\frac{\partial u}{\partial t} = \Delta^{\frac{\alpha}{2}} u + f(u)\,\dot{W} \tag{2.55}$$

with $\mu^{(1)}$ and $\mu^{(2)}$ as the initial datum, respectively. Then

$$\mathbb{P}\big(u^{(1)}(t,x) \leq u^{(2)}(t,x) \quad \text{for all } (t,x) \in (0,\infty)\times\mathbb{R}\big) = 1.$$

We see that in [Chen, Kim 14] the authors proves the weak comparison principle under a stronger assumption for $\alpha \in (1,2)$ on the initial measure (indeed it is the assumption with which they proved existence, uniqueness and regularity, cfr. [Chen, Dalang 15 B]). Actually this is all we need, since we are going to show the non-negativity for the fundamental solution, for which the initial datum is a Dirac delta measure.

However, to be complete, we can say that the weak comparison principle, as stated above, can be easily carry to the case in which the measure satisfies the weaker assumption (1.11) thanks to the continuity of the solution with respect to the initial datum (see Proposition 1.10).

Proposition 2.24 (Non-negativity)**.** For all $s \geq 0$, $y \in \mathbb{R}$,

$$\mathbb{P}\Big(U(s,y;t,x) \geq 0 \quad \text{for all } t > s,\ x \in \mathbb{R}\Big) = 1. \tag{2.56}$$

Proof of Proposition 2.24. Fix $(s,y) \in [0,\infty)\times\mathbb{R})$, then the process $(U(s,y;t,x))_{(t,x)\in(s,\infty)\times\mathbb{R}}$ is a mild solution of (2.55) with $\mu^{(2)} = \delta_y$ as the initial datum. At the same time, the process given by $(V(t,x)=0)_{(t,x)\in(s,\infty)\times\mathbb{R}}$ is a mild solution of (2.55) with $\mu^{(1)} \equiv 0$ as the initial datum. Since $\mu^{(1)} \leq \mu^{(2)}$, applying Theorem 2.23, we get (2.56). □

Part II

A 1d Rough Fractional Equation

Introduction to Part II

In the second part of this thesis, we study the *rough differential equation* (RDE) written as follows

$$Y_t = \xi + \int_0^t F(Y_u)\, p_{t-u}\, \mathrm{d}W_u, \tag{RDE}$$

where ξ is the starting point, $F : \mathbb{R} \to \mathbb{R}$ is a smooth function, W is a (possibly random) Hölder function and p is a *singular kernel*.

In the particular case where W is a (typical path of) Brownian motion, $p_u = u^{H-\frac{1}{2}}$ for some $H \in (0, \frac{1}{2})$, and $F \equiv 1$, the solution Y of (RDE) represents the so-called Riemann-Liouville fractional Brownian motion.

This kind of integral equation of convolution type is usually called "Volterra equation" and it has been already studied in literature, mostly with the classical theory of stochastic differential equations.

In [Prömel, Trabs 18], the authors proved the existence and uniqueness of a solution of (RDE) using the theory of paracontrolled distributions. In [Bayer et al. 17], the authors solve (RDE) by using the theory of *regularity structures* ([Hairer 14]).

Our aim is to prove existence and uniqueness of the solution Y of (RDE) with a more elementary approach, by using the theory of rough paths, initiated by [Lyons 98] and extended by [Gubinelli 04], which provides an alternative approach to the theory of stochastic differential equations.

In Chapter 3, we give some reminders as well as extensions of the classical rough path theory: we first recall the definition of rough path, controlled path and rough integral and then we define the integration with respect to singular kernels. This ingredient, which is one of the cornerstones in the theory of regularity structures [Hairer 14], is necessary to study Volterra equations (RDE) in the framework of rough paths.

In Chapter 4, we prove the existence and uniqueness of the solution to (RDE). In the case of $p_u = u^{H-\frac{1}{2}}$, we are able to prove existence and uniqueness of the solution under the assumption $H > \frac{1}{4}$, which is weaker than the assumption $H > \frac{1}{3}$ of [Prömel, Trabs 18].

In Chapter 5, we give some finer properties of the solution, which lead to a finite reformulation of (RDE).

Chapter 3

Controlled Paths and Rough Integrals

Introduction

This chapter is a quick remainder, in the simplest setting, of rough path theory, originally developed by Lyons in [Lyons 98] and later extended in [Gubinelli 04] (see also [Friz, Hairer 14]). The motivation arises from *rough differential equations* of the form

$$Y_t = \xi + \int_a^t F(Y_u)\,\mathrm{d}W_u \qquad \text{for } t \in (a,b], \tag{RDE1}$$

which can also be written as a rough differential problem:

$$\begin{cases} \mathrm{d}Y_t = F(Y_t)\,\mathrm{d}W_t & \text{for } t \in (a,b] \\ Y_a = \xi, \end{cases} \tag{RDE2}$$

where $[a,b]$ is a subset of $\mathbb{R}$, $\xi \in \mathbb{R}$ is the starting point, $F : \mathbb{R} \to \mathbb{R}$ is a (say, smooth) function and $W : [a,b] \to \mathbb{R}$ is a continuos Hölder function, which might be non differentiable and so classical theory cannot be applied.

Note that, if W is differentiable, then the integral (RDE1) is well defined and equals to $\int_a^t F(Y_u)\,W'_u\,\mathrm{d}u$. Then the solution is a path $Y : [a,b] \to \mathbb{R}$ such that

$$Y_t = \xi + \int_a^t F(Y_u)\,W'_u\,\mathrm{d}u.$$

Under suitable and classical assumptions on F, there exists a unique well-defined solution Y.

However, if W is not differentiable, say $W \in C^\theta$ with $\theta < 1$, then the integral with respect to W is not well-defined. More generally, one wishes to define integrals of the form

$$\int X_u\,\mathrm{d}W_u,$$

when X and W are continuous Hölder functions, say $X \in C^\beta$ and $W \in C^\theta$, with $\beta < 1$ and $\theta < 1$. Remarkably, when $\beta + \theta > 1$, we have a canonical definition of the integral, coming from Young ([Young 36]). If instead $\beta + \theta \le 1$, there is no canonical definition: actually, different "approximations" of X and W may yield different integrals of X with respect to W. A "solution", first conceived by Terry Lyons ([Lyons 98]), is to enrich the path X, giving more "information", which leads to the notion of *rough paths* defined below. Then Gubinelli introduced the important notion of *controlled path* (see [Gubinelli 04]).

Although rough path theory is a "deterministic" theory of integration, equations like (RDE1) are very common in probability where the driving signal W might be a Brownian motion. So, rough path theory can be used as a new path-wise approach to solve SDE, alternative to the Itô's approach which constructs solutions as limits of random variables. Rough path theory can also be used to solve SDE driven more generally by semimartingales (for the literature, see

[Gubinelli 04] and [Friz, Hairer 14]). In [Gubinelli 10] there is a generalization of rough path theory, which permits to study rough integrals through a larger algebraic structure. Hairer used rough path theory to find a robust solution of KPZ equation ([Hairer 13]) and he also generalized this theory to construct *regolarity structures* ([Hairer 14]).

In this chapter, after recalling the basic notions of rough paths and controlled paths, we define the integration with respect to singular kernels in the framework of rough paths theory. This can be done in a simple way, taking inspiration from [Hairer 14], and seems not to have been considered in the literature.

Description of the chapter.

- In Section 3.1, we first fix some notation, then we recall the crucial Sewing Lemma.
- In Section 3.2 we give the definition of a generalized integral, and we recall the definition of controlled path and rough path.
- In Section 3.3 we give a meaning of integrals of the form $\int g\, p\, \mathrm{d}W$, where g is controlled by X and p is a singular function. These general results are very useful to solve the rough fractional SDE introduced in the next chapter (Chapter 4).
- In Section 3.4 we defer some minor results and/or proofs.

3.1. Notation and Basic Tools

Fix $-\infty < a < b < +\infty$. We use the following notation: if $f : [a,b] \to \mathbb{R}$ is a function, for any $a \le s < t \le b$, we write

$$\delta f(s,t) := f(t) - f(s).$$

We recall the definition of Hölder functions.

Definition 3.1. We say that a function $f : [a,b] \to \mathbb{R}$ is a *Hölder function* of index β, with $\beta \in (0,1)$, if

$$\|f\|_\beta := \sup_{a \le s < t \le b} \frac{|\delta f(s,t)|}{|t-s|^\beta} < \infty, \tag{3.1}$$

and we write $f \in C^\beta$.

$\|\cdot\|_\beta$ is a semi-norm on C^β, which is null if and only if f is constant. We will also use the standard notation

$$\|f\|_\infty := \sup_{a \le t \le b} |f(t)|,$$

that is the standard norm on the space C^0 of the continuous functions.

We will deal also with functions of two ordered variables. We denote by $[a,b]^2_<$ the following subset of $\mathbb{R}^2$:

$$[a,b]^2_< := \{(s,t) \in [a,b]^2 \,\text{such that}\, s < t\}.$$

We say that a function $A : [a,b]^2_< \to \mathbb{R}$ is in C_2^γ if

$$\|A\|_\gamma := \sup_{a \le s < t \le b} \frac{|A(s,t)|}{|t-s|^\gamma} < \infty.$$

We also define the "δ" operator for a function of two variables in the following way:

$$\delta A(s,u,t) := A(s,t) - A(s,u) - A(u,t),$$

and we write

$$\|\delta A\|_\gamma := \sup_{a \le s < u < t \le b} \frac{|\delta A(s,u,t)|}{|t-s|^\gamma}.$$

Here we state a very simple Lemma which characterizes the functions for which the "δ operator" has some particular property. The proof is deferred to Section 3.4.

Lemma 3.2. A one-variable function f satisfies

$$\delta f(s,t) = \mathrm{o}(|t-s|) \quad \text{uniformly for } |t-s| \to 0 \tag{3.2}$$

if and only if it is constant.

A two-variable function A satisfies $\delta A \equiv 0$ if and only if there exists a one-variable function f such that $A(s,t) = \delta f(s,t)$.

Moreover we state the following useful computational tool, whose proof is immediate.

Lemma 3.3. If $A : [a,b]^2_< \to \mathbb{R}$ can be written as

$$A(s,t) = c(s)\, D(s,t),$$

for some functions $c : [a,b] \to \mathbb{R}$ and $D : [a,b]^2_< \to \mathbb{R}$, then

$$\delta A(s,u,t) = c(s)\, \delta D(s,u,t) - \delta c(s,u)\, D(u,t). \tag{3.3}$$

We end this section by stating the following fundamental result that we will often use and whose proof can be found for instance in [Gubinelli], [Friz, Hairer 14].

Theorem 3.4 (The Sewing Map)**.** If $A : [a,b]^2_< \to \mathbb{R}$ is a function such that $\|\delta A\|_\gamma < \infty$ for some $\gamma > 1$, then there exists a unique function $I : [a,b] \to \mathbb{R}$ with $I(a) = 0$ and such that

$$\delta I(s,t) = A(s,t) + \mathrm{o}(|t-s|) \quad \text{uniformly for } |t-s| \to 0, \tag{3.4}$$

More precisely, I is the limit of "Riemann sums":

$$I_t = \lim_{|\mathcal{P}| \to 0} \sum_{[t_i,t_{i+1}] \in \mathcal{P}} A(t_i, t_{i+1}) \quad \text{for every } t \in [a,b], \tag{3.5}$$

along arbitrary partitions $\mathcal{P}$ of the interval $[a,t]$ with mesh $|\mathcal{P}| = \max_{[t_i,t_{i+1}] \in \mathcal{P}} |t_{i+1} - t_i| \to 0$.

The remainder

$$R := \delta I - A, \quad \text{that is} \quad R(s,t) = I(t) - I(s) - A(s,t), \tag{3.6}$$

is a linear function of δA and it satisfies the following inequality:

$$\|R\|_\gamma \leq c_\gamma \, \|\delta A\|_\gamma, \quad \text{where} \quad c_\gamma = \frac{1}{1 - 2^{-(\gamma-1)}}. \tag{3.7}$$

3.2. Generalized Integral and Controlled Path

We still fix the interval $[a,b] \subset \mathbb{R}$ and we work with continuous functions $X, W : [a,b] \to \mathbb{R}$. We want to give a meaning to the "integral" of X with respect to W, that is

$$I(t) := \int_a^t X(s) \, \mathrm{d}W(s).$$

We know that, if $W \in C^1$, then

$$I(t) = \int_a^t X(s) \, W'(s) \, \mathrm{d}s.$$

Moreover, in this case we have

$$\delta I(s,t) = X_s \, \delta W(s,t) + \mathrm{o}(|t-s|), \qquad \text{uniformly for } |t-s| \to 0. \tag{3.8}$$

Our goal is to extend the definition of integral when $X \in C^\beta$ and $W \in C^\theta$ for some fixed $\beta, \theta \in (0,1)$. However, we cannot expect that condition (3.8) holds generally for any $\beta, \theta \in (0,1)$. We are going to give a more general definition.

Definition 3.5 (Generalized Integral). Fix β and θ in $(0,1)$ and let us consider two functions $X \in C^\beta$ and $W \in C^\theta$. We say that $I : [a,b] \to \mathbb{R}$ is a $(\beta+\theta)$-*integral* of X with respect to W, and we write $I = \int X \, \mathrm{d}W$, if the following holds

$$\delta I(s,t) = X(s) \, \delta W(s,t) + \mathrm{O}(|t-s|^{\beta+\theta}) \qquad \text{uniformly for } |t-s| \to 0. \tag{3.9}$$

We say that X is the derivative of I with respect to W.

Sometimes we will omit the $(\beta+\theta)$ writing from the integral, even if the definition (3.9) actually depends on $\beta+\theta$.

In Sections 3.2.1 and 3.2.2, we are going to show that this definition is not empty. In Section 3.2.1 we deal with the case $\beta+\theta > 1$, while in Section 3.2.2 we consider the case $\beta+\theta \leq 1$.

3.2.1. Young Integral. We notice that, when $\beta+\theta > 1$, (3.9) implies (3.8). Then, by Lemma 3.2, when $\beta+\theta > 1$, there exists at most one such a function I (up to the addition of a constant). This is called the *Young Integral* of X with respect to W and its existence was first studied and proved by Young in [Young 36] (in a slightly different context).

In the following theorem, we summarise the principal properties of the Young integral.

Theorem 3.6. Fix β, θ in $(0,1)$ with $\beta+\theta > 1$. Let $X, W : [a,b] \to \mathbb{R}$ be functions such that $X \in C^\beta$ and $W \in C^\theta$. Then there exists a unique integral $I : [a,b] \to \mathbb{R}$ of X with respect to W such that $I(a) = 0$ and (3.9) holds. Moreover, I satisfies

$$\|I\|_\theta \leq \left(\|X\|_\infty + c_{\beta+\theta} \|X\|_\beta \, \tau^\beta \right) \|W\|_\theta, \tag{3.10}$$

where the norms are considered in the interval $[a,b]$, $\tau := b-a$ is the length of this interval and $c_{\beta+\theta} := \frac{1}{1-2^{-(\beta+\theta-1)}}$ as in (3.7).

A proof of Theorem 3.6 can be found in [Gubinelli]; here we just give a brief sketch.

Uniqueness. If there exist I and I' integrals of X with respect to W according to Definition 3.5, then, by (3.9),

$$\delta(I - I')(s,t) = \mathrm{O}(|t-s|^{\beta+\theta}) = \mathrm{o}(|t-s|),$$

since $\beta+\theta > 1$. Hence, by Lemma 3.2, $I - I'$ is constant and then it is identically null, since $I(a) = I'(a) = 0$.

Existence. The existence follows easily from the Sewing Map (Theorem 3.4). Indeed, we just have to consider the function $A : [a,b]^2_< \to \mathbb{R}$ defined by

$$A(s,t) := X(s)\,\delta W(s,t).$$

Thanks to Lemma 3.3, we have $\delta A(s,u,t) = -\delta X(s,u)\,\delta W(u,t)$ and then, since $\gamma := \beta+\theta > 1$,

$$\|\delta A\|_\gamma = \|X\|_\beta \, \|W\|_\theta < \infty,$$

which implies the existence of the $(\beta+\theta)$-integral I. Moreover, we have

$$\begin{aligned} |\delta I(s,t)| &\leq |\delta I(s,t) - X(s)\,\delta W(s,t)| + |X(s)\,\delta W(s,t)| \\ &\leq \|R\|_{\beta+\theta} \, (t-s)^{\beta+\theta} + \|X\|_\infty \, \|W\|_\theta \, (t-s)^\theta. \end{aligned}$$

Thanks to the relation (3.7)) in Theorem 3.4 we know that $\|R\|_{\beta+\theta} \leq c_{\beta+\theta} \, \|A\|_{\beta+\theta} \leq c_{\beta+\theta} \|X\|_\beta \, \|W\|_\theta$, and then, for the Young integral, (3.10) holds.

3.2.2. Beyond Young.

When $\beta+\theta \leq 1$ there is no more uniqueness: if I satisfies (3.9), then also $I + f$ satisfy (3.9), for any Hölder function f in $C^{\beta+\theta}$.

The existence of such a function I is not obvious, but we notice that the proof of this is equivalent to proving that there exists a two-variable function R, that will play the role of "remainder".

Proposition 3.7. Fix β, θ in $(0,1)$ with $\beta+\theta \leq 1$. Let X, W be functions such that $X \in C^\beta$ and $W \in C^\theta$.

The two following are equivalent.

(i) There exists a integral I of X with respect to W, according to Definition 3.5.

(ii) There exists a function $R : [a,b]^2_< \to \mathbb{R}$ such that $R \in C_2^{\beta+\theta}$ and it satisfies the *Chen relation*

$$\delta R(s,u,t) = \delta X(s,u)\,\delta W(u,t). \tag{3.11}$$

PROOF. If there exists a integral I of X with respect to W, by (3.9), we have

$$\delta I(s,t) - X(s)\,\delta W(s,t) = \mathrm{O}(|t-s|^{\beta+\theta}).$$

If we set $R(s,t) := \delta I(s,t) - X(s)\,\delta W(s,t)$, then it is a function in $C_2^{\beta+\theta}$ and, by Lemma 3.2 and Lemma 3.3,

$$\delta R(s,u,t) = \delta X(s,u)\,\delta W(u,t),$$

that is (3.11).

Vice versa, if there exists $R : [a,b]^2_< \to \mathbb{R}$ in $C_2^{\beta+\theta}$ such that (3.11) holds, then we define the map $I : [a,b] \to \mathbb{R}$ with $I(a) = 0$ and

$$I(t) = X(0)\,\delta W(0,t) + R(0,t).$$

It follows by (3.11) that

$$\delta I(s,t) = X(s)\,\delta W(s,t) + R(s,t).$$

Then I is an integral of X with respect to W. □

Hence, to prove the existence of a integral I of X with respect to W, we should prove the existence of a 2-variable function R such that $|R(s,t)| = \mathrm{O}(|t-s|^{\beta+\theta})$ and (3.11) holds. This is actually always possible, by the Lyons-Victoir extension theorem [Friz, Hairer 14]; see also Sheet 3 of [Gubinelli]. We note that there is a one-to-one correspondence between an integral I and the two-variable function, also called remainder, R.

ITÔ STOCHASTIC INTEGRAL AS ROUGH INTEGRAL. Since in the case of our interest W is a path of a Brownian motion, we can show that Itô stochastic integral with respect to a Brownian motion is an integral in the sense of (3.9).

Let us fix a probability space $(\Omega, \mathcal{A}, P)$ with a filtration $(\mathcal{F}_t)_{t\in[0,1]}$. Let $(W_t)_{t\in[0,1]}$ be a Brownian motion and let $(X_t)_{t\in[0,1]}$ be an adapted process with continuous paths. We recall that a.s. $W \in C^\theta$ for any $\theta < \frac{1}{2}$. We know that

$$I_t = \int_0^t X_u \,\mathrm{d}W_u \tag{3.12}$$

is well defined as a random variable and the stochastic process $(I_t)_{t\in[0,1]}$ admits a version with continuous paths.

We define

$$R(s,t) = I_t - I_s - X_s\,\delta W(s,t),$$

then R is a two-variable random continuous function. Moreover, it satisfies the Chen relation (3.11), because of the linearity of the stochastic integral. We have the following theorem, whose proof can be found in [Gubinelli 04].

THEOREM 3.8. Assume that a.s. $X \in C^\beta$ for some $\beta \in (0,1)$. Then there is an a.s. finite constant C such that

$$|R(s,t)| \le C\,|t-s|^{\beta+\theta}, \qquad \text{for all } 0 \le s \le t \le 1. \tag{3.13}$$

In particular, a.s. the Itô integral in (3.12) is a integral of X with respect to W in the sense of (3.9).

3.2.3. Controlled Paths and Rough Integral. We now introduce the notion of *controlled paths.*

Definition 3.9. We say that $G = (g, g') \in C^\beta \times C^\beta$ is a *controlled path* by X if g is a 2β-integral of g' with respect to X, that is

$$\delta g(s,t) = g'(s)\,\delta X(s,t) + \mathrm{O}(|t-s|^{2\beta}) \tag{3.14}$$

uniformly for $|t-s| \to 0$, according to Definition 3.5.

We denote with $\mathcal{D}_X^{(\beta,\beta)}$ the set of all the paths $G \in C^\beta \times C^\beta$ controlled by X and we introduce the following seminorm

$$\|G\|_{\mathcal{D}_X^{(\beta,\beta)}} = \|(g,g')\|_{\mathcal{D}_X^{(\beta,\beta)}} = \|g'\|_\beta + \|g^R\|_{2\beta}, \tag{3.15}$$

where

$$g^R(s,t) := \delta g(s,t) - g'(s)\,\delta X(s,t), \tag{3.16}$$

is the remainder in $C_2^{2\beta}$, that is

$$\|g^R\|_{2\beta} = \sup_{a \le s < t \le b} \frac{|g^R(s,t)|}{|t-s|^{2\beta}} < \infty,$$

thanks to (3.14).

Fix $X \in C^\beta$. Classical rough path theory ensures that, if we fix a choice of the rough integral "$\int X\,\mathrm{d}X$", or equivalently if we fix the function $\mathbb{X}(s,t) =$ "$\int_s^t \delta X(s,u)\,\mathrm{d}X(u)$", and we suppose that $\beta > \frac{1}{3}$, then, for any $G \in \mathcal{D}_X^{(\beta,\beta)}$, there exists a unique function $Z : [a,b] \to \mathbb{R}$, the *rough integral* of G with respect to X, $Z(t) =$ "$\int_a^t G\,\mathrm{d}X$", such that $Z(a) = 0$ and

$$\delta Z(s,t) = g(s)\,\delta X(s,t) + g'(s)\,\mathbb{X}(s,t) + \mathrm{O}(|t-s|^{3\beta})$$

uniformly for $|t-s| \to 0$. Moreover, Z is the sum of "Riemann sums":

$$Z_t = \lim_{|\mathcal{P}| \to 0} \sum_{[t_i,t_{i+1}] \in \mathcal{P}} \Big(g(t_i)\,\delta X(t_i,t_{i+1}) + g'(t_i)\,\mathbb{X}(t_i,t_{i+1}) \Big) \quad \text{for every } t \in [a,b],$$

along partitions $\mathcal{P}$ of the interval $[a,t]$ with mesh $|\mathcal{P}| = \max_{[t_i,t_{i+1}] \in \mathcal{P}} |t_{i+1} - t_i| \to 0$.

In our setting, we need a slight extension: we need to define the integral of $G \in \mathcal{D}_X^{(\beta,\beta)}$ with respect to another path W, by fixing the value of "$\int X\,\mathrm{d}W$" . Motivated by the classical definition of rough paths and rough integrals (see for instance [Gubinelli 04]), we give the following definition.

Definition 3.10. Fix $\beta, \theta \in (0,1)$ with $2\beta + \theta > 1$. A (β,θ)- rough path is a triplet $(X, W, \mathbb{XW})$ such that

- $X : [a,b] \to \mathbb{R}$ is in C^β;
- $W : [a,b] \to \mathbb{R}$ is in C^θ;

- $\mathbb{XW} : [a,b]^2_< \to \mathbb{R}$ is a two-variable function in $C_2^{\beta+\theta}$ such that Chen relation holds, that is

$$\delta\mathbb{XW}(s,u,t) = \delta X(s,u)\,\delta W(u,t) \tag{3.17}$$

In the classical definition of *rough path* X and W coincide, $\mathbb{XW}$ is denoted by $\mathbb{X}$ and the rough path is just the pair $(X, \mathbb{X})$, with the assumption that $\beta > \frac{1}{3}$ (that coincides $2\beta + \theta > 1$, for $\beta = \theta$). Hence, Definition 3.10 is a generalization of the classical definition for rough paths.

Shortly we keep in mind these relations:

- $\beta, \theta \in (0,1)$ such that $\theta + \beta < 1$ and $\theta + 2\beta > 1$;
- $(X, W, \mathbb{XW}) \in C^\beta$ is a (β,θ)-rough path, that is $\in C^\beta \times C^\theta \times C_2^{\beta+\theta}$ and such that (3.17) holds.
- $G = (g, g') \in C^\beta \times C^\beta$ is a path controlled by X.

We know that, since $\beta+\theta < 1$, there exist infinite integrals of X with respect to W, according to Definition 3.5. However, thanks to Proposition 3.7, we can fix a choice of the integral of X with respect to W, corresponding to the "remainder" $\mathbb{XW}$.

We now want to define a rough integral of g with respect to W, by using the path X, since $G = (g, g')$ is a path controlled by X and we have fixed the integral of X with respect to W by fixing $\mathbb{XW}$. Heuristically, we write $I(t) =$ "$\int_a^t g(u)\,\mathrm{d}W(u)$", and then

$$\begin{aligned}
\delta I(s,t) - g(s)\delta W(s,t) &= \int_s^t \delta g(s,u)\,\mathrm{d}W(u) \\
&= g'(s)\int_s^t \delta X(s,u)\,\mathrm{d}W(u) + \int_s^t \mathrm{O}(|u-s|^{2\beta})\,\mathrm{d}W(u) \\
&= g'(s)\,\mathbb{XW}(s,t) + \mathrm{O}(|t-s|^{\theta+2\beta}).
\end{aligned}$$

We now give the following theorem, which gives a precise definition of such integral I. The proof of the following is based on the Sewing Map (see Theorem 3.4) and is postponed to Section 3.4.

Theorem 3.11. Let us fix $\beta, \theta \in (0,1)$ such that $\beta + \theta < 1$ and $2\beta + \theta > 1$. Also fix $(X, W) \in C^\beta \times C^\theta$ and $\mathbb{XW} \in C_2^{\beta+\theta}$ such that (3.17) holds. Then, for any controlled path $G = (g, g') \in \mathcal{D}_X^{(\beta,\beta)}$, there is a unique $I : [a,b] \to \mathbb{R}$ with $I(a) = 0$ such that

$$\delta I(s,t) = g(s)\,\delta W(s,t) + g'(s)\,\mathbb{XW}(s,t) + \mathrm{O}(|t-s|^{2\beta+\theta}), \quad \text{uniformly for } |t-s| \to 0. \tag{3.18}$$

We call informally I the "rough integral of G, or even g, with respect to W", and write $I(t) = \int_a^t g\,\mathrm{d}W$ (even though I depends also on g' and $\mathbb{XW}$).

One can obtain I as the limit of "Riemann sums":

$$I(t) = \lim_{|\mathcal{P}|\to 0} \sum_{[t_i,t_{i+1}]} \Big(g(t_i)\delta W(t_i,t_{i+1}) + g'(t_i)\,\mathbb{XW}(t_i,t_{i+1}) \Big)$$

along partitions $\mathcal{P}$ of the interval $[a,t]$ with mesh $|\mathcal{P}| = \max_{[t_i,t_{i+1}]\in\mathcal{P}} |t_{i+1} - t_i| \to 0$.

Moreover, we have the following estimates:

$$\|\delta I - g\delta W - g'\mathbb{XW}\|_{2\beta+\theta} \leq c_{2\beta+\theta}\, C_{W,\mathbb{XW}}\, \|G\|_{\mathcal{D}_X^{(\beta,\beta)}} \tag{3.19}$$

$$\|\delta I - g\delta W\|_{\beta+\theta} \leq c_{2\beta+\theta}\, C_{W,\mathbb{XW}}\, (\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\, \tau^{\beta} + \|g'\|_\infty) \tag{3.20}$$

$$\|\delta I\|_{\theta} \leq c_{2\beta+\theta}\, C_{W,\mathbb{XW}}\, (\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\, \tau^{2\beta} + \|g'\|_\infty\, \tau^{\beta} + \|g\|_\infty) \tag{3.21}$$

where $c_{2\beta+\theta} = \frac{1}{1-2^{-(2\beta+\theta-1)}}$, $\tau = b-a$ is the length of the interval, and

$$C_{W,\mathbb{XW}} := \max\{\|W\|_\theta, \|\mathbb{XW}\|_{\beta+\theta}\}. \tag{3.22}$$

.

3.2.4. Associativity. We can also extend the results when we deal with an integral of a controlled path G multiplied by a function p with a higher regularity. We always work with β, θ fixed with $\beta + \theta < 1$ and $2\beta + \theta > 1$, and $(X, W, \mathbb{XW}) \in C^\beta \times C^\theta \times C^{\beta+\theta}$.

We first see that, given a controlled path $G = (g, g')$ and a function p with a higher regularity (at least $C^{2\beta}$), it is always possible to construct another controlled path in a canonical way by multiplication. The proof is just computational and is deferred to Section 3.4.

Lemma 3.12. For any $p \in C^{2\beta}$, if $G = (g, g') \in \mathcal{D}_X^{(\beta,\beta)}$, then $(p\,g, p\,g') \in \mathcal{D}_X^{(\beta,\beta)}$.

Thanks to Theorem 3.11, we can define the rough integral of pg with respect to W, that is the map $I_{pg} : [a, b] \to \mathbb{R}$ with $I_{pg}(a) = 0$ and such that

$$\begin{aligned}\delta I_{pg}(s,t) &= (pg)(s)\, \delta W(s,t) + (pg')(s)\, \mathbb{XW}(s,t) + \mathrm{O}(|t-s|^{2\beta+\theta}) \\ &= p(s)g(s)\, \delta W(s,t) + p(s)\, g'(s)\, \mathbb{XW}(s,t) + \mathrm{O}(|t-s|^{2\beta+\theta}),\end{aligned}$$

uniformly for $|t-s| \to 0$.

We are going to show that the rough integral of the controlled path pg with respect to W, coincides with the rough integral of p with respect to the rough integral of g with respect to W. We can use the following notation

$$I_g = \int g\, \mathrm{d}W, \quad \text{and} \quad I_{pg} = \int pg\, \mathrm{d}W$$

even if these functions depend also on g' and $\mathbb{XW}$.

Proposition 3.13. We have

$$I_{pg} = \int p\, \mathrm{d}I_g, \tag{3.23}$$

that is

$$\int_a^t p(s)\, g(s)\, \mathrm{d}W(s) = \int_a^t p(s)\, \mathrm{d}I_g(s) \quad \text{for every } t \in [a,b].$$

Moreover,

$$\|I_{pg}\|_\theta \leq \left(\|p\|_\infty + c_{2\beta+\theta}\|p\|_{2\beta}\, \tau^{2\beta}\right) \|I_g\|_\theta, \tag{3.24}$$

where $c_{2\beta+\theta} = \frac{1}{1-2^{-(2\beta+\theta-1)}}$.

PROOF. First of all, we notice that $I_g \in C^\theta$, thanks to Theorem 3.11. Hence, it is well defined the integral

$$\int p \, \mathrm{d}I_g,$$

which is a Young Integral, and hence it is uniquely defined, since $2\beta + \theta > 1$ by assumptions. By Definition 3.5, we recall that $\int p \, \mathrm{d}I_g$ is the unique function such that

$$\begin{aligned}\delta\Big(\int p \, \mathrm{d}I_g\Big)(s,t) &= p(s)\, \delta I_g(s,t) + \mathrm{O}(|t-s|^{2\beta+\theta}) \\ &= p(s)\, g(s)\, \delta W(s,t) + p(s)\, g'(s)\, \mathbb{XW}(s,t) + \mathrm{O}(|t-s|^{2\beta+\theta})\end{aligned}$$

by using the property (3.18) of $\delta I_g(s,t)$. Then (3.23) holds. Relation (3.24) follows from relation (3.10), which follows from the Sewing Map (see Theorem 3.4) and can be applied in this case with $X = p$ and $W = I_g$ (recall that $2\beta + \theta > 1$). □

To simplify notations, given a function $p : [0,\infty) \to \mathbb{R}$, $C^{2\beta}$ over the interval $[a,b]$, we define the following norm:

$$|||p|||_{[a,b]} := \|p\|_{\infty,[a,b]} + \|p\|_{2\beta,[a,b]}\, (b-a)^{2\beta}. \tag{3.25}$$

Then Proposition 3.13 ensures that

$$\|I_{pg}\|_\theta \leq c_{2\beta+\theta}\, |||p|||\, \|I_g\|_\theta, \tag{3.26}$$

and then

$$|I_{pg}(b) - I_{pg}(a)| \leq c_{2\beta+\theta}\, (b-a)^\theta\, |||p|||_{[a,b]}\, \|I_g\|_{\theta,[a,b]}. \tag{3.27}$$

These relations will be used also in the next Chapters.

3.3. Singular Kernels and Rough Integral

In Section 3.2, we gave a meaning of the *rough integral* $\int g \, \mathrm{d}W$, when $g \in C^\beta$ and $W \in C^\theta$ are Hölder continuous functions with $2\beta + \theta > 1$, g is a path controlled by X and we fix the function $\mathbb{XW} \in C_2^{\beta+\theta}$ such that (3.17) holds.

In this section, that is the novel content of this chapter, we are going to study integrals in the form

$$I_{g\bar{p}}(t) := \int_0^t g\, \bar{p} \, \mathrm{d}W, \tag{3.28}$$

where $g \in C^\beta$, $W \in C^\theta$ and $\bar{p} : [0,\infty) \to \mathbb{R}$ is a function in $C^{2\beta}$ except for a point $s \in [0,\infty)$, where $\bar{p}$ has a singularity of order $\bar{\eta} \in \mathbb{R}^+$, for some $\bar{\eta}$ such that $\bar{\eta} < \theta$. We call $\bar{p}$ a singular kernel. With the order of singularity, we mean that

$$|\bar{p}_u| \leq \frac{c}{|u-s|^{\bar{\eta}}} \qquad \text{and} \qquad |\bar{p}'_u| \leq \frac{c}{|u-s|^{\bar{\eta}+1}}, \tag{3.29}$$

for some $c > 0$.

We are going to show that, under suitable assumptions and according to rough path theory, it is possible to give a well-posed definition of the integral in (3.28). The regularity of the integral function $t \mapsto I(t)$ will not be the same as the path W, as it happens in classical theory, but it will be a $(\theta - \bar{\eta})$-Hölder function; this is the reason for requiring that $\theta > \bar{\eta}$, that is the regularity of the integrator function W has to be higher than the order of the singularity of the integrand function. We still use the notation above and fix the following:

- $\beta, \theta \in (0,1)$ such that $\theta + \beta < 1$ and $2\beta + \theta > 1$;
- $\bar{\eta}$ such that $0 < \bar{\eta} < \theta$;
- $(X, W, \mathbb{XW}) \in C^\beta \times C^\theta \times C_2^{\beta+\theta}$ a rough path according to Definition 3.10;
- $G = (g, g') \in \mathcal{D}_X^{(\beta,\beta)}$, a controlled path by X;
- $\bar{p} \in C^{2\beta}$ except a singularity at s $\bar{\eta}$, such that (3.29) holds.

We first give the definition for the integral in (3.28). If $s > t$, then (3.28) is a well defined integral (see Section 3.2.4), since in $[0,t]$ $\bar{p}$ is in $C^{2\beta}$. The problem arises when $s \in [0,t]$: in this case we can write

$$\int_0^t g_u \, \bar{p}_u \, \mathrm{d}W_u = \int_0^s g_u \, \bar{p}_u \, \mathrm{d}W_u + \int_s^t g_u \, \bar{p}_u \, \mathrm{d}W_u$$

and now we are going to define the integral (3.28) when the singularity is one extreme of the integral itself.

Definition 3.14. Given G and $\bar{p}$ as above, we define

$$\int_0^s g_u \, \bar{p}_u \, \mathrm{d}W_u := \lim_{n\to\infty} \int_0^{s-\frac{1}{2^n}} g_u \, \bar{p}_u \, \mathrm{d}W_u \tag{3.30}$$

and

$$\int_s^t g_u \, \bar{p}_u \, \mathrm{d}W_u := \lim_{n\to\infty} \int_{s+\frac{1}{2^n}}^t g_u \, \bar{p}_u \, \mathrm{d}W_u. \tag{3.31}$$

This definition is well posed, since in any interval of the form $[0, s - \frac{1}{2^n}]$ and $[s + \frac{1}{2^n}, t]$, the kernel $\bar{p}$ is in $C^{2\beta}$ and then the integrals $\int_0^{s-\frac{1}{2^n}} g \, \bar{p} \, \mathrm{d}W$ and $\int_{s+\frac{1}{2^n}}^t g \, \bar{p} \, \mathrm{d}W$ are well defined thanks to Proposition 3.13. The limit exists, as we will show in Theorem 3.16 below.

We recall the definition of the norm $|||\cdot|||$(3.25):

$$\begin{aligned} |||\bar{p}|||_{[a,b]} &:= \|\bar{p}\|_{\infty,[a,b]} + (b-a)^{2\beta} \|\bar{p}\|_{2\beta,[a,b]} \\ &= \sup_{u\in[a,b]} |\bar{p}(u)| + (b-a)^{2\beta} \sup_{a\le v<u\le b} \frac{|\bar{p}(u) - \bar{p}(v)|}{(u-v)^{2\beta}}. \end{aligned} \tag{3.32}$$

Clearly this norm can be defined only on the intervals $[a,b]$ where $\bar{p}$ has no singularities.

The following lemma gives an estimate for $|||\bar{p}|||$ that will be used to get estimates for the norm of the integral $I_{g\bar{p}}$.

Lemma 3.15. Let $\bar{p} : [0, \infty)$ be a $C^{2\beta}$ function with a singularity at s of order η (in the sense of (3.29)). Then, for all the intervals $[a, b]$ with $0 \le a < b < s$, we have

$$\|\!|\bar{p}|\!\|_{[a,b]} \le \frac{c}{(s-b)^{\bar{\eta}}} \Big(1 + \frac{b-a}{s-b}\Big). \tag{3.33}$$

For all the intervals $[a, b]$ with $0 \le s < a < b$, we have

$$\|\!|\bar{p}|\!\|_{[a,b]} \le \frac{c}{(a-s)^{\bar{\eta}}} \Big(1 + \frac{b-a}{a-s}\Big). \tag{3.34}$$

If $[a, b]$ is an interval with no singularities for $\bar{p}$, then we know by assumptions that $\bar{p} \in C^{2\beta}$ and, by Proposition 3.13, we have

$$|I_{pg}(b) - I_{pg}(a)| \le c_{2\beta+\theta} \, \|\!|\bar{p}|\!\|_{[a,b]} \, \|I_g\|_{\theta,[a,b]} \, (b-a)^{\theta}, \tag{3.35}$$

(see also (3.27)).

However, we need to have an estimate for $|I_{pg}(b) - I_{pg}(a)|$ even when the singularity s is in $[a, b]$ and may coincide also with an end point of the interval. In this case, the Hölder regularity of the integral decreases, as we now show.

We recall the following constant, as we defined above:

$$c_{2\beta+\theta} = \frac{1}{1 - 2^{-(2\beta+\theta-1)}} \tag{3.36}$$

and we also define

$$c_{\beta,\theta,\bar{\eta}} := 4 \, \frac{1}{1 - 2^{-(2\beta+\theta-1)}} \, \frac{1}{2^{\theta-\bar{\eta}} - 1} = 4 \, c_{2\beta+\theta} \, \frac{1}{2^{\theta-\bar{\eta}} - 1}. \tag{3.37}$$

Theorem 3.16. Suppose that the function $\bar{p} : [0, \infty) \to \mathbb{R}$ has a singularity at $s \in [0, \infty]$ of order $\bar{\eta}$, according to (3.29). Then the integral of Definition 3.14 is well defined. Moreover, for $0 \le a < b < \infty$ with $s \in [a, b]$ (included a and b), we have

$$|I_{\bar{p}g}(b) - I_{\bar{p}g}(a)| \le c_{\beta,\theta,\bar{\eta}} \, c \, \|I_g\|_{\theta,[a,b]} \, (b-a)^{\theta-\bar{\eta}}, \tag{3.38}$$

where c is the same positive constant that appears in (3.29) and $c_{\beta,\theta,\bar{\eta}}$ is defined in (3.37).

Proof. We distinguish three cases, depending on the position of s in the interval $[a, b]$.

Case 1: $s = a$. In this case, we write

$$\begin{aligned} |I_{\bar{p}g}(b) - I_{\bar{p}g}(s)| &= \Big| \int_s^b \bar{p}_u \, g_u \, \mathrm{d}W_u \Big| \\ &= \Big| \lim_{n\to\infty} \int_{s+\frac{b-s}{2^n}}^b \bar{p}_u \, g_u \, \mathrm{d}W_u \Big| \le \lim_{n\to\infty} \sum_{i=0}^n \Big| \int_{s+\frac{b-s}{2^{i+1}}}^{s+\frac{b-s}{2^i}} \bar{p}_u \, g_u \, \mathrm{d}W_u \Big|, \end{aligned}$$

which is consistent with Definition 3.14.

For each integral in the intervals $[s+\frac{b-s}{2^{i+1}}, s+\frac{b-s}{2^i}]$, we notice that the function $\bar{p}$ is in $C^{2\beta}$ for assumption, and we can apply relation (3.26) and write

$$\Big|\int_{s+\frac{b-s}{2^{i+1}}}^{s+\frac{b-s}{2^i}} g_u\,\bar{p}_u\,\mathrm{d}W_u\Big| \leq c_{2\beta+\theta}\left(\frac{b-s}{2^{i+1}}\right)^{\theta} |||\bar{p}|||_{[s+\frac{b-s}{2^{i+1}},s+\frac{b-s}{2^i}]}\,\|I_g\|_{\theta,[s+\frac{b-s}{2^{i+1}},s+\frac{b-s}{2^i}]}. \tag{3.39}$$

Now we can use Lemma 3.15, and we have

$$|||\bar{p}|||_{[s+\frac{b-s}{2^{i+1}},s+\frac{b-s}{2^i}]} \leq 2c\left(\frac{b-s}{2^{i+1}}\right)^{-\bar{\eta}},$$

and then, from (3.39),

$$\Big|\int_{s+\frac{b-s}{2^{i+1}}}^{s+\frac{b-s}{2^i}} g_u\,\bar{p}_u\,\mathrm{d}W_u\Big| \leq c_{2\beta+\theta}\left(\frac{b-s}{2^{i+1}}\right)^{\theta-\bar{\eta}} 2c\,\|I_g\|_{\theta,[s,b]}.$$

Then

$$\lim_{n\to\infty}\sum_{i=0}^{n}\Big|\int_{s+\frac{b-s}{2^{i+1}}}^{s+\frac{b-s}{2^i}} g_u\,\bar{p}_u\,\mathrm{d}W_u\Big| \leq 2\,c_{2\beta+\theta}\,c\,\|I_g\|_{\theta,[s,b]}(b-s)^{\theta-\bar{\eta}}\sum_{i=0}^{\infty}2^{-(i+1)(\theta-\eta)}.$$

Since $\theta-\eta>0$, the series is geometric and we can calculate the sum:

$$\sum_{i=0}^{\infty}2^{-(i+1)(\theta-\eta)} = \frac{1}{2^{\theta-\bar{\eta}}}\,\frac{1}{1-\frac{1}{2^{\theta-\bar{\eta}}}} = \frac{1}{2^{\theta-\bar{\eta}}-1}.$$

Recalling the definition (3.37) of $c_{\beta,\theta,\bar{\eta}}$, we finally get

$$\Big|\int_s^b g_u\,\bar{p}_u\,\mathrm{d}W_u\Big| \leq \frac{1}{2}\,c_{\beta,\theta,\bar{\eta}}\,c\,\|I_g\|_{\theta,[s,b]}\,(b-s)^{\theta-\bar{\eta}},$$

and then (3.38) follows for $s=a$.

Case 2: $s=b$. This case is symmetrical to Case 1 and can be done in the same way, getting

$$\Big|\int_a^s g_u\,\bar{p}_u\,\mathrm{d}W_u\Big| \leq \frac{1}{2}\,c_{\beta,\theta,\bar{\eta}}\,c\,\|I_g\|_{\theta,[a,s]}\,(s-a)^{\theta-\bar{\eta}},$$

Case 3: $a<s<b$. In this case, we write

$$\begin{aligned}\Big|\int_a^b g_u\,\bar{p}_u\,\mathrm{d}W_u\Big| &\leq \Big|\int_a^s g_u\,\bar{p}_u\,\mathrm{d}W_u\Big| + \Big|\int_s^b g_u\,\bar{p}_u\,\mathrm{d}W_u\Big| \\ &\leq \frac{1}{2}c_{\beta,\theta,\bar{\eta}}\,c\,\|I_g\|_{\theta,[a,s]}\,(s-a)^{\theta-\bar{\eta}} + \frac{1}{2}c_{\beta,\theta,\bar{\eta}}\,c\,\|I_g\|_{\theta,[s,b]}\,(b-s)^{\theta-\bar{\eta}} \\ &\leq c_{\beta,\theta,\bar{\eta}}\,c\,\|I_g\|_{\theta,[a,b]}\,(b-a)^{\theta-\bar{\eta}},\end{aligned}$$

thanks to the proofs for cases 1 and 2. □

From Theorem 3.16 and Theorem 3.11 (in particular, see relation (3.21)), we can write the following.

Corollary 3.17. Suppose that the function $\bar{p} : [0, \infty) \to \mathbb{R}$ has a singularity in s and/or in t, of order $\bar{\eta}$ (according to (3.29)). Then, for any $0 \le a < b < \infty$, we have

$$|I_{\bar{p}g}(b) - I_{\bar{p}g}(a)| \le \tilde{c}_{\beta,\theta,\bar{\eta}}\, c\, C_{W,\mathbb{XW}} \left(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}} (b-a)^{2\beta} + \|g'\|_\infty (b-a)^\beta + \|g\|_\infty \right) (b-a)^{\theta-\bar{\eta}}, \tag{3.40}$$

where c is the same positive constant that appears in (3.29), $C_{W,\mathbb{XW}}$ is defined in (3.22) and $\tilde{c}_{\beta,\theta,\bar{\eta}}$ is a constant that depends only on the coefficients $\beta, \theta, \bar{\eta}$ and can be chosen as

$$\tilde{c}_{\beta,\theta,\bar{\eta}} = 4\, c_{2\beta+\theta}^2 \, \frac{1}{2^{\theta-\bar{\eta}} - 1} = 4 \left(\frac{1}{1 - 2^{-(2\beta+\theta-1)}} \right)^2 \frac{1}{2^{\theta-\bar{\eta}} - 1}. \tag{3.41}$$

3.4. Technical Proofs

In this Section we put some technical and/or minor proofs of some results we stated above.

Proof of Lemma 3.2.

Proof. Relation (3.2) says that, for any $\varepsilon > 0$, there exists a $k > 0$ such that, if $|t-s| < k$, then $|f(t) - f(s)| < \varepsilon\, |t-s|$. For any $t \in (a, b]$, we fix a partition of (a, t), say $t_0 = a < t_1 < \ldots < t_n = t$ with $\max_{i=0,\ldots,n-1} |t_{i+1} - t_i| < k$. We write

$$\begin{aligned} |f(t) - f(a)| &= \Big| \sum_{i=0}^{n-1} f(t_{i+1}) - f(t_i) \Big| \le \sum_{i=0}^{n-1} \Big| f(t_{i+1}) - f(t_i) \Big| \\ &\le \varepsilon \sum_{i=0}^{n-1} |t_{i+1} - t_i| = \varepsilon\, (t-a). \end{aligned}$$

Sending $\varepsilon \to 0$, we get $f(t) = f(a)$ for every $t \in [a, b]$.

For the second part, we notice that, if $A(s,t) = \delta f(s,t)$, then

$$\begin{aligned} \delta A(s,u,t) &= \delta f(s,t) - \delta f(s,u) - \delta f(u,t) \\ &= f(t) - f(s) - f(u) + f(s) - f(t) + f(u) = 0. \end{aligned}$$

Now let us suppose that $\delta A \equiv 0$. We define a function f such that $A(a,t) = f(t)$ for every $t \in (a, b]$. Since $\delta A \equiv 0$,for any $s < t$ in (a, b), we can write

$$A(a,t) = A(a,s) + A(s,t) \quad \text{that is} \quad A(s,t) = A(a,t) - A(a,s) = f(t) - f(s).$$

□

Proof of Theorem 3.11.

Proof. The proof is just an application of the Sewing Map (Theorem 3.4). We define the function $A : [a,b]^2_< \to \mathbb{R}$ as

$$A(s,t) = g(s)\, \delta W(s,t) + g'(s)\, \mathbb{XW}(s,t).$$

In order to apply Theorem 3.4, we have to prove that

$$\|\delta A\|_{2\beta+\theta} < \infty, \tag{3.42}$$

which yield to the existence and uniquess of the function I such that (3.18) holds, since $\theta+2\beta > 1$ by assumption. By applying Lemma 3.3, we write

$$\begin{aligned}\delta A(s,u,t) &= -\delta g(s,u)\,\delta W(u,t) - \delta g'(s,u)\,\mathbb{XW}(u,t) + g'(s)\,\delta\mathbb{XW}(s,u,t)\\ &= -g'(s)\,\delta X(s,u)\,\delta W(u,t) - g^R(s,u)\,\delta W(u,t) + \\ &\qquad - \delta g'(s,u)\,\mathbb{XW}(u,t) + g'(s)\,\delta X(s,u)\,\delta W(u,t)\\ &= -g^R(s,u)\,\delta W(u,t) - \delta g'(s,u)\,\mathbb{XW}(u,t),\end{aligned}$$

having used also Lemma 3.2, the fact that $G=(g,g')$ is a controlled path by X (see (3.14)) and the Chen relation (3.17). Then, we have

$$\begin{aligned}|\delta A(s,u,t)| &\le |g^R(s,u)\,\delta W(u,t)| + |\delta g'(s,u)\,\mathbb{XW}(u,t)|\\ &\le \|g^R\|_{2\beta}\,|u-s|^{2\beta}\,\|W\|_\theta\,|t-u|^\theta + \|g'\|_\beta\,|u-s|^\beta\,\|\mathbb{XW}\|_{\beta+\theta}\,|t-u|^{\beta+\theta}\\ &\le C_{W,\mathbb{XW}}\left(\|g^R\|_{2\beta} + \|g'\|_\beta\right)|t-s|^{2\beta+\theta}.\end{aligned}$$

Hence, we get

$$\|\delta A\|_{2\beta+\theta} \le C_{W,\mathbb{XW}}\,\|G\|_{\mathcal{D}_X^{(\beta,\beta)}},$$

recalling the definition of $\|\cdot\|_{\mathcal{D}_X^{(\beta,\beta)}}$ in (3.15).

Applying the Theorem 3.4 of the Sewing Map, we have

$$\|\delta I - g\delta W - g'\,\mathbb{XW}\|_{2\beta+\theta} = \|\delta I - A\|_{2\beta+\theta} \le c_{2\beta+\theta}\,\|\delta A\|_{2\beta+\theta}, \quad \text{where } c_{2\beta+\theta} = \frac{1}{1-2^{-(2\beta+\theta-1)}},$$

which yields to (3.19).

Now we write

$$\begin{aligned}|\delta I(s,t) - g(s)\,\delta W(s,t)| &\le |\delta I(s,t) - A(s,t)| + |g'(s)\,\mathbb{XW}(s,t)|\\ &\le \|\delta I - A\|_{2\beta+\theta}\,|t-s|^{2\beta+\theta} + \|g'\|_\infty\,\|\mathbb{XW}\|_{\beta+\theta}\,|t-s|^{\theta+\beta}\\ &\le \left(\|\delta I - A\|_{2\beta+\theta}\,|t-s|^\beta + \|g'\|_\infty\,\|\mathbb{XW}\|_{\beta+\theta}\right)|t-s|^{\beta+\theta}\\ &\le c_{2\beta+\theta}\,C_{W,\mathbb{XW}}\left(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,(t-s)^\beta + \|g'\|_\infty\right)|t-s|^{\beta+\theta},\end{aligned}$$

which yields to (3.20).

Finally, to prove (3.21), we write

$$\begin{aligned}|\delta I(s,t)| &\le |\delta I(s,t) - g(s)\,\delta W(s,t)| + |g(s)\,\delta W(s,t)|\\ &\le \|\delta I - g\,\delta W\|_{\theta+\beta}\,|t-s|^{\theta+\beta} + \|g\|_\infty\,\|W\|_\theta\,|t-s|^\theta\\ &= \left(\|\delta I - g\,\delta W\|_{\theta+\beta}\,|t-s|^\beta + \|g\|_\infty\,\|W\|_\theta\right)|t-s|^\theta.\end{aligned}$$

Then,

$$\begin{aligned}\|\delta I\|_\theta &\le \|\delta I - g\,\delta W\|_{\beta+\theta}\,\tau^\beta + \|g\|_\infty\,\|W\|_\theta\\ &\le c_{2\beta+\theta}\,C_{W,\mathbb{XW}}\,(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,\tau^{2\beta} + \|g'\|_\infty\,\tau^\beta + \|g\|_\infty),\end{aligned}$$

and we have done. $\square$

Proof of Lemma 3.12.

Proof. We are going to prove that

$$\delta(pg)(s,t) = (p\,g')(s)\,\delta X(s,t) + \mathrm{O}(|t-s|^{2\beta}) \quad \text{uniformly for } |t-s| \to 0, \tag{3.43}$$

which yields to conclusion, thanks to Definition 3.9. We can write

$$\begin{aligned}\delta(pg)(s,t) &= p(t)g(t) - p(s)g(s) = p(t)g(t) - p(s)g(t) + p(s)g(t) - p(s)g(s)\\ &= p(s)\,\delta g(s,t) + g(t)\,\delta p(s,t)\\ &= p(s)\left(g'(s)\,\delta X(s,t) + g^R(s,t)\right) + \mathrm{O}(|t-s|^{2\beta})\\ &= p(s)\,g'(s)\,\delta X(s,t) + \mathrm{O}(|t-s|^{2\beta}),\end{aligned}$$

since g is a controlled path by X (see (3.14)) and p is in $C^{2\beta}$. □

Proof of Lemma 3.15.

Proof. We can prove just relation (3.33), since relation (3.34) can be proved analogously.
We write

$$\|\bar{p}\|_{\infty,[a,b]} = \sup_{u\in[a,b]} |\bar{p}(u)| \le \sup_{u\in[a,b]} \frac{c}{|u-s|^{\bar{\eta}}} = \frac{c}{(s-b)^{\bar{\eta}}},$$

and, at the same time,

$$\|\bar{p}\|_{2\beta,[a,b]} = \sup_{a\le v<u\le b} \frac{|\bar{p}_u - \bar{p}_v|}{|u-v|^{2\beta}} \le \sup_{a\le z\le b} |\bar{p}'_z|\,(b-a)^{1-2\beta} \le \frac{c}{(s-b)^{\bar{\eta}+1}}\,(b-a)^{1-2\beta},$$

and these calculations hold directly to (3.33). □

Chapter 4

Rough Fractional SDE

Introduction

In this chapter we are going to use rough path theory and, in particular, the results in Chapter 3, to prove the existence and uniqueness of the solution to the following *stochastic integral equatation*:

$$Y_t = \xi + \frac{1}{\Gamma\Big(H+\frac{1}{2}\Big)} \int_0^t F(Y_u)\,(t-u)^{H-\frac{1}{2}}\,\mathrm{d}W_u \qquad \text{for all } t \in [0,T], \tag{4.1}$$

for some $H \in (0, \frac{1}{2})$, where $\xi \in \mathbb{R}$ is the starting point, $F : \mathbb{R} \to \mathbb{R}$ is a smooth function and, in this case, $(W_u)_{u \geq 0}$ is a Brownian motion with stochastic integration in the Itô sense. Setting $\alpha := H + \frac{1}{2}$, equation (4.1) can be formally written in the differential form as follows:

$$\begin{cases} D^\alpha Y_t = F(Y_t)\,\dot{W}_t & \text{for } t > 0 \\ Y_0 = \xi, \end{cases} \tag{4.2}$$

where D^α is the *fractional differential operator*, i.e. the inverse of the integral operator J^α defined by

$$J^\alpha Z_t = \frac{1}{\Gamma(\alpha)} \int_0^t Z_u\,(t-u)^{\alpha-1}\,\mathrm{d}u.$$

This kind of stochastic differential equation is called "fractional" because of the presence of the fractional operator D^α. We will solve this equation in the framework of rough path theory. The solution Y of (4.2) turns to be a path controlled by the process X given by

$$X_t = \frac{1}{\Gamma\Big(H+\frac{1}{2}\Big)} \int_0^t (t-u)^{H-\frac{1}{2}}\,\mathrm{d}W_u, \tag{4.3}$$

which is the so-called *Riemann-Liouville (RL) fractional Brownian motion* with Hurst parameter $H \in (0, \frac{1}{2})$, close to the usual fractional Brownian motion.

In applications, problems in the form of (4.2) can be found in finance (see [Bayer et al. 17]). Indeed, these differential equations appear in the study of the dynamics of an important class of stochastic "rough" volatility models. Also in [Prömel, Trabs 18], the authors deal with the Volterra integral equation (4.1) and they prove existence and uniqueness of the solution in the case of $H > \frac{1}{3}$.

Description of the chapter.

- In Section 4.1 we present a more general problem than (4.1) and we state our main results of existence and uniqueness, under suitable assumptions.
- In Section 4.2 we prove our main result.
- In Section 4.3 we prove some auxiliary technical results.

4.1. Main Results

In this section we present a more general form of the rough fractional differential equation (4.1) and we prove existence and uniqueness in this general formulation.

Now we fix

(i) the starting point $\xi \in \mathbb{R}$;

(ii) a sufficiently smooth function $F : \mathbb{R} \to \mathbb{R}$ (see below for details);

(iii) two exponents $\theta, \eta \in \mathbb{R}^+$, with $\theta > \eta$ and $\beta := \theta - \eta$;

(iv) a function $W \in C^\theta$;

(v) a function $p : [0, \infty) \to \mathbb{R}$ which is C^2 except for a singularity in zero of order η such that

$$|p_u| \leq \frac{c}{u^\eta}, \qquad |p'_u| \leq \frac{c}{u^{\eta+1}}, \qquad |p''_u| \leq \frac{c}{u^{\eta+2}}, \tag{4.4}$$

for some constant $c > 0$, for all $u < 1$;

(vi) a function X defined by

$$X_t = \int_0^t p_{t-u} \, \mathrm{d}W_u; \tag{4.5}$$

(vii) a function $\mathbb{XW} \in C_2^{\beta+\theta}$ such that the Chen relation (3.17) holds, that is

$$\delta\mathbb{XW}(s, u, t) = \delta X(s, u) \, \delta W(u, t).$$

Remark 4.1. We know that, for any $t \in [0, T]$, X is well-defined as an easy consequence of Theorem 3.16 with $g \equiv 1$ (for any t, we write $\hat{p}_u = p_{t-u}$, which has a singularity in t of order η). The following proposition, which will be proved below in Section 4.3, says that $X \in C^\beta$, where $\beta = \theta - \eta$, and then the definition of $\mathbb{XW}$ is well posed.

Proposition 4.2. The function X defined in (4.5) is a Hölder function with exponent given by $\beta = \theta - \eta$.

Remark 4.3. As we have seen in Chapter 3, if (g, g') is a path controlled by X, then we can uniquely define the integral of g with respect to W once we have fixed an integral of X with respect to W, which is equivalent to fix a map $\mathbb{XW} \in C_2^{\beta+\theta}$ that satisfies the Chen relation (see Definition 3.5). If W is a Brownian motion, the existence of $\mathbb{XW}$ is ensured by Proposition 3.7 and Theorem 3.8.

We want to solve the following rough fractional integral equation:

$$Y_t = \xi + \int_0^t F(Y_u) \, p_{t-u} \, \mathrm{d}W_u, \qquad \text{for } t \in [0, T] \tag{4.6}$$

Because of the presence of the singular kernel p_{t-u} in the integral, we cannot treat (4.6) as a classic rough integral equation as in [Friz, Hairer 14]. So, we are going to proceed in a different way, but always connected to the classical theory.

Let us give a meaningful definition of solution.

DEFINITION 4.4. A function $Y : [0,T] \to \mathbb{R}$ is called solution of (4.6) if

(i) $(Y, F(Y))$ is a path controlled by the function X defined in (4.5);

(ii) Y satisfies (4.6), where the integral that appears is a rough integral with the singular kernel p, as defined in Chapter 3.

Hence, the space of the solution to (4.6) is given by the space the paths controlled by the path X in (4.5).

REMARK 4.5. In the special case of the rough differential equation (4.1), we recall that W is a Brownian motion and X, as defined in (4.5) with $p_u = \frac{u^{\frac{1}{2}-H}}{\Gamma(H+\frac{1}{2})}$, is a RL-fractional Brownian motion with Hurst parameter H, and hence $\theta = \frac{1}{2} - \varepsilon$, $\eta = \frac{1}{2} - H$ and then rightly $\beta = H - \varepsilon$, for any ε small enough.

Our goal is to ensure existence and uniqueness of a solution to (4.6). Before stating the main theorem, we show the assumptions we need for the function F.

ASSUMPTIONS ON F. We now set two differents assumptions on F: the first one is enough to prove local existence and uniqueness, while the second one is needed when we want to prove a global existence.

The first set of assumption is

$$F'' \text{ is locally Lipschitz} \tag{4.7}$$

and for the second one we set

$$F \text{ is bounded and } F, F', F'' \text{ are globally Lipschitz} \tag{4.8}$$

We note that, in order to have condition (4.7), it is sufficient to request that $F \in C^3$ and, in order to have condition (4.8), we can just require that $F \in C^3$ and F, F', F'', F''' are bounded.

THEOREM 4.6 (EXISTENCE AND UNIQUENESS). We fix $\xi, F, \theta, \eta, W, p, X, \mathbb{XW}$ as above. Consider the rough integral equation in (4.6). If θ, η are such that

$$\theta + (\theta - \eta) < 1, \qquad \theta + 2(\theta - \eta) > 1, \tag{4.9}$$

We have

- **Uniqueness**: if F satisfies (4.7), then there exists at most one solution;
- **Local Existence**: if F satisfies (4.7), then there exists a solution on a short time interval, i.e. on the interval $[0,T]$, for T small enough;
- **Global Existence**: if F satisfies (4.8), then there exists a global solution on an arbitrary time interval.

REMARK 4.7. We just prove global existence and uniqueness under assumption (4.8), since local existence and uniqueness under assumption (4.7) follow by a localization argument.

Indeed, if F satisfies the weak assumption (4.7), then F satisfies the strong assumption (4.8) restricted to a closed ball B with center in ξ, which is a compact subset of $\mathbb{R}$. Then there exists a function $\tilde{F}$ which satisfies the strong assumption (4.8) on the whole $\mathbb{R}$ and coincides with F on B. By global existence, there exists a solution $Y : [0, T] \to \mathbb{R}$ of (4.6) with F replaced by $\tilde{F}$. Since Y is continuous with $Y(0) = \xi$, we can find $T' < T$ such that $Y(t) \in B$ for all $t \in [0, T']$. Then $F(Y_t) = \tilde{F}(Y_t)$ for all $t \in [0, T']$ (recall that $F = \tilde{F}$ on B), so Y is a solution of the original rough differential equation (4.6) on the shorter time interval $[0, T']$. We have proved local existence under the weak assumption (4.7), assuming global existence under the strong assumption (4.8). The same can be done for the uniqueness.

Remark 4.8. For the orginal Volterra equation (4.1), recalling that in this case $\theta = \frac{1}{2} - \varepsilon$ for any ε small enough and $\eta = \frac{1}{2} - H$, the assumption (4.9) corresponds to

$$\frac{1}{4} < H < \frac{1}{2}. \tag{4.10}$$

Sketch of the proof. To prove Theorem 4.6, we use the Banach contraction principle, the standard technique to solve SDE. We need the estimate that we got in Chapter 3, Section 3.3.

We first show that the integral that appears in (4.6) is a well defined rough integral controlled by the path X defined in (4.5). To do this, we need to show that $F(Y)$ is still a controlled path by X and we prove that, if (g, g') is a controlled path by X, then (I_g, g) is still a controlled path by X, where $I_g(t) := \int_0^t g_u \, p_{t-u} \, \mathrm{d}W_u$.

Then we prove that the integral operator

$$(Y, Y') \mapsto (\xi + I_{F(Y)}, F(Y))$$

is a contraction in a suitable Banach space, and then it has a fixed point, which turns out to be the solution Y to (4.6).

4.2. Proof of Theorem 4.6

The proof, based on the contraction mapping theorem, is given in Subsection 4.2.3. We first derive useful estimates on the composition and rough integral operators (Subsections 4.2.1 and 4.2.2).

The framework is the following:

$$\begin{aligned}
&\xi \in \mathbb{R} \\
&F : \mathbb{R} \to \mathbb{R} \qquad \text{satisfies the strong condition (4.8)} \\
&\theta, \eta > 0, \qquad \theta > \eta \\
&\beta := \theta - \eta, \qquad \theta + \beta < 1, \qquad \theta + 2\beta > 1 \\
&p \in C^2(0, \infty) \quad \text{satisfies (4.4)} \\
&W \in C^\theta,\ X \in C^\beta \quad \text{defined by (4.5),} \\
&\mathbb{XW} \in C_2^{\beta+\theta} \quad \text{that satisifes the Chen relation (3.17).}
\end{aligned}$$

We now are going to prove global existence and uniqueness under the strong assumption (4.8). We recall that this condition says that $F : \mathbb{R} \to \mathbb{R}$ is twice differentiable and F, F', F'' are globally Lipschitz. In particular, from now on we assume that $C_F > 0$ is such that

$$\max_{y,z \in \mathbb{R}} \{|F(y) - F(z)|, |F'(y) - F'(z)|, |F''(y) - F''(z)|\} \le C_F \, |y - z|.$$

We note that the integral equation (4.6) can be written as

$$Y = \xi + I_{F(Y)},$$

where

$$I_{F(Y)}(t) := \int_0^t F(Y_u)\, p_{t-u}\, \mathrm{d}W_u. \tag{4.11}$$

Then, in order to prove the existence and uniqueness of a solution to (4.6), we need to prove that there exists a unique fixed point of the following integral operator:

$$(Y, Y') \mapsto \mathcal{I}(Y) := (\xi + I_{F(Y)}, F(Y)). \tag{4.12}$$

However, since we are interested in defining $\mathcal{I}$ over the space of a possible solution of (4.6), we focus on the following subspace of $\mathcal{D}_X^{(\beta,\beta)}$:

$$\mathcal{E} = \{\mathcal{Y} = (Y, Y') \in \mathcal{D}_X^{(\beta,\beta)} \quad \text{such that} \quad \|\mathcal{Y}\|_{\mathcal{D}_X^{(\beta,\beta)}} \leq \mathfrak{C},\, Y(0) = \xi, \quad \text{and} \quad Y'(0) = F(\xi)\}, \tag{4.13}$$

where

$$\mathfrak{C} := 4\, \tilde{c}_{\beta,\theta,\eta}\, C_{X,W,\mathbb{XW}}\, C_F\, (1 + \|F\|_\infty)\, (1 + c) \tag{4.14}$$

and we recall that c is the same that appears in (4.4), $\tilde{C}_{X,W,\mathbb{XW}}$ is defined as

$$C_{X,W,\mathbb{XW}} := 3\, C_{W,\mathbb{XW}}\, (1 + \|X\|_\beta) = 3\, \max\{\|W\|_\theta, \|\mathbb{XW}\|_{\theta+\beta}\}\, (1 + \|X\|_\beta), \tag{4.15}$$

and $\tilde{c}_{\beta,\theta,\eta}$ is a positive constant which depends only on θ, η (and $\beta = \theta - \eta$) which can be defined as in (3.41).

We are going to show that there exists a unique solution of the rough integral equation (4.6), provided T small enough: we require that

$$T^\beta \leq \frac{1}{4(1 + \mathfrak{C})\, (1 + \|F\|_\infty)^2 (1 + \|X\|_\beta)^2}, \tag{4.16}$$

where $\mathfrak{C}$ is defined in (4.14).

To complete the proof for an arbitrary time interval $[0, T]$, we can split it into a finite number of sub-intervals for which (4.16) holds, and apply our existence and uniqueness result to all sub-intervals.

We proceed by three steps:

- In Subsection 4.2.1, we prove that, given Y, a path controlled by X, then also $F(Y)$ (with its appropriate derivative with respect to X) is a path controlled by X (see Definition 3.9), in order to make sense of the rough integral that appears in the right side of the equation (4.6). Moreover, we find useful estimates that link the norms $\|F(Y)\|_{\mathcal{D}_{\mathcal{X}}^{(\beta,\beta)}}$ and $\|Y\|_{\mathcal{D}_{\mathcal{X}}^{(\beta,\beta)}}$ and also the norms of the differences $\|F(Y) - F(\hat{Y})\|_{\mathcal{D}_{\mathcal{X}}^{(\beta,\beta)}}$ and $\|Y - \hat{Y}\|_{\mathcal{D}_{\mathcal{X}}^{(\beta,\beta)}}$.
- In Subsection 4.2.2, we show that also $t \mapsto \int F(Y_u)\, p_{t-u}\, \mathrm{d}W_u$ is a path controlled by X with derivative given by $F(Y)$ and we find some relations for the norms. This implies that $\mathcal{I}$ is a well defined operator over the space $\mathcal{E}$.
- In Subsection 4.2.3 we prove that the integral $\mathcal{I}$ is a contraction over the closed space $\mathcal{E} \subset \mathcal{D}_X^{(\beta,\beta)}$.

4.2.1. Composition Operator. Since we are dealing with integrals of $F(Y)$, where Y is a controlled path and F is a function, then we need some statements on the rough path $\mathcal{F}(Y) := (F(Y), F'(Y)\,Y)$. Now we show that $\mathcal{F}(Y)$ is still a path controlled by X and we get a relation to control $\|\mathcal{F}(Y)\|_{\mathcal{D}_X^{(\beta,\beta)}}$ with $\|\mathcal{Y}\|_{\mathcal{D}_X^{(\beta,\beta)}}$, where we use the notation $\mathcal{Y} = (Y, Y')$.

Lemma 4.9. Let $F : \mathbb{R} \to \mathbb{R}$ be a Lipschitz differentiable function with a Lipschitz derivative F'; in particular, we suppose that there exists $C_F > 0$ such that

$$\max\{|F(x) - F(y)|, |F'(x) - F'(y)|\} \leq C_F |x - y|. \tag{4.17}$$

If $\mathcal{Y} = (Y, Y')$ is in $\mathcal{D}_X^{(\beta,\beta)}$, then $\mathcal{F}(Y) := (F(Y), F'(Y)\,Y')$ is in $\mathcal{D}_X^{(\beta,\beta)}$. Moreover

$$\|\mathcal{F}(Y)\|_{\mathcal{D}_X^{(\beta,\beta)}} \leq C_F \Big(\|\mathcal{Y}\|_{\mathcal{D}_X^{(\beta,\beta)}} + \|Y\|_\beta \|Y'\|_\infty + \frac{1}{2} \|Y\|_\beta^2 \Big). \tag{4.18}$$

Proof. We have the following:

- $F(Y) \in C^\beta$, which can be easily proved by observing that

  $$|F(Y)(t) - F(Y)(s)| = |F(Y(t)) - F(Y(s))| \leq C_F\, |Y(t) - Y(s)| \leq C_F\, \|Y\|_\beta\, |t - s|^\beta,$$

  which implies

  $$\|F(Y)\|_\beta \leq C_F\, \|Y\|_\beta.$$

- $F'(Y)\,Y' \in C^\beta$, indeed:

  $$\begin{aligned}
  &|(F'(Y)\,Y')(t) - (F'(Y)\,Y')(s)| = |F'(Y(t))\,Y'(t) - F'(Y(s))\,Y'(s)| \\
  &= |F'(Y(t))\,Y'(t) - F'(Y(s))\,Y'(t) + F'(Y(s))\,Y'(t) - F'(Y(s))\,Y'(s)| \\
  &\leq |F'(Y(t)) - F'(Y(s))|\,|Y'(t)| + |F'(Y(s))|\,|Y'(t) - Y'(s)| \\
  &\leq C_F\, \|Y\|_\beta\, |t - s|^\beta\, \|Y'\|_\infty + C_F\, \|Y'\|_\beta\, |t - s|^\beta,
  \end{aligned}$$

  having used the Lipschitz property (4.17) of F and F'.

  Then

  $$\|F'(Y)\,Y'\|_\beta \leq C_F \Big(\|Y\|_\beta \|Y'\|_\infty + \|Y'\|_\beta \Big). \tag{4.19}$$

Now it remains to prove that

$$|F(Y)(t) - F(Y)(s) - (F'(Y)\,Y')(s)\,\delta X(s,t)| = \mathrm{O}(|t - s|^{2\beta}) \quad \text{uniformly for } |t - s| \to 0. \tag{4.20}$$

Since Y is a controlled path by X, then $Y'(s)\,\delta X(s,t) = \delta Y(s,t) - Y^R(s,t)$, where $Y^R(s,t)$ is in $C_2^{2\beta}$. We can write

$$\begin{aligned}
&F(Y(t)) - F(Y(s)) - F'(Y(s))\,Y'(s)\,\delta X(s,t) \\
&= F(Y(t)) - F(Y(s)) - F'(Y(s))\,(Y(t) - Y(s)) - F'(Y(s))\,Y^R(s,t).
\end{aligned}$$

The last term is $\mathrm{O}(|t-s|^{2\beta})$, because so is Y^R and F' is bounded. For the first three terms, we note that, since F and F' are Lipschitz with constant C_F, we have

$$|F(Y(t)) - F(Y(s)) - F'(Y(s))\,(Y(t) - Y(s))| = \int_{Y(s)}^{Y(t)} (F'(z) - F'(Y_s))\,\mathrm{d}z \le \frac{1}{2} C_F\,|Y(t) - Y(s)|^2.$$

This proves (4.20), noting that

$$(Y(t) - Y(s))^2 \le \|Y\|_\beta^2\,|t-s|^{2\beta}.$$

Moreover, we have just shown that

$$\begin{aligned} |F^R(Y)(s,t)| &= |\delta F(Y)(s,t) - F'(Y(s))\,Y'(s)\,\delta X(s,t)| \\ &\le \frac{1}{2} C_F\,\|Y\|_\beta^2\,|t-s|^{2\beta} + C_F\,\|Y^R\|_{2\beta}\,|t-s|^{2\beta}, \end{aligned}$$

which implies

$$\|F(Y)^R\|_{2\beta} \le C_F\left(\frac{1}{2}\,\|Y\|_\beta^2 + \|Y^R\|_{2\beta}\right). \tag{4.21}$$

Relation (4.18) follows easily from (4.19) and (4.21):

$$\begin{aligned} \|\mathcal{F}(Y)\|_{\mathcal{D}_X^{\beta,\beta}} &= \|F'(Y)\,Y'\|_\beta + \|F(Y)^R\|_{2\beta} \\ &\le C_F\left(\|Y\|_\beta \|Y'\|_\infty + \frac{1}{2}\,\|Y\|_\beta^2 + \left(\|Y'\|_\beta + \|Y^R\|_{2\beta}\right)\right) \\ &= C_F\left(\|\mathcal{Y}\|_{\mathcal{D}_X^{(\beta,\beta)}} + \|Y\|_\beta\|Y'\|_\infty + \frac{1}{2}\,\|Y\|_\beta^2\right). \end{aligned}$$

□

We need also estimates in order to control $\|\mathcal{F}(Y) - \mathcal{F}(\hat{Y})\|_{\mathcal{D}_X^{(\beta,\beta)}}$ by $\|Y - \hat{Y}\|_{\mathcal{D}_X^{(\beta,\beta)}}$, where both $\mathcal{Y} = (Y, Y')$ and $\hat{\mathcal{Y}} = (\hat{Y}, \hat{Y}')$ are controlled paths in $\mathcal{E}$ which is a subset of $\mathcal{D}_X^{(\beta,\beta)}$ defined in (4.13). The proof of the following is deferred to Section 4.3.

LEMMA 4.10. Let $F : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function Lipschitz, with also F' and F'' Lipschitz: in particular, suppose that there exists $C_F > 0$ such that

$$\max\{|F(x) - F(y)|, |F'(x) - F'(y)|, |F''(x) - F''(y)|\} \le C_F\,|x-y|.$$

Take $\mathcal{Y} = (Y, Y')$ and $\hat{\mathcal{Y}} = (\hat{Y}, \hat{Y}')$ in $\mathcal{D}_X^{(\beta,\beta)}$ such that $\|\mathcal{Y}\|_{\mathcal{D}^{(\beta,\beta)}} \le M$ and $\|\hat{\mathcal{Y}}\|_{\mathcal{D}^{(\beta,\beta)}} \le M$, for some $M > 0$ in the interval $[0,T]$. Then there exists a positive constant $C = C(M, T, X, C_F)$ such that

$$\|\mathcal{F}(Y) - \mathcal{F}(\hat{Y})\|_{\mathcal{D}_X^{(\beta,\beta)}} \le C\,\|\mathcal{Y} - \hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}.$$

More precisely, if $\mathcal{Y}$ and $\hat{\mathcal{Y}}$ are controlled paths in $\mathcal{E} \subset \mathcal{D}_X^{(\beta,\beta)}$, then

$$\|\mathcal{F}(Y) - \mathcal{F}(\hat{Y})\|_{\mathcal{D}_X^{(\beta,\beta)}} \le 3\,C_F\,(1 + \|X\|_\beta)\,\|\mathcal{Y} - \hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}.$$

4.2.2. Rough Integral Operator. We now want to show that the rough integral operator that we defined in (4.12) is well-defined.

In the following Proposition, we show that, if (g, g') is a path controlled by X, in notations $(g, g') \in \mathcal{D}_X^{(\beta,\beta)}$, then also the rough integral I_g, defined by (4.22) below, is a path controlled by X with derivative given by g. The proof of the following is deferred to Section 4.3.

Proposition 4.11. For every $G = (g, g') \in \mathcal{D}_X^{(\beta,\beta)}$, let us define the rough integral

$$I_g(t) := \int_0^t g_u \, p_{t-u} \, \mathrm{d}W_u. \tag{4.22}$$

Then (I_g, g) is in $\mathcal{D}_X^{(\beta,\beta)}$, that is I_g is a path controlled by X in C^β with g as a derivative with respect to X.

More precisely, for all $0 \le s < t \le T$,

$$\begin{aligned} \Big|\delta I_g(s,t) - g_s \, \delta X(s,t)\Big| &= \Big| \int_0^t (g_u - g_s)\,(p_{t-u} - p_{s-u})\,\mathrm{d}W_u \Big| \\ &\le c\,(t-s)^{2\beta}\,\tilde{c}_{\theta,\beta,\eta}\,C_{W,\mathbb{XW}}\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,t^\beta + \|g'\|_\infty + \|g\|_\beta\Big). \end{aligned} \tag{4.23}$$

where c is the same that appears in (4.4), $C_{W,\mathbb{XW}} = \max\{\|W\|_\theta, \|\mathbb{XW}\|_{\beta+\theta}\}$ (the same defined in (3.22)) and $\tilde{c}_{\theta,\beta,\eta}$ is a positive constant which depends only on θ, η (and $\beta = \theta - \eta$) and can be chosen as the one in (3.41).

By Lemma 4.16 below (see Section 4.3 which contains technical proofs) and thanks to relation (4.23) we got above, we obtain an estimates of $\|I_g\|_{\mathcal{D}_X^{(\beta,\beta)}}$ (where I_g is defined by (4.22)).

Corollary 4.12. For any $T > 0$, we have

$$\|I_g\|_{\mathcal{D}_X^{(\beta,\beta)},[0,T]} \le \tilde{c}_{\beta,\theta,\eta}\,C_{X,W,\mathbb{XW}}\,(1+c)\,\big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,T^\beta + |g'(0)|\big), \tag{4.24}$$

where $c_{\beta,\theta,\eta}$ is defined in (3.37), c is the same that appears in (4.4) and $C_{X,W,\mathbb{XW}}$ is defined as (4.15).

Proof. By definition,

$$\begin{aligned} \|I_g\|_{\mathcal{D}_X^{(\beta,\beta)}} &= \|g\|_\beta + \|I_g^R\|_{2\beta} \\ &\le \|g\|_\beta + \tilde{c}_{\theta,\beta,\eta}\,C_{W,\mathbb{XW}}\,c\,\big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,T^\beta + \|g'\|_\infty + \|g\|_\beta\big), \end{aligned}$$

where we used Proposition 4.11 to get an estimate for $\|I_g^R\|_{2\beta}$ (see (4.23)). Then, by applying

(4.31) and (4.32) (see Lemma 4.16 below), we can write (recalling (4.15))

$$
\begin{aligned}
&\|I_g\|_{\mathcal{D}_X^{(\beta,\beta)},[0,T]} \le (1+\|X\|_\beta)\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,T^\beta + |g'(0)|\Big) + \\
&+ \tilde c_{\theta,\beta,\eta}\, C_{W,\mathbb{XW}}\, c\left[2\,\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,T^\beta + |g'(0)| + (1+\|X\|_\beta)\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,T^\beta + |g'(0)|\Big)\right] \\
&\le (1+\|X\|_\beta)\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,T^\beta + |g'(0)|\Big) + \tilde c_{\theta,\beta,\eta}\, C_{W,\mathbb{XW}}\, c\,(1+\|X\|_\beta)\Big(3\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,T^\beta + 2|g'(0)|\Big) \\
&\le (1+\|X\|_\beta)\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,T^\beta + |g'(0)|\Big) + \tilde c_{\theta,\beta,\eta}\, C_{X,W,\mathbb{XW}}\, c\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,T^\beta + |g'(0)|\Big) \\
&\le \tilde c_{\theta,\beta,\eta}\, C_{X,W,\mathbb{XW}}\,(1+c)\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,T^\beta + |g'(0)|\Big),
\end{aligned}
$$

that is (4.24).

□

Thanks to Proposition 4.11, the rough integral operator $\mathcal{I}$ (see (4.12)) is well defined over all the space $\mathcal{D}_X^{(\beta,\beta)}$. Indeed, in Lemma 4.9 we proved that, given (Y,Y') in $\mathcal{D}_X^{(\beta,\beta)}$, also $(F(Y),F'(Y)\,Y')$ is in $\mathcal{D}_X^{(\beta,\beta)}$. Then, by applying Proposition 4.11, we have that the rough path $(I_{F(Y)},F(Y))$ is in $\mathcal{D}_X^{(\beta,\beta)}$ and then $(\xi + I_{F(Y)},F(Y))$ is still in $\mathcal{D}_X^{(\beta,\beta)}$.

However, as we say above, we restrict our study to a close subspace $\mathcal{E}$ of $\mathcal{D}_X^{(\beta,\beta)}$, defined in (4.13), where we find the solution Y of (4.6).

Remark 4.13. The space $\mathcal{D}_X^{(\beta,\beta)}$ is a Banach space with the norm $\|Y\|_\infty + \|Y'\|_\infty + \|\mathcal{Y}\|_{\mathcal{D}_X^{(\beta,\beta)}}$, which is equivalent to $|Y(0)| + |Y'(0)| + \|\mathcal{Y}\|_{\mathcal{D}_X^{(\beta,\beta)}}$. Hence, in the space $\mathcal{E}$, we can just consider the distance given by

$$d(\mathcal{Y},\hat{\mathcal{Y}}) = \|\mathcal{Y}-\hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}.$$

Since $\mathcal{E}$ is a closed subset of $\mathcal{D}_X^{(\beta,\beta)}$, we can say that $\mathcal{E}$ is a complete metric space with the distance $\|\mathcal{Y}-\hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}$.

Remark 4.14. $\mathcal{E}$ is not empty. Indeed, let us define

$$Y_t = \xi + F(\xi)\,X_t \qquad \text{for } t\in[0,T].$$

Then Y is a controlled path by X with $Y(0)=\xi$, derivative identically equals to $F(\xi)$ and with null remainder; indeed, for any s,t,

$$\delta Y(s,t) = F(\xi)\,\delta X(s,t).$$

Then $\|\mathcal{Y}\|_{\mathcal{D}_X^{(\beta,\beta)}} = 0$ and hence $\mathcal{Y}$ belongs to $\mathcal{E}$.

4.2.3. Fix Point Solution of RDE. Proving Theorem 4.6 is equivalent to prove the following.

Proposition 4.15. $\mathcal{I}$ is a contraction in $(\mathcal{E},d)$, where, for any $\mathcal{Y},\hat{\mathcal{Y}}\in\mathcal{E}$,

$$d(\mathcal{Y},\hat{\mathcal{Y}}) = \|\mathcal{Y}-\hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}. \tag{4.25}$$

If we assume this result, then we can prove directly Theorem 4.6.

Proof of Theorem 4.6. Thanks to Remarks 4.13 and 4.14, we know that $(\mathcal{E}, d)$ is a non-empty complete metric space and, thanks to Proposition 4.15, we know that $\mathcal{I}$ is a contraction on (E, d). Then, by the Banach fixed-point theorem, $\mathcal{I}$ admits a unique fixed-point $\mathcal{Y}^\star$ in $\mathcal{E}$, i.e. $\mathcal{I}(\mathcal{Y}^\star) = \mathcal{Y}^\star$. In particular, $\mathcal{Y}^\star = (Y^\star, Y'^\star)$ is such that

$$Y^\star(t) = \xi + \int_0^t F(Y^\star)(u)\, p_{t-u}\, \mathrm{d}W_u,$$

and then it is the unique solution to (4.6). □

Now it remains to prove Proposition 4.15 and then we are done. In the following proof we use some technical results whose proofs are deferred to Section 4.3.

Proof of Proposition 4.15. In order to prove Propposition 4.15, we proceed in the following way:

- first, we show that $\mathcal{I}$ maps $\mathcal{E}$ to $\mathcal{E}$, that is: for any $\mathcal{Y} \in \mathcal{E}$, we have $\mathcal{I}(\mathcal{Y}) \in \mathcal{E}$, that is: $\mathcal{I}(\mathcal{Y})(0) = (\xi, F(\xi))$ and
$$\forall\, Y \in \mathcal{E}: \;\; \|\mathcal{I}(\mathcal{Y})\|_{\mathcal{D}_X^{(\beta,\beta)}} \leq \mathfrak{C}; \tag{4.26}$$
- second, we show that $\mathcal{I}$ is a contraction: we will prove that, for all $\mathcal{Y}, \hat{\mathcal{Y}}$ in E,
$$\|\mathcal{I}(\mathcal{Y}) - \mathcal{I}(\hat{\mathcal{Y}})\|_{\mathcal{D}_X^{(\beta,\beta)}} \leq \frac{1}{2}\, \|\mathcal{Y} - \hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}. \tag{4.27}$$

Recalling the definition (4.13) of $\mathcal{E}$ and the definition of the operator $\mathcal{I}(\mathcal{Y}) = (\xi + I_{F(Y)}, F(Y))$, we know that $\xi + I_{F(Y)}(0) = \xi$ and its derivative in zero is $F(Y(0)) = F(\xi)$, and so, in order to say that $\mathcal{I}(\mathcal{Y}) \in \mathcal{E}$, we just have to show (4.26).

Thanks to Corollary 4.12, for $Y \in \mathcal{E}$, we write

$$\begin{aligned}
\|\mathcal{I}(\mathcal{Y})\|_{\mathcal{D}_X^{(\beta,\beta)}} &= \|I_{F(Y)}\|_{\mathcal{D}_X^{(\beta,\beta)}} \\
&\leq \tilde{c}_{\beta,\theta,\eta}\, C_{X,W,\mathbb{XW}}\, (1+c)\left(\|\mathcal{F}(\mathcal{Y})\|_{\mathcal{D}_X^{(\beta,\beta)}}\, T^\beta + |F'(\xi)F(\xi)|\right) \\
&= \underbrace{\tilde{c}_{\beta,\theta,\eta}\, C_{X,W,\mathbb{XW}}\, (1+c)\, \|\mathcal{F}(\mathcal{Y})\|_{\mathcal{D}_X^{(\beta,\beta)}}\, T^\beta}_{A_1} + \underbrace{\tilde{c}c_{\beta,\theta,\eta}\, C_{X,W,\mathbb{XW}}\, (1+c)\, C_F\, \|F\|_\infty}_{A_2},
\end{aligned}$$

where, in A_2 we used the fact that $\|F'\|_\infty \leq C_F$ and $|F(\xi)| \leq \|F\|_\infty$.

Now we show that

$$A_1 \leq \frac{1}{2}\mathfrak{C} \qquad \text{and} \qquad A_2 \leq \frac{1}{2}\mathfrak{C},$$

which lead to (4.26).

For A_2, easily, we have

$$A_2 = \frac{1}{4}\, \frac{\|F\|_\infty}{1 + \|F\|_\infty}\, \mathfrak{C} \leq \frac{1}{2}\, \mathfrak{C},$$

directly from the definifion of $\mathfrak{C}$ in (4.14).

To prove the relation for A_1, we first show that, if $\mathcal{Y} \in \mathcal{E}$, then

$$\|\mathcal{F}(Y)\|_{\mathcal{D}_X^{(\beta,\beta)},[0,T]} T^\beta \leq \frac{5}{8} C_F. \tag{4.28}$$

If $\mathcal{Y} \in \mathcal{E}$, from relation (4.18) proved in Lemma 4.9 and by Lemma 4.17 below, we can write

$$\|\mathcal{F}(Y)\|_{\mathcal{D}_X^{(\beta,\beta)},[0,T]} \leq C_F \left(\mathfrak{C} + \frac{3}{2} (1 + \|X\|_\beta)^2 (1 + \|F\|_\infty)^2\right). \tag{4.29}$$

Then

$$\begin{aligned}
\|\mathcal{F}(Y)\|_{\mathcal{D}_X^{(\beta,\beta)},[0,T]} T^\beta &\leq C_F \left(\mathfrak{C}\, T^\beta + \frac{3}{2} (1 + \|X\|_\beta)^2 (1 + \|F\|_\infty)^2 T^\beta\right) \\
&\leq C_F \left(\mathfrak{C} \frac{1}{4(1+\mathfrak{C})} + \frac{3}{2} (1 + \|X\|_\beta)^2 (1 + \|F\|_\infty)^2 \frac{1}{4(1+\|X\|_\beta)^2(1+\|F\|_\infty)^2}\right) \\
&\leq \frac{5}{8} C_F,
\end{aligned}$$

since T satisfies (4.16). Recalling the definition of $\mathfrak{C}$ (see (4.14)), we have

$$A_1 \leq \frac{5}{8} \tilde{c}_{\beta,\theta,\eta}\, C_{X,W,\mathbb{XW}}\, C_F = \frac{5}{32} \frac{\mathfrak{C}}{1 + \|F\|_\infty} \leq \frac{1}{2} \mathfrak{C}.$$

Hence we have proved (4.26).

Now we have to show (4.27). Thanks to Corollary 4.12, for $\mathcal{Y}, \hat{\mathcal{Y}} \in \mathcal{E}$, we write

$$\|\mathcal{I}(\mathcal{Y}) - \mathcal{I}(\hat{\mathcal{Y}})\|_{\mathcal{D}_X^{(\beta,\beta)}} = \|I_{F(Y)-F(\hat{Y})}\|_{\mathcal{D}_X^{(\beta,\beta)}} \leq \tilde{c}_{\beta,\theta,\eta}\, C_{X,W,\mathbb{XW}}\, (1 + c)\, \|\mathcal{F}(Y) - \mathcal{F}(\hat{Y})\|_{\mathcal{D}_X^{(\beta,\beta)}} T^\beta, \tag{4.30}$$

because (I_g, g) is a controlled path by X, where

$$g = F(Y) - F(\hat{Y}) \qquad \text{and} \qquad g' = F'(Y)\, Y' - F'(\hat{Y})\hat{Y}'.$$

In particular, $g'(0) = F'(Y(0))\, Y'(0) - F'(\hat{Y}(0))\, \hat{Y}'(0) = 0$, since both $\mathcal{Y}$ and $\hat{\mathcal{Y}}$ are in $\mathcal{E}$ (see (4.13)) and then (4.30) holds.

Thanks to Lemma 4.10, which gives an estimate on $\|\mathcal{F}(Y) - \mathcal{F}(\hat{Y})\|_{\mathcal{D}_X^{(\beta,\beta)}}$, for $\mathcal{Y}, \hat{\mathcal{Y}} \in \mathcal{E}$, we can write

$$\begin{aligned}
\|\mathcal{I}(\mathcal{Y}) - \mathcal{I}(\hat{\mathcal{Y}})\|_{\mathcal{D}_X^{(\beta,\beta)}} &\leq \tilde{c}_{\beta,\theta,\eta}\, C_{X,W,\mathbb{XW}}\, (1 + c)\, 3\, C_F\, (1 + \|X\|_\beta)\, \|\mathcal{Y} - \hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}\; T^\beta \\
&= \mathcal{K}\, \|\mathcal{Y} - \hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}},
\end{aligned}$$

where

$$\mathcal{K} = 3\, \tilde{c}_{\beta,\theta,\eta}\, C_{X,W,\mathbb{XW}}\, (1 + c)\, C_F\, (1 + \|X\|_\beta)\, T^\beta \leq \frac{3}{4} \mathfrak{C}\, (1 + \|X\|_\beta)\, T^\beta,$$

having used the definition (4.14) of $\mathfrak{C}$. Since T^β satisfies (4.16),

$$\mathcal{K} \leq \frac{3}{16} \frac{\mathfrak{C}}{1 + \mathfrak{C}} \frac{1}{1 + \|X\|_\beta} \leq \frac{3}{16} < \frac{1}{2}.$$

We have just proved that

$$\|\mathcal{I}(\mathcal{Y}) - \mathcal{I}(\hat{\mathcal{Y}})\|_{\mathcal{D}_X^{(\beta,\beta)}} < \frac{1}{2} \|\mathcal{Y} - \hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}$$

and then the operator $\mathcal{I}$ is a contraction on the close set $\mathcal{E}$ of $\mathcal{D}_X^{(\beta,\beta)}$. $\square$

4.3. Technical Proofs

Proof of Proposition 4.2.

Proof. For any $0 \le s < t < \infty$, write

$$X_t - X_s = \int_{\mathbb{R}} (p_{t-u} - p_{s-u})\, \mathrm{d}W_u,$$

with the convention that $p_u \equiv 0$ when $u \le 0$. Now we use Proposition 4.18 (in particular (4.42) with $g \equiv 1$), and we can write

$$|X_t - X_s| \le K\ (t-s)^{\theta-\eta},$$

where K is a constant which depends on the data of the problem, but not on s, t. □

Relations between norms. In this section, we report some useful relations between the norms.

Lemma 4.16. Let $G = (g, g') \in \mathcal{D}^{(\beta,\beta)}$ be a path controlled by X. Then we have

$$\|g'\|_{\infty,[s,t]} \le |g'(s)| + \|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\, (t-s)^{\beta} \tag{4.31}$$

and

$$\|g\|_{\beta,[s,t]} \le (1 + \|X\|_{\beta}) \left(|g'(s)| + \|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\, (t-s)^{\beta} \right). \tag{4.32}$$

Proof. Relation (4.31) follows by

$$|g'_u| \le |g'_s| + |g'_u - g'_s| \le |g'_s| + \|g'\|_{\beta}\, (t-s)^{\beta}.$$

To get relation (4.32), we recall that G is controlled by X and we can write

$$\begin{aligned} |\delta g(s,t)| &= |g'_s\, \delta X(s,t) + g^R(s,t)| \\ &\le \|g'\|_{\infty}\, \|X\|_{\beta}\, (t-s)^{\beta} + \|g^R\|_{2\beta}\, (t-s)^{2\beta} \\ &\le (t-s)^{\beta} \left(|g'(s)|\, \|X\|_{\beta} + \|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\, \|X\|_{\beta}\, (t-s)^{\beta} + \|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\, (t-s)^{\beta} \right), \end{aligned}$$

having used also (4.31). □

Lemma 4.17. If $\mathcal{Y} \in \mathcal{E}$ and T satisfies (4.16), then

$$\|Y\|_{\beta} \le (1 + \|X\|_{\beta})\, (1 + \|F\|_{\infty}) \tag{4.33}$$

and

$$\|Y'\|_{\infty} \le (1 + \|F\|_{\infty}). \tag{4.34}$$

Proof. The proof follows from Lemma 4.16. If $\mathcal{Y} \in \mathcal{E}$, thanks to (4.32), we can write

$$\begin{aligned} \|Y\|_{\beta} &\le (1 + \|X\|_{\beta})\, (|F(\xi)| + \mathfrak{C}\, T^{\beta}) \\ &\le (1 + \|X\|_{\beta})\, (\|F\|_{\infty} + 1), \end{aligned}$$

since $T^\beta \le \frac{1}{1+\mathfrak{C}}$ (see (4.16)).

From (4.31), we can write

$$\|Y'\|_\infty \le |F(\xi)| + \mathfrak{C}\, T^\beta \le \|F\|_\infty + 1.$$

□

PROOF OF LEMMA 4.10.

PROOF. We define

$$H = (h, h') = (F(Y) - F(\hat{Y}), F'(Y)\, Y' - F'(\hat{Y})\, \hat{Y}'). \tag{4.35}$$

We know that $H \in \mathcal{D}_X^{(\beta,\beta)}$ with $H^R := \delta h - h'\, \delta X$ in $C_2^{2\beta}$. By calculation, one gets

$$\|h'\|_\beta \le$$
$$C_F \left[\|Y\|_\beta \|Y' - \hat{Y}'\|_\infty + \|Y'\|_\infty \|Y - \hat{Y}\|_\beta + \|Y' - \hat{Y}'\|_\beta + \left(\|Y'\|_\beta + \|Y'\|_\infty \|Y\|_\beta\right) \|Y - \hat{Y}\|_\infty\right], \tag{4.36}$$

and

$$\|H^R\|_{2\beta}$$
$$\le C_F \left[\|Y^R - \hat{Y}^R\|_{2\beta} + \left(\|Y^R\|_{2\beta} + \frac{1}{2}\|Y\|_\beta^2\right) \|Y - \hat{Y}\|_\infty + \|Y\|_\beta \|Y - \hat{Y}\|_\beta + \frac{1}{2}\|Y - \hat{Y}\|_\beta^2\right] \tag{4.37}$$

(the proofs of these two relations are postponed at the end of the proof).

By (4.36) and (4.37), we have

$$\|H\|_{\mathcal{D}_X^{(\beta,\beta)}} = \|h'\|_\beta + \|H^R\|_{2\beta}$$
$$\le C_F \Bigg[\|Y^R - \hat{Y}^R\|_{2\beta} + \|Y' - \hat{Y}'\|_\beta + \|Y - \hat{Y}\|_\infty \left(\|Y^R\|_{2\beta} + \frac{1}{2}\|Y\|_\beta^2 + \|Y'\|_\beta + \|Y'\|_\infty \|Y\|_\beta\right) +$$
$$+ \|Y - \hat{Y}\|_\beta \left(\|Y'\|_\infty + \|Y\|_\beta\right) + \|Y' - \hat{Y}'\|_\infty \|Y\|_\beta + \frac{1}{2}\|Y - \hat{Y}\|_\beta^2\Bigg]$$

Since $Y(0) = \hat{Y}(0)$ and $Y'(0) = \hat{Y}'(0)$, then we have the follwing relations:

$$\|Y - \hat{Y}\|_\infty \le \|Y - \hat{Y}\|_\beta\, T^\beta$$
$$\|Y' - \hat{Y}'\|_\infty \le \|Y' - \hat{Y}'\|_\beta\, T^\beta \le \|\mathcal{Y} - \hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}\, T^\beta$$

and

$$\|Y - \hat{Y}\|_\beta \le \|Y' - \hat{Y}'\|_\infty \|X\|_\beta + \|Y^R - \hat{Y}^R\|_{2\beta}\, T^\beta \le \|\mathcal{Y} - \hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}\, T^\beta\, (1 + \|X\|_\beta).$$

We also use the fact that

$$\|Y - \hat{Y}\|_\beta^2 \le \|Y - \hat{Y}\|_\beta \left(\|Y\|_\beta + \|\hat{Y}\|_\beta\right),$$

and we write

$$
\begin{aligned}
&\|H\|_{\mathcal{D}_X^{(\beta,\beta)}} \le C_F \Big[\|\mathcal{Y}-\hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}+\\
&+\|\mathcal{Y}-\hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}\, T^{2\beta}\,(1+\|X\|_\beta)\,\big(\|\mathcal{Y}\|_{\mathcal{D}_X^{(\beta,\beta)}}+\frac{1}{2}\,\|Y\|_\beta^2+\|Y'\|_\infty\,\|Y\|_\beta\big)+\\
&+\|\mathcal{Y}-\hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}\, T^{\beta}\,(1+\|X\|_\beta)\,\big(\|Y'\|_\infty+\|Y\|_\beta\big)+\|\mathcal{Y}-\hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}\, T^{\beta}\,\|Y\|_\beta+\\
&+\frac{1}{2}\,\|\mathcal{Y}-\hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}\, T^{\beta}\,(1+\|X\|_\beta)\,(\|Y\|_\beta+\|\hat{Y}\|_\beta)\Big]\\
&\le C_F\,(1+\|X\|_\beta)\,\|\mathcal{Y}-\hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}\times\\
&\times\Big(1+T^{\beta}\,(\frac{5}{2}\|Y\|_\beta+\|Y'\|_\infty+\|\hat{Y}\|_\beta)+T^{2\beta}\,(\|\mathcal{Y}\|_{\mathcal{D}_X^{(\beta,\beta)}}+\frac{1}{2}\,\|Y\|_\beta^2+\|Y'\|_\infty\,\|Y\|_\beta)\Big).
\end{aligned}
$$

In Lemma 4.17, we proved that

$$
\max\{\|Y\|_\beta,\|Y'\|_\infty\}\le(1+\|X\|_\beta)\,(1+\|F\|_\infty), \tag{4.38}
$$

for any $\mathcal{Y}\in\mathcal{E}$. Then, since $\mathcal{Y},\hat{\mathcal{Y}}\in\mathcal{E}$ and T satisfies (4.16), by using also (4.31) and (4.32), we have

$$
T^{\beta}\,\Big(\frac{5}{2}\|Y\|_\beta+\|Y'\|_\infty+\|\hat{Y}\|_\beta\Big)\le\frac{9}{2}\,T^{\beta}\,(1+\|X\|_\beta)\,(1+\|F\|_\infty)\le\frac{9}{8}
$$

and

$$
\begin{aligned}
T^{2\beta}\,(\|\mathcal{Y}\|_{\mathcal{D}_X^{(\beta,\beta)}}+\frac{1}{2}\,\|Y\|_\beta^2+\|Y'\|_\infty\,\|Y\|_\beta)&\le T^{2\beta}\,(\mathfrak{C}+\frac{3}{2}\,(1+\|X\|_\beta)^2\,(1+\|F\|_\infty)^2)\\
&\le\frac{5}{32}.
\end{aligned}
$$

Then

$$
\|H\|_{\mathcal{D}_X^{(\beta,\beta)}}\le\Big(1+\frac{9}{8}+\frac{5}{32}\Big)\,C_F\,(1+\|X\|_\beta)\,\|\mathcal{Y}-\hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}<3\,C_F\,(1+\|X\|_\beta)\,\|\mathcal{Y}-\hat{\mathcal{Y}}\|_{\mathcal{D}_X^{(\beta,\beta)}}.
$$

Proof of (4.36)**.** By definition of h' in (4.35), we have

$$
h'(t)-h'(s)=(F'(Y_t)\,Y_t'-F'(\hat{Y}_t)\,\hat{Y}_t')-(F'(Y_s)\,Y_s'-F'(\hat{Y}_s)\,\hat{Y}_s').
$$

By adding and subtracting $F'(Y_t)\,Y_s'$ and $F'(\hat{Y}_t)\,\hat{Y}_s'$, we can write

$$
\begin{aligned}
h'(t)-h'(s)&=(F'(Y_t)-F'(Y_s))\,Y_s'+F'(Y_t)\,(Y_t'-Y_s')+\\
&\quad-(F'(\hat{Y}_t)-F'(\hat{Y}_s))\,\hat{Y}_s'-F'(\hat{Y}_t)\,(\hat{Y}_t'-\hat{Y}_s')\\
&:=g_{s,t}\,Y_s'+F'(Y_t)\,\delta Y_{s,t}'-\hat{g}_{s,t}\,\hat{Y}_s'+F'(\hat{Y}_t)\,\delta\hat{Y}_{s,t}',
\end{aligned}
$$

where we use the notations

$$
g_{s,t}:=F'(Y_t)-F'(Y_s),\qquad \hat{g}_{s,t}:=F'(\hat{Y}_t)-F'(\hat{Y}_s).
$$

By triangle inequality,

$$
|h'(t)-h'(s)|\le|g_{s,t}-\hat{g}_{s,t}|\,|Y_s'|+|\hat{g}_{s,t}|\,|Y_s'-\hat{Y}_s'|+|F'(Y_t)|\,|\delta Y_{s,t}'-\delta\hat{Y}_{s,t}'|+|F'(Y_t)-F'(\hat{Y}_t)|\,|\delta Y_{s,t}'|.
$$

We use the following relations:

$$
\begin{aligned}
&|\hat g_{s,t}| = |F'(\hat Y_t) - F'(\hat Y_s)| \le C_F\,|\hat Y_t - \hat Y_s| \le C_F\,\|\hat Y\|_\beta\,|t-s|^\beta \\
&|Y'_s - \hat Y'_s| \le \|Y' - \hat Y'\|_\infty \\
&|F'(Y_t)| \le \|F'\|_\infty \le C_F \\
&|\delta Y'_{s,t} - \delta \hat Y'_{s,t}| \le \|Y' - \hat Y'\|_\beta\,|t-s|^\beta \\
&|F'(Y_t) - F'(\hat Y_t)| \le C_F\,|Y_t - \hat Y_t| \le C_F\,\|Y - \hat Y\|_\infty \\
&|\delta Y'_{s,t}| \le \|Y'\|_\beta\,|t-s|^\beta .
\end{aligned}
$$

We can write

$$
\frac{|h'(t)-h'(s)|}{|t-s|^\beta} \le \frac{|g_{s,t} - \hat g_{s,t}|}{|t-s|^\beta}\,\|Y'\|_\infty + C_F\,\|\hat Y\|_\beta\,\|Y' - \hat Y'\|_\infty + C_F\,\|Y' - \hat Y'\|_\beta + C_F\,\|Y - \hat Y\|_\infty\,\|Y'\|_\beta .
$$

Now we focus on $|g_{s,t} - \hat g_{s,t}|$: we can write

$$
|g_{s,t} - \hat g_{s,t}| = |(F'(Y_s + \delta Y_{s,t}) - F'(\hat Y_s + \delta \hat Y_{s,t})) - (F'(Y_s) - F'(\hat Y_s))|.
$$

By summing and subtracting the term $F'(\hat Y_s + \delta Y_{s,t})$ and by triangle inequality, we get

$$
\begin{aligned}
|g_{s,t} - \hat g_{s,t}| &\le |F'(\hat Y_s + \delta Y_{s,t}) - F'(\hat Y_s + \delta \hat Y_{s,t})| + |F'(Y_s + \delta Y_{s,t}) - F'(Y_s) - (F'(\hat Y_s + \delta Y_{s,t}) - F'(\hat Y_s))| \\
&\le C_F|\delta Y_{s,t} - \delta \hat Y_{s,t}| + \Big|\int_0^1 \big(F''(Y_s + u\,\delta Y_{s,t}) - F''(\hat Y_s + u\,\delta Y_{s,t})\big)\,\delta Y_{s,t}\,\mathrm{d}u\Big| \\
&\le C_F|\delta Y_{s,t} - \delta \hat Y_{s,t}| + C_F\,\|Y - \hat Y\|_\infty\,|\delta Y_{s,t}|.
\end{aligned}
$$

Hence

$$
\frac{|g_{s,t} - \hat g_{s,t}|}{|t-s|^\beta} \le C_F\,(\|Y - \hat Y\|_\beta + \|Y - \hat Y\|_\infty\,\|Y\|_\beta).
$$

Proof of (4.37). We have

$$
\begin{aligned}
H^R(s,t) &= h(t) - h(s) - h'(s)\,\delta X(s,t) \\
&= (F(Y_t) - F(\hat Y_t)) - (F(Y_s) - F(\hat Y_s)) - (F'(Y_s)\,Y'_s - F'(\hat Y_s)\,\hat Y_s)\,\delta X(s,t) \\
&= (F(Y_t) - F(Y_s) - F'(Y_s)\,Y'_s\,\delta X(s,t)) - (F(\hat Y_t) - F(\hat Y_s) - F'(\hat Y_s)\,\hat Y'_s\,\delta X(s,t)).
\end{aligned}
$$

By summing and subtracting $F'(Y_s)\,\delta Y(s,t)$ and $F'(\hat Y_s)\,\delta \hat Y(s,t)$, we write

$$
\begin{aligned}
H^R(s,t) = &\Big(F(Y_s + \delta Y(s,t)) - F(Y_s) - F'(Y_s)\,\delta Y(s,t)\Big) + \\
&- \Big(F(\hat Y_s + \delta \hat Y(s,t)) - F(\hat Y_s) - F'(\hat Y_s)\,\delta \hat Y(s,t)\Big) \\
&+ F'(Y_s)\,(\delta Y(s,t) - Y'{s}\,\delta X(s,t)) - F'(\hat Y_s)\,(\delta \hat Y(s,t) - \hat Y'{s}\,\delta X(s,t)).
\end{aligned}
\tag{4.39}
$$

For the last two terms, we can write

$$
\begin{aligned}
&|F'(Y_s)\,(\delta Y(s,t) - Y'{s}\,\delta X(s,t)) - F'(\hat Y_s)\,(\delta \hat Y(s,t) - \hat Y'{s}\,\delta X(s,t))| \\
&= |F'(Y_s)\,Y^R(s,t) - F'(\hat Y_s)\,\hat Y^R(s,t)| \\
&\le |F'(Y_s)|\,|Y^R(s,t) - \hat Y^R(s,t)| + |Y^R(s,t)|\,|F'(Y_s) - F'(\hat Y_s)| \\
&\le C_F\,\|Y^R - \hat Y^R\|_{2\beta}\,|t-s|^{2\beta} + \|Y^R\|_{2\beta}\,|t-s|^{2\beta}\,C_F\,\|Y - \hat Y\|_\infty \\
&= C_F\,\Big(\|Y^R - \hat Y^R\|_{2\beta} + \|Y^R\|_{2\beta}\,\|Y - \hat Y\|_\infty\Big)\,|t-s|^{2\beta}.
\end{aligned}
$$

Now we look at the first two terms of (4.39). If we define

$$G(Y,\delta) := F(Y+\delta) - F(Y) - F'(Y)\,\delta, \tag{4.40}$$

we write these two first terms of (4.39) as

$$\begin{aligned} &G(Y_s,\delta Y(s,t)) - G(\hat Y_s,\delta\hat Y(s,t))\\ &= \underbrace{\big(G(Y_s,\delta Y(s,t)) - G(\hat Y_s,\delta Y(s,t))\big)}_{A} + \underbrace{\big(G(\hat Y_s,\delta Y(s,t)) - G(\hat Y_s,\delta\hat Y(s,t))\big)}_{B}. \end{aligned}$$

By (4.40), we have

$$G(Y,\delta) = \int_0^1 (F'(Y+u\,\delta) - F'(Y))\,\delta\,\mathrm{d}u,$$

hence

$$|G(Y,\delta)| \le \frac{1}{2}\,C_F\,|\delta|^2, \tag{4.41}$$

and we write

$$A = \int_0^1\int_0^1 (F''(Y_s + u\,v\,\delta Y(s,t)) - F''(\hat Y_s + u\,v\,\delta Y(s,t)))\,\delta^2\,u\,\mathrm{d}u\,\mathrm{d}v,$$

which yiels that

$$|A| \le \frac{1}{2}\,C_F\,\|Y-\hat Y\|_\infty\,|\delta Y(s,t)|^2 \le \frac{1}{2}\,C_F\,\|Y-\hat Y\|_\infty\,\|Y\|_\beta^2\,|t-s|^{2\beta}.$$

For B, we can write

$$B = F(\hat Y_s + \delta Y(s,t)) - F(\hat Y_s + \delta\hat Y(s,t)) - F'(\hat Y_s)\,(\delta Y(s,t) - \delta\hat Y(s,t)).$$

By adding and subtracting the term $F'(\hat Y_s + \delta Y(s,t))\,(\delta Y(s,t) - \delta\hat Y(s,t))$, we write

$$\begin{aligned} B &= F(\hat Y_s + \delta Y(s,t)) - F(\hat Y_s + \delta\hat Y(s,t)) - F'(\hat Y_s + \delta Y(s,t))\,(\delta Y(s,t) - \delta\hat Y(s,t)) +\\ &\quad + (F'(\hat Y_s + \delta Y(s,t)) - F'(\hat Y_s))\,(\delta Y(s,t) - \delta\hat Y(s,t))\\ &= G(\hat Y_s + \delta\hat Y(s,t), \delta Y(s,t) - \delta\hat Y(s,t)) + (F'(\hat Y_s + \delta Y(s,t)) - F'(\hat Y_s))\,(\delta Y(s,t) - \delta\hat Y(s,t)). \end{aligned}$$

Then, recalling also (4.41), we have

$$\begin{aligned} |B| &\le \frac{1}{2}\,C_F\,|\delta Y(s,t) - \delta\hat Y(s,t)|^2 + C_F\,|\delta Y(s,t)|\,|\delta Y(s,t) - \delta\hat Y(s,t)|\\ &\le \frac{1}{2}\,C_F\,\|Y-\hat Y\|_\beta^2\,|t-s|^{2\beta} + C_F\,\|Y\|_\beta\,\|Y-\hat Y\|_\beta\,|t-s|^\beta. \end{aligned}$$

Then, summing up the estimates, we get

$$\begin{aligned} &\|H^R\|_{2\beta}\\ &\le C_F\left(\frac{1}{2}\,\|Y-\hat Y\|_\beta^2 + \|Y\|_\beta\,\|Y-\hat Y\|_\beta + \frac{1}{2}\,\|Y\|_\beta^2\,\|Y-\hat Y\|_\infty + \|Y^R - \hat Y^R\|_{2\beta} + \|Y^R\|_{2\beta}\,\|Y-\hat Y\|_\infty\right), \end{aligned}$$

that is (4.37). □

Proof of Proposition 4.11. Before the proof of Proposition 4.11, we need the following result, whose proof is postponed to Chapter 5 (see Proposition 5.7).

PROPOSITION 4.18. Let us suppose that $G = (g, g')$ is a path controlled by X in $\mathcal{D}_X^{(\beta,\beta)}$ and $p : (0, \infty) \to \mathbb{R}$ is a function with a singularity of order η for $u = 0$ and is a C^2 function elsewhere such that satisfies (4.4) for some $c > 0$.

Then, if $\theta - \eta < 1$,

$$\Big| \int_{\mathbb{R}} g_u \left(p_{t-u} - p_{s-u} \right) \mathrm{d}W_u \Big| \le c_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\, (t-s)^{\theta-\eta} \left(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\, t^{2\beta} + \|g'\|_\infty\, t^\beta + \|g\|_\infty \right). \tag{4.42}$$

If $\theta + \beta - \eta < 1$, we have

$$\begin{aligned} &\Big| \int_{\mathbb{R}} (g_u - g_s) \left(p_{t-u} - p_{s-u} \right) \mathrm{d}W_u \Big| \\ &\le c_{\beta,\theta,\eta}\, C_{X,W,\mathbb{XW}}\, c\, (t-s)^{\theta+\beta-\eta} \left(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\, t^\beta + \|g'\|_\infty + \|g\|_\beta \right), \end{aligned} \tag{4.43}$$

where $c_{\beta,\theta,\eta}$ is a constant which depends only on β, θ, η and we use the convention that $p_u \equiv 0$ when $u \le 0$.

Thanks to this result, we can easily prove Proposition 4.11.

PROOF OF PROPOSITION 4.11. We know that, if (g, g') controlled path of X in $\mathcal{D}_X^{(\beta,\beta)}$, then, for any $t \in [0, T]$,

$$I_g(t) = \int_0^t g_u\, p_{t-u}\, \mathrm{d}W_u,$$

is well defined. In order to prove that (I_g, g) is in $\mathcal{D}_X^{(\beta,\beta)}$, we have to prove the following relation:

$$I_g(t) - I_g(s) = g_s\, \delta X_{s,t} + \mathrm{O}(|t-s|^{2\beta}),$$

uniformly for $|t - s| \to 0$.

By using the definition (4.22) of I_g, we can write

$$\begin{aligned} I_g(t) - I_g(s) &= \int_0^t g_u \left(p_{t-u} - p_{s-u} \right) \mathrm{d}W_u \\ &= g_s \int_0^t \left(p_{t-u} - p_{s-u} \right) \mathrm{d}W_u + \int_0^t (g_u - g_s) \left(p_{t-u} - p_{s-u} \right) \mathrm{d}W_u. \end{aligned}$$

The first integral equals $\delta X_{s,t}$, while, thanks to Proposition 4.18, the second integral is $\mathrm{O}(|t - s|^{2\beta})$, and, in particular, relation (4.23) follows. $\square$

Chapter 5

Finer Estimate for the Solution to a Linear Rough Fractional SDE

Introduction

In Chapter 4 we proved the existence and uniqueness of a solution to a general class of rough fractional stochastic differential equations. In this Chapter, we are going to find finer estimates of the behavior of the solution, which will provide an equivalent characterization bound on its increments. For simplicity, we will focus on the *linear* case, that is:

$$Y_t = \xi + \int_0^t Y_u \, p_{t-u} \, \mathrm{d}W_u, \tag{5.1}$$

where we recall that:

$$\begin{aligned}
&\xi \in \mathbb{R} \\
&\beta, \theta, \eta \in (0,1) \qquad \text{with } \beta := \theta - \eta \\
&\beta + \theta < 1, \quad 2\beta + \theta > 1, \quad \eta < \theta, \\
&u \mapsto p_u \quad \text{is a } C^{2\beta} \text{ function in } (0,\infty) \text{ with a discontinuity of order } \eta \text{ in zero} \\
&W \in C^{\theta}
\end{aligned}$$

and the solution Y turns to be a path controlled by the function X, where

$$X_t := \int_0^t p_{t-u} \, \mathrm{d}W_u, \text{ and } X \in C^{\beta} = C^{\theta-\eta}.$$

We also fix $\mathbb{XW}$, the "remainder" of the integral of X with respect to W, defined as a two-variable function in $C_2^{\beta+\theta}$ such that satisfies the Chen relation, that is:

$$\delta\mathbb{XW}(s,u,t) = \delta X(s,u) \, \delta W(u,t)$$

(cfr. (3.17)).

We recall that our motivation was the special case when W is a Brownian motion and $p_{t-u} = (t-u)^{H-\frac{1}{2}}$. In this case, X is a so-called Riemann-Liouville fractional Brownian motion with Hurst exponent H where, with the notation used above,

$$\theta = \frac{1}{2} - \varepsilon, \qquad \eta = \frac{1}{2} - H, \qquad \beta = \theta - \eta = H - \varepsilon, \qquad \text{for any } \varepsilon > 0,$$

and then, in order to satisfies the relations between θ, β and η, it turns out that we need to consider

$$H \in \Big(\frac{1}{4}, \frac{1}{2}\Big).$$

Since this is the case of our interest, in this Chapter we stick to it, even if, for convienience, we mantain the notation written as in (5.1).

In order to search for finer estimate of the solution, we find a new writing for $\delta Y(s,t)$. Starting from (5.1), we write for arbitrary $s, t \geq 0$

$$Y_t - Y_s = \int_{\mathbb{R}} Y_u \,(p_{t-u} - p_{s-u})\, \mathrm{d}W_u = Y_s\, \delta X(s,t) + \tilde{Y}^R(s,t), \tag{5.2}$$

with the convention that $p_u \equiv 0$ when $u \leq 0$ and where $\tilde{Y}^R : [0,\infty)^2 \to \mathbb{R}$ is a function in $C_2^{2\beta}$ with the arguments (s,t) not necessary ordered. It is defined as

$$\tilde{Y}^R(v,u) = \delta Y(v,u) - Y_v\, \delta X(v,u) = \begin{cases} Y^R(v,u) & \text{if } v \leq u \\ \delta Y(u,v)\, \delta X(u,v) - Y^R(u,v) & \text{if } v > u. \end{cases} \tag{5.3}$$

where $Y^R : [0,\infty)^2_< \to \mathbb{R}$ is the remainder with ordered arguments defined after Definitin 3.9 (see (3.16)). The equation (5.2) is the starting point to find a new finer expression. We recall that

$$|\delta X(s,t)| = |X_t - X_s| = \mathrm{O}(|t-s|^\beta) \qquad \text{and} \qquad |\tilde{Y}^R(s,t)| = \mathrm{O}(|t-s|^{2\beta}),$$

where $\beta < \frac{1}{2}$. The idea is to expand the remainder $\tilde{Y}^R$ until we find a term which is $\mathrm{o}(|t-s|)$. Of course, the expansion will be more and more complicated when β becomes smaller. Recall that, in order to make sense of the SDE driven by a fractional Brownian motion, the Hurst parameter $H = \beta - \varepsilon$ should be in $(\frac{1}{4}, \frac{1}{2})$ and $\theta < \frac{1}{2}$, since it represents the Hölder regularity of a path of a Brownian motion process. Then, for our interests, we will treat the case $\beta \in (\frac{1}{4}, \frac{1}{2})$.

Description of the Chapter.

- In Section 5.1 we present the results and we state the main theorem.
- In Section 5.2, we state and prove the key result to get the finer estimates for the solution. Proposition 5.6 permits to get important bounds on the rough integrals with singular kernels.
- In Section 5.3, we prove the theorem for the "characterization" of the solution. The proof is divided in three steps: the first one is a general result; the secon step permits to find a characterization when $\beta \in (\frac{1}{3}, \frac{1}{2})$; the third step gives a characterization when $\beta \in (\frac{1}{4}, \frac{1}{3})$.
- In Section 5.4, we collect some technical proofs.

5.1. Main result

Let us consider the case of the linear rough differential equation (see (4.6) in Chapter 4), that can be written as

$$Y_t = \xi + \int_0^t Y_u\, p_{t-u}\, \mathrm{d}W_u \qquad \text{for } t \in [0,T]. \tag{5.4}$$

Thanks to Theorem 4.6, we know that there exists a unique global solution Y in the space of paths controlled by $X_t = \int_0^t p_{t-u}\, \mathrm{d}W_u$, where the integral in (5.4) is a rough integral.

We need the following further condition: $p : [0,\infty) \to \mathbb{R}$ is a function which is C^3 except a singularity at zero of order η, more precisely there exists some positive $c > 0$ such that

$$|p_u| \le \frac{c}{u^\eta}, \qquad |p'_u| \le \frac{c}{u^{\eta+1}}, \qquad |p''_u| \le \frac{c}{u^{\eta+2}}, \qquad |p'''_u| \le \frac{c}{u^{\eta+3}} \tag{5.5}$$

We start from (5.2) and find new expansions for the increment $Y_t - Y_s$, when $|t-s| \to 0$. To this purpose, we need to define the following objects, which will provide a finer description of $Y_t - Y_s$:

$$\tilde{\mathbb{XW}}(s,t) := \int_{\mathbb{R}} \delta X(s,u)\,(p_{t-u} - p_{s-u})\,\mathrm{d}W_u, \tag{5.6}$$

$$C(Y,s) := \int_{\mathbb{R}} \tilde{Y}^R(s,u)\,p'_{s-u}\,\mathrm{d}W_u, \tag{5.7}$$

$$\mathfrak{XW}(s,t) = \int_{\mathbb{R}} \tilde{\mathbb{XW}}(s,u)\,(p_{t-u} - p_{s-u} - (t-s)\,p'_{s-u})\,\mathrm{d}W_u \tag{5.8}$$

where $\tilde{Y}^R$ is defined in (5.3).

A key step for the main Theorem 5.2 is the following Proposition.

Proposition 5.1.

$$|\tilde{\mathbb{XW}}(s,t)| = \mathrm{O}(|t-s|^{2\beta})$$
$$|C(Y,s)| = \mathrm{O}(s^{3\beta-1})$$
$$|\mathfrak{XW}(s,t)| = \mathrm{O}(|t-s|^{3\beta})$$

Theorem 5.2. $Y \in \mathcal{D}_X^{(\beta,\beta)}$ is a solution of (5.4) if and only if, for any $0 < s < t$:

(i) for $\frac{1}{3} < \beta < \frac{1}{2}$,

$$\delta Y(s,t) = Y_s\,\delta X(s,t) + Y_s\,\tilde{\mathbb{XW}}(s,t) + (t-s)\,C(Y,s) + \mathrm{o}(|t-s|); \tag{5.9}$$

(ii) for $\frac{1}{4} < \beta < \frac{1}{3}$,

$$\delta Y(s,t) = Y_s\,\delta X(s,t) + Y_s\,\tilde{\mathbb{XW}}(s,t) + (t-s)\,C(Y,s) + Y_s\,\mathfrak{XW}(s,t) + \mathrm{o}(|t-s|). \tag{5.10}$$

It is enough to prove that the solution Y of (5.4) satisfies (5.9)-(5.10), because there can be at most one Y which satisfies (5.9)-(5.10). This follows by Lemma 3.2, because the r.h.s. of these equations are linear in Y (note that $C(Y,s)$ is linear in Y).

Remark 5.3. The "characterization" given by Theorem 5.2 is not local in the sense that the r.h.s. of (5.9)-(5.10) is not a local function of Y, because the "coefficient" $C(Y,s)$ depends on the whole path Y and not only on the value Y_s. This is in contrast with the case of the usual SDE

$$\mathrm{d}Y_t = F(Y_t)\,\mathrm{d}W_t$$

which, in our language, would correspond to taking $p_t := \mathbb{1}_{[0,\infty)}(t)$ as the Heaviside function.

Sketch of the proof of Theorem 5.2. We will prove Theorem 5.2 and Proposition 5.1 simultaneously.

First, in Section 5.2, we present the key result to prove Theorem 5.2, that is the fundametal Proposition 5.6. This proposition permits to have finer estimates for rough integrals with singular kernels and is formulated in a more general way. We used the first part of that proposition to prove Proposition 4.18 whose proof we have postponed in this chapter and which we used in turn to prove Proposition 4.11.

Once we get the estimates from Proposition 5.6, we can prove Proposition 5.1 and then the equations (5.9) and (5.10) by using repeatedly the integral formulation of the solution (5.4).

5.2. Fundamental Proposition for Rough Integral with Singular Kernels

In this section we are going to prove a fundamental proposition that will be used in the study of the rough fractional SDE of Chapter 4.

In Chapter 3, Section 3.3, we defined the integral in the form

$$I_{g\bar{p}}(t) := \int_0^t g\,\bar{p}_u\,\mathrm{d}W_u,$$

where $\bar{p}$ is a kernel with a singularity of order $\bar{\eta}$ and $G = (g, g')$ is a controlled path by $X \in C^\beta$. In particular, by Corollary 3.17, we got the following:

$$|I_{\bar{p}g}(t) - I_{\bar{p}g}(s)| \le \tilde{c}_{\beta,\theta,\bar{\eta}}\, c\, C_{W,\mathbb{XW}} \left(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}} (t-s)^{2\beta} + \|g'\|_\infty (t-s)^\beta + \|g\|_\infty \right) (t-s)^{\theta-\bar{\eta}}, \tag{5.11}$$

where $\tilde{c}_{\beta,\theta,\bar{\eta}}$ is a positive constant which depends only on $\theta, \bar{\eta}$ and $\beta := \theta - \bar{\eta}$ and can be defined as in (3.41) and $C_{W,\mathbb{XW}} := \max\{\|W\|_\theta, \|\mathbb{XW}\|_{\beta+\theta}\}$, as defined in (3.22).

We give two different and stronger assumptions on the function $\bar{p}$. We assume that $\bar{p}$ has a singularity in some $s \in [0, \infty)$ of order $\bar{\eta}$ in the sense of (3.29), but also such that its behavior improves "far" from s, in the following sense:

$$|\bar{p}_u| \le \begin{cases} \dfrac{c}{|u-s|^{\bar{\eta}}} & \text{if } |u-s| < \delta \\ \dfrac{c\,\delta}{|u-s|^{\bar{\eta}+1}} & \text{if } |u-s| \ge \delta. \end{cases} \tag{5.12}$$

and

$$|\bar{p}'_u| \le \begin{cases} \dfrac{c}{|u-s|^{\bar{\eta}+1}} & \text{if } |u-s| < \delta \\ \dfrac{c\,\delta}{|u-s|^{\bar{\eta}+2}} & \text{if } |u-s| \ge \delta. \end{cases} \tag{5.13}$$

The reason for the above conditions (5.12) and (5.13) is that we will deal with rough integrals $\int g\,\bar{p}\,\mathrm{d}W_u$ where $\bar{p}$ is a "remainder" in the following sense:

$$\bar{p}_u = p_{t-u} - p_{s-u} \qquad \text{or} \qquad \bar{p}_u = p_{t-u} - p_{s-u} - (t-s)\,p'_{s-u}, \qquad \text{with} \qquad p_u \sim \frac{c}{u^\eta}.$$

These functions satisfies (5.12) and (5.13) with $\bar{\eta} = \eta$ and $\delta = (t-s)$ for the first case and, in the second case, with $c = c\,(t-s)$, $\bar{\eta} = \eta + 1$ and $\delta = (t-s)$ (see Corollaries 5.7 and 5.8).

The following Lemma can be compared with Lemma 3.15; in the following one, we use the stronger assumptions (5.12) and (5.13).

LEMMA 5.4. Fix $\beta \in (0, \frac{1}{2})$. If $\bar{p} : [0, T] \to \mathbb{R}$ is a $C^{2\beta}$ function with a singularity at s and such that (5.12) and (5.13) hold for some $c > 0$ and $\delta > 0$. Then, for all the intervals $[a, b]$, with $0 \le a < b < s$, and $s - b \ge \delta$, we have

$$|||\bar{p}|||_{[a,b]} \le \frac{c\delta}{(s-b)^{\bar{\eta}+1}} \Big(1 + \frac{b-a}{s-b}\Big). \tag{5.14}$$

PROOF. We recall the definition of $|||\bar{p}|||$:

$$|||\bar{p}|||_{[a,b]} := \|\bar{p}\|_\infty|_{[a,b]} + (b-a)^{2\beta} \, \|\bar{p}\|_{2\beta}|_{[a,b]}.$$

We can write

$$\sup_{u \in [a,b]} |\bar{p}_u| \le \sup_{u \in [a,b]} \frac{c\delta}{(s-u)^{\bar{\eta}+1}} \le \frac{c\,\delta}{(s-b)^{\bar{\eta}+1}},$$

having used condition (5.12), and noting that $s - u \ge s - b$ for any $u \in [a, b]$. Also we can write

$$\sup_{a \le v < u \le b} \frac{|\bar{p}_u - \bar{p}_v|}{(u-v)^{2\beta}} \le \sup_{a \le v < u \le b} \sup_{z \in [v,u]} |\bar{p}'_z| \, (u-v)^{1-2\beta} \le \frac{c\delta}{(s-b)^{\bar{\eta}+2}} \, (b-a)^{1-2\beta}.$$

Then

$$|||\bar{p}|||_{[a,b]} \le \frac{c\delta}{(s-b)^{\bar{\eta}+1}} \Big(1 + \frac{b-a}{s-b}\Big).$$

□

REMARK 5.5. If $\bar{p} : [0, T] \to \mathbb{R}$ has a singularity at s of order $\bar{\eta}$, i.e. it satisfies (3.29), then, for any $[a, b]$ with $0 \le a < b < s$, we can write

$$|||\bar{p}|||_{[a,b]} \le \frac{c}{(s-b)^{\bar{\eta}}} \Big(1 + \frac{b-a}{s-b}\Big),$$

thanks to Lemma 3.15. We note that this relation is weaker than (5.14) when $s - b \ge \delta$. Then assumptions (5.12) and (5.13) are needed to prove the following proposition, where (5.14) is used repeatedly.

PROPOSITION 5.6. Fix $s \in [0, \infty)$. Let $g : [0, s] \to \mathbb{R}$ be a controlled path of X and let $\bar{p} : [0, s) \to \mathbb{R}$ be a $C^{2\beta}$ function, with a singularity at s, and which satisfies (5.12) and (5.13).
For any $G = (g, g')$ path controlled by X, we have

$$\Big| \int_0^s g_u \, \bar{p}_u \, \mathrm{d}W_u \Big| \le \hat{c}_{\beta,\theta,\bar{\eta}} \, C_{W,\mathbb{XW}} \, c \, \delta^{\theta-\bar{\eta}} \Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}} \, s^{2\beta} + \|g'\|_\infty \, s^\beta + \|g\|_\infty \Big). \tag{5.15}$$

Moreover:

(i) If g satisfies

$$|g_u| \leq K\,|s-u|^{\beta}, \tag{5.16}$$

then, if $\theta+\beta-\bar{\eta}<1$,

$$\Big|\int_0^s g_u\,\bar{p}_u\,\mathrm{d}W_u\Big| \leq \hat{c}_{\beta,\theta,\bar{\eta}}\,C_{W,\mathbb{XW}}\,c\,\delta^{\theta+\beta-\bar{\eta}}\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,s^{\beta}+\|g'\|_{\infty}+2K\Big). \tag{5.17}$$

(ii) If g satisfies:

$$\begin{aligned} |g_u| &\leq K\,|s-u|^{2\beta} \\ |g'_u| &\leq K\,|s-u|^{\beta}, \end{aligned} \tag{5.18}$$

then, if $\theta+2\beta-\bar{\eta}<1$,

$$\Big|\int_0^s g_u\,\bar{p}_u\,\mathrm{d}W_u\Big| \leq \hat{c}_{2\beta,\theta,\bar{\eta}}\,C_{W,\mathbb{XW}}\,c\,\delta^{\theta+2\beta-\bar{\eta}}\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}+4K\Big) \tag{5.19}$$

(iii) If g satisfies:

$$\begin{aligned} |g_u| &\leq K\,|s-u|^{3\beta} \\ |g'_u| &\leq K\,|s-u|^{2\beta} \\ \|G\|_{\mathcal{D}_X^{(\beta,\beta)}}|_{[u,s]} &\leq K\,(s-u)^{\beta} \end{aligned} \tag{5.20}$$

then, if $\theta+3\beta-\bar{\eta}<1$,

$$\Big|\int_0^s g_u\,\bar{p}_u\,\mathrm{d}W_u\Big| \leq \hat{c}_{3\beta,\theta,\bar{\eta}}\,C_{W,\mathbb{XW}}\,c\,\delta^{\theta+3\beta-\bar{\eta}}\,8K. \tag{5.21}$$

In (5.17), (5.19) and (5.21), the constants $c>0$ and $\delta>0$ are the same that appear in (5.12)-(5.13) and we have used the following notations:

$$\hat{c}_{n\beta,\theta,\bar{\eta}} := \frac{1}{1-2^{-(2\beta+\theta-1)}}\,\frac{1}{1-2^{n\beta+\theta-\bar{\eta}-1}} \qquad \text{for } n=1,2,3,$$

and

$$C_{W,\mathbb{XW}} := \max\{\|W\|_{\theta},\|\mathbb{XW}\|_{\beta+\theta}\}.$$

Proof. We define

$$I_{[0,s]} := \int_0^s g_u\,\bar{p}_u\,\mathrm{d}W_u.$$

If we use Corollary 3.17 (note that (5.12)-(5.13) imply (3.29)), then we can write

$$|I_{[0,s]}| \leq \tilde{c}_{\beta,\theta,\bar{\eta}}\,C_{W,\mathbb{XW}}\,c\,s^{\theta-\bar{\eta}}\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,s^{2\beta}+\|g'\|_{\infty}\,s^{\beta}+\|g\|_{\infty}\Big).$$

Then, if $s\leq 2\delta$, we have proved (5.15). Moreover, if g satisfies (5.16), (5.18) or (5.20), we obtain (5.17), (5.19) or (5.21), respectively.

Now we just have to prove the proposition in the case of $s > 2\delta$. In this case, there exists $n = n(\delta) \in \mathbb{N}$ such that $2^n\,\delta < s \le 2^{n+1}\,\delta$. We can divide the integral and write

$$
\begin{aligned}
|I_{[0,s]}| &\le \Big| \int_0^{\frac{s}{2}} g_u\,\bar{p}_u\,\mathrm{d}W_u \Big| + \sum_{i=1}^{n} \Big| \int_{s_{i+1}}^{s_i} g_u\,\bar{p}_u\,\mathrm{d}W_u \Big| + \Big| \int_{s-\delta}^{s} g_u\,\bar{p}_u\,\mathrm{d}W_u \Big| \\
&= |I_{[0,\frac{s}{2}]}| + \sum_{i=1}^{n} \Big| I_{[s_{i+1},s_i]}| + |I_{[s-\delta,s]}|,
\end{aligned}
\tag{5.22}
$$

where $s_i = s - 2^{i-1}\,\delta$. For the last integral, $I_{[s-\delta,s]}$, we use Corollary 3.17 and we write

$$
\Big| \int_{s-\delta}^{s} g_u\,\bar{p}_u\,\mathrm{d}W_u \Big| \le \tilde{c}_{\beta,\theta,\bar{\eta}}\, C_{W,\mathbb{XW}}\, c\,\delta^{\theta-\bar{\eta}} \Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,\delta^{2\beta} + \|g'\|_\infty\,\delta^\beta + \|g\|_\infty \Big).
$$

If g satisfies (5.16), and then $\|g\|_\infty|_{[s-\delta,s]} \le K\delta^\beta$, we obtain (5.17):

$$
\Big| \int_{s-\delta}^{s} g_u\,\bar{p}_u\,\mathrm{d}W_u \Big| \le c_{\beta,\theta,\bar{\eta}}\, C_{W,\mathbb{XW}}\, c\,\delta^{\theta+\beta-\bar{\eta}} \Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,\delta^{\beta} + \|g'\|_\infty + K \Big).
$$

If g satisfies (5.18), we obtain (5.19):

$$
\Big| \int_{s-\delta}^{s} g_u\,\bar{p}_u\,\mathrm{d}W_u \Big| \le c_{\beta,\theta,\bar{\eta}}\, C_{W,\mathbb{XW}}\, c\,\delta^{\theta+2\beta-\bar{\eta}} \Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}} + K \Big),
$$

and, if g satisfies (5.20), we obtain (5.21):

$$
\Big| \int_{s-\delta}^{s} g_u\,\bar{p}_u\,\mathrm{d}W_u \Big| \le c_{\beta,\theta,\bar{\eta}}\, C_{W,\mathbb{XW}}\, c\,\delta^{\theta+3\beta-\bar{\eta}}\, K.
$$

Hence it remains to prove the estimates for the remaining integrals in (5.22). If we use (3.26), we can write

$$
\begin{aligned}
&\Big| \int_0^{\frac{s}{2}} g_u\,p_u\,\mathrm{d}W_u \Big| + \sum_{i=1}^{n} \Big| \int_{s_{i+1}}^{s_i} g_u\,p_u\,\mathrm{d}W_u \Big| \\
&\qquad \le \Big(\frac{s}{2}\Big)^\theta\, \|\!|p|\!\|_{[0,\frac{s}{2}]}\, \|I_g\|_\theta|_{[0,\frac{s}{2}]} + \sum_{i=1}^{n} (s_i - s_{i+1})^\theta\, \|\!|p|\!\||_{[s_{i+1},s_i]}\, \|I_g\|_\theta|_{[s_{i+1},s_i]},
\end{aligned}
\tag{5.23}
$$

We are going to give an estimate to the norms in this relation.

For the norms in the interval $[0,\frac{s}{2}]$ we can use condition (5.12), since $\frac{s}{2} > \delta$, and, by Lemma 5.4, we can write

$$
\|\!|p|\!\||_{[0,\frac{s}{2}]} \le c \Big(\frac{s}{2}\Big)^{-\eta-1} \delta \le c\,2^{\bar{\eta}+1}\,2^{n(-\bar{\eta}-1)}\,\delta^{-\bar{\eta}},
\tag{5.24}
$$

recalling also that $s > 2^n\,\delta$. For I_g, thanks to Theorem 3.11 (in particular, relation (3.21)), we write

$$
\|I_g\|_\theta|_{[0,\frac{s}{2}]} \le c_{2\beta+\theta}\, C_{W,\mathbb{XW}} \Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)},[0,\frac{s}{2}]} \Big(\frac{s}{2}\Big)^{2\beta} + \|g'\|_{\infty,[0,\frac{s}{2}]} \Big(\frac{s}{2}\Big)^{\beta} + \|g\|_{\infty,[0,\frac{s}{2}]} \Big).
\tag{5.25}
$$

If g satisfies (5.16), then

$$\|g\|_{\infty,[0,\frac{s}{2}]} \le K \sup_{u\in[0,\frac{s}{2}]} (s-u)^{\beta} = K\, s^{\beta} \le 2\,K \left(\frac{s}{2}\right)^{\beta},$$

and we can write

$$\begin{aligned}\|I_g\|_{\theta}|_{[0,\frac{s}{2}]} &\le c_{2\beta+\theta}\, C_{W,\mathbb{XW}} \left(\frac{s}{2}\right)^{\beta} \left(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}} \left(\frac{s}{2}\right)^{\beta} + \|g'\|_{\infty,[0,\frac{s}{2}]} + 2K\right)\\ &\le c_{2\beta+\theta}\, C_{W,\mathbb{XW}}\, 2^{n\beta}\, \delta^{\beta} \left(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\, s^{\beta} + \|g'\|_{\infty} + K\right),\end{aligned}$$

since $s \le 2^{n+1}\,\delta$.

If g satisfies (5.18), then we have the following estimates

$$\|g\|_{\infty,[0,\frac{s}{2}]} \le K \sup_{u\in[0,\frac{s}{2}]} (s-u)^{2\beta} \le K\, 2^{2\beta} \left(\frac{s}{2}\right)^{2\beta} \le 2\,K \left(\frac{s}{2}\right)^{2\beta}$$

and

$$\|g'\|_{\infty,[0,\frac{s}{2}]} \le K \sup_{u\in[0,\frac{s}{2}]} (s-u)^{\beta} \le K\, s^{\beta} \le 2\,K \left(\frac{s}{2}\right)^{\beta}.$$

Then, by (5.25), we can write in this case

$$\begin{aligned}\|I_g\|_{\theta}|_{[0,\frac{s}{2}]} &\le c_{2\beta+\theta}\, C_{W,\mathbb{XW}} \left(\frac{s}{2}\right)^{2\beta} \left(\|G\|_{\mathcal{D}_X^{(\beta,\beta)},[0,\frac{s}{2}]} + 4K\right)\\ &\le c_{2\beta+\theta}\, C_{W,\mathbb{XW}}\, 2^{2n\beta}\, \delta^{\beta} \left(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}} + 4K\right).\end{aligned}$$

If g satisfies (5.20), then we know that

$$\begin{aligned}\|g\|_{\infty,[0,\frac{s}{2}]} &\le K \sup_{u\in[0,\frac{s}{2}]} (s-u)^{3\beta} \le K\, 2^{3\beta} \left(\frac{s}{2}\right)^{3\beta} \le 4\,K \left(\frac{s}{2}\right)^{3\beta},\\ \|g'\|_{\infty,[0,\frac{s}{2}]} &\le K \sup_{u\in[0,\frac{s}{2}]} (s-u)^{2\beta} \le K\, 2^{2\beta} \left(\frac{s}{2}\right)^{2\beta} \le 2\,K \left(\frac{s}{2}\right)^{2\beta}\end{aligned}$$

and

$$\|G\|_{\mathcal{D}_X^{(\beta,\beta)},[0,\frac{s}{2}]} \le \|G\|_{\mathcal{D}_X^{(\beta,\beta)},[0,s]} \le K\, s^{\beta} = K\, 2^{\beta} \left(\frac{s}{2}\right)^{\beta} \le 2\,K \left(\frac{s}{2}\right)^{\beta}.$$

Then, we can write in this case

$$\begin{aligned}\|I_g\|_{\theta}|_{[0,\frac{s}{2}]} &\le c_{2\beta+\theta}\, C_{W,\mathbb{XW}} \left(\frac{s}{2}\right)^{3\beta} K\\ &\le c_{2\beta+\theta}\, C_{W,\mathbb{XW}}\, 2^{3n\beta}\, 8K.\end{aligned}$$

For the integrals $I_{[s_{i+1},s_i]}$, we first notice that

$$s - s_i = s_i - s_{i+1} = 2^{i-1}\,\delta \qquad \text{and} \qquad s - s_{i+1} = 2^{i}\,\delta \qquad \text{for } i = 1,\dots,n.$$

Since also $s - s_i \geq \delta$, for $i = 1, \ldots, n$, we have by Lemma 5.4

$$|||\bar{p}|||_{[s_{i+1},s_i]} \leq \frac{2c}{(s-s_i)^{\bar{\eta}+1}}\,\delta \leq c\,2^{(i-1)(-\bar{\eta}-1)}\,\delta^{-\bar{\eta}}. \tag{5.26}$$

By using relation (3.21), for $\|I_g\|_{\theta,[s_{i+1},s_i]}$, we write

$$\begin{aligned}
&\|I_g\|_{\theta,[s_{i+1},s_i]} \\
&\leq c_{2\beta+\theta}\,C_{W,\mathbb{XW}}\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)},[s_{i+1},s_i]}\,(s_i - s_{i+1})^{2\beta} + \|g'\|_{\infty,[s_{i+1},s_i]}\,(s_i - s_{i+1})^{\beta} + \|g\|_{\infty,[s_{i+1},s_i]}\Big) \\
&= c_{2\beta+\theta}\,C_{W,\mathbb{XW}}\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)},[s_{i+1},s_i]}\,(2^{i-1}\,\delta)^{2\beta} + \|g'\|_{\infty,[s_{i+1},s_i]}\,(2^{i-1}\,\delta)^{\beta} + \|g\|_{\infty,[s_{i+1},s_i]}\Big).
\end{aligned} \tag{5.27}$$

By proceeding in a similar way as we have done above for $I_{[0,\frac{s}{2}]}$, if g satisfies (5.16), then

$$\|g\|_{\infty,[s_{i+1},s_i]} \leq K \sup_{u\in[s_{i+1},s_i]}(s-u)^{\beta} = K\,(s-s_{i+1})^{\beta} = K\,(2^i\,\delta)^{\beta} \leq 2\,K\,(2^{i-1}\,\delta)^{\beta}$$

and we can write

$$\begin{aligned}
\|I_g\|_{\theta,[s_{i+1},s_i]} &\leq c_{2\beta+\theta}\,C_{W,\mathbb{XW}}\,\delta^{\beta}\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,(2^{i-1}\delta)^{\beta} + \|g'\|_{\infty} + 2K\Big)\,2^{(i-1)\beta} \\
&\leq c_{2\beta+\theta}\,C_{W,\mathbb{XW}}\,\delta^{\beta}\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,s^{\beta} + \|g'\|_{\infty} + 2K\Big)\,2^{(i-1)\beta}.
\end{aligned}$$

If g satisfies (5.18), then

$$\begin{aligned}
\|g\|_{\infty,[s_{i+1},s_i]} &\leq 2\,K\,(2^{i-1}\delta)^{2\beta} \\
\|g'\|_{\infty,[s_{i+1},s_i]} &\leq 2\,K\,(2^{i-1}\delta)^{\beta}
\end{aligned}$$

and we have

$$\|I_g\|_{\theta,[s_{i+1},s_i]} \leq c_{2\beta+\theta}\,C_{W,\mathbb{XW}}\,\delta^{2\beta}\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}} + 4K\Big)\,2^{(i-1)2\beta}.$$

If g satisfies (5.20), then

$$\begin{aligned}
\|g\|_{\infty,[s_{i+1},s_i]} &\leq 4\,K\,(2^{i-1}\delta)^{3\beta} \\
\|g'\|_{\infty,[s_{i+1},s_i]} &\leq 2\,K\,(2^{i-1}\delta)^{2\beta}
\end{aligned}$$

and

$$\|G\|_{\mathcal{D}_X^{(\beta,\beta)},[s_{i+1},s_i]} \leq 2\,K\,(2^{i-1}\delta)^{\beta},$$

and we can write

$$\|I_g\|_{\theta,[s_{i+1},s_i]} \leq c_{2\beta+\theta}\,C_{W,\mathbb{XW}}\,\delta^{3\beta}\,8K\,2^{(i-1)3\beta}.$$

Incidentally, if we plug $i = n+1$ in the previous bound, we find the estimate that we obtained for $I_{[0,\frac{s}{2}]}$. Then, summing up, recalling (5.23) and bounding $(\frac{s}{2})^{\theta} \leq 2^{n\theta}s^{\theta}$, it follows by (5.24), (5.25) and (5.26), (5.27) that

$$
\begin{aligned}
&|I_{[0,\frac{s}{2}]}| + \sum_{i=1}^{n} |I_{[s_{i+1},s_i]}| \\
&\le \Big(\frac{s}{2}\Big)^{\theta} \,|||\bar{p}|||_{[0,\frac{s}{2}]} \,\|I_g\|_{\theta}|_{[0,\frac{s}{2}]} + \sum_{i=1}^{n} (s_i - s_{i+1})^{\theta} \,|||\bar{p}|||_{[s_{i+1},s_i]} \,\|I_g\|_{\theta}|_{[s_{i+1},s_i]} \\
&\le c_{2\beta+\theta}\, C_{W,\mathbb{XW}}\, c\, \delta^{\theta-\bar{\eta}} \Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}} s^{2\beta} + \|g'\|_{\infty} s^{\beta} + \|g\|_{\infty} \Big) \Big(\sum_{i=1}^{n+1} 2^{(i-1)(\theta-\bar{\eta}-1)} \Big) \\
&= \hat{c}_{\beta,\theta,\bar{\eta}}\, C_{W,\mathbb{XW}}\, c\, \delta^{\theta-\bar{\eta}} \Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}} s^{2\beta} + \|g'\|_{\infty} s^{\beta} + \|g\|_{\infty} \Big),
\end{aligned}
$$

where we set

$$
\hat{c}_{\beta,\theta,\bar{\eta}} = c_{2\beta+\theta} \sum_{i=1}^{\infty} 2^{(i-1)(\theta-\bar{\eta}-1)} = \frac{1}{1-2^{-(2\beta+\theta-1)}} \frac{1}{1-2^{\theta-\bar{\eta}-1}}.
$$

Moreover:

- if g satisfies (5.16),

$$
\begin{aligned}
&|I_{[0,\frac{s}{2}]}| + \sum_{i=1}^{n} |I_{[s_{i+1},s_i]}| \\
&\le \Big(\frac{s}{2}\Big)^{\theta} \,|||\bar{p}|||_{[0,\frac{s}{2}]} \,\|I_g\|_{\theta}|_{[0,\frac{s}{2}]} + \sum_{i=1}^{n} (s_i - s_{i+1})^{\theta} \,|||\bar{p}|||_{[s_{i+1},s_i]} \,\|I_g\|_{\theta}|_{[s_{i+1},s_i]} \\
&\le c_{2\beta+\theta}\, C_{W,\mathbb{XW}}\, c\, \delta^{\theta+\beta-\bar{\eta}} \Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}} s^{\beta} + \|g'\|_{\infty} + 2K \Big) \Big(\sum_{i=1}^{n+1} 2^{(i-1)(\theta+\beta-\bar{\eta}-1)} \Big).
\end{aligned}
$$

  If $\theta + \beta - \bar{\eta} < 1$, the sum is convergent and we have

$$
\Big| \int_0^{\frac{s}{2}} g_u\, \bar{p}_u \,\mathrm{d}W_u \Big| + \sum_{i=1}^{n} \Big| \int_{s_{i+1}}^{s_i} g_u\, \bar{p}_u \,\mathrm{d}W_u \Big| \le \hat{c}_{\beta,\theta,\bar{\eta}}\, C_{W,\mathbb{XW}}\, c\, \delta^{\theta+\beta-\bar{\eta}} \Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}} s^{\beta} + \|g'\|_{\infty} + 2K \Big),
$$

  where

$$
\hat{c}_{\beta,\theta,\bar{\eta}} = c_{2\beta+\theta} \sum_{i=0}^{\infty} 2^{i(\theta+\beta-\bar{\eta}-1)} = \frac{1}{1-2^{-(2\beta+\theta-1)}} \frac{1}{1-2^{\theta+\beta-\bar{\eta}-1}}.
$$

- If g satisfies (5.18), we have

$$
\begin{aligned}
&|I_{[0,\frac{s}{2}]}| + \sum_{i=1}^{n} |I_{[s_{i+1},s_i]}| \\
&\le c_{2\beta+\theta}\, C_{W,\mathbb{XW}}\, c\, \delta^{\theta+2\beta-\bar{\eta}} \Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}} + 4K \Big) \Big(\sum_{i=1}^{n+1} 2^{(i-1)(\theta+2\beta-\bar{\eta}-1)} \Big).
\end{aligned}
$$

  Hence, if $\theta + 2\beta - \bar{\eta} < 1$, the sum is convergent and we have

$$
\Big| \int_0^{\frac{s}{2}} g_u\, \bar{p}_u \,\mathrm{d}W_u \Big| + \sum_{i=1}^{n} \Big| \int_{s_{i+1}}^{s_i} g_u\, \bar{p}_u \,\mathrm{d}W_u \Big| \le \hat{c}_{2\beta,\theta,\bar{\eta}}\, C_{W,\mathbb{XW}}\, c\, \delta^{\theta+2\beta-\bar{\eta}} \Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}} + 4K \Big).
$$

- If g satisfies (5.20) we have

$$|I_{[0,\frac{s}{2}]}| + \sum_{i=1}^{n} |I_{[s_{i+1},s_i]}| \leq c_{\beta,\theta,\bar\eta}\, C_{W,\mathbb{XW}}\, c\; \delta^{\theta+3\beta-\bar\eta}\, 8K \Big(\sum_{i=1}^{n+1} 2^{(i-1)(\theta+3\beta-\bar\eta-1)} \Big).$$

Hence, if $\theta + 3\beta - \bar\eta < 1$, the sum is convergent and we have

$$\Big| \int_0^{\frac{s}{2}} g_u\, \bar p_u \,\mathrm{d}W_u \Big| + \sum_{i=1}^{n} \Big| \int_{s_{i+1}}^{s_i} g_u\, \bar p_u \,\mathrm{d}W_u \Big| \leq \hat c_{3\beta,\theta,\bar\eta}\, C_{W,\mathbb{XW}}\, c\; \delta^{\theta+3\beta-\bar\eta}\, 8\, K.$$

□

Here we can prove Proposition 4.18, which we used in Chapter 4 and now turns out to be a particular and useful case for which Proposition 5.6 can be applied. We restate here the Proposition as a Corollary of Proposition 5.6.

Proposition 5.7. Let us suppose that $G = (g, g')$ is a controlled path by X in $\mathcal{D}_X^{(\beta,\beta)}$ and $p : (0,\infty) \to \mathbb{R}$ is a function with a singularity of order η for $u = 0$ and is a C^2 function elsewhere such that satisfies (4.4) for all u close to 0 and for some $c > 0$.

Then, if $\theta - \eta < 1$,

$$\Big| \int_{\mathbb{R}} g_u\, (p_{t-u} - p_{s-u}) \,\mathrm{d}W_u \Big| \leq \hat c_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\, (t-s)^{\theta-\eta} \Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\, t^{2\beta} + \|g'\|_\infty\, t^{\beta} + \|g\|_\infty \Big) \tag{5.28}$$

Then, if $\theta + \beta - \eta < 1$,

$$\begin{aligned} &\Big| \int_{\mathbb{R}} (g_u - g_s)\, (p_{t-u} - p_{s-u}) \,\mathrm{d}W_u \Big| \\ &\leq \hat c_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\, (t-s)^{\theta+\beta-\eta} \Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\, t^{\beta} + \|g'\|_\infty + \|g\|_\beta \Big), \end{aligned} \tag{5.29}$$

where we use the convention that $p_u \equiv 0$ when $u \leq 0$.

Proof. We can write

$$\int_0^t g_u\, (p_{t-u} - p_{s-u}) \,\mathrm{d}W_u = \int_0^s g_u\, (p_{t-u} - p_{s-u}) \,\mathrm{d}W_u + \int_s^t g_u\, p_{t-u} \,\mathrm{d}W_u.$$

For the second integral, we can use Corollary 3.17 with $\bar p_u = p_{t-u}$, which has a singularity in t of order η, and then we can write

$$\Big| \int_s^t g_u\, p_{t-u} \,\mathrm{d}W_u \Big| \leq \tilde c_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\, (t-s)^{\theta-\eta} \Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\, (t-s)^{2\beta} + \|g'\|_\infty\, (t-s)^{\beta} + \|g\|_\infty \Big).$$

It remains to prove (5.28) for the first integral over the interval $[0, s]$. This follows from (5.15) of Theorem 5.6 with $\bar p_u = p_{t-u} - p_{s-u}$, once we have proved that such a $\bar p$ satisfies conditions (5.12) and (5.13).

For $u \in [0, s)$, we can write

$$|p_{t-u} - p_{s-u}| \leq \frac{2c}{(s-u)^{\eta}}$$

and, at the same time,

$$|p_{t-u} - p_{s-u}| \leq \sup_{r \in [s,t]} |p'_{r-u}| \, (t-s) \leq \frac{c}{(s-u)^{\eta+1}} \, (t-s).$$

Then condition (5.12) is satisfied with $\delta = t - s$. The same can be done with $p'_{t-u} - p'_{s-u}$, under the assumption that $|p''_u| \leq \frac{c}{u^{\eta+2}}$. Then (5.28) follows.

To prove (5.29), we proceed in a similar way: we write

$$\begin{aligned}
&\int_0^t (g_u - g_s)\,(p_{t-u} - p_{s-u})\,\mathrm{d}W_u \\
&= \int_0^s (g_u - g_s)\,(p_{t-u} - p_{s-u})\,\mathrm{d}W_u + \int_s^t (g_u - g_s)\,p_{t-u}\,\mathrm{d}W_u.
\end{aligned}$$

For the second integral, we use Corollary 3.17 and, for the first one, relation (5.29) follows from the part (i) of Proposition 5.6 with $g_u = g_u - g_s$ and $\bar{p}_u = p_{t-u} - p_{s-u}$. Indeed, such $\bar{p}$ satisfies (5.12) and (5.13), as we have shown above, and clearly,

$$|g_u - g_s| \leq \|g\|_\beta \,(u-s)^\beta,$$

and then condition (5.16) holds with $K = \|g\|_\beta$.

□

We also give the following Corollary of Proposition 5.6, which is used below.

COROLLARY 5.8. Let $X \in C^\beta$ and $W \in C^\theta$ as above. Let $p : (0, \infty) \to \mathbb{R}$ be a C^3 function such that satisfies (5.5) for some $c > 0$ (and we use the convention that $p_u \equiv 0$ when $u \leq 0$). Let $G = (g, g)$ be a controlled path of X.

(i) If g satisfies (5.16), then, if $\theta + \beta - \eta < 2$,

$$\begin{aligned}
&\Big| \int_{\mathbb{R}} g_u \,(p_{t-u} - p_{s-u} - (t-s)\,p'_{s-u})\,\mathrm{d}W_u \Big| \\
&\leq \hat{c}_{\beta,\theta,\eta+1}\, C_{W,\mathbb{XW}}\, c\,(t-s)^{\theta+\beta-\eta} \left(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\, t^\beta + \|g'\|_\infty + 2K \right).
\end{aligned} \tag{5.30}$$

(ii) If g satisfies (5.18), then, if $\theta + 2\beta - \eta < 2$,

$$\Big| \int_{\mathbb{R}} g_u \,(p_{t-u} - p_{s-u} - (t-s)\,p'_{s-u})\,\mathrm{d}W_u \Big| \leq \hat{c}_{2\beta,\theta,\eta+1}\, C_{W,\mathbb{XW}}\, c\,(t-s)^{\theta+2\beta-\eta} \left(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}} + 4K \right) \tag{5.31}$$

(iii) If g satisfies (5.20) then, if $\theta + 3\beta - \eta < 2$,

$$\Big| \int_{\mathbb{R}} g_u \,(p_{t-u} - p_{s-u} - (t-s)\,p'_{s-u})\,\mathrm{d}W_u \Big| \leq \hat{c}_{3\beta,\theta,\eta+1}\, C_{W,\mathbb{XW}}\, c\,(t-s)^{\theta+3\beta-\eta}\, 8K. \tag{5.32}$$

PROOF. In the three cases, we can write

$$\int_{\mathbb{R}} g_u \,(p_{t-u} - p_{s-u} - (t-s)\,p'_{s-u})\,\mathrm{d}W_u = \int_0^s g_u \,(p_{t-u} - p_{s-u} - (t-s)\,p'_{s-u})\,\mathrm{d}W_u + \int_s^t g_u\, p_{t-u}\,\mathrm{d}W_u.$$

For the second integral, we can apply Theorem 3.16, and we write

$$\Big|\int_s^t g_u\, p_{t-u}\, \mathrm{d}W_u\Big|$$
$$\leq c_{\beta,\theta,\eta}\, C_{X,W,\mathbb{XW}}\, c\,(t-s)^{\theta-\eta}\,\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)},[s,t]}\,(t-s)^{2\beta}+\|g'\|_{\infty,[s,t]}\,(t-s)^{\beta}+\|g\|_{\infty,[s,t]}\Big).$$

- If g satisfies (5.16), then
$$\|g\|_{\infty,[s,t]}\leq K\,(t-s)^{\beta}$$
and then
$$\Big|\int_s^t g_u\, p_{t-u}\, \mathrm{d}W_u\Big|\leq c_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\,(t-s)^{\theta+\beta-\eta}\,\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}\,(t-s)^{\beta}+\|g'\|_{\infty}+K\Big).$$
- If g satisfies (5.18), then
$$\|g\|_{\infty,[s,t]}\leq K\,(s-t)^{2\beta}\qquad\text{and}\qquad\|g'\|_{\infty,[s,t]}\leq K\,(t-s)^{\beta},$$
and then
$$\Big|\int_s^t g_u\, p_{t-u}\, \mathrm{d}W_u\Big|\leq c_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\,(t-s)^{\theta+2\beta-\eta}\,\Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)}}+2K\Big).$$
- If g satisfies (5.20), then
$$\|g\|_{\infty,[s,t]}\leq K\,(s-t)^{3\beta},\|g'\|_{\infty,[s,t]}\leq K\,(t-s)^{2\beta},\qquad\text{and}\qquad\|G\|_{\mathcal{D}_X^{(\beta,\beta)},[s,t]}\leq K\,(t-s)^{\beta}$$
and then
$$\Big|\int_s^t g_u\, p_{t-u}\, \mathrm{d}W_u\Big|\leq c_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\,(t-s)^{\theta+3\beta-\eta}\,3K.$$

Hence we just have to show that (5.30), (5.31) and (5.32) holds for
$$\Big|\int_0^s g_u\,(p_{t-u}-p_{s-u}-(t-s)\,p'_{s-u})\,\mathrm{d}W_u\Big|.$$
We define
$$\bar{p}_u=\bar{p}_u^{(s,t)}:=p_{t-u}-p_{s-u}-(t-s)\,p'_{s-u}.$$
Then, $\bar{p}:[0,s)\to\infty$ has a singularity at s of order $\bar{\eta}=\eta+1$. Moreover,
$$|\bar{p}_u|=|p_{t-u}-p_{s-u}-(t-s)\,p'_{s-u}|\leq(t-s)^2\sup_{x\in[s-u,t-u]}|p''_x|\leq\frac{c\,(t-s)^2}{(s-u)^{\eta+2}}$$
and, if $s-u\leq t-s$,
$$\begin{aligned}|\bar{p}_u|&=|p_{t-u}-p_{s-u}-(t-s)\,p'_{s-u}|\leq|p_{t-u}-p_{s-u}|+(t-s)\,|p'_{s-u}|\\&\leq 2\,(t-s)\sup_{x\in[s-u,t-u]}|p'_x|\leq\frac{2c\,(t-s)}{(s-u)^{\eta+1}}.\end{aligned}$$
Then condition (5.12) holds with
$$c=2\,c\,(t-s),\qquad\delta=t-s,\qquad\bar{\eta}=\eta+1.\tag{5.33}$$
The same can be proved for (5.13), since we have supposed that $p\in C^3$ and (5.5) holds. Thus we can apply Proposition 5.6 with data given by (5.33) and we have done.

□

5.3. Proof of Theorem 5.2

In this section we are going to prove Theorem 5.2 by using mainly Proposition 5.6.

5.3.1. Step 1. The first step to prove Theorem 5.2 is to start from the definition of solution of (5.4) and get new estimates about the remainder of the controlled path Y. Moreover, we point out the the remainder defined below does not have to have ordered arguments.

Proposition 5.9. Let Y be the solution of (5.4). For any $s,t \in [0,T]$, we write

$$\delta Y(s,t) = Y_s\, \delta X(s,t) + \tilde{Y}^R(s,t), \tag{5.34}$$

where $|\tilde{Y}^R(s,t)| = \mathrm{O}(|t-s|^{2\beta})$.

For any fixed $s \geq 0$, the function $u \mapsto \tilde{Y}^R(s,u)$ is in $\mathcal{D}_X^{(\beta,\beta)}$, with derivative $(\tilde{Y}^R(s,u)) = Y_u - Y_s$. Then, for all $u \in [0,T]$,

$$|\tilde{Y}^R(s,u)| \leq (\|Y\|_\beta \,\|X\|_\beta + \|Y^R\|_{2\beta})|s-u|^{2\beta} \leq \|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}\,(1 + \|X\|_\beta)\,|s-u|^{2\beta} \tag{5.35}$$

$$|(\tilde{Y}^R)'(s,u)| \leq \|Y\|_\beta\,|s-u|^\beta \tag{5.36}$$

$$\|\tilde{Y}^R(s,\cdot)\|_{\mathcal{D}_X^{(\beta,\beta)}} = \|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}. \tag{5.37}$$

Proof. The first part follows directly from (5.4), that is the integral formulation of the solution. Indee, we can write

$$\begin{aligned} Y_t - Y_s &= \int_{\mathbb{R}} Y_u\,(p_{t-u} - p_{s-u})\,\mathrm{d}W_u \\ &= Y_s\,(X_t - X_s) + \int_{\mathbb{R}} (Y_u - Y_s)\,(p_{t-u} - p_{s-u})\,\mathrm{d}W_u \\ &=: Y_s\,(X_t - X_s) + \tilde{Y}^R(s,t), \end{aligned}$$

recalling our convention to set $p_u \equiv 0$ when $u \leq 0$. This equation holds for any s,t, not necessarily ordered.

We already proved (see Proposition 4.11) that

$$|\tilde{Y}^R(s,t)| = \Big|\int_{\mathbb{R}} (Y_u - Y_s)\,(p_{t-u} - p_{s-u})\,\mathrm{d}W_u\Big| = \mathrm{O}(|t-s|^{2\beta}),$$

which shows that Y is a controlled path with derivative $Y' = Y$ with respect to X, by the definition of controlled path (see Definition 3.9). Relation (5.35) is a simple link between $\tilde{Y}^R$, whose arguments are unordered, and Y^R, whose arguments are ordered, and follows immediately from (5.3). (The second inequality holds because $Y = Y'$ and then $\|Y\|_\beta \leq \|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}$.)

We now pass to prove the second part of Proposition 5.9. We are going to show that $\tilde{Y}^R(s,\cdot)$ is a controlled path by X. For $0 \leq v < u \leq t$, we can write

$$\tilde{Y}^R(s,u) - \tilde{Y}^R(s,v) = (Y_u - Y_v) - Y_s(X_u - X_v) = (Y_v - Y_s)\,\delta X(v,u) + Y^R(v,u), \tag{5.38}$$

having applied the fact that (Y,Y) is in $\mathcal{D}_X^{(\beta,\beta)}$ with reminder Y^R.

This implies that $(\tilde{Y}^R(s,\cdot), Y_{\cdot} - Y_s)$ is in $\mathcal{D}_X^{(\beta,\beta)}$ with reminder Y^R. In particular,

$$|(\tilde{Y}^R)'(s,u)| = |Y_s - Y_u| \leq \|Y\|_\beta \, |s-u|^\beta,$$

that is (5.36). Moreover, since the remainder is the same of Y, we have

$$\|\tilde{Y}^R(s,\cdot)\|_{\mathcal{D}_X^{(\beta,\beta)}} = \|Y\|_\beta + \|Y^R\|_{2\beta} = \|Y\|_{\mathcal{D}_X^{(\beta,\beta)}},$$

that proves (5.37). □

LEMMA 5.10. We have

$$\begin{aligned}&\Big|\int_{\mathbb{R}} \tilde{Y}^R(s,u)\,(p_{t-u} - p_{s-u} - (t-s)\,p'_{s-u})\,\mathrm{d}W_u\Big| \\ &\leq \tilde{c}_{2\beta,\theta,\eta+1}\, C_{W,\mathbb{XW}}\, c\,(t-s)^{3\beta}\;\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}\,(1+\|X\|_\beta)\end{aligned} \tag{5.39}$$

PROOF. We apply the second part of Corollary 5.8 where, in this case, $g_u = \tilde{Y}^R(s,u)$. Thanks to (5.35) and (5.36), we know that $\tilde{Y}^R(s,u)$ satisfies (5.18). Moreover, $\theta+2\beta-\eta = 3\beta < 2$, since $\beta < \frac{1}{2}$, and then, from (5.31), we get (5.39), recalling that $\theta - \eta = \beta$ and relation (5.37). □

5.3.2. STEP 2: THEOREM 5.2 FOR $\beta \in (\frac{1}{3}, \frac{1}{2})$.

Here we are going to recall and prove Theorem 5.2 for $\beta \in (\frac{1}{3}, \frac{1}{2})$.

PROPOSITION 5.11. For any $s,t \in [0,T]$, we write

$$\delta Y(s,t) = Y_s\,\delta X(s,t) + Y_s\,\tilde{\mathbb{XW}}(s,t) + (t-s)\,C(Y,s) + \mathcal{Y}^R(s,t), \tag{5.40}$$

where we used the notations:

$$\tilde{\mathbb{XW}}(s,t) := \int_{\mathbb{R}} \delta X(s,u)\,(p_{t-u} - p_{s-u})\,\mathrm{d}W_u, \tag{5.41}$$

$$C(Y,s) := \int_{\mathbb{R}} \tilde{Y}^R(s,u)\,p'_{s-u}\,\mathrm{d}W_u, \tag{5.42}$$

$$\mathcal{Y}^R(s,t) := \int_{\mathbb{R}} \tilde{Y}^R(s,u)\,(p_{t-u} - p_{s-u} - (t-s)\,p'_{s-u})\,\mathrm{d}W_u. \tag{5.43}$$

Moreover,

$$|\tilde{\mathbb{XW}}(s,t)| \leq \tilde{c}_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\,(1+\|X\|_\beta)\,|t-s|^{2\beta} \tag{5.44}$$

$$|C(Y,s)| \leq \tilde{c}_{\beta,\theta,\eta+1}\, C_{W,\mathbb{XW}}\, c\,(2+\|X\|_\beta)\,(\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}} + \|Y\|_\beta)\,s^{3\beta-1} \tag{5.45}$$

$$|\mathcal{Y}^R(s,t)| \leq \tilde{c}_{2\beta,\theta,\eta+1}\, C_{W,\mathbb{XW}}\, c\,\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}(1+\|X\|_\beta)\,|t-s|^{3\beta}. \tag{5.46}$$

PROOF. By Proposition 5.9, we know that

$$\begin{aligned}\delta Y(s,t) - Y_s\,\delta X(s,t) &= \tilde{Y}^R(s,t) \\ &= \int_{\mathbb{R}} (Y_u - Y_s)\,(p_{t-u} - p_{s-u})\,\mathrm{d}W_u \\ &= \int_{\mathbb{R}} (Y_s\,\delta X(s,u) + \tilde{Y}^R(s,u))\,(p_{t-u} - p_{s-u})\,\mathrm{d}W_u,\end{aligned}$$

having used again (5.34) for $Y_u - Y_s$. Then,

$$\begin{aligned}\delta Y(s,t) - Y_s\,\delta X(s,t) &= Y_s \int_{\mathbb{R}} \delta X(s,u)\,(p_{t-u} - p_{s-u})\,\mathrm{d}W_u + \int_{\mathbb{R}} \tilde{Y}^R(s,u)\,(p_{t-u} - p_{s-u})\,\mathrm{d}W_u \\ &= Y_s\,\mathbb{XW}(s,t) + (t-s)\,C(Y,s) + \int_{\mathbb{R}} \tilde{Y}^R(s,u)\,(p_{t-u} - p_{s-u} - (t-s)\,p'_{s-u})\,\mathrm{d}W_u,\end{aligned}$$

by adding and subtracting $(t-s)\,p'_{s-u}$ in the last integral. This is precisely (5.40).

To prove relation (5.44), we just apply relation Proposition 5.7 with $g_u = X_u$, which is trivially a controlled path by X with derivative 1 and remainder identically zero. By (5.29), recalling that $\beta = \theta - \eta$ and $2\beta < 1$, we get

$$\begin{aligned}|\tilde{\mathbb{XW}}(s,t)| &= \Big| \int_{\mathbb{R}} (X_u - X_s)\,(p_{t-u} - p_{s-u})\,\mathrm{d}W_u \Big| \\ &\le \tilde{c}_{\beta,\theta,\eta}\,C_{W,\mathbb{XW}}\,c\,\big(\|X\|_{\mathcal{D}_X^{(\beta,\beta)}}\,t^\beta + \|X'\|_\infty + \|X\|_\beta\big)\,(t-s)^{2\beta} \\ &= \tilde{c}_{\beta,\theta,\eta}\,C_{W,\mathbb{XW}}\,c\,(1 + \|X\|_\beta)\,(t-s)^{2\beta}.\end{aligned}$$

(Note that $\|X\|_{\mathcal{D}_X^{(\beta,\beta)}} = 0$).

Relation (5.45) follows from Theorem 3.16 with $\bar{p}_u = p'_{s-u}$ (then $\bar{\eta} = \eta + 1$) and $g_u = \tilde{Y}^R(s,u)$; see more precisely Corollary 3.17 with $[a,b] = [0,s]$. Then we can write

$$\begin{aligned}&|C(Y,s)| = \Big| \int_{\mathbb{R}} \tilde{Y}^R(s,u)\,p'_{s-u}\,\mathrm{d}W_u \Big| \\ &\le \tilde{c}_{\beta,\theta,\eta+1}\,C_{W,\mathbb{XW}}\,c\,s^{\theta-\eta-1}\Big(\|\tilde{Y}^R(s,\cdot)\|_{\mathcal{D}_X^{(\beta,\beta)}}\,s^{2\beta} + \|(\tilde{Y}^R(s,\cdot))'\|_{\infty,[0,s]}\,s^\beta + \|\tilde{Y}^R(s,\cdot)\|_{\infty,[0,s]}\Big) \\ &= \tilde{c}_{\beta,\theta,\eta+1}\,C_{W,\mathbb{XW}}\,c\,s^{\beta-1}\Big(\|\tilde{Y}^R(s,\cdot)\|_{\mathcal{D}_X^{(\beta,\beta)}}\,s^{2\beta} + \|(\tilde{Y}^R(s,\cdot))'\|_{\infty,[0,s]}\,s^\beta + \|\tilde{Y}^R(s,\cdot)\|_{\infty,[0,s]}\Big),\end{aligned}$$

recalling that $\theta - \eta = \beta$.

Moreover, recalling Proposition 5.9,

$$\|(\tilde{Y}^R(s,\cdot))'\|_{\infty,[0,s]} = \sup_{u\in[0,s]} |Y_u - Y_s| \le \|Y\|_\beta\,s^\beta$$

and

$$\|\tilde{Y}^R(s,\cdot)\|_{\infty,[0,s]} = \sup_{u\in[0,s]} |\tilde{Y}^R(s,u)| \le K\,s^{2\beta}, \qquad \text{with} \qquad K = (1 + \|X\|_\beta)\,\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}.$$

We can write

$$|C(Y,s)| \le \tilde{c}_{\beta,\theta,\eta+1}\,C_{W,\mathbb{XW}}\,c\,s^{3\beta-1}\Big(\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}} + \|Y\|_\beta + \|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}(1 + \|X\|_\beta)\Big) \tag{5.47}$$

$$\le \tilde{c}_{\beta,\theta,\eta+1}\,C_{W,\mathbb{XW}}\,c\,s^{3\beta-1}\,(2 + \|X\|_\beta)\,(\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}} + \|Y\|_\beta), \tag{5.48}$$

that is (5.45).

Finally, relation (5.46) was already obtained in Lemma 5.10. □

Notice that, when $\beta > \frac{1}{3}$, then $3\beta > 1$ and, by (5.46), we have that $|\mathcal{Y}^R(s,t)| = \mathrm{o}(|t-s|)$, and then (5.40) can be written as

$$\delta Y(s,t) = Y_s\,\delta X(s,t) + Y_s\,\mathbb{XW}(s,t) + (t-s)\,C(Y,s) + \mathrm{o}(|t-s|),$$

that is the characterization (5.9) for the solution Y to the differential problem (5.4) when $\beta = \theta - \eta > \frac{1}{3}$.

5.3.3. Step 3: Theorem 5.2 for $\beta \in (\frac{1}{4}, \frac{1}{3})$. When $\beta < \frac{1}{3}$, relation (5.46) does not give us a characterization, since $3\beta < 1$. Then we have to go further in the rewriting of the solution Y, by using (5.40).

Proposition 5.12. For any $s, t \in [0, T]$, we have

$$\begin{aligned} \delta Y(s,t) = {} & Y_s\, \delta X(s,t) + Y_s\, \mathbb{X}\tilde{\mathbb{W}}(s,t) + (t-s)\, C(Y,s) + Y_s\, \mathfrak{X}\mathfrak{W}(s,t) + \\ & + C(Y,s) \int_{\mathbb{R}} (u-s)\,(p_{t-u} - p_{s-u} - (t-s)p'_{s-u})\, \mathrm{d}W_u + \\ & + \int_{\mathbb{R}} \mathcal{Y}^R(s,u)\,(p_{t-u} - p_{s-u} - (t-s)p'_{s-u})\, \mathrm{d}W_u, \end{aligned} \tag{5.49}$$

where $\mathbb{X}\tilde{\mathbb{W}}$, $C(Y, \cdot)$, $\mathcal{Y}^R$ and $\mathfrak{X}\mathfrak{W}$ are defined in (5.41), (5.42) and (5.43) and (5.8).

Moreover,

$$|\mathfrak{X}\mathfrak{W}(s,t)| = \left| \int_{\mathbb{R}} \mathbb{X}\tilde{\mathbb{W}}(s,u)\,(p_{t-u} - p_{s-u} - (t-s)p'_{s-u})\, \mathrm{d}W_u \right| = \mathrm{O}(|t-s|^{3\beta}), \tag{5.50}$$

and, if $\beta < \frac{1}{3}$, then

$$\left| \int_{\mathbb{R}} (u-s)\,(p_{t-u} - p_{s-u} - (t-s)p'_{s-u})\, \mathrm{d}W_u \right| = \mathrm{O}(|t-s|^{4\beta}), \tag{5.51}$$

$$\left| \int_{\mathbb{R}} \mathcal{Y}^R(s,u)\,(p_{t-u} - p_{s-u} - (t-s)p'_{s-u})\, \mathrm{d}W_u \right| = \mathrm{O}(|t-s|^{4\beta}). \tag{5.52}$$

We first need to prove that the maps $t \mapsto \mathbb{X}\tilde{\mathbb{W}}(s,t)$ and $t \mapsto \mathcal{Y}^R(s,t)$ are controlled paths by X, otherwise the rough integrals given by the definition (5.8) of $\mathfrak{X}\mathfrak{W}$ and by the last integral in (5.49) would not be well defined. At the same time, we find useful relations on their norms. We divide the results in two different lemmas whose proofs are deferred to Section 5.4.

Lemma 5.13. Given a fixed $s \in [0, T]$, the map $t \mapsto \mathbb{X}\tilde{\mathbb{W}}(s,t)$ defined in (5.41) is a path controlled by X with derivative given by $(\mathbb{X}\tilde{\mathbb{W}}(s,t))' = X_t - X_s$.

We have the following estimates, for suitable K that will be explicit from the proof.

$$|\mathbb{X}\tilde{\mathbb{W}}(s,u)| \leq K\, |s-u|^{2\beta} \tag{5.53}$$

$$|(\mathbb{X}\tilde{\mathbb{W}}(s,t))'| \leq K\, |s-u|^{\beta}. \tag{5.54}$$

Lemma 5.14. Given a fixed $s \in [0, T]$, the map $t \mapsto \mathcal{Y}^R(s,t)$ defined in (5.43) is a path controlled by X with derivative given by $(\mathcal{Y}^R(s,t))' = \tilde{Y}^R(s,t)$. Then we have the following relations, for suitable K:

$$|\mathcal{Y}^R(s,u)| \leq K\, |s-u|^{3\beta} \tag{5.55}$$

$$|(\mathcal{Y}^R)'(s,u)| \leq K |s-u|^{2\beta} \tag{5.56}$$

$$\|\mathcal{Y}^R\|_{\mathcal{D}_X^{(\beta,\beta)}}|_{[v,s]} \leq K\, |s-v|^{\beta}. \tag{5.57}$$

With this results, we can proceed with the proof of Proposition 5.12.

Proof of Proposition 5.12. We start from (5.40), that says that

$$\tilde{Y}^R(s,u) = \delta Y(s,u) - Y_s\,\delta X(s,u) = Y_s\,\tilde{\mathbb{XW}}(s,u) + (u-s)\,C(Y,s) + \mathcal{Y}^R(s,u).$$

Then, we can write, by (5.43),

$$\begin{aligned}
&\mathcal{Y}^R(s,t) = \int_{\mathbb{R}} \tilde{Y}^R(s,u)\,(p_{t-u} - p_{s-u} - (t-s)p'_{s-u})\,\mathrm{d}W_u \\
&= Y_s \int_{\mathbb{R}} \tilde{\mathbb{XW}}(s,u)\,(p_{t-u} - p_{s-u} - (t-s)p'_{s-u})\,\mathrm{d}W_u + \\
&+ C(Y,s)\int_{\mathbb{R}} (u-s)\,(p_{t-u} - p_{s-u} - (t-s)p'_{s-u})\,\mathrm{d}W_u + \int_{\mathbb{R}} \mathcal{Y}^R(s,u)\,(p_{t-u} - p_{s-u} - (t-s)p'_{s-u})\,\mathrm{d}W_u.
\end{aligned}$$

Then, from (5.40), recalling the definition (5.8) of $\mathfrak{XW}$,

$$\begin{aligned}
&\delta Y(s,t) = Y_s\,\delta X(s,t) + Y_s\,\tilde{\mathbb{XW}}(s,t) + (t-s)\,C(Y,s) + \mathcal{Y}^R(s,t) \\
&= Y_s\,\delta X(s,t) + Y_s\,\tilde{\mathbb{XW}}(s,t) + (t-s)\,C(Y,s) + Y_s\,\mathfrak{XW}(s,t) + \\
&+ C(Y,s)\int_{\mathbb{R}} (u-s)\,(p_{t-u} - p_{s-u} - (t-s)p'_{s-u})\,\mathrm{d}W_u + \int_{\mathbb{R}} \mathcal{Y}^R(s,u)\,(p_{t-u} - p_{s-u} - (t-s)p'_{s-u})\,\mathrm{d}W_u,
\end{aligned}$$

that is (5.49).

Now we prove the second part, by applying Corollary 5.8. In particular, by Lemma 5.13, $g_u = \tilde{\mathbb{XW}}(s,u)$ satisfies (5.18), and then by (5.31), recalling that $\beta = \theta - \eta$,

$$|\mathfrak{XW}(s,t)| = \Big|\int_{\mathbb{R}} \tilde{\mathbb{XW}}(s,u)\,(p_{t-u} - p_{s-u} - (t-s)p'_{s-u})\,\mathrm{d}W_u\Big| = \mathrm{O}(|t-s|^{3\beta}).$$

The function $g_u = u - s$ is a controlled path of X with zero derivative with respect to X. Then we can apply Corollary 5.8 also for the second integral with the linear term. In particular, when $3\beta < 1$, for all u such that $|u-s| < 1$, we can easily show that g satisfies (5.20)

$$\begin{aligned}
&|g_u| = |u-s| \le |u-s|^{3\beta} \\
&|g'_u| = 0 \le |u-s|^{2\beta} \\
&\|G\|_{\mathcal{D}_X^{(\beta,\beta)},[u,s]} = \|g'\|_{\beta,[u,s]} + \|g^R\|_{2\beta,[u,s]} = |u-s|^{1-2\beta} \le |u-s|^{\beta}.
\end{aligned}$$

Then, from the third part of Corollary 5.8, we get

$$\Big|\int_{\mathbb{R}} (u-s)\,(p_{t-u} - p_{s-u} - (t-s)p'_{s-u})\,\mathrm{d}W_u\Big| = \mathrm{O}(|t-s|^{4\beta}).$$

By Lemma 5.14, $g_u = \mathcal{Y}^R(s,u)$ satisfies (5.20), and then we have

$$\Big|\int_{\mathbb{R}} \mathcal{Y}^R(s,u)\,(p_{t-u} - p_{s-u} - (t-s)p'_{s-u})\,\mathrm{d}W_u\Big| = \mathrm{O}(|t-s|^{4\beta})$$

□

5.4. Technical Proofs

Proof of Lemma 5.13.

Proof. Recalling the definition (5.41) of $\mathbb{X}\tilde{\mathbb{W}}$, we write

$$\begin{aligned}\mathbb{X}\tilde{\mathbb{W}}(s,t) - \mathbb{X}\tilde{\mathbb{W}}(s,t') &= \int_{\mathbb{R}} (X_u - X_s)\,(p_{t-u} - p_{t'-u})\,\mathrm{d}W_u \\ &= (X_{t'} - X_s) \int_{\mathbb{R}} (p_{t-u} - p_{t'-u})\,\mathrm{d}W_u + \int_{\mathbb{R}} (X_u - X_{t'})\,(p_{t-u} - p_{t'-u})\,\mathrm{d}W_u \\ &= (X_{t'} - X_s)\delta X(t',t) + \int_{\mathbb{R}} (X_u - X_{t'})\,(p_{t-u} - p_{t'-u})\,\mathrm{d}W_u.\end{aligned}$$

Then we can use Proposition 4.18 with $g_u = X_u$ and we have

$$\Big| \int_{\mathbb{R}} (X_u - X_{t'})\,(p_{t-u} - p_{t'-u})\,\mathrm{d}W_u \Big| = \mathrm{O}(|t-t'|^{2\beta}),$$

and this implies that $t \mapsto \mathbb{X}\tilde{\mathbb{W}}(s,t)$, for any fixed s, is a controlled path by X with derivative given by $(\mathbb{X}\tilde{\mathbb{W}}(s,t))' = X_t - X_s$.

Relation (5.53) has already proved in (5.44), while relation (5.54) follows from the derivative we have just found. □

Proof of Lemma 5.14. For the proof of Lemma 5.14 and then of Proposition 5.12, and, in particular, the fact that $|\mathfrak{Y}^R(s,t)| = \mathrm{O}(|t-s|^{4\beta})$ when $\beta < \frac{1}{3}$, we need the following proposition, whose proof can be compared to the proof of the above Proposition 5.6.

Lemma 5.15. Let $p : (0,T] \to \mathbb{R}$ be a C^2 function with a singularity at 0 of order η, such that the following conditions hold:

$$|p_u| \le \frac{c}{u^\eta}, \qquad |p'_u| \le \frac{c}{u^{\eta+1}}, \qquad |p''_u| \le \frac{c}{u^{\eta+2}}. \tag{5.58}$$

Let $s > 0$ be fixed. Then, for any t, t' with $0 \le t' < t \le s$, we have

$$\begin{aligned}&\Big| \int_{\mathbb{R}} (\tilde{Y}^R(s,u) - \tilde{Y}^R(s,t'))\,(p_{t-u} - p_{t'-u})\,\mathrm{d}W_u \Big| \\ &\qquad\qquad \le \hat{c}_{\beta,\theta,\eta}\, C_{W,\mathbb{X}\mathbb{W}}\, c\, \|Y\|_{\mathcal{D}_X^{(\beta,\beta)}} (1 + \|X\|_\beta)\,(t-t')^{2\beta}\,(s-t')^\beta,\end{aligned} \tag{5.59}$$

where $\hat{c}_{\beta,\theta,\eta}$ is a constant which depends only on β, θ, η, the constant $C_{X,W,\mathbb{X}\mathbb{W}} = \|W\|_\theta + \|\mathbb{X}\mathbb{W}\|_{\beta+\theta}$ and c is the same that appears in (5.58).

Proof. First of all, we divide the integral in two parts: we can write

$$\begin{aligned}&\int_{\mathbb{R}} (\tilde{Y}^R(s,u) - \tilde{Y}^R(s,t'))\,(p_{t-u} - p_{t'-u})\,\mathrm{d}W_u = \\ &\int_0^{t'} (\tilde{Y}^R(s,u) - \tilde{Y}^R(s,t'))\,(p_{t-u} - p_{t'-u})\,\mathrm{d}W_u + \int_{t'}^t (\tilde{Y}^R(s,u) - \tilde{Y}^R(s,t'))\,p_{t-u}\,\mathrm{d}W_u\end{aligned}$$

We first look after the second integral. By using Theorem 3.16, more precisely Corollary 3.17, we have

$$\begin{aligned}
&\Big|\int_{t'}^{t} (\tilde{Y}^R(s,u) - \tilde{Y}^R(s,t'))\, p_{t-u}\, \mathrm{d}W_u\Big| \\
&\le c_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\, (t-t')^{\theta-\eta} \left(\|G\|_{\mathcal{D}_X^{(\beta,\beta)},[t',t]} (t-t')^{2\beta} + \|g'\|_{\infty,[t',t]} (t-t')^{\beta} + \|g\|_{\infty,[t',t]} \right),
\end{aligned}$$

where, in this case, $g_u = \tilde{Y}^R(s,u) - \tilde{Y}^R(s,t')$ and $g'_u = (\tilde{Y}^R(s,u))' = Y_u - Y_s$, and then we can improve this inequality by using the relations above. In particular, recalling (5.38), we have

$$\begin{aligned}
\|g\|_{\infty,[t',t]} &= \sup_{u\in[t',t]} |\tilde{Y}^R(s,u) - \tilde{Y}^R(s,t')| = \sup_{u\in[t',t]} |\delta Y(s,t')\, \delta X(t',u) + Y^R(t',u)| \\
&\le (t-t')^{\beta} \left(\|Y\|_{\beta}\, \|X\|_{\beta}\, (s-t')^{\beta} + \|Y^R\|_{2\beta}\, (t-t')^{\beta} \right),
\end{aligned}$$

and, recalling (5.36),

$$\|g'\|_{\infty,[t',t]} \le (s-t')^{\beta}\, \|Y\|_{\beta}.$$

Recalling also that $\|\tilde{Y}^R(s,\cdot)\|_{\mathcal{D}_X^{(\beta,\beta)},[t,t']} = \|Y\|_{\mathcal{D}_X^{(\beta,\beta)},[t,t']}$ by (5.37), we can write

$$\begin{aligned}
&\Big|\int_{t'}^{t} (\tilde{Y}^R(s,u) - \tilde{Y}^R(s,t'))\, p_{t-u}\, \mathrm{d}W_u\Big| \\
&\le c_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\, (t-t')^{\theta+\beta-\eta} \times \\
&\times \left(\|Y\|_{\mathcal{D}_X^{(\beta,\beta)},[t',t]} (t-t')^{\beta} + \|Y\|_{\beta}\, (s-t')^{\beta} + \|Y\|_{\beta}\, \|X\|_{\beta} (s-t')^{\beta} + \|Y^R\|_{2\beta}\, (t-t')^{\beta} \right) \\
&\le c_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\, (s-t')^{\beta} (t-t')^{2\beta}\, \|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}\, (1 + \|X\|_{\beta}),
\end{aligned}$$

since $s > t > t'$ for assumptions, recalling that for our case we have $\theta - \eta = \beta$ and having used the definition of norm in $\mathcal{D}_X^{(\beta,\beta)}$ and the fact that $Y = Y'$.

Now we pass to the first integral over the interval $[0,t']$:

$$I_{[0,t']} = \int_0^{t'} (\tilde{Y}^R(s,u) - \tilde{Y}^R(s,t'))\, (p_{t-u} - p_{t'-u})\, \mathrm{d}W_u.$$

In this case, we would like to use the first part of Proposition 5.6, but it is not enough. Indeed, since $\bar{p} := p_{t-u} - p_{t'-u}$ satisfies (5.12)-(5.13) with $\delta = (t-s)$, by (5.17) we would obtain

$$\begin{aligned}
&\Big|\int_0^{t'} (\tilde{Y}^R(s,u) - \tilde{Y}^R(s,t'))\, (p_{t-u} - p_{t'-u})\, \mathrm{d}W_u\Big| \\
&\le \tilde{c}_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\, (t-t')^{2\beta} \left(\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}} (t-t')^{\beta} + \|(\tilde{Y}^R(s,\cdot))'\|_{\infty,[0,t']} + \|\tilde{Y}^R(s,\cdot)\|_{\beta,[0,t']} \right),
\end{aligned}$$

which misses the factor $(s-t')^{\beta}$ in (5.59). Hence, we have to decompose the proof of that proposition and then recompose it using the properties of the special function "g" of this case, that is $\tilde{Y}^R(s,\cdot)$.

If $t' \le 2(t-t')$, then we can use Theorem 3.16, more precisely Corollary 3.17, and we have

$$\begin{aligned}
|I_{[0,t']}| &\le c_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\, (t')^{\theta-\eta} \left(\|G\|_{\mathcal{D}_X^{(\beta,\beta)},[0,t']} (t')^{2\beta} + \|g'\|_{\infty,[0,t']} (t')^{\beta} + \|g\|_{\infty,[0,t]} \right) \\
&\le c_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\, (t')^{\theta+2\beta-\eta}\, \|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}\, (1 + \|X\|_{\beta}),
\end{aligned}$$

having used the facts that for $0 < u \le t' < s$, by (5.38)

$$\sup_{u\in[0,t']} |-\tilde{Y}^R(s,u) - \tilde{Y}^R(s,t')| = \sup_{u\in[0,t']} |\delta Y(s,u)\,\delta X(u,t') + Y^R(u,t')|$$
$$\le \|Y\|_\beta\,\|X\|_\beta \sup_{u\in[0,t']} (s-u)^\beta\,(t'-u)^\beta + \|Y^R\|_{2\beta} \sup_{u\in[0,t']} (t'-u)^{2\beta} \le s^\beta\,(t')^\beta\,(\|Y\|_\beta\,\|X\|_\beta + \|Y^R\|_{2\beta})$$
$$\le s^\beta\,(t')^\beta\,(1+\|X\|_\beta)\,\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}.$$

Then, in the case of $t' \le 2(t-t')$, which implies $s = (s-t')+t' \le (s-t')+2(t-t') \le 3(s-t')$, we have

$$\begin{aligned}|I_{[0,t']}| &\le c_{\beta,\theta,\eta}\,C_{W,\mathbb{XW}}\,c\,(t-t')^{2\beta}\,(s-t')^\beta\,\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}\,(1+\|X\|_\beta)\\ &\le c_{\beta,\theta,\eta}\,C_{W,\mathbb{XW}}\,c\,(t-t')^{2\beta}\,(s-t')^\beta\,\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}\,(1+\|X\|_\beta),\end{aligned}$$

and then (5.59) holds in this case.

Now we pass to the case $t' > 2\,(t-t')$. Following the scheme of the proof of Proposition 5.6, in this case we consider $n = n(t,t') \in \mathbb{N}$, such that $2^n\,(t-t') < t' \le 2^{n+1}\,(t-t')$ and we write

$$|I_{[0,t']}| \le |I_{[0,\frac{t'}{2}]}| + \sum_{i=1}^n |I_{t_{i+1},t_i}| + |I_{[t'-(t-t'),t']}|,$$

with $t_i = t' - 2^i\,(t-t')$. We now analyze these integrals.

For the last integral, we can use Theorem 3.16 (see Corollary 3.17) and we write

$$|I_{[t'-(t-t'),t']}| \le c_{\beta,\theta,\eta}\,C_{X,W,\mathbb{XW}}\,c\,(t-t')^{\theta-\eta}\,\times$$
$$\times\,\Big(\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}\,(t-t')^{2\beta} + \|(\tilde{Y}^R(s,\cdot))'\|_{\infty,[t'-(t-t'),t']}\,(t-t')^\beta + \|\tilde{Y}^R(s,\cdot) - \tilde{Y}^R(s,t')\|_{\infty,[t'-(t-t'),t']}\Big)$$

Now we use the fact that

$$\begin{aligned}\|(\tilde{Y}^R(s,\cdot))'\|_{\infty,[t'-(t-t'),t']} &= \sup_{u\in[t'-(t-t'),t']} |Y_u - Y_s|\\ &\le \|Y\|_\beta \sup_{u\in[t'-(t-t'),t']} (s-u)^\beta = \|Y\|_\beta\,(s-t'+(t-t'))^\beta\\ &\le 2\,\|Y\|_\beta\big[(s-t')^\beta + (t-t')^\beta\big],\end{aligned}$$

and, thanks to (5.38),

$$\begin{aligned}&\|\tilde{Y}^R(s,\cdot) - \tilde{Y}^R(s,t')\|_{\infty,[t'-(t-t'),t']}\\ &\le \|Y\|_\beta\,\|X\|_\beta \sup_{u\in[t'-(t-t'),t']} (t'-u)^\beta\,(s-u)^\beta + \|Y^R\|_{2\beta} \sup_{u\in[t'-(t-t'),t']} (t'-u)^{2\beta}\\ &\le \|Y\|_\beta\,\|X\|_\beta(t-t')^\beta\,(s-t'+(t-t'))^\beta + \|Y^R\|_{2\beta}\,(t-t')^{2\beta}\\ &\le \big[\|Y\|_\beta\,\|X\|_\beta + \|Y^R\|_{2\beta}\big]\,(t-t')^{2\beta} + \|Y\|_\beta\,\|X\|_\beta\,(t-t')^\beta\,(s-t')^\beta.\end{aligned}$$

Then

$$\begin{aligned}&|I_{[t'-(t-t'),t']}|\\ &\le c_{\beta,\theta,\eta}\,C_{W,\mathbb{XW}}\,c\,(t-t')^{\theta-\eta}\Big[(t-t')^{2\beta}\Big(\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}} + \|Y\|_\beta + \|Y\|_\beta\,\|X\|_\beta + \|Y^R\|_{2\beta}\Big)\\ &\qquad + (t-t')^\beta\,(s-t')^\beta\Big(\|Y\|_\beta + \|Y\|_\beta\,\|X\|_\beta\Big)\Big]\\ &\le c_{\beta,\theta,\eta}\,C_{W,\mathbb{XW}}\,c\,\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}\,(1+\|X\|_\beta)\,[(t-t')^{3\beta} + (s-t')^\beta\,(t-t')^{2\beta}],\end{aligned} \tag{5.60}$$

recalling that $\theta - \eta = \beta$ in this case.

For $i = 1, \dots, n$, we notice that:

$$t' - t_{i+1} = 2^i \,(t - t'), \qquad \text{and} \qquad t' - t_i = t_i - t_{i+1} = 2^{i-1}\,(t - t').$$

Thanks to (3.27), we can write

$$\begin{aligned} |I_{[t_{i+1},t_i]}| &\le (t_i - t_{i+1})^{\theta} \; |||p|||_{[t_{i+1},t_i]} \, \|I_g\|_{\theta,[t_{i+1},t_i]} \\ &= 2^{(i-1)\theta}\,(t-t')^{\theta}\; |||p|||_{[t_{i+1},t_i]} \, \|I_g\|_{\theta,[t_{i+1},t_i]}. \end{aligned}$$

In the proof of Corollary 5.7, we proved that the function $u \mapsto \bar{p}^{(t',t)}_u = p_{t-u} - p_{t'-u}$ satisfies (5.12) and (5.13) with $\delta = (t - t')$ and $\bar{\eta} = \eta$; then, thanks to Lemma 5.4, we have

$$|||\bar{p}^{(t',t)}|||_{[t_{i+1},t_i]} \le \frac{c(t-t')}{(t'-t_i)^{\eta+1}} \left(1 + \frac{(t_i - t_{i+1})}{(t'-t_i)}\right) = 2\,c\,2^{(i-1)(-\eta-1)}\,(t-t')^{-\eta}$$

and, thanks to (3.21),

$$\begin{aligned} &\|I_g\|_{\theta,[t_{i+1},t_i]} \\ &\le c_{2\beta+\theta}\, C_{W,\mathbb{XW}} \Big(\|G\|_{\mathcal{D}_X^{(\beta,\beta)},[t_{i+1},t_i]}\,(t_i - t_{i+1})^{2\beta} + \|g'\|_{\infty,[t_{i+1},t_i]}\,(t_i-t_{i+1})^{\beta} + \|g\|_{\infty,[t_{i+1},t_i]}\Big), \end{aligned}$$

where $g_u = \tilde{Y}^R(s,u) - \tilde{Y}^R(s,t')$ and then,

$$\begin{aligned} \|g'\|_{\infty,[t_{i+1},t_i]} &= \sup_{u\in[t_{i+1},t_i]} |Y_u - Y_s| \le \|Y\|_{\beta} \sup_{u\in[t_{i+1},t_i]} |u-s|^{\beta} = \|Y\|_{\beta}\,(s - t_{i+1})^{\beta} \\ &= \|Y\|_{\beta}\,(s - t' + t' - t_{i+1})^{\beta} \le \|Y\|_{\beta}\Big((s-t')^{\beta} + 2^{i\beta}(t-t')^{\beta}\Big), \end{aligned}$$

and

$$\begin{aligned} \|g\|_{\infty,[t_{i+1},t_i]} &= \sup_{u\in[t_{i+1},t_i]} |\tilde{Y}^R(s,u) - \tilde{Y}^R(s,t')| = \sup_{u\in[t_{i+1},t_i]} |\delta Y(u,s)\,\delta X(u,t') - Y^R(u,t')| \\ &\le \|Y\|_{\beta}\,\|X\|_{\beta} \sup_{u\in[t_{i+1},t_i]} (s-u)^{\beta}\,(t'-u)^{\beta} + \|Y^R\|_{2\beta} \sup_{u\in[t_{i+1},t_i]} (t'-u)^{2\beta} \\ &\le \|Y\|_{\beta}\,\|X\|_{\beta}(s-t_{i+1})^{\beta}\,(t'-t_{i+1})^{\beta} + \|Y^R\|_{2\beta}\,(t'-t_{i+1})^{2\beta} \\ &\le \|Y\|_{\beta}\,\|X\|_{\beta}\,(s-t')^{\beta}(t'-t_{i+1})^{\beta} + \big(\|Y\|_{\beta}\,\|X\|_{\beta} + \|Y^R\|_{2\beta}\big)\,(t'-t_{i+1})^{2\beta} \\ &= \|Y\|_{\beta}\,\|X\|_{\beta}\,(s-t')^{\beta}(2^i(t-t'))^{\beta} + \big(\|Y\|_{\beta}\,\|X\|_{\beta} + \|Y^R\|_{2\beta}\big)\,(2^i(t-t'))^{2\beta}. \end{aligned}$$

Then,

$$\begin{aligned} &\|I_g\|_{\theta,[t_{i+1},t_i]} \le c_{2\beta+\theta}\, C_{W,\mathbb{XW}} \times \\ &\times \Big(\|Y\|_{\mathcal{D}_X^{(\beta,\beta)},[t_{i+1},t_i]}\, 2^{(i-1)2\beta}(t-t')^{2\beta} + \|Y\|_{\beta}\Big((s-t')^{\beta} + 2^{i\beta}(t-t')^{\beta}\Big)\, 2^{(i-1)\beta}(t-t')^{\beta} + \\ &\qquad + \|Y\|_{\beta}\,\|X\|_{\beta}\,(s-t')^{\beta} 2^{i\beta}(t-t')^{\beta} + \big(\|Y\|_{\beta}\,\|X\|_{\beta} + \|Y^R\|_{2\beta}\big)\, 2^{i(2\beta)}(t-t')^{2\beta}\Big) \\ &\le c_{2\beta+\theta}\, C_{W,\mathbb{XW}}\, 2^{(i-1)\beta}(t-t')^{\beta}\Big(\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}\, 2^{(i-1)\beta}(t-t')^{\beta} + \|Y\|_{\beta}\big((s-t')^{\beta} + 2^{i\beta}(t-t')^{\beta}\big) + \\ &\qquad + \|Y\|_{\beta}\,\|X\|_{\beta}\,(s-t')^{\beta} + \big(\|Y\|_{\beta}\,\|X\|_{\beta} + \|Y^R\|_{2\beta}\big)\, 2^{i\beta}(t-t')^{\beta}\Big) \\ &\le c_{2\beta+\theta}\, C_{W,\mathbb{XW}}\, 2^{(i-1)\beta}(t-t')^{\beta}\, \|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}\,(1 + \|X\|_{\beta})\,\Big[2^{(i-1)\beta}(t-t')^{\beta} + (s-t')^{\beta}\Big]. \end{aligned}$$

Now we look at the integral in $[0, \frac{t'}{2}]$. In this case, we can proceed as above. We get

$$\|\!|\bar{p}^{(t',t)}\|\!|_{[0,\frac{t'}{2}]} \leq 2c\,\frac{t-t'}{\left(\frac{t'}{2}\right)^{\eta+1}} = 2^{2+\eta}c\,2^{n(-\eta-1)}\,(t-t')^{-\eta},$$

recalling that

$$t' > 2^n\,(t-t').$$

At the same time,

$$\|I_g\|_{\theta,[0,\frac{t'}{2}]} \leq c_{2\beta+\theta}\,C_{X,W,\mathbb{XW}}\left(\|G\|_{\mathcal{D}_X^{(\beta,\beta)},[0,\frac{t'}{2}]}\left(\frac{t'}{2}\right)^{2\beta} + \|g'\|_{\infty,[0,\frac{t'}{2}]}\left(\frac{t'}{2}\right)^{\beta} + \|g\|_{\infty,[0,\frac{t'}{2}]}\right),$$

where, as above, we can prove that

$$\|g'\|_{\infty,[0,\frac{t'}{2}]} \leq \|Y\|_\beta\,s^\beta \leq \|Y\|_\beta\left((s-t')^\beta + t'^\beta\right),$$

and

$$\begin{aligned}\|g\|_{\infty,[0,\frac{t'}{2}]} &= \sup_{u\in[0,\frac{t'}{2}]}|\tilde{Y}^R(s,u)-\tilde{Y}^R(s,t')| = \sup_{u\in[0,\frac{t'}{2}]}|\delta Y(u,s)\,\delta X(u,t')-Y^R(u,t')|\\ &\leq \|Y\|_\beta\,\|X\|_\beta\,(s)^\beta\,(t')^\beta + \|Y^R\|_{2\beta}\,(t')^{2\beta}\\ &\leq \|Y\|_\beta\,\|X\|_\beta\,(s-t')^\beta\,(t')^\beta + \left(\|Y\|_\beta\,\|X\|_\beta + \|Y^R\|_{2\beta}\right)(t')^{2\beta}.\end{aligned}$$

Then, since $t' \leq 2^{n+1}\,(t-t')$,

$$\begin{aligned}\|I_g\|_{\theta,[0,\frac{t'}{2}]} &\leq c_{2\beta+\theta}\,C_{W,\mathbb{XW}}\,t'^\beta\,\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}\,(1+\|X\|_\beta)\left[(s-t')^\beta + t'^\beta\right]\\ &\leq c_{2\beta+\theta}\,C_{W,\mathbb{XW}}\;2^{n\beta}(t-t')^\beta\,\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}\,(1+\|X\|_\beta)\left[(s-t')^\beta + 2^{n\beta}(t-t')^\beta\right].\end{aligned}$$

We can use Theorem 3.16 and write

$$\begin{aligned}&|I_{[0,\frac{t'}{2}]}| + \sum_{i=1}^{n}|I_{[t_{i+1},t_i]}|\\ &\leq \left(\frac{t'}{2}\right)^\theta\|\!|\bar{p}^{(t,t')}\|\!|_{[0,\frac{t'}{2}]}\,\|I_g\|_{\theta,[0,\frac{t'}{2}]} + \sum_{i=1}^{n}(t_i-t_{i+1})^\theta\,\|\!|\bar{p}^{(t,t')}\|\!|_{[t_{i+1},t_i]}\,\|I_g\|_{\theta,[t_{i+1},t_i]}.\end{aligned}$$

By using the relations we got above, we can write

$$\begin{aligned}|I_{[0,\frac{t'}{2}]}| + \sum_{i=1}^{n}|I_{[t_{i+1},t_i]}| &\leq c_{2\beta+\theta}\,C_{X,W,\mathbb{XW}}\,c\,\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}(1+\|X\|_\beta)\times\\ &\times\;(t-t')^{\theta+\beta-\eta}\left[(s-t')^\beta\sum_{i=1}^{n+1}2^{(i-1)(\theta+\beta-\eta-1)} + (t-t')^\beta\sum_{i=1}^{n+1}2^{(i-1)(\theta+2\beta-\eta-1)}\right].\end{aligned}$$

In our case, we have $\theta-\eta=\beta<\frac{1}{3}$, and then, both the sum converge. We can write

$$|I_{[0,\frac{t'}{2}]}| + \sum_{i=1}^{n}|I_{[t_{i+1},t_i]}| \leq \hat{c}_{\beta,\theta,\eta}\,C_{W,\mathbb{XW}}\,c\,\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}(1+\|X\|_\beta)\;[(s-t')^\beta\,(t-t')^{2\beta} + (t-t')^{3\beta}],$$

where

$$\hat{c}_{\beta,\theta,\eta} = c_{2\beta+\theta}\left[\frac{1}{1-2^{\theta+\beta-\eta-1}} + \frac{1}{1-2^{\theta+2\beta-\eta-1}}\right]$$

and then is the same estimate that we got for $|I_{[t'-(t-t'),t']}|$ (see (5.60)). Then

$$|I_{[0,t']}| \leq \hat{c}_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\, \|Y\|_{\mathcal{D}_X^{(\beta,\beta)}} (1+\|X\|_\beta)\, [(s-t')^\beta\,(t-t')^{2\beta} + (t-t')^{3\beta}],$$

and, since $v \leq t' < t \leq s$, this implies (5.59). □

Now we pass to prove Lemma 5.14.

PROOF OF LEMMA 5.14. Relation (5.55) is proved in Lemma 5.10.

Now we pass to prove that the map $t \mapsto \mathcal{Y}^R(s,t)$ is a path controlled by X. For any t, t', we can write

$$\begin{aligned}
&\mathcal{Y}^R(s,t) - \mathcal{Y}^R(s,t') = \int_{\mathbb{R}} \tilde{Y}^R(s,u)\,(p_{t-u} - p_{t'-u} - (t-t')\,p'_{s-u})\,\mathrm{d}W_u \\
&= \tilde{Y}^R(s,t')\,\delta X(t',t) + \int_{\mathbb{R}} (\tilde{Y}^R(s,u) - \tilde{Y}^R(s,t'))\,(p_{t-u} - p_{t'-u})\,\mathrm{d}W_u - (t-t')\int_{\mathbb{R}} \tilde{Y}^R(s,u)\,p'_{s-u}\,\mathrm{d}W_u.
\end{aligned}$$

We can apply Proposition 4.18, and, recalling that $\theta + 2\beta - \eta = 3\beta < 1$, we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} (\tilde{Y}^R(s,u) - \tilde{Y}^R(s,t'))\,(p_{t-u} - p_{t'-u})\,\mathrm{d}W_u\Big| \\
&\leq c_{\beta,\theta,\eta}\, C_{W,\mathbb{XW}}\, c\,(t-t')^{2\beta}\Big(\|\tilde{Y}^R(s,\cdot)\|_{\mathcal{D}_X^{(\beta,\beta)}}\, t^\beta + \|(\tilde{Y}^R(s,\cdot))'\|_\infty + \|\tilde{Y}^R(s,\cdot)\|_\beta\Big).
\end{aligned}$$

In particular, this implies that we can write

$$\mathcal{Y}^R(s,t) - \mathcal{Y}^R(s,t') = \tilde{Y}^R(s,t')\,\delta X(t',t) + \mathrm{O}(|t-t'|^{2\beta}),$$

which implies that $t \mapsto$ is a controlled path by X with derivative given by $t \mapsto \tilde{Y}^R(s,t)$ and the remainder term "$\mathrm{O}(|t-t'|^{2\beta})$" is given by

$$R(t',t) := \int_{\mathbb{R}} (\tilde{Y}^R(s,u) - \tilde{Y}^R(s,t'))\,(p_{t-u} - p_{t'-u})\,\mathrm{d}W_u - (t-t')\int_{\mathbb{R}} \tilde{Y}^R(s,u)\,p'_{s-u}\,\mathrm{d}W_u.$$

Thanks to (5.35), we can write

$$|(\mathcal{Y}^R)'(s,u)| = |\tilde{Y}^R(s,u)| \leq K(s-u)^{2\beta},$$

that is (5.56).

Relation (5.57) can be proved by noting that

$$\begin{aligned}
\|R\|_{2\beta}|_{[v,s]} &= \sup_{v\leq t'<t\leq s} \frac{|R(t',t)|}{(t-t')^{2\beta}} \\
&\leq \sup_{v\leq t'<t\leq s} \frac{\Big|\int_{\mathbb{R}} (\tilde{Y}^R(s,u) - \tilde{Y}^R(s,t'))\,(p_{t-u} - p_{t'-u})\,\mathrm{d}W_u\Big|}{(t-t')^{2\beta}} + \\
&\qquad + \sup_{v\leq t'<t\leq s} (t-t')^{1-2\beta}\Big|\int_{\mathbb{R}} \tilde{Y}^R(s,u)\,p'_{s-u}\,\mathrm{d}W_u\Big|.
\end{aligned}$$

For the first part, we use Lemma 5.15, and we get

$$\sup_{v\le t'<t\le s} \frac{\left|\int_{\mathbb{R}}(\tilde{Y}^R(s,u)-\tilde{Y}^R(s,t'))\,(p_{t-u}-p_{t'-u})\,\mathrm{d}W_u\right|}{(t-t')^{2\beta}} \le c_{\beta,\theta,\eta}\,C_{W,\mathbb{XW}}\,c\,(s-v)^{\beta}\,(1+\|X\|)\,\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}.$$

For the second part, we can use Corollary 3.17 with $[a,b]=[0,s]$, $g_u=\tilde{Y}^R(s,u)$, $\bar{p}_u=p'_{s-u}$ and $\bar{\eta}=\eta+1$, and we get

$$\begin{aligned}&\sup_{v\le t'<t\le s}(t-t')^{1-2\beta}\left|\int_{\mathbb{R}}\tilde{Y}^R(s,u)\,p'_{s-u}\,\mathrm{d}W_u\right|\\ &\le \tilde{c}_{\beta,\theta,\eta+1}\,C_{X,W,\mathbb{XW}}\,c\,(s-v)^{1-2\beta}\,s^{3\beta-1}\,(\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}+\|Y\|_{\beta}+\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}\,(1+\|X\|_{\beta}))\end{aligned}$$

having used relations (5.35) and (5.36). Now we can write

$$(s-v)^{1-2\beta}\,s^{3\beta-1}=\Big(\frac{s-v}{s}\Big)^{1-2\beta}s^{\beta}\le\Big(\frac{s-v}{s}\Big)^{\beta}s^{\beta}=(s-v)^{\beta},$$

when $\beta<\frac{1}{3}$, and then, recalling also that $\|Y\|_{\beta}=\|Y'\|_{\beta}\le\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}$,

$$\sup_{v\le t'<t\le s}(t-t')^{1-2\beta}\left|\int_{\mathbb{R}}\tilde{Y}^R(s,u)\,p'_{s-u}\,\mathrm{d}W_u\right|\le\tilde{c}_{\beta,\theta,\eta+1}\,C_{W,\mathbb{XW}}\,c\,(s-v)^{\beta}\,(1+\|X\|_{\beta})\,\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}.$$

Moreover, thanks to (5.38), we have

$$\begin{aligned}\|(\mathcal{Y}^R)'(s,\cdot)\|_{\beta}|_{[v,s]}&=\|\tilde{Y}^R(s,\cdot)\|_{\beta}|_{[v,s]}\le(s-v)^{\beta}\,[\|Y\|_{\beta}\,\|X\|_{\beta}+\|Y^R\|_{2\beta}]\\&\le(s-v)^{\beta}\,(1+\|X\|_{\beta})\,\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}}.\end{aligned}$$

Then

$$\begin{aligned}\|\mathcal{Y}^R\|_{\mathcal{D}_X^{(\beta,\beta)}}|_{[v,s]}&=\|(\mathcal{Y}^R)'(s,\cdot)\|_{\beta}|_{[v,s]}+\|\mathrm{Remainder}\|_{2\beta}|_{[v,s]}\\&\le c_{\beta,\theta,\eta}\,C_{W,\mathbb{XW}}\,(c+1)\,(s-v)^{\beta}\,(1+\|X\|_{\beta})\,\|Y\|_{\mathcal{D}_X^{(\beta,\beta)}},\end{aligned}$$

which completes the proof of (5.57). $\square$

Appendix A

Fractional Heat Kernel

In this section, we will define and study the (fractional) heat kernel, whose properties are key tools in our problem. We fix $\alpha \in (1,2]$; in some case, results and proofs are separated for $\alpha = 2$ and $\alpha \in (1,2)$.

Definition A.1. We denote by $(t,x) \mapsto g_t(x)$ the solution to the following non random differential problem:

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,x) = \Delta^{\frac{\alpha}{2}} g(t,x) \quad \text{for } t>0,\, x \in \mathbb{R}, \\ g(0,\cdot) = \delta_0. \end{cases} \tag{A.1}$$

We will call this function *heat kernel* for the case $\alpha = 2$, and *fractional heat kernel* for the case $\alpha \in (1,2)$.

When $\alpha = 2$, we have an explicit formula for g:

$$g_t(x) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}, \quad \text{for all } (t,x) \in (0,\infty) \times \mathbb{R}. \tag{A.2}$$

When $\alpha \in (1,2)$, we do not have an explicit formula for g, but we now show some properties that are all we need in order to study the equation (1.1).

The following proposition contains the first elementary properties of the function g.

Proposition A.2. The map $(t,x) \mapsto g_t(x)$, solution to (A.1), is the density of a symmetric α-stable Lévy process. In particular, the following properties hold:

1. *regularity*: $(t,x) \mapsto g_t(x)$ is a C^∞ function defined over $(0,\infty) \times \mathbb{R}$.
2. $g_t(x) > 0$ for all $(t,x) \in (0,\infty) \times \mathbb{R}$ and $\int_{\mathbb{R}} g_t(x) = 1$, for every $t > 0$ (g is a *density*).
3. *scaling property*: for every $t > 0$,
$$g_t(x) = \frac{1}{t^{\frac{1}{\alpha}}} g\Big(\frac{x}{t^{\frac{1}{\alpha}}}\Big),$$
where $g(\cdot) := g_1(\cdot)$.
4. g is symmetric over $\mathbb{R}$, that is $g(x) = g(-x)$ for all $x \in \mathbb{R}$.
5. $x \mapsto g(x)$ is strictly decreasing in $[0,\infty)$ and $\|g\|_\infty = g(0)$.
6. g satisfies the *semigroup property*: $g_t \star g_s = g_{t+s}$, that is
$$g_{t+s}(x) = \int_{\mathbb{R}} g_t(x-y)\, g_s(y)\, \mathrm{d}y.$$

7. *characteristic function*: the following relation holds:

$$\int_{\mathbb{R}} e^{-i\theta z}\, g_t(z)\, \mathrm{d}z = e^{-t|\theta|^\alpha}.$$

8. *asymptotic behaviour*: for $\alpha \in (1, 2)$, we have

$$g(x) \sim \frac{c_\alpha}{|x|^{1+\alpha}} \qquad \text{when } |x| \to \infty \tag{A.3}$$

for some $c_\alpha > 0$.

9. We have $g'(0) = 0$ and:
   (a) for $\alpha = 2$, $g'(x) = -\frac{x}{2}\, g(x)$
   (b) for $\alpha \in (1, 2)$,

$$g'(x) \sim -sign(x)\, c_\alpha \frac{(\alpha+1)}{|x|^{2+\alpha}} \quad \text{as } |x| \to \infty; \tag{A.4}$$

   moreover, $|x\, g'(x)| \leq c_1\, g(x)$ and also $|g'(x)| \leq c_2\, g(x)$ for some $c_1, c_2 > 0$.

Now we give some preliminary estimates on the square of g and its integral with respect to time and space.

Lemma A.3. The following properties hold:

1. for any $(t, x) \in (0, \infty) \times \mathbb{R}$,

$$g_t(x)^2 \leq \|g\|_\infty \frac{1}{t^{\frac{1}{\alpha}}} g_t(x). \tag{A.5}$$

2. for all $(t, x) \in (0, \infty) \times \mathbb{R}$,

$$\int_0^t \int_{\mathbb{R}} g_{t-s}^2(x-y)\, \mathrm{d}s\, \mathrm{d}y = \int_0^t \int_{\mathbb{R}} g_s^2(y)\, \mathrm{d}s\, \mathrm{d}y \leq \|g\|_\infty \frac{\alpha}{\alpha-1}\, t^{\frac{\alpha-1}{\alpha}}. \tag{A.6}$$

   In particular, $g \in L^2([0, T] \times \mathbb{R})$, for any $T > 0$.

Proof. Clearly $g_t(x)^2 \leq \|g_t\|_\infty\, g_t(x)$, and (A.5) follows from property (3) of Proposition A.2.

Thanks to (A.5), we can write

$$\int_0^t \int_{\mathbb{R}} g_s^2(y)\, \mathrm{d}s\, \mathrm{d}y \leq \|g\|_\infty \int_0^t \mathrm{d}s \frac{1}{s^{\frac{1}{\alpha}}} \int_{\mathbb{R}} \mathrm{d}y\, g_s(y),$$

which gives exactly $\|g\|_\infty \frac{\alpha}{\alpha-1} t^{\frac{\alpha-1}{\alpha}}$, since the integral over the space is 1 (we recall that g is a density). □

Remark A.4. In the case $\alpha = 2$, we could get the exact values:

$$g_t^2(x) = \frac{1}{\sqrt{8\pi}} \frac{1}{\sqrt{t}}\, g_{\frac{t}{2}}(x) \tag{A.7}$$

and

$$\int_0^t \int_{\mathbb{R}} g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y = \sqrt{\frac{t}{4\pi}}.$$

We now state a basic but crucial estimate, that will be used frequently. This estimate is easy to show for the case $\alpha = 2$, but it is more difficult and requires more efforts in the case $\alpha \in (1,2)$.

Proposition A.5. For any $x \in \mathbb{R}$ and $0 < r < t < \infty$, we have

$$\int_{\mathbb{R}} \frac{g_r^2(z)\, g_{t-r}^2(x-z)}{g_t^2(x)}\,\mathrm{d}z \le c(\alpha)\,\frac{t^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(t-r)^{\frac{1}{\alpha}}} \tag{A.8}$$

where $c(\alpha)$ is a constant which may depend only on α.

In the case $\alpha = 2$, thanks to (A.7) and the semigroup property,

$$\begin{aligned}
\int_{\mathbb{R}} g_r^2(z)\, g_{t-r}^2(x-z)\,\mathrm{d}z &= \frac{1}{\sqrt{8\pi r}}\,\frac{1}{\sqrt{8\pi(t-r)}}\int_{\mathbb{R}} g_{r/2}(z)\, g_{(t-r)/2}(x-z)\mathrm{d}z \\
&= \frac{1}{8\pi}\,\frac{1}{\sqrt{r(t-r)}}\, g_{t/2}(x) \\
&= \frac{1}{\sqrt{8\pi}}\,\frac{\sqrt{t}}{\sqrt{r(t-r)}}\, g_t^2(x),
\end{aligned} \tag{A.9}$$

which proves (A.8) for $\alpha = 2$ with $c(2) = \frac{1}{\sqrt{8\pi}}$. For $\alpha \in (1,2)$, it is not so straight forward. Indeed, if we used the same argument, by (A.5), we would get

$$\int_{\mathbb{R}} g_r^2(z)\, g_{t-r}^2(x-z)\,\mathrm{d}z \le \|g\|_\infty^2\,\frac{1}{r^{\frac{1}{\alpha}}(t-r)^{\frac{1}{\alpha}}}\int_{\mathbb{R}} g_r(z)\, g_{t-r}(x-z)\,\mathrm{d}z \tag{A.10}$$

$$= \|g\|_\infty^2\,\frac{1}{r^{\frac{1}{\alpha}}(t-r)^{\frac{1}{\alpha}}}\, g_t(x). \tag{A.11}$$

The point is that we cannot bound $g_t(x) \le c_1\, g_{c_2 t}^2(x)$ for $\alpha < 2$ (recall (A.3)). Instead, we need the following lemma.

Lemma A.6. Let $\alpha \in (1,2)$ and define $m = m_\alpha \in (0,\infty)$ as the largest $y > 0$ such that $g(y) + y\,g'(y) = 0$. Then, for every $x \in \mathbb{R}^+$, the map

$$t \mapsto g_t(x)$$

is incrasing over $t \in (0, (\frac{x}{m})^\alpha]$.

Proof. Thanks to (A.3) and (A.4), we have

$$g(y) + y\,g'(y) \sim \frac{1}{y^{\alpha+1}} + \frac{-(\alpha+1)}{y^{\alpha+1}} < 0 \qquad \text{for large } y > 0\,.$$

Since the function $g(y) + y\,g'(y)$ is continuous, and positive for $y = 0$, there is indeed a largest $y > 0$ for which $g(y) + y\,g'(y) = 0$, so m is well-defined.

We take the derivative of $g_t(x)$ with respect to the time:

$$\begin{aligned}
\frac{\partial}{\partial t} g_t(x) &= \frac{\partial}{\partial t}\left[\frac{1}{t^{\frac{1}{\alpha}}}\, g\Big(\frac{x}{t^{\frac{1}{\alpha}}}\Big)\right] \\
&= -\frac{1}{\alpha}\,\frac{1}{t^{1+\frac{1}{\alpha}}}\, g\Big(\frac{x}{t^{\frac{1}{\alpha}}}\Big) + \frac{1}{t^{\frac{1}{\alpha}}}\, g'\Big(\frac{x}{t^{\frac{1}{\alpha}}}\Big)\,\Big(-\frac{1}{\alpha}\,\frac{x}{t^{1+\frac{1}{\alpha}}}\Big) \\
&= -\frac{1}{\alpha}\,\frac{1}{t^{1+\frac{1}{\alpha}}}\left[g\Big(\frac{x}{t^{\frac{1}{\alpha}}}\Big) + \frac{x}{t^{\frac{1}{\alpha}}}\, g'\Big(\frac{x}{t^{\frac{1}{\alpha}}}\Big)\right].
\end{aligned}$$

For $t \in (0, (x/m)^\alpha]$ we have $x/t^{\frac{1}{\alpha}} > m$, hence the term in square bracket is negative, by definition of m. This concludes the proof. $\square$

Now we are able to proof Proposition A.5 also for the case $\alpha \in (1,2)$.

Proof of Proposition A.5. We have already proved (A.8) in the case $\alpha = 2$, hence we focus on the case $\alpha \in (1,2)$. Fix $(t,x) \in (0,\infty) \times \mathbb{R}^+$ (thanks to the symmetry of g, we just consider $x \geq 0$); if $\frac{x}{t^{\frac{1}{\alpha}}} \leq 2m$, where m is defined in Lemma A.6, we have

$$\frac{1}{t^{\frac{1}{\alpha}}}\, g\Big(\frac{x}{t^{\frac{1}{\alpha}}}\Big) \geq \frac{1}{t^{\frac{1}{\alpha}}}\, g(2m),$$

and, by (A.10) (see also (A.5)), we can write

$$\int_{\mathbb{R}} \frac{g_r^2(z)\, g_{t-r}^2(x-z)}{g_t^2(x)}\, \mathrm{d}z \leq \frac{\|g\|_\infty^2}{g(2m)}\, \frac{t^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(t-r)^{\frac{1}{\alpha}}}.$$

Hence, we now have to prove (A.8) for $\alpha \in (1,2)$ and for (t,x) such that $\frac{x}{t^{\frac{1}{\alpha}}} > 2m$, that is $t \in (0, (\frac{x}{2m})^\alpha]$ and then, thanks to Lemma A.6, in this case the function $s \mapsto g_s(\frac{x}{2})$ is increasing for any $s \leq t$. We divide the integral:

$$\int_{\mathbb{R}} g_r^2(z)\, g_{t-r}^2(x-z)\, \mathrm{d}z = \int_{-\infty}^{\frac{x}{2}} g_r^2(z)\, g_{t-r}^2(x-z)\, \mathrm{d}z + \int_{\frac{x}{2}}^{+\infty} g_r^2(z)\, g_{t-r}^2(x-z)\, \mathrm{d}z.$$

For $z < \frac{x}{2}$, we use the estimate

$$\begin{aligned}
&g_{t-r}^2(x-z) \leq g_{t-r}^2\Big(\frac{x}{2}\Big) \leq g_t^2\Big(\frac{x}{2}\Big) && \text{by the increasing property} \\
&g_r^2(z) \leq \|g\|_\infty\, \frac{1}{r^{\frac{1}{\alpha}}}\, g_r(z) && \text{(see (A.5))}.
\end{aligned}$$

We get

$$\int_{-\infty}^{\frac{x}{2}} g_r^2(z)\, g_{t-r}^2(x-z)\, \mathrm{d}z \leq \frac{\|g\|_\infty}{r^{\frac{1}{\alpha}}}\, g_t^2\Big(\frac{x}{2}\Big) \int_{-\infty}^{\frac{x}{2}} g_r(z) \leq \frac{\|g\|_\infty}{r^{\frac{1}{\alpha}}}\, g_t^2\Big(\frac{x}{2}\Big).$$

For $z > \frac{x}{2}$, we use the estimates

$$\begin{aligned}
&g_r^2(z) \leq g_r^2\Big(\frac{x}{2}\Big) \leq g_t^2\Big(\frac{x}{2}\Big) && \text{by the increasing property} \\
&g_{t-r}^2(x-z) \leq \frac{\|g\|_\infty}{(t-r)^{\frac{1}{\alpha}}}\, g_{t-r}(x-z) && \text{(see (A.5))},
\end{aligned}$$

and we have

$$\int_{\frac{x}{2}}^{+\infty} g_r^2(z)\, g_{t-r}^2(x-z)\,\mathrm{d}z \le \frac{\|g\|_\infty}{(t-r)^{\frac{1}{\alpha}}}\, g_t^2\Big(\frac{x}{2}\Big) \int_{\frac{x}{2}}^{+\infty} g_{t-r}(x-z)\,\mathrm{d}z \le \frac{\|g\|_\infty}{(t-r)^{\frac{1}{\alpha}}}\, g_t^2\Big(\frac{x}{2}\Big).$$

Summing up,

$$\begin{aligned}\int_{\mathbb{R}} g_r^2(z)\, g_{t-r}^2(x-z)\,\mathrm{d}z &\le \|g\|_\infty \left[\frac{1}{r^{\frac{1}{\alpha}}} + \frac{1}{(t-r)^{\frac{1}{\alpha}}}\right] g_t^2\Big(\frac{x}{2}\Big)\\ &\le 2\,\|g\|_\infty \frac{t^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(t-r)^{\frac{1}{\alpha}}}\, g_t^2\Big(\frac{x}{2}\Big),\end{aligned}$$

since both $\frac{t}{r}$ and $\frac{t}{t-r}$ are greater than 1. Then

$$\int_{\mathbb{R}} \frac{g_r^2(z)\, g_{t-r}^2(x-z)}{g_t^2(x)}\,\mathrm{d}z \le 2\,\|g\|_\infty\, C \frac{t^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(t-r)^{\frac{1}{\alpha}}}$$

where $C := \sup_{x\in\mathbb{R}} \frac{g^2(x/2)}{g^2(x)} < \infty$ is a positive constant (recall (A.3)). □

With the following lemma, we estimate the derivatives of g in the case $\alpha \in (1,2)$.

Lemma A.7. Let $\alpha \in (1,2)$. For all $z \in \mathbb{R}$ and $t \in (0,\infty)$, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}c} g_c(z)\right| \le \frac{2}{\alpha}\,\frac{1}{c}\, g_c(z). \tag{A.12}$$

and

$$\left|\frac{\mathrm{d}}{\mathrm{d}w} g_t(w)\right| \le \frac{1}{t^{\frac{1}{\alpha}}}\, g_t(w) \tag{A.13}$$

Proof. To prove (A.12), we can write

$$\frac{\mathrm{d}}{\mathrm{d}c} g_c(z) = \frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{1}{c^{\frac{1}{\alpha}}} g\Big(\frac{z}{c^{\frac{1}{\alpha}}}\Big)\right) = -\frac{1}{\alpha}\,\frac{1}{c^{1+\frac{1}{\alpha}}}\left[g\Big(\frac{z}{c^{\frac{1}{\alpha}}}\Big) + \frac{z}{c^{\frac{1}{\alpha}}}\, g'\Big(\frac{z}{c^{\frac{1}{\alpha}}}\Big)\right].$$

Since $|zg'(z)| \le g(z)$, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}c} g_c(z)\right| \le \frac{2}{\alpha}\,\frac{1}{c}\, g_c(z).$$

For (A.13), we notice that

$$\frac{\mathrm{d}}{\mathrm{d}w} g_t(w) = \frac{1}{t^{\frac{1}{\alpha}}}\,\frac{\mathrm{d}}{\mathrm{d}w}\left[g\Big(\frac{w}{t^{\frac{1}{\alpha}}}\Big)\right] = \frac{1}{t^{\frac{2}{\alpha}}}\, g'\Big(\frac{w}{t^{\frac{1}{\alpha}}}\Big).$$

By using the fact that $|g'(z)| \le g(z)$, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}w} g_t(w)\right| \le \frac{1}{t^{\frac{2}{\alpha}}}\, g\Big(\frac{w}{t^{\frac{1}{\alpha}}}\Big) = \frac{1}{t^{\frac{1}{\alpha}}}\, g_t(w),$$

that is (A.13). □

The following gives an estimate on the derivative of g in the case $\alpha = 2$, in which the heat kernel has an explicit representation and then the following relations can be proved just by computing.

Lemma A.8. Fix $\alpha = 2$ (Gaussian case). For all $z \in \mathbb{R}$ and $t \in (0, \infty)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}c} g_c(z) = \frac{1}{2c}\Big(\frac{z^2}{2c} - 1\Big) g_c(z). \tag{A.14}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}w} g_t(w) = -\frac{w}{2t}\, g_t(w) \tag{A.15}$$

We also need the following lemma, that holds when $\alpha = 2$.

Lemma A.9. Fix $\alpha = 2$ (Gaussian case). Let $0 \le s < t$ and $a, b \in \mathbb{R}$. For every $r \in (s, t)$,

$$\frac{g_{r-s}(z-y) g_{t-r}(x-z)}{g_{t-s}(x-y)} = g_{\frac{(t-r)(r-s)}{(t-s)}}\Big(z - \big(y + \frac{r-s}{t-s}(x-y)\big)\Big).$$

Proof. By direct calculation,

$$\frac{g_{r-s}(z-y) g_{t-r}(x-z)}{g_{t-s}(b-a)} = \sqrt{\frac{t-s}{2\pi(t-r)(r-s)}} e^{-\frac{(z-y)^2}{2(r-s)}} e^{-\frac{(x-z)^2}{2(t-r)}} e^{\frac{(x-y)^2}{2(t-s)}}$$

$$= \sqrt{\frac{t-s}{2\pi(t-r)(r-s)}} e^{-\frac{1}{2}\frac{(t-s)}{(t-r)(r-s)}\left(z - (y\frac{t-r}{t-s} + x\frac{r-s}{t-s})\right)^2}$$

$$= g_{\frac{(t-r)(r-s)}{(t-s)}}\Big(z - \big(y + \frac{r-s}{t-s}(x-y)\big)\Big).$$

□

A.0.1. Beta and Gamma function and integrals over time.

Now we recall some properties of the Beta and Gamma function; indeed, when we integrate with respect to the time, we would face expression like this:

$$\int_0^t \frac{t^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(t-r)^{\frac{1}{\alpha}}},$$

and it is useful to recall and state how to get exact expression of it.

Definition A.10 (Beta function). The *Beta function* is defined as

$$\mathrm{Beta}(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\, \mathrm{d}u,$$

for all $(\alpha, \beta) \in (0, \infty)^2$.

We recall that, for all $(\alpha, \beta) \in (0, \infty)^2$,

$$\mathrm{Beta}(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{A.16}$$

where the Gamma function is given by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,\mathrm{d}x,$$

for which we recall the following properties: The following properties hold:

1. $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ for all $\alpha > 0$;
2. $\Gamma(1) = 1$ and $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$;
3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$;

We have the following

Lemma A.11. For all $0 \le s < t$ and $\alpha > 1$, we have

$$\int_0^t \frac{t^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(t-r)^{\frac{1}{\alpha}}}\,\mathrm{d}r = t^{\frac{\alpha-1}{\alpha}}\,\mathrm{Beta}\left(\frac{\alpha-1}{\alpha}, \frac{\alpha-1}{\alpha}\right) = t^{\frac{\alpha-1}{\alpha}} \frac{\Gamma\left(\frac{\alpha-1}{\alpha}\right)^2}{\Gamma\left(2\,\frac{\alpha-1}{\alpha}\right)}, \tag{A.17}$$

$$\int_s^t \frac{t^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(t-r)^{\frac{1}{\alpha}}}\,\mathrm{d}r \le (t-s)^{\frac{\alpha-1}{\alpha}}\,\mathrm{Beta}\left(\frac{\alpha-1}{\alpha}, \frac{\alpha-1}{\alpha}\right) = (t-s)^{\frac{\alpha-1}{\alpha}} \frac{\Gamma\left(\frac{\alpha-1}{\alpha}\right)^2}{\Gamma\left(2\,\frac{\alpha-1}{\alpha}\right)} \tag{A.18}$$

Proof. With a change of the variable $u = \frac{r}{t}$, we write

$$\begin{aligned}
\int_0^t \frac{t^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(t-r)^{\frac{1}{\alpha}}} &= t \int_0^1 \frac{t^{\frac{1}{\alpha}}}{t^{\frac{1}{\alpha}} u^{\frac{1}{\alpha}} t^{\frac{1}{\alpha}} (1-u)^{\frac{1}{\alpha}}} \mathrm{d}u \\
&= t^{1-\frac{1}{\alpha}} \int_0^1 u^{\frac{\alpha-1}{\alpha}-1} (1-u)^{\frac{\alpha-1}{\alpha}-1}\,\mathrm{d}u \\
&= t^{\frac{\alpha-1}{\alpha}}\,\mathrm{Beta}\left(\frac{\alpha-1}{\alpha}, \frac{\alpha-1}{\alpha}\right).
\end{aligned}$$

Relation (A.18) follows from (A.17) by a simple translation and the fact that $\frac{t}{r} \le \frac{t-s}{r-s}$ for any $r \in (s,t)$:

$$\int_s^t \frac{t^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(t-r)^{\frac{1}{\alpha}}} \le \int_s^t \frac{(t-s)^{\frac{1}{\alpha}}}{(r-s)^{\frac{1}{\alpha}}(t-r)^{\frac{1}{\alpha}}} = \int_0^{t-s} \frac{(t-s)^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(t-s-r)^{\frac{1}{\alpha}}}.$$

□

Appendix B

Technical tools

In this section, we put all the technical tools which are useful for the proof of existence, uniquess and regularity of the solution of the Fractional Stochastic Heat Equation in Theorems 1.3 and 1.4. Indeed, all the results in this section are the components of these proofs: we put them apart since they require quite long calculations and have their own independence. We fix $\alpha \in (1,2]$ and we recall that $g_t(x) = g_t^{(\alpha)}(x)$ denotes the fractional heat kernel, see Appendix A.

B.1. Gronwall-type inequalities

Lemma B.1. Let $\alpha \in (1,2]$. Let $k \in \mathbb{N}, x \in \mathbb{R}$ and $t \in (0,\infty)$. Then we have

$$\frac{1}{\Gamma\Big(\frac{k(\alpha-1)}{\alpha}+1\Big)} \int_0^t \int_{\mathbb{R}} s^{\frac{k(\alpha-1)}{\alpha}}\, g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y \le \|g\|_\infty\, t^{\frac{(k+1)(\alpha-1)}{\alpha}}\, \frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)}{\Gamma\Big((k+1)\frac{\alpha-1}{\alpha}+1\Big)} \tag{B.1}$$

and, for all $z \in \mathbb{R}$ and $r \in [0,\infty)$ with $r < t$,

$$\frac{1}{\Gamma\Big(\frac{(k+1)(\alpha-1)}{\alpha}\Big)} \int_r^t \int_{\mathbb{R}} (s-r)^{\frac{k(\alpha-1)}{\alpha}}\, g_{s-r}^2(y-z)\, g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y$$
$$\le (\text{const.})\,(t-r)^{\frac{(k+1)(\alpha-1)}{\alpha}}\, g_{t-r}^2(x-y)\, \frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)}{\Gamma\Big(\frac{(k+2)(\alpha-1)}{\alpha}\Big)} \tag{B.2}$$

where the constant may depend only on α.

Proof. To prove (B.1), using (A.5), we just write

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} s^{\frac{k(\alpha-1)}{\alpha}}\, g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y \le g(0) \int_0^t s^{\frac{k(\alpha-1)}{\alpha}}\, \frac{1}{(t-s)^{\frac{1}{\alpha}}}\\
&= g(0)\, t^{\frac{(k+1)(\alpha-1)}{\alpha}} \int_0^1 v^{\frac{k(\alpha-1)}{\alpha}+1-1}\,(1-v)^{\frac{\alpha-1}{\alpha}-1}\,\mathrm{d}v\\
&= g(0)\, t^{\frac{(k+1)(\alpha-1)}{\alpha}}\, \mathrm{Beta}\Big(\frac{k(\alpha-1)}{\alpha}+1, \frac{\alpha-1}{\alpha}\Big)
\end{aligned}$$

and conclusion follows from the relation between Beta and Gamma function (see (A.16)).

Now, for (B.2), we recall Proposition A.5 and write

$$\begin{aligned}\int_r^t \int_{\mathbb{R}} (s-r)^{\frac{k(\alpha-1)}{\alpha}}\, g^2_{s-r}(y-z)\, g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y \\ &\le C\, g^2_{t-r}(x-z) \int_r^t (s-r)^{\frac{k(\alpha-1)}{\alpha}} \frac{(t-r)^{\frac{1}{\alpha}}}{(s-r)^{\frac{1}{\alpha}}(t-s)^{\frac{1}{\alpha}}}\,\mathrm{d}s \\ &= C\,(t-r)^{\frac{1}{\alpha}} \int_r^t (s-r)^{\frac{(k+1)(\alpha-1)}{\alpha}-1}\,(t-s)^{\frac{\alpha-1}{\alpha}-1}\,\mathrm{d}s,\end{aligned}$$

and, using the definition of Beta integral as above, we get

$$C\, g^2_{t-r}(x-z)\,(t-r)^{\frac{(k+1)(\alpha-1)}{\alpha}}\,\mathrm{BETA}\Big(\frac{(k+1)(\alpha-1)}{\alpha}, \frac{\alpha-1}{\alpha}\Big),$$

and (B.2) follows. □

LEMMA B.2. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of measurable and non-negative functions defined on $(0,\infty)\times\mathbb{R}$ such that, for all $(t,x)\in(0,\infty)\times\mathbb{R}$

$$I_1(t,x) := \int_0^t \mathrm{d}t_1 \int_{\mathbb{R}} \mathrm{d}x_1\, \varphi_1(t_1,x_1) g^2_{t-t_1}(x-x_1) < \infty. \tag{B.3}$$

Suppose there exist constants A, $B\ge 0$ such that, for all $(t,x)\in(0,\infty)\times\mathbb{R}$,

$$\varphi_{n+1}(t,x) \le A + B\int_0^t \int_{\mathbb{R}} \varphi_n(s,y)\, g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y \tag{B.4}$$

for all $n\in\mathbb{N}$. Then

$$\varphi_{n+1}(t,x) \le A\sum_{k=0}^{n-1} g(0)^k\, B^k t^{\frac{k(\alpha-1)}{\alpha}} \frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^k}{\Gamma\Big(\frac{k(\alpha-1)}{\alpha}+1\Big)} + B^n\, c^{n-1} \frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^n}{\Gamma\Big(\frac{n(\alpha-1)}{\alpha}\Big)} t^{\frac{(n-1)(\alpha-1)}{\alpha}}\, I_1(t,x) \tag{B.5}$$

for all $(t,x)\in(0,\infty)\times\mathbb{R}$ and $n\in\mathbb{N}$, where $c>0$ is a constant which can be chosen as the one in (A.8), and then it may depend only on α.

PROOF. Let us prove by induction the following

$$\begin{aligned}\varphi_{n+1}(t,x) \le & A\sum_{k=0}^{n-1} g(0)^k\, B^k t^{\frac{k(\alpha-1)}{\alpha}} \frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^k}{\Gamma\Big(\frac{k(\alpha-1)}{\alpha}+1\Big)} + \\ & + B^n\, c^{n-1} \frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^n}{\Gamma\Big(\frac{n(\alpha-1)}{\alpha}\Big)} \int_0^t \mathrm{d}t_1\,(t-t_1)^{\frac{(n-1)(\alpha-1)}{\alpha}} \int_{\mathbb{R}} \mathrm{d}x_1\, \varphi_1(t_1,x_1) g^2_{t-t_1}(x-x_1),\end{aligned} \tag{B.6}$$

which easily implies (B.5).

If $n = 1$, by (B.4)

$$\varphi_2(t,x) \le A + B \int_0^t \int_{\mathbb{R}} \varphi_1(t_1,x_1) g^2_{t-t_1}(x-x_1) \mathrm{d}t_1 \, \mathrm{d}x_1,$$

and then (B.6) holds.

Let $n \ge 1$ and suppose that (B.6) holds for all $(t,x) \in (0,\infty) \times \mathbb{R}$. We have

$$\varphi_{n+2}(t,x) \le A + B \Bigg[A \sum_{k=0}^{n-1} g(0)^k \, B^k \, \frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^k}{\Gamma\Big(\frac{k(\alpha-1)}{\alpha}+1\Big)} \int_0^t \int_{\mathbb{R}} s^{\frac{k(\alpha-1)}{\alpha}} \, g^2_{t-s}(x-y) \, \mathrm{d}s \, \mathrm{d}y$$

$$+ B^n \, c^{n-1} \, \frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^n}{\Gamma\Big(\frac{n(\alpha-1)}{\alpha}\Big)} \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \int_0^s \mathrm{d}t_1 \, (s-t_1)^{\frac{(n-1)(\alpha-1)}{\alpha}} \int_{\mathbb{R}} \mathrm{d}x_1 \, \varphi_1(t_1,x_1) g^2_{s-t_1}(y-x_1) g^2_{t-s}(x-y) \Bigg].$$

Using (B.1) and (B.2), the latter with k and r replaced respectively by $n-1$ and t_1, this is

$$A + A \sum_{k=0}^{n-1} g(0)^{k+1} \, B^{k+1} \frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^{k+1}}{\Gamma\Big(\frac{(k+1)(\alpha-1)}{\alpha}+1\Big)} \, t^{\frac{(k+1)(\alpha-1)}{\alpha}} +$$

$$+ B^{n+1} \, c^n \, \frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^{n+1}}{\Gamma\big(\frac{(n+1)(\alpha-1)}{\alpha}\big)} \int_0^t \mathrm{d}t_1 \, (t-t_1)^{\frac{n(\alpha-1)}{\alpha}} \int_{\mathbb{R}} \mathrm{d}x_1 \, \varphi_1(t_1,x_1) g^2_{t-t_1}(x-x_1).$$

Then (B.6) holds for every $n \in \mathbb{N}$. □

Thanks to Proposition 1.5, we have the following corollaries: the first one can be derived from the proof of Proposition 1.5, while the other two are its direct consequences.

Corollary B.3. Let $\alpha \in (1,2]$. For all $0 < s < t < \infty$ and $x \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} \mathrm{d}y \, [I_0(s,y)]^2 g_{t-s}(x-y) \le c(\alpha) \, \frac{1}{s^{\frac{1}{\alpha}}} \, [I_0(t,x)]^2,$$

where $c(\alpha)$ is a positive constant which depends only on α.

Corollary B.4. For all $k \in \mathbb{N}$, $(t,x) \in (0,\infty) \times \mathbb{R}$, we have

$$\int_0^t \mathrm{d}s \, (t-s)^{\frac{(k-1)(\alpha-1)}{\alpha}} \int_{\mathbb{R}} [I_0(s,y)]^2 \, g^2_{t-s}(x-y) \le c(\alpha) \, t^{\frac{k(\alpha-1)}{\alpha}} \, [I_0(t,x)]^2 \, \mathrm{Beta}\Big(\frac{k(\alpha-1)}{\alpha}, \frac{\alpha-1}{\alpha}\Big).$$

Corollary B.5. For every positive and non-decreasing function $C : (0,\infty) \to \mathbb{R}$,

$$\int_0^t \int_{\mathbb{R}} C(s)\big(1 + |I_0(s,y)|^2\big) g^2_{t-s}(x-y) \, \mathrm{d}s \, \mathrm{d}y \le c(\alpha) \, C(t) \, t^{\frac{\alpha-1}{\alpha}} \, \big(1 + |I_0(t,x)|^2\big),$$

for all $(t,x) \in (0,\infty) \times \mathbb{R}$, where $c(\alpha)$ is a constant which depends only on α.

The following two lemmas are variants of Lemma B.2 and contains Gronwall's type inequalities.

Lemma B.6. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of measurable non-negative functions defined on $(0,\infty)\times\mathbb{R}$ and denote by $\varphi_0(t,x) := |I_0(t,x)|^2$, where I_0 is defined in (1.7) (or, more generally, is a function which satisfies Proposition 1.5). Suppose that there exist constants $A \geq 1$ and $B \geq 0$ such that, for all $(t,x) \in (0,\infty)\times\mathbb{R}$ and $n \in \mathbb{N}_0$,

$$\varphi_{n+1}(t,x) \leq A|I_0(t,x)|^2 + B\int_0^t \int_{\mathbb{R}} \varphi_n(s,y)\, g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y \tag{B.7}$$

Then

$$\varphi_{n+1}(t,x) \leq A|I_0(t,x)|^2\left(1 + \sum_{k=1}^{n+1} c^{k-1} B^k\, t^{\frac{k(\alpha-1)}{\alpha}} \frac{\Gamma\left(\frac{\alpha-1}{\alpha}\right)^{k+1}}{\Gamma\left(\frac{(k+1)(\alpha-1)}{\alpha}\right)}\right), \tag{B.8}$$

for all $(t,x) \in (0,\infty)\times\mathbb{R}$ and $n \in \mathbb{N}_0$, where we can choose $c \geq 1$ as a constant which may depend only on α (a linear combination between $\|g\|_\infty$ and $c(\alpha)$ of (A.8) and of Corollary B.4).

Proof. We will prove that, for every $n \in \mathbb{N}_0$,

$$\begin{aligned}\varphi_{n+1}(t,x) &\leq A|I_0(t,x)|^2 + \\ &+ A\sum_{k=1}^{n+1} c^{k-1} B^k \int_0^t \mathrm{d}s\,(t-s)^{\frac{(k-1)(\alpha-1)}{\alpha}} \int_{\mathbb{R}} \mathrm{d}y\,|I_0(s,y)|^2\, g^2_{t-s}(x-y)\, \frac{\Gamma\left(\frac{\alpha-1}{\alpha}\right)^k}{\Gamma\left(\frac{k(\alpha-1)}{\alpha}\right)}\end{aligned} \tag{B.9}$$

which yields (B.8), since

$$\int_0^t \mathrm{d}s\,(t-s)^{\frac{(k-1)(\alpha-1)}{\alpha}} \int_{\mathbb{R}} \mathrm{d}y\,|I_0(s,y)|^2\, g^2_{t-s}(x-y) \leq c(\alpha)\, t^{\frac{k(\alpha-1)}{\alpha}} \frac{\Gamma\left(\frac{\alpha-1}{\alpha}\right)\Gamma\left(\frac{k(\alpha-1)}{\alpha}\right)}{\Gamma\left(\frac{(k+1)(\alpha-1)}{\alpha}\right)}\, [I_0(t,x)]^2,$$

having used Corollary B.4.

By induction: if $n = 0$, by (B.10),

$$\varphi_1(t,x) \leq A|I_0(t,x)|^2 + B\int_0^t \int_{\mathbb{R}} |I_0(s,y)|^2\, g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y$$

and then (B.9) holds, since $c, A \geq 1$.

Suppose that $n \geq 1$ and, for all $(s,y) \in (0,\infty)\times\mathbb{R}$,

$$\begin{aligned}\varphi_n(s,y) &\leq A|I_0(s,y)|^2 + \\ &+ A\sum_{k=1}^{n} c^{k-1} B^k \int_0^s \mathrm{d}r\,(s-r)^{\frac{(k-1)(\alpha-1)}{\alpha}} \int_{\mathbb{R}} \mathrm{d}z\,|I_0(r,z)|^2\, g^2_{s-r}(y-z)\, \frac{\Gamma\left(\frac{\alpha-1}{\alpha}\right)^k}{\Gamma\left(\frac{k(\alpha-1)}{\alpha}\right)}.\end{aligned}$$

By (B.10), we have

$$\begin{aligned}\varphi_{n+1}(t,x) &\le A|I_0(t,x)|^2 + B\int_0^t\int_{\mathbb{R}} \varphi_n(s,y)\, g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y \\ &\le A|I_0(t,x)|^2 + A\,B\int_0^t\int_{\mathbb{R}} |I_0(s,y)|^2\, g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y+ \\ &+ A\,B\int_0^t \mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\, g^2_{t-s}(x-y)\times \\ &\times \Big[\sum_{k=1}^n c^{k-1}B^k\int_0^s \mathrm{d}r\,(s-r)^{\frac{(k-1)(\alpha-1)}{\alpha}}\int_{\mathbb{R}}\mathrm{d}z\,|I_0(r,z)|^2\, g^2_{s-r}(y-z)\,\frac{\Gamma\big(\frac{\alpha-1}{\alpha}\big)^k}{\Gamma\big(\frac{k(\alpha-1)}{\alpha}\big)}\Big].\end{aligned}$$

Let us focus on the third term: for $k = 1,\dots,n$ we can change the order of integration and write

$$\begin{aligned}&\int_0^t \mathrm{d}r\int_{\mathbb{R}}\mathrm{d}z|I_0(r,z)|^2\int_r^t \mathrm{d}s\,(s-r)^{\frac{(k-1)(\alpha-1)}{\alpha}}\int_{\mathbb{R}}\mathrm{d}y\, g^2_{t-s}(x-y)g^2_{s-r}(y-z)\frac{\Gamma\big(\frac{\alpha-1}{\alpha}\big)^k}{\Gamma\big(\frac{k(\alpha-1)}{\alpha}\big)} \\ &\le c\int_0^t \mathrm{d}r\int_{\mathbb{R}}\mathrm{d}z\,|I_0(r,z)|^2\,(t-r)^{\frac{k(\alpha-1)}{\alpha}}\, g^2_{t-r}(x-z)\frac{\Gamma\big(\frac{\alpha-1}{\alpha}\big)^{k+1}}{\Gamma\big(\frac{(k+1)(\alpha-1)}{\alpha}\big)},\end{aligned}$$

where we applied (B.2) with k replaced by $k-1$ (such relation was proved in Lemma B.1). Then

$$\begin{aligned}\varphi_{n+1}(t,x) &\le A|I_0(t,x)|^2 + A\,B\int_0^t\int_{\mathbb{R}} |I_0(s,y)|^2\, g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y+ \\ &+ A\,B\,c\sum_{k=1}^n c^{k-1}B^k\int_0^t \mathrm{d}r\,(t-r)^{\frac{k(\alpha-1)}{\alpha}}\int_{\mathbb{R}}\mathrm{d}z|I_0(r,z)|^2\, g^2_{t-r}(x-z)\frac{\Gamma\big(\frac{\alpha-1}{\alpha}\big)^{k+1}}{\Gamma\big(\frac{(k+1)(\alpha-1)}{\alpha}\big)},\end{aligned}$$

which is (B.9). □

Lemma B.7. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of measurable non-negative functions defined on $(0,\infty)\times\mathbb{R}$ and denote by $\varphi_0(t,x) := |I_0(t,x)|^2$, where I_0 is defined in (1.7) (or, more generally, is a function which satisfies Proposition 1.5). Suppose that there exist constants $A \ge 1$ and $B \ge 0$ such that, for all $(t,x) \in (0,\infty)\times\mathbb{R}$ and $n \in \mathbb{N}_0$,

$$\varphi_{n+1}(t,x) \le A|I_0(t,x)|^2 + B\int_0^t\int_{\mathbb{R}}[1+\varphi_n(s,y)]g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y \tag{B.10}$$

Then

$$\varphi_{n+1}(t,x) \le A|I_0(t,x)|^2 + (1 + A|I_0(t,x)|^2)\sum_{k=1}^{n+1} c^k\, B^k\, t^{\frac{k(\alpha-1)}{\alpha}}\,\frac{\Gamma\big(\frac{\alpha-1}{\alpha}\big)^{k+1}}{\Gamma\big(\frac{(k+1)(\alpha-1)}{\alpha}\big)}, \tag{B.11}$$

for all $(t,x) \in (0,\infty)\times\mathbb{R}$ and $n \in \mathbb{N}_0$, where we can choose $c \ge 1$ as a constant which may depend only on α (a linear combination between $g(0)$ and $c(\alpha)$ of (A.8)).

PROOF. We will prove that, for every $n \in \mathbb{N}_0$,

$$\begin{aligned}\varphi_{n+1}(t,x) &\le A|I_0(t,x)|^2+\\ &+\sum_{k=1}^{n+1} c^{k-1}\, B^k \int_0^t \mathrm{d}s\,(t-s)^{\frac{(k-1)(\alpha-1)}{\alpha}} \int_{\mathbb{R}} \mathrm{d}y\,[1+A|I_0(s,y)|^2] g^2_{t-s}(x-y)\,\frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^k}{\Gamma\Big(\frac{k(\alpha-1)}{\alpha}\Big)}\end{aligned} \tag{B.12}$$

which yields (B.11), since

$$\int_0^t \mathrm{d}s\,(t-s)^{\frac{(k-1)(\alpha-1)}{\alpha}} \int_{\mathbb{R}} \mathrm{d}y\, g^2_{t-s}(x-y) \le g(0)\,\frac{\alpha}{k(\alpha-1)}\,t^{\frac{k(\alpha-1)}{\alpha}},$$

$$\int_0^t \mathrm{d}s\,(t-s)^{\frac{(k-1)(\alpha-1)}{\alpha}} \int_{\mathbb{R}} \mathrm{d}y\,|I_0(s,y)|^2 g^2_{t-s}(x-y) \le c(\alpha)\,t^{\frac{k(\alpha-1)}{\alpha}}\,\frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)\Gamma\Big(\frac{k(\alpha-1)}{\alpha}\Big)}{\Gamma\Big(\frac{(k+1)(\alpha-1)}{\alpha}\Big)}\,[I_0(t,x)]^2,$$

having used the property (A.5) of g and Corollary B.4.

By induction: if $n=0$, by (B.10),

$$\varphi_1(t,x) \le A|I_0(t,x)|^2 + B\int_0^t\int_{\mathbb{R}} [1+|I_0(s,y)|^2] g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y$$

and then (B.12) holds, since $c, A \ge 1$.

Suppose that $n \ge 1$ and, for all $(s,y) \in (0,\infty)\times\mathbb{R}$,

$$\begin{aligned}\varphi_n(s,y) &\le A|I_0(s,y)|^2+\\ &+\sum_{k=1}^{n} c^{k}\, B^k \int_0^s \mathrm{d}r\,(s-r)^{\frac{(k-1)(\alpha-1)}{\alpha}} \int_{\mathbb{R}} \mathrm{d}z\,[1+A|I_0(r,z)|^2] g^2_{s-r}(y-z)\,\frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^k}{\Gamma\Big(\frac{k(\alpha-1)}{\alpha}\Big)}.\end{aligned}$$

By (B.10), we have

$$\begin{aligned}\varphi_{n+1}(t,x) &\le A|I_0(t,x)|^2 + B\int_0^t\int_{\mathbb{R}} \big[1+\varphi_n(s,y))\big] g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y\\ &\le A|I_0(t,x)|^2 + B\int_0^t\int_{\mathbb{R}} \big[1+A|I_0(s,y)|^2\big] g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y+\\ &+ B\int_0^t \mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\,g^2_{t-s}(x-y)\times\\ &\times\Big[\sum_{k=1}^{n} c^k\,B^k\int_0^s \mathrm{d}r\,(s-r)^{\frac{(k-1)(\alpha-1)}{\alpha}}\int_{\mathbb{R}}\mathrm{d}z\,[1+A|I_0(r,z)|^2] g^2_{s-r}(y-z)\,\frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^k}{\Gamma\Big(\frac{k(\alpha-1)}{\alpha}\Big)}\Big].\end{aligned}$$

Let us focus on the third term: for $k=1,\dots,n$ we can change the order of integration and write

$$\begin{aligned}&\int_0^t \mathrm{d}r\int_{\mathbb{R}}\mathrm{d}z[1+A|I_0(r,z)|^2]\int_r^t \mathrm{d}s\,(s-r)^{\frac{(k-1)(\alpha-1)}{\alpha}}\int_{\mathbb{R}}\mathrm{d}y\,g^2_{t-s}(x-y) g^2_{s-r}(y-z)\frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^k}{\Gamma\Big(\frac{k(\alpha-1)}{\alpha}\Big)}\\ &\le c\int_0^t \mathrm{d}r\int_{\mathbb{R}}\mathrm{d}z\,[1+A|I_0(r,z)|^2]\,(t-r)^{\frac{k(\alpha-1)}{\alpha}}\,g^2_{t-r}(x-z)\frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^{k+1}}{\Gamma\Big(\frac{(k+1)(\alpha-1)}{\alpha}\Big)},\end{aligned}$$

where we applied (B.2) with k replaced by $k-1$ (such relation was proved in Lemma B.1). Then

$$\varphi_{n+1}(t,x) \le A|I_0(t,x)|^2 + B\int_0^t \int_{\mathbb{R}} \big[1 + A|I_0(s,y)|^2\big] g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y +$$
$$+ B\,c\sum_{k=1}^n B^k\,c^k \int_0^t \mathrm{d}r\,(t-r)^{\frac{k(\alpha-1)}{\alpha}} \int_{\mathbb{R}} \mathrm{d}z[1+A|I_0(r,z)|^2] g^2_{t-r}(x-z) \frac{\Gamma\Big(\frac{\alpha-1}{\alpha}\Big)^{k+1}}{\Gamma\Big(\frac{(k+1)(\alpha-1)}{\alpha}\Big)},$$

which is (B.12). □

B.2. Progressive measurability and stochastic integral

In this section we recall some definitions and properties of measurability. Indeed, it is a fundamental property required to define the stochastic integral that appears in the Definition 1.1 of mild solution.

Definition B.8. A stochastic process $(X(t,x))_{(t,x)\in[0,\infty)\times\mathbb{R}^n}$ is called *measurable* if the map

$$\begin{aligned} X:\ ([0,\infty)\times\mathbb{R}^n\times\Omega, \mathcal{B}([0,\infty)\times\mathbb{R}^n)\otimes\mathcal{A}) &\to \mathbb{R} \\ (t,x,\omega) &\mapsto X(t,x)(\omega) \end{aligned}$$

is measurable

Definition B.9. A filtration on $(\Omega,\mathcal{A},\mathbb{P})$, with respect to the set $[0,\infty)$, is a collection $(\mathcal{F}_t)_{t\in[0,\infty)}$ of non-decreasing sub-σ algebras, that is

$$\begin{aligned} &\mathcal{F}_t \subset \mathcal{A}, \\ &\mathcal{F}_t \quad \text{is a } \sigma \text{ algebra}, \\ &\mathcal{F}_s \subseteq \mathcal{F}_t, \end{aligned}$$

for all $s,t\in[0,\infty)$ and $s\le t$.

If $X=(X_t)_{t\in[0,\infty)}$ is a stochastic process on $(\Omega,\mathcal{A},\mathbb{P})$, we denote by $(\mathcal{F}_t^X)_{t\in[0,\infty)}$ the natural filtration of X, given by

$$\mathcal{F}_t^X = \sigma\Big(\{X_s \,|\, 0\le s\le t\}\Big). \tag{B.13}$$

Definition B.10. A stochastic process $(X(t,x))_{(t,x)\in[0,\infty)\times\mathbb{R}^n}$ is *adapted* to the filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ if, for any $(t,x)\in[0,\infty)\times\mathbb{R}^n$, $X(t,x)$ is $\mathcal{F}_t$-measurable, that is

$$X(t,x):\ (\Omega,\mathcal{F}_t)\to\mathbb{R}$$

is measurable.

A stochastic process $(X(t,x))_{(t,x)\in[0,\infty)\times\mathbb{R}^n}$ is *progressively measurable* with respect to the filtration $\mathcal{F}$ if, for any $t\ge 0$, the map

$$X:\ \Big([0,t]\times\mathbb{R}^n\times\Omega, \mathcal{B}([0,t]\times\mathbb{R}^n)\otimes\mathcal{F}_t\Big)\to\mathbb{R}$$

is measurable.

The assumption on a stochastic process φ to be progressively measurable is not always easy to check. However, in Proposition B.12 below we present a sufficient condition which is borrowed from Proposition 3.1 in [Chen, Dalang 15 A].

First we need the following definition.

DEFINITION B.11. We say that a stochastic process $\varphi = (\varphi(s,y))_{(s,y)\in(0,\infty)\times\mathbb{R}}$ is $L^2(\Omega)$-*continuous* if the map

$$\begin{aligned}(0,\infty)\times\mathbb{R} &\to L^2(\Omega)\\ (t,x) &\mapsto \varphi(t,x)\end{aligned}$$

is continuous. That is

$$\lim_{(t',x')\to(t,x)} \|\varphi(t,x)-\varphi(t',x')\|_2 = 0 \quad \text{for all } (t,x)\in(0,\infty)\times\mathbb{R}.$$

For the notion of adaptedness and progressive measurability, see Definition B.10.

PROPOSITION B.12. Let $t>0$ be fixed and let $(\varphi(s,y))_{(s,y)\in(0,t)\times\mathbb{R}}$ be a stochastic process such that

1. φ is adapted;
2. for all $(s,y)\in(0,t)\times\mathbb{R}$, $\|\varphi(s,y)\|_2<\infty$ and φ is L^2-continuous;
3. $\int_0^t\int_{\mathbb{R}}\mathbb{E}(|\varphi(s,y)|^2)\,\mathrm{d}s\,\mathrm{d}y<\infty$.

Then φ is is progressively measurable.

We now define the class of stochastic processes for which we can define the stochastic integral.

DEFINITION B.13. We denote by $\mathcal{M}^2([0,T]\times\mathbb{R}^N)$ the set of stochastic processes indexed by $[0,T]\times\mathbb{R}^N$, $\varphi = (\varphi(s,y))_{(s,y)\in[0,T]\times\mathbb{R}^N}$, such that

1. φ is progressively measurable;
2. $$\mathbb{E}\Big(\int_0^T\int_{\mathbb{R}^N}\varphi(s,y)^2\,\mathrm{d}s\,\mathrm{d}y\Big)<\infty.$$

If $\varphi\in\mathcal{M}^2([0,T]\times\mathbb{R}^N)$, then we can define

$$W_T(\varphi) := \int_0^T\int_{\mathbb{R}^N}\varphi(s,y)\,W(\mathrm{d}s,\mathrm{d}y),$$

as a stochastic process in $L^2(\Omega,\mathcal{F}_T,\mathbb{P})$. We report the following:

PROPOSITION B.14. There exists a unique linear isometry

$$W_T:\ \mathcal{M}^2([0,T]\times\mathbb{R}^N)\to L^2(\Omega,\mathcal{F}_T,\mathbb{P})$$

such that

$$W_T(X\,\mathbb{1}_{[a,b)}\,\mathbb{1}_A) = X\,[W_b(A)-W_a(A)]$$

for all $A \in \mathcal{B}^*(\mathbb{R}^N)$, $0 \le a < b \le T$ and $X \in L^2(\Omega, \mathcal{F}_a, \mathbb{P})$. This isometry is called *stochastic integral with respect to the white noise on* $[0,T] \times \mathbb{R}^N$, or *Walsh integral.* In particular,

- $W_T(\varphi)$ is a random variable in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$;
- $\mathbb{E}(W_T(\varphi)^2) = \int_0^T \int_{\mathbb{R}^N} \mathbb{E}(\varphi(s,y)^2)\, \mathrm{d}s\, \mathrm{d}y$;
- $W_T(\alpha\varphi + \beta\psi) = \alpha W_T(\varphi) + \beta W_T(\psi)$;
- $\langle W_T(\varphi), W_T(\psi)\rangle_{L^2(\Omega,\mathcal{F}_T,\mathbb{P})} = \langle \varphi, \psi \rangle_{\mathcal{M}^2([0,T]\times\mathbb{R}^N)}$ i.e.

$$\mathbb{E}\Big(W_T(\varphi)W_T(\psi)\Big) = \int_0^T \int_{\mathbb{R}^N} \mathbb{E}(\varphi(s,y)\psi(s,y))\, \mathrm{d}s\, \mathrm{d}y, \tag{B.14}$$

for all $\varphi, \psi \in \mathcal{M}^2([0,T] \times \mathbb{R}^N)$ and for all $\alpha, \beta \in \mathbb{R}$.

The following result will be used to prove that the *Picard iteration scheme* is well-defined (see (1.21)). For the proof, we refer to Lemma 3.3 and Proposition 3.4 in [Chen, Dalang 15 A].

Proposition B.15. Let $\varphi = (\varphi(t,x))_{(t,x)\in(0,\infty)\times\mathbb{R}}$ be a stochastic process such that

1. φ is adapted ;
2. φ is $L^2(\Omega)$-continuous;
3. for all $(t,x) \in (0,\infty) \times \mathbb{R}$,

$$\mathbb{E}(|\varphi(t,x)|^2) \le C(t)\Big(1 + |I_0(t,x)|^2\Big) \tag{B.15}$$

for some non-decreasing function $C : (0,\infty) \to \mathbb{R}$.

Then the process given by

$$(f(\varphi)\dot{W}) \star g := \Big(\int_0^t \int_{\mathbb{R}} f(\varphi(s,y)) g_{t-s}(x-y)\, W(\mathrm{d}s, \mathrm{d}y)\Big)_{(t,x)\in(0,\infty)\times\mathbb{R}} \tag{B.16}$$

is well-defined and satisfies the following properties:

(i) $(f(\varphi)\dot{W}) \star g$ is adapted;

(ii) $(f(\varphi)\dot{W}) \star g$ is $L^2(\Omega)$-continuous;

(iii) for all $(t,x) \in (0,\infty) \times \mathbb{R}$,

$$\mathbb{E}\Big(\Big|\big[(f(\varphi)\dot{W}) \star g\big](t,x)\Big|^2\Big) \le \tilde{C}(t)\Big(1 + |I_0(t,x)|^2\Big) \tag{B.17}$$

for some non-decreasing function $\tilde{C} : (0,\infty) \to \mathbb{R}$.

B.3. Tools for the regularity

To prove the regularity of the mild solution of (1.4), we recall some definitions and the main results.

DEFINITION B.16. Let $I \subset \mathbb{R}^N$; we say that a stochastic process $X = (X_i)_{i\in I}$ is continuous if, for every $\omega \in \Omega$, the trajectory $i \mapsto X_i(\omega)$ is a continuous function on I.

Now we give some sufficient condition for the existence of a continuous modification of a given process X, in the special case of $I \subset \mathbb{R}^N$. The classical Kolmogorov continuity theorem states that, if $X = (X_t)_{t\in I}$ is a stochastic process indexed by a compact cube $I = [a_1, b_1] \times \cdots \times [a_N, b_N] \subset \mathbb{R}^N$, and there exist constants $c > 0$, $p > 0$ and $\gamma > N$ such that

$$\mathbb{E}\Big(\big|X_t - X_s\big|^p\Big) \leq c\|t - s\|^\gamma \quad \text{for all } s, t \in I,$$

for some norm $\|\cdot\|$ on $\mathbb{R}^d$, then X has a continuous version which is in fact β-Hölder continuous, for any $\beta \in (0, \frac{\gamma-N}{p})$.

However, we are interested in having a sufficient condition which involves a "metric" on $\mathbb{R}^N$ of the following form

$$\|t - s\|_\alpha = \sum_{i=1}^N |t_i - s_i|^{\alpha_i}, \tag{B.18}$$

for some $\alpha = (\alpha_1, \dots, \alpha_N) \in (0, \infty)^N$, whose components are different. This is because we work on the heat equation which is by its nature anisotropic, since time and space variables have different natural scaling exponents. In particular, we shall deal with the following condition

$$\mathbb{E}\Big(\big|X(t,x) - X(s,y)\big|^p\Big) \leq C\Big(|t - s|^{p\frac{\alpha-1}{2\alpha}} + |x - y|^{p\frac{\alpha-1}{2}}\Big).$$

for all $t, s \in [0, T]$ and $x, y \in [-M, M]$ In principle we could still apply Kolmogorov's theorem, e.g. bounding $|x - y|^{p/2} \leq C(M)|x - y|^{p/(2\alpha)}$, since $\alpha \in (1, 2)$, but we would get non-optimal exponents. Instead, we will use the following generalization of Kolomorov's theorem; for a proof, we refer to [Kunita 90], Theorem 1.4.1.

THEOREM B.17 (GENERALIZED KOLMOGOROV CONTINUITY THEOREM). Let $X = (X_t)_{t\in I}$ be a stochastic process indexed by a compact cube $I = [a_1, b_1] \times \cdots \times [a_N, b_N] \subset \mathbb{R}^N$. Suppose that there exist positive constants c, p and $\alpha_1, \dots, \alpha_n$ with $\sum_{i=1}^N \frac{1}{\alpha_i} < 1$, such that

$$\mathbb{E}\Big(\big|X_t - X_s\big|^p\Big) \leq c \sum_{i=1}^N |t_i - s_i|^{\alpha_i} \quad \text{for all } t, s \in I.$$

Then X has a continuous version $\tilde{X}$ which is $(\beta_1, \dots, \beta_N)$-Hölder continuous, that is,

$$|\tilde{X}_t - \tilde{X}_s| \leq C\|t - s\|_\beta$$

for all $s, t \in I$, where C is a finite (random) constant such that $\mathbb{E}(C^p) < \infty$, and $\|\cdot\|_\beta$ is the "metric" defined in (B.18), for all $(\beta_1, \dots, \beta_N)$ such that

$$0 < \beta_i < \alpha_i \frac{\alpha_0 - N}{\alpha_0\, p}, \quad \text{for } i = 1, \dots, N \text{ and } \alpha_0 = \frac{N}{\sum_{i=1}^N \frac{1}{\alpha_i}}.$$

In many cases of our interest, the set of indices I is thought of as an interval of time, for instance $I = [0, \infty)$. More generally, we will consider the setting when $I = [0, \infty) \times \mathbb{R}^n$, since we shall work with processes $(X(t, x))_{(t,x) \in [0,\infty) \times \mathbb{R}^n}$ that are indexed by both "time" and "space".

Theorem B.18 (Burkholder-Davis-Gundy inequality)**.** For all $p \geq 2$ there exist universal positive constants b_p, c_p (which depend only on p) such that, for every continuous L^2 martingale $(M_t)_{t \geq 0}$, with $M_0 = 0$, and for all $t \geq 0$,

$$b_p \mathbb{E}(\langle M \rangle_t^{\frac{p}{2}}) \leq \mathbb{E}(|M_t|^p) \leq c_p \mathbb{E}(\langle M \rangle_t^{\frac{p}{2}}), \tag{B.19}$$

where $(\langle M \rangle_t)_{t \geq 0}$ denotes the quadratic variation of M.

The following is the application of the BDG inequality to the case of our interest. For a proof see Theorem 5.26 in [Khoshnevisan 09].

Corollary B.19. For all $p > 2$ there exists a constant $c_p > 0$ (the same of the BDG inequality) such that, for all $t > 0$,

$$\mathbb{E}\Big(\Big|\int_0^t \int_{\mathbb{R}^N} \varphi(s, y)\, W(\mathrm{d}s, \mathrm{d}y)\Big|^p\Big) \leq c_p\, \mathbb{E}\Big(\Big|\int_0^t \int_{\mathbb{R}^N} |\varphi(s, y)|^2\, \mathrm{d}s\, \mathrm{d}y\Big|^{p/2}\Big)$$

for every $\varphi \in \mathcal{M}^2([0, \infty) \times \mathbb{R}^N)$.

Note that, by the Ito isometry, when $p = 2$, we have

$$\mathbb{E}\Big(\Big|\int_0^t \int_{\mathbb{R}^N} \varphi(s, y)\, W(\mathrm{d}s, \mathrm{d}y)\Big|^2\Big) = \int_0^t \int_{\mathbb{R}^N} \mathbb{E}(|\varphi(s, y)|^2)\, \mathrm{d}s\, \mathrm{d}y.$$

Looking at the representation formula (1.6), we have to deal with a process of the form

$$(f(u)\dot{W}) \star g := \Big(\Big[(f(u)\dot{W}) \star g\Big](t, x) := \int_0^t \int_{\mathbb{R}} f(u(s, y)) g_{t-s}(x - y)\, W(\mathrm{d}s, \mathrm{d}y)\Big)_{(t,x) \in (0,\infty) \times \mathbb{R}}. \tag{B.20}$$

In order to get some useful bounds on the p-norms of a stochastic process of the form (B.20), we need the following result, which is a consequence of Corollary B.19.

Corollary B.20. Let $(\varphi(s, y))_{(s,y) \in [0,t] \times \mathbb{R}}$ be a progressively measurable stochastic process; then, for all even integers $p \geq 2$, and $(t, x) \in [0, \infty) \times \mathbb{R}$,

$$\Big\| \int_0^t \int_{\mathbb{R}} \varphi(s, y) g_{t-s}(x - y)\, W(\mathrm{d}s, \mathrm{d}y)\Big\|_p^2 \leq \tilde{c}_p \int_0^t \int_{\mathbb{R}} \|\varphi(s, y)\|_p^2\, g_{t-s}^2(x - y)\, \mathrm{d}s\, \mathrm{d}y, \tag{B.21}$$

where $\tilde{c}_p = c_p^{2/p}$ and c_p is the constant in the BDG inequality.

PROOF. By Corollary B.19, and using the fact that p is even,

$$\mathbb{E}\Big(\Big|\int_0^t\int_{\mathbb{R}}\varphi(s,y)g_{t-s}(x-y)\,W(\mathrm{d}s,\mathrm{d}y)\Big|^p\Big)\le c_p\,\mathbb{E}\Big(\Big|\int_0^t\int_{\mathbb{R}}|\varphi(s,y)|^2\,g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y\Big|^{p/2}\Big)$$
$$=c_p\,\mathbb{E}\Big(\prod_{i=1}^{p/2}\int_0^t\int_{\mathbb{R}}|\varphi(s_i,y_i)|^2\,g_{t-s_i}^2(x-y_i)\,\mathrm{d}s_i\,\mathrm{d}y_i\Big)$$
$$=c_p\int_{[0,t]^{p/2}}\int_{\mathbb{R}^{p/2}}\mathbb{E}\Bigg[\prod_{i=1}^{p/2}|\varphi(s_i,y_i)|^2\Bigg]\,g_{t-s_i}^2(x-y_i)\,\mathrm{d}\vec{s}\,\mathrm{d}\vec{y}.$$

Applying the generalized Hölder inequality,

$$\mathbb{E}\Bigg[\prod_{i=1}^{p/2}|\varphi(s_i,y_i)|^2\Bigg]\le\prod_{i=1}^{p/2}\mathbb{E}(|\varphi(s_i,y_i)|^p)^{\frac{2}{p}}=\prod_{i=1}^{p/2}\|\varphi(s_i,y_i)\|_p^2.$$

Then

$$\mathbb{E}\Big(\Big|\int_0^t\int_{\mathbb{R}}\varphi(s,y)g_{t-s}(x-y)\,W(\mathrm{d}s,\mathrm{d}y)\Big|^p\Big)\le c_p\,\Big(\int_0^t\int_{\mathbb{R}}\|\varphi(s,y)\|_p^2\,g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y\Big)^{\frac{p}{2}}.$$

□

Now we are going to prove Theorem 1.13.

THEOREM B.21. For all $x,x'\in\mathbb{R}$ and t,t' with $0<t'<t$, we have

$$\int_{t'}^t\int_{\mathbb{R}}g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y\le K_1\,|t-t'|^{\frac{\alpha-1}{\alpha}}\tag{B.22}$$
$$\int_0^t\int_{\mathbb{R}}\big(g_{t-s}(x-y)-g_{t-s}(x'-y)\big)^2\,\mathrm{d}s\,\mathrm{d}y\le K_2\,|x-x'|^{\alpha-1},\tag{B.23}$$
$$\int_0^{t'}\int_{\mathbb{R}}\big(g_{t-s}(x-y)-g_{t'-s}(x-y)\big)^2\,\mathrm{d}s\,\mathrm{d}y\le K_3\,|t-t'|^{\frac{\alpha-1}{\alpha}},\tag{B.24}$$

where K_1,K_2 and K_3 are positive constants which depend only on α.

PROOF OF THEOREM B.21. The proof of (B.22) is straightforward; indeed,

$$\int_{t'}^t\int_{\mathbb{R}}g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y\le\|g\|_\infty\int_{t'}^t\mathrm{d}s\,\frac{1}{(t-s)^{\frac{1}{\alpha}}}\int_{\mathbb{R}}\mathrm{d}y\,g_{t-s}(x-y)$$
$$=\|g\|_\infty\int_{t'}^t\mathrm{d}s\,\frac{1}{(t-s)^{\frac{1}{\alpha}}}=\|g\|_\infty\,\frac{\alpha}{\alpha-1}\,(t-t')^{\frac{\alpha-1}{\alpha}}.$$

In a similar way, it is easy to show that

$$\Diamond_{0,t}:=\int_0^t\int_{\mathbb{R}}\big(g_{t-s}(x-y)-g_{t-s}(x'-y)\big)^2\,\mathrm{d}s\,\mathrm{d}y$$
$$\le 2\int_0^t\int_{\mathbb{R}}g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y+2\int_0^t\int_{\mathbb{R}}g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y$$
$$\le 4\,\|g\|_\infty\frac{\alpha}{\alpha-1}\,t^{\frac{\alpha-1}{\alpha}},$$

which satisfies (B.23) in the case $t < (\text{const.})\,|x-x'|^\alpha$. Then, we can prove (B.23) considering the case $t > 2\,|x-x'|^\alpha$. We can write

$$\begin{aligned}\diamondsuit_{0,t} &= \int_0^{t-|x-x'|^\alpha}\int_{\mathbb{R}} \big(g_{t-s}(x-y)-g_{t-s}(x'-y)\big)^2\,\mathrm{d}s\,\mathrm{d}y + \int_{t-|x-x'|^\alpha}^t \int_{\mathbb{R}} \big(g_{t-s}(x-y)-g_{t-s}(x'-y)\big)^2\,\mathrm{d}s\,\mathrm{d}y \\ &=: \diamondsuit_{0,t-|x-x'|^\alpha} + \diamondsuit_{t-|x-x'|^\alpha,t}\end{aligned}$$

For the second part, as above,

$$\begin{aligned}&\diamondsuit_{t-|x-x'|^\alpha,t} \\ &\le 2\int_{t-|x-x'|^\alpha}^t\int_{\mathbb{R}} g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y + 2\int_{t-|x-x'|^\alpha}^t\int_{\mathbb{R}} g^2_{t-s}(x-y)\,\mathrm{d}s\,\mathrm{d}y \\ &\le 4\,\|g\|_\infty \int_{t-|x-x'|^\alpha}^t \mathrm{d}s\,\frac{1}{(t-s)^{\frac{1}{\alpha}}} = 4\,\|g\|_\infty\,\frac{\alpha}{\alpha-1}\,|x-x'|^{\alpha-1}.\end{aligned}$$

For the first part, we write

$$\begin{aligned}\diamondsuit_{0,t-|x-x'|^\alpha} &= \int_0^{t-|x-x'|^\alpha}\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\left(\int_{x'}^x \frac{\mathrm{d}}{\mathrm{d}w}g_{t-s}(w-y)\,\mathrm{d}w\right)^2 \\ &\le |x-x'|\int_0^{t-|x-x'|^\alpha}\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\int_{x'}^x\left(\frac{\mathrm{d}}{\mathrm{d}w}g_{t-s}(w-y)\right)^2\end{aligned}$$

having used Jensen inequality. For $\alpha\in(1,2)$, we use the estimate (A.13) for the derivative and we get

$$\begin{aligned}\diamondsuit^{\alpha\in(1,2)}_{0,t-|x-x'|^\alpha} &\le |x-x'|\int_0^{t-|x-x'|^\alpha}\mathrm{d}s\frac{1}{(t-s)^{\frac{2}{\alpha}}}\int_{\mathbb{R}}\mathrm{d}y\int_{x'}^x\mathrm{d}w\,g^2_{t-s}(w-y) \\ &\le \|g\|_\infty\,|x-x'|^2\int_0^{t-|x-x'|^\alpha}\mathrm{d}s\frac{1}{(t-s)^{\frac{3}{\alpha}}} = \|g\|_\infty\,|x-x'|^2\,\frac{\alpha}{3-\alpha}\,|x-x'|^{\alpha-3} \\ &= \|g\|_\infty\,\frac{\alpha}{3-\alpha}\,|x-x'|^{\alpha-1}.\end{aligned}$$

When $\alpha=2$, we use the estimate (A.15) of the derivative, getting

$$\begin{aligned}\diamondsuit^{\alpha=2}_{0,t-|x-x'|^2} &\le \frac{1}{4}|x-x'|\int_0^{t-|x-x'|^2}\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\int_{x'}^x\mathrm{d}w\,\frac{(w-y)^2}{(t-s)^2}g^2_{t-s}(w-y) \\ &= \frac{1}{4}|x-x'|\int_0^{t-|x-x'|^2}\mathrm{d}s\int_{x'}^x\mathrm{d}w\int_{\mathbb{R}}\mathrm{d}y\,\frac{1}{(t-s)}\Big(\frac{w-y}{\sqrt{t-s}}\Big)^2\,\frac{1}{t-s}\,g^2\Big(\frac{w-y}{t-s}\Big),\end{aligned}$$

having changer the order of integrals and using the scaling property of the heat kernel. Defining the new variable $z=\frac{w-y}{\sqrt{t-s}}$, we have

$$\begin{aligned}\diamondsuit^{\alpha=2}_{0,t-|x-x'|^2} &\le \frac{1}{4}|x-x'|\int_0^{t-|x-x'|^2}\mathrm{d}s\,\frac{1}{\sqrt{t-s}}\int_{x'}^x\mathrm{d}w\int_{\mathbb{R}}\mathrm{d}z\,z^2\,g^2(z) \\ &\le (\text{const.})\,|x-x'|^2\int_0^{t-|x-x'|^2}\mathrm{d}s\,\frac{1}{(t-s)^{\frac{3}{2}}} \\ &\le (\text{const.})\,|x-x'|.\end{aligned}$$

We now prove (B.24). By a similar argument used above:

$$\begin{aligned}\diamondsuit_{0,t'} &:= \int_0^{t'} \int_{\mathbb{R}} \big(g_{t-s}(x-y) - g_{t'-s}(x-y)\big)^2 \,\mathrm{d}s\,\mathrm{d}y \\ &\le 2\int_0^{t'} \int_{\mathbb{R}} g^2_{t-s}(x-y), \mathrm{d}s\,\mathrm{d}y + 2\int_0^{t'} \int_{\mathbb{R}} g^2_{t'-s}(x-y), \mathrm{d}s\,\mathrm{d}y \\ &\le 4\,\|g\|_\infty \int_0^{t'} \mathrm{d}s\, \frac{1}{(t'-s)^{\frac{1}{\alpha}}} = 4\,\|g\|_\infty\, \frac{\alpha}{\alpha-1}(t')^{\frac{\alpha-1}{\alpha}}.\end{aligned}$$

Then, if $t' \le |t-t'|$, (B.24) follows easily. It remains to prove (B.24) when $t' > |t-t'|$; in this case, we write

$$\begin{aligned}&\diamondsuit_{0,t'} \\ &= \int_0^{t'-|t-t'|} \int_{\mathbb{R}} \big(g_{t-s}(x-y) - g_{t'-s}(x-y)\big)^2 \,\mathrm{d}s\,\mathrm{d}y + \int_{t'-|t-t'|}^{t'} \int_{\mathbb{R}} \big(g_{t-s}(x-y) - g_{t'-s}(x-y)\big)^2 \,\mathrm{d}s\,\mathrm{d}y \\ &=: \diamondsuit_{0,t'-|t-t'|} + \diamondsuit_{t'-|t-t'|,t'}.\end{aligned}$$

For the second integral, we use the triangle inequality and the argument used above:

$$\diamondsuit_{t'-|t-t'|,t'} \le 4\,\|g\|_\infty \int_{t'-|t-t'|}^{t'} \mathrm{d}s\, \frac{1}{(t'-s)^{\frac{1}{\alpha}}} = 4\,\|g\|_\infty\, \frac{\alpha}{\alpha-1}\, |t-t'|^{\frac{\alpha-1}{\alpha}}.$$

For the first integral,

$$\begin{aligned}\diamondsuit_{0,t'-|t-t'|} &= \int_0^{t'-|t-t'|} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \left(\int_{t'}^{t} \frac{\mathrm{d}}{\mathrm{d}c}\, g_{c-s}(x-y)\,\mathrm{d}c\right)^2 \\ &\le |t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \int_{t'}^{t} \mathrm{d}c \left(\frac{\mathrm{d}}{\mathrm{d}c}\, g_{c-s}(x-y)\right)^2,\end{aligned}$$

having used Jensen inequality. Now we use the estimates of the derivative of g, with respect to the time, and we shall distinguish the case $\alpha \in (1,2)$ and $\alpha = 2$. If $\alpha \in (1,2)$, by using (A.12), we get

$$\begin{aligned}\diamondsuit^{\alpha\in(1,2)}_{0,t'-|t-t'|} &\le \frac{4}{\alpha^2}\,|t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \int_{t'}^{t} \mathrm{d}c\, \frac{1}{(c-s)^2}\, g^2_{c-s}(x-y) \\ &\le \frac{4\|g\|_\infty}{\alpha^2}\,|t-t'| \int_{t'}^{t} \mathrm{d}c \int_0^{t'-|t-t'|} \mathrm{d}s \frac{1}{(c-s)^{2+\frac{1}{\alpha}}}\end{aligned} \tag{B.25}$$

by changing the order of the integrals and using (A.5). If $\alpha = 2$, we use (A.14), getting

$$\begin{aligned}\diamondsuit^{\alpha=2}_{0,t'-|t-t'|} &\le |t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \int_{t'}^{t} \mathrm{d}c\, \frac{1}{4(c-s)^2} \left[\frac{(x-y)^2}{2(c-s)} - 1\right]^2 g^2_{c-s}(x-y) \\ &= \frac{1}{4}|t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{1}{(c-s)^2} \int_{\mathbb{R}} \mathrm{d}y \left[\frac{1}{2}\Big(\frac{x-y}{\sqrt{c-s}}\Big)^2 - 1\right]^2 \frac{1}{c-s} g^2\Big(\frac{x-y}{\sqrt{c-s}}\Big),\end{aligned}$$

by changing the integral and the scaling property of g. Now, by changing the variable $z = \frac{x-y}{\sqrt{c-s}}$, we write

$$\begin{aligned}\diamondsuit^{\alpha=2}_{0,t'-|t-t'|} &\leq \frac{1}{4}|t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c \, \frac{1}{(c-s)^{2+\frac{1}{2}}} \int_{\mathbb{R}} \mathrm{d}z \left(\frac{1}{2}z^2 - 1\right)^2 g^2(z) \\ &\leq (\text{const.})\, |t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c \, \frac{1}{(c-s)^{2+\frac{1}{2}}},\end{aligned} \tag{B.26}$$

since the integral over z is a real number. Looking at (B.25) and (B.26), we have just proved that for every $\alpha \in (1,2]$, we have

$$\begin{aligned}\diamondsuit_{0,t'-|t-t'|} &\leq (\text{const.})\, |t-t'| \int_{t'}^{t} \mathrm{d}c \int_0^{t'-|t-t'|} \mathrm{d}s \frac{1}{(c-s)^{2+\frac{1}{\alpha}}} = (\text{const.})\, |t-t'| \int_{t'}^{t} \mathrm{d}c \, \frac{1}{(c-2t'+t)^{1+\frac{1}{\alpha}}} \\ &\leq (\text{const.})\, \frac{\alpha}{\alpha+1} |t-t'| \, \alpha \, |t-t'|^{-\frac{1}{\alpha}} \\ &= (\text{const.})\, |t-t'|^{\frac{\alpha-1}{\alpha}}.\end{aligned}$$

□

The following theorem coincides with Theorem 1.14: here we show the proof.

Theorem B.22. For all $x, x' \in \mathbb{R}$, with $x' < x$, and t, t' with $0 < t' < t$, we have

$$\int_{t'}^{t} \int_{\mathbb{R}} I_0^2(s,y)\, g_{t-s}^2(x-y)\, \mathrm{d}s\, \mathrm{d}y \leq \tilde{K}_1\, I_0^2(t,x)\, |t-t'|^{\frac{\alpha-1}{\alpha}}, \tag{B.27}$$

$$\begin{aligned}&\int_0^{t} \int_{\mathbb{R}} I_0^2(s,y) \left(g_{t-s}(x-y) - g_{t-s}(x'-y)\right)^2 \mathrm{d}s\, \mathrm{d}y \\ &\qquad \leq \tilde{K}_2 \left(1 + \frac{1}{\sqrt{t}} 1_{\alpha=2}\right) \left(\max_{w\in[x',x]} I_0^2(t,w)\right) |x-x'|^{\alpha-1},\end{aligned} \tag{B.28}$$

$$\begin{aligned}&\int_0^{t'} \int_{\mathbb{R}} I_0^2(s,y) \left(g_{t-s}(x-y) - g_{t'-s}(x-y)\right)^2 \mathrm{d}s\, \mathrm{d}y \\ &\qquad \leq \tilde{K}_3 \left(1 + \frac{1}{\sqrt{t}} 1_{\alpha=2}\right) \left(\max_{w\in[t',t]} I_0^2(c,x)\right) |t-t'|^{\frac{\alpha-1}{\alpha}}.\end{aligned} \tag{B.29}$$

Proof of Theorem B.22. Thanks to Proposition 1.5, we have

$$\begin{aligned}\int_{t'}^{t} \int_{\mathbb{R}} I_0^2(s,y)\, g_{t-s}^2(x-y)\, \mathrm{d}s\, \mathrm{d}y &\leq c\, I_0^2(t,x) \int_{t'}^{t} \mathrm{d}s \frac{t^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}} (t-s)^{\frac{1}{\alpha}}} \\ &\leq c\, I_0^2(t,x)\, \mathrm{Beta}\Big(\frac{\alpha-1}{\alpha}, \frac{\alpha-1}{\alpha}\Big) |t-t'|^{\frac{\alpha-1}{\alpha}}\end{aligned}$$

(see also (A.18)) and (B.27) is proved.

We now pass to (B.28). The idea is similar to the one used to prove (B.23). We first write

$$
\begin{aligned}
\diamondsuit_{0,t} :=& \int_0^t \int_{\mathbb{R}} I_0^2(s,y)\big(g_{t-s}(x-y) - g_{t-s}(x'-y)\big)^2 \,\mathrm{d}s\,\mathrm{d}y \\
\leq& \, 2\int_0^t \int_{\mathbb{R}} I_0^2(s,y)\, g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y + 2\int_0^t \int_{\mathbb{R}} I_0^2(s,y) g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y \\
\leq& \, 4\, c\, I_0^2(t,x)\,\mathrm{B\small{ETA}}\Big(\frac{\alpha-1}{\alpha},\frac{\alpha-1}{\alpha}\Big), t^{\frac{\alpha-1}{\alpha}},
\end{aligned}
$$

as done previously, by using Proposition 1.5 and the definition of Beta function (see Definition A.10). Hence, we can suppose that $t > 2|x-x'|^\alpha$, otherwise (B.28) would follow easily from the above calculation. We can write

$$
\begin{aligned}
\diamondsuit_{0,t} =& \int_0^{t-|x-x'|^\alpha} \int_{\mathbb{R}} I_0^2(s,y)\,\big(g_{t-s}(x-y) - g_{t-s}(x'-y)\big)^2 \,\mathrm{d}s\,\mathrm{d}y + \\
&+ \int_{t-|x-x'|^\alpha}^t \int_{\mathbb{R}} I_0^2(s,y)\,\big(g_{t-s}(x-y) - g_{t-s}(x'-y)\big)^2 \,\mathrm{d}s\,\mathrm{d}y \\
=:& \, \diamondsuit_{0,t-|x-x'|^\alpha} + \diamondsuit_{t-|x-x'|^\alpha,t}
\end{aligned}
$$

For the second part, as above,

$$
\begin{aligned}
&\diamondsuit_{t-|x-x'|^\alpha,t} \\
&\leq 2\int_{t-|x-x'|^\alpha}^t \int_{\mathbb{R}} I_0^2(s,y)\, g_{t-s}^2(x-y)\,\mathrm{d}s\,\mathrm{d}y + 2\int_{t-|x-x'|^\alpha}^t \int_{\mathbb{R}} I_0^2(s,y)\, g_{t-s}^2(x'-y)\,\mathrm{d}s\,\mathrm{d}y \\
&\leq 4\, c\, \max(I_0^2(t,x), I_0^2(t,x')) \int_{t-|x-x'|^\alpha}^t \mathrm{d}s\, \frac{t^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(t-s)^{\frac{1}{\alpha}}} \\
&= 4\, c\, \max(I_0^2(t,x), I_0^2(t,x'))\,\mathrm{B\small{ETA}}\Big(\frac{\alpha-1}{\alpha},\frac{\alpha-1}{\alpha}\Big)\,|x-x'|^{\alpha-1},
\end{aligned}
$$

by using Proposition 1.5.

For the first part, we write

$$
\begin{aligned}
\diamondsuit_{0,t-|x-x'|^\alpha} =& \int_0^{t-|x-x'|^\alpha} \int_{\mathbb{R}} I_0^2(s,y)\,\Big(\int_{x'}^x \frac{\mathrm{d}}{\mathrm{d}w} g_{t-s}(w-y)\,\mathrm{d}w\Big)^2 \mathrm{d}s\,\mathrm{d}y \\
\leq& \, |x-x'| \int_0^{t-|x-x'|^\alpha} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \int_{x'}^x \mathrm{d}w\, \Big(\frac{\mathrm{d}}{\mathrm{d}w} g_{t-s}(w-y)\Big)^2,
\end{aligned}
$$

having used Jensen inequality. In the case $\alpha \in (1,2)$, we use the estimate (A.13) for the derivative and, with a change of integrals in y, w, and Proposition 1.5, we get

$$
\begin{aligned}
\diamondsuit_{0,t-|x-x'|^\alpha} \leq& \, |x-x'| \int_0^{t-|x-x'|^\alpha} \mathrm{d}s\, \frac{1}{(t-s)^{\frac{2}{\alpha}}} \int_{x'}^x \mathrm{d}w \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y)\, g_{t-s}^2(w-y) \\
\leq& \, c\,|x-x'| \int_0^{t-|x-x'|^\alpha} \mathrm{d}s \frac{t^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(t-s)^{\frac{3}{\alpha}}} \int_{x'}^x \mathrm{d}w\, I_0^2(t,w) \qquad \text{(B.30)} \\
\leq& \, c\,\Big(\max_{w\in[x',x]} I_0^2(t,w)\Big)\,|x-x'|^2 \int_0^{t-|x-x'|^\alpha} \mathrm{d}s \frac{t^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(t-s)^{\frac{3}{\alpha}}}.
\end{aligned}
$$

In the case of $\alpha = 2$, we use estimate (A.15), and we write

$$\diamondsuit_{0,t-|x-x'|^2} \leq |x - x'| \int_0^{t-|x-x'|^2} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \, I_0^2(s,y) \int_{x'}^{x} \mathrm{d}w \, \frac{(w-y)^2}{4(t-s)^2} \, g_{t-s}^2(w-y). \tag{B.31}$$

We use the fact that $x^2 \, g(x) \leq 1$ (it is easy to see by plotting the function), that is $x^2 \, g^2(x) \leq g(x)$. In particular, we have

$$\begin{aligned}(w-y)^2 \, g_{t-s}^2(w-y) &= \frac{(w-y)^2}{(t-s)} \, g^2\Big(\frac{w-y}{\sqrt{t-s}}\Big) \leq g\Big(\frac{w-y}{\sqrt{t-s}}\Big) \\ &= \sqrt{t-s} \, g_{t-s}(w-y).\end{aligned}$$

Then, from (B.31), we can write

$$\begin{aligned}\diamondsuit_{0,t-|x-x'|^2} &\leq \frac{1}{4} \, |x - x'| \int_0^{t-|x-x'|^2} \mathrm{d}s \, \frac{1}{(t-s)^{\frac{3}{2}}} \int_{x'}^{x} \mathrm{d}w \int_{\mathbb{R}} \mathrm{d}y \, I_0^2(s,y) \; g_{t-s}(w-y) \\ &\leq \frac{c(2)}{4} \, |x - x'| \int_0^{t-|x-x'|^2} \frac{1}{\sqrt{s}(t-s)^{\frac{3}{2}}} \int_{x'}^{x} \mathrm{d}w \, I_0^2(t,w) \\ &\leq \frac{1}{\sqrt{t}} \, \frac{c(2)}{4} \, |x - x'|^2 \Big(\max_{w \in [x',x]} I_0^2(t,w) \Big) \int_0^{t-|x-x'|^2} \frac{\sqrt{t}}{\sqrt{s}(t-s)^{\frac{3}{2}}},\end{aligned} \tag{B.32}$$

having used Lemma B.3.

Looking at (B.30) and (B.32), we can write

$$\diamondsuit_{0,t-|x-x'|^\alpha} \leq (\text{const.}) \, c(\alpha) \Big(1 + \frac{1}{\sqrt{t}} \, 1_{\alpha=2}\Big) \Big(\max_{w \in [x',x]} I_0^2(t,w) \Big) \, |x - x'|^2 \int_0^{t-|x-x'|^\alpha} \mathrm{d}s \frac{t^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}} (t-s)^{\frac{3}{\alpha}}}.$$

Since we are in the case $t > 2|x - x'|^\alpha$, then $\frac{t}{2} < t - |x - x'|^\alpha$, and we divide the integral by convenience:

$$\begin{aligned}&\diamondsuit_{0,t-|x-x'|^\alpha} \\ &\leq c \Big(1 + \frac{1}{\sqrt{t}} \, 1_{\alpha=2}\Big) \Big(\max_{w \in [x',x]} I_0^2(t,w) \Big) \, |x - x'|^2 \left[\int_0^{\frac{t}{2}} \mathrm{d}s \, \frac{t^{\frac{1}{\alpha}}}{(t-s)^{\frac{3}{\alpha}} s^{\frac{1}{\alpha}}} + \int_{\frac{t}{2}}^{t-|x-x'|^\alpha} \mathrm{d}s \, \frac{t^{\frac{1}{\alpha}}}{(t-s)^{\frac{3}{\alpha}} s^{\frac{1}{\alpha}}} \right] \\ &\leq c \Big(1 + \frac{1}{\sqrt{t}} \, 1_{\alpha=2}\Big) \Big(\max_{w \in [x',x]} I_0^2(t,w) \Big) \, |x - x'|^2 \left[\frac{t^{\frac{1}{\alpha}}}{(\frac{t}{2})^{\frac{3}{\alpha}}} \int_0^{\frac{t}{2}} \mathrm{d}s \, \frac{1}{r^{\frac{1}{\alpha}}} + \frac{t^{\frac{1}{\alpha}}}{(\frac{t}{2})^{\frac{1}{\alpha}}} \int_{\frac{t}{2}}^{t-|x-x'|^\alpha} \mathrm{d}s \, \frac{1}{(t-s)^{\frac{3}{\alpha}}} \right] \\ &\leq c \Big(1 + \frac{1}{\sqrt{t}} \, 1_{\alpha=2}\Big) \Big(\max_{w \in [x',x]} I_0^2(t,w) \Big) \, |x - x'|^2 \left[t^{\frac{\alpha-3}{\alpha}} + |x - x'|^{\alpha-3} \right] \\ &\leq c \Big(1 + \frac{1}{\sqrt{t}} \, 1_{\alpha=2}\Big) \Big(\max_{w \in [x',x]} I_0^2(t,w) \Big) \, |x - x'|^{\alpha-1},\end{aligned}$$

and we have done with (B.28).

We now pass to prove (B.29). We write

$$\begin{aligned}
\diamondsuit_{0,t'} &:= \int_0^{t'} \int_{\mathbb{R}} I_0^2(s,y) \left(g_{t-s}(x-y) - g_{t'-s}(x-y)\right)^2 \mathrm{d}s\, \mathrm{d}y \\
&\leq 2 \int_0^{t'} \int_{\mathbb{R}} I_0^2(s,y)\, g_{t-s}^2(x-y) \mathrm{d}s\, \mathrm{d}y + 2 \int_0^{t'} \int_{\mathbb{R}} I_0^2(s,y)\, g_{t'-s}^2(x-y) \mathrm{d}s\, \mathrm{d}y \\
&\leq 4\, c \max\left(I_0^2(t,x), I_0^2(t',x)\right) \int_0^{t'} \mathrm{d}s\, \frac{(t')^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(t'-s)^{\frac{1}{\alpha}}} \\
&= 4\, c \max\left(I_0^2(t,x), I_0^2(t',x)\right) \textsc{Beta}\Big(\frac{\alpha-1}{\alpha}, \frac{\alpha-1}{\alpha}\Big) (t')^{\frac{\alpha-1}{\alpha}}.
\end{aligned}$$

If $t' < 2\,|t-t'|$, (B.29) follows easily. It remains to prove (B.29) when $t' > 2\,|t-t'|$ (that is $3t' > 2t$); in this case, we write

$$\begin{aligned}
&\diamondsuit_{0,t'} \\
&= \int_0^{t'-|t-t'|} \int_{\mathbb{R}} I_0^2(s,y) \left(g_{t-s}(x-y) - g_{t'-s}(x-y)\right)^2 \mathrm{d}s\, \mathrm{d}y + \\
&\qquad + \int_{t'-|t-t'|}^{t'} \int_{\mathbb{R}} I_0^2(s,y) \left(g_{t-s}(x-y) - g_{t'-s}(x-y)\right)^2 \mathrm{d}s\, \mathrm{d}y \\
&=: \diamondsuit_{0,t'-|t-t'|} + \diamondsuit_{t'-|t-t'|,t'}.
\end{aligned}$$

For the second integral, we use the triangle inequality and the argument used above:

$$\begin{aligned}
\diamondsuit_{t'-|t-t'|,t'} &\leq 4\, c \max\left(I_0^2(t,x), I_0^2(t',x)\right) \int_{t'-|t-t'|}^{t'} \mathrm{d}s\, \frac{(t')^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(t'-s)^{\frac{1}{\alpha}}} \\
&= 4\, c \max\left(I_0^2(t,x), I_0^2(t',x)\right) \textsc{Beta}\Big(\frac{\alpha-1}{\alpha}, \frac{\alpha-1}{\alpha}\Big)\, |t-t'|^{\frac{\alpha-1}{\alpha}}.
\end{aligned}$$

For the first integral,

$$\begin{aligned}
\diamondsuit_{0,t'-|t-t'|} &= \int_0^{t'-|t-t'|} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \left(\int_{t'}^{t} \frac{\mathrm{d}}{\mathrm{d}c}\, g_{c-s}(x-y)\, \mathrm{d}c\right)^2 \\
&\leq |t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \int_{t'}^{t} \mathrm{d}c \left(\frac{\mathrm{d}}{\mathrm{d}c}\, g_{c-s}(x-y)\right)^2,
\end{aligned}$$

having used Jensen inequality. In the case of $\alpha \in (1,2)$, we use the estimate (A.12) for the derivative and, changing the order of the integrals and using Proposition 1.5, we get

$$\begin{aligned}
\diamondsuit_{0,t'-|t-t'|} &\leq \frac{1}{\alpha^2}\, |t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \int_{t'}^{t} \mathrm{d}c\, \frac{1}{(c-s)^2}\, g_{c-s}^2(x-y) \\
&\leq \frac{c(\alpha)}{\alpha^2}\, |t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, I_0^2(c,x)\, \frac{c^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(c-s)^{2+\frac{1}{\alpha}}} \\
&\leq \frac{c(\alpha)}{\alpha^2}\, |t-t'| \Big(\max_{c\in[t',t]} I_0^2(c,x)\Big) \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{c^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}(c-s)^{2+\frac{1}{\alpha}}}.
\end{aligned} \tag{B.33}$$

If $\alpha = 2$, we use the estimate (A.14) for the derivative, and we get

$$\diamondsuit_{0,t'-|t-t'|} \leq \frac{1}{4}\,|t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s\, \frac{1}{(c-s)^2} \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y) \int_{t'}^{t} \mathrm{d}c \left(\frac{(x-y)^2}{2(c-s)} - 1 \right)^2 g_{c-s}^2(x-y).$$

We now use the fact that the function $z \mapsto (\frac{z^2}{2} - 1)^2 g(z)$ is limited, say $(\frac{z^2}{2} - 1)^2 g(z) \leq M$ for any z, then $(\frac{z^2}{2} - 1)^2 g^2(z) \leq M g(z)$ for any z. In particular,

$$\begin{aligned}\left(\frac{(x-y)^2}{2(c-s)} - 1 \right)^2 g_{c-s}^2(x-y) &= \frac{1}{c-s} \left(\frac{(x-y)^2}{2(c-s)} - 1 \right)^2 g^2\left(\frac{x-y}{c-s} \right) \\ &\leq M\, \frac{1}{c-s}\, g\left(\frac{x-y}{c-s} \right) = \frac{M}{\sqrt{c-s}}\, g_{c-s}(x-y)\end{aligned}$$

Then, we can write

$$\begin{aligned}\diamondsuit_{0,t'-|t-t'|} &\leq \frac{M}{4}\,|t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{1}{(c-s)^{2+\frac{1}{2}}} \int_{\mathbb{R}} \mathrm{d}y\, I_0^2(s,y)\, g_{c-s}(x-y) \\ &\leq \frac{M\,c(2)}{4}\,|t-t'| \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{1}{(c-s)^{2+\frac{1}{2}}}\, \frac{1}{\sqrt{s}} I_0^2(c,x) \qquad \text{(B.34)} \\ &\leq \frac{M\,c(2)}{4}\, \frac{1}{\sqrt{t}}\,|t-t'|\,\sqrt{t} \left(\max_{c\in[t',t]} I_0^2(c,x) \right) \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{1}{\sqrt{s}(c-s)^{2+\frac{1}{2}}}\end{aligned}$$

having used Lemma B.3.

Summing up (B.33) and (B.34), we can write

$$\diamondsuit_{0,t'-|t-t'|} \leq (\text{const.}) \left(1 + \frac{1}{\sqrt{t}}\,\mathbf{1}_{\alpha=2} \right) |t-t'|\, t^{\frac{1}{\alpha}} \left(\max_{c\in[t',t]} I_0^2(c,x) \right) \int_0^{t'-|t-t'|} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{1}{s^{\frac{1}{\alpha}}(c-s)^{2+\frac{1}{\alpha}}}.$$

Now we split the integral over s, recalling that $t' > 2(t-t')$; we write

$$\begin{aligned}& t^{\frac{1}{\alpha}} \int_{t'}^{t} \mathrm{d}c\, \frac{1}{s^{\frac{1}{\alpha}}(c-s)^{2+\frac{1}{\alpha}}} \\ &\leq t^{\frac{1}{\alpha}} \left[\int_0^{\frac{t'}{2}} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{1}{s^{\frac{1}{\alpha}}(c-s)^{2+\frac{1}{\alpha}}} + \int_{\frac{t'}{2}}^{t'-(t-t')} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{1}{s^{\frac{1}{\alpha}}(c-s)^{2+\frac{1}{\alpha}}} \right] \\ &\leq |t-t'| \left[\frac{t^{\frac{1}{\alpha}}}{(t'/2)^{2+\frac{1}{\alpha}}} \int_0^{\frac{t'}{2}} \mathrm{d}s\, \frac{1}{s^{\frac{1}{\alpha}}} \int_{t'}^{t} \mathrm{d}c + \frac{t^{\frac{1}{\alpha}}}{(t'/2)^{\frac{1}{\alpha}}} \int_{\frac{t'}{2}}^{t'-(t-t')} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c\, \frac{1}{(c-s)^{2+\frac{1}{\alpha}}} \right] \\ &\leq (\text{const.})\,|t-t'| \left[\frac{t^{\frac{1}{\alpha}}}{(t')^{2+\frac{1}{\alpha}}}\,(t')^{1-\frac{1}{\alpha}}\,|t-t'| + \frac{t^{\frac{1}{\alpha}}}{(t')^{\frac{1}{\alpha}}}\,(t-t')^{-\frac{1}{\alpha}} \right];\end{aligned}$$

indeed,

$$\begin{aligned}\int_{\frac{t'}{2}}^{t'-(t-t')} \mathrm{d}s \int_{t'}^{t} \mathrm{d}c \, \frac{1}{(c-s)^{2+\frac{1}{\alpha}}} &= \int_{\frac{t'}{2}}^{t'-(t-t')} \mathrm{d}s \, \frac{1}{(-1-\frac{1}{\alpha})} \Big[(t-s)^{-1-\frac{1}{\alpha}} - (t'-s)^{-1-\frac{1}{\alpha}} \Big] \\ &= \frac{1}{(1+\frac{1}{\alpha})(\alpha)} \Big[-(t-s)^{-\frac{1}{\alpha}} + (t'-s)^{-\frac{1}{\alpha}} \Big]_{s=\frac{t'}{2}}^{s=t'-(t-t')} \\ &\leq (\mathrm{const.}) \Big[(t-\frac{t'}{2})^{-\frac{1}{\alpha}} + (t-t')^{-\frac{1}{\alpha}} \Big] \leq (\mathrm{const.}) \, (t-t')^{-\frac{1}{\alpha}}.\end{aligned}$$

Hence

$$\begin{aligned}\diamondsuit_{0,t'-|t-t'|} &\leq (\mathrm{const.}) \Big(1 + \frac{1}{\sqrt{t}} \mathbf{1}_{\alpha=2} \Big) \Big(\max_{c\in[t',t]} I_0^2(c,x) \Big) \frac{t^{\frac{1}{\alpha}}}{(t')^{\frac{1}{\alpha}}} \Big[\frac{(t-t')^2}{(t')^{1+\frac{1}{\alpha}}} + (t-t')^{1-\frac{1}{\alpha}} \Big] \\ &\leq (\mathrm{const.}) \Big(1 + \frac{1}{\sqrt{t}} \mathbf{1}_{\alpha=2} \Big) \Big(\max_{c\in[t',t]} I_0^2(c,x) \Big) |t-t'|^{\frac{\alpha-1}{\alpha}},\end{aligned}$$

reminding always that $t' > 2|t-t'|$. $\square$

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