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Monodromy of projections

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Introduction

The shadows cast by opaque objects and motion pictures displayed on a screen are examples of projections that everyone has in mind. In mathematics, linear projections are one of the most important morphisms in projective geometry, whose name originated from the visual effect of perspective. Linear projections have also useful applications to computer vision.

Given a point P in the complex projective space $\mathbb{P}^{n+1}$ and a hyperplane $H \cong \mathbb{P}^n \subset \mathbb{P}^{n+1}$ not containing P, a linear projection from P is the rational map

$$\pi_P : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$$

sending a point $Q \in \mathbb{P}^{n+1} \setminus \{P\}$ to the point $\langle Q, P\rangle \cap \mathbb{P}^n$, where $\langle Q, P\rangle$ is the line through P and Q. The map is not defined on P. We will restrict π_P to a variety X and, when it is clear from the context, we will just write π_P instead of $(\pi_P)_{|X}$. If we require $P \notin X$, the map we obtain is regular. One can also define projections from linear subspaces L of dimension $k > 1$ in an analogous way; those maps may also be realized as the composition of a sequence of projections from points spanning L.

In particular, we will focus on projections of hypersurfaces. Let X be a irreducible and reduced complex projective hypersurface, i.e. a subvariety of $\mathbb{P}^{n+1}$ of dimension n, and consider its projection from a point P not in X. It is a finite morphism of degree equal to the degree of X.

We can associate to this morphism a topological invariant: the monodromy group. If we consider an Zariski open $U \subset Y$ over which the map is unramified, the projection can be seen as a topological covering. The monodromy map on $\pi_P : X \to \mathbb{P}^n$ is

$$\mu : \pi_1(U, q) \to \mathrm{Aut}(\pi_P^{-1}(q)) \simeq S_d.$$

We denoted by S_d the symmetric group on d elements, where d is the degree of the map. The image

$$\mu\left(\pi_1(U, q)\right) =: M(\pi_P) \leq S_d$$

is a subgroup of the symmetric group, called monodromy group. It is a transitive subgroup since X is irreducible.

The goal of this work is to study the projections of an irreducible and reduced hyperurface $X \subset \mathbb{P}^{n+1}$ in terms of their monodromy group.

A classical result in projective geometry is the so-called Uniform position principle, which was introduced by Castelnuovo and later stated in more modern terms by Harris ([Ha4]). One of the several applications of this principle is the following

Lemma. *The projection of an irreducible and reduced hyperurface $X \subset \mathbb{P}^{n+1}$ from a general point P has monodromy group isomorphic to the symmetric group.*

We will say that a point P is uniform for X if $M(\pi_P) = S_d$, non uniform otherwise. We will denote by $\mathcal{W}(X)$ the locus in $\mathbb{P}^{n+1}$ of non uniform points for X.

Hence, projections from non uniform points are in some sense special. The problem we want to address in this work is to find a bound on the dimension of $\mathcal{W}(X)$.

In 1999, Cukierman [Cu] has shown that for a general smooth plane curve of degree d, every point $P \notin X$ is uniform. Moreover, in 2005 Pirola and Schlesinger [PS] proved that for an irreducible and reduced curve $X \subset \mathbb{P}^{c+1}$, the non uniform locus, that can be analogously defined in the Grassmannian $\mathbb{G}(c-1, \mathbb{P}^{c+1})$, has codimension at least two. Examples show that this bound is sharp ([PS, Remark 3.6]). In particular, they proved that a plane curve admits at most a finite number of non uniform points. In 2013 Cuzzucoli, Moschetti and Serizawa [CMS] proved the same statement for projections of smooth surfaces in $\mathbb{P}^3$.

The main Theorem of this thesis is a generalization of those results.

Theorem. *Let X be an irreducible, reduced, complex hypersurface of $\mathbb{P}^{n+1}$. Then, the locus $\mathcal{W}(X)$ is contained in a finite union of linear subspaces of codimension at least* 2.

Cones constructed over an irreducible and reduced plane curve admitting at least a non uniform point, are examples of hypersurfaces with exactly a finite union of codimension two linear subspaces of non uniform points (Proposition 3.3.4).

This bound can be improved if we add some hypothesis on the variety X.

Theorem. *Let X be a smooth projective hypersurface in $\mathbb{P}^{n+1}$. Then the locus of non uniform points is finite.*

The same is true for hypersurfaces $X \subset \mathbb{P}^{n+1}$ that are image of general projections of smooth varieties X in $\mathbb{P}^N$ with $N > n+1$ (Proposition 3.5.1) and for irreducible and reduced hypersurfaces of prime degree (Corollary 3.3.14).

Projections can be studied from several different point of views, in the context of algebra, topology and differential geometry.

An algebraic description of the monodromy group comes from Galois theory: the Galois group G_{π_P} of π_P is the Galois closure of the extension field $\mathbb{C}(X) : \mathbb{C}(\mathbb{P}^n)$, the fields of rational functions. The Galois group is isomorphic to the monodromy group [Ha, Section I]. Thanks to this description, we have that the monodromy group is independent on the choice of the open U and the base point q.

We study generators of the monodromy group by means of classical results as Bertini's Theorem on linear sections and Zariski Lefschetz type theorem on fundamental groups. In particular, Zariski has intensively studied the fundamental group of a complement of a hypersurface ([Zar1]), and this knowledge is very useful for our case when applied to the complement of the branch locus of the projection.

The differential of the map π_P allows us to define the locus in which the map is ramified or not. Moreover, differential geometry techniques as the theory of focal loci are used to study families of lines through the centre of projection P. It is a classical differential geometry theory in projective geometry introduced by Segre ([Se1], [Se]); it has been rewritten in modern terms for instance by Sernesi, Ciliberto and Flamini ([Ser], [CF], [CS]). In the classical definition, it is a locus, proper under some additional assumptions, in a family of linear subspaces in a projective space.

A complete classification of irreducible and reduced hypersurfaces admitting infinitely many non uniform points is not known. However, a consequence of the main theorem in this case is the following.

Theorem. *Assume X is an irreducible reduced hypersurface in $\mathbb{P}^{n+1}$ with $\dim \mathcal{W}(X) > 0$, and X not a cone. Then, the projection from all but finitely many points in $\mathcal{W}(X)$ must be decomposable.*

We recall that a map $X \to Y$ is *decomposable* if it factors non trivially as

$$X \xrightarrow{f} Z \xrightarrow{g} Y$$

with $\deg(f), \deg(g) > 1$. This property is deeply related with the monodromy group: a point P is uniform if the monodromy group $M(\pi_P)$ contains a transposition and if π_P is non decomposable. This suggest us the following conjecture

Conjecture. *Let $X \subset \mathbb{P}^{n+1}$ be a reduced and irreducible hypersurface that is not a cone. Then $\mathcal{W}(X)$ is at most finite.*

The conjecture is true when the degree of X is a prime number.
In order to approach the conjecture, the previous Theorem tells that a possible way to the study of the varieties admitting infinitely many non uniform points, is to understand better the notion of decomposable morphisms.

In Section 3.4 we introduce a technique based on fundamental groups. The first step is the generalisation to hypersurfaces of a result of Nori ([No, Prop. 4.1]).

Lemma. *Let Γ be a irreducible and reduced curve in $\mathbb{P}^n$ and $R \subset \mathbb{P}^n$ be a closed subset such that $\Gamma \nsubseteq R$ and the intersection between Γ and R is transverse. Then we have a surjective map*

$$\pi_1(\Gamma \setminus R) \to \pi_1(\mathbb{P}^n \setminus R).$$

If $\mathcal{W}(X)$ has dimension greater than zero, we apply the Lemma to Γ being the image, via a general projection from a point P, of a curve $\mathcal{S}$ inside $\mathcal{W}(X)$. The hypothesis of transverse intersection is verified for instance when we consider projections of smooth hypersurfaces (Remark 3.4.5).

An example of non uniform points are the so called Galois points. A point P is Galois if the extension of fields $\mathbb{C}(X) : \mathbb{C}(\mathbb{P}^n)$ of the projection of X from P is Galois. Fukasawa [Fu] classified the number of Galois points for smooth plane curves of degree $d \geq 4$. More in general, Yoshihara ([Yo2, Proposition 11]) proved that there is at most a finite number of Galois points $P \notin X$ for a smooth hypersurface $X \subset \mathbb{P}^n$. He also provided a bound for such a finite number. In the case of normal hypersurfaces X, Fukasawa and Takahashi [FT, Theorem 2, Proposition 6] proved that the number of Galois points is finite unless X is a cone.

The last part of the Thesis is devoted to the study of projections of smooth varieties. Using Theorem 3.3.3, we get the following bound on the dimension of $\mathcal{W}$ in the corresponding Grassmannian.

Proposition. *Let $\tilde{X}$ be a smooth irreducible complex projective variety of dimension n in $\mathbb{P}^{n+c}$, $c \geq 1$. The locus of non-uniform L not intersecting $\tilde{X}$ has codimension at least $n+1$ in the Grassmannian $\mathbb{G}(c-1, \mathbb{P}^{n+c})$.*

As before, this bound is sharp: the family of linear subspaces L passing through a non birational point (see [CaCi]) is an example of codimension $n+1$ family of non uniform elements in $\mathbb{G}(c-1, \mathbb{P}^{n+c})$. Moreover, we present a classification of smooth curves in any dimension admitting a family of lines of codimension exactly two in the Grassmannian.

The basic definitions and results on Galois theory, linear projections and monodromy are contained in the first chapter of this thesis. In the second chapter we introduce the main ingredients we used in the proof of our results. Finally, Chapter three is devoted to the proof of the main results concerning the monodromy of projections.

Chapter 1

Preliminaries

1.1 Generalities on permutation groups

We start recalling some basics definitions on permutations groups that we will use in the following. For a more complete treatment, see for instance [Is, Chapter 8].

Let $\Omega := \{1, \dots, d\}$ be a set of indices, $d \geq 2$. The set of all the permutation of Ω is a group called the *symmetric group* of d elements and we will denote it by S_d.

Let H be a group acting on Ω. Recall that such an action defines a natural homomorphism, called permutation representation, from H into the symmetric group S_d. We say that H is a permutation group if the action is faithful and we write $H \leq S_d$.

We list some basic definitions that will be useful later.

Definition 1.1.1. *The action of H on Ω is transitive if for every pair of indices (i, j), with $i, j \in \Omega$ $i \neq j$, there is a permutation $\sigma \in H$ such that $\sigma(i) = j$. Equivalently, H is transitive if it has only one orbit.*

In general, H is said k-transitive, with $1 \leq k \leq d$, if it is transitive on ordered k-ple of distinct indices in Ω.

Observe that if H is k-transitive on Ω then it is automatically m-transitive for all integers $1 \leq m \leq k$. The group S_d is d-transitive, while the alternating group A_d is $(d-2)$-transitive.

Let H acting transitively on Ω and let $\Delta \subseteq \Omega$ be non empty. The subset Δ is called a block for H if, for every $\sigma \in H$, either $\sigma \cdot \Delta = \Delta$ or $\sigma \cdot \Delta \cap \Delta = \emptyset$. A block is called trivial if $\Delta = \{i\}$ for some $i \in \Omega$ or $\Delta = \Omega$.

Definition 1.1.2. *A transitive subgroup H of S_d is called primitive if it has only trivial blocks. Otherwise H is imprimitive.*

The group S_d is primitive for every $d \geq 2$ and the alternating group A_d is primitive for every $d \geq 3$. Moreover, we can show that all the blocks have the same cardinality when the action is transitive.

Lemma 1.1.3. *Let H acting transitively on Ω and let Δ be a block. Then $|\Delta|$ divides $|\Omega|$ and in Ω there are exactly $|\Omega|/|\Delta|$ disjoint blocks, all with the same cardinality.*

Proof. Let σ, γ in H and suppose that $\sigma \cdot \Delta \neq \gamma \cdot \Delta$. Since Δ is a block, $\sigma \cdot \Delta \cap \gamma \cdot \Delta = \emptyset$. The group H acts transitively on Ω, and so the union of all disjoint blocks of the type $\sigma \cdot \Delta$, $\sigma \in H$ is the whole Ω. Moreover, since they are disjoint, they have equal cardinality and cover Ω, there must be exactly $|\Omega|/|\Delta|$ of them. □

Corollary 1.1.4. *If H acts transitively on a set Ω whose cardinality is prime, then H is primitive.*

Proof. Let $d := |\Omega| > 1$ be a prime number. If $\Delta \subset \Omega$ is a block, then by Lemma 1.1.3 $|\Delta|$ divides $|\Omega|$, and so $|\Delta| = |\Omega|$ or $|\Delta| = 1$. In both cases Δ is a trivial block and thus the given action of H on Ω is primitive. □

We can relate primitivity with double transitivity.

Lemma 1.1.5. *If $H \leq S_d$ be a 2-transitive subgroup, then it is primitive.*

Proof. We show that if H is imprimitive, then it is not 2-transitive. Indeed, if H is imprimitive, there exists at least a non trivial blocks Δ. Consider two distinct elements $i, j \in \Delta$ and $k \in \Omega \setminus \Delta$, that is non empty because Δ is non trivial. By assumptions, Δ is a block and so there is no permutation $\sigma \in H$ such that $\sigma(i) = j \in \Delta$ and $\sigma(j) = k \notin \Delta$. Hence H is not 2-transitive, that is a contradiction. □

Remark 1.1.6. A transitive subgroup of S_d generated by transpositions is S_d ([Cu, Lemma 2.5]).

If d is prime and H acts transitively on Ω, Lemma 1.1.3 implies that there can only be one block, namely the whole Ω; if moreover H contains at least one transposition, then it contains all the transpositions. As the transpositions generate S_d (Remark 1.1.6), it follows that $H = S_d$.

More in general, we have the following:

Theorem 1.1.7. *(Jordan) Let H be a transitive subgroup of S_d. If H is primitive and contains a transposition, then $H = S_d$.*

Proof. Since every element of S_d is a product of transpositions, it suffices to show that H contains all transpositions. To do this, consider the (undirected) graph with vertex set Ω, in which distinct points i and j are joined by an edge when the transposition (i, j) is an element of H.

Let $\sigma \in H$ be a permutation, it is easy to see that if i and j are two points joined by an edge, then their images under σ are also joined by an edge. In other words, the permutations in H define automorphisms of the graph.

Now, if G is a connected component of the graph, then $\sigma \cdot G$ is also a component. If G meets $\sigma \cdot G$ nontrivially, then $G \cup \sigma \cdot G$ is a connected set that contains the two components, and thus $G = \sigma \cdot G$. In other words, G is a block. The action of H is primitive, hence G must be a trivial block. By hypothesis, the graph has at least one edge, and so there is a component consisting of more than one point. Then $G = \Omega$, which means that the graph is connected.

Each two points i, j are joined by a path, and we define the distance $d(i,j)$ to be the length of the shortest path from i to j, where we set $d(i,j) = 1$ when the group H contains the transposition (i,j).

We need to prove that H contains all the transpositions, and so it suffices to show that $d(i,j) \leq 1$ for every $i, j \in \Omega$.

Suppose that there exist i, j such that the distance $\delta := d(i,j) > 1$ is as small as possible. There must exist a point $k \in \Omega$ such that $d(i,k) = d(k,j) = 1$ and thus the transposition $(k,j) \in H$. Then $(i,k) \cdot (k,j) = (i,j) \in H$ that is a contradiction. $\square$

1.2 Tangent spaces

We now introduce some basic definitions on tangent lines and tangent spaces to a projective variety. We will follow [Ha, Lecture 14-15] and [Sh, Chapter II.1].

Tangency is a local property, i.e. a property of a point $x \in X$ that remains unchanged if X is replaced by an open neighborhood of x. Since any point has an affine neighbourhood, we can restrict ourselves to affine varieties.

Suppone that $X \subset \mathbb{A}^n$ is given by $\{f_1 = \ldots = f_m = 0\}$, let $I(X)$ be its defining ideal and choose coordinates such that $x = (0, \ldots, 0)$. Every polynomial $f \in I(X)$ has a Taylor expansion

$$f(t) = f(0) + f^1(t) + \ldots + f^k(t)$$

where $f(0) = 0$ and f^i are homogeneous polynomials in $(t_1, \ldots, t_n)$ of degree i. The equations of the tangent space to X at x are

$$\sum_{i=1}^{n} \frac{\partial f}{\partial t_i}(0) t_i = 0$$

for all $f \in I(X)$.

Remark 1.2.1. The Zariski tangent space $T_x X$ to X at x is isomorphic to $(\mathrm{m}_x/\mathrm{m}_x^2)^*$, where m_x is the maximal ideal of the local ring $\mathcal{O}_x$.

Let $X \subset \mathbb{P}^n$ is a projective variety and let $x \in X$ be a point. The tangent space $T_x X$ is an affine linear subspace of $\mathbb{A}^n_i$, an open affine neighborhood of x. The closure $\mathbb{T}_x X$ of $T_x X$ in $\mathbb{P}^n$ does not depend on the choice of the affine charts on the projective space; it is called the projective tangent space.

Definition 1.2.2. *A point $x \in X$ is non singular (or smooth) if* $\dim T_x X = \dim_x X$.

The Zariski tangent space to a variety $X \subset \mathbb{A}^n$ at a point x does not give us a very precise picture of the local geometry of X when x is a singular point of X. We will introduce the refined notion of tangent cone.

In the definition of the tangent space of an affine variety X at a point x we took all $f \in I(X)$, expanded around x and took their linear parts; we defined $T_x(X)$ to be the zero loci of these homogeneous linear forms. In the definition of the tangent cone $TC_x(X)$ we look to the leading terms, i.e. the lowest degree terms.

Consider again X affine, $f \in I(X)$ with Taylor expansions around the point x,

$$f = f^l + \ldots + f_j^k$$

where $f^l \neq 0$. The form f^l is the leading form of f.

The equations of the tangent cone TC_xX are $f^l = 0$ for all $f \in I(X)$. Since TC_xX is defined by homogeneous polynomials, it is a cone with vertex x. One can see that $TC_xX \subseteq T_xX$, and that the two notions coincide if x is a smooth point.

Example 1.2.3. Let $X \subset \mathbb{A}^n$ be an hypersurface defined by $f = f(x_1, \ldots, x_n) = 0$ around $x = (0, \ldots, 0)$. Expanding f around x we get

$$f(t) = f_m(t) + f_{m+1}(t) + \ldots$$

where $t = (t_1, \ldots, t_n)$ and the f_j's are homogeneous polynomials of degree j. Thus the tangent cone to X at x is given by $f_m = 0$.

If $X \subset \mathbb{P}^n$ is a projective variety, we can associate a projective variety $\mathbb{T}C_x(X)$ called projective tangent cone. We can also realize $\mathbb{T}C_x(X) \subset \mathbb{T}_xX$ as the set of the tangent lines to X at x.

Remark 1.2.4. The most important fact about the tangent cone is that its dimension is always the local dimension of X at x.

1.2.1 Tangent lines

Consider a reduced and irreducible hypersurface X and a line $l \not\subseteq X$. The intersection $X \cap l$ consists of a finite number of points $P_1, \ldots, P_k$ counted with multiplicities $m_1, \ldots, m_k$. Notice that $\sum_{i=1}^k m_i$ is equal to the degree of X. We call the contact order of l with X at P_i the number $m_i - 1$, and we denote it by $\mathrm{ord}_{P_i}(l \cap X)$.

The line l is transverse to X at P_i if $\mathrm{ord}_{P_i}(l \cap X) = 0$, and tangent to X at P_i if $\mathrm{ord}_{P_i}(l \cap X) \geq 1$. In the case of higher contact order, i.e. $\mathrm{ord}_{P_i}(l \cap X) \geq 2$, we say that the line l is an asymptotic tangent to X at P_i. The line l is called bitangent to X at two points $P_i \neq P_j$ of the intersection $l \cap X$, if l is tangent to X at both points P_i, P_j.

We say that l is *simply tangent* to X if there is a unique point $P_i \in l \cap X$ with $\mathrm{ord}_{P_i}(l \cap X) = 1$ and l is transverse to X for all the other $P_j \neq P_i$ in $l \cap X$. Finally, we will say l is *more than simply tangent* to X if $\sum_{i=1}^k ord_{P_i}(l \cap X) \geq 2$, i.e. the line l is not secant nor simply tangent.

1.2.2 Dual variety and Gauss maps

Let X be a irreducible, non degenerate projective variety in $\mathbb{P}^N$ of dimension n. Let $P_X \subseteq X \times (\mathbb{P}^N)^*$ be the Zariski closure of

$$\mathcal{P}_X := \{(x, H) \in X^{sm} \times (\mathbb{P}^N)^* \mid T_xX \subseteq H\}$$

with the projections $p_1 : P_X \to X$ and $p_2 : P_X \to (\mathbb{P}^n)^*$

Definition 1.2.5. *The image* $X^* := p_2(P_X) \subset (\mathbb{P}^N)^*$ *is called the dual variety of* X.

Definition 1.2.6. *The variety* X *is said to be reflexive if the natural isomorphism* $\mathbb{P}^N \simeq ((\mathbb{P}^N)^*)^*$ *induces an isomorphism* $(X^*)^* \simeq X$.

Since we work in characteristic zero, we have the following biduality theorem.

Theorem 1.2.7. *All projective irreducible varieties are reflexive.*

Moreover, we can define a map

$$\gamma : X \dashrightarrow \mathbb{G}(n, N)$$

that sends a smooth point $x \in X$ to the tangent $[T_x X] \in \mathbb{G}(n, N)$. The map γ is called the Gauss map and it is a regular map on the open subset of smooth points of X.

When $X \subset \mathbb{P}^N$ is a smooth hypersurface given by an homogeneous polynomial f, then the Gauss map is given by $\gamma(x) = \left[\frac{\partial f}{\partial z_0}(x) : \ldots : \frac{\partial f}{\partial z_N}(x)\right]$, where $z_0, \ldots, z_N$ are coordinates in $\mathbb{P}^N$.

If X is smooth the Gauss map is a regular map defined everywhere. In this case we have a stronger result due to Zak ([FL, Cor 7.2], [Za1]).

Proposition 1.2.8. *Let* $X \subset \mathbb{P}^N$ *be a smooth, iireducible, non degenerate projective variety of dimension* n. *Then the Gauss map* $\gamma : X \to \mathbb{G}(n, N)$ *is finite and birational onto the image.*

It follows from the following more general Theorem in the case $m = n$.

Theorem 1.2.9 (Zak)**.** *Let* L *be a linear subspace of dimension* m *with* $n \leq m \leq N - 1$. *Then*

$$\dim\{x \in X \mid T_x X \subset L\} \leq m - n.$$

Proof. By contradiction, assume that there exists an irreducible component $S \subseteq \{x \in X \mid T_x X \subset L\}$ of dimension strictly grater than $m - n$. Since X is non degenerate, then $X \not\subseteq L$. Fix two points $x \in X \setminus L$ and $s \in S$. Note that the line $\langle x, s \rangle$ does not lie on X. Moreover, choose a point $p \in \langle x, s \rangle$ such that $p \notin X$: one can take a general linear subspace V of dimension $N - m - 1$ through p, disjoint from X and L, such that the projection $\pi_V : X \to L$ is not birational onto $\pi_V(S)$. We want now to apply the connectness Theorem (see Theorem 2.2.4) to

$$f := \pi_V \times (\pi_V)_{|S} : X \times S \to L \times L \simeq \mathbb{P}^m \times \mathbb{P}^m$$

Since $f^{-1}(\Delta)$ is connected and it does not consist only on the diagonal $\delta \subset X \times S$, there exist a curve T and a morphism $T \to f^{-1}(\Delta)$ whose image intersects δ. This give rise to a family of pairs $\{(x_t, s_t)\}_{t \in T}$ such that $x_t \neq s_t$ for the general $t \in T$ and $x_{t_0} = s_{t_0} := s_0$ for some $t_0 \in T$. The secant lines $\langle x_t, s_t \rangle$ meet V and so the tangent line $l_0 \subset T_{s_0} X \subset L$, limit of those secant lines as $t \to t_0$. But this contradicts the assumption that L and V are disjoint. $\square$

We conclude this part by stating another consequence of Theorem 1.2.9.

Corollary 1.2.10. *Let $X \subset \mathbb{P}^N$ be a smooth, non degenerate projective variety of dimension n. Let X^* be its dual variety. Then*

$$\dim(X^*) \geq \dim(X).$$

Proof. Consider the incidence variety

$$I := \{(x, L) \mid T_x X \subset L\}$$

where $x \in X$ is a point and L is a linear subspace of dimension m with $n \leq m \leq N-1$. The first projection $p : I \to X$ realizes I as a $\mathbb{P}^{m-n-1}$ bundle over X, i.e. $\dim I = m-1$. Consider now the second projection $q : I \to (\mathbb{P}^m)^*$ and let X^* be its image. Theorem 1.2.9 implies that the fibres of q have all dimension lower or equal to $m-n-1$. Therefore $\dim(X^*) \geq n = \dim(X)$.

□

1.3 Topology of fibre spaces

We briefly recall some definitions and classical results (we follow notations in [FL], [No]).

Let X and Y be two manifolds. A topological fibre bundle with base Y is a surjective map $f : X \to Y$ such that there exist an open covering $\{U_j\}$ of Y with $f^{-1}(U_j) \simeq U_j \times F$, where the fibre F is a topological variety.

We have an exact homotopy sequence

$$\ldots \to \pi_1(F) \to \pi_1(X) \to \pi_1(Y) \to \pi_0(F) \to \pi_0(X) \to \pi_0(Y) \to 1$$

If X and Y are connected, then $\pi_0(X) = \pi_0(Y) = 1$. Moreover, if also the fibre F is connected then we have an exact sequence

$$\ldots \to \pi_1(F) \to \pi_1(X) \to \pi_1(Y) \to 1$$

i.e. the map $\pi_1(X) \to \pi_1(Y)$ is surjective. In particular, if X is connected and f admits a section, then F is connected.

We recall two important results that we will use. The first one is the so called Stein factorization.

Proposition 1.3.1. *[Hart, III.11.5] Let $f : X \to Y$ a proper and dominant morphism between complex varieties. Then f admits a factorization*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "f"'] & Z \arrow[d, "g"] \\ & Y \end{tikzcd}$$

where $h : X \to Z$ has connected fibres and $g : Z \to Y$ is finite. Moreover, if X is normal, then also Z is normal.

Lemma 1.3.2. *[No, Lemma 1.5] Let X and Y be smooth connected complex varieties and let $f : X \to Y$ be an arbitrary morphism. Then*

A) there is a non empty Zariski open $U \in Y$ such that $f^{-1}(U) \to U$ is a fibre bundle;

B) if f is dominant, the image of $\pi_1(X)$ has finite index in $\pi_1(Y)$;

C) if the general fibre F of f is connected and the set $S = \{y \in Y \mid f^{-1}(y)$ is not generically reduced in at least one irreducible component of $f^{-1}(y)\}$ is a codimension at least 2 subset in Y, then the sequence

$$\pi_1(F) \to \pi_1(X) \to \pi_1(Y) \to 1$$

is exact.

Proof. By Hironaka's resolution of singularities ([La, Theorem 4.1.3]), we may assume that $X = \tilde{X} \setminus D$ where $\tilde{f} : \tilde{X} \to Y$ is a proper map and $i : X \to \tilde{X}$ an open immersion such that $f = \tilde{f} \circ i$ and $D = \tilde{X} \setminus i(X)$ is a normal crossing divisor. Let $D_i,\ i = 1, \dots, r$ be the irreducible components of D and set $D_S = \bigcap_{i \in S} D_i$ for every subset $S \subset \{1, \dots, r\}$. By assumptions, D_S is smooth for every S. Moreover, by Sard's theorem ([Hi, Chapter 3.1], there is a Zariski open $U \subset Y$ such that $\tilde{f}$ restricted to $\tilde{f}^{-1}(U) \cap D_S$ is a submersion. By Ehresmann's theorem, we get that the map is a locally trivial fibration. This proves A).

Set $F = f^{-1}(p)$ for $p \in U$, the open defined in A). From the homotopy sequence we have

$$\begin{tikzcd}
\pi_1(F) \arrow[r] & \pi_1(f^{-1}(U)) \arrow[r] \arrow[d] & \pi_1(U) \arrow[r] \arrow[d] & \pi_0(F) \\
& \pi_1(X) \arrow[r] & \pi_1(Y) &
\end{tikzcd}$$

where $\pi_1(U) \to \pi_1(Y)$ and $\pi_1(f^{-1}(U)) \to \pi_1(X)$ are surjective. Moreover, by Stein factorization (Proposition 1.3.1) on f, $\pi_0(F)$ is finite. This proves B).

We are left with part C).

$$\begin{tikzcd}
\pi_1(F) \arrow[r] \arrow[dr, "\beta"] & \pi_1(f^{-1}(U)) \arrow[r, "i"] \arrow[d, "k"] & \pi_1(U) \arrow[d, "j"] \\
& \pi_1(X) \arrow[r, "\alpha"] & \pi_1(Y)
\end{tikzcd}$$

Let $T := \{q \in Y \mid \dim f^{-1}(q) > \dim F\}$ be the codimension at least two subset of Y. Let R be a irreducible component of $Y \setminus U$ of codimension one in Y and $r \in R$ a smooth point not lying in $S \cup T$. Then, $f^{-1}(r)$ contains at least a smooth point m and let M be an irreducible component in $f^{-1}(R)$ containig m; moreover, $\dim f^{-1}(R) = \dim F$ and so f induces a surjection at the level of tangents at m.

Following [No, Section 1] we will denote by $\gamma(M)$ the subset of $\pi_1(f^{-1}(U))$ of coniugacy classes of elements $f_{|S^1}$ that does not depend on the choice of f. Since i is surjective for what we already proved and thanks to facts 1.2 and 1.3 in [No], we get that $\gamma(M) = \gamma(R)$. The subset $\gamma(R)$ generates $\ker(j)$ so we get finally that $i(\ker(k)) = \ker(j)$. $\square$

In general, if we have a (locally irreducible in the usual topology) smooth complex variety X, for any closed subset analytic space $R \subsetneq X$ there exist a surjective map

$$\pi_1(U) \twoheadrightarrow \pi_1(X)$$

where we set $U = X \setminus R$. Moreover if R is a closed subset of X of codimension at least 2, then $\pi_1(U) \simeq \pi_1(X)$.

1.4 Galois theory

1.4.1 Galois extensions

Let L, K be two fields and $L : K$ an extension field.

Definition 1.4.1. *An extension $L : K$ is called algebraic if every element α of L is a root of some polynomial with coefficients in K. If this polynomial is monic and irreducible over K, it is called the minimal polynomial of α.*

When L is generated as a K-algebra by the elements $\alpha_1, \dots, \alpha_r \in L$, we write $L = K(\alpha_1, \dots, \alpha_r)$.

Let $L : K$ be an algebraic extension. Its degree over K, denoted by $[L : K]$, is its dimension as a K-vector space. If L is generated over K by a single element with minimal polynomial f, then $[L : K]$ is equal to the degree of f.

We begin recalling two basic properties of field extensions. The first is the normality.

Definition 1.4.2. *An algebraic extension $L : K$ is said to be* normal *if, given $f \in K[x]$ an irreducible polynomial, then either f splits over L or f has no roots in L.*

An algebraic extension $L : K$ is normal if and only if the minimal polynomial over K of each element of L splits over L.

The latter property is the separability. Let f be a irreducible polynomial of degree k in $K[x]$ and let $L : K$ be a splitting field extension for f.

Definition 1.4.3. *We say that f is* separable *over K if it has k distinct roots in L.*

If f is an arbitrary polynomial in $K[x]$, we say that it is separable over K if each of its irreducible factors is separable. Let $L : K$ be an extension and let $\alpha \in L$.

Definition 1.4.4. *We say that α is separable over K if it is algebraic over K and its minimal polynomial over K is separable. An extension $L : K$ is separable if each $\alpha \in L$ is separable over K.*

An important result concerning separable extension is the following, known as the *theorem of primitive element.*

Proposition 1.4.5. *A finite separable extension can be generated by a single element.*

The two properties, separability and normality, will lead us to the following characterization of field extensions.

Definition 1.4.6. *A finite, normal and separable extension is called a* Galois extension.

1.4.2 Galois group

Given a field L we will denote by $Aut(L)$ the set of all automorphisms of L. This is a group with the law of composition. If $L : K$ is a field extension, we denote by $\Gamma(L : K)$ the set of automorphisms of L that fix K, i.e. $\Gamma(L : K) := \{\sigma \in Aut(L) \mid \sigma(t) = t, \ \forall t \in K\}$. Clearly $\Gamma(L : K)$ is a subgroup of $Aut(L)$.

Conversely, if H is a subset of $Aut(L)$, we can define the fixed field of H as the subfield of L, $L^H := \{t \in L \mid \sigma(t) = t \ \forall \sigma \in H\}$. Hence we find a field extension $L : L^H$.

Definition 1.4.7. *We call $\Gamma(L : K)$ the Galois group of the extension of fields $L : K$.*

If $L : K$ is the splitting field extension for a polynomial $f \in K[x]$, we call $G(f) := \Gamma(L : K)$ the Galois group of f.

Theorem 1.4.8. *Let R be the set of roots of f on L. Each $\sigma \in G(f)$ defines a permutation of R, so that it is well defined a group homomorphism from $G(f)$ to the symmetric group S_d of permutations of R.*

Proof. An element $\sigma \in G(f)$ is an automorphism of L that fixes K, hence $\sigma(f) = f$ since its coefficients are in K. Let $\alpha \in R$ be a root, then $f(\sigma(\alpha)) = \sigma(f(\alpha)) = \sigma(0) = 0$, hence σ maps R into R. Thus σ restricted to R is a permutation. Moreover, the map restricting σ on R is a group homomorphism. If there exists $\tau \in G(f)$ such that $\sigma(\alpha) = \tau(\alpha)$ for every $\alpha \in R$, then $\tau^{-1}\sigma$ fixes L and so $\sigma = \tau$. □

Lemma 1.4.9. *Suppose that $f \in K[x]$ is irreducible and that $L : K$ is a splitting field extension for f. If α and β are roots for f in L, there is an automorphism $\sigma : L \to L$ such that σ fixes K and $\sigma(\alpha) = \beta$.*

Thus, if we have that f is a irreducible polynomial, then the Galois group $G(f)$ acts transitively on the set of roots R. Conversely, let f be a monic polynomial of degree d with d distinct roots in L and let $G(f)$ act transitively on R. Let $\alpha \in R$ and g be the minimal polynomial of α. Then, if β is any other root in R, $g(\beta) = 0$ giving that g has at least d distinct roots. The minimal polynomial g divides f then $g = f$ providing that f is irreducible.

We recall that L^H is the subfield of L fixed by H. The degree $[L : K]$ of the finite extension $L : K$ over K is its dimension as a K-vector space. If L is generated over K by a single element with minimal polynomial f, then $[L : K]$ is equal to the degree of f.

If $L : K$ is a Galois extension we have the following characterization.

Lemma 1.4.10. *A finite extension $L : K$ is Galois with group $G = Aut(L : K)$ if and only if G has order $[L : K]$.*

Proof. If $L : K$ is Galois, it is the splitting field of a polynomial. Thus G has order $[L : K]$ by construction. Conversely, the extension $L : L^G$ is Galois by definition, so G has order $[L : K]$. Thus $L^G = K$. □

1.4.3 Galois covers

Let X be a topological space.

Definition 1.4.11. *A (topological) cover of X is a couple (Y, f) where f is a continuous map $f : Y \to X$ between topological spaces such that f is surjective and, for each point $x \in X$ there is a neighborhood W of x such that $f^{-1}(W)$ consists of a disjoint union of open sets Y_j, each mapping via f homeomorphically onto W.*

We recall the definition of a Galois cover.

Definition 1.4.12. *We say that a topological cover $f : Y \to X$ is a* Galois cover *if Y is connected and $G := Aut(f)$ acts transitively on $f^{-1}(x)$ for any $x \in X$.*

1.4.4 Galois groups of finite maps

This section is devoted to the description of the relation between Galois groups and finite maps.

Let X and Y be two irreducible complex algebraic varieties of the same dimension n and let $f : X \to Y$ be a map of degree $d > 0$. Let $f^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ be the inclusion of function fields induced by f; by the primitive element theorem (Proposition 1.4.5), $\mathbb{C}(X)$ is generated over $\mathbb{C}(Y)$ by a single rational function $g \in \mathbb{C}(X)$ satisfying a minimal polynomial of degree d

$$P(g) = g^d + h_1 g^{d-1} + \ldots + h_d = 0$$

where $h_1, \ldots, h_d \in \mathbb{C}(Y)$.

Let $y \in Y$ be a point and let Δ be the field of germs of meromorphic functions around y. Let $\phi : \mathbb{C}(Y) \to \Delta$ be the inclusion obtained by restriction. Let Δ_i be the field of germs of meromorphic functions around x_i, with $x_1, \ldots, x_d$ points in the fibre $f^{-1}(y)$. Then, the map f induces isomorphisms $\pi_i : \Delta_i \to \Delta$. Let $\phi_i : \mathbb{C}(X) \to \Delta_i$ be the inclusion obtained by restriction to Δ_i composed with π_i. Let $K = \phi(\mathbb{C}(Y))$ and let L be the subfield of Δ generated by the subfields $K_i = \phi_i(\mathbb{C}(X))$. Set $\phi(h_j) = \tilde{h_j}$ for $j = 1, \ldots, d$ and $\phi_i(g) = \tilde{g}_i$ for $i = 1, \ldots, d$. Each element $\tilde{g}_i$ satisfies the polynomial

$$\tilde{P}(\tilde{g}_i) = \tilde{g_i}^d + \tilde{h}_1 \tilde{g_i}^{d-1} + \ldots + \tilde{h}_d = 0$$

To see that L is indeed the splitting field of P, it suffices to show that all the $\tilde{g}_i$ are distinct since P has degree d and L is by definition the smallest field containing all the $\tilde{g}_i$. But since g generates $\mathbb{C}(X)$ over $\mathbb{C}(Y)$, it must have all distinct values at the points x_i. Then $\tilde{g}_i$ are all the roots of $\tilde{P}$ since they are all distinct. The field $L \subset \Delta$ is then the normalization of the extension $\mathbb{C}(X)/\mathbb{C}(Y)$ and the Galois group $G_f = Gal(L/K)$ acts on the roots $\tilde{g}_i$ of $\tilde{P}$. Therefore we have an inclusion

$$G_f \hookrightarrow S_d$$

Remark 1.4.13. If $\mathbb{C}(X) : \mathbb{C}(Y)$ is a Galois extension, the degree of G_f is d (see Lemma 1.4.10), hence it is a proper subgroup of S_d when $d \geq 3$.

The Galois group G_f associated to f is a transitive subgroup of the Symmetric group S_d. We remark here that it is imprimitive if and only if there is an intermediate field in the extension $\mathbb{C}(X) : \mathbb{C}(Y)$.

Indeed, if the action of G_f on the set $\Omega = \{x_1, \dots, x_d\}$ of points in a general fibre of f is imprimitive, then there are $\Delta_1, \dots, \Delta_r$ non trivial blocks inside Ω. We want to show that therefore there is an intermediate field in the extension. Let x_1 be an element in Δ_1. Look at the two subgroups $G_1 = Stab_G(x_1) < G_2 = Stab_G(\Delta_1)$, the stabilizers of x_1 and of the block Δ_1 respectively. Note that G_1 is not the all G_2: an element $\sigma \in G_2$ sends $\sigma(x) \in \Delta_1$ for any $x \in \Delta_1$, but it non necessarely fixes x_1; on the other hand, an element $\sigma \in G_1$ must have $\sigma(\Delta_1) = \Delta_1$ since Δ_1 is a block of the action by assumption.

Moreover, G_2 is not the whole G_f because the action is imprimitive. These strict inclusions $G_1 < G_2 < G_f$ correspond to strict inclusions of subfields of automorphism

$$L^{G_f} < L^{G_2} < L^{G_1}.$$

If there is an an intermediate field in the extension $\mathbb{C}(X) : \mathbb{C}(Y)$, analogously it gives rise to a non trivial block, hence the action is imprimitive.

1.5 Monodromy group

In this section we will give the definition of the mondoromy group. It will turn out to be isomorphic to the Galois group.

We begin by recalling a few topological notions we will use in the following.

1.5.1 Fundamental group and coverings

Let U be a connected manifold and let $y \in U$ be a point. A path on U is a continuous map $\gamma : [0,1] \to U$; a loop at y is a path on U such that $\gamma(0) = \gamma(1) = y$. Two loops γ and δ are said to be homotopic if there is a continuous map $G : [0,1] \times [0,1] \to U$ such that $G(0,t) = \gamma(t)$ and $G(1,t) = \delta(t)$ for all $t \in [0,1]$, and $G(s,0) = G(s,1) = y$. The fundamental group of U is the set of homotopy classes of loops at y and is denoted by $\pi_1(U,y)$.

A continuous map $f : V \to U$ between manifolds is a covering space of U if f is surjective and, for each point $y \in U$, there is a neighborhood W such that $f^{-1}(W)$ consists of a disjoint union of open sets V_j, each mapping via f homeomorphically onto U. A covering space $f : V \to U$ enjoys the path-lifting property, i.e. for any path $\gamma : [0,1] \to U$ and any preimage v of $\gamma(0) = y$ there is a path $\tilde{\gamma}$ on V such that $\tilde{\gamma}(0) = v$ and $f \circ \tilde{\gamma} = \gamma$.

There exists a covering space $f : V \to U$ such that V is simply connected; it is called universal covering space. Moreover, it is unique up to isomorphism.

1.5.2 Branched coverings

We recall here some basics on branched coverings between smooth complex varieties (see [Na]). Let Y be a connected complex manifold of dimension n and let $f : X \to Y$ be a branched covering of Y (see Definition 1.4.11). We define

Definition 1.5.1. *A morphism $f : X \to Y$ between smooth complex varieties is called unramified if the natural homomorphism $f^*\Omega^1_Y \to \Omega^1_X$ is surjective. This is equivalent to ask that f is topologically an immersion, i.e. $df_x : T_xX \to T_{f(x)}Y$ is injective for every $x \in X$.*

The ramification divisor R_f of a finite morphism $f : X \to Y$ is the scheme of zeros on X of the determinant of the Jacobian of f. The branch divisor B_f on Y is the pushforward of R_f via f. They are divisors in X and Y respectively.

Recall that the degree of a finite covering $f : X \to Y$ is the degree of the unramified map $X \setminus f^{-1}(B_f) \to Y \setminus B_f$. Equivalently, it is the degree of the separable extension $\mathbb{C}(X) : \mathbb{C}(Y)$ of function fields. Let $f : X \to Y$ be a finite morphism of degree d between smooth complex varieties. Let $f^{-1}(y) = \{x_1, \dots, x_k\}$ with $k \leq d$ be the fibre of f over a point $y \in Y$. If we assume X to be smooth, we can define a branch point as a point in Y with less than d points in its fibre.

One can give an analytical description of the branched covering $f : X \to Y$ locally around a branch point p (see [Na, Theorem 1.1.8, Theorem 1.1.14]): let W be a sufficiently small open neighbourhood of $p \in Y$ with coordinates $(w_1, \dots, w_n)$ such that $p = (0, \dots, 0)$, let $q \in f^{-1}(p)$ be a point and $B_f \cap W = \{(w_1, \dots, w_n) \in W \mid w_n = 0\}$. Then there are coordinates $(z_1, \dots, z_n)$ in the connected component U of $f^{-1}(W)$ such that $q = (0, \dots, 0) \in U$ and there exist a positive integer m such that

$$f_U : (z_1, \dots, z_n) \mapsto (w_1, \dots, w_n) = (z_1, \dots, z_{n-1}, z_n^m).$$

The coeffient m is called ramification index of f at q. The point q is in R_f if and only if $m \geq 2$. More generally, if B_f is normal crossing at p and $f^{-1}(B_f)$ at q, and $B_f \cap W = \{(w_1, \dots, w_n) \in W \mid w_k w_{k+1} \dots w_n = 0\}$ for some integer $1 \leq k \leq n$, with the same notation as above we get

$$f_U : (z_1, \dots, z_n) \mapsto (w_1, \dots, w_n) = (z_1, \dots, z_{k-1}, z_k^{m_k}, \dots, z_n^{m_n})$$

where m_j, $j = k, \dots, n$, is the ramification index of the irreducible component C_j of $f^{-1}(B_f)$ such that $C_j \cap U = \{(z_1, \dots, z_n) \in U \mid z_j = 0\}$, where $f^{-1}(B_f) \cap U = \{(z_1, \dots, z_n) \in U \mid z_k \dots z_n = 0\}$.

Let $\operatorname{ord}(x)$ of f at x be defined as the degree of the map $U_x \to U$, i.e. as the number of preimages in U_x of a generic point of U, where U is an open neighbourhood of y in Y and U_x is the open in $f^{-1}(U)$ containing the preimage $x \in f^{-1}(y)$. When X is an irreducible complex projective variety of dimension n and $f : X \to \mathbb{P}^n$ a finite morphism of degree d, the following holds.

Theorem 1.5.2. *[FL, Theorem 6.1] Let X be an irreducible projective variety of dimension n and let $f : X \to \mathbb{P}^n$ be a branched covering of degree d. Then there exists at least one point $x \in X$ at which $\operatorname{ord}(x) \geq \min\{d, n+1\}$. More generally, if X is normal, every irreducible component of R_f has codimension at most one in X.*

1.5.3 Monodromy of finite maps

Let $f : X \to Y$ be a finite morphism of degree $d \geq 1$ between irreducible algebraic complex varieties of the same dimension n.

Let U be a Zariski open subset of Y such that the restriction of f to $V := f^{-1}(U)$ is an étale covering $f : V \to U$. Let $q \in U$ be a point, then its fibre $\Gamma := f^{-1}(q) = \{x_1, \dots, x_d\}$ has d distinct points. For any loop $\gamma : [0,1] \to U$ centred in q and for any $x_i \in \Gamma$ there exist a unique lifting $\tilde{\gamma}_i$ such that $\tilde{\gamma}_i(0) = x_i$. We may define a permutation σ_i of the elements of Γ by sending $x_i \mapsto \tilde{\gamma}_i(1)$. The permutation σ_i depends only on the homotopy class of γ_i, then it is well defined a group homomorphism

$$\mu : \pi_1(U, q) \to Aut(f^{-1}(q)) \simeq S_d$$

called monodromy map.

Definition 1.5.3. *The image*

$$M(f) := \mu\left(\pi_1(U, q)\right) \leq S_d$$

is the monodromy group of the map f.

In the following Lemma we prove that $M(f)$ is a transitive subgroup of the symmetric group.

Lemma 1.5.4. *Let $\mu : \pi_1(U, q) \to S_d$ be the monodromy map of a finite étale covering $f : V \to U$ where V connected. Then the image $M(f)$ is a transitive subgroup of S_d.*

Proof. Label the points in the fibre $\Gamma = f^{-1}(q)$ as $x_1, \dots, x_d$. Consider two points x_i and x_j in the fibre $\Gamma = f^{-1}(q)$. Since V is connected, we may find a path $\tilde{\gamma}$ starting at x_i and ending at x_j. Let $\gamma = f \circ \tilde{\gamma}$ be the image of $\tilde{\gamma}$ in U; note that γ is a loop based at q. Then by construction we have that $\mu([\gamma])$ is a permutation which sends x_i to x_j. □

We have the following correspondence

$$\left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of connected covers} \\ f : V \to U \text{ of degree d} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{group homomorphism } \mu : \pi_1(U, y) \to S_d \\ \text{with transitive image} \\ \text{up to conjugacy in } S_d \end{array} \right\}$$

The group S_d acts by conjugation on the set on the right; geometrically, this correspond to a relabeling of the points in the fibre of the covering over the base point.

We show that there is an isomorphism between the monodromy group and the Galois group defined before.

Proposition 1.5.5. *[Ha, Sec. I] Let $f : X \to Y$ as before. Then the monodromy group $M(f)$ is isomorphic to the Galois group G_f.*

Proof. In the same notation as above, let γ be any loop in U centred in q; let $\tilde{\gamma}_i$ be the lifting of γ to V with $\tilde{\gamma}_i(0) = x_i$ and let σ be the permutation induced on $f^{-1}(y)$, i.e. $\tilde{\gamma}_i(1) = x_{\sigma(i)}$. For any germ $h \in L \subset \Delta$ of a meromorphic function at y in the field L, the analytic continuation of h along the path γ is well defined. Such analytic continuation along γ allow us to define an automorphism of the field L fixing K and sending the function element g_i to $g_{\sigma(i)}$, so that σ is in G_f.

Conversely, we claim that any automorphism of the field L over K is obtained by analytic continuation along some arc γ in U. For this purpose, we define a meromorphic function $\tilde{h}$ on U by choosing for every $z \in U$ an arc η from y to z and letting the germ of $\tilde{h}$ at z be the analytic continuation of h along η. Note that choosing a different arc η' yield the same germ, since the analytic continuation of h along $\eta^{-1}\eta'$ is again h. We will write $H = q(h_1, \ldots, h_d)$, where h_i is the germ in Δ_i of a meromorphic function $\tilde{h}_i$ on X. We have that $\tilde{h}$ cannot have essential singularities. Therefore $\tilde{h}$ extends to a meromorphic function on Y with germ h at y.

□

From this isomoprhism follows that the definition of the monodromy group does not depend on the base point q and on the choice of the open U as long as $f_{|V} : V \to U$ is unbranched.

1.5.4 Finite maps of curves

In this section we will consider the case of finite morphisms from projective curves to $\mathbb{P}^1$. We will follow mainly the book of Miranda [Mi, Ch. 3 Sec. 4].

Let $X \subset \mathbb{P}^n$ be a smooth projective curve and let $f : X \to \mathbb{P}^1$ be a finite map of degree d. Let $\mu : \pi_1(U, q) \to S_d$ the monodromy map described above. This map associates to a loop in U around a branch point y a permutation of the d points in the fibre over a general point q. Let B be the branch locus of the map f.

The following Lemma gives a description of the element in the symmetric group associated to a certain branch point([Mi, Lemma 4.6]).

Lemma 1.5.6. *Suppose that $y \in B$ is a branch point whose preimages $f^{-1}(y) = \{x_1, \ldots, x_k\}$ are contained in X and have multiplicities $m_1, \ldots, m_k$, $k < d$. Then, the cycle structure of the permutation σ representing a loop around y (up to the identification via a certain path α) is $(m_1, \ldots, m_k)$.*

Proof. For a branch point $y \in Y$, choose a small open neighborhood in the analytic topology W of y. It is isomorphic to a small punctured disc. Let $x_1, \ldots, x_k$, $k < \deg(f)$ be the preimages of y in X; by assumptions, at least one of the x_i's is a ramification point. Choose W small enough so that $f^{-1}(W)$ decomposes as a disjoint union of open neighborhoods $U_1, \ldots, U_k$ of the points $x_1, \ldots, x_k$ respectively. Set $m_j := mult_{x_j} f$ to be the multiplicity of f at these points; by the Local Normal Form ([Mi, Proposition 4.1]), there are coordinates z_j on the U_j and z on W so that the map f has the form

$$z \mapsto z^{m_j}$$

on U_j. Now $U_j \setminus \{x_j\}$ is isomorphic to a punctured disc and the map f sends $U_j \setminus \{x_j\}$ to $W \setminus \{y\}$ via a power map of power m_j. A loop centred at $q \in U$ around the branch point y is given by a path α from the point q to a point $q_0 \in W \setminus \{y\}$ and a loop γ centred at q_0 in $W \setminus \{y\}$. The permutation σ of the fiber of f over q is actually determined (up to this identification) by the loop γ in $W \setminus \{y\}$. Moreover, the monodromy for each cover $U_j \setminus \{x_j\}$ to $W \setminus \{y\}$ with $j = 1, \dots, k$ induces a cyclic permutation of those m_j preimages of q_0 which lie in U_j. $\square$

In general, let $f : X \to \mathbb{P}^1$ be a holomorphic map. The monodromy of f is generated by local monodromies: said $b_1, \dots, b_k \in \mathbb{P}^1$ the branch points of the map f, $U = \mathbb{P}^1 \setminus \{b_1, \dots, b_k\}$ and $q \in U$, the fundamental group $\pi_1(U, q)$ is generated by loops $[\gamma_1], \dots, [\gamma_k]$ satisfying

$$[\gamma_1] \cdots [\gamma_k] = 1$$

where γ_i is a small loop around b_i for $i = 1, \dots, k$. Thus the monodromy group $M(f)$ is generated by permutations $\sigma_i = \mu([\gamma_i])$ satisfying the relation

$$\sigma_1 \cdots \sigma_k = 1$$

Hence we have the following correspondence ([Mi, Cor 4.10]).

Proposition 1.5.7. *There is a one-to-one correspondence between*

$$\left\{ \begin{array}{l} \textit{isomorphism classes} \\ \textit{of holomorphic maps} \\ f : X \to \mathbb{P}^1 \textit{ of degree } d \\ \textit{with branch points } b_1, \dots, b_t \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{coniugacy classes of } t\textit{-ple } (\sigma_1, \dots, \sigma_t) \\ \textit{of permutations in } S_d \\ \textit{with } \sigma_1 \cdot \ldots \cdot \sigma_k = 1 \\ \textit{and the subgroup generated by them is transitive} \end{array} \right\}$$

Moreover if σ_i has cycle structure $(m_1, \dots, m_k)$, then there are k preimages $x_1, \dots, x_k$ of b_i in the corresponding cover $f : X \to \mathbb{P}^1$, with $mult_{x_j} f = m_j$ for each $j = 1, \dots, k$.

1.6 Projections

1.6.1 Cones

Cones will paly an important role in our results. Here we recall the definition and some basic results. We use the notation in [Ha, Example 3.1].

Let $\mathbb{P}^n$ in $\mathbb{P}^{n+1}$ be an hyperplane, $P \notin \mathbb{P}^n$ a point and $Y \subset \mathbb{P}^n$ a variety. A cone X over Y with vertex P is the union of the lines joining P and points in Y, i.e.

$$X = \bigcup_{y \in Y} \langle P, y \rangle.$$

More generally,

Definition 1.6.1. *Let $V \cong \mathbb{P}^t$ and $L \cong \mathbb{P}^{n-t}$ be complementary linear subspaces of $\mathbb{P}^{n+1}$. Let $Y \subset L$ be a variety. The cone X over Y with vertex V is the union of the $(t+1)$-planes $\langle V, y\rangle$ with $y \in Y$.*

Equivalently, let X be an irreducible reduced variety in $\mathbb{P}^{n+1}$, and let $V \cong \mathbb{P}^t \subset X$. Then X is a cone with vertex V if and only if for all $x \in X \setminus V$, the linear subspace $\langle x, V\rangle$ is contained in X.

We will write $Vert(X)$ for the vertex V.

Proposition 1.6.2 ([Ru], Prop. 1.3.3)**.** *Let $X \subset \mathbb{P}^N$ be a irreducible variety of dimension n. Then*

$$V := Vert(X) = \bigcap_{x \in X} T_x X$$

Proof. Let $T = \bigcap_{x\in X} T_x X$. Set S to be the join of TL and X, i.e. S is the Zariski closure of $\bigcup_{t\in T, x\in X} \langle t, x\rangle$ with $t \neq x$. Therefore, $\dim S = \dim T + \dim X - \dim(T \cap T_x X) = \dim X$ for $x \in X$ general (see [Ru, Section 1.3]). By definition, $T \subseteq V$.

Conversely, it is enough to prove that, given $v, w \in V$, the line $\langle v, w\rangle$ is contained in X (see [Ru, Prop 1.2.2]). Let $x \in X \setminus V$ be a point and consider $z \in \langle v, w\rangle$ with $z \neq v, w$. The lines $\langle v, x\rangle$ and $\langle w, x\rangle$ are contained in X. By the definition of a cone, also the plane spanned by x, v, w is contained in X and so is the line $\langle z, x\rangle$ for every $z \in \langle v, w\rangle$. This concludes the proof. □

An irreducible projective variety X is a cone if and only if $Vert(X) \neq \emptyset$. We end this chapter defining the maps we are going to study.

1.6.2 Linear projections

Let $X \subset \mathbb{P}^{n+c}$, $c \geq 1$ be an irreducible projective variety of dimension n. Let L be a linear subspace of dimension $k \leq c-1$ and let $\mathbb{P}_L^{n+c-k-1} \simeq H$ be a linear space disjoint from L. Let

$$\pi_L : X \subset \mathbb{P}^{n+c} \dashrightarrow \mathbb{P}_L^{n+c-k-1}$$

be the rational map defined on $X \setminus (X \cap L)$ sending $x \in X \mapsto \langle x, L\rangle \cap H$. The subspace $\mathbb{P}_L^{n+c-k-1}$ parametrizes all the $(k+1)$ dimensional subspaces in $\mathbb{P}^{n+c}$ containing L.

This map may also be realized as the composition of a sequence of projections from points $p_0, \dots, p_k$ spanning L, which is also called the centre of the projection.

Remark 1.6.3. If $L \cap X = \emptyset$ the map π_L is a finite morphism and the image $\tilde{X} = \pi_L(X)$ is a projective variety of $\dim(X) = \dim(\tilde{X})$. Indeed, it is sufficient to show that π_L has finite fibres by Stein factorization. Given a point $y \in \mathbb{P}^n$, we have that $\pi_L^{-1}(y) = \langle L, y\rangle \cap X$. If there exist a curve C inside $\pi_L^{-1}(y)$, then $\emptyset \neq L \cap C \subseteq L \cap X = \emptyset$, that is a contradiction.

In particular, the degree of the map π_L is equal to the degree of X.

We can ask when a projection is birational or a closed embedding. We denote by $\mathcal{T}(X)$ the union of all tangent hyperplanes of points of X and $Sec(X)$ the secant variety to X, i.e. $\overline{\bigcup_{x\neq y} \langle x, y\rangle}$, $x, y \in X$.

Proposition 1.6.4. *[Ru, Prop. 1.3.5] Let L be a linear space of dimenison $k \leq c-1$ such that $L \cap X = \emptyset$. Let $\pi_L : X \to \mathbb{P}^n$ be the projection. Then*

- *π_L is one to one if and only if $L \cap Sec(X) = \emptyset$;*
- *π_L is unramified if and only if $L \cap \mathcal{T}(X) = \emptyset$;*
- *π_L is a closed embedding if and only if $L \cap \mathcal{T}(X) = L \cap Sec(X) = \emptyset$.*

Proof. The morphism is one to one if and only if there exist no couple of point $x \neq y$ such that $\pi_L(x) = \pi_L(y)$, i.e. $\langle x, y\rangle \cap L \neq \emptyset$. Therefore, π_L is one to one if and only if there is no secant line to X through the centre of projection L. The map π_L is ramified at a point $x \in X$ if $T_xX \cap L \neq \emptyset$ by looking at the projective differential of π_L. A finite morphism is a closed embedding if and only if it is one to one and unramified. $\square$

We now want to recall the relation between branch points and permutations in the monodromy group. Let X be a irreducible and reduced projective hypersurface and let $\pi_P : X \to \mathbb{P}^n$ be a projection map of degree d with $P \notin X$. The fibre over a point $y \in \mathbb{P}^n$ is $\pi_P^{-1}(y) = \{x_1, \ldots, x_k\}$ with $k \leq d$ distinct points.

Recall that, if X is smooth, the branch locus $\mathcal{B}$ of π_P corresponds to the locus of points $y \in \mathbb{P}^n$ such that the cardinality of the fibre $\pi_P^{-1}(y)$ is strictly lower than d. If X is singular, the image via π_P of the singular locus of X is contained in the branch locus. This follows from the fact that any line passing through $Q \in Sing(X)$ is tangent to X. However, we want to distinguish between the order of the general line passing through Q and the lines for which such order increase. At this purpose, consider the diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\nu"] \arrow[dr, "\varphi"'] & X \arrow[d, "\pi_P"] \\ & \mathbb{P}^n \end{tikzcd}$$

where $\nu : \tilde{X} \to X$ is the normalization map. Let y be a general point of an irreducible component of $\mathcal{B}$ and let l be the 0 dimensional scheme cut by the line $\langle P, y\rangle$ on X. Pulling it back to $\tilde{X}$, we write $\tilde{l} = m_1x_1 + \ldots + m_tx_t$ where $d \geq t \geq k$ and $\sum_{i=1}^t m_i = d$. By the genericity of y, the points $x_1, \ldots, x_t$ are smooth in $\tilde{X}$ since the normalization has codim $\left(sing(\tilde{X})\right) \geq 2$.

Consider an analytic neighbourhood $\mathcal{U}(y)$ and consider all the disks $U_1^i, \ldots, U_k^i$ dominating analytic neighbourhoods $\mathcal{U}(x_i)$ of the points $x_1, \ldots, x_k$ in the fibre of y via φ. We will say that an irreducible component of $\mathcal{B}$ is a branch component if the fibre of φ of a general point y has at least a smooth point x_i with $m_i \geq 2$.

We denote the branching weight of a branch point y by

$$b(y) := \sum_{i=1}^{t} (m_i - 1) \geq 1,$$

where $m_1x_1 + \ldots + m_tx_t$ is the fibre $\varphi^{-1}(y)$ on $\tilde{X}$.

We say that $y \in B$ is a simple branch point if $b(y) = 1$. Simple branch points correspond to transpositions in the monodromy group, see [Ha3, pag. 698]. More generally, there is a relation between the branching weight of a point and the type of the corresponding permutation. Up to taking general plane sections of our projection, we can restrict to the case of plane curves (recall that $\pi_1\left(\mathbb{P}^1 \setminus (\mathbb{P}^1 \cap B)\right) \twoheadrightarrow \pi_1(\mathbb{P}^n \setminus B)$ for a general $\mathbb{P}^1 \subset \mathbb{P}^n$, see [FL, Theorem 3.1]. In this case Lemma 1.5.6 gives the explicit relation between branch points and permutations in the monodromy group.

Moreover, in the case of curves we have the following characterization from group theory.

Lemma 1.6.5. *[PS, Proposition 2.5] Let C be a smooth curve and let $\pi : C \to \mathbb{P}^1$ be a finite morphism of degree d with branch points $b_1, \ldots, b_k \in \mathbb{P}^1$. To each b_i we associate a permutation σ_i in the Galois group such that*

- *the Galois group G_π is generated by any set of $n-1$ permutations among the σ_i;*
- *if the scheme theoretic fibre over b_i is $\sum_{j=1}^{t_i} m_j p_j$, then σ_i is a permutation with cycle structure $(m_1, \ldots, m_{t_i})$.*

In particular, if all but one of the branch points are simple, then G_π is generated by transpositions, hence $G_\pi = S_d$.

1.6.3 Hypersurfaces from general projections

Let $\tilde{X}$ be a smooth variety of dimension n in $\mathbb{P}^{n+c}$ and let $L \cap \tilde{X} = \emptyset$ be a linear subspace of dimension $c-2$. We say that a projection $\pi_L : \tilde{X} \to\subset X \subset \mathbb{P}^{n+1}$ is a general projection if L is general in the Grassmannian $\mathbb{G}(c-2, n+c)$. The image of $\tilde{X}$ via π_L is an hypersurface X in $\mathbb{P}^{n+1}$.

Remark 1.6.6. A projection $\pi_M : \tilde{X} \to \mathbb{P}^n$, with $\dim M = c-1$, can also be seen as the composition of a general projection $\pi_L : \tilde{X} \to X \subset \mathbb{P}^{n+1}$ and $\pi_p : X \to \mathbb{P}^n$, where $L \subset M$ is a general subspace of dimension $c-2$ and $p \in M \setminus L$ is a point.

We first remark that, even if $\tilde{X}$ is a smooth variety, the image of a general projection $X \subset \mathbb{P}^{n+1}$ is a singular hypersurface (see Proposition 1.6.4). The singularities arising from a general projection of a smooth variety are well studied only in low dimension.

In the case of curves, the image $X \subset \mathbb{P}^2$ of a smooth curve $\tilde{X}$ is a plane curve with at most nodes as singularieties. General projections of a smooth surface $\tilde{X} \subset \mathbb{P}^N$, $N \geq 6$ up to $\mathbb{P}^5$ are birational onto the image. A general projection of a smooth, irreducible, non degenerate surface to $\mathbb{P}^4$ is a general surface.

Definition 1.6.7. *A non-degenerate, irreducible, projective surface $X \subset \mathbb{P}^4$ is called general surface if either X is smooth or the singularities of X are at most a finite number of improper double points, i.e. the origin of two smooth branches with independent tangent planes.*

If we project to $\mathbb{P}^3$, we have the following results, known in litterature as General Projection Theorem (see for instance [Fr], [En] for classical results and [MP], [CF] for modern references).

Theorem 1.6.8. *Let $\tilde{X}$ be a smooth surface of $\mathbb{P}^n$, $n \geq 4$. Then a projection X of $\tilde{X}$ from a suitable general linear subspace of $\mathbb{P}^n$ to $\mathbb{P}^3$ is a surface with ordinary singularities.*

We recall the definition of ordinary singularities on a surface.

Definition 1.6.9. *A surface $X \subset \mathbb{P}^3$ is said to have* ***ordinary singularities*** *if the one-dimensional component of the singular locus is a curve γ satisfying the following:*

1. *The curve γ has at most a finite number of ordinary triple points as singularities. All these points are also triple points of X. In local coordinates such a point is described by $f := x_1x_2x_3+$ higher degree terms.*

2. *The general point of γ corresponds to a nodal point of X. In local coordinates such a point is described by $f := x_1x_2+$ higher degree terms.*

3. *A finite number of smooth points of γ are pinch points. In local coordinates such a point is described by $f := x_2^2 + x_1^2x_3+$ higher degree terms.*

Remark 1.6.10. The normalization of a surface with ordinary singularities is smooth.

Generalising such a classifications to higher dimension is more complicated. Doherty proved that the projected hypersurface X of dimension n has semi log canonical singularities if $n \leq 5$ ([Doh, Main Theorem]). There are many deep results on the properties of general projections which are very useful at our purposes. In the following, we will use a theorem of Mather, [AO]. We state it here in the general setting: let $X \subset \mathbb{P}^{n+c}$ be a smooth variety of dimension n and codimension c. Let T be any linear subspace of dimension $t \leq c-1$ such that $T \cap X = \emptyset$ and consider the linear projection $\pi_T : X \to \mathbb{P}^{n+c-t-1}$.

Theorem 1.6.11. *[AO, Theorem 2] Let X and T as above. For any $i_1 \leq t+1$, define $X_{i_1} := \{x \in X \mid \dim(T_xX \cap T) = i_1 - 1\}$. When X_{i_1} is smooth, define $X_{i_1,i_2} := \{x \in X_{i_1} \mid \dim(T_xX_{i_1} \cap T) = i_2 - 1\}$ and so on. For $i_k \leq \ldots \leq i_2 \leq i_1$, when possible, define $X_{i_1,\ldots,i_k}$. For T general, every $X_{i_1,\ldots,i_k}$ is smooth and, when non empty, its codimension in X is equal to $\nu_{i_1,\ldots,i_k}$ defined below.*

Set $\mu_{i_1,\ldots,i_k}$ the number of sequences $j_1 \geq j_2 \geq \ldots \geq j_k$ such that $j_1 > 0$ and $i_r \geq j_r$ for $k \geq r \geq 1$. Then

$$\nu_{i_1,\ldots,i_k} = (c - t - 1 + i_1)\mu_{i_1,\ldots,i_k} - (i_1 - i_2)\mu_{i_2,\ldots,i_k} - \ldots - (i_{k-1} - i_k)\mu_{i_k}.$$

For instance, if $i_1 = q$, $i_2 = \ldots = i_k = 0$ the codimension of X_{i_1} in X is $q(c - t - 1 + q)$.

1.6.4 Galois points

We introduce the notion of Galois points, which is very related to projections and Galois theory. They have been extensively studied in various papers, for instance [Yo2, FT, Fu, MY], with a particular focus on computing the number of Galois points. Let us recall the definition.

Definition 1.6.12. *A point $P \in \mathbb{P}^{n+1} \setminus X$ is called a* Galois point *if the field extension $k(X) : k(\mathbb{P}^n)$ associated with the projection of X from P is Galois (see Definition 1.4.6).*

We will just write Galois points for what the previous authors refer to as outer Galois points.

Miura and Yoshihara [MY] determined all the possible monodromy groups for plane quartic curves. Fukasawa [Fu] classified the number of Galois points for smooth plane curves of degree $d \geq 4$ and the characteristic of the ground field $p \geq 0$.

More in general, Yoshihara ([Yo2, Proposition 11]) proved that there is at most a finite number of Galois points $P \notin X$ for a smooth hypersurface $X \subset \mathbb{P}^n$; moreover, he gave a bound on the number of Galois points and equations for an hypersurface X admitting such number of Galois points. In particular, there are at most $n+2$ Galois points in $\mathbb{P}^{n+1} \setminus X$ and they are exactly $n+2$ if X is a Fermat hypersurface.

In the case of normal hypersurfaces X, Fukasawa and Takahashi [FT, Theorem 2, Proposition 6] proved that the number of Galois points is finite unless X is a cone.

Chapter 2

Tools

In this chapter we introduce the main tools we use to prove our main results on monodromy of projections.

2.1 Focal loci

The theory of focal loci is a classical differential geometry technique applied in projective geometry. It was introduced by C. Segre ([Se1], [Se2]) and rewritten in more modern terms for instance by Sernesi and Ciliberto [Ser, Sec 4.6.7], [CS] and [CiC].

Definition 2.1.1. *[Hart, Ch.III, Sec.9]Let $f : X \to Y$ be a morphism of schemes and let $\mathcal{F}$ be an $\mathcal{O}_X$ module. We say that $\mathcal{F}$ is flat over Y if, for every point $x \in X$, the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,y}$ module, where $y = f(x)$. We say that X is flat over Y if $\mathcal{O}_X$ is.*

A flat family of schemes is a family whose elements are fibres of a flat morphism. We recall that, for a flat family of closed subschemes of projective space (over an integral scheme), the Hilbert polynomial of all the fibres is the same.

Let $\mathcal{X}$ be a flat family of closed subschemes of a projective scheme Y parametrized by an integral base scheme D. It can be described by the following diagram, where the map i is the inclusion, p, q are the projections on the first and second factor respectively and f is the restriction of q to $\mathcal{X}$

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, "i"] \arrow[d, "p|_{\mathcal{X}}"'] \arrow[dr, "f"'] & D \times Y \arrow[d, "q"] \\ D & Y \end{tikzcd}$$

Consider the following diagram arising from the short exact sequence of sheaves on $\mathcal{X}$:

$$\begin{tikzcd} & & (p^*T_D)_{|\mathcal{X}} \arrow[r, "g"] \arrow[d] & N_{\mathcal{X}|D\times Y} \arrow[d, equal] & \\ 0 \arrow[r] & T_{\mathcal{X}} \arrow[r] \arrow[d, "df"] & (T_{D\times Y})_{|\mathcal{X}} \arrow[r] \arrow[d] & N_{\mathcal{X}|D\times Y} \arrow[r] & 0 \\ & f^*T_Y \arrow[r, equal] & (q^*T_Y)_{|\mathcal{X}} & & \end{tikzcd}$$

The map $g : (p^*T_D)_{|\mathcal{X}} \to N_{\mathcal{X}|D\times Y}$ is called the *global characteristic map* of the family $\mathcal{X}$ and is defined by the composition of the arrows in its corresponding square.

Moreover, for each $d \in D$, g induces a homomorphism

$$g_d : T_{D,d} \otimes \mathcal{O}_{X_d} \to N_{X_d|Y}$$

where X_d is the fibre of $p_{|\mathcal{X}}$ at the point d.

If Y and D are smooth, all the sheaves in the diagram are locally free and so, at the level of sheaves, the kernel of g is equal to the kernel of df; it is a sheaf $\mathcal{F}$ over $\mathcal{X}$. If moreover, $f : \mathcal{X} \to Y$ is dominant and generically finite, the sheaf $\mathcal{F}$ is torsion free.

Definition 2.1.2. *The sheaf $\mathcal{F}$ is called the focal sheaf of the family $\mathcal{X}$. The locus $\mathcal{F}(\mathcal{X})$, i.e. the support of the sheaf, is called the focal scheme of $\mathcal{X}$.*

Therefore we have

Proposition 2.1.3. $\dim(f(\mathcal{X})) = \dim(\mathcal{X}) - \mathrm{rk}\,(\ker(g))$.

We can define a closed subscheme F of $\mathcal{X}$ satisfying the condition

$$\mathrm{rk}(g) < \min\left\{\, \mathrm{rk}((p^*T_D)_{|\mathcal{X}}), \mathrm{rk}(N_{\mathcal{X}|D\times Y})\right\} = \min\left\{\, \dim(D), \mathrm{codim}_{D\times\mathbb{P}^n}(\mathcal{X})\right\}$$

The points in F are called *first order foci of the family* $\mathcal{X}$. It is a proper closed subscheme of $\mathcal{X}$ if g has maximal rank. One defines higher order foci inductively: second order foci are the first order foci of the family of first order foci, and so on.

One can define the first order foci at a point d in D by restricting g to a fibre X_d; it depends only on the geometry of the family $\mathcal{X}$ in a neighborhood of the point d. A focus $y \in F_d$ can be thought a point where there is an intersection between the fiber X_d and the infinitesimally near ones.

2.1.1 Families of linear subspaces

We are interested in the case in which $\mathcal{X}$ is a family of linear subspaces of dimension k of $\mathbb{P}^n$. Here D is the desingularization of a subscheme D' of the Grassmannian $\mathbb{G}(k,n)$ parametrising $\mathcal{X}$.

Let $d \in D$ be a point and let X_d be the fibre of $p_{|\mathcal{X}}$ over the point d. In this setting

$$g_d : T_{D,d} \otimes X_d \to N_{X_d|\mathbb{P}^{n+1}},$$

is called the *local characteristic map*.

Definition 2.1.4. *A family $\mathcal{X}$ of linear subspaces in $\mathbb{P}^n$ is called filling if*

- *the dimension of D is $n-k$.*
- *the map $f = i \circ q$ is dominant.*

Remark 2.1.5. We will consider filling families $\mathcal{X}$ where f is dominant and generically finite. Hence $\dim(\mathcal{F}(\mathcal{X})) < \dim(\mathcal{X})$. Moreover, let $\mathcal{F}(X_d)$ be the focal scheme restricted to $X_d \simeq \mathbb{P}^k$. We get $\dim(\mathcal{F}(X_d)) < \dim(X_d) = k$. We will refer to this fact saying that the family has the filling property.

We are able to describe more precisely the focal locus in the case of a filling family of linear subspaces:

Lemma 2.1.6. *Let $\mathcal{X}$ be a filling family of subspaces of dimension k in $\mathbb{P}^n$ and let $d \in D$ be a general point. Then the focal locus in the fibre $X_d \simeq \mathbb{P}^k$ is a hypersurface of degree $n-k$ in X_d.*

Proof. The family $\mathcal{X}$ is a filling family of linear subspaces of $\mathbb{P}^n$, therefore the local characteristic map becomes

$$g_d : (T_{D,d} \otimes X_d) \simeq \mathcal{O}_{X_d}^{\oplus(n-k)} \longrightarrow N_{X_d|\mathbb{P}^{n+1}} \simeq \mathcal{O}_{X_d}^{\oplus(n-k)}(1)$$

In particular, the map is described by a $(n-k) \times (n-k)$ matrix M_d with linear entries. The focal locus in a general fibre is given by the equation $\det(M_d) = 0$. From the filling property, the determinant cannot be identically zero. □

In the case $k = 1$, the focal locus is given by $n-1$ points counted with multiplicity, where *multiplicity* means as root of the equation of degree n.

A first example of focal point is a fundamental point

Definition 2.1.7. *A point P is called* fundamental *if there is a subfamily of $\mathcal{X}$ of dimension s passing through it.*

Remark 2.1.8. Recall that the set of second order foci is the set of ramification points of f restricted to the first order foci F. The locus of fundamental points ϕ is contained in the locus of second order foci, since the fibre of $f_{|F}$ at points in ϕ has dimension grater than the general one.

The following lemma on fundamental points is well known, and we can trace its origins back to Segre ([Se2]).

Lemma 2.1.9. *Consider a filling family of lines in $\mathbb{P}^{n+1}$ and assume that a subfamily of dimension s pass through a point P. Then P is a focus of multiplicity s.*

Using the focal machinery we can give an alternative proof of the so called Trisecant Lemma ([Ru, Proposition 1.4.3]) for curves in $\mathbb{P}^3$.

Lemma 2.1.10. *Let $X \subset \mathbb{P}^3$ be a non degenerate, irreducible projective curve. Then the general secant line is not trisecant.*

Proof. Let $\mathcal{X}$ be the family of lines in $\mathbb{P}^3$ secant to the curve X. This family is parametrized by a base scheme of dimension 2 since we have to choose two points in the curve to have a secant line. Moreover, the map $\mathcal{X} \to \mathbb{P}^3$ is dominant since the family of secant lines to a non degenerate curve fill the whole $\mathbb{P}^3$.

Therefore, the family $\mathcal{X}$ is a filling family of lines and by Lemma 2.1.6, there are two points counted with multiplicity in the focal locus in a general line of the family. By assumption, every line in $\mathcal{X}$ intersect X in at least two distinct points, each of which is a focus of multiplicity one for the line: there is a one dimensional subfamily of lines in $\mathcal{X}$ through every point in X, so they are fundamental points (Lemma 2.1.9).

If a general line of $\mathcal{X}$ is trisecant, then there are three focal points in it. This contradicts Lemma 2.1.6. □

2.2 Topology of a complement of $\mathbb{P}^n$

The study of the fundamental group of the complement of hypersurfaces in $\mathbb{P}^n$ dates back to Enriques [En], van Kampen [VaK] and Zariski [Zar2]. Their interest was mainly on the study of complements of curves in $\mathbb{P}^2$, also considered as branch curves of a map from a surface. One of the tools Zariski introduced was his celebrated theorem on fundamental groups of hyperplane sections extending Lefschetz homological results ([Zar1]). On the fundamental groups of the complements of plane curves with only nodes worked also Fulton [Ful], Deligne [De] and Nori [No]. For a reference on the theory of complement of hypersurfaces see the book of Dimca [Di].

2.2.1 Classical results

We list in this section some classical results on hyperplane sections of projective varieties and fundamental groups. We will mainly follow [FL] and [La].

We start stating the famous Bertini's Theorem with the second part due to Deligne ([FL, Theorem 1.1]). This Theorem has been widely studied, see for instance the works of [Jo], [Kl]; in [MNP] is presented a different proof that works in any characteristic.

Theorem 2.2.1 (Bertini/Deligne)**.** *Let X be a irreducible complex variety and $f : X \to \mathbb{P}^r$ a morphism. Fix an integer $d < \dim(\overline{f(X)})$. Then there is a non empty Zariski open subset $U \subset \mathbb{G}(r-d, r)$ such that for all $L \in U$*

1. *$f^{-1}(L)$ is irreducible*

2. *if moreover X is also locally irreducible as a complex analytic space,*

$$\pi_1(f^{-1}(L)) \twoheadrightarrow \pi_1(X)$$

Proof. Assume that f is generically finite (see [Jo] for the general case). Let $\dim(X)= n$. Fix a linear subspace M in $\mathbb{P}^r$ of codimension $n+1$ disjoint from $\overline{f(X)}$ and consider the linear projection from it $\pi_M : \mathbb{P}^m \to \mathbb{P}^n$. There is a natural isomorphism between $\mathbb{G}(n-d, n)$ and $\{L \in \mathbb{G}(r-d, r) \mid M \subset L\}$, hence we can consider $\pi_M \circ f : X \to \mathbb{P}^n$ and restrict to the case $r = n$.

Assume $r = n$, so that f is dominant. We may assume that there exists a hypersurface $B \subset \mathbb{P}^n$ such that $f : X \to \mathbb{P}^n \setminus B$ is a topological covering, where we may replace X with the open $X \setminus f^{-1}(B)$. Take a point $p \in \mathbb{P}^n \setminus B$ and consider the blow up at p. The exceptional divisor parametrizes the $\mathbb{P}^{n-1}$ of lines l_t through p, so let $B_p := \{(y,t) \mid y \in l_t\} \subset (\mathbb{P}^n \setminus B) \times \mathbb{P}^{n-1}$. Analogously, consider the blow up of X at the fibre $f^{-1}(p)$, let $\tilde{X} := \{(x,t) \mid f(x) \in l_t\} \subset X \times \mathbb{P}^{n-1}$. The map $\tilde{f} : \tilde{X} \to B_p$ sending $(x,t) \mapsto (f(x), t)$ is a topological covering. The second projection $p_2 : B_p \to \mathbb{P}^{n-1}$ restricts to a topologically locally trivial fibre space over the open T of lines through p meeting B transversally. Composing with $\tilde{f}$, we get that also $h := p_2 \circ \tilde{f} : \tilde{X} \to \mathbb{P}^{n-1}$ is locally trivial over T and moreover $h^{-1}(T)$ is irreducible since $\tilde{X}$ is.

The map h is a fibration between connected spaces that admits a section, hence has connected fibres (Lemma 2.2.2). For instance consider the map $t \mapsto (x', t)$, with $x' \in f^{-1}(p)$. Therefore, for every $t \in T$, $f^{-1}(l_t \setminus B)$ is connected. It follows that for a linear subspace $L \in \mathbb{G}(r-d, r)$ that contains a line l_t with $t \in T$, $f^{-1}(L \setminus B)$ is connected and non singular, hence irreducible.

To get part 2), apply the previous argument to $g : X^* \to \mathbb{P}^n \setminus B$, where X^* is the universal cover of X. From the general case, we get that for almost every $L \in \mathbb{G}(r-d, r)$ containing the linear subspace M, $f^{-1}(L)$ is irreducible and $\pi_1(f^{-1}(L)) \twoheadrightarrow \pi_1(X)$. It follows that there is a dense open set U in $\mathbb{G}(r-d, r)$ for which the statement holds. □

We recall here an important lemma concerning conditions for the irreducibility of the fibres of a dominant morphism (see [La, Lemma 3.3.2]).

Lemma 2.2.2. *Let $f : X \to Y$ be a dominant morphism between irreducible complex varieties. Assume that f admits a section $s : Y \to X$ whose image lie in the smooth locus of X, i.e. $s(y) \in X \setminus Sing(X)$ for a general $y \in Y$.*

Then the general fibre $X_y := f^{-1}(y)$ is irreducible.

Proof. Up to shrinking Y and replacing X by a non empty Zariski open subset, we can assume that f is a smooth morphism, by the theorem on generic smoothness. It suffices now to show that the fibres are connected. Up to shrinking again Y, we can suppose that f is topologically locally trivial. A locally trivial fibration between path-connected spaces that admits a section has connected fibres. □

The following is a generalization of the Bertini's Theorem in the case of an arbitrary linear sections.

Theorem 2.2.3. *Let X be a irreducible variety, let $f : X \to \mathbb{P}^r$ be a morphism and let L be an arbitrary linear subspace in $\mathbb{P}^r$ of codimension $d < \dim(\overline{f(X)})$.*

- *If X is complete, then $f^{-1}(L)$ is connected. More in general, if f is proper over some open V in $\mathbb{P}^r$ containing L, then $f^{-1}(L)$ is connected.*
- *If X is locally irreducible then, for any U neighborhood of L,*

$$\pi_1(f^{-1}(U)) \twoheadrightarrow \pi_1(X)$$

Proof. To prove the first part, let $\mathbb{G}(r-d, \mathbb{P}^r)$ be the Grassmannian parametrizing codimension d linear spaces in $\mathbb{P}^r$ and let $W \subset \mathbb{G}(r-d, r)$ be the open set of linear spaces contained in V. Consider

$$Z := \{(x, L') \in X \times W \mid x \in f^{-1}(L')\}$$

This is a Zariski open subset of a Grassmannian bundle over X, hence it is irreducible. The morphism f is proper providing that the second projection $pr_2 : Z \to W$ is proper too. Consider its Stein factorization (Proposition 1.3.1)

$$Z \xrightarrow{h} W' \xrightarrow{g} W$$

where $h : Z \to W'$ has connected fibres and $g : W' \to W$ is finite. Theorem 2.2.1 implies that the general fibre of pr_2 is irreducible, hence g is generically one to one. Since the map g is a surjective branched covering of W, it must be one to one everywhere. Therefore $f^{-1}(L')$ is connected for every $L' \in W$.

By Theorem 2.2.1 every open neighbourhood U of L contains a linear space L' for which $\pi_1(f^{-1}(L')) \twoheadrightarrow \pi_1(X)$ and so follows the second part of the Theorem. □

We end this section stating the Fulton-Hansen Theorem and some of its important consequences.

Theorem 2.2.4 (Fulton-Hansen)**.** *Let X be a (irreducible projective) complete variety and $f : X \to \mathbb{P}^r \times \mathbb{P}^r$ a morphism. Assume* $\dim(f(X)) > r$*, then*

- *$f^{-1}(\Delta)$ is connected, where $\Delta \subset \mathbb{P}^r \times \mathbb{P}^r$ is the diagonal;*
- *if X is locally irreducible, $\pi_1(f^{-1}(\Delta)) \twoheadrightarrow \pi_1(X)$.*

For a proof of this result see [FL, Theorem 3.1] or [MNP, Theorem 5.1].
This Theorem has several applications. Here we list some of them related to our problem.

Theorem 2.2.5. *Let X be a complete variety of dimension n and let $f : X \to \mathbb{P}^r$ a morphism unramified. If $2n > r$, f is a closed embedding.*

Corollary 2.2.6. *Let $X \subset \mathbb{P}^r$ be a closed subvariety of dimension n with $2n > r$. If X is not normal, then the normalization $\nu : \tilde{X} \to X$ must be ramified.*

For instance, if X is a surface with ordinary singularities, the normalization $\tilde{X}$ is smooth and the map $\tilde{X} \to X$ ramifies over the pinch points of X.

We recall also that a consequence of the Fulton-Hansen Theorem is the finiteness of the Gauss map (Proposition 1.2.8) stated before.

2.2.2 Fundamental groups

The complex projective space $\mathbb{P}^n$ is simply connected, i.e. $\pi_1(\mathbb{P}^n) = 1$, while open subsets $U = \mathbb{P}^n \setminus R$, with R closed, may have non trivial fundamental group. The following result implies that the only non trivial cases are the ones in which R has codimension one in $\mathbb{P}^n$.

Proposition 2.2.7. *[Di, Ch.4, Prop. 1.1] Let $X \subset \mathbb{P}^n$ be a projective variety. Then*

$$\pi_1(\mathbb{P}^n \setminus X) = 0 \quad \textit{if } \dim(X) \leq n-2.$$

Proof. Any element in $\pi_1(\mathbb{P}^n \setminus X)$ is represented by a smooth map $\alpha : S^1 \to \mathbb{P}^n \setminus X$, and any relation among the elements in $\pi_1(\mathbb{P}^n \setminus X)$ is represented by a smooth map $\beta : D^2 \to \mathbb{P}^n \setminus X$ of the 2-disc D^2 into $\mathbb{P}^n$. This follows from the fact that any continuous map between smooth manifolds can be approximated, without changing the homotopy class, by a smooth map. See, for instance [Hi, p.124]. Moreover we need work only with maps $\alpha : S^1 \to \mathbb{P}^n \setminus X$ and $\beta : D^2 \to \mathbb{P}^n \setminus X$ which are transversal to all the strata in the decompositions of X; to see this, use the transversality theorem (for instance, in [Hi, p.74]). Hence all the strata of complex codimension strictly greater than one play no role. □

Moreover, Zariski established an important result that extends Lefschetz homological results ([Zar1]; see [HT] for a more complete proof).

Theorem 2.2.8 (Zariski). *Let $X \subset \mathbb{P}^n$, $n > 2$, be a complex projective hypersurface. Let H be a general hyperplane in $\mathbb{P}^n$ and let X_H be the corresponding hyperplane section of X. Then*

$$\pi_1(\mathbb{P}^n \setminus X) = \pi_1(H \setminus X_H)$$

It is worth noticing that the isolated singularities of X have no effect upon the fundamental group of its residual space. Moreover, iterating we get the following

Corollary 2.2.9. *Let Π be a general plane in $\mathbb{P}^n$, $n \geq 3$ and $X \subset \mathbb{P}^n$ an hypersurface. Let X_Π be the corresponding plane section. Then*

$$\pi_1(\mathbb{P}^n \setminus X) = \pi_1(\Pi \setminus X_\Pi)$$

2.3 Uniform position principle

We end this chapter by recalling the so called Uniform Position Principle, introduced by Castelnuovo. It has been used later by Harris (see [ACGH, Chapter 3], [Ha2]).

Theorem 2.3.1 (Uniform position principle). *Let $C \subset \mathbb{P}^{c+1}$, $c \geq 1$ be a irreducible, non degenerate curve of degree d. Then a general hyperplane meets C in d points any $c+1$ of which are linearly independent.*

An equivalent version of this Theorem will be useful for our treatment ([ACGH, Chapter 3.1]). Let $C \subset \mathbb{P}^{c+1}$ be a irreducible, non degenerate curve of degree d. Let $U \subset (\mathbb{P}^{c+1})^*$ be the open set of transverse hyperplanes to C and let

$$I = \{(p, H) \in C \times U \mid p \in H \cap C\}$$

Observe firstly that I is irreducible of dimension $c+1$. The second projection $p_2 : I \to U$ is a topological covering of degree d.

Lemma 2.3.2. *The monodromy group of p_2 is the full symmetric group S_d.*

One can use this result to prove the following.

Proposition 2.3.3. *[PS, Prop. 2.3] Let $L \subset \mathbb{P}^{c+1}$ be a $c-1$ plane that does not meet C and let B_{red} be the branch divisor of p_2 taken with its reduced scheme structure. Let $\mathbb{P}^1_L$ be the line parametrizing hyperplanes containing L. If $\mathbb{P}^1_L$ intersect B_{red} transversally, then the monodromy group of p_2 restricted to $p_2^{-1}(\mathbb{P}^1_L)$ is the full symmetric group.*

We remark that p_2 restricted to $p_2^{-1}(\mathbb{P}^1_L)$ is the same as the projection $\pi_L : C \to \mathbb{P}^1$ of C from L.

Chapter 3

Monodromy of projections

Let $X \subset \mathbb{P}^{n+c}$, $c \geq 1$ be a irreducible, reduced projective variety of dimension n and let L be a linear subspace of dimension $c-1$ such that $L \cap X = \emptyset$. Consider the projection map

$$\pi_L : X \subset \mathbb{P}^{n+c} \to \mathbb{P}^n$$

To the morphism π_L we can associate the monodromy group $M(\pi_L)$ defined in Section 1.5.

Definition. An element $L \in \mathbb{G}(c-1, n+c)$ such that $M(\pi_L) = S_d$ is called uniform. Otherwise, L is non uniform. We set $\mathcal{W}(X)$ the locus of non uniform elements L for the variety X. When clear, we will just write $\mathcal{W}$.

Our aim is to give an estimate of the dimension of $\mathcal{W}$.

3.1 General projections

A first rough bound on the dimension of $\mathcal{W}$ is given by looking at general projections. We prove a more general version of the Lemma 2.3.2, equivalent to the Uniform position principle.

Let $X \subset \mathbb{P}^{n+c}$ be a irreducible, reduced variety of dimension n and degree d. Let $U \subset \mathbb{G}(c, n+c)$ be the open set of linear subspaces of dimension c that are transverse to X and let

$$I = \{(p, L) \in X \times U \mid p \in L \cap X\}$$

The second projection $p_2 : I \to U$ is a topological covering of degree d.

Following the proof of Lemma 2.3.2 we get:

Lemma 3.1.1. *The monodromy group of p_2 is the full symmetric group S_d.*

Proof. To show that $M(p_2)$ is S_d, we have to show that it is 2-transitive and contains a transposition. Saying that the monodromy group is 2-transitive is equivalent to saying that

$$I(2) := \{(p_1, p_2, L) \in X \times X \times U \mid p_1, p_2 \in L \cap X \}$$

is connected. Therefore, consider a slight modification of the previous definition. Let

$$\tilde{I}(2) := \{(p_1, p_2, L) \in X \times X \times \mathbb{G} \mid p_1, p_2 \in L \cap X,\ p_1 \neq p_2\}$$

that maps to $(X \times X) \setminus \Delta$, where Δ is the diagonal, with fibres that are linear of dimension $c-1$. Hence $\tilde{I}(2)$ is irreducible and $I(2)$ is a Zariski open inside $\tilde{I}(2)$, hence it is connected.

We are left to show that $M(p_2)$ contains a transposition. Take $L_0 \in \mathbb{G}(c, n+c)$ simply tangent to X at a smooth point. Consider a family of subspaces $\{L_t\}_{t \in \mathbb{C},\ |t|<\varepsilon}$, with $L_t \in U$ for $t \neq 0$. Let X^* be the dual of X in $\mathbb{G}(c, n+c)$ of c-planes containing tangent lines to X. The family $\{L_t\}$ meets X^* transversally at L_0, then $L_t \cap C$ contains two points that come together to the point of tangency when $t \to 0$. Thus the monodromy group contains a transposition.

□

Let $L \in \mathbb{G}(c-1, n+c)$ be such that $L \cap X = \emptyset$ and let $\pi_L : X \to \mathbb{P}^n_L$ be the projection, where $\mathbb{P}^n_L$ parametrizes all the c-planes in $\mathbb{G}(c, n+c)$ through L.

Proposition 3.1.2. *Let B be the branch of $p_2 : I \to \mathbb{G}(c, n+c)$ taken with the reduced scheme structure. If $\mathbb{P}^n_L$ meets B transversally, then L is uniform.*

Proof. In the proof we follow the same argument in [PS, Proposition 2.3]. The map π_L corresponds to the restriction of p_2 to $p_2^{-1}(\mathbb{P}^n_L) \simeq X$. Since $\mathbb{P}^n_L$ meets B transversally, by a Lefschetz-type theorem ([Di, Theorem 6.5]) we have a surjection $\pi_1(\mathbb{P}^n_L \setminus (\mathbb{P}^n_L \cap B)) \twoheadrightarrow \pi_1(\mathbb{G}(c, n+c) \setminus B)$. Thanks to Lemma 3.1.1 we know that the monodromy of p_2 is the full symmetric group. Combining with the previous surjection, we get that L is uniform. □

A general L correspond to a general $\mathbb{P}^n_L$ in $\mathbb{G}(c, n+c)$. Using this we can deduce a first bound on the dimension of the non uniform locus $\mathcal{W}(X)$.

Corollary 3.1.3. *A general L is uniform.*

The same is proven also in [Cu, Proposition 2.3] taking hyperplane sections up to reduce the variety to a curve. The procedure is described in the following Section.

3.1.1 The case of curves

In the proof of our main results, we will use an induction argument on the dimension n of the variety X. It is based on the following fact.

Lemma 3.1.4. *Let X be an irreducible and reduced projective variety and π_L a projection as above. Let H be a general hyperplane containing L and let π' be the restriction of the projection π_L to $H \cap X$. By Bertini's theorem 2.2.1, the section $X \cap H$ is again irreducible and reduced; so it makes sense to consider $M(\pi')$ and*

$$M(\pi') \leq M(\pi_L).$$

In particular, $\mathcal{W}(X \cap H) \supseteq \mathcal{W}(X) \cap H$.

Proof. We have the following diagram

$$\begin{tikzcd} X \cap H \arrow[r, "i"] \arrow[d, "\pi'=\pi_{L|H}"] & X \arrow[d, "\pi_L"] \\ \mathbb{P}^{n-1} \arrow[r, "i"] & \mathbb{P}^n \end{tikzcd}$$

where the map i is an inclusion. Let B the branch divisor of the projection π_L and let $\mu : \pi_1(\mathbb{P}^n \setminus B) \twoheadrightarrow M(\pi_L) \leq S_d$ be the monodromy map. At the level of fundamental groups, we have the following diagram

$$\begin{tikzcd} \pi_1(\mathbb{P}^{n-1} \setminus (\mathbb{P}^{n-1} \cap B)) \arrow[r] \arrow[d, "\mu"] & \pi_1(\mathbb{P}^n \setminus B) \arrow[d, "\mu"] \\ M(\pi') \arrow[r] \arrow[dr] & M(\pi_L) \arrow[d] \\ & S_d \end{tikzcd}$$

where the map $\pi_1(\mathbb{P}^{n-1} \setminus (\mathbb{P}^{n-1} \cap B)) \to \pi_1(\mathbb{P}^n \setminus B)$ is an inclusion by the genericity of H. Since the diagram commutes, we get that $M(\pi')$ is contained in $M(\pi_L)$. Moreover, if L is such that $M(\pi_L) \neq S_d$, then also $M(\pi')$ is so; in other words, $\mathcal{W}(X \cap H) \supseteq \mathcal{W}(X) \cap H$. $\square$

The base case for the induction is the case of curves. In 2005 Pirola and Schlesinger [PS, Theorem 3.5] proved the following bound on the dimension of $\mathcal{W}$.

Theorem 3.1.5. *Let $X \subset \mathbb{P}^{c+1}$, $c \geq 1$ be an irreducible, non degenerate curve. The locus $\mathcal{W}(X)$ of non uniform $(c-1)$-planes has codimension at least two in the Grassmannian $\mathbb{G}(c-1, c+1)$.*

This bound is sharp ([PS, Remark 3.6]).

Example 3.1.6. An example in $\mathbb{P}^3$ of a curve C admitting a codimension two family on non uniform lines is the twisted cubic curve ([PS, Ex 2.6]). Given two distinct points $p, q \in C$, denote by H_p, H_q the osculating planes at p and q respectively. Let $L := H_p \cap H_q$; then the line L does not meet C. By Riemann-Hurwitz, the projection π_L from L is ramified only over H_p and H_q and, by Lemma 1.6.5, L is non uniform. Letting p and q vary along the curve, we find a two dimensional irreducible family of lines L that are non uniform.

Remark 3.1.7. We remark that, if $X \subset \mathbb{P}^{n+1}$ is an hypersurface with $n \geq 2$, up to taking hyperplane sections (Lemma 3.1.4) and using Theorem 3.1.5, the codimension of $\mathcal{W}(X)$ must be at least 2 in $\mathbb{P}^{n+1}$.

3.2 Filling families of tangent lines

The theory of focal loci has been introduced in Section 2.1. Here we present some results on the focal locus of a filling family of tangent lines in $\mathbb{P}^{n+1}$, following [CF].

Let $\mathcal{X}$ be a filling family of lines in $\mathbb{P}^{n+1}$. Assume $\mathcal{X}$, which has dimension n by the filling property, is locally parametrized by $D = D(u_1, \ldots, u_n)$. The line l_d corresponding to a point $d \in D$ can be given by the intersection of n distinct hyperplanes

$$l_d := \{a_1(d) \cdot \underline{x} = \ldots = a_n(d) \cdot \underline{x} = 0\}.$$

Here $\underline{x} = (x_0 : \ldots : x_{n+1})$ are the coordinates of $\mathbb{P}^{n+1}$ and $a_i(d) = (a_i(d)_0 : \ldots : a_i(d)_{n+1})$ denotes the independent vectors determining the hyperplane. We will denote by $\partial_{u_k} a_i(d)_j$ the partial derivative of $a_i(d)_j$ with respect to the variable u_k, and inductively, for high order derivatives, $\partial_{u_k,u_l} a_i(d)_j$, and so on. Write just $\partial_{u_i} a_i$ for the vector $(\partial_{u_k} a_1(d)_0 : \ldots : \partial_{u_i} a_i(d)_{n+1})$. The equation of the focal locus on the line l_d is ([CF, Sec. 4.3], [Se1])

$$\det \begin{pmatrix} \partial_{u_1} a_1 \cdot \underline{x} & \cdots & \partial_{u_n} a_1 \cdot \underline{x} \\ \vdots & & \vdots \\ \partial_{u_1} a_n \cdot \underline{x} & \cdots & \partial_{u_n} a_n \cdot \underline{x} \end{pmatrix} = 0. \tag{3.1}$$

The following result is a generalisation of part (a) and (c) of Proposition 5.1 in [CF].

Lemma 3.2.1. *Consider a filling family $\mathcal{X}$ of lines in $\mathbb{P}^{n+1}$, and assume its general member l is tangent to a irreducible reduced hypersurface X at a general point P. Then P is a focus on l, and if the contact order of l with X at P is 2, then P is a focus with multiplicity at least two on l.*

Proof. We can assume that the surface X is parametrised locally around P by the same B which parametrises the family $\mathcal{X}$, so $P := P(u_1, \ldots, u_n)$. Moreover we can choose a_1 to give the tangent plane to X at P. So we have

$$a_1 \cdot P = a_1 \cdot (\partial_{u_1} P) = \ldots = a_1 \cdot (\partial_{u_n} P) = 0.$$

By taking partial derivatives and by using the previous relations, we get

$$(\partial_{u_1} a_1) \cdot P = \ldots = (\partial_{u_n} a_1) \cdot P = 0. \tag{3.2}$$

It immediately follows that P satisfies Equation (3.1), and so is a focus on l independently of the choices of the other $n - 1$ hyperplanes. This proves the first claim.

For the second part of the proof, notice that the line l is tangent to X, so we can assume l to be explicitly parametrised as follows:

$$l := \{P + \lambda \partial_{u_1} P\}. \tag{3.3}$$

The contact order of l with X at P is 2, hence we get

$$a_1 \cdot (\partial_{u_1,u_1} P) = 0.$$

Notice that, for any hyperplane $\{b \cdot \underline{x} = 0\}$ passing through P and containing l, we have that $b \cdot (\partial_{u_1} P) = 0$ from equation (3.3), and hence, by taking derivatives, we get that

$$(\partial_{u_1} b) \cdot P = 0. \tag{3.4}$$

We can choose other $n-1$ independent vectors $b_2, \ldots, b_n$ in $\mathbb{P}^{n+1}$ and the hyperplanes $b_i \cdot \underline{x} = 0$ defined by them pass through P and contain l. The line l is given by the equations:

$$\{a_1 \cdot \underline{x} = b_2 \cdot \underline{x} = \ldots = b_n \cdot \underline{x} = 0\},$$

and the focal scheme on l described by Equation (3.1) is explicitly given as a function of the parameter λ by

$$\det \begin{pmatrix} (\partial_{u_1} a_1) \cdot \underline{x} & \cdots & (\partial_{u_n} a_1) \cdot \underline{x} \\ (\partial_{u_1} b_2) \cdot \underline{x} & \cdots & (\partial_{u_n} b_2) \cdot \underline{x} \\ \vdots & & \vdots \\ (\partial_{u_1} b_n) \cdot \underline{x} & \cdots & (\partial_{u_n} b_n) \cdot \underline{x} \end{pmatrix} = 0. \tag{3.5}$$

Now we can use Equation (3.3) to express the focal scheme as a function of the parameter λ. Moreover, if we consider Equation (3.2) and Equation (3.4), we get a simplified form for our matrix:

$$\det \begin{pmatrix} 0 & \lambda(\partial_{u_2} a_1) \cdot (\partial_{u_1} P) & \cdots & \lambda(\partial_{u_n} a_1) \cdot (\partial_{u_1} P) \\ \lambda(\partial_{u_1} b_2) \cdot (\partial_{u_1} P) & \cdots & \cdots & \cdots \\ \vdots & & & \vdots \\ \lambda(\partial_{u_1} b_n) \cdot (\partial_{u_1} P) & \cdots & \cdots & \cdots \end{pmatrix} = 0. \tag{3.6}$$

Lemma 2.1.6 guarantees that the determinant is not identically zero whenever l is a general element of the family. Such a determinant is given by $\lambda^2 \cdot \alpha(\lambda) = 0$, where α is a polynomial depending on λ. Hence the point P is a focus of multiplicity at least 2. $\square$

We conclude this part by studying the focal locus for a different family of lines in $\mathbb{P}^{n+1}$. Let $\mathcal{S}$ be a curve in $\mathbb{P}^{n+1}$ and let X be an irreducible reduced hypersurface with $\operatorname{codim}(X^{sing}) = 1$ in X. Let $\mathcal{X}$ be the family of lines given by the join between $\mathcal{S}$ and X^{sing}.

Lemma 3.2.2. *Assume that the family $\mathcal{X}$ is filling and that its general member l is contained in the tangent cone to X at a general point P in X^{sing}, taken with its reduced scheme structure. Assume moreover that the line l is not contained in the Zariski tangent space to X^{sing} at P. Then P is a focus on l with multiplicity at least two.*

Proof. We will follow the same lines of the proof of Lemma 3.2.1. Assume $\mathcal{S}$ is parametrised by the coordinate u_1 around the point $l \cap \mathcal{S}$ and X^{sing} is parametrised by the coordinates $u_2, \ldots, u_n$ around P. The family $\mathcal{X}$ is filling, hence we can assume that it is locally parametrized by the coordinates $u_1, \ldots, u_n$ around the line l. The point P is general, hence smooth in X^{sing}, so we can consider $\{a_1 \cdot \underline{x} = 0\}$ to be the hyperplane containing the line l and the Zariski tangent space to X^{sing} at P. As before, we want to describe l as the intersection of $\{a_1 \cdot \underline{x} = 0\}$ and other $n-1$ independent hyperplanes:

$$\{a_1 \cdot \underline{x} = b_2 \cdot \underline{x} = \ldots = b_n \cdot \underline{x} = 0\}.$$

Every line joining $\mathcal{S}$ and P belongs to $\mathcal{X}$, so we have that

$$(\partial_{u_1} b_i) \cdot P = (\partial_{u_1} a_1) \cdot P = 0. \tag{3.7}$$

Moreover, since $\{a_1 \cdot \underline{x} = 0\}$ contains the tangent space to X^{sing} at P, we have that

$$(\partial_{u_j} a_1) \cdot P = 0, \ \forall j = 2, \dots, n. \tag{3.8}$$

By hypothesis, the line l is not contained in the tangent space to X^{sing} at P, hence we can parametrise the family $\mathcal{X}$ around P by choosing the independent coordinate u_1 such that the direction of the line l is $\partial_{u_1} P$. In other words, we assume l to be explicitly parametrized as:

$$l := \{P + \lambda \partial_{u_1} P\}. \tag{3.9}$$

The line l is in the tangent cone to X at P, hence we get $a_1 \cdot (\partial_{u_1,u_1} P) = 0$. Therefore, as for Lemma 3.2.1, we obtain

$$(\partial_{u_1} a_1) \cdot (\partial_{u_1} P) = 0. \tag{3.10}$$

The focal scheme on the line l is still given by the determinant in Equation (3.5). Once we express it as a function of the parameter λ through Equation (3.9), and we consider Equations (3.7), (3.8) and (3.10), we find the same matrix of Equation (3.6). Such a determinant is given by $\lambda^2 \cdot \alpha(\lambda) = 0$, where α is a polynomial depending on λ. Lemma 2.1.6 guarantees that the determinant is not identically zero whenever l is a general element of the family. Hence the point P is a focus of multiplicity at least 2. $\square$

3.3 Hypersurfaces

The Theorem 3.1.5 implies that the locus of non uniform points is finite for irreducible and reduced plane curves. This has been generalized in [CMS] to smooth surfaces in $\mathbb{P}^3$.

Theorem 3.3.1. *[CMS, Theorem 1.1] Let $X \subset \mathbb{P}^3$ be a smooth surface. Then the locus of non uniform points $\mathcal{W}(X)$ is at most finite.*

Example 3.3.2. Let $X \subset \mathbb{P}^3$ be the Fermat cubic surface, zero locus of $G(x_0 : \ldots : x_3) = x_0^3 + \ldots + x_3^3$. The point $P = (0 : 0 : 0 : 1) \in \mathbb{P}^3$ is not uniform since the monodromy group is generated by 3-cycles.

The main result of this thesis generalizes those results.

Theorem 3.3.3. *Let X be an irreducible, reduced, complex hypersurface of $\mathbb{P}^{n+1}$. Then, the locus $\mathcal{W}(X)$ is contained in a finite union of linear subspaces of codimension at least 2.*

Cones provide examples of varieties with a finite union of non uniform linear subspaces of codimension at 2.

Proposition 3.3.4. *Let $C \subset \mathbb{P}^{n+1}$ be an irreducible and reduced curve contained in a plane $H \cong \mathbb{P}^2$ and consider a linear space V of dimension $n-2$ disjoint from H. Let X be the cone on C with vertex V. Then every point in $\langle Q, V\rangle \setminus V$ is non uniform for X, where Q is a non uniform point for C in H. Moreover, the non uniform locus of X is a finite union of such linear subspaces of codimension 2.*

Proof. Let P be a point in $\langle Q, V\rangle \setminus V$. Generators of the monodromy group of the projection of X from P are obtained by taking a general plane section by a plane containig P ([Di, Prop. 3.1]). The linear projection from the vertex V of X induces an isomorphism from the general $\mathbb{P}^2$ to H which sends $X \cap \mathbb{P}^2$ to C. There are at most a finite number of non uniform points for an irreducibile and reduced plane curve (Theorem 3.1.5). Therefore, the non uniform locus for the cone X is a finite union of linear subspaces of codimension 2 containing the vertex V. □

In a similar flavour, cones $X \subset \mathbb{P}^{n+1}$ with vertex of dimension $k-1$, provide an example of irreducible and reduced varieties with $\mathcal{W}(X)$ being a finite union of $\mathbb{P}^k$, $k = 1, \ldots, n-2$.

3.3.1 Proof of the main theorem

We start by characterising a point $Q \in \mathcal{W}(X)$ with respect to a general, hence uniform, point $P \in \mathbb{P}^{n+1} \setminus X$.

Consider a general $\mathbb{P}^2$ passing through Q; the plane curve $X \cap \mathbb{P}^2$ is irreducible and reduced thanks to Bertini's Theorem 2.2.1. We can associate Q with the set of permutations corresponding to the branch points of π_Q restricted to the chosen $\mathbb{P}^2$ (Corollary 2.2.9), whose type is given by Lemma 1.5.6. Recall that from the generality of the choice of the $\mathbb{P}^2$, these permutations generate the monodromy group $M(\pi_Q)$.

The contact orders of lines from a general, hence uniform, point P are the smallest possible, so a non uniform point Q must have at least one permutation among the generators of $M(\pi_Q)$ that has order strictly greater than the ones that generate $M(\pi_P) \simeq S_d$.

We will denote by $\mathcal{V}_Q$ the family of lines through Q corresponding to those generators with high order. If $\mathcal{S}$ is a subvariety of $\mathcal{W}(X)$, we denote by $\mathcal{V}_{\mathcal{S}}$ the union of $\mathcal{V}_Q$ for all $Q \in \mathcal{S} \subset \mathcal{W}(X)$.

There are several ways in which the contact order of a general line l in $\mathcal{V}_Q$ can increase. We will subdivide them in four cases.

(C1) The line l is more than simply tangent on points in X^{sm}, i.e. l is bitangent or asymptotic tangent;

(C2) The line l is bisecant to X^{sing};

(C3) The line l pass through a point of X^{sing} and is tangent to a point of X^{sm};

(C4) The line l is in the tangent cone to X at a point in X^{sing}.

Lemma 3.3.5. *Let X be a irreducible, reduced hypersurface in $\mathbb{P}^{n+1}$, and let $Q \in \mathcal{W}(X)$ be a non uniform point. Then, the base locus parametrising the family $\mathcal{V}_Q$ has dimension $n-1$ in the Grassmannian $\mathbb{G}(1, \mathbb{P}^{n+1})$.*

Proof. For $n = 1$, [PS, Proposition 2.5] guarantees that a non uniform point must have at least two non-simple tangent lines passing through it. We proceed now by induction. Assume that the claim is true for a hypersurface of $\dim X = n-1$, $n \geq 2$ and prove it for the case $\dim X = n$. By contradiction, assume that the dimension of the base of $\mathcal{V}_Q$ is smaller than

$n-1$. Take a general hyperplane H in $\mathbb{P}^{n+1}$ passing through Q; by Bertini's Theorem, the section $X \cap H$ is irreducible and reduced since $Q \notin X$. The hyperplane H contains a subfamily of $\mathcal{V}_Q$ parametrised by a base of dimension strictly lower than $n-1$, but this contradicts the induction hypotesis. □

Lemma 3.3.6. *Let $\mathcal{S} \subset \mathcal{W}(X)$ be an irreducible curve not contained in a linear space of codimension 2. Then the family $\mathcal{V}_\mathcal{S}$ is a filling family of lines.*

Proof. We want to show that $\mathcal{V}_\mathcal{S}$ is a family whose base space has dimension n and the map $\mathcal{V}_\mathcal{S} \to \mathbb{P}^{n+1}$ of Definition 2.1.4 is dominant. For every choice of $Q \in \mathcal{S}$, the dimension of the base of $\mathcal{V}_Q$ is $n-1$ thanks to Lemma 3.3.5. Every line in $\mathcal{V}_\mathcal{S}$ belongs to $\mathcal{V}_Q$ for a certain $Q \in \mathcal{S}$, so the dimension of the base of $\mathcal{V}_\mathcal{S}$ is n.

If $\mathcal{V}_\mathcal{S}$ were not dominant, then the family $\mathcal{V}_\mathcal{S}$ would be contained in a finite union of irreducible hypersurfaces $V_j,\ j = 1, \dots, k$ in $\mathbb{P}^{n+1}$, distinct from X since $\mathcal{S} \nsubseteq X$. For a general $Q \in \mathcal{S}$, $\mathcal{V}_Q$ is the union of cones over $V_j \cap X$ with vertex Q. For each $j = 1, \dots, k$, the hypersurface $\mathcal{V}_Q$ over $V_j \cap X$ coincide with V_j since they have the same dimension. Let us consider $V_j := V$ for a $j \in \{1, \dots, r\}$. The cone $\mathcal{V}_Q = \mathcal{V}'_Q$ for every $Q, Q' \in \mathcal{S}$ and we will just write $V \cong \mathcal{V}_Q$ for every $Q \in \mathcal{S}$. We claim that V is linear. Consider a general line l passing through a general point $T \in \mathcal{S}$ and not contained in V. If V would not be a hyperplane, there should be at least a point $Z \in V \cap l,\ Z \neq T$. By hypothesis, V is the cone over $V \cap X$ with vertex T. The line $\langle Z, T \rangle$ with $Z \in V$ is contained in V. This is a contradiction. Hence V is linear.

The family $\mathcal{V}_\mathcal{S}$ is contained in a finite union of hyperplanes $H_1, \dots, H_k$ and a finite number of cones R_j with vertex on $\mathcal{S}$. As a consequence, the curve $\mathcal{S}$ must be contained in the intersection of $H_1, \dots, H_k$. We want to exclude the case $k = 1$. Take a general $\mathbb{P}^2$ passing through a general point Q of $\mathcal{S}$: it intersects X in an irreducible and reduced curve. Lemma 3.1.4 ensures that Q is non uniform also for $\mathbb{P}^2 \cap X$. The hyperplanes H_i intersect this $\mathbb{P}^2$ in lines which correspond to generators of the monodromy group π_Q. Since the point Q is not uniform, there must be at least two generators coming from the H_i that are not transpositions, hence $k > 1$.

□

Proof. (Proof of Theorem 3.3.3).

We can assume $n > 1$ thanks to Theorem 3.1.5. Let us assume by contradiction that there exists a component of $\mathcal{W}(X)$ not contained in a linear space of codimension 2. We can choose a irreducible curve $\mathcal{S} \subset \mathcal{W}(X)$ with the same property. We now want to apply the focal machinery to the family $\mathcal{V}_\mathcal{S}$ to get a contradiction. The hypothesis of Lemma 3.3.6 are satisfied, so we know that $\mathcal{V}_\mathcal{S}$ is filling. Notice that all of the conditions (C1), ..., (C4) are defined by Zariski-closed properties, as a consequence there exist a Zariski-open $\mathcal{U} \subset \mathcal{S}$ such the lines in $\mathcal{V}_Q$ for $Q \in \mathcal{U}$ belong to exactly one of the cases (C1), ..., (C4). Let us study such cases one by one.

The general element of $\mathcal{V}_\mathcal{S}$ belongs to Case (C1).

If l is an asymptotic tangent line at the point P, then it is a focal point with multiplicity 2; if l is bitangent at P and Q, $P \neq Q$ then both the two points are focal point for l (see Lemma 3.2.1).

The general element of $\mathcal{V}_{\mathcal{S}}$ belongs to Case (C2).

The family $\mathcal{V}_{\mathcal{S}}$ is made by lines bisecant to X^{sing} and passing through $\mathcal{W}(X)$. Hence, there is a one dimensional subfamily of lines of $\mathcal{V}_{\mathcal{S}}$ trough every point in X^{sing}. By Lemma 2.1.9 the points in $l \cap X^{sing}$ are focal points for l, each of multiplicity one.

The general element of $\mathcal{V}_{\mathcal{S}}$ belongs to Case (C3).

A point in $l \cap X^{sm}$ at which l is tangent is focal by Lemma 3.2.1, while a point in $l \cap X^{sing}$ is focal by Lemma 2.1.9 as observed before.

The general element of $\mathcal{V}_{\mathcal{S}}$ belongs to Case (C4). In this case, $l \in \mathcal{V}_{\mathcal{S}}$ belongs to the tangent cone to X at a point x in X^{sing}. As a consequence, $\mathcal{S}$ must be contained in the intersection of all the tangent cones to X at points in X^{sing}. The family $\mathcal{V}_{\mathcal{S}}$ is filling, so its general member is not contained in the tangent space to X^{sing}. We are in the situation of Lemma 3.2.2, and the point x is a focus of multiplicity at least two.

In each of the previous cases, there are at least two points, counted with the right multiplicity, in the focal locus of l. Moreover, l passes through a point of $\mathcal{S}$ and, by construction, for every such point there is a $n-1$ dimensional subfamily of $\mathcal{V}_{\mathcal{S}}$. Therefore, this point is a focus for l as well, its multiplicity being $n-1$ by Lemma 2.1.9. Note that the general line meets $\mathcal{S}$ in a point outside X. Thus, we have at least $n-1+2=n+1$ focal points in a general line l of the filling family $\mathcal{V}_{\mathcal{S}}$. But this is a contradiction because of Lemma 2.1.6, as the focal locus in a general line of a filling family of lines in $\mathbb{P}^{n+1}$ consists of n points counted with multiplicity.

In each of the cases (C1), ..., (C4) we found a contradiction. This concludes the proof. □

3.3.2 Special cases

The aim of this section is to prove stronger results on the dimension of the non uniform locus, under some assumptions on the variety X.

Firstly, we state a consequence of Bertini's Theorem (Thm 2.2.1) that we will use in the following.

Proposition 3.3.7. *Let X be irreducible and reduced projective hypersurface. Assume that $\mathcal{W}(X)$ is not finite and let $K \cong \mathbb{P}^k$ be the smallest linear space containing a component of $\mathcal{W}(X)$. Then, a general linear subspace $H \cong \mathbb{P}^{k+1}$ containing K, cuts X in either a reducible or not reduced hypersurface.*

Proof. If that were not true, since a non uniform point for X is also non uniform for $X \cap H$ in H (Lemma 3.1.4), the locus $\mathcal{W}(X \cap H)$ would span a space of codimension 1 in H, which contradicts Theorem 3.3.3. □

Notice that, by Bertini's Theorem, a general section $H \cap X$ cannot be not reduced since it must be not reduced away from the base locus $K \cap H$ for the assumptions on the hypersurface X.

Using this property, we can generalize the Theorem in [CMS] for smooth hypersurfaces of dimension $n \geq 3$.

Theorem 3.3.8. *Let X be a smooth hypersurface in $\mathbb{P}^{n+1}$. Then the locus of non uniform points is finite.*

Proof. Assume $\mathcal{W}(X)$ is not finite. Thanks to Theorem 3.3.3, there are at least two hyperplanes H_1, H_2 containing $\mathcal{W}(X)$. Let us consider an irreducible curve $\mathcal{S}$ in $\mathcal{W}(X)$ and H_1, H_2 as in the proof of Lemma 3.3.6. Let $Y \in H_1 \cap X$ and let $T_Y X$ be its tangent hyperplane. By construction, H_1 contains a $n-1$ dimensional family of non simple tangent lines to X through a general point of $\mathcal{S}$. Hence, since every line $\langle Q, Y\rangle$ with $Q \in \mathcal{S}$ is tangent at Y, since $H_1 \cap X$ has codimension one in X. Hence $\mathcal{S}$ is contained in $T_Y X$ for every $Y \in H_1 \cap X$ and so, by linearity, the linear span $\langle \mathcal{S}\rangle$ is contained in $T_Y X$, $\forall\, Y \in H_1 \cap X$.

We firstly show that $\mathcal{S}$ should be a line. To do this, we recall that, since X is smooth, the Gauss map is finite and birational onto the image X^* (Theorem 1.2.8).
If, by contradiction, $\dim\langle \mathcal{S}\rangle \geq 2$, then its dual in $(\mathbb{P}^{n+1})^*$ is a linear space of dimension at most $n-2$ containing $T_Y X \in X^*$ for every $Y \in H_1 \cap X$, while we expect $(H_1 \cap X)^*$ to be $n-1$ dimensional in X^*.
Now we assume that $\mathcal{S}$ is a line. The family of hyperplanes in $\mathbb{P}^{n+1}$ through it is of dimension $n-1$. Therefore, if we assume that all the $T_Y X$, $Y \in H_1 \cap X$ are distinct, the general hyperplane passing through $\mathcal{S}$ is a hyperplane tangent to X at a point Y. But this contradicts Bertini's Theorem 2.2.1. In any case, there exists a hyperplane tangent to a positive dimensional subvariety of X. But this contradicts Theorem 1.2.8.

□

Remark 3.3.9. We can rewrite our problem in the dual space $(\mathbb{P}^{n+1})^*$. Let Q be a point in $\mathcal{W}(X)$ and let B be the branch divisor of π_Q. Let $\tilde{B}$ be the intersection of the hyperplane $\mathbb{P}^n \cong Q^\perp$, parametrizing the hyperplanes through Q, and X^*, that is the dual variety of X. If we assume that X is not developable, then X^* is an hypersurface in $(\mathbb{P}^{n+1})^*$. The condition of being non uniform for a point Q in $\mathbb{P}^{n+1} \setminus X$ correspond to ask that at least an irreducible component of the branch divisor B is not reduced. This translate into asking that the intersection of $Q^\perp$ and the singular locus of X^* has an irreducible component that forms a component of $\tilde{B}$. Note that there could be components of B corresponding to points of $\tilde{B}$. This cannot happen if we assume X to be smooth since the Gauss map is finite (Theorem 1.2.8). To have infinitely many hyperplanes $Q^\perp$ containing at least a component of X^*_{sing}, this must contain a linear component L. We remark that a linear irreducible component in the singular locus of X^* cannot exist if X is smooth, otherwise there would exist a family of hyperplanes contradicting Bertini's Theorem (Thm 2.2.1). It gives a different argument to prove Theorem 3.3.8.

We conclude this section by showing another important consequence of Proposition 3.3.7.

Proposition 3.3.10. *If $\dim \mathcal{W}(X) > 0$ and X is not a cone, then the monodromy group relative to all but finitely many points of $\mathcal{W}(X)$ contains transpositions.*

Proof. Denote by $K \simeq \mathbb{P}^k$ the smallest linear subspace containing a positive dimensional component of $\mathcal{W}(X)$ and let H be a general $\mathbb{P}^{k+1}$ containing K. By Theorem 3.3.3 $k < n$ and by Proposition 3.3.7, $X \cap H$ is reducible. Assume there exists an irreducible reduced component X_1 in $X \cap H$ of degree at least 2.

Let $\mathcal{A}^o$ be the set of points P in $\mathbb{P}^{k+1} \setminus X_1$ such that all the lines tangent to X_1 and passing through P are more than simply tangent. Consider the subvariety $\mathcal{A}$ of $\mathbb{P}^{k+1}$, Zariski closure of the set $\mathcal{A}^o$. If we assume by contradiction that every line tangent to X and passing through points in $\mathcal{W}(X)$ is not simply tangent (except maybe for finitely many points in $\mathcal{W}(X)$), then $\dim \mathcal{A} > 0$. As in the proof of the main theorem, considering a curve in $\mathcal{A}$, we can construct a filling family of lines in $\mathbb{P}^{n+1}$; analogously, we would get a contradiction by using Lemma 2.1.6. As a consequence, all but finitely many points of $\mathcal{W}(X)$ have simply tangent lines passing trough them. Notice that these lines will not be tangent to other components X_i in $X \cap H$ since the dual varieties of the X_i are distinct.

Let now $X \cap H$ have only linear components $X_1, \dots, X_d$ and let $V := X \cap \overline{\mathcal{W}(X)}$. We are now left to prove that X is a cone. Notice that $V \cap X_i \neq \emptyset$, $i = 1, \dots, d$ for every H and so X turns out to be a union of cones with vertices in V (see Definition 1.6.1). By irreducibility of X, V must be a unique linear subspace and X is a cone of vertex V.

□

When the hypersurface X is not a cone, this result lets us to give a strong characterization of its projections with non uniform monodromy group. Firstly we recall some definition and properties.

Definition 3.3.11. *A projection $\pi_P : X \to \mathbb{P}^n$ is decomposable if there exists an open dense subset $U \subset \mathbb{P}^n$ over which π_P factors non birationally, i.e.*

$$\pi_P^{-1}(U) \xrightarrow{f} V \xrightarrow{g} U$$

where f, g are finite morphism of degree at least 2.

Remark 3.3.12. The map π_P is decomposable if and only if there is an intermediate field in the extension $\mathbb{C}(X) : \mathbb{C}(\mathbb{P}^n)$, i.e. the group $M(\pi_P)$ is imprimitive.

We recall that, if the monodromy group $M(\pi_P)$ is the full symmetric group, then the projection π_P is indecomposable, since S_d is primitive. The converse holds if we require that $M(\pi_P)$ contains a transposition (see [PS, Remark 2.2]). Indeed, consider the subgroup $N \leq M(\pi_P)$ generated by transpositions; it is a non trivial normal subgroup of $M(\pi_P)$. Moreover, it must be transitive otherwise it will form a system of non trivial blocks, i.e. H will be imprimitive. Therefore H is the whole S_d.

Finally we have the following consequence of Proposition 3.3.10.

Theorem 3.3.13. *Assume X is a irreducible reduced hypersurface in $\mathbb{P}^{n+1}$ with $\mathcal{W}(X)$ not finite and X not a cone. Then, the projection from all but finitely many points in $\mathcal{W}(X)$ must be decomposable.*

Proof. Let Q be a general point in $\mathcal{W}(X)$. By Proposition 3.3.10 the monodromy group $M(\pi_Q)$ contains a transposition. Hence, to be non uniform, the projection must be decomposable. □

Corollary 3.3.14. *Let X be an irreducible, reduced hypersurface of $\mathbb{P}^{n+1}$ of prime degree d. Then the locus $\mathcal{W}(X)$ is finite, unless X is a cone.*

Proof. The map $\pi_P : X \to \mathbb{P}^n$ is indecomposable for every $P \notin X$ because otherwise the degree of an intermediate non birational map would divide d. □

A special case of non uniform points are the *Galois points* introduced in Section 1.6.4 (see Remark 1.4.13). We recall the definition.

Definition 3.3.15. *Let $\pi_P : X \subset \mathbb{P}^{n+1} \to \mathbb{P}^n$ be a projection from a point P. A point $P \in \mathbb{P}^{n+1}$ is called a* Galois point *if the field extension $\mathbb{C}(X) : \mathbb{C}(\mathbb{P}^n)$ associated to π_P is Galois (see Definition 1.4.6).*

As a consequence of Proposition 3.3.10 we get the following corollary, which was proven in [FT, Proposition 6].

Corollary 3.3.16. *Let X be an irreducible, reduced, hypersurface in $\mathbb{P}^{n+1}$ of degree $d \geq 3$. Then the number of Galois points is finite, unless X is a cone.*

Proof. Following the notation of [FT], denote by $\Delta'(X)$ the locus of outer Galois points relative to X. Clearly, $\Delta'(X) \subset \mathcal{W}(X)$. If we assume $\Delta'(X)$ to be infinite and X not being a cone, Proposition 3.3.10 shows the existence of transpositions in the monodromy group of a general point Q in $\Delta'(X)$. As a consequence, the field extension given by π_Q is not Galois since the action of an element in the Galois group on a general fibre of π_Q has no fixed components. This is a contradiction. □

All these results suggest us the following conjecture

Conjecture. *Let $X \subset \mathbb{P}^{n+1}$ be a reduced and irreducible hypersurface that is not a cone. Then $\mathcal{W}(X)$ is at most finite.*

3.4 Decomposable maps

In order to understand better $\mathcal{W}(X)$ when X is not a cone, Theorem 3.3.13 says that we have to study decomposable projections. In this section we want to introduce a technique involving fundamental groups to study this property.

From an algebraic point of view, saying that the map π_P is decomposable is equivalent to saying that the monodromy group of π_P is imprimitive. We recall here the definition introduced in Section 1.1.

Definition 3.4.1. *Consider a transitive group G acting on a set $\Omega = \{1, \ldots, m\}$. A block for this action is a non empty subset $B \subset \Omega$ such that either $gB = B$ or $(gB) \cap B = \emptyset$ for all $g \in G$. The whole Ω and subsets of single elements $\{i\} \subset \Omega$ are trivial blocks. We say that G is imprimitive if its action preserves non trivial blocks; it is primitive otherwise.*

In Section 2.2 have been introduced classical results on fundamental groups. Here we want to prove a generalization of [No, Proposition 4.1], that follows the idea in [Zar1, Main Theorem] or [FL, Theorem 1.1] and present a surjectivity of fundamental groups.

Lemma 3.4.2. *Let Γ be a irreducible and reduced curve in $\mathbb{P}^n$ and $R \subset \mathbb{P}^n$ be a closed subset such that $\Gamma \nsubseteq R$ and the intersection between Γ and R is transverse. Then we have a surjective map*

$$\pi_1(\Gamma \smallsetminus R) \to \pi_1(\mathbb{P}^n \smallsetminus R).$$

Proof. Let D be the normalisation of Γ, so that we have a map

$$\begin{tikzcd} D \arrow[d, "\sigma"'] \arrow[dr, "h"] & \\ \Gamma \arrow[r, "i"] & \mathbb{P}^n \end{tikzcd}$$

where i denotes the inclusion, σ the normalisation map and $h = i \circ \sigma$.

As in the proof of [No, Proposition 4.1], choose $G := Sl(n+1)$, the special linear group of order $n+1$, and consider

$$\begin{aligned} \theta : G \times D &\to \mathbb{P}^n \\ (g, x) &\mapsto g \cdot h(x). \end{aligned}$$

Since $\mathbb{P}^n$ is simply connected, the map θ is a fibre bundle with smooth and connected fibres. We can restrict to the Zariski-open subset $V := \mathbb{P}^n \smallsetminus R$. Denote by X the preimage $\theta^{-1}(V)$. We have that

$$\pi_1(X) \to \pi_1(\mathbb{P}^n \smallsetminus R)$$

is surjective from the homotopy sequence. Consider now the projection on the first factor $G \times D \to G$ and let pr_1 be its restriction to X, whose fibres are non empty outside the set of points

$$S := \big\{g \in G |\ g \cdot (h(D)) \subset R\big\}.$$

Note that the non empty fibres of pr_1 are isomorphic to $\Gamma \smallsetminus R$ and so they are reduced. We want to apply Part C of Lemma 1.3.2 to pr_1. We claim that S has codimension at least 2 in G. Indeed, choose x, y two distinct points in D. The set S is contained in $I := \big\{g \in G \mid g \cdot x, g \cdot y \in R\big\}$. For every choice of such x, y, we can define a map $\varphi_{x,y} : I \to R \times R$ sending g to $(g \cdot x, g \cdot y)$. The fibres of this map are cosets of $stab_x \cap stab_y$, the intersection of the stabilisers of the two points x and y in G. Its order correspond to the dimension of the space of matrices in $\mathbb{P}SL(n+1)$ with two fixed columns, therefore we have

$$\begin{aligned} \dim(S) \le \dim(I) &\le \mathrm{ord}(stab_x \cap stab_y) + \dim(R \times R) \\ &\le n^2 + 2(n-1) = n^2 + 2n - 2. \end{aligned}$$

Hence we can apply Lemma 1.3.2 to pr_1. Since G is simply connected, the map

$$\pi_1(F) \to \pi_1(X)$$

is surjective for the general fibre F of pr_1. Combining the two surjections above, we get

$$\pi_1(F) \twoheadrightarrow \pi_1(X) \twoheadrightarrow \pi_1(\mathbb{P}^n \smallsetminus R).$$

The fibre of pr_1 over a general $g \in G$ is $(g \cdot D) \setminus h^{-1}(R)$. Thanks to the assumption on Γ of transverse intersection, Part C of Lemma 1.3.2 holds also for the fibre over the identity of G, i.e.

$$\pi_1(D \smallsetminus h^{-1}(R)) \twoheadrightarrow \pi_1(\mathbb{P}^n \smallsetminus R).$$

From the initial diagram we have $h = \sigma \circ i$, so finally we get

$$\pi_1(\Gamma \smallsetminus R) \twoheadrightarrow \pi_1(\mathbb{P}^n \smallsetminus R).$$

□

The previous result can be applied to the case of X being an irreducible, reduced hypersurface in $\mathbb{P}^{n+1}$ and $\mathcal{S} \subset \mathbb{P}^{n+1}$ being a curve not contained in X.

Corollary 3.4.3. *Let $P \in \mathbb{P}^{n+1} \smallsetminus X$ and assume $\mathcal{S}_P := \pi_P(\mathcal{S})$ intersects the branch locus B_P of π_P transversally. Then we have a surjection*

$$\pi_1(\mathcal{S}_P \smallsetminus B_P) \to \pi_1(\mathbb{P}^n \smallsetminus B_P).$$

Proof. Apply Lemma 3.4.2 with $\Gamma = \pi_P(\mathcal{S})$ and $R = B_P$. □

The transversality hypothesis of Corollary 3.4.3 is an open condition; hence, if there exists a point P that verifies such hypothesis, then the general point in $\mathbb{P}^{n+1}$ does. We use this to state the following result on monodromy groups.

Proposition 3.4.4. *If for $P \in \mathbb{P}^{n+1}$ general and $\mathcal{S} \subset \mathbb{P}^{n+1}$ a curve, $\mathcal{S}_P = \pi_P(\mathcal{S})$ and B_P satisfy hypothesis of Corollary 3.4.3, then the projection from the general point of $\mathcal{S}$ has primitive monodromy group.*

Proof. Assume by contradiction that the general point of $\mathcal{S}$ has non primitive monodromy group. Take a general point $P \in \mathbb{P}^{n+1}$. Thanks to Corollary 3.4.3 we have the following surjective map, where $t \in \mathcal{S}_P \smallsetminus B_P$.

$$\pi_1(\mathcal{S}_P \smallsetminus B_P, t) \xrightarrow{\tau} \pi_1(\mathbb{P}^n \smallsetminus B_P, t) \tag{3.11}$$

We call $\gamma_1, \ldots, \gamma_k$ the generators of $\pi_1(\mathbb{P}^n \smallsetminus B_P, t)$ and $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_k$ the corresponding lifts in $\pi_1(\mathcal{S}_P \smallsetminus B_P, t)$. Recall that the monodromy group of π_P is generated by the images $\gamma_1, \ldots, \gamma_k$. Up to a homotopy equivalence, also the $\tilde{\gamma}_i$, seen as elements in $\pi_1(\mathbb{P}^n \smallsetminus B_P)$, generate $M(\pi_P)$.

We can assume without loss of generality that each element $\tilde{\gamma}_i$ is a loop in $\mathcal{S}_P \smallsetminus B_P$ around a single branch point.

Since P is general, the line $\langle P, t \rangle$ intersects $\mathcal{S}$ in a unique point q and meets X in d distinct points. By assumption, $M(\pi_q)$ is non primitive, that is, the fibre of π_q over t, which is the same as the fibre of π_P over t, splits into non trivial blocks. Since we are assuming that the general point of $\mathcal{S}$ has non primitive monodromy group, by a continuity argument we can move the point q around to show that this subdivision in blocks stays the same for every non primitive point in $\mathcal{S}$.

We remark that we can take a loop η_i in $\mathcal{S}_P \smallsetminus B_P$, homotopy equivalent to $\tilde{\gamma}_i$, such that every line joining P with a point of η_i do not meet $\mathcal{S}$ in one of the finitely many point that are primitive. Up to homotopy equivalence, for every $\tilde{\gamma}_i$ the preimages of points of the loops on $\mathcal{S}$ are non primitive points, all with the same block subdivision. Considering all the loops $\tilde{\gamma}_i$ in $\mathcal{S}_P \smallsetminus B_P$ around t, we can show that the group $M(\pi_P)$, generated by $\mu(\tilde{\gamma}_1), \ldots, \mu(\tilde{\gamma}_k)$, is non primitive. Again, by a continuity argument, we can show that also the projection from a general point of $\mathcal{S}$ is non primitive. This gives a contradiction, because we chose the point P to be general, hence uniform. This concludes the proof. □

Remark 3.4.5. Recall that the locus of points with imprimitive monodromy group is contained in $\mathcal{W}(X)$.

We want to apply this result to the study of decomposable projections, i.e. projections with imprimitive monodromy group.

Corollary 3.4.6. *Let $X \subset \mathbb{P}^{n+1}$ be an irreducible and reduced hypersurface that is not a cone. Assume that $\mathcal{W}(X)$ has positive dimension. Then there is no curve $\mathcal{S} \subset \mathcal{W}(X)$ which satisfies the hypothesis of Proposition 3.4.4.*

Proof. Proposition 3.3.10 ensures the existence of a transposition for all but finitely many points in $\mathbb{P}^{n+1} \setminus X$. Let $\mathcal{S} \subset \mathcal{W}(X)$ be a curve. Since a general point of $\mathcal{S}$ is non uniform, the projection from it has imprimitive monodromy group. If $\mathcal{S}$ satisfies the hypothesis of Proposition 3.4.4, this would give a contradiction. □

We can use this Corollary to obtain another proof on the finitness of $\mathcal{W}(X)$ when X is smooth (Theorem 3.3.8).

Indeed, by contradiction, assume that $\mathcal{W}(X)$ is not finite and let $\mathcal{S} \subset \mathcal{W}(X)$ be a curve. We claim that $\mathcal{S}$ satisfies the hypotesis of Proposition 3.4.4.

Let P be a general point in $\mathbb{P}^{n+1}$ and let $\mathcal{S}_P = \pi_P(\mathcal{S})$. By the genericity of P we can assume that $\mathcal{S}_P$ is not contained in B_P: let $y \in \mathcal{S}$ be a general point and choose P not lying in a line through y tangent to X. Assume, morevore, that $\mathcal{S}_P$ does not intersect transversally B_P at a point z, i.e. the tangent line to $\mathcal{S}$ at $y = \pi_P^{-1}(z) \cap \mathcal{S}$ is contained in the hyperplane tangent to X at $x \in \pi_P^{-1}(z) \cap X$. Thanks to the construction in Proposition 3.3.10 there exists at least a simple tangent line to X trough a point of $\mathcal{S}$; let z be the image of such a line.

Dualizing, the variety $X^* \subset (\mathbb{P}^{n+1})^*$ is an hypersurface given by the tangent hyperplanes to (smooth) points of X, while $\mathcal{S}^* \subset (\mathbb{P}^{n+1})^*$ is the hypersurface given by the hyperplanes containing a line tangent to a (smooth) point of $\mathcal{S}$. Since $\mathcal{S} \not\subseteq X$, the biduality theorem implies that $\mathcal{S}^* \neq X^*$. Hence the intersection $\mathcal{S}^* \cap X^*$ is of dimension $n-1$. As a consequence, the family of lines

$$\mathcal{F} := \left\{ l := \langle y, x \rangle \text{ where } y \in X,\ x \in \mathcal{S} \text{ and } l \subset \Pi \text{ such that } \Pi^* \in X^* \cap \mathcal{S}^* \right\}$$

has dimension $n-1$ in $\mathbb{G}(1, n+1)$.

Therefore, considering the duals in $(\mathbb{P}^{n+1})^*$, a line corresponding to a non transverse intersection of $\mathcal{S}_P$ and B_P, lives in a family of dimension $n-1$ in $\mathbb{G}(1,\mathbb{P}^{n+1})$. Since P is general, we can assume that it does not belong to a line of that family.

Finally, we can compare Proposition 3.4.4 with the main result on the non uniform locus (Theorem 3.3.3) at the level of monodromy groups. Let us set $M' := \mu(\operatorname{im}\tau)$.

$$\begin{tikzcd} \pi_1(\mathcal{S}_P \setminus B_P, t) \arrow[r, "\tau"] \arrow[d] & \pi_1(\mathbb{P}^n \setminus B_P, t) \arrow[d, "\mu"] \\ M' \arrow[r] & M(\pi_P) = S_d \end{tikzcd} \tag{3.12}$$

The group M' is a subgroup of $M(\pi_P)$ and can be interpreted as the part of the monodromy group of the projection π_P which comes from generators that can be restricted to $\mathcal{S}_P$. If $\mathcal{S}_P$ satisfies the hypothesis of Proposition 3.4.4, we obtain that τ is surjective, which comes from the Nori's lemma 3.4.2. Therefore, M' is equal to $M(\pi_P) = S_d$.

On the other hand, if Proposition 3.4.4 does not apply to $\mathcal{S}_P$, it means that τ is not surjective. However, it is still possible to have a surjection at the level of monodromy groups.

3.5 Higher codimension varieties

Let $\tilde{X} \subset \mathbb{P}^{n+c}$, $c \geq 2$ be a smooth projective variety of dimension n and let $L \subset \mathbb{P}^{n+c}$ be a linear subspace of dimension $l = c-1$ such that $L \cap \tilde{X} = \emptyset$. Let

$$\pi_L : \tilde{X} \to \mathbb{P}^n$$

be the linear projection of $\tilde{X}$ from L. We denote by $\mathcal{W}(\tilde{X})$ the subset of the Grassmanian $\mathbb{G}(c-1,\mathbb{P}^{n+c})$ of non uniform $(c-1)$-planes for $\tilde{X}$.

As said in Section 1.6, the above projection can be factorized

$$\tilde{X} \xrightarrow{\pi_M} X \subset \mathbb{P}^{n+1} \xrightarrow{\pi_P} \mathbb{P}^n$$

where M is a general linear subspace of dimension $c-2$ such that $M \subset L$ and $P \in L \setminus M$ is a point such that $\langle M, P\rangle = L$.

We state a result on hypersurfaces that are the image of a general projections of a smooth variety, that is a generalization of Theorem 3.3.8.

Proposition 3.5.1. *Let $\tilde{X}$ be a smooth irreducible complex projective variety of dimension n in $\mathbb{P}^{n+c}$, $c \geq 1$. Take X to be the projection of $\tilde{X}$ from a general linear subspace A of $\mathbb{P}^{n+c}$ of dimension $c-2$. Then, the locus $\mathcal{W}(X)$ is at most finite.*

Proof. Assume $\mathcal{W}(X)$ is not finite, and denote by $K \cong \mathbb{P}^k$ the smallest linear subspace of $\mathbb{P}^{n+1}$ containing one of its components. By Proposition 3.3.7, $k < n$ and there exists a family of $\mathbb{P}^{k+1}$ containing K such that $X \cap H$ is reducible for the general element H of this family.

We have that $X \cap H$ is the linear projection from A of $\tilde{X} \cap \langle H, A\rangle$. Notice that $\tilde{X} \cap \langle H, A\rangle$ must be reducible as well, because the projection is a continuous map. So $\langle H, A\rangle$ gives a family of $\mathbb{P}^{k+c-1}$ whose general member cuts $\tilde{X}$ in a reducible variety. The base locus of this family is obtained by intersecting $\tilde{X}$ with $\Gamma := \langle K, A\rangle$. In order not to contradicts Bertini's theorem (see Theorem 2.2.1 and Lemma 2.2.3), this base locus must consists of singular points. So, the only possibility is that the space Γ is tangent to $\tilde{X}$. This contradicts Theorem 1.6.11: $\Gamma \cap \tilde{X}$ is contained in one of the $X_{i_1,\dots,i_k}$, which are smooth, since we assumed A general. □

We can use this Theorem and reason as in [PS, Theorem 3.5] to show the following.

Proposition 3.5.2. *Let $\tilde{X}$ be a smooth irreducible complex projective variety of dimension n in $\mathbb{P}^{n+c}$, $c \geq 1$. The locus of non-uniform L not intersecting $\tilde{X}$ has codimension at least $n+1$ in the Grassmannian $\mathbb{G}(c-1, \mathbb{P}^{n+c})$.*

Proof. We will follow the proof of [PS, Theorem 3.5]. When $c = 1$ we know from Corollary 3.5.1 that all but finitely many points $P \in \mathbb{P}^{n+1} \setminus \tilde{X}$ are uniform. Now assume $c \geq 2$. After projecting from a general $(c-2)$-subspace M, we get $X \subset \mathbb{P}^{n+1}_M$, where $\mathbb{P}^{n+1}_M$ parametrises all the $(c-1)$-planes L containing M. Notice that projecting to $\mathbb{P}^n$ from $\tilde{X}$ is the same as projecting from X, hence we can work with X in $\mathbb{P}^{n+1}$ and apply Corollary 3.5.1.

Assume by contradiction that $\mathcal{W}(\tilde{X})$ has codimension at most n in the Grassmannian $\mathbb{G}(c-1, \mathbb{P}^{n+c})$. In this case there would be an irreducible subvariety D of codimension at most n such that the general $L \in D$ is non-uniform, i.e. every $L \in D \setminus \Delta$ is non uniform for a proper Zariski closed subset Δ. We claim that for a general element $M \in \mathbb{G}(c-2, \mathbb{P}^{n+c})$, the dimension of $(D \setminus \Delta) \cap \mathbb{P}^{n+1}_M$ is greater than zero. Notice first that $D \cap \mathbb{P}^{n+1}_M$ is at least one dimensional: D has codimension at most n in $\mathbb{G}(c-1, \mathbb{P}^{n+c})$ and $\mathbb{P}^{n+1}_M$ is $n+1$ dimensional. Secondly, we have that

$$\dim(\Delta \cap \mathbb{P}^{n+1}_M) < \dim(D \cap \mathbb{P}^{n+1}_M).$$

This implies that there exist infinitely many non uniform such planes L containing a general $M \in \mathbb{G}(c-2, \mathbb{P}^{n+c})$, but this contradicts the base case $c = 1$. Hence, the locus of non-uniform $(c-1)$-planes has codimension at least $n+1$ in the Grassmannian $\mathbb{G}(c-1, \mathbb{P}^{n+c})$. □

Remark 3.5.3. The same argument of [PS, Remark 3.6] shows that the bound of Proposition 3.5.2 is sharp. There are varieties X for which there exist points $x \notin X$ such that the projection $\pi_x : X \subset \mathbb{P}^{n+c} \to \mathbb{P}^{n+c-1}$ is non birational onto the image. If this happens, a $(c-1)$-plane L cointaining such an x is non uniform because the map π_L factorises non trivially. Thus, the family of the $(c-1)$-planes passing through x is a codimension $n+1$ family in $\mathbb{G}(c-1, \mathbb{P}^{n+c})$ of non uniform elements. Non birational projections have been studied for instance in [Noma, Theorem 1, Theorem 3] and [CaCi].

3.6 Examples of families of non uniform subspaces

3.6.1 Curves

Let C be a smooth, non degenerate, irreducible curve in $\mathbb{P}^n$. We recall that a Schubert cycle $\sigma(x)$, with $x \in \mathbb{P}^n$ a point, is the subvariety of $\mathbb{G}(n-2, \mathbb{P}^n)$ given by the $(n-2)$-planes passing through x.

We introduce a definition that we will use in the following.

Definition 3.6.1. *We say that C has* special monodromy *if exists a subvariety $\Sigma \subset \mathbb{G}(n-2, \mathbb{P}^n)$ of codimension 2 in the Grassmannian, closed and irreducible that is not a Schubert cycle $\sigma(x)$, such that the general $L \in \Sigma$ does not meet C and the projection π_L factors:*

$$C \xrightarrow{\gamma} \mathbb{P}^1 \xrightarrow{\beta} \mathbb{P}^1$$

with $\deg(\gamma), \deg(\beta) \geq 2$.

Proposition 3.5.2 (already proved in [PS, Theorem 3.5]) says that

$$\operatorname{codim}_{\mathbb{G}(c-1,c+1)} (\mathcal{W}(C)) \geq 2.$$

As seen in Remark 3.5.3, the Schubert cycle of a non birational point gives an example of a codimension two subvariety of $\mathbb{G}(n-2, \mathbb{P}^n)$ of non birational elements. Moreover, Pirola and Schlesinger ([PS, Theorem 4.2]) classified all the smooth curves $X \subset \mathbb{P}^3$ admitting a family of non uniform lines of codimension 2. Here we follow their argument and we present a generalization of their result.

Theorem 3.6.2. *Let C be a smooth, non degenerate, irreducible curve in $\mathbb{P}^n$. Suppose $\Sigma \subset \mathbb{G}(n-2, n)$ is an irreducible Zariski closed codimension 2 subvariety such that the general $L \in \Sigma$ does not meet C and is not uniform. Then one of the following occurs:*

- *there exists a non-birational point $x \in \mathbb{P}^n$ such that $\Sigma = \sigma(x)$;*
- *C is a rational normal curve and the general element in Σ is the intersection of two osculating hyperplanes;*
- *C is rational and has special monodromy.*

Before starting the proof, we recall the Lemma of Strano on families of trisecant lines to a curve. We will focus on lines that intersect a variety in at least three distinct points; we will refer to them as *honest trisecant* lines.

Lemma 3.6.3. *([PS, Lemma 4.5]) Let $X \subset \mathbb{P}^3$ be a reduced curve of degree $d \geq 3$, possibly reducible. Let $\Sigma \subset \mathbb{G}(1, \mathbb{P}^3)$ be an irreducible Zariski closed subvariety of codimension 2. Assume that a general $L \in \Sigma$ is an honest trisecant line to X. Then $\Sigma = \sigma(H)$ is the family of all the lines lying on a plane H that contains a subcurve of X of degree at least 3.*

We need the following generalized version.

Lemma 3.6.4. *Let $X \subset \mathbb{P}^n$ be a reduced variety of dimension $n-2$ and degree $d \geq 3$. Let $\Sigma \subset \mathbb{G}(1, \mathbb{P}^n)$ be a irreducible Zariski closed subvariety of codimension 2. Assume that a general $L \in \Sigma$ is an honest trisecant line to X. Then $\Sigma = \sigma(H)$ is the family of lines lying on a hyperplane H that contains a subvariety of X of dimension $n-2$ and degree at least 3.*

Proof. Let K be a general 3-plane in $\mathbb{P}^n$. Then $K \cap X$ is a curve and K contains a subfamily of Σ of dimension two. Applying Lemma 3.6.3, we have that $X \cap K$ contains a plane curve of degree at least three, and the general line in that plane is an honest trisecant to $X \cap K$. Then

the variety X itself must have an equidimensional subvariety of degree at least 3 contained in a hyperplane H. Indeed, if all the irreducible components of X were non degenerate, a general hyperplane section of X must have the same property.
The 3-plane K intersects H in the plane containing the curve and the subfamily of Σ. Hence, Σ is the family of lines in H whose general element is an honest trisecant to X.

□

We are ready to prove the Theorem 3.6.2. We will call hyperosculating hyperplane an hyperplane that has intersection of multiplicity grater than $n-1$ with the curve C at a point $x \in C$.

Proof. By assumption, the subspace L is non uniform. Hence (see Lemma 1.6.5) there should be at least two distinct branch points b_1 and b_2 of π_L that are not simple, i.e. that do not correspond to a tranposition in the monodromy group $M(\pi_L)$. The hyperplanes corresponding to b_1 and b_2 are singular points of $C^* \subset (\mathbb{P}^n)^*$ and so the line $\mathbb{P}^1_L = L^*$ meets the singular locus C^*_{sing} of C^* in at least two distinct points.

Suppose first that $\mathbb{P}^1_L$ meets C^*_{sing} in more than two distinct points. Consider the curve C^*_{sing} taken with its reduced scheme structure, and apply lemma 3.6.4 to the variety $\Sigma^* = \{\mathbb{P}^1_L \mid L \in \Sigma\}$.
We're looking for honest trisecant to C^*_{sing}, therefore $\Sigma^* = \sigma(H)$. Coming back to the duals, $\Sigma = \sigma(x)$, i.e. all the $(n-2)$-planes passing trough a point x, where x is the dual of the hyperlpane H. This point is not birational due to Theorem 3.5 in [PS].

Suppose now that π_L ramifies exactly over b_1, b_2; the branching weight of the b_i is at most $d-1$, so by Riemann-Hurwitz

$$2g(C) - 2 = d\left(2g(\mathbb{P}^1) - 2\right) + R \Rightarrow R = 2d - 2 + 2g(C) \leq 2d - 2$$

providing that the curve C is rational and π_L is given in some coordinates by $z \mapsto z^d$. Thus L is the intersection of the two hyperplanes H_1 and H_2 corresponding to b_1 and b_2 respectively, each of them meeting C at a single point with multiplicity d.

As L vary in a family of dimension $2n-4$, there must be infinitely many such hyperplanes. We cannot have infinitely many hyperosculating hyperplanes to C and so $d \leq n$. The curve $C \subset \mathbb{P}^n$ is non degenerate, therefore $d = n$ and C is a rational normal curve. We are left with the case when, for a general L in Σ, the line $\mathbb{P}^1_L$ meets C^*_{sing} in exactly two points b_1, b_2, but it also contains a smooth point $b_3 \in C^*$, that correspond to a transpositon in the monodromy group of π_L. By assumptions L is non uniform and the monodromy group $M(\pi_L)$ contains a transposition, hence it is imprimitive (Theorem 1.1.7). Therefore the map factors non-trivially as

$$C \xrightarrow{\gamma} Y \xrightarrow{a} \mathbb{P}^1$$

If the map a ramifies over more than two distinct points of $\mathbb{P}^1_L$, these points would all be singular points of C^* contraddicting the assumptions. Therefore $Y \cong \mathbb{P}^1$ is rational and, up to a choice of coordinates, a is the map $z \mapsto z^\alpha$ where $\alpha = \deg(a) \geq 2$. As before, we cannot have $\alpha \geq n+1$ because this would imply the existence of infinitely many hyperosculating planes to C. The degree of γ is at least 2, then the degree of a should be strictly lower than n

because hyperplanes cannot osculate in more than one point. All the integers $2 \leq \alpha \leq n-1$ are possible values for the degree of a.

If Σ is not a Schubert cycle $\sigma(x)$, then C is a curve with special monodromy. We claim that also C is rational. Let H be a general hyperplane. For every couple of effective divisors E_1, E_2 of degree $g = \deg(\gamma)$ on C, there exist $L \in \Sigma$ such that $L \subset H$ and the projection from L factors $C \xrightarrow{\gamma} Y \xrightarrow{a} \mathbb{P}^1$, where E_1, E_2 are fibres of γ. In particular, they are linearly equivalent. Given P_1, P_2 distinct points in $C \cap H$, we choose F_1, F_2 of degree $g-1$ such that $P_1 + P_2 + F_1 + F_2 \leq D_H$, where $D_H = P_1 + \ldots + P_d$, $P_1, \ldots, P_d \in C \cap H$. Then $P_i + F_i \sim P_j + F_j$ $i, j \in \{1, 2\}$, so $2P_1 \sim 2P_2$. This holds for a general hyperplane H and every pair of distinct points in $C \cap H$, so we can find a point $P \in C$ such that $Q - P$ is a 2-torsion point in the Jacobian for infinitely many points Q. Therefore C is a rational curve with special monodromy. $\square$

3.6.2 Rational curves with special monodromy

The aim of this part is to look for examples of smooth rational curves with special monodromy. Pirola e Schlesinger proved that the only examples in $\mathbb{P}^3$ are the general quartic and special sextics ([PS, Theorem 5.9]). The study of smooth curves with special monodromy in higher dimension is not complete. Here we present some result in the direction of a classification of such curves in $\mathbb{P}^4$.

What follows in this Section is a generalization of [PS, Section 5].

Let $C \subset \mathbb{P}^n$ be a smooth rational curve with special monodromy of degree $\alpha d \geq n$. Let $V \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\alpha d))$ be the $(n+1)$-dimensional space that defines the embedding

$$C \hookrightarrow \mathbb{P}^n = \mathbb{P}(V^*)$$

i.e. to each point we associate the polynomials in V that vanish at the point.

Set $\mathbb{P}_d := \mathbb{P}\left(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))\right)$ the projective space of polynomials of degree d and

$$q_\alpha : \mathbb{P}_d \to \mathbb{P}_{\alpha d}$$

the embedding given by $q_\alpha(f) = f^\alpha$ with $2 \leq \alpha \leq n-1$. This are the only cases that we have to study thanks to the proof of Theorem 3.6.2. Let $X := q_\alpha(\mathbb{P}_d)$ be the image of the map q_α.

Proposition 3.6.5. *Let $C \subset \mathbb{P}(V)$ be a smooth rational curve of degree $\alpha d \geq n$ with special monodromy. Then the intersection $(X \cap \mathbb{P}(V)) \subset \mathbb{P}_{\alpha d}$ has dimension at least $n-2$.*

Proof. Since C has special monodromy, there exists a codimension two family $\Sigma \subset \mathbb{G}(1, \mathbb{P}(V^*))$ of $(n-2)$ planes such that, for every $L \in \Sigma$, the projection π_L factors through a degree α morphism $\beta_L : \mathbb{P}^1 \to \mathbb{P}^1$. Up to a choice of coordinates, β_L is the map $z \mapsto z^\alpha$ and so the line $\mathbb{P}^1_L \subset \mathbb{P}(V) \subset \mathbb{P}_{\alpha d}$ intersects X in at least two points, that are points in the branch. Thus the surface $\Sigma' := \{\mathbb{P}^1_L \mid L \in \Sigma\}$ is contained in the set of lines secant to $X \cap \mathbb{P}(V)$.

If we take two points on $(X \cap \mathbb{P}(V))$, then there exists a secant line passing through them; therefore $2\dim(X \cap \mathbb{P}(V)) \geq \dim\{\text{secants to}(X \cap \mathbb{P}(V))\ \}$. Furthermore $\dim(\Sigma') = 2(n-2)$, hence $X \cap \mathbb{P}(V)$ is at least $n-2$ dimensional:

$$\dim(X \cap \mathbb{P}(V)) \geq \frac{1}{2}\dim(\Sigma') = n-2.$$

□

Proposition 3.6.5 for $n = 4$ says that the $\dim(X \cap \mathbb{P}(V)) \geq 2$. Therefore we will consider the case in which a two dimensional subvariety is contained in $X \cap \mathbb{P}(V)$.

α=**2.** Let us consider $C \subset \mathbb{P}^4$ of degree $2d$ and $q := q_2 : \mathbb{P}_d \to \mathbb{P}_{2d}$.

Lemma 3.6.6. *Suppose $S \subset \mathbb{P}_d$ is a reduced irreducible surface such that $q(S)$ spans a four-dimensional linear subspace $\mathbb{P}(V)$ of $\mathbb{P}_{2d}$. Then $S = \mathbb{P}^2$ and $d = 2$.*

Proof. Consider a surface $S \subset \mathbb{P}_d$ such that $\langle q(S) \rangle \cong \mathbb{P}^4 = \mathbb{P}(V)$. A general hyperplane H in $\mathbb{P}_{2d}$ cuts $q(S)$ on a curve that spans a $\mathbb{P}^3$ by the hypotesis on S. The pullback of the hyperplane H is a quadric Q since $q^*\mathcal{O}(1) = \mathcal{O}(2)$. Therefore $Y = S \cap Q$ is an irreducible curve by Bertini's Theorem 2.2.1. Via this construction, the curve Y is one of those classified in [PS, Proposition 5.6]. Using that classification, the only two possibilities are $\deg(Y) = 2$ or $\deg(Y) = 3$:

- $\deg(Y) = 3$ is not possible since $\deg(Q) = 2$, so we cannot have a odd degree for Y;
- if $\deg(Y) = 2$, then $S = \mathbb{P}^2$ because $\deg(Y) = 2\deg(S)$.

Therefore we are left with $S = \mathbb{P}^2 \subseteq \mathbb{P}_d$ and we claim that it must be $d = 2$.

According to the Eisenbud/Hopf proposition in [PS, Prop 5.5], if $\mathbb{P}^2 \subset \mathbb{P}_d$ is such that $\dim\langle q(\mathbb{P}^2)\rangle = 4$, then there exist two linearly independent forms F, G of degree a such that $\mathbb{P}^2 = \mathbb{P}(< hF^2, hFG, hG^2 >)$ where $h \in \mathbb{P}_{d-2a}$.
Hence $< q(\mathbb{P}^2) >= \mathbb{P}(< h^2F^4, h^2F^3G, h^2F^2G^2, h^2FG^3, h^2G^4 >)$ is a linear system that defines a morphism $\mathbb{P}^1 \to \mathbb{P}^4$ whose image is a rational normal quartic T in $\mathbb{P}^4$. On the other hand, $\mathbb{P}(V) \cong \mathbb{P}^4 =< q(\mathbb{P}^2) >$ and V induces the embedding $C \hookrightarrow \mathbb{P}^4 = \mathbb{P}(V^*)$; hence $\deg(C) = \deg(T) = 4$, i.e. $d = 2$. □

The following Lemma is the converse of Proposition 3.6.5.

Lemma 3.6.7. *Let $C \subset \mathbb{P}(V^*)$ be a smooth rational curve of degree $2d \geq 4$ and let the dimension of $X \cap \mathbb{P}(V)$ be grater than one. Then C has special monodromy.*

Proof. Assume that there exists a surface S in $X \cap \mathbb{P}(V)$ such that $\langle S \rangle \cong \mathbb{P}(V)$. Hence, also the lines secant to S are not all contained in a hyperplane. Let Σ' be the codimension two subvariety in $\mathbb{G}(1, \mathbb{P}(V))$ of lines secant to S; let Σ be the corresponding subvariety in $\mathbb{G}(1, \mathbb{P}(V^*))$.

Since Σ' is not contained in a hyperplane, the dual Σ is not the Schubert cycle of a point. By construction, a general line in Σ' is base point free, so its dual L in Σ does not meet C and the projection from L is decomposable (Theorem 3.6.2). Therefore, C has special monodromy. □

Finally we have:

Lemma 3.6.8. *Let $C \subset \mathbb{P}^4$ be a rational curve of degree $2d \geq 4$ with special monodromy. Then $\deg(C) = 4$. Moreover, every rational quartic has special monodromy.*

Proof. Since C is a quartic, $d = 2$ and $q_2(\mathbb{P}_2) = X \subset \mathbb{P}^4$. Therefore $X \cap \big(\mathbb{P}(V) \cong \mathbb{P}^4\big) = X$ that is a surface. Then we can apply Lemma 3.6.7 and obtain that C has special monodromy. □

α=**3.** The other case we have to consider for smooth rational curves in $\mathbb{P}^4$ is $\alpha = 3$. We were not able to have a complete classification in this situation, but we present some result in that direction.

Let $C \subset \mathbb{P}^4$ be a rational curve of degree $3d \geq 4$ and let $q_3 : \mathbb{P}_d \to \mathbb{P}_{3d}$ be a map and let as before X be its image. Similarly to Lemma 5.3 [PS] we can prove the following property.

Lemma 3.6.9. *The image $X := q_\alpha(\mathbb{P}_d)$ has no trisecants for $\alpha \geq 3$.*

Proof. By contradiction, let $f^\alpha, g^\alpha, h^\alpha \in X$ be three alligned points, where f, g, h are polynomials in $\mathbb{P}_d$. We can assume that the trisecant line, defined by f, g and h, is a base point free linear series $g^1_{\alpha d}$ and so it defines a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree αd.

Since we are considering all the roots to be distinct, f, g and h are in the ramification divisor of f, each with multiplicity $(\alpha - 1)d$. By Riemman-Hurwitz we get

$$-2 \geq -2\alpha d + 3(\alpha - 1)d$$

Therefore we have $-2 \geq (\alpha - 3)d \geq 0$ since by assumption $\alpha \geq 3$. But this is a contradiction. □

By proposition 3.6.5 we look for a surface $S \subset \mathbb{P}_d$ such that $q_3(S) \subset \mathbb{P}_{3d}$.

Lemma 3.6.10. *The span $\langle q_3(S) \rangle$ is $\mathbb{P}(V) \cong \mathbb{P}^4$.*

Proof. If the surface $q_3(S)$ spans a $\mathbb{P}^3$ inside $\mathbb{P}(V)$, we can find infinitely many trisecant to $q_3(S) \subset X$ since it has degree grater than or equal to 3. But this contradicts Lemma 3.6.8. Therefore $\langle q_3(S) \rangle \cong \mathbb{P}^(V)$. □

Let $S \subset \mathbb{P}_d$ be a surface such that $\tilde{S} := q_3(S) \subset X \cap \mathbb{P}(V)$ with $\langle \tilde{S} \rangle \cong \mathbb{P}(V) \cong \mathbb{P}^4$. Given a point $s \in \mathbb{P}^1$, we denote by H_s in $\mathbb{P}_{3d}$ the hyperplane of forms vanishing at s.

Taking a general hyperplane section of $\tilde{S}$, we get a curve $\tilde{Y}$ whose span is a $\mathbb{P}^3$. Therefore, it makes sense to study curves $Y \subset \mathbb{P}_d$ with $\langle q_3(Y) \rangle \cong \mathbb{P}^3$.

Remark 3.6.11. The pullback of a general hyperplane via q_3 is a cubic Z in $\mathbb{P}_d$. If we pullback the hyperplane section, the curve Y cut by Z on S must have degree at least 3.

If we look for a curve $Y \subset \mathbb{P}_d$ with $\langle q_3(Y) \rangle \cong \mathbb{P}^3$, we claim that the degree of Y is at most 6. Indeed, reasoning as in Proposition 5.6 of [PS], for a general $s \in \mathbb{P}^1$, the intersection $H_s \cap Y$ is transverse. The tangent line $T_{f(s)}Y$ meets Y only at $f(s)$ and with multiplicity two. Hence projection from a general plane containing the tangent line is a degree $3d - 2$ morphism $\mathbb{P}^1 \to \mathbb{P}^1$. By Riemann Hurwitz

$$-2 = -2(3d - 2) + \deg(R) \geq -2(3d - 2) + \deg(Y)(d - 1)$$

so that $\deg(Y) \leq 6$.

Combining it with the previous observation, the degree of the curve Y on S is $3 \leq \deg(Y) \leq 6$ and it is cut by cubics. Therefore $\deg(Y) = 3$ or $\deg(Y) = 6$. This implies that

- $\deg(S) = 1$, i.e. $S \cong \mathbb{P}^2$ is linear in $\mathbb{P}_d$.
- $\deg(S) = 2$, i.e. S is a quadric surface in $\mathbb{P}_d$.

To have a complete classification of the smooth rational curves $C \subset \mathbb{P}^4$ of degree $3d \geq 4$ with special monodromy, it is left to verify if this degrees for S are in fact admissible.

Bibliography

[AO] Alzati A., Ottaviani G., *The theorem of Mather on generic projections in the setting of Algebraic Geometry*, Manuscr. Math., 74, 391-412, 1992.

[ACGH] Arbarello E., Cornalba M., Griffiths P.A. and Harris J., *Geometry of algebraic curves. Vol. I*, volume 267 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, New York, 1985.

[AP] Artebani M., Pirola G.P., *Algebraic functions with even monodromy*, Proceedings of the American Mathematical Society 133(2), January 2004

[CaCi] Calabri A., Ciliberto C., *On special projections of varieties: epitome to a theorem of Beniamino Segre*, Adv. Geom. 1 (2001), 97-106

[Ch] Chisini O., *Courbes de diramation des planes multiples et tresses algebrique*, Deuxieme Colloque de Géométrie Algebrique (Liege, 1952), Masson, Paris, 1952.

[CiC] Ciliberto C., Chiantini L., *A few remarks on the lifting problem*, Journées de géometrie algébrique d'Orsay, 1992; Astérisque, no. 218, Société Matématique de France, 1993, pp. 95-109

[CF] Ciliberto C., Flamini F., *On the branch curve of a general projection of a surface to a plane*, Trans. Amer. Math. Soc., 363(7):3457–3471, 201.

[CS] Ciliberto C., Sernesi E., *Singularities of the theta divisor and congruences of planes*, J. Alg. Geom. 1, 1992, n.2, 231-250.

[Cu] Cukiermann F., *Monodromy of projections*, Mat. Contemp. 16:9–30, 1999. 15th School of Algebra (Portuguese), Canela, 1998.

[CMS] Cuzzucoli A., Moschetti R. and Serizawa M., *Non-uniform projections of surfaces in* $\mathbb{P}^3$, Le matematiche, LXXII, 2017.

[De] Deligne P., *Le groupe fondamental du complement d'une courbe plane n'ayant que des points doubles ordinaires est abelien (d'apres W. Fulton)*. Bourbaki Seminar, Vol. 1979/80, pp. 1–10, Lecture Notes in Math., 842, Springer, Berlin-New York, 1981.

[Di] Dimca A., *Singularity and Topology of Hypersurfaces*, Springer-Verlag, New York, 1992.

[Doh] Doherty D.C., *Singularities of generic projection hypersurfaces*, Proceedings of the American Math. Soc., Vol. 136, no. 7, 2008, pag. 2407-2415.

[Do] Dolgachev I. V., *Classical Algebraic Geometry: a modern view*, Cambridge University Press, 2012

[En] Enriques F., *Le superficie algebriche*, Zanichelli, Bologna, 1949.

[En2] Enriques F., *Sulla construzione delle funzioni algebriche di due variabili possendenti una data curve diramazione*, Annali di Matematica pure ed applicata, Ser. 4, Vol. 1, 1923, p.185-198.

[Fr] Franchetta A., *Sulla curva doppia della proiezione della superficiegenerale dell'S^4, da un punto generico su un S^3*,Rend. Acc. d'ItaliaSer. VII2(1940), 282–288 andLincei - Rend. Sc. fis. mat.nat.2(1947), 276–279.

[Ful] Fulton W., *On the fundamental group of the complement of a node curve*, Ann. of Math. (2) 111 (1980), no. 2, 407–409.

[FL] Fulton W., Lazarsfeld R., *Connectivity and its applications in Algebraic Geometry*, Algebraic Geometry, Lecture Notes in Math. 862, Springer-Verlag, New York, 1981, pp. 26-92.

[Fu] Fukasawa S., *Complete determination of the number of Galois points for a smooth plane curve*, Rend. Semin. Mat. Univ. Padova, 129:93–113, 2013.

[FT] Fukasawa S. and Takahashi T., *Galois points for a normal hypersurface*, Trans. Amer. Math. Soc., 366(3):1639–1658, 2014.

[GH] Griffiths P., Harris J., *Principles of Algebraic Geoemtry*, New York, Wiley, p.813, 1978.

[HT] Hamm H.A., Trang L.D., *Un Théorème de Zariski du type de Lefschetz*, Annales scientifiques de l'E.N.S., 4 série, tome 6, no. 3 (1973), p. 317-355.

[Ha] Harris J., *Algebraic Geometry: A First Course*, Graduate Texts in Math. 133, Springer-Verlag, New York, 1992.

[Ha2] Harris J., *Curves in Projective Space*, Se. de Mathématiques Superieures. Université de Montreal, 1982 (with the collaboration of D. Eisenbud).

[Ha3] Harris J., *Galois groups of enumerative problems*, Duke Math. Journal, 46(4):685–724, 1979.

[Ha4] Harris J., *The genus of space curves*, Math. Ann., 249 (1980), 191-204

[Hart] Hartshorne R., *Algebraic Geometry*, Springer-Verlag, New York - Heidelberg, 1977, Graduate Texts in Mathematics, No. 52

[Hi] Hirsch M.W., *Differential Topology*, Graduate Texts in Mathematics 33, Springer-Verlag, Berlin, 1976.

[Is] Isaacs M. I., *Finite Group Theory*, Vol.92, Graduate studies in mathematics, American Mathematical Soc., 2008.

[Jo] Jouanolou J.P., *Théorèmes de Bertini et applications*, Series de Mathématiques Pures et Aplliquées, I.R.M.A. Strasbourg, 1979.

[Kl] Kleiman S., *Bertini and his two fundamental Theorems*, Rend. Circ. Mat. Palermo.

[La] Lazarsfeld R., *Positivity in Algebraic Geometry I*, Springer-Verlag, vol.48.

[MNP] Martinelli D., Naranjo J.C., Pirola G.P., *Connectedness Bertini Theorem via Numerical equivalence*, Advances in Geometry, Vol.17, (2017), no.1, 31-38.

[Ma] Mather J.N., *Generic projections*, Ann. of Math. 2, 98:226–245, 1973.

[MM] Matsumura H., Monsky P., *On the automorphisms of hypersurfaces*, J. Math. Kyoto Univ. 3,(1963/1964), 347–361.MR 29

[MP] Mezzetti M.E., Portelli D., *A tour through some classical theorems on algebraic surfaces.* An. Stiint. Univ. Ovidius Constanta Ser. Mat.,5(1997), no.2, 51–78

[Mi] Miranda R., *Algebraic curves and Riemann surfaces*, volume 5 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1995.

[Miu] Miura K., *Field theory for function fields of plane quintic curves*, Algebra Colloq., 9(3):303–312, 2002.

[MY] Miura K., Yoshihara H., *Field theory for function fields of plane quartic curves*, J. Algebra, 226(1):283–294, 2000.

[Na] Namba M., *Branched coverings and algebraic functions*, Pitman Res. Notes Math. Ser., Vol. 161.

[Noma] Noma A., *Projective varieties with non birational linear projections and applications*, Transactions of the American Mathematical Society, Volume 370, Number 4, April 2018, Pages 2299–2320.

[No] Nori M.V., *Zariski's conjecture and related problems*, Ann. Sci. École Norm. Sup., 16(2):305–344, 1983.

[PS] Pirola G.P., Schlesinger E., *Monodromy of projective curves*, J. Algebraic Geom., 14(4):623–642, 200.

[Ru] Russo F., *On the Geometry of Some Special Projective Varieties* , Springer-Verlag, 2016.

[Se] Segre C., *Un' osservazione sui sistemi di rette degli spazi superiori*, Opere, a cura dell'UMI, IV, 1963.

[Se1] Segre C., *Sui fochi di 2 ordine dei sistemi infiniti di piani e sulle curve iperspaziali con una doppia infinità di piani plurisecanti*, Rend. R. Acc. Naz. Lincei, Vol. 30 (1921), p. 67–71.

[Se2] Segre C., *Un' osservazione sui sistemi di rette degli spazi superiori.* Opere, a cura dell'UMI, IV, 1963.

[Ser] Sernesi E., *Deformation of algebraic schemes*, volume 334 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 2006.

[Sh] Shafarevic I.R., *Basic Algebraic Geometry 1*, Springer-Verlag, Berlin, 1994.

[VaK] van Kampen E. R., *On the fundamental group of an algebraic curve*, Amer. J. Math. 55, 1933.

[Yo] Yoshihara H., *Function field theory of plane curves by dual curves*, J. Algebra, 239(1), 340–355, 2001.

[Yo2] Yoshihara H., *Galois points for smooth hypersurfaces*, J. Algebra,264(2):520–534, 2003.

[Za] Zak F.L., *Tangents and secants of algebraic varieties*, volume 127 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1993. Translated from the Russian manuscript by the author.

[Za1] Zak F. L., letter, December 26 1979.

[Za2] Zak F. L., *Tangents and secants of algebraic varieties.* Translations of Mathematical Monographs, vol. 127 (American Mathematical Society, Providence, RI, 1993).

[Zar] Zariski O., *A theorem on the Poincaré group of an algebraic hypersurface*, Ann. of Math. (2), 38(1):131–141, 1937.

[Zar1] Zariski O., *A Theorem on the Poincare Group of an Algebraic Hypersurface.* Annals of Mathematics, Second Series, Vol.38, No.1 (Jan., 1937), pp.131-141.

[Zar2] Zariski O., *On the Problem of Existence of Algebraic Functions of Two Variables Possessing a Given Branch Curve*, American Journal of Mathematics, Vol. 51, No. 2 (Apr., 1929), pp. 305-328.

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