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An intrinsic approach to the c-map

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To my grandparents and my granduncle,
who passed away during this journey.

Contents

Introduction

The main field around which this work revolves is quaternion Kähler geometry, the area of differential geometry studying quaternion Kähler manifolds. Such manifolds are defined by the property of being Riemannian with holonomy contained in $\mathrm{Sp}(n)\mathrm{Sp}(1)$ but not in $\mathrm{Sp}(n)$. Here $\mathrm{Sp}(n)$ is the unitary quaternionic Lie group, and $\mathrm{Sp}(1)$ acts by quaternionic scalar multiplication on the fibre of the tangent bundle, when said fibre is identified with $\mathbb{H}^n \cong \mathbb{R}^{4n}$.

These Riemannian manifolds are interesting for several reasons: first of all they are Einstein, so the Ricci curvature is a scalar multiple of the metric. Einstein metrics have many significant properties: they generalise metrics with constant sectional curvature, and more generally irreducible symmetric spaces; they arise as critical points of a natural functional, namely the total scalar curvature, and are often regarded as optimal elements in the space of metrics on a given manifold (see [8]). Einstein manifolds also play an important role in general relativity, as they solve Einstein's field equation in vacuum, and the scaling factor between the Ricci tensor and the metric is a strictly related to the cosmological constant. In particular, for quaternion Kähler manifolds, this constant is either positive or negative, but never zero.

The sign of this constant, which coincides with the scalar curvature, produces two entirely different geometries. In particular, for positive scalar curvature, we know from [35] that in every dimension there is a finite number of complete quaternion Kähler manifolds. Moreover, in this case there is a conjecture by LeBrun and Salamon, claiming that every complete quaternion Kähler manifold with positive scalar curvature is also a symmetric space (i.e. what is called a Wolf space). Said conjecture has so far been proven only up to dimension 8 ([42]).

Quaternion Kähler manifolds first make their appearance in 1955 in the classification theorem of Berger ([7], see Theorem 1.3.48 below). This theorem classifies simply connected, locally irreducible, non-locally symmetric Riemannian manifolds by their holonomy group. Said manifolds can be considered as fundamental building blocks for non-locally symmetric Riemannian manifolds. It turns out that not all compact Lie groups can be the holonomy of one of these blocks; instead, the ones that can are grouped in seven families: two generic ones (special orthogonal and unitary group) and five special ones. Among the special ones is where we can find $\mathrm{Sp}(n)\mathrm{Sp}(1)$, corresponding to quaternion Kähler manifolds. Whilst for the manifolds having as holonomy one of the two generic groups we do not have any particular condition on the curvature, manifolds with special holonomy are necessarily Einstein; however, among these, only quaternion Kähler manifolds have non vanishing Ricci tensor.

The problem with quaternion Kähler manifolds is that it is difficult to find them. Early examples appear in Cartan's classification of symmetric spaces, a most notable example being the so-called quaternionic projective space:

$$\mathbb{P}^n_{\mathbb{H}} = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\mathrm{Sp}(1)}.$$

In 1965, in [48], Wolf provides an explicit construction and a characterisation of quaternion Kähler symmetric spaces. For the first non symmetric spaces, we have to wait until Alekseevsky and Cortés ([1], [14]) classify the so called Alekseevskian spaces, i.e. quaternion Kähler manifolds with a simply transitive real solvable group of isometries. In this classification, the first homogeneous non-symmetric example is provided.

An important tool used to build quaternion Kähler manifolds of negative scalar curvature is the so called c-map ([10],[18], [5], [4], [37], [31]). It is a construction arising from supergravity and string theory, which produces a quaternion Kähler manifolds from a projective special Kähler one. It was first introduced by Cecotti in [10]; later, Ferrara and Sabharwal produced a coordinate description of the metrics involved in the construction [21]. The c-map played an important role in the completion of the classification of Alekseevskian spaces and recently it was used by Cortés et al. to obtain examples of quaternion Kähler manifolds in cohomogeneity one ([20]).

The c-map was first mathematically formalised in its local form only in 2009 by the hands of Hitchin ([30]). In [5] and [4], it is proved that the c-map can be described in terms of the so-called rigid c-map and a more general construction introduced by Haydys, called the HK-QK correspondence [26], and extended in [5] and [4] to manifolds with indefinite metrics. A global description was finally given in 2015 by Macia and Swann in [37]. Here, they also suggest a different approach to the last step of the c-map, as they show that the HK-QK correspondence can be replaced by a different general construction due to Swann, called the twist [46].

As mentioned before, the initial data of the c-map are the so called projective special Kähler manifolds. They are a special class of Kähler quotients of conic special Kähler manifolds, which is a class of pseudo-Kähler manifolds endowed with a symplectic, flat, torsion-free connection and an infinitesimal homothety. Explicit examples can be found in [2], where homogeneous projective special Kähler manifolds of semisimple Lie groups are classified. A notable case appearing in this list is the complex hyperbolic n-space. Many projective special Kähler manifolds can be constructed via the so called r-map ([18]), which is a construction also arising from supergravity and string theory allowing to build a projective special Kähler manifold starting essentially from a homogeneous cubic polynomial. We refer to [16] for a classification of 6-dimensional manifolds that can be constructed via the r-map. Another example of projective special Kähler manifold is obtained by taking the Weil-Petersson metric on the space of complex structure deformations on a Calabi-Yau 3-dimensional manifold [15].

Projective special Kähler manifolds appear in the study of supergravity and mirror symmetry with the name of *local special Kähler manifolds* (see [22] and [23] for more details on their story and applications to physics, and in particular [9] for their importance in mirror symmetry). The name projective special Kähler was given by Freed in [23] where he also shows how such manifolds are quotients of special Kähler ones ([23, Proposition 4.6, p.20]) (see e.g. [3] for the relation between this definition and the one we will use in this work). There is in particular an extrinsic construction ([3]) of simply connected projective special Kähler manifold of a certain dimension n, where the latter can be realised via an immersion in $T^*(\mathbb{C}^n)$ with its flat connection and standard complex structure.

The ultimate motivation behind this work is the construction of new quaternion Kähler manifolds. we aim to do so by applying the c-map to projective special Kähler manifolds; since the definition of special Kähler manifolds is rather unwieldy, this motivates us to look for a better way to describe them.

In this work, we manage to reduce the definition of projective special Kähler manifold to the data of a circle bundle $S \to M$ with a certain connection, and a 2-homogeneous bundle map $\gamma\colon S \to \sharp_2 S_{3,0}$. The bundle $\sharp_2 S_{3,0}$ is isomorphic to the bundle of symmetric holomorphic tensors, so γ can be interpreted locally as a homogeneous polynomial of degree 3 with complex functional coefficients. In the examples built via the r-map, this polynomial appears to be related to the polynomial forming the initial data for the r-map. We call γ the *deviance*, as it represents the obstruction of a projective special Kähler manifold to being the complex hyperbolic n-space, which now can be thought of as a distinguished case. Using the deviance, we manage to classify 4-dimensional Lie groups with a projective special Kähler structure. Moreover, we observe that for all $\beta\colon M \to \mathrm{U}(1)$, $\beta\gamma$ provides another projective special Kähler structure on the same manifold, which does not necessarily induce an isomorphism on the respective conic special Kähler manifolds.

Notice that giving γ is equivalent to giving a suitable family of compatible local sections for $\sharp_2 S_{3,0}M$. Thus, our characterisation is intrinsic in the sense that we reduce the projective special Kähler structure to data solely defined on the manifold itself and satisfying conditions on the manifold itself. We prove a lower bound for the scalar curvature, which is reached exactly when the deviance is zero; this condition characterises projective special Kähler manifolds isomorphic to the complex hyperbolic n-space if one assumes the manifold to be complete, connected and simply connected. Moreover, this characterisation provides a simpler way to build projective special Kähler manifolds, and we display this by classifying all possible projective special Kähler structures on 4-dimensional Lie groups. We note that an intrinsic characterisation of projective special Kähler Lie groups has been obtained independently in a very recent paper by Macia and Swann [38]. In this paper it is also shown that projective special Kähler Lie groups determine quaternion Kähler Lie groups via the c-map, if one assumes the exactness of the Kähler form and the invariance of the flat connection. A similar result, holding in the case that the projective special Kähler Lie group is the quotient of an affine special Kähler domain, can be deduced from the more general result [20, Corollary 24, p. 33].

Concerning the c-map, we build explicit invariant coframes for three quaternion Kähler Lie groups obtained via the c-map, that are isometric respectively to the following quaternion Kähler solvmanifolds

$$\mathrm{G}_2^*/\mathrm{SO}(4), \qquad \mathrm{SO}_0(4,3)/\mathrm{SO}(4)\mathrm{SO}(3), \qquad \mathrm{SU}(3,2)/S(\mathrm{U}(3)\mathrm{U}(2)).$$

Consistently with the results of [38], we obtain left-invariant quaternion Kähler (in particular Einstein) metrics on three solvable Lie groups. We also give an explicit realisation of said Lie groups as the Borel subgroups of the respective groups of isometries; thus inducing the isomorphism with the respective symmetric space. Finally, we manage to modify the invariant coframe in order to obtain one which is not only invariant, but also adapted to the quaternion Kähler structure. We also study the general case of the c-map applied to a complex hyperbolic n-space; if we write the latter with respect to an invariant coframe, we manage to show the structure constants of one of the possible quaternion Kähler Lie groups. We show the explicit structure constants only for $n = 2$.

Content of chapters

Chapter 1

In the first chapter, we present the general theory, starting from a brief resume of representation theory and the relation between vector bundles and representations. Then we proceed to present projective special Kähler and related manifolds, along with quaternion Kähler ones, with an introduction on quaternionic groups, in order to set the notation and the immersions in the group of invertible real matrices. There are also two presentations of the c-map: an intrinsic and global one, following the presentation of [37] and a local coordinate one that follows [18]. Then we present the r-map and recall a completeness result presented in [18]. After we have recalled all the necessary preliminaries, we can now use these constructions in order to build examples of invariant coframes on the resulting quaternion Kähler manifolds, and use the induced structure constants to explicitly identify these manifolds. We then modify the resulting coframe in order to obtain one which is also adapted to the quaternion Kähler structure.

Chapter 2

Here we go through the details of the author's paper [39]. We start with the analysis of the symmetries of the difference tensor between the flat and the Levi-Civita connection associated with a projective special Kähler manifold. Then we describe the metric of a conic special Kähler manifold in terms of the pullback of the projective special Kähler metric, and we observe that it has the form of a conic metric up to changes in signature. We proceed by giving a procedure to lift a unitary coframe on a projective special Kähler manifold to a unitary coframe on the corresponding conic special Kähler manifold. Then we use this lift to write the Levi-Civita connection on the conic special Kähler manifold and its curvature in terms of the ones on the projective special Kähler manifold. Locally, we encode the data of the difference tensor into tensors defined on open subsets of the projective special Kähler manifold. These data or their global version will be what we call deviance. We then transfer the conditions of conic special Kähler manifold to new conditions on the projective special Kähler manifold, which allows us to formulate a characterisation theorem (Theorem 2.5.6). We also compute the Ricci tensor and scalar curvature of a projective special Kähler manifold with respect to the deviance, giving a lower bound on the scalar curvature of a projective special Kähler manifold. Finally we use the deviance and the characterisation theorem to generate new projective special Kähler metrics, not necessarily equivalent to the original one. The chapter ends with a study of the complex hyperbolic n-space, characterised as the only projective special Kähler manifold with zero deviance.

Chapter 3

In the last chapter we apply the characterisation of projective special Kähler manifolds in terms of deviance to classify projective special Kähler Lie groups in dimension 4. Then we compute the deviance in known examples and finally we give a way to lift the coframe from a complex hyperbolic n-space to the corresponding quaternion Kähler manifold obtained via the c-map. The chapter ends with the explicit computation in the case of the complex hyperbolic 2-space.

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Notation

We denote by $\mathbb{R}^+$ the set of strictly positive real numbers.

Matrices

Given a ring R, we denote the space of p, q matrices with entries in R by $\mathrm{Mat}(m, n, R)$ and if $m = n$, we simply write $\mathrm{Mat}(n, R)$. The group of invertible matrices of a unitary ring R will be denoted by $\mathrm{GL}(n, R)$. If $R = \mathbb{R}$, we call $\mathrm{GL}^+(n, \mathbb{R})$ the subgroup of $\mathrm{GL}(n, \mathbb{R})$ of matrices with positive determinant. The identity $n \times n$-matrix is denoted with I_n. Given a matrix $A = (A^h_k)_{h,k}$, we denote by A^h_k the entry in row h and column k. We denote the transpose of a matrix A by A^t and if A has complex or quaternionic entries, we will denote by $A^\star$ its adjoint, i.e. its conjugate transpose.

Tensors

We will mostly adopt the Einstein notation for repeated indices, meaning that whenever we have a repeated index in a single term appearing as upper and lower index, then summation over that index is implied, that is

$$\alpha_k \beta^k = \sum_k \alpha_k \beta^k.$$

Let A, B be tensors of the same type; we define

$$\begin{aligned} A \wedge B &:= A \otimes B - B \otimes A \\ AB &:= \frac{1}{2}(A \otimes B + B \otimes A). \end{aligned}$$

We adopt the following notation for differential forms: $e^{i,j} := e^i \wedge e^j$. Given a vector field X on a manifold M and a differential form $\omega \in \Omega^k(M)$, we denote by $\iota_X \omega := \omega(X, \cdot, \dots, \cdot)$ the contraction of ω with X. We use the same notation with a tensor α with at least one covariant component and $\iota_X \alpha$ will be the contraction of α with X in the first component.

Given $\alpha \otimes A \in \Omega^k(M, \mathfrak{g})$ and $\beta \otimes B \in \Omega^h(M, \mathfrak{g})$, we define the form $[(\alpha \otimes A) \wedge (\beta \otimes B)] := \alpha \wedge \beta \otimes [A, B] \in \Omega^{k+h}(M, \mathfrak{g})$.

Groups and actions

Given a group G and a set X, if G acts on X on the right (respectively on the left) we will denote by R_g (respectively L_g) the action of the element $g \in G$ on X, that is

$$\begin{aligned} R_g \colon X &\longrightarrow X \\ x &\longmapsto x.g \end{aligned} \qquad\qquad \begin{aligned} L_g \colon X &\longrightarrow X \\ x &\longmapsto g.x \end{aligned}$$

On a Lie group G with Lie algebra $\mathfrak{g}$, we denote the *adjoint representation* of G by $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$. Namely, for all $g \in G$, $\mathrm{Ad}(g)$ is the differential of the map

$$\begin{aligned} G &\longrightarrow G \\ x &\longmapsto gxg^{-1} \end{aligned}$$

The differential of Ad, hence the adjoint representation of $\mathfrak{g}$, is instead denoted by $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$, that is $\mathrm{ad}(X) = [X, \cdot]$.

Given a Lie group G acting on a manifold M, let A an element of the Lie algebra of G, then we denote by A° the fundamental vector field associated to A, that is, for all $p \in M$:

$$A^\circ_p = \frac{d}{dt}(p \exp(tA))|_{t=0}.$$

Principal bundles

Let M be a differentiable manifold of dimension n, we denote by $\mathrm{GL}(M)$ the principal $\mathrm{GL}(n, \mathbb{R})$-bundle of frames, simply called *frame bundle*, namely

$$\mathrm{GL}(M) := \{u_p : \mathbb{R}^n \to T_pM | u_p \text{ is a linear isomorphism}\},$$

equipped with the projection mapping u_p to p and the right action of $A \in \mathrm{GL}(n, \mathbb{R})$ defined by $u_p.A := u_p \circ A$.

Given a generic bundle $p\colon P \to M$ and an open subset with inclusion map $\iota_U \colon U \hookrightarrow M$, we will denote the set of sections $U \to P$ by

$$\Gamma(U, P) := \{s\colon U \to P | p \circ s = \iota_U\}.$$

Complex vector bundles

We denote by $T^{1,0}M$ the complex holomorphic tangent bundle and by $T^{0,1}M$ the antiholomorphic tangent bundle of the manifold M; by $T^*_{1,0}M = \Lambda_{1,0}M$ the complex holomorphic cotangent bundle and by $T^*_{0,1}M = \Lambda_{0,1}M$ the anti-holomorphic cotangent bundle. Finally, if we denote by $\Lambda^k E$ the k-th skew-symmetric tensor power of vector bundles, representations or vector spaces and by $S^k E$ the symmetric one, we can also define

$$\Lambda_{p,q}M := \Lambda^p T^*_{1,0}M \otimes \Lambda^q T^*_{0,1}M$$
$$S_{p,q}M := S^p T^*_{1,0}M \otimes S^q T^*_{0,1}M$$

We adopt a similar notation for the corresponding representations in $\mathfrak{u}(p,q)$ (see Section 1.1).

Chapter 1

From projective special Kähler to quaternion Kähler manifolds

In this chapter we are introducing the basic objects that we are going to discuss in this work, namely projective special Kähler, conic special Kähler, hyperKähler and quaternion Kähler manifolds. In the central part of this chapter we are going to discuss the constructions known as the r-map and the c-map and in the final part we use these constructions to build quaternion Kähler manifolds. In particular, we find an invariant coframe on the resulting manifolds, and from its structure constants we are able to directly identify the manifolds from the list of Alekseevskian spaces. Finally, on these manifolds, we build an invariant coframe which is also adapted to the quaternion Kähler structure.

1.1 Representations and bundles

We start by giving some basic notions of representation theory that we are using for this work, and we take the opportunity to introduce and fix some notations. We mostly refer to [44] for this section, with a few changes in the notation.

Let $\mathbb{K}$ be a field and consider a $\mathbb{K}$-vector space V. Let $G \to \mathrm{Aut}_{\mathbb{K}}(V)$ be a $\mathbb{K}$-representation of G, then we will often denote the representation by V, where the action of G is implied. We denote the left action of $g \in G$ on $v \in V$ by $g.v$.

Given a $\mathbb{K}$-representation V, we denote the *dual representation* by V^*, namely $\mathrm{Hom}_{\mathbb{K}}(V,\mathbb{K})$ with left action of $A \in G$ on $\alpha \in V^*$ defined as $A.\alpha := \alpha \circ A^{-1}$. If $V = \mathbb{K}^n$, we can canonically identify $\mathrm{Aut}_{\mathbb{K}}(V)$ with $\mathrm{GL}(n,\mathbb{K})$ and thus for every $G \subseteq \mathrm{GL}(n,\mathbb{K})$ we have the so-called *standard representation* $\mathbb{K}^n$, where the action is defined by matrix multiplication.

Suppose now $\mathbb{K} = \mathbb{C}$, we can also define the *conjugate representation* $\overline{V}$ by taking the the conjugate vector space with the same group action.

On a generic complex representation, a *real structure* is a $\mathbb{C}$-antilinear involution $\sigma\colon V \to V$. On the complex tensorial algebra of $\mathbb{C}^n$ there is a canonical real structure obtained by extending the complex conjugation $\mathbb{C}^n \to \overline{\mathbb{C}}^n$ and its inverse. A subspace of the tensorial algebra which is stable by the canonical real structure inherits the real structure by restriction. If $\mathbb{K} = \mathbb{R}$, we can transform a real representation V of G to a complex representation $V \otimes_{\mathbb{R}} \mathbb{C}$ of G by extension of scalars. In this case we have a natural real structure:

$$\sigma = \mathrm{id}_V \otimes_{\mathbb{R}} \overline{(\cdot)}\colon V \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow V \otimes_{\mathbb{R}} \mathbb{C}.$$

We now present the following notation taken from [44]: if V is a complex representation with a real structure σ, we define

$$[V] := \{v \in V | \sigma(v) = v\}.$$

Otherwise, for any complex representation V,

$$[\![V]\!] := [V \oplus \overline{V}]$$

. In particular, the following isomorphisms of complex representations hold:

$$[V] \otimes_{\mathbb{R}} \mathbb{C} \cong V$$
$$[\![V]\!] \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus \overline{V}$$

Unitary representations

We write U for either $\mathrm{U}(n)$ or $\mathrm{U}(p,q)$ depending on the signature. Notice that, at the level of Lie algebras, $\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{u} \otimes_{\mathbb{R}} \mathbb{C}$, so in order to study a unitary representation we can study a complex $\mathrm{GL}(n,\mathbb{C})$-representation with a suitable real structure.

We fix the following notation

$$\Lambda_{p,q} := \Lambda^p(\mathbb{C}^n)^* \otimes \Lambda^q(\overline{\mathbb{C}^n})^*;$$

$$S_{p,q} := S^p(\mathbb{C}^n)^* \otimes S^q(\overline{\mathbb{C}^n})^*.$$

In particular $\Lambda_{1,1} \cong S_{1,1}$.

Remark 1.1.1. *We consider only forms, since the hermitian product $h = g + i\omega$ (preserved U), provides the identifications $\mathbb{C}^n \cong \overline{(\mathbb{C}^n)}^*$ and $\overline{\mathbb{C}^n} \cong (\mathbb{C}^n)^*$.*

Notice that when $p = q$, both $\Lambda_{p,q}$ and $S_{p,q}$ have a real structure induced by the one on the complex tensorial algebra. We thus have real representations $[\Lambda_{p,p}], [S_{p,p}]$ for all p and $[\![\Lambda_{p,q}]\!], [\![S_{p,q}]\!]$ for all $p \neq q$.

These U-representations, in general, are not irreducible. Consider in fact the U-equivariant map $L\colon \Lambda_{p-1,q-1} \to \Lambda_{p,q}$ defined by wedging with ω. This map provides a splitting of representations

$$\Lambda_{p,q} = L(\Lambda_{p,q}) \oplus \Lambda^0_{p,q},$$

where $\Lambda^0_{p,q}$ is the orthogonal complement of $L(\Lambda_{p,q})$. The same can be done for symmetric forms with the map $S_{p-1,q-1} \to S_{p,q}$, by taking the symmetric product with g. We denote the complement of the image by $S^0_{p,q}$, which is an irreducible representation.

Lemma 1.1.2. *[44, Lemma 3.1, p. 33] We have the following decompositions in U-irreducible components for $p < q$, $p + q \leq n$:*

$$\begin{aligned}
[\![\Lambda_{p,q}]\!] &\cong [\![\Lambda^0_{p,q}]\!] \oplus [\![\Lambda^0_{p-1,q-1}]\!] \oplus \cdots \oplus [\![\Lambda_{0,q-p}]\!] \\
[\![\Lambda_{q,p}]\!] &\cong [\![\Lambda^0_{q,p}]\!] \oplus [\![\Lambda^0_{q-1,p-1}]\!] \oplus \cdots \oplus [\![\Lambda_{q-p,0}]\!] \\
[\Lambda_{p,p}] &\cong [\Lambda^0_{p,p}] \oplus [\Lambda^0_{p-1,p-1}] \oplus \cdots \oplus [\Lambda^0_{1,1}] \oplus \mathbb{R}\omega.
\end{aligned}$$

We obtain a similar result for symmetric representations $[\![S^0_{p,q}]\!]$ and $[S^0_{p,p}]$.

Fibre bundles

Let M be an n-dimensional manifold, G a Lie group, P a principal G-bundle and F a manifold with a smooth left action of G. We can build the *fibre bundle associated to* F:

$$P \times_G F := (P \times F)/\sim,$$

where $\sim$ is defined by $(u, v) \sim (u.g, g^{-1}.v)$ for all $g \in G$. We denote the class of (u, v) by $[u, v]$, and the projection $P \times_G F \to M$ maps $[u, v]$ to $p(u)$. Notice that $P \times_G F \to M$ is a bundle with fibre diffeomorphic to F.

A useful case is when $F = V$ is a representation of G. In this case, $P \times_G V$ is a vector bundle with fibre isomorphic to V. Examples are the tangent bundle $TM \cong \mathrm{GL}(M) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n$ and the cotangent bundle $T^*M \cong \mathrm{GL}(M) \times_{\mathrm{GL}(n,\mathbb{R})} (\mathbb{R}^n)^*$.

Remark 1.1.3. *[33, I,Example 5.2, p. 76] There is a bijective correspondence between sections of $P \times_G F \to M$ and equivariant maps $P \to F$, that is $s : P \to F$ such that for all $u \in P$ and $g \in G$, $s(ug) = g^{-1}s(u)$. The correspondence is realised by mapping an equivariant map $s : P \to F$ to $M \ni p \mapsto [u, s(u)]$ for any choice of $u \in P$ projecting to p.*

In particular, if $F = V$ is a representation of G, this correspondence is an isomorphism of vector spaces.

Given the bundle $\mathrm{GL}(M) \to M$, a linear connection ∇ on M induces on $\mathrm{GL}(M)$ a principal connection that we call $\omega^\nabla \in \Omega^1(\mathrm{GL}(M), \mathfrak{gl}(n,\mathbb{R}))$, unless explicitly stated otherwise. The principal connection ω^∇ is defined by the property that for every smooth frame $u : U \to \mathrm{GL}(M)$ with $U \subseteq M$ open,

$$u^*(\omega^\nabla) = (u^h(\nabla u_k))_{h,k}.$$

We often work in the case where $\iota_P \colon P \to \mathrm{GL}(M)$ is a reduction for the group $G \leq \mathrm{GL}(n,\mathbb{R})$, that is a G-structure. In case $\omega^\nabla|_P = \iota_P^*\omega^\nabla$ has image in $\mathfrak{g}$, the Lie algebra of G, then $\omega^\nabla|_P$ is a principal connection form on P. In this case we say that ∇ is *adapted* to the G-structure P.

The correspondence between linear connections adapted to a G-structure P and principal connections on P is invertible. Given a principal connection ω on P, for all $X = x^k u_k$ written with respect to a frame u, we define $\nabla^\omega(x^k u_k) = dx^k u_k + u^*(\omega)^h_k x_h u^k$. In the correspondence of Remark 1.1.3, the covariant derivative of a section of $P \times_G V$ associated to an equivariant map $\nu : P \to V$, corresponds to the section associated to $D^\nabla \nu := d\nu + \omega^\nabla.\nu$. In the same way, if $\theta \in \Omega^1(P, \mathbb{R}^n)$ is the tautological form, the torsion and the curvature of ∇ correspond respectively to the equivariant forms

$$\Theta^\nabla = d\theta + \omega^\nabla \wedge \theta \colon P \longrightarrow (\mathbb{R}^n)^* \otimes \mathbb{R}^n,$$
$$\Omega^\nabla = d\omega^\nabla + \frac{1}{2}[\omega^\nabla \wedge \omega^\nabla] \colon P \longrightarrow \Lambda^2(\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes \mathbb{R}^n.$$

Alternatively, as differential forms we have $\Theta^\nabla \in \Omega^1(P, \mathbb{R}^n)$ and $\Omega^\nabla \in \Omega^2(P, \mathfrak{g})$. We denote in the same way the corresponding tensors.

Curvature decomposition

On a Riemannian manifold, $(\Omega^\nabla)^\flat$ has values in $\Lambda^2(\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*$. Define the following map

$$\begin{aligned}\mathfrak{B}\colon S^2(\Lambda^2(\mathbb{R}^n)^*) &\longrightarrow \Lambda^4(\mathbb{R}^n)^*\\ (\alpha\wedge\beta)(\gamma\wedge\delta) &\longmapsto \alpha\wedge\beta\wedge\gamma\wedge\delta.\end{aligned}$$

The symmetries of the curvature tensor lead to the following:

Lemma 1.1.4. *Let (M,g) be a Riemannian manifold, then $(\Omega^\nabla)^\flat$, seen as an eqivariant map, has image in $S^2\Lambda^2(\mathbb{R}^n)^* \cap \ker(\mathfrak{B})$.*

Proof. See [44, Lemma 4.2, p. 47]. □

If we take the trace of the curvature tensor, we obtain the *Ricci curvature* Ric_M, which is a tensor in $S^2(T^*M)$. Explicitly, $\mathrm{Ric}_M(X,Y) = \mathrm{tr}(R^\nabla(\cdot,Y)X)$. Using the metric, we can raise one index of Ric_M obtaining $(\mathrm{Ric}_M)_\sharp$. If we take the trace of this tensor and divide by the dimension n of M, we obtain the *scalar curvature* scal_M which is a function on M. Explicitly $\mathrm{scal}_M = \frac{1}{n}\mathrm{tr}(\mathrm{Ric}_M)_\sharp$.

In general, given a tensor R of curvature type, we call its Ricci and scalar components the tensors $\mathrm{Ric}(R)$ and $\mathrm{scal}(R)$ respectively, defined as

$$\mathrm{Ric}(R)(X,Y) := \mathrm{tr}(R(\cdot,Y)X), \qquad \mathrm{scal}(R) = \frac{1}{n}\mathrm{tr}(\mathrm{Ric}(R)_\sharp).$$

On Riemannian and Kähler manifolds, the space of curvature tensors decomposes in three components if the dimension is $n \geq 4$. We recall the definition of a Kähler manifold

Definition 1.1.5. *Let (M,g) be a (pseudo-)Riemannian manifold endowed with a compatible complex structure I, that is such that*

$$g(I\cdot, I\cdot) = g.$$

If the 2-form $\omega := g(I\cdot,\cdot)$ is closed, we say that (M,g,I,ω) is a (pseudo-)Kähler manifold*, and ω is called* Kähler form*.*

For Kähler manifolds, we have

Lemma 1.1.6. *Let (M, g, I, ω) be a Kähler manifold of complex dimension $n \geq 2$, then $(\Omega^\nabla)^\flat$, seen as an equivariant map, has image in $S^2[\Lambda_{1,1}] \cap \ker(\mathfrak{B})$.*

Proof. Based on the fact that the holonomy is $\mathfrak{u}(n) \cong [\Lambda_{1,1}]$. See e.g. [44, Proposition 4.6, p. 50]. □

Proposition 1.1.7. *We have the following* $\mathrm{U}(n)$*-decomposition in irreducible components:*

$$S^2[\Lambda_{1,1}] \cap \ker(\mathfrak{B}) \cong [S^0_{2,2}] \oplus [\Lambda^0_{1,1}] \oplus \mathbb{R}.$$

Proof. See e.g. [44, Proposition 4.7, p. 51]. □

In particular, Ric defined above corresponds to the projection to the components $[\Lambda^0_{1,1}] \oplus \mathbb{R} \cong [\Lambda_{1,1}]$ and scal to the projection on the component $\mathbb{R}$. Therefore, the Ricci and scalar tensors can be seen as the components of the curvature tensor in $[\Lambda_{1,1}]$ and $\mathbb{R}$ respectively. The third component is called the *Bochner tensor*.

1.2 Special Kähler manifolds

The coming definitions involve a flat connection ∇ and its exterior covariant derivative operator d^∇.

Definition 1.2.1. *A* special Kähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{I}, \widetilde{\omega}, \nabla)$ *is the data of a pseudo-Kähler manifold* $(\widetilde{M}, \widetilde{g}, \widetilde{I}, \widetilde{\omega})$ *and a flat, torsion free, symplectic connection* ∇ *such that*

$$d^\nabla \widetilde{I} = 0 \tag{1.1}$$

where we interpret $\widetilde{I}$ *as a* 1*-form with values in* $T\widetilde{M}$

We say that a special Kähler manifold is conic *if there exists a vector field* ξ *such that*

1. $\widetilde{g}(\xi, \xi)$ *is nowhere vanishing;*
2. $\nabla \xi = \widetilde{\nabla}^{LC} \xi = \mathrm{id}$*;*
3. $\widetilde{g}$ *is negative definite on* $\langle \xi, I\xi \rangle$ *and positive definite on its orthogonal complement.*

Where $\widetilde{\nabla}^{LC}$ *is the Levi-Civita connection.*

We will adopt the convention $\widetilde{\omega} = \widetilde{g}(\widetilde{I}\cdot, \cdot)$. Definition 1.2.1 is identical to Definition 3 in [18] if we take $-g$ as metric.

We start by showing in a conic special Kähler manifold how the Lie derivative along ξ and $I\xi$ behaves with respect to the Kähler structure.

Lemma 1.2.2 (Lemma 3.2, p. 1336 in [37]). *Let $(\widetilde{M}, \widetilde{g}, \widetilde{I}, \widetilde{\omega}, \nabla, \xi)$ be a conic special Kähler manifold, then:*

1. *ξ is a homothety of scaling factor 2 preserving $\widetilde{I}$;*
2. *$\widetilde{I}\xi$ preserves the Kähler structure.*

Proof. 1. For any $X, Y \in \mathfrak{X}\left(\widetilde{M}\right)$ we get:

$$\begin{aligned}
\mathcal{L}_\xi \widetilde{g}(X,Y) &= \xi(\widetilde{g}(X,Y)) - \widetilde{g}(\mathcal{L}_\xi X, Y) - \widetilde{g}(X, \mathcal{L}_\xi Y) \\
&= \widetilde{\nabla}^{LC}_\xi(\widetilde{g}(X,Y)) - \widetilde{g}([\xi, X], Y) - \widetilde{g}(X, [\xi, Y]) \\
&= \widetilde{g}(\widetilde{\nabla}^{LC}_\xi X, Y) + \widetilde{g}(X, \widetilde{\nabla}^{LC}_\xi Y) - \widetilde{g}(\widetilde{\nabla}^{LC}_\xi X - \widetilde{\nabla}^{LC}_X \xi, Y) \\
&\quad - \widetilde{g}(X, \widetilde{\nabla}^{LC}_\xi Y - \widetilde{\nabla}^{LC}_Y \xi) \\
&= \widetilde{g}(\widetilde{\nabla}^{LC}_X \xi, Y) + \widetilde{g}(X, \widetilde{\nabla}^{LC}_Y \xi) \\
&= \widetilde{g}(X,Y) + \widetilde{g}(X,Y) = 2\widetilde{g}(X,Y)
\end{aligned}$$

and also

$$\begin{aligned}
\mathcal{L}_\xi \widetilde{I}(X) &= \mathcal{L}_\xi(\widetilde{I}X) - \widetilde{I}\mathcal{L}_\xi X = [\xi, \widetilde{I}X] - \widetilde{I}[\xi, X] \\
&= \widetilde{\nabla}^{LC}_\xi(\widetilde{I}X) - \widetilde{\nabla}^{LC}_{\widetilde{I}X}\xi - \widetilde{I}\widetilde{\nabla}^{LC}_\xi X + \widetilde{I}\widetilde{\nabla}^{LC}_X \xi \\
&= \widetilde{I}\widetilde{\nabla}^{LC}_\xi X - \widetilde{I}X - \widetilde{I}\widetilde{\nabla}^{LC}_\xi X + \widetilde{I}X = 0
\end{aligned}$$

Therefore $\mathcal{L}_\xi \widetilde{g} = 2\widetilde{g}$, $\mathcal{L}_\xi \widetilde{I} = 0$ and thus $\mathcal{L}_\xi \widetilde{\omega} = 2\widetilde{\omega}$.

2. Similarly,

$$\begin{aligned}
\mathcal{L}_{\widetilde{I}\xi} \widetilde{g}(X,Y) &= (\widetilde{I}\xi)(\widetilde{g}(X,Y)) - \widetilde{g}(\mathcal{L}_{\widetilde{I}\xi} X, Y) - \widetilde{g}(X, \mathcal{L}_{\widetilde{I}\xi} Y) \\
&= \widetilde{\nabla}^{LC}_{\widetilde{I}\xi}(\widetilde{g}(X,Y)) - \widetilde{g}([\widetilde{I}\xi, X], Y) - \widetilde{g}(X, [\widetilde{I}\xi, Y]) \\
&= \widetilde{g}(\widetilde{\nabla}^{LC}_{\widetilde{I}\xi} X, Y) + \widetilde{g}(X, \widetilde{\nabla}^{LC}_{\widetilde{I}\xi} Y) - \widetilde{g}(\widetilde{\nabla}^{LC}_{\widetilde{I}\xi} X - \widetilde{\nabla}^{LC}_X(\widetilde{I}\xi), Y) \\
&\quad - \widetilde{g}(X, \widetilde{\nabla}^{LC}_{\widetilde{I}\xi} Y - \widetilde{\nabla}^{LC}_Y(\widetilde{I}\xi)) \\
&= \widetilde{g}(\widetilde{I}\widetilde{\nabla}^{LC}_X \xi, Y) + \widetilde{g}(X, \widetilde{I}\widetilde{\nabla}^{LC}_Y \xi)
\end{aligned}$$

$$= \widetilde{\omega}(X,Y) + \widetilde{\omega}(Y,X) = 0;$$

$$\begin{aligned}\mathcal{L}_{\widetilde{I}\xi}\widetilde{I}(X) &= \mathcal{L}_{\widetilde{I}\xi}(\widetilde{I}X) - \widetilde{I}\mathcal{L}_{\widetilde{I}\xi}X = [\widetilde{I}\xi, \widetilde{I}X] - \widetilde{I}[\widetilde{I}\xi, X] \\ &= \widetilde{\nabla}^{LC}_{\widetilde{I}\xi}(\widetilde{I}X) - \widetilde{\nabla}^{LC}_{\widetilde{I}X}(\widetilde{I}\xi) - \widetilde{I}\widetilde{\nabla}^{LC}_{\widetilde{I}\xi}X + \widetilde{I}\widetilde{\nabla}^{LC}_{X}(\widetilde{I}\xi) \\ &= \widetilde{I}\widetilde{\nabla}^{LC}_{\widetilde{I}\xi}X + X - \widetilde{I}\widetilde{\nabla}^{LC}_{\widetilde{I}\xi}X - X = 0.\end{aligned}$$

From which $\mathcal{L}_{\widetilde{I}\xi}\widetilde{g} = 0$, $\mathcal{L}_{\widetilde{I}\xi}\widetilde{I} = 0$ and $\mathcal{L}_{\widetilde{I}\xi}\widetilde{\omega} = 0$. □

Before proceeding, we write the following lemma for future reference.

Lemma 1.2.3. *In a conic special Kähler manifold* $(\widetilde{M}, \widetilde{g}, \widetilde{I}, \widetilde{\omega}, \nabla, \xi)$*, we have* $\nabla(\widetilde{I}\xi) = \widetilde{I}$.

Proof. For all $X \in \mathfrak{X}\left(\widetilde{M}\right)$

$$\begin{aligned}\nabla_X(\widetilde{I}\xi) - \widetilde{I}X &= (\nabla_X\widetilde{I})\xi + \widetilde{I}\nabla_X\xi - \widetilde{I}X = (\nabla_X\widetilde{I})\xi = (\nabla_\xi\widetilde{I})X \\ &= \nabla_\xi(\widetilde{I}X) - \widetilde{I}\nabla_\xi X = \nabla_{\widetilde{I}X}(\xi) + [\xi, \widetilde{I}X] - \widetilde{I}\left(\nabla_X\xi + [\xi, X]\right) \\ &= \widetilde{I}X + \mathcal{L}_\xi(\widetilde{I}X) - \widetilde{I}X - \widetilde{I}\mathcal{L}_\xi X = (\mathcal{L}_\xi\widetilde{I})X = 0\end{aligned}$$ □

If we compare Definition 1.2.1 with Definition 3.1 in [37], we notice that the main difference is the signature of the metric. In order to obtain two equivalent definitions, it is enough to require the metric to be negative definite on $\langle\xi, I\xi\rangle$ and positive definite on its orthogonal complement (condition 3 in Definition 1.2.1), and to set $X = -I\xi$. The proof of the equivalence is obtained by Lemma 1.2.3.

Definition 1.2.4. *A* projective special Kähler *manifold is a Kähler manifold* M *endowed with a* $\mathbb{C}^*$*-bundle* $\pi\colon \widetilde{M} \to M$ *such that* $(\widetilde{M}, \widetilde{g}, \widetilde{I}, \widetilde{\omega}, \nabla, \xi)$ *is conic special Kähler. Moreover,* ξ *and* $I\xi$ *are the fundamental vector fields associated to* $1, i \in \mathbb{C}$ *respectively and* M *is the Kähler quotient with respect to the induced* $\mathrm{U}(1)$*-action. In this case we say that* M *has a projective special Kähler structure.*

For brevity, we will often denote a projective special Kähler manifold by $(\pi\colon \widetilde{M} \to M, \nabla)$*.*

Remark 1.2.5. *We shall see later that by construction, the action is always Hamiltonian with moment map* $-\widetilde{g}(\xi,\xi)$*, and the choice of the level set affects the quotient only up to scaling.*

Remark 1.2.6. *Despite the name, projective special Kähler manifolds are not necessarily special Kähler, as the existence of a flat connection is in general not granted. See for example Proposition 2.6.9.*

Concerning the notation for projective special Kähler manifolds as in Definition 1.2.4, when a tensor or a connection is possessed by both $\widetilde{M}$ and M, we will write them and everything concerning them (torsion, curvature forms, covariant exterior differentials) on $\widetilde{M}$ with $\widetilde{(\cdot)}$ above, whereas the corresponding objects on M will be denoted without it.

Extrinsic construction of special Kähler manifolds

For further reference, we briefly mention an extrinsic construction of conic special Kähler manifolds provided by [3].

The aim of this section is to give an immersion of a special Kähler manifold $(M, g, I, \omega, \nabla)$ of dimension n in the complex vector space $V = T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$. We call $(z^1, \dots, z^n, w_1, \dots, w_n)$ the canonical coordinates on $T^*\mathbb{C}^n$, such that $z^k = x^k + iu^k$ and $w_k = y_k + iv_k$ for all $k = 1, \dots, n$. Consider on V the standard complex structure $dz_k \mapsto idw^k$ and the flat torsion free connection ∇ vanishing on the canonical coordinate coframe. In particular, the restriction to $T^*\mathbb{R}^n$ has a chart $(x^1, \dots, x^n, y_1, \dots, y_n)$, and the flat connection satisfies $\nabla dx^k = \nabla dy_k = 0$. Moreover, we have a complex symplectic form on V

$$\Omega = dz^k \wedge dw_k,$$

which, restricted to $dx^k \wedge dy_k$ on $T^*\mathbb{R}^n$, that is the canonical 2-form (see Section 1.4).

We have the following result

Theorem 1.2.7. *Let $(M, g, I, \omega, \nabla)$ be a simply connected special Kähler manifold of complex dimension n, then there exists a holomorphic immersion $\phi\colon M \to V = T^*\mathbb{C}^n$ such that*

$$\phi^*\Omega = 0, \qquad \phi^*(dx^k \wedge dy_k) \text{ is non-degenerate.}$$

The map ϕ induces by pullback the Kähler metric g, the connection ∇ and the symplectic form $\omega = 2\phi^*(dx^k \wedge dy_k) = g(I\cdot, \cdot)$ on M.

Moreover, if $(M, g, I, \omega, \nabla)$ is conic special Kähler, then for all $p \in M$ and U open neighbourhood of p, there exist a neighbourhood $U_1 \subseteq \mathbb{C}^*$ of 1 and $U_p \subseteq M$ of p such that $U_1 \cdot \phi(U_p) \subseteq \phi(U)$ holds.

Proof. For the first part see [3, Theorem 4(iii), p. 94] and for the second part see [3, Theorem 9(iii), p. 100]. □

An immersion $\phi : M \to T^*\mathbb{C}^n$ can be seen as a holomorphic form $\widetilde{\phi} = \widetilde{\phi}_k d\widetilde{z}^k$ on M, where $\widetilde{\phi}_k = w_k(\phi)$ and $\widetilde{z}^k = z^k \circ \phi$. In other words, $\widetilde{\phi} = \phi^*(w_k dz^k)$.

The condition $\phi^*\Omega = 0$ corresponds to $d\widetilde{\phi} = 0$, as

$$\phi^*\Omega = \phi^*(-d(w_k dz^k)) = -d\phi^*(w_k dz^k) = -d\widetilde{\phi}.$$

Therefore, locally we can find a holomorphic map F satisfying $dF = \widetilde{\phi}$. The condition appearing in the second part of Theorem 1.2.7 instead corresponds to a homogeneity condition on F. We can infer the following

Corollary 1.2.8. *Let $(M, g, I, \omega, \nabla, \xi)$ be a simply connected conic special Kähler manifold of complex dimension n, and let $\phi\colon M \to V = T^*\mathbb{C}^n$ be the immersion of Theorem 1.2.7, then it corresponds to a form dF for $F \in \mathcal{C}^\infty(M)$. Moreover, with respect to the chart $(z^1, \dots, z^n) \circ \phi$, F is homogeneous of degree 2 and ξ corresponds to the position vector.*

Proof. See [19, Theorem 2, p. 8] for the first part. For the second see [19, Proposition 5, p. 7] and [19, Proposition 6, p. 10]. □

We call such a function F the *holomorphic prepotential*.

Remark 1.2.9. *In particular, if (M, g, I, ω) is defined by a holomorphic prepotential F, then $\widetilde{\phi} = \partial_k F dz^k$ where $\partial_k F := \frac{\partial F}{\partial z^k}$. Thus, if $\partial^2_{h,k} = \partial_h \partial_h F$, we can describe the symplectic form:*

$$\begin{aligned}
\omega = 2\phi^*(dx^k \wedge dy_k) &= \frac{1}{2}(dz^k + d\overline{z}^k) \wedge (d\widetilde{\phi}_k + d\overline{\widetilde{\phi}_k}) \\
&= \frac{1}{2}(dz^k + d\overline{z}^k) \wedge (d(\partial_k F) + d\overline{(\partial_k F)}) \\
&= \frac{1}{2}(dz^k + d\overline{z}^k) \wedge (\partial^2_{k,h} F dz^h + \overline{\partial^2_{k,h} F} d\overline{z}^h) \\
&= \frac{1}{2}(\partial^2_{k,h} F dz^k \wedge dz^h + \overline{\partial^2_{k,h} F} dz^k \wedge d\overline{z}^h + \partial^2_{k,h} F d\overline{z}^k \wedge dz^h + \overline{\partial^2_{k,h} F} d\overline{z}^k \wedge d\overline{z}^h) \\
&= \frac{1}{2}(\partial^2_{k,h} F - \overline{\partial^2_{k,h} F}) d\overline{z}^k \wedge dz^h = i \operatorname{Im}(\partial^2_{k,h} F) d\overline{z}^k \wedge dz^h.
\end{aligned}$$

The metric is instead

$$g = \omega(\cdot, I\cdot) = i\,\mathrm{Im}(\partial^2_{k,h}F)(d\overline{z}^k \otimes i dz^h - dz^h \otimes (-i d\overline{z}^k) = -2\,\mathrm{Im}(\partial^2_{k,h}F) d\overline{z}^k dz^h,$$

so the Hermitian form is

$$h = -2\,\mathrm{Im}(\partial^2_{k,h}F) d\overline{z}^k \otimes dz^h.$$

A projective special Kähler manifold $(\pi : \widetilde{M} \to M, \nabla)$ such that $\widetilde{M}$ is defined by a holomorphic prepotential is called *projective special Kähler domain*.

1.3 Quaternion Kähler manifolds

In this section we are going to define hyperKähler and quaternion Kähler manifolds and we are going to state some of their properties. However, before dealing with manifolds we present some theory concerning quaternions.

1.3.1 Quaternions

For this part we will mostly refer to the notation of [43]. Let $\mathbb{H}$ be the $\mathbb{R}$-algebra of quaternions, namely

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$$

where the multiplication is $\mathbb{R}$-bilinear and satisfies

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1.2}$$

For $n \in \mathbb{N}$, we can view $\mathbb{H}^n$ as a right $\mathbb{H}$-module, where the scalar multiplication by $q \in \mathbb{H}$ is the matrix multiplication by (q) on the right.

Remark 1.3.1. *We adopt this convention so that the left multiplication by a quaternionic matrix A is $\mathbb{H}$-linear. Explicitly, for all $u, v \in \mathbb{H}^n$, we get $A(u+v) = Au + Av$ by distributivity and for all $q \in \mathbb{H}$, we get $A(uq) = (Au)q$ by associativity.*

As for complex numbers, there is a notion of conjugation also for quaternions:

$$(\cdot)^\star : \mathbb{H} \longrightarrow \mathbb{H}$$

$$q = a + bi + cj + dk \longmapsto q^\star = a - bi - cj - dk.$$

The only difference with complex numbers is that the conjugation reverses the order of multiplication, namely $(ab)^\star = b^\star a^\star$ for all $a, b \in \mathbb{H}$. This phenomenon justifies the choice to denote it by the symbol usually employed for the complex conjugate transposition. We use the same symbol also for the quaternionic conjugate transposition of matrices. As for complex numbers, $q \in \mathbb{H}$ is real if and only if $q = q^\star$ and we say that $q \in \mathbb{H}$ is an *imaginary quaternion* if $q = -q^\star$. As for complex numbers, we have $\mathrm{Re}(q) := \frac{1}{2}(q + q^\star)$ and $\mathrm{Im}(q) := \frac{1}{2}(q - q^\star)$ called respectly *real* and *imaginary* part of a quaternion. We denote the subset of imaginary quaternions by $\mathrm{Im}(\mathbb{H}) = \{ai + bj + ck | a, b, c \in \mathbb{R}\} \cong \mathbb{R}^3$.

We will now introduce two standard representations of $\mathbb{H}$. The definition of $\mathbb{H}$ provides an identification of $\mathbb{H}$ with $\mathbb{R}^4$, namely

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathbb{R}^4 \\ a + bi + cj + dk &\longmapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \end{aligned} \tag{1.3}$$

Using (1.3), we can obtain two suitable identifications of $\mathbb{H}$ with subgroups of $\mathrm{Mat}(4, \mathbb{R})$. An element $q \in \mathbb{H}$ can act in two ways on $\mathbb{H}$: by left multiplication $x \mapsto qx$ or by right multiplication of the conjugate $x \mapsto xq^\star$. Both these maps are $\mathbb{R}$-linear, so they provide two ways of mapping a quaternion in $\mathrm{Mat}(4, \mathbb{R})$ which we call respectively $\mathcal{M}_L$ and $\mathcal{M}_R$. We define the following matrices representing the images of i, j, k via $\mathcal{M}_R$ and $\mathcal{M}_L$:

$$i_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad j_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad k_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix};$$
$$i'_4 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad j'_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad k'_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, an explicit description of the two $\mathbb{R}$-algebra homomorphisms is:

$$\begin{aligned} \mathcal{M}_R : \mathbb{H} &\longrightarrow \mathrm{Mat}(4, \mathbb{R}) \\ a + bi + cj + dk &\longmapsto aI_4 + bi_4 + cj_4 + dk_4 = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}; \end{aligned} \tag{1.4}$$

$$\begin{aligned} \mathcal{M}_L : \mathbb{H} &\longrightarrow \mathrm{Mat}(4,\mathbb{R}) \\ a+bi+cj+dk &\longmapsto aI_4 + bi'_4 + cj'_4 + dk'_4 = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}. \end{aligned} \tag{1.5}$$

Remark 1.3.2. *Since left and right multiplication commute, the images of the two homomorphisms commute, namely* $[\mathcal{M}_L(\mathbb{H}), \mathcal{M}_R(\mathbb{H})] = \{0\}$*, that is* $[\mathcal{M}_L(a), \mathcal{M}_R(b)] = 0$ *for all* $a, b \in \mathbb{H}$*.*

We now present a characterisation of 4×4 real matrices representing a quaternionic left multiplication, which will be useful in the next section to characterise some Lie groups.

Lemma 1.3.3. *Let* $A : \mathbb{H} \to \mathbb{H}$ *be an* $\mathbb{R}$*-linear map and call* M_A *its matrix representation with respect to the basis* $1, i, j, k$*. The following are equivalent:*

1. *There exists* $a \in \mathbb{H}$ *such that* $M_A = \mathcal{M}_L(a)$*;*
2. M_A *commutes with* $\mathcal{M}_R(\mathbb{H})$*;*
3. M_A *commutes with* i_4*,* j_4*,* k_4*.*

Proof.

$1 \Rightarrow 2$ See Remark 1.3.2.

$2 \Rightarrow 1$ The condition $[M_A, \mathcal{M}_R(q)] = 0$ is equivalent to the fact that A commutes with the right multiplication by q. Condition 2 then implies that for all $x, q \in \mathbb{H}$, $A(xq) = A(x)q$, meaning that A is $\mathbb{H}$-linear. Let $a = A(1)$, then $A(q) = A(1q) = A(1)q = aq$, thus A is the left multiplication by a.

$2 \Leftrightarrow 3$ It is a consequence of $\mathbb{R}$-linearity and the fact that $\mathcal{M}_R(1) = I_4$ is in the centre of $\mathrm{Mat}(4, \mathbb{H})$. □

Remark 1.3.4. *Notice also that given* $q = a + bi + cj + dk \in \mathbb{H}$*, by a straightforward computation we have*

$$\det(\mathcal{M}_L(q)) = \det(\mathcal{M}_R(q)) = (a^2 + b^2 + c^2 + d^2)^2 \geq 0.$$

1.3.2 Quaternionic groups

In this section we use quaternions to define some useful Lie groups and their Lie algebras. Since $\mathbb{H}$ is a division algebra, all $\mathbb{H}$-modules are free, that is isomorphic, as $\mathbb{H}$-modules, to some direct sum of copies of $\mathbb{H}$ (See e.g. [25, Theorem 5.2, p. 335]). We can identify an $\mathbb{H}$-linear map from $\mathbb{H}^n$ to $\mathbb{H}^m$ with a $n \times m$ matrix with quaternionic entries in a canonical way. Explicitly, let $e_1, \dots, e_n \in \mathbb{H}^n$ be the canonical basis of $\mathbb{H}^n$ and $e^1, \dots, e^n \colon \mathbb{H}^n \to \mathbb{H}$ its dual basis as a right $\mathbb{H}$-module, that is $e^k \colon \mathbb{H}^n \to \mathbb{H}$ is an $\mathbb{H}$-linear map such that $e^k(e_h) = \delta^k_h$; then we map an $\mathbb{H}$-homomorphism to the matrix $(A^h_k = e^h(A(e_k)))_{h,k}$. In particular, $\mathrm{End}_{\mathbb{H}}(\mathbb{H}^n)$ is identified with $\mathrm{Mat}(n, \mathbb{H})$ and via this identification, $\mathrm{Aut}_{\mathbb{H}}(\mathbb{H}^n)$ corresponds to $\mathrm{GL}(n, \mathbb{H})$.

We can extend the correspondence (1.3) to

$$\begin{aligned} \mathbb{H}^n &\longrightarrow \mathbb{R}^{4n} \\ q = e_h(a^h + b^h i + c^h j + d^h k) &\longmapsto q_{\mathbb{R}} = a^h e_{4h-3} + b^h e_{4h-2} + c^h e_{4h-1} + d^h e_{4h}, \end{aligned} \tag{1.6}$$

There is a natural $\mathbb{R}$-algebra left action of $\mathrm{Mat}(n, \mathbb{H})$ on $\mathbb{H}^n$ given by matrix multiplication and there is one on the right given by quaternionic scalar multiplication. We can extend $\mathcal{M}_L$ of (1.5) in order to embed $\mathrm{Mat}(m, n, \mathbb{H})$ in $\mathrm{Mat}(4m, 4n, \mathbb{R})$:

$$\begin{aligned} &\mathcal{M}_L \colon \mathrm{Mat}(n, m, \mathbb{H}) \longrightarrow \mathrm{Mat}(4n, 4m, \mathbb{R}) \\ &\begin{pmatrix} a^1_1 & a^1_2 & \cdots & a^1_m \\ a^2_1 & a^2_2 & \cdots & a^2_m \\ \vdots & \vdots & \ddots & \vdots \\ a^n_1 & a^n_2 & \cdots & a^n_m \end{pmatrix} \longmapsto \begin{pmatrix} \mathcal{M}_L(a^1_1) & \mathcal{M}_L(a^1_2) & \cdots & \mathcal{M}_L(a^1_m) \\ \mathcal{M}_L(a^2_1) & \mathcal{M}_L(a^2_2) & \cdots & \mathcal{M}_L(a^2_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M}_L(a^n_1) & \mathcal{M}_L(a^n_2) & \cdots & \mathcal{M}_L(a^n_m) \end{pmatrix} \end{aligned}$$

Remark 1.3.5. *Given a unitary ring R, we can define the* category of matrices with entries in *R:* $\mathrm{Mat}(R)$ *with natural numbers as objects, and* $\mathrm{Mat}(n, m, R)$ *as set of morphisms from m to n. We have thus defined a functor $\mathcal{M}_L : \mathrm{Mat}(\mathbb{H}) \to \mathrm{Mat}(\mathbb{R})$ mapping the object n to $4n$ and defined by $\mathcal{M}_L : \mathrm{Mat}(n, m, \mathbb{H}) \to \mathrm{Mat}(4n, 4m, \mathbb{R})$ on arrows. Namely, for $A \in \mathrm{Mat}(n, m, \mathbb{H})$, $B \in \mathrm{Mat}(m, l, \mathbb{H})$*

$$\begin{gathered} \mathcal{M}_L(I_n) = I_{4n} \\ \mathcal{M}_L(A)\mathcal{M}_L(B) = \mathcal{M}_L(AB). \end{gathered}$$

Remark 1.3.6. *Notice that the map* (1.6) *is just* $\mathcal{M}_L(q)$ *restricted to the first column, if we see* q *as a column vector. In other terms, if* $e_1 \in \mathbb{R}^4$ *is the first element of the canonical basis, we have*

$$q_{\mathbb{R}} = \mathcal{M}_L(q)e_1.$$

The map $\mathcal{M}_L$ gives an inclusion of $\mathrm{Mat}(n, \mathbb{H})$ in $\mathrm{Mat}(4n, \mathbb{R})$ by mapping a matrix A to the real representation of the action of A by left multiplication.

We can also generalise the map $\mathcal{M}_R \colon \mathbb{H} \to \mathrm{Mat}(4, \mathbb{R})$ to a map $\mathcal{M}_R^n \colon \mathbb{H} \to \mathrm{Mat}(4n, \mathbb{R})$ by mapping a quaternion q in the real matrix representing the right action of $q^\star$ on $\mathbb{H}^n$ identified with $\mathbb{R}^n$ via (1.6). Explicitly, the images of i, j, k via $\mathcal{M}_R^n \colon \mathbb{H} \to \mathrm{Mat}(4n, \mathbb{R})$ are respectively

$$i_{4n} := \begin{pmatrix} i_4 & & & \\ & i_4 & & \\ & & \ddots & \\ & & & i_4 \end{pmatrix}, \quad j_{4n} := \begin{pmatrix} j_4 & & & \\ & j_4 & & \\ & & \ddots & \\ & & & j_4 \end{pmatrix}, \quad k_{4n} := \begin{pmatrix} k_4 & & & \\ & k_4 & & \\ & & \ddots & \\ & & & k_4 \end{pmatrix}.$$

Notice that i_{4n}, j_{4n}, k_{4n} satisfy quaternionic equations (1.2) in $\mathrm{Mat}(4n, \mathbb{R})$. Explicitly, the complete map is then

$$\begin{aligned} \mathcal{M}_R^n \colon \mathbb{H} &\longrightarrow \mathrm{Mat}(4n, \mathbb{R}) \\ q = a + bi + cj + dk &\longmapsto aI_{4n} + bi_{4n} + cj_{4n} + dk_{4n} \end{aligned}$$

Remark 1.3.7. *By definition, for all* $x \in \mathbb{H}^n$*,* $A \in \mathrm{Mat}(m, n, \mathbb{H})$ *and* $q \in \mathbb{H}$*, we have*

$$\begin{aligned} \mathcal{M}_L(A)x_{\mathbb{R}} &= (Ax)_{\mathbb{R}} \\ \mathcal{M}_R^n(q)x_{\mathbb{R}} &= (xq^\star)_{\mathbb{R}}. \end{aligned}$$

Remark 1.3.8. *When working in dimension* $4n$ *with* $n \geq 2$*, unless stated otherwise, we will use the inclusion* $\mathcal{M}_L$ *to map a subgroup* H *of* $\mathrm{Mat}(n, \mathbb{H})$ *to* $\mathcal{M}_L(H)$ *in* $\mathrm{Mat}(4n, \mathbb{R})$ *so, where it will not generate confusion, we will commit an abuse of notation by identifying* H *with* $\mathcal{M}_L(H)$*.*

We now present two useful results for further reference:

Lemma 1.3.9. $\mathrm{Mat}(n, \mathbb{H})$ *is the subset of* $\mathrm{Mat}(4n, \mathbb{R})$ *containing exactly the matrices commuting with* i_{4n}*,* j_{4n}*,* k_{4n}*.*

Moreover, $\mathrm{GL}(n, \mathbb{H}) = \mathrm{GL}(4n, \mathbb{R}) \cap \mathrm{Mat}(n, \mathbb{H})$*.*

Proof. Let $A \in \mathrm{Mat}(4n, \mathbb{R})$ and let $[A]^u_v$ be the 4×4-block in position u, v, then $[A, i_{4k}] = ([[A]^u_v, i_4])_{u,v}$, so A commutes with i_{4n} if and only if each $[A]^u_v$ commutes with i_4 and the same is true for j_{4n} and k_{4n}. Therefore, by Lemma 1.3.3, A commutes with i_{4n}, j_{4n}, k_{4n} if and only if its blocks represent quaternions and thus if and only if $A \in \mathrm{Mat}(n, \mathbb{H})$.

The last statement follows since $\mathcal{M}_L$ is an injective $\mathbb{R}$-algebra homomorphism, so invertible elements are mapped to invertible elements. □

Lemma 1.3.10. *Matrices in* $\mathcal{M}_L(\mathrm{Mat}(n, \mathbb{H}))$ *and* $\mathcal{M}^n_R(\mathbb{H})$ *have non-negative determinant.*

In particular, $\mathrm{GL}(n, \mathbb{H}) \leq \mathrm{GL}^+(4n, \mathbb{R})$.

Proof. For this proof we use the fact that, as we shall see later, a quaternion can be represented by a pair of complex numbers, and thus we can see quaternionic matrices as complex ones, which always have positive determinant. However, since we have not treated this part yet, we will be more explicit.

Let $A = (A^u_v) \in \mathrm{Mat}(n, \mathbb{H})$, and consider $\mathcal{M}_L(A)$, which is a matrix with blocks $\mathcal{M}_L(A^u_v)$ of the form $\begin{pmatrix} a^u_v & -b^u_v & -c^u_v & -d^u_v \\ b^u_v & a^u_v & -d^u_v & c^u_v \\ c^u_v & d^u_v & a^u_v & -b^u_v \\ d^u_v & -c^u_v & b^u_v & a^u_v \end{pmatrix}$. Notice that if we divide this matrix in 2×2 blocks, it is of the form $\left(\begin{array}{c|c} X^u_v & -Y^u_v \\ \hline Y^u_v & X^u_v \end{array}\right)$ which is the usual real representation of a complex matrix as a real one. The idea is to represent $\mathcal{M}_L(A)$ in a different basis, so that the determinant stays invariant, but it is easier to deduce its non-negativity. Consider the canonical basis of $\mathbb{R}^{4n}$ and, for the sake of this proof, subdivide it in n blocks of four adjacent elements by writing it as $e_{u,k}$ with $1 \leq u \leq n$ representing the u-th block and $1 \leq k \leq 4$ representing the position in this block. We build a new basis by moving the first two elements of each block to the left, that is

$$e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}, \ldots, e_{n,1}, e_{n,2}, e_{1,3}, e_{1,4}, e_{2,3}, e_{2,4}, \cdots, e_{n,3}, e_{n,4}.$$

This base change modifies $\mathcal{M}_L(A)$ so that it can be divided in four blocks as $\left(\begin{array}{c|c} X & -Y \\ \hline Y & X \end{array}\right)$, where $X = (X^u_v)_{u,v}$ and $Y = (Y^u_v)_{u,v}$. Now we know how to compute its determinant, which is $|\det(X + iY)|^2$ (see e.g. [32, property (d), p. 60]) and thus non-negative.

The second part is easier as, given $q \in \mathbb{H}$, the matrix $\mathcal{M}_R^n(q)$ is built with n identical blocks $\mathcal{M}_R(q)$ on the diagonal, thus $\det(\mathcal{M}_R^n(q)) = \det(\mathcal{M}_R(q))^n$ and we know by Lemma 1.3.10 that $\det(\mathcal{M}_R(q))$ is non-negative. □

Remark 1.3.11. *The space* $\mathrm{Mat}(n, \mathbb{H})$ *has a structure of differentiable manifold as a real vector space. The group* $\mathrm{GL}(n, \mathbb{H})$ *is an open subset of* $\mathrm{Mat}(n, \mathbb{H})$ *and inherits its differentiable structure, that makes it a Lie group whose Lie algebra is* $\mathfrak{gl}(n, \mathbb{H}) = \mathrm{Mat}(n, \mathbb{H})$.

For all $p, q \in \mathbb{N}$ such that $p+q = n$, let $I_{p,q} := \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, then we define an inner product $G_{p,q}$ on $\mathbb{H}^n$ with signature $(4p, 4q)$ as follows:

$$G_{p,q}(x, y) = \mathrm{Re}\left(\sum_{k=1}^{p} x_k^\star y_k - \sum_{k=p+1}^{n} x_k^\star y_k\right) = \mathrm{Re}\left(x^\star I_{p,q} y\right)$$

for all $x, y \in \mathbb{H}^n$. In this case we say that (p, q) is the *quaternionic signature*.

Remark 1.3.12. *The quaternionic conjugate transpose of matrices corresponds via the functor* $\mathcal{M}_L$ *to the transposition of real matrices. In particular then, the real part of a quaternion* q *corresponds by* $\mathcal{M}_L$ *to the symmetric part of* $\mathcal{M}_L(a)$*, which is its diagonal and of the form* $\mathrm{Re}(q)I_4$.

The same holds for $\mathcal{M}_R$ *in* (1.4).

Lemma 1.3.13. *The homomorphism* (1.6) *maps* $G_{p,q}$ *to the standard metric with signature* $(4p, 4q)$ *on* $\mathbb{R}^{4n}$.

Proof. Explicitly, the statement of this proposition means that for all $x, y \in \mathbb{H}^n$ and for all quaternionic signatures (p, q), we have

$$\mathrm{Re}(x^\star I_{p,q} y) = (x_\mathbb{R})^t I_{4p,4q} y_\mathbb{R}.$$

Notice that $I_{4p,4q} = \mathcal{M}_L(I_{p,q})$, thus by Remark 1.3.6, if $e_1 \in \mathbb{R}^4$ is the first element of the canonical basis, we have

$$(x_\mathbb{R})^t I_{4p,4q} y_\mathbb{R} = (\mathcal{M}_L(x)e_1)^t \mathcal{M}_L(I_{p,q}) \mathcal{M}_L(y) e_1.$$

By Remarks 1.3.12 and 1.3.5, we have

$$(\mathcal{M}_L(x)e_1)^t \mathcal{M}_L(I_{p,q}) \mathcal{M}_L(y) e_1 = e_1^t \mathcal{M}_L(x)^t \mathcal{M}_L(I_{p,q}) \mathcal{M}_L(y) e_1$$

$$= e_1^t \mathcal{M}_L(x^\star)\mathcal{M}_L(I_{p,q})\mathcal{M}_L(y)e_1$$
$$= e_1^t \mathcal{M}_L(x^\star I_{p,q} y)e_1,$$

which is the entry at row 1 and column 1 of $\mathcal{M}_L(x^\star I_{p,q} y)$, so it stays the same if we take the transpose and, again by Remark 1.3.12,

$$\begin{aligned} e_1^t \mathcal{M}_L(x^\star I_{p,q} y)e_1 &= e_1^t \frac{1}{2}(\mathcal{M}_L(x^\star I_{p,q} y) + \mathcal{M}_L(x^\star I_{p,q} y)^t)e_1 \\ &= e_1^t \mathcal{M}_L(\mathrm{Re}(x^\star I_{p,q} y))e_1 = \mathrm{Re}(x^\star I_{p,q} y). \end{aligned}$$ □

Definition 1.3.14. *For $n \in \mathbb{N}$, we define the group*

$$\mathrm{Sp}(n) = \{Q \in \mathrm{Mat}(n, \mathbb{H}) | Q^\star Q = I_n\},$$

called quaternionic unitary group.

More generally, for $p, q \in \mathbb{N}$ with $p + q = n$, we define the group

$$\mathrm{Sp}(p, q) = \{Q \in \mathrm{Mat}(n, \mathbb{H}) | Q^\star I_{p,q} Q = I_{p,q}\},$$

called indefinite quaternionic unitary group *of quaternionic signature (p, q).*

In particular, $\mathrm{Sp}(1) = \{q \in \mathbb{H} | q^\star q = 1\} \cong S^3$ is the unit 3-sphere in $\mathbb{H}$ with respect to the norm induced by $G_{1,0}$.

Remark 1.3.15. *Equivalently,* $\mathrm{Sp}(p, q)$ *can be described as*

$$\mathrm{Sp}(p, q) = \{Q \in \mathrm{Mat}(n, \mathbb{H}) | G_{p,q}(Q\cdot, Q\cdot) = G_{p,q}\}.$$

If $Q^\star I_{p,q} Q = I_{p,q}$, in particular

$$G_{p,q}(Qx, Qy) = \mathrm{Re}(x^\star Q^\star I_{p,q} Q y) = \mathrm{Re}(x^\star I_{p,q} y) = G_{p,q}(x, y),$$

for all $x, y \in \mathbb{H}^n$.

Conversely, notice that if $a = x + iy + jz + kw$, then

$$\mathrm{Re}(a) = x, \qquad \mathrm{Re}(ai) = -y, \qquad \mathrm{Re}(aj) = -z, \qquad \mathrm{Re}(ak) = -w,$$

so $a = \mathrm{Re}(a) - i\,\mathrm{Re}(ai) - j\,\mathrm{Re}(aj) - k\,\mathrm{Re}(ak)$. Applied to our case then,

$$\begin{aligned} &x^\star Q^\star I_{p,q} Q y \\ &= G_{p,q}(Qx, Qy) - iG_{p,q}(Qx, Qyi) - jG_{p,q}(Qx, Qyj) - kG_{p,q}(Qx, Qyk) \\ &= G_{p,q}(x, y) - iG_{p,q}(x, yi) - jG_{p,q}(x, yj) - kG_{p,q}(x, yk) = x^\star I_{p,q} y, \end{aligned}$$

so $Q^\star I_{p,q} Q = I_{p,q}$

Proposition 1.3.16. *For all n and $p+q=n$, the set* $\mathrm{Sp}(p,q)$ *is a closed Lie subgroup of* $\mathrm{GL}(n,\mathbb{H})$ *with Lie algebra*

$$\begin{aligned}\mathfrak{sp}(p,q) &= \{X \in \mathrm{Mat}(n,\mathbb{H}) | X^\star I_{p,q} + I_{p,q}X = 0\} \\ &= \left\{ \left(\begin{array}{c|c} A & B^\star \\ B & D \end{array}\right) \in \mathrm{Mat}(n,\mathbb{H}) | B \in \mathrm{Mat}(q,p,\mathbb{H}), A^\star = -A, D^\star = -D \right\}.\end{aligned}$$

In particular,

$$\begin{aligned}\mathfrak{sp}(n) &= \{X \in \mathrm{Mat}(n,\mathbb{H}) | X^\star = -X\}, \\ \mathfrak{sp}(1) &= \mathrm{Im}(\mathbb{H}).\end{aligned}$$

Proof. We will only treat the case with generic quaternionic signature (p,q) since the other case follows directly for $q=0$.

First of all notice that $\mathrm{Sp}(p,q)$ is actually a subset of $\mathrm{GL}(n,\mathbb{H})$, in fact, if $A \in \mathrm{Sp}(p,q)$, then $Q^\star I_{p,q} Q = I_{p,q}$ and thus $Q^{-1} = I_{p,q} Q^\star I_{p,q}$.

This subgroup is a closed subset, since it is the preimage of $I_{p,q}$ via the continuous map

$$\begin{aligned}\rho \colon \mathrm{Mat}(n,\mathbb{H}) &\longrightarrow \mathrm{Mat}(n,\mathbb{H}) \\ Q &\longmapsto Q^\star I_{p,q} Q.\end{aligned}$$

Thus, $\mathrm{Sp}(p,q)$ with the induced topology is a closed Lie subgroup of $\mathrm{GL}(n,\mathbb{H})$ (see e.g. [28, Theorem 2.3, p. 115]).

Moreover, its Lie algebra is $\ker(d_{I_n}\rho)$, so let $X \in \mathfrak{gl}(n,\mathbb{H})$, then

$$\begin{aligned}d_{I_n}\rho(X) &= d_{I_n}\rho\left(\frac{d}{dt}\exp(tX)|_{t=0}\right) = \frac{d}{dt}\rho(\exp(tX))|_{t=0} \\ &= \frac{d}{dt}(\exp(tX)^\star I_{p,q}\exp(tX))|_{t=0} = \frac{d}{dt}(\exp(tX^\star) I_{p,q}\exp(tX))|_{t=0} \\ &= X^\star I_{p,q} + I_{p,q}X.\end{aligned}$$

The matrix X can be subdivided in blocks by separating the first p lines and columns from the last q. Then $X = \left(\begin{array}{c|c} A & B \\ C & D \end{array}\right)$ must satisfy

$$0 = X^\star I_{p,q} + I_{p,q}X = \left(\begin{array}{c|c} A & B \\ C & D \end{array}\right)^\star I_{p,q} + I_{p,q}\left(\begin{array}{c|c} A & B \\ C & D \end{array}\right)$$

$$= \left(\begin{array}{c|c} A^\star & C^\star \\ \hline B^\star & D^\star \end{array}\right) \left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array}\right) + \left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array}\right) \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$
$$= \left(\begin{array}{c|c} A^\star & -C^\star \\ \hline B^\star & -D^\star \end{array}\right) + \left(\begin{array}{c|c} A & B \\ \hline -C & -D \end{array}\right) = \left(\begin{array}{c|c} A^\star + A & -C^\star + B \\ \hline B^\star - C & -D^\star - D \end{array}\right),$$

which translates to the conditions $A^\star = -A$, $B^\star = C$, $D^\star = -D$. □

Remark 1.3.17. *Observing the Lie algebras, for for $n = p + q$, we can compute the dimensions of these Lie groups, obtaining*

$$\dim(\mathrm{Sp}(p,q)) = \dim(\mathrm{Sp}(n)) = 2n^2 + n.$$

As for $\mathrm{GL}(n,\mathbb{H})$, we identify $\mathrm{Sp}(p,q)$ with $\mathcal{M}_L(\mathrm{Sp}(p,q))$, however, for $\mathrm{Sp}(1)$ it will be useful to identify it with $\mathcal{M}_R^n(\mathrm{Sp}(1)) \subset \mathrm{Mat}(4n,\mathbb{R})$ instead.

Remark 1.3.18. *As observed before,* $\mathrm{Sp}(1)$ *is the subset of elements with norm* 1 *in* $\mathbb{H}$ *and thus all its elements are of the form $a + bi + cj + dk$ with $a^2 + b^2 + c^2 + d^2 = 1$. In particular, 1, i, j, k are in* $\mathrm{Sp}(1)$.

Identifying $\mathrm{Sp}(1)$ *with* $\mathcal{M}_R^n(\mathrm{Sp}(1))$*, we have that $i_{4n}, j_{4n}, k_{4n} \in \mathrm{Sp}(1)$, in fact they are the image via $\mathcal{M}_R^n$ of i, j, k respectively.*

Proposition 1.3.19. *We can characterise* $\mathrm{Sp}(p,q)$ *in the following equivalent ways*

1. $\mathrm{Sp}(p,q) = \mathrm{GL}(n,\mathbb{H}) \cap \mathrm{SO}(4p,4q)$*;*
2. $\mathrm{Sp}(p,q)$ *is the subset of matrices in* $\mathrm{SO}(4p,4q)$ *commuting with* $\mathrm{Sp}(1)$.

Both conditions are still valid with $\mathrm{O}(4p,4q)$ *instead of* $\mathrm{Sp}(p,q)$.

Proof. We will prove both conditions for $\mathrm{O}(4p,4q)$, since we already know by Lemma 1.3.10 that the determinant of the inclusion is non-negative, and thus, such a matrix is in $\mathrm{O}(p,q)$ if and only if it is in $\mathrm{SO}(p,q)$.

1. By Proposition 1.3.16, we know that $\mathrm{Sp}(p,q) \leq \mathrm{GL}(n,\mathbb{H})$, so consider a matrix $Q \in \mathrm{GL}(n,\mathbb{H})$; explicitly, we have to prove that $\mathcal{M}_L(Q)$ is in $\mathrm{O}(p,q)$ if and only if $Q \in \mathrm{Sp}(p,q)$. Lemma 1.3.13 states that for all $x, y \in \mathbb{H}^n$, we have

$$\mathrm{Re}(x^\star I_{p,q} y) = x_{\mathbb{R}}^t I_{4p,4q} y_{\mathbb{R}}.$$

Thus, for all $x, y \in \mathbb{H}^n$, using Remark 1.3.7 we have

$$\begin{aligned}\operatorname{Re}(x^\star(Q^\star I_{p,q}Q)y) &= \operatorname{Re}((Qx)^\star I_{p,q}Qy) = (Qx)^t_{\mathbb{R}} I_{4p,4q}(Qy)_{\mathbb{R}} \\ &= (\mathcal{M}_L(Q)x_{\mathbb{R}})^t I_{4p,4q}\mathcal{M}_L(Q)y_{\mathbb{R}} \\ &= x^t_{\mathbb{R}}\mathcal{M}_L(Q)^t I_{4p,4q}\mathcal{M}_L(Q)y_{\mathbb{R}}.\end{aligned}$$

Thus, if $Q^\star I_{p,q}Q = I_{p,q}$,

$$\begin{aligned}x^t_{\mathbb{R}} I_{4p,4q} y_{\mathbb{R}} &= \operatorname{Re}(x^\star I_{p,q}y) = \operatorname{Re}(x^\star(Q^\star I_{p,q}Q)y) \\ &= x^t_{\mathbb{R}}\mathcal{M}_L(Q)^t I_{4p,4q}\mathcal{M}_L(Q)y_{\mathbb{R}}\end{aligned}$$

so $\mathcal{M}_L(Q)^t I_{4p,4q}\mathcal{M}_L(Q) = I_{4p,4q}$. Conversely, if this happens

$$\begin{aligned}\operatorname{Re}(x^\star I_{p,q}y) &= x^t_{\mathbb{R}} I_{4p,4q} y_{\mathbb{R}} = x^t_{\mathbb{R}}\mathcal{M}_L(Q)^t I_{4p,4q}\mathcal{M}_L(Q)y_{\mathbb{R}} \\ &= \operatorname{Re}(x^\star(Q^\star I_{p,q}Q)y),\end{aligned}$$

so $G_{p,q}(Q\cdot, Q\cdot) = G_{p,q}$ and thus, by Remark 1.3.15, $Q^\star I_{p,q}Q = I_{p,q}$.

2. This follows from the first point and Lemma 1.3.9. Explicitly, if Q belongs to $\mathrm{Sp}(p,q) = \mathrm{GL}(n,\mathbb{H}) \cap \mathrm{O}(p,q)$, then by Lemma 1.3.9, it also commutes with i_{4n}, j_{4n} and k_{4n} and clearly with I_{4n}. Then, Q commutes with $\langle I_{4n}, i_{4n}, j_{4n}, k_{4n}\rangle_{\mathbb{R}} = \mathcal{M}_{\mathbb{R}}(\mathbb{H})$ which contains in particular $\mathrm{Sp}(1)$. Conversely, if $Q \in \mathrm{O}(p,q)$ commutes with $\mathrm{Sp}(1)$, then in particular it commutes with i_{4n}, j_{4n} and k_{4n}, so by Lemma 1.3.9 we infer that Q belongs to $\mathrm{GL}(n,\mathbb{H})$ and thus to $\mathrm{GL}(n,\mathbb{H}) \cap \mathrm{O}(p,q) = \mathrm{Sp}(n)$.

□

We have a similar characterisation for $\mathrm{Sp}(1)$ as well:

Proposition 1.3.20. $\mathrm{Sp}(1)$ *is the subgroup of matrices in* $\mathrm{SO}(4p,4q)$ *(equivalently in* $\mathrm{O}(4p,4q)$*) commuting with* $\mathrm{Sp}(p,q)$.

Proof. We will prove this statement only for the case $\mathrm{O}(p,q)$, since by Lemma 1.3.10, matrices in the image of $\mathcal{M}_R$ have positive determinant, so they belong to $\mathrm{O}(p,q)$ if and only if they belong to $\mathrm{SO}(p,q)$.

Let $a \in \mathrm{Sp}(1)$ and notice that for all $x \in \mathbb{H}$, we have $(axa^\star)^\star = ax^\star a^\star$ and thus $\operatorname{Re}(axa^\star) = a\operatorname{Re}(x)a^\star = aa^\star\operatorname{Re}(x) = \operatorname{Re}(x)$, as real numbers are in the centre of $\mathbb{H}$. It follows that for all $x, y \in \mathbb{H}^n$,

$$\operatorname{Re}((xa^\star)^\star I_{p,q}(ya^\star)) = \operatorname{Re}(ax^\star I_{p,q}ya^\star) = \operatorname{Re}(x^\star I_{p,q}y) = x^t_{\mathbb{R}} I_{4p,4q} y_{\mathbb{R}},$$

but by Remark 1.3.7, we also have

$$\begin{aligned}\mathrm{Re}((xa^\star)^\star I_{p,q}(ya^\star)) &= (xa^\star)^t_{\mathbb{R}} I_{4p,4q}(ya^\star)_{\mathbb{R}} = x^t_{\mathbb{R}}\mathcal{M}_R(a)^t I_{p,q}\mathcal{M}_R(a)y_{\mathbb{R}}\\ &= x^t_{\mathbb{R}}\mathcal{M}_R(a)^t I_{4p,4q}\mathcal{M}_R(a)y_{\mathbb{R}},\end{aligned}$$

thus $\mathcal{M}_R(a)^t I_{4p,4q}\mathcal{M}_R(a) = I_{4p,4q}$ implying that $\mathcal{M}_R(a)$ belongs to $\mathrm{O}(4p,4q)$. We already know by Proposition 1.3.19 that $\mathrm{Sp}(p,q)$ commutes with $\mathrm{Sp}(1)$, so one direction is proven.

Conversely, let A be a matrix in $\mathrm{O}(4p,4q)$ commuting with $\mathrm{Sp}(p,q)$. We subdivide Q in 4×4 blocks and call $[A]^u_v$ the one in the u-th row and v-th column of blocks. This matrix corresponds to a subdivision of $\mathbb{R}^{4n}$ in blocks of dimension 4, each corresponding via (1.6) to a different component of $\mathbb{H}^n$. Consider the transformation $S_{h,l}$ switching the h-th with the l-th component of $\mathbb{R}^{4n}$. The matrix $S_{h,l}$ is the transpose and the inverse of itself, and if $1\le h,l\le p$ or $p<h,l\le n$, it belongs to $\mathrm{Sp}(p,q)$. Commutativity of A with $S_{h,l}$ then implies that $A = S_{h,l}AS_{h,l} = S^{-1}_{h,l}AS_{h,l}$. The effect of $A\mapsto S^{-1}_{h,l}AS_{h,l}$ is to swap columns and rows of blocks in position h and l. Thus the matrix stays invariant if

$$\begin{cases}[A]^h_v = [A]^l_v & \text{if } v\neq h,l\\ [A]^u_h = [A]^u_l & \text{if } u\neq h,l\\ [A]^u_u = [A]^v_v & \\ [A]^h_l = [A]^l_h & \text{if } v\neq h,l\end{cases}$$

Applying these conditions for all $S_{h,l}\in\mathrm{Sp}(p,q)$, we get that A is built by only 6 types of blocks: $B_1,\dots,B_6\in\mathrm{Mat}(4,\mathbb{R})$:

$$\left(\begin{array}{cccc|cccc} B_1 & B_2 & \cdots & B_2 & B_3 & \cdots & B_3 & B_3\\ B_2 & B_1 & \cdots & B_2 & B_3 & \cdots & B_3 & B_3\\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots\\ B_2 & B_2 & \cdots & B_1 & B_3 & \cdots & B_3 & B_3\\ \hline B_4 & B_4 & \cdots & B_4 & B_5 & \cdots & B_6 & B_6\\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots\\ B_4 & B_4 & \cdots & B_4 & B_6 & \cdots & B_5 & B_6\\ B_4 & B_4 & \cdots & B_4 & B_6 & \cdots & B_6 & B_5\end{array}\right).$$

Consider now the two quaternionic matrices in $\mathrm{Sp}(p,q)$:

$$T_1 = \left(\begin{array}{cc|c} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \hline 0 & 0 & I_{n-2}\end{array}\right) \qquad T_2 = \left(\begin{array}{c|cc} I_{n-2} & 0 & 0\\ \hline 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{array}\right),$$

then $[A, T_1] = 0$ and thus A is fixed by $A \mapsto \mathcal{M}_L(T_1)^{-1} A \mathcal{M}_L(T_1)$ which modifies only the first two rows and columns. In particular, the top left 8×8 block, the top right 8×4 and the bottom left 4×8 become

$$\begin{pmatrix} B_1 + B_2 & 0 \\ 0 & B_1 - B_2 \end{pmatrix}, \qquad \begin{pmatrix} B_3 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} B_4 & 0 \end{pmatrix}.$$

So the commutativity of A with $\mathcal{M}_L(T_1)$ implies $B_2, B_3, B_4 = 0$. The same operation with T_2 implies also that $B_6 = 0$. Therefore, the matrix A has only blocks on the diagonal: B_1 for the first p blocks and B_5 on the last q.

Consider the quaternionic matrix $U_a = aI_n$ for $a \in \mathrm{Sp}(1)$, then $U_a \in \mathrm{Sp}(p, q)$ as one can check directly that $U_a^\star I_{p,q} U_a = a^\star a I_{p,q} = I_{p,q}$. The commutativity $[A, U_a] = 0$ forces B_1 and B_5 to commute with $\mathcal{M}_L(a) \in \mathrm{Mat}(4, \mathbb{R})$. By seeing $B_1, B_5 \in \mathrm{End}_{\mathbb{R}}(\mathbb{H})$, this is equivalent to ask that for all $x \in \mathbb{H}$, $B_1(ax) = aB_1(x)$ and thus, it is $\mathbb{H}$-linear on the left. Let $b_1 = B_1(1)^\star$ and $b_5 = B_5(1)^\star$, then for $k = 1, 5$ we get $B_k(x) = x b_k^\star$, so $B_k = \mathcal{M}_R(b_k)$. The proof is then complete if we show that $b_1 = b_5$ in the case where neither p nor q is zero. In this case consider the quaternionic matrix

$$V = \left(\begin{array}{c|cc|c} I_{p-1} & & & \\ \hline & 0 & i & \\ & -i & 0 & \\ \hline & & & I_{q-1} \end{array}\right);$$

it is in $\mathrm{Sp}(p, q)$ and the map $A \mapsto V^{-1} A V = V A V$ modifies only the central blocks at rows and columns $p, p+1$ of A, which become

$$\begin{pmatrix} \mathcal{M}_L(-i) B_5 \mathcal{M}_L(i) & 0 \\ 0 & \mathcal{M}_L(-i) B_1 \mathcal{M}_L(i) \end{pmatrix} = \begin{pmatrix} B_5 & 0 \\ 0 & B_1 \end{pmatrix}$$

by Remark 1.3.2 and the fact that $\mathcal{M}_L$ is a homomorphism. Commutativity of A with V implies that $B_1 = B_5$ and thus $b_1 = b_5 = b$ and $A = \mathcal{M}_R^n(b)$. We are left to prove that $b \in \mathrm{Sp}(1)$ and since $A = \mathcal{M}_R^n(b)$ belongs to $\mathrm{O}(p, q)$, we have in particular $\mathcal{M}_R(b)^t \mathcal{M}_R(b) = I_4$. By Remark 1.3.12 and the fact that $\mathcal{M}_R$ is a homeomorphism, we have

$$\mathcal{M}_R(b^\star b) = \mathcal{M}_R(b)^t \mathcal{M}_R(b) = I_4 = \mathcal{M}_R(1).$$

Since $\mathcal{M}_R$ is injective, we infer $b^\star b = 1$, ending the proof. $\square$

Remark 1.3.21. *We have* $\mathrm{Sp}(p,q) \cap \mathrm{Sp}(1) = \{-I_{4n}, I_{4n}\} \cong \mathbb{Z}_2$*, in fact by Proposition 1.3.19, an element is in the intersection if and only if it is in the centre of* $\mathrm{Sp}(1)$ *which is* $\{\pm 1\}$ *and thus, via* $\mathcal{M}_R$ *we get the result. In particular, this copy of* $\mathbb{Z}_2$ *is also contained in the centre of* $\mathrm{Sp}(p,q)$*.*

Notice that every $l \in \mathbb{H}$ such that $l^2 = -1$ defines an immersion

$$\begin{aligned} \iota_l \colon \mathbb{C} &\longrightarrow \mathbb{H} \\ a + ib &\longmapsto a + lb. \end{aligned}$$

This map is an $\mathbb{R}$-algebra homomorphism, so we can define a structure of complex vector space by restriction of scalars along ι_l.

Remark 1.3.22. *The quaternionic solutions of the equation* $q^2 + 1 = 0$ *are the quaternions in*

$$\mathrm{Im}(\mathbb{H}) \cap \mathrm{Sp}(1) = \{ai + bj + ck \in \mathbb{H} | a^2 + b^2 + c^2 = 1\} \cong S^2.$$

An explicit computation shows that the elements in this set satisfy the equation. Notice that in general, $q^\star q q^\star = \|q\|^2 q^\star = q^\star \|q\|^2$*, so if* $q \neq 0$*, we can cancel* $q^\star$ *in order to obtain* $qq^\star = \|q\| = q^\star q$*, which is also satisfied by* $q = 0$*.*

For the opposite inclusion, let q *be a solution, then we can write*

$$\|q\|^4 = q^\star q^2 q^\star = -(q^\star)^2 = 1$$

Thus $\|q\| = 1$*, so* $q \in \mathrm{Sp}(1)$*.*

If we now square the double of the real part of q *we obtain*

$$(q + q^\star)^2 = q^2 + qq^\star + q^\star q + (q^\star)^2 = -1 + 1 + 1 - 1 = 0,$$

implying $\mathrm{Re}(q) = 0$ *as claimed.*

Definition 1.3.23. *We call* $\mathrm{U}_l(2p, 2q) := \{A \in \mathrm{SO}(2p, 2q) | [A, \mathcal{M}_R(l) = 0\}$ *the* unitary group *with signature* $(4p, 4q)$ *and we say that the complex signature is* $(2p, 2q)$*.*

Remark 1.3.24. *By Proposition 1.3.19, we have*

$$\mathrm{Sp}(p,q) = \bigcap_{l \in \mathrm{Sp}(1) \cap \mathrm{Im}(\mathbb{H})} U_l(p,q) = \mathrm{U}_i(p,q) \cap \mathrm{U}_j(p,q) \cap \mathrm{U}_k(p,q)$$

Remark 1.3.25. $\mathrm{Sp}(1)$ *is never contained in any of the* $\mathrm{U}_l(2p, 2q)$*, as* $\mathrm{Sp}(1) \cap \mathrm{U}_l(2p, 2q) = \{a + bl | a^2 + b^2 = 1\}$*; this is a subgrup isomorphic to* $\mathrm{U}(1)$*, which will be denoted by* $\mathrm{U}_l(1)$*.*

Another quaternionic group that has a central role in this thesis is the one generated by $\mathrm{Sp}(p, q)$ and $\mathrm{Sp}(1)$ in $\mathrm{SO}(4p, 4q)$, that is $\mathrm{Sp}(p, q)\mathrm{Sp}(1)$. The case with positive signature is denoted by $\mathrm{Sp}(n)\mathrm{Sp}(1)$.

Remark 1.3.26. *By the second theorem of isomorphism and Remark 1.3.21, we have* $\mathrm{Sp}(p, q)\mathrm{Sp}(1) = (\mathrm{Sp}(p, q) \times \mathrm{Sp}(1))/\mathbb{Z}_2$*.*

It follows that the Lie algebra of $\mathrm{Sp}(p, q)\mathrm{Sp}(1)$ *is isomorphic to*

$$\mathfrak{sp}(p, q) \oplus \mathfrak{sp}(1),$$

and thus, by Remark 1.3.17:

$$\dim(\mathrm{Sp}(p, q)\mathrm{Sp}(1)) = \dim(\mathrm{Sp}(n)\mathrm{Sp}(1)) = 2n^2 + n + 3.$$

Remark 1.3.27. *If* $n = 1$*, we have* $\dim(\mathrm{Sp}(1)\mathrm{Sp}(1)) = 6 = \dim(SO(4))$ *and since* $\mathrm{SO}(4)$ *is connected, we must have* $\mathrm{Sp}(1)\mathrm{Sp}(1) = \mathrm{SO}(4)$*.*

If $n \geq 2$ *instead*

$$\dim(\mathrm{Sp}(p, q)\mathrm{Sp}(1)) = 2n^2 + n + 3 < 2n(4n - 1) = \dim(\mathrm{SO}(4p, 4q)),$$

so $\mathrm{Sp}(p, q)\mathrm{Sp}(1)$ *is a proper subgroup of* $\mathrm{SO}(4p, 4q)$*.*

On quaternions, this group is interpreted as follows: we consider our usual left action of $\mathrm{Sp}(p, q) \times \mathrm{Sp}(1)$ on $\mathbb{H}^n$ given by $x \mapsto Qxq^\star$ for $(Q, q) \in \mathrm{Sp}(p, q) \times \mathrm{Sp}(1)$, which induces a map $\mathrm{Sp}(p, q) \times \mathrm{Sp}(1) \to \mathrm{Aut}_\mathbb{R}(\mathbb{H}^n)$ and $\mathrm{Sp}(p, q)\mathrm{Sp}(1)$ is the image of this map, whose kernel is $\{(I_n, 1), (-I_n, -1)\}$.

Remark 1.3.28. *A consequence of Remark 1.3.25,* $\mathrm{Sp}(p, q)\mathrm{Sp}(1)$ *is not contained in* $\mathrm{U}_l(2p, 2q)$ *for any* l*.*

We summarise the inclusions found in this section in the subgroup diagram in Figure 1.1.

1.3.3 Quaternionic manifolds

For this part, we follow the theory as presented in [44]. In view of the construction of the c-map, we now introduce the following manifolds.

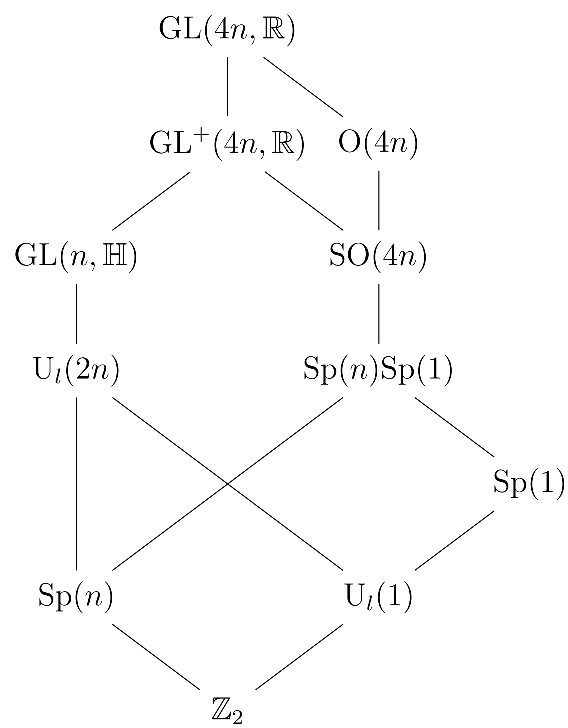

Figure 1.1: Diagram of subgroups defined in section 1.3.2

Definition 1.3.29. *A pseudo-Riemannian manifold with holonomy group contained in* $\mathrm{Sp}(p,q)$ *is called* pseudo-hyperKähler *with quaternionic signature* (p,q).

A Riemannian manifold with holonomy group contained in $\mathrm{Sp}(n)$ *is called* hyperKähler *of quaternionic dimension* n.

In general, let (M,∇) be a manifold of dimension n with holonomy group $\mathrm{Hol}(\nabla)$. If $H(M)$ is the holonomy bundle, then for any intermediate subgroup $\mathrm{Hol}(\nabla)\leq G\leq \mathrm{GL}(n,\mathbb{R}^n)$, we can build a principal G-bundle $P=H(M)\cdot G\subseteq \mathrm{GL}(M)$. In particular, on a pseudo-hyperKähler manifold (M,g) of quaternionic signature (p,q) with $n=p+q$, we can always reduce the frame bundle $\mathrm{GL}(M)$ to a principal $\mathrm{Sp}(p,q)$-bundle $P=H(M)\cdot\mathrm{Sp}(p,q)$, as $\mathrm{Hol}(\nabla^{LC})\leq \mathrm{Sp}(p,q)$. Moreover, we can describe the vector bundle of endomorphisms of TM as a fibre bundle with fibre $\mathrm{End}(\mathbb{R}^{4n})$ where, for

$A \in \mathrm{Sp}(p,q)$ and $\phi \in \mathrm{End}(\mathbb{R}^{4n})$, the action is defined as $A.\phi = A\phi A^{-1}$. Thus

$$\mathrm{End}(TM) \cong P \times_{\mathrm{Sp}(p,q)} \mathrm{End}(\mathbb{R}^{4n}),$$

where $\phi_x : T_xM \to T_xM$ is mapped to $[u, u^{-1} \circ \phi_x \circ u]$ for all $u : \mathbb{R}^{4n} \to T_xM$ in P_x.

The correspondence between sections of a fibre bundle and equivariant maps of P to the fibre (Remark 1.1.3) is realised by mapping $\phi : M \to \mathrm{End}(TM)$ to $u_x \mapsto u_x^{-1}\phi_x u_x$ for any $u_x \in P_x$. The inverse maps $s : P \to \mathrm{End}(\mathbb{R}^{4n})$ to the section $M \ni x \mapsto u_x \circ s(u_x) \circ u_x^{-1} \in \mathrm{End}(T_xM)$ for any choice of frame $u_x : \mathbb{R}^{4n} \to T_xM$.

Remark 1.3.30. *Notice also that the correspondence between sections of* $\mathrm{End}(TM)$ *and equivariant maps* $P \to \mathrm{End}(\mathbb{R}^{4n})$ *preserves pointwise composition, so the correspondence is an* $\mathbb{R}$*-algebra homomorphism.*

Notice that for all $l \in \mathbb{H}$ such that $l^2 = -1$ and thus $l \in \mathrm{Sp}(1) \cap \mathrm{Im}(\mathbb{H})$ by Remark 1.3.22, $\mathcal{M}_R(l)$ commutes with $\mathrm{Sp}(p,q)$ by Proposition 1.3.19. Equivalently, for all $A \in \mathrm{Sp}(p,q)$, we have $A\mathcal{M}_R(l)A^{-1} = \mathcal{M}_R(l)$, so $\mathcal{M}_R(l)$ is an $\mathrm{Sp}(p,q)$-invariant endomorphism and thus the constant map $P \to \mathrm{End}(\mathbb{R}^{4n})$ mapping every point to $\mathcal{M}_R(l)$ is equivariant. Let $L : TM \to TM$ be the section of $\mathrm{End}(TM)$ corresponding to $\mathcal{M}_R(l)$, then by Remark 1.3.30, we have $L^2 = -\mathrm{id}_{TM}$, so L is an almost complex structure. Notice also that since $\mathrm{Sp}(1) \subseteq \mathrm{SO}(4p,4q)$ (Proposition 1.3.20), L is compatible with the metric. Explicitly, for all $u \in P \subset H(M)\mathrm{SO}(4p,4q)$, the metric is $g = (u^{-1})^t I_{4p,4q} u^{-1}$, so

$$\begin{aligned} g(L\cdot, L\cdot) &= (u^{-1})^t I_{4p,4q} u^{-1} \circ (u_x \mathcal{M}_R(l) u_x^{-1} \otimes u_x \mathcal{M}_R(l) u_x^{-1}) \\ &= (u^{-1})^t \mathcal{M}_R(l)^t I_{4p,4q} \mathcal{M}_R(l) u^{-1} = (u^{-1})^t I_{4p,4q} u^{-1} = g. \end{aligned}$$

Notice also that the $\mathrm{Sp}(p,q)$-invariance of $\mathcal{M}_R(l)$ implies that the infinitesimal action of $\mathfrak{sp}(p,q)$ maps $\mathcal{M}_R(l)$ to 0. It follows that

$$\nabla^{LC} L = u(u^*\omega^{LC}.\mathcal{M}_R(l))u^{-1} = 0.$$

and thus L is integrable, i.e. a complex structure by the Newlander-Nirenberg theorem [33, II, Theorem 2.5, p. 124]), as the *Nijenhuis tensor*

$$\begin{aligned} N_L(X,Y) &= [LX, LY] - [X,Y] - L[X, LY] - L[LX, Y] \\ &= \nabla^{LC}_{LX}(LY) - \nabla^{LC}_{LY}(LX) - \nabla^{LC}_X(Y) + \nabla^{LC}_Y(X) - L\nabla^{LC}_X(LY) \end{aligned}$$

$$+ L\nabla^{LC}_{LY}(X) - L\nabla^{LC}_{LX}(Y) + L\nabla^{LC}_{Y}(LX)$$

vanishes. Moreover, we can define a corresponding symplectic form

$$\omega_L := g(L\cdot, \cdot).$$

In fact, we have $\nabla^{LC}\omega_L = \nabla^{LC}g(L\cdot,\cdot) = g(\nabla^{LC}L\cdot,\cdot) = 0$, so taking its alternating part gives $d\omega_L = 0$. Since L is a complex structure, ω_L is a pseudo-Kähler form.

Remark 1.3.31. *On a (pseudo-)hyperKähler manifold we have a* 2*-sphere of (pseudo-)Kähler structures, each constructed as above from an element* $l \in \mathrm{Sp}(1) \cap \mathrm{Im}(\mathbb{H})$.

Consider in particular $l = i, j, k$ and let $I, J, K : TM \to TM$ be the respective sections of $\mathrm{End}(TM)$. Again by Remark 1.3.30, the quaternionic equations (1.2) translate into

$$I^2 = J^2 = K^2 = IJK = -\mathrm{id}_{TM}. \tag{1.7}$$

We deduce that $I, J, K = IJ$ are complex structures on M compatible with the metric. We call the three corresponding symplectic forms $\omega_I, \omega_J, \omega_K$. It turns out that these three symplectic forms characterise pseudo-hyperKähler manifolds by using the following result of Hitchin.

Lemma 1.3.32. *[29, Lemma 6.8, p. 91] Given a (pseudo-)Riemannian manifold* (M, g) *with* I, J, K *almost complex structures such that* $K = IJ$*, let* $\omega_A = g(A\cdot,\cdot)$ *for* $A = I, J, K$ *be the corresponding symplectic forms, then* I, J, K *are integrable if* $d\omega_I = d\omega_J = d\omega_K = 0$.

Merging this lemma with the previous construction, we get a full characterisation of (pseudo-)hyperKähler manifolds.

Proposition 1.3.33. *A manifold* M *is (pseudo-)hyperKähler if and only if it is a (pseudo-)Riemannian manifold* (M, g) *endowed with three almost complex structures* I, J, K *compatible with the metric, such that* $IJ = K$ *and such that the three* 2*-forms*

$$\omega_A = g(A\cdot,\cdot),$$

for $A = I, J, K$ *are closed.*

Proof. We have already shown one implication. For the converse, if we have I, J, K such that $d\omega_I = d\omega_J = d\omega_K = 0$, then by Lemma 1.3.33 they are integrable and hence they provide three Kähler structures. In particular this means that the holonomy group $\mathrm{Hol}(\nabla^{LC})$ is contained in the intersection of $U_i(2p, 2q)$, $U_j(2p, 2q)$ and $U_k(2p, 2q)$, which by Remark 1.3.24 is $\mathrm{Sp}(p, q)$, thus proving that M must be hyperKähler. $\square$

Definition 1.3.34. *A $4n$-dimensional manifold, with $n \geq 2$ is called* quaternion Kähler *if it is a Riemannian manifold whose holonomy group is contained in* $\mathrm{Sp}(n)\mathrm{Sp}(1)$ *but not in* $\mathrm{Sp}(n)$*. In this case we say that the manifold has* quaternionic dimension *n.*

This definition generalises in a straightforward manner to the pseudo-Riemannian case, but we will not explore this case, so we omit it.

Remark 1.3.35. *We require $n \geq 2$ because if $n = 1$,* $\mathrm{Sp}(1)\mathrm{Sp}(1) = \mathrm{SO}(4n)$ *(Remark 1.3.27) and thus the holonomy condition does not impose restrictions. We also explicitly require that the holonomy group is not contained in* $\mathrm{Sp}(n)$ *in order not to fall into the hyperKähler case.*

Let (M, g) be a quaternion Kähler manifold and let $H(M) \subset \mathrm{GL}(M)$ be its holonomy bundle, then we have a principal $\mathrm{Sp}(n)\mathrm{Sp}(1)$-bundle $P = H(M).\mathrm{Sp}(n)\mathrm{Sp}(1) \subseteq \mathrm{GL}(M)$ with induced projection $p : P \to M$. For quaternion Kähler manifolds we cannot repeat the construction of the almost complex structures, since this time $\mathcal{M}_R(l)$ is no more invariant by the structure group. However, at each point $x \in M$ we have a trivialising open neighbourhood U, that is, such that $P|_U := p^{-1}(U) \cong U \times \mathrm{Sp}(n)\mathrm{Sp}(1)$. Here, for all $l \in \mathbb{H}$ such that $l^2 = -1$, we can define an equivariant map

$$\begin{aligned} U \times \mathrm{Sp}(n)\mathrm{Sp}(1) &\longrightarrow \mathrm{Mat}(4n, \mathbb{R}) \cong \mathrm{End}(\mathbb{R}^{4n}) \\ (y, H) &\longmapsto H^{-1}\mathcal{M}_R(l)H \end{aligned}$$

where H is seen as matrix in $\mathrm{GL}(4n, \mathbb{R})$. This map gives rise to an almost complex structure L on U, but the correspondence $l \mapsto L$ is not well defined, as it depends on the choice of the isomorphism $P|_U \cong U \times \mathrm{Sp}(n)\mathrm{Sp}(1)$.

Remark 1.3.36. *The images of such locally defined almost complex structures define globally a subbundle Z of* $\mathrm{End}(TM)$ *with fibre diffeomorphic to S^2. Explicitly,*

$$Z = P \times_{\mathrm{Sp}(n)\mathrm{Sp}(1)} \{\mathcal{M}_R(l) | l \in \mathbb{H}, l^2 = -1\} \subseteq P \times_{\mathrm{Sp}(n)\mathrm{Sp}(1)} \mathrm{End}(\mathbb{R}^{4n}).$$

By choosing i, j, k in the previous construction, we can find local sections $I, J, K \in \Gamma(U, Z)$ satisfying the quaternionic equations (1.7) *and generating at each point any other section $L \in \Gamma(U, Z)$, meaning that $L = aI + bJ + cK$ with $a, b, c \in \mathcal{C}^\infty(U)$ such that $a^2 + b^2 + c^2 = 1$.*

Remark 1.3.37. *The isomorphism* $\operatorname{End}(TM) \to \Lambda^2 T^*M$ *mapping L to $\omega_L = g(L\cdot, \cdot)$ maps Z to the subbundle $Z^\flat \subseteq \Lambda^2 T^*M$, also with fibre S^2. The three 2-forms $\omega_I, \omega_J, \omega_K \in \Omega^2(U)$ corresponding to I, J, K of Remark 1.3.36 generate all sections of $Z^\flat$.*

Definition 1.3.38. *On a quaternion Kähler manifold M, we have the so-called* fundamental 4-form $\Phi \in \Omega^4(M)$ *defined locally as follows*

$$\Phi = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K.$$

Remark 1.3.39. *Although the three 2-forms are only defined locally and up to a quaternionic rotation, the fundamental 4-form is globally defined and uniquely determined (see e.g. [44, Lemma 9.1, p. 126]).*

The fundamental 4-form can be used to determine whether a manifold is quaternion Kähler. First however, some definitions are necessary.

Definition 1.3.40. *Let M be a smooth manifold, and consider the* exterior algebra *of M, namely the space $\Omega^\bullet(M) = \bigoplus_{k=0}^\infty \Omega^k(M)$ endowed with the graded-antisymmetric operation*

$$\wedge\colon \Omega^\bullet(M) \times \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$$

and the exterior differential

$$d\colon \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$$

defined at each degree by $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$.

An algebraic ideal *is an $\mathbb{R}$-subspace $\mathcal{I} \subseteq \Omega^\bullet(M)$ such that for all $\alpha \in \Omega^\bullet(M)$, $\mathcal{I} \wedge \alpha \subseteq \mathcal{I}$.*

The algebraic ideal generated by a subset $\mathcal{A}$ *is the smallest algebraic ideal containing $\mathcal{A}$.*

A differential ideal *is an algebraic ideal $\mathcal{I}$ such that $d(\mathcal{I}) \subseteq \mathcal{I}$.*

We have the following theorem adapted from [45, Theorem 2.2, p. 423].

Theorem 1.3.41 (Swann 1990). *Let (M, g) be a Riemannian manifold of dimension $4n$ endowed with an S^2-subbundle $Z \subseteq \mathrm{End}(TM)$ such that at every $p \in M$ there is an open neighbourhood U such that sections of $Z|_U$ are generated by almost complex structures $\{I, J, K\}$, such that $IJ = K$. Suppose also that for all $A \in Z$ we have $g(A\cdot, A\cdot) = g$. Let $\Phi \in \Omega^4(M)$ be the fundamental 4-form.*

If $n \geq 3$, then $d\Phi = 0$ implies $\nabla\Phi = 0$ and therefore M has holonomy contained in $\mathrm{Sp}(n)\mathrm{Sp}(1)$. If $n = 2$, M has holonomy contained in $\mathrm{Sp}(n)\mathrm{Sp}(1)$ if and only if the fundamental 4-form Φ is closed and for every open set $U \subseteq M$, the algebraic ideal generated by the sections $U \to Z$ is a differential ideal of $\Omega^\bullet(U)$.

We can express the differential condition in a more explicitly manner in the following corollary.

Corollary 1.3.42. *In the situation of Theorem 1.3.41 it is enough to locally verify that for the generators $\omega_I, \omega_J, \omega_K$ of sections $U \to Z^\flat$ there are $\alpha_I, \alpha_J, \alpha_K \in \Omega(U)$ such that*

$$\begin{aligned} d\omega_I &= \alpha_K \wedge \omega_J - \alpha_J \wedge \omega_K; \\ d\omega_J &= \alpha_I \wedge \omega_K - \alpha_K \wedge \omega_I; \\ d\omega_K &= \alpha_J \wedge \omega_I - \alpha_I \wedge \omega_J. \end{aligned}$$

Proof. If the three conditions are satisfied, then for all $L = aI + bJ + cK$ with $a, b, c \in \mathcal{C}^\infty(U)$ such that $a^2 + b^2 + c^2 = 1$, we have that $d\omega_L$ is a linear combination of $\omega_I, \omega_J, \omega_K$. Moreover, for all $\beta \in \Omega^\bullet(U)$, $d(\omega_L \wedge \beta) = d\omega_L \wedge \beta + \omega_L \wedge d\beta$, which is in the algebraic ideal generated by sections $U \to Z^\flat$. Therefore this ideal is differential. A straightforward computation shows that condition $d\Phi = 0$ is also satisfied.

Conversely, suppose the manifold is quaternion Kähler. For all $L \in \Gamma(U, Z)$ and $X \in \mathfrak{X}(U)$, we have

$$g(LX, X) = g(L^2X, LX) = -g(X, LX) = -g(LX, X)$$

and thus $g(LX, X) = 0$. To simplify the notation, we call $I = I_1$, $J = I_2$, $K = I_3$ and the corresponding forms $\omega_I = \omega_1$, $\omega_J = \omega_2$, and $\omega_K = \omega_3$. These three forms are linearly independent, since given a non-zero $X \in \mathfrak{X}(U)$, the function $\omega_j(I_kX, X) = g(I_jI_kX, X)$ vanishes if and only if $j = k$. Notice that since the wedge product of 2-forms is commutative, the map

$$\Lambda^2T^*U \otimes \Lambda^2T^*U \longrightarrow \Lambda^4T^*U \qquad \alpha \otimes \beta \longmapsto \alpha \wedge \beta \tag{1.8}$$

factors through $S^2(\Lambda^2 T^*U)$ and thus in particular $\mathcal{B} = \{\omega_j \wedge \omega_k | 1 \leq j \leq k \leq 3\}$ is a basis for the image of (1.8). Now let ∇ be the Levi-Civita connection. Since the space $\{l \in \mathbb{H} | l^2 = -1\} \cong \mathrm{Sp}(1) \cap \mathrm{Im}(\mathbb{H})$ is stable under the action of $\mathrm{Sp}(n)\mathrm{Sp}(1)$ (acting by conjugation), in particular there exist forms $\alpha_j^k \in \Omega^1(U)$ for $1 \leq j, k \leq 3$ such that

$$\nabla I_j = \alpha_j^k \otimes I_k$$

and since $\nabla g = 0$,

$$\nabla \omega_j = \alpha_j^k \otimes \omega_k.$$

Since M has holonomy contained in $\mathrm{Sp}(n\mathrm{Sp}(1)$, $\nabla \Phi = 0$ and thus

$$\begin{aligned}\nabla \Phi = \nabla\Big(\sum_{j=1}^{3} \omega_j \wedge \omega_j\Big) &= \sum_{j=1}^{3} (\nabla(\omega_j) \wedge \omega_j + \omega_j \wedge \nabla(\omega_j)) \;= 2\sum_{j=1}^{3} \nabla(\omega_j) \wedge \omega_j \\ = 2\sum_{j,k=1}^{3} \alpha_j^k \otimes \omega_k \wedge \omega_j &= 2\sum_{j=1}^{3} \alpha_j^j \otimes \omega_j \wedge \omega_j + 2\sum_{j<k=1}^{3} (\alpha_j^k + \alpha_k^j) \otimes \omega_j \wedge \omega_k.\end{aligned}$$

Since every coefficient must vanish for the linear independence of $\mathcal{B}$, we get

$$\alpha_j^j = 0, \qquad \alpha_k^j = -\alpha_j^k.$$

Thus we have

$$\begin{array}{llll} \nabla\omega_1 = & & \alpha_2^1 \otimes \omega_2 & -\alpha_1^3 \otimes \omega_3 \\ \nabla\omega_2 = & -\alpha_2^1 \otimes \omega_1 & & +\alpha_3^2 \otimes \omega_3 \\ \nabla\omega_3 = & \alpha_1^3 \otimes \omega_1 & -\alpha_3^2 \otimes \omega_2 & \end{array}$$

The statement follows if we consider the antisymmetric part of these equations. $\square$

Remark 1.3.43. *Notice that in the situation of Theorem 1.3.41 and Corollary 1.3.42, the manifold is hyperKähler if and only if*

$$\alpha_I = \alpha_J = \alpha_K = 0.$$

Remark 1.3.44. *Despite the name, quaternion Kähler manifolds are not necessarily Kähler; in particular, no section of $Z^\flat$ can be Kähler, as they are not closed. This can also be deduced from Remark 1.3.28.*

Examples of quaternion Kähler solvmanifolds are described by [48, Theorem 5.4, p. 1043]. For an explicit list see [8, Table 14.52, p. 409]. An important case is the so-called *quaternionic projective space* in dimension n. Denoted by $\mathbb{P}^n_{\mathbb{H}}$, it can be defined as the projectivisation of $\mathbb{H}^{n+1}$ under the scalar multiplication by $\mathbb{H}$, that is

$$\mathbb{P}^n_{\mathbb{H}} := (\mathbb{H}^{n+1} \setminus \{0\})/ \sim$$

where $x \sim y$ if there is a quaternion $q \in \mathbb{H} \setminus \{0\}$ such that $x = yq$. The action of $\mathrm{Sp}(n+1)$ on $\mathbb{H}^{n+1} \setminus \{0\}$ by left multiplication factors to the quotient and here it is transitive. Consider the point $[e_{n+1}] \in \mathbb{P}^n_{\mathbb{H}}$ class of the $n+1$-th element of the canonical basis of $\mathbb{H}^n$, then its stabiliser consists of matrices of the form

$$\left(\begin{array}{c|c} A & \\ \hline & a \end{array}\right)$$

with $A \in \mathrm{Sp}(n)$ and $a \in \mathrm{Sp}(1)$.

Thus, the quaternionic projective space can be identified with the symmetric space

$$\mathbb{P}^n_{\mathbb{H}} := \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n) \times \mathrm{Sp}(1)}$$

In analogy with the real and complex projective spaces, we have:

$$\mathbb{P}^n_{\mathbb{R}} = \frac{\mathrm{O}(n+1)}{\mathrm{O}(n) \times \mathrm{O}(1)} = \frac{\mathrm{SO}(n+1)}{\mathrm{SO}(n)}, \quad \mathbb{P}^n_{\mathbb{C}} = \frac{\mathrm{U}(n+1)}{\mathrm{U}(n) \times \mathrm{U}(1)} = \frac{\mathrm{SU}(n+1)}{S(\mathrm{U}(n) \times \mathrm{U}(1))}.$$

An important property of quaternion Kähler manifolds is that they are Einstein.

Definition 1.3.45. *A Riemannian manifold (M, g) is called* Einstein *if its Ricci tensor is a multiple of the metric*

$$\mathrm{Ric}_M = \lambda g, \tag{1.9}$$

with $\lambda \in \mathbb{R}$ constant.

Remark 1.3.46. *For connected manifolds in dimension greater than* 2*, it is enough to verify* (1.9) *at each point, and thus assume $\lambda \in \mathcal{C}^\infty(M)$ (See for instance [8, Theorem 1.97, p. 44]), then it coincides with the scalar curvature of M which must therefore be constant.*

We have the following

Theorem 1.3.47 (Berger 1966)**.** *A quaternion Kähler manifold is Einstein.*

Proof. See for instance [8, Theorem 14.39, p. 403]. □

Another reason why quaternion Kähler manifolds are relevant is that they appear in Berger's classification theorem [7, Theorem 3, chapter III, p. 318] which we state as 1.3.48 and therefore they can be thought of as building blocks for Riemannian manifolds. First we recall some definitions: a *locally irreducible Riemannian manifold* is a manifold such that the holonomy representation is irreducible. Moreover, a manifold is *non-locally-symmetric* if $\nabla R \neq 0$, where R is the Riemannian curvature tensor.

Theorem 1.3.48 (Berger's classification)**.** *The holonomy group of a simply connected, locally irreducible, non-locally-symmetric Riemannian manifold must necessarily be one of the following*

1. $\mathrm{SO}(n)$*: generic orientable Riemannian manifolds of dimension* n*;*
2. $\mathrm{U}(n)$*: generic Kähler manifold of real dimension* $2n$*;*
3. $\mathrm{SU}(n)$*: Calabi-Yau manifold of real dimension* $2n$ $(\mathrm{Ric} = 0)$*;*
4. $\mathrm{Sp}(n)\mathrm{Sp}(1)$*: generic quaternion Kähler manifold of dimension* $4n$ *(Einstein,* $\mathrm{Ric} \neq 0$*);*
5. $\mathrm{Sp}(n)$*: generic hyperKähler manifold of dimension* $4n$ $(\mathrm{Ric} = 0)$*;*
6. G_2*: called* G_2*-manifold, has dimension* 7 $(\mathrm{Ric} = 0)$*;*
7. $\mathrm{Spin}(7)$*: called* $\mathrm{Spin}(7)$*-manifold, has dimension* 8 $(\mathrm{Ric} = 0)$*.*

The first two cases are generic orientable, Riemannian and Kähler respectively, while the others are necessarily Einstein. Among these cases, quaternion Kähler manifolds are the only one where the Ricci tensor does not vanish.

1.4 C-map

We are now presenting the so-called c-map, which allows to build quaternion Kähler manifolds starting from projective special Kähler ones. In the first part of this section we present an intrinsic description of this map and in the second we provide a coordinate description.

1.4.1 Intrinsic construction

We start with an intrinsic formulation of the c-map by following [37]. As showed in [5] and [4], the c-map can be divided in two stages: a first one called the rigid c-map which, from a projective special Kähler manifold, produces a hyperKähler one and a second one called HK/QK-correspondence, which allows to change the geometry of the resulting hyperKähler manifold in order to obtain a quaternion Kähler one (See different approaches in [26], [31], [4]). The HK/QK-correspondence has been described in [37] using the twist construction introduced in [46], which will be our approach as well.

The general idea is resumed in Figure 1.2: we start from a projective special Kähler manifold M and we take its associated conic special Kähler manifold $\widetilde{M}$. We then take the associated cotangent bundle $H = T^*\widetilde{M}$ and apply the twist by giving an S^1-bundle P which has suitable quotients Q with a quaternion Kähler structure.

$$\begin{tikzcd}
\widetilde{M} \arrow[dd] & H = T\widetilde{M} \arrow[l] & \\
 & & P \arrow[ul] \arrow[dl] \\
M & Q &
\end{tikzcd}$$

Figure 1.2: c-map: M is projective special Kähler, $\widetilde{M}$ the corresponding conic special Kähler, H hyperKähler, P the so-called twistor bundle and Q quaternion Kähler ([37, Fig.1, p. 1331]).

Rigid c-map

The rigid c-map, first described mathematically in [23], is a construction allowing to build a hyperKähler manifold from a special Kähler one. This part is based on [37, Section 2].

Let M be a generic manifold of dimension n and consider its frame bundle $\pi : \mathrm{GL}(M) \to M$. We can define the *tautological form* $\theta \in \Omega^1(\mathrm{GL}(M), \mathbb{R}^n)$ such that for $X \in T_u\mathrm{GL}(M)$, we have $\theta(X) = u^{-1}\pi_*(X)$. A connection ∇ on M induces on $\mathrm{GL}(M)$ a principal connection form $\omega^\nabla \in \Omega^1(\mathrm{GL}(M), \mathfrak{gl}(n, \mathbb{R}))$ such that for all sections $u : U \to \mathrm{GL}(M)$ defined on an open $U \subseteq M$,

$u^*\omega^\nabla(X)$ corresponds to the endomorphism mapping u_k to $\nabla_X u_k$ for all $X \in TU$. In other words,

$$\nabla(x^k u_k) = dx^k u_k + x^k (u^*\omega^\nabla)^h_k u_h = (dx^k + x^h (u^*\omega^\nabla)^k_h) u_k.$$

Notice that $R_g^*\omega^\nabla = \mathrm{ad}(g^{-1})(\omega^\nabla) = g^{-1}\omega^\nabla g$, where R_g is the action of the element g on the right.

Let now V be a representation of $\mathrm{GL}(n, \mathbb{R})$, we naturally have a right $\mathrm{GL}(n, \mathbb{R})$-action $\mathrm{GL}(M) \times V$ such that $(u, v).g = (ug, g^{-1}.v)$. The quotient by this action is the fibre bundle $\mathrm{GL}(M) \times_{\mathrm{GL}(n,\mathbb{R})} V$ associated to V, already described in Section 1.3.3 and it has a differential structure (see [33, I, Section I.5]) that makes $\mathrm{GL}(M) \times V \to \mathrm{GL}(M) \times_{\mathrm{GL}(n,\mathbb{R})} V$ a principal $\mathrm{GL}(n, \mathbb{R})$-bundle. Moreover, $\mathrm{GL}(M) \times_{\mathrm{GL}(n,\mathbb{R})} V \to M$ is a vector bundle with fibre isomorphic to V.

Consider the map $\nu \colon \mathrm{GL}(M) \times V \to V$ defined as the second projection. Let $g \in \mathrm{GL}(n, \mathbb{R})$, then $(R_g)^*\nu(u, v) = \nu(R_g(u, v)) = \nu(ug, g^{-1}.v) = g^{-1}.v$, thus

$$(R_g)^*\nu = g^{-1}.\nu \tag{1.10}$$

Being V a vector space, $TV \cong V \times V$; explicitly, let $p, v \in V$, then we can define $v_p \in TV$ as the tangent vector at $p \in V$ tangent to the path $\gamma_t = p+tv$ and the mapping $(p, v) \mapsto v_p$ defines the isomorphism. Consider a smooth function $f \in \mathcal{C}^\infty(N, V)$ on a smooth manifold N, take its differential

$$f_* : TN \to TV$$

According to the identification $TV \cong V \times V$, we can see f_* as having two components $TN \to V$. The first one is forced to be $f \circ \pi$ whereas we call the second one df. When $V = \mathbb{R}$, df coincides with df as differential form.

Consider the differential of the map ν, than in particular $\nu_* = (\nu \circ \pi, d\nu)$ and $d\nu \in \Omega^1(\mathrm{GL}(M) \times V, V)$. If we apply the right action, we obtain $R_g^* d\nu(X, v_p) = d\nu((R_g)_*(X), (R_g)_* v_p)$, but

$$\begin{aligned}(R_g)_* v_p &= \frac{d}{dt}(R_g(p + tv))|_{t=0} = \frac{d}{dt}(R_g(p) + tR_g(v))|_{t=0} = R_g(v)_{R_g(p)} \\ &= (g^{-1}.v)_{g^{-1}.p},\end{aligned}$$

so $R_g^* d\nu(X, v_p) = g^{-1}.v$, and thus

$$R_g^* d\nu = g^{-1}.d\nu \tag{1.11}$$

Consider now the cotangent bundle T^*M over an n-dimensional manifold, it can be seen as a fibred product $\mathrm{GL}(M) \times_{\mathrm{GL}(n,\mathbb{R})} (\mathbb{R}^n)^*$, where an element $g \in \mathrm{GL}(n,\mathbb{R})$ acts on the left with the representation dual to the standard one, namely $g.\beta = \beta g^{-1}$. Explicitly, the isomorphism maps a form $\sigma \in T_p^*M$ to $[u,\beta]$ where u is a generic frame and $\beta = \sigma \circ u : \mathbb{R}^n \to \mathbb{R}$. In this particular case $\nu\colon \mathrm{GL}(M) \times (\mathbb{R}^n)^* \to (\mathbb{R}^n)^*$, so (1.10) and (1.11) become $R_g^*\nu = \nu.g$ and $R_g^* d\nu = d\nu g$ respectively, for $g \in \mathrm{GL}(n,\mathbb{R})$.

Suppose we have a connection ∇ on M and let $\omega^\nabla \in \Omega^1(\mathrm{GL}(M), \mathfrak{gl}(n,\mathbb{R}))$ be the corresponding principal connection form. We can define a 1-form $\alpha \in \Omega^1(\mathrm{GL}(M) \times (\mathbb{R}^n)^*, (\mathbb{R}^n)^*)$ as

$$\alpha = d\nu - \nu\omega^\nabla$$

Notice that $R_g^*\alpha = d\nu g - \nu g g^{-1}\omega^\nabla g = \alpha g$, so its kernel is preserved by the action of $\mathrm{GL}(n,\mathbb{R})$, and thus it passes to the quotient $T(\mathrm{GL}(M) \times_{\mathrm{GL}(n,\mathbb{R})} (\mathbb{R}^n)^*)$ as the distribution $\mathrm{Hor}^\nabla \subset T(T^*M)$. This distribution is a complement for the vertical one $\mathrm{Ver} = \ker((\pi_{T^*M})_*) \subset T(T^*M)$, in fact, a vector in $\ker(\alpha)$ is a pair of tangent vectors $(X, v_\sigma) \in T(\mathrm{GL}(M)) \times T(\mathbb{R}^n)^*$ such that $v = \sigma\omega^\nabla(X)$. If (X, v_σ) projects also to the quotient in Ver, then $X \in T(\mathrm{GL}(M))$ is of the form A^* for some $A \in \mathfrak{gl}(n,\mathbb{R})$, so it is of the form $(A^*, \sigma A) = \frac{d}{dt}(u\exp(tA), \sigma\exp(tA))|_{t=0} = \frac{d}{dt}R_{\exp(tA)}(u,\sigma)|_{t=0}$. It is then tangent to the orbits, so it projects to 0 on $T(T^*M)$. The distribution Ver has constant rank n and

$$\begin{aligned}
\mathrm{rk}(\ker(\alpha_{(u,v)})) &= \dim(T(T_u\mathrm{GL}(M) \times T_v(\mathbb{R}^n)^*)) - \dim(\mathrm{Im}(\alpha_{(u,v)})) \\
&\geq \dim(M) + \dim(T_u\mathrm{GL}(M)) + \dim(T_v(\mathbb{R}^n)^*) - \dim((\mathbb{R}^n)^*) \\
&= n + n^2 + n - n = n^2 + n,
\end{aligned}$$

so the quotient Hor^∇ by the action of the n^2-dimensional group $\mathrm{GL}(n,\mathbb{R})$ has dimension at least n. By dimensional reasoning we have that $T(T^*M) = \mathrm{Ver} \oplus \mathrm{Hor}^\nabla$.

On a cotangent bundle $\pi_{T^*M}\colon T^*M \to M$ there is a canonical 1-form λ_{can} defined for any vector X tangent to T^*M in $\beta \in T_p^*M$ as

$$\lambda_{\mathrm{can}}(X) := \beta((\pi_{T^*M})_*X)$$

In the notation of the associated bundle, λ_{can} lifts by pullback to $\mathrm{GL}(M) \times (\mathbb{R}^n)^*$ as the right-invariant form $\nu\theta$, in fact, given a coframe u and a form $\sigma = \sigma_k u^k =$ at $p \in M$, then $(\lambda_{\mathrm{can}})_\sigma = \sigma_k \pi^* u^k$ and

$$(\nu\theta)_{(u,\sigma\circ u)} = \sigma \circ u \circ u^{-1} \circ \pi_* = \sigma_k \pi^* u^k.$$

We can then define a canonical 2-form $\omega_{\mathrm{can}} := -d\lambda_{\mathrm{can}} \in \Omega^2(T^*M)$.

A local frame $u\colon U \to \mathrm{GL}(M)$ on an open $U \subseteq M$ induces a trivialisation

$$\begin{aligned}\pi^{-1}(U) &\longrightarrow U \times \mathrm{GL}(n,\mathbb{R})\\ a &\longmapsto (\pi(a), u^{-1}a),\end{aligned}$$

which in turn induces a trivialisation

$$\begin{aligned}\pi_{T^*M}^{-1}(U) &\longrightarrow U \times \mathrm{GL}(n,\mathbb{R}) \times_{\mathrm{GL}(n,\mathbb{R})} (\mathbb{R}^n)^* \cong U \times (\mathbb{R}^n)^*\\ \sigma = \sigma_k u^k &\longmapsto (\pi_{T^*M}(\beta), (\sigma_1, \dots, \sigma_n)).\end{aligned}$$

Remark 1.4.1. *Given u, we also have an induced local coframe on $\pi_{T^*M}^{-1}(U)$ by taking $\pi_{T^*M}^* u^*\theta = (u^1, \dots, u^n)$ for the horizontal part, and for the vertical ones, consider the form $(u, \mathrm{id}_{(\mathbb{R}^n)^*})^*\alpha \in \Omega^1(U \times (\mathbb{R}^n)^*, (\mathbb{R}^n)^*)$. Explicitly, $(u, \mathrm{id}_{(\mathbb{R}^n)^*})^*\alpha = d\nu - \nu u^*\omega^\nabla$.*

Notice that ω_{can} is closed, being exact, and it is non-degenerate thanks to the following lemma which gives an explicit description of ω_{can} when lifted to $\mathrm{GL}(M) \times (\mathbb{R}^n)^*$:

Lemma 1.4.2. *Given a manifold M endowed with a connection ∇, in the notation above, the lift to $\mathrm{GL}(M) \times (\mathbb{R}^n)^*$ of ω_{can} is $-\alpha \wedge \theta$ if and only if ∇ is torsion free.*

Proof. The torsion form of a connection ∇ is $\Theta = d\theta + \omega^\nabla \wedge \theta = 0$ and since λ_{can} lifts to $\nu\theta$, so $-\omega_{\mathrm{can}}$ lifts to

$$\begin{aligned}d(\nu\theta) &= d\nu \wedge \theta + \nu d\theta = d\nu \wedge \theta + \nu(\Theta - \omega^\nabla \wedge \theta) = (d\nu - \nu\omega^\nabla) \wedge \theta + \nu\Theta\\ &= \alpha \wedge \theta + \nu\Theta,\end{aligned}$$

and $\nu\Theta$ vanishes if and only if $\Theta = 0$. $\square$

We can now prove that ω_{can} is non-degenerate. Let $X \in TT^*M$, then it splits in $X = X_V + X_H$ with $X_V \in \mathrm{Ver}$ and $X_H \in \mathrm{Hor}^\nabla$. Let $\widetilde{X_V}$ and $\widetilde{X_H}$ be related lifts to $\mathrm{GL}(M) \times (\mathbb{R}^n)^*$ of X_V and X_H respectively. Since ω_{can} lifts to $-\alpha \wedge \theta$, we can pull back $\omega_{\mathrm{can}}(X, \cdot)$ to $\mathrm{GL}(M) \times (\mathbb{R}^n)^*$ obtaining

$$-\alpha \wedge \theta(\widetilde{X_V} - \widetilde{X_H}, \cdot) = -\alpha(\widetilde{X_V})\theta + \theta(\widetilde{X_H})\alpha.$$

In particular, $\alpha(\widetilde{X_V})\theta$ factors to a horizontal 1-form, i.e. it vanishes on Ver, and $\theta(\widetilde{X_H})\alpha$ to a vertical one, i.e. it vanishes on Hor^∇. These two forms are linearly independent and thus $\omega_{\mathrm{can}}(X,\cdot)$ vanishes at a point if and only if they both do. Since α and θ do not vanish, necessarily $\theta(\widetilde{X_H})$ and $\alpha(\widetilde{X_V})$ vanish, so X_V and X_H, and thus X, are both vertical and horizontal, implying that X vanishes.

Suppose now M is (pseudo-)Kähler of dimension $2n$, then we have a (pseudo-)Riemannian metric g of signature $(2p,2q)$ and is endowed with a compatible almost complex structure I.

If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, we can see $\mathrm{GL}(n,\mathbb{C})$ inside $\mathrm{GL}(2n,\mathbb{R})$ and in particular we have an action of i on $\mathbb{R}^{2n}$ given by

$$I_0 = \left(\begin{array}{cc|c|cc} 0 & -1 & & & \\ 1 & 0 & & & \\ \hline & & \ddots & & \\ \hline & & & 0 & -1 \\ & & & 1 & 0 \end{array}\right), \tag{1.12}$$

corresponding to iI_n as a complex matrix. Explicitly, $\mathrm{GL}(n,\mathbb{C})$ corresponds to the subset of matrices in $\mathrm{GL}(2n,\mathbb{R})$ that commute with I_0. Intersect this subset with

$$\mathrm{O}(2p,2q) = \{A \in \mathrm{Mat}(2n,\mathbb{R}) | A^t I_{2p,2q} A = I_{2p,2q}\}$$

in order to obtain an inclusion of $\mathrm{U}(p,q)$ in $\mathrm{GL}(2n,\mathbb{R})$.

We say that a frame $u : \mathbb{R}^{2n} \to T_pM$ is adapted to the complex structure if $uI_0 = Iu$. In this case we can use the standard identification and view u as an isomorphism $\mathbb{C}^n \to T_pM$, so that $u(iI_n) = Iu$; we call such frames *complex frames*. The frame u is adapted to the Riemannian structure, or equivalently orthonormal, if $g(u,u) = I_{2p,2q}$. Consider the subbundle of $\mathrm{GL}(M)$ of frames adapted to both the Riemannian and the complex structure:

$$\mathrm{U}(M,g,I) = \{u \in \mathrm{GL}(M) | Iu = uI_0, g(u,u) = I_{2p,2q}\}.$$

This is a principal bundle with structure group $\mathrm{U}(p,q)$.

As we did for $\mathrm{GL}(M)$, we can write the cotangent space over M as the fibred product $\mathrm{U}(M,g,I) \times_{\mathrm{U}(p,q)} (\mathbb{R})^*$, where the representation of $\mathrm{U}(p,q)$ on $\mathbb{R}^{2n}$ is induced by the standard representation of $\mathrm{GL}(2n,\mathbb{R})$.

Let (M,g,I,∇) be a special Kähler manifold, let $H = T^*M$, we now build on it a natural pseudo-hyperKähler structure. Suppose g has signature

$(2p,2q)$, then $\pi^* g = \theta^t I_{2p,2q}\theta$, where, recall $I_{2p,2q} = \left(\begin{array}{c|c} I_{2p} & 0 \\ \hline 0 & -I_{2q} \end{array}\right)$. Consider the covariant tensor $\theta^t I_{2p,2q}\theta + \alpha I_{2p,2q}\alpha^t$. For all $A \in \mathrm{U}(p,q)$ we have

$$\begin{aligned} R_A^*(\theta^t I_{2p,2q}\theta + \alpha I_{2p,2q}\alpha^t) &= (A^{-1}\theta)^t I_{2p,2q} A^{-1}\theta + \alpha A I_{2p,2q}(\alpha A)^t \\ &= \theta^t (A^{-1})^t I_{2p,2q} A^{-1}\theta + \alpha A I_{2p,2q} A^t \alpha^t = \theta^t I_{2p,2q}\theta + \alpha I_{2p,2q}\alpha^t, \end{aligned}$$

as both A, A^{-1} belong to $\mathrm{O}(2p,2q)$. We infer that $\theta^t I_{2p,2q}\theta + \alpha I_{2p,2q}\alpha^t$ passes to the quotient $\mathrm{U}(M,g,I) \times_{\mathrm{U}(p,q)} (\mathbb{R}^n)^* = H$ as the tensor $\widehat{g} \in T_2 H$.

Consider now

$$\frac{1}{2}(\theta^t \wedge I_0^t I_{2p,2q}\theta - \alpha \wedge I_0^t I_{2p,2q}\alpha^t)$$

This is a 2-form, since $I_0^t I_{2p,2q}$ is antisymmetric. Again, for $A \in \mathrm{U}(p,q)$, we have

$$\begin{aligned} &\frac{1}{2}R_A^*(\theta^t \wedge I_0^t I_{2p,2q}\theta - \alpha \wedge I_0^t I_{2p,2q}\alpha^t) \\ &\quad = \frac{1}{2}(\theta^t \wedge (A^{-1})^t I_0^t I_{2p,2q} A^{-1}\theta - \alpha A \wedge I_0^t I_{2p,2q} A^t \alpha^t) \\ &\quad = \frac{1}{2}(\theta^t \wedge (I_0 A^{-1})^t I_{2p,2q} A^{-1}\theta - \alpha \wedge (I_0 A^t)^t I_{2p,2q} A^t \alpha^t) \\ &\quad = \frac{1}{2}(\theta^t \wedge (A^{-1} I_0)^t I_{2p,2q} A\theta - \alpha \wedge (A^t I_0)^t I_{2p,2q} A^t \alpha^t) \\ &\quad = \frac{1}{2}(\theta^t \wedge I_0^t (A^{-1})^t I_{2p,2q} A^{-1}\theta - \alpha \wedge I_0^t A I_{2p,2q} A^t \alpha^t) \\ &\quad = \frac{1}{2}(\theta^t \wedge I_0^t I_{2p,2q}\theta - \alpha \wedge I_0^t I_{2p,2q}\alpha^t), \end{aligned}$$

as both A and A^{-1} belong to $\mathrm{U}(p,q)$. Being right invariant, it passes to the quotient H as a 2-form $\omega_I \in \Omega^2(H)$.

We then define $\omega_J := -\omega_{\mathrm{can}}$, which pulls back to $\alpha \wedge \theta$ as we saw earlier. Finally, consider $-\alpha \wedge I_0\theta$ and let $A \in \mathrm{U}(p,q)$.

$$-R_A^*(\alpha \wedge I_0\theta) = -\alpha A \wedge I_0 A^{-1}\theta = -\alpha \wedge A A^{-1} I_0 \theta = -\alpha \wedge I_0\theta.$$

By invariance, this form passes to the quotient as $\omega_K \in \Omega^2(M)$.

Remark 1.4.3. *Consider $u \in \mathrm{U}(M,g,I)$, then as observed in Remark 1.4.1, we get an induced coframe $\widehat{u}^{-1} = (\widehat{u}^k)_{k=1,\dots,2n}$. With respect to this coframe, the forms $\omega_I, \omega_J, \omega_K$ are represented by the matrices*

$$\left(\begin{array}{c|c} I_0^t I_{2p,2q} & \\ \hline & -I_0^t I_{2p,2q} \end{array}\right), \qquad \left(\begin{array}{c|c} & -I_{2n} \\ \hline I_{2n} & \end{array}\right), \qquad \left(\begin{array}{c|c} & -I_0 \\ \hline -I_0 & \end{array}\right).$$

The metric instead is the block matrix $\left(\begin{array}{c|c} I_{2p,2q} & \\ \hline & I_{2p,2q} \end{array}\right)$*, so we can find endomorphisms L such that $\omega_L = g(L\cdot, \cdot)$ and write them with respect to the frame $\widehat{u}$ obtaining for I, J, K respectively:*

$$\left(\begin{array}{c|c} I_0 & \\ \hline & -I_0 \end{array}\right), \qquad \left(\begin{array}{c|c} & I_{2p,2q} \\ \hline -I_{2p,2q} & \end{array}\right), \qquad \left(\begin{array}{c|c} & I_0 I_{2p,2q} \\ \hline I_0 I_{2p,2q} & \end{array}\right).$$

Notice that the quaternionic equations (1.7) *hold.*

We can now state the following

Proposition 1.4.4. *Let $(M, g, I, \omega, \nabla)$ be a special Kähler manifold, then on $H = T^*M$, the previously defined $\omega_I, \omega_J, \omega_K$ determine a pseudo-hyperKähler structure.*

Proof. The idea is to use Proposition 1.3.33, in fact, by Remark 1.4.3 we know that the quaternionic equations (1.7) are satisfied, and by a straightforward computation, one checks also the compatibility of I, J, K with the metric.

We are only left to prove that $\omega_I, \omega_J, \omega_K$ are closed. We start with the easiest case: $\omega_J = -\omega_{\text{can}} = d\lambda_{\text{can}}$ which is exact and therefore closed. For ω_I, we need some preliminary result first. Notice that $d\alpha = -d\nu \wedge \omega^\nabla - \nu d\omega^\nabla = -\alpha \wedge \omega^\nabla - \nu\Omega^\nabla$ where Ω^∇ is the curvature of ∇ and thus it is zero, implying

$$d\alpha = -\alpha \wedge \omega^\nabla. \tag{1.13}$$

Notice also that since ∇ is symplectic with respect to the form represented by $S = I_0^t I_{2p,2q}$, we have

$$S\omega^\nabla + (\omega^\nabla)^t S = 0, \tag{1.14}$$

and since $S^2 = -I_{2n}$, if we conjugate (1.14) by S, we also obtain

$$\omega^\nabla S = -S(\omega^\nabla)^t.$$

Finally, notice that

$$(\beta_1 \wedge \beta_2)^t = (-1)^{d_1, d_2} \beta_2^t \wedge \beta_1^t, \tag{1.15}$$

where β_1 and β_2 are differential forms of degree respectively d_1, d_2.

Now consider the pullback of $2d\omega_I$ to $\mathrm{U}(M,g,I) \times (\mathbb{R}^{2n})^*$, we have

$$\begin{aligned} &d(\theta^t \wedge S\theta - \alpha \wedge S\alpha^t) \\ &\quad = d\theta^t \wedge S\theta - \theta^t \wedge Sd\theta - d\alpha \wedge S\alpha^t + \alpha \wedge Sd\alpha^t \\ &\quad = -(\omega^\nabla \wedge \theta)^t \wedge S\theta - \theta^t \wedge S(-\omega^\nabla \wedge \theta) + \alpha \wedge \omega^\nabla \wedge S\alpha^t - \alpha \wedge S(\alpha \wedge \omega^\nabla)^t \\ &\quad = \theta^t \wedge ((\omega^\nabla)^t S + S\omega^\nabla) \wedge \theta + \alpha \wedge (\omega^\nabla S + S(\omega^\nabla)^t) \wedge \alpha^t = 0, \end{aligned}$$

thus proving that ω_I is closed.

Finally, for ω_K, one can prove that the special condition (1.1) implies $\omega_K = d(-\nu I_0 \theta)$ (see e.g. [37, Lemma 2.3, p. 1333]). Therefore, being exact, ω_K is closed, proving that H is pseudo-hyperKähler. □

Remark 1.4.5. *The statement of Proposition 1.4.4 can be strengthened, as the implication can be reversed. In [37, Proposition 2.4, p. 1334], Macia and Swann prove that if we have a pseudo-Hermitian manifold $(M, g, I, \omega, \nabla)$ and a symplectic connection on it, then the 3-forms $\omega_I, \omega_J, \omega_K$ determine a pseudo-hyperKähler structure if and only if $(M, g, I, \omega, \nabla)$ is a special Kähler manifold.*

Moreover, if the special Kähler manifold is conic, then we can lift an infinitesimal isometry to the cotangent space using the connection.

Proposition 1.4.6. *If $(M, g, I, \omega, \nabla, \xi)$ is conic special Kähler, then let $\widehat{X}$ be the horizontal lift of $-I\xi$ to a vector field $\widehat{X}$ on T^*M with the hyperKähler structure defined in Proposition 1.4.4 by the Kähler forms $\omega_I, \omega_J, \omega_K$. Then, $\widehat{X}$ is an infinitesimal isometry such that*

$$\mathcal{L}_{\widehat{X}} \omega_I = 0, \qquad \mathcal{L}_{\widehat{X}} \omega_J = \omega_K, \qquad \mathcal{L}_{\widehat{X}} \omega_K = -\omega_J.$$

Proof. Let $\widetilde{X}$ be a lift of $\widehat{X}$ to $\mathrm{U}(M,g,I) \times_{\mathrm{U}(p,q)} (\mathbb{R}^{2n})^*$. Since $\widehat{X}$ is horizontal we have

$$\alpha(\widetilde{X}) = 0, \qquad \pi_*(\widetilde{X}) = -I\xi. \tag{1.16}$$

Let $\chi := \theta(\widetilde{X})$, then $\chi(u) = u^{-1}\pi_*\widetilde{X} = u^{-1}(-I\xi)$, so $\chi : \mathrm{U}(M,g,I) \to \mathbb{R}^{2n}$ is the equivariant map corresponding to the vector $-I\xi$. Thus, by Lemma 1.2.3 we have $\nabla(-I\xi) = -I$, which in the language of principal connections corresponds to $d\chi + \omega^\nabla \chi = -I_0\theta$, since for all $u \in \mathrm{U}(M,g,I)$, we have $-Iu\theta_u = u(-I_0\theta_u)$. Thus,

$$d\chi = -\omega^\nabla \chi - I_0\theta.$$

Therefore, we can compute

$$\begin{aligned}\mathcal{L}_{\widetilde{X}}\theta &= d\iota_{\widetilde{X}}\theta + \iota_{\widetilde{X}}d\theta = d\chi + \iota_{\widetilde{X}}(-\omega^\nabla \wedge \theta)\\ &= -\omega^\nabla\chi - I_0\theta - \omega^\nabla(\widetilde{X})\theta + \omega^\nabla\theta(\widetilde{X}) = -I_0\theta - \omega^\nabla(\widetilde{X})\theta;\\ \mathcal{L}_{\widetilde{X}}\alpha &= d\iota_{\widetilde{X}}\alpha + \iota_{\widetilde{X}}d\alpha = 0 + \iota_{\widetilde{X}}(-\alpha\wedge\omega^\nabla) = \alpha\omega^\nabla(\widetilde{X}).\end{aligned}$$

We compute $\mathcal{L}_{\widehat{X}}\omega_I$ by lifting it to $\mathrm{U}(M, g, I)$, obtaining, for $S = I_0^t I_{2p,2q}$:

$$\begin{aligned}&\mathcal{L}_{\widetilde{X}}(\theta^t \wedge S\theta - \alpha\wedge S\alpha^t)\\ &\quad= (\mathcal{L}_{\widetilde{X}}\theta)^t \wedge S\theta + \theta^t\wedge S\mathcal{L}_{\widetilde{X}}\theta - \mathcal{L}_{\widetilde{X}}\alpha\wedge S\alpha^t - \alpha\wedge S\mathcal{L}_{\widetilde{X}}\alpha^t\\ &\quad= (-I_0\theta - \omega^\nabla(\widetilde{X})\theta)^t\wedge S\theta + \theta^t\wedge S(-I_0\theta - \omega^\nabla(\widetilde{X})\theta)\\ &\qquad - \alpha\omega^\nabla(\widetilde{X})\wedge S\alpha^t - \alpha\wedge S\omega^\nabla(\widetilde{X})^t\alpha^t\\ &\quad= -\theta^t\wedge(I_0^tS + SI_0)\theta - \theta^t\wedge(\omega^\nabla(\widetilde{X})^tS + S\omega^\nabla(\widetilde{X}))\theta\\ &\qquad - \alpha\wedge(\omega^\nabla(\widetilde{X})S + S\omega^\nabla(\widetilde{X})^t)\alpha^t = 0\end{aligned}$$

where we used (1.15), compatibility of ω with I, (1.14) and (1.4.1).

We compute $\mathcal{L}_{\widehat{X}}\omega_J$ in the same way, obtaining

$$\begin{aligned}\mathcal{L}_{\widetilde{X}}(\alpha\wedge\theta) &= d\iota_{\widetilde{X}}(\alpha\wedge\theta) + 0 = d(\alpha(\widetilde{X})\theta) - \alpha\theta(\widetilde{X}) = 0 - d(\alpha\chi)\\ &= -d\alpha\chi + \alpha\wedge d\chi = \alpha\wedge\omega^\nabla\chi + \alpha\wedge(-I_0\theta - \omega^\nabla\chi) = -\alpha\wedge I_0\theta\end{aligned}$$

which is the lift of ω_K. Similarly, for ω_K:

$$\begin{aligned}\mathcal{L}_{\widetilde{X}}(-\alpha\wedge I_0\theta) &= -d\iota_{\widetilde{X}}(\alpha\wedge I_0\theta) + 0 = -d(\alpha(\widetilde{X})I_0\theta) + d(\alpha I_0\theta(\widetilde{X}))\\ &= 0 + d(\alpha I_0\chi) = d\alpha I_0\chi - \alpha\wedge d(I_0\chi)\\ &= -\alpha\wedge(d(I_0\chi) + \omega^\nabla I_0\chi).\end{aligned}$$

Notice that $I_0\chi = I_0\theta(-I\xi) = \theta(\xi); \mathrm{U}(M, g, I) \to \mathbb{R}^{2n}$ is the equivariant map corresponding to ξ in TM and thus $d(I_0\chi) + \omega^\nabla I_0\chi$ corresponds to $\nabla\xi = \mathrm{id}$, therefore $d(I_0\chi) + \omega^\nabla I_0\chi = \theta$. This proves $\mathcal{L}_{\widehat{X}}\omega_K = -\omega_J$.

Finally, we compute $\mathcal{L}_{\widehat{X}}\widehat{g}$

$$\begin{aligned}&\mathcal{L}_{\widetilde{X}}(\theta^t I_{2p,2q}\theta + \alpha I_{2p,2q}\alpha^t)\\ &\quad= (\mathcal{L}_{\widetilde{X}}\theta)^t I_{2p,2q}\theta + \theta^t I_{2p,2q}\mathcal{L}_{\widetilde{X}}\theta + \mathcal{L}_{\widetilde{X}}\alpha I_{2p,2q}\alpha^t + \alpha I_{2p,2q}(\mathcal{L}_{\widetilde{X}}\alpha)^t\\ &\quad= (-I_0\theta - \omega^\nabla(\widetilde{X})\theta)^t I_{2p,2q}\theta + \theta^t I_{2p,2q}(-I_0\theta - \omega^\nabla(\widetilde{X})\theta)\\ &\qquad + \alpha\omega^\nabla(\widetilde{X})I_{2p,2q}\alpha^t + \alpha I_{2p,2q}\omega^\nabla(\widetilde{X})^t\alpha^t\end{aligned}$$

$$\begin{aligned}
&= -\theta^t(I_0 I_{2p,2q} + I_{2p,2q} I_0)\theta - \theta^t \wedge (\omega^\nabla(\widetilde{X})^t I_{2p,2q} + I_{2p,2q}\omega^\nabla(\widetilde{X}))\theta \\
&\quad + \alpha(\omega^\nabla(\widetilde{X}) I_{2p,2q} + I_{2p,2q}\omega^\nabla(\widetilde{X})^t)\alpha^t \\
&= 0 - \theta^t(\omega^\nabla(\widetilde{X})^t S I_0 + S I_0 \omega^\nabla(\widetilde{X}))\theta - \alpha(\omega^\nabla(\widetilde{X}) I_0^t S + I_0^t S \omega^\nabla(\widetilde{X})^t)\alpha^t \\
&= \theta^t \wedge S(\omega^\nabla(\widetilde{X}) I_0 - I_0\omega^\nabla(\widetilde{X}))\theta + \alpha(\omega^\nabla(\widetilde{X}) I_0 - I_0\omega^\nabla(\widetilde{X}))S\alpha^t = 0
\end{aligned}$$

It vanishes because $\omega^\nabla(\widetilde{X})I_0 = I_0\omega^\nabla(\widetilde{X})$ and this is due to the fact that for every vector field Y, we have

$$\begin{aligned}
\nabla_{-I\xi}(IY) - I\nabla_{-I\xi}Y = \nabla_{-I\xi}I(Y) = \nabla_Y I(-I\xi) &= \nabla_Y\xi - I\nabla_Y(-I\xi) \\
= Y + I^2Y = 0
\end{aligned}$$

and thus on the frame bundle,

$$\begin{aligned}
(\omega^\nabla(\widetilde{X})I_0 - I_0\omega^\nabla(\widetilde{X}))\theta &= \omega^\nabla(\widetilde{X})I_0\theta - I_0\omega^\nabla(\widetilde{X})\theta \\
&= dI_0\theta + \omega^\nabla(\widetilde{X})I_0\theta - (I_0 d\theta + I_0\omega^\nabla(\widetilde{X}))\theta = 0.
\end{aligned}$$

This concludes the proof. □

Twist

Now we present the the construction that takes a pseudo-hyperKähler manifold with a suitable infinitesimal isometry as in Proposition 1.4.6, and modifies it in order to obtain a quaternion Kähler manifold. We will describe this construction in terms of the so-called twist construction, following [37].

Let (H, g, I, J, K) be a pseudo-hyperKähler manifold with a vector field X such that

$$\mathcal{L}_X g = 0, \qquad \mathcal{L}_X I = 0, \qquad \mathcal{L}_X J = K, \qquad \mathcal{L}_X K = -J \tag{1.17}$$

The idea of the twist is to consider a suitable S^1-bundle $P \to H$, lift the isometry to P in a suitable way and then quotient by said isometry in order to get another manifold Q which, with suitable modifications of the structure, will be quaternion Kähler.

Let us start with the first ingredient, that is $p : P \to H$; a principal S^1-bundle endowed with a principal connection ϑ with curvature

$$\Psi \in \Omega^2(H).$$

Then we need the so-called *twisting function*, that is $a \in \mathcal{C}^\infty(H)$ such that

$$da = -\iota_X \Psi.$$

Let now Y be the generator of the principal action on P and consider the horizontal lift $\widehat{X}_0$ to P. We define a lift $\widehat{X} = \widehat{X}_0 + aY$ of X.

If we now quotient P by the action of $\widehat{X}$, we obtain a manifold $Q := P/\langle \widehat{X} \rangle$ which is the twisted manifold.

A tensor on H can be mapped to a tensor on Q by taking the corresponding invariant tensor on P, pulling it back to the distribution

$$\mathrm{Hor}^\vartheta = \ker(\vartheta) \hookrightarrow P,$$

and then factoring the resulting tensor to the quotient Q. The initial tensor in H and the final in Q will be said to be Hor^ϑ-related and we will denote this relation by $\sim_\vartheta$. In order for this procedure to work, it is required that $\widehat{X} \notin \mathrm{Hor}^\vartheta$, otherwise we will lose some non-trivial tensors in the process. The condition of being transverse to Hor^ϑ at a point p is equivalent to $a(p) \neq 0$ by the way we defined $\widehat{X}$.

It turns out that the twist construction does not preserve closed forms, so we need to change our structure on H in order to obtain the structure we want on Q. Before doing so, we define the following 1-forms:

$$\alpha_0 := X^\flat = g(X, \cdot), \qquad \alpha_L := (LX)^\flat = \iota_X \omega_L = L\alpha_0, \text{ for } L = I, J, K. \tag{1.18}$$

We require the new metric g^Q to be such that

$$g^Q \sim_\vartheta fg + h(\alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2),$$

for $f, h \in \mathcal{C}^\infty(H)$.

In [37], Macia and Swann prove the following theorem describing exactly for which Ψ, a, f, h we obtain a twisted manifold Q which is quaternion Kähler.

Theorem 1.4.7. *[37, Theorem 4.1, p. 1339] Let (H, g, I, J, K) be a hyperKähler manifold of dimension at least 8 endowed with a vector field X satisfying (1.17). Then the only twists (Q, g_Q) that are quaternion Kähler are obtained by the data*

$$\Psi = k(dX^\flat + \omega_I), \qquad\qquad a = k(g(X,X) - \mu + c)$$

$$f = \frac{B}{\mu - c}, \qquad h = -\frac{B}{(\mu - c)^2},$$

where μ is the moment map for X on (M, g, I) and $c, k, B \in \mathbb{R}$ are constants.

Proof. In dimension greater than 8 see [37, Section 4.1, p. 1340] and for dimension 8 see [37, Section 4.2, p. 1344]. □

For reference, we also give the following result that allows us to compute exterior differentials of differential forms on Q in terms of differential forms in H, when these are invariant with respect to the infinitesimal isometry.

Proposition 1.4.8. *Let (H, g, I, J, K) be a hyperKähler manifold of dimension at least 8 endowed with a vector field X satisfying (1.17) and let (Q, g_Q) be its twist with respect to an S^1-bundle with curvature Ψ and a twisting function a. Let $\sigma \in \Omega^p(H)$ be X-invariant and let $\sigma^Q \in \Omega^p(Q)$ be such that $\sigma^Q \sim_\vartheta \sigma$, then*

$$d\sigma^Q \sim_\vartheta d\sigma - \frac{1}{a}\Psi \wedge \iota_X \sigma.$$

Proof. See [46, Corollary 3.6, p. 412]. □

1.4.2 Coordinate description

We now proceed by presenting the explicit coordinate construction of the c-map as illustrated in [18]. This construction was introduced in [21] (with a different notation), where an explicit metric is put on the final quaternion Kähler metric called Ferrara-Sabharwal metric. This extrinsic construction can be related, via [4], to the intrinsic one, presented in Section 1.4.1.

The following construction is carried out on projective special Kähler domains, and thus on projective special Kähler manifolds $(\pi : \widetilde{M} \to M, \nabla)$ endowed with a holomorphic prepotential for $\widetilde{M}$, so we have $F : \widetilde{M} \to \mathbb{C}$ homogeneous of degree 2 in the sense of Corollary 1.2.8.

Consider the Hessian matrix $\partial^2 F = (\partial^2_{h,k} F)_{h,k}$. Notice that $\partial^2 F$ has now entries that are homogeneous of degree 0. In particular they are invariant by the action generated by ξ by Corollary 1.2.8. This implies that $\partial^2 F$ is well defined on M.

Let $B = 2\,\mathrm{Im}(\partial^2 F)$, then it is also well defined on M. Using this matrix, we can build on $\widetilde{M}$ the following

$$\mathcal{N} = \overline{\partial^2 F} + i\frac{Bzz^tB^t}{z^tBz} = \mathcal{R} + i\mathcal{I},$$

where $\mathcal{R}$ and $\mathcal{I}$ are the real and imaginary part of $\mathcal{N}$ respectively. Notice that this map is still homogeneous of degree 0, and thus both $\mathcal{R}$ and $\mathcal{I}$ are well defined on M.

We are now ready to build the final quaternion Kähler manifold as

$$Q = M \times G,$$

where $G \cong \mathbb{R}^{2n+3} \times \mathbb{R}^+$ is a Lie group. Explicitly, we call the coordinates on G

$$(\widetilde{\zeta}, \zeta, \widetilde{\phi}, \phi) = (\widetilde{\zeta}_0, \dots, \widetilde{\zeta}_n, \zeta_0, \dots, \zeta_n, \widetilde{\phi}, \phi).$$

The group multiplication is then defined as

$$\begin{aligned}&(\widetilde{\zeta}, \zeta, \widetilde{\phi}, \phi) \cdot (\widetilde{\zeta}', \zeta', \widetilde{\phi}', \phi')\\ &\quad = (\widetilde{\zeta} + e^{\frac{\phi}{2}}\widetilde{\zeta}', \zeta + e^{\frac{\phi}{2}}\zeta', \widetilde{\phi} + e^{\phi}\widetilde{\phi}' + e^{\frac{\phi}{2}}(\zeta^t\widetilde{\zeta}' - \zeta'^t\widetilde{\zeta}), \phi + \phi').\end{aligned}$$

Now let $\pi : Q \to M$ be the projection on the first component, then Q is a trivial principal bundle. Let $p \in M$, then on $\pi^{-1}(p) \cong G$ we can put the following metric

$$\begin{aligned}g_G = {}& \frac{1}{4\phi^2}d\phi^2 + \frac{1}{4\phi^2}\left(d\widetilde{\phi} + \sum_{h=0}^{n}(\zeta_h d\widetilde{\zeta}_h - \widetilde{\zeta}_h d\zeta_h\right)^2 + \frac{1}{2\phi}\sum_{u,v=0}^{n}\mathcal{I}_{u,v}(p)d\zeta_u d\zeta_v\\ &+ \frac{1}{2\phi}\sum_{u,v=0}^{n}(\mathcal{I}^{-1})_{u,v}(p)\left(d\widetilde{\zeta}_u + \sum_{h=0}^{n}\mathcal{R}_{u,h}(p)d\zeta_h\right)\left(d\widetilde{\zeta}_v + \sum_{k=0}^{n}\mathcal{R}_{v,k}(p)d\zeta_k\right),\end{aligned}$$

where the coefficients are evaluated at p. Notice that g_G varies smoothly with respect to $p \in M$ and it can then be defined on all of Q.

We can now define on Q the metric

$$g_Q = \pi^* g_M + g_G, \tag{1.19}$$

where g_M is the metric of M. The metric (1.19) is called the *Ferrara-Sabharwal metric*.

From the group multiplication G, we can find left invariant forms. It follows that the following coframe is left invariant with respect to G:

$$\begin{aligned}\xi^k &= \sqrt{\frac{2}{\phi}}\left(d\widetilde{\zeta}_k + \sum_{h=0}^{n}\mathcal{R}_{k,h}d\zeta_h\right), \qquad \xi^{n+1} = \frac{d\phi}{\phi},\\ \eta^k &= \sqrt{\frac{2}{\phi}}d\zeta_k, \qquad \eta^{n+1} = \frac{1}{\phi}\left(d\widetilde{\phi} + \sum_{h=0}^{n}(\zeta_h d\widetilde{\zeta}_h - \widetilde{\zeta}_h d\zeta_h\right).\end{aligned} \tag{1.20}$$

With respect to (1.20), the metric g_G can be written as

$$g_G = \frac{1}{4}\left(\sum_{k,h=0}^{n}(\mathcal{I}^{-1})_{h,k}\xi^h\xi^k + (\xi^{n+1})^2 + \sum_{k,h=0}^{n}\mathcal{I}_{h,k}\eta^h\eta^k + (\eta^{n+1})^2\right). \quad (1.21)$$

We can also explicitly compute the differential of the invariant coframe (1.20)

$$\begin{aligned} d\xi^k &= \frac{1}{2}\xi^k \wedge \xi^{n+1}, \qquad d\eta^{n+1} = -\frac{1}{2}\sum_{h=0}^{n+1}\xi_h \wedge \eta^h, \\ d\eta^k &= -\frac{1}{2}\xi^{n+1} \wedge \eta^k, \qquad d\xi^{n+1} = 0. \end{aligned} \quad (1.22)$$

Let $\mathfrak{g}$ be the Lie algebra of G. From (1.22) we deduce that the direction ξ_{n+1} is such that $\mathrm{ad}(\xi_{n+1})$ acts diagonally on $\mathfrak{g}$. It follows that $\mathfrak{g}$ can be seen as $\mathfrak{h} \rtimes \langle\xi_{n+1}\rangle$, where $\mathfrak{h} = \langle\xi_0, \ldots, \xi_n, \eta_0, \ldots, \eta_{n+1}\rangle$ and the the endomorphism defining the semidirect product is $\mathrm{ad}(\xi_{n+1})|_{\mathfrak{h}}$. Notice also that the only non-zero structure constants of $\mathfrak{h}$ are

$$[\xi_h, \eta_k] = \frac{\delta_{h,k}}{2}\eta_{n+1},$$

for $h, k = 0, \ldots, n$.

Recall that the Heisenberg Lie algebra $\mathfrak{h}_m$ has a basis

$$A_1, \ldots, A_m, B_1, \ldots, B_m, D$$

such that the only non-zero Lie brackets of elements of the basis are

$$[A_k, B_k] = -[B_k, A_k] = D,$$

for all $k = 1, \ldots, m$. We deduce that $\mathfrak{h}$ is isomorphic to $\mathfrak{h}_{n+1}$, the Heisenberg Lie algebra of dimension $2n+3$. In conclusion, $G = H_{n+1} \rtimes \mathbb{R}^+$ where H_{n+1} is the Heisenberg Lie group of dimension $2n+3$.

Remark 1.4.9. *Notice that the quaternion Kähler structure is not directly visible from this description of the c-map. This fact will become apparent in the explicit examples of Section 1.6.*

We conclude this section with the following result by Cortés, Han and Mohaupt [18] proving that this construction preserves completeness

Theorem 1.4.10. *Let $(\pi : \widetilde{M} \to M, \nabla)$ be a complete projective special Kähler domain, then the quaternion Kähler manifold (Q, g_Q) is complete and has negative scalar curvature.*

Proof. See [18, Theorem 5, p. 199]. $\square$

The construction can also be generalised to projective special Kähler manifolds covered by projective special Kähler domains [18, Section 6]. In this case one repeats the same construction for each projective special Kähler domain obtaining a family of quaternion Kähler manifolds. These manifolds then are glued together in a unique quaternion Kähler manifold. Also in this case we have that the c-map preserves completeness [18, Theorem 10, p. 205].

1.5 R-map

In Section 1.4, we have seen how to build a quaternion Kähler from a given projective special Kähler one. In this section we present a way to construct projective special Kähler manifolds starting from a polynomial function of degree 3 . This construction is called *supergravity r-map*, or r-map for short.

The idea is to take an open subset $U \subseteq \mathbb{R}^n$ invariant by scalar multiplication under positive numbers and a polynomial function $h : U \to \mathbb{R}$ of degree 3. Then we endow the tangent bundle of U with a suitable metric which will render it projective special Kähler. For the construction we will mostly follow [18].

Let $U \subseteq \mathbb{R}^n$ be an open subset closed under scalar multiplication by $\mathbb{R}^+$ and let $h \colon U \to \mathbb{R}^+$ be a polynomial function of degree 3. Let now

$$\mathcal{H} = \{y \in U | h(y) = 1\}.$$

Since U is closed by scalar multiplication by $\mathbb{R}^+$ and h is homogeneous, we can deduce that $U = \mathbb{R}^+ \cdot \mathcal{H}$. Explicitly, we realise $x \in U$ as $\sqrt[3]{h(y)}\frac{y}{\sqrt[3]{h(y)}}$.

Let now ∂_y be the connection on $\mathbb{R}^n$ associated to the standard coordinate system $y = (y_1, \dots, y_n) = \mathrm{id}_{\mathbb{R}^n}$, that is such that $\partial_y dy^k = 0$ for all $k = 1, \dots, n$. In particular then, given $f \in \mathcal{C}^\infty(U)$, we have $\partial_y^2 f = \frac{\partial^2}{\partial y^h \partial y^k} dy^h \otimes dy^k$. We can now endow U with the following metric

$$g_U = -\frac{1}{4}\partial_y^2 \log(h)$$

Remark 1.5.1. *In order to simplify the exposition, we have chosen a different scaling factor than the corresponding metric in [18]. Note that the forthcoming metric g_M built from this metric agrees with the one in [18].*

Notice that being homogeneous, h is also regular on $\mathcal{H} \subseteq U$. Explicitly, suppose dh vanishes at a point $p = (p_1, \dots, p_n) \in \mathcal{H}$, then in particular $\frac{\partial h}{\partial y^k}(p) = 0$ for all $k = 1, \dots, n$, but then by Euler's formula on homogeneous functions we would have:

$$0 = \sum_{k=1}^{n} \frac{\partial h}{\partial y^k} p^k = 3h(p) = 3.$$

If we put on $\mathcal{H}$ the induced metric as a submanifold of (U, g_U), we obtain a Riemannian manifold $(\mathcal{H}, g_{\mathcal{H}} = g_U|_{\mathcal{H}})$ called *projective special real*.

The last step of the r-map consists in taking $M := TU$ with a suitable metric. Let $y = (y^1, \dots, y_n)$ be the the canonical coordinate chart on $U \subseteq \mathbb{R}^n$ (i.e. id_U), and let $x_1, \dots, x_n$ be the induced coordinates on the tangent space, meaning that for all $X \in T_y U$ and for all $p \in U$, we have

$$X = x^k(X) \frac{\partial}{\partial y_k}|_p.$$

If the metric on U is $g_U = g_{h,k}(y) dy^h \otimes dy^k$, then the metric on M is defined to be

$$g_M = g_{h,k}(y)(dx^h \otimes dx^k + dy^h \otimes dy^k).$$

Since U is parallelisable, its tangent bundle is trivial, and thus it can be seen as immersed in $\mathbb{C}^n$ as

$$M := TU \cong \mathbb{R}^n + iU;$$

this induces on M a complex structure compatible with the metric, and thus a Kähler structure.

Moreover, we can realise it as a projective special Kähler manifold by choosing as the associated conic special Kähler manifold the following

$$\widetilde{M} := \{\lambda(1, \zeta) | \lambda \in \mathbb{C}^*, \zeta \in M \subseteq \mathbb{R}^n + iU\} \subseteq \mathbb{C}^{n+1}.$$

Also, it turns out that $\widetilde{M}$ we have a prepotential if we take

$$F(z_0, \dots, z_n) = h(z_1, \dots, z_n)/z_0.$$

Remark 1.5.2. *Manifolds obtained via the r-map are actually projective special Kähler domains.*

Theorem 1.5.3. *Let $h\colon U \to \mathbb{R}^+$ be a polynomial function of degree 3, $(\mathcal{H}, g_{\mathcal{H}})$ be the corresponding projective special real manifold, and (M, g_M) the projective special Kähler manifold obtained via the r-map, then (M, g_M) is complete if $\mathcal{H}$ is.*

Proof. See [18, Theorem 4, p. 197]. □

The composition of the r-map with the c-map is called *q-map*, which then allows to build quaternion Kähler manifolds starting from polynomials of degree three. By combining Theorem 1.5.3 and Theorem 1.4.10, by virtue of Remark 1.5.2, we obtain

Corollary 1.5.4. *Given a polynomial function $h\colon U \to \mathbb{R}^+$ of degree 3, let $(\mathcal{H}, g_{\mathcal{H}})$ be the corresponding projective special real manifold, and (Q, g_Q) the quaternion Kähler manifold obtained via the r-map, then (Q, g_Q) is complete if $\mathcal{H}$ is.*

1.6 Examples

In this section we follow the q-map for the polynomials y^3 and $y_1^2 y_2$ using the coordinate description given in Section 1.4.2.

1.6.1 Example in dimension 8

For our first example of quaternionic Kähler manifold obtained via the q-map, we consider the polynomial function y^3 defined on $U := \mathbb{R}^+$. Then, following the r-map construction, we have $\mathcal{H} = \{y \in U | y^3 = 1\} = \{1\}$ and thus $U = \mathbb{R}^+ \cdot \{1\}$. On U we put the metric $g_U = -\frac{1}{4}\partial^2 \log(y^3) = \frac{3}{4y^2}dy^2$. Being a point, $\mathcal{H}$ is a complete submanifold of U. As last step of the r-map, we build $M := TU \cong \mathbb{R} + i\mathbb{R}^+ \subset \mathbb{C}$ with the metric $g_M = \frac{3}{4y^2}dx^2 + \frac{3}{4y^2}dy^2$. Notice that, up to rescaling, M is isometric to the hyperbolic plane. Via the c-map, we can now obtain from M a quaternion Kähler manifold Q which is complete by Corollary 1.5.4, since $\mathcal{H}$ is complete.

We now follow the construction given in section 1.4.2 in order to compute an orthonormal coframe on the final quaternion Kähler manifold Q. Let

$F(z_0, z_1) := \frac{h(z_1)}{z_0} = \frac{(z_1)^3}{z_0}$ and let $\partial^2 F$ be the Hessian matrix of F, then if $\frac{z_1}{z_0} = x + iy$,

$$\begin{aligned}\partial^2 F = \begin{pmatrix} 2\frac{z_1^3}{z_0^3} & -3\frac{z_1^2}{z_0^2} \\ -3\frac{z_1^2}{z_0^2} & 6\frac{z_1}{z_0} \end{pmatrix} &= \begin{pmatrix} 2(x+iy)^3 & -3(x+iy)^2 \\ -3(x+iy)^2 & 6(x+iy) \end{pmatrix} \\ &= \begin{pmatrix} 2x^3 - 6xy^2 & -3x^2 + 3y^2 \\ -3x^2 + 3y^2 & 6x \end{pmatrix} + i \begin{pmatrix} 6x^2 y - 2y^3 & -6xy \\ -6xy & 6y \end{pmatrix}.\end{aligned}$$

The matrix $B = 2\,\mathrm{Im}(\partial^2 F)$ has the form

$$B = \begin{pmatrix} 12x^2 y - 4y^3 & -12xy \\ -12xy & 12y \end{pmatrix}$$

and now we can use it to compute the matrix

$$\mathcal{N} = \overline{\partial^2 F} + i\frac{Bzz^tB^t}{z^tBz} = \begin{pmatrix} 2x^3 & -3x^2 \\ -3x^2 & 6x \end{pmatrix} + i \begin{pmatrix} 3x^2 y + y^3 & -3xy \\ -3xy & 3y \end{pmatrix}.$$

If we denote by $\mathcal{R}$ and $\mathcal{I}$ the real and imaginary part of $\mathcal{N}$ respectively, the construction of section 1.4.2 provides us with a technique to build a coframe $\tilde{e} = (dx, dy, \xi^0, \xi^1, \eta^0, \eta^1)$ such that $g_G = \frac{1}{4}\sum_{h,k=0}^{1}((\mathcal{I}^{-1})_{h,k}\xi^h\xi^k + (\xi^2)^2 + \mathcal{I}_{h,k}\eta^h\eta^k + (\eta^2)^2)$ as (1.21). In order to find an orthonormal coframe, we compute the Cholesky decomposition of $\mathcal{I}^{-1} = U^tU$, obtaining

$$U = \begin{pmatrix} \frac{1}{\sqrt{y^3}} & \frac{x}{\sqrt{y^3}} \\ 0 & \frac{1}{\sqrt{3y}} \end{pmatrix}, \qquad U^{-1} = \begin{pmatrix} \sqrt{y^3} & -\sqrt{3y}x \\ 0 & \sqrt{3y} \end{pmatrix}.$$

We choose $\mathcal{I}^{-1}$ instead of $\mathcal{I}$ just because it gives entries that are easier to work with. We can produce the following transformation, represented in the coframe $\tilde{e}$ in order to obtain an orthonormal coframe e

$$B = \begin{pmatrix} \frac{\sqrt{3}}{2y} I_2 & & & & \\ & \frac{1}{2}U & & & \\ & & \frac{1}{2} & & \\ & & & \frac{1}{2}(U^{-1})^t & \\ & & & & \frac{1}{2} \end{pmatrix}.$$

Explicitly, we want $e^k = B^k_h \tilde{e}^h$, so with respect to the chart

$$(x, y, \zeta_0, \zeta_1, \phi, \tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\phi}),$$

the new coframe e on Q has the form:

$$
\begin{aligned}
e^1 &= \frac{\sqrt{3}}{2y}dx \qquad\qquad e^2 = \frac{\sqrt{3}}{2y}dy \\
e^3 &= -\frac{x^3}{\sqrt{2\phi y^3}}d\zeta_0 + 3\frac{x^2}{\sqrt{2\phi y^3}}d\zeta_1 + \frac{1}{\sqrt{2\phi y^3}}d\tilde{\zeta}_0 + \frac{x}{\sqrt{2\phi y^3}}d\tilde{\zeta}_1 \\
e^4 &= -\frac{\sqrt{3}x^2}{\sqrt{2\phi y}}d\zeta_0 + \frac{\sqrt{6}x}{\sqrt{\phi y}}d\zeta_1 + \frac{1}{\sqrt{6\phi y}}d\tilde{\zeta}_1 \\
e^5 &= \frac{1}{2\phi}d\phi \qquad\qquad e^6 = \frac{\sqrt{y^3}}{\sqrt{2\phi}}d\zeta_0 \\
e^7 &= -\frac{\sqrt{3y}x}{\sqrt{2\phi}}d\zeta_0 + \frac{\sqrt{3y}}{\sqrt{2\phi}}d\zeta_1 \\
e^8 &= -\frac{\tilde{\zeta}_0}{2\phi}d\zeta_0 - \frac{\tilde{\zeta}_1}{2\phi}d\zeta_1 + \frac{\zeta_0}{2\phi}d\tilde{\zeta}_0 + \frac{\zeta_1}{2\phi}d\tilde{\zeta}_1 + \frac{1}{2\phi}d\tilde{\phi}
\end{aligned}
\tag{1.23}
$$

If we now compute the differentials, we obtain

$$
\begin{aligned}
de^1 &= \frac{2}{\sqrt{3}}e^{1,2} & de^2 &= 0 \\
de^3 &= 2e^{1,4} - \sqrt{3}e^{2,3} + e^{3,5} & de^4 &= \frac{4}{\sqrt{3}}e^{1,7} - \frac{1}{\sqrt{3}}e^{2,4} + e^{4,5} \\
de^5 &= 0 & de^6 &= \sqrt{3}e^{2,6} - e^{5,6} \\
de^7 &= -2e^{1,6} + \frac{1}{\sqrt{3}}e^{2,7} - e^{5,7} & de^8 &= -2e^{3,6} - 2e^{4,7} - 2e^{5,8}
\end{aligned}
\tag{1.24}
$$

On the Lie algebra

Let $\mathfrak{q}$ be the Lie algebra of Q. With the structure constants (1.24), we can compute the derived subalgebra

$$\mathfrak{n} := \mathfrak{q}^{(1)} = [\mathfrak{q}, \mathfrak{q}] = \langle e_1, e_3, e_4, e_6, e_7, e_8 \rangle$$

The algebras of the derived series $\mathfrak{q}^{(n)}$, with $\mathfrak{q}^{(0)} = \mathfrak{q}$ and $\mathfrak{q}^{(n+1)} = [\mathfrak{q}^{(n)}, \mathfrak{q}^{(n)}]$ continues with

$$\mathfrak{q}^{(2)} = \langle e_3, e_4, e_7, e_8 \rangle\,, \qquad \mathfrak{q}^{(3)} = \langle e_8 \rangle\,, \qquad \mathfrak{q}^{(4)} = 0.$$

Therefore, Q is a solvmanifold.

In addition, $\mathrm{ad}(X)$ has real eigenvalues for all $X \in \mathfrak{q}$, so in particular it is an Alekseevskian space. By the Alekseevsky-Cortés classification, there are only two possibilities: $\mathrm{SU}(2,2)/S(\mathrm{U}(2) \times \mathrm{U}(2))$ and $\mathrm{G}_2^*/\mathrm{SO}(4)$. We can verify that the second case occurs as follows.

Notice first that the orthogonal complement $\mathfrak{a} = \langle e_2, e_5 \rangle$ of $\mathfrak{n}$ is abelian, i.e. $\mathfrak{q}$ is *standard* in the sense of [27]. This is consistent with a general result of Lauret [34, Theorem 3.1, p. 1874] stating that any Einstein solvmanifold is standard.

Given a semisimple Lie group G, there are closed subgroups K, A, N, such that

- K is compact maximal;
- A is abelian;
- N is nilpotent;
- $G = KAN$.

This decomposition is called *Iwasawa decomposition* (e.g. [32, Theorem 6.46, p. 374]). The Lie algebras of these groups are then such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ with $\mathfrak{a}$ abelian and $\mathfrak{n}$ nilpotent. In particular $\mathfrak{a} + \mathfrak{n}$ is called *Borel subalgabra*.

Back to our solvmanifold Q, it is not difficult to check that $\mathfrak{q} = \mathfrak{a} \ltimes \mathfrak{n}$ is the Borel subalgebra of $\mathfrak{g}_2^*$ (see e.g. [12, Example 4.3, p. 17]). By [47, Proposition 4.4, p. 59], Q has an Einstein metric which makes it isometric to the symmetric space $\frac{\mathrm{G}_2^*}{\mathrm{SO}(4)}$. Since the (standard) Einstein metric on a solvmanifold is unique ([27, Theorem E, p. 283]), the quaternion Kähler submanifold Q is isometric to $\mathrm{G}_2^*/SO(4)$. In particular, if we compute its scalar curvature, we get -8.

Quaternionic Kähler structure

By substituting the structure constants from (1.24) in the Koszul formula, we can obtain the Levi-Civita connection and the corresponding curvature tensor Ω.

The aim is now to find three almost complex structures I, J, K giving the quaternion Kähler structure, without using the identification with the Wolf space $\mathrm{G}_2^*/\mathrm{SO}(4)$ ([48, Theorem 5.4, p. 1043]). The idea is to look for them in the holonomy algebra, since we know the latter is inside $\mathfrak{sp}(2) + \mathfrak{sp}(1)$ and $\mathfrak{sp}(1)$ contains them. In order to compute the holonomy algebra, we use the so-called "holonomy theorem" by Cartan and Ambrose and Singer (see e.g. [44, Theorem 2.8, p. 27]) that tells us it coincides with the image of the curvature form. Interpreting Ω as an endomorphism R of $T^1_1Q = \mathrm{End}(TQ)$ mapping $X^\flat \otimes Y$ to $\Omega(X,Y) : TQ \to TQ$, the map R is symmetric with respect to the metric g for the symmetries of the Riemannian tensor, in fact at every point $p \in Q$ for $X, Y, Z, W \in T_pQ$

$$\begin{aligned} g(R(X^\flat \otimes Y), Z^\flat \otimes W) &= g(\Omega(X,Y)Z, W) = g(\Omega(Z,W)X, Y) \\ &= g(R(Z^\flat \otimes W), X^\flat \otimes Y). \end{aligned}$$

Since R is symmetric, it is also diagonalisable. In particular, fixing the coframe e, R is a map from matrices to matrices and its image corresponds to the image of the curvature form Ω. Therefore, in order to find the image of the curvature form, it is sufficient to find the eigenvectors (eigenmatrices) corresponding to non-zero eigenvalues. The map R has three different eigenvalues: $-\frac{20}{3}, -4, 0$. Since the Levi-Civita connection is invariant, so is Ω and thus, as an endomorphism, R is equivariant. In particular, since $\mathfrak{sp}(1)$ is irreducible, by Schur's Lemma, R will act on it as a scalar multiplication, so we expect to find it inside an eigenspace.

We can find a basis I', J', K' for the eigenspace corresponding to $-\frac{20}{3}$ such that by sending $i, j, k \in \mathfrak{sp}(1)$ in I', J', K' we obtain an isomorphism between $\mathfrak{sp}(1)$ and this eigenspace seen as subalgebra of $\mathfrak{gl}(8,\mathbb{R})$. However, no linear combination of I', J', K' is such that its square is $-I_8$, and thus these cannot be almost complex structures.

Therefore, we must look for them in the eigenspace corresponding to -4. We can take in particular the three eigenmatrices I, J, K corresponding to the three 2-forms:

$$\begin{aligned} \omega_I &= -\frac{\sqrt{3}}{2}e^{1,3} - \frac{1}{2}e^{1,7} + \frac{1}{2}e^{2,4} - \frac{\sqrt{3}}{2}e^{2,6} + \frac{1}{2}e^{3,8} - \frac{\sqrt{3}}{2}e^{4,5} + \frac{1}{2}e^{5,6} - \frac{\sqrt{3}}{2}e^{7,8}; \\ \omega_J &= \frac{1}{2}e^{1,4} - \frac{\sqrt{3}}{2}e^{1,6} + \frac{\sqrt{3}}{2}e^{2,3} + \frac{1}{2}e^{2,7} - \frac{1}{2}e^{3,5} - \frac{\sqrt{3}}{2}e^{4,8} - \frac{\sqrt{3}}{2}e^{5,7} - \frac{1}{2}e^{6,8}; \\ \omega_K &= -e^{1,2} - \frac{\sqrt{3}}{2}e^{3,4} + \frac{1}{2}e^{3,6} - \frac{1}{2}e^{4,7} + e^{5,8} - \frac{\sqrt{3}}{2}e^{6,7}. \end{aligned}$$

We can check that $I^2 = J^2 = K^2 = IJK = -I_8$, so these are three matrices acting on $\mathbb{R}^8$ as $i, j, k \in \mathrm{Sp}(1)$. In particular, this correspondence provides an isomorphism between the eigenspace of -4 and $\mathfrak{sp}(1)$. As a consequence of Schur's lemma, the eigenspaces of $-\frac{20}{3}$ and -4 commute as subalgebras of

$\mathfrak{gl}(8,\mathbb{R})$, so the holonomy Lie algebra of Q is isomorphic to $\mathfrak{sp}(1)+\mathfrak{sp}(1)\cong\mathfrak{so}(3)+\mathfrak{so}(3)\cong\mathfrak{so}(4)$. We denote by the same symbols the almost complex structures corresponding to the matrices I, J, K.

In order to check that I, J, K are the almost complex structures providing the quaternion-Kähler structure, we apply Corollary 1.3.42, so we need need to check the differential condition on the corresponding differential 2-forms $\omega_I, \omega_J, \omega_K$.

In order to simplify computations, we first change coframe to an adapted one u. Explicitly, we choose it so that I, J, K are represented by the matrices i_8, j_8, k_8 respectively in the basis u, that is

$$\begin{aligned}
&u_1 = e_1, && u_2 = -Ie_1, && u_3 = -Je_1, && u_4 = -Ke_1,\\
&u_5 = e_5, && u_6 = -Ie_5, && u_7 = -Je_5, && u_8 = -Ke_5.
\end{aligned}$$

Explicitly, the matrix of frame change is $A = (A^h_k)_{h,k}$ such that $u_k = A^h_k e_h$

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0\\
0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0\\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}.$$

Notice that this is an orthogonal matrix, which tells us that also the frame u is orthonormal with respect to g_Q.

The following are the differentials with respect to the new dual coframe:

$$\begin{aligned}
du^1 &= \frac{2}{\sqrt{3}}u^{1,4}\\
du^2 &= -\sqrt{3}u^{1,3} - u^{1,6} + \frac{2}{\sqrt{3}}u^{2,4} + u^{2,5} + u^{4,7}\\
du^3 &= -\frac{1}{\sqrt{3}}u^{1,2} - u^{1,7} - \frac{2}{\sqrt{3}}u^{3,4} + u^{3,5} - u^{4,6}\\
du^4 &= 0 \qquad\qquad du^5 = 0\\
du^6 &= -u^{1,2} - \sqrt{3}u^{1,7} + u^{3,4} - u^{5,6}\\
du^7 &= -u^{1,3} + \sqrt{3}u^{1,6} - u^{2,4} - u^{5,7}
\end{aligned}$$

$$du^8 = 2u^{2,3} - 2u^{5,8} - 2u^{6,7}$$

With respect to this new frame, the three 2-forms are written as

$$\begin{aligned}
\omega_I &= -u^{1,2} + u^{3,4} - u^{5,6} + u^{7,8};\\
\omega_J &= -u^{1,3} - u^{2,4} - u^{5,7} - u^{6,8};\\
\omega_K &= -u^{1,4} + u^{2,3} - u^{5,8} + u^{6,7}.
\end{aligned}$$

Now we can see how these three forms generate a differential ideal, and in particular, if $\alpha_I := 2u^6$, $\alpha_J := 2u^7$ and $\alpha_K := -\sqrt{3}u^1 + u^8$, then

$$\begin{aligned}
d\omega_I &= \alpha_K \wedge \omega_J - \alpha_J \wedge \omega_K;\\
d\omega_J &= \alpha_I \wedge \omega_K - \alpha_K \wedge \omega_I;\\
d\omega_K &= \alpha_J \wedge \omega_I - \alpha_I \wedge \omega_J.
\end{aligned}$$

Therefore, by Corollary 1.3.42, we verify that Q has a quaternionic Kähler structure given by I, J, K. We can also write the quaternionic 4-form explicitly:

$$\begin{aligned}
\Phi = &-6u^{1,2,3,4} + 2u^{1,2,5,6} - 2u^{1,2,7,8} + 2u^{1,3,5,7} + 2u^{1,3,6,8} + 2u^{1,4,5,8} - 2u^{1,4,6,7}\\
&- 2u^{2,3,5,8} + 2u^{2,3,6,7} + 2u^{2,4,5,7} + 2u^{2,4,6,8} - 2u^{3,4,5,6} + 2u^{3,4,7,8} - 6u^{5,6,7,8}.
\end{aligned}$$

1.6.2 Example in dimension 12

The second example will be obtained from the polynomial function $y_1^2 y_2$ defined on $U := \mathbb{R}^+ \times \mathbb{R}^+$. Let $\mathcal{H} = \{(y_1, y_2) \in U | y_1^2 y_1 = 1\}$, that is the graph of $x \mapsto 1/x^2$ on positive real numbers, so in particular $U = \mathbb{R}^+ \cdot \mathcal{H}$. On U we define $g_U = -\frac{1}{4}\partial^2 \log(y_1^2 y_2) = \frac{1}{2y_1^2} dy_1^2 + \frac{1}{4y_1^2} dy_2^2$. Consider now the following embedding of $\mathcal{H}$:

$$\begin{aligned}
\gamma\colon \mathbb{R}^+ &\longrightarrow \mathcal{H}\\
t &\longmapsto (t, \frac{1}{t^2})
\end{aligned}$$

In particular, if $\iota_{\mathcal{H}}\colon \mathcal{H} \to U$ is the inclusion, so that $g_{\mathcal{H}} = g_U$, then $(\mathcal{H}, g_{\mathcal{H}})$ is isometric to $(\mathbb{R}^+, \gamma^* g_{\mathcal{H}})$, but

$$\gamma^* g_{\mathcal{H}} = \gamma^* \iota^* g_{\mathcal{U}} = \frac{(dt)^2}{2t^2} + \frac{\left(d\left(\frac{1}{t^2}\right)\right)^2}{\frac{4}{t^4}} = \frac{3(dt)^2}{2t^2} = \left(d\left(\sqrt{\frac{3}{2}}\log(t)\right)\right)^2$$

so $t \mapsto \sqrt{\frac{3}{2}}\log(t)$ is an isometry between the euclidean real line and $\mathbb{R}^+$, and therefore $\mathcal{H}$. In particular, since $\mathbb{R}$ is complete, so is $\mathcal{H}$, implying again that the resulting quaternionic Kähler manifold Q will be complete by Corollary 1.5.4.

Now let $M := TU \cong \mathbb{R}^2 + i(\mathbb{R}^+ \times \mathbb{R}^+) \subset \mathbb{C}^2$ equipped with $g_M = \frac{dx_1^2}{2y_1^2} + \frac{dy_1^2}{2y_1^2} + \frac{dx_2^2}{4y_2^2} + \frac{dy_2^2}{4y_2^2}$. Notice that M is isometric to the product of two rescalings of the hyperbolic plane.

Following the explicit construction of the c-map we define $F(z_0, z_1, z_2) := \frac{h(z_1,z_2)}{z_0} = \frac{(z_1)^2 z_2}{z_0}$. Setting $\frac{z_1}{z_0} = x_1 + iy_1$ and $\frac{z_2}{z_0} = x_2 + iy_2$, the Hessian matrix $\partial^2 F$ is of the form:

$$
\begin{aligned}
\partial^2 F &= \begin{pmatrix} \frac{2z_1^2 z_2}{z_0^3} & -\frac{2z_1 z_2}{z_0^2} & -\frac{z_1^2}{z_0^2} \\ -\frac{2z_1 z_2}{z_0^2} & \frac{2z_2}{z_0} & \frac{2z_1}{z_0} \\ -\frac{z_1^2}{z_0^2} & \frac{2z_1}{z_0} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 2(x_1+iy_1)^2(x_2+iy_2) & -2(x_1+iy_1)(x_2+iy_2) & -(x_1+iy_1)^2 \\ -2(x_1+iy_1)(x_2+iy_2) & 2(x_2+iy_2) & 2(x_1+iy_1) \\ (x_1+iy_1)^2 & 2(x_1+iy_1) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 2x_2x_1^2 - 4y_1y_2x_1 - 2x_2y_1^2 & 2y_1y_2 - 2x_1x_2 & y_1^2 - x_1^2 \\ 2y_1y_2 - 2x_1x_2 & 2x_2 & 2x_1 \\ y_1^2 - x_1^2 & 2x_1 & 0 \end{pmatrix} \\
&\quad + i \begin{pmatrix} 2y_2x_1^2 + 4x_2y_1x_1 - 2y_1^2y_2 & -2x_2y_1 - 2x_1y_2 & -2x_1y_1 \\ -2x_2y_1 - 2x_1y_2 & 2y_2 & 2y_1 \\ -2x_1y_1 & 2y_1 & 0 \end{pmatrix}.
\end{aligned}
$$

Following the construction as in the previous example, we obtain

$$
\begin{aligned}
\mathcal{N} &= \mathcal{R} + i\mathcal{I} \\
&= \begin{pmatrix} 2x_1^2x_2 & -2x_1x_2 & -x_1^2 \\ -2x_1x_2 & 2x_2 & 2x_1 \\ -x_1^2 & 2x_1 & 0 \end{pmatrix} + i \begin{pmatrix} \frac{2x_1^2y_2^2 + y_1^2y_2^2 + x_2^2y_1^2}{y_2} & -2x_1y_2 & -\frac{x_2y_1^2}{y_2} \\ -2x_1y_2 & 2y_2 & 0 \\ -\frac{x_2y_1^2}{y_2} & 0 & \frac{y_1^2}{y_2} \end{pmatrix}.
\end{aligned}
$$

We use this matrix to build an intermediate coframe as in (1.20), that is $\tilde{e} = (dx, dy, \xi_0, \xi_1, \xi_2, \eta_0, \eta_1, \eta_2)$ such that

$$g_G = \frac{1}{4}\left(\mathcal{I}_{h,k}\xi^h\xi^k + (\xi^3)^2 + (\mathcal{I}^{-1})^{h,k}\eta^h\eta^k + (\eta^3)^2\right).$$

As for the previous example, we compute the Cholesky decomposition of $\mathcal{I}^{-1} = U^t U$, obtaining

$$U = \begin{pmatrix} \frac{1}{y_1\sqrt{y_2}} & \frac{x_1}{y_1\sqrt{y_2}} & \frac{x_2}{y_1\sqrt{y_2}} \\ 0 & \frac{1}{\sqrt{2}y_2} & 0 \\ 0 & 0 & \frac{\sqrt{y_2}}{y_1} \end{pmatrix}.$$

The following is the transformation represented with respect to the coframe $\tilde{e}$ in order to obtain an orthonormal coframe e

$$\begin{pmatrix} \frac{1}{\sqrt{2}y_1} &&&&&&& \\ & \frac{1}{2y_2} &&&&&& \\ && \frac{1}{\sqrt{2}y_1} &&&&& \\ &&& \frac{1}{2y_2} &&&& \\ &&&& \frac{1}{2}U &&& \\ &&&&& \frac{1}{2} && \\ &&&&&& \frac{1}{2}(U^t)^{-1} & \\ &&&&&&& \frac{1}{2} \end{pmatrix}.$$

The new coframe can be written in the chart

$$(x_1, x_2, y_1, y_2, \zeta_0, \zeta_1, \zeta_2, \phi, \tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2, \tilde{\phi})$$

(see section 1.4.2) as follows:

$$\begin{aligned} e^1 &= \frac{1}{\sqrt{2}y_1} dx_1; \qquad & e^2 &= \frac{1}{2y_2} dx_2; \\ e^3 &= \frac{1}{\sqrt{2}y_1} dy_1; \qquad & e^4 &= \frac{1}{2y_2} dy_2; \end{aligned} \tag{1.26}$$

$$\begin{aligned} e^5 &= -\frac{{x_1}^2 x_2}{\sqrt{2\phi y_2} y_1} d\zeta_0 + \frac{\sqrt{2} x_1 x_2}{\sqrt{\phi y_2} y_1} d\zeta_1 + \frac{{x_1}^2}{\sqrt{2\phi y_2} y_1} d\zeta_2 + \frac{1}{\sqrt{2\phi y_2} y_1} d\tilde{\zeta}_0 \\ &\quad + \frac{x_1}{\sqrt{2\phi y_2} y_1} d\tilde{\zeta}_1 + \frac{x_2}{\sqrt{2\phi y_2} y_1} d\tilde{\zeta}_2; \\ e^6 &= -\frac{x_1 x_2}{\sqrt{\phi y_2}} d\zeta_0 + \frac{x_2}{\sqrt{\phi y_2}} d\zeta_1 + \frac{x_1}{\sqrt{\phi y_2}} d\zeta_2 + \frac{1}{2\sqrt{\phi y_2}} d\tilde{\zeta}_1; \\ e^7 &= -\frac{x_1^2 \sqrt{y_2}}{\sqrt{2\phi} y_1} d\zeta_0 + \frac{\sqrt{2} x_1 \sqrt{y_2}}{\sqrt{\phi} y_1} d\zeta_1 + \frac{\sqrt{y_2}}{\sqrt{2\phi} y_1} d\tilde{\zeta}_2; \\ e^8 &= \frac{1}{2\phi} d\phi; \qquad e^9 = \frac{y_1 \sqrt{y_2}}{\sqrt{2\phi}} d\zeta_0; \end{aligned}$$

$$e^{10} = -\frac{x_1\sqrt{y_2}}{\sqrt{\phi}}d\zeta_0 + \frac{\sqrt{y_2}}{\sqrt{\phi}}d\zeta_1; \qquad e^{11} = -\frac{x_2 y_1}{\sqrt{2\phi y_2}}d\zeta_0 + \frac{y_1}{\sqrt{2\phi y_2}}d\zeta_2;$$
$$e^{12} = -\frac{\tilde{\zeta}_0}{2\phi}d\zeta_0 - \frac{\tilde{\zeta}_1}{2\phi}d\zeta_1 - \frac{\tilde{\zeta}_2}{2\phi}d\zeta_2 + \frac{\zeta_0}{2\phi}d\tilde{\zeta}_0 + \frac{\zeta_1}{2\phi}d\tilde{\zeta}_1 + \frac{\zeta_2}{2\phi}d\tilde{\zeta}_2 + \frac{1}{2\phi}d\tilde{\phi}.$$

The differentials are then

$$\begin{aligned}
de^1 &= \sqrt{2}e^{1,3} \qquad\qquad de^2 = 2e^{2,4}\\
de^3 &= 0 \qquad\qquad\qquad de^4 = 0\\
de^5 &= 2e^{1,6} + 2e^{2,7} - \sqrt{2}e^{3,5} - e^{4,5} + e^{5,8}\\
de^6 &= 2e^{1,11} + 2e^{2,10} - e^{4,6} + e^{6,8}\\
de^7 &= 2e^{1,10} - \sqrt{2}e^{3,7} + e^{4,7} + e^{7,8}\\
de^8 &= 0\\
de^9 &= \sqrt{2}e^{3,9} + e^{4,9} - e^{8,9}\\
de^{10} &= -2e^{1,9} + e^{4,10} - e^{8,10}\\
de^{11} &= -2e^{2,9} + \sqrt{2}e^{3,11} - e^{4,11} - e^{8,11}\\
de^{12} &= -2e^{5,9} - 2e^{6,10} - 2e^{7,11} - 2e^{8,12}
\end{aligned}$$

With these structure constants we compute the Levi-Civita connection and its curvature tensor Ω.

On the Lie algebra

Let $\mathfrak{q}$ be the Lie algebra of Q. We can compute its derived algebra:

$$\mathfrak{n} := \mathfrak{q}^{(1)} = \langle u_1, u_2, u_3, u_5, u_6, u_7, u_{10}, u_{11}, u_{12} \rangle .$$

By continuing the derived series, one gets $\mathfrak{q}^{(4)} = 0$, so $\mathcal{L}_Q$ is a solvmanifold.

An explicit computation shows that for all $X \in \mathfrak{n}$, $\mathrm{ad}_X : \mathfrak{q} \to \mathfrak{q}$ has real eigenvalues, so $\mathfrak{q}$ is real solvable. By definition, $\mathfrak{q}$ is an Alekseevskian Lie algebra, so Q is an Alekseevskian space. By the Alekseevsky-Cortés classification, there are only three possibilities: $\mathrm{Sp}(3,1)/\mathrm{Sp}(3)\mathrm{Sp}(1)$, $\mathrm{SU}(3,2)/S(\mathrm{U}(3)\mathrm{U}(2))$ and $\mathrm{SO}_0(4,3)/(\mathrm{SO}(4)\mathrm{SO}(3))$. It follows the verification that we are dealing with the third case.

The orthogonal complement of $\mathfrak{n}$ is $\mathfrak{a} = \langle u_4, u_8, u_9 \rangle$ and is abelian, so $\mathfrak{q}$ is standard ([27]).

We can check that $\mathfrak{q} = \mathfrak{a} \ltimes \mathfrak{n}$ is the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{so}(4,3)$ as follows. The geometry of the roots of B_3 implies that $\mathfrak{b}$ has a basis

$$H_1, H_2, H_3, X_{\epsilon_1}, X_{\epsilon_2}, X_{\epsilon_3}, X_{\epsilon_1-\epsilon_3}, X_{\epsilon_2-\epsilon_3}, X_{\epsilon_1+\epsilon_3}, X_{\epsilon_2+\epsilon_3}, X_{\epsilon_1+\epsilon_2},$$

with $[H, X_\alpha] = \alpha(H)X_\alpha$ and $[X_\alpha, X_\beta]$ a non-zero multiple of $X_{\alpha+\beta}$.

The nilradical of $\mathfrak{b}$, $\langle X_\alpha | \alpha \rangle$, is uniquely determined by this condition by the classification of nice nilpotent Lie groups [13], where it appears as $96421 : 426$. Since the same condition is satisified by $\mathfrak{b}$, the two Lie algebras coincide.

As for the previous example, the metric on Q makes it isometric to the symmetric space $\mathrm{SO}_0(4,3)/\mathrm{SO}(4)\mathrm{SO}(3)$ ([47, Proposition 4.4, p. 59] and [27, Theorem E, p. 283]). In particular, in our case, the metric has scalar curvature -10.

On the quaternionic structure

As in the previous example, we aim to find $\mathfrak{sp}(1)$ containing the three almost complex structures I, J, K giving the quaternion Kähler structure by looking at the eigenmatrices of the curvature tensor seen as endomorphism. The curvature endomorphism R has three different eigenvalues: $-8, -6, 0$, the eigenspace of -8 has dimension 3 and the eigenspace of -6 has dimension 6. Since the curvature endomorphism is equivariant and $\mathfrak{sp}(1)$ is irreducible, we expect to find its representation inside one of the eigenspaces.

We can find an isomorphism between the eigenspace of -8 with commutator as Lie brackets and the Lie algebra $\mathfrak{sp}(1)$; however, none of its elements squares to $-I_{12}$, so it cannot contain any almost complex structure.

As for the eigenspace V_{-6} corresponding to the eigenvalue -6, since the holonomy Lie algebra is contained in $\mathfrak{sp}(3) + \mathfrak{sp}(1)$, by uniqueness of the decomposition in irreducible representation, we expect an orthogonal decomposition as $V_{-6} \cong \mathfrak{sp}(1) + \mathfrak{sp}(1)^\perp$. Now the strategy is the following:

- we look for a matrix I in V_{-6} such that $I^2 = -I_{12}$;
- since the action of I on $\mathfrak{sp}(1)$ is an isomorphism when restricted to the orthogonal complement of I, the space $\ker([I.\cdot]) \cap \langle I \rangle^\perp$ is the orthogonal complement of $\mathfrak{sp}(1)$ in V_{-6};
- we take the orthogonal complement of $\mathfrak{sp}(1)^\perp$ in V_{-6}, which necessarily is the representation of $\mathfrak{sp}(1)$ we are looking for;

- we pick a J in the orthogonal complement of I in $\mathfrak{sp}(1)$ and we normalise it so that $J^2 = -I_{12}$;
- finally we define $K = IJ$ and we check whether K is still in $\mathfrak{sp}(1)$.

In this way we obtain three forms I, J, K corresponding to the following 2-forms on Q:

$$\begin{aligned}
\omega_I = & -\frac{1}{\sqrt{2}}e^{1,5} - \frac{1}{\sqrt{2}}e^{1,11} - \frac{1}{2}e^{2,5} - \frac{1}{\sqrt{2}}e^{2,10} + \frac{1}{2}e^{2,11} + \frac{1}{\sqrt{2}}e^{3,7} - \frac{1}{\sqrt{2}}e^{3,9} \\
& + \frac{1}{\sqrt{2}}e^{4,6} - \frac{1}{2}e^{4,7} - \frac{1}{2}e^{4,9} + \frac{1}{2}e^{5,12} - \frac{1}{\sqrt{2}}e^{6,8} - \frac{1}{2}e^{7,8} + \frac{1}{2}e^{8,9} \\
& - \frac{1}{\sqrt{2}}e^{10,12} - \frac{1}{2}e^{11,12}; \\
\omega_J = & \frac{1}{\sqrt{2}}e^{1,7} - \frac{1}{\sqrt{2}}e^{1,9} + \frac{1}{\sqrt{2}}e^{2,6} - \frac{1}{2}e^{2,7} - \frac{1}{2}e^{2,9} + \frac{1}{\sqrt{2}}e^{3,5} + \frac{1}{\sqrt{2}}e^{3,11} \\
& + \frac{1}{2}e^{4,5} + \frac{1}{\sqrt{2}}e^{4,10} - \frac{1}{2}e^{4,11} - \frac{1}{2}e^{5,8} - \frac{1}{\sqrt{2}}e^{6,12} - \frac{1}{2}e^{7,12} \\
& - \frac{1}{\sqrt{2}}e^{8,10} - \frac{1}{2}e^{8,11} - \frac{1}{2}e^{9,12}; \\
\omega_K = & -e^{1,3} - e^{2,4} - \frac{1}{\sqrt{2}}e^{5,6} - \frac{1}{2}e^{5,7} + \frac{1}{2}e^{5,9} - \frac{1}{\sqrt{2}}e^{6,11} - \frac{1}{\sqrt{2}}e^{7,10} + \frac{1}{2}e^{7,11} \\
& + e^{8,12} - \frac{1}{\sqrt{2}}e^{9,10} - \frac{1}{2}e^{9,11}.
\end{aligned}$$

We can check on the corresponding matrices that $K^2 = IJK = -I_{12}$, so we have found three matrices acting on $\mathbb{R}^{12}$ as $i, j, k \in \mathrm{Sp}(1)$ and they generate the desired $\mathfrak{sp}(1)$.

It turns out that also the complement of $\mathfrak{sp}(1)$ in V_{-6} has three almost complex structures I', J', K' satisfying the quaternionic identities, so in particular $V_{-6} \cong \mathfrak{sp}(1) + \mathfrak{sp}(1) \cong \mathfrak{so}(3) + \mathfrak{so}(3) \cong \mathfrak{so}(4)$, and thus the holonomy algebra is isomorphic to $\mathfrak{so}(3) + \mathfrak{so}(4)$.

In order to simplify computations, we change coframe as follows:

$$\begin{array}{llll}
u_1 = e_1, & u_2 = -Ie_1, & u_3 = -Je_1, & u_4 = -Ke_1, \\
u_5 = e_2, & u_6 = -Ie_2, & u_7 = -Je_2, & u_8 = -Ke_2, \\
u_9 = e_8, & u_{10} = -Ie_8, & u_{11} = -Je_8, & u_{12} = -Ke_8.
\end{array}$$

Explicitly, the matrix of frame change is $A = (A^h_k)_{h,k}$ such that $u_k = A^h_k e_h$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

This matrix is orthonormal as expected, so the new frame u is orthonormal with respect to g_Q.

The following are the structure constants relative to the dual coframe u:

$$\begin{aligned} du^1 &= \sqrt{2}u^{1,4} \\ du^2 &= -u^{1,7} - u^{1,10} + u^{2,8} + u^{2,9} + 2u^{3,5} - u^{4,6} + u^{4,11} \\ du^3 &= -u^{1,6} - u^{1,11} - u^{3,8} + u^{3,9} + u^{4,7} - u^{4,10} \\ du^4 &= 0 \\ du^5 &= 2u^{5,8} \\ du^6 &= -u^{1,3} - \sqrt{2}u^{1,7} + u^{2,4} + u^{5,7} - u^{5,10} + u^{6,9} + u^{8,11} \\ du^7 &= -u^{1,2} + \sqrt{2}u^{1,6} - u^{3,4} - u^{5,6} - u^{5,11} + u^{7,9} - u^{8,10} \\ du^8 &= 0 \\ du^9 &= 0 \\ du^{10} &= -u^{1,2} - \sqrt{2}u^{1,11} + u^{3,4} - u^{5,6} - u^{5,11} + u^{7,8} - u^{9,10} \\ du^{11} &= -u^{1,3} + \sqrt{2}u^{1,10} - u^{2,4} - u^{5,7} + u^{5,10} - u^{6,8} - u^{9,11} \\ du^{12} &= 2u^{2,3} + 2u^{6,7} - 2u^{9,12} - 2u^{10,11} \end{aligned}$$

With respect to this new frame, the three 2-forms are written as

$$\omega_I = -u^{1,2} + u^{3,4} - u^{5,6} + u^{7,8} - u^{9,10} + u^{11,12};$$

$$\omega_J = -u^{1,3} - u^{2,4} - u^{5,7} - u^{6,8} - u^{9,11} - u^{10,12};$$
$$\omega_K = -u^{1,4} + u^{2,3} - u^{5,8} + u^{6,7} - u^{9,12} + u^{10,11}.$$

Now we can see how these three forms generate a differential ideal, so that in particular, for $\alpha_I := 2u^{10}$, $\alpha_J := 2u^{11}$ and $\alpha_K := -\sqrt{2}u^1 - u^5 + u^{12}$, the positive cyclic permutations in I, J, K of the following formula are satisfied

$$d\omega_I = \alpha_K \wedge \omega_J - \alpha_J \wedge \omega_K.$$

Therefore, by Corollary 1.3.42, I, J, K provide the quaternion Kähler structure.

Actually, the same property holds for I', J' and K', so this manifold has two different quaternionic Kähler structures where the almost complex structures of the first commute with the one of the second.

Notice that since the manifold has dimension $12 > 8$, by Theorem 1.3.41 it would have been enough to compute the exterior differential of the fundamental 4-form Φ.

Chapter 2

Construction of projective special Kähler manifolds

In this chapter we present the content of the author's paper [39]. Namely, we provide a characterisation of projective special Kähler manifolds that will hopefully shed more light on this type of structure. Our characterisation is intrinsic in the sense that we reduce the projective special Kähler structure to data solely defined on the manifold itself. The characterisation is obtained by means of a locally defined symmetric tensor that we call deviance, satisfying certain conditions: a differential one and an algebraic one. Moreover, this characterisation provides a simpler way to build projective special Kähler manifolds, and we display this in Chapter 3 by classifying all possible projective special Kähler structures on 4-dimensional Lie groups. Since we are ultimately interested in the c-map, throughout this chapter we adopt the same convention as [18], where we only consider projective special Kähler manifolds obtained from conic special Kähler manifolds with signature $(2n, 2)$. Nonetheless, our characterisation can be generalised to generic signatures. It is worth mentioning that the deviance, being a symmetric tensor of type (3,0), can often be seen as a homogeneous polynomial of degree three, which may have a role in providing a partial inversion of the r-map.

2.1 Difference tensor

This section is devoted to the tensor obtained as difference between the flat and Levi-Civita connection on a special Kähler manifold. We present the known symmetry of this tensor and write the flatness condition in terms of it [23, p. 9-11].

Let $(\widetilde{M}, \widetilde{g}, \widetilde{I}, \widetilde{\omega}, \nabla)$ be a special Kähler manifold of dimension $n+1$. We define $\widetilde{\eta}$ as the (1,2)-tensor such that for all vector fields X, Y on $\widetilde{M}$ we have $\widetilde{\eta}_X Y = \nabla_X Y - \widetilde{\nabla}^{LC}_X Y$, where the employed notation $\widetilde{\eta}_X Y$ means $\widetilde{\eta}(X, Y)$.

Consider frames adapted to the pseudo-Kähler structure, hence such that the linear model is $(\mathbb{R}^{2n+2}, g_0, I_0, \omega_0)$, where $g_0 = \sum_{k=1}^{2k}(e^k)^2 - (e^{2n+1})^2 - (e^{2n+2})^2$, $Ie_{2k-1} = e_{2k}$ for $k = 1, \dots, n+1$ (see (1.12)) and $\omega_0 = g_0(I_0\cdot, \cdot)$. Let ω^∇ and $\widetilde{\omega}^{LC}$ be the connection forms corresponding respectively to the flat and the Levi-Civita connections represented with respect to an adapted frame. Thus we have

$$\omega^\nabla = \widetilde{\omega}^{LC} + \widetilde{\eta}$$

Since both connections are symplectic, the corresponding forms, and thus $\widetilde{\eta}$, will have values in $\mathfrak{sp}(2n+2, \mathbb{R})$ which can be described as

$$\{A \in \mathfrak{gl}(2n+2, \mathbb{R}) | A^t I_0 + I_0 A = 0\},$$

where A^t is the transposed of A with respect to g_0, that is such that for all $X, Y \in \mathbb{R}^{2n+2}$, $g_0(AX, Y) = g_0(X, A^t Y)$. It follows that $\widetilde{\eta}$ corresponds to a section of $T^* \otimes \mathfrak{sp}(2n+2, \mathbb{R})$ where T is the standard real representation of $\mathrm{U}(n, 1)$.

The Lie algebra $\mathfrak{sp}(2n+2, \mathbb{R})$ is closed with respect to transposition, and thus it is also closed with respect to symmetrisation and antisymmetrisation. As a consequence, we have the following splitting:

$$\mathfrak{sp}(2n+2, \mathbb{R}) = (\mathfrak{sp}(2n+2, \mathbb{R}) \cap \mathfrak{sym}(2n, 2)) \oplus (\mathfrak{sp}(2n+2, \mathbb{R}) \cap \mathfrak{so}(2n, 2))$$

The first summand consists of symmetric matrices $A \in \mathfrak{gl}(2n+2, \mathbb{R})$ such that $0 = A^t I_0 + I_0 A = AI_0 + I_0 A$ and thus, as complex endomorphisms, its elements are all the real, $\mathbb{C}$-antilinear and symmetric ones and therefore it is $[\![S_{2,0}]\!]$. By contrast, the second summand is $\mathfrak{u}(n, 1)$, which is isomorphic to $[\Lambda_{1,1}]$. As a $\mathfrak{u}(n, 1)$-representation, the subspace containing $\widetilde{\eta}$ is then isomorphic to

$$[\![\Lambda_{1,0}]\!] \otimes ([\![S_{2,0}]\!] \oplus [\Lambda_{1,1}]) = ([\![\Lambda_{1,0}]\!] \otimes [\![S_{2,0}]\!]) \oplus ([\![\Lambda_{1,0}]\!] \otimes [\Lambda_{1,1}]).$$

The condition $d^\nabla \widetilde{I} = 0$ is equivalent to requiring the symmetry of $\nabla \widetilde{I} = \widetilde{\nabla}^{LC} \widetilde{I} + [\widetilde{\eta}, \widetilde{I}] = [\widetilde{\eta}, \widetilde{I}]$ in the covariant indices. Consider the splitting of $\widetilde{\eta} = \widetilde{\eta}^S + \widetilde{\eta}^A$ in its symmetric and antisymmetric part in the last two indices.

Then $\widetilde{\eta}^A \in \mathfrak{u}(n,1)$ so in particular it commutes with $\widetilde{I}$ giving $[\widetilde{\eta}^A, \widetilde{I}] = 0$, whereas $[\widetilde{\eta}^S, \widetilde{I}] = \widetilde{\eta}^S \widetilde{I} - \widetilde{I}\widetilde{\eta}^S = (\widetilde{\eta}^S)^t \widetilde{I} - \widetilde{I}\widetilde{\eta}^S = -2\widetilde{I}\widetilde{\eta}^S$. Thus $[\widetilde{\eta}, \widetilde{I}] = [\widetilde{\eta}^S, \widetilde{I}] + [\widetilde{\eta}^A, \widetilde{I}] = -2\widetilde{I}\widetilde{\eta}^S$ which then needs to be symmetric in the two covariant indices. However, the composition with $\widetilde{I}$ is an isomorphism acting only on the contravariant index, so $[\widetilde{\eta}, \widetilde{I}]$ is symmetric in the two covariant indices if and only if $\widetilde{\eta}^S$ itself is. Consider now the linear map that anti-symmetrises the two covariant indices

$$\begin{aligned} \mathfrak{A}\colon T^* \otimes T \otimes T^* &\longrightarrow \Lambda^2 T^* \otimes T \\ \alpha \otimes X \otimes \beta &\longmapsto \alpha \wedge \beta \otimes X. \end{aligned} \tag{2.1}$$

By a straightforward computation on the irreducible components of $[\![\Lambda_{1,0}]\!] \otimes [\![S_{2,0}]\!]$, one can see that applying $\mathfrak{A}$, the only vanishing component is $[\![S_{3,0}]\!]$ and hence this is where $\widetilde{\eta}^S$ must be.

Both the Levi-Civita and the flat connection are torsion-free, therefore $\widetilde{\eta}$ must be symmetric in the two covariant indices. We already know that $\mathfrak{A}(\widetilde{\eta}^S)$ vanishes thanks to the previous condition and moreover, (2.1) is injective (actually an isomorphism) when restricted to $T^* \otimes \mathfrak{so}(2n+2, \mathbb{R})$, so $\mathfrak{A}(\widetilde{\eta}^A) = 0$ if and only if $\widetilde{\eta}^A = 0$. The torsion-free condition is then equivalent to $\widetilde{\eta} = \widetilde{\eta}^S$, so in conclusion, $\widetilde{\eta}$ is in the irreducible component isomorphic to $[\![S_{3,0}]\!]$. The isomorphism is constructed with the musical isomorphisms $\flat$ and $\sharp$ corresponding to the metric; explicitly, it is a restriction of

$$\flat_2 = \mathrm{id} \otimes \flat \otimes \mathrm{id}\colon T^* \otimes T \otimes T^* \longrightarrow T_3$$

with inverse $\sharp_2 := \mathrm{id} \otimes \sharp \otimes \mathrm{id}$. We have then recovered the following result (see [23, Proposition 1.34, p. 39] or [6, Lemma 3, p.1745]).

Lemma 2.1.1. *On a special Kähler manifold* $(\widetilde{M}, \widetilde{g}, \widetilde{I}, \widetilde{\omega}, \nabla)$*, the tensor* $\widetilde{\eta}$ *is a section of* $\sharp_2[\![S_{3,0}\widetilde{M}]\!]$.

Notice that in the process we have also proven

$$\nabla \widetilde{I} = [\widetilde{\eta}, \widetilde{I}] = -2\widetilde{I}\eta. \tag{2.2}$$

By using the flatness of ∇, we observe:

$$0 = \Omega^\nabla = d\omega^\nabla + \frac{1}{2}[\omega^\nabla \wedge \omega^\nabla]$$

$$\begin{aligned}
&= d\widetilde{\omega}^{LC} + d\widetilde{\eta} + \frac{1}{2}\left([\widetilde{\omega}^{LC} \wedge \widetilde{\omega}^{LC}] + [\widetilde{\omega}^{LC} \wedge \widetilde{\eta}] + [\widetilde{\eta} \wedge \widetilde{\omega}^{LC}] + [\widetilde{\eta} \wedge \widetilde{\eta}]\right)\\
&= d\widetilde{\omega}^{LC} + \frac{1}{2}[\widetilde{\omega}^{LC} \wedge \widetilde{\omega}^{LC}] + d\widetilde{\eta} + [\widetilde{\omega}^{LC} \wedge \widetilde{\eta}] + \frac{1}{2}[\widetilde{\eta} \wedge \widetilde{\eta}]\\
&= \widetilde{\Omega}^{LC} + \widetilde{d}^{LC}\widetilde{\eta} + \frac{1}{2}[\widetilde{\eta} \wedge \widetilde{\eta}],
\end{aligned}$$

where $\widetilde{\Omega}^{LC}$ and $\widetilde{d}^{LC}$ are respectively the curvature and exterior covariant derivative of the Levi-Civita connection on $\widetilde{M}$.

Arguing as in [23, Proposition 1.34, p. 39] (see also [6, Proposition 4, p. 1743]), we can prove

Proposition 2.1.2. *For a Kähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{I}, \widetilde{\omega})$ with a tensor $\widetilde{\eta}$ in $T^*M \otimes TM \otimes T^*M$ such that $\flat_2\widetilde{\eta}$ is a section of $[\![S_{3,0}\widetilde{M}]\!]$ and with a connection ∇ with connection form $\omega^\nabla = \widetilde{\omega}^{LC} + \widetilde{\eta}$, then*

$$\Omega^\nabla = 0 \qquad \textit{if and only if} \qquad \begin{cases} \widetilde{\Omega}^{LC} + \frac{1}{2}[\widetilde{\eta} \wedge \widetilde{\eta}] = 0\\ \widetilde{d}^{LC}\widetilde{\eta} = 0\end{cases}$$

Proof. The Levi-Civita connection form takes values in $\mathfrak{u}(n,1)$, so $\widetilde{\Omega}^{LC}$ is of type $S^2(\mathfrak{u}(n,1))$ and therefore, if $\flat$ is the map lowering the contravariant index, we get that $\flat\widetilde{\Omega}^{LC}$ belongs to $\Omega^2(\widetilde{M}, [\Lambda_{1,1}\widetilde{M}])$. Now, since $\flat\widetilde{\eta} = \flat_2\widetilde{\eta}$ is a section of $[\![S_{3,0}\widetilde{M}]\!]$, it is in particular in $\Omega^1(\widetilde{M}, [\![S_{2,0}\widetilde{M}]\!])$, so $\flat\widetilde{d}^{LC}\widetilde{\eta} = \widetilde{d}^{LC}\flat_2\widetilde{\eta}$ belongs to $\Omega^2(\widetilde{M}, [\![S_{2,0}\widetilde{M}]\!])$. Finally, computations on a basis, show that $\flat[\widetilde{\eta} \wedge \widetilde{\eta}]$ belongs to $\Omega^2(\widetilde{M}, [\Lambda_{1,1}\widetilde{M}])$. Since $[\![S_{2,0}\widetilde{M}]\!]$ and $[\Lambda_{1,1}\widetilde{M}]$ intersect trivially, the quantities $\widetilde{d}^{LC}\widetilde{\eta}$ and $\widetilde{\Omega}^{LC} + \frac{1}{2}[\widetilde{\eta} \wedge \widetilde{\eta}]$ are independent, so their sum is 0 if and only if they vanish separately. □

2.2 Conic and projective special Kähler metrics

In this section we will consider the case of a projective special Kähler manifold $(\pi \colon \widetilde{M} \to M, \nabla)$ and we will give the explicit relation between the metric on $\widetilde{M}$ and the one on M (see e.g. [17, Section 1.1]).

The mapping $\pi \colon \widetilde{M} \to M$ is a $\mathbb{C}^*$-principal bundle with infinitesimal principal action generated by ξ and $\widetilde{I}\xi$. We can always build the function

$r = \sqrt{-\widetilde{g}(\xi,\xi)} \colon \widetilde{M} \to \mathbb{R}^+$ and define $S = r^{-1}(1) \subseteq \widetilde{M}$ with inclusion map $\iota_S \colon S \hookrightarrow \widetilde{M}$. Now r has no critical points, since

$$\begin{aligned} dr &= \frac{d(r^2)}{2r} = \frac{\widetilde{\nabla}^{LC}(r^2)}{2r} = -\frac{\widetilde{\nabla}^{LC}(\widetilde{g}(\xi,\xi))}{2r} \\ &= -\frac{2\widetilde{g}(\widetilde{\nabla}^{LC}\xi,\xi)}{2r} = -\frac{\widetilde{g}(\cdot,\xi)}{r} = -\frac{1}{r}\xi^\flat \end{aligned} \tag{2.3}$$

and $\widetilde{g}$ is non-degenerate. It follows that S is a submanifold of dimension $2n+1$ whose tangent bundle corresponds to $\ker(dr) \subset T\widetilde{M}$. Notice that $dr(\widetilde{I}\xi) = -\frac{\widetilde{g}(\widetilde{I}\xi,\xi)}{r} = -\frac{\widetilde{\omega}(\xi,\xi)}{r} = 0$, so $\widetilde{I}\xi$ is a vector field tangent to S and it induces a principal U(1)-action. The induced metric on S is $g_S = \iota_S^*\widetilde{g}$ and thus $\mathcal{L}_{\widetilde{I}\xi} g_S = \iota_S^* \mathcal{L}_{\widetilde{I}\xi}\widetilde{g} = 0$.

The principal action of $\mathbb{C}^*$ on $\widetilde{M}$ induces by inclusion an $\mathbb{R}^+$-action, and in addition we have

Lemma 2.2.1. *The map $r \colon \widetilde{M} \to \mathbb{R}^+$ is degree-one homogeneous with respect to the action of $\mathbb{R}^+ \subseteq \mathbb{C}^*$ on $\widetilde{M}$, i.e. for all $s \in \mathbb{R}^+$ and $p \in \widetilde{M}$*

$$r(ps) = r(p)s$$

Proof. Define the map

$$f \colon \mathbb{R} \longrightarrow \mathbb{R}^+, \qquad t \longmapsto r(pe^t)$$

Notice that $t \mapsto e^t$ is the exponential map of the Lie group $\mathbb{R}^+$, so $pe^t = \phi_\xi^t(p)$, where ϕ_ξ is the flow of ξ. The derivative of f is then

$$\begin{aligned} \frac{df}{dt}(t_0) &= \frac{d}{dt}(r \circ \phi_\xi^t(p))|_{t=t_0} = \frac{d}{dt}(r \circ \phi_\xi^t \phi_\xi^{t_0}(p))|_{t=0} = (\mathcal{L}_\xi r)(\phi_\xi^{t_0}(p)) \\ &= dr(\xi)(\phi_\xi^{t_0}(p)) = -\frac{\widetilde{g}(\xi,\xi)}{r}(\phi_\xi^{t_0}(p)) = r(\phi_\xi^{t_0}(p)) = r(pe^{t_0}) = f(t_0) \end{aligned}$$

moreover, $f(0) = r(p)$, so f is a solution of the following initial value problem:

$$\begin{cases} f' = f \\ f(0) = r(p) \end{cases}$$

which has a unique solution, namely $f(t) = r(p)e^t$ and thus $r(pe^t) = r(p)e^t$. Replacing e^t with s gives the statement. $\square$

As a consequence of this lemma, we can now define a retraction

$$p\colon \widetilde{M} \longrightarrow S, \qquad u \longmapsto u\frac{1}{r(u)}.$$

It is well defined, since $r(p(u)) = r\left(u\frac{1}{r(u)}\right) = \frac{r(u)}{r(u)} = 1$. Moreover, $p\iota_S = \mathrm{id}_S$ implies the surjectivity of p, which allows us to see $p\colon \widetilde{M} \to S$ as a principal $\mathbb{R}^+$-bundle and $\pi_S := \pi\iota_S\colon S \to M$ as a principal S^1-bundle; the composition of the two gives π.

Lemma 2.2.2. *If $(\pi\colon \widetilde{M} \to M, \nabla)$ is projective special Kähler, then $\widetilde{M}$ is diffeomorphic to $S \times \mathbb{R}^+$, and moreover*

$$\widetilde{g} = r^2 p^* g_S - dr^2$$

Proof. Let $a\colon S \times \mathbb{R}^+ \to \widetilde{M}$ be the restriction of the principal right action $\widetilde{M} \times \mathbb{R}^+ \to \widetilde{M}$ to $S \times \mathbb{R}^+$ and consider also $(p, r)\colon \widetilde{M} \to S \times \mathbb{R}^+$. These maps are smooth and each an inverse to the other, in fact if $u \in \widetilde{M}$, $a(p, r)(u) = a(p(u), r(u)) = u\frac{1}{r(u)}r(u) = u$ and for all $(q, s) \in S \times \mathbb{R}^+$, $(\pi_S, r)a(q, s) = (p(qs), r(qs)) = (q\frac{s}{r(qs)}, r(q)s) = (q, s)$.

For the second part of the statement consider the symmetric tensor

$$g' = \frac{1}{r^2}(\widetilde{g} + dr^2)$$

We want to prove it is basic, that is horizontal and invariant with respect to the principal $\mathbb{R}^+$-action.

Since there is only one vertical direction, and since g' is symmetric, it is enough to check whether g' vanishes when evaluated on the fundamental vector field ξ in one component. Using (2.3) we obtain

$$g'(\xi, \cdot) = \frac{1}{r}(\widetilde{g}(\xi, \cdot) + dr(\xi)dr) = \frac{1}{r}(-rdr + rdr) = 0.$$

And now for the $\mathbb{R}^+$-invariance:

$$\begin{aligned}\mathcal{L}_\xi g' &= -2\frac{\mathcal{L}_\xi r}{r^3}(\widetilde{g} + dr^2) + \frac{1}{r^2}(\mathcal{L}_\xi \widetilde{g} + 2\mathcal{L}_\xi(dr)dr) \\ &= -2\frac{dr(\xi)}{r^3}(\widetilde{g} + dr^2) + \frac{1}{r^2}(2\widetilde{g} + 2(d\iota_\xi dr + \iota_\xi d^2 r)dr)\end{aligned}$$

$$= -2\frac{r}{r^3}(\widetilde{g} + dr^2) + \frac{1}{r^2}(2\widetilde{g} + 2dr^2) = 0.$$

Therefore g' is basic, which in turn implies it is of the form p^*g'' for some tensor $g'' \in T_2S$, so that

$$\widetilde{g} = r^2 p^* g'' - dr^2.$$

The proof is ended by the following observation:

$$g_S = \iota_S^*\widetilde{g} = \iota_S^*\left(r^2 p^* g'' - dr^2\right) = \iota_S^* p^* g'' - \iota_S^* dr^2 = (p\iota_S)^* g'' = g''. \qquad \square$$

The $\mathbb{C}^*$-bundle $\pi\colon \widetilde{M} \to M$ has a unique principal connection orthogonal to the fibres with respect to $\widetilde{g}$; the connection form can be written as

$$\frac{dr}{r} + i\widetilde{\varphi}. \tag{2.4}$$

Explicitly, we can describe $\widetilde{\varphi}$ using the metric:

$$\widetilde{\varphi} = \frac{\widetilde{g}(\widetilde{I}\xi, \cdot)}{\widetilde{g}(\widetilde{I}\xi, \widetilde{I}\xi)} = -\frac{1}{r^2} I\xi^\flat = -\frac{1}{r^2}\iota_\xi\widetilde{\omega}.$$

If we restrict it to S, we obtain a connection form $\varphi = \iota_S^*\widetilde{\varphi} = -\iota_S^*(\iota_\xi\omega)$ corresponding to the S^1-action on S.

Notice that $p^*\varphi = \widetilde{\varphi}$, because the connection form (2.4) is right-invariant, so $\widetilde{\varphi} = p^*\varphi'$ for some φ', and thus $\varphi = \iota_S^*\widetilde{\varphi} = \iota_S^* p^*\varphi' = (p\iota_S)^*\varphi' = \varphi'$.

The moment map for the action generated by $\widetilde{I}\xi$ is $\mu\colon \widetilde{M} \to \mathfrak{u}(1) \cong \mathbb{R}$ s.t. $d\mu = \iota_{\widetilde{I}\xi}\omega = -\xi^\flat = r\,dr = d\left(\frac{r^2}{2}\right)$, so up to an additive constant, we can assume

$$\mu = \frac{r^2}{2}.$$

Since $S = \mu^{-1}(\frac{1}{2})$ is a level set of the moment map and M is the Kähler quotient, $\pi_S\colon S \to M$ is a pseudo-Riemannian submersion and thus we can write $g_S = \pi_S^* g - \varphi^2$.

Proposition 2.2.3. *A projective special Kähler manifold* $(\pi\colon \widetilde{M} \to M, \nabla)$ *satisfies*

$$\widetilde{g} = r^2\pi^* g - r^2\widetilde{\varphi}^2 - dr^2;$$

$$\widetilde{\omega} = r^2\pi^*\omega_M + r\widetilde{\varphi} \wedge dr.$$

Proof. From the previous arguments

$$\begin{aligned}\widetilde{g} &= r^2 p^* g_S - dr^2 = r^2 p^*(\pi_S^* g - \varphi^2) - dr^2 \\ &= r^2(\pi_S p)^* g - r^2 \widetilde{\varphi}^2 - dr^2 = r^2 \pi^* g - r^2 \widetilde{\varphi}^2 - dr^2.\end{aligned}$$

For the Kähler form it is enough to notice that π is holomorphic, M being a Kähler quotient, and that

$$(r\widetilde{\varphi}) \circ \widetilde{I} = -\frac{1}{r}\widetilde{I}\xi^\flat \widetilde{I} = -\frac{1}{r}\xi^\flat = dr. \qquad \square$$

For future reference we give the following

Remark 2.2.4. *The curvature of φ is computed using Lemma 1.2.2:*

$$d\varphi = -d\iota_S^* \iota_\xi \widetilde{\omega} = \iota_S^*(-\mathcal{L}_\xi \widetilde{\omega} + \iota_\xi d\widetilde{\omega}) = -2\iota_S^* \widetilde{\omega} = -2\pi_S^* \omega_M$$

in fact, the restriction to S of $\widetilde{\omega}$ maps fixes $r = 1$ and thus kills dr.

It will also be useful to compute

$$d\widetilde{\varphi} = -2\pi^* \omega_M.$$

2.3 Lifting the coframe

The purpose of this section is to lift a generic unitary coframe on a projective special Kähler manifold to one on the corresponding conic special Kähler. This will enable us to give a more explicit formulation of the Levi-Civita connection and associated curvature tensor on the conic special Kähler manifold.

In our convention, on a Kähler manifold (M, g, I, ω), the Hermitian form is $h = g + i\omega$. Given a projective special Kähler manifold $(\pi\colon \widetilde{M} \to M, \nabla)$ and an open subset $U \subseteq M$, consider a unitary coframe $\theta = (\theta^1, \ldots, \theta^n) \in \Omega^1(U, \mathbb{C}^n)$ on M, then we can build a coframe $\widetilde{\theta} \in \Omega^1(\pi^{-1}(U), \mathbb{C}^{n+1})$ on $\widetilde{M}$ as follows:

$$\widetilde{\theta}^k = \begin{cases} r\pi^*\theta^k & \text{if } k \leq n \\ dr + ir\widetilde{\varphi} & \text{if } k = n+1 \end{cases} \tag{2.5}$$

This coframe is compatible with the $\mathrm{U}(n,1)$-structure because it takes complex values and

$$\sum_{k=1}^{n} \overline{\widetilde{\theta}^k}\widetilde{\theta}^k - \overline{\widetilde{\theta}^{n+1}}\widetilde{\theta}^{n+1} = r^2\pi^*\left(\sum_{k=1}^{n} \overline{\theta^k}\theta^k\right) - dr^2 - r^2\widetilde{\varphi}^2 = \widetilde{g}$$

We will denote the dual frame to a given coframe by the same symbol, but with lower indices.

Remark 2.3.1. *Given a connection on a Kähler manifold, it can be represented by a connection form ω with values in $\mathfrak{u}(n,1)$, whose complexification is $\mathfrak{gl}(n+1,\mathbb{C}) \cong T^{1,0} \otimes T_{1,0} \oplus T^{0,1} \otimes T_{1,0}$, so we obtain projections in each component, respectively $\omega^{1,0}_{1,0}$ and $\omega^{0,1}_{0,1}$ such that $\omega = \omega^{1,0}_{1,0} + \omega^{0,1}_{0,1}$. Notice that $\omega^{0,1}_{0,1} = \overline{\omega^{1,0}_{1,0}}$ because ω comes from a real representation and to give the first component is equivalent to give the whole form. Notice also that $([\![T]\!], I)$, as complex representation, is isomorphic to $T^{1,0}$ and the component $A^{1,0}_{1,0}$ of an endomorphism A gives the corresponding endomorphism of $T^{1,0}$. We will often present connection forms by giving only the $T^{1,0}_{1,0}$ component.*

We will call $\mathfrak{R}$ the projection from the complex tensor algebra to the real representation, defined so that $\mathfrak{R}(\alpha) = \alpha + \overline{\alpha}$, where the conjugate is the real structure.

Proposition 2.3.2. *Let $(\pi\colon \widetilde{M} \to M, \nabla)$ be a projective special Kähler manifold, let (U, θ) be a local unitary coframe on M lifted as in (2.5) to a coframe $\widetilde{\theta}$ adapted to the $\mathrm{U}(n,1)$-structure on $\widetilde{M}$. With respect to $\widetilde{\theta}$, the Levi-Civita connection form on $\widetilde{M}$ is represented by*

$$\widetilde{\omega}^{LC} = \left(\begin{array}{c|c} \pi^*\omega^{LC} & 0 \\ \hline 0 & 0 \end{array}\right) + \frac{1}{r}\left(\begin{array}{ccc|c} i\operatorname{Im}\left(\widetilde{\theta}^{n+1}\right) & & 0 & \widetilde{\theta}^1 \\ & \ddots & & \vdots \\ 0 & & i\operatorname{Im}\left(\widetilde{\theta}^{n+1}\right) & \widetilde{\theta}^n \\ \hline \overline{\widetilde{\theta}^1} & \cdots & \overline{\widetilde{\theta}^n} & i\operatorname{Im}\left(\widetilde{\theta}^{n+1}\right) \end{array}\right),$$

that is

$$\widetilde{\omega}^{LC} = \left(\begin{array}{c|c} \pi^*\omega^{LC} + i\widetilde{\varphi} \otimes I_n & \pi^*\theta \\ \hline \pi^*\theta^\star & i\widetilde{\varphi} \end{array}\right) \tag{2.6}$$

and its curvature form is

$$\widetilde{\Omega}^{LC} = \left(\begin{array}{c|c} \pi^*(\Omega^{LC} + \theta \wedge \theta^\star - 2i\omega_M \otimes \mathrm{id}) & 0 \\ \hline 0 & 0 \end{array}\right).$$

Proof. The connection form (2.6) is metric if and only if the matrix is antihermitian with respect to $\widetilde{g}$ and since ω^{LC} is antihermitian with respect to g, we get

$$(\widetilde{\omega}^{LC})^\star = \left(\begin{array}{c|c} \pi^*(\omega^{LC})^\star - i\widetilde{\varphi} \otimes I_n & -\pi^*\theta \\ \hline -\pi^*\theta^\star & -i\widetilde{\varphi} \end{array}\right) = -\widetilde{\omega}^{LC}.$$

The torsion form of this connection is $\widetilde{\Theta}^{LC} = d\widetilde{\theta} + \widetilde{\omega}^{LC} \wedge \widetilde{\theta}$, so for $1 \leq k \leq n$

$$
\begin{aligned}
\left(\widetilde{\Theta}^{LC}\right)^k &= d\widetilde{\theta}^k + \sum_{j=1}^{n} \left(\widetilde{\omega}^{LC}\right)_j^k \wedge \widetilde{\theta}^j + \left(\widetilde{\omega}^{LC}\right)_{n+1}^k \wedge \widetilde{\theta}^{n+1} \\
&= d\left(r\pi^*\theta^k\right) + \sum_{j=1}^{n} \left(\pi^*(\omega^{LC})_j^k + i\widetilde{\varphi}\delta_j^k\right) \wedge \left(r\pi^*\theta^j\right) + \pi^*\theta^k \wedge \widetilde{\theta}^{n+1} \\
&= dr \wedge \pi^*\theta^k + r\pi^* d\theta^k + r\pi^*((\omega^{LC})_j^k \wedge \theta^j) + ir\widetilde{\varphi} \wedge \pi^*\theta^k + \pi^*\theta^k \wedge \widetilde{\theta}^{n+1} \\
&= r\pi^*(\Theta^{LC})^k + (dr + ir\widetilde{\varphi}) \wedge \pi^*\theta^k + \pi^*\theta^k \wedge \widetilde{\theta}^{n+1} \\
&= 0 + \widetilde{\theta}^{n+1} \wedge \pi^*\theta^k + \pi^*\theta^k \wedge \widetilde{\theta}^{n+1} = 0.
\end{aligned}
$$

In the last component

$$
\begin{aligned}
(\widetilde{\Theta}^{LC})^{n+1} &= d\widetilde{\theta}^{n+1} + \sum_{j=1}^{n} \pi^*\overline{\theta^j} \wedge r\pi^*\theta^j + i\widetilde{\varphi} \wedge \widetilde{\theta}^{n+1} \\
&= d(dr + ir\widetilde{\varphi}) + r\pi^* \left(\sum_{j=1}^{n} \overline{\theta^j} \wedge \theta^j \right) + i\widetilde{\varphi} \wedge \widetilde{\theta}^{n+1} \\
&= idr \wedge \widetilde{\varphi} + ird\widetilde{\varphi} + 2ir\pi^*\omega_M + i\widetilde{\varphi} \wedge (dr + ir\widetilde{\varphi}) \\
&= idr \wedge \widetilde{\varphi} + ir(d\widetilde{\varphi} + 2\pi^*\omega_M) + i\widetilde{\varphi} \wedge dr = 0.
\end{aligned}
$$

Since $\widetilde{\omega}^{LC}$ is metric and torsion-free, by uniqueness we infer that it must be the Levi-Civita connection.

Let us now compute its curvature form $\widetilde{\Omega}^{LC} = d\widetilde{\omega}^{LC} + \widetilde{\omega}^{LC} \wedge \widetilde{\omega}^{LC}$. For $1 \leq k, h \leq n$ we have

$$
\begin{aligned}
\left(\widetilde{\Omega}^{LC}\right)_k^h &= d(\widetilde{\omega}^{LC})_k^h + (\widetilde{\omega}^{LC})_j^h \wedge (\widetilde{\omega}^{LC})_k^j \\
&= d\pi^*(\omega^{LC})_k^h + id\widetilde{\varphi}\delta_k^h + \sum_{j=1}^{n} (\pi^*(\omega^{LC})_j^h + i\widetilde{\varphi}\delta_j^h) \wedge (\pi^*(\omega^{LC})_k^j + i\widetilde{\varphi}\delta_k^j) \\
&\quad + \pi^*\theta^h \wedge \pi^*\overline{\theta^k} \\
&= \pi^* d(\omega^{LC})_k^h - 2i\pi^*\omega_M \delta_k^h + \pi^*((\omega^{LC})_j^h \wedge (\omega^{LC})_k^j) \\
&\quad + i\widetilde{\varphi} \wedge \pi^*(\omega^{LC})_k^h + \pi^*(\omega^{LC})_k^h \wedge i\widetilde{\varphi} - \widetilde{\varphi} \wedge \widetilde{\varphi}\delta_k^h + \pi^*\theta^h \wedge \pi^*\overline{\theta^k} \\
&= \pi^*(\Omega^{LC})_k^h - 2i\pi^*\omega_M\delta_k^h + \pi^*(\theta^h \wedge \overline{\theta^k});
\end{aligned}
$$

$$\begin{aligned}\left(\widetilde{\Omega}^{LC}\right)^h_{n+1} &= d\pi^*\theta^h + \sum_{j=1}^n (\pi^*(\omega^{LC})^h_j + i\widetilde{\varphi}\delta^h_j) \wedge \pi^*\theta^j + \pi^*\theta^h \wedge i\widetilde{\varphi} \\ &= \pi^* d\theta^h + \pi^*\left((\omega^{LC})^h_j \wedge \theta^j\right) + i\widetilde{\varphi} \wedge \pi^*\theta^h + \pi^*\theta^h \wedge i\widetilde{\varphi} \\ &= \pi^*\left(\Theta^{LC}\right)^h = 0.\end{aligned}$$

Since the curvature form must also be antihermitian, we get

$$\left(\widetilde{\Omega}^{LC}\right)^{n+1}_k = -\left(\left(\widetilde{\Omega}^{LC}\right)^\star\right)^{n+1}_k = \overline{\left(\widetilde{\Omega}^{LC}\right)^k_{n+1}} = 0.$$

Finally,

$$\left(\widetilde{\Omega}^{LC}\right)^{n+1}_{n+1} = id\widetilde{\varphi} + \sum_{j=1}^n \pi^*\overline{\theta^j} \wedge \pi^*\theta^j - \widetilde{\varphi} \wedge \widetilde{\varphi} = id\widetilde{\varphi} + 2i\pi^*\omega_M = 0. \qquad \square$$

Remark 2.3.3. *The tensor $\theta \wedge \theta^\star - 2i\omega_M \otimes id$, or explicitly*

$$\Omega_{\mathbb{P}^n_{\mathbb{C}}} := \mathfrak{R}\left((\theta^k \wedge \overline{\theta^h}) \otimes \theta_k \otimes \theta^h - (\overline{\theta^k} \wedge \theta^k) \otimes \theta_h \otimes \theta^h\right)$$

is a curvature tensor of the complex projective space of dimension n. It is of curvature type, in fact if we lower the contravariant index we get

$$\begin{aligned}\Omega^\flat_{\mathbb{P}^n_{\mathbb{C}}} &= \mathfrak{R}\left(\frac{1}{2}\left((\theta^k \wedge \overline{\theta^h}) \otimes \overline{\theta^k} \otimes \theta^h - (\overline{\theta^k} \wedge \theta^k) \otimes \overline{\theta^h} \otimes \theta^h\right)\right) \\ &= \frac{1}{2}\Big((\theta^k \wedge \overline{\theta^h}) \otimes \overline{\theta^k} \otimes \theta^h - (\overline{\theta^k} \wedge \theta^k) \otimes \overline{\theta^h} \otimes \theta^h \\ &\quad + (\overline{\theta^k} \wedge \theta^h) \otimes \theta^k \otimes \overline{\theta^h} - (\theta^k \wedge \overline{\theta^k}) \otimes \theta^h \otimes \overline{\theta^h}\Big) \\ &= \frac{1}{2}\left((\theta^k \wedge \overline{\theta^h}) \otimes (\overline{\theta^k} \wedge \theta^h) + (\theta^k \wedge \overline{\theta^k}) \otimes (\overline{\theta^h} \wedge \theta^h)\right) \\ &= \frac{1}{4}\left((\theta^k \wedge \overline{\theta^h}) \otimes (\overline{\theta^k} \wedge \theta^h) + (\theta^h \wedge \overline{\theta^k}) \otimes (\overline{\theta^h} \wedge \theta^k)\right) + \frac{1}{2}(\theta^k \wedge \overline{\theta^k})^2 \\ &= \frac{1}{2}(\theta^k \wedge \overline{\theta^h})(\overline{\theta^k} \wedge \theta^h) + \frac{1}{2}(\theta^k \wedge \overline{\theta^k})^2\end{aligned}$$

and if we apply the map

$$\begin{aligned}\mathfrak{B}\colon S^2(\Lambda^2 T^*) &\longrightarrow \Lambda^4 T^* \\ (\alpha \wedge \beta)(\gamma \wedge \delta) &\longmapsto \alpha \wedge \beta \wedge \gamma \wedge \delta,\end{aligned}$$

we obtain

$$\begin{aligned}\mathfrak{B}(\Omega^\flat_{\mathbb{P}^n_\mathbb{C}}) &= \frac{1}{2}\left(\theta^k \wedge \overline{\theta^h} \wedge \overline{\theta^k} \wedge \theta^h + \theta^k \wedge \overline{\theta^k} \wedge \overline{\theta^h} \wedge \theta^h\right) \\ &= \frac{1}{2}\left(\theta^k \wedge \overline{\theta^h} \wedge \overline{\theta^k} \wedge \theta^h - \theta^k \wedge \overline{\theta^h} \wedge \overline{\theta^k} \wedge \theta^h\right) = 0,\end{aligned}$$

so $\Omega_{\mathbb{P}^n_\mathbb{C}}$ satisfies the Bianchi identity.

In fact, $\Omega_{\mathbb{P}^n_\mathbb{C}}$ is the curvature with respect to the Fubini-Study metric (see for example [33, II, p. 277]). In order to verify that $\Omega_{\mathbb{P}^n_\mathbb{C}}$ is exactly the curvature of the Fubini-Study rather than a multiple, we compute the Ricci tensor:

$$\mathrm{Ric}_{\mathbb{P}^n_\mathbb{C}} = \mathfrak{R}\left(n\theta^h \otimes \overline{\theta^h} + \delta_{h,k}\theta^h \otimes \overline{\theta^k}\right) = \mathfrak{R}\left((n+1)h\right) = 2(n+1)g. \tag{2.7}$$

Then,

$$\mathrm{scal}_{\mathbb{P}^n_\mathbb{C}} = 2(n+1). \tag{2.8}$$

Thus $\Omega_{\mathbb{P}^n_\mathbb{C}}$ corresponds exactly to the curvature of $\mathbb{P}^n_\mathbb{C}$ with the Fubini-Study metric.

Now, whenever we have a smooth map $f\colon M \to N$ between Riemannian manifolds, we can extend the pullback $f^*\colon T_\bullet N \to T_\bullet M$ on the covariant tensor algebra to the whole tensor algebra, using the musical isomorphisms in each contravariant component. Explicitly, for X vector field on N, we define $f^*X := \sharp f^*\flat X = (f^*X^\flat)_\sharp$. Notice that this extension of the pullback is still functorial, since if $f\colon M \to N$, $g\colon N \to L$ are smooth maps, then $f^*g^*X = \sharp f^*\flat\sharp g^*\flat X = \sharp f^*g^*\flat X = \sharp(gf)^*\flat X = (gf)^*X$.

Since $\widetilde{M}$ and M are Riemannian manifolds, we have $\pi^*\colon T^\bullet_\bullet M \to T^\bullet_\bullet \widetilde{M}$, and in particular, for $1 \leq k \leq n$ we have

$$\pi^*\theta_k = (\pi^*\theta^\flat_k)_\sharp = \frac{1}{2}(\pi^*\overline{\theta^k})_\sharp = \frac{1}{2r}(\overline{\widetilde{\theta}^k})_\sharp = \frac{1}{r}\widetilde{\theta}_k.$$

Remark 2.3.4. *In this notation,*

$$\widetilde{\Omega}^{LC} = r^2\pi^*(\Omega^{LC} + \Omega_{\mathbb{P}^n_\mathbb{C}}).$$

2.4 Deviance

In this section we will continue the analysis of the tensor $\widetilde{\eta}$ started in section 2.1. The aim is to reduce it to a locally defined tensor on M that we call deviance. We will then use it to give an explicit local description of the Ricci tensor and the scalar curvature.

Lemma 2.4.1. *On a projective special Kähler manifold* $(\pi\colon \widetilde{M} \to M, \nabla)$, *if* $\widetilde{\eta}_X Y = \nabla_X Y - \widetilde{\nabla}^{LC}_X Y$, *then* $\flat_2 \widetilde{\eta}$ *is horizontal with respect to* π.

In other words, $\flat_2(\widetilde{\eta})$ *is a section of* $\pi^*[\![\sharp_2 S_{3,0} M]\!] \subset [\![S_{3,0}\widetilde{M}]\!]$. *Explicitly,* $\widetilde{\eta}_v$, $\widetilde{\eta} v$ *and* $\widetilde{g}(\widetilde{\eta}, v)$ *vanish for all* $v \in \langle \xi, I\xi \rangle$.

Proof. First notice that $\widetilde{\eta}(\xi) = \nabla\xi - \widetilde{\nabla}^{LC}\xi = 0$, so by symmetry $\widetilde{\eta}_\xi = 0$ and $g(\eta, \xi) = 0$, so $\flat_2(\widetilde{\eta})$ in each component when evaluated at ξ. From this fact and (2.2), we also deduce $\widetilde{\eta}(\widetilde{I}\xi) = \widetilde{I}\widetilde{\eta}(\xi) + [\widetilde{\eta}, \widetilde{I}]\xi = 0 - 2\widetilde{I}\widetilde{\eta}(\xi) = 0$. By symmetry, we conclude that $\flat_2\widetilde{\eta}$ vanishes in every component on $I\xi$. Linearity then completes the proof. $\square$

Lemma 2.4.2. *Let* $(\widetilde{M}, \widetilde{g}, \widetilde{I}, \widetilde{\omega}, \nabla, \xi)$ *be a conic special Kähler manifold and* $\widetilde{\eta}$ *be as above, then*

1. $\mathcal{L}_\xi \widetilde{\eta} = 0$;
2. $\mathcal{L}_{\widetilde{I}\xi} \widetilde{\eta} = -2\widetilde{I}\widetilde{\eta}$.

Proof. The proof relies on a generic formula satisfied by a torsion-free connection D (see e.g. [37, equation (3.1), p. 1336]), that is:

$$\begin{aligned}
\mathcal{L}_A(D_X Y) &- D_{\mathcal{L}_A X} Y - D_X \mathcal{L}_A Y \\
&= [A, D_X Y] - D_{[A,X]} Y - D_X [A, Y] \\
&= D_A D_X Y - D_{D_X Y} A - D_{[A,X]} Y - D_X D_A Y + D_X D_Y A \\
&= \Omega^D(A, X) Y - D_{D_X Y} A + D_X D_Y A.
\end{aligned}$$

1. We check the formula on vector fields $X, Y \in \mathfrak{X}\left(\widetilde{M}\right)$

$$\begin{aligned}
(\mathcal{L}_\xi \widetilde{\eta})_X Y &= \mathcal{L}_\xi(\widetilde{\eta}_X Y) - \widetilde{\eta}_{\mathcal{L}_\xi X} Y - \widetilde{\eta}_X \mathcal{L}_\xi Y \\
&= \mathcal{L}_\xi \nabla_X Y - \mathcal{L}_\xi \widetilde{\nabla}^{LC}_X Y - \nabla_{\mathcal{L}_\xi X} Y + \widetilde{\nabla}^{LC}_{\mathcal{L}_\xi X} Y \\
&\quad - \nabla_X \mathcal{L}_\xi Y + \widetilde{\nabla}^{LC}_X \mathcal{L}_\xi Y \\
&= \Omega^\nabla(\xi, X) Y - \nabla_{\nabla_X Y} \xi + \nabla_X \nabla_Y \xi - \widetilde{\Omega}^{LC}(\xi, X) Y \\
&\quad + \widetilde{\nabla}^{LC}_{\widetilde{\nabla}^{LC}_X Y} \xi - \widetilde{\nabla}^{LC}_X \widetilde{\nabla}^{LC}_Y \xi \\
&= -\nabla_X Y + \nabla_X Y - \widetilde{\Omega}^{LC}(\xi, X) Y + \widetilde{\nabla}^{LC}_X Y - \widetilde{\nabla}^{LC}_X Y
\end{aligned}$$

$$= -\widetilde{\Omega}^{LC}(\xi, X)Y.$$

Lowering the contravariant index of the curvature form, for $Z \in \mathfrak{X}\left(\widetilde{M}\right)$, thanks to the symmetries of the Riemannian tensor we obtain

$$\begin{aligned}
\widetilde{g}\left(\widetilde{\Omega}^{LC}(\xi, X)Y, Z\right) &= \widetilde{g}\left(\widetilde{\Omega}^{LC}(Y, Z)\xi, X\right) \\
&= \widetilde{g}\left(\widetilde{\nabla}^{LC}_Y\widetilde{\nabla}^{LC}_Z\xi - \widetilde{\nabla}^{LC}_Z\widetilde{\nabla}^{LC}_Y\xi - \widetilde{\nabla}^{LC}_{[Y,Z]}\xi, X\right) \\
&= \widetilde{g}\left(\widetilde{\nabla}^{LC}_Y Z - \widetilde{\nabla}^{LC}_Z Y - [Y, Z], X\right) \\
&= \widetilde{g}\left(\Theta^{LC}(Y, Z), X\right) = 0,
\end{aligned}$$

proving that $\widetilde{\Omega}^{LC}(\xi, X)Y = 0$, which implies the statement.

2. As before

$$\begin{aligned}
(\mathcal{L}_{\widetilde{I}\xi}\widetilde{\eta})_X Y &= \Omega^{\nabla}(\widetilde{I}\xi, X)Y - \nabla_{\nabla_X Y}(\widetilde{I}\xi) + \nabla_X\nabla_Y(\widetilde{I}\xi) - \widetilde{\Omega}^{LC}(\widetilde{I}\xi, X)Y \\
&\quad + \widetilde{\nabla}^{LC}_{\widetilde{\nabla}^{LC}_X Y}(\widetilde{I}\xi) - \widetilde{\nabla}^{LC}_X\widetilde{\nabla}^{LC}_Y(\widetilde{I}\xi) \\
&= -\widetilde{I}\nabla_X Y + \nabla_X(\widetilde{I}Y) - \widetilde{\Omega}^{LC}(\widetilde{I}\xi, X)Y + \widetilde{I}\widetilde{\nabla}^{LC}_X Y - \widetilde{\nabla}^{LC}_X(\widetilde{I}Y) \\
&= (\nabla\widetilde{I})(X, Y) - \widetilde{\Omega}^{LC}(\widetilde{I}\xi, X)Y.
\end{aligned}$$

Proceeding as in the previous point

$$\begin{aligned}
&\widetilde{g}\left(\widetilde{\Omega}^{LC}(\widetilde{I}\xi, X)Y, Z\right) = \widetilde{g}\left(\widetilde{\Omega}^{LC}(Y, Z)(\widetilde{I}\xi), X\right) \\
&= \widetilde{g}\left(\widetilde{\nabla}^{LC}_Y\widetilde{\nabla}^{LC}_Z(\widetilde{I}\xi) - \widetilde{\nabla}^{LC}_Z\widetilde{\nabla}^{LC}_Y(\widetilde{I}\xi) - \widetilde{\nabla}^{LC}_{[Y,Z]}(\widetilde{I}\xi), X\right) \\
&= \widetilde{g}\left(\widetilde{I}\widetilde{\Omega}^{LC}(Y, Z)\xi, X\right) = -\widetilde{g}\left(\widetilde{\Omega}^{LC}(Y, Z)\xi, IX\right) \\
&= -\widetilde{g}\left(\widetilde{\Omega}^{LC}(\xi, \widetilde{I}X)Y, Z\right).
\end{aligned}$$

This quantity is zero as shown in the previous point, so it follows that $\mathcal{L}_{\widetilde{I}\xi}\widetilde{\eta} = \nabla\widetilde{I}$, so (2.2) ends the proof. □

We can now use a coframe $\widetilde{\theta}$ as in section 2.3 in order to progress in the study of $\widetilde{\eta}$. We then write

$$\widetilde{\eta} = \mathfrak{R}(\widetilde{\eta}^j_{k,h}\widetilde{\theta}^k \otimes \overline{\widetilde{\theta}_j} \otimes \widetilde{\theta}^h).$$

Since every operator we use is $\mathbb{C}$-linear, we can study only the component in $T_{1,0} \otimes T^{0,1} \otimes T_{1,0}$, that is $\widetilde{\eta}^j_{k,h}\widetilde{\theta}^k \otimes \overline{\widetilde{\theta}_j} \otimes \widetilde{\theta}^h$. Because of Lemma 2.4.1, the coefficients $\widetilde{\eta}^j_{k,h}$ vanish if any one of the indices is $n+1$; moreover, $\widetilde{\eta}^j_{k,h}$ is completely symmetric in its indices. The last statement follows from the fact that $\flat_2\widetilde{\eta}$ is a tensor in $\pi^*S_{3,0}M$, and such tensors are expressed using only $\pi^*\theta^k$ for $1 \leq k \leq n$, where the metric is positive definite, and thus $\flat_2$ does not change the signs of the coefficients of $\widetilde{\eta}$.

We are now ready to reduce $\widetilde{\eta}$ to an object defined locally on the base space M.

Proposition 2.4.3. *Given a projective special Kähler $(\pi\colon \widetilde{M} \to M, \nabla)$ and a section $s\colon U \to S \subseteq \widetilde{M}$ inducing a trivialisation $(p|_{\pi^{-1}(U)}, z)\colon \pi^{-1}(U) \to U \times \mathbb{C}^*$, there exists a tensor $\eta \in T_{1,0}U \otimes T^{0,1}U \otimes T_{1,0}U$ such that $\flat_2\eta$ is a tensor in $S_{3,0}U$ and*

$$\widetilde{\eta} = \Re(z^2\pi^*\eta) = r^2\cos(2\vartheta)2\operatorname{Re}\pi^*\eta + r^2\sin(2\vartheta)2\operatorname{Im}\pi^*\eta$$

where $z = re^{i\vartheta}$.

Proof. For every point $p \in M$ we can find a local unitary coframe θ defined on an open set containing p, and the corresponding coframe $\widetilde{\theta}$ on $\widetilde{M}$ as in (2.5).

For the coming arguments we first compute the following Lie derivatives

$$\begin{aligned}
\mathcal{L}_\xi\widetilde{\theta}^k &= d\iota_\xi\widetilde{\theta}^k + \iota_\xi d\widetilde{\theta}^k = d\iota_\xi(r\pi^*\theta^k) + \iota_\xi d(r\pi^*\theta^k)\\
&= 0 + \iota_\xi(dr \wedge \pi^*\theta^k) + r\iota_\xi d\pi^*\theta^k\\
&= dr(\xi)\pi^*\theta^k + r\iota_\xi\pi^*d\theta^k = r\pi^*\theta^k + 0 = \widetilde{\theta}^k;
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_\xi\widetilde{\theta}_k &= \widetilde{g}(\mathcal{L}_\xi\widetilde{\theta}_k, \cdot)_\sharp = \mathcal{L}_\xi\left(\widetilde{g}(\widetilde{\theta}_k, \cdot)\right)_\sharp - \left(\mathcal{L}_\xi\widetilde{g}(\widetilde{\theta}_k, \cdot)\right)_\sharp\\
&= \frac{1}{2}\left(\mathcal{L}_\xi\overline{\widetilde{\theta}^k}\right)_\sharp - 2\widetilde{g}(\widetilde{\theta}_k, \cdot)_\sharp = \frac{1}{2}\overline{\widetilde{\theta}^k}_\sharp - 2\widetilde{\theta}_k = -\widetilde{\theta}_k;
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{\widetilde{I}\xi}\widetilde{\theta}^k &= d\iota_{\widetilde{I}\xi}\widetilde{\theta}^k + \iota_{\widetilde{I}\xi}d\widetilde{\theta}^k = d\iota_{\widetilde{I}\xi}(r\pi^*\theta^k) + \iota_{\widetilde{I}\xi}d(r\pi^*\theta^k)\\
&= 0 + r\iota_{\widetilde{I}\xi}d\pi^*\theta^k = r\iota_{\widetilde{I}\xi}\pi^*d\theta^k = 0;
\end{aligned}$$

$$\mathcal{L}_{\widetilde{I}\xi}\widetilde{\theta}_k = \widetilde{g}(\mathcal{L}_{\widetilde{I}\xi}\widetilde{\theta}_k,\cdot)_\sharp = \mathcal{L}_{\widetilde{I}\xi}\left(\widetilde{g}(\widetilde{\theta}_k,\cdot)\right)_\sharp = \frac{1}{2}\left(\mathcal{L}_\xi\overline{\widetilde{\theta}^k}\right)_\sharp = 0.$$

Lemma 2.4.2 implies

$$\begin{aligned}
0 = \mathcal{L}_\xi\widetilde{\eta} &= \mathcal{L}_\xi\mathfrak{R}\left(\widetilde{\eta}^j_{k,h}\widetilde{\theta}^k\otimes\overline{\widetilde{\theta}_j}\otimes\widetilde{\theta}^h\right)\\
&= \mathfrak{R}\Big(\mathcal{L}_\xi\widetilde{\eta}^j_{k,h}\widetilde{\theta}^k\otimes\overline{\widetilde{\theta}_j}\otimes\widetilde{\theta}^h + \widetilde{\eta}^j_{k,h}\mathcal{L}_\xi\widetilde{\theta}^k\otimes\overline{\widetilde{\theta}_j}\otimes\widetilde{\theta}^h\\
&\quad + \widetilde{\eta}^j_{k,h}\widetilde{\theta}^k\otimes\mathcal{L}_\xi\overline{\widetilde{\theta}_j}\otimes\widetilde{\theta}^h + \widetilde{\eta}^j_{k,h}\widetilde{\theta}^k\otimes\overline{\widetilde{\theta}_j}\otimes\mathcal{L}_\xi\widetilde{\theta}^h\Big)\\
&= \mathfrak{R}\left(\mathcal{L}_\xi\widetilde{\eta}^j_{k,h}\widetilde{\theta}^k\otimes\overline{\widetilde{\theta}_j}\otimes\widetilde{\theta}^h + \widetilde{\eta}^j_{k,h}\widetilde{\theta}^k\otimes\overline{\widetilde{\theta}_j}\otimes\widetilde{\theta}^h\right)\\
&= \mathfrak{R}\left(\left(\mathcal{L}_\xi\widetilde{\eta}^j_{k,h} + \widetilde{\eta}^j_{k,h}\right)\widetilde{\theta}^k\otimes\overline{\widetilde{\theta}_j}\otimes\widetilde{\theta}^h\right)
\end{aligned}$$

and

$$\begin{aligned}
0 = \mathcal{L}_{\widetilde{I}\xi}\widetilde{\eta} + 2\widetilde{I}\widetilde{\eta} &= \mathcal{L}_{\widetilde{I}\xi}\mathfrak{R}\left(\widetilde{\eta}^j_{k,h}\widetilde{\theta}^k\otimes\overline{\widetilde{\theta}_j}\otimes\widetilde{\theta}^h\right) + \mathfrak{R}\left(2\widetilde{\eta}^j_{k,h}\widetilde{\theta}^k\otimes\widetilde{I}\left(\overline{\widetilde{\theta}_j}\right)\otimes\widetilde{\theta}^h\right)\\
&= \mathfrak{R}\left(\mathcal{L}_\xi\widetilde{\eta}^j_{k,h}\widetilde{\theta}^k\otimes\overline{\widetilde{\theta}_j}\otimes\widetilde{\theta}^h - 2i\widetilde{\eta}^j_{k,h}\widetilde{\theta}^k\otimes\overline{\widetilde{\theta}_j}\otimes\widetilde{\theta}^h\right)\\
&= \mathfrak{R}\left(\left(\mathcal{L}_{I\xi}\widetilde{\eta}^j_{k,h} - 2i\widetilde{\eta}^j_{k,h}\right)\widetilde{\theta}^k\otimes\overline{\widetilde{\theta}_j}\otimes\widetilde{\theta}^h\right).
\end{aligned}$$

Independent components must vanish, so we obtain a family of differential equations for $1 \le j,k,h \le n$

$$\begin{cases}\mathcal{L}_\xi\widetilde{\eta}^j_{k,h} = -\widetilde{\eta}^j_{k,h}\\ \mathcal{L}_{\widetilde{I}\xi}\widetilde{\eta}^j_{k,h} = 2i\widetilde{\eta}^j_{k,h}\end{cases}. \tag{2.9}$$

We define η, as the component in $T_{1,0}M\otimes T^{0,1}M\otimes T_{1,0}M$ of $s^*\widetilde{\eta}$, so that $\mathfrak{R}(\eta) = s^*\widetilde{\eta}$.

Notice that since $\pi s = \mathrm{id}_M$, the pullbacks satisfy $s^*\pi^* = \mathrm{id}_{T^\bullet_\bullet M}$, so

$$\begin{aligned}
s^*\widetilde{\eta} = s^*\mathfrak{R}(\widetilde{\eta}^j_{k,h}\widetilde{\theta}^k\otimes\overline{\widetilde{\theta}_j}\otimes\widetilde{\theta}^h) &= \mathfrak{R}(s^*(r^3\widetilde{\eta}^j_{k,h}\pi^*\theta^k\otimes\pi^*\overline{\theta_j}\otimes\pi^*\theta^h))\\
&= \mathfrak{R}((r\circ s)^3(\widetilde{\eta}^j_{k,h}\circ s)s^*\pi^*\theta^k\otimes s^*\pi^*\overline{\theta_j}\otimes s^*\pi^*\theta^h))\\
&= \mathfrak{R}((\widetilde{\eta}^j_{k,h}\circ s)\theta^k\otimes\overline{\theta_j}\otimes\theta^h).
\end{aligned}$$

Thus $\eta = s^*\widetilde{\eta}^j_{k,h}\theta^k\otimes\overline{\theta_j}\otimes\theta^h$ and we define $\eta^j_{k,h} := s^*\widetilde{\eta}^j_{k,h}$.

Now we will use (2.9) to find $\widetilde{\eta}^j_{k,h}$ at a point of π^*U. We define the function $f\colon \mathbb{R} \to \mathbb{C}$ such that $f(t) := \widetilde{\eta}^j_{k,h}(s(u)e^t)$ for $u \in U$ and compute its derivative at $t_0 \in \mathbb{R}$.

$$\begin{aligned}\frac{d}{dt}f|_{t_0} &= \frac{d}{dt}\widetilde{\eta}^j_{k,h}(s(u)e^t)|_{t=t_0} = \frac{d}{dt}\widetilde{\eta}^j_{k,h}(s(u)e^{t_0+t})|_{t=0} = \frac{d}{dt}\widetilde{\eta}^j_{k,h}(\phi^t_\xi(s(u)e^{t_0}))|_{t=0}\\ &= (\mathcal{L}_\xi\widetilde{\eta}^j_{k,h})(s(u)e^{t_0}) = -\widetilde{\eta}^j_{k,h}(s(u)e^{t_0}) = -f(t_0).\end{aligned}$$

Moreover, $f(0) = \widetilde{\eta}^j_{k,h}(s(u)) = \eta^j_{k,h}(u)$, so f satisfies the following initial value problem

$$\begin{cases} f' = -f \\ f(0) = \eta^j_{k,h}(u) \end{cases}$$

which has a unique solution, that is $f(t) = \eta^j_{k,h}(u)e^{-t}$. This means that $\widetilde{\eta}^j_{k,h}(s(u)e^t) = \eta^j_{k,h}(u)e^{-t}$ or equivalently, for all $\rho \in \mathbb{R}^+$ we have $\widetilde{\eta}^j_{k,h}(s(u)\rho) = \frac{1}{\rho}\eta^j_{k,h}(u) = (\frac{1}{r}\pi^*\eta^j_{k,h})(s(u)\rho)$.

Similarly, consider the function $f\colon \mathbb{R} \to \mathbb{C}$ such that $f(t) := \widetilde{\eta}^j_{k,h}(s(u)\rho e^{it})$ and compute its derivative at $t_0 \in \mathbb{R}$.

$$\begin{aligned}\frac{d}{dt}f|_{t_0} &= \frac{d}{dt}\widetilde{\eta}^j_{k,h}(s(u)\rho e^{it})|_{t=t_0} = \frac{d}{dt}\widetilde{\eta}^j_{k,h}(s(u)\rho e^{it_0+it})|_{t=0}\\ &= \frac{d}{dt}\widetilde{\eta}^j_{k,h}(\phi^t_{I\xi}(s(u)\rho e^{it_0}))|_{t=0} = (\mathcal{L}_{I\xi}\widetilde{\eta}^j_{k,h})(s(u)\rho e^{t_0})\\ &= 2i\widetilde{\eta}^j_{k,h}(s(u)\rho e^{t_0}) = 2if(t_0),\end{aligned}$$

and this time, $f(0) = \widetilde{\eta}^j_{k,h}(s(u)\rho) = \frac{1}{\rho}\eta^j_{k,h}(u)$, so that for f

$$\begin{cases} f' = 2if \\ f(0) = \frac{1}{\rho}\eta^j_{k,h}(u) \end{cases}.$$

Its unique solution is $f(t) = \eta^j_{k,h}(u)\frac{e^{2it}}{\rho}$, which implies

$$\widetilde{\eta}^j_{k,h}(s(u)\rho e^{it}) = \eta^j_{k,h}(u)\frac{e^{2it}}{\rho} = \left(\frac{\pi^*\eta^j_{k,h}}{r^3}\right)(s(u)\rho e^{it})\rho^2 e^{2it}.$$

Let now $z\colon \pi^{-1}(U) \to \mathbb{C}^*$ be as in the statement, then in particular for all $w \in \pi^{-1}(u)$, we have $w = s(u)z(u)$. Thus $\widetilde{\eta}^j_{k,h}(w) = z^2\frac{\pi^*\eta^j_{k,h}}{r^3}(w)$, so finally we have

$$\begin{aligned}\widetilde{\eta} &= \mathfrak{R}(\widetilde{\eta}^j_{k,h}\widetilde{\theta}^k \otimes \overline{\widetilde{\theta}_j} \otimes \widetilde{\theta}^h) = \mathfrak{R}\left(z^2 \frac{\pi^*\eta^j_{k,h}}{r^3}(r\pi^*\theta^k \otimes r\pi^*\overline{\theta_j} \otimes r\pi^*\theta^h)\right)\\ &= \mathfrak{R}(z^2\pi^*\eta^j_{k,h}(\pi^*\theta^k \otimes \pi^*\overline{\theta_j} \otimes \pi^*\theta^h)) = \mathfrak{R}(z^2\pi^*\eta).\end{aligned}$$ □

Definition 2.4.4. *Given a section $s\colon U \to S$ with U open subset of M, we will call the corresponding tensor η found in Proposition 2.4.3 the* deviance tensor *with respect to s.*

We can give a more global formulation of Proposition 2.4.3 in the following terms

Proposition 2.4.5. *Given a projective special Kähler manifold $(\pi\colon \widetilde{M} \to M, \nabla)$, there exists a map $\gamma\colon \widetilde{M} \to \sharp_2 S_{3,0}M \subset T_{1,0}M \otimes T^{0,1}M \otimes T_{1,0}M$ of bundles over M, such that $\gamma(ua) = a^2\gamma(u)$ and for every local section $s\colon U \to S \subset \widetilde{M}$, the deviance induced by s is $\eta = \gamma \circ s$.*

Let $L := \widetilde{M} \times_{\mathbb{C}^*} \mathbb{C}$, then γ can be identified with a homomorphism of complex vector bundles $\widehat{\gamma}\colon L \otimes L \to \sharp_2 S_{3,0}M$ such that $\gamma(u) = \widehat{\gamma}([u,1] \otimes [u,1])$.

Proof. Let $u \in \widetilde{M}$, then there exists an open neighbourhood $U \subseteq M$ of u and local trivialisation $(\pi|_{\pi^{-1}(U)}, z)\colon \pi^{-1}(U) \to U \times \mathbb{C}^*$ induced by a section $s\colon U \to S$ so, for all $w \in \pi^{-1}(U)$ we have $w = s(\pi(w))z(w)$. Let now $\eta\colon U \to S_{3,0}M$ be the deviance corresponding to s; we define $\gamma(u) := z(u)^2\eta(p)$ where $p = \pi(u)$. This definition is independent on the choice of s. In order to prove it take another $s'\colon U' \to S$ with $p \in U'$ and the corresponding z' and η', then, on $U \cap U'$, there is a map $c := z \circ s'\colon U \cap U' \to \mathbb{C}$ whose image is in S^1, as both s and s' are sections of S. By definition, $s' = s \cdot c$. Since $sz = s'z'$, $z(u) = z(s'(p)z'(u)) = z(s'(p))z'(u) = c(p)z'(u)$, so $z = z'\pi^*c$. Now, by construction, $\mathfrak{R}(z'^2\pi^*\eta') = \widetilde{\eta} = \mathfrak{R}(z^2\pi^*\eta) = \mathfrak{R}(z'^2\pi^*c^2\pi^*\eta')$, so $\eta' = c^2\eta$. Thus $z(u)^2\eta(p) = z'(u)^2c(p)^2\eta(p) = z'(u)^2\eta'(p)$ and hence γ is well defined. Moreover, $\gamma(ua) = z(ua)^2\eta(\pi(ua)) = z(u)^2a^2\eta(p) = a^2\gamma(u)$.

We can define the homomorphism $L \otimes L \to \sharp_2 S_{3,0}M$ locally: given a section $s\colon U \to S$, we map $[u,w] \otimes [u',w']$ to $z(u)z(u')ww' \cdot \eta^s_p$ where $p = \pi(u) = \pi(u')$. This map does not depend on the choice of the section as one can see from the relations above, and it is also independent on the representatives chosen for these classes; for the first class for example $z(ua)w = z(u)aw$.

This map commutes with the projections on M and it is $\mathbb{C}$-linear on the fibres, so it is a complex vector bundle map. □

Definition 2.4.6. *We call* $\gamma\colon S \to \sharp_2 S_{3,0}M$ *of Proposition 2.4.5 the* intrinsic deviance *of the projective special Kähler manifold.*

Remark 2.4.7. *Given a section* $s\colon U \to S$ *and the corresponding function* $z \in \mathcal{C}^\infty(\pi^{-1}(U), \mathbb{C}^*)$ *such that* $sz = \mathrm{id}_{\pi^{-1}(U)}$*, we can compute* $dz = z(\frac{1}{r}dr + id\vartheta)$*, since locally* $z = re^{i\vartheta}$*. Notice that* ϑ *is not globally defined on* $\pi^{-1}(U)$*, but* $d\vartheta$ *and* $e^{i\vartheta}$ *are. Moreover,*

$$\frac{1}{z}dz = \frac{1}{r}dr + id\vartheta \in \Omega^1(\pi^{-1}(U), \mathbb{C}) \tag{2.10}$$

is a principal connection form, in fact it is equivariant for the action of $\mathbb{C}^*$ *as* $z(ua) = az(u)$ *for all* $a \in \mathbb{C}$ *and, given a complex number* a *and its corresponding fundamental vector field* $a^\circ \in \mathfrak{X}\left(\widetilde{M}\right)$*,*

$$\frac{1}{z}dz(a^\circ)_u = \frac{1}{z}dz(\frac{d}{dt}ue^{at}|_{t=0}) = \frac{1}{z(u)}\frac{d}{dt}z(ue^{at})|_{t=0} = \frac{1}{z(u)}\frac{d}{dt}z(u)e^{at}|_{t=0} = a.$$

Remark 2.4.8. *A local section* $s\colon U \to S$ *induces* $\tau := s^*\widetilde{\varphi} = s^*\varphi \in \Omega^1(U)$ *such that on* $\pi^{-1}(U)$

$$\widetilde{\varphi} = d\vartheta + \pi^*\tau$$

and thus on $\pi_S^{-1}(U)$*:*

$$\varphi = d\vartheta|_S + \pi_S^*\tau.$$

If we consider in fact the form $\widetilde{\varphi} - d\vartheta$*, we notice that it is basic, as it can also be seen as the difference of two connection forms on* $\pi^{-1}(U)$ *(namely* (2.4) *and* (2.10)*) up to a multiplication by* i*. Therefore,* $\widetilde{\varphi} - d\vartheta = \pi^*\tau$ *for some* $\tau \in \Omega^1(U)$*. The second equation is simply obtained from the first by restriction to* $S \subseteq \widetilde{M}$*.*

2.5 Characterisation theorem

In this section we prove the main theorem of this chapter, characterising projective special Kähler manifolds in terms of the deviance. We start by deriving necessary conditions on the deviance, reflecting the curvature conditions of Proposition 2.1.2.

Proposition 2.5.1. *Let $(\pi\colon \widetilde{M} \to M, \nabla)$ be a projective special Kähler manifold with corresponding conic special Kähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{I}, \widetilde{\omega}, \nabla, \xi)$. Consider a local section $s\colon U \to S$, then the corresponding deviance η satisfies*

$$d^{LC}\eta = 2i\tau \wedge \eta$$

where $\tau = s^*\varphi \in \Omega^1(U)$.

Proof. Thanks to Proposition 2.4.3, we know that there exists $z = r^2 e^{2i\vartheta}$ and $\eta \in T_{1,0}U \otimes T^{0,1}U \otimes T_{1,0}U$ such that on $\pi^{-1}(U)$ we have $\widetilde{\eta} = \mathfrak{R}(z\pi^*\eta)$.

Now we would like to describe $\widetilde{d}^{LC}\widetilde{\eta}$ in terms of $d^{LC}\eta$. Notice that

$$\begin{aligned}
\widetilde{d}^{LC}\widetilde{\eta} &= \widetilde{d}^{LC}\mathfrak{R}(z^2\pi^*\eta) = \mathfrak{R}(\widetilde{d}^{LC}(z^2\pi^*\eta)) \\
&= \mathfrak{R}(2zdz \wedge \pi^*\eta + z^2\widetilde{d}^{LC}\pi^*\eta) \\
&= \mathfrak{R}(2(\frac{1}{r}dr + id\vartheta)z^2 \wedge \pi^*\eta + z^2\widetilde{d}^{LC}\pi^*\eta) \\
&= \mathfrak{R}\left(z^2\left(2(\frac{1}{r}dr + id\vartheta) \wedge \pi^*\eta + \widetilde{d}^{LC}\pi^*\eta\right)\right). \qquad (2.11)
\end{aligned}$$

The next step is to compute $\widetilde{d}^{LC}\pi^*\eta$, but since we are using the Levi-Civita connection, it is equivalent to compute $\sharp_2(\widetilde{d}^{LC}\pi^*\sigma)$, where $\sigma = \flat_2\eta \in S_{3,0}U$. Let us consider a local coframe θ in M and the corresponding lifting $\widetilde{\theta}$ as in (2.5), so that we can denote explicitly $\sigma = \sigma_{k,j,h}\theta^k \otimes \theta^j \otimes \theta^h$. We have

$$\begin{aligned}
\widetilde{\nabla}^{LC}\pi^*\theta^k &= \widetilde{\nabla}^{LC}\frac{\widetilde{\theta}^k}{r} = -\frac{dr}{r^2} \otimes \widetilde{\theta}^k - \frac{1}{r}\left((\widetilde{\omega}^{LC})^k_j \otimes \widetilde{\theta}^j\right) \\
&= -\frac{dr}{r} \otimes \pi^*\theta^k - \frac{1}{r}\left(\sum_{j=1}^{n} \pi^*(\omega^{LC})^k_j \otimes \widetilde{\theta}^j + i\widetilde{\varphi} \otimes \widetilde{\theta}^j + \pi^*\theta^k \otimes \theta^{n+1}\right) \\
&= -\frac{dr}{r} \otimes \pi^*\theta^k - \pi^*\left((\omega^{LC})^k_j \otimes \theta^j\right) - i\widetilde{\varphi} \otimes \pi^*\theta^j - \pi^*\theta^k \otimes \frac{1}{r}\theta^{n+1} \\
&= \pi^*\left(\nabla^{LC}\theta^k\right) - \frac{1}{r}\theta^{n+1} \otimes \pi^*\theta^k - \pi^*\theta^k \otimes \frac{1}{r}\theta^{n+1}.
\end{aligned}$$

We can now compute the following for $X \in \mathfrak{X}(\pi^{-1}(U))$:

$$\widetilde{\nabla}^{LC}_X\pi^*\sigma = \widetilde{\nabla}^{LC}_X\pi^*(\sigma_{k,j,h}\theta^k \otimes \theta^j \otimes \theta^h) = \widetilde{\nabla}^{LC}_X(\pi^*\sigma_{k,j,h}\pi^*\theta^k \otimes \pi^*\theta^j \otimes \pi^*\theta^h)$$

$$
\begin{aligned}
&= d\pi^*\sigma_{k,j,h}(X)\theta^k \otimes \theta^j \otimes \theta^h + \pi^*\sigma_{k,j,h}\Big(\widetilde{\nabla}^{LC}_X \pi^*\theta^k \otimes \pi^*\theta^j \otimes \pi^*\theta^h \\
&\quad + \pi^*\theta^k \otimes \widetilde{\nabla}^{LC}_X \pi^*\theta^j \otimes \pi^*\theta^h + \pi^*\theta^k \otimes \pi^*\theta^j \otimes \widetilde{\nabla}^{LC}_X \pi^*\theta^h\Big) \\
&= \pi^* d\sigma_{k,j,h}(X)\theta^k \otimes \theta^j \otimes \theta^h + \pi^*\sigma_{k,j,h}\pi^*\left(\nabla^{LC}\theta^k\right)_X \otimes \pi^*\theta^j \otimes \pi^*\theta^h \\
&\quad + \pi^*\sigma_{k,j,h}\pi^*\theta^k \otimes \pi^*\left(\nabla^{LC}\theta^j\right)_X \otimes \pi^*\theta^h \\
&\quad + \pi^*\sigma_{k,j,h}\pi^*\theta^k \otimes \pi^*\theta^j \otimes \pi^*\left(\nabla^{LC}\theta^j\right)_X - \frac{3}{r}\widetilde{\theta}^{n+1}(X)\pi^*\sigma \\
&\quad - \frac{1}{r}\Big(\pi^*\sigma_{k,j,h}\pi^*\theta^k(X)\widetilde{\theta}^{n+1} \otimes \pi^*\theta^j \otimes \pi^*\theta^h \\
&\quad + \pi^*\sigma_{k,j,h}\pi^*\theta^k \otimes \pi^*\theta^j(X)\widetilde{\theta}^{n+1} \otimes \pi^*\theta^h \\
&\quad + \pi^*\sigma_{k,j,h}\pi^*\theta^k \otimes \pi^*\theta^j \otimes \pi^*\theta^h(X)\widetilde{\theta}^{n+1}\Big) \\
&= \pi^*\left(\nabla^{LC}\sigma\right)_X - \frac{2}{r}\widetilde{\theta}^{n+1}(X)\pi^*\sigma - \frac{1}{r}\widetilde{\theta}^{n+1}(X)\pi^*\sigma - \frac{1}{r}\widetilde{\theta}^{n+1} \otimes \pi^*\sigma(X,\cdot,\cdot) \\
&\quad - \frac{1}{r}\pi^*\sigma(\cdot, X \otimes \widetilde{\theta}^{n+1}, \cdot) - \frac{1}{r}\pi^*\sigma(\cdot,\cdot, X \otimes \widetilde{\theta}^{n+1}).
\end{aligned}
$$

In general then, if $\sigma = \theta^k \otimes \sigma_k$, where $\sigma_k = \sigma_{k,j,h}\theta^j, \theta^h \in S^{2,0}U$, we have by symmetry

$$
\begin{aligned}
\widetilde{\nabla}^{LC}\pi^*\sigma = \pi^*\left(\nabla^{LC}\sigma\right) - \frac{2}{r}\widetilde{\theta}^{n+1} \otimes \pi^*\sigma - \frac{2}{r}((\widetilde{\theta}^{n+1})(\pi^*\theta^k)) \otimes \pi^*(\sigma_{k,j,h}\theta^j \otimes \theta^h) \\
- \frac{2}{r}\left(\pi^*(\sigma_{k,j,h}\theta^k \otimes \theta^j) \otimes ((\widetilde{\theta}^{n+1})(\pi^*\theta^h))\right).
\end{aligned}
$$

Notice in particular that the last two rows are symmetric in the first two indices.

In order to compute $\widetilde{d}^{LC}\pi^*\sigma$, we need to antisymmetrise $\widetilde{\nabla}^{LC}\pi^*\sigma$ in the first two indices and multiply by 2, so only the first row survives and we get

$$\widetilde{d}^{LC}\pi^*\sigma = \pi^*(d^{LC}\sigma) - \frac{2}{r}\widetilde{\theta}^{n+1} \wedge \pi^*\sigma$$

and therefore

$$\widetilde{d}^{LC}\pi^*\eta = \pi^*(d^{LC}\eta) - \frac{2}{r}\widetilde{\theta}^{n+1} \wedge \pi^*\eta.$$

Substituting this value in (2.11), we obtain

$$\widetilde{d}^{LC}\widetilde{\eta} = \mathfrak{R}\left(z^2\left(2(\frac{1}{r}dr + id\vartheta) \wedge \pi^*\eta + \pi^*(d^{LC}\eta) - \frac{2}{r}\widetilde{\theta}^{n+1} \wedge \pi^*\eta\right)\right)$$

$$= \mathfrak{R}\left(z^2\left(\pi^* d^{LC}\eta - 2i(\widetilde{\varphi} - d\vartheta) \wedge \pi^*\eta\right)\right).$$

As observed in Remark 2.4.8, $\widetilde{\varphi} - d\vartheta = \pi^*\tau$, so we have

$$\widetilde{d}^{LC}\widetilde{\eta} = \mathfrak{R}\left(z^2\pi^*\left(d^{LC}\eta - 2i\tau \wedge \eta\right)\right).$$

From Proposition 2.1.2, we know that $\widetilde{d}^{LC}\widetilde{\eta} = 0$, and since η belongs to $\Omega^1(U, T_{0,1} \otimes T^{1,0})$, η and $\overline{\eta}$ are linearly independent, so this quantity vanishes if and only if $z^2\pi^*\left(d^{LC}\eta - 2i\tau \wedge \eta\right)$ does. Therefore,

$$d^{LC}\eta - 2i\tau \wedge \eta = 0,$$

ending the proof. $\square$

Let us now look at the final ingredient of the curvature tensor, that is $\frac{1}{2}[\widetilde{\eta} \wedge \widetilde{\eta}]$. In the setting of Proposition 2.4.3, given a section $s\colon U \to S$, and the induced deviance η, then

$$\begin{aligned}\frac{1}{2}[\widetilde{\eta} \wedge \widetilde{\eta}] &= \frac{1}{2}[\mathfrak{R}(z^2\pi^*\eta) \wedge \mathfrak{R}(z^2\pi^*\eta)] = \frac{1}{2}[z^2\pi^*\eta + \overline{z}^2\pi^*\overline{\eta} \wedge z^2\pi^*\eta + \overline{z}^2\pi^*\overline{\eta}] \\ &= \frac{1}{2}\mathfrak{R}\left(z^4[\pi^*\eta \wedge \pi^*\eta]\right) + |z|^4[\pi^*\eta \wedge \pi^*\overline{\eta}].\end{aligned}$$

We can compute this tensor for a local coframe θ on M. Since we have

$$\pi^*\theta^k \circ \pi^*\theta_h = \frac{1}{r}\widetilde{\theta}^k(\frac{1}{r}\widetilde{\theta}_h) = \frac{1}{r^2}\widetilde{\theta}^k(\widetilde{\theta}_h) = \frac{1}{r^2}\delta^k_h = \frac{1}{r^2}\pi^*(\theta^k \circ \theta_h)$$

and $\pi^*\theta^k \circ \pi^*\overline{\theta_h} = \pi^*\overline{\theta^k} \circ \pi^*\theta_h = 0$, then

$$\begin{aligned}[\pi^*\eta \wedge \pi^*\eta] &= [\pi^*\eta^j_{k,h}\pi^*\theta^k \otimes \pi^*\overline{\theta_j} \otimes \pi^*\theta^h \wedge \pi^*\eta^{j'}_{k',h'}\pi^*\theta^{k'} \otimes \pi^*\overline{\theta_{j'}} \otimes \pi^*\theta^{h'}] \\ &= \pi^*\eta^j_{k,h}\pi^*\theta^k \wedge \pi^*\eta^{j'}_{k',h'}\pi^*\theta^{k'} \otimes [\pi^*\overline{\theta_j} \otimes \pi^*\theta^h, \pi^*\overline{\theta_{j'}} \otimes \pi^*\theta^{h'}] = 0\end{aligned}$$

and

$$\begin{aligned}[\pi^*\eta \wedge \pi^*\overline{\eta}] &= [\pi^*\eta^j_{k,h}\pi^*\theta^k \otimes \pi^*\overline{\theta_j} \otimes \pi^*\theta^h \wedge \pi^*\overline{\eta^{j'}_{k',h'}}\pi^*\overline{\theta^{k'}} \otimes \pi^*\theta_{j'} \otimes \pi^*\overline{\theta^{h'}}] \\ &= \pi^*\eta^j_{k,h}\pi^*\theta^k \wedge \pi^*\overline{\eta^{j'}_{k',h'}}\pi^*\overline{\theta^{k'}} \otimes [\pi^*\overline{\theta_j} \otimes \pi^*\theta^h, \pi^*\theta_{j'} \otimes \pi^*\overline{\theta^{h'}}] \\ &= \pi^*(\eta^j_{k,h}\theta^k \wedge \overline{\eta^{j'}_{k',h'}}\overline{\theta^{k'}}) \otimes \frac{1}{r^2}\pi^*(\overline{\theta_j} \otimes \theta^h(\theta_{j'}) \otimes \overline{\theta^{h'}} - \theta_{j'} \otimes \overline{\theta^{h'}}(\overline{\theta_j}) \otimes \theta^h)\end{aligned}$$

$$= \frac{1}{r^2}\pi^*[\eta \wedge \overline{\eta}].$$

Therefore,

$$\frac{1}{2}[\widetilde{\eta} \wedge \widetilde{\eta}] = \frac{|z|^4}{r^2}\pi^*[\eta \wedge \overline{\eta}] = r^2\pi^*[\eta \wedge \overline{\eta}]. \tag{2.12}$$

Remark 2.5.2. *Note that $[\eta \wedge \overline{\eta}]$ is independent on the local coframe, and if we consider another section such that $s' = sa$ on the intersection of their domains, with a taking values in S^1, if η' is the deviance corresponding to s, then $[\eta' \wedge \overline{\eta'}] = [\eta a \wedge \overline{\eta a}] = |a|^2[\eta \wedge \overline{\eta}] = [\eta \wedge \overline{\eta}]$. So, there is a globally defined section $M \to S^2(\mathfrak{u}(n))$ mapping p to $[\eta_p \wedge \overline{\eta_p}]$.*

For a projective special Kähler manifold $(\pi\colon \widetilde{M} \to M, \nabla)$ of real dimension $2n$, the statement of Proposition 2.1.2, interpreted in the light of the last observations and the ones made in Section 2.3 (see Remark 2.3.4), says that $0 = r^2\pi^*(\Omega^{LC} + \Omega_{\mathbb{P}^n_{\mathbb{C}}} + [\eta \wedge \overline{\eta}])$. Thus we have the following equation:

$$\Omega^{LC} + \Omega_{\mathbb{P}^n_{\mathbb{C}}} + [\eta \wedge \overline{\eta}] = 0 \tag{2.13}$$

This is a curvature tensor, so we can compute its Ricci and scalar component.

Proposition 2.5.3. *Let $(\pi\colon \widetilde{M} \to M, \nabla)$ be a projective special Kähler manifold of dimension $2n$, then*

$$\mathrm{Ric}_M(X,Y) + 2(n+1)g(X,Y) - \mathfrak{R}(h(\overline{\eta_X}, \eta_Y)) = 0; \tag{2.14}$$

$$\mathrm{scal}_M + 2(n+1) - \frac{2}{n}\left\|\eta\right\|_h^2 = 0. \tag{2.15}$$

Proof. The first summand in (2.13) gives the Ricci tensor of M, the second gives the Ricci tensor of the projective space (2.7). In order to compute the last term, consider a unitary frame θ; from previous computations,

$$\begin{aligned}
[\eta \wedge \overline{\eta}] &= (\eta^j_{k,h}\theta^k \wedge \overline{\eta^{j'}_{k',h'}}\overline{\theta^{k'}}) \otimes (\delta^h_{j'}\overline{\theta_j} \otimes \overline{\theta^{h'}} - \delta^{h'}_j\theta_{j'} \otimes \theta^h) \\
&= \mathfrak{R}\left(\eta^j_{k,h}\overline{\eta^h_{k',h'}}\theta^k \wedge \overline{\theta^{k'}} \otimes \overline{\theta_j} \otimes \overline{\theta^{h'}}\right),
\end{aligned}$$

then the Ricci component $\mathrm{Ric}([\eta \wedge \overline{\eta}])$ evaluated on $X = \mathfrak{R}(X^k\theta_k)$ and $Y = \mathfrak{R}(Y^k\theta_k)$ is the trace of $[\eta \wedge \overline{\eta}](\cdot, Y)X$, which is

$[\eta\wedge\overline{\eta}](\cdot, Y)X$

$$= \eta^j_{k,h}\overline{\eta^h_{u,v}}(\theta^k\overline{Y^u} - Y^k\overline{\theta^u}) \otimes \overline{\theta_j} \otimes \overline{X^v} + \overline{\eta^j_{k,h}}\eta^h_{u,v}(\overline{\theta^k}Y^u - \overline{Y^u}\theta^k) \otimes \theta_j \otimes X^v$$
$$= \mathfrak{R}\left(\eta^j_{k,h}\overline{\eta^h_{u,v}}(\theta^k\overline{Y^u} - Y^k\overline{\theta^u}) \otimes \overline{\theta_j} \otimes \overline{X^v}\right).$$

Its trace is therefore

$$-\mathfrak{R}\left(\eta^j_{k,h}\overline{\eta^h_{j,v}}Y^k\overline{X^v}\right) = -\mathfrak{R}\left(\eta^j_{k,h}\overline{\eta^h_{u,j}}Y^k\overline{X^u}\right) = -\mathfrak{R}(h(\overline{\eta_X}, \eta_Y)),$$

or equivalently, $\mathrm{Ric}([\eta \wedge \overline{\eta}]) = -\mathfrak{R}\left(\overline{\eta^h_{u,j}}\eta^j_{k,h}\overline{\theta^u}\theta^k\right)$. Thus we obtain (2.14).

From this tensor we can now compute the scalar component by taking the trace, raising the indices with g and then dividing it by the dimension of M. Hence, the first summand gives scal_M, the second gives $2(n+1)$ and the third

$$\begin{aligned}\frac{1}{2n}\mathrm{tr}\left(-\mathfrak{R}\left(\overline{\eta^h_{u,j}}\eta^j_{k,h}(\overline{\theta^u})_\sharp\theta^k\right)\right) &= -\frac{1}{2n}\mathrm{tr}\left(\mathfrak{R}\left(\overline{\eta^h_{u,j}}\eta^j_{k,h}(2\theta_u)\theta^k\right)\right)\\ &= -\frac{1}{n}\sum_{j,h,k}\mathfrak{R}\left(\eta^j_{k,h}\overline{\eta^h_{k,j}}\right) = -\frac{2}{n}\left\|\eta\right\|^2_h,\end{aligned}$$

proving (2.15). □

In particular, since the norm of η is non-negative, we obtain a lower bound for the scalar curvature:

Corollary 2.5.4. *Let $(\pi\colon \widetilde{M} \to M, \nabla)$ be a projective special Kähler manifold, then*

$$\mathrm{scal}_M \geq -2(n+1).$$

Equality holds at a point it and only if the deviance vanishes at that point.

Remark 2.5.5. *The lower bound is reached by projective special Kähler manifolds with zero deviance; we will see that this condition characterises the complex hyperbolic space (Proposition 2.6.8).*

We can now state the main result:

Theorem 2.5.6. *On a $2n$-dimensional Kähler manifold (M, g, I, ω), to give a projective special Kähler structure is equivalent to give an S^1-bundle*

$$\pi_S\colon S \longrightarrow M$$

endowed with a connection form φ and a bundle map $\gamma\colon S \to \sharp_2 S_{3,0}M$ such that:

1. *$d\varphi = -2\pi_S^*\omega$;*
2. *$\gamma(ua) = a^2\gamma(u)$ for all $a \in S^1$;*
3. *for a certain choice of an open covering $\{U_\alpha | \alpha \in \mathcal{A}\}$ of M and a family $\{s_\alpha \colon U_\alpha \to S\}_{\alpha\in\mathcal{A}}$ of sections, denoting by η_α the local 1-form taking values in $T^{0,1}M \otimes T_{1,0}M$ determined by $\gamma \circ s_\alpha$, for all $\alpha \in \mathcal{A}$:*

   D.1 $\quad \Omega^{LC} + \Omega_{\mathbb{P}^n_{\mathbb{C}}} + [\eta_\alpha \wedge \overline{\eta_\alpha}] = 0$

   D.2 $\quad d^{LC}\eta_\alpha = 2is_\alpha^*\varphi \wedge \eta_\alpha$

In this case, 3 is satisfied by every such family of sections.

Proof. Given a projective special Kähler manifold, we define $S := r^{-1}(1) \subset \widetilde{M}$ and $\varphi := -\iota_\xi\omega|_S$. The principal action on S is generated by $I\xi$ which is tangent to S since $T_uS = \ker(dr)$ and $dr(I\xi) = -\frac{1}{r}\xi^\flat(I\xi) = -\frac{\widetilde{g}(\xi, I\xi)}{r}$. The curvature is then $d\varphi = -2\pi_S^*\omega$ as shown in Remark 2.2.4, so the first point is satisfied. The second condition holds thanks to Proposition 2.4.5. For the third point, we get **D.1** from the arguments leading to equation (2.13) and **D.2** from Proposition 2.5.1.

In order to prove the other direction, define $\widetilde{M} := S \times \mathbb{R}^+$, $\pi := \pi_S \circ \pi_1 \colon \widetilde{M} \to M$, and $t := \pi_2 \in \mathcal{C}^\infty\left(\widetilde{M}, \mathbb{R}^+\right)$, where $\pi_1 \colon S \times \mathbb{R}^* \to S$ and $\pi_2 \colon S \times \mathbb{R}^+ \to \mathbb{R}^+$ are the projections. Let $\widetilde{\varphi} := \pi_1^*\varphi$, in particular $d\widetilde{\varphi} = \pi_1^*d\varphi = -2\pi^*\omega$ as expected. Define now

$$\widetilde{g} := t^2\pi^*g - t^2\widetilde{\varphi}^2 - dt^2 \tag{2.16}$$

which is non-degenerate, since $r\widetilde{\varphi}$ and dt are linearly independent and transverse to π, so we can form a basis for the 1-forms according to which we can see that $\widetilde{g}$ has signature $(2n, 2)$. Extend now I to $\widetilde{I}$ so that $\widetilde{I} \cdot (\pi^*\alpha) = \pi^*I\alpha$ for all $\alpha \in T^*M$ and $\widetilde{I} \cdot (dt) = t\widetilde{\varphi}$.

The metric $\widetilde{g}$ is compatible with $\widetilde{I}$ since $\widetilde{I}\cdot\widetilde{g} = t^2\widetilde{I}\cdot\pi^*g - (\widetilde{I}\cdot t\widetilde{\varphi})^2 - (\widetilde{I}\cdot dt)^2 = t^2\pi^*(I \cdot g) - (-dt)^2 - (t\widetilde{\varphi})^2 = t^2\pi^*(I \cdot g) - dt^2 - t^2\widetilde{\varphi}^2 = \widetilde{g}$.

We thus have a Kähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{I}, \widetilde{\omega})$, where

$$\widetilde{\omega} := t^2\pi^*\omega + t\widetilde{\varphi} \wedge dt.$$

Let $\xi := t\partial_t$ where ∂_t is the vector field corresponding to the coordinate derivation on $\mathbb{R}^+$. Notice that the function $r = \sqrt{-\widetilde{g}(\xi, \xi)}$ coincides with t, as $\sqrt{-\widetilde{g}(t\partial_t, t\partial_t)} = \sqrt{-t^2\widetilde{g}(\partial_t, \partial_t)} = t$. In particular $\widetilde{g}(\xi, \xi) = -t^2 \neq 0$ and $\widetilde{g}(\widetilde{I}\xi, \widetilde{I}\xi) = \widetilde{g}(\xi, \xi) < 0$, so $\widetilde{g}$ is negative definite on $\langle \xi, I\xi \rangle$ and hence positive definite on the orthogonal complement.

Let now θ be a unitary coframe on an open subset $U \subseteq M$, then we can lift it to a complex coframe $\widetilde{\theta}$ on $\pi^{-1}(U)$ defined as in (2.5). It is straightforward to check that $\widetilde{\theta}$ is adapted to the pseudo-Kähler structure of $\widetilde{M}$. Notice that the proof of Proposition 2.3.2 is still valid in this situation even though we do not know whether $\widetilde{M} \to M$ has a structure of projective special Kähler manifold; this gives us a description of the Levi-Civita connection form on $\widetilde{M}$ with respect to $\widetilde{\theta}$. Notice that $\widetilde{\theta}^k(\xi) = 0$ for $k \leq n$ and $\widetilde{\theta}^{n+1}(\xi) = dt(t\partial_t) + i\widetilde{\varphi}(t\partial_t) = t$ so $\xi = \mathfrak{R}(t\widetilde{\theta}_{n+1})$. We can thus compute

$$\begin{aligned}
\widetilde{\nabla}^{LC}\xi &= dt \otimes \mathfrak{R}(\widetilde{\theta}_{n+1}) + t\widetilde{\nabla}^{LC}\mathfrak{R}(\widetilde{\theta}_{n+1}) \\
&= \mathfrak{R}(dt \otimes \widetilde{\theta}_{n+1}) + \frac{t}{r}\mathfrak{R}\left(\sum_{k=1}^{n} \widetilde{\theta}^k \otimes \widetilde{\theta}_k + i\,\mathrm{Im}(\widetilde{\theta}^{n+1}) \otimes \widetilde{\theta}_{n+1}\right) \\
&= \mathfrak{R}\left(\sum_{k=1}^{n+1} \widetilde{\theta}^k \otimes \widetilde{\theta}_k\right) = \mathrm{id}.
\end{aligned}$$

Define on $\widetilde{M}$ the tensor $\widetilde{\eta} := \mathfrak{R}(t^2\pi^* \circ \gamma \circ \pi_1)$ section of $\sharp_2 S_{3,0}\widetilde{M}$. Each section $s_\alpha \colon U_\alpha \to S$ corresponds to the trivialisation $(\pi|_{\pi^{-1}(U_\alpha)}, z_\alpha) \colon \pi^{-1}U_\alpha \to U_\alpha \times \mathbb{C}^*$ in the sense that $s(\pi(u)) \cdot z_\alpha(u) = u$ for all $u \in \pi^{-1}(U_\alpha)$. For all α on $\pi^{-1}(U_\alpha)$, let $\widetilde{\eta}_\alpha := \widetilde{\eta}|_{U_\alpha}$, then locally we have

$$\begin{aligned}
\widetilde{\eta}_\alpha &= \mathfrak{R}(t^2\pi^* \circ \gamma \circ \pi_1(z_\alpha(s_\alpha \circ \pi))) = \mathfrak{R}\left(t^2\pi^* \circ \gamma \circ \left(\frac{z_\alpha}{|z_\alpha|} s_\alpha \circ \pi\right)\right) \\
&= \mathfrak{R}\left(t^2 \frac{z_\alpha^2}{|z_\alpha|^2}\pi^* \circ \gamma \circ s_\alpha \circ \pi\right) = \mathfrak{R}(z_\alpha^2\pi^* \circ \eta_\alpha \circ \pi)\mathfrak{R}(z_\alpha^2\pi^*\eta_\alpha).
\end{aligned}$$

We can build another connection $\nabla := \widetilde{\nabla}^{LC} + \widetilde{\eta}$. Notice that $\nabla\xi = \widetilde{\nabla}^{LC}\xi + \widetilde{\eta}(\xi) = \mathrm{id} + \mathfrak{R}(z_\alpha^2\pi^*\eta_\alpha)(\xi) = \mathrm{id}$ because locally η_α is horizontal for all α.

In order to prove that ∇ is symplectic, since the Levi-Civita connection is symplectic, it is enough to prove that $\widetilde{\omega}(\widetilde{\eta}, \cdot) + \widetilde{\omega}(\cdot, \widetilde{\eta}) = 0$. Locally, $\widetilde{\omega} = \frac{1}{2i}\sum_{k=1}^{n+1} \overline{\widetilde{\theta}^k} \wedge \widetilde{\theta}^k$ and in fact, for all $X = \mathfrak{R}(X^k\widetilde{\theta}_k)$, $Y = \mathfrak{R}(Y^k\widetilde{\theta}_k)$, $Z = \mathfrak{R}(Z^k\widetilde{\theta}_k)$ vector fields on $\widetilde{M}$:

$$2i(\widetilde{\omega}(\widetilde{\eta}_X Y, Z) + \widetilde{\omega}(Y, \widetilde{\eta}_X Z)) = \sum_{k=1}^{n+1} \left(\overline{\widetilde{\theta}^k}(\widetilde{\eta}_X Y)\widetilde{\theta}^k(Z) - \widetilde{\theta}^k(\widetilde{\eta}_X Y)\overline{\widetilde{\theta}^k}(Z)\right.$$

$$
\begin{aligned}
&+\overline{\widetilde{\theta}^k}(Y) \wedge \widetilde{\theta}^k(\widetilde{\eta}_X Z) - \widetilde{\theta}^k(Y) \wedge \overline{\widetilde{\theta}^k}(\widetilde{\eta}_X Z)\Big) \\
=& \sum_{k=1}^{n+1} \Big(z\pi^* \eta^k_{u,v} X^u Y^v Z^k - \overline{Z^k} \overline{z}^2 \overline{\pi^* \eta^k_{u,v}}\, \overline{X^u Y^v} \\
&+ \overline{Y}^k \overline{z}^2 \overline{\pi^* \eta^k_{u,v}}\, \overline{X^u Z^v} - z^2 \pi^* \eta^k_{u,v} X^u Z^v Y^k \Big) \\
=& \sum_{k=1}^{n+1} \mathfrak{R}\left(z^2 \pi^* \eta^k_{u,v} X^u Y^v Z^k - z^2 \pi^* \eta^k_{u,v} X^u Z^v Y^k \right) \\
=& \sum_{k=1}^{n+1} \mathfrak{R}\left(z^2 \pi^* (\eta^k_{u,v} - \eta^v_{u,k}) X^u Y^v Z^k \right).
\end{aligned}
$$

By the symmetry of η, this quantity vanishes.

Proving $d^{\nabla}\widetilde{I} = 0$, is equivalent to proving that $\nabla \widetilde{I}$ is symmetric in the two covariant indices, and thus $\nabla I = \widetilde{\nabla}^{LC} \widetilde{I} + [\eta, \widetilde{I}] = [\eta, \widetilde{I}]$. Since $I = \mathfrak{R}(i\widetilde{\theta}_k \widetilde{\theta}^k)$, we have

$$
\begin{aligned}
[\widetilde{\eta}, \widetilde{I}] = & \, iz^2 \pi^* \eta^u_{v,w} \widetilde{\theta}^v \otimes \overline{\widetilde{\theta}_u} \otimes \widetilde{\theta}^w - \overline{iz^2 \pi^* \eta^u_{v,w} \widetilde{\theta}^v \otimes \overline{\widetilde{\theta}_u} \otimes \widetilde{\theta}^w} \\
&+ iz^2 \pi^* \eta^u_{v,w} \widetilde{\theta}^v \otimes \overline{\widetilde{\theta}_u} \otimes \widetilde{\theta}^w - \overline{iz^2 \pi^* \eta^u_{v,w} \widetilde{\theta}^v \otimes \overline{\widetilde{\theta}_u} \otimes \widetilde{\theta}^w} = 2i\widetilde{\eta} = -2I\widetilde{\eta}
\end{aligned}
$$

which is symmetric, proving $d^{\nabla} I = 0$.

For the flatness of ∇, we compute the curvature locally:

$$
\Omega^{\nabla} = d\omega^{\nabla} + [\omega^{\nabla} \wedge \omega^{\nabla}] = \widetilde{\Omega}^{LC} + \widetilde{d^{LC}}\widetilde{\eta} + [\widetilde{\eta}, \widetilde{\eta}].
$$

By Proposition 2.3.2, $\widetilde{\Omega}^{LC} = r^2 \pi^*(\Omega^{LC} + \Omega_{\mathbb{P}^n_{\mathbb{C}}})$. For the same reasoning exposed in the proof of Proposition 2.5.1, $\widetilde{d^{LC}}\widetilde{\eta} = 0$ if and only if $d^{LC}\eta - 2is^*\varphi \wedge \eta = 0$, which is granted by **D.2**.

Finally, the computations leading to equation (2.12) still apply and thus we can deduce that

$$
\Omega^{\nabla} = r\pi^*(\Omega^{LC} + \Omega_{\mathbb{P}^n_{\mathbb{C}}} + [\eta \wedge \eta]) = 0,
$$

making the connection ∇ flat.

Notice that $\pi \colon \widetilde{M} \to M$ is a principal $\mathbb{C}^*$-bundle, where for all $le^{i\theta} \in \mathbb{C}^*$ and $(u, t) \in \widetilde{M}$:

$$
(u, t)le^{i\theta} := (ue^{i\theta}, tl).
$$

The infinitesimal vector field corresponding to 1 at (u, t_0) is

$$\frac{d}{dt}((u,t_0)\exp(t))|_{t=0} = \frac{d}{dt}((u,t_0e^t))|_{t=0} = t_0(\partial_t)_{(u,t_0)} = \xi_{(u,t_0)}.$$

The one corresponding to i is $X := \frac{d}{dt}((u,t_0)\exp(it))|_{t=0} = \frac{d}{dt}(ue^{it},t_0)|_{t=0}$, which is vertical and such that $\widetilde{\varphi}(X) = \varphi(p_*X) = \varphi(\frac{d}{dt}(ue^{it})|_{t=0}) = 1$ and $dr(X) = 0$. This means that $X = I\xi$ since $\widetilde{g}(X,\cdot) = -r^2\widetilde{\varphi} = -rIdr = I\xi^\flat$.

We are only left to prove that M is the Kähler quotient or $\widetilde{M}$ with respect to the U(1)-action and in order to do so, notice that $\widetilde{\omega}(I\xi,\cdot) = -\widetilde{g}(\xi,\cdot) = rdr = d\left(\frac{r^2}{2}\right)$, so $\mu := \frac{r^2}{2}$ is a moment map for $I\xi$. Notice that $\mu^{-1}(\frac{1}{2}) = S \times \{1\}$ and S is a principal bundle so, by definition of $\widetilde{g}$ and $\widetilde{\omega}$, $S/\mathrm{U}(1)$ is isometric to M and this ends the proof. □

Remark 2.5.7. *Starting from the family $\{\eta_\alpha\}_\alpha$, we can build a bundle map $\gamma\colon S \to M$ as long as the η_α's are linked by the relation $\eta_\alpha = g^2_{\alpha,\beta}\eta_\beta$, where $g_{\alpha,\beta}$ is a cocycle defining S.*

Remark 2.5.8. *Instead of requiring the existence of an S^1-bundle we could require a complex line bundle L with first Chern class $c_1(L) = [\frac{1}{\pi}\omega]$ and a map of complex vector bundles $L \otimes L \to \sharp_2 S_{3,0}M$.*

Remark 2.5.9. *Let (M, g, I) be a Kähler manifold, then if $H^2(M,\mathbb{Z}) = 0$, in particular, every complex line bundle and every circle bundle are trivial. Moreover, by de Rham's theorem, $H^2_{dR}(M) = H^2(M,\mathbb{R}) = H^2(M,\mathbb{Z})\otimes\mathbb{R} = 0$, so in particular $\omega = d\lambda$ for some $\lambda \in \Omega^1(M)$.*

Corollary 2.5.10. *A Kähler manifold (M, g, I, ω) of dimension $2n$ such that $H^2(M,\mathbb{Z}) = 0$, has a projective special Kähler structure if and only if there exists a section $\eta\colon M \to \sharp_2 S_{3,0}M$ such that*

$$\Omega^{LC} + \Omega_{\mathbb{P}^n_{\mathbb{C}}} + [\eta \wedge \overline{\eta}] = 0 \tag{2.17}$$

and

$$d^{LC}\eta = -4i\lambda \wedge \eta \tag{2.18}$$

for some $\lambda \in \Omega^1(M)$ such that $d\lambda = \omega$.

Proof. If M has a projective special Kähler structure, then from Theorem 2.5.6 we obtain an S^1-bundle $p\colon S \to M$ and the map $\gamma\colon S \to \sharp_2 S_{3,0}M$. Consider the corresponding line bundle $L = S \times_{\mathrm{U}(1)} \mathbb{C}$. As noted in Remark

2.5.9, we can assume $L = M \times \mathbb{C}$ and $S = M \times S^1$. In particular, there is a global section $s\colon M \to S$ and if we call $\eta = \gamma \circ s\colon M \to \sharp_2 S_{3,0}M$, it is a global section satisfying the curvature equation thanks to Theorem 2.5.6. Defining $\lambda := -\frac{1}{2}s^*\phi$, we have $d\lambda = -\frac{1}{2}s^*(-2\pi_S^*\omega) = (\pi_S s)^*\omega = \omega$ and thus also the differential condition is satisfied by Theorem 2.5.6.

Conversely, by de Rham's Theorem, we have $\lambda \in \Omega^1(M)$ such that $d\lambda = \omega$. We define $\pi_S = \pi_1\colon S = M \times S^1 \to M$ and choose as connection the form $\varphi = \pi_2^* d\vartheta - 2\pi_S^*\lambda$, where $d\vartheta$ is the fundamental 1-form on $S^1 = \mathrm{U}(1)$. Then $d\varphi = 0 - 2\pi_S^* d\lambda = -2\pi_S^*\omega$, so $S \to M$ has the desired curvature. Moreover, it is trivial, so we have a global section $s\colon M \to S$ mapping p to $(p, 1)$.

Given $\eta\colon M \to \sharp_2 S_{3,0}M$ as in the statement, we define $\gamma\colon S \to \sharp_2 S_{3,0}M$ such that $\gamma(p, a) := a^2\eta(p)$ for all $p \in M$ and $a \in \mathrm{U}(1)$. Notice that $\gamma \circ s = \gamma(\cdot, 1) = \eta$, so the curvature equation of this corollary gives the curvature equation in Theorem 2.5.6 and the same is true for the differential condition, since $s^*\varphi = s^*\pi_2^* d\vartheta - 2s^*\pi_S^*\lambda = 0 - 2\lambda$. By Theorem 2.5.6, M is thus projective special Kähler. $\square$

Remark 2.5.11. *Instead of requiring a section η, as in Corollary 2.5.10, we could use a section σ of $S_{3,0}M$ such that $\sharp_2\sigma = \eta$.*

Theorem 2.5.6 allows to find a whole class of projective special Kähler structures from a given one, as shown in the following

Proposition 2.5.12. *Let $(\pi\colon \widetilde{M} \to M, \nabla)$ be a projective special Kähler manifold, let $\gamma\colon S \to \sharp_2 S_{3,0}M$ be its intrinsic deviance and $\varphi \in \Omega^1(S)$ the principal connection form on $\pi_S\colon S \to M$, then for all $\beta \in \mathcal{C}^\infty(M, \mathrm{U}(1))$ there is a new projective special Kähler manifold $(\pi\colon \widetilde{M}^\beta \to M, \nabla^\beta)$ with intrinsic deviance $\gamma^\beta = \beta\gamma\colon S \to \sharp_2 S_{3,0}M$, and with principal connection form $\varphi^\beta = \pi_S^*\left(\frac{d\beta}{2i\beta}\right) + \varphi$ on the same bundle.*

Proof. We want to use Theorem 2.5.6, so consider the same bundle $\pi_S\colon S \to M$, but with the new connection form φ^β. Notice that φ^β is a real form, in fact $\overline{\beta}\beta = 1$, so

$$0 = \beta d\overline{\beta} + \overline{\beta} d\beta = \overline{\beta}\beta\left(\frac{d\overline{\beta}}{\overline{\beta}} + \frac{d\beta}{\beta}\right) = \left(\overline{\left(\frac{d\beta}{\beta}\right)} + \frac{d\beta}{\beta}\right) = 2\,\mathrm{Re}\left(\frac{d\beta}{\beta}\right),$$

and thus $\mathrm{Im}\left(\frac{d\beta}{2i\beta}\right) = -\frac{1}{2}\mathrm{Re}\left(\frac{d\beta}{\beta}\right) = 0$. Moreover $d\varphi^\beta = -\pi_S^*\left(\frac{d\beta \wedge d\beta}{\beta^2}\right) + d\varphi = d\varphi = -2\pi^*\omega$, so this is an acceptable principal connection form. The bundle map γ^β is still homogeneous of degree 2. We are only left to prove the two conditions of point 3, so consider a family of sections $\{(U_\alpha, s_\alpha)\}_{\alpha\in\mathcal{A}}$ corresponding to a trivialisation of S and let $\eta_\alpha^\beta := \gamma^\beta \circ s_\alpha = \beta\gamma \circ s_\alpha = \beta\eta_\alpha$. We thus have

$$\begin{aligned} d^{LC}\eta_\alpha^\beta &= d^{LC}(\beta\eta_\alpha) = d\beta \wedge \eta_\alpha + \beta 2is_\alpha^*\varphi \wedge \eta_\alpha = 2i\left(\frac{d\beta}{2i\beta} + s_\alpha^*\varphi\right) \wedge e^{2i\beta}\eta_\alpha \\ &= 2is_\alpha^*\left(d\pi_S^*\left(\frac{d\beta}{2i\beta}\right) + s_\alpha^*\varphi\right) \wedge \eta_\alpha^\beta = 2is_\alpha^*\varphi^\beta \wedge \eta_\alpha^\beta. \end{aligned}$$

As for the curvature condition **D.1**, it still holds because

$$[\eta_\alpha^\beta \wedge \overline{\eta_\alpha^\beta}] = [\beta\eta_\alpha \wedge \overline{\beta\eta_\alpha}] = [\eta_\alpha \wedge \overline{\eta_\alpha}]. \qquad \square$$

Before continuing, we recall the following elementary result.

Lemma 2.5.13. *Let M be a smooth manifold and G a Lie group with Lie algebra $\mathfrak{g}$ such that there is a smooth right action*

$$r\colon M \times G \longrightarrow M.$$

Then, the differential of r at a point (x, a) is

$$r_*(X, A) = (R_a)_*(X) + A^\circ,$$

for all $X \in T_xM$, $A \in \mathfrak{g}$.

Proof. We split the computation on each component of $T_{x,a}(M\times G) = T_xM \times T_aG \cong T_xM \times \mathfrak{g}$.

Consider the vector $(X, 0)$ and let x_t be a smooth path such that $x_0 = x$ and $\frac{d}{dt}x_t|_{t=0} = X$, then (x_t, a) is an integral curve for $(X, 0)$ in (x, a). Thus,

$$r_*(X, 0) = \frac{d}{dt}r(x_t, a)|_{t=0} = \frac{d}{dt}R_a(x_t)|_{t=0} = (R_a)_*(X).$$

For the vector $(0, A)$ instead, consider the curve $a_t = (x, a\exp(tA))$, then for $t = 0$ we are in (x, a) and its tangent vector for $t = 0$ is $(0, A)$. Hence,

$$r_*(0, A) = \frac{d}{dt}r(x, a\exp(tA))|_{t=0} = \frac{d}{dt}xa\exp(tA)|_{t=0} = (A^\circ)_{xa}.$$

The statement follows from the linearity of r_*. $\square$

We now present the following isomorphism result:

Proposition 2.5.14. *In the setting of Proposition 2.5.12, if moreover β has a square root, meaning that $\beta = b^2$ for some $b : M \to \mathrm{U}(1)$, then the map*

$$\begin{aligned} m_b \colon S &\longrightarrow S \\ u &\longmapsto u.b(\pi_S(u)) = R_{b(\pi_S(u))}(u) \end{aligned}$$

induces a bundle isomorphism preserving connection and deviance, that is

$$\varphi^\beta = m_b^*(\varphi), \qquad \gamma^\beta = \gamma \circ m_b.$$

In particular, if $H^1_{dR}(M) = 0$, then every β has a square root.

Proof. The preservation of γ follows from its 2-homogeneity, in fact, for all $u \in S$:

$$\gamma \circ m_b(u) = \gamma(ub(\pi_S(u))) = b(\pi_S(u))^2\gamma(u) = (\beta \circ \pi_S)\gamma(u) = \gamma^\beta.$$

For the connection instead, we first compute the differential of m_b. Let $r : S \times \mathrm{U}(1) \to S$ be the principal right action, then we can see m_b as $r \circ (\mathrm{id}_S \times (b \circ \pi_S))$. The differential of $(\mathrm{id}_S \times (u \circ \pi_S))$ is $\mathrm{id}_{TS} \times \pi_S^* db$, where db has values in $\mathfrak{u}(1) = i\mathbb{R}$. Lemma 2.5.13 gives us the differential of the action. We have

$$((m_b)_*)_u = (R_{b\pi_S(u)})_* + (d_{\pi_S(u)}b)^\circ$$

Now let us compute the pullback of φ, using the fact that φ is right invariant and $d\beta = db^2 = 2bdb$

$$\begin{aligned} m_b^*(\varphi) = \varphi \circ (m_b)_* &= \varphi \circ (R_{b\pi_S(u)})_* + \varphi((d_{\pi_S(u)}b)^\circ) = R^*_{b\pi_S(u)}\varphi + \frac{1}{ib} d_{\pi_S(u)}b \\ &= \varphi + \frac{1}{i2b^2} d_{\pi_S(u)}\beta = \varphi + \frac{1}{i2\beta} d_{\pi_S(u)}\beta = \varphi^\beta. \end{aligned}$$

In order to prove the last statement, let $a : \mathrm{U}(1) \to \mathbb{C}$ be the standard identification of $\mathrm{U}(1)$ with the unit circle. Notice that we can write the fundamental form of $\mathrm{U}(1)$ as

$$\frac{1}{ia} da.$$

Locally in fact it is $a = e^{i\alpha}$ and thus $\frac{1}{ia}da = d\alpha$.

Now let $\beta\colon M \to \mathrm{U}(1)$, and consider the pullback

$$\beta^*\left(\frac{1}{ia}da\right) = \frac{1}{i\beta}d\beta.$$

We have $\beta^*\colon H^1_{dR}(\mathrm{U}(1)) \to H^1_{dR}(M) = 0$, so in particular $\frac{1}{i\beta}d\beta$ is exact. Let $\lambda \in \mathcal{C}^\infty(M)$ be such that $d\lambda = \frac{1}{i\beta}d\beta$, then $e^{-i\lambda}\beta$ is a smooth function with image in $\mathrm{U}(1)$ and differential

$$-ie^{i\lambda}\beta d\lambda + e^{i\lambda}d\beta = -\frac{ie^{i\lambda}\beta}{i\beta}d\beta + e^{i\lambda}d\beta = -e^{i\lambda}d\beta + e^{i\lambda}d\beta = 0.$$

So up to a locally constant function k, $ke^{i\lambda} = \beta$. Without loss of generality, we can assume $k = 1$ (take $\lambda' = \lambda - i\log(k)$). Then let $b = e^{\frac{i\lambda}{2}}$ and $b^2 = \beta$. $\square$

Remark 2.5.15. *In the statement of Proposition 2.5.12, if $H^1_{dR}(M) = 0$, then, up to isomorphism, there is a unique projective special Kähler structure on M, once we fix the Kähler structure.*

2.6 Complex hyperbolic n-space

In this section we are going to describe a special family of projective special Kähler manifolds, which can be thought of as the simplest possible model in a given dimension.

Let $\mathbb{C}^{n,1}$ be the Hermitian space $\mathbb{C}^{n+1}$ endowed with the Hermitian form

$$\langle z, w\rangle = \overline{z_1}w_1 + \cdots + \overline{z_n}w_n - \overline{z_{n+1}}w_{n+1}.$$

It is a complex vector space, so it makes sense to consider the projective space associated to it, that is $\mathbb{P}(\mathbb{C}^{n,1}) = (\mathbb{C}^{n,1}\setminus\{0\})/\mathbb{C}^*$ with the quotient topology and the canonical differentiable structure, where $\mathbb{C}^*$ acts by scalar multiplication. We will denote the quotient class corresponding to an element $z \in \mathbb{C}^{n,1}$ by $[z]$. We can define the following open subset:

$$\mathcal{H}^n_{\mathbb{C}} := \{[v] \in \mathbb{P}(\mathbb{C}^{n,1}) | \langle v, v\rangle < 0\}.$$

Let $v = (v_1, \ldots, v_{n+1}) \in \mathbb{C}^{n,1}$, notice that if $[v] \in \mathcal{H}^n_{\mathbb{C}}$, then $|v_1|^2 + \cdots + |v_n|^2 - |v_{n+1}|^2 < 0$ so $|v_{n+1}|^2 > |v_1|^2 + \cdots + |v_n|^2 \geq 0$ which implies $v_{n+1} \neq 0$. We thus have a global differentiable chart $\mathcal{H}^n_{\mathbb{C}} \to \mathbb{C}^n$ by restricting the projective chart $[v] \mapsto (\frac{v_1}{v_{n+1}}, \ldots, \frac{v_n}{v_{n+1}})$.

Remark 2.6.1. *The inverse of this chart is the map* $\mathbb{C}^n \to \mathbb{P}(\mathbb{C}^{n,1})$ *such that* $\mathbb{C}^n \ni z = (z_1, \ldots, z_n) \mapsto [(z_1, \ldots, z_n, 1)]$*, which is in* $\mathcal{H}^n_{\mathbb{C}}$ *if and only if* $\|z\|^2 < 1$*. We have proven that* $\mathcal{H}^n_{\mathbb{C}}$ *is diffeomorphic to the complex unit ball and thus in particular it is contractible.*

Consider now the Lie group $\mathrm{SU}(n,1)$ of the matrices with determinant 1 that are unitary with respect to the Hermitian metric on $\mathbb{C}^{n,1}$. We define a left action of $\mathrm{SU}(n,1)$ on $\mathcal{H}_{\mathbb{C}}$ such that $A[v] = [Av]$; it is well defined by linearity and invertibility and it is smooth.

This action is also transitive, in fact given $[v],[w] \in \mathcal{H}^n_{\mathbb{C}}$, without loss of generality, we can assume that $\langle v, v\rangle = -1 = \langle w, w\rangle$. Because of this, we can always complete v and w to an orthonormal basis with respect to the Hermitian product, obtaining $\{v_1, \ldots, v_n, v\}$ and $\{w_1, \ldots, w_n, w\}$. Consider the following block matrices $V = (v_1|\ldots|v_n|v)$ and $W = (w_1|\ldots|w_n|w)$ which, up to permuting two of the first n-columns, belong to $\mathrm{SU}(n,1)$. The matrix $A = WV^{-1} \in \mathrm{SU}(n,1)$ maps v in w and thus $[v]$ in $[w]$.

We shall now compute the stabiliser of the last element of the canonical basis e_{n+1} for this action, that is, the set of matrices $A \in \mathrm{SU}(n,1)$ such that $Ae_{n+1} = \lambda e_{n+1}$ for $\lambda \in \mathbb{C}$. Observe that $\lambda \in \mathrm{U}(1)$ since

$$-1 = \langle e_{n+1}, e_{n+1}\rangle = \langle Ae_{n+1}, Ae_{n+1}\rangle = \langle \lambda e_{n+1}, \lambda e_{n+1}\rangle = -|\lambda|^2.$$

Moreover, the last column of A is $A_{n+1} = Ae_{n+1} = \lambda e_{n+1}$. This forces A to assume the form

$$\begin{pmatrix} B & 0 \\ 0 & \lambda \end{pmatrix}.$$

Since A belongs to $\mathrm{SU}(n,1)$, we must infer that B belongs to $\mathrm{U}(n)$ and $\lambda = \det(B)^{-1}$. The stabiliser of e_{n+1} is thus $S(\mathrm{U}(n)\mathrm{U}(1))$, which is isomorphic to $\mathrm{U}(n)$. We deduce the diffeomorphism $\mathcal{H}^n_{\mathbb{C}} \cong \mathrm{SU}(n,1)/S(\mathrm{U}(n)\mathrm{U}(1))$.

Proposition 2.6.2. $\mathcal{H}^n_{\mathbb{C}}$ *is a symmetric space.*

Proof. Consider the Lie algebra

$$\mathfrak{g} := \mathfrak{su}(n,1) = \left\{ \begin{pmatrix} A & b \\ b^\star & -\mathrm{tr}(A) \end{pmatrix} | A \in \mathfrak{u}(n), b \in \mathbb{C}^n \right\}.$$

It can be decomposed as $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ where:

$$\mathfrak{h} := \left\{ \begin{pmatrix} A & 0 \\ 0 & -\mathrm{tr}(A) \end{pmatrix} | A \in \mathfrak{u}(n) \right\};$$

$$\mathfrak{m} := \left\{ \begin{pmatrix} 0 & b \\ b^\star & 0 \end{pmatrix} | b \in \mathbb{C}^n \right\}.$$

Notice that $\mathfrak{h}$ is the Lie algebra corresponding to the subgroup $S(\mathrm{U}(n)\mathrm{U}(1))$. Furthermore, we have $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$, proving that the space $\mathrm{SU}(n,1)/S(\mathrm{U}(n)\mathrm{U}(1))$ is symmetric. □

Notice that $\mathfrak{h} \cong \mathfrak{u}(n)$ as a Lie algebra and $\mathfrak{m} \cong \mathbb{C}^n$ as a vector space, with isomorphism $\begin{pmatrix} 0 & b \\ b^\star & 0 \end{pmatrix} \mapsto b$.

We will adopt the nomenclature of [24] for the following

Definition 2.6.3. *We call the Kähler manifold $\mathcal{H}^n_{\mathbb{C}}$ of complex dimension n, the* complex hyperbolic n-space*.*

There is a natural Kähler structure on $\mathcal{H}^n_{\mathbb{C}}$ coming from its representation as a symmetric space G/H.

On a symmetric space, there is a one-to-one correspondence between Riemannian metrics and $\mathrm{Ad}(H)$-invariant positive definite symmetric bilinear forms on $\mathfrak{m}$ (See [33, II, Corollary 3.2, p. 200]).

Let $\theta\colon T_{[e_{n+1}]}\mathcal{H}^n_{\mathbb{C}} \cong \mathfrak{m} \to \mathbb{C}^n$ be the identification mapping to x the tangent vector corresponding to $\begin{pmatrix} 0 & x \\ x^\star & 0 \end{pmatrix}$. With this identification, for $A \in \mathrm{U}(n)$ we see that the $\mathrm{Ad}(A)$-action on $\mathfrak{m}$ corresponds on $\mathbb{C}^n$ to the $x \mapsto \det(A)Ax$. The metric is induced by the Killing form on $\mathfrak{su}(n,1)$ given by the following Lemma ([28]).

Lemma 2.6.4. *The Killing form on $\mathfrak{su}(n,1)$ is*

$$B(X,Y) = 2(n+1)\mathrm{tr}(XY), \quad \forall X, Y \in \mathfrak{u}(n,1).$$

Proof. The Killing form on $\mathfrak{su}(n,1)$ is the same as the Killing form of its complexification $\mathfrak{sl}(n+1,\mathbb{C})$ (see [28, Lemma 6.1, p. 180]). In turn, the Killing form of $\mathfrak{sl}(n+1,\mathbb{C})$ evaluated on $X, Y \in \mathfrak{sl}(n+1,\mathbb{C})$ is $2(n+1)\mathrm{tr}(XY)$ ([28, (5), p. 187]), ending the proof. □

We restrict the Killing form to $\mathfrak{m}$ in order to define an $\mathrm{Ad}(H)$-invariant bilinear form, that is, given $x, y \in \mathbb{C}^n$, if X, Y are the corresponding tangent vectors,

$$B(X,Y) = 2(n+1)\mathrm{tr}\left(\begin{pmatrix} 0 & x \\ x^\star & 0 \end{pmatrix} \begin{pmatrix} 0 & y \\ y^\star & 0 \end{pmatrix} \right) = 2(n+1)\mathrm{tr}\begin{pmatrix} xy^\star & 0 \\ 0 & x^\star y \end{pmatrix}$$

$$= 2(n+1)\,\mathrm{Re}(x^\star y) = 2(n+1)(\theta^\star\theta)(X,Y).$$

We define $g_{[e_{n+1}]} := \theta^\star\theta$, which is $\mathrm{Ad}(\mathrm{U}(n))$-invariant, so it extends to a global Riemannian metric g. By using the same idea, we can also define an almost complex structure I on $\mathfrak{m}$ as the map corresponding to the scalar multiplication by i on $\mathbb{C}^n$. This structure is compatible with the metric and it is $\mathrm{Ad}(\mathrm{U}(n))$-invariant, so it defines a Kähler structure (see [33, II, Proposition 9.3, p. 260]).

The Kähler form ω is then:

$$\omega(X,Y) = g(IX,Y) = \mathrm{Re}(x^\star i^\star y) = \mathrm{Im}(x^\star y) = \mathrm{Im}(\theta^\star \otimes \theta)(X,Y).$$

We shall now compute the curvature of the complex hyperbolic n-space.

Proposition 2.6.5. *The curvature tensor of $\mathcal{H}^n_{\mathbb{C}}$ is $-\Omega_{\mathbb{P}^n_{\mathbb{C}}}$.*

Proof. Since $\mathcal{H}^n_{\mathbb{C}}$ is a symmetric space G/H, we can compute its Riemannian curvature tensor at the point $p = [e_{n+1}]$, corresponding to the coset H, with the formula $\Omega_{\mathcal{H}^n_{\mathbb{C}}}(X,Y)Z = -[[X,Y],Z]$ (see [28, Theorem 4.2, p. 215]) using the usual identification of $T_p\mathcal{H}^n_{\mathbb{C}}$ with $\mathfrak{m}$. Explicitly, if $x,y,z \in \mathbb{C}^n$ and X,Y,Z are the respective tangent vectors, $\Omega_{\mathcal{H}^n_{\mathbb{C}}}(X,Y)Z$ at p corresponds to

$$\begin{aligned}
&-\left[\left[\begin{pmatrix} 0 & x \\ x^\star & 0 \end{pmatrix}, \begin{pmatrix} 0 & y \\ y^\star & 0 \end{pmatrix}\right], \begin{pmatrix} 0 & z \\ z^\star & 0 \end{pmatrix}\right] \\
&= -\left[\begin{pmatrix} xy^\star - yx^\star & 0 \\ 0 & x^\star y - y^\star x \end{pmatrix}, \begin{pmatrix} 0 & z \\ z^\star & 0 \end{pmatrix}\right] \\
&= -\begin{pmatrix} 0 & (xy^\star - yx^\star)z - z(x^\star y - y^\star x) \\ ((xy^\star - yx^\star)z - z(x^\star y - y^\star x))^\star & 0 \end{pmatrix} \\
&= -\begin{pmatrix} 0 & (xy^\star - yx^\star - x^\star y + y^\star x)z \\ ((xy^\star - yx^\star - x^\star y + y^\star x)z)^\star & 0 \end{pmatrix}.
\end{aligned}$$

It follows that

$$\Omega_{\mathcal{H}^n_{\mathbb{C}}} = -\mathfrak{R}(\theta^k \wedge \overline{\theta^h} \otimes \theta_k \otimes \theta^h - \theta^h \wedge \overline{\theta^h} \otimes \theta_k \otimes \theta^k) = -\Omega_{\mathbb{P}^n_{\mathbb{C}}}.$$

The curvature of the complex hyperbolic n-space is thus opposite of the curvature of the complex projective space of the same dimension, as computed in Remark 2.3.3. □

Proposition 2.6.6. *The manifold $\mathcal{H}^n_{\mathbb{C}}$ is a projective special Kähler manifold for all $n \geq 1$ with constant zero deviance.*

Proof. By Remark 2.6.1, we know that $\mathcal{H}^n_{\mathbb{C}}$ is contractible, so in particular $H^2(M, \mathbb{Z}) = 0$, and thus we can apply Corollary 2.5.10. If we choose as tensor η of type $\sharp_2 S_{3,0}M$ the 0-section, then the differential condition (2.18) is trivially satisfied, while the condition (2.17) follows from the computation of the curvature tensor in Proposition 2.6.5. □

Notice that the deviance measures the difference of a projective special Kähler manifold of dimension $2n$ from being the complex hyperbolic n-space. More precisely, we have

Proposition 2.6.7. *At a point p of a projective special Kähler manifold M with intrinsic deviance $\gamma\colon S \to \sharp_2 S_{3,0}M$, the curvature tensor Ω_M coincides with the one of $\mathcal{H}^n_{\mathbb{C}}$ exactly in those points p where $\gamma|_p$ vanishes.*

In particular, for any section of S defined on an open neighbourhood of p, the corresponding local deviance vanishes at p whenever the two curvatures coincide.

Proof. One direction follows from condition **D.1**. For the opposite one, if $\Omega_M = \Omega_{\mathcal{H}^n_{\mathbb{C}}} = -\Omega_{\mathbb{P}^n_{\mathbb{C}}}$, then $\mathrm{scal}_M = -2(n+1)$ and the intrinsic deviance vanishes as the norm of any local deviance vanishes by (2.15). □

We can also prove

Proposition 2.6.8. *The only complete connected and simply connected projective special Kähler manifold of dimension $2n$ with zero deviance is $\mathcal{H}^n_{\mathbb{C}}$.*

Proof. Let $(\pi\colon \widetilde{M} \to M, \nabla)$ be such a projective special Kähler manifold. Consider a point $p \in M$, then (T_pM, g, I) can be seen as a complex vector space compatible with the metric and can thus be identified with the tangent space at a point of $\mathcal{H}^n_{\mathbb{C}}$ via an isomorphism F as they are both isomorphic to $\mathbb{C}^n$ with the standard metric. Being complex manifolds, $\mathcal{H}^n_{\mathbb{C}}$ and M are analytic, and since the curvature of M is forced to be $-\Omega_{\mathbb{P}^n_{\mathbb{C}}}$, which corresponds to a $\mathfrak{u}(n)$-invariant map from the bundle of unitary frames to $S^2(\mathfrak{u}(n))$, it is also parallel with respect to the Levi-Civita connection. It follows that the linear isomorphism F preserves the curvature tensors and their covariant derivatives. It follows that F can be extended to a diffeomorphism $f\colon M \to \mathcal{H}^n_{\mathbb{C}}$ (See [33, I, Corollary 7.3, p. 261]) such that F is its differential at p.

Since F preserves I and ω which are parallel, f is an isomorphism of Kähler manifolds, as the latter maps parallel tensors to parallel tensors.

Since the deviance of both manifolds is zero, we also have an isomorphism of projective special Kähler manifolds. □

The complex hyperbolic n-space provides an example of projective special Kähler manifold which is not special Kähler. This is due to the fact that any complete special Kähler manifold is flat ([36, Theorem 2, p. 712]), which is a consequence (see [6, Corollary 3.3, p. 2406]) of the Calabi-Pogorelov Theorem ([40, Theorem 7.7, p. 125]). If $\mathcal{H}^n_{\mathbb{C}}$ were a special Kähler manifold, it would be flat, which is in contradiction with the statement of Proposition 2.6.5. We can prove the same result directly.

Proposition 2.6.9. *The complex hyperbolic n-space is never special Kähler.*

Proof. Assume by contradiction that $\mathcal{H}^n_{\mathbb{C}}$ has a special Kähler structure, then there is a connection ∇ with the required properties. By Lemma 2.1.1, we know that $\nabla = \nabla^{LC} + \widetilde{\eta}$ with $\widetilde{\eta}$ section of $\sharp_2[\![S_{3,0}\mathcal{H}^n_{\mathbb{C}}]\!]$. By Proposition 2.1.2, flatness of ∇ implies in particular

$$\Omega^{LC} + \frac{1}{2}[\widetilde{\eta} \wedge \widetilde{\eta}] = 0. \tag{2.19}$$

The form of the first summand is provided by Proposition 2.6.5 which states that $\Omega^{LC} = -\Omega_{\mathbb{P}^n_{\mathbb{C}}}$. Since $\widetilde{\eta}$ is a section of $\sharp_2[\![S_{3,0}\mathcal{H}^n_{\mathbb{C}}]\!]$, we can write it as $\widetilde{\eta} = \widehat{\eta} + \overline{\widehat{\eta}}$ for a unique tensor $\widehat{\eta}$ of type $\sharp_2 S_{3,0}\mathcal{H}^n_{\mathbb{C}}$. Therefore, we have

$$\frac{1}{2}[\widetilde{\eta} \wedge \widetilde{\eta}] = \frac{1}{2}[\widehat{\eta} + \overline{\widehat{\eta}} \wedge \widehat{\eta} + \overline{\widehat{\eta}}] = \frac{1}{2}\mathfrak{R}[\widehat{\eta} \wedge \widehat{\eta}] + [\widehat{\eta} \wedge \overline{\widehat{\eta}}].$$

If we write $\widehat{\eta}$ on a complex frame θ, we observe that $[\widehat{\eta} \wedge \widehat{\eta}]$ vanishes, as $[\overline{\theta_k} \otimes \theta^j, \overline{\theta_k} \otimes \theta^j] = 0$, so $\frac{1}{2}[\widetilde{\eta} \wedge \widetilde{\eta}] = [\widehat{\eta} \wedge \widehat{\eta}]$ and (2.19) becomes

$$[\widehat{\eta} \wedge \widehat{\eta}] = \Omega_{\mathbb{P}^n_{\mathbb{C}}}.$$

In particular then, the scalar components of these two tensors must equal each other. The scalar curvature of the projective space however is $2(n+1) > 0$ (See (2.8)), whereas the scalar component of $[\widehat{\eta} \wedge \widehat{\eta}]$ can be computed as in the proof of Proposition 2.5.3, obtaining $-\frac{2}{n}\left\|\widehat{\eta}\right\|^2 \leq 0$. Being real functions with different sign, they can never be the same, giving a contradiction. □

We conclude this section by giving the Iwasawa decomposition of $\mathrm{SU}(n, 1)$ (see e.g. [28, Theorem 1.3, p. 403]), which will imply in particular that $\mathcal{H}^n_{\mathbb{C}}$

can be identified with a solvable Lie group; this group will appear in our classification of projective special Kähler Lie groups of dimension 4 in the next chapter.

Let E^j_k be the $(n+1)\times(n+1)$ matrix with entry 1 at row j and column k and 0 everywhere else. Let us define the following matrices for $1 \le k \le n$ and $1 \le j < h \le n$:

$$\begin{aligned} P_k &= E^k_{n+1} + E^{n+1}_k; & Q_k &= iE^k_{n+1} - iE^{n+1}_k; \\ H_{j,h} &= E^j_h - E^h_j; & K_{j,h} &= iE^j_h + iE^h_j; \\ D_k &= iE^k_k - iE^{n+1}_{n+1}. \end{aligned}$$

These matrices form a real basis for $\mathfrak{su}(n,1)$.

Consider the Cartan involution θ mapping $A \in \mathfrak{su}(n,1)$ to $-A^\star$. Its action on the basis is

$$\theta(P_k) = -P_k \quad \theta(Q_k) = -Q_k \quad \theta(H_{j,h}) = H_{j,h} \quad \theta(K_{j,h}) = K_{j,h} \quad \theta(D_k) = D_k.$$

We can then define $\mathfrak{k} = \langle H_{j,h}, K_{j,h}, D_k | 1 \le j < h \le n, 1 \le k \le n\rangle_{\mathbb{R}}$ and $\mathfrak{p} = \langle P_k, Q_k | 1 \le k \le n\rangle_{\mathbb{R}}$.

Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. We choose $P_1 \in \mathfrak{a}$, then $[P_1, Q_1] = -2D_1$ for all $k \neq 1$, $[P_1, P_k] = H_{1,k}$ and $[P_1, Q_k] = -K_{1,k}$, thus $\mathfrak{a} = \langle P_1\rangle_{\mathbb{R}}$ is a maximal abelian subalgebra of $\mathfrak{p}$ containing P_1. Consider $C_{\mathfrak{k}}(\mathfrak{a}) = \{B \in \mathfrak{k} | [A,B] = 0, \text{ for all } A \in \mathfrak{a}\} = \{B \in \mathfrak{k} | [P_1, B] = 0\}$. Computations show that the matrices A that commute with P_1 must have a zero at positions $A^1_k, A^{n+1}_k, A^k_1, A^k_{n+1}$ for all $2 \le k \le n$ and moreover $A^1_{n+1} = A^{n+1}_1$ and $A^1_1 = A^{n+1}_{n+1}$. The only matrices of $\mathfrak{k}$ that satisfy this condition are $D_1 - 2D_k$ for $2 \le k \le n$, and $H_{j,h}$ or $K_{j,h}$ for $2 \le j < h \le n$. We also know that $\mathfrak{g}_0 = \mathfrak{a} + C_{\mathfrak{k}}(\mathfrak{a}) = \langle P_1, D_1 - 2D_k, H_{j,h}, K_{j,h} | 2 \le k \le n, 2 \le j < h \le n\rangle_{\mathbb{R}}$ (e.g. [28, IX.§1, p. 401]).

On $\mathfrak{a}_0$ we can define a linear form α mapping P_1 to 1, and thus we can compute the root spaces

$$\begin{aligned} \mathfrak{g}_{2\alpha} &= \langle Q_1 - D_1\rangle_{\mathbb{R}}\,; \\ \mathfrak{g}_{-2\alpha} &= \langle Q_1 + D_1\rangle_{\mathbb{R}}\,; \\ \mathfrak{g}_{\alpha} &= \langle P_k + H_{1,k}, Q_k - K_{1,k} | 2 \le k \le n\rangle_{\mathbb{R}}\,; \\ \mathfrak{g}_{-\alpha} &= \langle P_k - H_{1,k}, Q_k + K_{1,k} | 2 \le k \le n\rangle_{\mathbb{R}}\,. \end{aligned}$$

The positive roots are thus $\{\alpha, 2\alpha\}$ and

$$\mathfrak{n} = \langle Q_1 - D_1, P_k + H_{1,k}, Q_k - K_{1,k} | 2 \le k \le n\rangle_{\mathbb{R}}\,.$$

Explicitly, the Lie brackets on the elements of the basis are

$$\begin{aligned}
&[Q_1 - D_1, P_k + H_{1,k}] = [Q_1 - D_1, Q_k - K_{1,k}] = 0;\\
&[P_h + H_{1,h}, P_k + H_{1,k}] = [Q_h - K_{1,h}, Q_k - K_{1,k}] = 0;\\
&[P_h + H_{1,h}, Q_k - K_{1,k}] = 2\delta_{h,k}(Q_1 - D_1).
\end{aligned}$$

for all $2 \leq h, k \leq n$ so, by rescaling the generating elements of the basis, we get that $\mathfrak{n}$ is isomorphic to the Heisenberg Lie algebra of dimension $2n-1$. We now have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ which is the Iwasawa decomposition.

Notice that $\mathfrak{k}$ is the Lie subalgebra corresponding to $S(\mathrm{U}(n)\mathrm{U}(1))$, so in particular the tangent space is isomorphic to $\mathfrak{a} \oplus \mathfrak{n}$ and thus we have proven

Proposition 2.6.10. *The complex hyperbolic n-space is a solvmanifold isomorphic to $H_{n-1} \rtimes \mathbb{R}$, where H_{n-1} is the Heisenberg group of dimension $2n-1$.*

Chapter 3

Applications of the deviance

In this chapter we give some applications of the deviance. The first one is the classification of 4-dimensional projective special Kähler Lie groups. Here we can see how the introduction of the deviance in substitution to the classical definition of projective special Kähler manifold, simplifies computations. In fact, all it is left to do is to solve the algebraic condition **D.1** of Theorem 2.5.6 for a generic deviance tensor defined in a neighbourhood of the identity element, and then refine the solutions by imposing the differential condition **D.2**.

In the following section, we compute the deviance for the examples of Section 1.6. In particular, in the first part we also give a classification result for projective special Kähler Lie groups in dimension 2.

The final section contains the computation of the coframe of a quaternion Kähler manifold obtained from the intrinsic c-map construction as presented in Section 1.4.1, starting from the complex hyperbolic n-space. An explicit computation is done for $n = 2$.

3.1 Classification of 4-dimensional projective special Kähler Lie groups

If M is a Lie group, the conditions of Theorem 2.5.6 are simpler, because a Lie group is always parallelisable. As a consequence, the bundle $\sharp_2 S_{3,0}(M)$ is trivial, and in particular we have a global coordinate system to write the local deviances.

Definition 3.1.1. *A* projective special Kähler Lie group *is a Lie group with projective special Kähler structure such that the Kähler structure is left-invariant.*

Notice that we do not require the deviance to be left-invariant.

An example is $\mathcal{H}^n_{\mathbb{C}}$, since the Iwasawa decomposition $\mathrm{SU}(n,1) = KAN$ gives a left-invariant Kähler structure on the solvable Lie group AN. We denote by $\mathcal{H}_\lambda$ the hyperbolic plane with curvature $-\lambda^2$, which is actually just a rescaling of $\mathcal{H}^1_{\mathbb{C}}$, since there is only one non-abelian Lie algebra. Consider in fact such a Lie algebra and a unitary frame e, such that $[e_1, e_2] = ae_1 + be_2$ with $(a,b) \neq 0$. By defining $u_1 = \frac{1}{\sqrt{a^2+b^2}}(ae_1 + be_2)$ and $u_2 = Iu_1$, we get $[u_1, u_2] = \sqrt{a^2+b^2}u_1$, which is a rescaling of the hyperbolic plane.

With Definition 3.1.1, we are able to classify 4-dimensional projective special Kähler Lie groups; we obtain exactly two, which coincide with the two 4-dimensional cases appearing in the classification of projective special Kähler manifolds homogeneous under the action of a semisimple Lie groups ([2]).

Theorem 3.1.2. *Up to isomorphisms of projective special Kähler manifolds, there are only two connected simply connected projective special Kähler Lie groups of dimension 4: $\mathcal{H}_{\sqrt{2}} \times \mathcal{H}_2$ and the complex hyperbolic plane. Up to isomorphisms that also preserve the Lie group structure, there are four projective special Kähler connected and simply connected Lie groups of dimension 4, listed in Table 3.4.*

Proof. We will start from the classification of pseudo-Kähler Lie groups provided by [41]. Table 3.1 displays the eighteen families of pseudo-Kähler Lie algebras in dimension 4.

Among these families, only for the ones in Table 3.2 the metric can be positive definite i.e. Kähler. It is now straightforward to find a unitary frame u for each case, that is such that $g = \sum_{k=1}^4 (u^k)^2$, $Iu_1 = u_2$, $Iu_3 = u_4$ and $\omega = u^{1,2} + u^{3,4}$. With respect to u, we can write the new structure constants and compute the Levi-Civita connection form ω^{LC} and the corresponding curvature form Ω^{LC}. We write

$$H_1 := \left(\begin{array}{cc|cc} & -u^{1,2} & & \\ u^{1,2} & & & \\ \hline & & & \\ & & & \end{array}\right) \qquad H_2 = \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & & & -u^{3,4} \\ & & u^{3,4} & \end{array}\right)$$

$\mathfrak{g}$	I	ω
$\mathfrak{rh}_3$	$Ie_1 = e_2, Ie_3 = e_4$	$a_1(e^{1,3} + e^{2,4}) + a_2(e^{1,4} - e^{2,3}) + a_3e^{1,2}$, $a_1^2 + a_2^2 \neq 0$
$\mathfrak{rr}_{3,0}$	$Ie_1 = e_2, Ie_3 = e_4$	$a_1e^{1,2} + a_2e^{3,4}$, $a_1a_2 \neq 0$
$\mathfrak{rr}'_{3,0}$	$Ie_1 = e_4, Ie_2 = e_3$	$a_1e^{1,4} + a_2e^{2,3}$, $a_1a_2 \neq 0$
$\mathfrak{r}_2\mathfrak{r}_2$	$Ie_1 = e_2, Ie_3 = e_4$	$a_1e^{1,2} + a_2e^{3,4}$, $a_1a_2 \neq 0$
$\mathfrak{r}'_2$	$Ie_1 = e_3, Ie_2 = e_4$	$a_1(e^{1,3} - e^{2,4}) + a_2(e^{1,4} + e^{2,3})$, $a_1^2 + a_2^2 \neq 0$
$\mathfrak{r}'_2$	$Ie_1 = -e_2, Ie_3 = e_4$	$a_1(e^{1,3} - e^{2,4}) + a_2(e^{1,4} + e^{2,3}) + a_3e^{1,2}$, $a_1^2 + a_2^2 \neq 0$
$\mathfrak{r}_{4,-1,-1}$	$Ie_4 = e_1, Ie_2 = e_3$	$a_1(e^{1,2} + e^{3,4}) + a_2(e^{1,3} - e^{2,4}) + a_3e^{1,4}$, $a_1^2 + a_2^2 \neq 0$
$\mathfrak{r}'_{4,0,\delta}$	$Ie_4 = e_1, Ie_2 = e_3$	$a_1e^{1,4} + a_2e^{2,3}$, $a_1a_2 \neq 0, \delta > 0$
$\mathfrak{r}'_{4,0,\delta}$	$Ie_4 = e_1, Ie_2 = -e_3$	$a_1e^{1,4} + a_2e^{2,3}$, $a_1a_2 \neq 0, \delta > 0$
$\mathfrak{d}_{4,1}$	$Ie_1 = e_4, Ie_2 = e_3$	$a_1(e^{1,2} - e^{3,4}) + a_2e^{1,4}$, $a_1 \neq 0$
$\mathfrak{d}_{4,2}$	$Ie_4 = -e_2, Ie_1 = e_3$	$a_1(e^{1,4} + e^{2,3}) + a_2e^{2,4}$, $a_1 \neq 0$
$\mathfrak{d}_{4,2}$	$Ie_4 = -2e_1, Ie_2 = e_3$	$a_1e^{1,4} + a_2e^{2,3}$, $a_1a_2 \neq 0$
$\mathfrak{d}_{4,1/2}$	$Ie_4 = e_3, Ie_1 = e_2$	$a_1(e^{1,2} - e^{3,4})$, $a_1 \neq 0$
$\mathfrak{d}_{4,1/2}$	$Ie_4 = e_3, Ie_1 = -e_2$	$a_1(e^{1,2} - e^{3,4})$, $a_1 \neq 0$
$\mathfrak{d}'_{4,\delta}$	$Ie_4 = e_3, Ie_1 = e_2$	$a_1(e^{1,2} - \delta e^{3,4})$, $a_1 \neq 0, \delta > 0$
$\mathfrak{d}'_{4,\delta}$	$Ie_4 = -e_3, Ie_1 = e_2$	$a_1(e^{1,2} - \delta e^{3,4})$, $a_1 \neq 0, \delta > 0$
$\mathfrak{d}'_{4,\delta}$	$Ie_4 = -e_3, Ie_1 = -e_2$	$a_1(e^{1,2} - \delta e^{3,4})$, $a_1 \neq 0, \delta > 0$
$\mathfrak{d}'_{4,\delta}$	$Ie_4 = e_3, Ie_1 = -e_2$	$a_1(e^{1,2} - \delta e^{3,4})$, $a_1 \neq 0, \delta > 0$

Table 3.1: Classification of pseudo-Kähler Lie algebras in dimension 4 [41, Table 5.1]

From the computations in Table 3.3 we notice that the curvature tensors are of two types:

(i) $a^2H_1 + b^2H_2$ for $a, b \geq 0$;

(ii) $-a^2(\Omega_{\mathbb{P}^2_{\mathbb{C}}} + 6bH_2)$ for $a > 0$ and $b \in \{0, 1\}$.

Consider now the globally defined complex coframe $\theta^1 = u^1 + iu^2$, $\theta^2 = u^3 + iu^4$. If M has a projective special Kähler structure, thanks to Theorem 2.5.6, there is an S^1-bundle $\pi_S \colon S \to M$ and a suitable family of sections.

Case	$\mathfrak{g}$	I	ω	Conditions
I	$\mathfrak{rr}_{3,0}$	$Ie_1 = e_2, Ie_3 = e_4$	$a_1e^{1,2} + a_2e^{3,4}$	$a_1, a_2 > 0$
II	$\mathfrak{rr}'_{3,0}$	$Ie_1 = e_4, Ie_2 = e_3$	$a_1e^{1,4} + a_2e^{2,3}$	$a_1, a_2 > 0$
III	$\mathfrak{r}_2\mathfrak{r}_2$	$Ie_1 = e_2, Ie_3 = e_4$	$a_1e^{1,2} + a_2e^{3,4}$	$a_1, a_2 > 0$
IV	$\mathfrak{r}'_{4,0,\delta}$	$Ie_4 = e_1, Ie_2 = e_3$	$a_1e^{1,4} + a_2e^{2,3}$	$a_1 < 0; a_2, \delta > 0$
V	$\mathfrak{r}'_{4,0,\delta}$	$Ie_4 = e_1, Ie_2 = -e_3$	$a_1e^{1,4} + a_2e^{2,3}$	$a_1, a_2 < 0; \delta > 0$
VI	$\mathfrak{d}_{4,2}$	$Ie_4 = -2e_1, Ie_2 = e_3$	$a_1e^{1,4} + a_2e^{2,3}$	$a_1, a_2 > 0$
VII	$\mathfrak{d}_{4,1/2}$	$Ie_4 = e_3, Ie_1 = e_2$	$a_1(e^{1,2} - e^{3,4})$	$a_1 > 0$
VIII	$\mathfrak{d}'_{4,\delta}$	$Ie_4 = e_3, Ie_1 = e_2$	$a_1(e^{1,2} - \delta e^{3,4})$	$a_1, \delta > 0$
IX	$\mathfrak{d}'_{4,\delta}$	$Ie_4 = -e_3, Ie_1 = -e_2$	$a_1(e^{1,2} - \delta e^{3,4})$	$a_1 < 0; \delta > 0$

Table 3.2: Kähler Lie algebras of dimension 4

Choose in this family a section $s\colon U \to S$ with U containing the identity element of M. Without loss of generality, we can assume U simply connected. Let $\eta = \gamma \circ s$ which is a section of $\sharp_2 S_{3,0}U$, then applying $\flat_2$ we obtain a section σ of $S_{3,0}U$ which better displays the symmetry.

We write σ in its generic form with respect to θ:

$$\sigma = c_1(\theta^1)^3 + c_2(\theta^1)^2\theta^2 + c_3\theta^1(\theta^2)^2 + c_4(\theta^2)^3$$

for some functions $c_1, c_2, c_3, c_4 \in \mathcal{C}^\infty(U, \mathbb{C})$. By raising the second index, we obtain $\eta = \sharp_2\sigma$ which is

$$\begin{aligned}\eta = 2c_1\theta^1 \otimes \overline{\theta_1} \otimes \theta^1 + \frac{2c_2}{3}\left(\theta^1 \otimes \overline{\theta_1} \otimes \theta^2 + \theta^1 \otimes \overline{\theta_2} \otimes \theta^1 + \theta^2 \otimes \overline{\theta_1} \otimes \theta^1\right) \\ + \frac{2c_3}{3}\left(\theta^1 \otimes \overline{\theta_2} \otimes \theta^2 + \theta^2 \otimes \overline{\theta_1} \otimes \theta^2 + \theta^2 \otimes \overline{\theta_2} \otimes \theta^1\right) + 2c_4\theta^2 \otimes \overline{\theta_2} \otimes \theta^2.\end{aligned}$$

With respect to this generic section, we can compute $[\eta \wedge \overline{\eta}]$ explicitly:

$$\begin{aligned}[\eta \wedge \overline{\eta}] = \frac{4}{9}\mathfrak{R}\Bigg(&\overline{\theta^1} \wedge \theta^1 \otimes \begin{pmatrix} 9|c_1|^2 + |c_2|^2 & 3\overline{c_1}c_2 + \overline{c_2}c_3 \\ 3\overline{c_2}c_1 + \overline{c_3}c_2 & |c_2|^2 + |c_3|^2 \end{pmatrix} \\ &+ \overline{\theta^1} \wedge \theta^2 \otimes \begin{pmatrix} 3\overline{c_1}c_2 + \overline{c_2}c_3 & \overline{c_1}c_3 + \overline{c_2}c_4 \\ |c_2|^2 + |c_3|^2 & \overline{c_2}c_3 + 3\overline{c_3}c_4 \end{pmatrix} \\ &+ \overline{\theta^2} \wedge \theta^1 \otimes \begin{pmatrix} 3\overline{c_2}c_1 + \overline{c_3}c_2 & |c_2|^2 + |c_3|^2 \\ \overline{c_3}c_1 + \overline{c_4}c_2 & \overline{c_3}c_2 + 3\overline{c_4}c_3 \end{pmatrix} \\ &+ \overline{\theta^2} \wedge \theta^2 \otimes \begin{pmatrix} |c_2|^2 + |c_3|^2 & \overline{c_2}c_3 + \overline{c_3}c_4 \\ \overline{c_3}c_2 + 3\overline{c_4}c_3 & |c_3|^2 + 9|c_4|^2 \end{pmatrix}\Bigg).\end{aligned}$$

Case	$\mathfrak{g}$	Str. constants	Ω^{LC}
I	$\mathfrak{rr}_{3,0}$	$[u_1, u_2] = au_2$ $a > 0$	$a^2 H_1$
II	$\mathfrak{rr}'_{3,0}$	$[u_1, u_3] = -u_4$ $[u_1, u_4] = u_3$	0
III	$\mathfrak{r}_2\mathfrak{r}_2$	$[u_1, u_2] = au_2$ $[u_3, u_4] = bu_4$ $a, b > 0$	$a^2 H_1 + b^2 H_2$
IV	$\mathfrak{r}'_{4,0,\delta}$	$[u_1, u_2] = au_2$ $[u_1, u_3] = -\delta a u_4$ $[u_1, u_4] = \delta a u_3$ $a, \delta > 0$	$a^2 H_1$
V	$\mathfrak{r}'_{4,0,\delta}$	$[u_1, u_2] = au_2$ $[u_1, u_3] = \delta a u_4$ $[u_1, u_4] = -\delta a u_3$ $a, \delta > 0$	$a^2 H_1$
VI	$\mathfrak{d}_{4,2}$	$[u_1, u_2] = -2au_1$ $[u_1, u_3] = 2au_4$ $[u_2, u_3] = -au_3$ $[u_2, u_4] = au_4$ $a > 0$	$-a^2 \Omega_{\mathbb{P}^2_{\mathbb{C}}} - 6a^2 H_2$
VII	$\mathfrak{d}_{4,1/2}$	$[u_1, u_2] = 2au_4$ $[u_1, u_3] = -au_1$ $[u_2, u_3] = -au_2$ $[u_3, u_4] = 2au_4$ $a > 0$	$-a^2 \Omega_{\mathbb{P}^2_{\mathbb{C}}}$
VIII	$\mathfrak{d}'_{4,\delta}$	$[u_1, u_2] = 2a\sqrt{\delta} u_4$ $[u_1, u_3] = -a\sqrt{\delta} u_1 + \frac{2a}{\sqrt{\delta}} u_2$ $[u_2, u_3] = -\frac{2a}{\sqrt{\delta}} u_1 - a\sqrt{\delta} u_2$ $[u_3, u_4] = 2a\sqrt{\delta} u_4$ $a, \delta > 0$	$-\delta a^2 \Omega_{\mathbb{P}^2_{\mathbb{C}}}$
IX	$\mathfrak{d}'_{4,\delta}$	$[u_1, u_2] = -2a\sqrt{\delta} u_3$ $[u_1, u_4] = -a\sqrt{\delta} u_1 - \frac{2a}{\sqrt{\delta}} u_2$ $[u_2, u_4] = \frac{2a}{\sqrt{\delta}} u_1 - a\sqrt{\delta} u_2$ $[u_3, u_4] = -2a\sqrt{\delta} u_3$ $a, \delta > 0$	$-\delta a^2 \Omega_{\mathbb{P}^2_{\mathbb{C}}}$

Table 3.3: Curvature tensors

Notice that if we define $v_1, v_2, v_3 \in \mathcal{C}^\infty(U, \mathbb{C}^2)$ such that

$$v_1 := \begin{pmatrix} 2c_1 \\ \frac{2c_2}{3} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, v_2 := \begin{pmatrix} \frac{2c_2}{3} \\ \frac{2c_3}{3} \end{pmatrix} = \begin{pmatrix} y \\ z \end{pmatrix}, v_3 := \begin{pmatrix} \frac{2c_3}{3} \\ 2c_4 \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix}, \tag{3.1}$$

then we have

$$\begin{aligned}[][\eta \wedge \overline{\eta}] = \mathfrak{R}\Bigg(&\overline{\theta^1} \wedge \theta^1 \otimes \begin{pmatrix} \|v_1\|^2 & \langle v_1, v_2\rangle \\ \overline{\langle v_1, v_2\rangle} & \|v_2\|^2 \end{pmatrix} + \overline{\theta^1} \wedge \theta^2 \otimes \begin{pmatrix} \langle v_1, v_2\rangle & \langle v_1, v_3\rangle \\ \|v_2\|^2 & \langle v_2, v_3\rangle \end{pmatrix} \\ &+\overline{\theta^2} \wedge \theta^1 \otimes \begin{pmatrix} \overline{\langle v_1, v_2\rangle} & \|v_2\|^2 \\ \overline{\langle v_2, v_3\rangle} & \overline{\langle v_2, v_3\rangle} \end{pmatrix} + \overline{\theta^2} \wedge \theta^2 \otimes \begin{pmatrix} \|v_2\|^2 & \langle v_2, v_3\rangle \\ \overline{\langle v_2, v_3\rangle} & \|v_3\|^2 \end{pmatrix}\Bigg)\end{aligned}$$

In other words, the coefficients of $[\eta \wedge \overline{\eta}]$ are the pairwise Hermitian products of v_1, v_2, v_3.

Returning to the classification, if we write $H_1, H_2, \Omega_{\mathbb{P}^2_{\mathbb{C}}}$ with respect to the complex coframe, we notice that the positions corresponding to the mixed Hermitian products are always zero.

$$H_1 = \mathfrak{R}\left(\overline{\theta^1} \wedge \theta^1 \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}\right), \qquad H_2 = \mathfrak{R}\left(\overline{\theta^2} \wedge \theta^2 \otimes \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right)$$

$$\begin{aligned}\Omega_{\mathbb{P}^2_{\mathbb{C}}} = \mathfrak{R}\Bigg(&\overline{\theta^1} \wedge \theta^1 \otimes \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} + \overline{\theta^1} \wedge \theta^2 \otimes \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \\ &+\overline{\theta^2} \wedge \theta^1 \otimes \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \overline{\theta^2} \wedge \theta^2 \otimes \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}\Bigg)\end{aligned}$$

As a consequence, for all cases, if (2.13) holds, then v_1, v_2, v_3 must be orthogonal.

Now we will treat each case of possible curvature tensor separately.

(i) Let $a, b \geq 0$ and $\Omega^{LC} = a^2 H_1 + b^2 H_2$, then

$$\Omega^{LC} = \mathfrak{R}\left(\overline{\theta^1} \wedge \theta^1 \otimes \begin{pmatrix} \frac{a^2}{2} & 0 \\ 0 & 0 \end{pmatrix} + \overline{\theta^2} \wedge \theta^2 \otimes \begin{pmatrix} 0 & 0 \\ 0 & \frac{b^2}{2} \end{pmatrix}\right)$$

So, by (2.13), $[\eta \wedge \overline{\eta}] = -\Omega^{LC} - \Omega_{\mathbb{P}^2_{\mathbb{C}}}$, which implies

$$\|v_1\|^2 = 2 - \frac{a^2}{2}, \qquad \|v_2\|^2 = 1, \qquad \|v_3\|^2 = 2 - \frac{b^2}{2}.$$

These equalities translate to a linear system in the squared norms of x, y, z, w introduced in (3.1), namely

$$\begin{cases} |x|^2 + |y|^2 = 2 - \frac{a^2}{2} \\ |y|^2 + |z|^2 = 1 \\ |z|^2 + |w|^2 = 2 - \frac{b^2}{2} \end{cases}.$$

Its solutions are

$$\begin{cases} |x|^2 = 1 - \frac{a^2}{2} + s \\ |y|^2 = 1 - s \\ |z|^2 = s \\ |w|^2 = 2 - \frac{b^2}{2} - s \end{cases} \quad \text{for } s \in [0,1]. \tag{3.2}$$

Imposing the orthogonality conditions $\langle v_1, v_2 \rangle = \langle v_2, v_3 \rangle = \langle v_3, v_4 \rangle = 0$, we get:

$$\begin{cases} \overline{x}y + \overline{y}z = 0 \\ \overline{y}z + \overline{z}w = 0 \\ \overline{x}z + \overline{y}w = 0 \end{cases}. \tag{3.3}$$

Notice that because of (3.2), y and z cannot vanish simultaneously, so we have (at each point) three different cases:

- Suppose at first that $z = 0$, then $s = 0$ and $\|y\| = 1$, so $y \neq 0$ and (3.3) becomes

  $$\begin{cases} \overline{x}y = 0 \\ 0 = 0 \\ \overline{y}w = 0 \end{cases}$$

  Implying $x = w = 0$, so the solutions are $(x, y, z, w) = (0, y, 0, 0)$ for $y \in \mathcal{C}^\infty(U, S^1)$ if we identify S^1 with complex numbers with module 1. Notice that $y = e^{i\alpha}$ for some $\alpha \in \mathcal{C}^\infty(U)$ since we chose U simply connected. Explicitly, if ψ is the fundamental 1-form of S^1, then $y^*\psi$ is closed, and thus represents a class in the first de Rham cohomology group of U, but $H^1_{dR}(U) = 0$, so $y^*\psi = d\alpha$ for some $\alpha \in \mathcal{C}^\infty(U)$. Since ψ is locally represented

by the differential of the argument of the complex number, up to adding a constant to α, we can assume $y = e^{i\alpha}$. Thus we have $(c_1, c_2, c_3, c_4) = (0, \frac{3}{2}e^{i\alpha}, 0, 0)$ for some $\alpha \in \mathcal{C}^\infty(U)$.
Finally, (3.2) gives

$$\begin{cases} 1 - \frac{a^2}{2} = 0 \\ 2 - \frac{b^2}{2} = 0 \end{cases}$$

and thus $a = \sqrt{2}$ and $b = 2$.

- Suppose now that $z \neq 0$ and $y = 0$, then (3.3) becomes

$$\begin{cases} 0 = 0 \\ \overline{z}w = 0 \\ \overline{x}z = 0 \end{cases}$$

and then $w = x = 0$ so, similarly to the previous case, the solutions are $(c_1, c_2, c_3, c_4) = (0, 0, e^{i\alpha}, 0)$ for $\alpha \in \mathcal{C}^\infty(U)$ and this time, (3.2) implies $a = 2$ and $b = \sqrt{2}$.

- The remaining case has $z \neq 0$ and $y \neq 0$. In order to solve it, let us call $t := \overline{y}z \neq 0$, then (3.2) and (3.3) give

$$\begin{aligned} z &= \frac{ty}{|y|^2} = \frac{ty}{1-s} \\ x &= -\frac{\bar{t}y}{|y|^2} = -\frac{\bar{t}y}{1-s} \\ w &= -\frac{tz}{|z|^2} = -\frac{t^2y}{s(1-s)} \\ 0 = \overline{x}z + \overline{y}w &= \left(-\frac{t\overline{y}}{1-s}\right)\left(\frac{ty}{1-s}\right) + \overline{y}\left(-\frac{t^2y}{s(1-s)}\right) \\ &= -t^2\left(\frac{1}{1-s} + \frac{1}{s}\right) = -\frac{t^2}{s(1-s)}, \end{aligned}$$

in contradiction with $t \neq 0$.

In conclusion, for this class of curvature tensors, the only solutions are for

$$a = \sqrt{2}, \qquad b = 2, \qquad \sigma = \frac{3}{2}e^{i\alpha}(\theta^1)^2\theta^2 \qquad \text{for } \alpha \in \mathcal{C}^\infty(U)$$

and

$$a = 2, \qquad b = \sqrt{2}, \qquad \sigma = \frac{3}{2} e^{i\alpha} \theta^1 (\theta^2)^2 \qquad \text{for } \alpha \in \mathcal{C}^\infty(U).$$

We deduce that in Table 3.3 there are no solutions for cases I, II, IV, V, and the only solutions in case III are the ones mentioned before. Moreover, these solutions are isomorphic to one another and the isomorphism is obtained by swapping u_1 with u_3 and u_2 with u_4. The simply connected Lie group corresponding to this case is $\mathcal{H}_{\sqrt{2}} \times \mathcal{H}_2$.

(ii) Let now $a > 0, b \in \{0, 1\}$ and $\Omega^{LC} = -a^2(\Omega_{\mathbb{P}^2_{\mathbb{C}}} + 6bH_2)$, then

$$\begin{aligned}
[\eta \wedge \overline{\eta}] &= -\Omega^{LC} - \Omega_{\mathbb{P}^n_{\mathbb{C}}} = (a^2 - 1)\Omega_{\mathbb{P}^n_{\mathbb{C}}} + 6a^2 b H_2 \\
&= \Re \left(\overline{\theta^1} \wedge \theta^1 \otimes \begin{pmatrix} 2(1-a^2) & 0 \\ 0 & 1-a^2 \end{pmatrix} + \overline{\theta^1} \wedge \theta^2 \otimes \begin{pmatrix} 0 & 0 \\ 1-a^2 & 0 \end{pmatrix} \right. \\
&\quad \left. + \overline{\theta^2} \wedge \theta^1 \otimes \begin{pmatrix} 0 & 1-a^2 \\ 0 & 0 \end{pmatrix} + \overline{\theta^2} \wedge \theta^2 \otimes \begin{pmatrix} 1-a^2 & 0 \\ 0 & 2 - 2a^2 + 3a^2 b \end{pmatrix} \right).
\end{aligned}$$

Therefore we obtain the following equations

$$\|v_1\|^2 = 2 - 2a^2, \qquad \|v_2\|^2 = 1 - a^2, \qquad \|v_3\|^2 = 2 - 2a^2 + 3a^2 b.$$

Giving the conditions

$$\begin{cases} |x|^2 + |y|^2 = 2 - 2a^2 \\ |y|^2 + |z|^2 = 1 - a^2 \\ |z|^2 + |w|^2 = 2 - 2a^2 + 3a^2 b \end{cases}$$

with solutions

$$\begin{cases} |x|^2 = 1 - a^2 + s \\ |y|^2 = 1 - a^2 - s \\ |z|^2 = s \\ |w|^2 = 2 - 2a^2 + 3a^2 b - s \end{cases} \qquad \text{for } s \in [0, 1 - a^2] \tag{3.4}$$

We now impose the vanishing of $\langle v_1, v_2 \rangle$, $\langle v_2, v_3 \rangle$, $\langle v_3, v_4 \rangle$, that is (3.3).

We have four different cases:

- Suppose at first that $y = z = 0$, then $s = 0$ and $a = 1$, so (3.3) is always satisfied, while (3.4) becomes

$$\begin{cases} |x|^2 = 0 \\ |y|^2 = 0 \\ |z|^2 = 0 \\ |w|^2 = 3b \end{cases}$$

  that has solutions $(x, y, z, w) = (0, 0, 0, \sqrt{3b}e^{i\alpha})$ for $\alpha \in \mathcal{C}^\infty(U)$ and thus $(c_1, c_2, c_3, c_4) = (0, 0, 0, \frac{\sqrt{3b}}{2}e^{i\alpha})$. In conclusion, $a = 1$ and $\sigma = \frac{\sqrt{3b}}{2}e^{i\alpha}(\theta_2)^3$.

- Suppose now that $z = 0$ but $y \neq 0$, then $s = 0$ and $a^2 - 1 \neq 0$. The system (3.3) implies $x = w = 0$, but then by (3.4), $0 = |x|^2 = 1 - a^2 \neq 0$, so in this case there are no solutions.

- Analogously, if $z \neq 0$ but $y = 0$, then $s = 1 - a^2$ and (3.3) gives $w = x = 0$, so from (3.4) we get $0 = |x|^2 = 2 - 2a^2 = 2|z|^2 \neq 0$ leaving no solutions.

- The remaining case has $z \neq 0$ and $y \neq 0$. In order to solve it, let us call $t := \overline{y}z \neq 0$, then (3.4) and (3.3) give

$$\begin{aligned} z &= \frac{ty}{|y|^2} = \frac{ty}{1 - a^2 - s} \\ x &= -\frac{\bar{t}y}{|y|^2} = -\frac{\bar{t}y}{1 - a^2 - s} \\ w &= -\frac{tz}{|z|^2} = -\frac{t^2 y}{s(1 - a^2 - s)} \\ 0 &= \overline{x}z + \overline{y}w \\ &= \left(-\frac{t\overline{y}}{1 - a^2 - s}\right)\left(\frac{ty}{1 - a^2 - s}\right) + \overline{y}\left(\frac{-t^2 y}{s(1 - a^2 - s)}\right) \\ &= -t^2\left(\frac{1}{1 - a^2 - s} + \frac{1}{s}\right) = -\frac{t^2(1 - a^2)}{s(1 - a^2 - s)} \end{aligned}$$

  The latter implies $a = 1$, and from (3.4) we deduce a contradiction: $0 < |y|^2 = -s < 0$.

In conclusion, the only solutions for this type of curvature tensors are obtained for

$$a = 1, b = 0, \qquad\qquad \sigma = 0;$$

and

$$a = 1, b = 1 \qquad \sigma = \frac{\sqrt{3}}{2} e^{i\alpha} (\theta^2)^3 \qquad \text{for } \alpha \in \mathcal{C}^\infty(U).$$

In Table 3.3, these results correspond to the cases: VI for $a = 1$ and $\sigma = \frac{\sqrt{3}}{2} e^{i\alpha}(\theta^2)^3$ for $\alpha \in \mathcal{C}^\infty(U)$; VII for $a = 1$ and $\sigma = 0$; VIII and IX for $a = \frac{1}{\sqrt{\delta}}$, $\delta > 0$ and $\sigma = 0$.

Table 3.4 summarises (up to isomorphisms) the cases satisfying the curvature condition, showing the non-vanishing differentials of the coframe and the Levi-Civita connection. We know that these Lie groups are all solvable,

Case	Structure constants	ω^{LC}	PSK
III	$du^2 = -\sqrt{2} u^{1,2}$ $du^4 = -2u^{3,4}$	$\left(\begin{array}{cc\|cc} & \sqrt{2}u^2 & & \\ -\sqrt{2}u^2 & & & \\ \hline & & & 2u^4 \\ & & -2u^4 & \end{array}\right)$	✓
VI	$du^1 = 2u^{1,2}$ $du^3 = u^{2,3}$ $du^4 = -2u^{1,3} - u^{2,4}$	$\left(\begin{array}{cc\|cc} 0 & -2u^1 & u^4 & u^3 \\ 2u^1 & 0 & -u^3 & u^4 \\ \hline -u^4 & u^3 & 0 & -u^1 \\ -u^3 & -u^4 & u^1 & 0 \end{array}\right)$	
VII	$du^1 = u^{1,3}$ $du^2 = u^{2,3}$ $du^4 = -2u^{1,2} - 2u^{3,4}$	$\left(\begin{array}{cc\|cc} 0 & u^4 & -u^1 & u^2 \\ -u^4 & 0 & -u^2 & -u^1 \\ \hline u^1 & u^3 & 0 & 2u^4 \\ -u^2 & -u^4 & -2u^4 & 0 \end{array}\right)$	✓
VIII	$du^1 = u^{1,3} + \frac{2}{\delta} u^{2,3}$ $du^2 = -\frac{2}{\delta} u^{1,3} + u^{2,3}$ $du^4 = -2u^{1,2} - 2u^{3,4}$ $\delta > 0$	$\left(\begin{array}{cc\|cc} 0 & \frac{2}{\delta}u^3 + u^4 & -u^1 & u^2 \\ -\frac{2}{\delta}u^3 - u^4 & 0 & -u^2 & -u^1 \\ \hline u^1 & u^2 & 0 & 2u^4 \\ -u^2 & u^1 & -2u^4 & 0 \end{array}\right)$	✓
IX	$du^1 = u^{1,4} - \frac{2}{\delta} u^{2,4}$ $du^2 = \frac{2}{\delta} u^{1,4} + u^{2,4}$ $du^3 = 2u^{1,2} + 2u^{3,4}$ $\delta > 0$	$\left(\begin{array}{cc\|cc} 0 & -\frac{2}{\delta}u^4 - u^3 & -u^2 & -u^1 \\ \frac{2}{\delta}u^4 + u^3 & 0 & u^1 & -u^2 \\ \hline u^2 & -u^1 & 0 & -2u^3 \\ u^1 & u^2 & 2u^3 & 0 \end{array}\right)$	✓

Table 3.4: Cases satisfying the curvature condition

and this implies that they are the product of a torus (product of circumference) and a euclidean space [11, Theorem 2ª, p.675]. In particular, they have trivial second cohomology group. We can thus apply directly Corollary 2.5.10, assuring that the previously found σ and α are actually global. Notice that for cases III, VII, VIII, IX, the Kähler form is exact with invariant potentials; respectively $-\frac{1}{\sqrt{2}}u^2 - \frac{1}{2}u^4$, $-\frac{1}{2}u^4$, $-\frac{1}{2}u^4$, $\frac{1}{2}u^3$.

Now we must check whether condition **D.2** holds for the cases left. We can immediately say that cases VII, VIII, IX are all projective special Kähler because $\sigma = 0$, and thus the differential condition is trivially satisfied.

Concerning case III, we can compute $d^{LC}\sigma$ by understanding how the Levi-Civita connection behaves on the unitary complex coframe θ.

$$\begin{aligned}\nabla^{LC}\theta^1 &= \nabla^{LC}u^1 + i\nabla^{LC}u^2 = -(\omega^{LC})^1_k \otimes u^k - i(\omega^{LC})^2_k \otimes u^k \\ &= -\sqrt{2}u^2 \otimes u^2 + i\sqrt{2}u^2 \otimes u^1 = \sqrt{2}iu^2 \otimes \theta^1;\end{aligned}$$

$$\begin{aligned}\nabla^{LC}\theta^2 &= \nabla^{LC}u^3 + i\nabla^{LC}u^4 = -(\omega^{LC})^3_k \otimes u^k - i(\omega^{LC})^4_k \otimes u^k \\ &= -2u^4 \otimes u^4 + i2u^4 \otimes u^3 = 2iu^4 \otimes \theta^2.\end{aligned}$$

Now we can compute

$$\begin{aligned}\nabla^{LC}\sigma &= \nabla^{LC}\left(\frac{3}{2}e^{i\alpha}(\theta^1)^2\theta^2\right) \\ &= \frac{3}{2}id\alpha \otimes e^{i\alpha}(\theta^1)^2\theta^2 + 3\sqrt{2}iu^2 e^{i\alpha}(\theta^1)^2\theta^2 + \frac{3}{2}2iu^4 \otimes e^{i\alpha}(\theta^1)^2\theta^2 \\ &= -4i\left(-\frac{1}{4}d\alpha - \frac{1}{\sqrt{2}}u^2 - \frac{1}{2}u^4\right) \otimes \sigma.\end{aligned}$$

If we define $\lambda := -\frac{1}{4}d\alpha - \frac{1}{\sqrt{2}}u^2 - \frac{1}{2}u^4$, we have that $d\lambda = \omega$ and $d^{LC}\sigma = -4i\lambda \wedge \sigma$. Thanks to Corollary 2.5.10, we have proven that also case III has a projective special Kähler structure for every choice of $\alpha \in \mathcal{C}^\infty(M)$.

Suppose that VI is projective special Kähler, than by Theorem 2.5.6, locally we must have the differential condition **D.2**. Consider the unitary global complex coframe θ.

$$\begin{aligned}\nabla^{LC}\theta^2 &= \nabla^{LC}u^3 + i\nabla^{LC}u^4 \\ &= u^4 \otimes u^1 - u^3 \otimes u^2 + u^1 \otimes u^4 + i(u^3 \otimes u^1 + u^4 \otimes u^2 - u^1 \otimes u^3) \\ &= u^4 \otimes \theta^1 + iu^3 \otimes \theta^1 - iu^1 \otimes \theta^2 = i\overline{\theta^2} \otimes \theta^1 - iu^1 \otimes \theta^2.\end{aligned}$$

Thus

$$\nabla^{LC}\sigma = \nabla^{LC}\left(\frac{\sqrt{3}}{2}e^{i\alpha}(\theta^2)^3\right)$$

$$
\begin{aligned}
&= id\alpha \otimes \frac{\sqrt{3}}{2} e^{i\alpha} (\theta^2)^3 + 3\frac{\sqrt{3}}{2} e^{i\alpha} (\nabla^{LC}\theta^2)(\theta^2)^2 \\
&= id\alpha \otimes \sigma + 3\frac{\sqrt{3}}{2} e^{i\alpha} (i\overline{\theta^2} \otimes \theta^1 - iu^1 \otimes \theta^2)(\theta^2)^2 \\
&= i(d\alpha - 3u^1) \otimes \sigma + 3i\overline{\theta^2} \otimes \frac{\sqrt{3}}{2} e^{i\alpha} \theta^1 (\theta^2)^2;
\end{aligned}
$$

$$
d^{LC}\sigma = i(d\alpha - 3u^1) \wedge \sigma + 3i\overline{\theta^2} \wedge \frac{\sqrt{3}}{2} e^{i\alpha} \theta^1 (\theta^2)^2.
$$

Notice that this is never of the form required by condition **D.2** for any available choice of σ, since evaluating the last component at θ_1, we obtain $i\frac{\sqrt{3}}{2}\overline{\theta^2} \wedge \theta^2 \otimes \theta^2$ whereas the same operation on a form of type $i\tau \wedge \sigma$ would evaluate to zero. We deduce that VI does not admit a projective special Kähler structure.

We are now left with cases III, VII, VIII, IX. At the level of Lie groups, case III corresponds to the connected simply connected Lie group $\mathcal{H}_{\sqrt{2}} \times \mathcal{H}_2$ with $\sigma = \frac{3}{2}(\theta^1)^2\theta^2$ up to isomorphism. The other deviances are in fact obtained by taking $e^{i\alpha}\sigma$ and thus we are in the situation of Proposition 2.5.12. The Lie groups corresponding to the cases VII, VIII and IX, are in particular homogeneous, and they all have zero deviance, so by Proposition 2.6.8 we deduce that they are all isomorphic to $\mathcal{H}^2_{\mathbb{C}}$ as projective special Kähler manifolds. □

Remark 3.1.3. *Notice that case VII coincides with the description of $\mathcal{H}^2_{\mathbb{C}}$ given in Proposition 2.6.10.*

Remark 3.1.4. *It is striking that in case III, which is obtained via the r-map from the polynomial x^2y, the deviance is a global tensor which is a multiple of this polynomial with respect to a Kähler holomorphic coframe.*

It turns out that all 4-dimensional projective special Kähler Lie groups are simply connected, so this theorem already presents all possible cases.

Proposition 3.1.5. *Let $(\pi\colon \widetilde{M} \to M, \nabla)$ be a projective special Kähler manifold, then the universal cover $p\colon U \to M$ admits a projective special Kähler structure. In particular, if $\gamma\colon S \to \sharp_2 S_{3,0}M$ is the intrinsic deviance for M, then $p^*S \to U$ is an S^1-bundle and if we call p' the canonical map $p^*S \to S$, then U has deviance $p^* \circ \gamma \circ p'\colon p^*S \to \sharp_2 S_{3,0}U$ on U.*

If M is a projective special Kähler Lie group, then so is U.

Proof. Since $p\colon U \to M$ is a cover, we can lift the whole Kähler structure of M to U by pullback $(U, p^*g, p^*I, p^*\omega)$ (the pullback of I makes sense, since p is a local diffeomorphism). We will now use Theorem 2.5.6. The S^1-bundle S lifts to an S^1-bundle $\pi_{p^*S}\colon p^*S \to U$, where the right action can be defined locally, since p is a local diffeomorphism. The principal connection φ on S lifts to $\varphi' = p'^*\varphi$ and its curvature is, as expected, $d\varphi' = p'^*d\varphi = -2p'\pi_S^*\omega = -2\pi_{p^*S}^*p^*\omega$. Let $\gamma' = p^* \circ \gamma \circ p'\colon p^*S \to \sharp_2 S_{3,0}U$, then $\gamma'(ua) = a^2\gamma'(u)$ holds, as the action is defined on the fibres, which are preserved by the pullback. The remaining properties also follow from the fact p is a local diffeomorphism.

Finally, if M is a Lie group with left invariant Kähler structure, then U is a Lie group and its Kähler structure is also left invariant. □

Given a universal cover $p\colon U \to M$ of a projective special Kähler Lie group, $\ker(p)$ is a discrete subgroup and when M is connected, $\ker(p)$ is in the center $Z(U)$ of U. From this observation we obtain the following corollary

Corollary 3.1.6. *A connected 4-dimensional projective special Kähler Lie group is isomorphic to one of the following:*

- $\mathcal{H}_{\sqrt{2}} \times \mathcal{H}_2$ *with deviance* $\flat_2(\frac{3}{2}(\theta^1)^2\theta^2)$ *in the standard complex unitary coframe* θ*;*
- *complex hyperbolic* n*-space with zero deviance.*

Proof. The proof follows from Theorem 3.1.2 with Proposition 3.1.5, as a connected group M with universal cover $p\colon U \to M$ is isomorphic to $U/\ker(p)$ and, if M is a projective special Kähler Lie group, so is U by Proposition 3.1.5. Since U is also simply connected, Theorem 3.1.2 provides all the possibilities up to isomorphisms preserving the Lie structure. Therefore, it suffices to show that these possibilities for U have trivial centre.

Starting with $\mathcal{H}_{\sqrt{2}} \times \mathcal{H}_2$, its centre is $Z(\mathcal{H}_{\sqrt{2}}) \times Z(\mathcal{H}_2)$, so it is enough to prove that $Z(\mathcal{H}_\lambda)$ is trivial and, up to scaling, we can assume $\lambda = 1$. This Lie group is isomorphic to the following Lie group of orientation preserving affinities on the 1-dimensional real affine space, that is

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a \in \mathbb{R}^+, b \in \mathbb{R} \right\}$$

A generic matrix of this form $X = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ is in the center if and only if for every other orientation preserving affinity A, $AX = XA$, that is if and only if, for all $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$ we have $xb + y = ay + b$, that is if and only if $x = 1$ and $y = 0$. Therefore $X = I_2$, so the centre is trivial.

For the complex holomorphic 2-space we have three Lie groups, corresponding to cases VII, VIII and IX in Table 3.3. Notice that cases VIII and IX have isomorphic Lie algebras, so it is enough to study the cases VII and VIII.

In both cases, the derived Lie algebra is $\langle u_1, u_2, u_4\rangle_{\mathbb{R}}$ and the only non-vanishing bracket on the elements of the basis is $[u_1, u_2] = 2u_4$, so in both cases, the derived Lie algebra is isomorphic to the Heisenberg Lie algebra $\mathfrak{h}_1$ of dimension 3. For each case, the element u_3 acts on this Lie algebra as the following derivations represented in the basis u_1, u_2, u_4

$$\varphi_{\mathrm{VII}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad \varphi_{\mathrm{VIII}} = \begin{pmatrix} 1 & \frac{2}{\delta} & 0 \\ -\frac{2}{\delta} & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

We can then see both Lie algebras as a semidirect product; namely, case VII is isomorphic to $\mathfrak{h}_1 \rtimes_{\varphi_{\mathrm{VII}}} \mathbb{R}$ and case VIII to $\mathfrak{h}_1 \rtimes_{\varphi_{\mathrm{VIII}}} \mathbb{R}$. The Lie algebra corresponding to the semidirect product of two Lie groups is the semidirect product of the associated Lie algebras. From the Lie groups-Lie algebras correspondence, we can see both groups as semidirect products $H_1 \rtimes \mathbb{R}^+$, described by different automorphisms $\tau_{VII}, \tau_{VIII} \colon \mathbb{R}^+ \to \mathrm{Aut}(H_1)$.

Given a semidirect product $H_1 \rtimes_\tau \mathbb{R}^+$, the multiplication is defined as

$$(x_1, y_1)(x_2, y_2) = (x_1\tau(y_1)(x_2), y_1y_2).$$

Since $\mathbb{R}^+$ is abelian, two such elements commute if and only if $x_1\tau(y_1)(x_2) = x_2\tau(y_2)(x_1)$. An element (x, y) is then in the center if and only if

$$x\tau(y)(u) = u\tau(v)(x) \tag{3.5}$$

for all $(u, v) \in H_1 \rtimes \mathbb{R}^+$. In particular, for $u = 1$ we get

$$\tau(v)(x) = x, \tag{3.6}$$

so x is a fixed point for every $\tau(v)$. By applying (3.6) to (3.5), we obtain that $\tau(y)$ must be the conjugation by x.

In order to compute τ_{VII} and τ_{VIII}, we first need to consider the following automorphisms of $\mathfrak{h}_1$

$$\exp(t\varphi_{\mathrm{VII}}) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}, \quad \exp(t\varphi_{\mathrm{VIII}}) = \begin{pmatrix} e^t\cos(t\frac{\delta}{2}) & e^t\sin(t\frac{\delta}{2}) & 0 \\ -e^t\sin(t\frac{\delta}{2}) & e^t\cos(t\frac{\delta}{2}) & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}.$$

For both cases, the differential at the identity of $\tau_k(e^t)$ is $\exp(t\varphi_k)$, and since $\exp\colon \mathfrak{h}_1 \to H_1$ is invertible with inverse log, we can compute $\tau_k(e^t) = \exp \circ \exp(t\varphi_k) \circ \log$.

Explicitly, given a generic $x = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in H_1$,

$$\log(x) = \begin{pmatrix} 0 & a & c - \frac{ab}{2} \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = au_1 + bu_2 + \frac{1}{2}\left(c - \frac{ab}{2}\right)u_4.$$

We analyze the two cases:

VII) If we apply $\exp(t\varphi_{\mathrm{VII}})$ it becomes $e^t au_1 + e^t bu_2 + \frac{e^{2t}}{2}(c - \frac{ab}{2})u_4$ which exponentiates to $\begin{pmatrix} 1 & e^t a & e^{2t}c \\ 0 & 1 & e^t b \\ 0 & 0 & 1 \end{pmatrix}$, which equals x for all t if and only if $a = b = c = 0$, implying that the centre of case VII is trivial.

VIII) If we apply $\exp(t\varphi_{\mathrm{VIII}})$ it becomes

$$e^t(a\cos(t\frac{\delta}{2}) + b\sin(t\frac{\delta}{2}))u_1 + e^t(b\cos(t\frac{\delta}{2}) - a\sin(t\frac{\delta}{2}))u_2 + \frac{e^{2t}}{2}(b - \frac{ac}{2})u_4.$$

If we exponentiate it, it is equal to x if and only if

$$\begin{cases} e^t\left(a\cos\left(\frac{t\delta}{2}\right) + b\sin\left(\frac{t\delta}{2}\right)\right) = a \\ e^t\left(b\cos\left(\frac{t\delta}{2}\right) - a\sin\left(\frac{t\delta}{2}\right)\right) = b \\ e^{2t}\left(c + \frac{ab}{2} - \cos(t\delta) + \frac{b^2-a^2}{4}\sin(t\delta)\right) = c \end{cases}.$$

If this holds for all t, then in particular if we take the limit of this system for $t \to -\infty$, the equations must still be satisfied, but this implies $a = b = c = 0$.

In either case the centre is trivial, ending the proof. □

3.2 Deviance in the examples

We now compute the deviance in the projective special Kähler manifolds appearing in the examples of Section 1.6.

3.2.1 Example $\mathcal{H}_{\frac{2}{\sqrt{3}}}$

This is the projective special Kähler manifold used in Example 1.6.1, that is the hyperbolic plane with curvature $-\frac{4}{3}$. We use the coframe (1.23) restricted to M, that is

$$e^1 = \frac{\sqrt{3}}{2y}dx \qquad\qquad e^2 = \frac{\sqrt{3}}{2y}dy$$

with structure constants

$$de^1 = \frac{2}{\sqrt{3}}e^{1,2} \qquad\qquad de^2 = 0.$$

We have a global complex coordinate chart $z = x + iy \colon M \to \mathbb{C}$ and we can consider the complex coframe $\theta^1 = e^1 + ie^2 = \frac{\sqrt{3}}{2y}(dx + idy) = \frac{\sqrt{3}}{2\operatorname{Im}(z)}dz$ in $\Lambda_{1,0}M$. The Levi-Civita connection form is then

$$\omega^{LC} = \begin{pmatrix} 0 & -\frac{2}{\sqrt{3}}e^1 \\ \frac{2}{\sqrt{3}}e^1 & 0 \end{pmatrix} = \mathfrak{R}\left(-\frac{i}{\sqrt{3}}(\theta^1 + \overline{\theta^1}) \otimes \overline{\theta_1} \otimes \theta^1\right),$$

and the curvature form

$$\Omega^{LC} = \begin{pmatrix} 0 & -\frac{4}{3}e^{1,2} \\ \frac{4}{3}e^{1,2} & 0 \end{pmatrix} = \mathfrak{R}\left(\frac{2}{3}(\overline{\theta^1} \wedge \theta^1) \otimes \theta_1 \otimes \theta^1\right).$$

In particular, the Ricci tensor and scalar curvature are

$$\operatorname{Ric}_M = -\frac{4}{3}g = -\frac{4}{3}\overline{\theta^1}\theta^1, \qquad\qquad \operatorname{scal}_M = -\frac{4}{3}.$$

The manifold M is contractible, so we can apply Corollary 2.5.10 which tells us that there is a global η section of $\sharp_2 S_{3,0}M$. In particular its Kähler form is exact: $e^{1,2} = d\left(\frac{\sqrt{3}}{2}e^1\right)$. Since M has complex dimension 1, the bundle $S_{3,0}M$ has complex dimension 1 and is generated by $(\theta^1)^3$, so there exists some $c \in \mathcal{C}^\infty(M, \mathbb{C})$ such that

$$\eta = \sharp_2(c\theta^1)^3 = 2c\theta^1 \otimes \overline{\theta_1} \otimes \theta^1.$$

From Proposition 2.5.3 we can compute the norm of c, in fact from (2.15) we have

$$8|c|^2 = 2\,\|\eta\|_h^2 = \operatorname{scal}_M + 4 = -\frac{4}{3} + 4 = \frac{8}{3},$$

implying $c = \frac{e^{i\alpha}}{\sqrt{3}}$ for some $\alpha \in \mathcal{C}^\infty(M)$. Without loss of generality, we can assume $\alpha = 0$ as a consequence of Proposition 2.5.12, so a possible deviance is

$$\eta = \frac{2}{\sqrt{3}}\theta^1 \otimes \overline{\theta_1} \otimes \theta^1.$$

The corresponding symmetric tensor obtained by lowering the second index is

$$\flat_2\eta = \frac{1}{\sqrt{3}}(\theta^1)^3.$$

Notice that we have found another example of projective special Kähler Lie group, different from $\mathcal{H}^1_{\mathbb{C}} \cong \mathcal{H}_2$ which is projective special Kähler by Proposition 2.6.6. These examples already appear in [37, Section 6], where it is also proven that these are the only $\mathcal{H}_\lambda$ with a projective special Kähler structure. Moreover, we can now state the following:

Proposition 3.2.1. *Up to isomorphisms, there are only two connected projective special Kähler Lie groups in dimension* 2*, namely* $\mathcal{H}_2 \cong \mathcal{H}^n_{\mathbb{C}}$ *and* $\mathcal{H}_{\frac{2}{\sqrt{3}}}$ *with deviances respectively* 0 *and* $\sharp_2\frac{1}{\sqrt{3}}(\theta^1)^3$ *with respect to the standard unitary coframe.*

Proof. We have already seen that the two examples are projective special Kähler manifolds, so we are left to prove that they are the only ones. Consider a 2-dimensional projective special Kähler Lie group (M, ∇). In the non-abelian case, the Lie algebra of M is isometrically isomorphic to the one of $\mathcal{H}_\lambda$ for a suitable λ. In fact, the center of $\mathcal{H}_\lambda$ is trivial (see e.g. the proof of Corollary 3.1.6), so $\mathcal{H}_\lambda$ is the only complete connected Lie group with this Lie algebra.

The case of the hyperbolic plane is treated in [37, Section 6] leaving only the cases in the statement.

It remains to show that the abelian case does not occur. If M is abelian it must be flat, so at any point p, we can always find a contractible neighbourhood U where we can define a local unitary coframe e such that $\nabla^{LC}e^1 = \nabla^{LC}e^2 = 0$. Consider U as a projective special Kähler manifold; being contractible, we can apply Corollary 2.5.10 on U. In particular, $\omega|_U = d\lambda$ for some $\lambda \in \Omega^1(U)$. We define $\theta^1 := e^1 + ie^2$ and since $S_{3,0}M$ has complex rank 1, the deviance must necessarily be $\eta = f\sharp_2((\theta^1)^3)$ for some $f \in \mathcal{C}^\infty(U)$. Since the scalar curvature of M is 0, by (2.15) we deduce that $\|\eta\| = n(n+1)$, so $f = n(n+1)e^{i\alpha}$ for $\alpha \in \mathcal{C}^\infty(U)$. We can now compute

$$d^{LC}\eta = d^{LC}(f\sharp_2((\theta^1)^3)) = df \wedge \sharp_2(\theta^1)^3 + fd^{LC}\sharp_2(\theta^1)^3$$

$$= in(n+1)e^{i\alpha}d\alpha \wedge \sharp_2(\theta^1)^3 + f(d\theta^1 \otimes 2\overline{\theta_1} \otimes \theta^1 - \theta^1 \otimes \nabla^{LC}(2\overline{\theta_1} \otimes \theta^1))$$

$$= id\alpha \wedge \eta + 0$$

If (2.18) holds, then $i(d\alpha - 4\lambda) \wedge \eta = 0$, so in particular $(d\alpha - 4\lambda) \wedge \theta^1 = 0$. Notice that the operation $\cdot \wedge \theta^1 : \Omega^1(U, \mathbb{C}) \to \Omega^2(U, \mathbb{C})$ is injective, as $(a_1 e^1 + a_2 e^2) \wedge \theta^1 = (ia_1 - a_2)e^{1,2}$ vanishes if and only if $a_1 = a_2 = 0$. It follows that $d\alpha = 4\lambda$, but then $\omega|_U = d\lambda = 0$, in contradiction with the non-degeneracy of ω. Therefore, the abelian case does not occur. □

3.2.2 Example $\mathcal{H}_{\sqrt{2}} \times \mathcal{H}_2$

This second projective special Kähler manifold is the one used in Example 1.6.2. We use the coframe (1.26) restricted to M where we invert e^2 and e^3, that is

$$e^1 = \frac{1}{\sqrt{2}y_1}dx_1 \qquad e^2 = \frac{1}{\sqrt{2}y_1}dy_1 \qquad e^3 = \frac{1}{2y_2}dx_2 \qquad e^4 = \frac{1}{2y_2}dy_2$$

with structure constants

$$de^1 = \sqrt{2}e^{1,2} \qquad de^2 = 0 \qquad de^3 = 2e^{3,4} \qquad de^4 = 0.$$

We follow the construction adopted for the previous example, so we find a global complex coordinate chart $(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \colon M \to \mathbb{C}^2$ and the complex coframe $\theta = (\theta^1, \theta^2)$ defined as

$$\begin{aligned} \theta^1 &= e^1 + ie^2 = \frac{1}{\sqrt{2}y_1}(dx_1 + idy_1) = \frac{1}{\sqrt{2}\operatorname{Im}(z_1)}dz_1; \\ \theta^2 &= e^3 + ie^4 = \frac{1}{2y_2}(dx_2 + idy_2) = \frac{1}{2\operatorname{Im}(z_2)}dz_2. \end{aligned}$$

The Levi-Civita connection is then

$$\omega^{LC} = \left(\begin{array}{cc|cc} 0 & -\sqrt{2}e^1 & & \\ \sqrt{2}e^1 & 0 & & \\ \hline & & 0 & -2e^3 \\ & & 2e^3 & 0 \end{array}\right)$$

$$= \Re\left(-\frac{i}{\sqrt{2}}(\theta^1 + \overline{\theta^1}) \otimes \overline{\theta_1} \otimes \theta^1 - i(\theta^2 + \overline{\theta^2}) \otimes \overline{\theta_2} \otimes \theta^2\right),$$

and the curvature form

$$\Omega^{LC} = \left(\begin{array}{cc|cc} 0 & -2e^{1,2} & & \\ 2e^{1,2} & 0 & & \\ \hline & & 0 & -4e^{3,4} \\ & & 4e^{3,4} & 0 \end{array}\right)$$
$$= \Re\left((\overline{\theta^1} \wedge \theta^1) \otimes \overline{\theta_1} \otimes \theta^1 + 2(\overline{\theta^2} \wedge \theta^2) \otimes \overline{\theta_2} \otimes \theta^2\right).$$

The manifold M is isomorphic as a metric Lie group to $\mathcal{H}_{\sqrt{2}} \times \mathcal{H}_2$, so we are in case III of table 3.4. More specifically, the two coframes are one opposite to the other, so the deviance can be chosen to be

$$\eta = \sharp_2\left(\frac{3}{2}(\theta^1)^2\theta^2\right) = \frac{1}{2}(\theta^1 \otimes \overline{\theta_1} \otimes \theta^2 + \theta^1 \otimes \overline{\theta_2} \otimes \theta^1 + \theta^2 \otimes \overline{\theta_1} \otimes \theta^1).$$

3.3 Coframe lift through the c-map

In the work by Macia and Swann [38], published as preprint while this thesis was being written, it is shown that applying the c-map to a projective special Kähler Lie groups results in a Lie group with a left invariant quaternion Kähler structure. In this section, we prove the result in a special case by applying the intrinsic construction of the c-map described in Section 1.4.1 to the case of the complex hyperbolic n-space. We will first explore the construction in the general case with zero deviance, i.e. for any projective special Kähler Lie group isomorphic to $\mathcal{H}^n_{\mathbb{C}}$ as a projective special Kähler manifold, then apply it to case VII in the classification of Section 3.1, obtaining the Wolf space $\mathrm{SU}(4,2)/S(\mathrm{U}(4)\mathrm{U}(2))$.

Complex Hyperbolic n-space

We start from a projective special Kähler Lie group M of dimension $2n$ with principal S^1-bundle $\pi_S\colon S \to M$, connection $\varphi \in \Omega^1(S)$ and zero deviance. Then by Proposition 2.6.8, necessarily $M \cong \mathcal{H}^n_{\mathbb{C}}$. Suppose e is a unitary left invariant global coframe on M, so orthogonal and such that $Ie^{2k-1} = e^{2k}$ for all $k = 1, \dots n$. Call $c^h_{k,j} \in \mathbb{R}$ its structure constants, that is

$$de^h = c^h_{k,j}e^{k,j}, \qquad c^h_{k,j} \in \mathbb{R}.$$

Consider the principal bundle $\widetilde{M} := S \times \mathbb{R}^+$ with projection to M defined by $\pi := \pi_S \pi_1$ where $\pi_1 \colon S \times \mathbb{R}^+ \to S$ is the projection on the first component.

Let $r : \mathbb{R}^+ \to \mathbb{R}$ be the inclusion; we can extend e to a coframe $\widetilde{e}$ on $\widetilde{M} = S \times \mathbb{R}^+$ by taking for $k = 1, \dots, 2n$:

$$\widetilde{e}^k = r\pi^* e^2, \qquad \widetilde{e}^{2n+1} = dr, \qquad \widetilde{e}^{2n+2} = r\pi_1^* \varphi.$$

Notice that the differentials in the new coframe become

$$\begin{aligned}
d\widetilde{e}^k &= dr \wedge \pi^* e^k + r\pi^* de^k = \frac{1}{r}\widetilde{e}^{2n+1} \wedge \widetilde{e}^k + r\pi^*(c^k_{j,h} e^{j,h}) \\
&= -\frac{1}{r}\widetilde{e}^{k,2n+1} + r c^k_{j,h} \pi^* e^{j,h} = \frac{c^k_{j,h}}{r}\widetilde{e}^{j,h} - \frac{1}{r}\widetilde{e}^{k,2n+1}; \\
d\widetilde{e}^{2n+1} &= d^2 r = 0; \\
d\widetilde{e}^{2n+2} &= dr \wedge \pi_1^* \varphi + r\pi_1^* d\varphi = \widetilde{e}^{2n+1} \wedge \pi_1^* \varphi - 2r\pi^* \omega \\
&= -\frac{2}{r}\sum_{k=1}^{n} (\widetilde{e}^{2k-1,2k}) + \frac{1}{r}\widetilde{e}^{2n+1,2n+2}.
\end{aligned}$$

In the proof of Theorem 2.5.6 we show that $(\pi \colon \widetilde{M} \to M, \nabla)$ is a conic special Kähler manifold with metric (2.16), that is $\sum_{k=1}^{2n} (\widetilde{e}^k)^2 - (\widetilde{e}^{2n-1})^2 - (\widetilde{e}^{2n})^2$ and complex structure $\widetilde{I}$ such that $\widetilde{I}\widetilde{e}^{2n-1} = \widetilde{e}^{2n}$ for all $k = 1, \dots, n+1$. It follows that $\widetilde{e}$ is a unitary coframe on $\widetilde{M}$.

Let ω^{LC} be the Levi-Civita connection form of (M, g, I, ω), then by Proposition 2.3.2 we know how to write the Levi-Civita connection on $\widetilde{M}$ with respect to $\widetilde{e}$:

$$\begin{aligned}
&\widetilde{\omega}^{LC} = \\
&= \frac{1}{r}\Bigg(\sum_{h,k=1}^{n} \big(\mathrm{Re}(\omega^{LC})^h_{k,j} \widetilde{e}^{2j-1} - \mathrm{Im}(\omega^{LC})^h_{k,j} \widetilde{e}^{2j} \big) \otimes (\widetilde{e}_{2h-1} \otimes \widetilde{e}^{2k-1} + \widetilde{e}_{2h} \otimes \widetilde{e}^{2k}) \\
&\quad + \sum_{h,k=1}^{n} \big(\mathrm{Im}(\omega^{LC})^h_{k,j} \widetilde{e}^{2j-1} + \mathrm{Re}(\omega^{LC})^h_{k,j} \widetilde{e}^{2j} \big) \otimes (\widetilde{e}_{2h-1} \otimes \widetilde{e}^{2k} - \widetilde{e}_{2h} \otimes \widetilde{e}^{2k-1}) \\
&\quad - \widetilde{e}^{2n+2} \otimes (\widetilde{e}_{2h-1} \otimes \widetilde{e}^{2h} - \widetilde{e}_{2h} \otimes \widetilde{e}^{2h-1}) \\
&\quad + \sum_{h=1}^{n} \widetilde{e}^{2h-1} \otimes (\widetilde{e}_{2h-1} \otimes \widetilde{e}^{2n+1} + \widetilde{e}_{2h} \otimes \widetilde{e}^{2n+2})
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{h=1}^{n} \widetilde{e}^{2h} \otimes (\widetilde{e}_{2h-1} \otimes \widetilde{e}^{2n+2} - \widetilde{e}_{2h} \otimes \widetilde{e}^{2n+1}) \\
&+\sum_{h=1}^{n} \widetilde{e}^{2h-1} \otimes (\widetilde{e}_{2n+1} \otimes \widetilde{e}^{2h-1} + \widetilde{e}_{2n+2} \otimes \widetilde{e}^{2h}) \\
&+\sum_{h=1}^{n} \widetilde{e}^{2h} \otimes (\widetilde{e}_{2n+1} \otimes \widetilde{e}^{2h} - \widetilde{e}_{2n+2} \otimes \widetilde{e}^{2h-1}) \Bigg).
\end{aligned}
$$

Remark 3.3.1. *Notice that the coefficients of $r\widetilde{\omega}^{LC}$ are constant because the coframe is orthonormal and $c^k_{h,j}$ is constant for $h, k, j = 1, \dots, n$.*

Since the deviance is zero, we know that the flat connection coincides with the Levi-Civita one. From the proof of Theorem 2.5.6, we also have $\xi = r\partial_r = r\widetilde{e}_{2n+1}$ and thus we define $X = -I\xi = -r\widetilde{e}_{2n+2}$.

Proceeding with the rigid c-map, let $\pi_H \colon H = T^*\widetilde{M} \to \widetilde{M}$. We can lift the coframe $\widetilde{e}$ to a coframe $\widehat{e}$ on H, following Section 1.4.1 and [37]. In the notation of that section, the first $2n+2$ components of the coframe are $\pi_H^*\widetilde{e} = \pi_H^*\widetilde{e}^*\theta$ and the last $2n+2$ are $\widetilde{e}^*\alpha$. The differentials of the first $2n+2$ components are the pullback of the differentials of $\widetilde{e}$ whereas for the remaining $2n+2$ we use (1.13). We thus have, for $k = 1, \dots, n$ and $h = 1, \dots, 2n+2$

$$
\begin{aligned}
d\widehat{e}^k &= \frac{c^k_{j,l}}{r \circ \pi_H}\widehat{e}^{j,l} - \frac{1}{r \circ \pi_H}\widehat{e}^{k,2n+1}; \\
d\widehat{e}^{2n+1} &= 0; \\
d\widehat{e}^{2n+2} &= -\frac{2}{r \circ \pi_H}\sum_{k=1}^{n}(\widehat{e}^{2k-1,2k}) + \frac{1}{r \circ \pi_H}\widehat{e}^{2n+1,2n+2} \\
d\widehat{e}^{2n+2+h} &= -\widehat{e}^{2n+2+j} \wedge \pi_H^*(\widetilde{\omega}^{LC})^h_j.
\end{aligned}
$$

For brevity, from now on we will simply denote $r \circ \pi_H$ by r.

We want to lift X horizontally as a vector $\widehat{X}$ tangent to H. By (1.16), we know that $\widehat{e}^{2n+2+h}(\widehat{X}) = 0$ and $\widehat{e}^h(\widehat{X}) = \widetilde{e}^h((\pi_H)_*\widehat{X}) = \widetilde{e}^h(X)$, for all $h = 1, \dots, 2n+2$. This implies $\widehat{X} = -r\widehat{e}_{2n+2}$.

Notice that on H, the hyperKähler structure is given by Remark 1.4.3, so in particular the metric and the Kähler form fixed by $\widehat{X}$ are

$$
g = \sum_{k=1}^{2n}(\widehat{e}^k)^2 - \sum_{k=2n+1}^{2n+2}(\widehat{e}^k)^2 + \sum_{k=2n+3}^{4n+2}(\widehat{e}^k)^2 - \sum_{k=4n+3}^{4n+4}(\widehat{e}^k)^2;
$$

$$\omega_I = \sum_{k=1}^{n} \widehat{e}^{2k-1,2k} - \widehat{e}^{2n+1,2n+2} - \sum_{k=n+2}^{2n} \widehat{e}^{2k-1,2k} + \widehat{e}^{2n+1,2n+2}.$$

We have

$$\iota_{\widehat{X}}\omega_I = \omega_I(-r\widehat{e}_{2n+2},\cdot) = -r\widehat{e}^{2n+1} = -rdr = d\Big(-\frac{r^2}{2}\Big).$$

so $\mu := -\frac{r^2}{2}$ is a moment map for $\widehat{X}$.

We can also compute the 1-forms (1.18), obtaining:

$$\alpha_0 = r\widehat{e}^{2n+2}, \qquad \alpha_1 = -r\widehat{e}^{2n+1}, \qquad \alpha_2 = r\widehat{e}^{4n+4}, \qquad \alpha_3 = r\widehat{e}^{4n+3}.$$

We can now compute the data of Theorem 1.4.7:

$$\begin{aligned}
\Psi &= k(dX^\flat + \omega_I) = k(d(r\widehat{e}^{2n+2}) + \omega_I) = k(d(r)\wedge\widehat{e}^{2n+2} + rd\widehat{e}^{2n+2} + \omega_I)\\
&= k\Big(\widehat{e}^{2n+1,2n+2} - 2\sum_{j=1}^{n}(\widehat{e}^{2j-1,2j}) + \widehat{e}^{2n+1,2n+2} + \omega_I\Big)\\
&= k\Big(2\widehat{e}^{2n+1,2n+2} - 2\sum_{j=1}^{n}(\widehat{e}^{2j-1,2j})\\
&\qquad + \sum_{j=1}^{n}\widehat{e}^{2j-1,2j} - \widehat{e}^{2n+1,2n+2} - \sum_{j=n+2}^{2n}\widehat{e}^{2j-1,2j} + \widehat{e}^{4n+3,4n+4}\Big)\\
&= -k\Big(\sum_{j=1}^{n}\widehat{e}^{2j-1,2j} - \widehat{e}^{2n+1,2n+2} + \sum_{j=n+2}^{2n}\widehat{e}^{2j-1,2j} - \widehat{e}^{4n+3,4n+4}\Big);
\end{aligned}$$

$$a = k\big(g(\widehat{X},\widehat{X}) - \mu + c\big) = k\Big(-r^2 + \frac{r^2}{2} + c\Big) = -k\Big(\frac{r^2}{2} - c\Big);$$

$$f = \frac{B}{\mu - c} = -\frac{B}{\frac{r^2}{2}+c}, \qquad h = -\frac{B}{(\mu-c)^2} = -\frac{B}{\big(\frac{r^2}{2}+c\big)^2}.$$

In order to simplify the following steps, we fix the constants

$$c = 0, \qquad B = -\frac{1}{2}.$$

We know by Theorem 1.4.7 that the manifold Q resulting from the twist induced by these data is quaternion Kähler with metric g_Q such that

$$\begin{aligned}g_Q &\sim_\vartheta fg + h(\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2)\\ &= \left(\frac{1}{r^2}g + \frac{r^2}{r^4}(\widehat{e}^{2n+1})^2 + \frac{r^2}{r^4}(\widehat{e}^{2n+2})^2 + \frac{r^2}{r^4}(\widehat{e}^{4n+3})^2 + \frac{r^2}{r^4}(\widehat{e}^{4n+4})^2\right)\\ &= \frac{1}{r^2}\sum_{k=1}^{4n+4}(\widehat{e}^k)^2.\end{aligned}$$

Consider the new coframe $\breve{e} = \frac{1}{r}\widehat{e}$, which is then orthonormal with respect to this new metric in H. Suppose $d\widehat{e}^k = \frac{\widehat{c}^k_{j,h}}{r}\widehat{e}^{j,h}$, then the differentials of $\breve{e}$ become

$$d\breve{e}^k = d\left(\frac{1}{r}\widehat{e}^k\right) = -\frac{dr}{r^2}\wedge\widehat{e}^k + \frac{1}{r^2}\widehat{c}^k_{j,h}\widehat{e}^{j,h} = -\breve{e}^{k,2n+1} + \widehat{c}^k_{j,h}\breve{e}^{j,h}$$

And thus, for $k = 1,\dots,2n$ and $h = 1,\dots,2n+2$, we get:

$$\begin{aligned}&d\breve{e}^k = c^k_{j,l}\breve{e}^{j,l};\\ &d\breve{e}^{2n+1} = 0;\\ &d\breve{e}^{2n+2} = -2\sum_{k=1}^{n}(\breve{e}^{2k-1,2k});\\ &d\breve{e}^{2n+2+h} = -\breve{e}^{2n+2+h,2n+1} + \left(\breve{\omega}^{LC}\right)^h_{j,l}\breve{e}^{l,2n+2+j},\end{aligned}$$

where $(\breve{\omega}^{LC})^h_{j,l} = r\pi_H^*\left(\widetilde{\omega}^{LC}\right)^h_j(\widehat{e}_l)$.

Remark 3.3.2. *By Remark 3.3.1 the coefficients of the differentials of $\breve{e}$ are constant.*

Notice that M is contractible by Remark 2.6.1. This implies that the bundle $S \to M$ is trivial, and then there is a global section $s : M \to S$. Let $(\pi, z)\colon S \to M \times \mathrm{U}(1)$ be the induced trivialisation.

Let us now compute the Lie derivatives of the $\breve{e}^k$'s with respect to $\widehat{X}$. Notice that in the new frame, $\widehat{X} = -\breve{e}_{2n+2}$. Let $k = 1,\dots,2n$, then

$$\begin{aligned}&\mathcal{L}_{\widehat{X}}\breve{e}^k = d\iota_{\widehat{X}}\breve{e}^k + \iota_{\widehat{X}}d\breve{e}^k = 0;\\ &\mathcal{L}_{\widehat{X}}\breve{e}^{2n+1} = 0 + \iota_{\widehat{X}}d\breve{e}^k = 0;\end{aligned}$$

$$\mathcal{L}_{\widehat{X}}\breve{e}^{2n+2} = -d1 + 0 = 0.$$

The first $2n+2$ components of the coframe are thus invariant. For the remaining components instead, let $h = 1, \dots, n+1$:

$$\begin{aligned}
\mathcal{L}_{\widehat{X}}\breve{e}^{2n+2+2h-1} &= 0 + (\breve{\omega}^{LC})^{2h-1}_{j,2n+2}\breve{e}^{2n+2+j} = -\breve{e}^{2n+2+2h};\\
\mathcal{L}_{\widehat{X}}\breve{e}^{2n+2+2h} &= 0 + (\breve{\omega}^{LC})^{2h}_{j,2n+2}\breve{e}^{2n+2+j} = \breve{e}^{2n+2+2h-1}.
\end{aligned}$$

The second part of the coframe is not $\widehat{X}$-invariant, but it is rotated by the infinitesimal action of $\widehat{X}$, so we now want to build a new orthonormal coframe which is also invariant. Since we have a global section $s : M \to S$, let $(\pi, z)\colon S \to M \times \mathrm{U}(1)$ be the induced trivialisation. Consider $b = \pi_H^*\pi_1^*(z^{-1})\colon H \to \mathrm{U}(1)$, then if we compute the Lie derivative of b as a function in $\mathbb{C} \cong \mathbb{R}^2$, we get

$$\begin{aligned}
\mathcal{L}_{\widehat{X}}b = db(\widehat{X}) &= d\pi_H^*\pi_1^*z^{-1}(\widehat{X}) = \pi_H^*\pi_1^*dz^{-1}(\widehat{X})\\
&= dz^{-1}((\pi_1)_*(\pi_H)_*\widehat{X}) \circ (\pi_H\pi_1)\\
&= dz^{-1}(X) \circ (\pi_H\pi_1) = \frac{d}{dt}(z^{-1} \circ R_{e^{-it}})|_{t=0} \circ (\pi_H\pi_1)\\
&= \frac{d}{dt}(z^{-1}e^{it})|_{t=0} \circ (\pi_H\pi_1) = iz^{-1} \circ (\pi_H\pi_1) = ib.
\end{aligned}$$

Thus, if we let $z^{-1} = x + iy$, then $\mathcal{L}_{\widehat{X}}(x+iy) = -y + ix$, so $\mathcal{L}_{\widehat{X}}$ rotates x and y. Now consider the following 1-forms:

$$x\breve{e}^{2n+2+2h-1} + y\breve{e}^{2n+2+2h}, \qquad -y\breve{e}^{2n+2+2h-1} + x\breve{e}^{2n+2+2h};$$

if we compute the Lie derivatives, we can show they are $\widehat{X}$-invariant:

$$\begin{aligned}
&\mathcal{L}_{\widehat{X}}(x\breve{e}^{2n+2+2h-1} + y\breve{e}^{2n+2+2h})\\
&\qquad = -y\breve{e}^{2n+2+2h-1} - x\breve{e}^{2n+2+2h} + x\breve{e}^{2n+2+2h} + y\breve{e}^{2n+2+2h-1} = 0;\\
&\mathcal{L}_{\widehat{X}}(-y\breve{e}^{2n+2+2h-1} + x\breve{e}^{2n+2+2h})\\
&\qquad = -x\breve{e}^{2n+2+2h-1} + y\breve{e}^{2n+2+2h} - y\breve{e}^{2n+2+2h} + x\breve{e}^{2n+2+2h-1} = 0.
\end{aligned}$$

Moreover, the transformation is orthonormal, as

$$\det\begin{pmatrix} x & y\\ -y & x\end{pmatrix} = x^2 + y^2 = |z^{-1}|^2 = 1.$$

Notice also that these 1-forms are rotated by I.

Define now on the twist Q, the coframe u such that

$$\begin{aligned} &u^k \sim_\vartheta \breve{e}^k && \text{for } 1 \leq k \leq 2n+2 \\ &u^{2n+2+2h-1} \sim_\vartheta x\breve{e}^{2n+2+2h-1} + y\breve{e}^{2n+2+2h} && \text{for } 1 \leq h \leq n+1 \\ &u^{2n+2+2h} \sim_\vartheta -y\breve{e}^{2n+2+2h-1} + x\breve{e}^{2n+2+2h} && \text{for } 1 \leq h \leq n+1 \end{aligned}$$

Since every component of u corresponds to a $\widehat{X}$-invariant form, we can compute its differential using Proposition 1.4.8. Notice that for all $1 \leq k \leq 4n+4$, we have $\iota_{\widehat{X}}\breve{e}^k = \breve{e}^k(\breve{e}_{2n+2}) = \delta^k_{2n+2}$. The differentials of u are going to have the same coefficients as the corresponding forms on H, except for u^{2n+2}, for which

$$\begin{aligned} du^{2n+2} \sim_\vartheta d\breve{e}^{2n+2} - \frac{1}{a}\Psi &= -2\sum_{k=1}^{n}(\breve{e}^{2k-1,2k}) \\ &\quad + \frac{2}{r^2}\left(\sum_{k=1}^{n}\widehat{e}^{2k-1,2k} - \widehat{e}^{2n+1,2n+2} + \sum_{k=n+2}^{2n}\widehat{e}^{2k-1,2k} - \widehat{e}^{2n+1,2n+2}\right) \\ &= -2\sum_{k=1}^{n}(\breve{e}^{2k-1,2k}) \\ &\quad + 2\left(\sum_{k=1}^{n}\breve{e}^{2k-1,2k} - \breve{e}^{2n+1,2n+2} + \sum_{k=n+2}^{2n}\breve{e}^{2k-1,2k} - \breve{e}^{2n+1,2n+2}\right) \\ &= -2\breve{e}^{2n+1,2n+2} + 2\sum_{k=n+2}^{2n}\breve{e}^{2k-1,2k} - 2\breve{e}^{2n+1,2n+2}. \end{aligned}$$

So for the first $2n+2$ components, for $k = 1, \dots, 2n$,

$$\begin{aligned} du^k &= c^k_{j,h}u^{j,h} \\ du^{2n+1} &= 0 \\ du^{2n+2} &= -2u^{2n+1,2n+2} + 2\sum_{k=n+2}^{2n}u^{2k-1,2k} - 2u^{2n+1,2n+2}. \end{aligned}$$

In order to compute the last components of u, we need to know the differentials of x and y in terms of the elements of the coframe. In order to compute the differential of z^{-1}, we compute dz and then $dz^{-1} = d\overline{z}$. Notice that $(\mathrm{id}_S)_* = (z(s\circ\pi_S))_*$ is the composition of $(z, s\circ\pi_S)_* = (dz, s_*(\pi_S)_*)$ and the differential of the action $S\times \mathrm{U}(1)\to S$; see Lemma 2.5.13. Hence, we get $(\mathrm{id}_{\widetilde{M}})_* = (dz)^\circ + s_*(\pi_S)_*$, so if we compose this tensor with φ, we obtain

$$\varphi = \varphi\circ(\mathrm{id}_{\widetilde{M}})_* = \varphi((dz)^\circ + s_*(\pi_S)_*) = \frac{1}{iz}dz + \pi_S^* s^*\varphi.$$

It follows that

$$dz^{-1} = d\overline{z} = \overline{dz} = \overline{iz(\varphi - \pi_S^* s^*\varphi)} = iz^{-1}(\pi_S^* s^*\varphi - \varphi).$$

In particular, $db = ib(\pi_H^*\pi^* s^*\varphi - \pi_H^*\pi_1^*\varphi)$, so

$$\begin{aligned} dx &= -y(\pi_H^*\pi^* s^*\varphi - \pi_H^*\pi_1^*\varphi);\\ dy &= x(\pi_H^*\pi^* s^*\varphi - \pi_H^*\pi_1^*\varphi). \end{aligned}$$

If now $s^*\varphi$ is invariant, as is the cases of the classification of Section 3.1, then $s^*\varphi = \sum_{k=1}^{2n}\lambda_k e^k$ with λ_k constant for all $k = 1,\dots,n$. Therefore,

$$\begin{aligned} dx &= -y\left(\sum_{k=1}^{2n}\lambda_k\breve{e}^k - \breve{e}^{2n+2}\right);\\ dy &= x\left(\sum_{k=1}^{2n}\lambda_k\breve{e}^k - \breve{e}^{2n+2}\right). \end{aligned}$$

We can finally compute the differentials of the last components of the coframe.

$$\begin{aligned} &du^{2n+2+2h-1} \qquad (3.7)\\ &\quad\sim_\vartheta dx\wedge\breve{e}^{2n+2+2h-1} + xd\breve{e}^{2n+2+2h-1} + dy\wedge\breve{e}^{2n+2+2h} + yd\breve{e}^{2n+2+2h}\\ &\quad= \left(\sum_{k=1}^{2n}\lambda_k\breve{e}^k - \breve{e}^{2n+2}\right)\wedge(-y\breve{e}^{2n+2+2h-1} + x\breve{e}^{2n+2+2h})\\ &\qquad + x\left(-\breve{e}^{2n+2+2h-1,2n+1} + \left(\breve{\omega}^{LC}\right)_{j,l}^{2h-1}\breve{e}^{l,2n+2+j}\right)\\ &\qquad + y\left(-\breve{e}^{2n+2+2h,2n+1} + \left(\breve{\omega}^{LC}\right)_{j,l}^{2h}\breve{e}^{l,2n+2+j}\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{k=1}^{2n} \lambda_k \breve{e}^k - \breve{e}^{2n+2} \right) \wedge (-y\breve{e}^{2n+2+2h-1} + x\breve{e}^{2n+2+2h}) \\
&\quad - (x\breve{e}^{2n+2+2h-1} + y\breve{e}^{2n+2+2h}) \wedge \breve{e}^{2n+1} \\
&\quad x\big(\breve{\omega}^{LC}\big)^{2h-1}_{2j-1,l}\breve{e}^{l,2n+2+2j-1} + x\big(\breve{\omega}^{LC}\big)^{2h-1}_{2j,l}\breve{e}^{l,2n+2+2j} \\
&\quad + y\big(\breve{\omega}^{LC}\big)^{2h}_{2j-1,l}\breve{e}^{l,2n+2+2j-1} + y\big(\breve{\omega}^{LC}\big)^{2h}_{2j,l}\breve{e}^{l,2n+2+2j}.
\end{aligned}
$$

From the symmetries of the Levi-Civita connection form (in particular from the fact that $\nabla^{LC} I = 0$), we have

$$
A^h_{j,l} = \big(\breve{\omega}^{LC}\big)^{2h-1}_{2j-1,l} = \big(\breve{\omega}^{LC}\big)^{2h}_{2j,l}, \qquad B^h_{j,l} = \big(\breve{\omega}^{LC}\big)^{2h}_{2j-1,l} = -\big(\breve{\omega}^{LC}\big)^{2h-1}_{2j,l}.
$$

The sum of the four terms in (3.7) whose coefficients involve the Levi-Civita connection is then

$$
A^h_{j,l}\breve{e}^l \wedge \big(x\breve{e}^{2n+2+2j-1} + y\breve{e}^{2n+2+2j}\big) + B^h_{j,l}\breve{e}^l \wedge \big(yx\breve{e}^{2n+2+2j-1} - x\breve{e}^{2n+2+2j}\big)
$$

In conclusion, the differential of $u^{2n+2+2h-1}$ is

$$
\begin{aligned}
du^{2n+2+2h-1} &= -\left(\sum_{k=1}^{2n} \lambda_k u^k - u^{2n+2} \right) \wedge u^{2n+2+2h} - u^{2n+2+2h-1} \wedge u^{2n+1} \\
&\quad + A^h_{j,l} u^l \wedge u^{2n+2+2j-1} - B^h_{j,l} u^l \wedge u^{2n+2+2j} \\
&= -\sum_{k=1}^{2n} \lambda_k u^{k,2n+2+2h} + u^{2n+2,2n+2+2h} + u^{2n+1,2n+2+2h-1} \\
&\quad + A^h_{j,l} u^{l,2n+2+2j-1} - B^h_{j,l} u^{l,2n+2+2j}.
\end{aligned}
$$

The same reasoning can be applied to $u^{2n+2+2h}$, for which

$$
\begin{aligned}
&du^{2n+2+2h} \qquad (3.8) \\
&\quad \sim_\vartheta -dy \wedge \breve{e}^{2n+2+2h-1} - yd\breve{e}^{2n+2+2h-1} + dx \wedge \breve{e}^{2n+2+2h} + xd\breve{e}^{2n+2+2h} \\
&\quad = -\left(\sum_{k=1}^{2n} \lambda_k \breve{e}^k - \breve{e}^{2n+2} \right) \wedge (x\breve{e}^{2n+2+2h-1} + y\breve{e}^{2n+2+2h}) \\
&\qquad - y\left(-\breve{e}^{2n+2+2h-1,2n+1} + \big(\breve{\omega}^{LC}\big)^{2h-1}_{j,l}\breve{e}^{l,2n+2+j} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ x\Big(- \breve{e}^{2n+2+2h,2n+1} + \big(\breve{\omega}^{LC}\big)^{2h}_{j,l}\breve{e}^{l,2n+2+j}\Big)\\
= &-\Big(\sum_{k=1}^{2n}\lambda_k\breve{e}^k - \breve{e}^{2n+2}\Big)\wedge\big(x\breve{e}^{2n+2+2h-1} + y\breve{e}^{2n+2+2h}\big)\\
&- \big(-y\breve{e}^{2n+2+2h-1} + x\breve{e}^{2n+2+2h}\big)\wedge\breve{e}^{2n+1}\\
&- y\big(\breve{\omega}^{LC}\big)^{2h-1}_{2j-1,l}\breve{e}^{l,2n+2+2j-1} - y\big(\breve{\omega}^{LC}\big)^{2h-1}_{2j,l}\breve{e}^{l,2n+2+2j}\\
&+ x\big(\breve{\omega}^{LC}\big)^{2h}_{2j-1,l}\breve{e}^{l,2n+2+2j-1} + x\big(\breve{\omega}^{LC}\big)^{2h}_{2j,l}\breve{e}^{l,2n+2+2j}.
\end{aligned}
$$

By symmetry, the sum of the four terms in (3.8) with coefficients involving the Levi-Civita connection becomes

$$
A^h_{j,l}\breve{e}^l\wedge\big(- y\breve{e}^{2n+2+2j-1} + x\breve{e}^{2n+2+2j}\big) + B^h_{j,l}\breve{e}^l\wedge\big(x\breve{e}^{2n+2+2j-1} + y\breve{e}^{2n+2+2j}\big)
$$

In conclusion, the differential of $u^{2n+2+2h}$ is

$$
\begin{aligned}
du^{2n+2+2h} &= -\Big(\sum_{k=1}^{2n}\lambda_k u^k - u^{2n+2}\Big)\wedge u^{2n+2+2h-1} - u^{2n+2+2h}\wedge u^{2n+1}\\
&\quad + A^h_{j,l}u^l\wedge u^{2n+2+2j} + B^h_{j,l}u^l\wedge u^{2n+2+2j-1}\\
&= -\sum_{k=1}^{2n}\lambda_k u^{k,2n+2+2h-1} + u^{2n+2,2n+2+2h-1} + u^{2n+1,2n+2+2h}\\
&\quad + A^h_{j,l}u^{l,2n+2+2j} + B^h_{j,l}u^{l,2n+2+2j-1}.
\end{aligned}
$$

We have thus computed all of the differentials of the coframe, for which we are able to describe the structure constants starting from the entries of the Levi-Civita connection with respect to a suitable invariant coframe e on the initial Lie group.

Case VII: $\mathcal{H}^2_{\mathbb{C}}$

Consider case VII of Section 3.1. This case corresponds to $\mathcal{H}^2_{\mathbb{C}}$ and thus it has zero deviance. Let Q be the quaternion Kähler manifold obtained from $\mathcal{H}^2_{\mathbb{C}}$ by the c-map.

If we now apply the general construction we just mentioned, we get on Q the following differentials:

$$
du^1 = u^{1,3}; \qquad du^2 = u^{2,3}; \qquad du^3 = 0;
$$

$$
\begin{aligned}
du^4 &= -2u^{1,2} - 2u^{3,4}; \qquad du^5 = 0;\\
du^6 &= -2u^{5,6} + 2u^{7,8} + 2u^{9,10} - 2u^{11,12};\\
du^7 &= u^{1,9} + u^{1,11} - u^{2,10} - u^{2,12} - u^{5,7};\\
du^8 &= u^{1,10} + u^{1,12} + u^{2,9} + u^{2,11} - u^{5,8};\\
du^9 &= -u^{1,7} - u^{2,8} + u^{3,11} - u^{4,10} - u^{4,12} - u^{5,9};\\
du^{10} &= -u^{1,8} + u^{2,7} + u^{3,12} + u^{4,9} + u^{4,11} - u^{5,10};\\
du^{11} &= +u^{1,7} + u^{2,8} + u^{3,9} + u^{4,10} + u^{4,12} - u^{5,11};\\
du^{12} &= +u^{1,8} - u^{2,7} + u^{3,10} - u^{4,9} - u^{4,11} - u^{5,12}.
\end{aligned}
$$

Denoting by $\mathfrak{q}$ the Lie algebra of Q, its derived algebra is

$$\mathfrak{q}^{(1)} = \langle u_1, u_2, u_4, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12} \rangle .$$

The derived series has $\mathfrak{q}^{(5)} = 0$, so Q is solvable. One can check that $\mathfrak{q}$ is also real solvable.

Since we know that Q is a quaternion Kähler symmetric space with dimension 12, it must necessarily be one of the following (see [8, Table 14.52, p. 409]):

$$\frac{\mathrm{Sp}(3,1)}{\mathrm{Sp}(3)\mathrm{Sp}(1)} = \mathbb{P}^3_{\mathbb{H}}, \qquad \frac{\mathrm{SU}(3,2)}{\mathrm{S}(U(3)U(2))}, \qquad \frac{\mathrm{SO}_0(4,3)}{\mathrm{SO}(4)\mathrm{SO}(3)}.$$

One could apply the reasoning done for the examples in Section 1.6 and show directly that $Q \cong \frac{\mathrm{SU}(3,2)}{\mathrm{S}(U(3)U(2))}$.

As an alternative, we can construct the curvature operator R as in Section 1.6 and if we compute its eigenvalues, we obtain

$$\underbrace{-10}_{1}, \underbrace{-6,-6,-6}_{3}, \underbrace{-4,-4,-4,-4,-4,-4,-4,-4}_{8}, \underbrace{0,\ldots,0}_{54};$$

so, we infer that the dimension of the holonomy algebra of Q is 12. The dimension of their holonomy algebras of the possible symmetric spaces are 24, 12 and 9 respectively, so necessarily $Q \cong \frac{\mathrm{SU}(3,2)}{\mathrm{S}(U(3)U(2))}$.

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