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Poly-free constructions for some Artin groups

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List of Symbols

The next list describes several symbols that will be later used within this document.

$A \cap B$	Set-theoretic intersection of sets A and B
$A \cup B$	Set-theoretic union of sets A and B
$A \sqcup B$	Set-theoretic disjoint union of sets A and B
$A \subseteq B$	Set-theoretic inclusion of set A into set B
$A \setminus B$	Set-theoretic difference of set A and B
$\mathscr{P}(A)$	Power set of the set A
$\langle X \rangle$	Subgroup generated by the elements of X
$\langle\langle X \rangle\rangle_G$	Normal subgroup generated by the elements of X inside G
$[g, h]$	The element $ghg^{-1}h^{-1}$ of a group
G'	Commutator subgroup of G
G^{ab}	Abelianization of G
$H \trianglelefteq G$	Normal subgroup H of G
$H \triangleleft G$	Means $H \trianglelefteq G$ and $H \neq G$
$H \times K$	Direct product of H and K
$H * K$	Free product of H and K
$\coprod_{i \in I} G_i$	Free product of the family of groups $\{ G_i \}_{i \in I}$
$\mathrm{Cay}(G, X)$	Cayley graph of G with respect to the set of generators X
LHS	Abbreviation of “left hand side”
RHS	Abbreviation of “right hand side”

Abstract

A group G is said to be *poly-free* if there exists a subnormal series of subgroups

$$\{1_G\} = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_{k-1} \triangleleft G_k = G$$

whose factors $\frac{G_i}{G_{i-1}}$ are non-trivial free groups. In this thesis we will explore poly-freeness and some stronger variants of this property, such as requiring that all the factors are finitely generated or that each G_i is normal in G (and in this last case we will say that G is *strongly poly-free*).
An Artin group $\mathcal{A}(\Gamma)$ is a group defined via a presentation whose structure is encoded inside a Coxeter graph Γ. In this thesis we achieve the following new results about poly-freeness of certain families of Artin groups.

1. In Chapter 1 we set up the basic definitions we need and survey the most important properties of Artin groups. We also explain why poly-freeness is an interesting property for groups, which are its implications for the structure of a group and we also provide an original result where we show that strong poly-freeness is preserved by free products.

2. In Chapter 2 we focus on irreducible Artin groups of finite type and we establish that the only poly-free Artin groups in this family are those of type $\mathcal{I}_2(m)$ $(m \geq 3)$, $\mathcal{A}_3$, $\mathcal{B}_3$, $\mathcal{B}_4$ and $\mathcal{D}_4$, with the only possible exception of $\mathcal{F}_4$ which remains undetermined at the moment.

3. In Chapter 3 we study Artin groups built on Coxeter trees. We prove that these groups are strongly poly-free with finitely generated factors and we deduce that also Artin groups built on Coxeter forests are strongly poly-free.

4. In the last section of Chapter 3 we define a (new) costruction for a family of Coxeter graphs that we call '2-join' and we prove that the associated Artin group is strongly poly-free when each graph in the family we begin with has at most 2 vertices.

Contents

Chapter 1

Introduction

1.1 Preface

In this section we simply want to make a brief digression about the history of Artin groups without the aim to be overly precise, but rather to give a general idea of how they arise, what are the most important results regarding them and how they are connected with other active fields of research in mathematics.
Although usually referred as “Artin groups” by Emil Artin who worked on braid groups in the first half of the XX century, the definition of this whole family of groups (which includes braid groups) is due to Jacques Tits in [21] (1966) so that the name “Artin-Tits groups” would be much more appropriate even if it rarely occurs in the literature (except for this section, however, we will stick with the established convention). Tits introduced these groups as extensions of Coxeter groups which had already been intensively studied since the thirties, when H. S. M. Coxeter discovered them during the study of reflection groups (i.e., subgroups of $\mathrm{Aut}(\mathbb{R}^n)$ generated by elements of order two).
Artin-Tits groups are defined in a purely algebraic way by means of an explicit presentation in terms of generators and relators and the description of such presentation is commonly encoded in a simplicial graph Γ whose edges are labelled with integers. This leads to wonder whether there exists any connection between the structure of the combinatorial data that is written in the graph and the algebraic properties of Artin-Tits groups. Not surprisingly, this is sometimes the case and a number of results we will present in this thesis (either original or already known in the literature) state hypotheses on the structure of the graph Γ in order to infer algebraic information regarding the Artin-Tits group associated to it. For example, in [2] (2017) the authors Blasco-García, Martínez-Pérez and Paris prove that if Γ has only even labels and for any triangular subgraph at least two edges are labelled with ‘2’, then the associated Artin-Tits group is poly-free. However, the “slightest

variation" in the structure of the graph may change a lot the properties of the related Artin-Tits group; e.g., all results contained in the work we have just cited heavily rely on the hypothesis that all edges are labelled with an even integer.
This situation has the consequence that many general questions about Artin-Tits groups have been answered only for some more or less extended families using tools specifically developed for that cases. For example, the following questions are still open in the general case.

1. Do Artin-Tits group have torsion?
2. What is the center of Artin-Tits groups?
3. Are Artin-Tits groups orderable?
4. Does the $K(\pi, 1)$ conjecture hold for all Artin-Tits groups?

We find pretty astonishing the lack of an answer especially for the first question, which is widely believed to be true although no one has been able to prove it or provide a counterexample (that, if exists, should not be that hard to find).
However, Artin-Tits groups are not just interesting in themselves. Their relation to Coxeter groups is not limited to the definition, but extends to topological applications. The easiest case is the one of braid groups (and pure braid groups) which can be shown to be the fundamental groups of certain topological spaces (built thanks to the action of the related Coxeter group on a vector space of dimension equal to the number of vertices of Γ). Such approach has been greatly generalized to all Artin-Tits groups which in [22] (1983) by Van der Lek are shown to be the fundamental groups of spaces of the form

$$M = (T \times V) \setminus \left(\bigcup_{r \in R} H_r \times H_r \right)$$

where V is the finite dimensional vector space on which the Coxeter group acts, R is the set of all its elements of order two, H_r is the hyperplane of V fixed by $r \in R$ and T is a particular open cone inside V called the "Tits cone". The $K(\pi, 1)$ conjecture says that the manifold M is a classifying space for the associated Artin-Tits group and an affirmative answer is known only for some families such as finite type Artin-Tis groups (see [9] (1972) by Pierre Deligne).

In this thesis we will not deal with topological aspects and we will obtain all results using algebraic techniques. In particular, we want to study poly-freeness, a strong structural property of groups that in turn implies an affirmative answer to questions 1 and 3 above, together with other properties that we will explain in Section 1.3. We will show that the only poly-free irreducible Artin-Tits groups of finite type are those of type $I_2(n)$ ($n \geq 3$), A_3, B_3, B_4 and D_4 (with the only exception of F_4 which remains unknown at the moment). Moreover, we will prove that all Artin-Tits groups whose associated graph is a tree or a forest are poly-free and we give a couple of results that will hopefully help to study other families.

Finally, although this thesis has entirely been edited by myself, I must greatly thank my advisors Thomas Stefan Weigel and Conchita Martínez-Pérez for the huge support I received.

1.2 Basic facts and definitions

This section is devoted to set some elementary definitions and state some very well-known results for later reference. In particular we provide a couple of short paragraphs where we recall some results that we will need from the theory of sigma invariants and Bass-Serre theory.

1.2.1 Presentation of groups

Since Artin groups are defined by means of their presentation, in this section we recall what it means to give the presentation of a group by generators and relators as well as some well-known properties that we will use extensively throughout this thesis.

Definition 1.2.1. *A group F is said to be free over a set X if there exists an injection $\iota\colon X \to F$ such that for any group H and any map of sets $f\colon X \to H$ there exists a unique morphism of groups $\varphi\colon F \to H$ such that $f = \varphi \circ \iota$; equivalently there exists a unique morphisms of groups φ that makes the following diagram commute.*

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & F \arrow[d, dashed, "\exists!\,\varphi"] \\ & H \end{tikzcd}$$

There are other equivalent definitions of what a free group should be: the one we chose above is what - from a categorical point of view - is called the universal property of free groups. Although in this thesis we will not adopt the point of view of category theory the above is nevertheless a remarkable property that allows us to define morphisms from a free group F to an arbitrary group H by simply specifying the images of the standard generators $\iota(X)$ of F without any further concern about the existence and uniqueness of such map. Anyway, we still have to show that for any set X such an algebraic object exists.

Proposition 1.2.2. *For any set X there exists a group F which satisfies the definition of free group over the set X together with a suitable inclusion of X into*

F. Moreover, if F and G are two such groups, they are isomorphic. Finally, if X and Y are equipotent sets, then any two groups F and G free over X and Y, respectively, are isomorphic.

Proof. If $X = \emptyset$ the trivial group satisfies the definition of free group over X, otherwise let X^+ and X^- be sets each containing exactly a copy of each element x in X that we will mark with an upper '+' or '−' respectively, i.e.

$$X^+ \coloneqq \left\{ x^+ \mid \forall x \in X \right\}, \qquad X^- \coloneqq \left\{ x^- \mid \forall x \in X \right\}.$$

For ease of notation we will henceforth identify the elements inside X with the corresponding ones inside X^+ so that $X^+ = X$. Let W be the set of words in the alphabet $S = X^+ \sqcup X^-$, i.e. all possible finite sequences $x_0^{\varepsilon_0} x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_k^{\varepsilon_k}$ with $x_i \in X$ and $\varepsilon_i \in \{ +, \ - \}$ for all $i = 0, \dots, k$, including the empty sequence $\emptyset \in W$. Given a word $w = x_0^{\varepsilon_0} x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_k^{\varepsilon_k} \in W$ we say that w is *reduced* if for each $0 \leq i \leq k-1$ either $x_i \neq x_{i+1}$ or $x_i = x_{i+1}$ but $\varepsilon_i \neq -\varepsilon_{i+1}$. We also define the empty word $\emptyset$ to be reduced. We say that a word w' comes from an *elementary reduction* of another word w if it can be obtained from the latter after the deletion of a subsequence of type x^+x^- or x^-x^+ for some $x \in X$. If a word w'' can be obtained from w after a (possibily empty) finite sequence of elementary reductions we say that w'' comes from w by reduction. Given any two words $v, w \in W$ we say that they are equivalent if there exists a sequence of words $w = w_0, w_1, \dots, w_m = v$ such that for each $0 \leq j \leq m-1$ one of w_j and w_{j+1} comes from the other by an elementary reduction. It's readily checked that being equivalent is an equivalence relation on the set W and we will denote it by '$\sim$'. We are therefore allowed to consider the quotient set $F \coloneqq \frac{W}{\sim}$ and it can be shown that there is a unique reduced word inside each equivalence class (see, for instance, [4, Chapter I, Theorem 4]).
On W we define a binary operation $\circ \colon W \times W \to W$ which sends any two words $w_1 = x_0^{\varepsilon_0} \dots x_m^{\varepsilon_m}$ and $w_2 = y_0^{\gamma_0} \dots y_n^{\gamma_n}$ to the word $w_1 \circ w_2 = x_0^{\varepsilon_0} \dots x_m^{\varepsilon_m} y_0^{\gamma_0} \dots y_n^{\gamma_n}$ given by juxtaposing the sequence of symbols of w_2 after those of w_1. This operation induces a well-defined binary operation $* \colon F \times F \to F$ by setting $[w_1]_\sim * [w_2]_\sim = [w_1 \circ w_2]_\sim$. We claim that the pair $(F, *)$ is a group that satisfies the definition of free group over the set X with the inclusion $\iota \colon X \to F$ sending any element x inside X to the equivalence class of the word of length one made just of the symbol x^+. The operation $*$ is associative, the identity element is the equivalence class represented by the empty word (since for any $[w]_\sim \in F$ we have $[w]_\sim * [\emptyset]_\sim = [w \circ \emptyset]_\sim = [w]_\sim = [\emptyset \circ w]_\sim = [\emptyset]_\sim * [w]_\sim$) and for each element $[w] \in F$ with $w = x_0^{\varepsilon_0} \dots x_k^{\varepsilon_k}$ its inverse is the equivalence class $[x_k^{\delta(\varepsilon_k)} \dots x_0^{\delta(\varepsilon_0)}]_\sim$, where δ is the function that sends $+ \mapsto -$ and $- \mapsto +$, in particular $[x^+]_\sim^{-1} = [x^-]_\sim$. Therefore $(F, \ *)$ is a group and it is generated by the set of elements $\{ [x^+]_\sim \mid x \in X \}$ in a one-to-one correspondence with the elements of X.
We are left to show that for any group H and for any map $f \colon X \to H$ there exists a

unique morphism of groups $\varphi\colon F \to H$ such that $f = \varphi \circ \iota$. Denote with $h_x \in H$ the image of $x \in X$ under f. Since ι sends $x \mapsto [x^+]_\sim$ and $[x^+x^-]_\sim = [\emptyset]_\sim$, then $1_H = \varphi([\emptyset]_\sim) = \varphi([x^+x^-]) = \varphi([x^+]_\sim)\varphi([x^-]_\sim)$ which implies $\varphi([x^-]_\sim) = \varphi([x^+]_\sim)^{-1}$. Since the set of all $[x^+]_\sim$ generates F, for each $g = [x_0^{\varepsilon_0} \dots x_k^{\varepsilon_k}]_\sim \in F$ the only choice we have for a morphism of groups φ subject to the above restrictions is to set $\varphi(g) \coloneqq \varphi([x_0^{\varepsilon_0}]_\sim) \dots \varphi([x_k^{\varepsilon_k}])_\sim = h_{x_0}^{\overline{\varepsilon}_0} \dots h_{x_k}^{\overline{\varepsilon}_k}$ (where $\overline{\varepsilon}_i = -1$ if $\varepsilon_i =$ '$-$' and $\overline{\varepsilon}_i = 1$ otherwise), provided that such map is well-defined over the set of equivalence classes of F. Indeed, this is the case since two words in any equivalence class differ only by subsequences of type x^+x^- or x^-x^+ which both get mapped to the product of an element of H and its inverse, which is the identity in H. Therefore F is a free group over X.

Now, suppose that F and G are free groups over the same set X with respect to the inclusions $\iota\colon X \to F$ and $\lambda\colon X \to G$. By the defining property of F there exists a unique morphism $\varphi\colon F \to G$ such that $\lambda = \varphi \circ \iota$ and by the same argument for G there exists a unique morphism $\psi\colon G \to F$ such that $\iota = \psi \circ \lambda$. This implies $\iota = (\psi \circ \varphi) \circ \iota$, but the definition of free groups says that F admits a unique morphism that can replace the function inside the parenthesis in the last equality. Since Id_F trivially satisfies such condition it must be $\psi \circ \varphi = \mathrm{Id}_F$. With a completely analogous argument one shows that $\varphi \circ \psi = \mathrm{Id}_G$, so φ and ψ are one the inverse of the other and F is isomorphic to G.

$$\begin{tikzcd}
 & F \arrow[d, "\varphi"] \arrow[dd, bend left, "\mathrm{Id}_F"] \\
X \arrow[ur, hook, "\iota"] \arrow[r, hook, "\lambda"] \arrow[dr, "\iota"'] & G \arrow[d, "\psi"] \\
 & F
\end{tikzcd}$$

Finally, let X and Y be sets, $f\colon X \to Y$ a bijection and let F and G be free groups on X and Y, respectively, together with inclusions $\iota\colon X \to F$ and $\lambda\colon Y \to G$. Arguing in a very similar fashion as we have just done, with regard to the diagram below the equality

$$\psi \circ \varphi \circ \iota = \psi \circ \lambda \circ f = \iota \circ f^{-1} \circ f = \iota$$

implies that (by the universal property of the free group F) $\psi \circ \varphi = \mathrm{Id}_F$ and with an analogous argument for G we also get $\varphi \circ \psi = \mathrm{Id}_G$. Therefore F and G are isomorphic.

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[d, "f"'] \arrow[dr, "\lambda\circ f"] \arrow[ddr, bend right=60, "\mathrm{Id}_F"'] & F \arrow[d, "\varphi"] \\
Y \arrow[r, "\lambda"] \arrow[dr, "\iota\circ f^{-1}"'] & G \arrow[d, "\psi"] \\
 & F
\end{tikzcd}$$

■

Although in the construction we gave above elements of free groups are equivalence classes of a convenient set of words, for the sake of convenience from now on we will always write x instead of $[x^+]_\sim$ even if we are actually referring to the equivalence class of the symbol x. Of course we will pay special attention when there could be the chance of confusion.
Moreover, since abstract algebra is concerned with the study of groups up to isomorphism, the previous proposition allows us to speak about *the* free group built from a set X and we will denote it by the symbol $F(X)$. Since what actually matters in the construction of $F(X)$ is the cardinality of X, when $|X| = n < +\infty$ we will also write F_n instead of $F(X)$.

Definition 1.2.3. *Let G be a group and let X be a set, $F(X)$ the free group over X and $N \trianglelefteq F(X)$ a normal subgroup. If $G \simeq \frac{F(X)}{N}$ then we say that the pair $(F(X), N)$ is a presentation for the group G.*

Quite obviously a group may have many presentations over a lot of different sets X (i.e., its set of generators), however the remarkable result is the following.

Proposition 1.2.4. *Every group G admits a presentation.*

Proof. Consider $F(G)$ and consider Id_G also as the inclusion map from G to $F(G)$ (so far G is regarded just as a set). By the definition of free group there exists a unique morphisms of groups $\varphi\colon F(G) \to G$ such that $\mathrm{Id}_G = \varphi \circ \mathrm{Id}_G$, in particular φ is surjective and the first homomorphism theorem tells us that $\frac{F(G)}{\operatorname{Ker}\varphi} \simeq \operatorname{Im}(\varphi) = G$. The pair $(F(G), \operatorname{Ker}(\varphi))$ is a presentation for G. ■

However, most of the time we will deal with the presentation of a group G we will not depict it as a pair $(F(X), N)$ but rather we will adopt the following striking notation

$$G = \langle X \mid R \rangle$$

where R is a set of elements of N that generates it as a normal subgroup inside $F(X)$. If the elements of X or R are indexed by some set of indexes we will also adopt the convention of explicitly listing them inside brackets, e.g.

$$G = \langle x_1, x_2, x_3, \ldots \mid r_1, r_2, r_3, \ldots \rangle .$$

Finally, since the (equivalence classes of) words r inside R represent elements that are "sent to the identity" when passing to the quotient $\frac{F(X)}{N} \simeq G$ we will also write them as '$r = 1$' to stress this fact and we will called them *relators*. Recalling that the r's inside R are just elements of $F(X)$ (and hence equivalence classes of finite sequences of symbols) we will sometimes manipulate the equality $r = 1$ inside $F(X)$ to get some other equality that we will more generally call a *relation* as it depicts an equality that must hold once the LHS and RHS are projected over G (e.g., we may rewrite the relator $ab^{-1}c = 1$ to the relation $b = ca$).

In general, given two presentations it is an extremely difficult problem to establish if they represent isomorphic groups, however there are some easy moves, called *Tietze transformations*, that allow to manipulate the presentation of a group without changing its isomorphism type.

Theorem 1.2.5 (H. Tietze)**.** *Let $G \simeq \langle a, b, c, \ldots \mid P, Q, R, \ldots \rangle$ be a presentation of groups, then the following transformations leave the group unchanged.*

i) Adding or deleting a generator: if $w(a, b, c, \ldots)$ is a word in the defining generators of G, then

$$\langle a, b, c, \ldots \mid P, Q, R, \ldots \rangle \simeq \langle a, b, c, \ldots, x \mid P, Q, R, \ldots, x = w(a, b, c, \ldots) \rangle .$$

ii) Adding or deleting a relation: if $W(P, Q, R, \ldots)$ is a word in the defining relators of G, then

$$\langle a, b, c, \ldots \mid P, Q, R, \ldots \rangle \simeq \langle a, b, c, \ldots \mid P, Q, R, \ldots, U = W(P, Q, R, \ldots) \rangle .$$

Proof. See [14, Chapter I, Theorem 1.5]. ■

Since we will work a lot with morphisms between presentations of groups we will make extensive use of the following proposition.

Proposition 1.2.6. *Let X be a set, let $N \trianglelefteq F(X)$ be a normal subgroup of $F(X)$ and let H be any group. Suppose we have a map of sets $\psi \colon X \to H$ such that*[1]

[1] The universal property of free groups ensures that there exists a unique morphism $\varphi \colon F(X) \to H$ such that $\varphi \circ \iota = \psi$ where $\iota \colon X \to F(X)$ is the standard embedding of X inside $F(X)$. By abuse of notation in what follows we identify the set X with its image through ι so that its elements can be regarded as elements of the group $F(X)$ and whenever we write $\psi(g)$ for any $g \in F(X)$ we actually mean $(\varphi \circ \iota)(g)$ with no chance of ambiguity since by construction $\psi(x) = (\varphi \circ \iota)(x)$ for all x in X.

$\psi(N) = \{ 1_H \}$, then there exists a unique morphism of groups $\overline{\psi}\colon \frac{F(X)}{N} \to H$ such that $\psi(x) = \overline{\psi}(xN)$ for each x in X.
In particular, if $G = \langle X \mid R \rangle$ and ψ is such that $\psi(R) = \{ 1_H \}$, then there exists a unique morphism of group $\overline{\psi}\colon G \to H$ such that $\psi(x) = \overline{\psi}(xN)$ for each x in X, where $G \simeq \frac{F(X)}{N}$ and $N = \langle\langle R \rangle\rangle_{F(X)}$.

Proof. By the definition of free group there exists a unique morphisms of groups $\varphi\colon F(X) \to H$ such that $\varphi(x) = \psi(x)$ for each x in $X \subset F(X)$. By the first homomorphism theorem the hypothesis $\psi(N) = \{ 1_H \}$ allows to build a well-defined morphism of groups $\overline{\psi}\colon \frac{F(X)}{N} \to H$ sending a coset xN to $\varphi(x)$.
If $G = \langle X \mid R \rangle$ is given by a presentation, the hypothesis $\psi(R) = \{ 1_H \}$ implies that also the image of $N = \langle\langle R \rangle\rangle_{F(X)}$ is trivial since each element in the normal closure of R can be written as a product of elements conjugate to elements of R or their inverses, i.e. for each n in N there exist $r_1, \ldots, r_k$ in R, $\varepsilon_1, \ldots, \varepsilon_k \in \{ \pm 1 \}$ and $g_1, \ldots, g_k$ in $F(X)$ such that $n = (r_1^{\varepsilon_1})^{g_1} \cdot \ldots \cdot (r_k^{\varepsilon_k})^{g_k}$, so that

$$\overline{\psi}(n) = \overline{\psi}(r_1^{\varepsilon_1})^{g_1} \cdot \ldots \cdot \overline{\psi}(r_k^{\varepsilon_k})^{g_k} = 1_H, \qquad \forall n \in N$$

and the first part of the proposition applies. ■

Whenever a map ψ satisfies the hypotheses of previous proposition we will say that ψ *preserves the relators* (or relations) of G. Moreover, with a slight abuse of notation we will denote the morphism $\overline{\psi}$ with the same plain letter ψ that denotes the initial map defined only on the set of generators.

Proposition 1.2.7. *Given two groups H and K, suppose that a presentation is known for each of them, say $H = \langle X|R \rangle$ and $K = \langle Y|S \rangle$. When taking various kind of products between H and K it may be possible to describe the resulting group by means of a presentation as well. We will make use of the following results.*

1. *If $G \coloneqq H * K$, then $G \simeq \langle X \sqcup Y \mid R \sqcup S \rangle$.*
2. *If $G \coloneqq H \times K$, then $G \simeq \langle X \sqcup Y \mid R \sqcup S \sqcup \{ xy = yx,\ \forall x \in X,\ y \in Y \} \rangle$.*
3. *If $G \coloneqq H \rtimes_\varphi K$ for some morphism of groups $\varphi\colon K \to \mathrm{Aut}(H)$, then*
$$G \simeq \left\langle X \sqcup Y \;\middle|\; R \sqcup S \sqcup \{ yxy^{-1} = \varphi(y)(x),\ \forall x \in X,\ y \in Y \} \right\rangle.$$
4. *Another common type of construction is the amalgamated product of H and K over a common subgroup A (this construction, for example, is central in the well-known Seifert-Van Kampen theorem and corresponds to the pushout in the category of groups). If A is a group with injections $\iota\colon A \to H$, $\lambda\colon A \to K$,*

we write $G \coloneqq H *_A K$ for the amalgamated product of H and K over A and it can be shown that

$$G \simeq \langle X \sqcup Y \mid R \sqcup S \sqcup \{\, \iota(a) = \lambda(a),\ \forall a \in A \,\}\rangle\,.$$

Of course, the above presentation can also be taken as a definition for $H *_A K$.

We take the chance to state a couple of results about free products with amalgamation and semidirect products. Since both are very well-known we refer to external sources for the proofs.

Lemma 1.2.8. *Let $G = G_1 *_H G_2$ and let T_i $(i = 1, 2)$ be a set of representatives of the right cosets of G_i in G containing the identity. Set $T_1^\times \coloneqq T_1 \setminus \{\,1\,\}$ and $T_2^\times \coloneqq T_2 \setminus \{\,1\,\}$, then each element g of G can be written in an unique way as a word of the form $g_0 \dots g_k$ $(k \in \mathbb{N})$ where*

- *$g_0 \in H$,*
- *either $g_j \in T_1^\times,\ g_{j+1} \in T_2^\times$ or $g_j \in T_2^\times,\ g_{j+1} \in T_1^\times$ for each $j = 1, \dots, k-1$.*

*The word $g_0 \dots g_k$ is called the H-*normal form *of g.*

Proof. See [13, Theorem 2.6, Chapter IV]. ■

Proposition 1.2.9. *Let $1 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 1$ be a short exact sequence of groups, the following statements are equivalent.*

1. *The exists a homomorphism of groups $\gamma\colon K \to G$ such that $\beta \circ \gamma = \mathrm{Id}_K$ (γ is said to be a* section *for β),*
2. *There exists a homomorphism of groups $\varphi\colon K \to \mathrm{Aut}(H)$ and an isomorphism $\theta\colon G \to H \rtimes_\varphi K$ such that*

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r, "\alpha"] \arrow[d, "\mathrm{Id}"'] & G \arrow[r, "\beta"] \arrow[d, "\theta"'] & K \arrow[r] \arrow[d, "\mathrm{Id}"'] & 1 \\
1 \arrow[r] & H \arrow[r] & H \rtimes_\rho K \arrow[r] & K \arrow[r] & 1
\end{tikzcd}$$

commutes, where the bottom short exact sequence is the standard one for semidirect products.

Proof. See [5, Theorem 3.3]. ■

1.2.2 Graphs

Definition 1.2.10. *Let V be a set and let $E \subset \mathscr{P}(V)$ such that each element e of E has exactly two (distinct) elements, called the* endings *of e. A* simplicial graph Γ *is a pair (V, E), where the elements of V are called* vertices of Γ *and the elements of E are called* edges of Γ*. To avoid confusion we will often denote V and E by $V(\Gamma)$ and $E(\Gamma)$, respectively.*

Throughout this thesis we always suppose that all graphs we consider have a finite number of vertices and edges. Given a graph $\Gamma = (V, E)$ we say that $\Lambda = (V', E')$ is a *subgraph* of Γ (and we will write $\Lambda \subseteq \Gamma$) if $V' \subseteq V$ and $E' \subseteq E$. A *clique* Λ of Γ is a subgraph of Γ such that each vertex of Λ is connected to any other vertex in $V(\Lambda)$ (i.e., Λ is a complete graph if regarded on its own). We say that Λ is a *full subgraph* of Γ if for each $e = \{ v_1, v_2 \} \in E(\Gamma)$ with $v_1, v_2 \in V(\Lambda)$ we have $e \in E(\Lambda)$. We denote with $\Gamma \setminus \Lambda$ the graph given by $(V \setminus V', E \setminus E')$. Finally, given a subset W of $V(\Gamma)$ we call the *(full) subgraph spanned by W* the subgraph of Γ having W as vertices and edges the edges of Γ with both endings in W.

Definition 1.2.11. *Given a graph Γ and a vertex $v \in V(\Gamma)$, a* path *p inside Γ of length $n \in \mathbb{N}$ based at the vertex v is a sequence of vertices $p := (v_i)_{0 \le i \le n}$ such that $v = v_0$ and $\{ v_i, v_{i+1} \} \in E(\Gamma)$ for each $i = 0, \dots n-1$.*
Clearly, to any path $(v_i)_{0 \le i \le n}$ based at v we can associate

- *the symbol $\emptyset_v$, if $n = 0$,*
- *the sequence of edges $(e_j)_{1 \le i \le n}$ where $e_j := \{ v_{j-1}, v_j \}$, if $n \ge 1$.*

Since the above correspondence is one-to-one we will tacitly switch between the sequences $(v_i)_{0 \le i \le n}$ and $(e_j)_{1 \le j \le n}$ when talking about paths.
A path p is said to be reduced *if either $n = 0$ or $e_j \neq e_{j+1}$ for all $j = 1, \dots, n-1$.*

Let $v, w \in V(\Gamma)$, we say that $v \sim w$ if there exists a path $p = (v_i)_{0 \le i \le n}$ in Γ such that $v = v_0$ and $w = v_n$. It's easy to check that '$\sim$' is an equivalence relation over the set of vertices of Γ and the full subgraphs spanned inside Γ by its equivalence classes are called the *connected components of* Γ.

Definition 1.2.12. *A graph Γ is a* tree *if for any pair of vertices there exists a unique reduced path connecting them. A graph Γ is a* forest *if all its connected components are trees.*

Lemma 1.2.13. *For any connected graph Γ there exists at least one subgraph T which is a tree and such that $V(T) = V(\Gamma)$. Such subtrees T are called* maximal subtrees of Γ.

1.2.3 Bass-Serre theory

Despite the definition we have just given (that will be implicitly adopted everywhere else in this thesis unless otherwise stated), only in this section we need a more refined definition of graph in order to state the fundamental theorem of Bass-Serre theory. Moreover, since we are only interested in the applicative results that this theory provides, we will refer to external references for all proofs but a couple of them (for a complete introduction to the topic see [19]).

Definition 1.2.14. *A graph $\Gamma = (V, E, o, t, ^-)$ in the sense of Serre is a pair of sets (V, E) together with three maps $o, t\colon E \to V$ and $^-\colon E \to E$ such that*

- $\overline{\overline{e}} = e$ *and* $\overline{e} \neq e$, *for each* $e \in E$,
- $o(\overline{e}) = t(e)$, *for each* $e \in E$.

An orientation E^+ for Γ is a subset of E such that $|E^+ \cap \{ e, \overline{e} \} | = 1$ for each $e \in E$.
We say that a group G acts on the graph $\Gamma = (V, E, o, t, ^-)$ if V and E are G-sets and $o, t, ^-$ are maps of G-sets. The action is said to be without inversion of edges *if for all $g \in G$ and $e \in E$ we have $g \cdot e \neq \overline{e}$.*

Lemma 1.2.15. *Let $\Gamma = (V, E, o, t, ^-)$ be a graph in the sense of Serre and let G be a group acting on Γ. Denote with $\mathcal{O}_V$ and $\mathcal{O}_E$ the set of orbits of G on V and E, respectively. Then the tuple $G \mathbin{\backslash\backslash} \Gamma \coloneqq (\mathcal{O}_V, \mathcal{O}_E, \mathfrak{o}, \mathfrak{t}, ^-)$ where*

- $\forall e \in E,\ \mathfrak{o}(G \cdot e) \coloneqq G \cdot o(e)$,
- $\forall e \in E,\ \mathfrak{t}(G \cdot e) \coloneqq G \cdot t(e)$,
- $\forall e \in E,\ \overline{G \cdot e} \coloneqq G \cdot \overline{e}$,

is a well-defined graph in the sense of Serre, called the quotient graph of Γ by G.

Proof. The proof is just an easy check that the maps $\mathfrak{o}, \mathfrak{t}\colon \mathcal{O}_E \to \mathcal{O}_V$ and $^-\colon \mathcal{O}_E \to \mathcal{O}_E$ are well-defined. The other requirements are trivial. ■

Definition 1.2.16. *A* graph of groups $(\Gamma, \mathcal{G})$ *is a connected graph $\Gamma = (V, E, o, t, ^-)$ together with two families of groups $\mathcal{G} = (\mathcal{G}_V, \mathcal{G}_E)$ indexed over the set of vertices and edges of Γ and a family of injective morphisms $\{ \alpha_e\colon G_e \to G_{t(e)} \mid e \in E \}$ such that $G_e = G_{\overline{e}}$ for all $e \in E(\Gamma)$.*

Definition 1.2.17. *The fundamental group of a graph of groups $(\Gamma, \mathcal{G})$ with respect to a maximal subtree T of Γ is*

$$\pi_1(\Gamma, \mathcal{G}, T) \coloneqq \left\langle \begin{matrix} G_v, & v \in V(\Gamma), \\ t_e, & e \in E(\Gamma), \end{matrix} \;\middle|\; \begin{matrix} t_e^{-1}\alpha_e(g)t_e = \alpha_{\overline{e}}(g), & \forall e \in E(\Gamma),\ \forall g \in G_e, \\ t_e t_{\overline{e}} = 1, & \forall e \in E(\Gamma), \\ t_e = 1, & \forall e \in E(T), \end{matrix} \right\rangle. \tag{1.1}$$

It can be shown that the isomorphism type of $\pi_1(\Gamma, \mathcal{G}, T)$ does not depend on the choice of the maximal subtree T in Γ, therefore we will also denote this group simply by $\pi_1(\Gamma, \mathcal{G})$ when the choice of a specific tree T is not relevant.

Theorem 1.2.18 (Bass-Serre fundamental theorem)**.** *If G is the fundamental group of a graph of groups $(\Gamma, \mathcal{G})$, then G acts without inversion of edges on a tree T (called the* Serre tree of $(\Gamma, \mathcal{G})$*) such that $\Gamma \simeq G \backslash\backslash T$ and the stabilisers in G of the vertices and edges of the tree T are isomorphic to G_v $(v \in V(\Gamma))$ and G_e $(e \in E(\Gamma))$, respectively.*
Conversely, if G acts on a tree T without inversion of edges, then G is isomorphic to the fundamental group of the graph of groups $(G \backslash\backslash T, \mathcal{G})$ whose vertex and edge groups are (up to isomorphism) the stabilisers of the vertices and edges of T, respectively.

Proof. See [4, Chapter 8, Theorem 24 and Theorem 26]. ■

The following statement provides a concrete example of the above theorem when the group G is a free product with amalgamation.

Proposition 1.2.19. *Let $G = G_1 *_H G_2$, then there exists a tree T on which G acts without inversion of edges such that $\Gamma = G \backslash\backslash T$ is a segment*[2]*. Moreover, this segment can be lifted to a segment in T such that the stabilisers in G of its vertices and edge are equal to G_1, G_2 and H, respectively.*

Proof. Given $G = G_1 *_H G_2$ we choose the set of vertices of $T = (V, E, o, t, ^-)$ to be the left cosets of G_1 and G_2 in G. As edges we choose the cosets of H in G together with the maps $o(gH) \coloneqq gG_1$ and $t(gH) \coloneqq gG_2$. The action of G on T is given by left multiplication and it follows straight from the definition of T that this action is without inversion of edges.
To show that T is connected observe that the vertices indexed by the cosets G_1 and G_2 are connected by the edge associated to H. For $i = 1, 2$ we show that all vertices indexed by cosets of G_i are connected to G_i: this will imply that T is a connected graph. Fix $i = 1$ (the case for $i = 2$ being analogous) and take any left coset gG_1. The normal form theorem for elements of an amalgamated product tells us that $g = g_0 \dots g_k$ where $g_0 \dots g_k$ is the unique H-normal form of g described in Lemma 1.2.8. We will prove the claim by induction on the length of the H-normal form of g. If $k = 0$, then $g_0 \in H$ implies that $g_0G_1 = G_1$ and the claim is trivial. Now suppose the statement true for $k - 1$: if $g_k \in G_1$, then $g_0 \dots g_{k-1}g_kG_1 = g_0 \dots g_{k-1}G_1$ are the same vertex which is connected to G_1 by the inductive hypothesis; otherwise $g_k \in G_2$ and there is an edge between $g_0 \dots g_{k-1}G_1$

[2] A graph Γ in the sense of Serre is called a *segment* if $V(\Gamma) = \{ v, w \}$, $E(\Gamma) = \{ e, \overline{e} \}$, $o(e) = v$ and $t(e) = w$.

and $g_0 \dots g_{k-1}G_2 = g_0 \dots g_{k-1}g_kG_2$ and from this vertex to $g_0 \dots g_{k-1}g_kG_1$, the inductive hypothesis allows us to conclude again that gG_1 is connected to G_1. In any case we have shown that the vertices G_i and gG_i $(i = 1, 2)$ are connected and therefore T is connected.
To show that T is a tree assume by contradiction that there exists a closed reduced path $e_0, \dots, e_h$ in T of length $h > 0$. Since T is connected we may assume without loss of generality that such path is based at the vertex G_1, i.e. $o(e_0) = t(e_h) = G_1$. For each $j = 1, \dots, h-1$ we have

$$\begin{aligned} o(e_j) \in {}^{G}\!/_{G_1} \quad &\Rightarrow \quad t(e_j) \in {}^{G}\!/_{G_2}, \\ o(e_j) \in {}^{G}\!/_{G_2} \quad &\Rightarrow \quad t(e_j) \in {}^{G}\!/_{G_1}, \end{aligned}$$

therefore h must be odd. Set $t \coloneqq \frac{h-1}{2}$, there exist elements $a_1, \dots, a_t \in G_1 \setminus H$ and elements $b_1, \dots, b_t \in G_2 \setminus H$ such that

$$\begin{array}{llll} o(e_0) = G_1, & o(e_1) = a_1G_2, & o(e_2) = a_1b_1G_1, & \dots \\ \dots & o(e_h) = a_1b_1 \dots a_tG_2, & t(e_h) = a_1b_1 \dots a_tb_tG_1, & \end{array}$$

but $o(e_0) = t(e_h)$ by construction and this is a contradiction with the uniqueness of the H-normal forms for $G = G_1 *_H G_2$. Therefore T is a tree on which G acts without inversion of edges and the set of orbits of the vertices $\mathcal{O}_V = \{ {}^{G}\!/_{G_1}, {}^{G}\!/_{G_2} \}$ has only two elements that are connected by the only orbit ${}^{G}\!/_{H}$ of all edges of T. Therefore $G \setminus\!\!\setminus T$ is a segment. ■

Finally, we provide a couple of applications of what is stated in Theorem 1.2.18 that we will need later.

Theorem 1.2.20 (Nielsen-Schreier). *Let $F = F(X)$ be a free group and $H \leq F$, then H is free.*

Proof. The Cayley graph $T \coloneqq \mathrm{Cay}(F, X)$ of F with respect to the set of generators X is a tree. Indeed, T is connected since X is a generating set for F, therefore T is a tree if and only if there exists a unique reduced path connecting 1_F to g for any $g \in F$. Since reduced paths in T based at 1_F correspond to reduced words in the alphabet X and each element g has a unique reduced representative (see proof of Proposition 1.2.2), T is a tree.
F acts freely on it by left multiplication and without inversion of edges. The induced action of H on T is without inversion of edges as well and Theorem 1.2.18 implies that H is isomorphic to the fundamental group of a graph of groups with trivial edge and vertex groups, therefore it is free. ■

Theorem 1.2.21 (Kurosh). *Let H, K be groups and set $G \coloneqq H * K$, then for any subgroup S of G there exist two families of subgroups $\{ H_i \}_{i \in I}$ and $\{ K_j \}_{j \in J}$ of H*

and K, respectively, together with two families $\{ g_i \}_{i\in I}$, $\{ g_j \}_{j\in J}$ of elements in G and a subset $X \subseteq G$ such that

$$S \simeq F(X) * \coprod_{i\in I}(K_i)^{g_i} * \coprod_{j\in J}(K_j)^{g_j}.$$

Proof. Let T be the Serre tree associated to the free product $G = H*K$ as described in the proof of Proposition 1.2.19. Since G acts on T (by left multiplication), then also S acts on T and by Theorem 1.2.18 S is isomorphic to the fundamental group of the graph of groups $(\Xi, \mathcal{S})$ with $\Xi \coloneqq S \mathbin{\backslash\!\backslash} T$. Each vertex group of $(\Xi, \mathcal{S})$ is, by construction, a subgroup of the stabiliser in G of some vertex of T, then either $G_v \cong H_i^{g_i}$ or $G_v \cong K_j^{g_j}$ for suitable subgroups $H_i \leq H$, $K_j \leq K$ and elements $g_i, g_j \in G$. Moreover, since the edge stabilsers of $(\Gamma, \mathcal{G})$ are trivial, so are the edge stabilisers of $(\Xi, \mathcal{S})$. By Theorem 1.2.18 S is isomorphic to the fundamental group of the graph of groups $(\Xi, \mathcal{S})$ which reads

$$\pi_1(\Xi, \mathcal{G}, R) = \left\langle \begin{matrix} G_v, & v \in V(\Xi), \\ t_e, & e \in E(\Xi), \end{matrix} \;\middle|\; \begin{matrix} t_e t_{\overline{e}} = 1, & \forall e \in E(\Xi), \\ t_e = 1, & \forall e \in E(R), \end{matrix} \right\rangle = \pi_1(\Xi) * \coprod_{v\in V(\Xi)} G_v,$$

where R is any maximal subtree of Ξ and $\pi_1(\Xi)$ is the fundamental group of Ξ in the standard topological sense. Since we observed that each vertex group is conjugated to a subgroup of either H or K through an element of G, the thesis follows. ■

1.2.4 Sigma invariant

In this section we briefly define the Σ^1 invariant of a group G and a couple of theorems that we will use in Section 2.2 to achieve a preliminary result about dihedral Artin groups. However, since the main results in this thesis do not depend on the theory of sigma invariants we will give just the main definitions and the statements we need, referring to external references for all proofs.
Let G be any group (although the typical case is when G is infinite) and denote by $\mathrm{Hom}(G, \mathbb{R})$ the set of homomorphism of groups from G to the additive group of real numbers. The elements of $\mathrm{Hom}(G, \mathbb{R})$ are called *characters*, the set $\mathrm{Hom}(G, \mathbb{R})$ has a natural structure of real vector space and to each $\mu \in \mathrm{Hom}(G, \mathbb{R})$ we associate the monoid

$$G_\mu \coloneqq \{ g \in G \mid \mu(g) \geq 0 \}.$$

Clearly G_μ does not change if μ is replaced by a positive multiple so that the collection of these submonoids can be thought as the set of open rays in $\mathrm{Hom}(G, \mathbb{R})$.

Definition 1.2.22. *Let G be any group, we define the* character sphere $S(G)$ *as the quotient of* $\operatorname{Hom}(G,\mathbb{R})$ *under the equivalence relation '$\sim$' defined as follows.*

$$\forall \mu_1, \mu_2 \in \operatorname{Hom}(G,\mathbb{R})\colon \quad \mu_1 \sim \mu_2 \quad \iff \quad \exists \lambda \in \mathbb{R}^+\colon \mu_1 = \lambda\mu_2,$$
$$S(G) \coloneqq \{\, [\mu]_\sim \mid \mu \in \operatorname{Hom}(G,\mathbb{R}) \setminus \{\,0\,\}\,\}\,.$$

If G is finitely generated, the real vector space $\operatorname{Hom}(G,\mathbb{R})$ has dimension equal to the torsion-free rank of G^{ab} (see [20, Lemma A1.1]), which is finite. Therefore $\operatorname{Hom}(G,\mathbb{R})$ can be endowed with the (essentially unique) topology induced by the standard norm of $\mathbb{R}^n$. As a consequence the space $S(G)$ equipped with the quotient topology is homeomorphic to the unit sphere in a Euclidean vector space of the corresponding dimension.

Definition 1.2.23. *Let G be a finitely generated group, $X \subset G$ a set of generators and $\Gamma = \operatorname{Cay}(G,X)$ the Cayley graph of G with respect to X. For each non-zero homomorphism $\mu \in \operatorname{Hom}(G,\mathbb{R})$ denote by Γ_μ the full subgraph of Γ spanned by the monoid G_μ, then*

$$\Sigma^1(G) \coloneqq \{\, [\mu]_\sim \in S(G) \mid \Gamma_\mu \text{ is connected}\,\}$$

is called the sigma 1 invariant of G *and does not depend on the choice of the generating set X.*

For each $\mu \in \operatorname{Hom}(G,\mathbb{R})$ we define its *rank* as the $\mathbb{Z}$-rank of $\mu(G) \subset \mathbb{R}$. Since the rank is constant over equivalence classes of $\sim$ we define the rank of $[\mu]_\sim \in S(G)$ as the rank of his representative. We are now ready to state the results we are interested in.

Theorem 1.2.24. *The kernel of a rank 1 character $\mu\colon G \twoheadrightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ is finitely generated if and only if $\{\, [\mu]_\sim, [-\mu]_\sim \,\} \in \Sigma^1(G)$.*

Proof. See [20, Corollary A4.3]. ∎

Theorem 1.2.25 (Brown)**.** *Let G be a group given by a presentation $\langle a, b | r\rangle$ where $r = c_1c_2\dots c_k$ is a cyclically reduced, non-empty word containing both generators. Then a non-zero character $\mu\colon G \to \mathbb{R}$ represents a point of $\Sigma^1(G)$ if and only if the sequence $f_r(\mu) = (\mu(c_1),\ \mu(c_1c_2),\ \dots,\ \mu(c_1c_2\dots c_k))$ satisfies the following conditions:*

- *if one of $\mu(a)$ and $\mu(b)$ is zero, then $f_r(\mu)$ achieves its minimum twice,*
- *otherwise $f_r(\mu)$ achieves its minimum once.*

Proof. See [20, Theorem B4.1]. ∎

1.3 Poly-freeness

Definition 1.3.1. *A group G is called* poly-free *if there exists an integer k and a subnormal series of subgroups $(G_i)_{0\leq i\leq k}$ such that*

$$\{ 1_G \} = G_0 \lhd G_1 \lhd \ldots \lhd G_{k-1} \lhd G_k = G$$

and for each $i \in \{ 1, \ldots, k \}$ the factors $\frac{G_i}{G_{i-1}}$ are non-trivial free groups. We will call such subnormal series a poly-free series *for G. We say that G is* poly-fg-free *if there exists a poly-free series for G such that each factor is a (non-trivial) finitely generated free group. In general, poly-(fg)-free series, if they exist, are not unique and we define the poly-(fg)-free length of a poly-(fg)-free group G as the minimum integer k such that G admits a subnormal series as the above one, i.e.*

$$\begin{aligned} \mathrm{pfl}(G) &\coloneqq \min \left\{ k \in \mathbb{N} \;\middle|\; (G_i)_{0\leq i\leq k} \text{ is a poly-free series for } G \right\}, \\ \mathrm{pfl}_{\mathrm{fg}}(G) &\coloneqq \min \left\{ k \in \mathbb{N} \;\middle|\; (G_i)_{0\leq i\leq k} \text{ is a poly-fg-free series for } G \right\}. \end{aligned}$$

We call G strongly poly-(fg)-free *(or* normally poly-(fg)-free*) if there exists a poly-(fg)-free series of G such that each term is a normal subgroup of G. We define* $\mathrm{spfl}(G)$ *and* $\mathrm{spfl}_{\mathrm{fg}}(G)$ *correspondingly.*

Example 1.3.2. *The trivial group $\{ 1 \}$ is (the only) poly-free group of length 0 and non-trivial free groups F are the only poly-free groups of length 1; they are also strongly poly-free. It is also easy to give an example of a group G that admits poly-free series of different length. Let $G = F(\{ a, b \})$ be the free group on two elements. The series $\{ 1_G \} \lhd G$ is obviously strongly poly-free of length 1. If we consider the series of normal subgroups $\{ 1 \} \lhd G' \lhd G$ we have that the first factor is the commutator subgroup of G (which is free since it is the subgroup of a free group, see Theorem 1.2.20) while the second factor is $\frac{G}{G'} = G^{ab} \simeq \mathbb{Z} \times \mathbb{Z}$ which is poly-free (as it is the direct product of two free groups, see Lemma 1.3.6). Starting from this we can build the subnormal series*

$$\{ 1_G \} \lhd G' \lhd \langle\langle G', a \rangle\rangle_G \lhd G$$

which is a strongly poly-free series of length 3 with free factors G' (of infinite countable rank), $\mathbb{Z}$ and $\mathbb{Z}$.
In Corollary 1.3.14 we will show that the length of a strongly poly-fg-free series of a group G is actually unique.

In the following statements we show that poly-freeness has some basic algebraic properties that one may expect when studying properties of a group, e.g. subgroups of a poly-free group are poly-free and the extension of a poly-free group by another poly-free group is poly-free as well.

Lemma 1.3.3. *Let G be a poly-free (respectively, strongly poly-free) group and let $H \leq G$, then H is poly-free (respectively, strongly poly-free) and* $\mathrm{pfl}(H) \leq \mathrm{pfl}(G)$ *(respectively,* $\mathrm{spfl}(H) \leq \mathrm{spfl}(G)$*).*

Proof. Given a poly-free series of minimal length $(G_i)_{0 \leq i \leq n}$ for G, then the sequence $(G_i \cap H)_{0 \leq i \leq n}$ can be refined (after deleting potentially equal consecutive terms) to a poly-free series for H of length at most n. Indeed, for each $i \in \{ 1, \ldots, n \}$, $G_{i-1} \cap H$ is the kernel of the projection of $G_i \cap H$ over $\frac{G_i}{G_{i-1}}$ (which is not surjective in general), therefore $\frac{G_i \cap H}{G_{i-1} \cap H}$ is isomorphic to a subgroup of the free group $\frac{G_i}{G_{i-1}}$ and hence is free (see Theorem 1.2.20). Clearly, if $(G_i)_{0 \leq i \leq n}$ is a strongly poly-free series, so is also $(G_i \cap H)_{0 \leq i \leq n}$. ∎

Remark 1.3.4. *Observe that in the previous lemma we cannot say anything about the rank of the free factors of the poly-free series we build for H since the factors $\frac{G_i \cap H}{G_{i-1} \cap H}$ are just subgroups of the free factors $\frac{G_i}{G_{i-1}}$ and may have arbitrary rank.*

Lemma 1.3.5. *Let G be a group and $H \trianglelefteq G$ a poly-free normal subgroup of G such that $\frac{G}{H}$ is also poly-free, then G is poly-free and* $\mathrm{pfl}(G) \leq \mathrm{pfl}(H) + \mathrm{pfl}(\frac{G}{H})$*. Moreover, if H and $\frac{G}{H}$ are both strongly poly-free, then G is also strongly poly-free and* $\mathrm{spfl}(G) \leq \mathrm{spfl}(H) + \mathrm{spfl}(\frac{G}{H})$*.*

Proof. Let $(P_i)_{0 \leq i \leq h}$ and $(Q_i)_{0 \leq i \leq m}$ be poly-free series of minimal length for H and $\frac{G}{H}$, respectively, and let $\pi \colon G \to \frac{G}{H}$ be the canonical projection on the quotient. For each $i = 0, \ldots, m$ set $R_i \coloneqq \pi^{-1}(Q_i)$, then

$$\{ 1_G \} \lhd P_0 \lhd \ldots \lhd P_h = H = R_0 \lhd \ldots \lhd R_m = G$$

is a poly-free series for G. Indeed, for each $i = 0, \ldots, m$ we have $Q_i \simeq \frac{R_i}{H}$ and by the third homomorphism theorem we get

$$\frac{R_{i+1}}{R_i} \simeq \frac{R_{i+1}/H}{R_i/H} \simeq \frac{Q_{i+1}}{Q_i}, \qquad \forall i = 0, \ldots, m-1,$$

hence $\frac{R_{i+1}}{R_i}$ is free as well and if each Q_i is normal in $\frac{G}{H}$ then each R_i is normal in G. Since the poly-free series we have built has $h + m$ terms it follows that $\mathrm{pfl}(G) \leq \mathrm{pfl}(H) + \mathrm{pfl}(\frac{G}{H})$. Finally, if H and $\frac{G}{H}$ are both strongly poly-free, the same series for G is also strongly poly-free and $\mathrm{spfl}(G) \leq \mathrm{spfl}(H) + \mathrm{spfl}(\frac{G}{H})$. ∎

Notice that the inequality in the previous lemma cannot be made into an equality since given any non-trivial poly-free normal subgroup N of a non-trivial free group F such that $\frac{G}{N}$ is poly-free we have $\mathrm{pfl}(F) = 1$, $\mathrm{pfl}(N) = 1$ (since all subgroups of a free group are free as well, see Theorem 1.2.20) and $\mathrm{pfl}\left(\frac{G}{N}\right) \geq 1$.

Lemma 1.3.6. *Let H, K be poly-free (respectively, strongly poly-free) groups, then $G = H \times K$ is poly-free (respectively, strongly poly-free) and* $\operatorname{pfl}(G) \leq \operatorname{pfl}(H) + \operatorname{pfl}(K)$ *(respectively,* $\operatorname{spfl}(G) \leq \operatorname{spfl}(H) + \operatorname{spfl}(K)$*). Moreover, if H, K are poly-fg-free groups, then G is poly-fg-free.*

Proof. Since G can be seen as an extension of H and K the first part of the thesis follows straight from previous lemma; however in this case a (strongly) poly-free series for H and K can be written explicitly in an easy way. Let $(H_i)_{0 \leq i \leq h}$ and $(K_i)_{0 \leq i \leq k}$ be (strongly) poly-free series of minimal length for H, K respectively, then

$$\{ 1_G \} = H_0 \times K_0 \triangleleft H_1 \times K_0 \triangleleft \ldots \triangleleft H_h \times K_0 \triangleleft H_h \times K_1 \triangleleft \ldots \triangleleft H_h \times K_k = G$$

is a (strongly) poly-free series for G and $\operatorname{pfl}(G) \leq h + k$. This construction also shows that if H, K are poly-fg-free, then G is also poly-fg-free. ■

Definition 1.3.7. *A group G is called* indicable *if it admits a surjective homomorphism onto $\mathbb{Z}$. A group G is called* locally indicable *if each non-trivial finitely generated subgroup $H \leq G$ is indicable.*

Lemma 1.3.8. *A poly-free group G is locally indicable.*

Proof. Let $H \leq G$ be any non-trivial subgroup of G, by Lemma 1.3.3 H is poly-free, hence it admits a poly-free series $(H_i)_{0 \leq i \leq k}$ with $k > 0$. Then $\frac{H}{H_{k-1}}$ is a non-trivial free group F, hence it contains a copy of $\mathbb{Z}$. Let $\pi \colon H \to F$ be the canonical projection, which is surjective, then composing π with the surjective map $F \to \mathbb{Z}$ gives the desired surjection from H to $\mathbb{Z}$. Therefore G is locally indicable. ■

Corollary 1.3.9. *If G is poly-free, then G is torsion-free.*

Proof. By previous Lemma G is locally indicable, hence torsion-free since any torsion element would generate a finite subgroup that could not be mapped onto $\mathbb{Z}$. ■

Corollary 1.3.10. *Every poly-free group G is right orderable (i.e., for every $a, b, g \in G$ there exists a total order '$\leq$' on G such that $a \leq b \Rightarrow ag \leq bg$).*

Proof. It is a consequence of Lemma 1.3.8 and [18, Proposition 1.1] by Rhemtulla and Rolfsen where it is shown that every locally indicable group is right orderable (which in turns is an easy consequence of a result contained in [3] by Burns and Hale). ■

Lemma 1.3.11. *Let H, K be strongly poly-free groups. Set $h := \operatorname{spfl}(H)$ and $k := \operatorname{spfl}(K)$, then the free product $G = H * K$ is strongly poly-free and*

$$\operatorname{spfl}(G) \leq \max \{ h, k \} + 1.$$

Proof. We proceed by induction on $n = \max\{\, h, k \,\}$.
If $n = 1$, then H and K themselves are (possibly trivial) free groups and $G = H * K$ is free as well, hence strongly poly-free.
Now let H, K be groups with $(H_i)_{0\le i\le h}$ and $(K_j)_{0\le j\le k}$ strongly poly-free series of minimal length for H and K, respectively. Consider the projections

$$f\colon H \to \frac{H}{H_1}, \qquad g\colon K \to \frac{K}{K_1}$$

and observe that $\frac{H}{H_1}$ is strongly poly-free with a series given by

$$\{\, 1 \,\} = \frac{H_1}{H_1} \lhd \frac{H_2}{H_1} \lhd \ldots \lhd \frac{H_h}{H_1} = \frac{H}{H_1}.$$

Indeed, by the third homomorphism theorem its factors are isomorphic to the last $h-1$ factors of the strongly poly-free serie of H

$$\frac{{}^{H_i}\!/_{H_1}}{{}^{H_{i-1}}\!/_{H_1}} \simeq \frac{H_i}{H_{i-1}}, \qquad \forall i = 2, \ldots, h.$$

Hence $\frac{H}{H_1}$ is strongly poly-free of length less than or equal to $h-1$. Similarly $\frac{K}{K_1}$ is poly-free of length less then or equal to $k-1$. By the inductive hypothesis we may assume $\frac{H}{H_1} * \frac{K}{K_1}$ strongly poly-free and

$$\operatorname{spfl}\left(\frac{H}{H_1} * \frac{K}{K_1}\right) = \max\{\, h-1, k-1 \,\} + 1 = \max\{\, h, k \,\}.$$

Consider the kernel of the map induced by f and g on the free product of H and K, namely

$$f * g\colon H * K \to \frac{H}{H_1} * \frac{K}{K_1}.$$

By Theorem 1.2.21 there exist a family $\{\, H_i \,\}_{i\in I}$ of subgroups of H, a family $\{\, K_j \,\}_{j\in J}$ of subgroups of K, elements $g_i, g_j \in G$ and a subset $X \subset H * K$ such that

$$\operatorname{Ker}(f * g) = F(X) * \coprod_{i\in I} (H_i)^{g_i} * \coprod_{j\in J} (K_j)^{g_j}. \tag{1.2}$$

Moreover, in our case all subgroups H_i and K_j must be contained inside the free groups $H_1 = \operatorname{Ker}(f)$ and $K_1 = \operatorname{Ker}(g)$, respectively, otherwise there would be some element in Equation (1.2) that does not get sent to the identity by $f * g$. Since H_1 and K_1 are normal subgroups, $\overline{H}_i := H_i^{g_i}$ and $\overline{K}_j := K_j^{g_j}$ still lie inside H_1 and K_1 for each $i \in I$ and $j \in J$, in particular they are free groups. This implies that $\operatorname{Ker}(f * g)$ is free.
In conclusion we have proven that $H * K$ is an extension of $\frac{H}{H_1} * \frac{K}{K_1}$ by the free subgroup $\operatorname{Ker}(f * g)$ of G. Proposition 1.3.5 allows to conclude that the group $H * K$ is strongly poly-free and $\operatorname{spfl}(G) \le \max\{\, h, k \,\} + 1$. ∎

Remark 1.3.12. *The proof of the above lemma is original and we do not know whether it holds under the weaker hypothesis of H, K being just poly-free, however we have not been able to find any counterexample.*

While the length of general poly-free series of a group G may actually behave unexpectedly as shown in Example 1.3.2, under the assumption of restricting to strongly poly-fg-free series the associated length is unique, this is a consequence of a result by Meier which reads as follows.

Theorem 1.3.13 (Meier, [15, Theorem 16])**.** *A group G is poly-locally free if there exists a subnormal series*

$$\{ 1_G \} = N_0 \lhd N_1 \lhd \ldots \lhd N_{n-1} \lhd N_n = G$$

whose factors are locally free groups (i.e., each finitely generated subgroup is free). If all subgroups N_i are normal in G and the abelianized factors $(\frac{N_i}{N_{i-1}})^{ab}$ are non-zero and of finite torsion-free rank as abelian groups, then the homological dimension of G over a field $\mathbb{K}$ of characteristic 0 is equal to n, the number of locally free factors.

Corollary 1.3.14. *All the strongly poly-fg-free series $(G_i)_{0\leq i\leq n}$ of a group G have the same length, which coincides with the homological dimension of G over any field of characteristic 0.*

Proof. Every poly-free group G is poly-locally free since all subgroups of a free group are free as well. Since all factors in a poly-fg-free serie for G are non-trivial finitely generated free groups, the abelian rank of their abelianization equals their free rank which is finite and non-zero. Hence we can apply the theorem above to deduce that the length of a strongly poly-fg-free serie is unique since it equals the homological dimension of G over any field of characteristic 0. ■

1.4 Artin groups

In this section we are going to define the main object of interest for this thesis together with some examples and a brief summary of what is known and what is not. Since Artin groups are closely related to Coxeter groups we will start giving the definition of the latter, whose definition relies on a combinatorial type of data represented by a Coxeter matrix or a Coxeter graph.

Definition 1.4.1. *A square matrix $M = (m_{ij})$ of dimension $n \in \mathbb{N}^+$ is called a* Coxeter matrix *if it is symmetric and*

i) $m_{ii} = 1$ *for each* $i = 1, \ldots, n$, *ii)* $m_{ij} = m_{ji} \in \{ 2, 3, \dots \} \cup \{ \infty \}$ *for each* $i, j = 1, \dots, n$ *and* $i \neq j$.

Definition 1.4.2. *A* Coxeter graph *is a simplicial graph* Γ *with a finite number of vertices and edges, each of the latter labelled with integers greater than or equal to '2'. For later purpose we define* Γ_{odd} *as the Coxeter graph obtained from* Γ *after the deletion of all edges labelled with an even integer.*

When drawing a Coxeter graph we adopt the convention to mark each edge with its associated label except for those with label '2', which will be tacitly understood. A Coxeter matrix brings the same amount of information as a Coxeter graph. The bijection between the former and the latter objects is given by sending a Coxeter matrix M to the Coxeter graph Γ having the set of vertices indexed over the set of columns (or rows) of M. Two distinct vertices i, j in Γ are linked by an edge e labelled with $m_e \coloneqq m_{ij}$ if and only if $m_{ij} \neq \infty$ in M. Clearly this correspondence is a bijection.
However, depending on the which families of Coxeter groups one is dealing with, it could be more convenient to use another (equivalent) convention for Coxeter graphs. From a Coxeter graphs Γ as defined above we can build the labelled graph $\dot{\Gamma}$ adding an edge labelled with '∞' for any pair of unlinked vertices in Γ and deleting all edges labelled with '2' in Γ. When drawing this type of Coxeter graph we adopt the convention to mark each edge with its associated label except for those with label '3', which will be implicit.

Notation 1.4.3. *Whenever we will consider a Coxeter graph denoted by a Greek upper case letter with a dot above it (e.g.,* $\dot{\Gamma}$*) it will be implicit that such graph has to be understood following the alternative definition we have just given. Instead, whenever we denote a graph by a plain Greek upper case letter, it has to be understood as described in Definition 1.4.2.*
In particular, in Chapter 2 we will always adopt the second convention we described (the one denoted by $\dot{\Gamma}$*), while in Chapter 3 we will always use the first definition we gave for a Coxeter graph* Γ.

Clearly this construction can be reversed and there is a bijection between the two types of Coxeter graphs: we are introducing this alternative version just to ease the representation of some families of Coxeter graphs with a lot of edges labelled with '2'.

Definition 1.4.4. *Let* Γ *be a Coxeter graph with vertex set* $\{ s_1, \dots, s_n \}$*, we define the Coxeter group* $\mathcal{W}(\Gamma)$ *of type* Γ *as the group having the following presentation*

$$\mathcal{W}(\Gamma) \coloneqq \left\langle s_i, \ \forall s_i \in V(\Gamma) \ \middle| \ \begin{array}{c} s_i^2 = 1, \quad \forall s_i \in V(\Gamma), \\ (s_i s_j)^{m_e} = 1, \quad \forall e = \{ s_i, s_j \} \in E(\Gamma) \end{array} \right\rangle .$$

Coxeter groups have been introduced in the thirties by H. S. M. Coxeter. They arose during the study of real reflection groups, i.e. subgroups of the automorphisms of Euclidean vector spaces generated by elements of order 2 called reflections. Indeed, the relator $s_i^2 = 1$ that holds for each $s_i \in \mathcal{W}(\Gamma)$ can be geometrically interpreted saying that each s_i is an automorphisms of a vector space fixing an hyperplane τ_i pointwise and reflecting the remaining points across it. Given such action of the s_i's on the vector space, the product $s_i s_j$ of two distinct reflections corresponds to a rotation of twice the angle α between the hyperplanes τ_i and τ_j. If $m := \frac{\pi}{\alpha}$ is an integer, then $s_i s_j$ is an element of $\mathrm{Aut}(\mathbb{R}^n)$ of order m and the vertices s_i, s_j of Γ are connected by an edge labelled with m; otherwise, if m is not an integer, the product $s_i s_j$ is of infinite order and we set no conditions on it in the presentation of $\mathcal{W}(\Gamma)$. This geometric interpretation explains the second type of relations that appear in the definition of Coxeter groups. In particular, when a pair of vertices is linked by an edge labelled with '2' it means that they commute (since each generator is also a reflection)

$$(s_i s_j)^2 = 1 \implies s_i s_j = s_j^{-1} s_i^{-1} \implies s_i s_j = s_j s_i.$$

For a Coxeter graph Γ, consider its related graph $\dot{\Gamma}$: if $\dot{\Gamma}$ has $\dot{\Gamma}_1, \ldots, \dot{\Gamma}_d$ connected components, then the presentation of $\mathcal{W}(\dot{\Gamma})$ is the same presentation as $\mathcal{W}(\dot{\Gamma}_1) \times \ldots \times \mathcal{W}(\dot{\Gamma}_d)$ since each pair of generators belonging to different connected components commutes. This leads to the following definition.

Definition 1.4.5. *Let Γ be a Coxeter graph. If $\dot{\Gamma}$ is connected, then $\mathcal{W}(\Gamma)$ is called* irreducible.

The main result in this area is due to Coxeter who showed that each real reflection group admits a presentation in the form of a Coxeter group and each Coxeter group admits a linear representation over $\mathrm{Aut}(\mathbb{R}^n)$ for some $n \in \mathbb{N}^+$ (the first part of this statement is proven in [6] (1934) while the second is contained in [7] (1935)). Moreover, in the case of finite reflection groups such correspondence is one-to-one, whereas there exist infinite Coxeter groups that do not admit a faithful representation as a Euclidean reflection group.
H. S. M. Coxeter in [7] also achieved the classification of all finite irreducible Coxeter groups shown in Table 1.1.

Artin groups are obtained from Coxeter groups removing the condition that generators must be elements of order 2. Before giving the definition of Artin groups let us establish the following notation for the sake of simplicity.

Notation 1.4.6. *For any finite set of symbols $a_1, a_2, \ldots, a_n$ and each integer $m \in \mathbb{N}$ we set*

$$\Pi(a_1, a_2, \ldots, a_n; m) := \underbrace{a_1\ a_2\ \ldots\ a_n\ a_1\ a_2\ \ldots}_{m \text{ symbols}}.$$

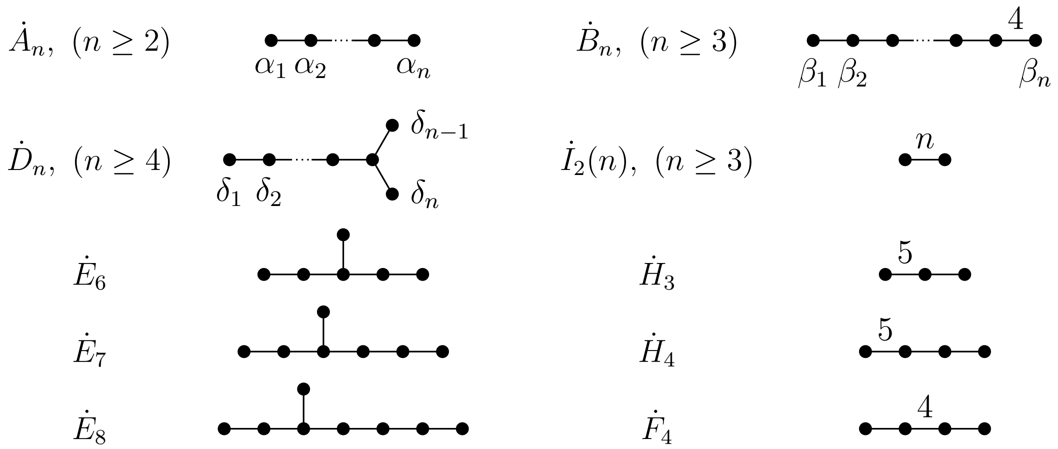

Table 1.1: Classification of all finite irreducible Coxeter groups.

Definition 1.4.7. *Let* Γ *be a Coxeter graph (as in Definition 1.4.2) with vertex set* $V(\Gamma) = \{ a_1, \ldots, a_n \}$*, we define the Artin group* $\mathcal{A}(\Gamma)$ *of type* Γ *as the group having the following presentation*

$$\mathcal{A}(\Gamma) \coloneqq \langle a_i,\ a_i \in V(\Gamma) \mid \Pi(a_i, a_j; m_e) = \Pi(a_j, a_i; m_e),\ \forall e = \{ a_i, a_j \} \in E(\Gamma) \rangle \,.$$

For a Coxeter graph of type $\dot{\Gamma}$ *(as described in Notation 1.4.3) we set* $\mathcal{A}(\dot{\Gamma}) \coloneqq \mathcal{A}(\Gamma)$*.*

Definition 1.4.8. *If* Γ *is any graph whose related graph* $\dot{\Gamma}$ *appears in Table 1.1 the associated Artin group* $\mathcal{A}(\Gamma)$ *is called of* finite type*. For simplicity we will denote such groups with the calligraphic version of the letter that represents the graph (e.g.,* $\mathcal{A}_n = \mathcal{A}(\dot{A}_n)$*,* $\mathcal{B}_n = \mathcal{A}(\dot{B}_n)$*,* $\mathcal{I}_2(n) = \mathcal{A}(\dot{I}_2(n))$*, etc...).*

Since Coxeter groups are closely related to vector spaces, identifying them with a Coxeter matrix instead of a Coxeter graph may sometimes be advantageous; however, since we study Artin groups, we will always use Coxeter graphs and with a slight abuse of notation we will denote with the same symbols (e.g., a_i) both the vertices of Γ and the generators of $\mathcal{A}(\Gamma)$. Moreover, when useful, we will write the defining relation of $\mathcal{A}(\Gamma)$ associated to the edge e with endings a_i and a_j as

$$\begin{aligned} R_{i,j} \coloneqq R_e &\coloneqq \Pi(a_i, a_j; m_e) \cdot \Pi(a_j, a_i; m_e)^{-1} \\ &= \begin{cases} a_i\ a_j \ldots\ a_i\ a_j^{-1}\ a_i^{-1}\ \ldots\ a_j^{-1}, & m_e \text{ odd}, \\ a_i\ a_j \ldots\ a_j\ a_i^{-1}\ a_j^{-1}\ \ldots\ a_j^{-1}, & m_e \text{ even}, \end{cases} \end{aligned}$$

so that we can write more compactly

$$\mathcal{A}(\Gamma) = \langle a_i,\ a_i \in V(\Gamma) \mid R_e = 1,\ \forall e \in E(\Gamma) \rangle \,.$$

If needed, to avoid confusion we will specify with a superscript to which Coxeter graph or Artin group the relation R_e refers (e.g., we will write either R_e^{Γ} or $R_e^{\mathcal{A}(\Gamma)}$).

Each Coxeter group $\mathcal{W}(\Gamma)$ is the quotient of the corresponding Artin group $\mathcal{A}(\Gamma)$ under the kernel of the unique morphism of groups defined extending the map φ which sends each generator a_i of $\mathcal{A}(\Gamma)$ to the generator s_i of $\mathcal{W}(\Gamma)$. The existence and uniqueness of such map is guaranteed by Proposition 1.2.6 since for each relator of $\mathcal{A}(\Gamma)$ we have

$$\varphi(R_e) = \begin{cases} s_i\ s_j \dots\ s_i\ s_j^{-1}\ s_i^{-1}\ \dots\ s_j^{-1}, & m_e \text{ odd},\\ s_i\ s_j \dots\ s_j\ s_i^{-1}\ s_j^{-1}\ \dots\ s_j^{-1}, & m_e \text{ even}, \end{cases} = (s_i s_j)^{m_e} = 1.$$

The kernel of $\varphi\colon \mathcal{A}(\Gamma) \to \mathcal{W}(\Gamma)$ is called the *pure Artin group of type* Γ and we will denote it with $\mathcal{P}(\Gamma)$. In the next chapters we will be interested in studying a few structure properties of some families of Artin groups, in particular poly-freeness.

Since each normal subgroup is the kernel of a morphism of groups, building a poly-free series for a group G essentially amounts to find the right morphisms going from G to another smaller poly-free group and iterate this process. With this in mind we will make extensive use of the following map defined for any Artin group.

Lemma 1.4.9. *Let Γ be a Coxeter graph, then the map $\chi_\Gamma\colon \mathcal{A}(\Gamma) \to \mathbb{Z}$ defined on the generators of $\mathcal{A}(\Gamma)$ in the following way*

$$a_i \mapsto 1, \qquad \forall a_i \in V(\Gamma)$$

extends (uniquely) to a well-defined morphism of groups and it is surjective if Γ is non-empty.
Moreover, let $\Lambda_1, \dots, \Lambda_d$ be the connected components of Γ_{odd}, then the maps $\chi_\Gamma^{(h)}$ $(h = 1, \dots, d)$ defined sending

$$\begin{cases} a_i \mapsto 1, & a_i \in V(\Lambda_h),\\ a_i \mapsto 0, & a_i \notin V(\Lambda_h), \end{cases}$$

extends (uniquely) to a well-defined epimorphism of groups.

Proof. We just have to apply Proposition 1.2.6 to check that χ_Γ is well-defined. For any edge e of Γ let R_e be the associated relator in the presentation of $\mathcal{A}(\Gamma)$. By construction each R_e is a word of even length containing the generators of $\mathcal{A}(\Gamma)$ in the first half and their inverses in the other half. This implies that $\chi_\Gamma(R_e) = 0_{\mathbb{Z}}$ for any edge e of Γ and by the proposition cited above χ_Γ is morphism of groups. Clearly χ_Γ is surjective if Γ is non-empty.
For $h = 1, \dots, d$ the proof for the map $\chi_\Gamma^{(h)}$ is analogous observing that if a_i, a_j are distinct vertices of Λ_h then the same argument as above applies; otherwise if two

vertices $a_i \in V(\Lambda_h)$ and $a_j \in V(\Lambda_k)$ (with $k \neq h$) are connected inside Γ the integer m_{ij} must be even and both a_i and a_j appear inside the relator R_{ij} as many times as their inverses, so that $\chi_\Gamma^{(h)}(R_{ij}) = 0$. We conclude that $\chi_\Gamma^{(h)}$ is a well-defined morphism of groups and since each connected component has at least one vertex by definition each $\chi_\Gamma^{(h)}$ is surjective. ■

From now on, given a Coxeter graph Γ, we will denote by $\chi_\Gamma \colon \mathcal{A}(\Gamma) \to \mathbb{Z}$ the homomorphism of groups defined in Lemma 1.4.9.

Notation 1.4.10. *For a Coxeter graph Γ we denote by $\mathcal{A}'(\Gamma)$ the commutator subgroup of $\mathcal{A}(\Gamma)$.*

Lemma 1.4.11. *Let Γ be a Coxeter graph, then $\mathcal{A}'(\Gamma) = \mathrm{Ker}(\chi_\Gamma)$ if and only if Γ_{odd} is connected.*

Proof. Since Artin groups are defined by means of their presentation it is easy to compute the abelianization

$$\mathcal{A}(\Gamma)^{\mathrm{ab}} = \frac{\mathcal{A}(\Gamma)}{\mathcal{A}'(\Gamma)} = \langle a_i \mid R_e = 1 \ (\forall e \in E(\Gamma)),\ a_i a_j = a_j a_i \ (\forall a_i, a_j \in V(\Gamma)) \rangle .$$

Given any edge $e = \{ a_i, a_j \} \in E(\Gamma)$ the abelianization of the corresponding relation R_e leads to $a_i = a_j$ if m_e is odd and to the identity if m_e is even, therefore

$$\mathcal{A}(\Gamma)^{\mathrm{ab}} \simeq \mathbb{Z}^d$$

where d is the number of connected components of Γ_{odd} and the isomorphism is given by sending any element $g = a_{i_1}^{k_1} \dots a_{i_t}^{k_t} \in \mathcal{A}(\Gamma)$ to $(\chi_\Gamma^{(1)}(g), \dots, \chi_\Gamma^{(d)}(g)) \in \mathbb{Z}^d$. Now, suppose $\mathcal{A}'(\Gamma) = \mathrm{Ker}(\chi_\Gamma)$, this means that

$$\mathcal{A}(\Gamma)^{\mathrm{ab}} = \frac{\mathcal{A}(\Gamma)}{\mathcal{A}'(\Gamma)} = \frac{\mathcal{A}(\Gamma)}{\mathrm{Ker}(\chi_\Gamma)} \simeq \mathbb{Z},$$

therefore $d = 1$ and Γ_{odd} is connected. Conversely, suppose Γ_{odd} is connected, by the observation above the isomorphism between $\mathcal{A}(\Gamma)^{\mathrm{ab}}$ and $\mathbb{Z}$ is given by $\chi_\Gamma^{(1)} = \chi_\Gamma$, so that $\mathcal{A}'(\Gamma) = \mathrm{Ker}\left(\chi_\Gamma^{(1)}\right) = \mathrm{Ker}(\chi_\Gamma)$. ■

For the same reason explained in previous paragraph we will also make wide usage of the following well-known property.

Proposition 1.4.12. *Let Γ be a Coxeter graph and let S be a (non-empty) subset of the vertices of Γ. Let Σ be the full subgraph spanned by S inside Γ, then the subgroup generated inside $\mathcal{A}(\Gamma)$ by the elements associated to the vertices in S is isomorphic to the abstract Artin group $\mathcal{A}(\Sigma)$ and the isomorphism is the most obvious one, namely*

$$S \ni a_i \mapsto a_i \in \mathcal{A}(\Sigma).$$

Proof. The original proof has a topological approach and is due to Van der Lek in [22, Chapter II, Theorem 4.13] (1983). Also, Paris gave an alternative exclusively algebraic proof in [17, Theorem 3.1] (1997). ■

1.5 An example

Before moving further we give an extensive example of all the definitions we gave above taking as Coxeter graphs the family of type $\dot{A}_n$ ($n \geq 2$). Artin groups of this type are called "braid groups" due to the geometric interpretation we will explain below. Also, these have been the very first family of Artin groups to be studied due to their interesting properties and connections with other fields of mathematics such as knot and link theory or mapping class groups; actually Artin groups may be regarded as some kind of generalisation of those. In particular we will show that pure Artin groups of this type are poly-fg-free by means of their connection with some topological objects. All we are going to show in this section is already well-known in the literature and can mostly be found in the book by Kassel and Turaev [12, Chapter I]. We will use it as a reference for some technical statements for which we will not provide proofs as well as the source for some images included below.

The explicit presentations of the Coxeter and Artin groups of type $\dot{A}_n$ are as follows

$$\mathcal{W}(\dot{A}_n) = \left\langle s_i,\ i = 1,\dots,n \;\middle|\; \begin{array}{cc} s_i^2 = 1, & i = 1,\dots,n, \\ s_i s_j = s_j s_i, & i,j = 1,\dots,n,\ i \neq j, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, & i = 1,\dots,n-1 \end{array} \right\rangle,$$
$$\mathcal{A}_n = \left\langle a_i,\ i = 1,\dots,n \;\middle|\; \begin{array}{cc} a_i a_j = a_j a_i, & i,j = 1,\dots,n,\ i \neq j, \\ a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}, & i = 1,\dots,n-1 \end{array} \right\rangle.$$

Let us start identifying the Coxeter group $\mathcal{W}(\dot{A}_n)$ as the symmetric group $\mathcal{S}_{n+1}$ on $n+1$ objects. By Proposition 1.2.6 it is a straightforward check that the map sending each generator s_i of $\mathcal{W}(\dot{A}_n)$ to the simple transposition $(i, i+1)$ inside $\mathcal{S}_{n+1}$ extends uniquely to a morphism $\varphi\colon \mathcal{W}(\dot{A}_n) \to \mathcal{S}_{n+1}$ (indeed, any two transpositions $(i, i+1)$ and $(j, j+1)$ with $|i-j| \geq 2$ commute and $(i, i+1)(i+1, i+2)(i, i+1) = (i+1, i+2)(i, i+1)(i+1, i+2)$ holds for each $i = 1, \dots, n$). Since the set of simple transpositions $\{\, (i, i+1) \mid i = 1, \dots, n \,\}$ generates $\mathcal{S}_{n+1}$ it follows that φ is surjective. This means that $\mathcal{S}_{n+1} \simeq \frac{\mathcal{W}(a_n)}{\mathrm{Ker}(\varphi)}$ admits a representation like

$$\left\langle s_i,\ i = 1,\dots,n \;\middle|\; \{\, s_i^2 = 1,\ s_i s_j = s_j s_i,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \,\} \cup R \right\rangle$$

where R is a (possibly empty) set of words (relations) whose normal closure generates $\mathrm{Ker}(\varphi)$ inside $\mathcal{W}(\dot{A}_n)$. However, it is possible to show (see [12, Theorem 4.1]) that

the presentation above with $R = \emptyset$ is already a presentation for $\mathcal{S}_{n+1}$, so that $\mathcal{W}(\dot{A}_n) \simeq \mathcal{S}_{n+1}$; in a more informal language each word in the simple transpositions of $\mathcal{S}_{n+1}$ that equals the identity can be reduced using only the relators that appear in the definition of $\mathcal{W}(\dot{A}_n)$. Therefore, $\mathcal{S}_{n+1}$ provides a concrete interpretation for the abstract Coxeter groups of type $\dot{A}_n$.
From this identification we can give the following description of the action of $\mathcal{W}(\dot{A}_n)$ on a real vector space of dimension n: each generator s_i (i.e., a simple transposition) acts by a permutation of the i^{th} and $(i+1)^{\text{th}}$ component of each vector with respect to any fixed basis for the vector space.

The description of $\mathcal{A}_n$ is even more geometric as it can be identified with the group of geometric braids up to isotopy, however we need a couple of preliminary definitions before giving the exact statement.

Definition 1.5.1. *A* geometric braid *b on $n \geq 1$ strings is a subset $b \subset \mathbb{R}^2 \times [0,1]$ formed by n disjoint topological intervals called the* strings *of b such that the projection $\mathbb{R}^2 \times [0,1] \to [0,1]$ maps each string homeomorphically onto $[0,1]$ and*

$$\begin{aligned} b \cap (\mathbb{R}^2 \times \{ 0 \}) &= \{ (1,0,0), (2,0,0), \dots, (n,0,0) \}, \\ b \cap (\mathbb{R}^2 \times \{ 1 \}) &= \{ (1,0,1), (2,0,1), \dots, (n,0,1) \}. \end{aligned}$$

We will denote with $\mathfrak{G}_n$ the set of geometric braids on n strings.

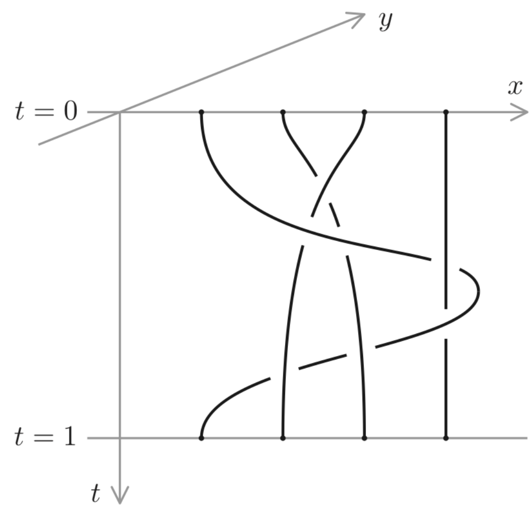

Figure 1.1: A geometric braid on 4 strings.

Two braids on the same number of strings b_1 and b_2 are isotopic if there exists a continuous map $F \colon b_1 \times [0,1] \to \mathbb{R}^2 \times [0,1]$ such that for each $t \in [0,1]$ the map $F_t \coloneqq F(\cdot, t) \colon b_1 \to \mathbb{R}^2 \times [0,1]$ is an embedding whose image is a geometric braid on

n strings, $F(\cdot, 0) = \mathrm{Id}_{b_1}$ and $F(b_1, 1) = b_2$. Notice that since $F_t(b_1)$ is a geometric braid on n strings for each $t \in [0, 1]$ and F is continuous by assumption, then F_t fixes pointwise all starting and ending points of each string. It is an easy check to verify that being isotopic is an equivalence relation on the set of geometric braids and we denote it with '$\sim$'. We will call $\mathcal{G}_n := \frac{\mathfrak{G}_n}{\sim}$ the set of *braids* whose objects, loosely speaking, are determined only by how strings are interlaced ignoring continuous deformations that fix their endings.
Also, for any pair of geometric braids b_1 and b_2 on the same number of strings we define their geometric product as their concatenation (together with a suitable reparametrization), i.e.

$$b_1 b_2 = \left\{ (x, y, t) \in \mathbb{R}^2 \times [0, 1] \;\middle|\; (x, y, 2t) \in b_1 \;\text{ or }\; (x, y, 2t - 1) \in b_2 \right\}.$$

The above geometric product induces a well-defined operation on the set of braids $\mathcal{G}_n := \frac{\mathfrak{G}_n}{\sim}$ just setting $[b_1]_\sim [b_2]_\sim := [b_1 b_2]_\sim$. The set $\mathcal{G}_n$ together with this product turns out to be a group since the operation is easily checked to be associative, the identity is given by the braid with n straight strings and the inverse of an element $[b]_\sim \in \mathcal{G}_n$ is the equivalence class represented by the braid $\overline{b}$ obtained reflecting b across the hyperplane with equation $z = \frac{1}{2}$.
With the help of braid diagrams and of the related Reidemeister moves (in a much similar way to the theory of Reidemeister moves for knots) it can be shown that $\mathcal{A}_n \simeq \mathcal{G}_{n+1}$ under the mapping sending a generator a_i $(i = 1, \ldots, n)$ of $\mathcal{A}_n$ to the equivalence class of the geometric braid σ_i with $n + 1$ strings obtained from the identity by twisting the i^{th} and $(i + 1)^{\text{th}}$ strings together as shown in Figure 1.2.

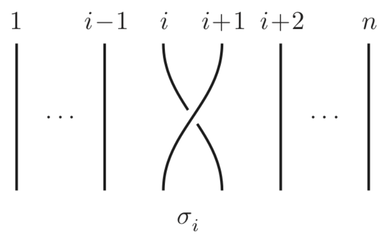

Figure 1.2: A representative of the braid σ_i.

A complete proof of this isomorphism is rather long and may get technical; we point the interested reader to [12, Section 1.2] and we restrict ourselves to observe that the images of the LHS and RHS of each defining relation of $\mathcal{A}_n$ do get sent to the same element of $\mathcal{G}_n$ as shown in Figures 1.3 and 1.4.

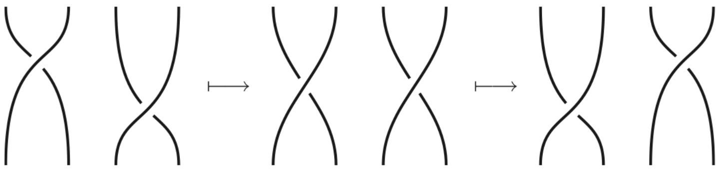

Figure 1.3: Isotopy transforming $\sigma_i\sigma_j$ to $\sigma_j\sigma_i$.

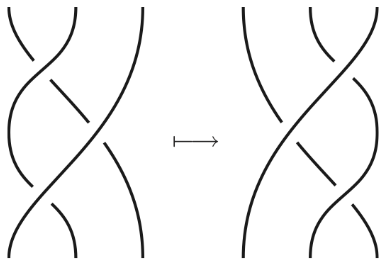

Figure 1.4: Isotopy transforming $\sigma_i\sigma_{i+1}\sigma_i$ to $\sigma_{i+1}\sigma_i\sigma_{i+1}$.

Under this geometric interpretation of $\mathcal{A}_n$ it is possible to give the following description of the standard projection to $\mathcal{W}(\dot{A}_n)$: each element $g = \sigma_{i_1}^{\varepsilon_1}\dots\sigma_{i_k}^{\varepsilon_k} \in \mathcal{G}_{n+1} \simeq \mathcal{A}_n$ is sent to the permutation

$$\begin{pmatrix} 1 & 2 & \dots & n & n+1 \\ t(1) & t(2) & \dots & t(n) & t(n+1) \end{pmatrix}$$

where $t(i)$ denotes the position of the ending of the string starting at the i^{th} place. The kernel of this projection is the group of pure braids $\mathcal{P}_n \coloneqq \mathcal{P}(\dot{A}_n)$. Its elements are exactly those braids whose underlying permutation is the identity. We are going to show that these groups are poly-free leveraging a topological argument involving certain spaces whose fundamental group is exactly $\mathcal{P}_n$.
Let $\iota_n \colon \mathcal{A}_n \to \mathcal{A}_{n+1}$ the morphism defined sending each generator σ_i $(i = 1, \dots, n)$ of $\mathcal{A}_n$ to the braid ς_i obtained from a geometric representative of σ_i adding a $(n+1)^{\text{th}}$ string unlinked from the previous (this operation is well-defined on the set of braids). If we restrict to the pure braid group $\mathcal{P}_n$ the map ι_n admits a section $f_n \colon \mathcal{P}_n \to \mathcal{P}_{n-1}$ which sends a braid b to the braid b' obtained deleting the last string of b (note that such map is well-defined since we are working with pure braids whose strings does not permute their endings). Thinking in terms of braids it is readily checked that $f_{n+1} \circ \iota_n = \mathrm{Id}_{\mathcal{P}_n}$, which in particular implies that ι_n is injective and f_n is surjective for all $n \geq 2$. The subgroups we are interested in to build a poly-free series for $\mathcal{P}_n$ are the kernels of the family of maps f_k

$$U_k \coloneqq \mathrm{Ker}(f_k \colon \mathcal{P}_k \to \mathcal{P}_{k-1}), \qquad k = 1, \dots, n.$$

Theorem 1.5.2. *For each $n \geq 2$ the group U_n is free on $n-1$ generators.*

Proof. The core idea of the proof is to build a topological space whose fundamental group is $\mathcal{P}_n$ and apply the long exact sequence of homotopy groups induced by a Serre fibration. The candidate for the space we need comes from looking at a geometrical braid as the one in Figure 1.1: consider the projection of each string on the plane with equation $z = 0$, their images already look like the collection of n closed paths in $\mathbb{R}^2$ drawn in the same plane. The idea at the base of the following definition is to realize the image of the projection of the n strings as a single closed path in an appropriate space (rather than the union of n distinct closed path inside the same plane). For $n \geq 2$ we define the configuration space $\mathcal{C}_n$ as follows

$$\mathcal{C}_n \coloneqq \left\{ (x_1, \dots, x_n) \in \left(\mathbb{R}^2\right)^n \;\middle|\; x_i \neq x_j,\ \forall i \neq j \in \{ 1, \dots, n \} \right\}.$$

Notice that each coordinate of a point in $\mathcal{C}_n$ must be distinct from the others (otherwise we would have $\mathcal{C}_n = \mathbb{R}^{2n}$ whose fundamental group is trivial): this restriction we adopt in the definition is harmless to our purpose since by definition a geometric braid excludes the possibility that two strings intersect each other in one or more points. In this way we have a bijection between the set of braids on n strings and the homotopy classes of closed paths in $\mathcal{C}_n$. To any geometric braid b we can associate a closed path $p\colon [0,1] \to \mathcal{C}_n$ starting at the point $((1,0), \dots, (n,0))$ and sending $t \in [0,1]$ to $(p_1(t), \dots, p_n(t))$ where $p_i(\cdot)$ denotes the projection on the first two coordinates of the i^{th} string of b. Conversely, given a closed path $p = (p_1(\cdot), \dots, p_n(\cdot))\colon [0,1] \to \mathcal{C}_n$ we associate to it the braid represented geometrically by the union of the strings $(p_i(t), t)$ inside $\mathbb{R}^3$ for $i = 1, \dots, n$ (notice that any pair of these strings does not intersect inside $\mathbb{R}^3$ because of the condition we set in the definition of configuration spaces). We also define $\mathcal{C}_{m,n} \coloneqq \mathcal{C}_n \setminus Q_m$ where Q_m is any set of m points in $\mathcal{C}_n$. Given these definitions the strategy is as follows.

- Prove that for $n > r \geq 1$, the projection map

$$\begin{aligned} p_{n,r}\colon \mathcal{C}_{m,n} &\to \mathcal{C}_{m,r} \\ (x_1, \dots, x_n) &\mapsto (x_1, \dots, x_r) \end{aligned}$$

  is a locally trivial fibration with fiber $\mathcal{C}_{m+r,n-r}$. This is quite technical and is proven in [12, Lemma 1.27 with $M = \mathbb{R}^2$].

- Prove that for any $m \geq 0,\ n \geq 1$, the group $\pi_i(\mathcal{C}_{m,n})$ is trivial for all $i \geq 2$. To achieve this it is enough to apply the long exact homotopy given by the fibration $p_{n,1}\colon \mathcal{C}_{m,n} \to \mathcal{C}_{m,1} = \mathbb{R}^2 \setminus Q_m$ with fiber $\mathcal{C}_{m+1,n-1}$

$$\dots \to \pi_{i+1}(\mathbb{R}^2 \setminus Q_m) \to \pi_{i+1}(\mathcal{C}_{m+1,n-1}) \to \pi_{i+1}(\mathcal{C}_{m,n}) \to \pi_i(\mathbb{R}^2 \setminus Q_m) \to \dots .$$

Since $\pi_i(\mathbb{R}^2 \setminus Q_m)$ is trivial for each $i \geq 2$, we have

$$\pi_i(\mathcal{C}_{m+1,n-1}) \simeq \pi_i(\mathcal{C}_{m,n}), \quad \forall i \geq 2,$$

which means that

$$\pi_i(\mathcal{C}_{m,n}) \simeq \pi_i(\mathcal{C}_{m+n-1,1}) = \pi_i(\mathbb{R}^2 \setminus Q_{m+n-1}) = \{ 1 \}, \quad \forall i \geq 2.$$

- Finally, since $\mathbb{R}^2$ minus a finite set of points is connected, then $\pi_0(\mathbb{R}^2 \setminus Q_m)$ has only one element (although it is not canonically a group) and the homotopy sequence induced by $p_{n,1}$ is non-trivial only for $i = 1$

$$1 \to \pi_1(\mathcal{C}_{n-1,1}) \to \pi_1(\mathcal{C}_{0,n}) \xrightarrow{p^*_{n,1}} \pi_1(\mathcal{C}_{0,n-1}) \to 1.$$

By the bijection we constructed between homotopy classes of closed paths in $\mathcal{C}_{0,n}$ and pure braids we can actually replace $\pi_1(\mathcal{C}_{0,n})$ and $\pi_1(\mathcal{C}_{0,n-1})$ with $\mathcal{P}_n$ and $\mathcal{P}_{n-1}$ respectively and obtain

$$1 \to \pi_1(\mathcal{C}_{n-1,1}) \to \mathcal{P}_n \xrightarrow{f_n} \mathcal{P}_{n-1} \to 1.$$

This means that $U_n = \operatorname{Ker}(f_n \colon \mathcal{P}_n \to \mathcal{P}_{n-1}) = \pi_1(\mathcal{C}_{n-1,1})$ is free, since $\mathcal{C}_{n-1,1}$ is $\mathbb{R}^2$ minus $n-1$ points, whose fundamental group is free on $n-1$ generators.

■

Corollary 1.5.3. *For each $n \geq 2$ the group $\mathcal{P}_n$ is strongly poly-free and it admits a poly-fg-free series*

$$\{ 1 \} = U_n^{(0)} \lhd U_n^{(1)} \lhd \ldots \lhd U_n^{(n-1)} = \mathcal{P}_n$$

where $\frac{U_n^{(i)}}{U_n^{(i-1)}}$ is free of rank $n-i$ for each $i = 1, \ldots, n-1$.

Proof. Choose $U_n^{(0)} = \{ 1 \}$ and for each $i = 1, \ldots, n-1$ set

$$U_n^{(i)} \coloneqq \operatorname{Ker}(f_{n-i+1} \circ \ldots \circ f_n \colon \mathcal{P}_n \to \mathcal{P}_{n-i}).$$

Thinking in terms of braids we see that the elements of $\frac{U_n^{(i)}}{U_n^{(i-1)}}$ are precisely those which get sent to the identity by the map f_{n-i+1}, therefore

$$\frac{U_n^{(i)}}{U_n^{(i-1)}} \simeq \operatorname{Ker}(f_{n-i+1} \colon \mathcal{P}_{n-i+1} \to \mathcal{P}_{n-i}) \simeq F_{n-i}$$

and the statement follows. ■

Obviously this proof of the poly-freeness of pure braid groups heavily relies on their connection with topological objects which allows to leverage a number of powerful tools. However, for general Artin groups such a proof cannot be carried out. Motivated by the result of [2, Theorem 3.17] (2017) where Blasco-García, Martínez-Pérez and Paris prove using an exclusively algebraic method that even Artin groups of FC type[3] are poly-free, the aim of this thesis is to find new results about the poly-freeness of Artin groups using only algebraic tools.

[3]An Artin group of type Γ is *even* if all the edges of Γ are labelled with even integers. Such Artin group is said to be of *FC type* if each tringular subgraph of Γ has at least two edges labelled with '2'.

Chapter 2

Finite type Artin groups

This section is devoted to classify which finite type Artin groups are poly-free and which are not. Since poly-freeness is preserved by subgroups and direct products (see Lemma 1.3.3 and Lemma 1.3.6) the study of poly-freeness for Artin groups of finite type can be reduced to the study of poly-freeness of the irreducible Artin groups inside this family. Recall that the irreducible Artin groups of finite type are those built starting from Coxeter graphs listed in Table 1.1 and that we denote them by the calligraphic version of the same letter used for the graph (e.g., $\mathcal{A}_n \coloneqq \mathcal{A}(\dot{A}_n)$, $\mathcal{I}_2(n) \coloneqq \mathcal{A}(\dot{I}_2(n))$, etc...). Also, when we are working on Artin groups with at most four generators we will denote them by the letters a, b, c, d rather than a_1, a_2, a_3, a_4 and when we are dealing with multiple Artin groups at the same time we will use the symbols α_i's, β_i's and δ_i's instead of a_i's to avoid confusion (as shown in Table 1.1).

2.1 Known obstructions

Mulholland and Rolfsen in [16, Theorem 1.1] show that the following groups are not locally indicable

$$\mathcal{A}_n\ (n \geq 4), \quad \mathcal{B}_n\ (n \geq 5), \quad \mathcal{D}_n\ (n \geq 5), \quad \mathcal{E}_n\ (n = 6, 7, 8), \quad \mathcal{H}_n\ (n = 3, 4).$$

By Lemma 1.3.8 they cannot be poly-free. We will show that all the remaining irreducible Artin groups of finite type, i.e.

$$\mathcal{I}_2(m)\ (m \geq 3), \qquad \mathcal{A}_3, \qquad \mathcal{B}_3, \qquad \mathcal{B}_4, \qquad \mathcal{D}_4, \qquad \mathcal{F}_4,$$

are poly-free up to the possible exception of $\mathcal{F}_4$ which remains undetermined at the moment. Moreover, we will construct an explicit poly-fg-free series for each of them (and in the case of $\mathcal{I}_2(m)$ and $\mathcal{B}_3$ it will also be a strongly poly-fg-free series).

2.2 Dihedral Artin groups

Set $m \geq 3$, we will prove that dihedral Artin groups

$$\mathcal{I}_2(m) \coloneqq \mathcal{A}\left(\underset{a\quad b}{\overset{m}{\bullet\!-\!\bullet}}\right) = \langle a, b \mid \Pi(a, b; m) = \Pi(b, a; m) \rangle$$

are poly-fg-free and their poly-fg-free length is 2, hence they are strongly poly-fg-free. In order to obtain this result we will consider the morphism $\chi_m \coloneqq \chi_{I_2(m)}$ defined in Lemma 1.4.9 and show that $\mathcal{I}_2(m) \cong K_m \rtimes \mathbb{Z}$ where $K_m \coloneqq \mathrm{Ker}(\chi_m)$ is a free group of finite rank, so

$$\{ 1 \} \lhd K_m \lhd \mathcal{I}_2(m)$$

is a poly-fg-free series for $\mathcal{I}_2(m)$ of length 2.

2.2.1 Preliminaries

In this section we establish some key lemmata that hold both for odd and even dihedral Artin groups.

Lemma 2.2.1. *Let χ_m as above and set $K_m \coloneqq \mathrm{Ker}\,\chi_m$, then*

$$\mathcal{I}_2(m) \cong K_m \rtimes \mathbb{Z}.$$

Proof. $\mathrm{Im}(\chi_m) = \mathbb{Z}$ is free, hence the short exact sequence of groups

$$1 \to K_m \hookrightarrow \mathcal{I}_2(m) \overset{\chi_m}{\twoheadrightarrow} \mathbb{Z} \to 1$$

splits since freeness returns a section of the map χ_m. By Proposition 1.2.9 this is equivalent to say that $\mathcal{I}_2(m) \cong K_m \rtimes \mathbb{Z}$. ■

We want to find a generating set for K_m. Observe that the character χ_m coincides with the group homomorphism $\varepsilon\colon \mathcal{I}_2(m) \to \mathbb{Z}$ defined by sending an element g of $\mathcal{I}_2(m)$ to the integer number $\sum_i \gamma_i$ where γ_i are the exponents of the letters a, b in a word representing g.

Lemma 2.2.2. *The subgroup K_m of $\mathcal{I}_2(m)$ is generated by the set $\{ k^{b^i} \mid i \in \mathbb{Z} \}$ where $k \coloneqq ab^{-1}$.*

Proof. Let $g \in \mathcal{I}_2(m)$, then g can be represented by a word in a, b belonging to one of these two cases:

(i) $g = a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \ldots a^{\alpha_t} b^{\beta_t}$ with $\alpha_i, \beta_i \in \mathbb{Z} \setminus \{ 0 \}$ and β_t possibly 0,

(ii) $g = b^{\beta_1}a^{\alpha_1}b^{\beta_2}a^{\alpha_2}\dots b^{\beta_t}a^{\alpha_t}$ with $\alpha_i, \beta_i \in \mathbb{Z} \setminus \{0\}$ and α_t possibly 0.

Without loss of generality we can suppose to be in case (ii), indeed suppose the word representing g is of type $a^{\alpha_1}b^{\beta_1}a^{\alpha_2}b^{\beta_2}\dots a^{\alpha_t}b^{\beta_t}$, then we can replace the leading letter a with

$$a = \begin{cases} \Pi(b, a; m)\ \Pi(a^{-1}, b^{-1}; m-1), & m \text{ odd}, \\ \Pi(b, a; m)\ \Pi(b^{-1}, a^{-1}; m-1), & m \text{ even}, \end{cases}$$

falling back in case (ii). For the sake of convenience define

$$\delta_l \coloneqq \sum_{i=1}^{l} \alpha_i + \beta_i, \qquad \forall l = 1, \dots, t,$$

$$\Delta_i \coloneqq \begin{cases} k^{b^{-\beta_1}} \ \dots \ k^{b^{-\delta_1+1}}, & \alpha_i > 0, \\ 1 & \alpha_i = 0, \\ (k^{-1})^{b^{-\beta_1+1}} \ \dots \ (k^{-1})^{b^{-\delta_1}}, & \alpha_i < 0. \end{cases}$$

We claim that any non-trivial element $g = b^{\beta_1}a^{\alpha_1}b^{\beta_2}a^{\alpha_2}\dots b^{\beta_t}a^{\alpha_t}$ of $\mathcal{I}_2(m)$ can be written as

$$g = \Delta_1 \cdot \ldots \cdot \Delta_t b^{\delta_t}.$$

We proceed by induction on t. If $t = 1$ we distinguish three cases.

- If $\alpha_1 > 0$, then

$$\begin{aligned} g &= b^{\beta_1}a^{\alpha_1} = b^{\beta_1}\left(ab^{-1}b\right)^{\alpha_1} = b^{\beta_1}(kb)^{\alpha_1} \\ &= \left[k^{b^{-\beta_1}} b^{\beta_1+1} (kb)^{\alpha_1-1}\right] = \left[k^{b^{-\beta_1}} \dots k^{b^{-\delta_1+1}}\right] b^{\delta_1} = \Delta_1 b^{\delta_1}. \end{aligned}$$

- If $\alpha_1 = 0$, then

$$g = b^{\beta_1} = b^{\delta_1} = \Delta_1 b^{\delta_1}.$$

- If $\alpha_1 < 0$, set $\gamma_1 \coloneqq -\alpha_1$, then

$$\begin{aligned} g &= b^{\beta_1}a^{\alpha_1} = b^{\beta_1}\left(ab^{-1}b\right)^{\alpha_1} = b^{\beta_1}(kb)^{-\gamma_1} = b^{\beta_1}\left(b^{-1}k^{-1}\right)^{\gamma_1} \\ &= \left[(k^{-1})^{b^{-\beta_1+1}} b^{\beta_1-1}\left(b^{-1}k^{-1}\right)^{\gamma_1-1}\right] = \left[(k^{-1})^{b^{-\beta_1+1}} \dots (k^{-1})^{b^{-\delta_1}}\right] b^{\delta_1} \\ &= \Delta_1 b^{\delta_1}. \end{aligned}$$

In any case the base step is verified.
Now suppose the statement true for $t-1$ ($t \geq 2$) and consider an element

$$g = b^{\beta_1}a^{\alpha_1}b^{\beta_2}a^{\alpha_2}\dots b^{\beta_t}a^{\alpha_t} = \Delta_1 \dots \Delta_{t-1}b^{\delta_{t-1}} \cdot b^{\beta_t}a^{\alpha_t} = \Delta_1 \dots \Delta_{t-1}b^{\delta_{t-1}+\beta_t}a^{\alpha_t}.$$

Applying the same procedure as we did in the base step to the term $b^{\delta_{t-1}+\beta_t}a^{\alpha_t}$ we obtain

$$b^{\delta_{t-1}+\beta_t}a^{\alpha_t} = \Delta_t b^{\delta_{t-1}+\beta_t+\alpha_t} = \Delta_t b^{\delta_t}$$

and the claim is proven.
If we restrict to consider $g = b^{\beta_1}a^{\alpha_1}b^{\beta_2}a^{\alpha_2}\dots b^{\beta_t}a^{\alpha_t} \in K_m$ this amounts to say that $\delta_t = \sum_{i=1}^{t}(\alpha_i+\beta_i) = \chi_m(g) = 0$ and the statement follows. ∎

Notice that the map $\chi_m\colon \mathcal{I}_2(m) \to \mathbb{Z} \hookrightarrow \mathbb{R}$ can be regarded as an element of $\mathrm{Hom}(\mathcal{I}_2(m),\mathbb{R})$ and it determines two different equivalence classes $[\chi_m], [-\chi_m] \in S(\mathcal{I}_2(m))$. With this in mind we now show that K_m is finitely generated, while in Propositions 2.2.11 and 2.2.14 we will show some finite generating sets explicitly.

Lemma 2.2.3. *K_m is finitely generated.*

Proof. Theorem 1.2.24 states that K_m is finitely generated if and only if

$$\{\,[\chi_m],[-\chi_m]\,\} \in \Sigma^1(\mathcal{I}_2(m)).$$

To achieve this we use Theorem 1.2.25. The defining relation of $\mathcal{I}_2(m)$ can be rewritten to form the cyclically reduced word

$$r := \Pi(a,b;m)\ \Pi(b,a;m)^{-1}$$

that we use to construct the sequences $\eta_i = \chi_m(r_i)$ and $\mu_i = -\chi_m(r_i)$ for $1 \le i \le 2m$, where r_i denotes the word given by the first i letters of r. Explicitly we have

$$\begin{aligned}
\eta_1 &= \chi_m(a) = 1,\\
&\vdots\\
\eta_m &= \chi_m(\Pi(a,b;m)) = m,\\
\eta_{m+1} &= \chi_m(\Pi(a,b;m)\xi^{-1}) = m-1,\\
&\vdots\\
\eta_{2m} &= \chi_m(\Pi(a,b;m)\ \Pi(b,a;m)^{-1}) = 0
\end{aligned}$$

and

$$\begin{aligned}
\mu_1 &= -\chi_m(a) = -1,\\
&\vdots\\
\mu_m &= -\chi_m(\Pi(a,b;m)) = -m,\\
\mu_{m+1} &= -\chi_m(\Pi(a,b;m)\xi^{-1}) = -m+1,\\
&\vdots
\end{aligned}$$

$$\mu_{2m} = -\chi_m(\Pi(a,b;m)\ \Pi(b,a;m)^{-1}) = 0,$$

where ξ is either b if m is odd or a if m is even. Both sequences assume their minimum only once, therefore $[\chi_m], [-\chi_m] \in \Sigma^1(\mathcal{I}_2(m))$ by the theorem cited above. We conclude that K_m is finitely generated. ■

The goal of the next two sections is to prove that K_m is actually a free group of rank $m-1$. In order to obtain such result we need to work directly on the relator defining $\mathcal{I}_2(m)$, hence the necessity to distinguish between the cases when m is even and m is odd.

2.2.2 Odd dihedral Artin groups

In this section let $m = 2n+1$ $(n \geq 1)$. Although not necessary before moving to the main result we compute the character sphere of $\mathcal{I}_2(m)$.

$$\begin{aligned}
\mathcal{I}_2(m)^{\mathrm{ab}} = \frac{\mathcal{I}_2(m)}{\mathcal{I}_2'(m)} &= \langle a, b \mid \Pi(a,b;2n+1) = \Pi(b,a;2n+1),\ ab = ba\rangle \\
&= \left\langle a, b \;\middle|\; a^{n+1}b^n = a^n b^{n+1},\ ab = ba \right\rangle \\
&= \langle a, b \mid a = b\rangle \\
&= \langle a \,|\, \emptyset\rangle \\
&\simeq \mathbb{Z}.
\end{aligned}$$

It follows that the torsion free rank of $\mathcal{I}_2(m)^{\mathrm{ab}}$ is 1 and by [20, Lemma A1.1] it is the dimension of $\mathrm{Hom}(\mathcal{I}_2(2n+1), \mathbb{R})$ as a real vector space. Therefore the character sphere $S(\mathcal{I}_2(m))$ is homeomorphic to S^0, i.e. it consists of two points, namely the equivalence classes $[\chi_m], [-\chi_m]$. Taking into account the proof for Lemma 2.2.3 we have

$$\Sigma^1(\mathcal{I}_2(2n+1)) = \{\, [\chi_{2n+1}], [-\chi_{2n+1}] \,\} = S(\mathcal{I}_2(2n+1)).$$

Remark 2.2.4. *In the odd case the character sphere $S(\mathcal{I}_2(2n+1))$ could also be determined by observing that a generic character $\theta\colon \mathcal{I}_2(m) \to \mathbb{R}$ sending the generator a to $\alpha \in \mathbb{R}$ and the generator b to $\beta \in \mathbb{R}$ must preserve the defining relation of $\mathcal{I}_2(m)$, hence*

$$\begin{gathered}
\theta(\Pi(a,b;2n+1)) = \theta(\Pi(b,a;2n+1)), \\
(n+1)\alpha + n\beta = (n+1)\beta + n\alpha, \\
\alpha = \beta.
\end{gathered}$$

This means that θ is a real multiple of the character χ_{2n+1} defined above and $S(\mathcal{I}_2(2n+1))$ contains exactly two equivalence classes.

Now let us resume our main goal.

First examples: $\mathcal{I}_2(3)$ and $\mathcal{I}_2(5)$

To achieve that

$$\mathcal{I}_2(3) = \langle a, b \mid aba = bab \rangle \tag{2.1}$$

is poly-free we work towards proving that K_3 is free. Clearly $k \coloneqq ab^{-1}$ belongs to K_3 and since K_3 is normal in $\mathcal{I}_2(3)$ we have $K_3^b = K_3$. In order to find a finite generating set for K_3 we study the action of conjugation by b on the element k.

Proposition 2.2.5. *K_3 is generated by the set $\{ k, k^b \}$, where $k \coloneqq ab^{-1}$ and a, b are as in Equation* (2.1).

Proof. By Lemma 2.2.2 we know that $\{ k^{b^i} \mid i \in \mathbb{Z} \}$ is a set of generators for K_3. Actually this set is much bigger than what is needed to generate K_3. Set $k_i \coloneqq k^{b^i}$, $i \in \mathbb{Z}$. We show by induction that for $i \leq -1$ and $i \geq 2$ each k_i can be expressed as a product of $k_0 = k$ and $k_1 = k^b$. The base step for induction is as follows.

$$k_1 = k^b = b^{-1}ab^{-1}b = b^{-1}a, \tag{2.2}$$

$$k_2 = k^{b^2} = b^{-2}ab = b^{-1}\left(b^{-1}ab\right) = b^{-1}\left(aba^{-1}\right) = \left(b^{-1}a\right)\left(ba^{-1}\right) = k^b k^{-1}, \tag{2.3}$$

$$k_{-1} = k^{b^{-1}} = (bab^{-1})b^{-1} = a^{-1}bab^{-1} = (k^b)^{-1}k.$$

For $i \geq 0$ suppose the claim true for i, i.e. $k_i = W_i(k_0, k_1)$ a word involving only k_0 and k_1, then $k_{i+1} = k_i^b = W_i(k_0^b, k_1^b) = W_i(k_1, k_1k_0^{-1})$ and the statement follows for $i \geq 0$. Analogously the case $i \leq -1$ can be checked. This implies that K_3 is generated by the set $\{ k, k^b \}$ and it is the normal closure of $\{ k \}$ inside $\mathcal{I}_2(3)$. ∎

Proposition 2.2.6. *K_3 is free of rank* 2.

Proof. The calculation carried out in the previous proof suggests that K_3 may be realized as the semidirect product of an infinite cyclic group $\langle t \rangle$ acting on a free group of rank 2, $F_2 = F(\{ k_0, k_1 \})$. We now prove this intuition. From now until the end of this proof k_0 and k_1 will be regarded as abstract generators of F_2. Eventually we will show an isomorphism connecting $k_0, k_1 \in F_2$ to $k_0 = ab^{-1}, k_1 = b^{-1}a \in \mathcal{I}_2(3)$ which justifies the abuse of notation. Start defining a homomorphism $\varphi \colon \langle t \rangle \to \mathrm{Aut}(F_2)$ by setting

$$\begin{aligned} \varphi(t) \coloneqq \varphi_t \colon F_2 &\to F_2, \\ k_0 &\mapsto k_1, \\ k_1 &\mapsto k_1k_0^{-1} \end{aligned}$$

as suggested by Equations (2.2) and (2.3) (here we want the action of t to mimic the action of conjugation through b). Clearly φ_t is a group homomorphism as it is defined on the generators of a free group, moreover it is invertible having as inverse the map

$$\begin{aligned} \varphi_{t^{-1}} \colon F_2 &\to F_2, \\ k_0 &\mapsto k_1^{-1} k_0, \\ k_1 &\mapsto k_0. \end{aligned}$$

Since we want φ to be an homomorphism we must set

$$\varphi(t^k) \coloneqq \begin{cases} \varphi_t^k, & k > 0, \\ \mathrm{Id}_{F_2}, & k = 0, \\ \varphi_{t^{-1}}^{-k}, & k < 0 \end{cases}$$

and we can build the semidirect product $\mathcal{H}_3 \coloneqq F_2 \rtimes_\varphi \langle t \rangle$. As it is clear from the construction the action of $\langle t \rangle$ on F_2 is uniquely determined by the action of the generating element t, therefore (taking into account Proposition 1.2.7, case 3) we have the presentation

$$\mathcal{H}_3 = \left\langle t, k_0, k_1 \;\middle|\; k_0^t = k_1,\ k_1^t = k_0^t k_0^{-1} \right\rangle.$$

Applying a sequence of Tietze transformations (see Theorem 1.2.5) we can rewrite such presentation in the following way

$$\begin{aligned} \mathcal{H}_3 &= \left\langle t, k_0, k_1 \;\middle|\; k_1 = k_0^t,\ k_0^{t^2} = k_0^t k_0^{-1} \right\rangle && \text{substitute } k_1, \\ &= \left\langle s, t, k_0 \;\middle|\; s = k_0 t,\ k_0^{t^2} = k_0^t k_0^{-1} \right\rangle && \text{remove } k_1, \text{ add } s = k_0 t, \\ &= \left\langle s, t, k_0 \;\middle|\; k_0 = st^{-1},\ \left(st^{-1}\right)^{t^2} = \left(st^{-1}\right)^t \left(st^{-1}\right)^{-1} \right\rangle && \text{solve for } k_0, \text{ substitute}, \\ &= \left\langle s, t \;\middle|\; \left(st^{-1}\right)^{t^2} = \left(st^{-1}\right)^t \left(st^{-1}\right)^{-1} \right\rangle && \text{remove } k_0, \\ &= \langle s, t \mid sts = tst \rangle && \text{simplify and rearrange.} \end{aligned}$$

This means that $\mathcal{H}_3 \cong \mathcal{I}_2(3)$ through the isomorphism sending $s \mapsto a$ and $t \mapsto b$. Such isomorphism maps the free group generated by $\{ k_0, k_1 \}$ inside $\mathcal{H}_3$ to the subgroup of $\mathcal{I}_2(3)$ generated by $\{ ab^{-1}, b^{-1}a \} = \{ k, k^b \}$ which is K_3, as shown in Proposition 2.2.5. Therefore K_3 is a free group of rank 2. ■

Theorem 2.2.7. *$\mathcal{I}_2(3)$ is strongly poly-fg-free with a strongly poly-fg-free series of length 2 and free factors given by*

$$1 \triangleleft K_3 \triangleleft \mathcal{I}_2(3),$$
$$K_3 \simeq F_2, \qquad \frac{\mathcal{I}_2(3)}{K_3} \simeq \mathbb{Z}.$$

Proof. Follows immediately from Lemma 2.2.1 and Proposition 2.2.6. ■

Before generalizing to arbitrary odd dihedral Artin groups (where computations become messy) let us try to apply the same procedure to the group

$$\mathcal{I}_2(5) = \langle a, b \mid ababa = babab \rangle \tag{2.4}$$

to check that the result obtained with $m = 3$ is not just a lucky case. As before, we want to show that the kernel K_5 of χ_5 is a free group. Although the technique we use is essentially the same as before, the "bigger" defining relation $ababa = babab$ requires to proceed in a more systematic way.

Proposition 2.2.8. *K_5 is generated by the set $\{ k, k^b, k^{b^2}, k^{b^3} \}$, where $k := ab^{-1}$ and a, b are as in Equation* (2.4).

Proof. By Lemma 2.2.2 we know that $\{ k^{b^i} \mid i \in \mathbb{Z} \}$ is a set of generators for K_5. Set $k_i := k^{b^i}$, $i \in \mathbb{Z}$. In order to reduce this set we exchange the generators $\{ a, b \}$ of $\mathcal{I}_2(5)$ with a new generating set $\{ k_0, b \}$ rewriting the only relation in the following way

$$\begin{aligned} ababa &= babab, \\ a\left(b^{-1}b\right)ba\left(b^{-1}b\right)ba\left(b^{-1}b\right) &= ba\left(b^{-1}b\right)ba\left(b^{-1}b\right)b, \\ \left(ab^{-1}\right)b^2\left(ab^{-1}\right)b^2\left(ab^{-1}\right)b &= b\left(ab^{-1}\right)b^2\left(ab^{-1}\right)b^2, \\ k_0b^2k_0b^2k_0b &= bk_0b^2k_0b^2. \end{aligned}$$

In turn this relation can be manipulated to obtain new ones expressing k_{-1} and k_4 in terms of k, k^b, k^{b^2} and k^{b^3}.

$$\begin{aligned} b^{-2}k_0b^2k_0b^2k_0b &= b^{-1}k_0b^2k_0b^2, && \text{multiply by } b^{-2} \text{ on the left,} \\ k_0^{b^2} &= k_0^b k_0^{b^{-1}} \left(k_0^{-1}\right)^{b^{-2}} k_0^{-1}, && \text{solve for } k_0^{b^2}, \\ k_0^{b^4} &= k_0^{b^3} k_0^b k_0^{-1} \left(k_0^{b^2}\right)^{-1}, && \text{conjugate by } b^2, \qquad (2.5) \\ k_0 &= \left(k_0^{b^2}\right)^{-1}\left(k_0^{b^4}\right)^{-1} k_0^{b^3} k_0^b, && \text{solve for } k_0, \\ k_0^{b^{-1}} &= \left(k_0^b\right)^{-1}\left(k_0^{b^3}\right)^{-1} k_0^{b^2} k_0, && \text{conjugate by } b^{-1}. \qquad (2.6) \end{aligned}$$

Arguing by induction as we did in Proposition 2.2.5 using Equations (2.5) and (2.6) as base step we conclude that for $i \leq -1$ and $i \geq 4$ each k_i can be expressed in terms of k_0, k_1, k_2 and k_3, therefore $\{ k, k^b, k^{b^2}, k^{b^3} \}$ is a generating set for K_5. ■

Proposition 2.2.9. *K_5 is free of rank* 4.

Proof. As we did for the group $\mathcal{I}_2(3)$ let us define an action of the infinite cyclic group $\langle t \rangle$ on $F_4 = F(\{ k_0, k_1, k_2, k_3 \})$ mimicking the action by conjugation through b on k inside $\mathcal{I}_2(5)$. Again, by abuse of notation, we will regard $k_0, \dots, k_3$ as abstract generators of F_4 and eventually we will connect them to the elements $k_0 = ab^{-1}, \ \dots, \ k_3 = b^{-3}ab^2 \in \mathcal{I}_2(5)$. Let $\varphi_t \colon F_4 \to F_4$ the homomorphism defined setting

$$\varphi_t(k_0) = k_1, \quad \varphi_t(k_1) = k_2, \quad \varphi_t(k_2) = k_3, \quad \varphi_t(k_3) = k_0^{t^3} k_0^t k_0^{-1} \left(k_0^{t^2}\right)^{-1},$$

as suggested by Equation (2.5). Such homomorphism is an automorphism having as inverse the homomorphism $\varphi_{t^{-1}}$ given by

$$\varphi_{t^{-1}}(k_0) = \left(k_0^t\right)^{-1} \left(k_0^{t^3}\right)^{-1} k_0^{t^2} k_0, \quad \varphi_{t^{-1}}(k_1) = k_0, \quad \varphi_{t^{-1}}(k_2) = k_1, \quad \varphi_{t^{-1}}(k_3) = k_2.$$

As we did in the proof of Proposition 2.2.6 we consider the homomorphism

$$\begin{aligned} \varphi \colon \langle t \rangle &\to \operatorname{Aut}(F_4) \\ t^k &\mapsto (\varphi_t)^k, \qquad k \geq 0, \\ t^k &\mapsto (\varphi_{t^{-1}})^{-k}, \quad k < 0, \end{aligned}$$

and we can define the semidirect product $\mathcal{H}_5 \coloneqq K_5 \rtimes_\varphi \langle t \rangle$ which has a presentation

$$\mathcal{H}_5 = \left\langle t, k_0, k_1, k_2, k_3 \;\middle|\; k_0^t = k_1, \ k_1^t = k_2, \ k_2^t = k_3, \ k_3^t = k_3 k_1 k_0^{-1} k_2^{-1} \right\rangle.$$

Applying a sequence of Tietze transformations we can rewrite such presentation in the following way

$$\begin{aligned} \mathcal{H}_5 &= \left\langle t, k_0, k_1, k_2, k_3 \;\middle|\; k_i = k_0^{t^i} \ (i = 1,2,3), \ k_3^t = k_3 k_1 k_0^{-1} k_2^{-1} \right\rangle \\ &= \left\langle s, t, k_0 \;\middle|\; s = k_0 t, \ k_0^{t^3} = k_0^{t^4} k_0^t k_0^{-1} \left(k_0^{-1}\right)^{t^2} \right\rangle \\ &= \left\langle s, t, k_0 \;\middle|\; k_0 = st^{-1}, \ \left(st^{-1}\right)^{t^4} = \left(st^{-1}\right)^{t^3} \left(st^{-1}\right)^t \left(st^{-1}\right)^{-1} \left(\left(st^{-1}\right)^{t^2}\right)^{-1} \right\rangle \\ &= \left\langle s, t \;\middle|\; \left(st^{-1}\right)^{t^4} = \left(st^{-1}\right)^{t^3} \left(st^{-1}\right)^t \left(st^{-1}\right)^{-1} \left(ts^{-1}\right)^{t^2} \right\rangle \\ &= \left\langle s, t \;\middle|\; t^{-4} s t^{-1} t^4 = t^{-3} s t^{-1} t^3 t^{-1} s t^{-1} t t s^{-1} t^{-2} t s^{-1} t^2 \right\rangle \\ &= \left\langle s, t \;\middle|\; ststs = tstst \right\rangle. \end{aligned}$$

Notice that the elements s, t in the last presentation satisfy exactly the same relations as the generators $\{ a, b \}$ in the standard presentation of $\mathcal{I}_2(5)$, so that the groups $\mathcal{I}_2(5)$ and $\mathcal{H}_5$ are isomorphic through the map sending $s \mapsto a$ and $t \mapsto b$. Therefore

$$t \mapsto b, \quad k_0 \mapsto ab^{-1}, \quad k_1 \mapsto \left(ab^{-1}\right)^b, \quad k_2 \mapsto \left(ab^{-1}\right)^{b^2}, \quad k_3 \mapsto \left(ab^{-1}\right)^{b^3}.$$

is an isomorphism that maps the free group generated by $\{ k_0, k_1, k_2, k_3 \}$ inside $\mathcal{H}_5$ to the subgroup of $\mathcal{I}_2(5)$ generated by

$$\{ ab^{-1}, b^{-1}a, b^{-2}ab, b^{-3}ab^2 \} = \left\{ k, k^b, k^{b^2}, k^{b^3} \right\}$$

which is K_5, as shown in Proposition (2.2.8). We conclude that K_5 is a free group of rank 4. ∎

Theorem 2.2.10. $\mathcal{I}_2(5)$ *is strongly poly-fg-free with a strongly poly-fg-free series of length* 2 *and free factors given by*

$$1 \lhd K_5 \lhd \mathcal{I}_2(5),$$
$$K_5 \simeq F_4, \qquad \frac{\mathcal{I}_2(5)}{K_5} \cong \mathbb{Z}.$$

Proof. Follows immediately from Lemma 2.2.1 and Proposition 2.2.9. ∎

The general case

Following the same process as in the previous examples we are able to prove that odd dihedral Artin groups are strongly poly-fg-free with $\mathrm{spfl}_{\mathrm{fg}}(\mathcal{I}_2(m)) = 2$. In Lemma 2.2.3 we showed that K_m is finitely generated for any integer $m \geq 3$ and now we give an explicit finite set of generators for K_m when m is odd.

Proposition 2.2.11. K_{2n+1} *is generated by the set* $\{ k, k^b, \dots, k^{b^{2n-1}} \}$.

Proof. By Lemma 2.2.2 we know that $\{ k^{b^i} \mid i \in \mathbb{Z} \}$ is a set of generators for K_{2n+1}. Set $k_i := k^{b^i},\ i \in \mathbb{Z}$. In order to further reduce this set we exchange the generators $\{ a, b \}$ of $\mathcal{I}_2(2n+1)$ with the generating set $\{ k_0, b \}$ rewriting the only relation in the following way.

$$\begin{aligned}
\Pi(a, b; 2n+1) &= \Pi(b, a; 2n+1), && \text{defining relation of } \mathcal{I}_2(2n+1),\\
\Pi(a, b^{-1}, b, b; 4n+3) &= \Pi(b, a, b^{-1}, b; 4n+1), && \text{insert } b^{-1}b \text{ after each } a,\\
\Pi(k_0, b, b; 3n+2) &= \Pi(b, k_0, b; 3n+1), && \text{substitute } k_0 = ab^{-1},\\
b^{-2}\ \Pi(k_0, b, b; 3n+2) &= b^{-2}\ \Pi(b, k_0, b; 3n+1), && \text{multiply on the left by } b^{-2},\\
k_0^{b^2}\ \Pi(k_0, b, b; 3n-1) &= b^{-1}\ \Pi(k_0, b, b; 3n), && \text{rearrange LHS, simplify RHS.}
\end{aligned}$$

Now solve for $k_0^{b^2}$ and further manipulate the right hand side to obtain a new relation involving only b and the conjugate of k_0 through positive powers of b

$$\begin{aligned}
k_0^{b^2} &= b^{-1}\ \Pi(k_0, b, b; 3n-1)\ \Pi(k_0^{-1}, b^{-1}, b^{-1}; 3n-2),\\
k_0^{b^2} &= k_0^{b}\ \Pi(b, k_0, b; 3n-3)\ \Pi(k_0^{-1}, b^{-1}, b^{-1}; 3n-2),
\end{aligned}$$

$$\begin{aligned}
k_0^{b^2} &= k_0^b k_0^{b^{-1}} b\ \Pi(b,b,k_0;3n-5)\ \Pi(k_0^{-1},b^{-1},b^{-1};3n-2),\\
k_0^{b^2} &= k_0^b k_0^{b^{-1}} k_0^{b^{-3}} b^3\ \Pi(b,b,k_0;3n-8)\ \Pi(k_0^{-1},b^{-1},b^{-1};3n-2),\\
k_0^{b^2} &= k_0^b k_0^{b^{-1}} k_0^{b^{-3}} \dots k_0^{b^{3-2n}} b^{-3+2n} b\ \Pi(k_0^{-1},b^{-1},b^{-1};3n-2),\\
k_0^{b^2} &= k_0^b k_0^{b^{-1}} k_0^{b^{-3}} \dots k_0^{b^{3-2n}} b^{-2+2n}\ \Pi(k_0^{-1},b^{-1},b^{-1};3n-2),\\
k_0^{b^2} &= k_0^b k_0^{b^{-1}} k_0^{b^{-3}} \dots k_0^{b^{3-2n}} \left(k_0^{-1}\right)^{b^{2-2n}} b^{2n-2}\ \Pi(b^{-1},b^{-1},k_0^{-1};3n-3),\\
k_0^{b^2} &= k_0^b k_0^{b^{-1}} k_0^{b^{-3}} \dots k_0^{b^{3-2n}} \left(k_0^{-1}\right)^{b^{2-2n}} \left(k_0^{-1}\right)^{b^{4-2n}} b^{2n-4}\ \Pi(b^{-1},b^{-1},k_0^{-1};3n-6),\\
k_0^{b^2} &= k_0^b k_0^{b^{-1}} k_0^{b^{-3}} \dots k_0^{b^{3-2n}} \left(k_0^{-1}\right)^{b^{2-2n}} \left(k_0^{-1}\right)^{b^{4-2n}} \dots \left(k_0^{-1}\right)^{b^{-2}} \left(k_0^{-1}\right)^{b^{0}},\\
k_0^{b^2} &= \prod_{i=0}^{n-1} k_0^{b^{1-2i}} \cdot \prod_{i=0}^{n-1} \left(k_0^{-1}\right)^{b^{2-2n+2i}}
\end{aligned}$$

and conjugating by b^{2n-2} both sides we obtain

$$k_0^{b^{2n}} = \prod_{i=0}^{n-1} k_0^{b^{2n-1-2i}} \cdot \prod_{i=0}^{n-1} \left(k_0^{-1}\right)^{b^{2i}}.$$

The last equality together with Lemma 2.2.2 implies that $\{\,k_0, k_0^b, \dots, k_0^{b^{2n-1}}\,\}$ is a generating set for K_{2n+1}. ■

Proposition 2.2.12. *K_{2n+1} is free of rank $2n$.*

Proof. Let the infinite cyclic group $\langle t\rangle$ act on F_{2n} mimicking the action by conjugation of b on k_0 inside $\mathcal{I}_2(2n+1)$. Let $\varphi_t\colon F_{2n}\to F_{2n}$ be the homomorphism defined by setting

$$\varphi_t(k_0)=k_1,\quad \varphi_t(k_1)=k_2,\quad \dots,\quad \varphi_t(k_{2n-2})=k_{2n-1},$$
$$\varphi_t(k_{2n-1}) = \prod_{i=0}^{n-1} k_0^{t^{2n-1-2i}} \cdot \prod_{i=0}^{n-1} \left(k_0^{-1}\right)^{t^{2i}}.$$

Such homomorphism is an automorphism having as inverse $\varphi_{t^{-1}}$ given by

$$\varphi_{t^{-1}}(k_0) = \prod_{i=0}^{n-1} \left(k_0^{-1}\right)^{2i+1} \cdot \prod_{i=0}^{n-1} k_0^{t^{2n-2-2i}},$$
$$\varphi_{t^{-1}}(k_1)=k_0,\quad \varphi_{t^{-1}}(k_2)=k_1,\quad \dots,\quad \varphi_{t^{-1}}(k_{2n-1})=k_{2n-2}$$

and we obtain a homomorphism $\varphi\colon\langle t\rangle\to \mathrm{Aut}(F_{2n})$ by setting $\varphi(t^{-1}) := \varphi_{t^{-1}}$, $\varphi(t^k) := \varphi_t^k$ for $k\geq 0$ and $\varphi(t^k) := \varphi_{t^{-1}}^{-k}$ for $k<0$. Hence we can define the semidirect product $\mathcal{H}_{2n+1} \coloneqq F_{2n} \rtimes_{\varphi} \langle t \rangle$ which has a presentation

$$\mathcal{H}_{2n+1} = \left\langle t, k_0, \ldots, k_{2n-1} \;\middle|\; \begin{array}{l} k_i = k_{i-1}^t, \quad (i = 1, \ldots, 2n-1), \\ k_{2n-1}^t = \prod_{i=0}^{i=n-1} k_0^{t^{2n-1-2i}} \cdot \prod_{i=0}^{i=n-1} \left(k_0^{-1}\right)^{t^{2i}} \end{array} \right\rangle.$$

Applying a sequence of Tietze transformations we can rewrite such presentation in the following way

$$\begin{aligned}
\mathcal{H}_{2n+1} &= \left\langle t, k_0, \ldots, k_{2n-1} \;\middle|\; \begin{array}{l} k_i = k_0^{t^i}, \quad (i = 1, \ldots, 2n-1), \\ k_0^{t^{2n}} = \prod_{i=0}^{n-1} k_0^{t^{2n-1-2i}} \cdot \prod_{i=0}^{n-1} \left(k_0^{-1}\right)^{t^{2i}} \end{array} \right\rangle \\
&= \left\langle s, t, k_0 \;\middle|\; s = k_0 t,\ k_0^{t^{2n}} = \prod_{i=0}^{n-1} k_0^{t^{2n-1-2i}} \cdot \prod_{i=0}^{n-1} \left(k_0^{-1}\right)^{t^{2i}} \right\rangle \\
&= \left\langle s, t, k_0 \;\middle|\; k_0 = st^{-1},\ \left(st^{-1}\right)^{t^{2n}} = \prod_{i=0}^{n-1} \left(st^{-1}\right)^{t^{2n-1-2i}} \cdot \prod_{i=0}^{n-1} \left(ts^{-1}\right)^{t^{2i}} \right\rangle \\
&= \left\langle s, t \;\middle|\; \left(st^{-1}\right)^{t^{2n}} = \prod_{i=0}^{n-1} \left(st^{-1}\right)^{t^{2n-1-2i}} \cdot \prod_{i=0}^{n-1} \left(ts^{-1}\right)^{t^{2i}} \right\rangle \\
&= \left\langle s, t \;\middle|\; t^{-2n} s t^{-1} t^{2n} = t^{-2n+1} \underbrace{st \ldots st}_{2n-2 \text{ letters}}\; s \cdot ts^{-1} \underbrace{t^{-1}s^{-1} \ldots t^{-1}s^{-1}}_{2n-2 \text{ letters}} t^{2n-2} \right\rangle \\
&= \left\langle s, t \;\middle|\; st = \underbrace{tst \ldots stst}_{2n+1 \text{ letters}} \underbrace{s^{-1}t^{-1}s^{-1} \ldots t^{-1}s^{-1}}_{2n-1 \text{ letters}} \right\rangle \\
&= \langle s, t \mid \Pi(s, t; 2n+1) = \Pi(t, s; 2n+1) \rangle .
\end{aligned}$$

The elements s, t in the last presentation satify exactly the same relations as the generators a, b in the standard presentation of $\mathcal{I}_2(2n+1)$, this means that the groups $\mathcal{I}_2(2n+1)$ and $\mathcal{H}_{2n+1}$ are isomorphic through the map sending $s \mapsto a$ and $t \mapsto b$. Therefore

$$t \mapsto b, \quad k_0 \mapsto ab^{-1}, \quad k_1 \mapsto \left(ab^{-1}\right)^b, \quad \ldots, \quad k_{2n-1} \mapsto \left(ab^{-1}\right)^{b^{2n-1}}.$$

is an isomorphism that maps the free group generated by $\{ k_0, \ldots, k_{2n-1} \}$ inside $\mathcal{H}_{2n+1}$ to the subgroup of $\mathcal{I}_2(2n+1)$ generated by

$$\left\{ ab^{-1}, \ldots, b^{-2n+1}ab^{2n-2} \right\} = \left\{ k, \ldots, k^{b^{2n-1}} \right\}$$

which is K_{2n+1}, as shown in Proposition (2.2.11). We conclude that K_{2n+1} is a free group of rank $2n$. ■

2.2.3 Even dihedral Artin groups

In this section let $m \coloneqq 2n$ ($n \geq 2$). As we did at the beginning of the previous section we compute the character sphere of $\mathcal{I}_2(2n)$.

$$\begin{aligned}\mathcal{I}_2(m)^{\mathrm{ab}} = \frac{\mathcal{I}_2(m)}{\mathcal{I}_2(m)'} &= \big\langle a, b \;\big|\; (ab)^{2n} = (ba)^{2n},\ ab = ba \big\rangle \\ &= \langle a, b \mid a^n b^n = a^n b^n,\ ab = ba \rangle \\ &= \langle a, b \mid ab = ba \rangle \\ &\simeq \mathbb{Z} \times \mathbb{Z}.\end{aligned}$$

It follows that the torsion free rank of $\mathcal{I}_2(2n)^{\mathrm{ab}}$ is 2 and by [20, Lemma A1.1] it is also the dimension of $\mathrm{Hom}(\mathcal{I}_2(2n), \mathbb{R})$ as a real vector space. Therefore the character sphere $S(\mathcal{I}_2(2n))$ is homeomorphic to S^1. Taking into account the proof for Lemma 2.2.3 we have

$$\{\, [\chi_{2n}], [-\chi_{2n}] \,\} \subseteq \Sigma^1(\mathcal{I}_2(2n)) \subseteq S(\mathcal{I}_2(2n)).$$

Remark 2.2.13. *In the even case the character sphere $S(\mathcal{I}_2(2n))$ can be determined by observing that a generic character $\theta\colon \mathcal{I}_2(2n) \to \mathbb{R}$ sending the generator a to $\alpha \in \mathbb{R}$ and the generator b to $\beta \in \mathbb{R}$ must preserve only the defining relation of $\mathcal{I}_2(2n)$, hence*

$$\theta((ab)^{2n}) = \theta((ba)^{2n}) \quad\Longrightarrow\quad n\alpha + n\beta = n\beta + n\alpha \quad\Longrightarrow\quad 0 = 0.$$

This means that $\mathrm{Hom}(\mathcal{I}_2(2n), \mathbb{R})$ *is a 2 dimensional real vector space with basis*

$$\left\{ \theta_1\colon \begin{array}{c} a \mapsto 1 \\ b \mapsto 0 \end{array},\ \theta_2\colon \begin{array}{c} a \mapsto 0 \\ b \mapsto 1 \end{array} \right\}.$$

Now let us resume our main goal moving directly to the general case. The strategy to achieve poly-freeness of even dihedral Artin groups will be the same as in the odd case (i.e., proving that K_{2n} is free), however computations has to be redone because the change of the parity of the lengths of LHS and RHS in the defining relator of $\mathcal{I}_2(2n)$ leads to slightly different results.

The general case

We remind that in Lemma 2.2.3 we showed that K_m is finitely generated for any integer $m \geq 3$ and now we give an explicit finite set of generators for K_m when m is even.

Proposition 2.2.14. *K_{2n} is generated by the set $\{\, k, k^b, \ldots, k^{b^{2n-2}} \,\}$.*

Proof. By Lemma 2.2.2 we know that $\{ k^{b^i} \mid i \in \mathbb{Z} \}$ is a set of generators for K_{2n}. Set $k_i := k^{b^i}$, $i \in \mathbb{Z}$. Start exchanging the generators a, b of $\mathcal{I}_2(2n)$ with the generating set $\{ k_0, b \}$ rewriting the only relation in the following way

$$\begin{aligned}
\Pi(a,b;2n) &= \Pi(b,a;2n), && \text{defining relation of } \mathcal{I}_2(2n),\\
\Pi(a,b^{-1},b,b;4n) &= \Pi(b,a,b^{-1},b;4n), && \text{insert } b^{-1}b \text{ after each } a,\\
\Pi(k_0,b,b;3n) &= \Pi(b,k_0,b;3n), && \text{substitute } k_0 = ab^{-1},\\
b^{-2}\,\Pi(k_0,b,b;3n) &= b^{-2}\,\Pi(b,k_0,b;3n), && \text{multiply on the left by } b^{-2},\\
k_0^{b^2}\,\Pi(k_0,b,b;3n-3) &= b^{-1}\,\Pi(k_0,b,b;3n-1), && \text{rearrange LHS, simplify RHS.}
\end{aligned}$$

Now solve for $k_0^{b^2}$ and further manipulate the right hand side to obtain a new relation involving only b and the conjugate of k_0 through positive powers of b

$$\begin{aligned}
k_0^{b^2} &= b^{-1}\,\Pi(k_0,b,b;3n-2)\,\Pi(b^{-1},k_0^{-1},b^{-1};3n-4),\\
k_0^{b^2} &= k_0^{b}\,\Pi(b,k_0,b;3n-4)\,\Pi(b^{-1},k_0^{-1},b^{-1};3n-4),\\
k_0^{b^2} &= k_0^{b}k_0^{b^{-1}}b\,\Pi(b,b,k_0;3n-6)\,\Pi(b^{-1},k_0^{-1},b^{-1};3n-4),\\
k_0^{b^2} &= k_0^{b}k_0^{b^{-1}}k_0^{b^{-3}}b^3\,\Pi(b,b,k_0;3n-9)\,\Pi(b^{-1},k_0^{-1},b^{-1};3n-4),\\
k_0^{b^2} &= k_0^{b}k_0^{b^{-1}}k_0^{b^{-3}}\dots k_0^{b^{3-2n}}b^{2n-3}\,\Pi(b^{-1},k_0^{-1},b^{-1};3n-4),\\
k_0^{b^2} &= k_0^{b}k_0^{b^{-1}}k_0^{b^{-3}}\dots k_0^{b^{3-2n}}b^{2n-4}\,\Pi(k_0^{-1},b^{-1},b^{-1};3n-5),\\
k_0^{b^2} &= k_0^{b}k_0^{b^{-1}}k_0^{b^{-3}}\dots k_0^{b^{3-2n}}\left(k_0^{-1}\right)^{b^{4-2n}}b^{2n-6}\,\Pi(k_0^{-1},b^{-1},b^{-1};3n-8),\\
k_0^{b^2} &= k_0^{b}k_0^{b^{-1}}k_0^{b^{-3}}\dots k_0^{b^{3-2n}}\left(k_0^{-1}\right)^{b^{4-2n}}\left(k_0^{-1}\right)^{b^{6-2n}}\dots\left(k_0^{-1}\right)^{b^{-2}}\left(k_0^{-1}\right)^{b^{0}},\\
k_0^{b^2} &= \prod_{i=0}^{n-1} k_0^{b^{1-2i}} \cdot \prod_{i=0}^{n-2} (k_0^{-1})^{b^{4-2n+2i}}
\end{aligned}$$

and conjugating both sides by b^{2n-3} we obtain

$$k_0^{b^{2n-1}} = \prod_{i=0}^{n-1} k_0^{b^{2n-2-2i}} \cdot \prod_{i=0}^{n-2} \left(k_0^{-1}\right)^{b^{2i+1}}.$$

The last equality together with Lemma 2.2.2 implies that $\{ k_0, k_0^b, \dots, k_0^{b^{2n-2}} \}$ is a generating set for K_{2n}. ∎

Proposition 2.2.15. *K_{2n} is free of rank $2n-1$.*

Proof. Let the infinite cyclic group $\langle t \rangle$ act on F_{2n-1} mimicking the action by conjugation of b on k_0 inside $\mathcal{I}_2(2n)$. Let $\varphi_t \colon F_{2n-1} \to F_{2n-1}$ be the homomorphism defined by setting

$$\varphi_t(k_0) = k_1, \quad \varphi_t(k_1) = k_2, \quad \dots, \quad \varphi_t(k_{2n-3}) = k_{2n-2},$$

$$\varphi_t(k_{2n-2}) = \prod_{i=0}^{n-1} k_0^{t^{2n-2-2i}} \cdot \prod_{i=0}^{n-2} \left(k_0^{-1}\right)^{t^{2i+1}}.$$

Such homomorphism is an automorphism having as inverse $\varphi_{t^{-1}}$ given by

$$\varphi_{t^{-1}}(k_0) = \prod_{i=0}^{n-2} \left(k_0^{-1}\right)^{t^{2i+1}} \cdot \prod_{i=0}^{n-1} k_0^{t^{2n-2-2i}},$$
$$\varphi_{t^{-1}}(k_1) = k_0, \quad \varphi_{t^{-1}}(k_2) = k_1, \quad \dots, \quad \varphi_{t^{-1}}(k_{2n-1}) = k_{2n-2}$$

and we obtain a homomorphism $\varphi\colon \langle t\rangle \to \mathrm{Aut}(F_{2n-1})$ by setting $\varphi(t^{-1}) := \varphi_{t^{-1}}$, $\varphi(t^k) := \varphi_t^k$ for $k \geq 0$ and $\varphi(t^k) := \varphi_{t^{-1}}^{-k}$ for $k < 0$. Hence we can define the semidirect product $\mathcal{H}_{2n} := F_{2n-1} \rtimes_\varphi \langle t\rangle$ which has a presentation

$$\mathcal{H}_{2n} = \left\langle t, k_0, \dots, k_{2n-2} \;\middle|\; \begin{array}{l} k_i = k_{i-1}^t, \quad (i = 1, \dots, 2n-2), \\ k_{2n-2}^t = \prod_{i=0}^{n-1} k_0^{t^{2n-2-2i}} \cdot \prod_{i=0}^{n-2} \left(k_0^{-1}\right)^{t^{2i+1}} \end{array} \right\rangle$$

Applying a sequence of Tietze transformations we can rewrite such presentation in the following way

$$\begin{aligned}
\mathcal{H}_{2n} &= \left\langle t, k_0, \dots, k_{2n-2} \;\middle|\; \begin{array}{l} k_i = k_0^{t^i}, \quad (i = 1, \dots, 2n-2), \\ k_0^{t^{2n-1}} = \prod_{i=0}^{n-1} k_0^{t^{2n-2-2i}} \cdot \prod_{i=0}^{n-2} \left(k_0^{-1}\right)^{t^{2i+1}} \end{array} \right\rangle \\
&= \left\langle s, t, k_0 \;\middle|\; s = k_0 t,\ k_0^{t^{2n-1}} = \prod_{i=0}^{n-1} k_0^{t^{2n-2-2i}} \cdot \prod_{i=0}^{n-2} \left(k_0^{-1}\right)^{t^{2i+1}} \right\rangle \\
&= \left\langle s, t, k_0 \;\middle|\; k_0 = st^{-1},\ \left(st^{-1}\right)^{t^{2n-1}} = \prod_{i=0}^{n-1} \left(st^{-1}\right)^{t^{2n-2-2i}} \cdot \prod_{i=0}^{n-2} \left(ts^{-1}\right)^{t^{2i+1}} \right\rangle \\
&= \left\langle s, t \;\middle|\; \left(st^{-1}\right)^{t^{2n-1}} = \prod_{i=0}^{n-1} \left(st^{-1}\right)^{t^{2n-2-2i}} \cdot \prod_{i=0}^{n-2} \left(ts^{-1}\right)^{t^{2i+1}} \right\rangle \\
&= \left\langle s, t \;\middle|\; t^{-2n+1} s t^{-1} t^{2n-1} = t^{-2n+2} \underbrace{st \dots st}_{2n-2 \text{ letters}}\ st^{-1} \cdot s^{-1} \underbrace{t^{-1}s^{-1} \dots t^{-1}s^{-1}}_{2n-4 \text{ letters}} t^{2n-3} \right\rangle \\
&= \left\langle s, t \;\middle|\; st = \underbrace{tst \dots sts}_{2n \text{ letters}} \underbrace{t^{-1}s^{-1} \dots t^{-1}s^{-1}}_{2n-2 \text{ letters}} \right\rangle \\
&= \left\langle s, t \;\middle|\; (st)^{2n} = (ts)^{2n} \right\rangle.
\end{aligned}$$

The elements s, t in the last presentation satisfy exactly the same relations as the generators a, b in the standard presentation of $\mathcal{I}_2(2n)$, this means that the groups $\mathcal{I}_2(2n)$ and $\mathcal{H}_{2n}$ are isomorphic through the map sending $s \mapsto a$ and $t \mapsto b$. Therefore

$$t \mapsto b, \quad k_0 \mapsto ab^{-1}, \quad k_1 \mapsto \left(ab^{-1}\right)^b, \quad \ldots, \quad k_{2n-2} \mapsto \left(ab^{-1}\right)^{b^{2n-2}}.$$

is an isomorphism that maps the free group generated by $\{ k_0, \ldots, k_{2n-2} \}$ inside $\mathcal{H}_{2n}$ to the subgroup of $\mathcal{I}_2(2n)$ generated by

$$\left\{ ab^{-1}, \ldots, b^{-2n-2}ab^{2n-3} \right\} = \left\{ k, \ldots, k^{b^{2n-2}} \right\}$$

which is K_{2n}, as shown in Proposition 2.2.14. We conclude that K_{2n} is a free group of rank $2n-1$. ■

Finally, we can gather the results from the previous sections in the following theorem.

Theorem 2.2.16. *For each $m \geq 3$, $\mathcal{I}_2(m)$ is strongly poly-fg-free with a strongly poly-fg-free series of length 2 and free factors given by*

$$1 \lhd K_m \lhd \mathcal{I}_2(m),$$
$$K_m \simeq F_{m-1}, \qquad \frac{\mathcal{I}_2(m)}{K_m} \simeq \mathbb{Z}.$$

Proof. Follows immediately from Lemma 2.2.1, Proposition 2.2.12 and Proposition 2.2.15. ■

2.3 Other finite type Artin groups

2.3.1 Artin group $\mathcal{A}_3$

While we cannot immediately apply the same approach used to prove poly-freeness of dihedral Artin groups (essentially because we suspect that the poly-free length of $\mathcal{A}_3$ is greater than 2), it is enough to analyse the quotient of

$$\mathcal{A}_3 = \mathcal{A}\left(\underset{a\quad b\quad c}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet} \right) = \langle a, b, c \mid aba = bab,\ bcb = cbc,\ ac = ca \rangle$$

by its commutator subgroup before applying the same kind of machinery we used in Section 2.2. We will obtain a poly-fg-free serie for $\mathcal{A}_3$ of length 3. In [16, Theorem

3.6] a description of the commutator subgroup $\mathcal{A}_3'$ is given in terms of generators and relations as follows[1]

$$\mathcal{A}_3' = \left\langle p_0,\ p_1,\ q_0,\ q_1 \;\middle|\; \begin{array}{ll} q_0^{p_0^{-1}} = q_1, & q_1^{p_0^{-1}} = q_1^2 q_0^{-1} q_1, \\ q_0^{p_1^{-1}} = q_0^{-1} q_1, & q_1^{p_1^{-1}} = \left(q_0^{-1} q_1\right)^3 q_0^{-2} q_1 \end{array} \right\rangle,$$

where p_0, p_1, q_0, q_1 are given with respect to the generators a, b, c of $\mathcal{A}_3$ by

$$p_0 = ba^{-1}, \quad p_1 = aba^{-2}, \quad q_0 = ca^{-1}, \quad q_1 = ba^{-1}ca^{-1}.$$

Let us start evaluating the abelianization of $\mathcal{A}_3$

$$\begin{aligned} \frac{\mathcal{A}_3}{\mathcal{A}_3'} &= \langle a, b, c \mid aba = bab,\ bcb = cbc,\ ac = ca,\ ab = ba,\ bc = cb \rangle \\ &= \langle a, b, c \mid a = b = c \rangle \\ &= \langle a \mid \emptyset \rangle \\ &\simeq \mathbb{Z}. \end{aligned}$$

Looking at the defining relations of $\mathcal{A}_3'$ it seems possible that it may have the structure of a semidirect product of two free groups of rank 2, namely we want to show that the presentation of $\mathcal{A}_3'$ is actually a presentation for the semidirect product given by $F(\{p_0, p_1\})$ acting on $F(\{q_0, q_1\})$. From now on p_0, p_1 and q_0, q_1 will denote abstract letters and we will justify the abuse of notation later on by identifying them with the corresponding elements inside $\mathcal{A}_3'$.
Set $A = F(\{p_0, p_1\})$ and $B = F(\{q_0, q_1\})$, in order to build a semidirect product we need to provide a morphism $\varphi\colon A \to \mathrm{Aut}(B)$; for $i = 1, 2$ we will denote by φ_{p_i} the image of p_i under φ (i.e., the action of p_i on B). Since A is a free group we only need to provide the values of φ on its generators and since φ takes values in the automorphism group of a free group this amounts to say that φ is uniquely determined once we provide the values of the action of p_0, p_1 on q_0, q_1 (or equivalently the values of the action of the inverses of p_0 and p_1). Looking at the defining relations of $\mathcal{A}_3'$ we are led to set

$$\varphi_{p_0^{-1}}(q_0) \coloneqq q_1, \tag{2.7}$$
$$\varphi_{p_0^{-1}}(q_1) \coloneqq q_1^2 q_0^{-1} q_1, \tag{2.8}$$
$$\varphi_{p_1^{-1}}(q_0) \coloneqq q_0^{-1} q_1, \tag{2.9}$$
$$\varphi_{p_1^{-1}}(q_1) \coloneqq \left(q_0^{-1} q_1\right)^3 q_0^{-2} q_1. \tag{2.10}$$

[1] The approach of the authors consists in the application of the Reidemeister-Schreier rewriting procedure (that we present in Section 3.1) followed by a non-trivial change of generators.

It remains to check that $\varphi_{p_0^{-1}}$ and $\varphi_{p_1^{-1}}$ are actually automorphisms of B. To do that let us show that the actions given by Equations (2.7) - (2.8) and Equations (2.9) - (2.10) have inverses (which will be the actions of the related p_i on the generators q_0, q_1 of B). From (2.7) it follows immediately that it must be

$$\varphi_{p_0}(q_1) = q_0. \tag{2.11}$$

Moreover, with a bit of manipulations we obtain

$$\begin{aligned}
\varphi_{p_0^{-1}}(q_1) &= q_1^2 q_0^{-1} q_1, && \text{Equation (2.8)},\\
\varphi_{p_0^{-2}}(q_0) &= \left(\varphi_{p_0^{-1}}(q_0)\right)^2 q_0^{-1}\varphi_{p_0^{-1}}(q_0), && \text{substitute } q_1 \text{ from Equation (2.7)},\\
\varphi_{p_0}(q_0^{-1}) &= q_0^{-2}\varphi_{p_0^{-1}}(q_0)q_0^{-1}, && \text{apply } \varphi_{p_0} \text{ and retrieve } \varphi_{p_0}(q_0^{-1}),\\
\varphi_{p_0}(q_0) &= q_0\varphi_{p_0^{-1}}(q_0^{-1})q_0^2, && \text{take inverses},\\
\varphi_{p_0}(q_0) &= q_0 q_1^{-1} q_0^2, && \text{use Equation (2.8).}
\end{aligned} \tag{2.12}$$

Further, $\varphi_{p_1^{-1}}(q_0) = q_0^{-1}q_1$ implies $\varphi_{p_1}(q_0) = \varphi_{p_1}(q_1)q_0^{-1}$, so that

$$\begin{aligned}
\varphi_{p_1^{-1}}(q_1) &= \left(q_0^{-1}q_1\right)^3 q_0^{-2}q_1, && \text{Equation (2.10)},\\
q_1 &= q_0^3\varphi_{p_1}(q_0^{-2})\varphi_{p_1}(q_1), && \text{substitute Equation (2.9) and apply } \varphi_{p_1},\\
q_1 &= q_0^3\left(\varphi_{p_1}(q_1)q_0^{-1}\right)^{-2}\varphi_{p_1}(q_1), && \text{use identity above},\\
q_1 &= q_0^4\varphi_{p_1}(q_1)^{-1}q_0, && \text{simplify},\\
\varphi_{p_1}(q_1) &= q_0 q_1^{-1} q_0^4, && \text{retrieve } \varphi_{p_1}(q_1),
\end{aligned} \tag{2.13}$$

and also

$$\varphi_{p_1}(q_0) = \varphi_{p_1}(q_1)q_0^{-1} = q_0 q_1^{-1} q_0^3. \tag{2.14}$$

So far we have recovered the action of φ_{p_i} $(i = 0, 1)$ on q_0, q_1 applying only necessary conditions in order for them to be the inverses of the corresponding $\varphi_{p_i^{-1}}$. Actually these conditions are also sufficient as the following full computation shows.

$$\begin{array}{lllll}
q_0 & \xmapsto{\varphi_{p_0}} & q_0q_1^{-1}q_0^2 & \xmapsto{\varphi_{p_0^{-1}}} & q_1(q_1^2q_0^{-1}q_1)^{-1}q_1^2 = q_0,\\
q_0 & \xmapsto{\varphi_{p_0^{-1}}} & q_1 & \xmapsto{\varphi_{p_0}} & q_0,\\
q_0 & \xmapsto{\varphi_{p_1}} & q_0q_1^{-1}q_0^3 & \xmapsto{\varphi_{p_1^{-1}}} & q_0^{-1}q_1\left[\left(q_0^{-1}q_1\right)^3 q_0^{-2}q_1\right]^{-1}\left(q_0^{-1}q_1\right)^3 = q_0,\\
q_0 & \xmapsto{\varphi_{p_1^{-1}}} & q_0^{-1}q_1 & \xmapsto{\varphi_{p_1}} & \left(q_0q_1^{-1}q_o^3\right)^{-1} q_0q_1{-}1q_0^4 = q_0,\\
q_1 & \xmapsto{\varphi_{p_0}} & q_0 & \xmapsto{\varphi_{p_0^{-1}}} & q_1,\\
q_1 & \xmapsto{\varphi_{p_0^{-1}}} & q_1^2q_0^{-1}q_1 & \xmapsto{\varphi_{p_0}} & q_0^2\left(q_0q_1^{-1}q_0^2\right)^{-1}q_0 = q_1,\\
q_1 & \xmapsto{\varphi_{p_1}} & q_0q_1^{-1}q_0^4 & \xmapsto{\varphi_{p_1^{-1}}} & q_0^{-1}q_1\left[\left(q_0^{-1}q_1\right)^3 q_0^{-2}q_1\right]^{-1}\left(q_0^{-1}q_1\right)^4 = q_1,\\
q_1 & \xmapsto{\varphi_{p_1^{-1}}} & \left(q_0^{-1}q_1\right)^3 q_0^{-2}q_1 & \xmapsto{\varphi_{p_1}} & \left[\left(q_0q_1^{-1}q_0^3\right)^{-1} q_0q_i^{-1}q_0^4\right]^3\left(q_0q_1^{-1}q_0^3\right)^{-2} q_0q_1^{-1}q_0^4 = q_1.
\end{array}$$

Therefore Equations (2.7) to (2.10) provide a unique homomorphism from A to $\mathrm{Aut}\,(B)$, hence the above presentation of $\mathcal{A}'_3$ is actually the same of $A \rtimes_\varphi B$. The isomorphism linking these groups is the obvious one, i.e. the one identifying the generators p_i, q_j $(i,j = 0,1)$ of $\mathcal{A}'_3$ to the generators of A and B (whence the previous abuse of notation).

Theorem 2.3.1. *$\mathcal{A}_3$ is poly-fg-free with a poly-fg-series of length 3 and free factors given by*

$$\{\,1\,\} \lhd \langle q_0, q_1\rangle \lhd \mathcal{A}'_3 \lhd \mathcal{A}_3,$$
$$\langle q_0, q_1\rangle \simeq F_2, \qquad \frac{\mathcal{A}'_3}{\langle q_0, q_1\rangle} \simeq F_2, \qquad \frac{\mathcal{A}_3}{\mathcal{A}'_3} \simeq \mathbb{Z}.$$

Proof. The above discussion shows that $\mathcal{A}'_3 \simeq F\left(\{\,q_0, q_1\,\}\right) \rtimes_\varphi F\left(\{\,p_0, p_1\,\}\right)$, hence the subgroup generated by q_0, q_1 inside $\mathcal{A}'_3$ is free, normal and its quotient is isomorphic to the group generated by p_0, p_1, which is free. ■

2.3.2 Artin groups $\mathcal{B}_3$ and $\mathcal{B}_4$

Mulholland and Rolfsen in [16, Theorem 1.1] prove that the Artin groups of type $\dot{B}_n$ are not locally indicable for $n \geq 5$. As already stated at the beginning of this chapter this result implies that they cannot be poly-free. In this section we will prove that the Artin groups of this type for $n = 3, 4$ are poly-free.
Obtaining a strongly poly-free series for

$$\mathcal{B}_3 := \mathcal{A}\left(\underset{a\quad b\quad c}{\bullet\!-\!\bullet\overset{4}{-}\bullet}\right) = \left\langle a, b, c \;\middle|\; aba = bab,\ (bc)^2 = (cb)^2,\ ac = ca \right\rangle$$

is straightforward using [16, Theorem 3.9] which proves[2] that its commutator subgroup is a free group of rank 4 generated by

$$p_0 = [a^{-1}, b^{-1}], \qquad\qquad p_1 = [c, b][a^{-1}, b^{-1}],$$
$$p_2 = [a, b][a^{-1}, b^{-1}], \qquad\qquad p_3 = [ac, b][a^{-1}, b^{-1}].$$

Theorem 2.3.2. *$\mathcal{B}_3$ is strongly poly-fg-free with a poly-fg-free series of length 3 and free factors given by*

$$\{\,1\,\} \lhd \mathcal{B}'_3 \lhd \langle\!\langle \mathcal{B}'_3, b\rangle\!\rangle_{\mathcal{B}_3} \lhd \mathcal{B}_3,$$
$$\mathcal{B}'_3 \simeq F_4, \qquad \frac{\langle\!\langle \mathcal{B}'_3, b\rangle\!\rangle_{\mathcal{B}_3}}{\mathcal{B}'_3} \simeq \mathbb{Z}, \qquad \frac{\mathcal{B}_3}{\langle\!\langle \mathcal{B}'_3, b\rangle\!\rangle_{\mathcal{B}_3}} \simeq \mathbb{Z}.$$

[2]The approach of the authors consists in the application of the Reidemeister-Schreier rewriting procedure (that we present in Section 3.1) followed by a non-trivial change of generators.

Proof. Follows from the result of Mulholland and Rolfsen contained in [16, Theorem 3.9] and the fact that $\mathcal{B}_3^{\mathrm{ab}} \simeq \mathbb{Z} \times \mathbb{Z}$ under the isomorphism

$$\begin{aligned}\psi\colon \mathcal{B}_3^{\mathrm{ab}} &\to \mathbb{Z} \times \mathbb{Z},\\ a\mathcal{B}_3' &\mapsto (1_{\mathbb{Z}}, 0_{\mathbb{Z}}),\\ b\mathcal{B}_3' &\mapsto (1_{\mathbb{Z}}, 0_{\mathbb{Z}}),\\ c\mathcal{B}_3' &\mapsto (0_{\mathbb{Z}}, 1_{\mathbb{Z}}).\end{aligned}$$

■

Although [16] also provides a presentation for $\mathcal{B}_4'$ we have not been able to use that result to build a poly-free series for $\mathcal{B}_4$. Instead, for $\mathcal{B}_4$ we rely on the paper [8] by Crisp and Paris where they show that $\mathcal{B}_n = F_n \rtimes \mathcal{A}_{n-1}$ for all $n \geq 3$, which in turn means that poly-freeness of $\mathcal{B}_4$ follows from poly-freeness of $\mathcal{A}_3$ that we achieved in Section 2.3.1. This result also allows to construct a poly-free series for $\mathcal{B}_3$ of length 3 whose terms are not all normal in the whole group. Although the central result in this section is due to Crisp and Paris we nevertheless give a complete proof for it since the authors do not provide computations in their article. To state the result of Crisp and Paris we need the following proposition due to Artin (see [1]).

Proposition 2.3.3. *Let $F_n = F(\{x_1, \ldots, x_n\})$ be the free group on n generators, then the map sending each generator α_i $(i = 1, \ldots, n-1)$ of $\mathcal{A}_{n-1}$ to the element ρ_i of* $\mathrm{Aut}(F_n)$

$$\alpha_i \mapsto \rho_i\colon \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\ x_j \mapsto x_j, \quad j \neq i, i+1, \end{cases} \quad, \qquad \forall i = 1, \ldots, n-1$$

extends (uniquely) to a well-defined homomorphism of groups $\rho\colon \mathcal{A}_{n-1} \to \mathrm{Aut}(F_n)$. Such map is called the Artin's representation of braid groups.

Proof. First of all we retrieve the action of ρ on α_i^{-1}:

$$\rho(\alpha_i^{-1}) = \rho(\alpha_i)^{-1} = \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i, \\ x_j \mapsto x_j, \quad j \neq i, i+1, \end{cases} \quad, \qquad \forall i = 1, \ldots, n-1.$$

By Proposition 1.2.6 the map ρ is well-defined if and only if it preserves all the relators in the definition of $\mathcal{A}_{n-1}$. Such presentation has exactly one relation for each pair of distinct vertices α_i, α_j inside the graph $\dot{A}_{n-1}$. If i, j are consecutive integers the relator is $R^{\mathcal{A}_{n-1}}_{i,i+1} = \alpha_i\alpha_{i+1}\alpha_i\alpha_{i+1}^{-1}\alpha_i^{-1}\alpha_{i+1}^{-1}$ and $R^{\mathcal{A}_{n-1}}_{i,j} = \alpha_i\alpha_j\alpha_j^{-1}\alpha_i^{-1}$ otherwise. In the former case for $i = 1, \ldots, n-2$ we have

$$\begin{aligned}
\rho(R^{\mathcal{A}_{n-1}}_{i,i+1}) &= \rho(\alpha_i\alpha_{i+1}\alpha_i\alpha_{i+1}^{-1}\alpha_i^{-1}) \circ \begin{cases} x_{i+1} \mapsto x_{i+1}x_{i+2}x_{i+1}^{-1}, \\ x_{i+2} \mapsto x_{i+1}, \\ x_j \mapsto x_j, \quad j \neq i+1, i+2, \end{cases} \\
&= \rho(\alpha_i\alpha_{i+1}\alpha_i\alpha_{i+1}^{-1}) \circ \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1} \\ x_{i+1} \mapsto x_i x_{i+2} x_i^{-1} \\ x_{i+2} \mapsto x_i, \\ x_j \mapsto x_j, \quad j \neq i, i+1, i+2, \end{cases} \\
&= \rho(\alpha_i\alpha_{i+1}\alpha_i) \circ \begin{cases} x_i \mapsto x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1} \\ x_{i+1} \mapsto x_i x_{i+1} x_i^{-1} \\ x_{i+2} \mapsto x_i, \\ x_j \mapsto x_j, \quad j \neq i, i+1, i+2, \end{cases} \\
&= \rho(\alpha_i\alpha_{i+1}) \circ \begin{cases} x_i \mapsto x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1} \\ x_{i+1} \mapsto x_i \\ x_{i+2} \mapsto x_{i+1}, \\ x_j \mapsto x_j, \quad j \neq i, i+1, i+2, \end{cases} \\
&= \rho(\alpha_i) \circ \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1} \\ x_{i+1} \mapsto x_i \\ x_{i+2} \mapsto x_{i+2}, \\ x_j \mapsto x_j, \quad j \neq i, i+1, i+2, \end{cases} \\
&= \begin{cases} x_i \mapsto x_i \\ x_{i+1} \mapsto x_{i+1} \\ x_{i+2} \mapsto x_{i+2}, \\ x_j \mapsto x_j, \quad j \neq i, i+1, i+2, \end{cases} \\
&= \mathrm{Id}_{F_n}.
\end{aligned}$$

Whereas the latter relators hold when $|i-j| \geq 2$, in such case the sets of generators of F_n on which $\rho(\alpha_i)$ and $\rho(\alpha_j)$ act non-trivially are disjoint, therefore $\rho(\alpha_i)$ and $\rho(\alpha_j)$ commute and

$$\rho(R^{\mathcal{A}_{n-1}}_{i,j}) = \rho(\alpha_i\alpha_j\alpha_i^{-1}\alpha_j^{-1}) = \mathrm{Id}_{F_n}.$$

Hence ρ is a well-defined homomorphism of groups. ■

Theorem 2.3.4 (Crisp and Paris, [8, Theorem 2.1]). *For all $n \geq 3$, let $\rho\colon \mathcal{A}_{n-1} \to \mathrm{Aut}(F_n)$ be the Artin's representation of braid groups defined in Proposition 2.3.3, then*

$$\mathcal{B}_n \simeq F_n \rtimes_\rho \mathcal{A}_{n-1}.$$

Proof. To prove the statement Crisp and Paris explicitly construct an homomorphism from $\mathcal{B}_n$ to $F_n \rtimes_\rho \mathcal{A}_{n-1}$ and show that there exists an inverse homomorphism. Let $F_n = F(\{x_1, \dots, x_n\})$ and $\varphi\colon \mathcal{B}_n \to F_n \rtimes_\rho \mathcal{A}_{n-1}$ be the homomorphism defined on the generators $\beta_1, \dots, \beta_n$ of $\mathcal{B}_n$ in the following way

$$\varphi\colon \begin{cases} \beta_i \mapsto \alpha_i, & i = 1, \dots, n-1, \\ \beta_n \mapsto x_n. \end{cases}$$

According to Proposition 1.2.6 this mapping on the generators extends uniquely to a homomorphism of groups if φ sends the relators of $\mathcal{B}_n$ to the identity of $F_n \rtimes_\rho \mathcal{A}_{n-1}$. For all $i, j = 1, \dots, n-1$ with $|i - j| \geq 2$ we have

$$\varphi(R^{\mathcal{B}_n}_{i,j}) = \varphi(\beta_i\beta_j\beta_i^{-1}\beta_j^{-1}) = (1_{F_n}, \alpha_i\alpha_j\alpha_i^{-1}\alpha_j^{-1}) = (1_{F_n}, 1_{\mathcal{A}_{n-1}}),$$
$$\varphi(R^{\mathcal{B}_n}_{i,i+1}) = \varphi(\beta_i\beta_{i+1}\beta_i\beta_{i+1}^{-1}\beta_i^{-1}\beta_{i+1}^{-1}) = (1_{F_n}, \alpha_i\alpha_{i+1}\alpha_i\alpha_{i+1}^{-1}\alpha_i^{-1}\alpha_{i+1}^{-1}) = (1_{F_n}, 1_{\mathcal{A}_{n-1}}).$$

If $j = n$, for all $i = 1, \dots, n-2$ we have

$$\begin{aligned}
\varphi(R^{\mathcal{B}_n}_{i,n}) &= \varphi(\beta_i\beta_n\beta_i^{-1}\beta_n^{-1}) \\
&= (1_{F_n}, \alpha_i)(x_n, 1_{\mathcal{A}_{n-1}})(1_{F_n}, \alpha_i)^{-1}(x_n, 1_{\mathcal{A}_{n-1}})^{-1} \\
&= (\rho_i(x_n), \alpha_i)(\rho_i^{-1}(x_n^{-1}), \alpha_i^{-1}) \\
&= (x_n, \alpha_i)(x_n^{-1}, \alpha_i^{-1}) \\
&= (x_n\ \rho_i(x_n^{-1}), 1_{\mathcal{A}_{n-1}}) \\
&= (1_{F_n}, 1_{\mathcal{A}_{n-1}}).
\end{aligned}$$

Finally, for $i = n-1$ and $j = n$ we have

$$\begin{aligned}
\varphi(R^{\mathcal{B}_n}_{n-1,n}) &= \varphi(\beta_{n-1}\beta_n\beta_{n-1}\beta_n\beta_{n-1}^{-1}\beta_n^{-1}\beta_{n-1}^{-1}\beta_n^{-1}) \\
&= (1_{F_n}, \alpha_{n-1})(x_n, 1_{\mathcal{A}_{n-1}})(1_{F_n}, \alpha_{n-1})(x_n, 1_{\mathcal{A}_{n-1}}) \cdot \\
&\quad \cdot (1_{F_n}, \alpha_{n-1}^{-1})(x_n^{-1}, 1_{\mathcal{A}_{n-1}})(1_{F_n}, \alpha_{n-1}^{-1})(x_n^{-1}, 1_{\mathcal{A}_{n-1}}) \\
&= (\rho_{n-1}(x_n), \alpha_{n-1})(\rho_{n-1}(x_n), \alpha_{n-1}) \cdot \\
&\quad \cdot (\rho_{n-1}^{-1}(x_n^{-1}), \alpha_{n-1}^{-1})(\rho_{n-1}^{-1}(x_n^{-1}), \alpha_{n-1}^{-1}) \\
&= (x_n^{-1}x_{n-1}x_n, \alpha_{n-1})(x_n^{-1}x_{n-1}x_n, \alpha_{n-1})(x_{n-1}^{-1}, \alpha_{n-1}^{-1})(x_{n-1}^{-1}, \alpha_{n-1}^{-1}) \\
&= (x_n^{-1}x_{n-1}x_n\ \rho_{n-1}(x_n^{-1}x_{n-1}x_n), \alpha_{n-1}^2)(x_{n-1}^{-1}\ \rho_{n-1}^{-1}(x_{n-1}^{-1}), \alpha_{n-1}^{-2}) \\
&= (x_n^{-1}x_{n-1}x_n((x_n^{-1}x_{n-1}x_n)^{-1}x_n(x_n^{-1}x_{n-1}x_n)), \alpha_{n-1}^2) \cdot \\
&\quad \cdot (x_{n-1}^{-1}(x_{n-1}x_n^{-1}x_{n-1}^{-1}), \alpha_{n-1}^{-2})
\end{aligned}$$

$$\begin{aligned}
&= (x_{n-1}x_n, \alpha_{n-1}^2)(x_n^{-1}x_{n-1}^{-1}, \alpha_{n-1}^{-2}) \\
&= (x_{n-1}x_n\ \rho_{n-1}^2(x_n^{-1}x_{n-1}^{-1}), 1_{\mathcal{A}_{n-1}}) \\
&= (x_{n-1}x_n\ \rho_{n-1}(x_n^{-1}x_{n-1}^{-1}), 1_{\mathcal{A}_{n-1}}) \\
&= (x_{n-1}x_n(x_n^{-1}x_{n-1}x_n)^{-1}x_n^{-1}, 1_{\mathcal{A}_{n-1}}) \\
&= (1_{F_n}, 1_{\mathcal{A}_{n-1}}).
\end{aligned}$$

Therefore φ is an homomorphism of groups. We claim that the function

$$\psi\colon F_n \rtimes_\rho \mathcal{A}_{n-1} \to \mathcal{B}_n$$

defined on the generators of $F_n \rtimes_\rho \mathcal{A}_{n-1}$ in the following way

$$\psi\colon \begin{cases} (1_{F_n}, \alpha_i) \mapsto \beta_i, & \forall i = 1, \dots, n-1, \\ (x_i, 1_{\mathcal{A}_n}) \mapsto \beta_i^{-1} \dots \beta_{n-1}^{-1}\beta_n\beta_{n-1} \dots \beta_i, & \forall i = 1, \dots, n, \end{cases}$$

is a homomorphism of groups. According to Proposition 1.2.7 a presentation for the semidirect product $F_n \rtimes_\rho \mathcal{A}_{n-1}$ is given by

$$\left\langle \begin{matrix} \alpha_i, & i = 1, \dots, n-1, \\ x_j, & j = 1, \dots, n \end{matrix} \ \middle|\ \begin{matrix} R_{i,j}^{\mathcal{A}_{n-1}} = 1, & \forall i, j = 1, \dots, n-1,\ i \neq j, \\ \alpha_i x_j \alpha_i^{-1} = \rho_i(x_j), & \forall i = 1, \dots, n-1,\ \forall j = 1, \dots, n \end{matrix} \right\rangle,$$

hence to check that ψ is a well-defined homomorphism of groups we have to check that it sends all relators in the previous presentation to $1_{\mathcal{B}_n}$. For relators of type $R_{i,j}^{\mathcal{A}_{n-1}}$ the check is straightforward

$$\begin{aligned}
&\psi(R_{i,j}^{\mathcal{A}_{n-1}}) = \beta_i\beta_j\beta_i^{-1}\beta_j^{-1} = 1_{\mathcal{B}_n}, && \forall i, j = 1, \dots, n-1,\ |i-j| \geq 2, \\
&\psi(R_{i,i+1}^{\mathcal{A}_{n-1}}) = \beta_i\beta_{i+1}\beta_i\beta_{i+1}^{-1}\beta_i^{-1}\beta_{i+1}^{-1} = 1_{\mathcal{B}_n}, && \forall i = 1, \dots, n-1.
\end{aligned}$$

For each $i = 1, \dots, n-1$ and $j = 1, \dots, n$ set $S_{i,j} \coloneqq \alpha_i x_j \alpha_i^{-1} \rho_i(x_j)^{-1}$, for relators of this type we have a few cases to discuss.

- If $i = j$ we have

$$\begin{aligned}
\psi(S_{i,i}) &= \psi(\alpha_i x_i \alpha_i^{-1} \rho_i(x_i)^{-1}) \\
&= \beta_i\ (\beta_i^{-1} \dots \beta_{n-1}^{-1}\beta_n \dots \beta_i)\ \beta_i^{-1}\ (\beta_{i+1}^{-1} \dots \beta_{n-1}^{-1}\beta_n \dots \beta_{i+1})^{-1} \\
&= 1_{\mathcal{B}_n}.
\end{aligned}$$

- If $j = i+1$ we proceed by induction to show that the elements of $\mathcal{B}_n$ represented by the words $W_i \coloneqq \psi(S_{n-i,n-i+1})$ are all trivial for all $i = 1, \dots, n-1$. For $i = 1$ we have

$$W_1 = \psi(\alpha_{n-1}x_n\alpha_{n-1}^{-1}\ \rho_{n-1}(x_n)^{-1})$$

$$
\begin{aligned}
&= \beta_{n-1}\beta_n\beta_{n-1}^{-1}\ \psi(x_n^{-1}x_{n-1}x_n)^{-1}\\
&= \beta_{n-1}\beta_n\beta_{n-1}^{-1}\ (\beta_n^{-1}\beta_{n-1}^{-1}\beta_n\beta_{n-1}\beta_n)^{-1}\\
&= \beta_{n-1}\beta_n\ (\beta_{n-1}^{-1}\beta_n^{-1}\beta_{n-1}^{-1}\beta_n^{-1})\ \beta_{n-1}\beta_n\\
&= \beta_{n-1}\beta_n\ (\beta_n^{-1}\beta_{n-1}^{-1}\beta_n^{-1}\beta_{n-1}^{-1})\ \beta_{n-1}\beta_n\\
&= 1_{\mathcal{B}_n}.
\end{aligned}
$$

Now suppose $W_i = 1_{\mathcal{B}_n}$ $(i \geq 1)$, then

$$
\begin{aligned}
W_{i+1} &= \psi(\alpha_{n-i-1}\ x_{n-i}\ \alpha_{n-i-1}^{-1}\ \rho_{n-i-1}(x_{n-i})^{-1})\\
&= \beta_{n-i-1}\ (\beta_{n-i}^{-1}\dots\beta_{n-1}^{-1}\beta_n\dots\beta_{n-i})\ \beta_{n-i-1}^{-1}\ (\beta_{n-i}^{-1}\dots\beta_n^{-1}\beta_{n-1}\dots\beta_{n-i})\ \cdot\\
&\quad\cdot(\beta_{n-i-1}^{-1}\dots\beta_n^{-1}\beta_{n-1}\dots\beta_{n-i-1})\ (\beta_{n-i}^{-1}\dots\beta_{n-1}^{-1}\beta_n\dots\beta_{n-i})\\
&= \beta_{n-i-1}\ (\beta_{n-i}^{-1}\dots\beta_{n-1}^{-1}\beta_n\dots\beta_{n-i+1})\ \beta_{n-i-1}^{-1}\beta_{n-i}^{-1}\beta_{n-i-1}\ \cdot\\
&\quad\cdot(\beta_{n-i+1}^{-1}\dots\beta_n^{-1}\beta_{n-1}\dots\beta_{n-i+1})\ \beta_{n-i-1}^{-1}\beta_{n-i}^{-1}\beta_{n-i-1}\ \cdot\\
&\quad\cdot(\beta_{n-i+1}^{-1}\dots\beta_n^{-1}\beta_{n-1}\dots\beta_{n-i+1})\ \beta_{n-i-1}^{-1}\beta_{n-i}\beta_{n-i-1}\ \cdot\\
&\quad\cdot(\beta_{n-i+1}^{-1}\dots\beta_{n-1}^{-1}\beta_n\dots\beta_{n-i})\\
&= \beta_{n-i-1}\beta_{n-i}^{-1}\beta_{n-i-1}^{-1}\ (\beta_{n-i+1}^{-1}\dots\beta_{n-1}^{-1}\beta_n\dots\beta_{n-i+1})\ \beta_{n-i}^{-1}\ \cdot\\
&\quad\cdot(\beta_{n-i+1}^{-1}\dots\beta_n^{-1}\beta_{n-1}\dots\beta_{n-i+1})\ \cdot\\
&\quad\cdot(\beta_{n-i}^{-1}\beta_{n-i+1}^{-1}\dots\beta_n^{-1}\beta_{n-1}\dots\beta_{n-i+1}\beta_{n-i})\ \cdot\\
&\quad\cdot(\beta_{n-i+1}^{-1}\dots\beta_{n-1}^{-1}\beta_n\dots\beta_{n-i+1})\ \beta_{n-i-1}\beta_{n-i}\\
&= \beta_{n-i}^{-1}\beta_{n-i-1}^{-1}\beta_{n-i}\ (\beta_{n-i+1}^{-1}\dots\beta_{n-1}^{-1}\beta_n\dots\beta_{n-i+1})\ \beta_{n-i}^{-1}\ \cdot\\
&\quad\cdot(\beta_{n-i+1}^{-1}\dots\beta_n^{-1}\beta_{n-1}\dots\beta_{n-i+1})\ \cdot\\
&\quad\cdot(\beta_{n-i}^{-1}\beta_{n-i+1}^{-1}\dots\beta_n^{-1}\beta_{n-1}\dots\beta_{n-i+1}\beta_{n-i})\ \cdot\\
&\quad\cdot(\beta_{n-i+1}^{-1}\dots\beta_{n-1}^{-1}\beta_n\dots\beta_{n-i+1})\ \beta_{n-i-1}\beta_{n-i}\\
&= W_i^{\beta_{n-i-1}\beta_{n-i}}\\
&= 1_{\mathcal{B}_n}.
\end{aligned}
$$

- If $j \neq i, i+1$, recalling that β_i and β_j commute, if $j > i+1$ we have immediately

$$
\begin{aligned}
\psi(S_{i,j}) &= \psi(\alpha_i x_j \alpha_i^{-1}\rho_i(x_j)^{-1})\\
&= \beta_i\ (\beta_j^{-1}\dots\beta_{n-1}^{-1}\beta_n\dots\beta_j)\ \beta_i^{-1}\ (\beta_j^{-1}\dots\beta_{n-1}^{-1}\beta_n\dots\beta_j)^{-1}\\
&= \beta_i\beta_i^{-1}\ (\beta_j^{-1}\dots\beta_{n-1}^{-1}\beta_n\dots\beta_j)\ (\beta_j^{-1}\dots\beta_{n-1}^{-1}\beta_n\dots\beta_j)^{-1}\\
&= 1_{\mathcal{B}_n}.
\end{aligned}
$$

Otherwise, when $j < i$ we have

$$
\psi(S_{i,j}) = \psi(\alpha_i x_j \alpha_i^{-1}\rho_i(x_j)^{-1})
$$

$$\begin{aligned}
&= \beta_i\ (\beta_j^{-1} \dots \beta_{n-1}^{-1}\beta_n \dots \beta_j)\ \beta_i^{-1}\ (\beta_j^{-1} \dots \beta_{n-1}^{-1}\beta_n \dots \beta_j)^{-1} \\
&= (\beta_j^{-1} \dots \beta_{i-2}^{-1})\ \beta_i\ (\beta_{i-1}^{-1}\beta_i^{-1}\beta_{i+1}^{-1} \dots \beta_{n-1}^{-1}\beta_n \dots \beta_{i+1}\beta_i\beta_{i-1})\ \beta_i^{-1} \cdot \\
&\quad \cdot (\beta_{i-2} \dots \beta_j)\ (\beta_j^{-1} \dots \beta_n^{-1}\beta_{n-1} \dots \beta_j) \\
&= (\beta_j^{-1} \dots \beta_{i-2}^{-1})\ \beta_i\ (\beta_{i-1}^{-1}\beta_i^{-1}\beta_{i+1}^{-1} \dots \beta_{n-1}^{-1}\beta_n \dots \beta_{i+1})\ (\beta_i\beta_{i-1}\beta_i^{-1}) \cdot \\
&\quad \cdot (\beta_{i-1}^{-1}\beta_i^{-1} \dots \beta_n^{-1}\beta_{n-1} \dots \beta_j) \\
&= (\beta_j^{-1} \dots \beta_{i-2}^{-1})\ \beta_i\ (\beta_{i-1}^{-1}\beta_i^{-1}\beta_{i+1}^{-1} \dots \beta_{n-1}^{-1}\beta_n \dots \beta_{i+1})\ \beta_{i-1}^{-1} \cdot \\
&\quad \cdot (\beta_{i+1}^{-1} \dots \beta_n^{-1}\beta_{n-1} \dots \beta_j) \\
&= (\beta_j^{-1} \dots \beta_{i-2}^{-1})\ (\beta_i\beta_{i-1}^{-1}\beta_i^{-1})\ (\beta_{i-1}^{-1}\ \beta_i\beta_{i-1} \dots \beta_j) \\
&= (\beta_j^{-1} \dots \beta_{i-2}^{-1})\ (\beta_{i-1}^{-1}\beta_i^{-1}\beta_{i-1})\ (\beta_{i-1}^{-1}\ \beta_i\beta_{i-1} \dots \beta_j) \\
&= 1_{\mathcal{B}_n},
\end{aligned}$$

where in the third to last equality we used the fact that β_{i-1}^{-1} commutes with each β_l for all $l \geq i+1$.

Hence ψ is a well-defined homomorphism of groups and it is trivial to verify on the generators that $\psi \circ \varphi = \mathrm{Id}_{\mathcal{B}_n}$ and $\varphi \circ \psi = \mathrm{Id}_{F_n \rtimes \mathcal{A}_{n-1}}$. The claim follows. ■

Theorem 2.3.5. *The group $\mathcal{B}_3$ admits a poly-fg-free series of length 3 as follows*

$$1 \lhd \mathcal{B}_3^{(1)} \lhd \mathcal{B}_3^{(2)} \lhd \mathcal{B}_3$$

where

$$\mathcal{B}_3^{(1)} := \left\langle \beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1,\ \beta_2^{-1}\beta_3\beta_2,\ \beta_3 \right\rangle, \qquad \mathcal{B}_3^{(2)} := \left\langle \mathcal{B}_3^{(1)},\ \beta_1\beta_2^{-1},\ \beta_2^{-1}\beta_1 \right\rangle,$$

and the free factors are

$$\mathcal{B}_3^{(1)} \simeq F_3, \qquad \frac{\mathcal{B}_3^{(2)}}{\mathcal{B}_3^{(1)}} \simeq F_2, \qquad \frac{\mathcal{B}_3}{\mathcal{B}_3^{(2)}} \simeq \mathbb{Z}.$$

Proof. Follows from the isomorphisms shown in Theorem 2.3.4 and Theorem 2.2.16 (since $\mathcal{A}_2 = \mathcal{I}_2(3)$). ■

Theorem 2.3.6. *The group $\mathcal{B}_4$ is poly-fg-free with a poly-fg-free series of length 4 given by*

$$1 \lhd \mathcal{B}_4^{(1)} \lhd \mathcal{B}_4^{(2)} \lhd \mathcal{B}_4^{(3)} \lhd \mathcal{B}_4,$$

where

$$\mathcal{B}_4^{(1)} := \left\langle \beta_1^{-1}\beta_2^{-1}\beta_3^{-1}\beta_4\beta_3\beta_2\beta_1,\ \beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1,\ \beta_2^{-1}\beta_3\beta_2, \beta_3 \right\rangle,$$
$$\mathcal{B}_4^{(2)} := \left\langle \mathcal{B}_4^{(1)},\ \beta_3\beta_1^{-1},\ \beta_2\beta_1^{-1}\beta_3\beta_1^{-1} \right\rangle,$$

$$\mathcal{B}_4^{(3)} := \left\langle \mathcal{B}_4^{(2)},\ \beta_2\beta_1^{-1},\ \beta_1\beta_2\beta_1^{-2} \right\rangle$$

and the free factors are

$$\mathcal{B}_4^{(1)} \simeq F_4, \qquad \frac{\mathcal{B}_4^{(2)}}{\mathcal{B}_4^{(1)}} \simeq F_2, \qquad \frac{\mathcal{B}_4^{(3)}}{\mathcal{B}_4^{(2)}} \simeq F_2, \qquad \frac{\mathcal{B}_4}{\mathcal{B}_4^{(3)}} \simeq \mathbb{Z}.$$

Proof. Follows from the isomorphisms shown in Theorem 2.3.4 and Theorem 2.3.1. ■

2.3.3 Artin group $\mathcal{D}_4$

The approach we will adopt to build a poly-fg-free series of length 3 for $\mathcal{D}_4$ is to use a result analogous to the one we used for $\mathcal{B}_4$, provided in the same article by Crisp and Paris [8]. They show that $\mathcal{D}_n = F_{n-1} \rtimes \mathcal{A}_{n-1}$ for all $n \geq 4$, which in turn means that poly-freeness of $\mathcal{D}_4$ follows from poly-freeness of $\mathcal{A}_3$. Also in this case we provide complete computations for the result of Crisp and Paris which are missing in their article. To state their result we need the following proposition.

Proposition 2.3.7. *Let $F_{n-1} = F(\{x_1, \dots, x_{n-1}\})$ be the free group on $n-1$ generators, then the map sending each generator α_i $(i = 1, \dots, n-1)$ of $\mathcal{A}_{n-1}$ to the element ρ_i of $\mathrm{Aut}(F_{n-1})$*

$$\alpha_i \mapsto \rho_i : \begin{cases} x_i \mapsto x_i x_{i+1}^{-1} x_i, \\ x_{i+1} \mapsto x_i, \\ x_j \mapsto x_j, \quad j \neq i, i+1, \end{cases} \qquad , \qquad \forall i = 1, \dots, n-2,$$

$$\alpha_{n-1} \mapsto \rho_{n-1} : \begin{cases} x_i \mapsto x_{n-1}^{-1} x_i, \quad i = 1, \dots, n-2, \\ x_{n-1} \mapsto x_{n-1}, \end{cases}$$

extends (uniquely) to a well-defined homomorphism[3] *of groups*

$$\rho\colon \mathcal{A}_{n-1} \to \mathrm{Aut}(F_{n-1}).$$

Proof. Let us begin retrieving the action of ρ on α_i^{-1}

$$\rho(\alpha_i^{-1}) = \rho_i^{-1} = \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_{i+1} x_i^{-1} x_{i+1}, \\ x_j \mapsto x_j, \quad j \neq i, i+1, \end{cases} \qquad , \qquad \forall i = 1, \dots, n-2,$$

[3] In the literature ρ is also referred as the “braid monodromy representation” since it has a topological interpretation explained in [8, Section 2.3].

$$\rho(\alpha_{n-1}^{-1}) = \rho_{n-1}^{-1} = \begin{cases} x_i \mapsto x_{n-1}x_i, & i = 1, \dots, n-2, \\ x_{n-1} \mapsto x_{n-1}, \end{cases}$$

By Proposition 1.2.6 the map ρ is well-defined if and only if it preserves all the relators in the definition of $\mathcal{A}_{n-1}$. Such presentation has exactly one relation for each pair of distinct vertices α_i, α_j inside the graph $\dot{A}_{n-1}$. If i, j are consecutive integers the relator is $R_{i,i+1}^{\mathcal{A}_{n-1}} = \alpha_i\alpha_{i+1}\alpha_i\alpha_{i+1}^{-1}\alpha_i^{-1}\alpha_{i+1}^{-1}$ and $R_{i,j}^{\mathcal{A}_{n-1}} = \alpha_i\alpha_j\alpha_j^{-1}\alpha_i^{-1}$ otherwise. To check the former relator we distinguish two cases.

- If $i = 1, \dots, n-3$, then

$$\begin{aligned}
\rho(R_{i,i+1}^{\mathcal{A}_{n-1}}) = \rho(\alpha_i\alpha_{i+1}\alpha_i\alpha_{i+1}^{-1}\alpha_i^{-1})\circ &\begin{cases} x_{i+1} \mapsto x_{i+2}, \\ x_{i+2} \mapsto x_{i+2}x_{i+1}^{-1}x_{i+2}, \\ x_j \mapsto x_j, \quad j \neq i+1, i+2, \end{cases} \\
= \rho(\alpha_i\alpha_{i+1}\alpha_i\alpha_{i+1}^{-1})\circ &\begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_{i+2}, \\ x_{i+2} \mapsto x_{i+2}x_{i+1}^{-1}x_ix_{i+1}^{-1}x_{i+2}, \\ x_j \mapsto x_j, \quad j \neq i, i+1, i+2, \end{cases} \\
= \rho(\alpha_i\alpha_{i+1}\alpha_i)\circ &\begin{cases} x_i \mapsto x_{i+2}, \\ x_{i+1} \mapsto x_{i+2}x_{i+1}^{-1}x_{i+2}, \\ x_{i+2} \mapsto x_{i+2}x_{i+1}^{-1}x_ix_{i+1}^{-1}x_{i+2}, \\ x_j \mapsto x_j, \quad j \neq i, i+1, i+2, \end{cases} \\
= \rho(\alpha_i\alpha_{i+1})\circ &\begin{cases} x_i \mapsto x_{i+2}, \\ x_{i+1} \mapsto x_{i+2}x_i^{-1}x_{i+2}, \\ x_{i+2} \mapsto x_{i+2}x_{i+1}^{-1}x_{i+2}, \\ x_j \mapsto x_j, \quad j \neq i, i+1, i+2, \end{cases} \\
= \rho(\alpha_i)\circ &\begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}x_i^{-1}x_{i+1}, \\ x_{i+2} \mapsto x_{i+2}, \\ x_j \mapsto x_j, \quad j \neq i, i+1, i+2, \end{cases} \\
= &\begin{cases} x_i \mapsto x_i, \\ x_{i+1} \mapsto x_{i+1}, \\ x_{i+2} \mapsto x_{i+2}, \\ x_j \mapsto x_j, \quad j \neq i, i+1, i+2, \end{cases} \\
= &\,\mathrm{Id}_{F_{n-1}}.
\end{aligned}$$

- If $i = n-2$, then

$$
\begin{aligned}
\rho(R^{\mathcal{A}_{n-1}}_{n-2,n-1}) &= \rho(\alpha_{n-2}\alpha_{n-1}\alpha_{n-2}\alpha_{n-1}^{-1}\alpha_{n-2}^{-1})\circ \begin{cases} x_{n-1} \mapsto x_{n-1}, \\ x_j \mapsto x_{n-1}x_j, \quad j \neq n-1, \end{cases} \\
&= \rho(\alpha_{n-2}\alpha_{n-1}\alpha_{n-2}\alpha_{n-1}^{-1})\circ \begin{cases} x_{n-2} \mapsto x_{n-1}x_{n-2}^{-1}x_{n-1}x_{n-1}, \\ x_{n-1} \mapsto x_{n-1}x_{n-2}^{-1}x_{n-1}, \\ x_j \mapsto x_{n-1}x_{n-2}^{-1}x_{n-1}x_j, \\ \qquad j \neq n-2, n-1, \end{cases} \\
&= \rho(\alpha_{n-2}\alpha_{n-1}\alpha_{n-2})\circ \begin{cases} x_{n-2} \mapsto x_{n-1}x_{n-2}^{-1}x_{n-1}, \\ x_{n-1} \mapsto x_{n-1}x_{n-2}^{-1}, \\ x_j \mapsto x_{n-1}x_{n-2}^{-1}x_{n-1}x_j, \\ \qquad j \neq n-2, n-1, \end{cases} \\
&= \rho(\alpha_{n-2}\alpha_{n-1})\circ \begin{cases} x_{n-2} \mapsto x_{n-1}, \\ x_{n-1} \mapsto x_{n-1}x_{n-2}^{-1}, \\ x_j \mapsto x_{n-1}x_j, \\ \qquad j \neq n-2, n-1, \end{cases} \\
&= \rho(\alpha_{n-2})\circ \begin{cases} x_{n-2} \mapsto x_{n-1}, \\ x_{n-1} \mapsto x_{n-1}x_{n-2}^{-1}x_{n-1}, \\ x_j \mapsto x_j, \quad j \neq n-2, n-1, \end{cases} \\
&= \begin{cases} x_{n-2} \mapsto x_{n-2}, \\ x_{n-1} \mapsto x_{n-1}, \\ x_j \mapsto x_j, \quad j \neq n-2, n-1, \end{cases} \\
&= \mathrm{Id}_{F_{n-1}}\,.
\end{aligned}
$$

Whereas the latter relators hold when $|i-j| \geq 2$ and we have again two cases.

- If $i, j \neq n-1$ the sets of generators of F_{n-1} on which $\rho(\alpha_i)$ and $\rho(\alpha_j)$ act non-trivially are disjoint, therefore $\rho(\alpha_i)$ and $\rho(\alpha_j)$ commute and we have

$$\rho(R^{\mathcal{A}_{n-1}}_{i,j}) = \rho(\alpha_i\alpha_j\alpha_i^{-1}\alpha_j^{-1}) = \mathrm{Id}_{F_{n-1}}\,.$$

- If either i or j is equal to $n-1$, say $j = n-1$, we still have to do a little check

$$\rho(R^{\mathcal{A}_{n-1}}_{i,n-1}) = \rho(\alpha_i\alpha_{n-1}\alpha_i^{-1})\circ \begin{cases} x_{n-1} \mapsto x_{n-1}, \\ x_j \mapsto x_{n-1}x_j, \quad j \neq n-1, \end{cases}$$

$$= \rho(\alpha_i\alpha_{n-1})\circ \begin{cases} x_i \mapsto x_{n-1}x_{i+1}, \\ x_{i+1} \mapsto x_{n-1}x_{i+1}x_i^{-1}x_{i+1}, \\ x_{n-1} \mapsto x_{n-1}, \\ x_j \mapsto x_{n-1}x_j, \quad j \neq i, i+1, n-1, \end{cases}$$

$$= \rho(\alpha_i)\circ \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}x_i^{-1}x_{i+1}, \\ x_{n-1} \mapsto x_{n-1}, \\ x_j \mapsto x_j, \qquad j \neq i, i+1, n-1, \end{cases}$$

$$= \mathrm{Id}_{F_{n-1}}.$$

We conclude that ρ is a well-defined homomorphism of groups. ■

Theorem 2.3.8 (Crisp and Paris, [8, Theorem 2.3])**.** *For all $n \geq 4$, let $\rho\colon \mathcal{A}_{n-1} \to \mathrm{Aut}(F_{n-1})$ be the representation described in Proposition 2.3.7, then*

$$\mathcal{D}_n \simeq F_{n-1} \rtimes_\rho \mathcal{A}_{n-1}.$$

Proof. Let $F_{n-1} = F(\{x_1, \dots, x_{n-1}\})$. To prove the theorem Crisp and Paris explicitly construct an isomorphism $\varphi\colon \mathcal{D}_n \to F_{n-1} \rtimes_\rho \mathcal{A}_{n-1}$ defined on the generators $\delta_1, \dots, \delta_n$ of $\mathcal{D}_n$ in the following way

$$\varphi\colon \begin{cases} \delta_i \mapsto (1_{F_{n-1}}, \alpha_i), \qquad i = 1, \dots, n-1, \\ \delta_n \mapsto (x_{n-1}, \alpha_{n-1}). \end{cases}$$

According to Proposition 1.2.6 this mapping on the generators extends uniquely to a homomorphism of groups if φ sends the relators of $\mathcal{D}_n$ to the identity of $F_{n-1} \rtimes_\rho \mathcal{A}_{n-1}$. For all $i = 1, \dots, n-2$ and $j = 1, \dots, n-1$ with $|i-j| \geq 2$ we have

$$\varphi(R^{\mathcal{D}_n}_{i,j}) = (1_{F_{n-1}}, \alpha_i\alpha_j\alpha_i^{-1}\alpha_j^{-1}) = (1_{F_{n-1}}, 1_{\mathcal{A}_{n-1}}),$$
$$\varphi(R^{\mathcal{D}_n}_{i,i+1}) = (1_{F_{n-1}}, \alpha_i\alpha_{i+1}\alpha_i\alpha_{i+1}^{-1}\alpha_i^{-1}\alpha_{i+1}^{-1}) = (1_{F_{n-1}}, 1_{\mathcal{A}_{n-1}}).$$

Instead, when $j = n$, for all $i = 1, \dots, n-3, n-1$ we have

$$\begin{aligned} \varphi(R^{\mathcal{D}_n}_{i,n}) = \varphi(\delta_i\delta_n\delta_i^{-1}\delta_n^{-1}) &= (1_{F_{n-1}}, \alpha_i)(x_{n-1}, \alpha_{n-1})(1_{F_{n-1}}, \alpha_i)^{-1}(x_{n-1}, \alpha_{n-1})^{-1} \\ &= (\rho_i(x_{n-1}), \alpha_i\alpha_{n-1})(\rho_i^{-1}(x_{n-1}^{-1}), \alpha_i^{-1}\alpha_{n-1}^{-1}) \\ &= (x_{n-1}, \alpha_i\alpha_{n-1})(x_{n-1}^{-1}, \alpha_i^{-1}\alpha_{n-1}^{-1}) \\ &= (x_{n-1}\ \rho_i\rho_{n-1}(x_{n-1}^{-1}), 1_{\mathcal{A}_{n-1}}) \\ &= (1_{F_{n-1}}, 1_{\mathcal{A}_{n-1}}) \end{aligned}$$

and for $i = n-2$ we have

$$\begin{aligned}
\varphi(R^{\mathcal{D}_n}_{n-2,n}) &= \varphi(\delta_{n-2}\delta_n\delta_{n-2}\delta_n^{-1}\delta_n^{-1}\delta_{n-2}^{-1}) \\
&= (1_{F_{n-1}}, \alpha_{n-2})(x_{n-1}, \alpha_{n-1})(1_{F_{n-1}}, \alpha_{n-2}) \cdot \\
&\quad \cdot (x_{n-1}, \alpha_{n-1})^{-1}(1_{F_{n-1}}, \alpha_{n-2})^{-1}(x_{n-1}, \alpha_{n-1})^{-1} \\
&= (\rho_{n-2}(x_{n-1}), \alpha_{n-2}\alpha_{n-1})(\rho_{n-2}(x_{n-1}^{-1}), \alpha_{n-2}\alpha_{n-1}^{-1}) \cdot \\
&\quad \cdot (\rho_{n-2}^{-1}(x_{n-1}^{-1}), \alpha_{n-2}^{-1}\alpha_{n-1}^{-1}) \\
&= (x_{n-2}, \alpha_{n-2}\alpha_{n-1})(x_{n-2}^{-1}, \alpha_{n-2}\alpha_{n-1}^{-1})(x_{n-1}^{-1}x_{n-2}x_{n-1}^{-1}, \alpha_{n-2}^{-1}\alpha_{n-1}^{-1}) \\
&= (x_{n-2}\ \rho_{n-2}\ \rho_{n-1}(x_{n-2}^{-1}), \alpha_{n-2}\alpha_{n-1}\alpha_{n-2}\alpha_{n-1}^{-1}) \cdot \\
&\quad \cdot (x_{n-1}^{-1}x_{n-2}x_{n-1}^{-1}, \alpha_{n-2}^{-1}\alpha_{n-1}^{-1}) \\
&= (x_{n-1}\ \rho_{n-2}\rho_{n-1}\rho_{n-2}\rho_{n-1}^{-1}(x_{n-1}^{-1}x_{n-2}x_{n-1}^{-1}), 1_{\mathcal{A}_{n-1}}) \\
&= (1_{F_{n-1}}, 1_{\mathcal{A}_{n-1}}).
\end{aligned}$$

Therefore φ is an homomorphism of groups. Next, we claim that the function

$$\psi\colon F_{n-1} \rtimes_\rho \mathcal{A}_{n-1} \to \mathcal{D}_n$$

defined on the generators of $F_{n-1} \rtimes_\rho \mathcal{A}_{n-1}$ in the following way

$$\psi\colon \begin{cases} (1_{F_{n-1}}, \alpha_i) \mapsto \delta_i, & \forall i = 1, \dots, n-1, \\ (x_i, 1_{\mathcal{A}_{n-1}}) \mapsto \delta_i\delta_{i+1}\dots\delta_{n-2}\ (\delta_n\delta_{n-1}^{-1})\ \delta_{n-2}^{-1}\dots, \delta_{i+1}^{-1}\delta_i^{-1}, & \forall i = 1, \dots, n-1, \end{cases}$$

is a homomorphism of groups and is the inverse of φ. According to Proposition 1.2.7 a presentation for the semidirect product $F_{n-1} \rtimes_\rho \mathcal{A}_{n-1}$ is given by

$$\left\langle \begin{matrix} \alpha_i, & i = 1, \dots, n-1, \\ x_j, & j = 1, \dots, n-1 \end{matrix} \;\middle|\; \begin{matrix} R^{\mathcal{A}_{n-1}}_{i,j} = 1, & \forall i, j = 1, \dots, n-1,\ i \neq j, \\ \alpha_i x_j \alpha_i^{-1} = \rho_i(x_j), & \forall i = 1, \dots, n-1,\ \forall j = 1, \dots, n \end{matrix} \right\rangle,$$

hence to check that ψ is a well-defined homomorphism of groups we have to check that it sends all relators in the previous presentation to $1_{\mathcal{D}_n}$. For relators of type $R^{\mathcal{A}_{n-1}}_{i,j}$ the check is straightforward

$$\begin{aligned}
\psi(R^{\mathcal{A}_{n-1}}_{i,j}) &= \delta_i\delta_j\delta_i^{-1}\delta_j^{-1} = 1, && \forall i, j = 1, \dots, n-1,\ |i-j| \geq 2, \\
\psi(R^{\mathcal{A}_{n-1}}_{i,i+1}) &= \delta_i\delta_{i+1}\delta_i\delta_{i+1}^{-1}\delta_i^{-1}\delta_{i+1}^{-1} = 1, && \forall i = 1, \dots, n-2.
\end{aligned}$$

For each $i, j = 1, \dots, n-1$ set $S_{i,j} \coloneqq \alpha_i x_j \alpha_i^{-1}\rho_i(x_j)^{-1}$, for this type of relators we have to make a few cases.

- If $j = n-1$ and $i = 1, \dots, n-2$, then

$$\psi(S_{i,n-1}) = \begin{cases} \delta_{n-1}\ (\delta_n\delta_{n-1}^{-1})\ \delta_{n-1}^{-1}\ (\delta_n\delta_{n-1}^{-1})^{-1} = 1_{\mathcal{D}_n}, & i = n-1, \\ \delta_{n-2}\ (\delta_n\delta_{n-1}^{-1})\ \delta_{n-2}^{-1}\ (\delta_{n-2}\delta_n\delta_{n-1}^{-1}\delta_{n-2}^{-1})^{-1} = 1_{\mathcal{D}_n}, & i = n-2, \\ \delta_i\ (\delta_n\delta_{n-1}^{-1})\ \delta_i^{-1}\ (\delta_n\delta_{n-1}^{-1})^{-1} = 1_{\mathcal{D}_n}, & 1 \leq i \leq n-3. \end{cases}$$

- If $j = 1, \dots, n-2$, then
  - If $i < j-1$, we have
  $$\psi(\alpha_i x_j \alpha_i^{-1} \rho_i(x_j)^{-1}) = \delta_i \psi(x_j) \delta_i^{-1} \psi(x_j)^{-1} = 1_{\mathcal{D}_n}$$
  since $\psi(x_j)$ is written using only δ_k $(k \geq j)$ and δ_i commutes with all of them.
  - If $i = j-1$, we have
  $$\psi(\alpha_{j-1} x_j \alpha_{j-1}^{-1} \rho_i(x_j)^{-1}) = \delta_{j-1}\ (\delta_j \dots \delta_j^{-1})\ \delta_{j-1}^{-1}\ (\delta_{j-1} \dots \delta_{j-1}^{-1})^{-1} = 1_{\mathcal{D}_n}.$$
  - If $i = j$, we proceed by induction on $W_i \coloneqq \psi(S_{n-i-1,n-i-1}),\ i \geq 1$. In the base case we have
  $$\begin{aligned} W_1 &= \psi(\alpha_{n-2} x_{n-2} \alpha_{n-2}^{-1} \rho_{n-2}(x_{n-2})^{-1}) \\ &= \delta_{n-2}\ (\delta_{n-2}\delta_n\delta_{n-1}^{-1}\delta_{n-2}^{-1})\ \delta_{n-2}^{-1}\ \cdot \\ &\quad \cdot (\delta_{n-2}\delta_n\delta_{n-1}^{-1}\delta_{n-2}^{-1})^{-1}\ \delta_n\delta_{n-1}^{-1}\ (\delta_{n-2}\delta_n\delta_{n-1}^{-1}\delta_{n-2}^{-1})^{-1} \\ &= \delta_{n-2}^2\delta_n\ (\delta_{n-1}^{-1}\delta_{n-2}^{-1}\delta_{n-1})\ (\delta_n^{-1}\delta_{n-2}^{-1}\delta_n)\ (\delta_{n-1}^{-1}\delta_{n-2}\delta_{n-1})\ \delta_n^{-1}\delta_{n-2}^{-1} \\ &= \delta_{n-2}^2\delta_n\ (\delta_{n-2}\delta_{n-1}^{-1}\delta_{n-2}^{-1})\ (\delta_{n-2}\delta_n^{-1}\delta_{n-2}^{-1})\ (\delta_{n-2}\delta_{n-1}\delta_{n-2}^{-1})\ \delta_n^{-1}\delta_{n-2}^{-1} \\ &= \delta_{n-2}^2\delta_n\delta_{n-2}\delta_{n-1}^{-1}\delta_n^{-1}\delta_{n-1}\delta_{n-2}^{-1}\delta_n^{-1}\delta_{n-2}^{-1} \\ &= \delta_{n-2}^2\delta_n\delta_{n-2}\delta_n^{-1}\delta_{n-2}^{-1}\delta_n^{-1}\delta_{n-2}^{-1} \\ &= \delta_{n-2}^2\delta_{n-2}^{-1}\delta_n\delta_{n-2}\delta_{n-2}^{-1}\delta_n^{-1}\delta_{n-2}^{-1} \\ &= 1_{\mathcal{D}_n}. \end{aligned}$$

  Now suppose the statement true for $i-1$ $(i \geq 2)$, we have
  $$\begin{aligned} W_i &= \psi(\alpha_{n-i} x_{n-i} \alpha_{n-i}^{-1} \rho_{n-i}(x_{n-i})^{-1}) \\ &= \psi(\alpha_{n-i} x_{n-i} \alpha_{n-i}^{-1} x_{n-i}^{-1} x_{n-i+1} x_{n-i}^{-1}) \\ &= \delta_{n-i}\ (\delta_{n-i} \dots \delta_{n-i}^{-1})\ \delta_{n-i}^{-1}\ (\delta_{n-i} \dots \delta_{n-i}^{-1})^{-1}\ \cdot \\ &\quad \cdot (\delta_{n-1+1} \dots \delta_{n-1+1}^{-1})\ (\delta_{n-i} \dots \delta_{n-i}^{-1})^{-1} \\ &= \delta_{n-i}\ (\delta_{n-i} \dots \delta_{n-i}^{-1})\ \delta_{n-i}^{-1}\delta_{n-i}\delta_{n-i+1}\ (\delta_{n-i+2} \dots \delta_{n-i+2}^{-1})^{-1}\ \cdot \\ &\quad \cdot \delta_{n-i+1}^{-1}\delta_{n-i}^{-1}\delta_{n-i+1}\ (\delta_{n-i+2} \dots \delta_{n-i+2}^{-1})\ \delta_{n-i+1}^{-1}\delta_{n-i}\delta_{n-i+1}\ \cdot \\ &\quad \cdot (\delta_{n-i+2} \dots \delta_{n-i+2}^{-1})^{-1}\ \delta_{n-i+1}^{-1}\delta_{n-i}^{-1} \\ &= \delta_{n-i}\delta_{n-i}\ (\delta_{n-i+1} \dots \delta_{n-i+1}^{-1})\ \delta_{n-i}^{-1}\delta_{n-i+1}\ (\delta_{n-i+2} \dots \delta_{n-i+2}^{-1})^{-1}\ \cdot \\ &\quad \cdot \delta_{n-i}\delta_{n-i+1}^{-1}\delta_{n-i}^{-1}\ (\delta_{n-i+2} \dots \delta_{n-i+2}^{-1})\ \delta_{n-i}\delta_{n-i+1}\delta_{n-i}^{-1}\ \cdot \\ &\quad \cdot (\delta_{n-i+2} \dots \delta_{n-i+2}^{-1})^{-1}\ \delta_{n-i+1}^{-1}\delta_{n-i}^{-1} \\ &= \delta_{n-i}\delta_{n-i}\ (\delta_{n-i+1} \dots \delta_{n-i+1}^{-1})\ \delta_{n-i}^{-1}\delta_{n-i+1}\delta_{n-i}\ (\delta_{n-i+2} \dots \delta_{n-i+2}^{-1})^{-1}\ \cdot \\ &\quad \cdot \delta_{n-i+1}^{-1}\delta_{n-i}^{-1}\delta_{n-i}\ (\delta_{n-i+2} \dots \delta_{n-i+2}^{-1})\ \delta_{n-i+1}\ \cdot \\ &\quad \cdot (\delta_{n-i+2} \dots \delta_{n-i+2}^{-1})^{-1}\ \delta_{n-i}^{-1}\delta_{n-i+1}^{-1}\delta_{n-i}^{-1} \end{aligned}$$

$$\begin{aligned}
&= \delta_{n-i}\delta_{n-i}\ (\delta_{n-i+1}\dots\delta_{n-i+1}^{-1})\ \delta_{n-i}^{-1}\delta_{n-i+1}\delta_{n-i}\delta_{n-i+1}^{-1}\ \cdot\\
&\quad\cdot (\delta_{n-i+1}\dots\delta_{n-i+1}^{-1})^{-1}\ (\delta_{n-i+2}\dots\delta_{n-i+2}^{-1})\ \cdot\\
&\quad\cdot (\delta_{n-i+1}\dots\delta_{n-i+1}^{-1})^{-1}\ \delta_{n-i}^{-1}\delta_{n-i+1}^{-1}\\
&= \delta_{n-i}\delta_{n-i}\delta_{n-i+1}\ (\delta_{n-i+2}\dots\delta_{n-i+2}^{-1})\ \delta_{n-i+1}^{-1}\delta_{n-i}^{-1}\delta_{n-i+1}\delta_{n-i}\delta_{n-i+1}^{-1}\ \cdot\\
&\quad\cdot (\delta_{n-i+1}\dots\delta_{n-i+1}^{-1})^{-1}\ (\delta_{n-i+2}\dots\delta_{n-i+2}^{-1})\ \cdot\\
&\quad\cdot (\delta_{n-i+1}\dots\delta_{n-i+1}^{-1})^{-1}\ \delta_{n-i}^{-1}\delta_{n-i+1}^{-1}\\
&= \delta_{n-i}\delta_{n-i}\delta_{n-i+1}\ (\delta_{n-i+2}\dots\delta_{n-i+2}^{-1})\ \delta_{n-i}\delta_{n-i+1}^{-1}\delta_{n-i}^{-1}\delta_{n-i}\delta_{n-i+1}^{-1}\ \cdot\\
&\quad\cdot (\delta_{n-i+1}\dots\delta_{n-i+1}^{-1})^{-1}\ (\delta_{n-i+2}\dots\delta_{n-i+2}^{-1})\ \cdot\\
&\quad\cdot (\delta_{n-i+1}\dots\delta_{n-i+1}^{-1})^{-1}\ \delta_{n-i}^{-1}\delta_{n-i+1}^{-1}\\
&= \delta_{n-i}\delta_{n-i}\delta_{n-i+1}\delta_{n-i}\ (\delta_{n-i+2}\dots\delta_{n-i+2}^{-1})\ \delta_{n-i+1}^{-1}\delta_{n-i+1}^{-1}\ \cdot\\
&\quad\cdot (\delta_{n-i+1}\dots\delta_{n-i+1}^{-1})^{-1}\ (\delta_{n-i+2}\dots\delta_{n-i+2}^{-1})\ \cdot\\
&\quad\cdot (\delta_{n-i+1}\dots\delta_{n-i+1}^{-1})^{-1}\ \delta_{n-i}^{-1}\delta_{n-i+1}^{-1}\\
&= \delta_{n-i+1}\delta_{n-i}\delta_{n-i+1}\ (\delta_{n-i+1}\dots\delta_{n-i+1}^{-1})\ \delta_{n-i+1}^{-1}\ (\delta_{n-i+1}\dots\delta_{n-i+1}^{-1})^{-1}\ \cdot\\
&\quad\cdot (\delta_{n-i+2}\dots\delta_{n-i+2}^{-1})\ (\delta_{n-i+1}\dots\delta_{n-i+1}^{-1})^{-1}\ \delta_{n-i}^{-1}\delta_{n-i+1}^{-1}\\
&= W_{i-1}^{\delta_{n-i}^{-1}\delta_{n-i+1}^{-1}}\\
&= 1_{\mathcal{D}_n}.
\end{aligned}$$

– If $i = j+1$, we have

$$\begin{aligned}
\psi(S_{i,i+1}) &= \psi(\alpha_{j+1}x_j\alpha_{j+1}^{-1}\rho_i(x_j)^{-1})\\
&= \delta_{j+1}\ (\delta_j\delta_{j+1}\dots\delta_{j+1}^{-1}\delta_j^{-1})\ \delta_{j+1}^{-1}\ (\delta_j\delta_{j+1}\dots\delta_{j+1}^{-1}\delta_j^{-1})^{-1}\\
&= \delta_{j+1}\delta_j\delta_{j+1}\ (\delta_{j+2}\dots\delta_{j+2}^{-1})\ \delta_{j+1}^{-1}\delta_j^{-1}\delta_{j+1}^{-1}\ (\delta_j\delta_{j+1}\dots\delta_{j+1}^{-1}\delta_j^{-1})^{-1}\\
&= \delta_j\delta_{j+1}\delta_j\ (\delta_{j+2}\dots\delta_{j+2}^{-1})\ \delta_j^{-1}\delta_{j+1}^{-1}\delta_j^{-1}\ (\delta_j\delta_{j+1}\dots\delta_{j+1}^{-1}\delta_j^{-1})^{-1}\\
&= \delta_j\delta_{j+1}\ \left[\delta_j\ (\delta_{j+2}\dots\delta_{j+2}^{-1})\ \delta_j^{-1}\ (\delta_{j+2}\dots\delta_{j+2}^{-1})^{-1}\right]\ \delta_{j+1}^{-1}\delta_j^{-1}\\
&= \psi(\alpha_j x_{j+2}\alpha_j^{-1}\rho_i(x_j)^{-1})^{\delta_{j+1}^{-1}\delta_j^{-1}}\\
&= 1_{\mathcal{D}_n}
\end{aligned}$$

where in the last equality we used the first case $i < j+1$.

– If $i > j+1$, we have

$$\begin{aligned}
\psi(S_{i,j}) &= \psi(\alpha_i x_j \alpha_i^{-1}\rho_i(x_j)^{-1})\\
&= \delta_i\ (\delta_j\dots\delta_j^{-1})\ \delta_i^{-1}\ (\delta_j\dots\delta_j^{-1})^{-1}\\
&= (\delta_j\dots\delta_{i-2})\ \delta_i\delta_{i-1}\delta_i\ (\delta_{i+1}\dots\delta_{i+1}^{-1})\ \delta_i^{-1}\delta_{i-1}^{-1}\delta_i^{-1}\ \cdot\\
&\quad\cdot (\delta_{i-2}^{-1}\dots\delta_j^{-1})\ (\delta_j\dots\delta_j^{-1})^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= (\delta_j \dots \delta_{i-2})\ \delta_i\delta_{i-1}\delta_i\ (\delta_{i+1} \dots \delta_{i+1}^{-1})\ \delta_i^{-1}\delta_{i-1}^{-1}\delta_i^{-1} \cdot \\
&\quad \cdot (\delta_{i-1}\delta_i \dots \delta_{n-2}(\delta_n\delta_{n-1}^{-1})^{-1}\delta_{n-2}^{-1} \dots \delta_j) \\
&= (\delta_j \dots \delta_{i-2})\ \delta_i\delta_{i-1}\delta_i\ (\delta_{i+1} \dots \delta_{i+1}^{-1})\ \delta_{i-1}^{-1}\delta_i^{-1}\delta_{i-1}^{-1} \cdot \\
&\quad \cdot (\delta_{i-1}\delta_i \dots \delta_{n-2}(\delta_n\delta_{n-1}^{-1})^{-1}\delta_{n-2}^{-1} \dots \delta_j^{-1}) \\
&= (\delta_j \dots \delta_{i-2})\ (\delta_i\delta_{i-1}\delta_i\delta_{i-1}^{-1}\delta_i^{-1}\delta_{i-1}^{-1})\ (\delta_{i-2}^{-1} \dots \delta_j^{-1}) \\
&= 1_{\mathcal{D}_n}.
\end{aligned}
$$

Therefore ψ is a well-defined homomorphism of groups and its trivial to verify on the generators that $\psi \circ \varphi = \mathrm{Id}_{\mathcal{D}_n}$ and $\varphi \circ \psi = \mathrm{Id}_{F_{n-1} \rtimes \mathcal{A}_{n-1}}$. The claim follows. ■

Theorem 2.3.9. *The group $\mathcal{D}_4$ is poly-fg-free with a poly-fg-free series of length 4 given by*

$$1 \triangleleft \mathcal{D}_4^{(1)} \triangleleft \mathcal{D}_4^{(2)} \triangleleft \mathcal{D}_4^{(3)} \triangleleft \mathcal{D}_4,$$

where

$$
\begin{aligned}
\mathcal{D}_4^{(1)} &:= \left\langle \delta_1\delta_2\delta_4\delta_3^{-1}\delta_2^{-1}\delta_1^{-1},\ \delta_2\delta_4\delta_3^{-1}\delta_2^{-1},\ \delta_4\delta_3^{-1} \right\rangle, \\
\mathcal{D}_4^{(2)} &:= \left\langle \mathcal{D}_4^{(1)},\ \delta_3\delta_1^{-1},\ \delta_2\delta_1^{-1}\delta_3\delta_1^{-1} \right\rangle, \\
\mathcal{D}_4^{(3)} &:= \left\langle \mathcal{D}_4^{(2)},\ \delta_2\delta_1^{-1},\ \delta_1\delta_2\delta_1^{-2} \right\rangle
\end{aligned}
$$

and the free factors are

$$\mathcal{D}_4^{(1)} \simeq F_3, \qquad \frac{\mathcal{D}_4^{(2)}}{\mathcal{D}_4^{(1)}} \simeq F_2, \qquad \frac{\mathcal{D}_4^{(3)}}{\mathcal{D}_4^{(2)}} \simeq F_2, \qquad \frac{\mathcal{D}_4}{\mathcal{D}_4^{(3)}} \simeq \mathbb{Z}.$$

Proof. Follows from the isomorphisms shown in Theorem 2.3.8 and Theorem 2.3.1. ■

2.3.4 Artin group $\mathcal{F}_4$

As we have already stated in the introduction we have not been able to prove or disprove that $\mathcal{F}_4 := \mathcal{A}(\bullet\!-\!\bullet\overset{4}{-}\bullet\!-\!\bullet)$ is poly-free. Our approach mainly consisted in projecting $\mathcal{F}_4$ over some smaller group G that we knew to be poly-free and try to detect the isomorphisms type of the kernel in the hope of finding it out to be poly-free (the main tool we employed for this step is the Reidemeister-Schreier rewriting procedure explained in Section 3.1). Unfortunately we did not succeed, mostly because we have not been able to simplify enough the presentations we obtained for the kernels mentioned above. Observe, however, that if $\mathcal{F}_4$ is poly-free this approach must surely work for some group G (e.g., the last factor of its poly-free series). More precisely the projections we took into account are the following (well-defined) maps.

$$
\begin{aligned}
\pi_1 \colon \mathcal{A}\left(\underset{a}{\bullet}\!-\!\underset{b}{\bullet}\!\overset{4}{-}\!\underset{c}{\bullet}\!-\!\underset{d}{\bullet}\right) &\to \mathcal{A}\left(\underset{a}{\bullet}\!-\!\underset{b}{\bullet}\right) \times \mathcal{A}\left(\underset{c}{\bullet}\!-\!\underset{d}{\bullet}\right), \\
a &\mapsto (a, 1), \\
b &\mapsto (b, 1), \\
c &\mapsto (1, c), \\
d &\mapsto (1, d),
\end{aligned}
$$

$$
\begin{aligned}
\pi_2 \colon \mathcal{A}\left(\underset{a}{\bullet}\!-\!\underset{b}{\bullet}\!\overset{4}{-}\!\underset{c}{\bullet}\!-\!\underset{d}{\bullet}\right) &\to \mathcal{A}\left(\underset{a}{\bullet}\!-\!\underset{b}{\bullet}\right) \times \mathbb{Z}, \\
a &\mapsto (a, 0_{\mathbb{Z}}), \\
b &\mapsto (b, 0_{\mathbb{Z}}) \\
c, d &\mapsto (1, 1_{\mathbb{Z}}),
\end{aligned}
$$

$$
\begin{aligned}
\pi_3 \colon \mathcal{A}\left(\underset{a}{\bullet}\!-\!\underset{b}{\bullet}\!\overset{4}{-}\!\underset{c}{\bullet}\!-\!\underset{d}{\bullet}\right) &\to \mathbb{Z} \times \mathbb{Z}, \\
a, b &\mapsto (1_{\mathbb{Z}}, 0_{\mathbb{Z}}), \\
c, d &\mapsto (0_{\mathbb{Z}}, 1_{\mathbb{Z}}),
\end{aligned}
$$

$$
\begin{aligned}
\pi_4 \colon \mathcal{A}\left(\underset{a}{\bullet}\!-\!\underset{b}{\bullet}\!\overset{4}{-}\!\underset{c}{\bullet}\!-\!\underset{d}{\bullet}\right) &\to \mathbb{Z}, \\
a, b, c, d &\mapsto 1_{\mathbb{Z}}.
\end{aligned}
$$

Summary

In Table 2.1 we provide a summary of the main results of this chapter.

Group	Poly-fg-free	Strongly poly-free	Poly-free length
$\mathcal{I}_2(m)$ $(m \geq 3)$	Yes	Yes	$\mathrm{spfl}_{\mathrm{fg}}(\mathcal{I}_2(m)) = 2$
$\mathcal{A}_3$	Yes	?	$\mathrm{pfl}_{\mathrm{fg}}(\mathcal{A}_3) \leq 3$
$\mathcal{B}_3$	Yes	Yes	$\mathrm{spfl}_{\mathrm{fg}}(\mathcal{B}_3) = 3$
$\mathcal{B}_4$	Yes	?	$\mathrm{pfl}_{\mathrm{fg}}(\mathcal{B}_4) \leq 4$
$\mathcal{D}_4$	Yes	?	$\mathrm{pfl}_{\mathrm{fg}}(\mathcal{D}_4) \leq 4$
$\mathcal{F}_4$	?	?	?

Table 2.1: List of all poly-free irreducible Artin groups of finite type: all the other irreducible Artin groups of finite type not listed in this table are not poly-free.

Chapter 3

Artin groups built on trees

In this chapter we show that Artin groups built on trees are strongly poly-fg-free of length 2 and we use this result to prove that Artin groups built on forests are also strongly poly-free. Finally, in the last section we describe another approach to study poly-freeness of Artin groups which leverage the theory of Bass and Serre for groups acting on trees.
To achieve the first result we will need to compute a presentation of certain subgroups starting from the presentation of the whole Artin group. The main tool to do this is the so called "Reidemeister-Schreier rewriting procedure" that we describe in the next section.

3.1 Reidemeister-Schreier rewriting procedure

Definition 3.1.1. *Let G be a group given by a presentation $\langle \mathcal{X} \mid \mathcal{R} \rangle$. Let H be a subgroup of G and let $\mathcal{K}$ be a set of words in the alphabet $\mathcal{X}$ such that*

i) $\emptyset \in \mathcal{K}$,

ii) the elements of G represented by words in $\mathcal{K}$ form a system of right coset representatives for H,

iii) for each word $w \in \mathcal{K}$ each initial segment of w is also in $\mathcal{K}$.

Then $\mathcal{K}$ is called a Schreier system for G modulo H.

Notation 3.1.2. *Let $\mathcal{K}$ be a Schreier system for G modulo a subgroup H. Given a word w in G we will denote with $\overline{w}$ the unique element of $\mathcal{K}$ such that $Hw = H\overline{w}$.*

Theorem 3.1.3 (Reidemeister-Schreier rewriting procedure)**.** *Let G be a group given by a presentation $\langle \mathcal{X} \mid \mathcal{R} \rangle$. Let H be a subgroup of G and let $\mathcal{K}$ be a Schreier* *system for G modulo H. Then a presentation for H is given by*

$$H \simeq \left\langle s_{K,a_\nu}, \begin{array}{l} \forall K \in \mathcal{K}, \\ \forall a_\nu \in \mathcal{X}, \end{array} \middle| \begin{array}{ll} s_{K,a_\nu} = 1, & \forall K \in \mathcal{K},\ \forall a_\nu \in \mathcal{X} \colon Ka_\nu \equiv \overline{Ka_\nu}, \\ \mathcal{T}\left(KR_\mu K^{-1}\right) = 1, & \forall K \in \mathcal{K},\ \forall R_\mu \in \mathcal{R}, \end{array} \right\rangle$$

where the generators s_{K,a_ν} are defined with respect to the generators of G as

$$s_{K,a_\nu} = (Ka_\nu)\left(\overline{Ka_\nu}\right)^{-1},$$

$Ka_\nu \equiv \overline{Ka_\nu}$ means that those two words are equivalent inside the free group $F(\mathcal{X})$ and $\mathcal{T}$ is the Reidemeister-Schreier rewriting function defined as follows

$$\begin{aligned} \mathcal{T} \colon F(\mathcal{X}) &\to F\left(\{ s_{K,a_\nu} \}\right), \\ a_{i_1}^{\varepsilon_i} \dots a_{i_m}^{\varepsilon_m} &\mapsto s_{K_{i_1},a_{i_1}}^{\varepsilon_1} \dots s_{K_{i_m},a_{i_m}}^{\varepsilon_m}, \end{aligned}$$

with

$$K_{i_j} := \begin{cases} \overline{a_{i_1}^{\varepsilon_i} \dots a_{i_{j-1}}^{\varepsilon_{j-1}}}, & \textit{if } \varepsilon_j = 1, \\ \overline{a_{i_1}^{\varepsilon_i} \dots a_{i_j}^{\varepsilon_j}}, & \textit{if } \varepsilon_j = -1. \end{cases}$$

Proof. See [14, Theorem 2.9]. ∎

3.2 Artin groups built on trees

Definition 3.2.1. *Let Γ be a graph. A vertex v of Γ is said to be a* cut vertex *if the full subgraph of Γ spanned by the set of vertices $V(\Gamma) \setminus \{ v \}$ has more connected components than Γ.*

Theorem 3.2.2. *Let Γ be a Coxeter graph. Suppose that Γ has a cut vertex a_1. Denote by Υ_t $(t = 1, \dots, d)$ the connected components of the full subgraph of Γ spanned by the set of vertices $V(\Gamma) \setminus \{ a_1 \}$ and denote by $\overline{\Upsilon}_t$ the full subgraph of Γ spanned by the set of vertices $V(\Upsilon_t) \cup \{ a_1 \}$. Let $\mathcal{A}(\Gamma)$ be the Artin group associated with Γ, then the kernel of the map χ_Γ defined in Lemma 1.4.9 admits a decomposition as follows*

$$\operatorname{Ker}(\chi_\Gamma) = \coprod_{t=1}^{d} \operatorname{Ker}(\chi_{\overline{\Upsilon}_t}).$$

Proof. Since χ_Γ is surjective, $\mathbb{Z}$ is isomorphic to the quotient of $\mathcal{A}(\Gamma)$ by the kernel of χ_Γ. Clearly, the map sending any integer k to a_1^k is a section for χ_Γ, hence the set $\mathcal{K} = \{ a_1^k \mid k \in \mathbb{Z} \}$ is a set of transversals for $\operatorname{Ker}(\chi_\Gamma)$ and it satisfies all requests to be a Schreier system for $\mathcal{A}(\Gamma)$ modulo $\operatorname{Ker}(\chi_\Gamma)$. Applying the Reidemeister-Schreier rewriting procedure with Schreier system $\mathcal{K}$ as described in Theorem 3.1.3 we obtain the following presentation

$$\operatorname{Ker}(\chi_\Gamma) = \left\langle s_{a_1^k,a_i}, \begin{array}{r} \forall k \in \mathbb{Z}, \\ 1 \le i \le n, \end{array} \middle| \begin{array}{l} s_{a_1^k,a_i} = 1, \quad \forall k \in \mathbb{Z},\ 1 \le i \le n \colon a_1^k a_i \equiv \overline{a_1^k a_i}, \\ \mathcal{T}\left(a_1^k R_e a_1^{-k}\right) = 1, \qquad \forall k \in \mathbb{Z},\ \forall e \in E(\Gamma), \end{array} \right\rangle \tag{3.1}$$

where

$$s_{a_1^k,a_i} = a_1^k a_i \left(\overline{a_1^k a_i}\right)^{-1} = a_1^k a_i a_1^{-k-1}.$$

The following steps point towards simplifying such presentation. First observe that

$$a_1^k a_i \equiv \overline{a_1^k a_i} = a_1^{k+1} \qquad \Longleftrightarrow \qquad i = 1,$$

hence

$$s_{a_1^k,a_1} = 1, \quad \forall k \in \mathbb{Z} \tag{3.2}$$

and we can delete such generators from the above presentation.
Next, we want to write explicitly the relators of type $\mathcal{T}(a_1^k R_e a_1^{-k})$, where, for each edge e of Γ with endings a_i and a_j, the relator R_e has shape

$$R_e = \underbrace{a_i a_j \dots a_i}_{m_e \text{ letters}} \underbrace{a_j^{-1} a_i^{-1} \dots a_j^{-1}}_{m_e \text{ letters}} \qquad \text{or} \qquad R_e = \underbrace{a_i a_j \dots a_j}_{m_e \text{ letters}} \underbrace{a_i^{-1} a_j^{-1} \dots a_i^{-1}}_{m_e \text{ letters}} \tag{3.3}$$

according to the parity of m_e. We remind that for each $K \in \mathcal{K}$ we have $K = a_1^k$ for some integer k and we denote by $R_{e,r}$ $(1 \le r \le 2m_e)$ the word given by the first r letters of R_e so we can compute the values of the Reidemeister-Schreier rewriting function $\mathcal{T}$ on the word $a_1^k R_e a_1^{-k}$ simply by applying its definition. In the following, when the edge e is understood, we will always write m instead of m_e; e.g., $R_{e,m}$ means R_{e,m_e}. If R_e is as the first expression of Equation (3.3) (i.e., if e has an odd label) and $k \ge 0$, then

$$\begin{aligned}
\mathcal{T}(a_1^k R_e a_1^{-k}) &= \mathcal{T}(a_1^k \cdot a_i a_j \dots a_i a_j^{-1} a_i^{-1} \dots a_j^{-1} \cdot a_1^{-k}) \\
&= \left(s_{\overline{1},a_1}\ s_{\overline{a_1},a_1} \dots\ s_{\overline{a_1^{k-1}},a_1} \right) \cdot \\
&\quad \cdot s_{\overline{a_1^k},a_i}\ s_{\overline{a_1^k R_{e,1}},a_j} \ \dots\ s_{\overline{a_1^k R_{e,m-1}},a_i}\ s^{-1}_{\overline{a_1^k R_{e,m+1}},a_j}\ s^{-1}_{\overline{a_1^k R_{e,m+2}},a_i} \dots\ s^{-1}_{\overline{a_1^k R_{e,2m}},a_j} \cdot \\
&\quad \cdot \left(s_{\overline{a_1^k R_e a_1^{-1}},a_1}\ s_{\overline{a_1^k R_e a_1^{-2}},a_1}\ \dots\ s_{\overline{a_1^k R_e a_1^{-k}},a_1} \right) \\
&= \left(s_{1,a_1}\ s_{a_1,a_1}\ \dots\ s_{a_1^{k-1},a_1} \right) \cdot \\
&\quad \cdot s_{a_1^k,a_i}\ s_{a_1^{k+1},a_j}\ \dots\ s_{a_1^{k+m-1},a_i}\ s^{-1}_{a_1^{k+m-1},a_j}\ s^{-1}_{a_1^{k+m-2},a_i} \dots\ s^{-1}_{a_1^k,a_j} \cdot \\
&\quad \cdot \left(s_{a_1^{k-1},a_1}\ s_{a_1^{k-2},a_1}\ \dots\ s_{1,a_1} \right)
\end{aligned}$$

$$= s_{a_1^k,a_i}\ s_{a_1^{k+1},a_j}\ \dots\ s_{a_1^{k+m-1},a_i}\ s^{-1}_{a_1^{k+m-1},a_j}\ s^{-1}_{a_1^{k+m-2},a_i} \dots\ s^{-1}_{a_1^k,a_j},$$

where in the second to last equality we use the fact that

$$\overline{R_{e,t}} = \begin{cases} a_1^{t-1}, & 1 \leq t \leq m_e, \\ a_1^{2m_e-t}, & m_e < t \leq 2m_e. \end{cases}$$

For the sake of convenience in the following we will denote the word we obtained above by $S^k_{i,j}$. If $k < 0$ then the computation does not change and we get again a word of type $S^k_{i,j}$.
Instead, if R_e is as the second expression of Equation (3.3) (i.e., if e has an even label), with completely analogous computations for any integer k we obtain

$$\mathcal{T}(a_1^k R_e a_1^{-k}) = s_{a_1^k,a_i}\ s_{a_1^{k+1},a_j}\ \dots\ s_{a_1^{k+m-1},a_j}\ s^{-1}_{a_1^{k+m-1},a_i}\ s^{-1}_{a_1^{k+m-2},a_j} \dots\ s^{-1}_{a_1^k,a_i} =: T^k_{i,j}.$$

Recalling Equation (3.2), if we consider an edge e having a_1 as one of its endings, say $a_i = a_1$, according to the type of R_e we obtain either

$$\mathcal{T}(a_1^k R_e a_1^{-k}) = s_{a_1^{k+1},a_j}\ s_{a_1^{k+3},a_j}\ \dots\ s_{a_1^{k+m-2},a_j}\ s^{-1}_{a_1^{k+m-1},a_j}\ s^{-1}_{a_1^{k+m-3},a_j}\ \dots\ s^{-1}_{a_1^k,a_j} =: U^k_j$$

or

$$\mathcal{T}(a_1^k R_e a_1^{-k}) = s_{a_1^{k+1},a_j}\ s_{a_1^{k+3},a_j}\ \dots\ s_{a_1^{k+m-1},a_j}\ s^{-1}_{a_1^{k+m-2},a_j}\ s^{-1}_{a_1^{k+m-4},a_j}\ \dots\ s^{-1}_{a_1^{k+1},a_j} =: V^k_j.$$

Since such words are going to be relators for our presentation and since $T^k_{i,j}$ and V^k_j are the inverses of $S^k_{i,j}$ and U^k_j respectively, we will use only the latter to denote the relation associated to an edge e of Γ. All the above implies that Presentation (3.1) can be written more explicitly as follows

$$\operatorname{Ker}(\chi_\Gamma) = \left\langle s_{a_1^k,a_i}, \begin{array}{r} \forall k \in \mathbb{Z}, \\ 2 \leq i \leq n, \end{array} \middle| \begin{array}{l} U^k_i = 1, \text{ if } \{\, a_i, a_1 \,\} \in E(\Gamma),\ \forall k \in \mathbb{Z}, \\ S^k_{i,j} = 1, \text{ if } \{\, a_i, a_j \,\} \in E(\Gamma),\ \forall k \in \mathbb{Z},\ i,j \neq 1 \end{array} \right\rangle. \tag{3.4}$$

Let $\mathcal{X}$ be the family of generators in the above presentation and notice that the index i describing this family ranges from 2 up to n. Since i originally indexed the set of vertices of Γ and each vertex different from a_1 belongs exactly to one $\overline{\Upsilon}_t$ for $t = 1, \dots, d$, this allows to say that, if we set

$$\mathcal{X}_t := \left\{ s_{a_1^k,a_i} \,\middle|\, a_i \in V(\overline{\Upsilon}_t),\ i \neq 1 \right\}, \qquad \forall t = 1, \dots, d,$$

then

$$\mathcal{X} = \bigsqcup_{t=1}^{d} \mathcal{X}_t. \tag{3.5}$$

Similarly, the set of relations $\mathcal{R}$ inside (3.4) can be written as the disjoint union

$$\mathcal{R} = \bigsqcup_{t=1}^{d} \mathcal{R}_t \tag{3.6}$$

where

$$\begin{aligned} \mathcal{R}_t \coloneqq & \left\{ U_i^k \,\middle|\, \{ a_i, a_1 \} \in E(\overline{\Upsilon}_t),\ \forall k \in \mathbb{Z} \right\} \cup \\ & \cup \left\{ S_{i,j}^k \,\middle|\, \{ a_i, a_j \} \in E(\overline{\Upsilon}_t),\ \forall i, j \neq 1,\ \forall k \in \mathbb{Z} \right\}, \qquad \forall t = 1, \dots, d. \end{aligned}$$

Moreover, each word in $\mathcal{R}_t$ is composed only of generators inside $\mathcal{X}_t$ since each vertex of Γ except a_1 is contained in exactly one subgraph $\overline{\Upsilon}_t$ and the generators of $\mathrm{Ker}(\chi_\Gamma)$ of type $s_{a_1^k, a_1}$ have been cancelled from Presentation (3.4). This allows us to consider the abstract groups defined by $\langle \mathcal{X}_t \,|\, \mathcal{R}_t \rangle$ for $t = 1, \dots, d$. Each of these groups contains the same generators and relators as those provided by the Reidemeister-Schreier rewriting procedure applied to each subgraph $\overline{\Upsilon}_t$ using the Schreier system $\{ a_1^k \mid k \in \mathbb{Z} \}$ for $\mathcal{A}(\overline{\Upsilon}_t)$ modulo $\mathrm{Ker}(\chi_{\overline{\Upsilon}_t})$, based at the copy of the vertex a_1 belonging to each $\overline{\Upsilon}_t$. This amounts to say that

$$\mathrm{Ker}(\chi_{\overline{\Upsilon}_t}) = \langle \mathcal{X}_t \,|\, \mathcal{R}_t \rangle, \quad \forall t = 1, \dots, d.$$

Using Equations (3.5) and (3.6) we have

$$\mathrm{Ker}(\chi_\Gamma) = \langle \mathcal{X} \,|\, \mathcal{R} \rangle = \left\langle \sqcup_{t=1}^d \mathcal{X}_t \,\middle|\, \sqcup_{t=1}^d \mathcal{R}_t \right\rangle = \coprod_{t=1}^{d} \mathrm{Ker}(\chi_{\overline{\Upsilon}_t}).$$

■

Remark 3.2.3. *We observe that in the case of dihedral Artin groups $\mathcal{I}_2(m) = \mathcal{A}(\overset{m}{\bullet\!\!-\!\!\bullet})$ the set of generators in the presentation of $\mathrm{Ker}\left(\chi_{\mathcal{I}_2(m)}\right)$ we obtained in the above proof is exactly the same as the one we studied in Section 2.2. Indeed, if we set $b \coloneqq a_1$ and $a \coloneqq a_2$, then*

$$\mathrm{Ker}(\chi_{\mathcal{I}_2(m)}) = \left\langle s_{a_1^k, a_2}\ (k \in \mathbb{Z}) \,\middle|\, U_2^k = 1\ (k \in \mathbb{Z}) \right\rangle \simeq F_{2m-1}$$

where, by construction, the generators correspond to the elements

$$s_{a_1^k, a_2} = a_1^k a_2 a_1^{-k-1} = \left(a_2 a_1^{-1}\right)^{a_1^{-k}} = (ab^{-1})^{b^{-k}},$$

i.e., $s_{a_1^0, a_2} = a_2 a_1^{-1} = ab^{-1}$ is precisely the element whose normal closure generates the subgroup $\mathrm{Ker}\left(\chi_{\mathcal{I}_2(m)}\right)$ in Proposition 2.2.11.

Poly-freeness of Artin groups built on trees comes straight as a corollary of the previous theorem.

Corollary 3.2.4. *Let Γ be a Coxeter tree on $n \geq 2$ vertices. Let $\mathcal{A}(\Gamma)$ be the associated Artin group, then $\mathrm{Ker}(\chi_\Gamma)$ is free of rank*

$$\sum_{e \in E(\Gamma)} (m_e - 1)\,,$$

where m_e denotes the label associated to the edge e of Γ.

Proof. If $n = 2$ the group $\mathcal{A}_\Gamma$ is a dihedral Artin group and the claim has already been proved in Theorems 2.2.12 and 2.2.15.
For $n \geq 3$ we proceed by induction on the number of vertices of Γ. Indeed, suppose the statement true for trees built on less than n vertices and let Γ be a tree with n vertices. Since $n \geq 3$ and Γ is a tree there exists at least a vertex, say a_1, that is a cut vertex for Γ. Applying Theorem 3.2.2 using a_1 as cut vertex we get

$$\mathrm{Ker}(\chi_\Gamma) = \coprod_{t=1}^{d} \mathrm{Ker}(\chi_{\overline{\Upsilon}_t}).$$

All subgraphs $\overline{\Upsilon}_t$ ($1 \leq t \leq d$) have strictly less vertices than Γ and being subgraphs of Γ they are trees as well. By the inductive hypothesis for each $t = 1, \ldots, d$ $\mathrm{Ker}(\chi_{\overline{\Upsilon}_t})$ is free of rank

$$\mathrm{rk}\left(\mathrm{Ker}(\chi_{\overline{\Upsilon}_t})\right) = \sum_{e \in E(\overline{\Upsilon}_t)} (m_e - 1).$$

Each edge of Γ belongs exactly to one of the connected components $\overline{\Upsilon}_t$ and the decomposition above gives us

$$\mathrm{rk}\left(\mathrm{Ker}(\chi_\Gamma)\right) = \sum_{t=1}^{d} \left(\sum_{e \in E(\overline{\Upsilon}_t)} (m_e - 1) \right) = \sum_{e \in E(\Gamma)} (m_e - 1)\,.$$

∎

Corollary 3.2.5. *Let Γ be a Coxeter tree, then $\mathcal{A}(\Gamma)$ is strongly poly-fg-free; more precisely:*

- *If $|V(\Gamma)| = 1$, then $\mathcal{A}(\Gamma) \simeq \mathbb{Z}$ is free, hence $\mathrm{spfl}_{\mathrm{fg}}(\mathcal{A}(\Gamma)) = 1$ and a strongly poly-fg-free series is trivially given by $\{1\} \lhd \mathcal{A}(\Gamma)$. Notice that in this case $\mathrm{Ker}(\chi_\Gamma) = \{1\}$.*

- *If* $|V(\Gamma)| \geq 2$, *a strongly poly-fg-free series for* $\mathcal{A}(\Gamma)$ *is given by*

$$\{ 1 \} \triangleleft \mathrm{Ker}(\chi_\Gamma) \triangleleft \mathcal{A}(\Gamma)$$

with free factors

$$\mathrm{Ker}(\chi_\Gamma) \simeq F_m, \qquad \frac{\mathcal{A}(\Gamma)}{\mathrm{Ker}(\chi_\Gamma)} \simeq F_1,$$

where $m \coloneqq \sum_{e\in E(\Gamma)}(m_e - 1)$ *and* $\mathrm{spfl}_{\mathrm{fg}}(\mathcal{A}(\Gamma)) = 2$.

Proof. It follows straight from Corollary 3.2.4. ■

Corollary 3.2.6. *Let* Γ *be a Coxeter forest with connected components* $\Gamma_1, \dots, \Gamma_d$, *then the Artin group* $\mathcal{A}(\Gamma)$ *is strongly poly-free with*

$$\mathrm{spfl}(\mathcal{A}(\Gamma)) \leq d + 1.$$

Proof. We prove the claim by induction on the number of connected components of Γ. If $d = 1$, then $\Gamma = \Gamma_1$ is a tree and the associated Artin group is strongly poly-free by Corollary 3.2.5 with $\mathrm{spfl}(\mathcal{A}(\Gamma)) = 2$.
Let the statement be true for forests with $d - 1$ connected components. Set $\Lambda \coloneqq \Gamma_1 \sqcup \ldots \sqcup \Gamma_{d-1}$, by the inductive hypothesis $\mathcal{A}(\Lambda)$ is strongly poly-free and $\mathrm{spfl}(\mathcal{A}(\Lambda)) \leq d$. Looking at the presentation of $\mathcal{A}(\Gamma)$ we have $\mathcal{A}(\Gamma) = \mathcal{A}(\Lambda) * \mathcal{A}(\Gamma_d)$ and by Lemma 1.3.11 the claim follows. ■

Under the hypothesis that Γ_{odd} is connected we can restate Theorem 3.2.2 and Corollay 3.2.4 as follows.

Corollary 3.2.7. *Let* Γ *be a Coxeter graph such that* Γ_{odd} *is connected. Suppose* Γ *has a cut vertex* a_1. *Denote by* Υ_t $(t = 1, \dots, d)$ *the connected components of the full subgraph of* Γ *spanned by the set of vertices* $V(\Gamma) \setminus \{ a_1 \}$ *and denote by* $\overline{\Upsilon}_t$ *the subgraph of* Γ *obtained by re-adding a copy of the vertex* a_1 *to the connected component* Υ_t. *Let* $\mathcal{A}(\Gamma)$ *be the Artin group associated with* Γ, *then its commutator subgroup admits a decomposition as follows*

$$\mathcal{A}'(\Gamma) = \coprod_{t=1}^{d} \mathcal{A}'(\overline{\Upsilon}_t).$$

Moreover, if Γ *is a tree its commutator subgroup* $\mathcal{A}'(\Gamma)$ *is free of rank*

$$\sum_{e\in E(\Gamma)} (m_e - 1)\,.$$

Proof. It follows immediately from Theorem 3.2.2, Corollary 3.2.4 and Lemma 1.4.11. ■

Next we give a generalized version of Theorem 3.2.2 with a much abridged proof. This is possible since the key steps that make the previous proof work are much more combinatorial than it could seem from our exposition. We chose to do so because the particularly easy shape of the Schreier system we had actually allowed us to make explicit computations for all relators. However, if all what we care about is just obtaining a decomposition for $\mathrm{Ker}(\chi_\Gamma)$ we can argue in a much more abstract way as follows.

Theorem 3.2.8. *Let Γ be a Coxeter graph. Let Γ_1 and Γ_2 be full subgraphs of Γ such that $\Delta \coloneqq \Gamma_1 \cap \Gamma_2$ is not empty and $\Gamma = \Gamma_1 \cup \Gamma_2$, then*

$$\mathrm{Ker}\left(\chi_\Gamma\right) = \mathrm{Ker}\left(\chi_{\Gamma_1}\right) *_{\mathrm{Ker}(\chi_\Delta)} \mathrm{Ker}\left(\chi_{\Gamma_2}\right).$$

Proof. Since $\Delta \neq \emptyset$, it has at least one vertex, say a_1. Let Λ be any full subgraph of Γ such that $a_1 \in V(\Lambda)$. Choose $\mathcal{K} = \{\, a_1^k \mid k \in \mathbb{Z} \,\}$ as a Schreier system for $\mathcal{A}(\Lambda)$ modulo $\mathrm{Ker}(\chi_\Lambda)$. The Reidemeister-Schreier procedure described in Theorem 3.1.3 applied using the Schreier system $\mathcal{K}$ tells us that the kernel of χ_Λ admits a presentation

$$\mathrm{Ker}(\chi_\Lambda) = \langle X_\Lambda \mid R_\Lambda \rangle\,,$$

where

$$X_\Lambda \coloneqq \left\{\, s^{\Lambda}_{a_1^k, a_i} \;\middle|\; \forall a_i \in V(\Lambda) \,\right\},$$

$$R_\Lambda \coloneqq \left\{ \begin{array}{c|c} s^{\Lambda}_{a_1^k, a_i}, & \forall k \in \mathbb{Z},\ \forall a_i \in V(\Lambda)\colon a_1^k a_i \equiv \overline{a_1^k a_i}, \\ \mathcal{T}\left(a_1^l R_e a_1^{-l}\right), & \forall l \in \mathbb{Z},\ \forall e \in E(\Lambda), \end{array} \right\}.$$

We are particularly interested in the cases when $\Lambda = \Gamma,\ \Gamma_1,\ \Gamma_2$, or Δ. Since $\Delta = \Gamma_1 \cap \Gamma_2$, Δ is a full subgraph of Γ and Proposition 1.4.12 guarantees that $\mathcal{A}(\Delta)$ injects into $\mathcal{A}(\Gamma)$, $\mathcal{A}(\Gamma_1)$ and $\mathcal{A}(\Gamma_2)$, therefore $\mathrm{Ker}(\chi_\Delta)$ injects inside $\mathrm{Ker}(\chi_{\Gamma_i})$ for $i = 1, 2$ through the identity. Finally

$$\begin{aligned}
\mathrm{Ker}(\chi_\Gamma) &= \langle X_\Gamma \mid R_\Gamma \rangle \\
&= \left\langle X_{\Gamma_1} \sqcup X_{\Gamma_2} \,\middle|\, R_{\Gamma_1} \sqcup R_{\Gamma_2} \sqcup \left\{\, s^{\Gamma_1}_{a_1^k, a_i} = s^{\Gamma_2}_{a_1^k, a_i},\ \forall a_i \in V(\Delta) \,\right\} \right\rangle \\
&= \langle X_{\Gamma_1} \mid R_\Gamma \rangle *_{\langle X_\Delta \mid R_\Delta \rangle} \langle X_{\Gamma_1} \mid R_\Gamma \rangle \\
&= \mathrm{Ker}\left(\chi_{\Gamma_1}\right) *_{\mathrm{Ker}(\chi_\Delta)} \mathrm{Ker}\left(\chi_{\Gamma_2}\right).
\end{aligned}$$

■

Remark 3.2.9.

- *The previous theorem generalises Theorem 3.2.2, indeed if a_1 is a cut vertex for Γ, then let Γ_1 be one of the connected components (including a_1) that* *arise when a_1 is removed from Γ and let Γ_2 be the union of the remaining components (including a_1 as well). Since $\Delta = \Gamma_1 \cap \Gamma_2 = \{ a_1 \}$, $\mathcal{A}(\Delta) \simeq \mathbb{Z}$ and clearly* $\mathrm{Ker}(\chi_\Delta) = \{ 1 \}$ *from Theorem 3.2.8 we have*

$$\mathrm{Ker}(\chi_\Gamma) = \mathrm{Ker}(\chi_{\Gamma_1}) * \mathrm{Ker}(\chi_{\Gamma_2})$$

and the statement of Theorem 3.2.2 follows from iterating such procedure on Γ_2. In particular, this means that all the computations done in the proof of Theorem 3.2.2 are not strictly necessary to obtain the conclusion, however they are interesting since they show that the rewriting function $\mathcal{T}$ can be actually computed (thanks to the extremely easy shape of the words inside the Schreier system we chose).

- *The hypothesis $\Delta \neq \emptyset$ is required. Indeed, consider*

$$\Gamma_1 = \underset{a\quad b}{\bullet\overset{2}{\text{—}}\bullet}, \qquad \Gamma_2 = \underset{c\quad d}{\bullet\overset{2}{\text{—}}\bullet}, \qquad a \in V(\Gamma_1), \qquad c \in V(\Gamma_2),$$

and let $\Gamma = \Gamma_1 \sqcup \Gamma_2$; then, by definition, $\mathcal{A}(\Gamma) = \mathcal{I}_2(2) * \mathcal{I}_2(2)$. The element $ac^{-1} \in \mathcal{A}(\Gamma)$ clearly belongs to $\mathrm{Ker}(\chi_\Gamma)$. *If it were true that such kernel admits a decomposition as* $\mathrm{Ker}(\chi_{\Gamma_1}) * \mathrm{Ker}(\chi_{\Gamma_2})$, *this would imply $a \in \mathrm{Ker}(\chi_{\Gamma_1})$, a contradiction.*

3.3 Further developments

In the last part of this thesis we propose an approach to deal with the study of Artin groups whose Coxeter graph may be "more connected" than a tree or a forest. This approach requires the use of results from Bass-Serre theory that we briefly recalled in Section 1.2.3.

Lemma 3.3.1. *Let Γ be a Coxeter graph and $\Lambda \subseteq \Gamma$ be a full subgraph. Denote by N_Λ ("neighbours" of Λ) the set of vertices of $\Gamma \setminus \Lambda$ that are connected to at least a vertex of Λ and denote by B_Λ ("bridges" of Λ) the set of all edges having exactly one of their endings in $V(\Lambda)$ (and the other in N_Λ). Suppose that $d := \gcd \{ m_e \mid e \in B_\Lambda \} \geq 2$. Let $\hat{\Gamma}$ be the labelled graph obtained by replacing all vertices of $\Lambda \subseteq \Gamma$ with a single vertex $\hat{v}$ linked to all vertices in N_Λ with edges labelled with d. Then $\hat{\Gamma}$ is a Coxeter graph and the map defined on the generators of $\mathcal{A}(\Gamma)$ sending*

$$\begin{cases} a_i \mapsto \hat{v}, & a_i \in \Lambda, \\ a_i \mapsto a_i, & a_i \notin \Lambda, \end{cases}$$

extends (uniquely) to a surjective homomorphism of groups $\psi_\Lambda \colon \mathcal{A}(\Gamma) \to \mathcal{A}(\hat{\Gamma})$.

Proof. The proof is just an application of Proposition 1.2.6. Let $e \in E(\Gamma)$, if $e \in E(\Lambda)$, then

$$\psi_\Lambda(R_e^\Gamma) = \underbrace{\hat{v}\dots\hat{v}}_{m_e \text{ letters}}\ \underbrace{\hat{v}^{-1}\dots\hat{v}^{-1}}_{m_e \text{ letters}} = 1_{\mathcal{A}(\hat{\Gamma})}.$$

If $e \in B_\Lambda$, then m_e is a multiple of d, say $m_e = kd$ ($k \in \mathbb{N}$) and

$$\begin{aligned}\psi_\Lambda(R_e^\Gamma) &= \Pi(\hat{v}, a_i; m_e) \cdot \Pi(a_i, \hat{v}; m_e)^{-1}\\ &= \begin{cases} \Pi(\hat{v}, a_i; d)^k \cdot \Pi(a_i, \hat{v}; d)^{-k}, & d \text{ even},\\ \underbrace{\Pi(\hat{v}, a_i; d) \cdot \Pi(a_i, \hat{v}; d) \dots}_{k \text{ factors}} \cdot \underbrace{\dots \Pi(\hat{v}, a_i; d)^{-1} \cdot \Pi(a_i, \hat{v}; d)^{-1}}_{k \text{ factors}}, & d \text{ odd}, \end{cases}\\ &= 1_{\mathcal{A}(\hat{\Gamma})}.\end{aligned}$$

In the remaining case

$$\psi_\Lambda(R_e^\Gamma) = R_e^{\hat{\Gamma}} = 1_{\mathcal{A}(\hat{\Gamma})}.$$

Therefore ψ_Λ is a well-defined homomorphism of groups. Clearly it is surjective. ■

Definition 3.3.2. *Let $\mathfrak{F} = \{\,\Gamma_1, \dots, \Gamma_k\,\}$ be a finite family of Coxeter graphs and let $J_{\mathfrak{F}}$ be a subset of $\mathscr{P}(\sqcup_{i=1}^k V(\Gamma_i))$ such that for each $e \in J_{\mathfrak{F}}$ we have $e = \{\,v_i, v_j\,\}$ where v_i, v_j are vertices belonging to distinct elements of $\mathfrak{F}$. We define the 2-join* $\mathrm{jn}_2(\mathfrak{F}, J_{\mathfrak{F}})$ *of $\mathfrak{F}$ along $J_{\mathfrak{F}}$ as the Coxeter graph with*

$$V\left(\mathrm{jn}_2(\mathfrak{F}, J_{\mathfrak{F}})\right) := \bigsqcup_{i=1}^k V(\Gamma_i), \qquad E\left(\mathrm{jn}_2(\mathfrak{F}, J_{\mathfrak{F}})\right) := \left(\bigsqcup_{i=1}^k E(\Gamma_i)\right) \sqcup J_{\mathfrak{F}}$$

where each edge in $J_{\mathfrak{F}}$ is labelled with '2'.

Remark 3.3.3. *A 2-join $\mathrm{jn}_2(\mathfrak{F}, J_{\mathfrak{F}})$ such that for each $e = \{\,v_i, v_j\,\} \in J_{\mathfrak{F}}$ we have $\{\,v_i', v_j'\,\} \in J_{\mathfrak{F}}$ for all v_i' in the same graph as v_i and all v_j' in the same graph as v_j coincides with the graph product*[1] *of the Coxeter groups $\mathcal{A}(\Gamma_1), \dots, \mathcal{A}(\Gamma_k)$.*

Observe that each element of $\mathfrak{F}$ is a full subgraph of $\mathrm{jn}_2(\mathfrak{F}, J_{\mathfrak{F}})$ for any set $J_{\mathfrak{F}}$ satisfying the conditions of previous definition.

Proposition 3.3.4. *Let $\mathfrak{F}$ be a finite family of Coxeter graphs, each of them having at most two vertices. Let $J_{\mathfrak{F}}$ be any set satisfying the conditions described in Definition 3.3.2 with respect to the family $\mathfrak{F}$ and let $\Gamma := \mathrm{jn}_2(\mathfrak{F}, J_{\mathfrak{F}})$. Suppose that the hypotheses of Lemma 3.3.1 are satisfied for a fixed subgraph $\Lambda \in \mathfrak{F}$ with exactly two vertices u, w and let $\hat{\Gamma}$ be the Coxeter graph obtained from Γ collapsing Λ to a single vertex $\hat{v}$ as described in Lemma 3.3.1, then the kernel of the map $\psi_\Lambda \colon \mathcal{A}(\Gamma) \to \mathcal{A}(\hat{\Gamma})$ is free.*

[1] For the definition of the *graph products of groups* see [10].

Proof. We proceed by induction on the number k of vertices of $\Theta := \Gamma \setminus \Lambda$ that are not connected to all vertices of Λ.
If $k = 0$, then each generator of $\mathcal{A}(\Lambda)$ commutes with each generator of $\mathcal{A}(\Theta)$, this means that $\mathcal{A}(\Gamma) = \mathcal{A}(\Lambda) \times \mathcal{A}(\Theta)$ and the map ψ_Λ factors as

$$\psi_\Lambda = \chi_\Lambda \times \mathrm{Id}_{\mathcal{A}(\Theta)} \colon \mathcal{A}(\Lambda) \times \mathcal{A}(\Theta) \to \mathbb{Z} \times \mathcal{A}(\Theta),$$

therefore $\mathrm{Ker}(\psi_\Lambda) = \mathrm{Ker}(\chi_\Lambda) \times \mathrm{Ker}(\mathrm{Id}_{\mathcal{A}(\Theta)}) \simeq \mathrm{Ker}(\chi_\Lambda)$ which is free by Corollary 3.2.4.
Now let $k \geq 1$ and suppose the statement true for $k - 1$. There exists at least one vertex v of Θ not linked to at least a vertex u of Λ and we proceed as follows: let Γ_1 be the full subgraph spanned inside Γ by the set of vertices $V(\Gamma) \setminus \{ u \}$ and let Γ_2 be the full subgraph spanned inside Γ by the set of vertices $V(\Gamma) \setminus \{ v \}$. Set $\Delta := \Gamma_1 \cap \Gamma_2$, since $\Gamma = \Gamma_1 \cup \Gamma_2$ then $\mathcal{A}(\Gamma) = \mathcal{A}(\Gamma_1) *_{\mathcal{A}(\Delta)} \mathcal{A}(\Gamma_2)$. Consider the Serre tree Σ (described in Proposition 1.2.19) associated to this decomposition of $\mathcal{A}(\Gamma)$ as a free product with amalgamation. Since Γ_1 can be embedded in $\hat{\Gamma}$ sending the other vertex w of Λ to $\hat{v}$ and each other vertex to its own copy, the restriction of ψ_Λ to the subgroup $\mathcal{A}(\Gamma_1)$ of $\mathcal{A}(\Gamma)$ is injective; therefore $\mathrm{Ker}(\psi_\Lambda) \cap \mathcal{A}(\Gamma_1) = \{ 1 \}$ and since a kernel is a normal subgroup we have

$$\mathrm{Ker}(\psi) \cap \mathcal{A}(\Gamma_1)^g = \{ 1 \}, \qquad \forall g \in \mathcal{A}(\Gamma). \tag{3.7}$$

Since $\mathcal{A}(\Delta)$ is a subgroup of $\mathcal{A}(\Gamma_1)$ we also have

$$\mathrm{Ker}(\psi) \cap \mathcal{A}(\Delta)^g = \{ 1 \}, \qquad \forall g \in \mathcal{A}(\Gamma). \tag{3.8}$$

Recalling the action by left multiplication of $\mathcal{A}(\Gamma)$ on its Serre tree Σ, Equations (3.7) and (3.8) amount to say that under the induced action of $\mathrm{Ker}(\psi_e)$ the stabilisers of the edges and the stabilisers of the vertices represented by the left cosets ${}^{\mathcal{A}(\Gamma)}/_{\mathcal{A}(\Gamma_1)}$ are trivial. In the language of the Bass-Serre theorem this means that the graph of groups associated to the action of $K = \mathrm{Ker}(\psi_e)$ on Σ has trivial edge groups K_e for all $e \in E(K \setminus \Sigma)$ and trivial vertex groups K_v for all $v \in \{ \mathcal{O}_x \mid x \in {}^{\mathcal{A}(\Gamma)}/_{\mathcal{A}(\Gamma_1)} \}$ (where $\mathcal{O}_x$ denotes the orbit of a coset x under the action of K). In particular, the first relation in Equation (1.1) becomes trivial, so that

$$\mathrm{Ker}(\psi) \simeq \left\langle \mathrm{Stab}(\mathcal{O}_x)\ (x \in {}^{\mathcal{A}(\Gamma)}/_{\mathcal{A}(\Gamma_2)}),\ t_e\ (e \in E^+(K \mathbin{\backslash\!\backslash} \Sigma) \setminus E(T)) \;\middle|\; \emptyset \right\rangle, \tag{3.9}$$

where T is a maximal subtree of the quotint graph $K \mathbin{\backslash\!\backslash} \Sigma$ (see Lemma 1.2.15) and $E^+(K \mathbin{\backslash\!\backslash} \Sigma)$ the choice of an orientation for $K \mathbin{\backslash\!\backslash} \Sigma$. Since Γ_2 contains Λ and $\Gamma_2 \setminus \Lambda$ has exactly $k - 1$ vertices not linked to all vertices of Λ, the inductive hypothesis ensures that $\mathrm{Ker}\left(\psi_\Lambda|_{\mathcal{A}(\Gamma_2)}\right)$ is free and

$$\mathrm{Stab}(v) = \mathrm{Ker}(\psi_\Lambda) \cap \mathcal{A}(\Gamma_2) = \mathrm{Ker}\left(\psi_\Lambda|_{\mathcal{A}(\Gamma_2)}\right), \qquad \forall v \in {}^{\mathcal{A}(\Gamma)}/_{\mathcal{A}(\Gamma_2)}$$

is free as well since they are subgroups of a free group. Therefore Equation (3.9) implies that $K = \mathrm{Ker}(\psi_\Lambda)$ is free since it decomposes as a free product of free groups. This proves the claim. ∎

Theorem 3.3.5. *Let $\mathfrak{F}$ be a finite family of Coxeter graphs, each of them having at most two vertices. Let $J_{\mathfrak{F}}$ be any set satisfying the conditions described in Definition 3.3.2 with respect to the family $\mathfrak{F}$ and let $\Gamma \coloneqq \mathrm{jn}_2(\mathfrak{F}, J_{\mathfrak{F}})$, then the associated Artin group $\mathcal{A}(\Gamma)$ is strongly poly-free.*

Proof. We proceed by induction on the number n of vertices of Γ.
The case $n = 0$ (i.e., $\mathfrak{F} = \emptyset$) is degenerate with $\mathcal{A}(\Gamma) = \{ 1 \}$ and the statement is trivial. If $n = 1$, then Γ is made of just a vertex, $\mathcal{A}(\Gamma) \simeq \mathbb{Z}$ and the statement follows immediately.
Now let $n \geq 2$. If Γ is not connected, then it is the disjoint union of its connected components $\Gamma_1, \ldots, \Gamma_d$ ($d \geq 2$), each of them having strictly less vertices than Γ. Lemma 1.3.11 and the inductive hypothesis imply that $\mathcal{A}(\Gamma) = \mathcal{A}(\Gamma_1) * \ldots * \mathcal{A}(\Gamma_d)$ is strongly poly-free. Therefore, to prove the statement for $n \geq 2$, we can suppose that Γ is connected. If $\mathfrak{F}$ contains graphs with two vertices, then call Λ one of such graphs, otherwise let Λ be the subgraph spanned by any connected pair of vertices in Γ. The hypotheses we made on the structure of Γ ensure that the surjective map $\psi_\Lambda \colon \mathcal{A}(\Gamma) \to \mathcal{A}(\hat{\Gamma})$ described in Lemma 3.3.1 is a well-defined homomorphism of groups. Since $|V(\hat{\Gamma})| = n - 1$ the inductive hypothesis tells us that $\mathcal{A}(\hat{\Gamma})$ is strongly poly-free and by Proposition 3.3.4 the map ψ_Λ has free kernel. Lemma 1.3.5 allows to conclude that $\mathcal{A}(\Gamma)$ is strongly poly-free. ∎

Chapter 4

Conclusion

In Chapter 2 we have been able to determine which irreducible Artin groups of finite type are poly-free and which are not (with the only exception of $\mathcal{F}_4$ which remains unknown at the moment) and to build explicit poly-fg-free series for each of them. Namely, we have proven that the only poly-free irreducible Artin groups of finite type are $\mathcal{I}_2(m)$ ($m \geq 3$), $\mathcal{A}_3$, $\mathcal{B}_3$, $\mathcal{B}_4$, and $\mathcal{D}_4$. However, to achieve this result we had to rely on very specific constructions and facts which are extremely (if not completely) unlikely to be generalised to other families of Artin groups. Still, if compared to the work of Hermiller and Šunić [11] (2007) who proved that all right angled[1] Artin groups are poly-free, it is interesting that the only irreducible finite type Artin groups which are poly-free are those built from the smallest graphs in this family. This also gives a wide set of obstructions against poly-freeness of a generic Artin group built on a Coxeter graph Γ: subgroups of poly-free groups are poly-free and since the Artin groups built on full subgraphs of Γ injects as subgroups inside $\mathcal{A}(\Gamma)$, if $\mathcal{A}(\Gamma)$ is poly-free then Γ cannot contain any full subgraph isomorphic to those of finite type that are not poly-free.

In Chapter 3, instead, we considered the case when Γ is a tree and leveraged the Reidemeister-Schreier rewriting procedure to obtain the explicit presentation of a suitable normal subgroup of $\mathcal{A}(\Gamma)$ and deduce that such subgroup is free. This allowed us to conclude that all Artin groups build on Coxeter trees are strongly poly-fg-free and we have also been able to compute the rank of each factor. However, in order to successfully apply this technique the hypothesis of Γ being a tree has revealed to be essential. For this reason in the last part of the chapter we changed approach and gave some partial results about poly-freeness of Artin groups built on graphs that can be "more connected" than a tree or a forest. Despite the hypotheses of Theorem 3.3.5 are still a bit restrictive, we think that a refinement of that kind of approach employing Bass-Serre theory could lead to more general

[1]An Artin group of type Γ is *right angled* if all the edges of Γ are labelled with '2'.

results.

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