Università degli Studi di Pavia

DIPARTIMENTO DI MATEMATICA

Corso di Dottorato di Ricerca consortile in Matematica
Università degli Studi di Pavia - Università degli Studi di Milano-Bicocca - Istituto Nazionale di Alta Matematica

XXXII Ciclo

Special almost-Kähler geometry of some homogeneous manifolds

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Dicembre 2019

Contents

Abstract

In this thesis we study metrics with special curvature properties on some homogeneous almost-Kähler manifolds. More precisely, given a symplectic manifold (M, ω) equipped with a compatible almost-complex structure J, we consider the homogeneous equation $\rho = \lambda\omega$, where ρ is the Chern-Ricci form of J, that we call speciality condition. In particular, we focus on two classes of symplectic manifolds: symplectic T^2-bundles over T^2 and adjoint orbits of semisimple Lie groups.

Symplectic T^2-bundles over T^2 are distributed in five classes. We prove that the ones belonging to four of these classes admit a special (Chern-Ricci flat) locally homogeneous compatible almost-complex structure, while the ones in the remaining class do not admit Chern-Ricci flat locally homogeneous compatible almost-complex structures. It is an open problem whether they admit non-locally homogeneous special compatible almost-complex structures.

Adjoint orbits of non-compact semisimple Lie groups turn out to be naturally almost-Kähler manifolds endowed with the Kirillov-Kostant-Souriau symplectic form and a canonically defined almost-complex structure. We give explicit formulae for the Chern-Ricci form, the Hermitian scalar curvature and the Nijenhuis tensor in terms of root data and we discuss the speciality condition, which may be translated in terms of root data as well. Moreover, we examine when compact quotients of these orbits are Kähler manifolds.

Aknowledgments

This project has received funding from the European Union's H2020 research and innovation programme under the MarieSkłowdoska-Curie grant No. 778010 *IPaDEGAN*.

Besides the formal aknowledgments, I would like to express my sincere gratitude to some people who have contributed to this thesis, both directly and indirectly.

First of all, to my advisor Alberto, who brought me in the world of symplectic geometry, even if I came from a completely different area. I appreciate his way of carrying the research on and also the infinite patience he had with me. I really liked to work with him and I learnt many things in doing it.

To Prof. Vestislav Apostolov and Prof. Fabio Podestà for the useful corrections, comments and remarks on the thesis.

To my mum, for her support during these years. She has always motivated me with precious advices and conversations.

Last but not least, to Bruno, for his special and constant presence. These years have been very hard, but we carried on together and we will always do so.

Introduction

One of the fundamental question that has been driving the research in the area of Riemannian geometry for years is the one formulated by René Thom in 1958, but probably older, *"Are there any best Riemannian structures on a smooth manifold?"* [10, Pg. 1]. It is an extremely vague question since there is no a precise meaning for the word *best* in this context. A motivation for examining this problem is that metrics with privileged structure should give information about topology or other geometric properties of a manifold. In dimension 2 the problem was solved in virtue of the *Uniformization theorem*, proved by Poincaré and Koebe independently in the first decade of the 900 [51, 70]. The theorem states that a compact Riemann surface admits a unique Riemannian metric of unit volume with constant Gauss curvature, or, equivalently, the universal cover of a Riemann surface is conformally equivalent to the unit disk, $\mathbb{C}$ or the Riemann sphere. Thus, a Riemann surface always has a canonical Riemannian metric: the one with constant Gauss curvature. In higher dimension, the situation is quite different and there are no general results. From the uniformization theorem, we can grasp that a preferred metric on a manifold should have special curvature properties, in particular its curvature needs to be constant. But which curvature? Saying just "curvature" makes sense for surfaces since all curvature tensors coincide with the Gauss curvature, which is a scalar. However, in higher dimension, a Riemannian metric induces many curvature tensors, each capturing different features of the metric, and it is not clear which one to choose. The constancy of the Riemann tensor or the sectional curvature imposes many constraints on the metric, producing few manifolds admitting this kind of metrics, the so-called *space forms* [49, Chapter 5, Section 3]. Three typical examples of space forms are the hyperbolic space $\mathbb{H}^n$, the Euclidean space $\mathbb{R}^n$ and the sphere S^n, with negative, zero and positive sectional curavature respectively. Since many other curvature tensors may be defined from the Riemann curvature, one may try to consider these ones. For the scalar curvature, one has that it is always possible to find a metric with constant scalar curvature on a compact manifold, as a consequence of the solution to the Yamabe problem [5, 80, 84]. In addition, in dimension greater than 3, the moduli space of metrics with constant scalar curvature is infinite-dimensional. From this, we infer that the scalar curvature gives a quite weak geometric characterization of the manifold. Taking the traceless component of the Riemann tensor gives the Weyl tensor, and manifolds in dimension greater than 3 having vanishing Weyl tensor are said *locally conformaly flat*. However, the vanishing of the Weyl tensor is quite hard to check in general. In the very special case of dimension 4, by Hodge duality, one may loosen the vanishing condition on the Weyl tensor and consider metrics having vanishing self-dual or antiself-dual part of the Weyl tensor. These metrics are called *antiself-dual* and *self-dual* respectively and they have been extensively studied [4]. Interesting intermediate candidates seem to be metrics with constant Ricci tensor, the *Einstein metrics* [10]. Actually, Einstein metrics are one of the central subjects in modern Riemannian geometry. Thus, a suitable condition for canonical metrics seems to be the Einstein condition.

When a manifold carries further structure beyond the Riemannian one, it is quite natural to look for best metrics which are also compatible with the existing structure. An important example of this situation is Kähler geometry, which has been a fruitful area about advancements on canonical metrics. This is due mostly to the Calabi's conjecture [14], which became a theorem after Yau's proof [85]. The *Calabi-Yau theorem* implies that a compact complex manifold of Kähler type admits a Ricci-flat Kähler metric if and only if it has vanishing first Chern class and that there is a unique Ricci-flat metric in each Kähler class. Hence Ricci-flat Kähler manifolds, the *Calabi-Yau manifolds*, are good examples of manifolds admitting preferred metrics which preserve the underlying geometric structure. Actually, canonical metrics in the Kähler case have been deeply studied, and there are many cornerstones theorems which say whether a manifold admits a Kähler-Einstein metric, i.e., with constant Ricci form, [6, 20–22, 77–79, 86] together with more recent works concerning Kähler metrics with constant scalar curvature [17–19]. In particular, the existence of Kähler-Einstein metrics imposes strong restrictions on the topology of a manifold as it gives a relation between the first two Chern numbers of the manifold [10, Chapter 2, Section E]. This constraint extends to the broader class of general type manifolds and, in this context, it takes the name of *Bogomolov-Miyaoka-Yau inequality* [11, 63, 85, 86].

Leaving the context of Kähler manifolds, a quite natural and interesting question is to ask what is the right condition that a best metric needs to satisfy on a symplectic manifold. In other words, *what is a canonical metric on a symplectic manifold?* More precisely, given a symplectic manifold, we ask if it admits a preferred metric which, at the same time, preserves the symplectic structure. This area is quite unexplored at the moment and, there are very few examples of symplectic manifolds admitting metrics with distinguished curvature properties, thus it seems significant to find new ones. On the contrary, in Kähler geometry, projective manifolds provide a wide class of examples of manifolds admitting metrics with special curvature properties. A convenient way of choosing a metric on a symplectic manifold is by picking the metric induced by the symplectic structure and a selected compatible almost-complex structure on it. Notice that finding a compatible almost-complex structure is not a big deal, as compatible almost-complex structures exist in abundance on each symplectic manifold. These triple structures consisting of a symplectic form, a compatible almost-complex structure and a metric, take the name of *almost-Kähler structures* and they have been previously considered in [37, 38, 74]. Manifolds carrying an almost-Kähler structure are called *almost-Kähler manifolds*. Observe that the chosen compatible almost-complex structure needs not to be integrable, thus this class of structures is quite more general than the ones of Kähler structures. In addition, very often there cannot exist Kähler structures at all on such symplectic manifolds. A suitable curvature tensor on a symplectic manifold that ties-up the properties of the symplectic form and the compatible almost-complex structure is the Chern-Ricci form, which generalizes the usual Ricci form on a Kähler manifold. Thus, given an almost-Kähler structure on a symplectic manifold, what may we say about its Chern-Ricci form? Is the symplectic topology of the manifold constrained by the existence of a metric having Chern-Ricci form equal to a multiple of the symplectic form? At the moment, answers to these questions are still partial and incomplete, mostly because of the fact that examples of symplectic non-Kähler manifolds are rare in literature. The first known example of non-Kähler compact symplectic manifold is the celebrated *Kodaira-Thurston manifold* [76]. This example has been generalized allowing to produce examples of simply-connected symplectic non-Kähler manifolds [25, 59]. Note that the condition on the Chern-Ricci form to be a multiple of the symplectic form may be considered as a natural generalization of the Kähler-Einstein condition to the non-integrable case.

In this thesis, we will focus on symplectic manifolds admitting compatible almost-complex structures whose Chern-Ricci form si a multiple of the symplectic form. More precisely, given a symplectic manifold (M, ω) equipped with a compatible almost-complex structure J and induced Riemannian metric g, define the Chern-Ricci form of J as $\rho = \mathrm{tr}(JR) \in \Omega^2(M, \mathbb{R})$, with R the curvature tensor of the Chern connection associated with J. We then say that the almost-complex structure is *special* [3, 27] if it has constant Chern-Ricci form, i.e., its Chern-Ricci form satisfies $\rho = \lambda\omega$, for some $\lambda \in \mathbb{R}$. In analogy with the Kähler case, (M, ω) is *symplectic general type*, *symplectic Calabi-Yau* and *symplectic Fano* when λ is negative, zero or positive respectively. In particular, special almost-complex structures have constant Hermitian scalar curvature, thus they are zeros of the moment map on the space of compatible almost-complex structures of a compact symplectic manifold acted on by the group of Hamiltonian diffeomorphisms introduced by Donaldson [29]. In connection with the problem of finding non-integrable zeros of this moment map, Lejmi showed that the complex structure of any locally toric Kähler-Einstein surface can be deformed to a non-integrable special almost-complex structure [56]. More recently, a moment map picture tailored specifically for Kähler-Einstein metrics has been proposed by Donaldson [30] and extended by García-Prada, Salamon and Trautwein [34] to non-integrable almost-complex structures. Even within this framework, one may see the problem of finding special almost-complex structures on a symplectic manifold as the problem of finding zeros of a moment map. Notice that metrics associated with such special (non necessarily integrable) almost-complex structures fit in the picture described above about canonical metrics since they provide preferred metrics for symplectic manifolds. The main goal of this thesis is to provide new classes of examples of compact symplectic manifolds admitting special compatible almost-complex structures, also pointing out some of their geometric and topological properties. In particular, all our examples turn out to be *locally homogeneous manifolds*, i.e., compact quotients of manifolds carrying a transitive action of a Lie group. This assumption allows to handle both geometric and algebraic tools to explore the problem. In this context, a systematic approach to the issue of finding symplectic manifolds admitting special compatible almost-complex structures has been provided by [2, 27]. The final hope is to shed some light on the topic of canonical metrics on symplectic manifolds.

The contents are developed as follows. The first chapter is preliminary and contains the tools and the notations we are going to use throughout the thesis. In particular, we review in some detail symplectic manifolds (Section 1.1) and almost-complex manifolds (Section 1.2) and we provide examples of them. Then we focus on the differential geometry of almost-Kähler structures (Section 1.3). We introduce all the geometric objects that we are going to study in the subsequent chapters and we discuss the main equation of the thesis: the speciality condition $\rho = \lambda\omega$. Then we specialize all the given definitions to homogeneous symplectic manifolds (Section 1.4). In the second chapter we study the homogeneous speciality condition on 4-dimensional symplectic torus bundles. In the first Section 2.1, we introduce symplectic Lie groups and compact solvmanifolds, and in the second one (Section 2.2) we concentrate on particular 4-dimensional solvmanifolds, which are torus bundles over T^2. These come in five classes and for each of them we study the homogeneous speciality condition (Section 2.3). We discover that manifolds belonging to four of these classes admit *Chern-Ricci flat* compatible almost-complex structures, i.e., special with $\lambda = 0$, while for manifolds belonging to the remaining one, we prove that in each cohomology class there exists a symplectic form not admitting Chern-Ricci flat locally homogeneous compatible almost-complex structures (Theorem 2.3.5, Corollary 2.3.6). It would be interesting to understand whether there exist non-locally homogeneous special compatible almost-complex structures, especially for a comparison with the Calabi-Yau theorem in Kähler geometry. We plan to come back on this point in the future. The subject of the last chapter is concerned with the study of a different class of homogeneous manifolds: adjoint orbits of semisimple Lie groups. The results are all contained in the work [28]. In the first Section 3.1 we set up the theoretical background on adjoint orbits, while in the second one (Section 3.2) we describe the Lie algebra structure of the Lie algebra underlying an adjoint orbit. Section 3.3 and Section 3.4 are dedicated to the definition of a canonical homogeneous almost-complex structure on adjoint orbits and to the reformulation of the speciality condition in this precise context. We find a necessary and sufficient condition for an adjoint orbit to admit special canonical compatible almost-complex structure (Corollary 3.4.8) and other results concerning uniqueness and finiteness of such adjoint orbits (Proposition 3.4.2, Proposition 3.4.3). In Section 3.5 we provide explicit formulae for the Hermitian scalar curvature (Lemma 3.5.1) and the Nijenhuis tensor associated with the canonical almost-complex structure (Lemma 3.5.3, Theorem 3.5.5), while in Section 3.6 we describe the locally homogeneous almost-Kähler structure induced by the canonical almost-complex structure on compact quotients of adjoint orbits. In Section 3.7 we translate the speciality condition in a combinatorial condition on the Vogan diagram of a real semisimple Lie algebra. This allows to find many infinite families of adjoint orbits of classical simple Lie groups admitting special canonical almost-complex structure (Theorem 3.7.5). More generally, it is possible to list algorithmically all adjoint orbits of simple Lie groups admitting special canonical compatible almost-complex structure. In particular, we implemented an algorithm that is able to do it (Appendix A) and we ran it for Vogan diagrams up to rank 8 (Appendix B), hence all adjoint orbits of exceptional simple Lie groups admitting special canonical almost-complex structure are classified (Section B.2). The intricate combinatorics of root systems prevented us to grasp a more direct way of checking whether an adjoint orbit admits special canonical almost-complex structure, but one may control it by following the steps of the algorithm in each case. In the last part of the section, we discuss integrability of the canonical almost-complex structure on adjoint orbits of classical simple Lie groups (Theorem 3.7.15) and exceptional ones (Theorem 3.7.16). Finally, we sum up the contents of the chapter and we discuss a couple of problems that are missing to complete this broad picture (Section 3.8).

Unless otherwise specified, the results presented in this thesis are original. In particular, the original results in Chapter 3 are contained in [28].

Chapter 1

Geometry of almost-Kähler manifolds

In this chapter we collect definitions, results and examples concerning symplectic, almost-complex and homogeneous manifolds in order to set up the background for future chapters. The first Section 1.1 is dedicated to symplectic manifolds. Examples and properties about the local structure of symplectic manifolds are given. Section 1.2 deals with almost-complex structures, important geometric objects which are closely related to symplectic structures. The interplay between symplectic and almost-complex structures leads to the notion of almost-Kähler manifold, treated in Section 1.3. As we are interested in homogeneous structures, Section 1.4 is about homogeneous symplectic manifolds and homogeneous almost-Kähler structures on them.

1.1 Symplectic manifolds

Symplectic geometry is concerned with the geometry of a closed differential 2-form on a smooth manifold. It arose in the classical mechanics of Hamilton and Lagrange, for the phase space of a closed system is naturally equipped with a symplectic structure. In the recent years, it has become an independent research area within differential geometry and topology, carrying many links to other branches of mathematics and theoretical physics. Despite the definition requires a smooth background, symplectic geometry turns out to be quite different from the Riemannian geometry we are used to. Indeed, the characterizing properties of a symplectic manifold are basically topological and free of the differential structure of the manifold. In this section we recall basic definitions, examples and results concerning symplectic manifolds and their geometric properties.

Let M be a smooth connected manifold of dimension m without boundary. A *symplectic form* on M is a 2-differential form $\omega \in \Omega^2(M)$ which is closed, i.e., $\mathrm{d}\omega = 0$, and non- degenerate, meaning that $\omega(X,Y) = 0$, for every vector field $X \in \mathfrak{X}(M)$, implies $Y = 0$. A manifold M equipped with a symplectic form is called a *symplectic manifold.* Some examples of symplectic manifolds are the following.

Example 1.1.1. The Euclidean space $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_{2n})$ is a symplectic manifold with symplectic form

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}x^{i+n}. \tag{1.1}$$

The symplectic form ω_0 is often called *standard*.

Example 1.1.2. Cotangent bundles make up a class of symplectic manifolds which is fundamental in classical mechanics. Indeed, they may be seen as phase spaces with coordinates p and q corresponding to momentum and position respectively. More precisely, given a smooth manifold M, its cotangent bundle T^*M is the vector bundle having as sections the 1-forms on M and it carries a canonical 1-form $\theta \in \Omega^1(T^*M)$, called the *Liouville form*. It is defined in local coordinates by $\theta = \sum_{i=1}^{\dim M} p_i \mathrm{d}q_i$ and it has the property that, called π the projection $\pi : T^*M \to M$ and chosen $x \in M$ and $\alpha \in T^*_x M$, one has

$$\theta|_{(x,\alpha)} = \pi^* \alpha. \tag{1.2}$$

A symplectic form on T^*M is then defined by $\omega = -\mathrm{d}\theta$ and, in local coordinates, it has the expression

$$\omega = \sum_{i=1}^{\dim M} \mathrm{d}q_i \wedge \mathrm{d}p_i. \tag{1.3}$$

Further details on this topic may be found in [60, Chapter 3].

Example 1.1.3. Every orientable surface is a symplectic manifold with symplectic form given by the volume form.

Example 1.1.4. The 2-sphere S^2 is the unique symplectic sphere. In other words, the $2n$-sphere S^{2n} admits a symplectic structure if and only if $n = 1$. To see this, let $\omega \in \Omega^2(S^{2n})$ be a symplectic form on S^{2n} and consider the integral

$$\int_{S^{2n}} \omega^n \neq 0, \tag{1.4}$$

by non-degeneracy of ω. Since ω is closed, it represents a 2-cohomology class in $H^2_{\mathrm{dR}}(S^{2n}, \mathbb{R})$. In particular,

$$0 \neq \int_{S^{2n}} \omega^n = \int_{S^{2n}} [\omega]^n, \tag{1.5}$$

showing that $0 \neq [\omega] \in H^2_{\mathrm{dR}}(S^{2n}, \mathbb{R})$. However, $H^2_{\mathrm{dR}}(S^{2n}, \mathbb{R})$ is non-trivial only for $n = 1$ and this shows that ω may be a symplectic form only for $n = 1$, as an exact symplectic form on a compact manifold would produce a vanishing volume. In local coordinates (x, y) on S^2, the symplectic form is defined by

$$\omega = 2\frac{\mathrm{d}x \wedge \mathrm{d}y}{(1 + x^2 + y^2)^2}. \tag{1.6}$$

Example 1.1.5. The complex projective space $\mathbb{CP}^n$ is a symplectic manifold with the *Fubini-Study form*, defined in local affine coordinates $z = (z_1, \dots, z_n)$ by

$$\omega_{FS} = i\partial\bar{\partial} \log(1 + |z|^2). \tag{1.7}$$

Observe that, for $n = 1$, $\mathbb{CP}^1$ is diffeomorphic to S^2 and the Fubini-Study form coincides exactly with (1.6).

Some properties of symplectic manifolds may be immediately deduced from the definition. Non-degeneracy of the symplectic form implies that a symplectic manifold is even-dimensional and orientable, for the top exterior power of the symplectic form is a volume form. Thus, in the following we will always assume that the dimension of M is even $m = 2n$. Moreover, non-degeneracy of the symplectic form provides a canonical identification between tangent and cotangent bundle via the map

$$TM \to T^*M, \quad X \mapsto \iota(X)\omega = \omega(X, \cdot). \tag{1.8}$$

On the other hand, closedness of ω reads that ω represents a 2-cohomology class $a = [\omega] \in H^2(M, \mathbb{R})$. In particular, one may see from Example 1.1.4 that a compact symplectic manifold cannot have exact symplectic form. Indeed, the cohomology class $a^n \in H^{2n}(M, \mathbb{R})$ is represented by the volume form ω^n and, if ω is exact, its integral over M vanishes, leading to a contradiction. This fact shows that a smooth compact manifold needs to have the right topology in order to be symplectic. For example, the second Betti number of M needs to satisfy $b_2 \geq 1$.

Two symplectic manifolds may be considered "the same" from the symplectic point of view if there exists a diffeomorphism between them that preserves the symplectic structure. More precisely, a *symplectomorphism* of a symplectic manifold (M, ω) is a diffeomorphism $\psi \in \mathrm{Diff}(M)$ such that $\psi^*\omega = \omega$. The symplectomorphisms of (M, ω) form a group

$$\mathrm{Symp}(M, \omega) = \{\psi \in \mathrm{Diff}(M) \mid \psi^*\omega = \omega\}, \tag{1.9}$$

which is generally infinite-dimensional. As we have seen above, the map (1.8) establishes a one-to-one correspondence between vector fields and 1-forms, and a vector field $X \in \mathfrak{X}(M)$ is said to be *symplectic* if $\iota(X)\omega$ is closed. Observe that this is equivalent to require that $\mathcal{L}_X\omega = 0$, where $\mathcal{L}$ denotes the Lie derivative, as a consequence of the Cartan formula

$$\mathcal{L}_X\omega = \iota(X)\mathrm{d}\omega + \mathrm{d}(\iota(X)\omega). \tag{1.10}$$

Among the symplectic vector fields we find *Hamiltonian vector fields*, for which the associated 1-form is not just closed, but exact. More precisely, a vector field $X_H \in \mathfrak{X}(M)$ is said to be Hamiltonian if there exists a smooth function H, called the *Hamiltonian function*, such that

$$\iota(X_H)\omega = dH. \tag{1.11}$$

The Hamiltonian vector fields generate the subgroup of Hamiltonian symplectomorphisms

$$\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}(M, \omega). \tag{1.12}$$

A feature of symplectic manifolds is that they locally look all the same, or, in other words, they have no local invariants. Globally the situation is slightly different as there exist global invariants, such as the cohomology class $[\omega] \in H^2_{\mathrm{dR}}(M, \mathbb{R})$ or the first Chern class c_1 of (M, ω) in $H^2_{\mathrm{dR}}(M, \mathbb{R})$. The key point for the lack of local invariants is *Moser's argument*, which says that for every family of symplectic forms $\omega_t \in \Omega^2(M)$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = \mathrm{d}\sigma_t, \tag{1.13}$$

there exists a family of diffeomorphisms $\psi_t \in \mathrm{Diff}(M)$ such that $\psi_t^*\omega_t = \omega_0$. Among the consequences of Moser's argument, we find Darboux's Theorem [60, Theorem 3.2.2] which states that symplectic manifolds are modeled on $(\mathbb{R}^{2n}, \omega_0)$, hence they have no local invariants.

Theorem 1.1.6 (Darboux). *Every symplectic form ω on M is locally diffeomorphic to the standard symplectic form ω_0 on $\mathbb{R}^{2n}$.*

As a corollary, we have that every symplectic manifold (M, ω) may be covered by charts $(U_\alpha, \alpha : U_\alpha \to \alpha(U_\alpha))_\alpha$ such that $\alpha^*\omega_0 = \omega$. Charts with this property are called *Darboux charts* and their transition maps consists of symplectic matrices,

$$d(\beta \circ \alpha^{-1})(x) \in Sp(n, \mathbb{R}), \quad x \in \alpha(U_\alpha \cap U_\beta), \tag{1.14}$$

where

$$Sp(n, \mathbb{R}) = \{A \in GL(2n, \mathbb{R}) \mid A^t J_0 A = J_0\} \tag{1.15}$$

and $J_0 = \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix}$. Darboux's theorem says also that the tangent bundle of a symplectic manifold carries a reduction of the structure group to $Sp(n, \mathbb{R})$.

The next section is concerned with almost-complex structures, which are geometric objects highly related with symplectic structures.

1.2 Almost-complex manifolds

Almost-complex geometry is the geometry of an endomorphism of the tangent bundle of a manifold which squares to $-\mathrm{id}$. This endomorphism may be thought as a generalization of the multiplication by the imaginary unit performed fiber by fiber on the tangent bundle. Generally the existence of almost-complex structures finds some obstructions which are topological in nature. Nevertheless, on symplectic manifolds there is plenty of almost-complex structures. Not only that, many of them also satisfy a compatibility relation with the symplectic form.

An *almost-complex structure* on a smooth manifold M is an endomorphism of the tangent bundle $J \in \mathrm{End}(TM)$ satisfying $J^2 = -\mathrm{id}$. A manifold M equipped with an almost-complex structure is called an *almost-complex manifold*. As in the symplectic case, almost-complex manifolds are orientable and have even dimension. In addition, the tangent bundle of an almost-complex manifold is a complex vector bundle, hence its structure group reduces to $GL(n, \mathbb{C}) = \{A \in GL(2n, \mathbb{R}) \mid AJ_0 = J_0 A\}$.

Example 1.2.1. The Euclidean space $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_{2n})$ and equipped with the almost-complex structure

$$J_0\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{i+n}}, \quad J_0\left(\frac{\partial}{\partial x_{i+n}}\right) = -\frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n, \tag{1.16}$$

is an almost-complex manifold. We often call the almost-complex structure J_0 *standard*.

Actually each even-dimensional smooth manifold may be equipped with an almost-complex structure defined in local charts, for example as in (1.16). When these charts may be patched together to give a globally defined almost-complex structure, we say that the almost-complex structure is *integrable* and the couple (M, J) is a *complex manifold*. Notice that the notion of almost-complex structure is weaker than the one of complex structure, for which each complex manifold is also almost-complex, but the vice versa is false, as the following example shows.

Example 1.2.2. A famous theorem of Borel and Serre [13] states that among the spheres S^{2n}, the only admitting an almost-complex structure are S^2 and S^6. Observe that S^2 carries the usual almost-complex strcture of the Riemann sphere, hence it is actually a complex manifold. On the other hand, S^6, which may be thought as the unitary sphere in the imaginary octonions $\mathrm{Im}\mathbb{O}$

$$S^6 = \{x \in \mathrm{Im}\mathbb{O} \mid |x|^2 = 1\}, \tag{1.17}$$

inherits an almost-complex structure from the octonion algebra $\mathbb{O}$, which is non-integrable. Whether if admits an integrable almost-complex structure is an open problem [1]. For more details about the proof of the result about S^6 see the self-contained paper [52].

Given an almost-complex manifold (M, J), one may associate with J a $(2,1)$ skew-symmetric tensor N_J defined by

$$N_J(X,Y) = \frac{1}{4}\left([JX,JY] - J[JX,Y] - J[X,JY] - [X,Y]\right), \tag{1.18}$$

for $X, Y \in \mathfrak{X}(M)$, which takes the name of *Nijenhuis tensor* of J. A celebrated theorem by Newlander and Niremberg [66] relates the integrability of an almost-complex structure with its Nijenhuis tensor.

Theorem 1.2.3 (Newlander-Niremberg)**.** *Let (M, J) be an almost-complex manifold. Then J is integrable if and only if $N_J = 0$.*

Given a symplectic manifold (M, ω) equipped with an almost-complex structure J, we say that J is *ω-compatible* (or, simply, *compatible* when there is no ambiguity) if it satisfies two conditions: ω is J-invariant, i.e., $\omega(JX, JY) = \omega(X, Y)$ for every $X, Y \in \mathfrak{X}(M)$, and

$$\omega(X, JX) > 0, \quad \forall X \in \mathfrak{X}(M) \setminus \{0\}. \tag{1.19}$$

Condition (1.19) allows to define a Riemannian metric g on M by putting

$$g(X,Y) = \omega(X, JY). \tag{1.20}$$

Notice that the induced metric g is J-orthogonal, meaning that $g(JX, JY) = g(X, Y)$ for every $X, Y \in \mathfrak{X}(M)$.

Example 1.2.4. On $\mathbb{R}^{2n}$, the standard symplectic form ω_0 defined in (1.1) and the standard almost-complex structure J_0 (1.16) are compatible. The induced Riemannian metric coincides with the identity.

Actually, compatible almost-complex structures are very common in symplectic geometry. Indeed, it holds that the space $\mathcal{J}(M, \omega)$ of ω-compatible almost-complex structures on M is always non-empty.

Lemma 1.2.5. *$\mathcal{J}(M, \omega)$ is non-empty and contractible.*

In short, non-emptiness of $\mathcal{J}(M, \omega)$ follows by the existence of a Riemannian metric on M and non-degeneracy of ω. For contractibility, note that $\mathcal{J}(M, \omega)$ is the space of sections of a fiber bundle on M with fiber at each point $x \in M$ the space of compatible linear complex structures on T_xM, called the *Siegel half-space*. Moreover, each fiber turns out to be isomorphic to $Sp(n, \mathbb{R})/U(n)$, which, in turns may be identified with the space of $2n \times 2n$ symmetric complex matrices with positive definite imaginary part via the map

$$\begin{aligned} \{X + iY \mid X, Y \text{ symm},\ Y > 0\} &\to Sp(n,\mathbb{R})/U(n) \\ X + iY &\to \begin{pmatrix} XY^{-1} & -(XY^{-1}X + Y) \\ Y^{-1} & -Y^{-1}X \end{pmatrix}. \end{aligned} \tag{1.21}$$

This explains the reason for the name "half-space". However, $Sp(n, \mathbb{R})/U(n)$ is contractible, as $U(n)$ is a maximal subgroup of $Sp(n, \mathbb{R})$ and the Cartan polar decomposition [48, Theorem 6.31(c)] induces the $U(n)$-equivariant diffeomorphism

$$\begin{aligned} \mathfrak{p} &\to Sp(n,\mathbb{R})/U(n), \\ X &\mapsto e^X \end{aligned} \tag{1.22}$$

where $\mathfrak{p}$ is a $U(n)$-invariant subspace of $\mathfrak{u}(n)$ in $\mathfrak{sp}(n,\mathbb{R})$. This shows that the fiber is contractible, hence $\mathcal{J}(M,\omega)$ is contractible.

A symplectic manifold (M,ω) together with a compatible almost-complex structure J is said to be an *almost-Kähler manifold*. An almost-Kähler manifold with an integrable almost-complex structure turns out to be a *Kähler manifold*. For a couple of decads from the discovery of almost-Kähler manifolds, it was an open problem whether each symplectic manifold was actually Kähler. However, Thurston provided the first examples of symplectic non-Kähler manifolds [76], among which we find the well-known *Kodaira-Thurston manifold*. The class of almost-Kähler manifolds is the subject of the next section.

1.3 Almost-Kähler manifolds

At the intersection between symplectic, almost-complex and Riemannian geometry we come across almost-Kähler manifolds. These manifolds turn out to be rich of structure and many interesting properties arise from the fact of being symplectic, Riemannian and almost-complex. In this class we find the well known Kähler manifolds, which share many properties but carry a more rigid framework. This section is a little survey about the differential geometry of almost-Kähler manifolds. In the final part, we will focus the attention on almost-Kähler manifolds admitting *special* compatible almost-complex structures, which are the main theme of this thesis.

A symplectic manifold (M,ω) of dimension $2n$ equipped with a compatible almost-complex structure J and induced Riemannian metric g is said to be an *almost-Kähler manifold* and they were firstly introduced in [74]. The presence of a symplectic structure together with a compatible almost-complex structure allows to further reduce the structure group of TM to $U(n)=\{A\in GL(n,\mathbb{C}) \mid AA^*=\mathrm{id}\}$, since $Sp(n,\mathbb{R})\cap GL(n,\mathbb{C})=U(n)$ (see for example [60, Lemma 2.2.1]). When the almost-complex structure is integrable, M turns out to be a Kähler manifold. Notice that a wide class of examples of Kähler manifolds is the one of complex projective manifolds. On the other hand, for *strictly* almost Kähler manifolds, i.e., with non-integrable compatible almost-complex structure, such a rich class of examples is not known. One of the best known examples of symplectic non-Kähler manifold is the *Kodaira-Thurston manifold*, a torus-bundle over a 2-dimensional torus discovered by Thurston in 1976 [76]. We recall its construction below and we will use this example as a guide as the various concept are introduced.

Example 1.3.1 (Kodaira-Thurston manifold)**.** Let $Heis(3,\mathbb{R})$ be the 3-dimensional Lie group defined by

$$Heis(3,\mathbb{R})=\left\{\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} \;\middle|\; x,y,z\in\mathbb{R}\right\}\subset GL(3,\mathbb{R}) \tag{1.23}$$

and consider its central extension

$$G=\left\{\begin{pmatrix}e^t & 0 & 0 & 0\\ 0 & 1 & x & z\\ 0 & 0 & 1 & y\\ 0 & 0 & 0 & 1\end{pmatrix} \;\middle|\; x,y,z,t\in\mathbb{R}\right\}\subset GL(4,\mathbb{R}). \tag{1.24}$$

The discrete subgroup $\Gamma=G\cap GL(4,\mathbb{Z})$ of G acts by left multiplication on G. In coordinates, if $\gamma=(a,b,c,d)\in\Gamma$ and $h=(x,y,z,t)\in G$, with $a,b,c,d\in\mathbb{Z}$ and $x,y,z,t\in\mathbb{R}$, then

$$\gamma\cdot h=(a+x,b+y,c+ay+z,d+t). \tag{1.25}$$

We define the Kodaira-Thurston manifold as the quotient $M = \Gamma \setminus G$ of G by the left action of Γ. By identifying the 2-torus T^2 with the quotient $T^2 = \mathbb{Z}^2 \setminus \mathbb{R}^2$, we may give M the structure of a T^2-bundle over T^2 through the projection

$$\pi : M \to T^2, \qquad [x, y, z, t] \mapsto [y, t], \tag{1.26}$$

where (x, y, z, t) are local coordinates on M. Indeed, π is well defined and the fibers are diffeomorphic to T^2. Define the local basis of vector fields

$$(e_1, e_2, e_3, e_4) = (\partial_x, \partial_y + x\partial_z, \partial_t, \partial_z), \tag{1.27}$$

and observe that these vector fields are *left-invariant*, meaning that $(\mathrm{d}L_g)(e_i) = e_i$, $1 \leq i \leq 4$, for every $g \in G$, where L_g denotes the left multiplication by g. Moreover, the unique non-vanishing commutator is $[e_1, e_2] = e_4$. A dual coframe is then given by

$$(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}t, \mathrm{d}z - x\mathrm{d}y). \tag{1.28}$$

An almost-Kähler structure on M may be defined by the symplectic form

$$\omega = \varphi_1 \wedge \varphi_3 + \varphi_2 \wedge \varphi_4, \tag{1.29}$$

the compatible almost-complex structure

$$J = e_3 \otimes \varphi_1 - e_1 \otimes \varphi_3 + e_4 \otimes \varphi_2 - e_2 \otimes \varphi_4, \tag{1.30}$$

and the induced Riemannian metric

$$g = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2. \tag{1.31}$$

The almost-Kähler structure defined in this way is G-invariant, in a sense that will be made more precise in Section 1.4.

Almost-Kähler manifolds carry three compatible structures and it would be useful to have connections which are compatible with all three of them. As shown below, this happens only if the almost-complex structure is integrable, hence the almost-Kähler manifold is actually a Kähler manifold, showing that this condition is too strong to be required on generic almost-Kähler manifold. Thus, some properties should be relaxed. In particular, on an almost-Kähler manifold one may define a connection which takes into account not only the metric g but also the almost-complex structure J. This connection is called *Chern connection* and it was firstly introduced by Gauduchon in [35].

Definition 1.3.2. The *Chern connection* ∇ on TM is the unique metric connection having as torsion tensor T^∇ the Nijenhuis tensor N_J. More explicitly, denoted by D the Levi-Civita connection of g and given two vector fields $X, Y \in \mathfrak{X}(M)$, ∇ is defined by

$$\nabla_X Y = D_X Y - \frac{1}{2} J(D_X J)Y. \tag{1.32}$$

The reason for defining this new connection is that, in general, the Levi-Civita connection on an almost-Kähler manifold does not behave very well with respect to J. Indeed, J is parallel with respect to the Levi-Civita connection only in the Kähler case.

Theorem 1.3.3 ([50, Chapter IX, Corollary 3.5]). *$DJ = 0$ if and only if $N_J = 0$.*

Since $T^\nabla = N_J$, the above definition shows that the Levi-Civita connection coincides with the Chern connection if and only if the manifold is Kähler. Thus, on a Kähler manifold, the geometry of the Chern connection is the same as the usual Riemannian geometry of the induced metric. On the other hand, both J and ω are parallel with respect to the Chern connection (see for example [67, Proposition 1.4.4]).

Lemma 1.3.4. *$\nabla J = 0$ and $\nabla\omega = 0$.*

Proof. Let $X, Y, Z \in \mathfrak{X}(M)$ be three vector fields on M. Then

$$\begin{aligned}
(\nabla_X J)Y =&\nabla_X(JY) - J\nabla_X Y \\
=&D_X(JY) - \frac{1}{2}J(D_X J)JY - JD_X Y - \frac{1}{2}(D_X J)Y \\
=&D_X(JY) - \frac{1}{2}(D_X J)Y - JD_X Y - \frac{1}{2}(D_X J)Y \\
=&D_X(JY) - JD_X Y - (D_X J)Y \\
=&0
\end{aligned} \tag{1.33}$$

where in the in the first and last equalities we used the formula for the covariant derivative of a $(1,1)$-tensor $(D_X J)Y = D_X(JY) - JD_X Y$, and in the second equality the fact that $J^2 = -\mathrm{id}$, hence J anticommutes with $D_X J$.

For the symplectic form, the connection ∇ acts on ω as follows

$$(\nabla_Z\omega)(X, Y) = Z(\omega(X, Y)) - \omega(\nabla_Z X, Y) - \omega(X, \nabla_Z Y). \tag{1.34}$$

By the definition of ω in terms of g, $\omega(X, Y) = g(JX, Y)$, and the fact that $\nabla J = 0$, one gets

$$\begin{aligned}
(\nabla_Z\omega)(X, Y) =&Z(\omega(X, Y)) - \omega(\nabla_Z X, Y) - \omega(X, \nabla_Z Y) \\
=&Z(g(JX, Y)) - g(J\nabla_Z X, Y) - g(JX, \nabla_Z Y) \\
=&Z(g(JX, Y)) - g(\nabla_Z(JX), Y) - g(JX, \nabla_Z Y),
\end{aligned} \tag{1.35}$$

which vanishes as ∇ is a metric connection. □

The curvature of the Chern connection is defined as usual by

$$R^\nabla(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} \tag{1.36}$$

for every $X, Y \in \mathfrak{X}(M)$. Observe that, by Lemma (1.3.4),

$$JR^\nabla = R^\nabla J. \tag{1.37}$$

Example 1.3.5 (continued)**.** By expanding Example 1.3.1, with some computations one may get the Nijenhuis tensor of J

$$N_J = \frac{1}{4}\left(e_2 \otimes (\varphi_{23} - \varphi_{14}) + e_4 \otimes (\varphi_{34} - \varphi_{12})\right), \tag{1.38}$$

where φ_{ij} is a short notation for $\varphi_i \wedge \varphi_j$. The Chern connection is given by $\nabla = \mathrm{d} + A$, where the connection 1-form $A \in \Gamma(\mathfrak{o}(4) \otimes T^*M)$ has the form

$$A = \frac{1}{4}\begin{pmatrix} 0 & \varphi_4 & 0 & \varphi_2 \\ -\varphi_4 & 0 & \varphi_2 & -2\varphi_1 \\ 0 & -\varphi_2 & 0 & \varphi_4 \\ -\varphi_2 & 2\varphi_1 & -\varphi_4 & 0 \end{pmatrix}, \tag{1.39}$$

while the Chern curvature R^∇ is

$$R^\nabla = \frac{1}{8}\begin{pmatrix} 0 & -3\varphi_{12} & -\varphi_{24} & \varphi_{14} \\ 3\varphi_{12} & 0 & \varphi_{14} & \varphi_{24} \\ \varphi_{24} & -\varphi_{14} & 0 & -3\varphi_{12} \\ -\varphi_{14} & -\varphi_{24} & 3\varphi_{12} & 0 \end{pmatrix}. \tag{1.40}$$

The presence of multiple structures on an almost-Kähler manifold allows to define tensors which tie-up the properties of J and ω. One of them is the *Chern-Ricci form* $\rho \in \Omega^2(M)$, defined by

$$\rho = \mathrm{tr}_{\mathbb{R}}(JR^\nabla). \tag{1.41}$$

More explicitly, given two vector fields $X, Y \in \mathfrak{X}(M)$ and an orthonormal basis $e_1, \ldots, e_n$, we may write

$$\rho(X,Y) = \sum_{i=1}^{2n} g(R^\nabla(X,Y)Je_i, e_i), \tag{1.42}$$

by (1.37). If J is integrable, $\rho(X,Y)$ coincides with $2\mathrm{Ric}(JX,Y)$, the usual Ricci form on a Kähler manifold. Observe that ∇ is a complex linear connection on TM, hence by Chern-Weil theory, the Chern classes of TM may be expressed in terms of the curvature R^∇. In particular, the first Chern class $c_1 \in H^2_{\mathrm{dR}}(M,\mathbb{R})$ is defined as

$$c_1 = \left[\frac{1}{2\pi}\mathrm{tr}_{\mathbb{C}}(iR^\nabla)\right]. \tag{1.43}$$

Since $2\mathrm{tr}_{\mathbb{C}}(iR^\nabla) = \mathrm{tr}_{\mathbb{R}}(JR^\nabla)$,

$$c_1 = \left[\frac{1}{4\pi}\mathrm{tr}_{\mathbb{R}}(JR^\nabla)\right] = \frac{1}{4\pi}[\rho]. \tag{1.44}$$

Thus the Chern-Ricci form ρ represents $4\pi c_1 \in H^2_{\mathrm{dR}}(M,\mathbb{R})$. Another object which comes with the Chern connection is a function called the *Hermitian scalar curvature* $s \in C^\infty(M,\mathbb{R})$, defined by

$$s\omega^n = n\rho \wedge \omega^{n-1}. \tag{1.45}$$

Our main interest in this thesis is the study of almost-complex structures inducing some specific curvature properties of the Chern connection. We will focus mainly on the following condition, which has been considered previously by Apostolov and Drăghici [3].

Definition 1.3.6. A compatible almost-complex structure J on (M,ω) is said to be *special* if there exists a constant $\lambda \in \mathbb{R}$ such that

$$\rho = \lambda\omega. \tag{1.46}$$

Condition (1.46) is sometimes called *Chern-Einstein condition* [2] or *Hermite-Einstein condition* [57]. The reason for the last terminology is the analogy with Hermite-Einstein metrics on holomorphic vector bundles. However, we prefer to avoid the terms "Hermite" or "Einstein" in our definition since metrics on symplectic manifolds satisfying (1.46) are neither Hermite nor Einstein in general. Indeed, it was proved first by Sekigawa [72] and then by Drăghici [31] that if a special compatible almost-complex structure J on a closed symplectic manifold has $\lambda \geq 0$, then the metric being Einstein implies that J is integrable. The general case is still open and it is called the *Goldberg conjecture.* Since we are principally interested in special almost-complex structures on strictly almost-Kähler manifolds, our manifolds will be neither Kähler nor Einstein in general. In addition, when J is integrable, M is Kähler-Einstein, thus condition (1.46) may be considered as the natural generalization of the Kähler-Einstein condition to the non-integrable case.

The existence of a special almost-complex structure on a symplectic manifold (M, ω) has some immediate topological consequences. First, if J is such that $\rho = \lambda\omega$, then the first Chern class c_1 of M and $[\omega]$ must satisfy

$$4\pi c_1 = \lambda[\omega], \tag{1.47}$$

as elements of $H^2_{\mathrm{dR}}(M, \mathbb{R})$. According to the sign of λ, (M, ω) is special *symplectic general type* if $\lambda < 0$, *symplectic Calabi-Yau* if $\lambda = 0$ and *symplectic Fano* if $\lambda > 0$. Moreover, if J is special with constant λ, it follows from the definition of Hermitian scalar curvature (1.45) that $s = n\lambda$.

Example 1.3.7 (continued)**.** By computing explicitly the trace of the endomorphism JR^∇ for the G-invariant almost-Kähler structure on the Kodaira-Thurston manifold defined in Example 1.3.5, one gets that $\rho = 0$ and so also the Hermitian scalar curvature vanishes. Thus, the almost-complex structure J on the Kodaira-Thurston manifold is special, with constant $\lambda = 0$. In particular, the Kodaira-Thurston manifold is special symplectic Calabi-Yau. When the constant λ is equal to 0, we will often say that J is *Chern-Ricci flat.* Actually, all G-invariant almost-Kähler structures are Chern-Ricci flat on the Kodaira-Thurston manifold, being it a 2-step *nilmanifold* [82]. More generally, we will explore Chern-Ricci flatness condition on certain *solvmanifolds* in Chapter 2.

Examples of Kähler-Einstein manifolds are numerous in the Kähler context. On the other hand, strictly almost-Kähler manifolds with special compatible almost-complex structure are quite rare in literature, thus having new examples turns out to be notable. The main goal of this thesis is to study new examples of symplectic manifolds admitting non-integrable special compatible almost-complex structures. In particular, all our examples will be compact quotients of almost-Kähler *homogeneous manifolds*, that is, manifolds carrying a transitive Lie group action and an invariant almost-Kähler structure. They will be the subject of the next section.

1.4 Homogeneous symplectic manifolds

Homogeneous manifolds constitute a class of important spaces both in mathematics and physics because of the large number of symmetries they have. They carry the action of a group which preserves some geometric structure, thus they "look the same" from each point. Because of this property, geometry and algebra are highly intertwined so that many geometric questions may be answered in an algebraic way and vice versa. In this section we review the geometric features of homogeneous manifolds, focusing on homogeneous symplectic manifolds equipped with a compatible almost-complex structure. We follow [10, Chapter 7] and [27, Section 3] for this discussion.

A smooth manifold M is said to be a *G-homogeneous manifold*, or, simply, *homogeneous manifold*, if there is a closed and connected Lie group G acting transitively on it. For example, when M is a Riemannian manifold G may be its group of isometries. Fix once for all a point $x \in M$. Transitivity of the action together with the orbit-stabilizer theorem imply that M is diffeomorphic to the quotient $G/\mathrm{Stab}(x)$. Notice that the stabilizer $V = \mathrm{Stab}(x)$ is a closed subgroup of G, thus compact as G is closed. We will often call it the *isotropy* of M. Before specializing our definitions to homogeneous symplectic manifolds, we give few of examples of homogeneous spaces.

Example 1.4.1. A Lie group is trivially a homogeneous manifold, as it acts transitively on itself by left multiplication and the stabilizer of each point is trivial.

Example 1.4.2. One of the typical examples of homogeneous space if the n-sphere S^n. The group $SO(n+1)$ acts transitively and by isometries on the n-sphere S^n with stabilizer of a point $SO(n)$. Hence S^n may be viewed as the homogeneous manifold $SO(n+1)/SO(n)$. Similarly, $\mathbb{R}^n$ may be identified with the quotient of $E(n)$, the group of isometries of the Euclidean space, by $O(n)$, and the hyperbolic space $\mathbb{H}^n$ with $SO(n,1)/SO(n)$.

Example 1.4.3 ([26] [33, Section 2.3.3] [27, Section 4.2.1]). By extending the previous example, the group $SO(2n,1)$ acts transitively on the *twistor space* of $\mathbb{H}^{2n}$, that is, the space of all orthogonal ortientation-preserving complex structures on $T_x\mathbb{H}^{2n}$ for $x \in \mathbb{H}^{2n}$, with isotropy $U(n)$. Thus, the twistor space of the hyperbolic space turns out to be diffeomorphic to the homogeneous space $SO(2n,1)/U(n)$.

Example 1.4.4 ([10, Example 7.15]). The group $SU(n+1)$ acts transitively on the projective space $\mathbb{CP}^n$ with stabilizer of a point $S(U(1) \times U(n))$. Therefore

$$\mathbb{CP}^n = SU(n+1)/S(U(1) \times U(n)). \tag{1.48}$$

Example 1.4.5. The *Grassmannian* $Gr(k,n)$ is defined as the set of all k-dimensional linear subspaces of an n-dimensional vector space V. We may give the Grassmannian $Gr(k,n)$ the structure of homogeneous space via the quotient $O(n)/O(k) \times O(n-k)$. Indeed $O(n)$ is the group of isometries of V endowed with the Euclidean scalar product, while $O(k)$ and $O(n-k)$ stabilize a k-dimensional vector subspace and its orthogonal complement respectively.

Remark 1.4.6 ([10, Note 7.12]). For the above definition of homogeneouos manifold to make sense, the group G needs to act *effectively* on G/V, i.e., V contains no non-trivial normal subgroups of G. However, in general, G does not necessarily act on G/V in an effective way. Denoted by C the maximal normal subgroup of G contained in V, then G/C acts on G/V effectively with isotropy V/C. Usually, in the examples it is not given the group acting effectively, as it is always possible to determine it. In the cases treated in this thesis, the action of G on G/V will be *almost-effective* on G/V, meaning that V contains no non-discrete normal subgroups of G. In this case, the Lie algebras of G and G/C are the same.

Let now (M,ω) be a G-homogeneous symplectic manifold endowed with a compatible almost-complex structure J and denote by g the induced Riemannian metric. We say that ω and J are *homogeneous*, or *G-invariant*, if G acts on M by symplectic and holomorphic transformations, i.e., for every element $h \in G$,

$$h^*\omega = \omega, \qquad J\mathrm{d}h = \mathrm{d}hJ. \tag{1.49}$$

By compatibility, also $h^*g = g$, hence h acts also as an isometry of M.

Homogeneous structures are quite important in the context of homogeneous manifolds, as they are completely determined from their value at a subspace of the Lie algebra of the acting Lie group. We make more precise this statement in the next lines. Notice that the Lie algebra $\mathfrak{g} = T_eG$ of G may be identified with a subalgebra of the Lie algebra $\mathfrak{X}(M)$ of vector fields on M, as each element $X \in \mathfrak{g}$ naturally induces a vector field on M in the following way. Let $X \in \mathfrak{g}$ and let ϕ_t be the one-parameter group of diffeomorphisms of M

$$\phi_t : M \to M, \quad \phi_t(y) = e^{tX}y. \tag{1.50}$$

The infinitesimal generator of the flow ϕ_t defines a vector field

$$X^M(y) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} e^{tX}y, \tag{1.51}$$

called the *induced vector field.* Then the map

$$\mathfrak{g} \to \mathfrak{X}(M), \quad X \mapsto X^M \tag{1.52}$$

defines an antihomomorphism of Lie algebras, meaning that $[X,Y]_{\mathfrak{g}} = -[X^M, Y^M]$. From now on we will identify $X \in \mathfrak{g}$ with the induced vector field $X^M \in \mathfrak{X}(M)$.

Let now $\mathfrak{v} \subset \mathfrak{g}$ be the Lie algebra of V in $\mathfrak{g}$. With the identification of $\mathfrak{g}$ with a subalgebra of $\mathfrak{X}(M)$, $\mathfrak{v}$ is isomorphic to the Lie subalgebra of vector fields which are symplectic, holomorphic, Killing and vanish at x. Then V acts on $\mathfrak{g}$ via the *adjoint representation*, that we recall here below. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and observe that G acts on itself by conjugation

$$\Phi_k = L_k \circ R_{k^{-1}} : G \to G, \quad h \mapsto khk^{-1}, \tag{1.53}$$

where L_k and $R_{k^{-1}}$ are the left and right multiplication by $k \in G$ and k^{-1} respectively. Define the homomorphism Ad_k as the differential of Φ_k at the identity $e \in G : \mathrm{Ad}_k = \mathrm{d}_e\Phi_k : \mathfrak{g} \to \mathfrak{g}$. Then the *adjoint representation* of G is defined as the representation

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}), \quad k \mapsto \mathrm{Ad}_k. \tag{1.54}$$

Taking again the derivative of Ad at the identity $e \in G$ one may define the *adjoint representation* of the Lie algebra $\mathfrak{g}$,

$$\begin{aligned} \mathrm{d}_e\mathrm{Ad} = \mathrm{ad} : \mathfrak{g} &\to \mathrm{Der}(\mathfrak{g}) \\ X &\mapsto \mathrm{ad}_X : Y \mapsto [X,Y], \end{aligned} \tag{1.55}$$

where $\mathrm{Der}(\mathfrak{g})$ is the set of derivations of $\mathfrak{g}$. As V is a subgroup of G, it acts on $\mathfrak{g}$ by restriction of the adjoint representation on it, that we denote $\mathrm{Ad}^V : V \to \mathrm{Aut}(\mathfrak{g})$. Moreover, as V is compact, Ad^V is completely reducible [40, Theorem 4.28] and there exists a V-invariant complement $\mathfrak{m} \subset \mathfrak{g}$ satisfying

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{m}, \tag{1.56}$$

since $\mathfrak{v}$ is V-invariant by definition. Fix once for all the complement $\mathfrak{m}$. Then, it may be identified with the tangent space to the orbit M at x in a canonical way. Indeed, consider the map which evaluates at x the induced vector field of an element $X \in \mathfrak{g}$,

$$\mathfrak{g} \to T_xM, \quad X \mapsto X^M(x). \tag{1.57}$$

Since $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{m}$ and the vector fields induced by elements in $\mathfrak{v}$ vanish at x, the evaluation (1.57) descends to an isomorphism between $\mathfrak{m}$ and T_xM. As a consequence, by evaluating the symplectic form ω and the compatible almost-complex structure J at x, one may define a linear symplectic form σ and a linear complex structure H on $\mathfrak{m}$ as

$$\sigma(X,Y) = \omega_x(X,Y), \quad HX = J_xX, \tag{1.58}$$

with $X, Y \in \mathfrak{m}$. Notice that closednenss of ω implies that σ satisfies the following relation

$$\sigma([X,Y]_{\mathfrak{m}}, Z) + \sigma([Y,Z]_{\mathfrak{m}}, X) + \sigma([Z,X]_{\mathfrak{m}}, Y) = 0, \tag{1.59}$$

for $X, Y, Z \in \mathfrak{m}$. By compatibility, it is also defined the scalar product $\langle X, Y \rangle = g_x(X, Y)$ on $\mathfrak{m}$ such that $\langle X, Y \rangle = \sigma(X, HY)$. The key point about σ and H is that they are V-invariant, for x is fixed by $K = \mathrm{Stab}(x)$, $\mathfrak{m}$ is V-invariant and ω and J are homogeneous. Again by compatibility, the induced scalar product is V-invariant too. Summing up, given an homogeneous almost-Kähler structure on a homomgeneous manifold (M, ω, J, g), one is allowed to study it just by looking at a suitable subspace of the Lie algebra of the acting Lie group. On the other hand, given a connected Lie group G and an even-dimensional compact subgroup $V \subset G$, a homogeneous almost-Kähler structure on the coset space G/V is fully determined by a suitable linear structure on the Lie algebra $\mathfrak{g}$. This statement is summarized rigorously in the following theorem.

Theorem 1.4.7 ([27, Theorem 20]). *Let G be a connected Lie group and let $V \subset G$ be an even dimensional compact subgroup of G which contains no non-discrete normal subgroups of G. Let M be the coset space G/V and denote by $\mathfrak{g}$ and $\mathfrak{v}$ the Lie algebras of G and V respectively. Fix a V-invariant subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{m}$. Then, given a V-invariant linear symplectic form σ on $\mathfrak{m}$ satisfying*

$$\sigma([X, Y]_{\mathfrak{m}}, Z) + \sigma([Y, Z]_{\mathfrak{m}}, X) + \sigma([Z, X]_{\mathfrak{m}}, Y) = 0, \tag{1.60}$$

for $X, Y, Z \in \mathfrak{m}$ and a V-invariant compatible complex structure H on $\mathfrak{m}$, it is defined a homogeneous almost-Kähler structure on M by letting

$$\omega(u, v) = \sigma(\mathrm{d}L_{h^{-1}} u, \mathrm{d}L_{h^{-1}} v), \quad Ju = \mathrm{d}L_h H \mathrm{d}L_{h^{-1}} u, \tag{1.61}$$

for all $u, v \in T_{[h]}M$.

Recall that the subgroup V as in the above theorem is called the isotropy of G/V.

Up to now, we have learnt that dealing with invariant objects allows to treat problems in an algebraic way, as all the geometric quantities related to the almost-Kähler structure may be studied at the Lie algebra level. More precisely, invariance of ω and J implies invariance of the Nijenhuis tensor N_J and the Chern-curvature R^∇. Thus, all tensors involving R^∇, J and ω may be completely written in terms of the Lie algebra $\mathfrak{g}$, the symplectic form σ and the complex structure H. Since we are principally interested in the study of the speciality condition (1.46) on homogeneous almost-Kähler manifolds, we recall formulae for the objects we are going to compute in the next chapters: the Chern-Ricci form, the Hermitian scalar curvature and the the Nijenhuis tensor of J for homogeneous almost-Käher structures. For the details about the proofs see [27, Section 3].

Proposition 1.4.8. *Let $X, Y \in \mathfrak{m}$. Then*

$$\rho_x(X, Y) = \mathrm{tr}(\mathrm{ad}_{H[X,Y]_{\mathfrak{g}}} - H \mathrm{ad}_{[X,Y]_{\mathfrak{g}}}). \tag{1.62}$$

Note that the above formula for the Chern-Ricci form at the Lie algebra level is completely analogous to the one for the homogeneous Ricci form in case the almost-complex structure induced by H is integrable [53]. Formula (1.62) may be written in terms of the Lie algebra cohomology of the acting group G. Following [47, Chapter IV, Section 3], we recall briefly few definitions and properties concerning the cohomology theory for Lie algebras, as we will use these tools for computations in the next chapters. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$. Define the vector space of *n-cochains* $C^n(\mathfrak{g}, \mathbb{R})$ as

$$C^n(\mathfrak{g}, \mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(\Lambda^n \mathfrak{g}, \mathbb{R}), \tag{1.63}$$

where $\Lambda^n\mathfrak{g}$ is the n-th exterior power of $\mathfrak{g}$, and the *coboundary operator* $\delta_n : C^n(\mathfrak{g},\mathbb{R}) \to C^{n+1}(\mathfrak{g},\mathbb{R})$ by

$$(\delta_n\alpha)(X_1,\dots,X_{n+1}) = \sum_{l=1}^{n+1}(-1)^{l+1}X_l\alpha(X_1,\dots,\hat{X}_l,\dots,X_{n+1})+ \\ + \sum_{r<s}(-1)^{r+s}\alpha([X_r,X_s],X_1,\dots,\hat{X}_r,\dots,\hat{X}_s,\dots,X_{n+1}), \quad (1.64)$$

where $\hat{X}_j$ denotes that the j-th entries is omitted. The coboundary operator δ_n is often called *Chevalley-Eilenberg differential.* Then define the spaces of n*-cocycles* and n*-coboundaries* respectively by

$$\begin{aligned} Z^n(\mathfrak{g},\mathbb{R}) =&\ker(\delta_n) \\ B^n(\mathfrak{g},\mathbb{R}) =&\mathrm{Im}(\delta_{n-1}). \end{aligned} \quad (1.65)$$

Since $\delta_n\delta_{n-1} = 0$, $B^n(\mathfrak{g},\mathbb{R}) \subset Z^n(\mathfrak{g},\mathbb{R})$, hence it makes sense to define the quotient

$$H^n(\mathfrak{g},\mathbb{R}) = Z^n(\mathfrak{g},\mathbb{R})/B^n(\mathfrak{g},\mathbb{R}), \quad (1.66)$$

called the n-th *cohomology* group of $\mathfrak{g}$ with coefficients in $\mathbb{R}$. In small dimension, the cohomology groups of $\mathfrak{g}$ are

$$\begin{aligned} H^0(\mathfrak{g},\mathbb{R}) =&\mathbb{R} \\ H^1(\mathfrak{g},\mathbb{R}) =&(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*, \end{aligned} \quad (1.67)$$

where equality (1.67) comes from the fact that $\delta_1 : \mathfrak{g}^* \to C^2(\mathfrak{g},\mathbb{R})$ is simply $(\delta_1\alpha)(X,Y) = -\alpha([X,Y])$. In the next chapter, we will repeatedly use formula (1.67) for studying the topology of certain torus bundles.

Coming back to the Chern-Ricci form, let $\zeta \in \mathfrak{g}^*$ be the linear form defined by

$$\zeta(X) = -\mathrm{tr}(\mathrm{ad}_{HX} - H\mathrm{ad}_X). \quad (1.68)$$

Then, by the discussion above, the Chern-Ricci form ρ_x at a point x turns out to be the Chevalley-Eilenberg differential of ζ, that is

$$\rho_x = \delta_1\zeta. \quad (1.69)$$

Expression (1.69) of ρ_x as differential of a linear form will be useful in the theory developed in the next chapters.

The Hermitian scalar curvature s, defined by $s\omega^n = n\rho\wedge\omega^{n-1}$, may be treated in a proper way in the context of homogeneouos spaces, since it has an explicit expression by mean of symplectic basis. Thus, let $e_1,\dots,e_n,e_{n+1},\dots,e_{2n}$ be a *symplectic basis* for σ, i.e., a basis of T_xM such that $\sigma(e_i,e_j) = \sigma(e_{i+n},e_{j+n}) = 0$ and $\sigma(e_i,e_{j+n}) = \delta_{ij}$, for $1 \le i,j \le n$. At a point $x \in M$, s has the following expression in terms of σ and ρ

$$s\sigma^n = n\rho_x \wedge \sigma^{n-1}. \quad (1.70)$$

In order to find s explicitly, we plug the symplectic basis into expression (1.70) getting

$$s\sigma^n(e_{i_1},e_{i_1+n},e_{i_2},e_{i_2+n},\dots,e_{i_n},e_{i_n+n}) = n\rho(e_{i_1},e_{i_1+n})\sigma^{n-1}(e_{i_2},e_{i_2+n},\dots,e_{i_n},e_{i_n+n}), \quad (1.71)$$

which, by definition of symplectic basis, simplifies as

$$s = n\rho(e_{i_1}, e_{i_1+n}). \tag{1.72}$$

Summing over all indices reads the formula

$$s = \sum_{i=1}^{n} \rho(e_i, e_{i+n}). \tag{1.73}$$

This explicit expression of s will be useful in the computations in Chapter 3.

The remaining geometric object that will come into our computations is the Nijenhuis tensor of J. The following proposition gives the formula of the Nijenhuis tensor for homogeneous almost-Kähler structures.

Proposition 1.4.9. *Let $X, Y \in \mathfrak{m}$. Then $N_J(X,Y)_x = N_H(X,Y)$, where*

$$N_H(X,Y) = \frac{1}{4}\left([HX, HY] - H[HX,Y] - H[X,HY] - [X,Y]\right) \tag{1.74}$$

defines the Nijenhuis tensor of the complex structure H.

To conclude this section, we stress that the aim of this thesis is to study the equation $\rho = \lambda\omega$ on *compact* symplectic manifolds endowed with a compatible almost-complex structure. However, in our context, homogeneous manifolds G/V are not compact in general. A way of making them compact, is to take the quotient of G/V by the action of a *discrete co-compact subgroup* $\Gamma \subset G$ (or *lattice*), i.e., a discrete subgroup of G without torsion such that the quotient $\Gamma\backslash G/V$ is compact. Notice that not all Lie groups admit such kinds of subgroups. Indeed, a necessary condition for G to admit lattices is to be *unimodular* [61, Lemma 6.2], meaning that the Haar measure on G is both left-invariant and right-invariant. Equivalently, G is unimodular if its Lie algebra $\mathfrak{g}$ satisfies $\mathrm{tr}(\mathrm{ad}_X) = 0 \quad \forall X \in \mathfrak{g}$. Among the unimodular Lie groups we find nilpotent and semisimple ones, which exhaust almost completely the classes of Lie groups we are going to examine.

Given a homogeneous space G/V, with G unimodular, equipped with a homogeneous almost-Kähler structure (ω, J, g), and a lattice $\Gamma \subset G$, the homogeneous almost-Kähler structure descends to an almost-Kähler structure $(\omega_\Gamma, J_\Gamma, g_\Gamma)$ on the quotient, which is called *locally homogeneous*. Also the quotient $\Gamma\backslash G/V$ takes the name of *locally homogeneous manifold*. Observe that $(\omega_\Gamma, J_\Gamma, g_\Gamma)$ is not homogeneous, as the action of Γ does not commute with the action of G in general. Since, by G-invariance of the almost-Kähler structure, also the Chern-Ricci form ρ is homogeneous on G/V, in order to study the speciality condition on locally homogeneous compact symplectic manifold, it suffices to study the homogeneous equation $\rho = \lambda\omega$ on G/V. Indeed, if it is satisfied on the covering G/V, then the induced equation $\rho_\Gamma = \lambda\omega_\Gamma$ is satisfied on the quotient $\Gamma\backslash G/V$. This will be the strategy to follow in the subsequent chapters.

Summing up, in the following chapters we will study the speciality condition on locally homogeneous compact symplectic manifolds of the form $\Gamma\backslash G/V$ by studying the homogeneous equation $\rho = \lambda\omega$ on the covering G/V.

Chapter 2

Symplectic T^2-bundles over T^2

The first class of compact symplectic manifolds that we consider is made of certain 4-dimensional fiber bundles having both base space and fiber diffeomorphic to the 2-torus T^2. They may be obtained by the action of a lattice on a 4-dimensional solvable Lie group, thus they are 4-dimensional solvmanifolds. In Section 2.1, we give basic definitions and results concerning symplectic Lie groups and solvmanifolds. In Section 2.2, we describe the geometries of symplectic T^2-bundles over T^2, which turn out to be distributed in five classes. Finally, in Section 2.3, we study the existence of special locally homogeneous compatible almost-complex structures for each of the aforementioned classes.

2.1 Compact solvmanifolds

In this section we introduce the manifolds on which we are going to study the speciality condition. They are compact symplectic manifolds that may be obtained as quotients of solvable Lie groups by discrete subgroups, and they are called solvmanifolds. We will see later on that they also admit a further bundle structure which is intertwined with the symplectic structure. For the first algebraic part we refer to [48, Chapter I].

We start with some algebraic background. A Lie algebra $\mathfrak{g}$ is said to be *solvable* if its *derived series*

$$\mathfrak{g} \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \cdots , \tag{2.1}$$

where $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}^{k-1}]$ and $\mathfrak{g}^0 = \mathfrak{g}$, terminates, i.e., there exists an integer k such that $\mathfrak{g}^k = \{0\}$. The Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if its *lower central series*

$$\mathfrak{g} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \cdots , \tag{2.2}$$

where $\mathfrak{g}^{(k)} = [\mathfrak{g}, \mathfrak{g}^{(k-1)}]$ and $\mathfrak{g}^0 = \mathfrak{g}$, terminates, i.e., there exists an integer k such that $\mathfrak{g}^{(k)} = \{0\}$. A nilpotent Lie algebra having lower central series terminating at step k is called a *k-step nilpotent* Lie algebra. Observe that a nilpotent Lie algebra is solvable, but the vice versa is false in general. Moreover, note that an abelian Lie algebra is nilpotent, as its lower central series terminates at the first step. A Lie group G is said to be solvable (respectively nilpotent or k-step nilpotent, if k is specified) if its Lie algebra $\mathfrak{g}$ is solvable (resp. nilpotent or k-step nilpotent).

Example 2.1.1. The Lie group G whose quotient gives the Kodaira-Thurston manifold (see Example 1.3.1) is a 4-dimensional 2-step nilpotent Lie group. Indeed, its Lie algebra $\mathfrak{rh}_3$ turns out to be the central extension $\mathbb{R} \oplus \mathfrak{h}_3$, where

$$\mathfrak{h}_3 = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}. \tag{2.3}$$

The generators of $\mathfrak{rh}_3$ may be chosen as

$$e_1 = E_{23}, \quad e_2 = E_{34}, \quad e_3 = E_{11}, \quad e_4 = E_{24}, \tag{2.4}$$

where E_{ij} denotes the matrix having 1 in the entry (i, j) and 0 elsewhere. One may check that the unique non-trivial commutation relation is

$$[e_1, e_2] = e_4, \tag{2.5}$$

showing that the Lie algebra $\mathfrak{rh}_3$ is nilpotent, as its lower central series terminates at step 2.

After this algebraic round up, we may introduce our principal objects of study: compact quotients of symplectic Lie groups. The goal is to study the speciality condition for compatible almost-complex structures on these manifolds. We start by setting the background. A *symplectic Lie group* G is a connected Lie group equipped with a G-invariant symplectic form ω. In particular, it is a homogeneous space with trivial isotropy. As we discussed in Section 1.4, the symplectic form ω is then completely determined by the linear symplectic form σ on the Lie algebra $\mathfrak{g}$ of G that satisfies

$$\sigma([X, Y]_{\mathfrak{g}}, Z) + \sigma([Y, Z]_{\mathfrak{g}}, X) + \sigma([Z, X]_{\mathfrak{g}}, Y) = 0. \tag{2.6}$$

Symplectic unimodular Lie groups have been extensively studied by Chu in [23]. He proved that unimodular symplectic Lie groups are solvable, and moreover that in dimension 4 the same result holds even without unimodularity assumption on G.

Remark 2.1.2. Since Lie groups are parallelizable, a symplectic Lie group has trivial first Chern class.

In order to get compact manifolds, we need a discrete lattice $\Gamma \subset G$ making the quotient $\Gamma \backslash G$ compact, as we explained in Section 1.4. Hence the group G needs to be unimodular. Recall that a Lie group G is unimodular if its Lie algebra $\mathfrak{g}$ satisfies

$$\mathrm{tr}(\mathrm{ad}_X) = 0 \quad \text{forall } X \in \mathfrak{g}. \tag{2.7}$$

In particular, as we observed in Section 1.4, nilpotent Lie groups are unimodular. Indeed, a nilpotent Lie group has a nilpotent Lie algebra that, by Engel's Theorem [48, Theorem 1.35], embeds in the Lie algebra of strictly upper-triangular matrices, which have vanishing trace. By Chu's results [23, Theorem 9, Corollary to Theorem 11], we may restrict our attention to compact quotients of solvable Lie groups. Quotients $\Gamma \backslash G$ of connected and simply-connected solvable Lie groups by lattices are called *solvmanifolds*, or *nilmanifolds* if the Lie group is nilpotent.

Remark 2.1.3. The original definition of solvmanifold is slightly different from the one given above. Indeed, a solvmanifold is usually defined as the quotient of a connected solvable Lie group by a closed subgroup [7, 64]. However, since we will consider only solvmanifolds of the form $\Gamma \backslash G$, we decided to include the more restrictive definition in this presentation. Notice that the original definition is not equivalent to the one involving just discrete co-cocompact subgroups, as not all solvmanifolds may be written as the quotient of a solvable Lie group by a lattice. A counterexample is given by the Klein bottle [7, Chapter 3]. On the other hand, the two definitions are equivalent for nilmanifolds [58].

Nilmanifolds were introduced by Mal'cev in the seminal paper [58], while solvmanifolds appeared few years later in Mostow's work [64]. Compact solvmanifolds and nilmanifolds are very rigid from the topological point of view, since they are completely determined, up to diffeomorphism, by their fundamental group [64], which is isomorphic to Γ. Moreover, they are *aspherical* spaces, meaning that their homotopy groups π_n vanish for $n \geq 2$. This may be seen from the long exact sequence in homotopy induced by the couple (G, Γ), by using connectedness and simply-connectedness of G and discreteness of Γ. For more details, a good survey concerning the structure of solvmanifolds and nilmanifolds is Auslander's work [7].

We conclude this section with a couple of examples.

Example 2.1.4. The n-dimensional torus T^n is a compact nilmanifold. Indeed, T^n may be defined as the quotient $\mathbb{Z}^n \backslash \mathbb{R}^n$ and $\mathbb{R}^n$ is an abelian, hence nilpotent, Lie group.

Example 2.1.5. The Kodaira-Thurston manifold (Example 1.3.1) is defined as the quotient of a nilpotent Lie group G by the action of a discrete co-compact subgroup Γ, thus it is a nilmanifold. Actually, as the commutation relation (2.5) shows, it is a 2-step nilmanifold, as G is a 2-step nilpotent Lie group.

2.2 T^2-bundles over T^2

Compact solvmanifolds and nilmanifolds are examples of *geometries* in the sense of Thurston. Such a geometry is a pair (X, G) where X is a complete and simply connected Riemannian manifold and G is a group of isometries acting transitively on X that contains a discrete subgroup Γ such that $\Gamma \backslash X$ has finite volume. A manifold $\Gamma \backslash X$ is then called a *geometric manifold* modelled on the geometry X. In our cases, $X = G$ is a connected and simply-connected solvable Lie group, Γ is a discrete co-compact subgroup of G and so $\Gamma \backslash G$ is a compact geometric manifold modelled on G. In particular, this chapter is dedicated to study these geometries in dimension 4, which is the smallest non-trivial dimension to consider. Geometries in dimension 4 have been classified in nineteen families by Filipkiewicz [32] and, among these, the ones that model symplectic manifolds are the ones listed in [36] and that we report below.

(a) $X = E^4$, the Euclidean space, with $G = \mathbb{R}^4 \ltimes \mathrm{SO}(4)$, the rigid motions of $\mathbb{R}^4$.

(b) $X = Nil^3 \times E^1$, $G = X$, where $Nil^3 \times E^1$ is the 2-step nilpotent Lie group defined in Example 1.3.1

$$Nil^3 \times E^1 = \left\{ \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & 1 & x & z \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z, t \in \mathbb{R} \right\} \subset GL(4, \mathbb{R}). \tag{2.8}$$

Notice that the Kodaira-Thurston manifold is a geometric manifold modelled on $Nil^3 \times E^1$.

(c) $X = Sol^3 \times E^1$, $G = X$, where $Sol^3 \times E^1$ is the solvable Lie group

$$Sol^3 \times E^1 = \left\{ \begin{pmatrix} e^z & 0 & 0 & 0 \\ 0 & e^x & 0 & y \\ 0 & 0 & e^{-x} & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z, t \in \mathbb{R} \right\}. \tag{2.9}$$

Given $x_0, x \in Sol^3 \times E^1$, $x_0 = (x_0, y_0, z_0, t_0)$, $x = (x, y, z, t)$, the left multiplication in coordinates is given by

$$x_0 \cdot x = (x_0 + x, y_0 + ye^{x_0}, z_0 + z, t_0 + te^{-x_0}). \tag{2.10}$$

(d) $X = Sol_1^3 \times E^1$, $G = X$, where $Sol_1^3 \times E^1$ is the solvable Lie group

$$Sol_1^3 \times E^1 = \left\{ \begin{pmatrix} e^z & 0 & 0 & 0 \\ 0 & \cos(x) & -\sin(x) & y \\ 0 & \sin(x) & \cos(x) & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, z, t \in \mathbb{R} \right\}. \tag{2.11}$$

Actually, this geometry does not appear in Geiges' list, but it needs to be considered in the aim of finding homogeneous Chern-Ricci flat almost-complex structures on symplectic Lie groups. However, geometries modelled on this group are all diffeomorphic to the 4-torus T^4, as we will see in Section 2.3.3.

(e) $X = Nil^4$, $G = X$, where Nil^4 is the 3-step nilpotent Lie group

$$Nil^4 = \left\{ \begin{pmatrix} 1 & t & \frac{t^2}{2} & y \\ 0 & 1 & t & z \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, z, t \in \mathbb{R} \right\}. \tag{2.12}$$

Given $x_0, x \in Nil^4$, $x_0 = (x_0, y_0, z_0, t_0)$, $x = (x, y, z, t)$, the coordinates for the left multiplication are

$$x_0 \cdot x = \left(x_0 + x, y_0 + y + t_0 z + \frac{1}{2} t_0^2 x, z_0 + z + t_0 x, t_0 + t \right) \tag{2.13}$$

Observe that the Lie algebras of the Lie groups itemized above correspond to the 4-dimensional symplectic unimodular Lie algebras classified by Ovando in [69]. We list them in Table 2.1, following the notation of the paper.

Lie algebra	Geometry	Relations	Type
$\mathbb{R}^4$	(a)	trivial	abelian
$\mathfrak{rh}_3$	(b)	$[e_1, e_2] = e_4$	2-step nilpotent
$\mathfrak{rr}_{3,-1}$	(c)	$[e_1, e_2] = e_2, \; [e_1, e_4] = -e_4$	solvable
$\mathfrak{rr}'_{3,0}$	(d)	$[e_1, e_2] = -e_4, \; [e_1, e_4] = e_2$	solvable
$\mathfrak{n}_4$	(e)	$[e_4, e_1] = e_3, \; [e_4, e_3] = e_2$	3-step nilpotent

Table 2.1: Unimodular symplectic Lie algebras of dimension 4. In column 2 we specify to which of the geometries listed above the Lie algebra corresponds to, while in column 3 we find the non-trivial commutation relations of the Lie algebra. The last column indicates whether the Lie algebra is abelian, solvable or nilpotent and in the last case we also write the step at which the lower central series vanishes.

$\mathbb{R}^4$ is the 4-dimensional abelian Lie algebra and we will not consider this case in our analysis, as compact quotients are all diffeomorphic to the 4-torus T^4. The Lie algebra $\mathfrak{rh}_3$ has been discussed in Example 2.1.1 and is the Lie algebra of the Lie group giving the Kodaira-Thurston manifold. $\mathfrak{rr}'_{3,0}$ is the central extension of the Lie algebra of the rigid motions of $\mathbb{R}^2$, while $\mathfrak{n}_4$ is often reffered in literature as *filiform* Lie algebra [46, Section 2].

Geometric manifolds modelled on the geometries (a), (b), (c), (d), (e) admit the structure of orientable T^2-bundle over T^2 and, except for (d), they have been classified in [81] through the classification of T^2-bundles over T^2 [71]. For the sake of completeness, we recall briefly the technical construction of T^2-bundles over T^2 [71]. Let $A, B \in GL(2,\mathbb{Z})$ be two commuting matrices and let m, n be two integers. A T^2-bundle over T^2 denoted by $\pi : \{A, B, (m, n)\} \to S$ is constructed as follows. Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the point of the torus $T^2 = \mathbb{Z}^2 \backslash \mathbb{R}^2$ corresponding to $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Let $F = T^2$, $S = T^2$ and define $\{A, B, (0, 0)\}$ to be $F \times \mathbb{R}^2/\sim$, where

$$\begin{aligned} \left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{pmatrix} x+1 \\ y \end{pmatrix}\right) &\sim \left(\left[A\begin{pmatrix} s \\ t \end{pmatrix}\right], \begin{pmatrix} x \\ y \end{pmatrix}\right) \\ \left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{pmatrix} x \\ y+1 \end{pmatrix}\right) &\sim \left(\left[B\begin{pmatrix} s \\ t \end{pmatrix}\right], \begin{pmatrix} x \\ y \end{pmatrix}\right). \end{aligned} \tag{2.14}$$

Denote the point of $\{A, B, (0, 0)\}$ corresponding to $\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right)$ by $\begin{bmatrix} s, & x \\ t, & y \end{bmatrix}$. Then $\pi : \{A, B, (0, 0)\} \to S$ is a T^2-bundle over T^2, with projection π defined by $\pi \begin{bmatrix} s, & x \\ t, & y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. Now let D be a small disk in S of radius ε centered at $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ and define $\{A, B, (m, n)\}$ by

$$\{A, B, (m, n)\} = (\{A, B, (0, 0)\} \setminus \pi^{-1}(\mathrm{Int}(D)) \cup (F \times D), \tag{2.15}$$

where $F \times \partial D$ is glued to $\pi^{-1}(\partial D)$ via the homeomorphism

$$\begin{aligned} \pi^{-1}(\partial D) &\to F \times \partial D \\ \left(\begin{bmatrix} s \\ t \end{bmatrix}, \varepsilon(\theta)\right) &\mapsto \left(\begin{bmatrix} s + m\theta/2\pi \\ t + n\theta/2\pi \end{bmatrix}, [\varepsilon(\theta)]\right), \end{aligned} \tag{2.16}$$

where $\varepsilon(\theta) = \begin{pmatrix} 1/2 + \varepsilon\cos(\theta) \\ 1/2 + \varepsilon\sin(\theta) \end{pmatrix}$. The projection $\pi : \{A, B, (m, n)\} \to S$ is then defined as

$$\begin{aligned} \pi \begin{bmatrix} s, & x \\ t, & y \end{bmatrix} &= \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{if } \begin{bmatrix} x \\ y \end{bmatrix} \notin D \\ \pi\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{if } \begin{bmatrix} x \\ y \end{bmatrix} \in D. \end{aligned} \tag{2.17}$$

Then $\pi : \{A, B, (m, n)\} \to S$ is a T^2-bundle over T^2. A and B are called the *monodromy matrices* of the bundle, while (m, n) is the *Euler class*.

Remark 2.2.1. As we will see in Section 2.3.3, geometries of type (d) are all diffeomorphic to the 4-torus T^4, thus T^2-bundles over T^2 modelled on $Sol_1^3 \times E^1$ are all equivalent to the ones modelled on $\mathbb{R}^4$. This is why the geometry $Sol_1^3 \times E^1$ does not appear in Ue's classification.

The reason why we are interested in orientable torus bundles is that they are compact manifolds admitting a symplectic structure [36, Theorem 1], hence they are suitable spaces for studying the speciality condition. In addition, these bundles have further geometric structure. Indeed most of them turn out to be *symplectic fibrations*, meaning that the symplectic structure on the total space restricts to a symplectic form on each fiber. More precisely, except for cases (b) and (e), the fibration as torus bundle is unique and it may always be chosen to be symplectic. In case (b) there are two distinct ways of writing the torus bundle and one of these is a symplectic fibration. Finally, if the total space is modelled on the geometry (e), then it may not be written as symplectic fibration. On the contrary, the torus fibration is always a *Lagrangian fibration*, meaning that the symplectic form vanishes on each fiber.

2.3 Chern-Ricci flatness condition on symplectic unimodular Lie algebras

Now that we have introduced our objects and described their structure, we may study the speciality conditions on them. Let $M = \Gamma\backslash G$ be an orientable T^2-bundle over T^2, where G is a connected and simply-connected solvable Lie group and $\Gamma \subset G$ is a discrete co-compact subgroup. As we observed in Section 1.4, in order to study the speciality condition for locally homogeneous almost-Kähler structures, it suffices to study the homogeneous equation $\rho = \lambda\omega$ on the covering. More precisely, given ω a homogeneous symplectic structure and J a homogeneous compatible almost-complex structure on G with Chern-Ricci form ρ, it holds that ρ is homogeneous too. Hence the condition $\rho = \lambda\omega$ on G descends to a locally homogeneous equation $\rho_\Gamma = \lambda\omega_\Gamma$ on $\Gamma\backslash G$, and if it is satisfied on the cover G, then it is satisfied also on the quotient. In addition, by G-invariance, the homogeneous equation $\rho = \lambda\omega$ on G may be studied as an algebraic equation on the Lie algebra $\mathfrak{g}$ of G. More precisely, there is an explicit formula for the Chern-Ricci form for homogeneous almost-Kähler structures (1.62) that we recall in a while. Following the notation of Section 1.4, call σ and H the symplectic linear form and the linear complex structure on $T_xG \cong \mathfrak{g}$ respectively obtained by evaluating at $x \in G$ the symplectic form ω and the almost-complex structure J. The Chern-Ricci form at the point x may be written as

$$\rho_x(X, Y) = \mathrm{tr}(\mathrm{ad}_{H[X,Y]} - H\mathrm{ad}_{[X,Y]}), \tag{2.18}$$

for all $X, Y \in \mathfrak{g}$. If in addition G is assumed to be unimodular, by (2.7), the Chern-Ricci form simplifies to

$$\rho_x(X, Y) = -\mathrm{tr}(H\mathrm{ad}_{[X,Y]}). \tag{2.19}$$

Observe that a solution to equation $\rho = \lambda\omega$ on G may exist only with $\lambda = 0$. Indeed, by Remark 2.1.2, ρ is an exact 2-form, hence its cohomology class is trivial in $H^2_{\mathrm{dR}}(G, \mathbb{R})$. If $\rho = \lambda\omega$ is satisfied, then also ω is exact on G. However, by G-invariance, also its projection on the locally homogeneous compact manifold $\Gamma\backslash G$ is exact, a contradiction as a symplectic form on a compact symplectic manifold cannot be exact. We are then reduced to study the homogeneous equation $\rho = 0$ on connected and simply-connected solvable Lie groups, or, in other words, the equation

$$\mathrm{tr}(H\mathrm{ad}_{[X,Y]}) = 0 \tag{2.20}$$

on symplectic unimodular Lie algebras, which are the ones reported in Table 2.1. We often call *Chern-Ricci flat* the symplectic manifolds having identically zero Chern-Ricci form.

Remark 2.3.1. Observe that all symplectic linear forms σ on the Lie algebra $\mathfrak{g}$ of G are linearly symplectomorphic, as one may always find a *σ-standard basis* [60, Theorem 2.1.3]. In order words, one may always find an isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ such that $\psi^*\sigma = \omega_0$, where ω_0 is the linear standard symplectic form (1.1). However, not all the symplectic forms induced on the quotient $\Gamma\backslash G$ are symplectomorphic, as ψ does not commute with action of Γ in general.

In the following sections, for each T^2-bundle over T^2 of geometric type (b), (c), (d), (e), we first describe the underlying Lie algebra and the de Rham cohomology of the bundle. Then we discuss whether, in each 2-cohomology class of the compact symplectic manifold $\Gamma\backslash G$, there exists a locally homogeneous almost-Kähler structure for which $\rho = 0$. In other words, we discuss whether in each 2-cohomology class of the given Lie group, there is a symplectic form ω and a ω-compatible almost-complex structure J, such that their corresponding evaluation σ and H satisfy the equation $\mathrm{tr}(H\mathrm{ad}_{[X,Y]}) = 0$. Finally, we make some remarks about integrability of compatible almost-complex structures.

2.3.1 Geometric type $Nil^3 \times E^1$

We amply discussed the geometry of $Nil^3 \times E^1$, mostly through examples. However, for the sake of clarity, we repeat definitions and results. The Lie algebra underlying the geometry $(Nil^3 \times E^1, Nil^3 \times E^1)$ is $\mathfrak{rh}_3$, the 4-dimensional 2-step nilpotent Lie algebra generated by e_1, e_2, e_3, e_4 with non-trivial commutation relation

$$[e_1, e_2] = e_4. \tag{2.21}$$

The generators may be represented through the matrices

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$e_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.22}$$

Let $\Gamma \subset Nil^3 \times E^1$ be a lattice. For the topology of $\Gamma\backslash Nil^3 \times E^1$ notice that

$$H^0_{\mathrm{dR}}(\Gamma\backslash Nil^3 \times E^1) \cong \mathbb{R}, \quad H^4_{\mathrm{dR}}(\Gamma\backslash Nil^3 \times E^1) \cong \mathbb{R}, \tag{2.23}$$

as $\Gamma\backslash Nil^3 \times E^1$ is connected and orientable, thus the 0-th and 4-th Betti numbers are $b_0 = b_4 = 1$. The cohomology group $H^1_{\mathrm{dR}}(\Gamma\backslash Nil^3 \times E^1, \mathbb{R})$ may be understood in terms of the Lie algebra cohomology of $\mathfrak{rh}_3$. Indeed, by Nomizu's results [68], the de Rham cohomology of the nilmanifold $\Gamma\backslash Nil^3 \times E^1$ is isomorphic to the Lie algebra cohomology of $\mathfrak{rh}_3$. So, by identity (1.67),

$$H^1(\mathfrak{rh}_3, \mathbb{R}) = (\mathfrak{rh}_3/[\mathfrak{rh}_3, \mathfrak{rh}_3])^* \cong (\mathrm{span}\{e_1, e_2, e_3\})^* \cong \mathbb{R}^3, \tag{2.24}$$

which implies $H^1_{\mathrm{dR}}(\Gamma\backslash Nil^3 \times E^1) \cong \mathbb{R}^3$ and $b_1 = 3$. By Poincarè duality, also $b_3 = 3$, as $\Gamma\backslash Nil^3 \times E^1$ is compact and orientable. Finally, since the quotient $\Gamma\backslash Nil^3 \times E^1$ is a T^2-bundle over T^2, the Euler characteristic $\chi(\Gamma\backslash Nil^3 \times E^1)$ of $\Gamma\backslash Nil^3 \times E^1$ turns out to be the product $\chi(\Gamma\backslash Nil^3 \times E^1) = \chi(T^2)\chi(T^2)$, by the fibration property of the Euler characteristic [62, Section 9]. However, the Euler characteristic of the torus vanishes, implying $\chi(\Gamma\backslash Nil^3 \times E^1) = 0$. By definition,

$$0 = \chi(\Gamma\backslash Nil^3 \times E^1) = \sum_{i=0}^{4} (-1)^i b_i = b_2 - 4, \tag{2.25}$$

hence $b_2 = 4$.

For what concerns Chern-Ricci flatness, notice that a result of Vezzoni [82] says that any homogeneous compatible almost-complex structure on a symplectic 2-step nilmanifold is Chern-Ricci flat. As $\Gamma\backslash Nil^3 \times E^1$ is a 2-step nilmanifold, every locally homogeneous compatible almost-complex structure on it is Chern-Ricci flat. Recall that, for $\Gamma = (Nil^3 \times E^1) \cap \mathbb{Z}^4$, the manifold $\Gamma\backslash Nil^3 \times E^1$ is the Kodaira-Thurston manifold discussed in Example 1.3.1.

Finally, as we proved above, the first Betti number b_1 of $\Gamma\backslash Nil^3 \times E^1$ is equal to 3, hence $\Gamma\backslash Nil^3 \times E^1$ cannot be a Kähler manifold. This shows that there are no integrable compatible almost-complex structures once we have fixed the symplectic form on $\Gamma\backslash Nil^3 \times E^1$. On the other hand, it is well known [8, Chapter V, Section *Fibre bundles*] that the Kodaira-Thurston manifold admits integrable complex structures, but they cannot be compatible with a chosen symplectic form.

2.3.2 Geometric type $Sol^3 \times E^1$

The Lie algebra underlying the geometry $(Sol^3 \times E^1, Sol^3 \times E^1)$ is $\mathfrak{r}\mathfrak{r}_{3,-1}$, the 4-dimensional solvable Lie algebra generated by e_1, e_2, e_3, e_4 with commutation relations

$$[e_1, e_2] = e_2, \quad [e_1, e_4] = -e_4. \tag{2.26}$$

The generators may be represented as

$$\begin{aligned} e_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ e_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.27}$$

and the adjoint representation of an element $Z \in \mathfrak{g}$, $Z = \sum_{i=1}^4 Z^i e_i$, has the form

$$\mathrm{ad}_Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -Z^2 & Z^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Z^4 & 0 & 0 & -Z^1 \end{pmatrix}. \tag{2.28}$$

The topology of $\Gamma\backslash Sol^3 \times E^1$ may be fully understood in terms of the Lie algebra $\mathfrak{r}\mathfrak{r}_{3,-1}$. Indeed, the Lie algebra $\mathfrak{r}\mathfrak{r}_{3,-1}$ is *split-solvable*, thus by the result of Hattori [43], the de Rham cohomology of the solvmanifold $\Gamma\backslash Sol^3 \times E^1$ is isomorphic to the Lie algebra cohomology of $\mathfrak{r}\mathfrak{r}_{3,-1}$. First, a Lie algebra $\mathfrak{g}$ is said to be split-solvable, or *completely solvable*, if the eigenvalues of ad_Z are real numbers for each $Z \in \mathfrak{g}$. In our case, the adjoint of an element $Z = \sum_{i=1}^4 Z^i e_i$ in $\mathfrak{r}\mathfrak{r}_{3,-1}$ has matrix representation (2.28) which has eigenvalues $\{0, \pm Z^1\} \subset \mathbb{R}$. As this holds for each Z, $\mathfrak{r}\mathfrak{r}_{3,-1}$ is split-solvable. Actually, we know that $H^0_{\mathrm{dR}}(\Gamma\backslash Sol^3 \times E^1, \mathbb{R})$ and $H^4_{\mathrm{dR}}(\Gamma\backslash Sol^3 \times E^1, \mathbb{R})$ are isomorphic to $\mathbb{R}$ as $\Gamma\backslash Sol^3 \times E^1$ is connected and symplectic, hence orientable, so $b_0 = b_4 = 1$. For the first cohomology group we need the Lie algebra cohomology of $\mathfrak{r}\mathfrak{r}_{3,-1}$. By the results of Hattori, $H^1_{\mathrm{dR}}(\Gamma\backslash Sol^3 \times E^1, \mathbb{R}) \cong H^1(\mathfrak{r}\mathfrak{r}_{3,-1}, \mathbb{R})$, so, by (1.67),

$$H^1(\mathfrak{r}\mathfrak{r}_{3,-1}, \mathbb{R}) = (\mathfrak{r}\mathfrak{r}_{3,-1}/[\mathfrak{r}\mathfrak{r}_{3,-1}, \mathfrak{r}\mathfrak{r}_{3,-1}])^* \cong (\mathrm{span}\{e_1, e_3\})^* \cong \mathbb{R}^2, \tag{2.29}$$

which implies $H^1_{\mathrm{dR}}(\Gamma\backslash Sol^3 \times E^1, \mathbb{R}) \cong \mathbb{R}^2$ and $b_1 = 2$. Since $\Gamma\backslash Sol^3 \times E^1$ is compact and orientable, Poincarè duality reads $b_1 = b_3$. Finally, as we argued for $\Gamma\backslash Nil^3 \times E^1$, the Euler characteristic $\chi(\Gamma\backslash Sol^3 \times E^1)$ vanishes as $\Gamma\backslash Sol^3 \times E^1$ is a T^2-bundles over T^2 and the 2-torus has vanishing Euler characteristic. Then

$$0 = \chi(\Gamma\backslash Sol^3 \times E^1) = \sum_{i=0}^{4} (-1)^i b_i = b_2 - 2, \tag{2.30}$$

reads $b_2 = 2$.

Now we may discuss Chern-Ricci flatness condition. First we need to parametrize the homogeneous symplectic forms for varying the 2-cohomology class in $H^2_{\mathrm{dR}}(\Gamma\backslash Sol^3 \times E^1, \mathbb{R})$. In order to do this, observe that a basis of $Sol^3 \times E^1$-invariant 1-forms on $Sol^3 \times E^1$ is given by

$$e^1 = e^{-t}\mathrm{d}x, \quad e^2 = \mathrm{d}y, \quad e^3 = e^t\mathrm{d}z, \quad e^4 = \mathrm{d}t, \tag{2.31}$$

where the coordinates are as in (2.9). Then

$$e^1 \wedge e^3 = \mathrm{d}x \wedge \mathrm{d}z, \quad e^2 \wedge e^4 = \mathrm{d}y \wedge \mathrm{d}t \tag{2.32}$$

are $Sol^3 \times E^1$-invariant closed 2-forms whose cohomology classes generate $H^2_{\mathrm{dR}}(\Gamma\backslash Sol^3 \times E^1, \mathbb{R}) \cong \mathbb{R}^2$. Thus, a cohomology class $a \in H^2_{\mathrm{dR}}(\Gamma\backslash Sol^3 \times E^1, \mathbb{R})$ may be written as

$$a = \alpha[e^1 \wedge e^3] + \beta[e^2 \wedge e^4], \tag{2.33}$$

and it contains a symplectic form if and only if $a^2 \neq 0$, that is, $\alpha\beta \neq 0$. Hence, the generic homogeneous symplectic form may be written as

$$\sigma = \alpha e^1 \wedge e^3 + \beta e^2 \wedge e^4, \tag{2.34}$$

with α and β different from 0.

Choose the standard linear complex structure on $\mathfrak{rr}_{3,-1}$

$$H = e_3 \otimes e^1 - e_1 \otimes e^3 + e_4 \otimes e^2 - e_2 \otimes e^4, \tag{2.35}$$

and observe that it is compatible with σ and the induced scalar product on $\mathfrak{rr}_{3,-1}$ is

$$\langle\ ,\ \rangle = \alpha e^1 \otimes e^1 + \beta e^2 \otimes e^2 + \alpha e^3 \otimes e^3 + \beta e^4 \otimes e^4. \tag{2.36}$$

Then, one may compute $\mathrm{tr}(H\mathrm{ad}_{[X,Y]})$ and observe that it vanishes identically for each $X, Y \in \mathfrak{rr}_{3,-1}$, showing that the homogeneous almost-complex structure J induced by H on the Lie group $Sol^3 \times E^1$ is Chern-Ricci flat. Thus, also the induced structure on the quotient $\Gamma\backslash Sol^3 \times E^1$, for a lattice $\Gamma \subset Sol^3 \times E^1$, is Chern-Ricci flat. As this result holds for each α and β different from 0, we have found that in each cohomology class of $H^2_{\mathrm{dR}}(\Gamma\backslash Sol^3 \times E^1, \mathbb{R})$ for which $\alpha\beta \neq 0$ there exists a symplectic form ω admitting a Chern-Ricci flat ω-compatible almost-complex structure.

As regards integrability, since $\mathfrak{rr}_{3,-1}$ is a split-solvable Lie algebra, we say that $\Gamma\backslash Sol^3 \times E^1$ is a solvmanifold of *completely solvable type*. A compact solvmanifold of completely solvable type has a Kähler structure if and only of it is a complex torus [42]. Thus, by Geiges' classification [36], $\Gamma\backslash Sol^3 \times E^1$ is not a complex torus and so it cannot be a Kähler manifold.

2.3.3 Geometric type $Sol_1^3 \times E^1$

The Lie algebra underlying the geometry $(Sol_1^3 \times E^1, Sol_1^3 \times E^1)$ is $\mathfrak{rr}'_{3,0}$, that is, the 4-dimensional solvable Lie algebra generated by e_1, e_2, e_3, e_4 with commutation relations

$$[e_1, e_2] = -e_4, \quad [e_1, e_4] = e_2. \tag{2.37}$$

The generators may be represented as

$$\begin{aligned} e_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ e_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.38}$$

The adjoint representation of an element $Z \in \mathfrak{g}$, $Z = \sum_{i=1}^4 Z^i e_i$, takes the form

$$\mathrm{ad}_Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -Z^4 & 0 & 0 & Z^1 \\ 0 & 0 & 0 & 0 \\ Z^2 & -Z^1 & 0 & 0 \end{pmatrix}, \tag{2.39}$$

while the commutator of two elements $X, Y \in \mathfrak{g}$, $X = \sum_{i=1}^4 X^i e_i$, $Y = \sum_{i=1}^4 Y^i e_i$ has coordinates

$$[X, Y] = (X^1Y^4 - X^4Y^1)e_2 + (-X^1Y^2 + X^2Y^1)e_4. \tag{2.40}$$

Let now e^1, e^2, e^3, e^4 be the dual basis of e_1, e_2, e_3, e_4 and consider the standard symplectic form $\sigma = e^1 \wedge e^3 + e^2 \wedge e^4$ defined in [69] and the compatible linear complex structure

$$H = e_3 \otimes e^1 - e_1 \otimes e^3 + e_4 \otimes e^2 - e_2 \otimes e^4 \tag{2.41}$$

on $\mathfrak{rr}_{3,-1}$. With some standard computations involving the formula in Proposition 1.4.9, one may check that the Nijenhuis tensor of H vanishes identically. So the almost-complex structure J induced by H on $Sol_1^3 \times E_1$ descends to an integrable almost-complex structure on $\Gamma\backslash Sol_1^3 \times E_1$, making it a Kähler manifold. In particular, $\Gamma\backslash Sol_1^3 \times E_1$ turns out to be diffeomorphic to the product $T^2 \times T^2$. This follows from Wall's result [83] (and the correction [54]) which says that the unique compact homogeneous Kähler solvmanifolds are complex tori. By concluding, $\Gamma\backslash Sol_1^3 \times E_1$ is diffeomorphic to the Kähler manifold T^4, and its Betti numbers are given by $b_i = \binom{4}{i}$: $b_0 = b_4 = 1$, $b_1 = b_3 = 4$ and $b_2 = 6$.

For Chern-Ricci flatness, choose the standard linear complex structure H which is compatible with the standard linear symplectic form σ. Then

$$\mathrm{tr}(H\mathrm{ad}_Z) = 2Z^1, \tag{2.42}$$

but one may check from (2.40) that the commutator of two elements has vanishing first component, so

$$\mathrm{tr}(H\mathrm{ad}_{[X,Y]}) = 0, \quad \forall X, Y \in \mathfrak{g}. \tag{2.43}$$

Thus, H is a Chern-Ricci flat linear complex structure inducing a locally homogeneous Chern-Ricci flat integrable complex structure on $\Gamma\backslash Sol_1^3 \times E_1$, which is diffeomorphic to the 4-torus T^4.

2.3.4 Geometric type Nil^4

The Lie algebra underlying the geometry (Nil^4, Nil^4) is $\mathfrak{n}_4$, the 4-dimensional 3-step nilpotent Lie algebra generated by e_1, e_2, e_3, e_4 with commutation relations

$$[e_4, e_1] = e_3, \quad [e_4, e_3] = e_2. \tag{2.44}$$

Therefore, the only non-trivial structure constants are

$$c^3{}_{41} = 1, \quad c^2{}_{43} = 1. \tag{2.45}$$

The generators may be represented as

$$\begin{aligned} e_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ e_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad e_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{2.46}$$

while the commutator of two elements $X, Y \in \mathfrak{g}$, $X = \sum_{i=1}^4 X^i e_i$, $Y = \sum_{i=1}^4 Y^i e_i$ has coordinates

$$[X, Y] = (X^4 Y^3 - X^3 Y^4) e_2 + (X^4 Y^1 - X^1 Y^4) e_3. \tag{2.47}$$

As for $Nil^3 \times E^1$, the topology of the nilmanifold $\Gamma \backslash Nil^4$ may be understood in terms of the Lie algebra cohomology of $\mathfrak{n}_4$. Actually, we know that $H^0_{\mathrm{dR}}(\Gamma \backslash Nil^4) \cong \mathbb{R}$ and $H^4_{\mathrm{dR}}(\Gamma \backslash Nil^4) \cong \mathbb{R}$ as $\Gamma \backslash Nil^4$ is connected and orientable. Thus, $b_0 = b_4 = 1$. In order to compute b_1, notice that

$$H^1(\mathfrak{n}_4, \mathbb{R}) = (\mathfrak{n}_4 / [\mathfrak{n}_4, \mathfrak{n}_4])^* \cong (\mathrm{span}\{e_1, e_4\})^* \cong \mathbb{R}^2, \tag{2.48}$$

so $H^1_{\mathrm{dR}}(\Gamma \backslash Nil^4) \cong \mathbb{R}^2$, by Nomizu's result [68], and $b_1 = 2$. By Poincarè duality, also $b_3 = 2$, since $\Gamma \backslash Nil^4$ is compact and orientable. For b_2, as in the previous cases, the Euler characteristic $\chi(\Gamma \backslash Nil^4)$ of $\Gamma \backslash Nil^4$ vanishes and equation

$$0 = \chi(\Gamma \backslash Nil^4) = \sum_{i=0}^{4} (-1)^i b_i = b_2 - 2, \tag{2.49}$$

reads $b_2 = 2$.

Remark 2.3.2. By Theorem [36, Theorem 1], Nil^4 is the unique geometry such that torus bundles modelled on it do not admit the structure of symplectic fibrations.

Now we discuss Chern-Ricci flatness condition. We anticipate that $\mathfrak{n}_4$ does not admit σ-compatible linear complex structures that are Chern-Ricci flat, for σ varying in the cohomology classes of $H^2_{\mathrm{dR}}(\Gamma \backslash Nil^4, \mathbb{R})$. In order to prove this fact, we have to show that for each σ-compatible linear complex structure H on $\mathfrak{n}_4$, the quantity $\mathrm{tr}(H \mathrm{ad}_{[X,Y]})$ is non zero for each $X, Y \in \mathfrak{n}_4$. First, as above, we need to parametrize the homogeneous symplectic structures on Nil^4, for varying the cohomology class. To do this, observe that a basis of Nil^4-invariant 1-forms on Nil^4 is given by

$$e^1 = \mathrm{d}x, \quad e^2 = \frac{t^2}{2} \mathrm{d}x + \mathrm{d}y - t \mathrm{d}z, \quad e^3 = -t \mathrm{d}x + \mathrm{d}z, \quad e^4 = \mathrm{d}t, \tag{2.50}$$

where the coordinates are as in (2.12). Then

$$e^1 \wedge e^3 = \mathrm{d}x \wedge \mathrm{d}z, \quad e^2 \wedge e^4 = \left(\frac{t^2}{2}\mathrm{d}x + \mathrm{d}y - t\mathrm{d}z\right) \wedge \mathrm{d}t \tag{2.51}$$

are Nil^4-invariant closed 2-forms whose cohomology classes generate $H^2_{\mathrm{dR}}(\Gamma\backslash Nil^4, \mathbb{R}) \cong \mathbb{R}^2$. Thus, a cohomology class $a \in H^2_{\mathrm{dR}}(\Gamma\backslash Nil^4, \mathbb{R})$ may be written as

$$a = \alpha[e^1 \wedge e^3] + \beta[e^2 \wedge e^4], \tag{2.52}$$

and it contains a symplectic form if and only if $a^2 \neq 0$, that is, $\alpha\beta \neq 0$. Thus, the generic homogeneous symplectic form may be written as

$$\sigma = \alpha e^1 \wedge e^3 + \beta e^2 \wedge e^4, \tag{2.53}$$

with α and β different from 0. By performing the change of coordinates (we always take the positive result of the square root)

$$e^1 = \frac{\mathrm{sgn}(\alpha)}{\sqrt{|\alpha|}} f^1, \quad e^2 = \frac{\mathrm{sgn}(\beta)}{\sqrt{|\beta|}} f^2, \quad e^3 = \frac{1}{\sqrt{|\alpha|}} f^3, \quad e^4 = \frac{1}{\sqrt{|\beta|}} f^4, \tag{2.54}$$

σ takes the standard form

$$\sigma = f^1 \wedge f^3 + f^2 \wedge f^4. \tag{2.55}$$

Moreover, the basis of $\mathfrak{n}_4$ and the commutation relations change as

$$e_1 = \frac{\sqrt{|\alpha|}}{\mathrm{sgn}(\alpha)} f_1, \quad e_2 = \frac{\sqrt{|\beta|}}{\mathrm{sgn}(\beta)} f_2, \quad e_3 = \sqrt{|\alpha|} f_3, \quad e_4 = \sqrt{|\beta|} f_4 \tag{2.56}$$

and

$$[f_4, f_1] = \frac{\mathrm{sgn}(\alpha)}{\sqrt{|\beta|}} f_3, \quad [f_4, f_3] = \frac{1}{\mathrm{sgn}(\beta)\sqrt{|\alpha|}} f_2, \tag{2.57}$$

so that the new structure coefficients are

$$c^3{}_{41} = \frac{\mathrm{sgn}(\alpha)}{\sqrt{|\beta|}}, \quad c^2{}_{43} = \frac{1}{\mathrm{sgn}(\beta)\sqrt{|\alpha|}}. \tag{2.58}$$

A suitable parametrization for compatible linear complex structures on a vector space is given by the one of the Siegel half-space (1.21). More precisely, a σ-compatible linear complex structure $H \in \mathcal{J}(\mathfrak{n}_4, \sigma)$ may be written as

$$H = \begin{pmatrix} XY^{-1} & -(XY^{-1}X + Y) \\ Y^{-1} & -Y^{-1}X \end{pmatrix}, \tag{2.59}$$

where X and Y are 2×2 symmetric matrices and Y is positive-definite. A fitting way of expressing X and Y is via the following matrices

$$\begin{aligned} X &= \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \\ Y &= e^s \left(\begin{pmatrix} \cosh(r) & 0 \\ 0 & \cosh(r) \end{pmatrix} + \sinh(r) \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix} \right), \end{aligned} \tag{2.60}$$

with $x_1, x_2, x_3, s, t \in \mathbb{R}$ and $r \geq 0$. In order to write Y, we used the fact that a matrix is positive definite if and only if it may be written as the exponential of another matrix, together with further computations.

Let $\zeta \in \mathfrak{n}_4^*$ be the linear form defined by $\zeta(X) = -\mathrm{tr}(H\mathrm{ad}_X)$, as in Section 1.4 (up to change the sign), and observe that

$$\rho(X,Y) = \zeta([X,Y]) = \frac{1}{\mathrm{sgn}(\beta)\sqrt{|\alpha|}}(X^4Y^3 - X^3Y^4)\zeta(f_2) + \frac{\mathrm{sgn}(\alpha)}{\sqrt{|\beta|}}(X^4Y^1 - X^1Y^4)\zeta(f_3), \tag{2.61}$$

by the form of the structure coefficients (2.58) in the new basis. In the basis of the f_i's the linear form ζ takes the values

$$\begin{aligned} \zeta(f_i) &= -\,\mathrm{tr}(H\mathrm{ad}_{f_i}) \\ &= -\sum_{j=1}^{4} f^j(H\mathrm{ad}_{f_i}(f_j)) \\ &= -\sum_{j,k=1}^{4} f^j(Hc^k{}_{ij}f_k) \\ &= -\sum_{j,k,l=1}^{4} c^k{}_{ij}f^j(H^l{}_k f_l) \\ &= -\sum_{j,k=1}^{4} c^k{}_{ij}H^j{}_k. \end{aligned} \tag{2.62}$$

By (2.61), in order to compute $\rho(X,Y)$ we need just to compute $\zeta(f_2)$ and $\zeta(f_3)$. However, $\zeta(f_2) = 0$ and

$$\zeta(f_3) = -c^2{}_{34}H^4{}_2 = \frac{1}{\mathrm{sgn}(\beta)\sqrt{|\alpha|}}H^4{}_2. \tag{2.63}$$

Hence

$$\rho(X,Y) = \frac{\mathrm{sgn}(\alpha)}{\mathrm{sgn}(\beta)\sqrt{|\alpha\beta|}}(X^4Y^1 - X^1Y^4)H^4{}_2, \tag{2.64}$$

which is zero for each X, Y if and only if $H^4{}_2 = 0$. However, the coefficient $H^4{}_2$ turns out to be

$$H^4{}_2 = \left(Y^{-1}\right)^2{}_2 = e^{-s}(\cosh(r) + \cos(t)\sinh(r)), \tag{2.65}$$

which never vanishes, as equation

$$\tanh(r) = -\frac{1}{\cos(t)} \tag{2.66}$$

has no real solutions. This shows that $\mathfrak{n}_4$ does not admit Chern-Ricci flat σ-compatible linear complex stuctures, thus neither $\Gamma\backslash Nil^4$ admits locally homogeneous Chern-Ricci flat ω-compatible almost-complex structures. As this result holds for each $\alpha, \beta \neq 0$ and in each such cohomology class of $H^2_{\mathrm{dR}}(\Gamma\backslash Nil^4, \mathbb{R})$ there exists a locally homogeneous symplectic form [36, Theorem 2], we have proved that in each such cohomology class of $H^2_{\mathrm{dR}}(\Gamma\backslash Nil^4, \mathbb{R})$ there exists a locally homogeneous symplectic form which does not admit Chern-Ricci flat compatible almost-complex structures.

Remark 2.3.3. An interesting problem is to establish whether the symplectic forms considered above on $\Gamma\backslash Nil^4$ admit *non locally homogeneous* Chern-Ricci flat compatible almost-complex structures. As there are no locally homogeneous ones, we are pushed to suspect that neither there are non locally homogeneous Chern-Ricci flat compatible almost-complex structures on $\Gamma\backslash Nil^4$. In other words, that there exist symplectic structures not admitting Chern-Ricci flat compatible almost-complex structures at all. This should be compared with the Kählerian case, in which the Calabi-Yau theorem [85] guarantees that condition $c_1 = 0$ is equivalent to have Ricci-flat compatible complex structures. We plan to come back on this point in the future.

Remark 2.3.4. Since there are no homogeneous special compatible almost-complex structures on $\Gamma\backslash Nil^4$, it is quite natural to ask whether there exist some other related objects on this manifold, namely *Chern-Einstein solitons*. A Chern-Einstein soliton is a compatible almost-complex structure J for which there exists a holomorphic vector field X (i.e., a real vector field satisfying $L_X J = 0$) such that equation

$$\rho - \lambda\omega = L_X\omega \tag{2.67}$$

is satisfied for a constant $\lambda \in \mathbb{R}$. Notice that in the case under examination the constant λ has to be 0. The answer to this question is again no if X is a holomorphic vector field generated by a derivation of $\mathfrak{n}_4$, which is a natural characteristic to require [55]. In this case, once a compatible complex structure H is fixed on $\mathfrak{n}_4$, the differential condition (2.67) may be translated in the following two algebraic equations on $\mathfrak{n}_4$:

$$\rho(X,Y) = \sigma(D(X),Y) + \sigma(X,D(Y)), \tag{2.68}$$
$$[H,D] = 0, \tag{2.69}$$

where $X, Y \in \mathfrak{n}_4$ and D is a derivation of $\mathfrak{n}_4$. With similar computations to the ones above, one may show that any derivation satisfying (2.68) actually does not satisfy condition (2.69), i.e., they are not compatible with H.

Finally, remember that a nilmanifold with a G-invariant almost-complex structure is a Kähler manifold if and only if it is a complex torus [9,41]. As $\Gamma\backslash Nil^4$ is a non-trivial bundle by Geiges' classification, compatible almost-complex structures on $\Gamma\backslash Nil^4$ cannot be integrable.

To conclude, we sum up the main results of this chapter.

Theorem 2.3.5. *Let G be a symplectic unimodular Lie group of dimension 4 and let ω be a symplectic form on it. If $G \neq Nil^4$, then there exist a homogeneous special compatible almost-complex structure on (G,ω). If $G = Nil^4$ there exist no homogeneous special compatible almost-complex structures on (G,ω).*

Corollary 2.3.6. *Let (M,ω) be a symplectic T^2-bundle over T^2. Then (M,ω) admits a special locally homogeneous compatible almost-complex structure if it admits the structure of symplectic fibration.*

Chapter 3

Adjoint orbits of semisimple Lie groups

A wide class of symplectic manifolds admitting a special compatible almost-complex structure is the one of adjoint orbits of semisimple Lie groups. The homogeneous almost-Kähler geometry of these manifolds is entirely determined by the geometry of a suitable vector that depends on the real Lie algebra structure of the underlying Lie algebra. In particular, all geometric quantities, such as the Chern-Ricci form, the Hermitian scalar curvature and the Nijenhuis tensor, may be expressed in terms of root data. This allows to deduce many properties about the geometry of adjoint orbits and algorithmically establish which ones admit a special homogeneous compatible almost-complex structure. Even if the intricate combinatorics of root systems prevents to spot a full classification of orbits admitting a special compatible almost-complex structure, many infinite families of adjoint orbits and all the exceptional ones have been classified.

The contents of this chapter are contained in [28] and are organized as follows. In Section 3.1 and Section 3.2, we recall the homogeneous almost-Kähler geometry of adjoint orbits and the structure properties of the underlying Lie algebra. In Section 3.3 and Section 3.4, we define a canonical compatible almost-complex structure on adjoint orbits of semisimple Lie groups and we discuss the speciality condition for it. Section 3.5 is dedicated to explicit formulae for the Hermitian scalar curvature and the Nijenhuis tensor of the canonical almost-complex structure, while in Section 3.6, we study compact quotients of adjoint orbits of semisimple Lie groups, with an eye on the integrability of the canonical almost-complex structure. Section 3.7 contains the main classifications results, while in Section 3.8 we sum up the results of the chapter and we discuss some open problems.

3.1 Adjoint orbits

In this section we recall some definitions and facts concerning adjoint orbits of semisimple Lie groups, mostly to introduce the main objects of study and to set up the notations. For this part we refer principally to [27].

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then G acts on the dual $\mathfrak{g}^*$ by *coadjoint action* Ad^*, defined by

$$\mathrm{Ad}^*_g\theta = \theta \circ \mathrm{Ad}_g^{-1} \tag{3.1}$$

for all $\theta \in \mathfrak{g}^*$, where $g \in G$ and Ad is the adjoint representation of G on $\mathfrak{g}$ (see Section 1.4 for the defintion of adjoint action). Fix $\theta \in \mathfrak{g}^*$ and consider its *coadjoint orbit*, that is, the orbit of θ under the coadjoint action. Observe that, by the orbit-stabilizer theorem, the coadjoint orbit of θ is diffeomorphic to the coset space $G/\mathrm{Stab}(\theta)$, with $\mathrm{Stab}(\theta)$ the stabilizer of θ under the coadjoint action. The important feature of coadjoint orbits is that they carry a canonically defined homogeneous symplectic form making them homogeneous symplectic manifolds. More precisely, let $\sigma \in \Lambda^2(\mathfrak{g})$ be the skew-symmetric bilinear form defined by

$$\sigma(X,Y) = \theta([X,Y]). \tag{3.2}$$

The canonical symplectic form $\omega \in \Omega^2(G/\mathrm{Stab}(\theta))$ on $G/\mathrm{Stab}(\theta)$ is then defined as the homogeneous symplectic form induced by σ and it is called the *Kirillov-Kostant-Souriau symplectic form* on $G/\mathrm{Stab}(\theta)$. Coadjoint orbits constitute an extensively studied class of manifolds and they are especially important in representation theory and geometric quantization. Moreover, they play an important role in the context of finding examples of homogeneous symplectic manifolds admitting special compatible almost-complex structures. Indeed, under certain hypotesis on the Lie group G and the isotropy, coadjoint orbits admit a homogeneous special ω-compatible almost-complex structure.

Theorem 3.1.1 ([27, Theorem 27])**.** *Let G be a connected semisimple Lie group and let $M \subset \mathfrak{g}^*$ be a coadjoint orbit equipped with the Kirillov-Kostant-Souriau symplectic form ω. Assume that the isotropy of M is compact and contains no non-discrete normal subgroups of G. If the first Chern class of (M,ω) satisfies $4\pi c_1 = \lambda\omega$ for some $\lambda \in \mathbb{R}$, then there exists a homogeneous special compatible almost-complex structure on (M,ω).*

In the following, we discuss some parts of the proof of Theorem 3.1.1, mostly to introduce some objects and results that we are going to use in the next sections. For more details, see the complete proof in [27, Theorem 27]. Recall that, a semisimple Lie group G is a Lie group having semisimple Lie algebra $\mathfrak{g}$, meaning that it splits as a direct sum of simple Lie algebras, i.e., non-abelian Lie algebras having no non-trivial ideals. Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$ and let (M,ω) be the coadjoint orbit of an element $\theta \in \mathfrak{g}^*$ equipped with the Kirillov-Kostant-Souriau symplectic form ω. Assume that the isotropy $V = \mathrm{Stab}(\theta)$ of M is compact. Remind that Lie algebras carry the G-invariant symmetric bilinear form defined by

$$B(X,Y) = \mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_Y), \tag{3.3}$$

for $X, Y \in \mathfrak{g}$, that takes the name of *Killing form*. This bilinear form plays an important role in the context of semisimple Lie algebras. Indeed, by Cartan's criterion, the Killing form B is non-degenerate on $\mathfrak{g}$, hence $\mathfrak{g}$ may be canonically identified with its dual $\mathfrak{g}^*$. Thus, one may associate with θ a unique element $v \in \mathfrak{g}$ such that

$$\theta(X) = B(v,X), \tag{3.4}$$

for all $X \in \mathfrak{g}$. By G-equivariance of this identification, the coadjoint action turns out to coincide with the adjoint one. Therefore, V is the isotropy subgroup of v with respect to the adjoint representation and its Lie algebra coincides with $\mathfrak{v} = \{X \in \mathfrak{g} \mid [v,X]_{\mathfrak{g}} = 0\}$. Being the Lie algebra of a compact Lie group, $\mathfrak{v}$ is a *compact Lie algebra*. Moreover, by the existence of the Cartan decomposition explained in Section 3.2, the Killing form B restricts to a negative-definite scalar product on it (see also [10, Lemma 7.36]). On the other hand, as we explained in Section 1.4, the orthogonal V-invariant complement

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid B(X,Y) = 0 \ \forall Y \in \mathfrak{v}\} \tag{3.5}$$

is canonically isomorphic to the tangent space at the identity coset $e \in G/V$ of the adjoint orbit G/V of v. Observe that $\mathfrak{v}$ is V-invariant by definition and, by compactness of V, also $\mathfrak{m}$

is V-invariant. In the next sections, we will always refer to adjoint orbits instead of coadjoint ones.

Let $2n$ be the dimension of $\mathfrak{m}$ and consider the induced splitting $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{m}$. We will denote by σ both the 2-form defined by (3.2) and its restriction to $\mathfrak{m}$. By the identification (3.4), σ may be expressed as $\sigma(X,Y) = B(v, [X,Y]_{\mathfrak{g}})$ or as

$$\sigma(X,Y) = B([v,X]_{\mathfrak{m}}, Y), \tag{3.6}$$

for $X, Y \in \mathfrak{m}$, by G-invariance of the Killing form. In particular, by V-invariance, the adjoint representation ad_v restricts to an endomorphism of $\mathfrak{m}$ which is also invertible, by definition of $\mathfrak{v}$. Thus B is non-degenerate on $\mathfrak{m}$ too. Let $T \subset V$ be the torus generated by the center of V. Since ad_v is invertible on $\mathfrak{m}$ and T is abelian, thus it has only irreducible representations of real dimension 2, the action of T on $\mathfrak{m}$ via ad_v induces the decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_n, \tag{3.7}$$

where each $\mathfrak{m}_i$ is a T-invariant symplectic subspace of dimension 2. In addition, B restricts to a definite symmetric bilinear form on each $\mathfrak{m}_i$. Put $\varepsilon_i = 1$ if B is positive-definite on $\mathfrak{m}_i$ and $\varepsilon_i = -1$ otherwise. Moreover, let $\{u_i, v_i\}$ be an orthogonal basis of $\mathfrak{m}_i$ such that $B(u_i,u_i) = B(v_i,v_i) = \varepsilon_i$. By skew-symmetry of ad_v with respect to B, one may find $\lambda_i \neq 0$ such that

$$[v,u_i] = \lambda_i v_i, \quad [v,v_i] = -\lambda_i u_i. \tag{3.8}$$

Up to exchange the role of u_i and v_i, we may assume that $\lambda_i > 0$. At this point, the endomorphism H of $\mathfrak{m}$ defined by

$$HX = \sum_{i=1}^{n} \frac{\varepsilon_i}{\lambda_i} \mathrm{ad}_v(X_i), \tag{3.9}$$

where $X \in \mathfrak{m}$ and each X_i is the component of X along $\mathfrak{m}_i$, is a V-invariant σ-compatible complex structure on $\mathfrak{m}$. In addition, the basis defined by $e_i = (1/\sqrt{\lambda_i})u_i$, $e_{i+n} = (\varepsilon_i/\sqrt{\lambda_i})v_i$, for $1 \leq i \leq n$, is a symplectic basis of $\mathfrak{m}$ satisfying $He_i = e_{i+n}$, $He_{i+n} = -e_i$. Then, by Theorem 1.4.7, H induces a homogeneous almost-complex structure J on M which is compatible with ω.

Remark 3.1.2. The homogeneous almost-complex structure J induced by H has been recently studied also by Alekseevsky and Podestà [2]. In that work, the authors classify special compatible almost-complex structures on adjoint orbits of the form G/L, with G a non-compact classical simple Lie group and L a maximal torus. In our notation, this means that the weight φ associated with the orbit lies in a general position, i.e., does not belong to any wall of the dominant Weyl chamber. In particular, they found that special compatible almost-complex structures on such orbits exist only when the Lie algebra $\mathfrak{g}$ of G is $\mathfrak{sl}(2,\mathbb{R})$, and $\lambda < 0$, or $\mathfrak{su}(p+1,p)$ with $p \geq 1$, and $\lambda = 0$ [2, Theorem 1.1]. Interestingly, they also proved that J is the unique homogeneous almost-complex structure that is compatible with ω.

As we showed in Section 1.4, the Chern-Ricci form ρ of J is determined by the linear form $\zeta \in \mathfrak{g}^*$ defined by $\zeta(X) = \mathrm{tr}(\mathrm{ad}_{HX} - H\mathrm{ad}_X)$. By semisimplicity assumption of G, $\mathfrak{g}$ is unimodular, that is, $\mathrm{tr}(\mathrm{ad}_X) = 0$ for each $X \in \mathfrak{g}$, and this implies that $\mathrm{tr}(\mathrm{ad}_{HX}) = 0$. Thus, the form ζ reduces to $\zeta(X) = -\mathrm{tr}(H\mathrm{ad}_X)$, $X \in \mathfrak{g}$. Using the symplectic basis $\{e_i\}$, by

G-invariance of the Killing form and the definitions of $\{u_i, v_i\}$ one has

$$\begin{aligned} \zeta(X) &= -\sum_{i=1}^{n} \sigma([X, e_i]_{\mathfrak{m}}, e_i) + \sigma([X, He_i]_{\mathfrak{m}}, He_i) \\ &= -\sum_{i=1}^{n} B([v, [X, e_i]_{\mathfrak{m}}]_{\mathfrak{m}}, e_i) + B([v, [X, He_i]_{\mathfrak{m}}]_{\mathfrak{m}}, He_i) \\ &= -\sum_{i=1}^{n} B(X, [[v, e_i]_{\mathfrak{m}}, e_i]_{\mathfrak{g}}) + B(X, [[v, He_i]_{\mathfrak{m}}, He_i]_{\mathfrak{g}}) \\ &= -\sum_{i=1}^{n} \frac{1}{\lambda_i}(B(X, [[v, u_i]_{\mathfrak{g}}, u_i]_{\mathfrak{g}}) + B(X, [[v, v_i]_{\mathfrak{g}}, v_i]_{\mathfrak{g}})) \\ &= 2\sum_{i=1}^{n} B(X, [u_i, v_i]_{\mathfrak{g}}). \end{aligned} \tag{3.10}$$

Hence the form ζ may be written as $\zeta(X) = B(v', X)$, with

$$v' = 2\sum_{i=1}^{n}[u_i, v_i]. \tag{3.11}$$

Observe that, by V-invariance of ζ and G-invariance of B, v' belongs to the center of $\mathfrak{v}$.

At this point, if $v' = \lambda v$ for some $\lambda \in \mathbb{R}$, then $\zeta = \lambda\theta$ and so, by the relation between ρ and ζ (1.69) (up to a sign), it holds that $\rho = \lambda\omega$, showing that J is a special compatible almost-complex structure on (M, ω). On the other hand, if J satisfies $\rho = \lambda\omega$, semisimplicity of $\mathfrak{g}$ together with compactness of V, reads that $v' = \lambda v$.

Summing up, the condition for the existence of a homogeneous special compatible almost-complex structure is translated in checking whether v' is a multiple of the given vector v:

$$\rho = \lambda\omega \iff v' = \lambda v, \tag{3.12}$$

for some $\lambda \in \mathbb{R}$. Thus, as in the previous chapter, we have rewritten the speciality condition as a pure algebraic equation on the Lie algebra $\mathfrak{g}$ of the semisimple Lie group G. In order to study this equation, we need to understand the structure of $\mathfrak{g}$, that is the topic of the next section.

3.2 The Lie algebra structure of $\mathfrak{g}$

The present section is a brief summary of the main important facts concerning the Lie algebra structure of $\mathfrak{g}$ and the structure theory of real semisimple Lie algebras. The contents we treat are discussed in [44, Chapter 3] and [48, Chapter II, Chapter VI] and we principally follow [39]. Even if the theory we develop covers the extensively studied case in which the Lie group G is compact (see for example [10, Chapter 8]), we will focus on real non-compact semisimple Lie groups.

Let G be a real non-compact semisimple Lie group with Lie algebra $\mathfrak{g}$. Then G acts on $\mathfrak{g}$ by adjoint action and let $v \in \mathfrak{g}$ be a chosen element with compact stabilizer $V \subset G$. The adjoint orbit G/V of v comes equipped with the Kirillov-Kostant-Souriau symplectic form ω. Observe that compactness of V ensures that the orbit G/V is not compact, thus it is not forced to be a Kähler manifold a priori. Indeed, orbits of compact semisimple Lie group turns out to be

Kähler manifolds by a result of Borel and Weil [73]. On the other hand, as semisimple Lie groups are unimodular, one may always get compact manifolds by modding out the orbit G/V by a discrete co-compact subgroup of G. Compact quotients of adjoint orbits will be treated extensively in Section 3.6.

In what follows, we are going to study the real Lie algebra structure of $\mathfrak{g}$. Let $\mathfrak{g}_{\mathbb{C}}$ be the complex Lie algebra obtained from $\mathfrak{g}$ by complexification, that is

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}. \tag{3.13}$$

Being $\mathfrak{g}$ a real form of $\mathfrak{g}_{\mathbb{C}}$, it is fixed by a unique complex conjugation τ on $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{k} \subset \mathfrak{g}$ be the maximal compact subalgebra of $\mathfrak{g}$ such that $\mathfrak{v} \subset \mathfrak{k}$. Then $\mathfrak{g}$ splits as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{3.14}$$

where $\mathfrak{p} = \mathfrak{p}_{\mathbb{C}} \cap \mathfrak{g}$, with $\mathfrak{p}_{\mathbb{C}}$ the $\mathrm{ad}(\mathfrak{k}_{\mathbb{C}})$-invariant complement of the complexification $\mathfrak{k}_{\mathbb{C}}$. The decomposition (3.14) is called *Cartan decomposition* of $\mathfrak{g}$ and the subspaces $\mathfrak{k}$ and $\mathfrak{p}$ satisfy the following relations

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}. \tag{3.15}$$

A *compact real form* of $\mathfrak{g}_{\mathbb{C}}$, i.e., a real form on which the restriction of the Killing form is negative definite, is given by

$$\mathfrak{g}_0 = \mathfrak{k} \oplus i\mathfrak{p}. \tag{3.16}$$

Indeed, by maximality of $\mathfrak{k}$, B is positive definite over $\mathfrak{p}$. Let τ_0 be the complex conjugation induced on $\mathfrak{g}_{\mathbb{C}}$ by $\mathfrak{g}_0$ and let $\mathfrak{h}_0$ be a maximal abelian subalgebra of $\mathfrak{k}$ such that $v \in \mathfrak{h}_0$. As the stabilizer V of v is compact, $\mathfrak{k}$ and $\mathfrak{g}$ have the same rank, thus complexifying $\mathfrak{h}_0$ provides a τ_0-invariant Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$.

The adjoint representation of $\mathfrak{h}_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$ induces a decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\mathbb{C}}^{\alpha} \tag{3.17}$$

where the *set of roots* $\Delta \subset \mathfrak{h}_{\mathbb{C}}^*$ is a finite subset of the dual $\mathfrak{h}_{\mathbb{C}}^*$ and the *root spaces*

$$\mathfrak{g}_{\mathbb{C}}^{\alpha} = \{X \in \mathfrak{g}_{\mathbb{C}} \mid [h, X] = \alpha(h)X \quad \forall h \in \mathfrak{h}_{\mathbb{C}}\} \tag{3.18}$$

have complex dimension 1. The decomposition (3.17) is called *root space decomposition*. In addition, for two roots $\alpha, \beta \in \Delta$, $[\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{\beta}] = \mathfrak{g}_{\mathbb{C}}^{\alpha+\beta}$ if $\alpha + \beta \in \Delta$, and $[\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{\beta}] = 0$ otherwise.

By compactness of $\mathfrak{g}_0$, all roots assume real values on the real vector space $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{h}_0$ [48, Corollary 6.49], hence one may regard Δ as a subset of $\mathfrak{h}_{\mathbb{R}}^*$ instead of the full dual space $\mathfrak{h}_{\mathbb{C}}^*$. Since $\mathfrak{h}_{\mathbb{R}}$ is a purely imaginary subspace of $\mathfrak{h}_{\mathbb{C}}$ with respect to the conjugation τ_0, the root spaces satisfy

$$\tau_0(\mathfrak{g}_{\mathbb{C}}^{\alpha}) = \mathfrak{g}_{\mathbb{C}}^{-\alpha}, \quad \alpha \in \Delta. \tag{3.19}$$

Before proceeding, we recall a couple of properties concerning the conjugations τ and τ_0 that will be useful to explain compact and non-compact roots.

Lemma 3.2.1. *The conjugations τ and τ_0 commute, hence $\theta = \tau\tau_0$ is an involutive automorphism of $\mathfrak{g}_{\mathbb{C}}$. Its 1 and -1 eigenspaces are $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ respectively. Moreover the adjoint action of the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ commutes with θ.*

Proof. Let $a_1, b_1 \in \mathfrak{k}$ and $a_2, b_2 \in \mathfrak{p}$. Put

$$A = a_1 + a_2 + i(b_1 + b_2) = a_1 + ib_2 + i(b_1 - ia_2) \in \mathfrak{g}_{\mathbb{C}}, \tag{3.20}$$

according to the real forms $\mathfrak{g}$ and $\mathfrak{g}_0$. Then

$$\begin{aligned} \tau\tau_0(A) =& \tau\tau_0(a_1 + ib_2 + i(b_1 - ia_2)) \\ =& \tau(a_1 + ib_2 - i(b_1 - ia_2)) \\ =& a_1 - a_2 + i(b_1 - b_2). \end{aligned} \tag{3.21}$$

On the other hand

$$\begin{aligned} \tau_0\tau(A) =& \tau_0\tau(a_1 + a_2 + i(b_1 + b_2)) \\ =& \tau_0(a_1 + a_2 - i(b_1 + b_2)) \\ =& \tau_0(a_1 - ib_2 - i(ia_2 + b_1)) \\ =& a_1 - ib_2 + i(ia_2 + b_1) \\ =& a_1 - a_2 + i(b_1 - b_2). \end{aligned} \tag{3.22}$$

Thus, $\tau_0\tau = \tau\tau_0$. The commutativity of τ and τ_0 together with the fact that they are involutions yields that the composition $\theta = \tau\tau$ is involutive too. Hence its eigenvalues are $\{\pm 1\}$. In order to compute the eigenspaces, take A defined as above. Then, it holds that $\theta(A) = A$ if and only if

$$a_1 - a_2 + i(b_1 - b_2) = a_1 + a_2 + i(b_1 + b_2), \tag{3.23}$$

hence if and only if $a_2 + ib_2 = 0$. Since $a_2 + ib_2 \in \mathfrak{p}_{\mathbb{C}}$, $\theta(A) = A$ if and only if $A \in \mathfrak{k}_{\mathbb{C}}$, showing that the 1-eigenspace of θ coincide with $\mathfrak{k}_{\mathbb{C}}$.

Similarly, $\theta(A) = -A$ if and only if

$$-a_1 + a_2 - i(b_1 - b_2) = a_1 + a_2 + i(b_1 + b_2), \tag{3.24}$$

hence if and only if $a_1 + ib_1 = 0$. Since $a_1 + ib_1 \in \mathfrak{k}_{\mathbb{C}}$, $\theta(A) = -A$ if and only if $A \in \mathfrak{p}_{\mathbb{C}}$, showing that the -1-eigenspace of θ coincide with $\mathfrak{p}_{\mathbb{C}}$.

For the last part of the statement, recall that $\mathfrak{h}_{\mathbb{C}} \subset \mathfrak{k}_{\mathbb{C}}$ and let $h \in \mathfrak{h}_{\mathbb{C}}$. We prove that θ commutes with ad_h on $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ separately, as this implies that they commute on $\mathfrak{g}_{\mathbb{C}}$. Let $X \in \mathfrak{k}_{\mathbb{C}}$. Then $\theta(X) = X$ and

$$\mathrm{ad}_h(\theta(X)) = \mathrm{ad}_h X \in \mathfrak{k}_{\mathbb{C}} \tag{3.25}$$

since $\mathfrak{k}_{\mathbb{C}}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$. But $\mathfrak{k}_{\mathbb{C}}$ is the 1-eigenspace of θ, so $\mathrm{ad}_h X = \theta(\mathrm{ad}_h X)$. This shows that $\mathrm{ad}_h(\theta(X)) = \theta(\mathrm{ad}_h(X))$. Now, let $X \in \mathfrak{p}_{\mathbb{C}}$. Then $\theta(X) = -X$ and

$$\mathrm{ad}_h(\theta(X)) = -\mathrm{ad}_h(X) \in \mathfrak{p}_{\mathbb{C}}, \tag{3.26}$$

by the properties of the Cartan decomposition (3.15). Since $\mathfrak{p}_{\mathbb{C}}$ is the -1-eigenspace of θ, $\mathrm{ad}_h(\theta(X)) = \theta(\mathrm{ad}_h X)$, and this concludes the proof. □

The involution θ defined in Lemma 3.2.1 is called the *Cartan involution* of $\mathfrak{g}$. With the previous results at hand, one has that each root space is contained in one of the eigenspaces of θ.

Proposition 3.2.2. *Each root space $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ is contained in either $\mathfrak{k}_{\mathbb{C}}$ or $\mathfrak{p}_{\mathbb{C}}$.*

Proof. Let X be a vector inside the root space $\mathfrak{g}_{\mathbb{C}}^{\alpha}$, $\alpha \in \Delta$, so that $\mathrm{ad}_h(X) = \alpha(h)X$ for every $h \in \mathfrak{h}_{\mathbb{C}}$. By Lemma 3.2.1, ad_h commutes with θ, thus

$$\begin{aligned} \mathrm{ad}_h\theta(X) =&\theta(\mathrm{ad}_h(x)) \\ =&\theta(\alpha(h)X) \\ =&\tau\tau_0(\alpha(h)X) \\ =&\alpha(h)\tau\tau_0(X) \\ =&\alpha(h)\theta(X). \end{aligned} \tag{3.27}$$

This shows that $\theta(X)$ is a root vector with respect to α. Since $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ is 1-dimensional, $\theta(X) = \mu X$, implying $\mu = \pm 1$, being θ an involution. As a consequence, X belongs to either $\mathfrak{k}_{\mathbb{C}}$ or $\mathfrak{p}_{\mathbb{C}}$. □

A root $\alpha \in \Delta$ is said to be *compact* if its root space $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ is contained in $\mathfrak{k}_{\mathbb{C}}$ and *non-compact* if its root space is contained in $\mathfrak{p}_{\mathbb{C}}$. Thus, to each root $\alpha \in \Delta$ one may associate a coefficient ε_α defined to be equal to -1 if α is compact and 1 otherwise. From the fact that θ is a Lie algebra automorphisms, we have that $\varepsilon_\alpha = \varepsilon_{-\alpha}$ and $\varepsilon_{\alpha+\beta} = -\varepsilon_\alpha\varepsilon_\beta$, for $\alpha, \beta, \alpha+\beta \in \Delta$.

A *positive root system* is a subset $\Delta^+ \subset \Delta$ satisfying:

1. For all $\alpha \in \Delta$, either α or $-\alpha$ belongs to Δ^+;
2. If $\alpha, \beta \in \Delta^+$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta^+$.

A positive root is called *simple* if it cannot be written as a sum of positive roots and we call Σ^+ the set of such roots. It turns out that Σ^+ is a basis of $\mathfrak{h}_{\mathbb{R}}$, once we have fixed a positive root system Δ^+. In addition, each root $\alpha \in \Delta$ may be written as

$$\alpha = \sum_{\gamma \in \Sigma^+} n_\gamma \gamma, \tag{3.28}$$

with $n_\gamma \in \mathbb{Z}$ all positive if α is positive and all negative if α is negative. The compactness of a root α expressed as in (3.28) is entirely determined by the compactness of $\gamma \in \Sigma^+$ and the coefficients n_γ. More precisely, it holds a formula for determining if a root is compact or non-compact.

Lemma 3.2.3. *Let $\alpha \in \Delta^+$ be a positive root expressed as $\alpha = \sum_{\gamma \in \Sigma^+} n_\gamma \gamma$, $n_\gamma \in \mathbb{Z}$ positive. Then*

$$\varepsilon_\alpha = (-1)^{1+\sum_{\gamma\in\Sigma^+} n_\gamma} \prod_{\gamma \in \Sigma^+} \varepsilon_\gamma^{n_\gamma}. \tag{3.29}$$

Proof. We prove the result by induction on the sum $\alpha = \sum_{\gamma \in \Sigma^+} n_\gamma \gamma$. Thus, for $\alpha = \gamma_i + \gamma_j$

$$\varepsilon_{\gamma_i+\gamma_j} = -\varepsilon_{\gamma_i}\varepsilon_{\gamma_j} = (-1)^{1+n_{\gamma_i}+n_{\gamma_j}} \varepsilon_{\gamma_i}\varepsilon_{\gamma_j}, \tag{3.30}$$

as stated. Now suppose that the formula (3.29) holds for $\alpha = \sum_{\gamma\in\Sigma^+} n_\gamma\gamma$ and we want to compute the coefficient ε for the root $\zeta + \sum_{\gamma\in\Sigma^+} n_\gamma\gamma$, with $\zeta \in \Sigma^+$ and $n_\zeta = 1$. Then

$$\begin{aligned} \varepsilon_{\zeta+\alpha} =& -\varepsilon_\zeta\varepsilon_\alpha = -\varepsilon_\zeta(-1)^{1+\sum_{\gamma\in\Sigma^+} n_\gamma} \prod_{\gamma\in\Sigma^+} \varepsilon_\gamma^{n_\gamma} \\ =&(-1)^{1+n_\zeta+\sum_{\gamma\in\Sigma^+} n_\gamma} \varepsilon_\zeta \prod_{\gamma\in\Sigma^+} \varepsilon_\gamma^{n_\gamma}, \end{aligned} \tag{3.31}$$

as stated. □

Observe that compactness of simple roots induces a splitting

$$\Sigma^+ = \Sigma_c^+ \cup \Sigma_n^+, \tag{3.32}$$

where $\Sigma_c^+ = \{\gamma \in \Sigma^+ \mid \varepsilon_\gamma = -1\}$ is the set of simple compact roots and $\Sigma_n^+ = \{\gamma \in \Sigma^+ \mid \varepsilon_\gamma = 1\}$ is the set of simple non-compact roots. As the product

$$\prod_{\gamma \in \Sigma_n^+} \varepsilon_\gamma^{n_\gamma} = 1, \tag{3.33}$$

formula (3.29) reduces to the following one.

Lemma 3.2.4. *A root α of the form $\alpha = \sum_{\gamma \in \Sigma^+} n_\gamma \gamma$ has*

$$\varepsilon_\alpha = (-1)^{1+\sum_{\gamma \in \Sigma_n^+} n_\gamma}. \tag{3.34}$$

More explicitly, lemma above says that in order to determine the compactness index of a root it suffices to count how many simple non-compact roots there are in its epression (3.28). If they come in an even number, the root is compact, otherwise it is non-compact.

By compactness of $\mathfrak{h}_0$, the Killing form restricts to a positive scalar product on $\mathfrak{h}_\mathbb{R}$. Indeed, for $X, Y \in \mathfrak{h}_0$, iX and iY belong to $\mathfrak{h}_\mathbb{R}$ and so

$$B(iX, iY) = -B(X, Y) > 0, \tag{3.35}$$

as B is negative definite on $\mathfrak{h}_0$. Therefore, one may define a positive scalar product on $\mathfrak{h}_\mathbb{R}^*$ by letting

$$(\psi, \psi') = B(h_\psi, h_{\psi'}), \tag{3.36}$$

where h_ψ and $h_{\psi'}$ are the elements in $\mathfrak{h}_\mathbb{R}$ corresponding to ψ and ψ' via the canonical isomorphism defined by the Killing form. The set of hyperplanes $P_\alpha = \{\psi \in \mathfrak{h}_\mathbb{R}^* \mid (\psi, \alpha) = 0\}$, for $\alpha \in \Delta$, divides $\mathfrak{h}_\mathbb{R}^*$ into a finite number of closed convex cones, called *Weyl chambers.* Moreover, each positive root system Δ^+ corresponds bijectively to a *dominant Weyl chamber* defined by

$$C = \{\psi \in \mathfrak{h}_\mathbb{R}^* \mid (\psi, \alpha) \geq 0 \ \ \forall \alpha \in \Delta^+\}. \tag{3.37}$$

We have chosen v inside $\mathfrak{h}_0$, so that $iv \in \mathfrak{h}_\mathbb{R}$, hence there exists a unique co-vector $\varphi \in \mathfrak{h}_\mathbb{R}$ such that $h_\varphi = -iv$. Moreover, one may always choose a positive root system Δ^+ such that φ belongs to the dominant Weyl chamber C. From now on, we will always assume to have picked the dominant Weyl chamber in such a way that it contains φ.

A proper way of expressing the elements in the fundamental Weyl chamber is through the fundamental dominant weights, which we are going to recall in a while. Denote by ℓ the *rank* of $\mathfrak{g}$, i.e., the dimension of the Cartan subalgebra $\mathfrak{h}_0$, so that one may label the simple roots such that $\Sigma^+ = \{\gamma_1, \dots, \gamma_\ell\}$. Let $A = (A_{ij})$ be the *Cartan matrix* associated with the root system of $\mathfrak{g}$, defined by

$$A_{ij} = 2\frac{(\gamma_i, \gamma_j)}{(\gamma_i, \gamma_i)}. \tag{3.38}$$

The *fundamental dominant weights* $\varphi_1, \dots, \varphi_\ell$ are the linear forms on $\mathfrak{h}_\mathbb{R}$ determined by

$$\varphi_j = \sum_{i=1}^{\ell} A^{ij} \gamma_i, \tag{3.39}$$

where $(A^{ij}) = A^{-1}$, and they constitute a basis of $\mathfrak{h}_\mathbb{R}$ with a nice property.

Lemma 3.2.5. *Fundamental dominant weights and simple roots satisfy the following relation*

$$2\frac{(\varphi_j, \gamma_i)}{(\gamma_i, \gamma_i)} = \delta_{ij}. \tag{3.40}$$

Proof. By definition, $\varphi_j = \sum_{k=1}^{\ell} A^{kj}\gamma_k$. Therefore

$$(\varphi_j, \gamma_i) = \sum_{k=1}^{\ell} A^{kj}(\gamma_k, \gamma_i) = \frac{1}{2}\sum_{k=1}^{\ell} A^{kj}A_{ik}(\gamma_i, \gamma_i) = \frac{1}{2}\delta_{ij}(\gamma_i, \gamma_i) \tag{3.41}$$

and the thesis follows. □

Actually, the basis of fundamental dominant weights turns out to be well suited to our purposes.

Lemma 3.2.6. *Let $\psi \in \mathfrak{h}_{\mathbb{R}}^*$ and write it as $\psi = \sum_{i=1}^{\ell} w^j\varphi_j$, for some reals $w^1, \dots, w^\ell$. Then $(\psi, \alpha) \geq 0$ for each $\alpha \in \Delta^+$ if and only if $w^i \geq 0$ for every i. Moreover $\Delta^+ \setminus \psi^\perp =$ span$\{\gamma_j \mid w^j \neq 0\} \cap \Delta^+$.*

Proof. For each $\alpha \in \Delta^+$ of the form $\alpha = \sum_{i=1}^{\ell} n^i\gamma_i$, for appropriate non-negative integers n^i, using Lemma 3.2.5, one computes

$$(\psi, \alpha) = \sum_{i,j=1}^{\ell} w^i n^j(\varphi_i, \gamma_j) = \frac{1}{2}\sum_{i=1}^{\ell} w^i n^i|\gamma_i|^2, \tag{3.42}$$

which is non-negative if and only if all the w^i's are non-negative. □

As a consequence of the above lemma, the dominant Weyl chamber C turns out to be the closed convex cone spanned by the fundamental dominant weights $\varphi_1, \dots, \varphi_n$, and co-vectors belonging to it are called *dominant weights*. In particular, our element φ such that $h_\varphi = -iv$ turns out to be a dominant weight, as it may be written as

$$\varphi = \sum_{i=1}^{\ell} v^i\varphi_i, \tag{3.43}$$

with $v^i \geq 0$ for every $i = 1, \dots, \ell$.

To conclude this part, we recall the following properties which will be useful in the subsequent sections (see for Example [48, Theorem 6.6], [39, Section 3]).

Theorem 3.2.7. *For each root $\alpha \in \Delta$ one may choose vectors $e_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$ such that:*

1. $[e_\alpha, e_{-\alpha}] = h_\alpha$;
2. $B(e_\alpha, e_\beta) = \delta_{\alpha,-\beta}$;
3. $B(h_\alpha, h) = \alpha(h)$ *for every* $h \in \mathfrak{h}_{\mathbb{C}}$;
4. $[e_\alpha, e_\beta] = 0$ *if* $\alpha + \beta \neq 0$ *and* $\alpha + \beta \notin \Delta$;
5. $[e_\alpha, e_\beta] = N_{\alpha,\beta}e_{\alpha+\beta}$ *if* $\alpha + \beta \in \Delta$, *where* $N_{\alpha,\beta} \in \mathbb{R}$ *are non-zero and satisfy*

$$N_{\beta,\alpha} = -N_{\alpha,\beta} = N_{-\alpha,-\beta} = N_{-\beta,\alpha+\beta} = N_{\alpha+\beta,-\alpha}; \tag{3.44}$$

6. $\tau_0(e_\alpha) = -e_{-\alpha}$;

7. $\tau(e_\alpha) = \varepsilon_\alpha e_{-\alpha}$.

A basis with these properties is sometimes reffered as *Weyl basis* [44, Chapter IX, Section 5]. Note that one may define $N_{\alpha,\beta} = 0$ whenever $\alpha + \beta \neq 0$ and $\alpha + \beta \notin \Delta$, so that, by abuse of notation, one may write $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$ for all $\alpha, \beta \in \Delta$ even when $\alpha + \beta$ is not a root (so $e_{\alpha+\beta}$ is not defined). This is going to be useful in some subsequent computations.

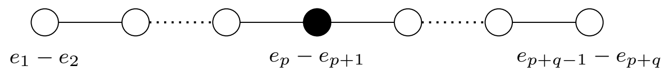
We end this section by recalling the definitions of some tools which are quite handy for studying real Lie algebras and that we will use extensively: Vogan diagrams. However, in order to define these diagrams, we need other graphs associated to Lie algebras, called Dynkin diagrams. For this part we refer to [48, Chapter II, Section 5 and Chapter VI, Section 8]. Let $\mathfrak{g}_\mathbb{C}$ be a complex semisimple Lie algebra with root system Δ, set of simple roots Σ^+ and let A be the associated Cartan matrix. We equip $\mathfrak{g}_\mathbb{C}$ with a graph defined as follows. To each simple root γ_i we associate a vertex of a graph and we attach to that vertex a weight proportional to $|\gamma_i|^2$. Two vertices, say γ_i, γ_j, are connected by $A_{ij}A_{ji}$ edges. The resulting graph is said to be the *Dynkin diagram* of $(\mathfrak{g}_\mathbb{C}, \Delta)$. Often we will draw an arrow above multiple edges, with the tip pointing towards the shortest root. Such diagrams played an important role in the classification of complex simple Lie algebras, as they determine completely the associated Lie algebra. The list of connected Dynkin diagram which led to the classification of complex simple Lie algebra may be found in [48, Chapter II, Figure 2.4]

Let now $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan involution θ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Cartan decomposition. A Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is said to be *θ-stable* if

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}), \tag{3.45}$$

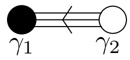
and in this case we put $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$. A θ-stable Cartan subalgebra $\mathfrak{h}$ is said to be *maximally compact* if the dimension of $\mathfrak{t}$ is as large as possibile. So let $\mathfrak{h} \in \mathfrak{g}$ be a maximally compact θ-stable Cartan subalgebra, denote by $\mathfrak{h}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C} \in \mathfrak{g}_\mathbb{C}$ its complexification and let Δ be the associated root system. Let Δ^+ be a set of positive roots chosen such that it takes $i\mathfrak{t}$ before $\mathfrak{a}_\mathbb{C}$ and call Σ^+ the set of simple roots. By definition of maximally compact θ-stable Cartan subalgebra, there are no real roots in Δ. Moreover, θ permutes the simple roots, hence it fixes the roots which are purely imaginary and permutes in 2-cycles the complex ones. The *Vogan diagram* of $(\mathfrak{g}, \mathfrak{h}, \Sigma^+)$ is then defined as the Dynkin diagram of Δ^+ with the 2-element orbits under θ so labeled and the 1-element orbits painted or not, according as the corresponding imaginary simple root is non-compact or compact.

Example 3.2.8. Let $\mathfrak{g} = \mathfrak{su}(p, q)$ with negative conjugate transposition as Cartan involution and take $\mathfrak{h}_0 = \mathfrak{t}_0$ be the diagonal subalgebra. Then θ is 1 on all the roots. With the standard ordering, the positive roots are $e_i - e_j$ with $1 \le i < j \le p+q$, where $e_1, \ldots, e_{p+q}$ is the standard orthonormal basis of $\mathbb{R}^{p+q}$. A positive root is compact if both i and j belong to $\{1, \ldots, p\}$ or $\{p+1, \ldots, p+q\}$ and non-compact if i is in $\{1, \ldots, p\}$ and j is in $\{p+1, \ldots, p+q\}$. Thus, among the simple roots $e_i - e_{i+1}$, $e_p - e_{p+1}$ is non-compact while all the others are compact. Thus the Vogan diagram of $\mathfrak{su}(p, q)$ is



Actually each Vogan diagram is equivalent to a Vogan diagram with one painted node. This result is known as the Borel and de Siebenthal Theorem [48, Theorem 6.96].

Example 3.2.9. Consider the Vogan diagram of G_2



Thus, γ_1 is non-compact and γ_2 is compact. Then, by Lemma 3.2.4, the compact and non-compact positive roots are respectively

$$\begin{aligned}\Delta_c^+ =&\{\gamma_2, 2\gamma_1+\gamma_2\}\\ \Delta_n^+ =&\{\gamma_1, \gamma_1+\gamma_2, 3\gamma_1+2\gamma_2, 3\gamma_1+\gamma_2\}.\end{aligned} \tag{3.46}$$

The root system of G_2 together with compact and non-compact roots is depicted in Figure 3.1.

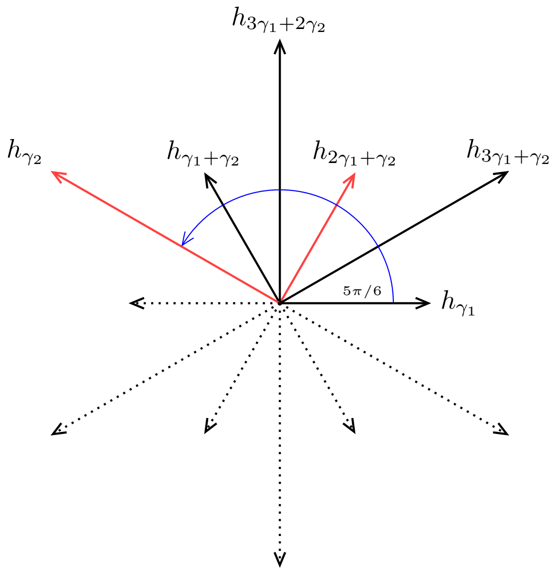


Figure 3.1: The root diagram of G_2. Thick black arrows represent the positive non-compact roots, while the red arrows represent the compact ones. The names at the end of each arrow indicates the vector corresponding to the given root. The negative roots are identified with dashed black arrows.

Now that we have set the necessary algebraic background, we are ready to study the almost-Kähler geometry of adjoint orbits, starting with the definition of a canonical almost-complex structure.

3.3 The canonical almost-complex structure

In this section we specialize some definitions given in Section 3.1 and we provide a homogeneous almost-complex structure for adjoint orbits which turns out to be compatible with the Kirillov-Kostant-Souriau symplectic form ω.

For each root $\alpha \in \Delta$ one may define the real number

$$\lambda_\alpha = s_\alpha(\alpha, \varphi), \tag{3.47}$$

where s_α is the *signum* of α, meaning that $s_\alpha = 1$ if α is a positive root and $s_\alpha = -1$ if it is negative. Remember that we have chosen φ inside the dominant Weyl chamber, thus $\lambda_\alpha \geq 0$ for every α and, in particular, $\lambda_\alpha = 0$ when α is orthogonal to φ. Then, define the vectors

$$u_\alpha = \frac{i^{\frac{1-\varepsilon_\alpha}{2}}}{\sqrt{2}}(e_\alpha + e_{-\alpha}), \quad v_\alpha = \frac{i^{\frac{3-\varepsilon_\alpha}{2}}}{\sqrt{2}} s_\alpha(e_\alpha - e_{-\alpha}) \tag{3.48}$$

and notice that $u_\alpha = u_{-\alpha}$ and $v_\alpha = v_{-\alpha}$. Moreover, these vectors satisfy some useful relations.

Lemma 3.3.1. *For all roots $\alpha, \beta \in \Delta$ the following hold:*

1. $B(u_\alpha, u_\beta) = B(v_\alpha, v_\beta) = (\delta_{\alpha,\beta} + \delta_{\alpha,-\beta})\varepsilon_\alpha$;
2. $B(u_\alpha, v_\beta) = 0$;
3. $u_\alpha, v_\alpha \in \mathfrak{g}$;
4. $[v, u_\alpha] = \lambda_\alpha v_\alpha$ *and* $[v, v_\alpha] = -\lambda_\alpha u_\alpha$.

Proof. The statements are consequences of Theorem 3.2.7. More precisely:

1. By item 2 of the theorem on has

$$\begin{aligned} B(u_\alpha, u_\beta) =& \frac{i^{\frac{2-\varepsilon_\alpha-\varepsilon_\beta}{2}}}{2} B(e_\alpha + e_{-\alpha}, e_\beta + e_{-\beta}) \\ =& \frac{i^{\frac{2-\varepsilon_\alpha-\varepsilon_\beta}{2}}}{2} (\delta_{\alpha,-\beta} + \delta_{\alpha,\beta} + \delta_{-\alpha,-\beta} + \delta_{-\alpha,\beta}) \\ =& i^{\frac{2-\varepsilon_\alpha-\varepsilon_\beta}{2}} (\delta_{\alpha,\beta} + \delta_{\alpha,-\beta}) \\ =& (\delta_{\alpha,\beta} + \delta_{\alpha,-\beta})\varepsilon_\alpha \end{aligned} \tag{3.49}$$

   and similarly for the scalar product of v_α and v_β.

2. Likewise the previous point,

$$B(u_\alpha, v_\beta) = \frac{i^{\frac{4-\varepsilon_\alpha-\varepsilon_\beta}{2}}}{2} (\delta_{\alpha,-\beta} - \delta_{\alpha,\beta} + \delta_{-\alpha,-\beta} - \delta_{-\alpha,\beta}) = 0. \tag{3.50}$$

3. In order to check whether u_α and v_α belong to $\mathfrak{g}$ it suffices to show that they are invariant under the conjugation τ. By point 7 of Theorem 3.2.7 one may write

$$\tau(u_\alpha) = \frac{\overline{i^{\frac{1-\varepsilon_\alpha}{2}}}}{\sqrt{2}}(\varepsilon_\alpha e_{-\alpha} + \varepsilon_\alpha e_\alpha), \quad \tau(v_\alpha) = \frac{\overline{i^{\frac{3-\varepsilon_\alpha}{2}}}}{\sqrt{2}} s_\alpha(\varepsilon_\alpha e_{-\alpha} - \varepsilon_\alpha e_\alpha) \tag{3.51}$$

   and the result follows by considering separately the cases $\varepsilon_\alpha = 1$ and $\varepsilon_\alpha = -1$.

4. Finally, by writing explicitly u_α, one has

$$[v, u_\alpha] = \frac{i^{\frac{1-\varepsilon_\alpha}{2}}}{\sqrt{2}}[v, e_\alpha + e_{-\alpha}]. \tag{3.52}$$

Since $v \in \mathfrak{h}_0$, by definition of root space, v satisfies $[v, e_\alpha] = \alpha(v)e_\alpha$. Moreover, by item 3 of Theorem 3.2.7 and recalling that $h_\varphi = -iv$,

$$\lambda_\alpha = s_\alpha(\alpha, \varphi) = s_\alpha B(h_\alpha, h_\varphi) = -is_\alpha B(h_\alpha, v) = -is_\alpha \alpha(v). \tag{3.53}$$

Putting everything together we get

$$[v, u_\alpha] = \alpha(v)\frac{i^{\frac{1-\varepsilon_\alpha}{2}}}{\sqrt{2}}(e_\alpha - e_{-\alpha}) = i\lambda_\alpha s_\alpha \frac{i^{\frac{1-\varepsilon_\alpha}{2}}}{\sqrt{2}}(e_\alpha - e_{-\alpha}) = \lambda_\alpha v_\alpha. \tag{3.54}$$

Similar computations prove that also $[v, v_\alpha] = -\lambda_\alpha u_\alpha$.

□

As a consequence of the lemma above, $\mathfrak{g}$ has the B-orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Delta^+} \operatorname{span}\{u_\alpha, v_\alpha\}. \tag{3.55}$$

Observe that this decompositon may be refined by splitting the set of positive roots as

$$\Delta^+ = (\Delta^+ \cap \varphi^\perp) \cup (\Delta^+ \setminus \varphi^\perp) \tag{3.56}$$

where $(\Delta^+ \cap \varphi^\perp)$ is the set of positive roots which are orthogonal to φ and $(\Delta^+ \setminus \varphi^\perp)$ is the set of roots which are not orthogonal to φ. By the definition of λ_α (3.47), a root α belongs to $\Delta^+ \setminus \varphi^\perp$ if $\lambda_\alpha > 0$, while it belongs to $\Delta^+ \cap \varphi^\perp$ if $\lambda_\alpha = 0$. Hence, by item 4 of Lemma 3.3.1, the Lie algebra $\mathfrak{v}$ of the stabilizer V of v decomposes as

$$\mathfrak{v} = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Delta^+ \cap \varphi^\perp} \operatorname{span}\{u_\alpha, v_\alpha\}. \tag{3.57}$$

Since V is compact, all roots belonging to $\Delta^+ \cap \varphi^\perp$ must be compact, while

$$\mathfrak{m} = \bigoplus_{\alpha \in \Delta^+ \setminus \varphi^\perp} \operatorname{span}\{u_\alpha, v_\alpha\}. \tag{3.58}$$

Notice that the subspaces $\operatorname{span}\{u_\alpha, v_\alpha\}$, for $\alpha \in \Delta^+ \setminus \varphi^\perp$, play the role of the $\mathfrak{m}_i$'s in Section 3.1. Putting together the last observations, we deduce a formula for the dimension of the adjoint orbit G/V of v in terms of φ and the positive roots.

Proposition 3.3.2. *The summands of the B-orthogonal decomposition $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{m}$ are given by*

$$\mathfrak{v} = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Delta^+ \cap \varphi^\perp} \operatorname{span}\{u_\alpha, v_\alpha\}, \qquad \mathfrak{m} = \bigoplus_{\alpha \in \Delta^+ \setminus \varphi^\perp} \operatorname{span}\{u_\alpha, v_\alpha\}. \tag{3.59}$$

Moreover, all roots of $\Delta^+ \cap \varphi^\perp$ are compact and the dimension of G/V is given by

$$\dim G/V = \dim \mathfrak{g} - \ell - 2|\{\alpha \in \Delta^+ \mid (\alpha, \varphi) = 0\}|, \tag{3.60}$$

where $\ell = \dim \mathfrak{h}_0$ is the rank of $\mathfrak{g}$.

At this point we are able to define a canonical homogeneous almost-complex structure J on G/V. In order to do this, let H be the complex structure on $\mathfrak{h}_0^\perp = \operatorname{span}\{u_\alpha, v_\alpha \mid \alpha \in \Delta^+\}$ defined on the basis by

$$Hu_\alpha = \varepsilon_\alpha v_\alpha, \qquad Hv_\alpha = -\varepsilon_\alpha u_\alpha, \tag{3.61}$$

for $\alpha \in \Delta^+$. Note that H makes complex the splitting

$$\mathfrak{h}_0^\perp = \operatorname{span}\{u_\alpha, v_\alpha \mid \alpha \in \Delta^+ \cap \varphi^\perp\} \oplus \operatorname{span}\{u_\alpha, v_\alpha \mid \alpha \in \Delta^+ \setminus \varphi^\perp\}, \tag{3.62}$$

where the second summand is exactly $\mathfrak{m}$. Then we define the *canonical almost-complex structure* J on G/V as the homogeneous almost-complex structure induced by H on $\mathfrak{m}$.

Remark 3.3.3. By decomposing $X \in \mathfrak{m}$ as $X = \sum_{\alpha\in\Delta^+\setminus\varphi^\perp} X_\alpha$, where $X_\alpha \in \operatorname{span}\{u_\alpha, v_\alpha\}$, and by using item 4 of Lemma 3.3.1, one has

$$HX = \sum_{\alpha\in\Delta^+\setminus\varphi^\perp} \frac{\varepsilon_\alpha}{\lambda_\alpha}[v, X_\alpha]. \tag{3.63}$$

Hence J on the orbit G/V coincides with the almost-complex structure defined in Section 3.1.

The basis $\{u_\alpha, v_\alpha\}$, $\alpha \in \Delta^+ \setminus \varphi^\perp$, on $\mathfrak{m}$ defined in (3.48), turns out to be useful as it allows to define a symplectic orthonormal basis for $\mathfrak{m}$.

Lemma 3.3.4. *The vectors* $(1/\sqrt{\lambda_\alpha})u_\alpha$, $(\varepsilon_\alpha/\sqrt{\lambda_\alpha})v_\alpha$, *for* $\alpha \in \Delta^+\setminus\varphi^\perp$, *constitute a symplectic basis of* $\mathfrak{m}$ *which is also orthonormal with respect to the scalar product induced by* H *and* σ.

Proof. Put $e_\alpha = (1/\sqrt{\lambda_\alpha})u_\alpha$ and $f_\alpha = (\varepsilon_\alpha/\sqrt{\lambda_\alpha})v_\alpha$. Recall that the basis $\{e_\alpha, f_\alpha\}$, for $\alpha \in \Delta^+ \setminus \varphi^\perp$ is symplectic if $\sigma(e_\alpha, e_\beta) = \sigma(f_\alpha, f_\beta) = 0$ and $\sigma(e_\alpha, f_\beta) = \delta_{\alpha,\beta}$, for $\alpha, \beta \in \Delta^+ \setminus \varphi^\perp$. Some computations together with Lemma 3.3.1 read

$$\sigma(e_\alpha, e_\beta) = \frac{1}{\sqrt{\lambda_\alpha\lambda_\beta}}\sigma(u_\alpha, u_\beta) = \frac{1}{\sqrt{\lambda_\alpha\lambda_\beta}}B([v, u_\alpha], u_\beta) = \frac{\lambda_\alpha}{\sqrt{\lambda_\alpha\lambda_\beta}}B(v_\alpha, u_\beta) = 0. \tag{3.64}$$

And similar computations show that $\sigma(f_\alpha, f_\beta) = 0$. Then

$$\sigma(e_\alpha, f_\beta) = \frac{\varepsilon_\beta}{\sqrt{\lambda_\alpha\lambda_\beta}}\sigma(u_\alpha, v_\beta) = \frac{\varepsilon_\beta}{\sqrt{\lambda_\alpha\lambda_\beta}}B([v, u_\alpha], v_\beta) = \frac{\varepsilon_\beta\lambda_\alpha}{\sqrt{\lambda_\alpha\lambda_\beta}}B(v_\alpha, v_\beta) = \delta_{\alpha,\beta}, \tag{3.65}$$

since α and β are both positive roots. This shows that the basis $\{e_\alpha, f_\alpha\}$ is symplectic.

For the orthonormality, we compute

$$\langle e_\alpha, e_\beta\rangle = \sigma(e_\alpha, He_\beta) = \frac{1}{\sqrt{\lambda_\alpha\lambda_\beta}}\sigma(u_\alpha, Hu_\beta) = \frac{\varepsilon_\beta}{\sqrt{\lambda_\alpha\lambda_\beta}}\sigma(u_\alpha, v_\beta) = \sigma(e_\alpha, f_\beta) = \delta_{\alpha,\beta}, \tag{3.66}$$

as the basis is symplectic. Similarly $\langle f_\alpha, f_\beta\rangle = \delta_{\alpha,\beta}$. We are left with the mixed terms

$$\langle e_\alpha, f_\beta\rangle = \frac{\varepsilon_\beta}{\sqrt{\lambda_\alpha\lambda_\beta}}\sigma(u_\alpha, Hv_\beta) = -\frac{1}{\sqrt{\lambda_\alpha\lambda_\beta}}\sigma(u_\alpha, u_\beta) = 0, \tag{3.67}$$

again since the basis is symplectic. □

We end this section by showing some results concerning fibrations of adjoint orbit. First, the canonical almost-complex structure is compatible with the projections of adjoint orbits. More precisely, one has the following property.

Proposition 3.3.5. *Let $\tilde{\varphi} \in \mathfrak{h}_{\mathbb{R}}^*$ such that for all roots $\alpha \in \Delta$ it holds that $(\alpha, \tilde{\varphi}) = 0$ whenever $(\alpha, \varphi) = 0$. Then the stabilizer $\tilde{V}$ of $\tilde{v} = ih_{\tilde{\varphi}}$ contains V and the induced projection $\pi : G/V \to G/\tilde{V}$ is pseudo-holomorphic with respect to the almost-complex structures J and $\tilde{J}$ induced by the canonical complex structures on the adjoint orbits of v and $\tilde{v}$.*

Proof. As a consequence of Proposition 3.3.2, the Lie algebra of the stabilizer of $\tilde{v}$ is given by

$$\tilde{\mathfrak{v}} = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Delta^+ \cap \tilde{\varphi}^\perp} \operatorname{span}\{u_\alpha, v_\alpha\} \tag{3.68}$$

and it contains $\mathfrak{v}$ since all roots which are orthogonal to φ are orthogonal also to $\tilde{\varphi}$. Therefore $V \subset \tilde{V}$. The induced projection $\pi : G/V \to G/\tilde{V}$ is G-equivariant, being defined at the Lie algebra level. We are left with proving that π is pseudo-holomorphic, i.e., its differential is a complex linear map between the two complex vector bundles $T\,G/V$ and $T\,G/\tilde{V}$. Since π is G-equivariant, it suffices to prove that the differential $\mathrm{d}_e\pi$ at the identity coset $e \in G/V$ is complex linear. To this end, observe that

$$\tilde{\mathfrak{m}} = \bigoplus_{\alpha \in \Delta^+ \setminus \tilde{\varphi}^\perp} \operatorname{span}\{u_\alpha, v_\alpha\} \tag{3.69}$$

is contained in $\mathfrak{m}$ and the differential $\mathrm{d}_e\pi$ is by definition the B-orthogonal projection $f : \mathfrak{m} \to \tilde{\mathfrak{m}}$, defined as

$$f\left(\sum_{\alpha \in \Delta^+ \setminus \varphi^\perp} X_u^\alpha u_\alpha + X_v^\alpha v_\alpha\right) = \sum_{\alpha \in \Delta^+ \setminus \tilde{\varphi}^\perp} X_u^\alpha u_\alpha + X_v^\alpha v_\alpha \tag{3.70}$$

with $\sum_{\alpha \in \Delta^+ \setminus \varphi^\perp} X_u^\alpha u_\alpha + X_v^\alpha v_\alpha \in \mathfrak{m}$ an element expressed on the basis of $\mathfrak{m}$. Then

$$\begin{aligned} fH\left(\sum_{\alpha \in \Delta^+ \setminus \varphi^\perp} X_u^\alpha u_\alpha + X_v^\alpha v_\alpha\right) &= \sum_{\alpha \in \Delta^+ \setminus \varphi^\perp} X_u^\alpha fH(u_\alpha) + X_v^\alpha fH(v_\alpha) \\ &= \sum_{\alpha \in \Delta^+ \setminus \tilde{\varphi}^\perp} X_u^\alpha \varepsilon_\alpha v_\alpha - X_v^\alpha \varepsilon_\alpha u_\alpha \\ &= H\left(\sum_{\alpha \in \Delta^+ \setminus \tilde{\varphi}^\perp} X_u^\alpha u_\alpha + X_v^\alpha v_\alpha\right) \\ &= Hf\left(\sum_{\alpha \in \Delta^+ \setminus \varphi^\perp} X_u^\alpha u_\alpha + X_v^\alpha v_\alpha\right). \end{aligned} \tag{3.71}$$

Thus f commutes with H, showing that the differential $\mathrm{d}\pi$ is complex linear. □

In the situation of the previous proposition, choosing $\tilde{\varphi} \in C$ being orthogonal to no roots, yields the following corollary.

Corollary 3.3.6. *There exists an element $w \in \mathfrak{g}$ whose stabilizer is a maximal torus $T \subset G$ such that $T \subset V$. Then the natural projection*

$$G/T \to G/V \tag{3.72}$$

is pseudo-holomorphic, where the orbit G/T is equipped with the homogeneous almost-complex structure induced by H.

Observe that, in general, the element w as in the statement above is far from being unique as it corresponds to the choice of $\tilde{\varphi}$ in an open dense subset of C. Instead, choosing $\tilde{\varphi} \in C$ in a very special position gives the next result.

Corollary 3.3.7. *If there exists a non-zero $\tilde{\varphi} \in C$ which is orthogonal to all compact roots, then there exists $\tilde{v} \in \mathfrak{g}$ whose stabilizer is a maximal compact subgroup $K \subset G$ such that $V \subset K$. Then the natural projection*

$$G/V \to G/K \tag{3.73}$$

is pseudo-holomorphic, where the orbit G/K is equipped with the homogeneous almost-complex structure induced by H.

At this point, we are ready to discuss the speciality condition for the canonical almost-complex structure J on adjoint orbits.

3.4 Speciality condition for the canonical almost-complex structure

Along this section we consider adjoint orbits G/V equipped with the homogeneous almost-Kähler structure defined by the canonical almost-complex stucture J and the Kirillov-Kostant-Souriau symplectic form ω. The main goal is to characterize when J is special, i.e., its Chern-Ricci form ρ satisfies $\rho = \lambda\omega$.

As shown in Section 3.1, as the symplectic form ω is determined by σ and v, so the Chern-Ricci form ρ of J is determined by the 2-form on $\mathfrak{m}$ defined by $B(v', [\cdot, \cdot])$, where

$$v' = 2 \sum_{\alpha \in \Delta^+ \setminus \varphi^\perp} [u_\alpha, v_\alpha]. \tag{3.74}$$

By writing explicitly u_α and v_α and by item 1 of Theorem 3.2.7, one may compute

$$\begin{aligned} v' =& 2 \sum_{\alpha \in \Delta^+ \setminus \varphi^\perp} \frac{i^{\frac{4-2\varepsilon_\alpha}{2}}}{2} [e_\alpha + e_{-\alpha}, e_\alpha - e_{-\alpha}] \\ =& -2 \sum_{\alpha \in \Delta^+ \setminus \varphi^\perp} i^{2-\varepsilon_\alpha} [e_\alpha, e_{-\alpha}] \\ =& -2 \sum_{\alpha \in \Delta^+ \setminus \varphi^\perp} i^{2-\varepsilon_\alpha} h_\alpha \\ =& -2i \sum_{\alpha \in \Delta^+ \setminus \varphi^\perp} \varepsilon_\alpha h_\alpha. \end{aligned} \tag{3.75}$$

Thus $v' = -2i \sum_{\alpha \in \Delta^+ \setminus \varphi^\perp} \varepsilon_\alpha h_\alpha$, hence the co-vector

$$\varphi' = -2 \sum_{\alpha \in \Delta^+ \setminus \varphi^\perp} \varepsilon_\alpha \alpha \ \in \mathfrak{h}^*_{\mathbb{R}} \tag{3.76}$$

defines an element of the root lattice such that $h_{\varphi'} = -iv'$. As we discussed in Section 3.1, the speciality condition for the Chern-Ricci form $\rho = \lambda\omega$ is equivalent to $v' = \lambda v$, hence also to $\varphi' = \lambda\varphi$:

$$\rho = \lambda\omega \ \Leftrightarrow \varphi' = \lambda\varphi. \tag{3.77}$$

Notice that φ' depends only discretely on φ, in the sense that it depends on which roots φ is orthogonal to, but not on the distance between φ and those roots.

The equation $\varphi' = \lambda\varphi$ turns out to be quite complicated from the combinatorial point of view. However, it becomes more tractable by introducing the element of the root lattice $\eta \in \mathfrak{h}^*_{\mathbb{R}}$ defined as the sum of all positive roots weighted by their compactness index

$$\eta = -2 \sum_{\alpha \in \Delta^+} \varepsilon_\alpha \alpha. \tag{3.78}$$

Observe that the element η depends on the semisimple Lie algebra $\mathfrak{g}$ and on the chosen set of positive roots only. Since the roots which are orthogonal to φ are necessarily compact, expressing φ' as

$$\varphi' = -2 \sum_{\alpha \in \Delta^+ \setminus \varphi^\perp} \varepsilon_\alpha \alpha = \eta - 2 \sum_{\alpha \in \Delta^+ \cap \varphi^\perp} \alpha, \tag{3.79}$$

allows to rewrite the condition $\rho = \lambda\omega$ as

$$\eta - 2 \sum_{\alpha \in \Delta^+ \cap \varphi^\perp} \alpha = \lambda\varphi. \tag{3.80}$$

Equation (3.80) permits to deduce the sign of λ, for it is entirely determined by the scalar product between φ and η.

Lemma 3.4.1. *If $\varphi' = \lambda\varphi$, then $\lambda = (\eta, \varphi)/|\varphi|^2$.*

Proof. Taking the scalar product of φ with equation (3.80), one gets

$$(\eta, \varphi) - 2 \sum_{\alpha \in \Delta^+ \cap \varphi^\perp} (\alpha, \varphi) = \lambda(\varphi, \varphi) \tag{3.81}$$

but the second summand on the left-hand side vanishes since the sum is performed over the roots which are orthogonal to φ. ☐

Up to now we have considered a fixed element v, hence a fixed φ. However, the main goal of this thesis is to solve the equation $\rho = \lambda\omega$, that is, to find φ such that $\varphi' = \lambda\varphi$. The next lemma shows that the sign of λ is uniquely determined by the semisimple Lie algebra $\mathfrak{g}$, hence by the group G, and by the choice of the dominant Weyl chamber, hence by the choice of the set of positive roots Δ^+.

Proposition 3.4.2. *Let $\psi, \varphi \in \mathfrak{h}^*_{\mathbb{R}}$ be elements belonging to the dominant Weyl chamber C. Suppose that both $ih_\varphi, ih_\psi \in \mathfrak{g}$ have compact isotropy and that $\varphi' = \lambda\varphi$, $\psi' = \mu\psi$ for some real constants λ, μ. Then λ and μ have the same sign. Moreover, φ and ψ are one multiple of the other whenever $\lambda, \mu < 0$.*

Proof. By (3.80) one may write conditions $\varphi' = \lambda\varphi$, $\psi' = \mu\psi$ as

$$\eta - 2 \sum_{\alpha \in \Delta^+ \cap \varphi^\perp} \alpha = \lambda\varphi, \qquad \eta - 2 \sum_{\alpha \in \Delta^+ \cap \psi^\perp} \alpha = \mu\psi. \tag{3.82}$$

By Lemma 3.4.1, taking the scalar products of the previous equations with ψ and φ respectively reads

$$\mu|\psi|^2 - \lambda(\varphi, \psi) = 2 \sum_{\alpha \in \Delta^+ \cap \varphi^\perp} (\alpha, \psi), \qquad \lambda|\varphi|^2 - \mu(\psi, \varphi) = 2 \sum_{\alpha \in \Delta^+ \cap \psi^\perp} (\alpha, \varphi). \tag{3.83}$$

Since φ and ψ are dominant, $(\alpha, \varphi) \geq 0$ and $(\alpha, \psi) \geq 0$ for each positive root α and $(\varphi, \psi) > 0$. Thus the right-hand sides of the above equations are non-negative, implying

$$\mu|\psi|^2 - \lambda(\varphi, \psi) \geq 0, \qquad \lambda|\varphi|^2 - \mu(\psi, \varphi) \geq 0, \tag{3.84}$$

which gives a contradiction as soon as λ and μ have different signs.

Finally, by taking a linear combination of the equations in (3.83), one gets

$$\sum_{\alpha \in \Delta^+ \cap \varphi^\perp} 2\mu(\alpha, \psi) + \sum_{\alpha \in \Delta^+ \cap \psi^\perp} 2\lambda(\alpha, \varphi) = |\lambda\varphi - \mu\psi|^2 \tag{3.85}$$

and this shows that $\lambda\varphi = \mu\psi$ when λ and μ are both negative. □

In particular, the above lemma says that if the equation $\varphi' = \lambda\varphi$ is satisfied with $\lambda < 0$, then it has a unique solution up to scaling. However, this is a special feature of the case $\lambda < 0$. Indeed when $\lambda > 0$ the set of solutions is just finite up to scaling, as the following proposition and the examples in the Appendix B show.

Proposition 3.4.3. *The set of all $\varphi \in \mathfrak{h}^*_{\mathbb{R}}$ belonging to the dominant Weyl chamber C, such that $v = ih_\varphi$ has compact isotropy and satisfying $\varphi' = \lambda\varphi$ for some $\lambda \neq 0$ is finite up to scaling.*

Proof. Given φ as in the statement, after having rescaled it to $|\lambda|\varphi$, by (3.80) one may write $\varphi' = \lambda\varphi$ in the form

$$\eta - 2 \sum_{\alpha \in \Delta^+ \cap \varphi^\perp} \alpha = \pm\varphi \tag{3.86}$$

and this shows that φ belongs to the root lattice. As a consequence, since no non-compact roots may be orthogonal to φ, triangle inequality reads

$$|\varphi| \leq |\eta| + 2 \sum_{\alpha \in \Delta_c} |\alpha|, \tag{3.87}$$

where $\Delta_c \subset \Delta$ is the set of compact roots. Since the right-hand side depends on $\mathfrak{g}$ only, we may conclude that φ belongs to a bounded subset of the root lattice, hence finite. □

Collecting together the results stated up to now, we get the following two results that give necessary and sufficient conditions for the existence of elements φ having orbits admitting special canonical almost-complex structure.

Theorem 3.4.4. *An element $\varphi \in \mathfrak{h}^*_{\mathbb{R}}$ belongs to the dominant Weyl chamber C, the stabilizer of $v = ih_\varphi$ is compact and $\varphi' = \lambda\varphi$ for some $\lambda \neq 0$ if and only if there exists a non-empty subset $S \subset \{1, \ldots, \ell\}$ such that:*

1. *all non-compact simple roots of Σ^+ belong to $\{\gamma_i \mid i \in S\}$;*
2. *The element*
$$\frac{1}{\lambda}\left(\eta - 2 \sum_{\alpha \in \mathrm{span}\{\gamma_i | i \in S^c\} \cap \Delta^+} \alpha\right) \tag{3.88}$$
belongs to the open convex cone spanned by $\{\varphi_i \mid i \in S\}$;
3. *The co-vector φ has the form*
$$\varphi = \frac{1}{\lambda}\left(\eta - 2 \sum_{\alpha \in \mathrm{span}\{\gamma_i | i \in S^c\} \cap \Delta^+} \alpha\right). \tag{3.89}$$

Proof. Assume that φ belongs to the dominant Weyl chamber $C \subset \mathfrak{h}_{\mathbb{R}}^*$, the stabilizer of $v = ih_\varphi$ is compact and $\varphi' = \lambda\varphi$ for some $\lambda \neq 0$. Since φ is dominant, it belongs to the convex cone spanned by $\varphi_1, \dots, \varphi_\ell$. In particular, $\varphi = \sum_{i=1}^{\ell} v^i \varphi_i$ for suitable non-negative coefficients v^i. Notice that the coefficients v^i must be non-zero whenever γ_i is non-compact, by our assumption on the compactness of the stabilizer of v. The set S constituted by all indices $1 \leq i \leq \ell$ such that $v^i \neq 0$ is non-empty, satisfies the first item of the statement and reads $\varphi = \sum_{i\in S} v^i \varphi_i$. On the other hand, Lemma 3.2.6 implies that $\Delta^+ \cap \varphi^\perp = \mathrm{span}\{\gamma_i \mid i \in S^c\}$. Hence, using expression (3.79) for φ', we have that the equation $\varphi' = \lambda\varphi$ may be written as in the third item of the statements and the second one follows.

Conversely, if $\varphi \in \mathfrak{h}_{\mathbb{R}}^*$ and there is a set $S \subset \{1, \dots, \ell\}$ satisfying all three items of the statement for some real $\lambda \neq 0$, then $(\varphi, \alpha) \geq 0$ for any positive root α, so $\varphi \in C$. In addition, $v = ih_\varphi$ has compact stabilizer and the relation $\varphi' = \lambda\varphi$ holds. □

In a similar fashion, the set of all φ's such that their φ' vanish may be described in terms of certain subsets of $\{1, \dots, \ell\}$. However, in contrast with the case $\lambda \neq 0$, such a set is very often infinite even up to scaling. More precisely, this happens when the set S in the statement below has cardinality bigger than one.

Theorem 3.4.5. *An element $\varphi \in \mathfrak{h}_{\mathbb{R}}^*$ belongs to the dominant Weyl chamber C, the stabilizer of $v = ih_\varphi$ is compact and $\varphi' = 0$ if and only if there exists a non-empty subset $S \subset \{1, \dots, \ell\}$ such that:*

1. *all non-compact simple roots of Σ^+ belong to $\{\gamma_i \mid i \in S\}$;*
2. *$\varphi = \sum_{i\in S} v^i \varphi_i$ with $v^i > 0$ for every $i \in S$;*
3. *$\eta - 2\sum_{\alpha\in \mathrm{span}\{\gamma_i \mid i\in S^c\}\cap\Delta^+} \alpha = 0$.*

Proof. Assume that φ belongs to the dominant Weyl chamber $C \subset \mathfrak{h}_{\mathbb{R}}^*$, the stabilizer of $v = ih_\varphi$ is compact and $\varphi' = 0$. Since φ is dominant, it belongs to the convex cone spanned by $\varphi_1, \dots, \varphi_\ell$. In particular, $\varphi = \sum_{i=1}^{\ell} v^i \varphi_i$ for suitable non-negative coefficients v^i. Notice that the coefficient v^i must be non-zero, hence positive whenever γ_i is non-compact, by our assumption on the compactness of the stabilizer of v. The set S constituted by all indices $1 \leq i \leq \ell$ such that $v^i \neq 0$ is non-empty, satisfies the first two items of the statement. By (3.80), the condition $\varphi' = 0$ may be written as

$$\eta - 2 \sum_{\alpha\in \mathrm{span}\{\gamma_i \mid v^i = 0\}\cap\Delta^+} \alpha = 0, \tag{3.90}$$

and by definition of S, we have that $v^i = 0$ if and only if $i \in S^c$. Thus, also the third item of the statement is satisfied.

On the other hand, if $\varphi \in \mathfrak{h}_{\mathbb{R}}^*$ and there is a set $S \subset \{1, \dots, \ell\}$ satisfying all three items of the statement, then $(\varphi, \alpha) \geq 0$ for any positive root α, implying $\varphi \in C$. In addition, $v = ih_\varphi$ has compact stabilizer and $\varphi' = 0$. □

Putting together Theorem 3.4.4 and Theorem 3.4.5, one has a summary result, which needs the following lemma.

Lemma 3.4.6. *The reflection σ_γ with respect to a compact simple root γ preserves compactness of the roots.*

Proof. Let $\alpha \in \Delta$ be a root and let $\gamma \in \Sigma^+$ be a compact simple root. Then

$$\sigma_\gamma(\alpha) = \alpha - 2\frac{(\alpha,\gamma)}{(\gamma,\gamma)}\gamma \tag{3.91}$$

and it is a root as reflections with respect to simple roots permute the roots [45, Section 10.2, Lemma B]. If α is compact, $\sigma_\gamma(\alpha)$ is the sum of two compact roots, hence it is compact as a consequence of Lemma 3.2.4. Using the same result, if α is non-compact, then $\sigma_\gamma(\alpha)$ is the sum of a non-compact and a compact root, thus it is non-compact. □

Corollary 3.4.7. *For any $\varphi \in \mathfrak{h}^*_{\mathbb{R}}$ and any real $\lambda \in \{-1, 0, 1\}$, the following conditions are equivalent:*

- *φ belongs to the dominant Weyl chamber C, the stabilizer of $v = ih_\varphi$ is compact and $\varphi' = \lambda\varphi$.*
- *There exists a subset $S \subseteq \{1, \ldots, \ell\}$ such that $i \in S$ if γ_i is a non-compact simple root, and*

$$\varphi_S = \eta - 2 \sum_{\alpha \in \operatorname{span}\{\gamma_i | i \in S^c\} \cap \Delta^+} \alpha \tag{3.92}$$

satisfies $(\varphi_S, \gamma_i) = \lambda|(\varphi_S, \gamma_i)|$ for all $i \in S$. Moreover

$$\varphi = \begin{cases} \lambda\varphi_S & \text{if} \quad \lambda = \pm 1 \\ \sum_{i \in S} v^i \varphi_i \text{ for some } v^i > 0 & \text{if} \quad \lambda = 0 \end{cases}. \tag{3.93}$$

Proof. First, we show that, given $S \subseteq \{1, \ldots, \ell\}$ such that $i \in S$ whenever γ_i is a non-compact simple root, the element φ_S belongs to the span of φ_i, $i \in S$. To see this, let $\varphi_S = \sum_{j=1}^{\ell} w^i \varphi_i$ and notice that, by equation (3.42), $w^j = 2(\varphi_S, \gamma_j)/|\gamma_j|^2$. We are then reduced to prove that $(\varphi_S, \gamma_j) = 0$ for each $j \in S^c$. In order to do this, we will show that, for every $j \in S^c$, φ_S is fixed by the reflection σ_{γ_j} with respect to the hyperplane orthogonal to γ_j. Observe that, by definition of η, one has

$$\eta = 2 \sum_{\alpha \in \Delta^+} \alpha - 4 \sum_{\alpha \in \Delta_n^+} \alpha. \tag{3.94}$$

Since any root is either positive or negative and the reflection of a root by σ_{γ_j} is still a root, one may conclude that the reflection $\sigma_{\gamma_j}(\alpha)$ of a positive root α is a positive root, unless $\alpha = \gamma_j$. On the other hand, by Lemma 3.4.6, a reflection with respect to a compact root preserve compactness, in particular $\sigma_{\gamma_j}(\alpha)$ has the same compactness as α. Hence

$$\sigma_{\gamma_j}(\eta) = \eta - 4\gamma_j. \tag{3.95}$$

By the properties of root systems, the vector $2\sum_{\alpha \in \operatorname{span}\{\gamma_i | i \in S^c\} \cap \Delta^+} \alpha$ is twice the δ-vector associated with the root subsystem $\operatorname{span}\{\gamma_i \mid i \in S^c\} \cap \Delta^+$, where the δ-vector of a root system is defined as the half sum of all positive roots. Similarly as for η, one has

$$\sigma_{\gamma_j}\left(2 \sum_{\alpha \in \operatorname{span}\{\gamma_i | i \in S^c\} \cap \Delta^+} \alpha\right) = 2 \sum_{\alpha \in \operatorname{span}\{\gamma_i | i \in S^c\} \cap \Delta^+} \alpha - 4\gamma_j. \tag{3.96}$$

Putting everything together,

$$
\begin{aligned}
\sigma_{\gamma_j}(\varphi_S) =& \sigma_{\gamma_j}\left(\eta - 2\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in S^c\}\cap\Delta^+}\alpha\right)\\
=& \eta - 4\gamma_j - 2\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in S^c\}\cap\Delta^+}\alpha + 4\gamma_j\\
=& \eta - 2\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in S^c\}\cap\Delta^+}\alpha\\
=& \varphi_S.
\end{aligned}
\tag{3.97}
$$

This proves that φ_S belongs to $\operatorname{span}\{\varphi_i \mid i \in S\}$. The condition $(\varphi_S, \gamma_i) = \lambda|(\varphi_S, \gamma_i)|$ for all $i \in S$, is then equivalent to $\varphi_S \in \lambda C$ if $\lambda = \pm 1$ and to $\varphi_S = 0$ if $\lambda = 0$. With all this at hand, the statement follows by Theorem 3.4.4 and Theorem 3.4.5. □

Actually, the statement of Corollary 3.4.7 may be slightly refined. Indeed, in order to establish whether there exists a vector having orbit with special canonical almost-complex structure, it suffices to check the second condition of the above statment for S the set of indices of simple non-compact roots.

Corollary 3.4.8. *For any $\varphi \in \mathfrak{h}^*_{\mathbb{R}}$ and any real $\lambda \in \{-1, 0, 1\}$, the following conditions are equivalent:*

- *φ belongs to the dominant Weyl chamber C, the stabilizer of $v = ih_\varphi$ is compact and $\varphi' = \lambda\varphi$;*
- *for $S = \{i \mid \gamma_i \in \Sigma_n^+\}$, $\varphi_S = \eta - 2\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in S^c\}\cap\Delta^+}\alpha$ satisfies $(\varphi_S, \gamma_i) = \lambda|(\varphi_S, \gamma_i)|$ for all $i \in S$. Moreover*

$$
\varphi = \begin{cases} \lambda\varphi_S & \text{if} \quad \lambda = \pm 1 \\ \sum_{i\in S} v^i\varphi_i \text{ for some } v^i > 0 & \text{if} \quad \lambda = 0 \end{cases}
\tag{3.98}
$$

Proof. In order to simplify the notation, put $P = \{i \mid \gamma_i \in \Sigma_n^+\}$. We discuss the cases $\lambda = -1, 0$ and $\lambda = 1$ separately. In the first case we will prove that if there exists a set $S \supseteq P$ such that $(\varphi_S, \gamma_i) = \lambda|(\varphi_S, \gamma_i)|$ for all $i \in S$, then $S = P$. For the case $\lambda = 1$, we will prove that if there exists a set $S \supseteq P$ such that $(\varphi_S, \gamma_i) = \lambda|(\varphi_S, \gamma_i)|$ for all $i \in S$, then the same condition is satisfied also for P and φ_P, thus it suffices to prove it for P. The statement then follows by Corollary 3.4.7.

Assume that there exists a set $S \supseteq P$ such that the vector

$$
\varphi_S = \eta - 2\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in S^c\}\cap\Delta^+}\alpha
\tag{3.99}
$$

satisfies $(\varphi_S, \gamma_i) = \lambda|(\varphi_S, \gamma_i)|$, for each $i \in S$, with $\lambda = -1, 0$. Thus, by Corollary 3.4.7, the adjoint orbit of φ_S is special symplectic general type or special symplectic Calabi-Yau. Choose an index $i \in S \setminus P$, so that the simple root γ_i is compact and consider the reflection of φ_S with respect to γ_i

$$\begin{aligned}\sigma_{\gamma_i}(\varphi_S) =&\sigma_{\gamma_i}\left(\eta - 2\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in S^c\}\cap\Delta^+}\alpha\right)\\ =&\eta - 4\gamma_i - 2\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in S^c\}\cap\Delta^+}\alpha + 4\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in S^c\}\cap\Delta^+}\frac{(\alpha,\gamma_i)}{(\gamma_i,\gamma_i)}\gamma_i\\ =&\varphi_S - \left(4 - 4\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in S^c\}\cap\Delta^+}\frac{(\alpha,\gamma_i)}{(\gamma_i,\gamma_i)}\right)\gamma_i.\end{aligned} \tag{3.100}$$

From the explicit formula for reflections, one has

$$2 - 2\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in S^c\}\cap\Delta^+}\frac{(\alpha,\gamma_i)}{(\gamma_i,\gamma_i)} = \frac{(\varphi_S,\gamma_i)}{(\gamma_i,\gamma_i)}, \tag{3.101}$$

which simplifies as

$$|\gamma_i|^2 - \sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in S^c\}\cap\Delta^+}(\alpha,\gamma_i) = \frac{1}{2}(\varphi_S,\gamma_i). \tag{3.102}$$

By writing the second summand of left-hand side in (3.102) in terms of simple roots

$$\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in S^c\}\cap\Delta^+}\alpha = \sum_{j\in S^c}a^j\gamma_j \tag{3.103}$$

one has

$$\left(\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in S^c\}\cap\Delta^+}\alpha,\gamma_i\right) = \left(\sum_{j\in S^c}a^j\gamma_j,\gamma_i\right) = \sum_{j\in S^c}a^j(\gamma_j,\gamma_i) \leq 0 \tag{3.104}$$

since $j \in S$ and the a^j's are positive. Thus

$$(\varphi_S,\gamma_i) = 2|\gamma_i|^2 - 2\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in S^c\}\cap\Delta^+}(\alpha,\gamma_i) > 0, \tag{3.105}$$

a contradiction by the assumption that the orbit is special symplectic general type or special symplectic Calabi-Yau. This shows that $S \setminus P = \emptyset$, hence $S = P$.

Assume now that there exists a set $S \supseteq P$ such that the vector φ_S as above satisfies $(\varphi_S,\gamma_i) = \lambda|(\varphi_S,\gamma_i)|$, for each $i \in S$, with $\lambda = 1$. Hence, by Corollary 3.4.7, the adjoint orbit of φ_S is special symplectic Fano. Notice that, since $S^c \subseteq P^c$, one has

$$\begin{aligned}\varphi_P =&\eta - 2\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in P^c\}\cap\Delta^+}\alpha\\ =&\eta - 2\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in S^c\}\cap\Delta^+}\alpha - 2\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in P^c\setminus S^c\}\cap\Delta^+}\alpha\\ =&\varphi_S - 2\sum_{\alpha\in\operatorname{span}\{\gamma_i|i\in P^c\setminus S^c\}\cap\Delta^+}\alpha.\end{aligned} \tag{3.106}$$

By writing $\sum_{\alpha\in\mathrm{span}\{\gamma_i|i\in P^c\setminus S^c\}\cap\Delta^+}\alpha$ in terms of simple roots

$$\sum_{\alpha\in\mathrm{span}\{\gamma_i|i\in P^c\setminus S^c\}\cap\Delta^+}\alpha=\sum_{i\in P^c\setminus S^c}b^i\gamma_i \tag{3.107}$$

and arguing as above, it turns out that for $j\in P$

$$(\varphi_P,\gamma_j)=(\varphi_S,\gamma_j)-2\sum_{i\in P^c\setminus S^c}b^i(\gamma_i,\gamma_j)>0. \tag{3.108}$$

This shows that P and φ_P satisfy $(\varphi_P,\gamma_i)=\lambda|(\varphi_P,\gamma_i)|$ for all $i\in P$, hence the adjoint orbit of φ_P is special symplectic Fano too. □

Corollary 3.4.8 shows that the condition for a vector to admit adjoint orbit with special compatible almost-complex structure does not depend on the choice of the set S, but on the algebraic structure of the underlying Lie algebra $\mathfrak{g}$. In particular, it depends on the Vogan diagram of $\mathfrak{g}$ only, in a sense that will be more clear in Section 3.7. Notice that corollary above gives a further proof of uniqueness for special symplectic general type adjoint orbits.

One may give a geometric flavour to Corollary 3.4.8. Let φ_P and φ_S be two distinct vectors belonging to the dominant Weyl chamber such that there exist ψ_P, ψ_S satisfying $\psi_P=\lambda_P\varphi_P$ and $\psi_S=\lambda_S\varphi_S$, with $S\supseteq P$. Called V_P and V_S their respective stabilizers, the adjoint orbits G/V_P and G/V_S of ψ_P and ψ_S respectively admit special canonical almost-complex structures. The inclusion $P\subset S$ induces an inclusion $V_S\subset V_P$ which, in turn, induces a fibration $G/V_S\to G/V_P$ with compact fibers V_S/V_P. In particular, the fibers are compact homogeneous complex manifolds admitting a Kähler-Einstein metric, by the result of Borel and Weil [73]. Moreover, they are compact and homogeneous of Fano type. If the associated metric on G/V_S is special, then the basis of the fibration should be symplectic Fano, that is $\lambda_S>0$, since the metric on G/V_S is locally a product of the ones on G/V_P and V_S/V_P.

The next section is concerned with some geometric objects attached to an almost-complex structure. In particular, we study them in case the almost-complex structure is special.

3.5 Geometric formulae

In this section we provide formulae for the Hermitian scalar curvature and the Nijenhuis tensor of the canonical almost-complex structure in terms of root data, as they are quite natural objects arising in the study almost-complex structures. We are going to use these formulae for computations on explicit examples (Appendix B).

3.5.1 Hermitian scalar curvature

In this section we give a formula for the Hermitian scalar curvature associated with the canonical almost-complex structure on the adjoint orbit G/V of φ. Since, in our context, both the symplectic form ω and the almost-complex structure J are homogeneous on G/V, the Hermitian scalar curvature s of J is constant. Thus, it suffices to compute it at the identity coset $e\in G/V$. By Lemma 3.3.4, a symplectic basis of $\mathfrak{m}$ is given by $(1/\sqrt{\lambda_\alpha})u_\alpha$, $(\varepsilon_\alpha/\sqrt{\lambda_\alpha})v_\alpha$, for $\alpha\in\Delta^+\setminus\varphi^\perp$. As we have seen at the end of Section 1.4, the Hermitian scalar curvature has a very explicit expression once one has a symplectic basis (1.73). So let $e_1,\dots,e_n,Je_1,\dots,Je_n\in T_e(G/V)$ be a symplectic basis at the identity coset $e\in G/V$. Then the Hermitian scalar curvature at e is given by

$$s(e)=\sum_{i=1}^n\rho(e_i,Je_i). \tag{3.109}$$

We observed in Section 3.4 that the Chern-Ricci form of J is defined by $\rho(X,Y) = B(v', [X,Y])$ where $v' = 2\sum_{\alpha\in\Delta^+\setminus\varphi^\perp}[u_\alpha, v_\alpha]$. Thus, the Hermitian scalar curvature is given by

$$s = \sum_{\alpha\in\Delta^+\setminus\varphi^\perp} \frac{1}{\lambda_\alpha}\rho(u_\alpha, \varepsilon_\alpha v_\alpha) = \sum_{\alpha\in\Delta^+\setminus\varphi^\perp} \frac{1}{\lambda_\alpha}B(v', [u_\alpha, \varepsilon_\alpha v_\alpha]). \tag{3.110}$$

Collecting all the terms inside B, one has $s = B(v', z)$, where

$$z = \sum_{\alpha\in\Delta^+\setminus\varphi^\perp} \frac{\varepsilon_\alpha}{\lambda_\alpha}[u_\alpha, v_\alpha]. \tag{3.111}$$

By the definition of u_α and v_α (3.48), the expression of z may be simplified as

$$z = -\sum_{\alpha\in\Delta^+\setminus\varphi^\perp} \frac{i}{\lambda_\alpha}h_\alpha. \tag{3.112}$$

However, writing the Hermitian scalar curvature in terms of weights is more suitable for our purposes.

Lemma 3.5.1. *The Hermitian scalar curvature of J is given by*

$$s = -2\sum_{\alpha,\beta\in\Delta^+\setminus\varphi^\perp} \varepsilon_\alpha\frac{(\alpha,\beta)}{(\varphi,\beta)}. \tag{3.113}$$

Proof. Write $z = ih_\zeta$ and $v' = ih_{\varphi'}$ where $\zeta, \varphi' \in \mathfrak{h}^*_{\mathbb{R}}$ are defined by

$$\zeta = -\sum_{\alpha\in\Delta^+\setminus\varphi^\perp} \frac{1}{\lambda_\alpha}\alpha, \qquad \varphi' = -2\sum_{\alpha\in\Delta^+\setminus\varphi^\perp}\varepsilon_\alpha\alpha. \tag{3.114}$$

Then

$$s = B(v', z) = -(\varphi', \zeta). \tag{3.115}$$

By writing explicitly φ' and ζ

$$s = -(\varphi', \zeta) = -2\sum_{\alpha,\beta\in\Delta^+\setminus\varphi^\perp} \frac{\varepsilon_\alpha}{\lambda_\beta}(\alpha,\beta). \tag{3.116}$$

Then the formula follows by recalling the definition of $\lambda_\beta = (\varphi, \beta)$, if β is a positive root. ☐

When an almost-complex structure on a symplectic manifold satisfies $\rho = \lambda\omega$, the Hermitian scalar curvature is constant and equal to $s = n\lambda$, where $2n$ is the dimension of the manifold. In particular, in our context, we have the following dimension formula.

Proposition 3.5.2. *If $\varphi' = \lambda\varphi$ with $\lambda \neq 0$, then*

$$\dim G/V = -\frac{4|\varphi|^2}{(\eta,\varphi)}\sum_{\alpha,\beta\in\Delta^+\setminus\varphi^\perp} \varepsilon_\alpha\frac{(\alpha,\beta)}{(\varphi,\beta)}. \tag{3.117}$$

Proof. Since $\varphi' = \lambda\varphi$, J satisfies $\rho = \lambda\omega$ and, by the observation above, $s = n\lambda$. Thus, $\dim G/V = 2n = 2s/\lambda$. By Lemma 3.4.1, $\lambda = \frac{(\eta,\varphi)}{|\varphi|^2}$, so that

$$\dim G/V = -\frac{4}{\lambda}\sum_{\alpha,\beta\in\Delta^+\setminus\varphi^\perp} \varepsilon_\alpha\frac{(\alpha,\beta)}{(\varphi,\beta)} = -\frac{4|\varphi|^2}{(\eta,\varphi)}\sum_{\alpha,\beta\in\Delta^+\setminus\varphi^\perp} \varepsilon_\alpha\frac{(\alpha,\beta)}{(\varphi,\beta)}. \tag{3.118}$$

☐

3.5.2 Nijenhuis tensor

In this section we provide some formulae expressing quantities related to the Nijenhuis tensor of the canonical almost-complex structure in terms of root data. Recall that the Nijenhuis tensor of an almost-complex structure J on a manifold M is a skew-symmetric $(2,1)$-tensor defined by

$$N(X,Y) = \frac{1}{4}\left([JX,JY] - J[JX,Y] - J[X,JY] - [X,Y]\right). \tag{3.119}$$

We recall the following identities

$$N(JX,Y) = N(X,JY) = -JN(X,Y), \tag{3.120}$$

that will be useful in the next computations.

If J is compatible with a symplectic form ω, then the pointwise norm $|N|^2$ of N with respect to the induced Riemannian metric defines a smooth function on M. Moreover, J is integrable if and only if $|N|^2 = 0$. In order to compute $|N|^2$ at a point $p \in M$, choose a symplectic basis of the form $e_1, Je_1, \ldots, e_n, Je_n$ of T_pM so that

$$|N|^2(p) = 2\sum_{i,j=1}^{n} |N(e_i,e_j)|^2. \tag{3.121}$$

In our context, the almost-complex structure J is homogeneous on the adjoint orbit G/V, thus its Nijenhuis tensor is homogeneous as well. In particular, it is completely determined by its value at the identity coset $e \in G/V$, where it may be described by the skew-symmetric bilinear map

$$N_H : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}. \tag{3.122}$$

As the symplectic form ω is homogeneous too, the squared norm $|N|^2$ is a constant function on G/V. In what follows, we compute N_H and $|N|^2$ in terms of root data.

Recall that we defined the basis $\{u_\alpha, v_\alpha\}$ on $\mathfrak{m}$ by (3.48) and the almost-complex structure H by (3.61). So, by Proposition 1.4.9, N_H is given by

$$N_H(u_\alpha,u_\beta) = \varepsilon_\alpha\varepsilon_\beta[v_\alpha,v_\beta]_{\mathfrak{m}} - \varepsilon_\alpha H[v_\alpha,u_\beta]_{\mathfrak{m}} - \varepsilon_\beta H[u_\alpha,v_\beta]_{\mathfrak{m}} - [u_\alpha,u_\beta]_{\mathfrak{m}}, \tag{3.123}$$

where the subscript $\mathfrak{m}$ denotes the projection of the commutator onto $\mathfrak{m}$. In addition, by the identities (3.120), N_H satisfies

$$N_H(u_\alpha,v_\beta) = -\varepsilon_\beta H N_H(u_\alpha,u_\beta), \quad N_H(v_\alpha,v_\beta) = -\varepsilon_\alpha\varepsilon_\beta N_H(u_\alpha,u_\beta). \tag{3.124}$$

Identities (3.124) together with skew-symmetry of N_H imply that N_H is entirely determined by $N_H(u_\alpha,u_\beta)$, for $\alpha,\beta \in \Delta^+ \setminus \varphi^\perp$, which has an explicit formula as the following lemma shows.

Lemma 3.5.3. *For all $\alpha,\beta \in \Delta^+ \setminus \varphi^\perp$ one has*

$$N_H(u_\alpha,u_\beta) = \frac{(\varepsilon_\alpha+1)(\varepsilon_\beta+1)}{4\sqrt{2}} N_{\alpha,\beta}\, v_{\alpha+\beta} + \frac{(\varepsilon_\alpha\varepsilon_\beta-1)s_{\alpha-\beta} + \varepsilon_\alpha - \varepsilon_\beta}{4\sqrt{2}} N_{\alpha,-\beta}\,(v_{\alpha-\beta})_{\mathfrak{m}}, \tag{3.125}$$

where $(v_{\alpha-\beta})_{\mathfrak{m}}$ denotes the component of $v_{\alpha-\beta}$ along $\mathfrak{m}$ according to the decomposition $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{m}$ and the constants $N_{\alpha,\beta}$, $N_{\alpha,-\beta}$ are as in Theorem 3.2.7.

Proof. The component $N_H(u_\alpha, u_\beta)$ has the explicit form

$$\begin{aligned} N_H(u_\alpha, u_\beta) =& \frac{1}{4}\left([Hu_\alpha, Hu_\beta]_{\mathfrak{m}} - H[Hu_\alpha, u_\beta]_{\mathfrak{m}} - H[u_\alpha, Hu_\beta]_{\mathfrak{m}} - [u_\alpha, u_\beta]_{\mathfrak{m}}\right) \\ =& \frac{1}{4}\left(\varepsilon_\alpha\varepsilon_\beta[v_\alpha, v_\beta]_{\mathfrak{m}} - \varepsilon_\alpha H[v_\alpha, u_\beta]_{\mathfrak{m}} - \varepsilon_\beta H[u_\alpha, v_\beta]_{\mathfrak{m}} - [u_\alpha, u_\beta]_{\mathfrak{m}}\right). \end{aligned} \tag{3.126}$$

We compute separately all summands of (3.126). Let $\alpha, \beta \in \Delta^+ \setminus \varphi^\perp$. In the computations several identities are used: the definintion of u_α and v_α, the identities characterizing the coefficients $N_{\alpha,\beta}$ and $\varepsilon_{\alpha+\beta} = -\varepsilon_\alpha\varepsilon_\beta$.

$$\begin{aligned} [v_\alpha, v_\beta]_{\mathfrak{m}} =& -\frac{i^{\frac{-1-\varepsilon_\alpha-\varepsilon_\beta-\varepsilon_\alpha\varepsilon_\beta}{2}}}{\sqrt{2}}(N_{\alpha,\beta}v_{\alpha+\beta} - s_{\alpha-\beta}N_{\alpha,-\beta}(v_{\alpha-\beta})_{\mathfrak{m}}) \\ [v_\alpha, u_\beta]_{\mathfrak{m}} =& -\frac{i^{\frac{-1-\varepsilon_\alpha-\varepsilon_\beta-\varepsilon_\alpha\varepsilon_\beta}{2}}}{\sqrt{2}}(N_{\alpha,\beta}u_{\alpha+\beta} + N_{\alpha,-\beta}(u_{\alpha-\beta})_{\mathfrak{m}}) \\ [u_\alpha, v_\beta]_{\mathfrak{m}} =& -\frac{i^{\frac{-1-\varepsilon_\alpha-\varepsilon_\beta-\varepsilon_\alpha\varepsilon_\beta}{2}}}{\sqrt{2}}(N_{\alpha,\beta}u_{\alpha+\beta} - N_{\alpha,-\beta}(u_{\alpha-\beta})_{\mathfrak{m}}) \\ [u_\alpha, u_\beta]_{\mathfrak{m}} =& \frac{i^{\frac{-1-\varepsilon_\alpha-\varepsilon_\beta-\varepsilon_\alpha\varepsilon_\beta}{2}}}{\sqrt{2}}(N_{\alpha,\beta}v_{\alpha+\beta} + s_{\alpha-\beta}N_{\alpha,-\beta}(v_{\alpha-\beta})_{\mathfrak{m}}) \end{aligned} \tag{3.127}$$

Before continuing, we make a couple of observations which clarify the notation. First, for $\alpha, \beta \in \Delta^+ \setminus \varphi^\perp$, the sum $\alpha+\beta$ may not be a root. In this case, $N_{\alpha,\beta} = 0$ so that we do not have to care about $v_{\alpha+\beta}$. On the other hand, if $\alpha+\beta$ is a root, then it belongs to $\Delta^+ \setminus \varphi^\perp$, as φ belongs to the dominant Weyl chamber and

$$(\varphi, \alpha+\beta) = (\varphi, \alpha) + (\varphi, \beta) > 0. \tag{3.128}$$

Then, for all $\alpha, \beta \in \Delta^+ \setminus \varphi^\perp$, the difference $\alpha - \beta$ is not necessarily a positive root and, moreover, it may be orthogonal to φ. The former is not a big deal, as $N_{\alpha,\beta}v_{\alpha-\beta}$ is invariant under switching α and β and it vanishes when $\alpha-\beta$ is not a root. On the other hand, one has

$$(\varphi, \alpha-\beta) = (\varphi, \alpha) - (\varphi, \beta), \tag{3.129}$$

thus one among $\alpha-\beta$ and $\beta-\alpha$ belongs to $\Delta^+ \setminus \varphi^\perp$ precisely when $(\varphi,\alpha) \neq (\varphi,\beta)$. In this case $s_{\alpha-\beta}(\varphi, \alpha-\beta) > 0$. Hence, either $(v_{\alpha-\beta})_{\mathfrak{m}} = 0$ and $(\varphi, \alpha-\beta) = 0$, or $(v_{\alpha-\beta})_{\mathfrak{m}} = v_{\alpha-\beta}$ and $(\varphi, \alpha-\beta) \neq 0$.

Plugging all summands in identity (3.126), we get

$$\begin{aligned} N_H(u_\alpha, u_\beta) =& \frac{i^{\frac{-1-\varepsilon_\alpha-\varepsilon_\beta-\varepsilon_\alpha\varepsilon_\beta}{2}}}{4\sqrt{2}}(N_{\alpha,\beta}v_{\alpha+\beta}(-\varepsilon_\alpha\varepsilon_\beta + \varepsilon_\alpha\varepsilon_{\alpha+\beta} + \varepsilon_\beta\varepsilon_{\alpha+\beta} - 1) + \\ &+ N_{\alpha,-\beta}(v_{\alpha-\beta})_{\mathfrak{m}}(s_{\alpha-\beta}\varepsilon_\alpha\varepsilon_\beta + \varepsilon_\alpha\varepsilon_{\alpha-\beta} - \varepsilon_\beta\varepsilon_{\alpha-\beta} - s_{\alpha-\beta})) \\ =& \frac{i^{\frac{-1-\varepsilon_\alpha-\varepsilon_\beta-\varepsilon_\alpha\varepsilon_\beta}{2}}}{4\sqrt{2}}(N_{\alpha,\beta}v_{\alpha+\beta}(-\varepsilon_\alpha\varepsilon_\beta - \varepsilon_\beta - \varepsilon_\alpha - 1) + \\ &+ N_{\alpha,-\beta}(v_{\alpha-\beta})_{\mathfrak{m}}(s_{\alpha-\beta}\varepsilon_\alpha\varepsilon_\beta - \varepsilon_\beta + \varepsilon_\alpha - s_{\alpha-\beta})) \\ =& \frac{i^{\frac{-1-\varepsilon_\alpha-\varepsilon_\beta-\varepsilon_\alpha\varepsilon_\beta}{2}}}{4\sqrt{2}}(-N_{\alpha,\beta}v_{\alpha+\beta}(\varepsilon_\alpha+1)(\varepsilon_\beta+1) + \\ &+ N_{\alpha,-\beta}(v_{\alpha-\beta})_{\mathfrak{m}}((\varepsilon_\alpha\varepsilon_\beta - 1)s_{\alpha-\beta} + \varepsilon_\alpha - \varepsilon_\beta). \end{aligned} \tag{3.130}$$

Hence

$$\begin{aligned} N_H(u_\alpha, u_\beta) =& \frac{i^{\frac{-1-\varepsilon_\alpha-\varepsilon_\beta-\varepsilon_\alpha\varepsilon_\beta}{2}}}{4\sqrt{2}}(-N_{\alpha,\beta}v_{\alpha+\beta}(\varepsilon_\alpha+1)(\varepsilon_\beta+1)+ \\ &+ N_{\alpha,-\beta}(v_{\alpha-\beta})_{\mathfrak{m}}((\varepsilon_\alpha\varepsilon_\beta-1)s_{\alpha-\beta}+\varepsilon_\alpha-\varepsilon_\beta)). \end{aligned} \tag{3.131}$$

The first summand is non-zero only if both α and β are non-compact and, in this case, the coefficient $\frac{i^{\frac{-1-\varepsilon_\alpha-\varepsilon_\beta-\varepsilon_\alpha\varepsilon_\beta}{2}}}{4\sqrt{2}}$ is equal to -1. On the other hand, the second summand may be non-zero if α and β have different compactness indices and, in this case, the coefficient $\frac{i^{\frac{-1-\varepsilon_\alpha-\varepsilon_\beta-\varepsilon_\alpha\varepsilon_\beta}{2}}}{4\sqrt{2}}$ is equal to 1. This concludes the proof. □

Remark 3.5.4. As we have seen in the above proof, the coefficients $(\varepsilon_\alpha+1)(\varepsilon_\beta+1)$ vanish as soon as one root among α and β is compact. In a similar way, $(\varepsilon_\alpha\varepsilon_\beta-1)s_{\alpha-\beta}+\varepsilon_\alpha-\varepsilon_\beta=0$ when α and β are both compact or non-compact. Summing up, we have

$$N_H(u_\alpha, u_\beta) = \begin{cases} 0 & \text{if} \quad \alpha,\beta\in\Delta_c^+ \\ \frac{1}{\sqrt{2}}N_{\alpha,\beta}v_{\alpha+\beta} & \text{if} \quad \alpha,\beta\in\Delta_n^+ \\ \frac{\varepsilon_\alpha-s_{\alpha-\beta}}{2\sqrt{2}}N_{\alpha,-\beta}(v_{\alpha-\beta})_{\mathfrak{m}} & \quad\text{otherwise} \end{cases} \tag{3.132}$$

where Δ_c^+ and Δ_n^+ are set set of positive compact and non-compact roots respectively.

At this point we may express the squared norm of N in terms of root data. Recall that, by Lemma 3.3.4, there is a symplectic orthonormal basis on $\mathfrak{m}$ constituted by $\{(1/\sqrt{\lambda_\alpha})u_\alpha, (\varepsilon_\alpha/\sqrt{\lambda_\alpha})v_\alpha\}$. Thus, by Lemma 3.5.3 and the formula for the norm of N at a point (3.121), one may compute explicitly the squared norm of N as follows.

Theorem 3.5.5. *The squared norm of the Nijenhuis tensor of J is given by*

$$|N|^2 = \sum_{\alpha,\beta\in\Delta_n^+}\frac{(\varphi,\alpha+\beta)}{(\varphi,\alpha)(\varphi,\beta)}{N^2}_{\alpha,\beta} + \sum_{\substack{\alpha\in\Delta_c^+\setminus\varphi^\perp\\ \beta\in\Delta_n^+}}(1+s_{\alpha-\beta})\frac{(\varphi,\alpha-\beta)}{(\varphi,\alpha)(\varphi,\beta)}{N^2}_{\alpha,-\beta} \tag{3.133}$$

where the coefficients ${N^2}_{\alpha,\beta}$ are as in Theorem 3.2.7.

Proof. Recalling the definition of $\lambda_\alpha=(\varphi,\alpha)$, for $\alpha\in\Delta^+\setminus\varphi^\perp$, by identity (3.121) and the defintion of the symplectic basis $\{(1/\sqrt{\lambda_\alpha})u_\alpha, (\varepsilon_\alpha/\sqrt{\lambda_\alpha})v_\alpha\}$ one has

$$|N|^2 = 2\sum_{\alpha,\beta\in\Delta^+\setminus\varphi^\perp}\frac{|N_H(u_\alpha,u_\beta)|^2}{\lambda_\alpha\lambda_\beta} = 2\sum_{\alpha,\beta\in\Delta^+\setminus\varphi^\perp}\frac{|N_H(u_\alpha,u_\beta)|^2}{(\varphi,\alpha)(\varphi,\beta)}. \tag{3.134}$$

Thus, by Lemma 3.5.3 and Remark 3.5.4, one gets

$$\begin{aligned} |N|^2 =& \sum_{\alpha,\beta\in\Delta_n^+\setminus\varphi^\perp}\frac{{N^2}_{\alpha,\beta}|v_{\alpha+\beta}|^2}{(\varphi,\alpha)(\varphi,\beta)} + \sum_{\substack{\alpha\in\Delta_c^+\setminus\varphi^\perp\\ \beta\in\Delta_n^+\setminus\varphi^\perp}}\frac{(1+s_{\alpha-\beta})^2}{4(\varphi,\alpha)(\varphi,\beta)}{N^2}_{\alpha,-\beta}|(v_{\alpha-\beta})_{\mathfrak{m}}|^2+ \\ &+\sum_{\substack{\alpha\in\Delta_n^+\setminus\varphi^\perp\\ \beta\in\Delta_c^+\setminus\varphi^\perp}}\frac{(1-s_{\alpha-\beta})^2}{4(\varphi,\alpha)(\varphi,\beta)}{N^2}_{\alpha,-\beta}|(v_{\alpha-\beta})_{\mathfrak{m}}|^2. \end{aligned} \tag{3.135}$$

By the identities ${N^2}_{\alpha,-\beta} = {N^2}_{\beta,-\alpha}$, $v_{\alpha-\beta} = v_{\beta-\alpha}$, $s_{\alpha-\beta} = -s_{\beta-\alpha}$ and $(1+s_{\alpha-\beta})^2 = 2(1+s_{\alpha-\beta})$, one may sum the last two summands in (3.135) to obtain

$$|N|^2 = \sum_{\alpha,\beta\in\Delta_n^+\setminus\varphi^\perp} \frac{{N^2}_{\alpha,\beta}|v_{\alpha+\beta}|^2}{(\varphi,\alpha)(\varphi,\beta)} + \sum_{\substack{\alpha\in\Delta_c^+\setminus\varphi^\perp \\ \beta\in\Delta_n^+\setminus\varphi^\perp}} \frac{1+s_{\alpha-\beta}}{(\varphi,\alpha)(\varphi,\beta)} {N^2}_{\alpha,-\beta}|(v_{\alpha-\beta})_{\mathfrak{m}}|^2. \tag{3.136}$$

To conclude, observe that, by Lemma 3.3.1 and the definitions of σ and H, one has

$$\begin{aligned} |v_{\alpha+\beta}|^2 =& \sigma(v_{\alpha+\beta}, Hv_{\alpha+\beta}) \\ =& -\varepsilon_{\alpha+\beta}B([v,v_{\alpha+\beta}],u_{\alpha+\beta}) \\ =& \varepsilon_{\alpha+\beta}\lambda_{\alpha+\beta}B(u_{\alpha+\beta},u_{\alpha+\beta}) \\ =& \lambda_{\alpha+\beta} \\ =& (\varphi,\alpha+\beta) \end{aligned} \tag{3.137}$$

and, similarly, $|(v_{\alpha-\beta})_{\mathfrak{m}}|^2 = s_{\alpha-\beta}(\varphi,\alpha-\beta)$. The thesis follows by substituting these norms in (3.136) and observing that $\Delta_n^+\setminus\varphi^\perp = \Delta_n^+$, since non-compact roots cannot be orthogonal to φ. □

Remark 3.5.6. As we observed in the proof of Lemma 3.5.3, $s_{\alpha-\beta}(\varphi,\alpha-\beta) \geq 0$, since φ belongs to the dominant Weyl chamber C. This shows that all summands of the formula for $|N|^2$ in Theorem 3.5.5 are positive. In particular, the almost-complex structure J turns out to be non-integrable as soon as there are two non-compact positive roots α and β such that $\alpha+\beta$ is a root.

3.6 Compact quotients

As in the previous sections, let $v \in \mathfrak{g}$ be an element with compact stabilizer $V \subset G$ and consider its adjoint orbit G/V endowed with the Kirillov-Kostant-Souriau symplectic form and the canonical almost-complex structure J. This section is concerned with locally homogeneous compact manifolds of the form

$$M = \Gamma\backslash G/V, \tag{3.138}$$

where $\Gamma \subset G$ is a discrete co-compact subgroup without torsion, whose existence is guaranteed by a theorem of Borel [12]. Since ω and J are G-invariant, they descend to a symplectic form and an almost-complex structure on M, that we denote ω_Γ and J_Γ respectively. Thus $(M,\omega_\Gamma,J_\Gamma)$ is a compact almost-Kähler manifold. Leaving out for a moment the almost-complex structure J_Γ we, ask the following question.

Question 3.6.1. *Does the compact symplectic manifold (M,ω_Γ) admit a (non-necessarily locally homogeneous) compatible complex structure?*

In other words, we ask if, on M, there exists an integrable almost-complex structure, say J', among all almost-complex structures which are compatible with ω_Γ. This would make (M,ω_Γ,J') a Kähler manifold.

The aim of this section is to give some partial answers to Question 3.6.1. Notice that all geometric objects on G/V such as the Chern-Ricci form, the Hermitian scalar curvature and the Nijenhuis tensor of J, descend to the corresponding objects on $(M,\omega_\Gamma,J_\Gamma)$, as they are G-invariant, hence Γ-invariant. In particular, answer to Question 3.6.1 is affirmative when J is yet integrable on G/V, so that $(M,\omega_\Gamma,J_\Gamma)$ is a Kähler manifold.

On the other hand, under certain hypothesis involving just the Lie group, answer to Question 3.6.1 is negative, as a consequence of the theorem due to Carlson and Toledo [15, Theorem 8.2].

Theorem 3.6.2. *Let $K \subset G$ be a maximal compact subgroup. If G/K is not Hermitian symmetric, then (M, ω_Γ) is not of the homotopy type of a compact Kähler manifold.*

Observe that, in order to fit with the statement of Theorem 3.6.2, one has to drop the symplectic form ω_Γ and to endow G/V with a homogeneous complex structure, say $\tilde{J}$, which always exists by the hypothesis on G and V [39, Section 2]. The key point is that $\tilde{J}$ is integrable, but it is not compatible with ω_Γ, as positiveness of the associated Riemannian metric fails.

Coming back to our adjoint orbit $(G/V, \omega, J)$, assume that J satisfies $\rho = \lambda\omega$, so that the same equation holds on the quotient M. Then one may conclude that the first Chern class c_1 of (M, ω_Γ), which is represented by $\rho_\Gamma/4\pi$ (see identity (1.44) or [27, Section 2]), satisfies

$$4\pi c_1 = \lambda[\omega_\Gamma] \in H^2_{\mathrm{dR}}(M, \mathbb{R}). \tag{3.139}$$

Therefore, if the canonical almost-complex structure J is special on G/V, i.e., $\rho = \lambda\omega$ for some $\lambda \in \mathbb{R}$, then the compact symplectic manifold (M, ω_Γ) turns out to be:

- symplectic general type if $\lambda < 0$,
- symplectic Calabi-Yau if $\lambda = 0$,
- symplectic Fano if $\lambda > 0$.

In the symplectic general type case, it might be that J itself is integrable. Hence J_Γ is integrable as well, providing many instances of positive answer to Question 3.6.1. In particular, this happens when G/V is Hermitian symmetric. We refer to Section 3.7 for many examples in the case G is simple. On the other hand, answer to Question 3.6.1 is always negative whenever M is special symplectic Fano. The obstruction has topological nature also in this case.

Lemma 3.6.3. *If (M, ω_Γ) is special symplectic Fano, then it is not of Kähler type.*

Proof. If J' were an integrable almost-complex structure compatible with ω_Γ, then (M, ω_Γ, J') would be a compact Kähler manifold with positive Ricci curvature. Thus, by Myers's Theorem [65], it would have finite fundamental group, contradicting the fact that M is covered by G/V, which is non-compact. □

With the techniques discussed in Section 3.7, we were able to produce several examples of adjoint orbits $(G/V, \omega, J)$ with G a non-compact simple Lie group. After having examined these examples, we are pushed to suspect that the answer to Question 3.6.1 is that a compact quotient (M, ω_Γ) is of Kähler type if and only if J_Γ is integrable. This proposal should be compared with the result of Carlson and Toledo [16, Theorem 0.1]. In that work, they consider a homogeneous complex structure $\tilde{J}$ on G/V (which always exists but it is rarely compatible with ω) and establish when the complex manifold $(M, \tilde{J}_\Gamma)$, with $M = \Gamma\backslash G/V$, admits a Kähler metric. Unfortunately, their approach seems to be hardly adaptable to our situation. We plan to come back to Question 3.6.1 in the future.

With the results developed through the previous sections, we are ready to give explicit examples of adjoint orbits admitting special canonical almost-complex structure. This is the goal of the next section.

3.7 Vogan diagrams

Vogan diagrams are combinatorial tools by which one may classify real semisimple Lie algebras [48, Chapter VI]. As we will see, they turn out to be useful objects for studying (and hopefully classify) adjoint orbits $(G/V, \omega, J)$ of non-compact simple Lie groups satisfying $\rho = \lambda\omega$. This final section is entirely dedicated to this purpose.

Let $\mathfrak{g}$ be a real semisimple Lie algebra. Recall that the Vogan diagram of $\mathfrak{g}$ is the Dynkin diagram of $\mathfrak{g}$ with some painted vertices (including no one and all) and some pairs of unpainted vertices related by an automorphism of order two that intertwines the elements of the pair. A vertex is painted when the corresponding simple root is non-compact, while paired unpainted vertices correspond to roots that are exchanged by the Cartan involution θ. Actually, not all Vogan diagrams are well behaved for our analysis. Indeed, the ones with *non-trivial automorphism*, meaning that there is at least one couple of automorphism-related vertices, will be ruled out form our study because of the following result.

Lemma 3.7.1. *Elements of a real semisimple Lie algebra having Vogan diagram with non-trivial automorphism cannot have compact isotropy.*

Proof. Let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan involution θ and corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The strategy is to prove that if $\mathfrak{g}$ has Vogan diagram with non-trivial automorphism, then its Cartan subalgebras are not contained in a maximally compact subalgebra. By [48, Proposition 6.59], it suffices to consider θ-stable Cartan subalgebras $\mathfrak{h} \subset \mathfrak{g}$, i.e., such that $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p})$. In this case, we put $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$, $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$ and, by Proposition [48, Proposition 6.70], purely imaginary roots are contained in $\mathfrak{t}$, while real ones are contained in $\mathfrak{a}$. So, assume that the Vogan diagram of $\mathfrak{g}$ has non-trivial automorphism and let $v \in \mathfrak{g}$ such that it is contained in a θ-stable Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. Then, $\mathfrak{a} \neq \{0\}$, as the Cartan involution acts non-trivially on the diagram, hence $\mathfrak{h}$ cannot be contained in a maximally compact subalgebra of $\mathfrak{g}$. Since $\mathfrak{h}$ is contained in the Lie algebra $\mathfrak{v}$ of the stabilizer of v, $\mathfrak{v}$ cannot be compact and so neither V. □

In view of our applications, by lemma above, we are reduced to consider Vogan diagrams with trivial automorphism. For this reason, from now on, a Vogan diagram will be simply a Dynkin diagram with some painted vertices. Moreover, dealing with adjoint orbits or Vogan diagrams is equivalent in this context, due to the following result.

Lemma 3.7.2. *Let G be a real semisimple Lie group with Lie algebra $\mathfrak{g}$ and let ℓ be the rank of $\mathfrak{g}$. To any $v \in \mathfrak{g}$ with compact stabilizer, one may associate a Vogan diagram and a vector $(v^1, \ldots, v^\ell) \in \mathbb{R}^\ell$ with $v^i \geq 0$. Moreover, $v^i > 0$ if the i-th node of the Vogan diagram is painted.*

Proof. As we explained in Section 3.2, given v as in the statement, it is possible to find a Cartan subalgebra $\mathfrak{h}_0 \in \mathfrak{g}$ containing v. Since the Killing form B restricts to a positive-definite scalar product on $\mathfrak{h}_\mathbb{R} = i\mathfrak{h}_0 \subset \mathfrak{g}_\mathbb{C}$, one may define $\varphi \in \mathfrak{h}_\mathbb{R}^*$ by $\varphi(h) = -B(iv, h)$, for all $h \in \mathfrak{h}_\mathbb{R}$. At this point one may choose a positive root system $\Delta^+ \subset \mathfrak{h}_\mathbb{R}^*$ so that φ belongs to the associated dominant Weyl chamber C or, equivalently, so that $(\varphi, \alpha) \geq 0$ for all $\alpha \in \Delta^+$. The dominant Weyl chamber also determines the set of simple roots $\Sigma^+ \subset \Delta^+$, which splits as $\Sigma^+ = \Sigma_c^+ \cup \Sigma_n^+$, where Σ_c^+ is the set of compact simple roots and Σ_n^+ is the set of non-compact ones. With all this at hand, one may define a Vogan diagram by taking the Dynkin diagram associated with Σ^+ and painting all the nodes corresponding to elements in Σ_n^+.

Actually, one may label the simple roots from 1 to ℓ, $\Sigma^+ = \{\gamma_1, \ldots, \gamma_\ell\}$, since the rank of $\mathfrak{g}$ is exactly ℓ. Denoted by $A = (A_{ij})$ the Cartan matrix of $\mathfrak{g}_\mathbb{C}$, the fundamental dominant weights $\varphi_1, \ldots, \varphi_\ell$ are given by

$$\varphi_j = \sum_{i=1}^{\ell} A^{ij} \gamma_i, \tag{3.140}$$

where $(A^{ij}) = A^{-1}$. As φ belongs to the dominant Weyl chamber C, by Lemma 3.2.6, one may write φ as

$$\varphi = \sum_{i=1}^{\ell} v^i \varphi_i \tag{3.141}$$

with all $v^i \geq 0$ for $i = 1, \ldots, \ell$ and $v^i > 0$ whenever $\gamma_i \in \Sigma_n^+$, by compactness assumption on the stabilizer of v. Thus, the vector $(v^1, \ldots, v^\ell) \in \mathbb{R}^\ell$ is the one claimed in the statement, and the thesis follows by the rule for painting the nodes in a Vogan diagram. □

Observe that one may associate different Vogan diagrams to the same element v. Indeed, by the above proof, the choice of a Cartan subalgebra containing v and the dominant Weyl chamber are not canonical at all. More precisely, uniqueness of the associated Vogan diagram fails when φ belongs to some walls of the dominant Weyl chamber C, so that there exists a different Weyl chamber C' that contains φ. Nevertheless, the vector $(v^1, \ldots, v^\ell)$ is uniquely determined once the Vogan diagram is chosen and the simple roots are labelled.

The correspondence established in Lemma 3.7.2 may be reversed. Indeed, starting with a connected Dynkin diagram, one may algorithmically find a positive root system Δ^+ for the associated complex Lie algebra. In a similar way, by Lemma 3.2.4, starting with a connected Vogan diagram for a real Lie algebra $\mathfrak{g}$, one may algorithmically find a positive root system Δ^+ and determine for each root $\alpha \in \Delta^+$ whether it is compact or not. Thus, denote by C the dominant Weyl chamber associated with Δ^+ and choose $\varphi \in C$ of the form $\varphi = \sum_{i=1}^{\ell} v^i \varphi_i$, where the φ_i's are the fundamental dominant weights and $v^i \geq 0$. Since we consider only Vogan diagrams with trivial automorphism, the element $v = ih_\varphi \in \mathfrak{g}$ has compact stabilizer V for the action of any Lie group having $\mathfrak{g}$ as Lie algebra. Actually, one may be more precise about the Lie algebra $\mathfrak{v}$ of the stabilizer V of v.

Proposition 3.7.3. *Up to isomorphism, the Lie algebra $\mathfrak{v}$ of the stabilizer of v decomposes as*

$$\mathfrak{v} = \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_r \oplus \mathbb{R}^m, \tag{3.142}$$

where $\mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_r$ is the compact Lie algebra associated with the Vogan diagram obtained by removing from the original Vogan diagram all vertices corresponding to the coefficients v^i that are non-zero, while m is the number of removed vertices. Moreover, r is the number of connected components of the obtained Vogan diagram and all the $\mathfrak{v}_i$'s are simple.

Proof. By Proposition 3.3.2, the Lie algebra of the stabilizer of v is given by

$$\mathfrak{v} = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Delta^+ \cap \varphi^\perp} \operatorname{span}\{u_\alpha, v_\alpha\}. \tag{3.143}$$

Consider the splitting

$$\Delta^+ \cap \varphi^\perp = \Delta_1^+ \cup \cdots \cup \Delta_r^+ \tag{3.144}$$

as union of irreducible positive root systems. It induces a decomposition of $\mathfrak{v}$ as in (3.142), where each $\mathfrak{v}_i$ is the Lie algebra corresponding to Δ_k^+ and $\mathbb{R}^m$ is the center of $\mathfrak{v}$. Since no non-compact root is orthogonal to φ, each $\mathfrak{v}_k$ is compact. We are left with proving that the dimension m of the center $\mathfrak{z}(\mathfrak{v})$ of $\mathfrak{v}$ is the same as the number of v^i's that are non-zero. Note that $\mathfrak{z}(\mathfrak{v})$ is a subalgebra of the Cartan subalgebra $\mathfrak{h}_0$. Thus, as a vector space, one may identify it with a subspace of $\mathfrak{h}_{\mathbb{R}}^*$, by mapping each element $z \in \mathfrak{z}(\mathfrak{v})$ to the co-vector ψ defined by $z = ih_\psi$. By definition of u_α and v_α (3.48), item 3 of Theorem 3.2.7 and the fact that $\mathfrak{z}(\mathfrak{v}) \in \mathfrak{h}_0$, one gets

$$\begin{aligned}[z, u_\alpha] =& \frac{i^{\frac{1-\varepsilon_\alpha}{2}}}{\sqrt{2}}[z, e_\alpha + e_{-\alpha}] \\ =& \frac{i^{\frac{1-\varepsilon_\alpha}{2}}}{\sqrt{2}}\alpha(z)(e_\alpha - e_{-\alpha}) \\ =& \frac{i^{\frac{3-\varepsilon_\alpha}{2}}}{\sqrt{2}}(\psi, \alpha)(e_\alpha - e_{-\alpha}) \\ =& (\psi, \alpha)v_\alpha.\end{aligned} \tag{3.145}$$

for $\alpha \in \Delta^+ \cap \varphi^\perp$. Thus, with similar computations for $[z, v_\alpha]$,

$$[z, u_\alpha] = (\psi, \alpha)v_\alpha, \qquad [z, v_\alpha] = -(\psi, \alpha)u_\alpha. \tag{3.146}$$

Since both commutators above have to be zero when z belongs to $\mathfrak{z}(\mathfrak{v})$, the center $\mathfrak{z}(\mathfrak{v})$ turns out to have the same dimension as the orthogonal complement to $\Delta^+ \cap \varphi^\perp$. As φ is expressed as a sum of fundamental dominant weights φ_i's which satisfy $2\frac{(\varphi_i,\gamma_j)}{(\gamma_j,\gamma_j)} = \delta_{ij}$, one may write

$$\Delta^+ \cap \varphi^\perp = \operatorname{span}\{\gamma_i \mid v^i = 0\} \cap \Delta^+. \tag{3.147}$$

Thus, the orthogonal complement of $\Delta^+ \cap \varphi^\perp$ coincides with $\operatorname{span}\{\varphi_i \mid v^i \neq 0\}$, which has cardinality equal to the number of nodes which have been removed from the Vogan diagram. □

At this point, we may consider the adjoint orbit G/V of v equipped with the Kirillov-Kostant-Souriau symplectic form ω and the canonical almost-complex structure J. As we discussed in Section 3.4, deciding whether J satisfies $\rho = \lambda\omega$ reduces to a combinatorial problem on φ (see Theorem 3.4.4 and Theorem 3.4.5) that may be treated at the Lie algebra level. In order to simplify the results below, we define φ to λ-*special* (or just *special*) if the orbit $(G/V, \omega, J)$ of $v = ih_\varphi \in \mathfrak{g}$ satisfies $\rho = \lambda\omega$. Hence, for any real non-compact simple Lie group G, all adjoint orbits G/V endowed with the Kirillov-Kostant-Souriau symplectic form ω and the canonical almost-complex structure J satisfying $\rho = \lambda\omega$, for some λ, may be algorithmically listed up to isomorphism and scaling. Indeed, by what we said above, this is equivalent to list (up to scaling) all special φ's for all possible connected Vogan diagrams.

As one may expect, the number of such φ's grows quite fast with the rank ℓ of the Vogan diagram. However, it is possible to implement an algorithm on a computer which lists all Vogan diagrams admitting a special φ. We did it and we ran the algorithm on a standard computer for Vogan diagrams of rank up to 11, thus including all diagrams of exceptional type. Actually, the program turned out to be extremely useful in order to make predictions and to understand which direction to follow. By making experiments, we learnt many aspects concerning the relations between special φ's and their Vogan diagrams that we were able to prove at a later time. All the details of the algorithm, including the flow chart and the full code, may be found in Appendix A, while in Appendix B we list all special φ's for all connected Vogan diagrams up to rank $\ell = 8$.

Even if the general pattern is still unclear, leaving little hope for a complete classification, some general existence and non-existence results may be red off directly from the Vogan diagram. The first result in this direction says that, under precise hypothesis, the fundamental dominant weights $\varphi_1, \ldots, \varphi_\ell$ are special.

Proposition 3.7.4. *Given a Vogan diagram with a unique painted node, let* $\Sigma^+ = \{\gamma_1, \dots, \gamma_\ell\}$ *be the associated set of simple roots. Assume that the unique painted node corresponds to the simple root* γ_p*. Then* φ_p *is special.*

Proof. Let C be the dominant Weyl chamber associated with Σ^+. By definition, φ_p belongs to C, as φ_p lies in a wall of C. Moreover, by Lemma 3.2.6, the set $\Delta^+ \setminus \varphi^\perp$ of all positive roots that are not orthogonal to φ_p is constituted by all positive roots $\alpha = \sum_{i=1}^{\ell} n^i \gamma_i$, with $n^p > 0$. Since the unique painted node is the p-th one, all simple roots γ_i with $i \neq p$ are compact. So let $i \neq p$ and call σ_{γ_i} the reflection with respect to the hyperplane orthogonal to γ_i. For any $\psi \in \mathfrak{h}_{\mathbb{R}}^*$ one has

$$(\varphi_p, \sigma_{\gamma_i}(\psi)) = (\varphi_p, \psi) - 2\frac{(\psi, \gamma_i)}{(\gamma_i, \gamma_i)}(\varphi_p, \gamma_i) = (\varphi_p, \psi) \tag{3.148}$$

since and (φ_p, γ_i) vanishes when $i \neq p$. Thus for all $\alpha \in \Delta^+ \setminus \varphi_p^\perp$

$$\sigma_{\gamma_i}(\alpha) \in \Delta^+ \setminus \varphi_p^\perp. \tag{3.149}$$

Consider now the co-vector $\varphi' = -2\sum_{\alpha\in\Delta^+\setminus\varphi_p^\perp} \varepsilon_\alpha \alpha$ and observe that, by (3.149), for $i \neq p$,

$$\sigma_{\gamma_i}(\varphi') = -2 \sum_{\alpha\in\Delta^+\setminus\varphi_p^\perp} \varepsilon_\alpha \sigma_{\gamma_i}(\alpha) = -2 \sum_{\alpha\in\Delta^+\setminus\varphi_p^\perp} \varepsilon_\alpha \alpha = \varphi'. \tag{3.150}$$

This shows that φ' is orthogonal to all compact simple roots, as the above equality holds for each $i \neq p$. As a consequence of Lemma 3.2.5, φ' must be a multiple of φ_p, thus φ_p is special by (3.77). □

Proposition 3.7.4 shows that Vogan diagrams with one painted node always admit a special φ. In addition, one may classify all the corresponding adjoint orbits. Here we prove this facts for adjoint orbits of classical non-compact simple Lie groups, while in Section B.2 one may find adjoint orbits of non-compact exceptional Lie groups corresponding to Vogan diagrams with one painted node. Recall that a simple Lie group is said to be *classical* if the underlying Dynkin diagram of its Lie algebra is of type A_ℓ, B_ℓ, C_ℓ, D_ℓ and *exceptional* if the underlying Dynkin diagram if of type G_2, F_4, E_6, E_7, E_8.

Theorem 3.7.5. *Let* G *be a real non-compact simple classical Lie group. There exists orbits* $(G/V, \omega, J)$ *satisfying* $\rho = \lambda\omega$ *if the Lie algebras of* G *and* V *and the constant* λ *are as in the following table*

$\mathfrak{g}$	$\mathfrak{v}$	λ	
$\mathfrak{su}(p,q)$	$\mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus \mathbb{R}$	-1	$p, q \geq 1$
$\mathfrak{so}(2p,q)$	$\mathfrak{su}(p) \oplus \mathfrak{so}(q) \oplus \mathbb{R}$	$p-q-1$	$p, q \geq 1$
$\mathfrak{so}^*(2\ell)$	$\mathfrak{su}(\ell) \oplus \mathbb{R}$	-1	$\ell \geq 4$
$\mathfrak{sp}(p,q)$	$\mathfrak{su}(p) \oplus \mathfrak{sp}(q) \oplus \mathbb{R}$	$p-2q+1$	$p, q \geq 1$
$\mathfrak{sp}(\ell,\mathbb{R})$	$\mathfrak{su}(\ell) \oplus \mathbb{R}$	-1	$\ell \geq 3$

Proof. By Proposition 3.7.4, we know that Vogan diagrams with one painted node admit a special vector. In order to understand which Lie algebra corresponds to a Vogan diagram it suffices to look at the diagrams in [48, Page 414], while for the stabilizers we use Lemma 3.7.3.

For the coefficients λ some work is required. Observe first that $\mathfrak{su}(p,q)$, $\mathfrak{so}^*(2\ell)$ and $\mathfrak{sp}(\ell,\mathbb{R})$ have the property that the quotient of their associated Lie groups by a maximal compact subgroup is Hermitian symmetric [48, Appendix C.3], hence $\lambda < 0$ (see for Example [50, Chapter XI, Section 9]). For the other real forms, some computations involving roots are necessary and in the following we show the ones made for $\mathfrak{sp}(p,q)$. The other cases may be treated similarly. By the diagrams in [48, Page 414], the Vogan diagram of $\mathfrak{sp}(p,q)$ corresponds to the Dynkin diagram of C_{p+q} with the p-th node painted and $q > 1$. The element providing the adjoint orbit is φ_p (up to scaling), by Proposition 3.7.4. By Lemma 3.2.5 and Lemma 3.4.1, λ may be determined from the equation

$$\lambda = \frac{(\eta, \varphi_p)}{|\varphi_p|^2} = -\frac{2}{|\varphi_p|^2}\sum_{\alpha\in\Delta^+}\varepsilon_\alpha(\alpha,\varphi_p) = -\frac{2}{|\varphi_p|^2}\sum_{\alpha\in\Delta^+}\varepsilon_\alpha n_\alpha^p\frac{(\gamma_p,\gamma_p)}{2}, \tag{3.151}$$

where $\alpha = \sum_{i=1}^{\ell} n_\alpha^i\gamma_i$. If one chooses the simple roots and φ_p to be normalized with unitary norm, the equation simplifies further to

$$\lambda = -\sum_{\alpha\in\Delta^+}\varepsilon_\alpha n_\alpha^p = -\sum_{\alpha\in\Delta^+\setminus\varphi_p^\perp}\varepsilon_\alpha n_\alpha^p, \tag{3.152}$$

as $n_\alpha^p = 0$ if $\alpha \in \Delta^+ \cap \varphi_p^\perp$. Since $\mathfrak{sp}(p,q)$ has an underlying diagram of type C_{p+q}, its set of positive roots is of the form

$$\Delta^+ = \{e_i - e_j \mid 1 \le i < j \le p+q\} \cup \{e_i + e_j \mid 1 \le i < j \le p+q\} \cup \{2e_i \mid 1 \le i \le p+q\}, \tag{3.153}$$

where $e_1, \ldots, e_{p+q}$ denotes the orthonormal basis of $\mathbb{R}^{p+q}$ (see for example [48, Chapter II, Section 5]). The set of simple roots is $\{\gamma_i = e_i - e_{i+1} \mid 1 \le i \le p+q-1\} \cup \{2e_{p+q}\}$. Thus the roots may be written in terms of simple roots as

$$\begin{aligned} e_i - e_j =& \sum_{k=i}^{j-1} e_k - e_{k+1} = \sum_{k=i}^{j-1}\gamma_k, \\ e_i + e_j =& \sum_{k=i}^{p+q-1}\gamma_k + \sum_{k=j}^{p+q-1}\gamma_k + \gamma_{p+q}, \\ 2e_i =& 2\sum_{k=1}^{p+q-1}\gamma_k + \gamma_{p+q} \end{aligned} \tag{3.154}$$

so that

$$\Delta^+ \setminus \varphi_p^\perp = R_1 \cup R_2 \cup R_3, \tag{3.155}$$

where

$$\begin{aligned} R_1 =& \{e_i - e_j \mid 1 \le i \le p < j \le p+q\}, \\ R_2 =& \{e_i + e_j \mid i \le p \text{ or } i < j \le p\}, \\ R_3 =& \{2e_i \mid 1 \le i \le p\}. \end{aligned} \tag{3.156}$$

Notice that, by Lemma 3.2.4, $R_1 \subset \Delta_n^+$, $R_3 \subset \Delta_c^+$, while

$$\begin{aligned} R_2^c =& R_2 \cap \Delta_c^+ = \{e_i + e_j \mid 1 \le i < j \le p\}, \\ R_2^n =& R_2^n \cap \Delta_n^+ = \{e_i + e_j \mid 1 \le i \le p < j \le p+q\}. \end{aligned} \tag{3.157}$$

Moreover, roots α belonging to R_1 and R_2^n have $n_\alpha^p = 1$, while they have $n_\alpha^p = 2$ if they belong to R_3 or R_2^c. Putting all these information together one gets

$$\begin{aligned}\lambda =& -\sum_{\alpha\in R_1} 1 + \sum_{\alpha\in R_2^c} 2 - \sum_{\alpha\in R_2^n} 1 + \sum_{\alpha\in R_3} 2 \\ =& -pq + 2\sum_{k=1}^{p}(p-k) - pq + 2p \\ =& -2pq + 2\left(\frac{1}{2}p^2 - \frac{1}{2}p\right) + 2p \\ =& p(p-2q+1).\end{aligned} \tag{3.158}$$

Hence, up to scaling, $\lambda = p - 2q + 1$. □

We highlight that the statement of Theorem 3.7.5 cannot be reversed. First of all, because rescaling the orbit has the effect of rescaling λ in a consistent way. In addition, a non-compact simple Lie group may have more than one orbit satisfying $\rho = \lambda\omega$ in general (see tables in Appendix B). In the general case of V being a torus, a classification of all orbits $(G/V, \omega, J)$ has been provided by Alekseevsky and Podestà [2, Theorem 1.1].

Many well known examples appearing in the general theory, are included in the table of Theorem 3.7.5, thus they are adjoint orbit with special almost-Kähler structure.

Example 3.7.6 (Hyperbolic Riemann surfaces)**.** The hyperbolic plane $\mathbb{H}^2$ may be represented as the adjoint orbit $SO(2,1)/U(1)$, so it sits in the class represented by the second line of the table. Since $SO(2,1)$ is semisimple, it admits lattices $\Gamma \subset SO(2,1)$ and the Riemann surfaces of genus strictly greater than 1 may be obtained as quotients $\Gamma\backslash SO(2,1)/U(1)$, as a consequence of the uniformization theorem. Due to dimensional reasons, the canonical almost-complex structure is always integrable on these manifolds. As an example, if $\Gamma = SL(2,\mathbb{Z}) \subset SO(2,1)$, then $\Gamma\backslash SO(2,1)/U(1)$ is the *modular curve*.

Example 3.7.7 (The Siegel half space)**.** Another space that we have seen throughout the thesis, is the Siegel half space $Sp(n)/U(n)$. This space turns out to be an adjoint orbit belonging to the class of the last line of the table, as $\mathfrak{u}(n) \cong \mathfrak{su}(n)$. The quotient of $Sp(n)/U(n)$ by the lattice $Sp(n,\mathbb{Z})$ is the *moduli space of principally polarized abelian varieties*. Observe that, for $n = 1$, this moduli space is exactly the modular curve defined above. We will prove later that, also in this case, the compatible almost-complex structure is always integrable.

Example 3.7.8 (Twistor space of $\mathbb{H}^{2n}$)**.** The twistor space of the hyperbolic space $\mathbb{H}^{2n}$ turns out to be diffeomorphic to the adjoint orbit $SO(2n,1)/U(n)$ (see Example 1.4.3), so it sits in the class represented by the second line of the table, for $p = n$, $q = 1$. The almost-complex structure on these adjoint orbits is integrable only for $p = q = 1$, as we will see at the end of the section. Moreover, taking compact quotients $\Gamma\backslash\mathbb{H}^{2n}$, for a lattice $\Gamma \in \mathbb{H}^{2n}$, one gets the so called *hyperbolic manifolds*. Then the twistor space of the hyperbolic manifold $\Gamma\backslash\mathbb{H}^{2n}$ is the quotient $\Gamma\backslash SO(2n,1)/U(n)$.

Example 3.7.9 (Period domains of weight 2 [27, Section 4.2.2])**.** More generally, *Griffiths period domains of weight 2*, i.e., homogeneous manifolds of the form $SO(2p,q)/U(p) \times SO(q)$, belong to the class represented by the second line of the table, for p and q greater than 1. The canonical almost-complex structure on these manifolds is not integrable in general. Notice that, for $p = q = 1$, one recovers the twistor space of the hyperbolic space discussed above.

Example 3.7.10 ($G_{2(2)}/U(2)$)**.** The non-compact split real form $G_{2(2)}$ of the complex Lie group G_2 may be seen as the automorphism group of the split-octonions $\mathbb{O}_s$. In particular $G_{2(2)} = \mathrm{Aut}(\mathbb{O}_s)$ is contained in $SO(3,4)$ and it has dimension 14. A possible representation of the Lie algebra $\mathfrak{g}_{2(2)}$ of $G_{2(2)}$ is through the 7×7 matrices $\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$ with

$$\begin{aligned} A &= \begin{pmatrix} 0 & x_{10} - x_{14} & x_{12} + x_{13} \\ -x_{10} + x_{14} & 0 & x_{11} - x_9 \\ -x_{12} - x_{13} & -x_{11} + x_9 & 0 \end{pmatrix}, \\ B &= \begin{pmatrix} x_4 - x_7 & -x_8 - x_3 & x_2 + x_5 & x_6 - x_1 \\ x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \end{pmatrix}, \\ C &= \begin{pmatrix} 0 & x_9 & x_{12} & x_{14} \\ -x_9 & 0 & x_{10} & x_{13} \\ -x_{12} & -x_{10} & 0 & x_{11} \\ -x_{14} & -x_{13} & -x_{11} & 0 \end{pmatrix}. \end{aligned} \tag{3.159}$$

This construction is similar to the one given in [45, Section 19.3], but we use split-octonions instead of octonions, as we are dealing with the split real form of G_2. The element $v = (0,0,0,0,0,0,0,0,0,0,0,1,1,0)$, written in coordinates, has stabilizer isomorphic to $U(2)$, since the matrices in the Lie algebra of the stabilizer have the form

$$\begin{pmatrix} 0 & 0 & a+b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a-b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & a & d \\ 0 & 0 & 0 & -c & 0 & d & b \\ 0 & 0 & 0 & -a & -d & 0 & c \\ 0 & 0 & 0 & -d & -b & -c & 0 \end{pmatrix}, \tag{3.160}$$

for $a, b, c, d \in \mathbb{R}$. Thus, the orbit of v is isomorphic to $G_{2(2)}/U(2)$ and it carries a special compatible almost-complex structure. To see this, one has to choose the positive roots of $\mathfrak{g}_2$ such that the weight φ satisfying $v = ih_\varphi$ belongs to the dominant Weyl chamber. Then one has to compute the vectors u_α and v_α, for each positive root α that is not orthogonal to v, and the canonical linear complex structure H. At this point the vectors v and $-2\sum_{\alpha \in \Delta^+ \setminus \varphi^\perp} \varepsilon_\alpha \alpha$ turn out to be one multiple of the other, thus the canonical almost-complex structure is special on the adjoint orbit of v, by (3.77).

The next lemma shows that when the canonical almost Kähler structure of an adjoint orbit satisfies $\rho = \lambda\omega$, the sign of λ is determined a priori by the Vogan diagram. Moreover, it gives an effective criterion for deciding whether a Vogan diagram admits no special φ.

Lemma 3.7.11. *Let a Vogan diagram admitting a λ-special element φ be given and let $\Sigma^+ = \{\gamma_1, \dots, \gamma_\ell\}$ be the associated set of simple roots. Assume that φ belongs to the dominant Weyl chamber. If $\gamma_i \in \Sigma_n^+$, then (η, φ_i) has the same sign as λ.*

Proof. Recall that the vector $\eta \in \mathfrak{h}_{\mathbb{R}}^*$ is defined by

$$\eta = -2 \sum_{\alpha \in \Delta^+} \varepsilon_\alpha \alpha \tag{3.161}$$

and it depends on the Vogan diagram only. By Theorem 3.4.4 and Theorem 3.4.5, the hypothesis of φ being λ-special may be rewritten as

$$\eta - 2 \sum_{\alpha \in \mathrm{span}\{\gamma_j | j \in S^c\} \cap \Delta^+} \alpha = \lambda \varphi, \tag{3.162}$$

where $S \subset \{1, \dots, \ell\}$ is the subset of indices such that $\varphi = \sum_{i \in S} v^i \varphi_i$ with $v^i > 0$. Let now $\gamma_i \in \Sigma_n^+$, so that $i \in S$, since indices of non-compact simple roots belong to S. Consider the scalar product of equation (3.162) with φ_i

$$(\eta, \varphi_i) - 2 \sum_{\alpha \in \mathrm{span}\{\gamma_j | j \in S^c\} \cap \Delta^+} (\alpha, \varphi_i) = \lambda(\varphi, \varphi_i). \tag{3.163}$$

The second summand of the left-hand side in (3.163) vanishes, as φ_i with $i \in S$ is orthogonal to all roots belonging to $\mathrm{span}\{\gamma_j \mid j \in S^c\}$, thus we are left with $(\eta, \varphi_i) = \lambda(\varphi, \varphi_i)$. However, (φ, φ_i) is always positive since φ belongs to the dominant Weyl chamber, hence the sign of (η, φ_i) coincides with the one of λ. □

Observe that the statement of theorem above cannot be reversed for special diagrams of type symplectic general type or symplectic Fano, as there exist many Vogan diagrams of these types for which (η, φ_i) have the same sign for each $i \in \{j \mid \gamma_j \in \Sigma_n^+\}$, but admitting no special vectors. On the other hand, Vogan diagrams such that $(\eta, \varphi_i) = 0$, for $i \in \{j \mid \gamma_j \in \Sigma_n^+\}$, always admit special vectors.

Corollary 3.7.12. *A Vogan diagram admits 0-special elements belonging to the dominant Weyl chamber if and only if $(\eta, \varphi_i) = 0$ for $i \in \{j \mid \gamma_j \in \Sigma_n^+\}$.*

Proof. The "only if" part is exactly the content of Lemma 3.7.11. For the other implication, if $(\eta, \varphi_i) = 0$ for all $i \in P = \{j \mid \gamma_j \in \Sigma_n^+\}$, then η is a linear combination of simple compact roots. Hence, the vector

$$\varphi_P = \eta - 2 \sum_{\alpha \in \mathrm{span}\{\gamma_j | j \in P^c\}} \alpha, \tag{3.164}$$

lies in the subspace spanned by the simple compact roots. Moreover, as we proved in Corollary 3.4.7, all simple compact roots belong to the stabilizer $\mathfrak{v}$ of φ_P, as it may be written as a linear combination of φ_i, with $i \in P$. Thus, φ_P is a vector belonging to the subspace spanned by the simple compact roots and it is orthogonal to each of them. As the Killing form is negative definite on $\mathfrak{v}$, φ_P must be 0. By Corollary 3.4.8, each $\varphi = \sum_{i \in P} v^i \varphi_i$, with $v^i > 0$, admits a special symplectic Calabi-Yau adjoint orbit. □

In the remaining part of this section, we consider adjoint orbits $(G/V, \omega, J)$ satisfying $\rho = \lambda\omega$ and we discuss integrability of the canonical almost-complex structure J. Again, by Theorem 3.5.5, to establish whether J is integrable or not reduces to a problem on the special co-vector φ. In order to simplify the statements of the next results, we give the following definition.

Definition 3.7.13. An element φ is *integrable* if the orbits $(G/V, \omega, J)$ of $v = ih_\varphi$ has integrable canonical almost-complex structure J.

Actually, adjoint orbits of semisimple Lie groups may be built from simple ones just by taking the product, as a semisimple Lie group decomposes as the product of simple ones. The canonical almost-complex structure on an adjoint orbit of a real non-compact semisimple Lie group is then integrable if and only if it is integrable on each factor. Moreover, it is special if and only if it is special on each factor, with the same constant λ. Hence, in order to study the canonical almost-complex structure on adjoint orbits of real non-compact semisimple Lie groups it suffices to study it on adjoint orbits of real non-compact simple Lie groups.

Remember that a Vogan diagram is said to be *classical* if the underlying Dynkin diagram is of type A_ℓ, B_ℓ, C_ℓ or D_ℓ. The next result says that integrable φ's for such diagrams appear quite seldom.

Theorem 3.7.14. *Given a Vogan diagram of classical type heaving at least two painted nodes, any φ belonging to the associated dominant Weyl chamber is not integrable.*

Proof. First, observe that the integrability of the canonical almost-complex structure depends quite weakly from φ itself. Indeed, by Theorem 3.5.5, the norm of the Nijenhuis tensor of the canonical almost-complex structure on G/V is non-zero as soon as there are two non-commuting non-compact roots. We are then reduced to show that classical Vogan diagrams with at least two painted nodes always admit a couple of non-commuting non-compact roots. The proof of this fact is done case-by-case. The explicit form of the roots has been taken from [48, Chapter II, Section 5].

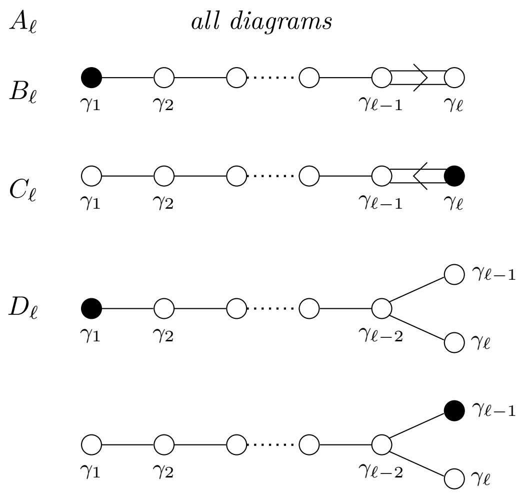
- A_ℓ: the roots of A_ℓ are $\Delta = \{e_i - e_j \mid i \neq j\}$, where $e_1, \dots, e_{\ell+1}$ form an orthonormal basis of $\mathbb{R}^{\ell+1}$. In particular, $\{e_i - e_{i+1} \mid 1 \leq i \leq \ell\}$ form a basis for Δ^+ and $\gamma_i = e_i - e_{i+1}$, $1 \leq i \leq \ell$, are the simple roots of Δ^+. Let $P = \{i_1, \dots, i_m\}$ be the (ordered) set of painted nodes of the given Vogan diagram, with $|P| \geq 2$, and put
$$\alpha = \sum_{j=i_1}^{i_2-1} \gamma_j = \sum_{j=i_1}^{i_2-1} e_j - e_{j+1} = e_{i_1} - e_{i_2} \in \Delta^+. \tag{3.165}$$
Observe that, by Lemma 3.2.4, α is non-compact since the sum of the coefficients relative to non-compact simple roots is just the coefficient of γ_{i_1} which is 1. Then the linear combination
$$\beta = \sum_{j=i_1}^{i_2} \gamma_j = \sum_{j=i_1}^{i_2} e_j - e_{j+1} = e_{i_1} - e_{i_2+1} \in \Delta^+ \tag{3.166}$$
gives a root such that $\beta = \alpha + \gamma_{i_2}$. This shows that $[\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{\gamma_{i_2}}] \subseteq \mathfrak{g}_{\mathbb{C}}^{\beta}$ and $[\alpha, \gamma_{i_2}] \neq 0$. Thus the commutator of the non-compact roots α and γ_{i_2} is non-zero.
- B_ℓ: the roots of B_ℓ are $\Delta = \{\pm e_k,\ \pm(e_i \pm e_j) \mid i \neq j\}$ and a basis is given by $\Sigma^+ = \{e_i - e_{i+1} \mid 1 \leq i \leq \ell - 1\} \cup \{e_\ell\}$. As above, put $\gamma_i = e_i - e_{i+1}$, $1 \leq i \leq \ell - 1$, and $\gamma_\ell = e_\ell$ and let $P = \{i_1, \dots, i_m\}$ be the (ordered) set of painted vertices of the given Vogan diagram, with $|P| \geq 2$. We have to distinguish two cases.
  - $i_1, i_2 \in \{1, \dots, \ell - 1\}$: it holds the same argument as for A_ℓ;
  - $i_1 \in \{1, \dots, \ell - 1\},\ i_2 = \ell$: put
  $$\alpha = \sum_{j=i_1}^{\ell-1} \gamma_j = \sum_{j=i_2}^{\ell-1} e_j - e_{j+1} = e_{i_1} - e_\ell \in \Delta^+ \tag{3.167}$$
  and observe that α is non-compact by Lemma 3.2.4. Then
  $$\beta = \alpha + \gamma_\ell = e_{i_1} - e_\ell + e_\ell = e_{i_1} \in \Delta^+ \tag{3.168}$$
  and so, as above, $[\alpha, \gamma_\ell] \neq 0$.

- C_ℓ: the roots of C_ℓ are $\Delta = \{\pm 2e_k,\ \pm(e_i \pm e_j) \mid i \neq j\}$ and a basis is given by $\Sigma^+ = \{e_i - e_{i+1} \mid 1 \leq i \leq \ell - 1\} \cup \{2e_\ell\}$. As above, put $\gamma_i = e_i - e_{i+1}$, $1 \leq i \leq \ell - 1$, and $\gamma_\ell = 2e_\ell$ and let $P = \{i_1, \dots, i_m\}$ be the (ordered) set of painted nodes of the given Vogan diagram, with $|P| \geq 2$. Again, we have to distinguish two cases.
  - $i_1, i_2 \in \{1, \dots, \ell - 1\}$: it holds the same argument as for A_ℓ;
  - $i_1 \in \{1, \dots, \ell - 1\}$, $i_2 = \ell$: put
  $$\alpha = \sum_{j=i_1}^{\ell-1} \gamma_j = \sum_{j=i_2}^{\ell-1} e_j - e_{j+1} = e_{i_1} - e_\ell \in \Delta^+ \tag{3.169}$$
  and observe that α is non-compact by Lemma 3.2.4. Then
  $$\beta = \alpha + \gamma_\ell = e_{i_1} - e_\ell + 2e_\ell = e_{i_1} + e_\ell \in \Delta^+, \tag{3.170}$$
  thus $[\alpha, \gamma_\ell] \neq 0$.
- D_ℓ: the roots of D_ℓ are $\Delta = \{\pm(e_i \pm e_j) \mid i \neq j\}$ and a basis is given by $\Sigma^+ = \{e_i - e_{i+1} | \ 1 \leq i \leq \ell - 1\} \cup \{e_{\ell-1} + e_\ell\}$. As above, put $\gamma_i = e_i - e_{i+1}$, $1 \leq i \leq \ell - 1$, and $\gamma_\ell = e_{\ell-1} + e_\ell$ and let $P = \{i_1, \dots, i_m\}$ be the (ordered) set of painted vertices of the given Vogan diagram with $|P| \geq 2$. We have to consider four cases.
  - $i_1, i_2 \in \{1, \dots, \ell - 1\}$: it holds the same argument as for A_ℓ;
  - $i_1 \in \{1, \dots, \ell - 2\}$, $i_2 = \ell - 1$: put
  $$\alpha = \sum_{j=i_1}^{\ell-2} \gamma_j = \sum_{j=i_2}^{\ell-2} e_j - e_{j+1} = e_{i_1} - e_{\ell-1} \in \Delta^+ \tag{3.171}$$
  and observe that α is non-compact by Lemma 3.2.4. Then
  $$\beta = \alpha + \gamma_{\ell-1} = e_{i_1} - e_{\ell-1} + e_{\ell-1} - e_\ell = e_{i_1} - e_\ell \in \Delta^+, \tag{3.172}$$
  hence $[\alpha, \gamma_{\ell-1}] \neq 0$;
  - $i_1 \in \{1, \dots, \ell - 2\}$, $i_2 = \ell$: put
  $$\alpha = \sum_{j=i_1}^{\ell-2} \gamma_j = \sum_{j=i_2}^{\ell-2} e_j - e_{j+1} = e_{i_1} - e_{\ell-1} \in \Delta^+ \tag{3.173}$$
  and observe that α is non-compact by Lemma 3.2.4. Then
  $$\beta = \alpha + \gamma_\ell = e_{i_1} - e_{\ell-1} + e_{\ell-1} + e_\ell = e_{i_1} + e_\ell \in \Delta^+, \tag{3.174}$$
  so, as above, $[\alpha, \gamma_\ell] \neq 0$;
  - $(i_1, i_2) = (\ell - 1, \ell)$: put
  $$\alpha = \gamma_{\ell-2} + \gamma_\ell = e_{\ell-2} - e_{\ell-1} + e_{\ell-1} + e_\ell = e_{\ell-2} + e_\ell \in \Delta^+ \tag{3.175}$$
  and observe that α is non-compact by Lemma 3.2.4. Then
  $$\beta = \alpha + \gamma_{\ell-1} = e_{\ell-2} + e_\ell + e_{\ell-1} - e_\ell = e_{\ell-2} + e_{\ell-1} \in \Delta^+ \tag{3.176}$$
  and this shows that $[\alpha, \gamma_\ell] \neq 0$.

□

As a consequence of Theorem 3.7.14, a classical Vogan diagram may admit an integrable element φ only when it has one painted node. If in addition φ is special, we can state which are the Vogan diagrams having integrable φ.

Theorem 3.7.15. *Given a Vogan diagram of classical type having only one painted node, assume that there exists φ belonging to the associated Weyl chamber that it is special and integrable. Then the Vogan diagram is one of the following:*



Moreover, up to scaling, φ coincides with the fundamental dominant weight corresponding to the painted node.

Proof. By Theorem 3.7.14, the considered Vogan diagram has a single painted node. So, assume that the unique painted node is the p-th one and the Lie algebra has rank ℓ. Since φ belongs to the dominant Weyl chamber, it is a non-negative linear combination of fundamental dominant weights $\varphi = \sum_{i=1}^{\ell} v^i \varphi_i$, with $v^p > 0$. Suppose that φ is not a multiple of φ_p. As φ is special, Proposition 3.4.2 and Theorem 3.4.5 force the orbit to be special symplectic Fano, contradicting the integrability condition by Lemma 3.6.3. Thus, φ is a multiple of φ_p. Finally, consider a maximal compact subgroup K of G. By Theorem 3.6.2, the integrability assumption of the canonical almost-complex structure on the orbit forces G/K to be Hermitian symmetric. The thesis follows by the fact that Vogan diagrams having G/K Hermitian symmetric are classified [48, Appendix C.3]. □

Theorem 3.7.15 classifies all classical Vogan diagrams admitting integrable and special φ. In the remaining part of the section, we treat the exceptional cases. Let $\mathfrak{g}$ be a real non-compact exceptional Lie algebra with trivial automorphism, that is, one among

$$\mathfrak{g}_{2(2)},\ \mathfrak{f}_{4(4)},\ \mathfrak{f}_{4(-20)},\ \mathfrak{e}_{6(2)},\ \mathfrak{e}_{6(-14)},\ \mathfrak{e}_{7(7)},\ \mathfrak{e}_{7(-5)},\ \mathfrak{e}_{7(-25)},\ \mathfrak{e}_{8(8)},\ \mathfrak{e}_{8(-24)}, \tag{3.177}$$

where we follow the notation of [44]. Let $v \in \mathfrak{g}$ be an element having compact stabilizer $V \subset G$, define $v = ih_\varphi$ and suppose that φ is integrable. Then the canonical complex structure J on G/V descends to an integrable almost-complex structure J_Γ on $M = \Gamma\backslash G/V$, for $\Gamma \subset G$ a discrete co-compact subgroup, making M a Kähler manifold. Hence, in force of Theorem 3.6.2, the quotient G/K, with K a maximal compact subgroup, has to be Hermitian symmetric. In particular, the only possibilities for $\mathfrak{g}$ are $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ and $\mathfrak{g} = \mathfrak{e}_{7(-25)}$ [48, Appendix C.4]. By the Borel and the Siebenthal Theorem [48, Theorem 6.96], in order to find exceptional Vogan diagrams admitting special and integrable φ, it suffices to look at the ones equivalent to the Vogan diagrams of $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ and $\mathfrak{g} = \mathfrak{e}_{7(-25)}$. To do this, we look at the tables of E_6 and E_7 in Section B.2 and we consider each case. In order to determine the equivalence class of a Vogan diagram, i.e., to which diagram with one painted node a Vogan diagram is equivalent to, we used the rules discussed in [24].

- $\mathfrak{g} = \mathfrak{e}_{6(-14)}$: there are three Vogan diagrams equivalent to the one of $\mathfrak{g} = \mathfrak{e}_{6(-14)}$. First, the diagram with γ_3 and γ_5 painted is excluded from our analysis since the associated orbit is special symplectic Fano, hence it cannot admit integrable complex structure by Lemma 3.6.3. For the diagram with γ_1 and γ_6 painted, observe that the roots

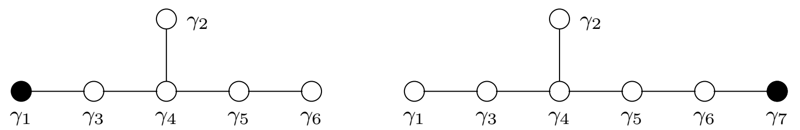
$$\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5, \qquad \gamma_6 \tag{3.178}$$

  are both non-compact and their sum $\sum_{i=1}^{6} \gamma_i$ is a root. Thus, we have found two non-compact non-commuting roots and, by Theorem 3.5.5 and Remark 3.5.6, the canonical almost-complex structure is not integrable. We are left with the Vogan diagram of $\mathfrak{e}_{6(-14)}$ which is special symplectic general type. Uniqueness of the special vector then follows by Theorem 3.7.4 and Proposition 3.4.2, while integrability follows from the fact that G/K is Hermitian symmetric.

- $\mathfrak{e}_{7(-25)}$: in this case there is only one diagram to consider. As above, since the Vogan diagram is special symplectic general type, the special vector is unique by Theorem 3.7.4 and it is integrable as G/K is Hermitian symmetric.

Summing up, we have the following result.

Theorem 3.7.16. *Given a Vogan diagram of exceptional type, assume that there exists φ belonging to the associated dominant Weyl chamber that is special and integrable. Then the Vogan diagram is one of the following*



Moreover, up to scaling, φ coincides with the fundamental dominant weight corresponding to the painted node.

In the next section we sum up the main results proved through this long chapter and we point out some remaining open problems.

3.8 Conclusions and open problems

In this chapter, we learnt how to understand whether an adjoint orbit of a real non-compact semisimple Lie group admits a special compatible homogeneous almost-complex structure. The speciality condition is encoded in the geometry of a particular vector which depends on the underlying Lie algebra only. More precisely, there is a canonically defined compatible almost-complex structure on these orbits (3.3, precisely (3.61)). We have necessary and sufficient conditions for an adjoint orbit to admit canonical special compatible almost-complex structure

(Theorem 3.4.4 and Theorem 3.4.5) which, put together, lead to Corollary 3.4.7 and Corollary 3.4.8. They say that speciality of the canonical almost-complex structure depends on the signs of the coefficients of a certain vector φ_P, determined by the Lie algebra. If some precise coefficients of φ_P, with respect to the basis of fundamental dominant weights, are all negative, zero or positive, then the orbit is special symplectic general type, symplectic Calabi-Yau and symplectic Fano respectively. Moreover, special vectors belonging to the same Lie algebra are of the same type (Proposition 3.4.2) and they come in a finite number (Proposition 3.4.3). Since knowing special adjoint orbits of simple Lie groups permits to build special adjoint orbits of semisimple Lie groups, just by taking products of special adjoint orbits of simple Lie groups having the same constant λ, it is quite natural to restrict the concrete analysis to real non-compact adjoint orbits of simple Lie groups. By using some combinatorial tools, such as Vogan diagrams, we were able to detect many infinite families of non-compact simple classical special adjoint orbits (Theorem 3.7.5) and to classify all the exceptional ones (see Section B.2 of the appendix). Actually, one may algorithmically list all special non-compact adjoint orbits, however, in practice, it is not possible to find a pattern among the examples of low rank. This suggests that the classification of all such orbits is quite complicated and very likely some new ideas are required in order to complete it. Nevertheless, there is an algorithmic method which allows to understand if a simple Lie algebra admits special vectors (see Section A.1), thus drawing us up to a classification. For convenience of the reader, we collect altogether the existence results obtained so far.

Theorem 3.8.1. *Let G be a real non-compact simple Lie group. If G is classical, there exists orbits $(G/V, \omega, J)$ satisfying $\rho = \lambda\omega$ if the Lie algebras of G and V and the constant λ are as in the following table*

$\mathfrak{g}$	$\mathfrak{v}$	λ	
$\mathfrak{su}(p,q)$	$\mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus \mathbb{R}$	-1	$p, q \geq 1$
$\mathfrak{so}(2p,q)$	$\mathfrak{su}(p) \oplus \mathfrak{so}(q) \oplus \mathbb{R}$	$p-q-1$	$p, q \geq 1$
$\mathfrak{so}^*(2\ell)$	$\mathfrak{su}(\ell) \oplus \mathbb{R}$	-1	$\ell \geq 4$
$\mathfrak{sp}(p,q)$	$\mathfrak{su}(p) \oplus \mathfrak{sp}(q) \oplus \mathbb{R}$	$p-2q+1$	$p, q \geq 1$
$\mathfrak{sp}(\ell,\mathbb{R})$	$\mathfrak{su}(\ell) \oplus \mathbb{R}$	-1	$\ell \geq 3$

If G is exceptional, then there exists orbits $(G/V, \omega, J)$ satisfying $\rho = \lambda\omega$ if the Lie algebras of G and V are as in the tables in Appendix B.2 and the sign of the constant λ is determined by the sign of the Hermitian scalar curvature s.

Notice that, among the simple classical special adjoint orbits we find also the compact ones, which are Kählerian as the formula for the norm of the Nijenhuis tensor (Theorem 3.5.5) and Remark 3.5.6 show. On the other hand, in the non-compact case, we recover many well known examples coming from the general theory (see the examples right after Theorem 3.7.5). The non-compact Kählerian special adjoint orbits of simple Lie groups, i.e., the ones for which the canonical almost-complex structure is integrable, are outnumbered beside the non-Kähler ones (Theorem 3.7.15 and Theorem 3.7.16), thus many special adjoint orbits of real non-compact simple Lie groups provide examples of symplectic non-Kähler manifolds carrying a "best metric".

To complete this wide picture, we are led to consider few problems. First, we know that special symplectic Fano adjoint orbits come in a finite number on a Vogan diagram, but we do not know exactly how many they are, neither which they are. From (3.87), one may deduce just a very rough estimate on the number of special symplectic Fano adjoint orbits

$$|\varphi| \leq |\eta| + 2 \sum_{\alpha \in \Delta_c^+} |\alpha|, \tag{3.179}$$

but it would be useful to know a more precise upper bound on it and to understand the form of the special vectors. This number should be related to the number of fibrations such that both the base and the fiber may be combined to give a total space which is a special symplectic Fano adjoint orbit. However, the intricate combinatorics of Lie algebras prevented us to clarify this relation.

For what concerns Vogan diagrams, in Section 3.7 we stressed out that Vogan diagrams for which the scalar products (η, φ_i) have the same signs for each i in the indices of simple non-compact roots, are not necessarily associated with special vectors, as this happens only for Vogan diagrams of type symplectic Calabi-Yau (Corollary 3.7.12). Still, there is an algorithmic method to know if a Vogan diagram has the above property, but a more direct rule would be preferable. It would be handy to shed some light on this point, since understanding the above property for Vogan diagrams may help to understand whether there exist special vectors.

Finally, Theorem 3.7.15 and Theorem 3.7.16 say which special adjoint orbits are Kähler manifolds and the remaining ones carry a non-integrable canonical almost-complex structure. However, it is not known whether they admit a non necessarily homogeneous compatible integrable almost-complex structure making them Kähler manifolds. Notice that adjoint orbits G/V for which the quotient G/K by a maximal compact subgroup is not Hermitian symmetric are ruled out from this discussion by Theorem 3.6.2. In particular, all special symplectic Fano and symplectic Calabi-Yau adjoint orbits need not to be considered in this context, as integrability of the almost-complex structure forces G/K to be Hermitian symmetric, thus to have negative curvature [50, Chapter XI, Proposition 9.7]. On the other hand, the question is completely open for special symplectic general type adjoint orbits for which G/K is Hermitian symmetric. It would be interesting to give an answer to this question and compare it to the results of Carlson and Toledo [16, Theorem 0.1]. A possible way to work out the problem may be to find some topological obstructions related to the first two Chern classes.

Appendix A

The classification algorithm

This appendix is dedicated to the explanation of the algorithm that we used to produce the tables in Appendix B. After we have coded it, we were able to make experiments and to understand many facts that we have proved at a later time. In the following, we give few technical details about the code and we explain the algorithm in general. Then we examine the original Code A.1 and we discuss a couple of examples.

A.1 The algorithm and the theory behind it

The program that we used to produce examples of adjoint orbits admitting special canonical almost-complex structure is called `specialAdjointOrbits.sage` and it is written in the SageMath sintax. SageMath is a free open-source mathematics software system licensed under the GNU GPL [75]. The code may be run as a SageMath script. Once the script is executed, one is asked to insert the Lie algebra type of the adjoint orbit (so one among the letters A, B, C, D, E, F, G) and the rank of the Lie algebra. Then, for each vector $v = ih_\varphi$ admitting a special adjoint orbit, the script prints:

- The normalized vector φ expressed in the basis of the fundamental dominant weights;
- The set of indices of simple non-compact roots and the set S of Theorem 3.4.4 and Theorem 3.4.5;
- The symplectic type of the special orbit, so *symplectic general type*, *symplectic Calabi-Yau* and *symplectic Fano*. If the canonical almost-complex structure on the orbit is integrable, then the word "symplectic" is omitted;
- The dimension of the stabilizer V and the orbit G/V of $v = ih_\varphi$;
- Whether φ is a root;
- The Hermitian scalar curvature of the canonical almost-complex structure (computed with the formula in Lemma 3.5.1);
- The Lie algebra $\mathfrak{g}$ and the Lie algebra of the stabilizer $\mathfrak{v}$.

The heart of the algorithm lies in Corollary 3.4.8. The key point is that a Vogan diagram $\mathcal{D}_P$ of a simple real Lie algebra, with P the set of indices of painted nodes, admits a special vector if and only if the vector

$$\varphi_P = \eta - 2 \sum_{\alpha \in \text{span}\{\gamma_i | i \in P^c\} \cap \Delta^+} \alpha, \tag{A.1}$$

written as $\varphi_P = \sum_{i \in P} w_P^i \varphi_i$ in the basis of fundamental dominant weights, with $w_P^i = 2\frac{(\varphi_P, \gamma_i)}{(\gamma_i, \gamma_i)}$, satisfies one of the following conditions:

1. $w_P^i < 0$ for all $i \in P$;

2. $w_P^i > 0$ for all $i \in P$;

3. $\varphi_P = 0$.

In case 1, the diagram admits a unique (up to scaling) special symplectic general type adjoint orbit associated with $\varphi = -\sum_{i \in P} w_P^i \varphi_i$. In case 2, the diagram admits a finite number (up to scaling) of special symplectic Fano adjoint orbits. In the last case, the diagram admits a continuous family of special symplectic Calabi-Yau adjoint orbits with special vectors of the form $v = ih_\varphi$, with $\varphi = \sum_{i \in P} v^i \varphi_i$. In particular, if $|P| = 1$, the diagram admits a unique (up to scaling) special symplectic Calabi-Yau adjoint orbit.

Corollary 3.4.8 gives then a practical way of finding special vectors once the Vogan diagram is given. Indeed, the compact and non-compact roots, the vector η and P come with the Vogan diagram. Hence, one is reduced to check whether the coefficients of the vector φ_P, expressed on the basis of the fundamental dominant weights, satisfy one of the conditions 1, 2, 3 above. In case 2, there may be more special vectors associated with the given Vogan diagram. In order to find them all, for each set $S \supseteq P$ one needs to check whether

$$\varphi_S = \eta - 2 \sum_{\alpha \in \text{span}\{\gamma_i | i \in S^c\} \cap \Delta^+} \alpha = \sum_{i \in S} w_S^i \varphi_i \tag{A.2}$$

has $w_S^i > 0$ for each $i \in S$. If this happens, $\sum_{i \in S} w_S^i \varphi_i$ gives a special vector. The flow chart of the algorithm is illustrated in Figure A.1 and it gives the idea of how the algorithm works.

A.2 The code

In the following, we comment briefly the lines of the Code A.1. We tried to keep the same notation as in the theory developed up to now.

1-171 : Two definitions of functions, one for determining the Lie algebra and the other for computing the isotropy of the orbit, are defined. More explicitly, the first function determines the equivalence class of the Vogan diagram through the rules in [24], while the second determines the stabilizer of the orbit using Proposition 3.7.3.

172-182 : The `lieType` and the `rank` of the Lie algebra are inserted by the user and the corresponding Dynkin diagram is printed (so the Vogan diagram without painted nodes). Then the positive roots `positiveRoots`, the Cartan matrix `C` and the scalar product between simple roots `B` are defined. Finally, the dimension of the Lie algebra of type `lieType` and rank `rank` is printed.

183-188 : A for cycle over the all possible Vogan diagrams of type `lieType` and rank `rank` is defined and, up to the end, the code sits inside the cycle. The compact and non-compact roots `compactroots`, `noncompactroots`, the compactness index `epsilon` and the η vector `eta` are defined. Then the vector `phiP` is defined as

$$\texttt{phiP} = \texttt{C} * \left(\eta - 2 \sum_{\alpha \in \mathrm{span}\{\gamma_i | i \in P^c\} \cap \Delta^+} \alpha \right). \tag{A.3}$$

Notice that `phiP` is expressed in the basis of fundamental dominant weights, as the Cartan matrix changes the basis.

189-201 : If `phiP=0`, the adjoint orbit is special symplectic Calabi-Yau. Then, the continuous family of special vectors is printed (in the basis of fundamental dominant weights), together with the set of non-compact simple roots. After that, the dimensions of the stabilizer and the adjoint orbit, whether `varphiP` is a root, the Hermitian scalar curvature, the Lie algebra and the Lie algebra of the stabilizer of the orbit are printed.

202-224 : If the coordinates of `phiP` corresponding to the indices in `P` are all negative, then the orbit is special symplectic general type and, as in the symplectic Calabi-Yau case, all the information are printed. Notice that the special vector `phiP` is printed normalized. Since this is the unique case for which the compatible almost-complex structure may be integrable, in the lines 205-215, using Remark 3.5.6, integrability is checked.

225-240 : If the coordinates of `phiP` corresponding to the indices in `P` are all positive, the orbit is special symplectic Fano. Then, another for cycle over the possible sets `S` containing `P` starts and, if the vector

$$\texttt{phiS} = \texttt{C} * \left(\eta - 2 \sum_{\alpha \in \mathrm{span}\{\gamma_i | i \in S^c\} \cap \Delta^+} \alpha \right), \tag{A.4}$$

has positive coefficients corresponding to the indices in `S`, its orbit is special symplectic Fano and all the information as above are printed.

241 : The time required for the computation is printed in seconds.

A.3 Examples and discussion

In this section we give some examples of how the algorithm works together with few comments about the outputs.

Suppose that we want to compute all Vogan diagrams of type A_3 admitting special vectors. The output of the program will be as follows.

```
Type: A
Rank: 3
O---O---O
1   2   3
A3
Dimension: 15

(1, 0, 0)    Non-compact simple roots: [0]     general type
Dimension V: 9     Dimension G/V: 6
```

```
Is phi a root? No
Hermitian scalar curvature: -24
Lie algebra: su(1,3)
Stabilizer: su(3) x R1

(0, 1, 0)     Non-compact simple roots: [1]     general type
Dimension V: 7     Dimension G/V: 8
Is phi a root? No
Hermitian scalar curvature: -32
Lie algebra: su(2,2)
Stabilizer: su(2) x su(2) x R1

(0, 0, 1)     Non-compact simple roots: [2]     general type
Dimension V: 9     Dimension G/V: 6
Is phi a root? No
Hermitian scalar curvature: -24
Lie algebra: su(1,3)
Stabilizer: su(3) x R1

(1, 0, 1)     Non-compact simple roots: [0, 2]     symplectic general type
Dimension V: 5     Dimension G/V: 10
Is phi a root? Yes
Hermitian scalar curvature: -10
Lie algebra: su(2,2)
Stabilizer: su(2) x R2

0.916521
```

There are 3 general type adjoint orbits with integrable canonical almost-complex structure corresponding to the diagrams with one painted node. By the symmetries of the Dynkin diagram of A_3, orbits associated with diagrams having painted nodes in symmetric position are diffeomorphic. Then there is one orbit corresponding to the Vogan diagram with two symmetric painted nodes which is special symplectic general type. The special vector φ is a root only in the last case. The Lie algebras are all of type $\mathfrak{su}(p,q)$, $p+q=4$, which are the real forms of $\mathfrak{sl}(4,\mathbb{C})$. Overall, the computations requires 0.916521 seconds to be completed.

Now suppose that we want to compute all Vogan diagrams of exceptional type G_2 admitting special vectors. The output will be as follows.

```
Type: G
Rank: 2
  3
0=<=0
1   2
G2
Dimension: 14

(1, 0)     Non-compact simple roots: [0]     symplectic general type
Dimension V: 4     Dimension G/V: 10
```

```
Is phi a root? Yes
Hermitian scalar curvature: -30
Lie algebra: g2(2)
Stabilizer: su(2) x R1

(0, 1)     Non-compact simple roots: [1]     symplectic general type
Dimension V: 4     Dimension G/V: 10
Is phi a root? Yes
Hermitian scalar curvature: -10
Lie algebra: g2(2)
Stabilizer: su(2) x R1

0.788584
```

In this case there are just two orbits corresponding to Vogan diagrams with one painted node and they are not of Kähler type. Notice that, despite they have the same dimension and they are both diffeomorphic to $G_{2(2)}/U(2)$, where $G_{2(2)}$ is the split real form of G_2, they are not diffeomorphic as the two stabilizers correspond to two non-conjugate copies of $\mathfrak{su}(2)$ in $\mathfrak{g}_{2(2)}$.

We conclude with special adjoint orbits of type B_4. The output is

```
Type: B
Rank: 4
0---0---0=>=0
1   2   3   4
B4
Dimension: 36

(1, 0, 0, 0)     Non-compact simple roots: [0]     general type
Dimension V: 22     Dimension G/V: 14
Is phi a root? Yes
Hermitian scalar curvature: -98
Lie algebra: so(2,7)
Stabilizer: so(7) x R1

(0, 1, 0, 0)     Non-compact simple roots: [1]     symplectic general type
Dimension V: 14     Dimension G/V: 22
Is phi a root? Yes
Hermitian scalar curvature: -88
Lie algebra: so(4,5)
Stabilizer: so(5) x su(2) x R1

(0, 0, 1, 0)     Non-compact simple roots: [2]     symplectic general type
Dimension V: 12     Dimension G/V: 24
Is phi a root? No
Hermitian scalar curvature: -24
Lie algebra: so(6,3)
Stabilizer: su(3) x su(2) x R1
```

```
(0, 0, 0, 1)     Non-compact simple roots:  [3]     S: [3]     symplectic Fano
Dimension V: 16     Dimension G/V: 20
Is phi a root? No
Hermitian scalar curvature:  80
Lie algebra:  so(8,1)
Stabilizer:  su(4) x R1

(2, 0, 0, 1)     Non-compact simple roots:  [3]     S: [0, 3]     symplectic Fano
Dimension V: 10     Dimension G/V: 26
Is phi a root? No
Hermitian scalar curvature:  52
Lie algebra:  so(8,1)
Stabilizer:  su(3) x R2

0.956499
```

There are four special adjoint orbits corresponding to Vogan diagrams with one painted node. The first one, corresponding to the Vogan diagram with the first node painted, is Kähler-Einstein general type and it is one of the cases appearing in Theorem 3.7.15. Notice that the stabilizer is obtained by removing the first node of the Vogan diagram and considering the remaining Dynkin diagram, which in this case is of type B_3. The remaining ones are symplectic general type and symplectic Fano. Notice that there are two symplectic Fano adjoint orbits corresponding to the Vogan diagram with the 4-th painted node, one having $S = P = \{4\}$ and the other having $S = \{1, 4\}$. In particular, the latter is fibered over the first. The Lie algebras are all of type $\mathfrak{so}(2p, q)$, with $2p + q = 9$, as they are real forms of $\mathfrak{so}(9, \mathbb{C})$. The time required is less than a second.

The runtimes depends strongly from the type and the rank of the chosen Lie type. Up to rank 8, the computations require less than 11 seconds on a 1.80 GHz 64-bit Intel Kaby Lake R processor. This allows to classify all special adjoint orbits carrying special canonical almost-complex structure up to rank 8. In particular, as one may see from the tables in Appendix B, all orbits of exceptional type are classified. However, the runtimes and also the number of special orbits grow very fast with the rank. Unfortunately, the outputs are often unpredictable and it is hard to catch a glimpse of a possible pattern.

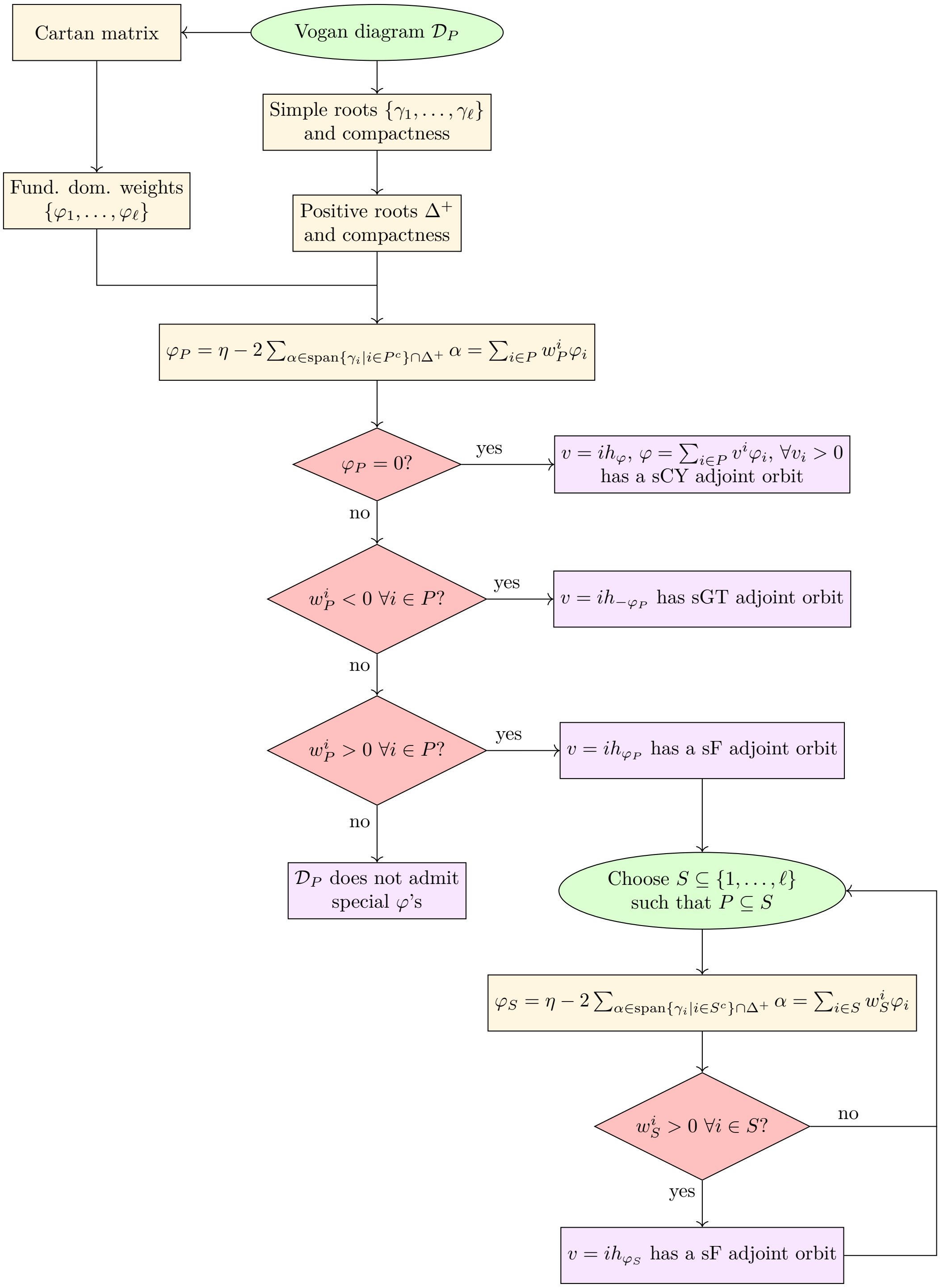


Figure A.1: Flow chart of the algorithm for the classification of Vogan diagrams admitting special adjoint orbits.

```
def lieAlgebraType(lieType,rank,P):
   lieAlgebra=''
   if lieType=='A':
      equivclass=sum((-1)^(len(P)-s)*(P[s-1]+1) for s in range(1,len(P)+1))
      if equivclass<=((rank+1)/2).floor():
         eqclass=equivclass
      else:
         eqclass=rank+1-equivclass
      lieAlgebra='su('+str(eqclass)+','+str(rank+1-eqclass)+')'
   elif lieType=='B':
      eqclass=sum((-1)^(len(P)-s)*(P[s-1]+1) for s in range(1,len(P)+1))
      lieAlgebra='so('+str(2*eqclass)+','+str(2*rank-2*eqclass+1)+')'
   elif lieType=='C':
      if rank-1 in P:
         lieAlgebra='sp('+str(rank)+',R)'
      else:
         N=sum((-1)^(len(P)-s)*(P[s-1]+1) for s in range(1,len(P)+1))
         if N<= rank/2:
            eqclass=N
         else:
            eqclass=rank-N
         lieAlgebra='sp('+str(eqclass)+','+str(rank-eqclass)+')'
   elif lieType=='D':
      if (rank-2 in P and rank-1 not in P) or (rank-2 not in P and rank-1 in P) :
         lieAlgebra='so*('+str(2*rank)+')'
      elif Set([rank-2,rank-1]) in P:
         N=sum((-1)^(len(P)-s)*(P[s-1]+1) for s in range(1,len(P)-1))
         if N<=rank/2:
            eqclass=N-1
         else:
            eqclass=rank-N-1
         lieAlgebra='so('+str(2*eqclass)+','+str(2*rank-2*eqclass)+')'
      else:
         N=sum((-1)^(len(P)-s)*(P[s-1]+1) for s in range(1,len(P)+1))
         if N<=rank/2:
            eqclass=N
         else:
            eqclass=rank-N
         lieAlgebra='so('+str(2*eqclass)+','+str(2*rank-2*eqclass)+')'
   elif lieType=='G':
      lieAlgebra='g2(2)'
   elif lieType=='F':
      if Set(P).intersection(Set([0,1]))!=Set([]):
         lieAlgebra='f4(4)'
      else:
         lieAlgebra='f4(-20)'
   elif lieType=='E' and rank==6:
      II=[j for j in P if j<=3 and j!=1]
      JJ=[j for j in P if j>3]
      if 1 in P:
         s=1
      else:
         s=0
      if II!=[] or JJ!=[]:
         if II!=[] and 0 not in II:
            I=sum((-1)^(len(II)-a-1)*(II[a]) for a in range(len(II)))
         elif 0 in II:
            I=(-1)^(len(II)-1)+sum((-1)^(len(II)-a)*(II[a]) for a in range(len(II)))
         else:
```

```
               I=0
            J=sum((-1)^(len(JJ)-a-1)*(JJ[a]) for a in range(len(JJ)))
            if P==[0] or P==[5] or P==[2,4] or P==[0,3,4] or P==[0,1] or P==[1,2] or P==[1,4] or P==[1,5] or P==[1,3,5] or (len(JJ)!=1 and J==2-I and (I+s)%2==1) or (len(JJ)!=1 and J==4-I and (I+s)%2==0) or (len(JJ)!=1 and J==1-I) or (len(JJ)==1 and ((J==4+I and (I+s)%2==1) or J==1+I)):
               lieAlgebra='e6(-14)'
            else:
               lieAlgebra='e6(2)'
         else:
            lieAlgebra='e6(2)'
      elif lieType=='E' and rank==7:
         II=[j for j in P if j<=3 and j!=1]
         JJ=[j for j in P if j>3]
         if 1 in P:
            s=1
         else:
            s=0
         if II!=[] or JJ!=[]:
            if II!=[] and 0 not in II:
               I=sum((-1)^(len(II)-a-1)*(II[a]) for a in range(len(II)))
            elif 0 in II:
               I=(-1)^(len(II)-1)+sum((-1)^(len(II)-a)*(II[a]) for a in range(len(II)))
            else:
               I=0
            J=sum((-1)^(len(JJ)-a-1)*(JJ[a]) for a in range(len(JJ)))
            if P==[0] or P==[3] or P==[5] or P==[3,5] or P==[3,4,6] or P==[1,4] or P==[1,6] or P==[1,2,4] or P==[0,1,3,4] or (len(JJ)!=1 and (((J==1-I or J==3-I) and (I+s)%2==1) or ((J==2-I or J==4-I) and (I+s)%2==0) )) or (len(JJ)==1 and (((J==1+I or J==2+I or J==3+I or J==5+I) and (I+s)%2==0) or (J==4+I and (I+s)%2==1))):
               lieAlgebra='e7(-5)'
            elif P==[6] or P==[2,4] or P==[0,3,4] or P==[0,1] or P==[1,2] or P==[1,5] or P==[1,3,5] or P==[1,3,4,6] or P==[1,3,4,5,6] or (len(JJ)!=1 and ((J==1-I and (I+s)%2==0) or (J==2-I and (I+s)%2==1))) or (len(JJ)==1 and ((J==1+I or J==2+I or J==5+I) and (I+s)%2==1)):
               lieAlgebra='e7(-25)'
            else:
               lieAlgebra='e7(7)'
         else:
            lieAlgebra='e7(7)'
      elif lieType=='E' and rank==8:
         II=[j for j in P if j<=3 and j!=1]
         JJ=[j for j in P if j>3]
         if 1 in P:
            s=1
         else:
            s=0
         if II!=[] or JJ!=[]:
            if II!=[] and 0 not in II:
               I=sum((-1)^(len(II)-a-1)*(II[a]) for a in range(len(II)))
            elif 0 in II:
               I=(-1)^(len(II)-1)+sum((-1)^(len(II)-a)*(II[a]) for a in range(len(II)))
            else:
               I=0
            J=sum((-1)^(len(JJ)-a-1)*(JJ[a]) for a in range(len(JJ)))
            if P==[7] or P==[2] or P==[3] or P==[0,2] or P==[1,2] or P==[1,5] or P==[1,6] or (len(JJ)!=1 and ((J==1-I or J==5-I) and (I+s)%2==0)) or (len(JJ)!=1 and (J==3-I and (I+s)%2==1)) or (len(JJ)!=1 and (J==2-I or J==6-I)) or (len(JJ)==1 and ((J==1+I or J==5+I) and (I+s)%2==1)) or (len(JJ)==1 and ((J==3+I and (I+s)%2==0) or J==2+I or J==6+I)):
               lieAlgebra='e8(-24)'
            else:
               lieAlgebra='e8(8)'
         else:
            lieAlgebra='e8(8)'
```

```
   return lieAlgebra
def stabilizer(lieType,rank,P):
   stabiliz=''
   if len(P)<rank:
      st=str(CartanType([lieType,rank]).subtype([i+1 for i in range(rank) if i not in P])).translate(None, "[]',' '")
      stab=''
      for i in range(len(st)):
          if st[i] in ['A','B','C','D','E']:
         if st[i]=='A':
             stab=stab+'su('
             for j in range(i,len(st)):
            if st[j]=='r' or st[j]=='x' or j==len(st)-1:
                if j==len(st)-1:
               stab=stab+str(int(st[i+1:len(st)])+1)+') x '
               break
                else:
               stop=j
               stab=stab+str(int(st[i+1:stop])+1)+') x '
                   break
         if st[i]=='B':
             stab=stab+'so('
             for j in range(i,len(st)):
            if st[j]=='r' or st[j]=='x' or j==len(st)-1:
                if j==len(st)-1:
               stab=stab+str(int(st[i+1:len(st)])*2+1)+') x '
               break
                else:
               stop=j
               stab=stab+str(int(st[i+1:stop])*2+1)+') x '
                   break
         if st[i]=='C':
             stab=stab+'sp('
             for j in range(i,len(st)):
            if st[j]=='r' or st[j]=='x' or j==len(st)-1:
                if j==len(st)-1:
               stab=stab+str(int(st[i+1:len(st)]))+') x '
               break
                else:
               stop=j
               stab=stab+str(int(st[i+1:stop]))+') x '
                   break
         if st[i]=='D':
             stab=stab+'so('
             for j in range(i,len(st)):
            if st[j]=='r' or st[j]=='x' or j==len(st)-1:
                if j==len(st)-1:
               stab=stab+str(int(st[i+1:len(st)])*2)+') x '
               break
                else:
               stop=j
               stab=stab+str(int(st[i+1:stop])*2)+') x '
                   break
         if st[i]=='E':
            if st[i+1]=='6':
               stab=stab+'e6 x '
            elif st[i+1]=='7':
               stab=stab+'e7 x '
      return stab[:-2]+'x R'+str(len(P))+'\n'
   else:
      return 'R'+str(len(P))+'\n'
[lieType,rank]=[raw_input('Type: '),input('Rank: ')]
```

```
print DynkinDiagram([lieType,rank])
W=WeylGroup([lieType,rank],implementation='permutation')
positiveRoots=W.positive_roots()
C=CartanMatrix([lieType,rank])
if lieType=='F':
   B=(1/36)*matrix([[4,-2,0,0],[-2,4,-2,0],[0,-2,2,-1],[0,0,-1,2]])
else:
   B=matrix(QQ,gap('BilinearFormMat(RootSystem(SimpleLieAlgebra("'+lieType+'",'+str(rank)+',Rationals)))'))
print "Dimension:",2*len(positiveRoots)+rank
print '  '
for P in [q for q in Combinations(range(rank)) if q!=[]]:
      compactroots=[root for root in positiveRoots if sum(root[k] for k in P)%2==0]
      noncompactroots=[root for root in positiveRoots if sum(root[k] for k in P)%2!=0]
      epsilon={root: (1 if root in noncompactroots else -1) for root in positiveRoots}
      eta=-2*sum(epsilon[alpha]*alpha for alpha in positiveRoots)
      phiP=(C)*(eta-2*sum(root for root in positiveRoots if all(root[k]==0 for k in P)))
      if all(phiP[k]==0 for k in range(len(phiP))):
         print [var('v'+str(k)) if k in P else 0 for k in range(len(phiP))],' for all vi>0     Non-compact simple roots:',P,'  ','symplectic Calabi-Yau'
         print "Dimension V:",(2*sum(all(root[k]==0 for k in P) for root in positiveRoots)+rank),"    Dimension G/V:",2*(len(positiveRoots)-sum(all(root[k]==0 for k in P) for root in positiveRoots))
         if len(P)==1:
            if C.inverse()*vector([1 if i in P else 0 for i in range(rank)]) in W.roots():
               print 'Is phi a root? Yes'
            else:
               print 'Is phi a root? No'
         else:
            print 'Is phi a root? No'
         print "Hermitian scalar curvature:",0
         print 'Lie algebra: '+lieAlgebraType(lieType,rank,P)
         print 'Stabilizer: '+stabilizer(lieType,rank,P)
      elif [sgn(phiP[k]) for k in range(len(phiP))]==[-1 if k in P else 0 for k in range(len(phiP))]:
         Omega0=[root for root in Set(positiveRoots).difference(Set([root for root in positiveRoots if all(root[k]==0 for k in P)]))]
         OmegaOnc=[root for root in Set(Omega0).intersection(Set(noncompactroots))]
         typeOrbit=''
         if len(P)>1:
            typeOrbit='symplectic general type'
         else:
            for alpha in OmegaOnc:
               for beta in OmegaOnc:
                  if alpha+beta in Omega0:
                     typeOrbit='symplectic general type'
                     break
            if typeOrbit!='symplectic general type':
               typeOrbit='general type'
         print -phiP/(gcd(phiP))   ,'    ','Non-compact simple roots:',P,'    ',typeOrbit
         print "Dimension V:",(2*(len(positiveRoots)-len(Omega0))+rank),"    Dimension G/V:",2*len(Omega0)
         if C.inverse()*(-phiP/(gcd(phiP))) in W.roots():
            print 'Is phi a root? Yes'
         else:
            print 'Is phi a root? No'
         print "Hermitian scalar curvature:",4*gcd(phiP)*sum(sum(epsilon[alpha]*((alpha*B*beta)/(sum(phiP[z]*beta[z]*B[z,z] for z in range(rank)))) for alpha in Omega0) for beta in Omega0)
         print 'Lie algebra: '+lieAlgebraType(lieType,rank,P)
         print 'Stabilizer: '+stabilizer(lieType,rank,P)
      elif [sgn(phiP[k]) for k in range(len(phiP))]==[1 if k in P else 0 for k in range(len(phiP))]:
         T=Combinations([i for i in range(rank) if i not in P])
         for i in range(T.cardinality()):
            S=T[i]+P
```

```
            phiS=(C)*(eta-2*sum(root for root in positiveRoots if all(root[k]==0 for k in S)))
            if [sgn(phiS[k]) for k in range(len(phiS))]==[1 if k in S else 0 for k in range(len(phiS))]:
               print phiS*(1/gcd(phiS)),'   ','Non-compact simple roots:',P,'   ','S:',S,'   ','symplectic Fano'
               Omega=[root for root in Set(positiveRoots).difference(Set([root for root in positiveRoots if all(root[k]==0 for k in S)]))]
               print "Dimension V:",(2*(len(positiveRoots)-len(Omega))+rank),"    Dimension G/V:",2*len(Omega)
               if C.inverse()*(phiS/(gcd(phiS))) in W.roots():
                  print 'Is phi a root? Yes'
               else:
                  print 'Is phi a root? No'
               print "Hermitian scalar curvature:",-4*gcd(phiS)*sum(sum(epsilon[alpha]*((alpha*B*beta)/(sum(phiS[z]*beta[z]*B[z,z] for z in range(rank)))) for alpha in Omega) for beta in Omega)
               print 'Lie algebra: '+lieAlgebraType(lieType,rank,P)
               print 'Stabilizer: '+stabilizer(lieType,rank,S)
print(cputime())
```

Code A.1: Source code of the classification algorithm.

Appendix B

Vogan diagrams with special φ

In this appendix we classify all special φ's associated with connected Vogan diagrams having rank at most $\ell = 8$, where notations and terminology are as in Section 3.7. This classification is a consequence of the results discussed in Chapter 3. In particular, we have the following result.

Proposition B.0.1. *Let G be a non-compact real simple Lie group with rank $\ell \leq 8$ and let $(G/V, \omega, J)$ be an adjoint orbit of G endowed with the canonical almost-Kähler structure. If $(G/V, \omega, J)$ satisfies $\rho = \lambda\omega$, then it is isomorphic up to scaling to the orbit of $v = ih_\varphi$ for some φ contained in the following tables.*

Few comments for reading the tables are in order. For each Lie algebra type, we specify the dimension and its real forms together with the fundamental dominant weights $\varphi_1, \ldots, \varphi_\ell$ written in terms of simple roots $\gamma_1, \ldots, \gamma_\ell$. For each Vogan diagram we list all special φ's expressed as a sum of fundamental dominant weights φ_i's and for each of them we provide the following data.

- Whether φ is a root;
- The symplectic type of the special orbit of v such that $v = ih_\varphi$ and the integrability of the canonical almost-complex structure J. More precisely, we write sGT, sCY, sF if the orbit satisfies $\rho = \lambda\omega$ with $\lambda < 0$, $\lambda = 0$, $\lambda > 0$ respectively and we remove the 's' when J is integrable;
- The Hermitian scalar curvature s of the canonical almost-complex structure J. Note that from s one may compute λ through the identity $s = \frac{\lambda}{2} \dim G/V$;
- The dimension of the stabilizer $\mathfrak{v}$ of V and of the orbit G/V;
- The Lie algebras of G and V.

The choice of the acronyms at the second point is motived as follows. Given an adjoint orbit $(G/V, \omega, J)$ satisfying $\rho = \lambda\omega$ and a discrete co-compact subgroup $\Gamma \subset G$, the quotient $(\Gamma\backslash G/V, \omega_\Gamma, J_\Gamma)$ is a compact almost-Kähler manifold satisfying $\rho_\Gamma = \lambda\omega_\Gamma$ as soon as $\Gamma\backslash G/V$ is smooth. Thus, as discussed in Section 3.6, $(\Gamma\backslash G/V, \omega_\Gamma, J_\Gamma)$ is special symplectic general type, symplectic Calabi-Yau or symplectic Fano according to the sign of λ. In other words, sGT, sCY, sF denote the symplectic type of any compact quotient of $(G/V, \omega)$.

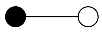
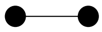
B.1 Special classical adjoint orbits

Rank 2

A_2

Dimension of $\mathfrak{g} = 8$
One non-compact real form with trivial automorphism: $\mathfrak{su}(1,2)$.
Fundamental dominant weights: $\varphi_1 = \frac{2}{3}\gamma_1 + \frac{1}{3}\gamma_2$

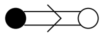
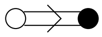
$$\varphi_2 = \frac{1}{3}\gamma_1 + \frac{2}{3}\gamma_2$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
 γ_1 γ_2	φ_1	no	GT	-12	4	4	$\mathfrak{su}(1,2)$	$\mathfrak{su}(2) \oplus \mathbb{R}$
 γ_1 γ_2	$t_1\varphi_1 + t_2\varphi_2$ for all $t_1, t_2 > 0$	no	sCY	0	2	6	$\mathfrak{su}(1,2)$	$\mathbb{R}^2$

B_2

Dimension of $\mathfrak{g} = 10$
Two non-compact real forms: $\mathfrak{so}(4,1)$, $\mathfrak{so}(2,3)$.
Fundamental dominant weights: $\varphi_1 = \gamma_1 + \gamma_2$

$$\varphi_2 = \frac{1}{2}\gamma_1 + \gamma_2$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
2 1  γ_1 γ_2	φ_1	yes	GT	-18	4	6	$\mathfrak{so}(2,3)$	$\mathfrak{su}(2) \oplus \mathbb{R}$
2 1  γ_1 γ_2	φ_2	no	sCY	0	4	6	$\mathfrak{so}(4,1)$	$\mathfrak{su}(2) \oplus \mathbb{R}$

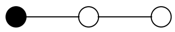
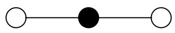
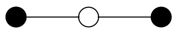
Rank 3

A_3

Dimension of $\mathfrak{g} = 15$
Two non-compact real forms with trivial automorphism: $\mathfrak{su}(1,3)$, $\mathfrak{su}(2,2)$.
Fundamental dominant weights: $\varphi_1 = \frac{3}{4}\gamma_1 + \frac{1}{2}\gamma_2 + \frac{1}{4}\gamma_3$

$$\varphi_2 = \frac{1}{2}\gamma_1 + \gamma_2 + \frac{1}{2}\gamma_3$$

$$\varphi_3 = \frac{1}{4}\gamma_1 + \frac{1}{2}\gamma_2 + \frac{3}{4}\gamma_3$$

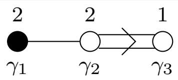
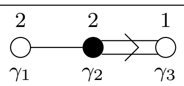
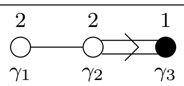
Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
 γ_1 γ_2 γ_3	φ_1	no	GT	-24	9	6	$\mathfrak{su}(1,3)$	$\mathfrak{su}(3) \oplus \mathbb{R}$
 γ_1 γ_2 γ_3	φ_2	no	GT	-32	7	8	$\mathfrak{su}(2,2)$	$(\mathfrak{su}(2))^2 \oplus \mathbb{R}$
 γ_1 γ_2 γ_3	$\varphi_1 + \varphi_3$	yes	sGT	-10	5	10	$\mathfrak{su}(2,2)$	$\mathfrak{su}(2) \oplus \mathbb{R}$

B_3

Dimension of $\mathfrak{g} = 21$
Three non-compact real forms: $\mathfrak{so}(6,1)$, $\mathfrak{so}(4,3)$, $\mathfrak{so}(2,5)$.
Fundamental dominant weights: $\varphi_1 = \gamma_1 + \gamma_2 + \gamma_3$

$$\varphi_2 = \gamma_1 + 2\gamma_2 + 2\gamma_3$$

$$\varphi_3 = \tfrac{1}{2}\gamma_1 + \gamma_2 + \tfrac{3}{2}\gamma_3$$

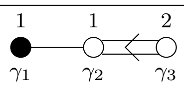
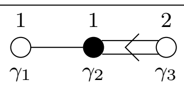
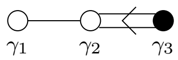
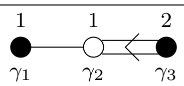
Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	yes	GT	-50	11	10	$\mathfrak{so}(2,5)$	$\mathfrak{so}(5) \oplus \mathbb{R}$
	φ_2	yes	sGT	-28	7	14	$\mathfrak{so}(4,3)$	$(\mathfrak{su}(2))^2 \oplus \mathbb{R}$
	φ_3	no	sF	24	9	12	$\mathfrak{so}(6,1)$	$\mathfrak{su}(3) \oplus \mathbb{R}$

C_3

Dimension of $\mathfrak{g} = 21$
Two non-compact real forms: $\mathfrak{sp}(1,2)$, $\mathfrak{sp}(3,\mathbb{R})$.
Fundamental dominant weights: $\varphi_1 = \gamma_1 + \gamma_2 + \tfrac{1}{2}\gamma_3$

$$\varphi_2 = \gamma_1 + 2\gamma_2 + \gamma_3$$

$$\varphi_3 = \gamma_1 + 2\gamma_2 + \tfrac{3}{2}\gamma_3$$

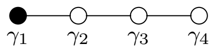
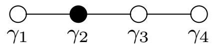
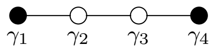
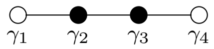
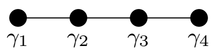
Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	sGT	-20	11	10	$\mathfrak{sp}(1,2)$	$\mathfrak{so}(5) \oplus \mathbb{R}$
	φ_2	yes	sF	14	7	14	$\mathfrak{sp}(1,2)$	$(\mathfrak{su}(2))^2 \oplus \mathbb{R}$
	φ_3	no	GT	-48	9	12	$\mathfrak{sp}(3,\mathbb{R})$	$\mathfrak{su}(3) \oplus \mathbb{R}$
	$\varphi_1 + \varphi_3$	no	sGT	-16	5	16	$\mathfrak{sp}(3,\mathbb{R})$	$\mathfrak{su}(2) \oplus \mathbb{R}^2$

Rank 4

A_4

Dimension of $\mathfrak{g} = 24$
Two non-compact real forms with trivial automorphism: $\mathfrak{su}(1,4)$, $\mathfrak{su}(2,3)$.
Fundamental dominant weights:

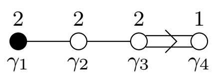
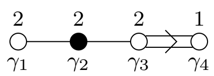
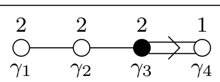
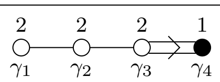
$$\begin{aligned}\varphi_1 &= \tfrac{4}{5}\gamma_1 + \tfrac{3}{5}\gamma_2 + \tfrac{2}{5}\gamma_3 + \tfrac{1}{5}\gamma_4\\ \varphi_2 &= \tfrac{3}{5}\gamma_1 + \tfrac{6}{5}\gamma_2 + \tfrac{4}{5}\gamma_3 + \tfrac{2}{5}\gamma_4\\ \varphi_3 &= \tfrac{2}{5}\gamma_1 + \tfrac{4}{5}\gamma_2 + \tfrac{6}{5}\gamma_3 + \tfrac{3}{5}\gamma_4\\ \varphi_4 &= \tfrac{1}{5}\gamma_1 + \tfrac{2}{5}\gamma_2 + \tfrac{3}{5}\gamma_3 + \tfrac{4}{5}\gamma_4\end{aligned}$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	GT	-40	16	8	$\mathfrak{su}(1,4)$	$\mathfrak{su}(4) \oplus \mathbb{R}$
	φ_2	no	GT	-60	12	12	$\mathfrak{su}(2,3)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
	$\varphi_1 + \varphi_4$	yes	sGT	-28	10	14	$\mathfrak{su}(2,3)$	$\mathfrak{su}(3) \oplus \mathbb{R}^2$
	$\varphi_2 + \varphi_3$	no	sF	16	8	16	$\mathfrak{su}(1,4)$	$(\mathfrak{su}(2))^2 \oplus \mathbb{R}$
	$\sum_{i=1}^4 t_i\varphi_i$ for all $t_i > 0$	no	sCY	0	4	20	$\mathfrak{su}(2,3)$	$\mathbb{R}^4$

B_4

Dimension of $\mathfrak{g} = 36$
Four non-compact real forms: $\mathfrak{so}(8,1)$, $\mathfrak{so}(6,3)$, $\mathfrak{so}(4,5)$, $\mathfrak{so}(2,7)$.
Fundamental dominant weights:

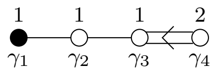
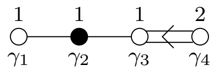
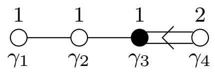
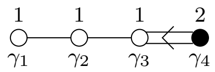
$$\begin{aligned}\varphi_1 &= \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4\\ \varphi_2 &= \gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4\\ \varphi_3 &= \gamma_1 + 2\gamma_2 + 3\gamma_3 + 3\gamma_4\\ \varphi_4 &= \tfrac{1}{2}\gamma_1 + \gamma_2 + \tfrac{3}{2}\gamma_3 + 2\gamma_4\end{aligned}$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	yes	GT	-98	22	14	$\mathfrak{so}(2,7)$	$\mathfrak{so}(7) \oplus \mathbb{R}$
	φ_2	yes	sGT	-88	14	22	$\mathfrak{so}(4,5)$	$\mathfrak{su}(2) \oplus \mathfrak{so}(5) \oplus \mathbb{R}$
	φ_3	no	sGT	-24	12	24	$\mathfrak{so}(6,3)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	φ_4	no	sF	80	16	20	$\mathfrak{so}(8,1)$	$\mathfrak{su}(4) \oplus \mathbb{R}$
	$2\varphi_1 + \varphi_4$	no	sF	52	10	26	$\mathfrak{so}(8,1)$	$\mathfrak{su}(3) \oplus \mathbb{R}^2$

C_4

Dimension of $\mathfrak{g} = 36$
Three non-compact real forms: $\mathfrak{sp}(1,3)$, $\mathfrak{sp}(2,2)$, $\mathfrak{sp}(4,\mathbb{R})$.
Fundamental dominant weights:

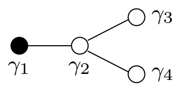
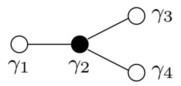
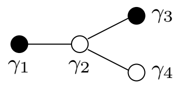
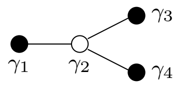
$$\varphi_1 = \gamma_1 + \gamma_2 + \gamma_3 + \tfrac{1}{2}\gamma_4$$
$$\varphi_2 = \gamma_1 + 2\gamma_2 + 2\gamma_3 + \gamma_4$$
$$\varphi_3 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + \tfrac{3}{2}\gamma_4$$
$$\varphi_4 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 2\gamma_4$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	sGT	-56	22	14	$\mathfrak{sp}(1,3)$	$\mathfrak{sp}(3) \oplus \mathbb{R}$
	φ_2	yes	sGT	-22	14	22	$\mathfrak{sp}(2,2)$	$\mathfrak{su}(2) \oplus \mathfrak{so}(5) \oplus \mathbb{R}$
	φ_3	no	sF	48	12	24	$\mathfrak{sp}(1,3)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$3\varphi_1 + \varphi_3$	no	sF	28	8	28	$\mathfrak{sp}(1,3)$	$(\mathfrak{su}(2))^2 \oplus \mathbb{R}^2$
	φ_4	no	GT	-100	16	20	$\mathfrak{sp}(4,\mathbb{R})$	$\mathfrak{su}(4) \oplus \mathbb{R}$

D_4

Dimension of $\mathfrak{g} = 28$
Two non-compact real forms with trivial automorphism: $\mathfrak{so}(2,6)$, $\mathfrak{so}(4,4)$.
Fundamental dominant weights:

$$\varphi_1 = \gamma_1 + \gamma_2 + \tfrac{1}{2}\gamma_3 + \tfrac{1}{2}\gamma_4$$
$$\varphi_2 = \gamma_1 + 2\gamma_2 + \gamma_3 + \gamma_4$$
$$\varphi_3 = \tfrac{1}{2}\gamma_1 + \gamma_2 + \gamma_3 + \tfrac{1}{2}\gamma_4$$
$$\varphi_4 = \tfrac{1}{2}\gamma_1 + \gamma_2 + \tfrac{1}{2}\gamma_3 + \gamma_4$$

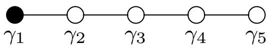
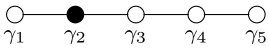
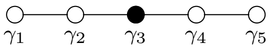
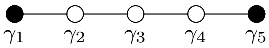
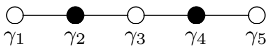
Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	GT	-72	16	12	$\mathfrak{so}(2,6)$	$\mathfrak{su}(4) \oplus \mathbb{R}$
	φ_2	yes	sGT	-54	10	18	$\mathfrak{so}(4,4)$	$(\mathfrak{su}(2))^3 \oplus \mathbb{R}$
	$t_1\varphi_1 + t_3\varphi_3$ for all $t_1, t_2 > 0$	no	sCY	0	10	18	$\mathfrak{so}(2,6)$	$\mathfrak{su}(3) \oplus \mathbb{R}^2$
	$\varphi_1 + \varphi_3 + \varphi_4$	no	sGT	-22	6	22	$\mathfrak{so}(4,4)$	$\mathfrak{su}(2) \oplus \mathbb{R}^3$

Rank 5

A_5

Dimension of $\mathfrak{g} = 35$
Three non-compact real forms with trivial automorphism: $\mathfrak{su}(1,5)$, $\mathfrak{su}(2,4)$, $\mathfrak{su}(3,3)$.
Fundamental dominant weights:

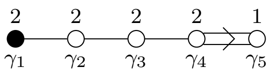
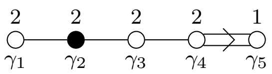
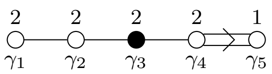
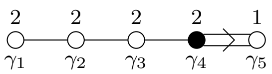
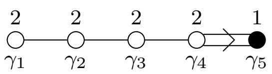
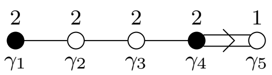
$$\varphi_1 = \tfrac{5}{6}\gamma_1 + \tfrac{2}{3}\gamma_2 + \tfrac{1}{2}\gamma_3 + \tfrac{1}{3}\gamma_4 + \tfrac{1}{6}\gamma_5$$
$$\varphi_2 = \tfrac{2}{3}\gamma_1 + \tfrac{4}{3}\gamma_2 + \gamma_3 + \tfrac{2}{3}\gamma_4 + \tfrac{1}{3}\gamma_5$$
$$\varphi_3 = \tfrac{1}{2}\gamma_1 + \gamma_2 + \tfrac{3}{2}\gamma_3 + \gamma_4 + \tfrac{1}{2}\gamma_5$$
$$\varphi_4 = \tfrac{1}{3}\gamma_1 + \tfrac{2}{3}\gamma_2 + \gamma_3 + \tfrac{4}{3}\gamma_4 + \tfrac{2}{3}\gamma_5$$
$$\varphi_5 = \tfrac{1}{6}\gamma_1 + \tfrac{1}{3}\gamma_2 + \tfrac{1}{2}\gamma_3 + \tfrac{2}{3}\gamma_4 + \tfrac{5}{6}\gamma_5$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	GT	-60	25	10	$\mathfrak{su}(1,5)$	$\mathfrak{su}(5) \oplus \mathbb{R}$
	φ_2	no	GT	-96	19	16	$\mathfrak{su}(2,4)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}$
	φ_3	no	GT	-108	17	18	$\mathfrak{su}(3,3)$	$(\mathfrak{su}(3))^2 \oplus \mathbb{R}$
	$\varphi_1 + \varphi_3$	yes	sGT	-54	17	18	$\mathfrak{su}(2,4)$	$\mathfrak{su}(4) \oplus \mathbb{R}^2$
	$t_2\varphi_2 + t_4\varphi_4$ for all $t_2, t_4 > 0$	no	sCY	0	11	24	$\mathfrak{su}(2,4)$	$(\mathfrak{su}(2))^3 \oplus \mathbb{R}^2$

B_5

Dimension of $\mathfrak{g} = 55$
Five non-compact real forms: $\mathfrak{so}(2,9)$, $\mathfrak{so}(4,7)$, $\mathfrak{so}(6,5)$, $\mathfrak{so}(8,3)$, $\mathfrak{so}(10,1)$.
Fundamental dominant weights: $\varphi_1 = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5$

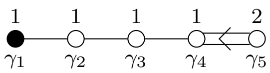
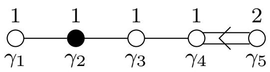
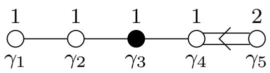
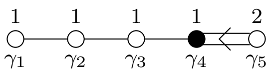
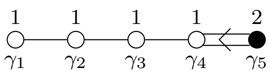
$$\varphi_2 = \gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5$$
$$\varphi_3 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 3\gamma_4 + 3\gamma_5$$
$$\varphi_4 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 4\gamma_5$$
$$\varphi_5 = \tfrac{1}{2}\gamma_1 + \gamma_2 + \tfrac{3}{2}\gamma_3 + 2\gamma_4 + \tfrac{5}{2}\gamma_5$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	yes	GT	-162	37	18	$\mathfrak{so}(2,9)$	$\mathfrak{so}(9) \oplus \mathbb{R}$
	φ_2	yes	sGT	-180	25	30	$\mathfrak{so}(4,7)$	$\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathbb{R}$
	φ_3	no	sGT	-180	19	36	$\mathfrak{so}(6,5)$	$\mathfrak{su}(3) \oplus \mathfrak{so}(5) \oplus \mathbb{R}$
	φ_4	no	sCY	0	19	36	$\mathfrak{so}(8,3)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	φ_5	no	sF	180	25	30	$\mathfrak{so}(10,1)$	$\mathfrak{su}(5) \oplus \mathbb{R}$
	$5\varphi_1 + 4\varphi_5$	no	sF	38	17	38	$\mathfrak{so}(10,1)$	$\mathfrak{su}(4) \oplus \mathbb{R}^2$
	$5\varphi_2 + 2\varphi_5$	no	sF	42	13	42	$\mathfrak{so}(10,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^2$
	$\varphi_1 + 2\varphi_2 + \varphi_5$	no	sF	88	11	44	$\mathfrak{so}(10,1)$	$\mathfrak{su}(3) \oplus \mathbb{R}^3$
	$2\varphi_1 + \varphi_4$	no	sGT	-42	13	42	$\mathfrak{so}(6,5)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$

C_5

Dimension of $\mathfrak{g} = 55$
Three non-compact real forms: $\mathfrak{sp}(1,4)$, $\mathfrak{sp}(2,3)$, $\mathfrak{sp}(5,\mathbb{R})$.
Fundamental dominant weights:

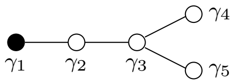
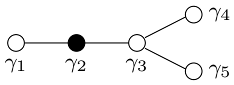
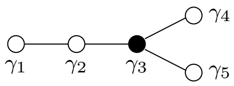
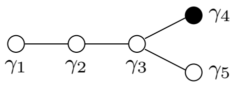
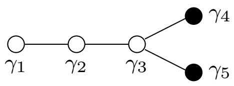
$$\varphi_1 = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \tfrac{1}{2}\gamma_5$$
$$\varphi_2 = \gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + \gamma_5$$
$$\varphi_3 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 3\gamma_4 + \tfrac{3}{2}\gamma_5$$
$$\varphi_4 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 2\gamma_5$$
$$\varphi_5 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + \tfrac{5}{2}\gamma_5$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	sGT	-108	37	18	$\mathfrak{sp}(1,4)$	$\mathfrak{sp}(4) \oplus \mathbb{R}$
	φ_2	yes	sGT	-90	25	30	$\mathfrak{sp}(2,3)$	$\mathfrak{su}(2) \oplus \mathfrak{sp}(3) \oplus \mathbb{R}$
	φ_3	no	sCY	0	19	36	$\mathfrak{sp}(2,3)$	$\mathfrak{su}(3) \oplus \mathfrak{so}(5) \oplus \mathbb{R}$
	φ_4	no	sF	108	19	36	$\mathfrak{sp}(1,4)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$2\varphi_1 + \varphi_4$	no	sF	84	13	42	$\mathfrak{sp}(1,4)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
	$4\varphi_2 + \varphi_4$	no	sF	44	11	44	$\mathfrak{sp}(1,4)$	$(\mathfrak{su}(2))^3 \oplus \mathbb{R}^2$
	$\varphi_4 + 2\varphi_5$	no	sF	38	17	38	$\mathfrak{sp}(1,4)$	$\mathfrak{su}(4) \oplus \mathbb{R}^2$
	$2\varphi_1 + 3\varphi_2 + \varphi_4$	no	sF	46	9	46	$\mathfrak{sp}(1,4)$	$(\mathfrak{su}(2))^2 \oplus \mathbb{R}^3$
	φ_5	no	GT	-180	25	30	$\mathfrak{sp}(5,\mathbb{R})$	$\mathfrak{su}(5) \oplus \mathbb{R}$

D_5

Dimension of $\mathfrak{g} = 45$
Three non-compact real forms with trivial automorphism: $\mathfrak{so}(2,8)$, $\mathfrak{so}(4,6)$, $\mathfrak{so}^*(10)$.
Fundamental dominant weights:

$$\varphi_1 = \gamma_1 + \gamma_2 + \gamma_3 + \tfrac{1}{2}\gamma_4 + \tfrac{1}{2}\gamma_5$$
$$\varphi_2 = \gamma_1 + 2\gamma_2 + 2\gamma_3 + \gamma_4 + \gamma_5$$
$$\varphi_3 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + \tfrac{3}{2}\gamma_4 + \tfrac{3}{2}\gamma_5$$
$$\varphi_4 = \tfrac{1}{2}\gamma_1 + \gamma_2 + \tfrac{3}{2}\gamma_3 + \tfrac{5}{4}\gamma_4 + \tfrac{3}{4}\gamma_5$$
$$\varphi_5 = \tfrac{1}{2}\gamma_1 + \gamma_2 + \tfrac{3}{2}\gamma_3 + \tfrac{3}{4}\gamma_4 + \tfrac{5}{4}\gamma_5$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	GT	-128	29	16	$\mathfrak{so}(2,8)$	$\mathfrak{so}(8) \oplus \mathbb{R}$
	φ_2	yes	sGT	-130	19	26	$\mathfrak{so}(4,6)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}$
	φ_3	no	sGT	-60	15	30	$\mathfrak{so}(4,6)$	$\mathfrak{su}(3) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}$
	φ_4	no	sGT	-160	25	20	$\mathfrak{so}^*(10)$	$\mathfrak{su}(5) \oplus \mathbb{R}$
	$\varphi_4 + \varphi_5$	no	sF	28	17	28	$\mathfrak{so}(2,8)$	$\mathfrak{su}(4) \oplus \mathbb{R}^2$

Rank 6

A_6

Dimension of $\mathfrak{g} = 48$
Three non-compact real forms with trivial automorphism: $\mathfrak{su}(1,6)$, $\mathfrak{su}(2,5)$, $\mathfrak{su}(3,4)$.
Fundamental dominant weights:

$$\begin{aligned}
\varphi_1 &= \tfrac{6}{7}\gamma_1 + \tfrac{5}{7}\gamma_2 + \tfrac{4}{7}\gamma_3 + \tfrac{3}{7}\gamma_4 + \tfrac{2}{7}\gamma_5 + \tfrac{1}{7}\gamma_6\\
\varphi_2 &= \tfrac{5}{7}\gamma_1 + \tfrac{10}{7}\gamma_2 + \tfrac{8}{7}\gamma_3 + \tfrac{6}{7}\gamma_4 + \tfrac{4}{7}\gamma_5 + \tfrac{2}{7}\gamma_6\\
\varphi_3 &= \tfrac{4}{7}\gamma_1 + \tfrac{8}{7}\gamma_2 + \tfrac{12}{7}\gamma_3 + \tfrac{9}{7}\gamma_4 + \tfrac{6}{7}\gamma_5 + \tfrac{3}{7}\gamma_6\\
\varphi_4 &= \tfrac{3}{7}\gamma_1 + \tfrac{6}{7}\gamma_2 + \tfrac{9}{7}\gamma_3 + \tfrac{12}{7}\gamma_4 + \tfrac{8}{7}\gamma_5 + \tfrac{4}{7}\gamma_6\\
\varphi_5 &= \tfrac{2}{7}\gamma_1 + \tfrac{4}{7}\gamma_2 + \tfrac{6}{7}\gamma_3 + \tfrac{8}{7}\gamma_4 + \tfrac{10}{7}\gamma_5 + \tfrac{5}{7}\gamma_6\\
\varphi_6 &= \tfrac{1}{7}\gamma_1 + \tfrac{2}{7}\gamma_2 + \tfrac{3}{7}\gamma_3 + \tfrac{4}{7}\gamma_4 + \tfrac{5}{7}\gamma_5 + \tfrac{6}{7}\gamma_6
\end{aligned}$$

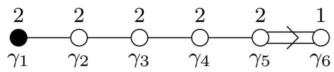
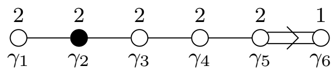
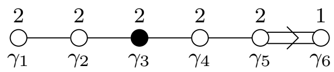
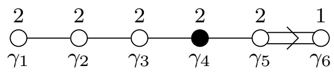
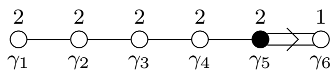
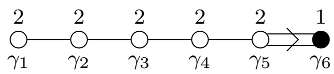
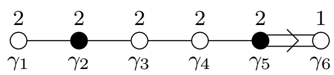
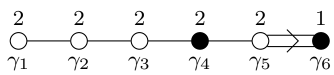
Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
●—○—○—○—○—○ ($\gamma_1 \ \gamma_2 \ \gamma_3 \ \gamma_4 \ \gamma_5 \ \gamma_6$)	φ_1	no	GT	-84	36	12	$\mathfrak{su}(1,6)$	$\mathfrak{su}(6) \oplus \mathbb{R}$
○—●—○—○—○—○ ($\gamma_1 \ \gamma_2 \ \gamma_3 \ \gamma_4 \ \gamma_5 \ \gamma_6$)	φ_2	no	GT	-140	28	20	$\mathfrak{su}(2,5)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \mathbb{R}$
○—○—●—○—○—○ ($\gamma_1 \ \gamma_2 \ \gamma_3 \ \gamma_4 \ \gamma_5 \ \gamma_6$)	φ_3	no	GT	-168	24	24	$\mathfrak{su}(3,4)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(4) \oplus \mathbb{R}$
●—○—○—○—●—○ ($\gamma_1 \ \gamma_2 \ \gamma_3 \ \gamma_4 \ \gamma_5 \ \gamma_6$)	$\varphi_1 + 4\varphi_5$	no	sGT	-28	20	28	$\mathfrak{su}(3,4)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
●—○—○—○—○—● ($\gamma_1 \ \gamma_2 \ \gamma_3 \ \gamma_4 \ \gamma_5 \ \gamma_6$)	$\varphi_1 + \varphi_6$	yes	sGT	-88	26	22	$\mathfrak{su}(2,5)$	$\mathfrak{su}(5) \oplus \mathbb{R}^2$
○—●—○—○—●—○ ($\gamma_1 \ \gamma_2 \ \gamma_3 \ \gamma_4 \ \gamma_5 \ \gamma_6$)	$\varphi_2 + \varphi_4$	no	sGT	-32	16	32	$\mathfrak{su}(3,4)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^2$
○—○—●—●—○—○ ($\gamma_1 \ \gamma_2 \ \gamma_3 \ \gamma_4 \ \gamma_5 \ \gamma_6$)	$\varphi_3 + \varphi_4$	no	sF	60	18	30	$\mathfrak{su}(1,6)$	$(\mathfrak{su}(3))^2 \oplus \mathbb{R}^2$
	$3\varphi_1 + \varphi_3 + 2\varphi_4$	no	sF	34	14	34	$\mathfrak{su}(1,6)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^3$
	$3\varphi_1 + \varphi_3 + \varphi_4 + 3\varphi_6$	no	sF	38	10	38	$\mathfrak{su}(1,6)$	$(\mathfrak{su}(2))^2 \oplus \mathbb{R}^4$
●—●—●—●—●—● ($\gamma_1 \ \gamma_2 \ \gamma_3 \ \gamma_4 \ \gamma_5 \ \gamma_6$)	$\sum_{i=1}^{6} t_i\varphi_i$ for all $t_i > 0$	no	sCY	0	6	42	$\mathfrak{su}(3,4)$	$\mathbb{R}^6$

B_6

Dimension of $\mathfrak{g} = 78$

Six non-compact real forms: $\mathfrak{so}(2,11)$, $\mathfrak{so}(4,9)$, $\mathfrak{so}(6,7)$, $\mathfrak{so}(8,5)$, $\mathfrak{so}(10,3)$, $\mathfrak{so}(12,1)$.

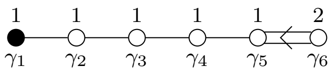
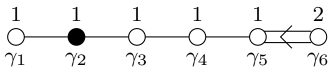
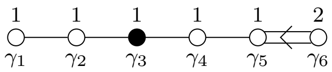
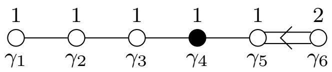
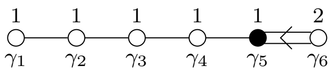
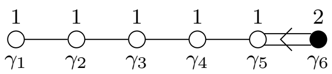
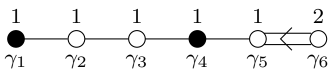
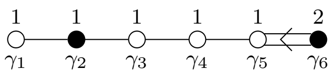
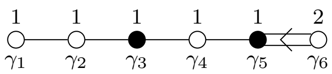
Fundamental dominant weights:
$$\begin{aligned}
\varphi_1 &= \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6\\
\varphi_2 &= \gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 2\gamma_6\\
\varphi_3 &= \gamma_1 + 2\gamma_2 + 3\gamma_3 + 3\gamma_4 + 3\gamma_5 + 3\gamma_6\\
\varphi_4 &= \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 4\gamma_5 + 4\gamma_6\\
\varphi_5 &= \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 5\gamma_6\\
\varphi_6 &= \tfrac{1}{2}\varphi_1 + \varphi_2 + \tfrac{3}{2}\varphi_3 + 2\varphi_4 + \tfrac{5}{2}\varphi_5 + 3\varphi_6
\end{aligned}$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	yes	GT	-242	56	22	$\mathfrak{so}(2,11)$	$\mathfrak{so}(11) \oplus \mathbb{R}$
	φ_2	yes	sGT	-304	40	38	$\mathfrak{so}(4,9)$	$\mathfrak{su}(2) \oplus \mathfrak{so}(9) \oplus \mathbb{R}$
	φ_3	no	sGT	-240	30	48	$\mathfrak{so}(6,7)$	$\mathfrak{su}(3) \oplus \mathfrak{so}(7) \oplus \mathbb{R}$
	φ_4	no	sGT	-104	26	52	$\mathfrak{so}(8,5)$	$\mathfrak{su}(4) \oplus \mathfrak{so}(5) \oplus \mathbb{R}$
	φ_5	no	sF	50	28	50	$\mathfrak{so}(10,3)$	$\mathfrak{su}(5) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	φ_6	no	sF	336	36	42	$\mathfrak{so}(12,1)$	$\mathfrak{su}(6) \oplus \mathbb{R}$
	$\varphi_1 + \varphi_6$	no	sF	312	26	52	$\mathfrak{so}(12,1)$	$\mathfrak{su}(5) \oplus \mathbb{R}^2$
	$3\varphi_3 + \varphi_6$	no	sF	120	18	60	$\mathfrak{so}(12,1)$	$(\mathfrak{su}(3))^2 \oplus \mathbb{R}^2$
	$3\varphi_2 + 2\varphi_6$	no	sF	116	20	58	$\mathfrak{so}(12,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^2$
	$2\varphi_1 + 5\varphi_2 + 4\varphi_6$	no	sF	60	18	60	$\mathfrak{so}(12,1)$	$\mathfrak{su}(4) \oplus \mathbb{R}^3$
	$3\varphi_1 + 5\varphi_3 + 2\varphi_6$	no	sF	64	14	64	$\mathfrak{so}(12,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^3$
	$3\varphi_2 + 4\varphi_3 + 2\varphi_6$	no	sF	64	14	64	$\mathfrak{so}(12,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^3$
	$\varphi_1 + \varphi_2 + 2\varphi_3 + \varphi_6$	no	sF	132	12	66	$\mathfrak{so}(12,1)$	$\mathfrak{su}(3) \oplus \mathbb{R}^4$
	$\varphi_2 + \varphi_5$	no	sGT	-62	16	62	$\mathfrak{so}(6,7)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^2$
	$t_4\varphi_4 + t_6\varphi_6$ for all $t_4, t_6 > 0$	no	sCY	0	20	58	$\mathfrak{so}(4,9)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$

C_6

Dimension of $\mathfrak{g} = 78$
Four non-compact real forms: $\mathfrak{sp}(1,5)$, $\mathfrak{sp}(2,4)$, $\mathfrak{sp}(3,3)$, $\mathfrak{sp}(6,\mathbb{R})$.
Fundamental dominant weights:

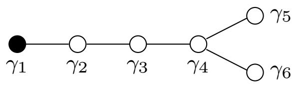
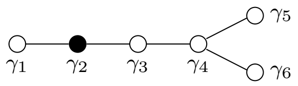
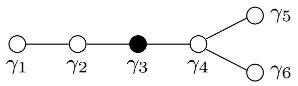
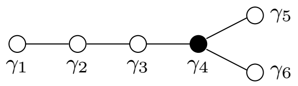
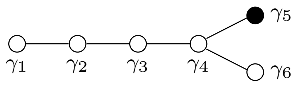
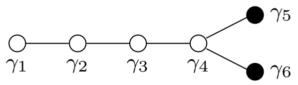
$$\begin{aligned}
\varphi_1 &= \gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5+\tfrac{1}{2}\gamma_6\\
\varphi_2 &= \gamma_1+2\gamma_2+2\gamma_3+2\gamma_4+2\gamma_5+\gamma_6\\
\varphi_3 &= \gamma_1+2\gamma_2+3\gamma_3+3\gamma_4+3\gamma_5+\tfrac{3}{2}\gamma_6\\
\varphi_4 &= \gamma_1+2\gamma_2+3\gamma_3+4\gamma_4+4\gamma_5+2\gamma_6\\
\varphi_5 &= \gamma_1+2\gamma_2+3\gamma_3+4\gamma_4+5\gamma_5+\tfrac{5}{2}\gamma_6\\
\varphi_6 &= \gamma_1+2\gamma_2+3\gamma_3+4\gamma_4+5\gamma_5+3\gamma_6
\end{aligned}$$

Vogan diagram	φ	$\varphi\in\Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	sGT	-176	56	22	$\mathfrak{sp}(1,5)$	$\mathfrak{sp}(5)\oplus\mathbb{R}$
	φ_2	yes	sGT	-190	40	38	$\mathfrak{sp}(2,4)$	$\mathfrak{su}(2)\oplus\mathfrak{sp}(4)\oplus\mathbb{R}$
	φ_3	no	sGT	-96	30	48	$\mathfrak{sp}(3,3)$	$\mathfrak{su}(3)\oplus\mathfrak{sp}(3)\oplus\mathbb{R}$
	φ_4	no	sF	52	26	52	$\mathfrak{sp}(2,4)$	$\mathfrak{su}(4)\oplus\mathfrak{so}(5)\oplus\mathbb{R}$
	φ_5	no	sF	200	28	50	$\mathfrak{sp}(1,5)$	$\mathfrak{su}(5)\oplus\mathfrak{su}(2)\oplus\mathbb{R}$
	$5\varphi_1+3\varphi_5$	no	sF	58	20	58	$\mathfrak{sp}(1,5)$	$\mathfrak{su}(4)\oplus\mathfrak{su}(2)\oplus\mathbb{R}^2$
	$5\varphi_2+2\varphi_5$	no	sF	62	16	62	$\mathfrak{sp}(1,5)$	$(\mathfrak{su}(2))^2\oplus\mathfrak{su}(3)\oplus\mathbb{R}^2$
	$5\varphi_3+\varphi_5$	no	sF	62	16	62	$\mathfrak{sp}(1,5)$	$\mathfrak{su}(3)\oplus(\mathfrak{su}(2))^2\oplus\mathbb{R}^2$
	$\varphi_5+\varphi_6$	no	sF	104	26	52	$\mathfrak{sp}(1,5)$	$\mathfrak{su}(5)\oplus\mathbb{R}^2$
	$\varphi_1+2\varphi_2+\varphi_5$	no	sF	128	14	64	$\mathfrak{sp}(1,5)$	$\mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathbb{R}^3$
	$3\varphi_1+4\varphi_3+\varphi_5$	no	sF	66	12	66	$\mathfrak{sp}(1,5)$	$(\mathfrak{su}(2))^3\oplus\mathbb{R}^3$
	$5\varphi_1+\varphi_5+2\varphi_6$	no	sF	60	18	60	$\mathfrak{sp}(1,5)$	$\mathfrak{su}(4)\oplus\mathbb{R}^3$
	$3\varphi_2+3\varphi_3+\varphi_5$	no	sF	66	12	66	$\mathfrak{sp}(1,5)$	$(\mathfrak{su}(2))^3\oplus\mathbb{R}^3$
	$2\varphi_1+2\varphi_2+3\varphi_3+\varphi_5$	no	sF	68	10	68	$\mathfrak{sp}(1,5)$	$(\mathfrak{su}(2))^2\oplus\mathbb{R}^4$
	φ_6	no	GT	-294	36	42	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(6)\oplus\mathbb{R}$
	$t_1\varphi_1+t_4\varphi_4$ for all $t_1,t_4>0$	no	sCY	0	20	58	$\mathfrak{sp}(3,3)$	$\mathfrak{su}(3)\oplus\mathfrak{so}(5)\oplus\mathbb{R}^2$
	$2\varphi_2+\varphi_6$	no	sGT	-58	20	58	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(2)\oplus\mathfrak{su}(4)\oplus\mathbb{R}^2$
	$\varphi_3+\varphi_5$	no	sF	62	16	62	$\mathfrak{sp}(2,4)$	$\mathfrak{su}(3)\oplus(\mathfrak{su}(2))^2\oplus\mathbb{R}^2$

D_6

Dimension of $\mathfrak{g} = 66$
Three non-compact real forms with trivial automorphism: $\mathfrak{so}(2,10)$, $\mathfrak{so}(4,8)$, $\mathfrak{so}^*(12)$.
Fundamental dominant weights:

$$\varphi_1 = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \tfrac{1}{2}\gamma_5 + \tfrac{1}{2}\gamma_6$$
$$\varphi_2 = \gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + \gamma_5 + \gamma_6$$
$$\varphi_3 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 3\gamma_4 + \tfrac{3}{2}\gamma_5 + \tfrac{3}{2}\gamma_6$$
$$\varphi_4 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 2\gamma_5 + 2\gamma_6$$
$$\varphi_5 = \tfrac{1}{2}\gamma_1 + \gamma_2 + \tfrac{3}{2}\gamma_3 + 2\gamma_4 + \tfrac{3}{2}\gamma_5 + \gamma_6$$
$$\varphi_6 = \tfrac{1}{2}\gamma_1 + \gamma_2 + \tfrac{3}{2}\gamma_3 + 2\gamma_4 + \gamma_5 + \tfrac{3}{2}\gamma_6$$

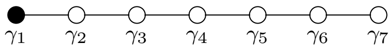
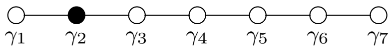
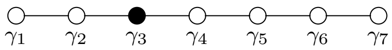
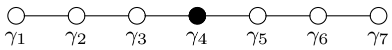
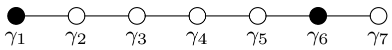
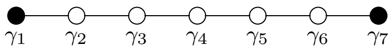
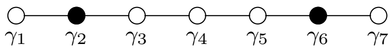
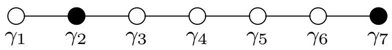
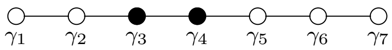
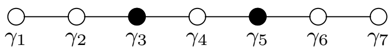
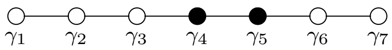
Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	GT	-200	46	20	$\mathfrak{so}(2,10)$	$\mathfrak{so}(10) \oplus \mathbb{R}$
	φ_2	yes	sGT	-238	32	34	$\mathfrak{so}(4,8)$	$\mathfrak{su}(2) \oplus \mathfrak{so}(8) \oplus \mathbb{R}$
	φ_3	no	sGT	-168	24	42	$\mathfrak{so}(6,6)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(4) \oplus \mathbb{R}$
	φ_4	no	sGT	-44	22	44	$\mathfrak{so}(4,8)$	$\mathfrak{su}(4) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}$
	φ_5	no	GT	-300	36	30	$\mathfrak{so}^*(12)$	$\mathfrak{su}(6) \oplus \mathbb{R}$
	$\varphi_5 + \varphi_6$	no	sF	80	26	40	$\mathfrak{so}(2,10)$	$\mathfrak{su}(5) \oplus \mathbb{R}^2$
	$5\varphi_1 + \varphi_5 + \varphi_6$	no	sF	48	18	48	$\mathfrak{so}(2,10)$	$\mathfrak{su}(4) \oplus \mathbb{R}^3$

Rank 7

A_7

Dimension of $\mathfrak{g} = 63$
Four non-compact real forms with trivial automorphism: $\mathfrak{su}(1,7)$, $\mathfrak{su}(2,6)$, $\mathfrak{su}(3,5)$, $\mathfrak{su}(4,4)$.
Fundamental dominant weights:

$$\varphi_1 = \tfrac{7}{8}\gamma_1 + \tfrac{3}{4}\gamma_2 + \tfrac{5}{8}\gamma_3 + \tfrac{1}{2}\gamma_4 + \tfrac{3}{8}\gamma_5 + \tfrac{1}{4}\gamma_6 + \tfrac{1}{8}\gamma_7$$
$$\varphi_2 = \tfrac{3}{4}\gamma_1 + \tfrac{3}{2}\gamma_2 + \tfrac{5}{4}\gamma_3 + \gamma_4 + \tfrac{3}{4}\gamma_5 + \tfrac{1}{2}\gamma_6 + \tfrac{1}{4}\gamma_7$$
$$\varphi_3 = \tfrac{5}{8}\gamma_1 + \tfrac{5}{4}\gamma_2 + \tfrac{15}{8}\gamma_3 + \tfrac{3}{2}\gamma_4 + \tfrac{9}{8}\gamma_5 + \tfrac{3}{4}\gamma_6 + \tfrac{3}{8}\gamma_7$$
$$\varphi_4 = \tfrac{1}{2}\gamma_1 + \gamma_2 + \tfrac{3}{2}\gamma_3 + 2\gamma_4 + \tfrac{3}{2}\gamma_5 + \gamma_6 + \tfrac{1}{2}\gamma_7$$
$$\varphi_5 = \tfrac{3}{8}\gamma_1 + \tfrac{3}{4}\gamma_2 + \tfrac{9}{8}\gamma_3 + \tfrac{3}{2}\gamma_4 + \tfrac{15}{8}\gamma_5 + \tfrac{5}{4}\gamma_6 + \tfrac{5}{8}\gamma_7$$
$$\varphi_6 = \tfrac{1}{4}\gamma_1 + \tfrac{1}{2}\gamma_2 + \tfrac{3}{4}\gamma_3 + \gamma_4 + \tfrac{5}{4}\gamma_5 + \tfrac{3}{2}\gamma_6 + \tfrac{3}{4}\gamma_7$$
$$\varphi_7 = \tfrac{1}{8}\gamma_1 + \tfrac{1}{4}\gamma_2 + \tfrac{3}{8}\gamma_3 + \tfrac{1}{2}\gamma_4 + \tfrac{5}{8}\gamma_5 + \tfrac{3}{4}\gamma_6 + \tfrac{7}{8}\gamma_7$$

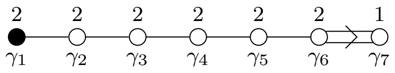
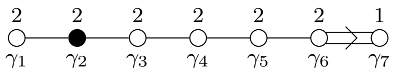
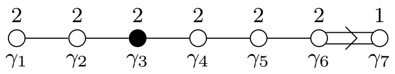
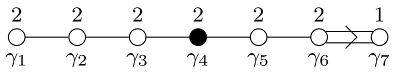
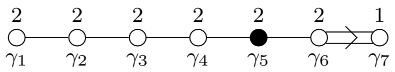
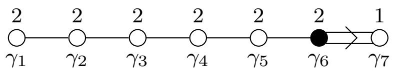
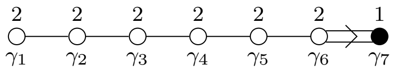
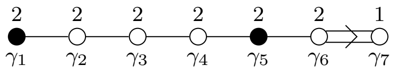
Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	GT	-112	49	14	$\mathfrak{su}(1,7)$	$\mathfrak{su}(7) \oplus \mathbb{R}$
	φ_2	no	GT	-192	39	24	$\mathfrak{su}(2,6)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(6) \oplus \mathbb{R}$
	φ_3	no	GT	-240	33	30	$\mathfrak{su}(3,5)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(5) \oplus \mathbb{R}$
	φ_4	no	GT	-256	31	32	$\mathfrak{su}(4,4)$	$(\mathfrak{su}(4))^2 \oplus \mathbb{R}$
	$2\varphi_1 + 5\varphi_6$	no	sGT	-34	29	34	$\mathfrak{su}(3,5)$	$\mathfrak{su}(5) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
	$\varphi_1 + \varphi_7$	yes	sGT	-130	37	26	$\mathfrak{su}(2,6)$	$\mathfrak{su}(6) \oplus \mathbb{R}^2$
	$\varphi_2 + \varphi_6$	no	sGT	-80	23	40	$\mathfrak{su}(4,4)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(4) \oplus \mathbb{R}^2$
	$5\varphi_2 + 2\varphi_7$	no	sGT	-34	29	34	$\mathfrak{su}(3,5)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \mathbb{R}^2$
	$4\varphi_3 + \varphi_4$	no	sF	38	25	38	$\mathfrak{su}(1,7)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^2$
	$3\varphi_1 + 3\varphi_3 + \varphi_4$	no	sF	42	21	42	$\mathfrak{su}(1,7)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^3$
	$3\varphi_2 + 2\varphi_3 + \varphi_4$	no	sF	42	21	42	$\mathfrak{su}(1,7)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^3$
	$2\varphi_1 + 2\varphi_2 + 2\varphi_3 + \varphi_4$	no	sF	44	19	44	$\mathfrak{su}(1,7)$	$\mathfrak{su}(4) \oplus \mathbb{R}^4$
	$\varphi_3 + \varphi_5$	no	sF	42	21	42	$\mathfrak{su}(2,6)$	$(\mathfrak{su}(3))^2 \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
	$\varphi_4 + 4\varphi_5$	no	sF	38	25	38	$\mathfrak{su}(1,7)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^2$
	$\varphi_4 + 2\varphi_5 + 3\varphi_6$	no	sF	42	21	42	$\mathfrak{su}(1,7)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^3$
	$\varphi_4 + 3\varphi_5 + 3\varphi_7$	no	sF	42	21	42	$\mathfrak{su}(1,7)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^3$
	$\varphi_4 + 2\varphi_5 + 2\varphi_6 + 2\varphi_7$	no	sF	44	19	44	$\mathfrak{su}(1,7)$	$\mathfrak{su}(4) \oplus \mathbb{R}^4$

B_7

Dimension of $\mathfrak{g} = 105$

Seven non-compact real forms: $\mathfrak{so}(2,13)$, $\mathfrak{so}(4,11)$, $\mathfrak{so}(6,9)$, $\mathfrak{so}(8,7)$, $\mathfrak{so}(10,5)$, $\mathfrak{so}(12,3)$, $\mathfrak{so}(14,1)$.

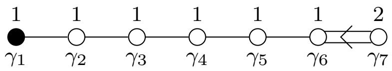
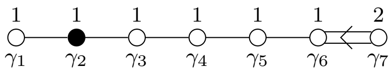
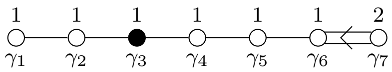
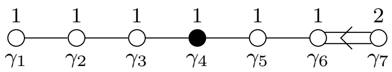
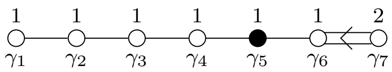
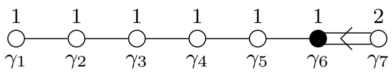
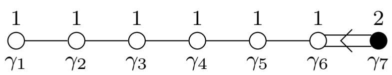
Fundamental dominant weights:
$$\varphi_1 = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7$$
$$\varphi_2 = \gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 2\gamma_6 + 2\gamma_7$$
$$\varphi_3 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 3\gamma_4 + 3\gamma_5 + 3\gamma_6 + 3\gamma_7$$
$$\varphi_4 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 4\gamma_5 + 4\gamma_6 + 4\gamma_7$$
$$\varphi_5 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 5\gamma_6 + 5\gamma_7$$
$$\varphi_6 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 6\gamma_6 + 6\gamma_7$$
$$\varphi_7 = \tfrac{1}{2}\varphi_1 + \varphi_2 + \tfrac{3}{2}\varphi_3 + 2\varphi_4 + \tfrac{5}{2}\varphi_5 + 3\varphi_6 + \tfrac{7}{2}\gamma_7$$

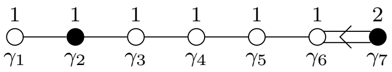
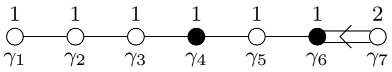
Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	yes	GT	-338	79	26	$\mathfrak{so}(2,13)$	$\mathfrak{so}(13) \oplus \mathbb{R}$
	φ_2	yes	sGT	-460	59	46	$\mathfrak{so}(4,11)$	$\mathfrak{su}(2) \oplus \mathfrak{so}(11) \oplus \mathbb{R}$
	φ_3	no	sGT	-420	45	60	$\mathfrak{so}(6,9)$	$\mathfrak{su}(3) \oplus \mathfrak{so}(9) \oplus \mathbb{R}$
	φ_4	no	sGT	-272	37	68	$\mathfrak{so}(8,7)$	$\mathfrak{su}(4) \oplus \mathfrak{so}(7) \oplus \mathbb{R}$
	φ_5	no	sGT	-70	35	70	$\mathfrak{so}(10,5)$	$\mathfrak{su}(5) \oplus \mathfrak{so}(5) \oplus \mathbb{R}$
	φ_6	no	sF	132	39	66	$\mathfrak{so}(12,3)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$6\varphi_1 + \varphi_6$	no	sF	76	29	76	$\mathfrak{so}(12,3)$	$\mathfrak{su}(5) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
	$\varphi_6 + 2\varphi_7$	no	sF	68	37	68	$\mathfrak{so}(12,3)$	$\mathfrak{su}(6) \oplus \mathbb{R}^2$
	φ_7	no	sF	560	49	56	$\mathfrak{so}(14,1)$	$\mathfrak{su}(7) \oplus \mathbb{R}$
	$7\varphi_1 + 8\varphi_7$	no	sF	68	37	68	$\mathfrak{so}(14,1)$	$\mathfrak{su}(6) \oplus \mathbb{R}^2$
	$7\varphi_2 + 6\varphi_7$	no	sF	76	29	76	$\mathfrak{so}(14,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \mathbb{R}^2$
	$7\varphi_3 + 4\varphi_7$	no	sF	80	25	80	$\mathfrak{so}(14,1)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^2$
	$7\varphi_4 + 2\varphi_7$	no	sF	80	25	80	$\mathfrak{so}(14,1)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^2$
	$\varphi_1 + 3\varphi_2 + 3\varphi_7$	no	sF	156	27	78	$\mathfrak{so}(14,1)$	$\mathfrak{su}(5) \oplus \mathbb{R}^3$
	$3\varphi_1 + 6\varphi_3 + 4\varphi_7$	no	sF	84	21	84	$\mathfrak{so}(14,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^3$
	$2\varphi_1 + 3\varphi_4 + \varphi_7$	no	sF	172	19	86	$\mathfrak{so}(14,1)$	$(\mathfrak{su}(3))^2 \oplus \mathbb{R}^3$
	$3\varphi_2 + 5\varphi_3 + 4\varphi_7$	no	sF	84	21	84	$\mathfrak{so}(14,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^3$
	$4\varphi_2 + 5\varphi_4 + 2\varphi_7$	no	sF	88	17	88	$\mathfrak{so}(14,1)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^3$
	$2\varphi_3 + 2\varphi_4 + \varphi_7$	no	sF	172	19	86	$\mathfrak{so}(14,1)$	$(\mathfrak{su}(3))^2 \oplus \mathbb{R}^3$
	$2\varphi_1 + 2\varphi_2 + 5\varphi_3 + 4\varphi_7$	no	sF	86	19	86	$\mathfrak{so}(14,1)$	$\mathfrak{su}(4) \oplus \mathbb{R}^4$
	$2\varphi_1 + 3\varphi_2 + 5\varphi_4 + 2\varphi_7$	no	sF	90	15	90	$\mathfrak{so}(14,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^4$
	$3\varphi_1 + 3\varphi_3 + 4\varphi_4 + 2\varphi_7$	no	sF	90	15	90	$\mathfrak{so}(14,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^4$
	$3\varphi_2 + 2\varphi_3 + 4\varphi_4 + 2\varphi_7$	no	sF	90	15	90	$\mathfrak{so}(14,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^4$
	$\varphi_1 + \varphi_2 + \varphi_3 + 2\varphi_4 + \varphi_7$	no	sF	184	13	92	$\mathfrak{so}(14,1)$	$\mathfrak{su}(3) \oplus \mathbb{R}^5$
	$\varphi_1 + 2\varphi_5$	no	sGT	-78	27	78	$\mathfrak{so}(8,7)$	$\mathfrak{su}(4) \oplus \mathfrak{so}(5) \oplus \mathbb{R}^2$

C_7

Dimension of $\mathfrak{g} = 105$
Four non-compact real forms: $\mathfrak{sp}(1,6)$, $\mathfrak{sp}(2,5)$, $\mathfrak{sp}(3,4)$, $\mathfrak{sp}(7,\mathbb{R})$.
Fundamental dominant weights:

$$\varphi_1 = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \tfrac{1}{2}\gamma_7$$
$$\varphi_2 = \gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 2\gamma_6 + \gamma_7$$
$$\varphi_3 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 3\gamma_4 + 3\gamma_5 + 3\gamma_6 + \tfrac{3}{2}\gamma_7$$
$$\varphi_4 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 4\gamma_5 + 4\gamma_6 + 2\gamma_7$$
$$\varphi_5 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 5\gamma_6 + \tfrac{5}{2}\gamma_7$$
$$\varphi_6 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 6\gamma_6 + 3\gamma_7$$
$$\varphi_7 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 6\gamma_6 + \tfrac{7}{2}\gamma_7$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	sGT	-260	79	26	$\mathfrak{sp}(1,6)$	$\mathfrak{sp}(6) \oplus \mathbb{R}$
	φ_2	yes	sGT	-322	59	46	$\mathfrak{sp}(2,5)$	$\mathfrak{su}(2) \oplus \mathfrak{sp}(5) \oplus \mathbb{R}$
	φ_3	no	sGT	-240	45	60	$\mathfrak{sp}(3,4)$	$\mathfrak{su}(3) \oplus \mathfrak{sp}(4) \oplus \mathbb{R}$
	φ_4	no	sGT	-68	37	68	$\mathfrak{sp}(3,4)$	$\mathfrak{su}(4) \oplus \mathfrak{sp}(3) \oplus \mathbb{R}$
	φ_5	no	sF	140	35	70	$\mathfrak{sp}(2,5)$	$\mathfrak{su}(5) \oplus \mathfrak{so}(5) \oplus \mathbb{R}$
	$5\varphi_1 + \varphi_5$	no	sF	78	27	78	$\mathfrak{sp}(2,5)$	$\mathfrak{su}(4) \oplus \mathfrak{so}(5) \oplus \mathbb{R}^2$
	φ_6	no	sF	330	39	66	$\mathfrak{sp}(1,6)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$3\varphi_1 + 2\varphi_6$	no	sF	152	29	76	$\mathfrak{sp}(1,6)$	$\mathfrak{su}(5) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
	$2\varphi_2 + \varphi_6$	no	sF	246	23	82	$\mathfrak{sp}(1,6)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(4) \oplus \mathbb{R}^2$
	$3\varphi_3 + \varphi_6$	no	sF	168	21	84	$\mathfrak{sp}(1,6)$	$(\mathfrak{su}(3))^2 \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
	$6\varphi_4 + \varphi_6$	no	sF	82	23	82	$\mathfrak{sp}(1,6)$	$\mathfrak{su}(4) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}^2$
	$3\varphi_6 + 2\varphi_7$	no	sF	68	37	68	$\mathfrak{sp}(1,6)$	$\mathfrak{su}(6) \oplus \mathbb{R}^2$
	$2\varphi_1 + 5\varphi_2 + 3\varphi_6$	no	sF	84	21	84	$\mathfrak{sp}(1,6)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^3$
	$3\varphi_1 + 5\varphi_3 + 2\varphi_6$	no	sF	88	17	88	$\mathfrak{sp}(1,6)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^3$
	$4\varphi_1 + 5\varphi_4 + \varphi_6$	no	sF	88	17	88	$\mathfrak{sp}(1,6)$	$\mathfrak{su}(3) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}^3$
	$3\varphi_1 + \varphi_6 + \varphi_7$	no	sF	156	27	78	$\mathfrak{sp}(1,6)$	$\mathfrak{su}(5) \oplus \mathbb{R}^3$
	$3\varphi_2 + 4\varphi_3 + 2\varphi_6$	no	sF	88	17	88	$\mathfrak{sp}(1,6)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^3$
	$4\varphi_2 + 4\varphi_4 + \varphi_6$	no	sF	90	15	90	$\mathfrak{sp}(1,6)$	$(\mathfrak{su}(2))^4 \oplus \mathbb{R}^3$
	$6\varphi_2 + \varphi_6 + 2\varphi_7$	no	sF	84	21	84	$\mathfrak{sp}(1,6)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^3$
	$4\varphi_3 + 3\varphi_4 + \varphi_6$	no	sF	88	17	88	$\mathfrak{sp}(1,6)$	$\mathfrak{su}(3) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}^3$
	$\varphi_1 + \varphi_2 + 2\varphi_3 + \varphi_6$	no	sF	180	15	90	$\mathfrak{sp}(1,6)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^4$
	$2\varphi_1 + 3\varphi_2 + 4\varphi_4 + \varphi_6$	no	sF	92	13	92	$\mathfrak{sp}(1,6)$	$(\mathfrak{su}(2))^3 \oplus \mathbb{R}^4$
	$2\varphi_1 + 5\varphi_2 + \varphi_6 + 2\varphi_7$	no	sF	86	19	86	$\mathfrak{sp}(1,6)$	$\mathfrak{su}(4) \oplus \oplus\mathbb{R}^4$
	$3\varphi_1 + 3\varphi_3 + 3\varphi_4 + \varphi_6$	no	sF	92	13	92	$\mathfrak{sp}(1,6)$	$(\mathfrak{su}(2))^3 \oplus \mathbb{R}^4$
	$3\varphi_2 + 2\varphi_3 + 3\varphi_4 + \varphi_6$	no	sF	92	13	92	$\mathfrak{sp}(1,6)$	$(\mathfrak{su}(2))^3 \oplus \mathbb{R}^4$
	$2\varphi_1 + 2\varphi_2 + 2\varphi_3 + 3\varphi_4 + \varphi_6$	no	sF	94	11	94	$\mathfrak{sp}(1,6)$	$(\mathfrak{su}(2))^2 \oplus \mathbb{R}^5$
	φ_7	no	GT	-448	49	56	$\mathfrak{sp}(7,\mathbb{R})$	$\mathfrak{su}(7) \oplus \mathbb{R}$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	$\varphi_2 + 2\varphi_7$	no	sGT	-76	29	76	$\mathfrak{sp}(7,\mathbb{R})$	$\mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \mathbb{R}^2$
	$2\varphi_4 + \varphi_6$	no	sF	82	23	82	$\mathfrak{sp}(2,5)$	$\mathfrak{su}(4) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}^2$
	$4\varphi_1 + \varphi_4 + \varphi_6$	no	sF	88	17	88	$\mathfrak{sp}(2,5)$	$\mathfrak{su}(3) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}^3$

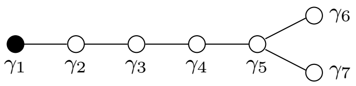
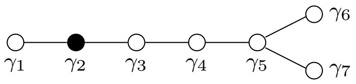
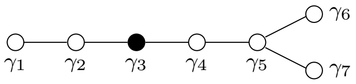
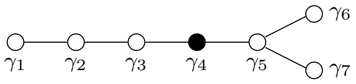
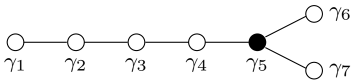
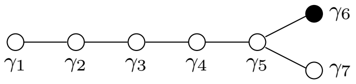
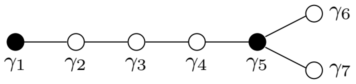
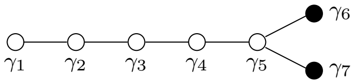
D_7

Dimension of $\mathfrak{g} = 91$

Four non-compact real forms with trivial automorphism: $\mathfrak{so}(2,12)$, $\mathfrak{so}(4,10)$, $\mathfrak{so}(6,8)$, $\mathfrak{so}^*(14)$.

Fundamental dominant weights: $\varphi_1 = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \frac{1}{2}\gamma_6 + \frac{1}{2}\gamma_7$

$$\varphi_2 = \gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + \gamma_6 + \gamma_7$$
$$\varphi_3 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 3\gamma_4 + 3\gamma_5 + \frac{3}{2}\gamma_6 + \frac{3}{2}\gamma_7$$
$$\varphi_4 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 4\gamma_5 + 2\gamma_6 + 2\gamma_7$$
$$\varphi_5 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + \frac{5}{2}\gamma_6 + \frac{5}{2}\gamma_7$$
$$\varphi_6 = \frac{1}{2}\gamma_1 + \gamma_2 + \frac{3}{2}\gamma_3 + 2\gamma_4 + \frac{5}{2}\gamma_5 + \frac{7}{4}\gamma_6 + \frac{5}{4}\gamma_7$$
$$\varphi_7 = \frac{1}{2}\gamma_1 + \gamma_2 + \frac{3}{2}\gamma_3 + 2\gamma_4 + \frac{5}{2}\gamma_5 + \frac{5}{4}\gamma_6 + \frac{7}{4}\gamma_7$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	GT	-288	67	24	$\mathfrak{so}(2,12)$	$\mathfrak{so}(12) \oplus \mathbb{R}$
	φ_2	yes	sGT	-378	49	42	$\mathfrak{so}(4,10)$	$\mathfrak{su}(2) \oplus \mathfrak{so}(10) \oplus \mathbb{R}$
	φ_3	no	sGT	-324	37	54	$\mathfrak{so}(6,8)$	$\mathfrak{su}(3) \oplus \mathfrak{so}(8) \oplus \mathbb{R}$
	φ_4	no	sGT	-180	31	60	$\mathfrak{so}(6,8)$	$(\mathfrak{su}(4))^2 \oplus \mathbb{R}$
	φ_5	no	sCY	0	31	60	$\mathfrak{so}(4,10)$	$\mathfrak{su}(5) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}$
	φ_6	no	GT	-504	49	42	$\mathfrak{so}^*(14)$	$\mathfrak{su}(7) \oplus \mathbb{R}$
	$3\varphi_1 + \varphi_5$	no	sGT	-68	23	68	$\mathfrak{so}(6,8)$	$\mathfrak{su}(4) \oplus (\mathfrak{su}(2)^2) \oplus \mathbb{R}^2$
	$\varphi_6 + \varphi_7$	no	sF	162	37	54	$\mathfrak{so}(2,12)$	$\mathfrak{su}(6) \oplus \mathbb{R}^2$
	$3\varphi_1 + \varphi_6 + \varphi_7$	no	sF	128	27	64	$\mathfrak{so}(2,12)$	$\mathfrak{su}(5) \oplus \mathbb{R}^3$
	$6\varphi_2 + \varphi_6 + \varphi_7$	no	sF	70	21	70	$\mathfrak{so}(2,12)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^3$
	$2\varphi_1 + 5\varphi_2 + \varphi_6 + \varphi_7$	no	sF	72	19	72	$\mathfrak{so}(2,12)$	$\mathfrak{su}(4) \oplus \mathbb{R}^4$

Rank 8

A_8

Dimension of $\mathfrak{g} = 80$
Four non-compact real forms with trivial automorphism: $\mathfrak{su}(1,8)$, $\mathfrak{su}(2,7)$, $\mathfrak{su}(3,6)$, $\mathfrak{su}(4,5)$.
Fundamental dominant weights:

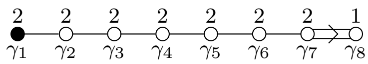
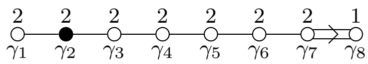
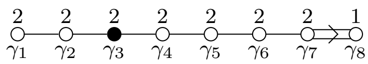
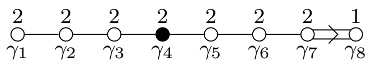
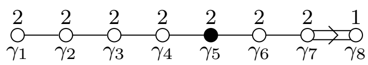
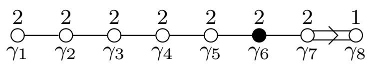
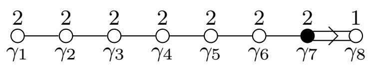
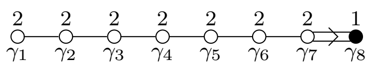
$$\varphi_1 = \tfrac{8}{9}\gamma_1 + \tfrac{7}{9}\gamma_2 + \tfrac{2}{3}\gamma_3 + \tfrac{5}{9}\gamma_4 + \tfrac{4}{9}\gamma_5 + \tfrac{1}{3}\gamma_6 + \tfrac{2}{9}\gamma_7 + \tfrac{1}{9}\gamma_8$$
$$\varphi_2 = \tfrac{7}{9}\gamma_1 + \tfrac{14}{9}\gamma_2 + \tfrac{4}{3}\gamma_3 + \tfrac{10}{9}\gamma_4 + \tfrac{8}{9}\gamma_5 + \tfrac{2}{3}\gamma_6 + \tfrac{4}{9}\gamma_7 + \tfrac{2}{9}\gamma_8$$
$$\varphi_3 = \tfrac{2}{3}\gamma_1 + \tfrac{4}{3}\gamma_2 + 2\gamma_3 + \tfrac{5}{3}\gamma_4 + \tfrac{4}{3}\gamma_5 + \gamma_6 + \tfrac{2}{3}\gamma_7 + \tfrac{1}{3}\gamma_8$$
$$\varphi_4 = \tfrac{5}{9}\gamma_1 + \tfrac{10}{9}\gamma_2 + \tfrac{5}{3}\gamma_3 + \tfrac{20}{9}\gamma_4 + \tfrac{16}{9}\gamma_5 + \tfrac{4}{3}\gamma_6 + \tfrac{8}{9}\gamma_7 + \tfrac{4}{9}\gamma_8$$
$$\varphi_5 = \tfrac{4}{9}\gamma_1 + \tfrac{8}{9}\gamma_2 + \tfrac{4}{3}\gamma_3 + \tfrac{16}{9}\gamma_4 + \tfrac{20}{9}\gamma_5 + \tfrac{5}{3}\gamma_6 + \tfrac{10}{9}\gamma_7 + \tfrac{5}{9}\gamma_8$$
$$\varphi_6 = \tfrac{1}{3}\gamma_1 + \tfrac{2}{3}\gamma_2 + \gamma_3 + \tfrac{4}{3}\gamma_4 + \tfrac{5}{3}\gamma_5 + 2\gamma_6 + \tfrac{4}{3}\gamma_7 + \tfrac{2}{3}\gamma_8$$
$$\varphi_7 = \tfrac{2}{9}\gamma_1 + \tfrac{4}{9}\gamma_2 + \tfrac{2}{3}\gamma_3 + \tfrac{8}{9}\gamma_4 + \tfrac{10}{9}\gamma_5 + \tfrac{4}{3}\gamma_6 + \tfrac{14}{9}\gamma_7 + \tfrac{7}{9}\gamma_8$$
$$\varphi_8 = \tfrac{1}{9}\gamma_1 + \tfrac{2}{9}\gamma_2 + \tfrac{1}{3}\gamma_3 + \tfrac{4}{9}\gamma_4 + \tfrac{5}{9}\gamma_5 + \tfrac{2}{3}\gamma_6 + \tfrac{7}{9}\gamma_7 + \tfrac{8}{9}\gamma_8$$

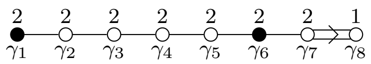
Vogan diagram	φ	$\varphi \in \Delta$	Type	s	$\dim V$	$\dim G/V$	$\mathfrak{g}$	$\mathfrak{v}$
●—○—○—○—○—○—○—○ ($\gamma_1 \ldots \gamma_8$)	φ_1	no	GT	-144	64	16	$\mathfrak{su}(1,8)$	$\mathfrak{su}(8) \oplus \mathbb{R}$
○—●—○—○—○—○—○—○ ($\gamma_1 \ldots \gamma_8$)	φ_2	no	GT	-252	52	28	$\mathfrak{su}(2,7)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(7) \oplus \mathbb{R}$
○—○—●—○—○—○—○—○ ($\gamma_1 \ldots \gamma_8$)	φ_3	no	GT	-324	44	36	$\mathfrak{su}(3,6)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(6) \oplus \mathbb{R}$
○—○—○—●—○—○—○—○ ($\gamma_1 \ldots \gamma_8$)	φ_4	no	GT	-360	40	40	$\mathfrak{su}(4,5)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(5) \oplus \mathbb{R}$
●—○—○—○—○—○—●—○ ($\gamma_1 \ldots \gamma_8$)	$\varphi_1 + 2\varphi_7$	no	sGT	-120	40	40	$\mathfrak{su}(3,6)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
●—○—○—○—○—○—○—● ($\gamma_1 \ldots \gamma_8$)	$\varphi_1 + \varphi_8$	yes	sGT	-180	50	30	$\mathfrak{su}(2,7)$	$\mathfrak{su}(7) \oplus \mathbb{R}^2$
○—●—○—○—○—○—●—○ ($\gamma_1 \ldots \gamma_8$)	$\varphi_2 + \varphi_7$	no	sGT	-144	32	48	$\mathfrak{su}(4,5)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(5) \oplus \mathbb{R}^2$
○—○—●—○—○—●—○—○ ($\gamma_1 \ldots \gamma_8$)	$\varphi_3 + \varphi_6$	no	sCY	0	26	54	$\mathfrak{su}(3,6)$	$(\mathfrak{su}(3))^3 \oplus \mathbb{R}^2$
○—○—○—●—●—○—○—○ ($\gamma_1 \ldots \gamma_8$)	$\varphi_4 + \varphi_5$	no	sF	144	32	48	$\mathfrak{su}(1,8)$	$(\mathfrak{su}(4))^2 \oplus \mathbb{R}^2$
	$4\varphi_1 + 2\varphi_4 + 3\varphi_5$	no	sF	54	26	54	$\mathfrak{su}(1,8)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^3$
	$4\varphi_2 + \varphi_4 + 3\varphi_5$	no	sF	56	24	56	$\mathfrak{su}(1,8)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(4) \oplus \mathbb{R}^3$
	$2\varphi_1 + 3\varphi_2 + \varphi_4 + 3\varphi_5$	no	sF	58	22	58	$\mathfrak{su}(1,8)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^4$
	$4\varphi_1 + 2\varphi_4 + \varphi_5 + 4\varphi_7$	no	sF	62	18	62	$\mathfrak{su}(1,8)$	$\mathfrak{su}(3) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}^4$
	$2\varphi_1 + \varphi_4 + \varphi_5 + 2\varphi_8$	no	sF	120	20	60	$\mathfrak{su}(1,8)$	$(\mathfrak{su}(3))^2 \oplus \mathbb{R}^4$
	$4\varphi_2 + \varphi_4 + \varphi_5 + 4\varphi_7$	no	sF	64	16	64	$\mathfrak{su}(1,8)$	$(\mathfrak{su}(2))^4 \oplus \mathbb{R}^4$
	$4\varphi_2 + \varphi_4 + 2\varphi_5 + 4\varphi_8$	no	sF	62	18	62	$\mathfrak{su}(1,8)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^4$
	$3\varphi_4 + \varphi_5 + 3\varphi_7 + 2\varphi_8$	no	sF	58	22	58	$\mathfrak{su}(1,8)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^4$
	$2\varphi_1 + 3\varphi_2 + \varphi_4 + \varphi_5 + 4\varphi_7$	no	sF	66	14	66	$\mathfrak{su}(1,8)$	$(\mathfrak{su}(2))^3 \oplus \mathbb{R}^5$
	$2\varphi_1 + 3\varphi_2 + \varphi_4 + 2\varphi_5 + 4\varphi_8$	no	sF	64	16	64	$\mathfrak{su}(1,8)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^5$
	$2\varphi_1 + 3\varphi_2 + \varphi_4 + \varphi_5 + 3\varphi_7 + 2\varphi_8$	no	sF	68	12	68	$\mathfrak{su}(1,8)$	$(\mathfrak{su}(2))^2 \oplus \mathbb{R}^6$
●—●—●—●—●—●—●—● ($\gamma_1 \ldots \gamma_8$)	$\sum_{i=1}^{8} t_i\varphi_i$ for all $t_i > 0$	no	sCY	0	8	72	$\mathfrak{su}(4,5)$	$\mathbb{R}^8$

B_8

Dimension of $\mathfrak{g} = 136$
Eight non-compact real forms: $\mathfrak{so}(2,15)$, $\mathfrak{so}(4,13)$, $\mathfrak{so}(6,11)$, $\mathfrak{so}(8,9)$, $\mathfrak{so}(10,7)$, $\mathfrak{so}(12,5)$, $\mathfrak{so}(14,3)$, $\mathfrak{so}(16,1)$.
Fundamental dominant weights:

$$\varphi_1 = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 + \gamma_8$$
$$\varphi_2 = \gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 2\gamma_6 + 2\gamma_7 + 2\gamma_8$$
$$\varphi_3 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 3\gamma_4 + 3\gamma_5 + 3\gamma_6 + 3\gamma_7 + 3\gamma_8$$
$$\varphi_4 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 4\gamma_5 + 4\gamma_6 + 4\gamma_7 + 4\gamma_8$$
$$\varphi_5 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 5\gamma_6 + 5\gamma_7 + 5\gamma_8$$
$$\varphi_6 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 6\gamma_6 + 6\gamma_7 + 6\gamma_8$$
$$\varphi_7 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 6\gamma_6 + 7\gamma_7 + 7\gamma_8$$
$$\varphi_8 = \tfrac{1}{2}\gamma_1 + \gamma_2 + \tfrac{3}{2}\gamma_3 + 2\gamma_4 + \tfrac{5}{2}\gamma_5 + 3\gamma_6 + \tfrac{7}{2}\gamma_7 + 4\gamma_8$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	yes	GT	-450	106	30	$\mathfrak{so}(2,15)$	$\mathfrak{so}(15) \oplus \mathbb{R}$
	φ_2	yes	sGT	-648	82	54	$\mathfrak{so}(4,13)$	$\mathfrak{su}(2) \oplus \mathfrak{so}(13) \oplus \mathbb{R}$
	φ_3	no	sGT	-648	64	72	$\mathfrak{so}(6,11)$	$\mathfrak{su}(3) \oplus \mathfrak{so}(11) \oplus \mathbb{R}$
	φ_4	no	sGT	-504	52	84	$\mathfrak{so}(8,9)$	$\mathfrak{su}(4) \oplus \mathfrak{so}(9) \oplus \mathbb{R}$
	φ_5	no	sGT	-270	46	90	$\mathfrak{so}(10,7)$	$\mathfrak{su}(5) \oplus \mathfrak{so}(7) \oplus \mathbb{R}$
	φ_6	no	sCY	0	46	90	$\mathfrak{so}(12,5)$	$\mathfrak{su}(6) \oplus \mathfrak{so}(5) \oplus \mathbb{R}$
	φ_7	no	sF	252	52	84	$\mathfrak{so}(14,3)$	$\mathfrak{su}(7) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$7\varphi_1 + 2\varphi_7$	no	sF	96	40	96	$\mathfrak{so}(14,3)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
	$7\varphi_2 + \varphi_7$	no	sF	104	32	104	$\mathfrak{so}(14,3)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(5) \oplus \mathbb{R}^2$
	$\varphi_7 + \varphi_8$	no	sF	172	50	86	$\mathfrak{so}(14,3)$	$\mathfrak{su}(7) \oplus \mathbb{R}^2$
	$2\varphi_1 + 6\varphi_2 + \varphi_7$	no	sF	106	30	106	$\mathfrak{so}(14,3)$	$\mathfrak{su}(5) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^3$
	$7\varphi_1 + \varphi_7 + 2\varphi_8$	no	sF	98	38	98	$\mathfrak{so}(14,3)$	$\mathfrak{su}(6) \oplus \mathbb{R}^3$
	φ_8	no	sF	864	64	72	$\mathfrak{so}(16,1)$	$\mathfrak{su}(8) \oplus \mathbb{R}$
	$4\varphi_1 + 5\varphi_8$	no	sF	172	50	86	$\mathfrak{so}(16,1)$	$\mathfrak{su}(7) \oplus \mathbb{R}^2$
	$\varphi_2 + \varphi_8$	no	sF	768	40	96	$\mathfrak{so}(16,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(6) \oplus \mathbb{R}^2$
	$4\varphi_3 + 3\varphi_8$	no	sF	204	34	102	$\mathfrak{so}(16,1)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(5) \oplus \mathbb{R}^2$
	$2\varphi_4 + \varphi_8$	no	sF	416	32	104	$\mathfrak{so}(16,1)$	$(\mathfrak{su}(4))^2 \oplus \mathbb{R}^2$
	$4\varphi_5 + \varphi_8$	no	sF	204	34	102	$\mathfrak{so}(16,1)$	$\mathfrak{su}(5) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^2$
	$2\varphi_1 + 7\varphi_2 + 8\varphi_8$	no	sF	98	38	98	$\mathfrak{so}(16,1)$	$\mathfrak{su}(6) \oplus \mathbb{R}^3$
	$3\varphi_1 + 7\varphi_3 + 6\varphi_8$	no	sF	106	30	106	$\mathfrak{so}(16,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \mathbb{R}^3$
	$4\varphi_1 + 7\varphi_4 + 4\varphi_8$	no	sF	110	26	110	$\mathfrak{so}(16,1)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^3$
	$5\varphi_1 + 7\varphi_5 + 2\varphi_8$	no	sF	110	26	110	$\mathfrak{so}(16,1)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^3$
	$\varphi_2 + 2\varphi_3 + 2\varphi_8$	no	sF	318	30	106	$\mathfrak{so}(16,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \mathbb{R}^3$
	$2\varphi_2 + 3\varphi_4 + 2\varphi_8$	no	sF	224	24	122	$\mathfrak{so}(16,1)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(4) \oplus \mathbb{R}^3$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	$4\varphi_3 + 5\varphi_4 + 4\varphi_8$	no	sF	110	26	110	$\mathfrak{so}(16,1)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^3$
	$5\varphi_3 + 5\varphi_5 + 2\varphi_8$	no	sF	114	22	114	$\mathfrak{so}(16,1)$	$(\mathfrak{su}(3))^2 \oplus \mathfrak{su}(2) \oplus \mathbb{R}^3$
	$5\varphi_4 + 4\varphi_5 + 2\varphi_8$	no	sF	110	26	110	$\mathfrak{so}(16,1)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^3$
	$\varphi_1 + \varphi_2 + 3\varphi_3 + 3\varphi_8$	no	sF	216	28	108	$\mathfrak{so}(16,1)$	$\mathfrak{su}(5) \oplus \mathbb{R}^4$
	$2\varphi_1 + 3\varphi_2 + 6\varphi_4 + 4\varphi_8$	no	sF	114	22	114	$\mathfrak{so}(16,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^4$
	$\varphi_1 + 2\varphi_2 + 3\varphi_5 + \varphi_8$	no	sF	232	20	116	$\mathfrak{so}(16,1)$	$(\mathfrak{su}(3))^2 \oplus \mathbb{R}^4$
	$3\varphi_1 + 3\varphi_3 + 5\varphi_4 + 4\varphi_8$	no	sF	114	22	114	$\mathfrak{so}(16,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^4$
	$3\varphi_1 + 4\varphi_3 + 5\varphi_5 + 2\varphi_8$	no	sF	118	18	118	$\mathfrak{so}(16,1)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^4$
	$2\varphi_1 + 2\varphi_4 + 2\varphi_5 + \varphi_8$	no	sF	232	20	116	$\mathfrak{so}(16,1)$	$(\mathfrak{su}(3))^2 \oplus \mathbb{R}^4$
	$3\varphi_2 + 2\varphi_3 + 5\varphi_4 + 4\varphi_8$	no	sF	114	22	114	$\mathfrak{so}(16,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^4$
	$3\varphi_2 + 3\varphi_3 + 5\varphi_5 + 2\varphi_8$	no	sF	118	18	118	$\mathfrak{so}(16,1)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^4$
	$4\varphi_2 + 3\varphi_4 + 4\varphi_5 + 2\varphi_8$	no	sF	118	18	118	$\mathfrak{so}(16,1)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^4$
	$2\varphi_3 + \varphi_4 + 2\varphi_5 + \varphi_8$	no	sF	232	20	116	$\mathfrak{so}(16,1)$	$(\mathfrak{su}(3))^2 \oplus \mathbb{R}^4$
	$2\varphi_1 + 2\varphi_2 + 2\varphi_3 + 5\varphi_4 + 4\varphi_8$	no	sF	116	20	116	$\mathfrak{so}(16,1)$	$\mathfrak{su}(4) \oplus \mathbb{R}^5$
	$2\varphi_1 + 2\varphi_2 + 3\varphi_3 + 5\varphi_5 + 2\varphi_8$	no	sF	120	16	120	$\mathfrak{so}(16,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^5$
	$2\varphi_1 + 3\varphi_2 + 3\varphi_4 + 4\varphi_5 + 2\varphi_8$	no	sF	120	16	120	$\mathfrak{so}(16,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^5$
	$3\varphi_1 + 3\varphi_3 + 2\varphi_4 + 4\varphi_5 + 2\varphi_8$	no	sF	120	16	120	$\mathfrak{so}(16,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^5$
	$3\varphi_2 + 2\varphi_3 + 2\varphi_4 + 4\varphi_5 + 2\varphi_8$	no	sF	120	16	120	$\mathfrak{so}(16,1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}^5$
	$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 + 2\varphi_5 + \varphi_8$	no	sF	244	14	122	$\mathfrak{so}(16,1)$	$\mathfrak{su}(3) \oplus \mathbb{R}^6$
	$5\varphi_2 + 6\varphi_5 + 2\varphi_8$	no	sF	114	22	114	$\mathfrak{so}(16,1)$	$\mathfrak{su}(2) \oplus (\mathfrak{su}(3))^2 \oplus \mathbb{R}^3$
	$4\varphi_1 + \varphi_6$	no	sGT	−100	36	100	$\mathfrak{so}(10,7)$	$\mathfrak{su}(5) \oplus \mathfrak{so}(5) \oplus \mathbb{R}^2$

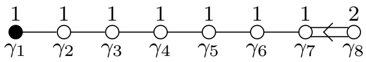
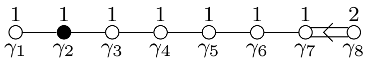
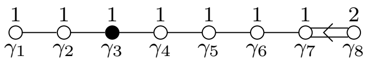
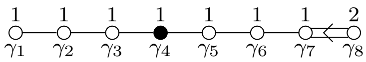
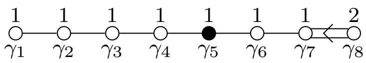
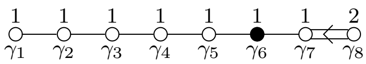
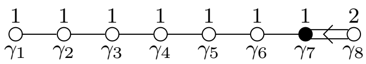
C_8

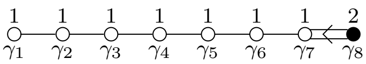
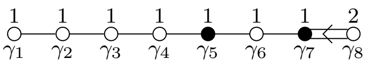
Dimension of $\mathfrak{g} = 136$

Five non-compact real forms: $\mathfrak{sp}(1,7)$, $\mathfrak{sp}(2,6)$, $\mathfrak{sp}(3,5)$, $\mathfrak{sp}(4,4)$, $\mathfrak{sp}(8,\mathbb{R})$.

Fundamental dominant weights:

$$\begin{aligned}
\varphi_1 &= \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 + \tfrac{1}{2}\gamma_8\\
\varphi_2 &= \gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 2\gamma_6 + 2\gamma_7 + \gamma_8\\
\varphi_3 &= \gamma_1 + 2\gamma_2 + 3\gamma_3 + 3\gamma_4 + 3\gamma_5 + 3\gamma_6 + 3\gamma_7 + \tfrac{3}{2}\gamma_8\\
\varphi_4 &= \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 4\gamma_5 + 4\gamma_6 + 4\gamma_7 + 2\gamma_8\\
\varphi_5 &= \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 5\gamma_6 + 5\gamma_7 + \tfrac{5}{2}\gamma_8\\
\varphi_6 &= \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 6\gamma_6 + 6\gamma_7 + 3\gamma_8\\
\varphi_7 &= \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 6\gamma_6 + 7\gamma_7 + \tfrac{7}{2}\gamma_8\\
\varphi_8 &= \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 6\gamma_6 + 7\gamma_7 + 4\gamma_8
\end{aligned}$$

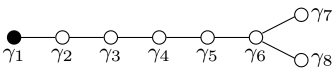
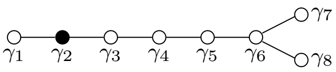
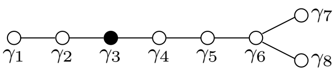
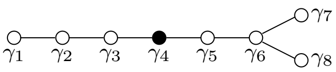
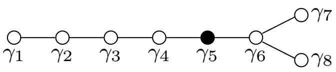
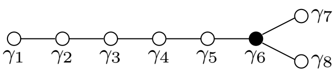
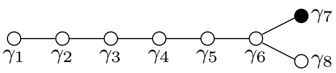
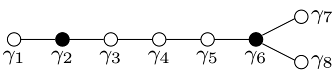
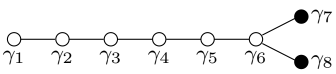
Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	sGT	−360	106	30	$\mathfrak{sp}(1,7)$	$\mathfrak{sp}(7) \oplus \mathbb{R}$
	φ_2	yes	sGT	−486	82	54	$\mathfrak{sp}(2,6)$	$\mathfrak{su}(2) \oplus \mathfrak{sp}(6) \oplus \mathbb{R}$
	φ_3	no	sGT	−432	64	72	$\mathfrak{sp}(3,5)$	$\mathfrak{su}(3) \oplus \mathfrak{sp}(5) \oplus \mathbb{R}$
	φ_4	no	sGT	−252	52	84	$\mathfrak{sp}(4,4)$	$\mathfrak{su}(4) \oplus \mathfrak{sp}(4) \oplus \mathbb{R}$
	φ_5	no	sCY	0	46	90	$\mathfrak{sp}(3,5)$	$\mathfrak{su}(5) \oplus \mathfrak{sp}(3) \oplus \mathbb{R}$
	φ_6	no	sF	270	46	90	$\mathfrak{sp}(2,6)$	$\mathfrak{su}(6) \oplus \mathfrak{so}(5) \oplus \mathbb{R}$
	$3\varphi_1 + \varphi_6$	no	sF	200	36	100	$\mathfrak{sp}(2,6)$	$\mathfrak{su}(5) \oplus \mathfrak{so}(5) \oplus \mathbb{R}^2$
	$6\varphi_2 + \varphi_6$	no	sF	106	30	106	$\mathfrak{sp}(2,6)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathfrak{so}(5) \oplus \mathbb{R}^2$
	$2\varphi_1 + 5\varphi_2 + \varphi_6$	no	sF	108	28	108	$\mathfrak{sp}(2,6)$	$\mathfrak{su}(4) \oplus \mathfrak{so}(5) \oplus \mathbb{R}^3$
	φ_7	no	sF	504	52	84	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(7) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$7\varphi_1 + 5\varphi_7$	no	sF	96	40	96	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
	$7\varphi_2 + 4\varphi_7$	no	sF	104	32	104	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(5) \oplus \mathbb{R}^2$
	$7\varphi_3 + 3\varphi_7$	no	sF	108	28	108	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
	$7\varphi_4 + 2\varphi_7$	no	sF	108	28	108	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
	$7\varphi_5 + \varphi_7$	no	sF	104	32	104	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(5) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}^2$
	$2\varphi_7 + \varphi_8$	no	sF	172	50	86	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(7) \oplus \mathbb{R}^2$
	$\varphi_1 + 3\varphi_2 + 2\varphi_7$	no	sF	212	30	106	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(5) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^3$
	$\varphi_1 + 2\varphi_3 + \varphi_7$	no	sF	336	24	112	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(4) \oplus \mathbb{R}^3$
	$2\varphi_1 + 3\varphi_4 + \varphi_7$	no	sF	228	22	114	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(3))^2 \oplus \mathfrak{su}(2) \oplus \mathbb{R}^3$
	$5\varphi_1 + 6\varphi_5 + \varphi_7$	no	sF	112	24	112	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(4) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}^3$
	$7\varphi_1 + 3\varphi_7 + 2\varphi_8$	no	sF	98	38	98	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(6) \oplus \mathbb{R}^3$
	$3\varphi_2 + 5\varphi_3 + 3\varphi_7$	no	sF	112	24	112	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(4) \oplus \mathbb{R}^3$
	$4\varphi_2 + 5\varphi_4 + 2\varphi_7$	no	sF	116	20	116	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^3 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^3$
	$5\varphi_2 + 5\varphi_5 + \varphi_7$	no	sF	116	20	116	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^3 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^3$
	$7\varphi_2 + 2\varphi_7 + 2\varphi_8$	no	sF	106	30	106	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \mathbb{R}^3$
	$2\varphi_3 + 2\varphi_4 + \varphi_7$	no	sF	228	22	114	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(3))^2 \oplus \mathfrak{su}(2) \oplus \mathbb{R}^3$
	$5\varphi_3 + 4\varphi_5 + \varphi_7$	no	sF	116	20	116	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(3) \oplus (\mathfrak{su}(2))^3 \oplus \mathbb{R}^3$
	$7\varphi_3 + \varphi_7 + 2\varphi_8$	no	sF	110	26	110	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^3$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	$5\varphi_4 + 3\varphi_5 + \varphi_7$	no	sF	112	24	112	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(4) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}^3$
	$2\varphi_1 + 2\varphi_2 + 5\varphi_3 + 3\varphi_7$	no	sF	114	22	114	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^4$
	$2\varphi_1 + 3\varphi_2 + 5\varphi_4 + 2\varphi_7$	no	sF	118	18	118	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^4$
	$2\varphi_1 + 4\varphi_2 + 5\varphi_5 + \varphi_7$	no	sF	118	18	118	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^4$
	$\varphi_1 + 3\varphi_2 + \varphi_7 + \varphi_8$	no	sF	216	28	108	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(5) \oplus \mathbb{R}^4$
	$3\varphi_1 + 3\varphi_3 + 4\varphi_4 + 2\varphi_7$	no	sF	118	18	118	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^4$
	$3\varphi_1 + 4\varphi_3 + 4\varphi_5 + \varphi_7$	no	sF	120	16	120	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^4 \oplus \mathbb{R}^4$
	$3\varphi_1 + 6\varphi_3 + \varphi_7 + 2\varphi_8$	no	sF	114	22	114	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^4$
	$4\varphi_1 + 4\varphi_4 + 3\varphi_5 + \varphi_7$	no	sF	118	18	118	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(3) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}^4$
	$3\varphi_2 + 2\varphi_3 + 4\varphi_4 + 2\varphi_7$	no	sF	118	18	118	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^4$
	$3\varphi_2 + 3\varphi_3 + 4\varphi_5 + \varphi_7$	no	sF	120	16	120	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^4 \oplus \mathbb{R}^4$
	$3\varphi_2 + 5\varphi_3 + \varphi_7 + 2\varphi_8$	no	sF	114	22	114	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^4$
	$4\varphi_2 + 3\varphi_4 + 3\varphi_5 + \varphi_7$	no	sF	120	16	120	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^4 \oplus \mathbb{R}^4$
	$4\varphi_3 + 2\varphi_4 + 3\varphi_5 + \varphi_7$	no	sF	118	18	118	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(3) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}^4$
	$\varphi_1 + \varphi_2 + \varphi_3 + 2\varphi_4 + \varphi_7$	no	sF	240	16	120	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^5$
	$2\varphi_1 + 2\varphi_2 + 3\varphi_3 + 4\varphi_5 + \varphi_7$	no	sF	122	14	122	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^3 \oplus \mathbb{R}^5$
	$2\varphi_1 + 2\varphi_2 + 5\varphi_3 + \varphi_7 + 2\varphi_8$	no	sF	116	20	116	$\mathfrak{sp}(1,7)$	$\mathfrak{su}(4) \oplus \mathbb{R}^5$
	$2\varphi_1 + 3\varphi_2 + 3\varphi_4 + 3\varphi_5 + \varphi_7$	no	sF	122	14	122	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^3 \oplus \mathbb{R}^5$
	$3\varphi_1 + 3\varphi_3 + 2\varphi_4 + 3\varphi_5 + \varphi_7$	no	sF	122	14	122	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^3 \oplus \mathbb{R}^5$
	$3\varphi_2 + 2\varphi_3 + 2\varphi_4 + 3\varphi_5 + \varphi_7$	no	sF	122	14	122	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^3 \oplus \mathbb{R}^5$
	$2\varphi_1 + 2\varphi_2 + 2\varphi_3 + 2\varphi_4 + 3\varphi_5 + \varphi_7$	no	sF	124	12	124	$\mathfrak{sp}(1,7)$	$(\mathfrak{su}(2))^2 \oplus \mathbb{R}^6$
	φ_7	no	GT	−648	64	72	$\mathfrak{sp}(8,\mathbb{R})$	$\mathfrak{su}(8) \oplus \mathbb{R}$
	$3\varphi_5 + \varphi_7$	no	sF	104	32	104	$\mathfrak{sp}(2,6)$	$\mathfrak{su}(5) \oplus (\mathfrak{su})^2 \oplus \mathbb{R}^2$
	$5\varphi_1 + 2\varphi_5 + \varphi_7$	no	sF	112	24	112	$\mathfrak{sp}(2,6)$	$\mathfrak{su}(4) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}^3$
	$5\varphi_2 + \varphi_5 + \varphi_7$	no	sF	116	20	116	$\mathfrak{sp}(2,6)$	$(\mathfrak{su}(2))^3 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^3$
	$2\varphi_1 + 4\varphi_2 + \varphi_5 + \varphi_7$	no	sF	118	18	118	$\mathfrak{sp}(2,6)$	$\mathfrak{su}(3) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}^4$

D_8

Dimension of $\mathfrak{g} = 120$
4 non-compact real forms with trivial automorphism: $\mathfrak{so}(2,14)$, $\mathfrak{so}(4,12)$, $\mathfrak{so}(6,10)$, $\mathfrak{so}^*(16)$.
Fundamental dominant weights:

$$\varphi_1 = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \tfrac{1}{2}\gamma_7 + \tfrac{1}{2}\gamma_8$$
$$\varphi_2 = \gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 2\gamma_6 + \gamma_7 + \gamma_8$$
$$\varphi_3 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 3\gamma_4 + 3\gamma_5 + 3\gamma_6 + \tfrac{3}{2}\gamma_7 + \tfrac{3}{2}\gamma_8$$
$$\varphi_4 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 4\gamma_5 + 4\gamma_6 + 2\gamma_7 + 2\gamma_8$$
$$\varphi_5 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 5\gamma_6 + \tfrac{5}{2}\gamma_7 + \tfrac{5}{2}\gamma_8$$
$$\varphi_6 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 6\gamma_6 + 3\gamma_7 + 3\gamma_8$$
$$\varphi_7 = \tfrac{1}{2}\gamma_1 + \gamma_2 + \tfrac{3}{2}\gamma_3 + 2\gamma_4 + \tfrac{5}{2}\gamma_5 + 3\gamma_6 + 2\gamma_7 + \tfrac{3}{2}\gamma_8$$
$$\varphi_8 = \tfrac{1}{2}\gamma_1 + \gamma_2 + \tfrac{3}{2}\gamma_3 + 2\gamma_4 + \tfrac{5}{2}\gamma_5 + 3\gamma_6 + \tfrac{3}{2}\gamma_7 + 2\gamma_8$$

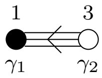
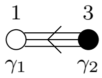
Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	GT	-392	92	28	$\mathfrak{so}(2,14)$	$\mathfrak{so}(14) \oplus \mathbb{R}$
	φ_2	yes	sGT	-550	70	50	$\mathfrak{so}(4,12)$	$\mathfrak{su}(2) \oplus \mathfrak{so}(12) \oplus \mathbb{R}$
	φ_3	no	sGT	-528	54	66	$\mathfrak{so}(6,10)$	$\mathfrak{su}(3) \oplus \mathfrak{so}(10) \oplus \mathbb{R}$
	φ_4	no	sGT	-380	44	76	$\mathfrak{so}(8,8)$	$\mathfrak{su}(4) \oplus \mathfrak{so}(8) \oplus \mathbb{R}$
	φ_5	no	sGT	-160	40	80	$\mathfrak{so}(6,10)$	$\mathfrak{su}(5) \oplus \mathfrak{su}(4) \oplus \mathbb{R}$
	φ_6	no	sF	78	42	78	$\mathfrak{so}(4,12)$	$\mathfrak{su}(6) \oplus (\mathfrak{su}(2))^2 \oplus \mathbb{R}$
	φ_7	no	GT	-784	64	56	$\mathfrak{so}^*(16)$	$\mathfrak{su}(8) \oplus \mathbb{R}$
	$2\varphi_2 + \varphi_6$	no	sGT	-94	26	94	$\mathfrak{so}(8,8)$	$(\mathfrak{su}(2))^3 \oplus \mathfrak{su}(4) \oplus \mathbb{R}^2$
	$\varphi_7 + \varphi_8$	no	sF	280	50	70	$\mathfrak{so}(2,14)$	$\mathfrak{su}(7) \oplus \mathbb{R}^2$
	$7\varphi_1 + 3\varphi_7 + 3\varphi_8$	no	sF	82	38	82	$\mathfrak{so}(2,14)$	$\mathfrak{su}(6) \oplus \mathbb{R}^3$
	$7\varphi_2 + 2\varphi_7 + 2\varphi_8$	no	sF	90	30	90	$\mathfrak{so}(2,14)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \mathbb{R}^3$
	$7\varphi_3 + \varphi_7 + \varphi_8$	no	sF	94	26	94	$\mathfrak{so}(2,14)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^3$
	$\varphi_1 + 3\varphi_2 + \varphi_7 + \varphi_8$	no	sF	184	28	92	$\mathfrak{so}(2,14)$	$\mathfrak{su}(5) \oplus \mathbb{R}^4$
	$3\varphi_1 + 6\varphi_3 + \varphi_7 + \varphi_8$	no	sF	98	22	98	$\mathfrak{so}(2,14)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^4$
	$3\varphi_2 + 5\varphi_3 + \varphi_7 + \varphi_8$	no	sF	98	22	98	$\mathfrak{so}(2,14)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}^4$
	$2\varphi_1 + 2\varphi_2 + 5\varphi_3 + \varphi_7 + \varphi_8$	no	sF	100	20	100	$\mathfrak{so}(2,14)$	$\mathfrak{su}(4) \oplus \mathbb{R}^5$

B.2 Special exceptional adjoint orbits

G_2

Dimension of $\mathfrak{g} = 14$
One non-compact real form: $\mathfrak{g}_{2(2)} = \mathrm{G}$.
Fundamental dominant weights: $\varphi_1 = 2\gamma_1 + \gamma_2$

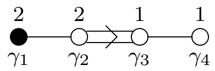
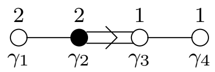
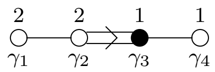
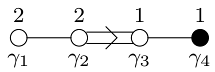
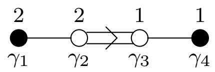
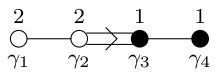
$$\varphi_2 = 3\gamma_1 + 2\gamma_2$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	yes	sGT	-30	4	10	$\mathfrak{g}_{2(2)}$	$\mathfrak{su}(2) \oplus \mathbb{R}$
	φ_2	yes	sGT	-10	4	10	$\mathfrak{g}_{2(2)}$	$\mathfrak{su}(2) \oplus \mathbb{R}$

F_4

Dimension of $\mathfrak{g} = 52$
Two non-compact real forms: $\mathfrak{f}_{4(4)} = \mathrm{FI}$, $\mathfrak{f}_{4(-20)} = \mathrm{FII}$.
Fundamental dominant weights: $\varphi_1 = 2\gamma_1 + 3\gamma_2 + 4\gamma_3 + 2\gamma_4$

$$\varphi_2 = 3\gamma_1 + 6\gamma_2 + 8\gamma_3 + 4\gamma_4$$
$$\varphi_3 = 2\gamma_1 + 4\gamma_2 + 6\gamma_3 + 3\gamma_4$$
$$\varphi_4 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 2\gamma_4$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	yes	sGT	-180	22	30	$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(3) \oplus \mathbb{R}$
	φ_2	no	sGT	-40	12	40	$\mathfrak{f}_{4(4)}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
	φ_3	no	sF	120	12	40	$\mathfrak{f}_{4(-20)}$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$3\varphi_1 + \varphi_3$	no	sF	44	8	44	$\mathfrak{f}_{4(-20)}$	$(\mathfrak{su}(2))^2 \oplus \mathbb{R}^2$
	$\varphi_3 + \varphi_4$	no	sF	84	10	42	$\mathfrak{f}_{4(-20)}$	$\mathfrak{su}(3) \oplus \mathbb{R}^2$
	φ_4	yes	sF	90	22	30	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(7) \oplus \mathbb{R}$
	$\varphi_1 + 2\varphi_4$	no	sGT	-40	12	40	$\mathfrak{f}_{4(4)}$	$\mathfrak{so}(5) \oplus \mathbb{R}^2$
	$\varphi_3 + \varphi_4$	no	sF	84	10	42	$\mathfrak{f}_{4(-20)}$	$\mathfrak{su}(3) \oplus \mathbb{R}^2$

E_6

Dimension of $\mathfrak{g} = 78$

Two non-compact real forms with trivial automorphism: $\mathfrak{e}_{6(2)} =$ EII, $\mathfrak{e}_{6(-14)}=$ EIII.

Fundamental dominant weights: $\varphi_1 = \frac{4}{3}\gamma_1 + \gamma_2 + \frac{5}{3}\gamma_3 + 2\gamma_4 + \frac{4}{3}\gamma_5 + \frac{2}{3}\gamma_6$

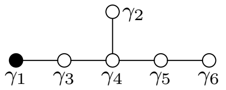
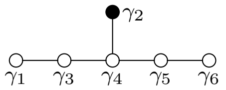
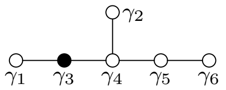
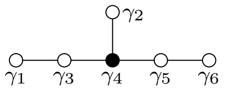
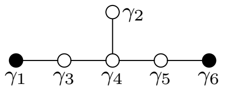
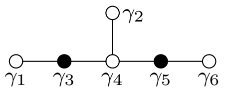
$$\varphi_2 = \gamma_1 + 2\gamma_2 + 2\gamma_3 + 3\gamma_4 + 2\gamma_5 + \gamma_6$$

$$\varphi_3 = \frac{5}{3}\gamma_1 + 2\gamma_2 + \frac{10}{3}\gamma_3 + 4\gamma_4 + \frac{8}{3}\gamma_5 + \frac{4}{3}\gamma_6$$

$$\varphi_4 = 2\gamma_1 + 3\gamma_2 + 4\gamma_3 + 6\gamma_4 + 4\gamma_5 + 2\gamma_6$$

$$\varphi_5 = \frac{4}{3}\gamma_1 + 2\gamma_2 + \frac{8}{3}\gamma_3 + 4\gamma_4 + \frac{10}{3}\gamma_5 + \frac{5}{3}\gamma_6$$

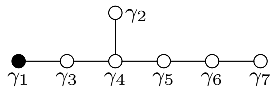
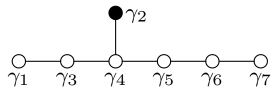
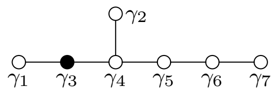
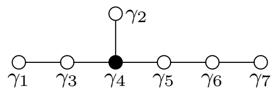
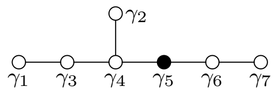
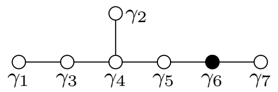
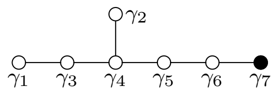
$$\varphi_6 = \frac{2}{3}\gamma_1 + \gamma_2 + \frac{4}{3}\gamma_3 + 2\gamma_4 + \frac{5}{3}\gamma_5 + \frac{4}{3}\gamma_6$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	GT	-384	46	32	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10) \oplus \mathbb{R}$
	φ_2	yes	sGT	-378	36	42	$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(6) \oplus \mathbb{R}$
	φ_3	no	sGT	-150	28	50	$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \mathbb{R}$
	φ_4	no	sGT	-58	20	58	$\mathfrak{e}_{6(2)}$	$(\mathfrak{su}(3))^2 \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$t_1\varphi_1 + t_6\varphi_6$ for all $t_1, t_6 > 0$	no	sCY	0	30	48	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(8) \oplus \mathbb{R}^2$
	$\varphi_3 + \varphi_5$	no	sF	62	16	62	$\mathfrak{e}_{6(-14)}$	$(\mathfrak{su}(2))^2 \oplus \mathfrak{su}(3) \oplus \mathbb{R}^2$

E_7

Dimension of $\mathfrak{g} = 133$
Three non-compact real forms: $\mathfrak{e}_{7(7)}$ = EV, $\mathfrak{e}_{7(-5)}$= EVI, $\mathfrak{e}_{7(-25)}$= EVII
Fundamental dominant weights: $\varphi_1 = 2\gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 3\gamma_5 + 2\gamma_6 + \gamma_7$

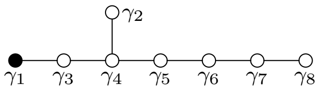
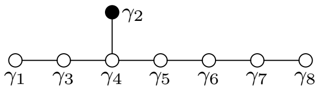
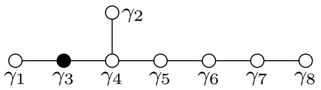
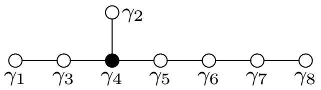
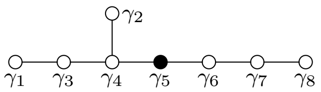
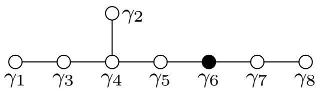
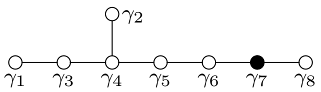
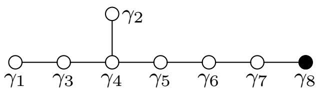
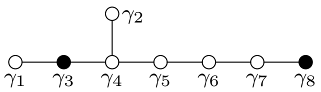
$$\varphi_2 = 2\gamma_1 + \tfrac{7}{2}\gamma_2 + 4\gamma_3 + 6\gamma_4 + \tfrac{9}{2}\gamma_5 + 3\gamma_6 + \tfrac{3}{2}\gamma_7$$
$$\varphi_3 = 3\gamma_1 + 4\gamma_2 + 6\gamma_3 + 8\gamma_4 + 6\gamma_5 + 4\gamma_6 + 2\gamma_7$$
$$\varphi_4 = 4\gamma_1 + 6\gamma_2 + 8\gamma_3 + 12\gamma_4 + 9\gamma_5 + 6\gamma_6 + 3\gamma_7$$
$$\varphi_5 = 3\gamma_1 + \tfrac{9}{2}\gamma_2 + 6\gamma_3 + 9\gamma_4 + \tfrac{15}{2}\gamma_5 + 5\gamma_6 + \tfrac{5}{2}\gamma_7$$
$$\varphi_6 = 2\gamma_1 + 3\gamma_2 + 4\gamma_3 + 6\gamma_4 + 5\gamma_5 + 4\gamma_6 + 2\gamma_7$$
$$\varphi_7 = \gamma_1 + \tfrac{3}{2}\gamma_2 + 2\gamma_3 + 3\gamma_4 + \tfrac{5}{2}\gamma_5 + 2\gamma_6 + \tfrac{3}{2}\gamma_7$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	yes	sGT	−990	67	66	$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}(12) \oplus \mathbb{R}$
	φ_2	no	sGT	−504	49	84	$\mathfrak{e}_{7(7)}$	$\mathfrak{su}(7) \oplus \mathbb{R}$
	φ_3	no	sGT	−94	39	94	$\mathfrak{e}_{7(-5)}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(6) \oplus \mathbb{R}$
	φ_4	no	sCY	0	27	106	$\mathfrak{e}_{7(-5)}$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathbb{R}$
	φ_5	no	sGT	−200	33	100	$\mathfrak{e}_{7(7)}$	$\mathfrak{su}(5) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
	φ_6	no	sGT	−252	49	84	$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}(10) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	φ_7	no	GT	−972	79	54	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_6 \oplus \mathbb{R}$

E_8

Dimension of $\mathfrak{g} = 248$
Two non-compact real forms: $\mathfrak{e}_{8(8)}$ = EVIII, $\mathfrak{e}_{8(-24)}$= EIX.
Fundamental dominant weights:

$$\varphi_1 = 4\gamma_1 + 5\gamma_2 + 7\gamma_3 + 10\gamma_4 + 8\gamma_5 + 6\gamma_6 + 4\gamma_7 + 2\gamma_8$$
$$\varphi_2 = 5\gamma_1 + 8\gamma_2 + 10\gamma_3 + 15\gamma_4 + 12\gamma_5 + 9\gamma_6 + 6\gamma_7 + 3\gamma_8$$
$$\varphi_3 = 7\gamma_1 + 10\gamma_2 + 14\gamma_3 + 20\gamma_4 + 16\gamma_5 + 12\gamma_6 + 8\gamma_7 + 4\gamma_8$$
$$\varphi_4 = 10\gamma_1 + 15\gamma_2 + 20\gamma_3 + 30\gamma_4 + 24\gamma_5 + 18\gamma_6 + 12\gamma_7 + 6\gamma_8$$
$$\varphi_5 = 8\gamma_1 + 12\gamma_2 + 16\gamma_3 + 24\gamma_4 + 20\gamma_5 + 15\gamma_6 + 10\gamma_7 + 5\gamma_8$$
$$\varphi_6 = 6\gamma_1 + 9\gamma_2 + 12\gamma_3 + 18\gamma_4 + 15\gamma_5 + 12\gamma_6 + 8\gamma_7 + 4\gamma_8$$
$$\varphi_7 = 4\gamma_1 + 6\gamma_2 + 8\gamma_3 + 12\gamma_4 + 10\gamma_5 + 8\gamma_6 + 6\gamma_7 + 3\gamma_8$$
$$\varphi_8 = 2\gamma_1 + 3\gamma_2 + 4\gamma_3 + 6\gamma_4 + 5\gamma_5 + 4\gamma_6 + 3\gamma_7 + 2\gamma_8$$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	dim$V$	dimG/V	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	sGT	-1404	92	156	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(14) \oplus \mathbb{R}$
	φ_2	no	sGT	-552	64	184	$\mathfrak{e}_{8(8)}$	$\mathfrak{su}(8) \oplus \mathbb{R}$
	φ_3	no	sF	196	52	196	$\mathfrak{e}_{8(-24)}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(7) \oplus \mathbb{R}$
	φ_4	no	sF	212	36	212	$\mathfrak{e}_{8(-24)}$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \mathbb{R}$
	φ_5	no	sGT	-208	40	208	$\mathfrak{e}_{8(8)}$	$\mathfrak{su}(5) \oplus \mathfrak{su}(4) \oplus \mathbb{R}$
	φ_6	no	sGT	-388	54	194	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(10) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
	φ_7	no	sGT	-166	82	166	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_6 \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	φ_8	yes	sGT	-3078	134	114	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_7 \oplus \mathbb{R}$
	$\varphi_3 + \varphi_8$	no	sGT	-208	40	208	$\mathfrak{e}_{8(8)}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(6) \oplus \mathbb{R}^2$

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