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On the volume of Fano K-moduli spaces

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Contents

Abstract

In this thesis we compute the CM volume, that is the degree of the descended CM line bundle, of the Fano K-moduli space of Quartic del Pezzo in any dimension, and of the K-moduli space of the log Fano hyperplane arrangements of dimension one and two. Furthermore, we relate these volumes to the Weil-Petersson volumes by extending the notion of Weil-Petersson metric in the log case.

Declaration

I declare that to the best of my knowledge the contents of this thesis are original and my work except where indicated otherwise.

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Introduction

In this thesis, we compute the *Weil Petersson Volume* of the K-moduli space of quartic del-Pezzo, and of the K-moduli space of log Fano hyperplane arrangements. The aim of this work is to provide a new direction into the study of some geometric properties of K-moduli spaces. Some K-moduli spaces are proper and projective, and it is possible to describe these explicitly, e.g. as GIT quotients, then we can try to do some intersection theory on these. Indeed, we may calculate the volume of such moduli spaces by using a *natural* class, i.e. the CM line bundle λ_{CM}. That is, a *functorial* line bundle descending on the K-moduli space as a natural $\mathbb{Q}$-polarization. Moreover, such volumes are *Weil-Petersson volumes.* Indeed, in [76] it is proven, in the absolute case, that the degree of the CM line bundle is related to the Weil-Petersson metric. In this thesis we extend the latter result to the case of moduli of pairs.

A brief outline of the developed machinery

From a naive approach when constructing the moduli space of Fano varieties we would look at

$$M_d := \{\text{Fano varieties with } d(k) = h^0(X, -kK_X)\}/\sim_{\text{biholomorphisms}}.$$

When one tries to give an algebraic/complex structure to M_d then there are some issues: the automorphism group of some Fano varieties is infinite, and M_d is in general non separable, a famous counterexample is the *Mukai-Umemura 3-fold* given in [45]. Thus, in the attempt of constructing at least a *coarse moduli space*, we should restrict the class of Fano varieties. The picture of K-stability gives a key ingredient for constructing the moduli space of Fano varieties. K-stability was firstly introduced by Tian in late '90s, and later Donaldson gave a reformulation of it in an algebro-geometric fashion. The main philosophy behind it, is encoded in the YTD-conjecture. If X is a Fano variety, then

$$\begin{cases} K\text{-stable} = \text{KE} + |\mathrm{Aut}(X)| < \infty, \\ K\text{-polystable} = \text{KE}. \end{cases}$$

The proof of the YTD conjecture involved deep analytical work. Nevertheless, many constructions of K-stability motivated the study of the subject in a purely algebro-geometric fashion. In the last ten years, a substantial progress was made towards the construction of the K-moduli space of Fano varieties. So far, in the general theory of K-moduli space a *good moduli space* in the sense of Alper [26] is constructed. Yet, it is still unclear if the latter is proper and projective. For smoothable Fanos, it is known that the K-moduli is proper and projective, but it is still unknown in the general case. The explicit examples considered in this work, are provided in [66] and [78]. In these examples, the K-moduli spaces are GIT quotients, whose semistable locus equals the stable locus. In order to briefly explain the machinery adopted in this thesis for calculating the CM volume, consider X to be a normal Fano variety endowed with an action of a complex reductive group G. Consider the GIT quotient $M\colon = X//_L G$ with linearization L. Suppose that the semistable locus of M, denoted by X^{ss}, and the stable locus of M, denoted by X^s, satisfy $X^{ss} = X^s$. The latter condition gives the surjectivity of the *Kirwan map*,

$$\kappa\colon H^\bullet_G(X) \longrightarrow H^\bullet(M).$$

With that map the cohomology ring of M, can be studied via the equivariant cohomology ring of X. When the Kirwan map is surjective, the Jeffrey-Kirwan residue formula [19]

$$\kappa(\eta)e^{\omega_0}[M] = \frac{C^G}{\mathrm{Vol(T)}}\mathrm{Res}\left(\mathcal{D}^2(Y)\sum_{F\in\mathcal{F}}\int_F \frac{i_F^*(\eta e^{\bar{\omega}}(Y))}{e_F(Y)}[dY]\right),$$

is the best candidate for computing intersection product of classes in this case. In the above formula ω_0, in the left hand side, denotes the symplectic form of the Marsdsen-Weinstein reduction coming from the symplectic picture of the GIT quotients [1, Chapter 5]. In the right hand side C^G is a constant depending on the G-action on X, $\bar{\omega}$ denotes the *equivariant extension* to the symplectic form on X [12, Chapter 9, section 9.3]. $\mathrm{Vol}(T)$, and $[dY]$ are the volume of the maximal torus T of G, and the measure on its Lie algebra $\mathfrak{t}$ induced by the restriction to $\mathfrak{t}$ of the fixed inner product on the Lie algebra of $G, \mathfrak{g}$. The set $\mathcal{F}$ is the fixed point set of the maximal torus action, $i_F\colon F \hookrightarrow M$ is the inclusion, and e_F is the equivariant Euler class of the normal bundle to F in M. Suppose that the *GIT-stability conditions* matches with the *K-stability conditions.* Then, the main strategy for calculating the volume can be outlined as follows:

- The descending CM line bundle on M, is calculated in two steps

    1. Take a proper and flat family in M, $\pi\colon \mathcal{X} \to \mathbb{P}^1$ with a π-ample

line bundle $\mathcal{L}$. Compute the following intersection number

$$[\lambda_{\mathrm{CM}}] \cdot (\mathcal{X} \to \mathbb{P}^1) = \pi_*(c_1(\lambda_{\mathrm{CM}})) \cdot [\mathbb{P}^1].$$

The first Chern class of the CM line bundle depends on $\mathcal{L}$.

2. Express the CM line bundle in terms of the generators of $\mathrm{Pic}(M)$, and use the first step to compute *the right power* of the linearization L which is equivalent to λ_{CM}. Call $\tilde{L}$ the linearization that matches with the CM line bundle by

- Compute the following intersection pairing via the Jeffrey-Kirwan non abelian localization theorem

$$[\tilde{L}]^{\dim M} \cdot [M] = \kappa(c_1(\tilde{L})^{\dim M}))e^{\omega_0}[M]$$

to get the CM volume of M.

A brief outline of this work

The first Chapter of this thesis gives the basic definitions of K-stability and provides a review of the past ten years in the construction of the K-moduli spaces of Fano varieties. We can divide the work in two main parts.The first one is devoted to K-moduli spaces in the absolute case, that is no pair (X, D). And, the second one is devoted to K-moduli spaces of pairs.
The first part consists of two chapters:

- In Chapter two we introduce the CM line bundle in the absolute case and we explain the relation between the degree of the latter and the Weil-Petersson metric by revisiting the work of Li, Wang and Xu [76].

- In Chapter three, motivated by the work of Spotti and Sun of 2015 [67], Odaka, Spotti and Sun of 2016 [66], and the pioneering work of Mabuchi and Mukai [61], we compute the CM volume of the K-moduli space of del Pezzo quartics in any dimension using the procedure described before. In Theorem 3.2.1 we express the CM volume of the K-moduli space of quartic del Pezzo, and by [23] we are able to extend the result also in the even case (Remark 3.2.1).

The second part consists also of two chapters:

- The fourth Chapter is devoted to provide a description of the CM line bundle in the log case by stating the general definition and the main properties. Using the outstanding effort of Guenancia and Paun[57] in understanding the *global Monge Ampere equation* in the conic Kähler-Einstein case, and the effort of Berman in [55], and lately by Tian and Wang in [58] on understanding the YTD conjecture in the log case we are able to define the Weil-Peterson metric for the smooth locus of K-polystable families of log Fano pairs along the line of Fujiki and Schumacher in [59]. Then, generalising the results given in [76] we are able to extend it to the whole family, and with the same arguments of [76] and [89] we are also able to extend it to the K-moduli space. More technically, in Theorem 4.0.3 we prove that for a family $f : (\mathcal{X}, \mathcal{D}) \to B$ of log K-polystable Fano varieties, there exists a continuous Deligne's Pairing metric h_{DP} on the log CM line bundle $\lambda_{\mathrm{CM},\mathcal{D}}$ such that the Weil-Petersson metric ω_{WP} extends as a positive current to the whole base B.

- In the fifth Chapter we provide some examples of the procedure described before, that is we compute the log CM volume of the K-moduli space of log Fano hyperplane arrangements. It was proven by Fujita in [78] that such K-moduli space coincides with a GIT quotient, and the latter is classically described by Dolgachev [8], and later by Alexeev [25], [11]. Revisiting the work of [78], in Theorem 5.2.1, we are able to compute the log Donaldson-Futaki invariant for those integral test configuration coming from some *dreamy prime divisor*, and then we related it to the volume of the moduli spaces of log Fano hyperplane arrangements of dimension one. That is, in Theorem 5.4.2 we prove that the CM volume of the moduli space of log Fano hyperplane arrangements can be expressed as the sum of the log Futaki invariants of certain integral test configurations. As a first example, we first examine the case of four points in the projective line relating our work also to previous works [32], and [37]. The last two sections are devoted to more technical works, i.e. the calculation of the volume of the moduli space of log Fano hyperplane arrangements using the Jeffrey-Kirwan non abelian localization theorem in dimension one, Theorem 5.6.1, and dimension two, Theorem 5.7.1. The adopted procedure for computing the volume of the moduli space of hyperplane arrangement in dimension 2, can be adopted in any dimension. However, already from the case of dimension 3 it is hard to see a general pattern emerging.

Chapter 1

K-stability and K-moduli spaces

In differential geometry the constructions of K-stability were originally related with the existence of Kähler-Einstein metrics on Fano manifolds. For an introduction to this topic see for instance [2], [10]. K-stability was firstly introduced by Tian in late '90s, and later Donaldson gave a reformulation of it in an algebro-geometric fashion. Many constructions of K-stability motivated the study of the subject in a purely algebro-geometric fashion. In the last ten years, a substantial progress was made towards the construction of the K-moduli space of Fano varieties. We subdivided this Chapter in two main sections, the first one is devoted to the understanding of K-stability in an algebro-geometric way, and the second one provides a short review of the constructions of K-moduli spaces.

1.1 The definition of K-stability

The definition of K-stability requires some work. We start with the classical definition of K-stability.

Definition 1.1.1. *A $\mathbb{Q}$-Fano variety X is a normal projective variety with* log terminal singularities *and the anticanonical divisor $-rK_X$ is a Cartier divisor for some $r \in \mathbb{N}$.*

Definition 1.1.2. *Let $(X, -rK_X)$ be a $\mathbb{Q}-$Fano variety. A* test configuration *$(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ for $(X, -rK_X)$ consists of*

- *A variety $\mathcal{X}$ endowed with a $\mathbb{G}_m-$action.*
- *A $\mathbb{G}_m-$equivariant line bundle $\mathcal{L} \to \mathcal{X}$*
- *A flat $\mathbb{G}_m-$equivariant morphism $\pi\colon (\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by multiplication, and π is such that $\forall t \in \mathbb{A}^1 \setminus \{0\}$, $\pi^{-1}(t)\colon = (\mathcal{X}_t, \mathcal{L}_t) \simeq (X, -rK_X)$*

Definition 1.1.3. *For a fixed $r \in \mathbb{Q}_{>0}$. A $\mathbb{Q}$-test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ of a $\mathbb{Q}$-Fano variety $(X, -rK_X)$ consist of a $\mathbb{Q}$-cartier divisor class $\mathcal{L}$ on $\mathcal{X}$ such that for some integer $m \geq 1$, $(\mathcal{X}, m\mathcal{L})$ yields a test configuration of $(X, -mrK_X)$ in the sense of Definition 1.1.2. We say that a $\mathbb{Q}$-test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ is normal if $\mathcal{L} \sim_{\mathbb{Q}} -K_{\mathcal{X}}$, and the central fiber $\mathcal{X}_0$ is a $\mathbb{Q}-$Fano variety.*

To any $\mathbb{Q}-$test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ we can associate a numerical quantity called the *Donaldson-Futaki invariant.* In order to define the latter, let $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ be a $\mathbb{Q}$-test configuration. Outside the zeroth fiber we have a $\mathbb{Q}$-Fano variety, so by the Hirzebruch-Riemann-Roch Theorem, for a sufficiently large $k \in \mathbb{Z}_{\geq 0}$ we can associate the *Hilbert polynomial* as follows

$$d(k) \colon = \dim H^o(\mathcal{X}_t, k\mathcal{L}_t) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}).$$

Where a_0, and a_1 are rational numbers. By definition, the multiplicative group $\mathbb{G}_m$ acts on the central fiber $(\mathcal{X}_0, \mathcal{L}_0)$, and since it acts linearly on $\mathcal{L}_0$, it also acts on the space of sections $H^0(\mathcal{X}_0, k\mathcal{L}_0)$. The dimension of $H^0(\mathcal{X}_0, k\mathcal{L}_0)$ yields to another polynomial $w(k)$ which is the *total weight* of the $\mathbb{G}_m$ action on sections. Then, by the equivariant Riemann-Roch Theorem we get

$$w(k) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}).$$

By taking the Laurent expansion we get

$$\frac{w(k)}{kd(k)} = F_0 + F_1 k^{-1} + O(k^{-2}). \tag{1.1}$$

Definition 1.1.4. *([46]) For any given $\mathbb{Q}-$ test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ we define the* Donaldson-Futaki *invariant of the given $\mathbb{Q}$-test configuration to be*

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) := -\frac{F_1}{a_0} = \frac{a_1 b_0 - a_0 b_1}{a_0^2}.$$

Let $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ be a $\mathbb{Q}-$test configuration, by adding the trivial fiber at $\infty \in \mathbb{P}^1$, we can obtain a so called compactified test configuration ([70],[71]) $(\overline{\mathcal{X}}, \overline{\mathcal{L}})/\mathbb{P}^1$. More precisely, a compactified test configuration is obtained by gluing $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ and $(X \times (\mathbb{P}^1 \setminus \{0\}), \mathrm{pr}_1^* L)$ along $(\mathcal{X} \setminus \mathcal{X}_0, \mathcal{L}_{\mathcal{X} \setminus \mathcal{X}_0})$ and $(X \times (\mathbb{P}^1 \setminus \{0, \infty\}), \mathrm{pr}_1^* L)$.

Remark 1.1.1. *The Donaldson-Futaki invariant can be written with an intersection theoretic formula [70], [71]. That is, let $f \colon \mathcal{X} \to C$ be a proper and flat family of $\mathbb{Q}$-Fano varieties over a smooth curve C. Let $\mathcal{L} \to \mathcal{X}$ be a relatively ample line bundle for f. Then the Donaldson Futaki invariant is defined as*

$$\mathrm{DF}(\mathcal{X}/C,\mathcal{L}) = \frac{1}{2(n+1)\cdot\mathcal{L}_t^n}\left(n\mu(\mathcal{X}_t,\mathcal{L}_t)\mathcal{L}^{n+1} + (n+1)\mathcal{L}^n\cdot K_{\mathcal{X}/C}\right) \quad (1.2)$$

where

$$\mu(\mathcal{X}_t,\mathcal{L}_t) = \frac{\mathcal{L}_t^n\cdot(-K_{\mathcal{X}_t})}{\mathcal{L}_t^n}, \forall t\in C.$$

Compactified test configurations can be thought as proper and flat families over a smooth curve $\mathbb{P}^1$. When a test configuration is also normal, the Donaldson-Futaki invariant does not change under its compactification, as the following result shows.

Proposition 1.1.1. *([72] Let $(\mathcal{X},\mathcal{L})/\mathbb{A}^1$ be a normal $\mathbb{Q}-$test configuration. Then*

$$\mathrm{DF}(\mathcal{X},\mathcal{L}) = \mathrm{DF}(\overline{\mathcal{X}},\overline{\mathcal{L}})$$

The Donaldson-Futaki invariant can be interpreted as the weight of a *functorial line bundle*, known as *the CM line bundle* [47].
We are now ready to give the definition of K-stability.

Definition 1.1.5. *Let $(X,-K_X)$ be a $\mathbb{Q}-$Fano variety.*

1. *X is called K-stable (resp. K-semistable), if for any normal $\mathbb{Q}-$test configuration $(\mathcal{X},\mathcal{L})/\mathbb{A}^1$ of $(X,-K_X)$, we have*

$$\mathrm{DF}(\mathcal{X},\mathcal{L}) > 0,\ \text{(resp. } \mathrm{DF}(\mathcal{X},\mathcal{L}) \geq 0).$$

2. *X is called K-polystable if for any normal $\mathbb{Q}-$test configuration $(\mathcal{X},\mathcal{L})$ of $(X,-K_X)$, X is K-semistable and*

$$\mathrm{DF}(\mathcal{X},\mathcal{L}) = 0 \Leftrightarrow \mathcal{X}\simeq X\times\mathbb{A}^1.$$

As we mentioned in the introduction of this Chapter, the study of K-stability was motivated by the existence of Kähler-Einstein metrics on Fano manifolds. Namely, the Yau-Tian-Donaldson (YTD) conjecture.

Theorem 1.1.1. *(YTD-conjecture) A Fano manifold admits a Kähler-Einstein metric $\Longleftrightarrow$ $(X,-K_X)$ is $K-$polystable.*

The above conjecture, i.e. Theorem 1.1.1, is not longer a conjecture. The $''\Rightarrow''$ was proven by Tian [45],Donaldson [46], Stoppa [48] and Berman [55], and the $''\Leftarrow''$ was proven in [51], [52], [53], by Chen, Donaldson and Sun. The Definition 1.1.5 of K-stability imitates a GIT stability notion given in [3]. Indeed, when defining K-stability, the role played by the Donaldson-Futaki invariant is similar to the numerical criterion that defines the notion of GIT-stability, known as *the Hilbert-Mumford criterion.* A great effort in the understanding of K-stability, during the last decade, was made by Fujita [79], [80], [77], [78], [81], [82], together with Odaka [86]. Fujita, in [77] introduced a new criterion for testing K-stability on $\mathbb{Q}$-Fano varieties, known as the β-invariant. We recall some definitions from the work of Fujita [77].

Definition 1.1.6. *Let X be a $\mathbb{Q}$-Fano variety, and let F be a prime divisor of X. Fix $f : Y \to X$ to be a normalization, and call $L = -K_X$. We define*

- *The* log discrepancy $A_X(F) := \text{ord}_F(K_{Y/X}) + 1$. *Since X is log terminal, $A_X(F)$ holds.*
- *$\forall r \in \mathbb{Z}_{>0}$ such that rL is Cartier, and for any $k \in \mathbb{Z}_{\geq 0}$, set*

$$V_k^r := H^0(X, krL)$$

- *$\forall j \in \mathbb{R}$ define the $\mathbb{C}$-vector subspace $\mathcal{F}_F^j V_k^r$ of V_k^r as $H^0(\tilde{X}, f^*(krL) + |-jF|)$ if $j \geq 0$,V_k^r if $j < 0$.*
- *For any $x \in \mathbb{R}_{\geq 0}$, we set*

$$\text{vol}_X(L - xF) = \text{vol}_Y(f^*L - xF)$$

- *Call the β-invariant the following number*

$$\beta(F) = 1 - \frac{\int_0^\infty \text{vol}_X(L - xF)dx}{A_X(F) \cdot L^n}$$

- *F is said to be* dreamy *over X if the graded $\mathbb{C}$-algebra*

$$\oplus_{k,j \in \mathbb{Z}_{\geq 0}} \mathcal{F}_F^j V_k^r$$

  is finitely generated for some (hence, for any) $r \in \mathbb{Z}_{\geq 0}$ with rL Cartier

Remark 1.1.2. *The function* $\text{vol}_X(L - xF)$ *is continuous and non increasing over $x \in [0, +\infty)$. Moreover, if $x >> 0$ then* $\text{vol}_X(L - xF)$ *is identically zero since $f^*L - xF$ is not pseudo-effective for very large x.*

A new characterization of K-stability is given in the setting of Definition 1.1.6.

Theorem 1.1.2. *([77]) Let X be a $\mathbb{Q}-$Fano variety. Then we have*

1. *X is $K-$stable $\Leftrightarrow$ $\beta(F) > 0$ holds for any dreamy prime divisor F over X.*

2. *X is $K-$semistable $\Leftrightarrow \beta(F) \geq 0$ holds for any (dreamy) prime divisor F over X.*

The *classical* Definition 1.1.5 of K-stability, can be restored in the case of *integral test configuration.* In [77] and [78], it shown that there is a bijection between integral $\mathbb{Q}-$test configurations and dreamy prime divisors.

Definition 1.1.7. *A $\mathbb{Q}-$test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ is said to be* integral *if its central fiber $\mathcal{X}_0$ is reduced and irreducible.*

Definition 1.1.8. *For a algebraic variety X we define the* divisorial valuation *on X as the group homomorphism*

$$c \cdot ord_F : \mathbb{C}(X)^* \to (\mathbb{Q}, +)$$

where $c \in \mathbb{Q}_{>0}$ and F is a prime divisor of X.

Theorem 1.1.3. *([77]) Let $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ be a non trivial integral test configuration. Then, for any $r \in \mathbb{Z}_{>0}$ such that rL is Cartier, we have*

1. *$\mathbb{C}(\mathcal{X}) = \mathbb{C}(X)[t]$. In particular, the divisorial valuation at the central fiber $v_{\mathcal{X}_0}$ is a multiple of the divisorial valuation at any $F \subset X$ dreamy prime divisor. Namely $v_{\mathcal{X}_0} = c \cdot ord_F$.*

2. $$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = c \cdot \beta(F) \cdot A_X(F).$$

3. *There exists a $d \in \mathbb{Q}$ with $dr \in \mathbb{Z}$ such that $\forall k \in \mathbb{Z}_{\geq 0}$ we have that $H^0(\mathcal{X}, kr\mathcal{L}) = \oplus_{j \in \mathbb{Z}} t^{-j} \cdot \mathcal{F}_F^{\frac{dkr+j}{c}} V_k^r$, is a $\mathbb{C}(t)$-module.*

The above result shows that given a test configuration with integral central fiber, also known as *integral test configuration*, there exist a dreamy prime divisor, and some rational number such that the divisorial valuation at the central fiber is a multiple of the divisorial valuation at the dreamy prime divisor. Therefore, given a test configuration with integral central fiber we can associate a dreamy prime divisor. Conversely, start with some F dreamy prime divisor, we can construct a test configuration with integral central fiber as follows: $\forall c \in \mathbb{Z}_{>0}$, define

$$\mathcal{X}^{F,c} := \mathrm{Proj}_{\mathbb{A}_1^t} \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \left(\bigoplus_{j \in \mathbb{Z}} t^{-j} \mathcal{F}_F^{\frac{j}{c}} V_k^r \right).$$

$$\mathcal{L}^{F,c} := \frac{1}{mr} (\mathrm{relative}\mathcal{O}(m))$$

for a sufficiently divisible non negative integer m. Set $\mathcal{X}^{F,1} = \mathcal{X}^F$, $\mathcal{L}^{F,1} = \mathcal{L}^F$ [78, Lemma 3.8].

This discussion shows that there is a bijection between test configurations with integral central fiber and dreamy prime divisors. Therefore, it will be sufficient to calculate $\mathrm{DF}(\mathcal{X}, \mathcal{L})$ with any dreamy prime divisor, as the second assertion of Theorem 1.1.3 suggests. However, not every $\mathbb{Q}$-test configurations are integral, therefore in the above framework the notion of β-invariant on a $\mathbb{Q}$-Fano variety yields to a more general criterion for testing K-stability. Towards the understanding of K-stability, another notion is given by Fujita and Odaka in [86], and Li in [75], i.e. the notion of *uniform* K*- stability.*

Definition 1.1.9. *Let X be a $\mathbb{Q}$-Fano variety. Let $F \subset X$ be a prime divisor of X, and let $f : Y \to X$ be a normalization of X. We define the δ-invariant of X to be*

$$\delta(X) := \inf_{F \subset X} \frac{A_X(F)}{S_X(F)},$$

where $A_X(F)$ is like in definition 1.1.6 and

$$S_X(F) = \frac{1}{(L)^n} \int_0^\infty vol(f^*L - tF)dt.$$

Using the δ- invariant, we can collect some fundamental results given in [86], [75] and [83]

- If $\delta(X) \geq 1$, then X is K-semistable.
- if $\delta(X) > 1$ then X it is uniformly K-stable.
- If X is uniformly K-stable then it is also K-semistable.

It is known that if the automorphism group of X is finite, then the converse of the above statements is also true. In general the converse of the above statements is still a big conjecture.
Log K-stability was firstly introduced by Li in [50] in an analytical way. An algebro-geometric approach is given in [77]. We begin with the following

Definition 1.1.10. *A pair (X, D), where X is a variety and D is a divisor of X, is said to be a* log Fano pair *if it is a projective klt pair with D effective $\mathbb{Q}$-divisor and $-(K_X + D)$ is ample and $\mathbb{Q}$-Cartier.*

We can define the β-invariant exactly as before, for completeness we report the main definitions
For the rest of this section we assume (X, D) to be a log Fano pair with ample log canonical line bundle $L = -(K_X + D)$.

Definition 1.1.11. *Let F be a prime divisor of X. Fix $f\colon \tilde{X} \to X$ to be a normalization. Define*

- $A_{(X,D)} := ord_F(K_{\tilde{X}} - f^*(K_X + D)) + 1$
- $\forall r \in \mathbb{Z}_{>0}$ *such that* rL *is Cartier, and for any* $k \in \mathbb{Z}_{\geq 0}$, *ser*

$$V_k^r := H^0(X, krL)$$

- $\forall j \in \mathbb{R}$ *define the* $\mathbb{C}-$*vector subspace* $\mathcal{F}_F^j V_k^r$ *of* V_k^r *as* $H^0(\tilde{X}, f^*(krL) + \lfloor -jF \rfloor)$ *if* $j \geq 0$,V_k^r *if* $j < 0$.
- *For any* $x \in \mathbb{R}_{\geq 0}$, *we set*

$$vol_X(L - xF) \colon := vol_{\tilde{X}}(f^*L - xF)$$

- *Call the* $\beta-$ *invariant*

$$\beta(F) = 1 - \frac{\int_0^\infty vol_X(L - xF)dx}{A_{(X,D)}(F) \cdot L^n}$$

- F *is said to be* dreamy *over* (X, D) *if the graded* $\mathbb{C}-$*algebra*

$$\oplus_{k,j \in \mathbb{Z}_{\geq 0}} \mathcal{F}_F^j V_k^r$$

  is finitely generated for some (hence, for any) $r \in \mathbb{Z}_{\geq 0}$ *with* rL *Cartier*

A test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ for $(X, L = -K_X - D)$ is defined as before. The definition of *integral test configuration* holds also in this case.

Definition 1.1.12. *The log Donaldson-Futaki invariant for a test configuration* $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ *is defined via the Odaka [71] and Wang [70] formula, as follows*

$$\mathrm{DF}_D(\mathcal{X}, \mathcal{L}) := \frac{n}{n+1} \cdot \frac{\overline{\mathcal{L}}^{n+1}}{L^n} + \frac{\overline{\mathcal{L}^n} \cdot \left(K_{\overline{\mathcal{X}}/\mathbb{P}^1} + D_{\overline{\mathcal{X}}}\right)}{L^n}$$

where $D_{\overline{\mathcal{X}}}$ *is the closure of* $D \times (\mathbb{A}^1 \setminus \{0\})$ *on* $\overline{\mathcal{X}}$.

Theorem 1.1.4. *([77]) Let* $(\mathcal{X}, \mathcal{L})$ *be a test configuration with integral central fiber* $\mathcal{X}_0$. *Then, for any* $r \in \mathbb{Z}_{>0}$ *such that* rL *is Cartier, we have*

1. $\mathbb{C}(\mathcal{X}) = \mathbb{C}(X)[t]$. *In particular, the divisorial valuation at the central fiber* $v_{\mathcal{X}_0}$ *is a multiple of the divisorial valuation at any* $F \subset X$ *dreamy prime divisor. Namely* $v_{\mathcal{X}_0} = c \cdot ord_F$.
2. $$DF_D(\mathcal{X}, \mathcal{L}) = c \cdot \beta(F) \cdot A_{(X,D)}(F).$$

3. *There exists a* $d \in \mathbb{Q}$ *with* $dr \in \mathbb{Z}$ *such that* $\forall k \in \mathbb{Z}_{\geq 0}$ *such that*

$$H^0(\mathcal{X}, kr\mathcal{L}) = \oplus_{j\in\mathbb{Z}} t^{-j} \cdot \mathcal{F}_F^{\frac{dkr+j}{c}} V_k^r$$

is a $\mathbb{C}(t)-$*module.*

The correspondence between integral test configuration and dreamy prime divisors extends also in the log case [77], [78]. One can define log K-stability classically exactly as in definition 1.1.5, an algebraic notion of log K-stability is given in [78, Theorem 1.8]

Theorem 1.1.5. *Let* (X, D) *be a log Fano pair.*

- *The following are equivalent*
  1. (X, D) *is* K*-stable,*
  2. $\beta_{(X,D)}(F) > 0$ *for any dreamy prime divisor* F *on* (X, D).
- *The following are equivalent*
  1. (X, D) *is* K*-polystable,*
  2. $\beta_{(X,D)}(F) \geq 0$ *holds for any dreamy prime divisor* F *over* (X, D), *and equality holds only if* F *is of "product type,"i.e. it is associated to a product type test configuration.*

Other notions of K-stability are given trough the notion of Ding stability [77], [78].

1.2 An introduction to K-moduli space

A central topic in modern algebraic geometry is the description of the moduli space of varieties. Let X be a variety that admits a canonical polarization K_X. When $K_X > 0$, namely when X is of general type, one constructs the moduli space of general type variety via the KSB moduli theory, where KSB stands for Kollar and Shepherd-Barron [15]. The KSB moduli theory can be thought as an higher dimensional analogy of the compactified moduli spaces of curves of genus g. In this section we are interested in the case when $K_X < 0$, namely the case of Fano varieties. Let $(X, -K_X)$ be a Fano variety, by the Kodaira-Nakano vanishing theorem, all the sheaf cohomology groups $H^i(X, -kK_X)$ vanish except in degree zero. The dimension of the degree zero sheaf cohomology group $h^0(X, -kK_X)$ equals a polynomial $d(k)$, called the Hilbert polynomial. A natural attempt when constructing the moduli space of Fano varieties is by considering the following

$$M_d := \{\text{Fano varieties with } d(k) = h^0(X, -kK_X)\}/\sim_{\text{biholomorphisms}}.$$

In order to classify *geometrically* the points of M_d one requires an algebraic/complex structure on M_d. That means, for any given flat family $f : \mathcal{X} \to A$ whose fibers are Fano varieties, the natural map $A \to M_d$ is holomorphic with respect to the analytic structure given in M_d. When we attempt to give a complex structure to M_d, then we have two main issues:

1. In general the moduli space of *all* Fano varieties is non separable, hence the analytic structure is not preserved. Indeed, it can be seen that in dimension $n \geq 3$, there exist flat families of smooth Fanos $\pi : \mathcal{X} \to \Delta$ over the complex disc Δ, such that $\mathcal{X}_t \simeq \mathcal{X}_s$ for any $t, s \neq 0$, but $X_0 \neq X_t$, for $t \neq 0$. Thus $[X_0] \in [X_t]$ (here the square bracket denotes the isomorphism class). Hence, $[X_t]$ would be a non-closed point in M_d, then M_d fails to be Hausdorff. A well-known concrete example of this phenomenon is given by deformations of Mukai-Umemura Fano 3-fold [45].

2. Fano varieties may have non finite Automorphism group.

Because of this two issues we shall restrict the class of Fano varieties that we want to encode in the moduli space. When restricting this class, the picture of K-stability is considered.

1.2.1 Some basic definitions in moduli theory

In order to introduce the topic, we firstly give some abstract definition. Euristically, a *moduli problem* can be thought as a pair $(A, \sim)$ where A is some set of objects and "$\sim$" is an equivalence relation for the elements of A. Given the quotient $A/\sim$ one wants to associate a scheme or a variety that gives the notion of point for that quotient. The set of objects and its equivalence relation $(A, \sim)$ belongs to some category $\mathbf{C}$. We say that $(A, \sim)$ define a class of objects $\mathcal{S}$ in some category $\mathbf{C}$.

Definition 1.2.1. *Let $\mathcal{S}$ be a class of objects in some category $\boldsymbol{C}$. $\forall A \in \mathrm{Ob}(\boldsymbol{C})$, a family of $\mathcal{S}$-objects over A is another object $\mathcal{X} \in \mathrm{Ob}(\boldsymbol{C})$ together with a surjective and flat morphism $f : \mathcal{X} \to A$, such that $\forall a \in A,\ f^{-1}(a) := \mathcal{X}_a \in \mathrm{Ob}(\mathcal{S})$.*

Definition 1.2.2. *Let S define a class of objects in some category $\boldsymbol{C}$. A* moduli problem *is a contravariant functor form the category $\boldsymbol{C}$ to the category of Sets* ***Set***

$$\begin{aligned} \mathcal{U}_S : & \boldsymbol{C} \to \boldsymbol{Set} \\ & A \longmapsto \mathcal{U}_S(A) := \{\textit{Set of isomorphism classes of } \mathcal{S}\textit{-objects in } A\} \end{aligned}$$

Such that $\forall f \in \mathrm{Hom}_{\boldsymbol{C}}(A, A')$ is associated the set map

$$\{\mathcal{X} \to A\} \to \{f^*\mathcal{X} \to A'\}.$$

Definition 1.2.3. *Let S define a class of objects in some category $\boldsymbol{C}$. An object $\mathcal{M}_{\mathcal{S}} \in \mathrm{Ob}(\boldsymbol{C})$ that represents the functor $\mathcal{U}_{\mathcal{S}}$ is called a* fine moduli space *for the moduli problem. Where, representing the functor means that the* Hom$-$*functor, also known as the* functor of points $\mathrm{Hom}_{\boldsymbol{C}}(-, \mathcal{M}_{\mathcal{S}})$*is isomorphic as a functor to the moduli problem functor via a* natural *transformation T.*

When a moduli functor is representable as a fine moduli space there is a very special object with mapping to the moduli space. The latter is such that fibers over each point are the objects parametrised by that point.

Definition 1.2.4. *Let $\mathcal{M}_{\mathcal{S}}$ be a fine moduli space. The* universal family $\mathfrak{U}_{\mathcal{S}}$ *is an object of $\mathcal{S}$ corresponding to $T(\mathrm{id}_{\mathcal{M}_{\mathcal{S}}})$, where T is the natural transformation between the moduli functor and the functor of points. This is a family, whose projection is called the universal function*

$$\pi : \mathfrak{U}_{\mathcal{S}} \to \mathcal{M}_{\mathcal{S}},$$

such that $\forall m \in \mathcal{M}_{\mathcal{S}},\ \pi^{-1}(m) \in \mathrm{Ob}(\mathcal{S})$.

When the universal family exists, then any other family of $\mathcal{S}$-objects is obtained from the universal family by pullback. Moduli spaces that parametrize objects that have automorphism, generally, do not admit the universal family. Therefore, one introduce the notion of *coarse moduli space*

Definition 1.2.5. *A* coarse moduli space *for a moduli problem functor is a pair $(\mathcal{M}, T)$ where $\mathcal{M}$ is a scheme and T is a natural transformation which is not necessarily an isomorphism. Such that*

- *$\mathcal{M}$ is universal satisfying the natural transformation T,*
- *For any algebraically closed field k $T_{A \in \mathrm{Ob}(\mathcal{S})} : \mathcal{U}_{\mathcal{S}}(A) \to \mathrm{Hom}(A, \mathcal{M})$ is a Set isomorphism.*

1.2.2 K-moduli spaces via the Hitchin-Kobayashi correspondence

This section is based on a review given by Spotti in [68]. In order to construct the moduli space of Fano varieties, due to some issues, it is reasonable to restrict the *class* of Fano manifolds, to those that are $K-$stable. The forthcoming construction was central in the years 2010-2015. Define the moduli space of Fano varieties that admit a KE-metric

$$\mathcal{E}M_d := \{(X, \omega, d) \text{ Fano manifold} \mid \omega \text{ is KE}\}/\text{isometries}.$$

Similarly, define

$$\mathcal{K}M_d := \{[X] \in M_d \mid X \text{is } K- \text{ polystable}\} \subset M_d.$$

Remark 1.2.1. *The above inclusion is strict. Indeed, as we have already explained there exists some classical obstructions to the existence of KE metrics on Fano manifolds. Besides K-stability, we shall mention also the obstructions that have been found by Futaki in [39], Matsushima in[38], and there is an observation in the work of Tian [45] where an example of a smooth Fano 3- fold with no non-trivial holomorphic vector fields is shown.*

From the *Hitchin-Kobayashi correspondence* we have the following *forgetful* map

$$\begin{aligned} \phi_d \colon \mathcal{E}M_d &\to \mathcal{K}M_d \\ [(X,\omega)] &\longmapsto \phi_d([(X,\omega)]) := [X] \end{aligned}$$

The map ϕ_d is well defined because of " $\Rightarrow$ " of the YTD conjecture, and it is surjective because of "$\Leftarrow$" of the YTD conjecture, and by the *Bando-Mabuchi uniqueness* [41] ϕ_d is also injective. Therefore ϕ_d is a bijection. By construction, on $\mathcal{E}M_d$ there is a natural structure of metric space. In order to understand the notion of distance of points in $\mathcal{E}M_d$, namely KE Fano manifolds, we need to introduce the following

Definition 1.2.6. *Let* $(S, d_s), (T, d_T)$ *be compact metric spaces, the* Gromov-Hausdorff distance *of* S *and* T *is defined as*

$$d_{GH}(S,T) := \inf_{S,T \hookrightarrow U} \inf\{C > 0 : S \subseteq N_C(S), T \subseteq N_C(T)\},$$

Where, $N_C(S)$ *is a neighbourhood of radius* C *isometrically embedded in a metric space* U*.*

The topology induced by the Gromov-Hausdorff distance is clearly Hausdorff, and it induces a metric structure on the space of isomorphism classes of compact metric spaces. There is a notion of convergence of such distance (see [6]). The Gromov-Hausdorff distance gives the possibility to study the degeneration of Riemannian manifolds into singular spaces. In the case of KE metrics we have an important property of *precompactness*: given sequence of Kähler-Einstein Fano manifolds $(X_m, \omega_m)_{m\in\mathbb{N}}$ of the same dimension, then there exist a compact metric space $S_\infty$for which $(X_m, \omega_m)_{m\in\mathbb{N}}$ *weakly converges* in the Gromov-Hausdorff sense. The *Cheeger-Colding theory* [44] provides S_∞ with a structure of *stratified* Einstein space. Later, Donaldson and Sun [65] proved that S_∞ admits a compatible structure of normal Fano variety. These results allow to endow $\mathcal{E}M_d$ with a natural structure of compact metric space. Thus, it makes sense to consider its closure in the space of isomorphism classes of compact metric spaces. With this in mind, and by a result of Berman [55] it is known that all the KE Fano manifolds are polystable holds also in the singular case, then the Hitchin-Kobayashi map can be extended

$$\tilde{\phi}_d : \overline{\mathcal{E}M_d} \longrightarrow \overline{\mathcal{K}M_d}. \tag{1.3}$$

Where, the closure $\overline{\mathcal{E}M_d}$ is in the Gromov-Hausdorff sense, the closure $\overline{\mathcal{K}M_d}$ denotes the set of all $\mathbb{Q}$-Gorenstein smoothable Fano varieties. A $\mathbb{Q}$-Fano variety X, is called $\mathbb{Q}$-Gorenstein smoothable if there exists a deformation over the complex disc Δ, $\pi : \mathcal{X} \to \Delta$, such that $\mathcal{X}_0 \simeq X$, $\mathcal{X}_t$ is smooth for all $t \neq 0$, and $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier. The surjectivity of $\tilde{\phi}_d$ was proven in [66]. Explicit examples of this construction are given in the case of del Pezzo surfaces, the pioneer of these examples are Mabuchi and Mukai [61], where they proved the existence of such compactification for del Pezzo of degree four. Tian in [42] proved that all smooth del Pezzo surfaces admit a KE metrics, besides the well-known obstructed cases of the blow-up of the plane in one or two points. Explicit construction of such moduli space in degree 1,2,3 and 4 can be found in[67], [66]. Other examples occur for the moduli space of cubic threefolds in [90], and most recently the moduli space of cubic fourfolds in [91]. In the log case examples are provided in [78], [88], and [89].

1.2.3 Towards the construction of a purely algebraic K-moduli space

We have seen in the introduction to this section that when trying to construct a general moduli space M_d for Fano varieties one of the tecnical issues was the separateness of the space. Therefore, this leads to restrict the class of Fano manifolds to class of the $K-$(poly)stable one. We aim to construct at least a coarse moduli space, to do so one involves more technicalities, namely the notion of *Artin stack.*

Definition 1.2.7. *We say that $\mathcal{M}$ is an* Artin stack *of $\mathbb{Q}$-Gorenstein Fano varieties when the following hold:*

1. *A categorical moduli space M is defined;*
2. *There exist a etale $\{[U_i/G_i]\}$ cover such that U_i are algebraic scheme of finite type and G_i reductive group, where there is a G_i-equivariant family of $\mathbb{Q}$-Fano;*
3. *The closed orbits of the action $G_i \circlearrowleft U_i$ correspond to geometric points of M that parametrizes $\mathbb{Q}$-Fano varieties that are $\mathbb{Q}$-Gorenstein.*

We have the following moduli problem functor from the category of Schemes **Sch** to the category of Sets **Set**:

$$\mathbf{Sch} \ni S \longmapsto M_{n,V}^{K-ss}(S) \in \mathbf{Set}$$

Where $M_{n,V}^{K-ss}(S)$ is defined as the set of families whose geometric fibers are $\mathbb{Q}$-Fano varieties of dimension n and volume V. The following result, is a local statement, and shows that $M_{n,V}^{K-ss}$ is representable by an *Artin stack.*

Theorem 1.2.1. *The set of all $\mathbb{Q}$-Gorenstein K-semistable (K-ss) Fano varieties of fixed dimension n, and of fixed volume V form a good Artin stack $\mathcal{M}_{n,V}^{K-ss}$ of finite type in the sense of [26]*

Theorem 1.2.2. *There exists a separated* good moduli space $\mathcal{M}_{n,V}^{K-ps}$

In theorem 1.2.2 the word *good* means that $\overline{\mathcal{M}}_{n,V}^{K-ss}$ is uniquely determined by the functor

$$\overline{\mathcal{M}}_{n,V}^{K-ss} \longrightarrow \mathcal{M}_{n,V}^{K-ps}$$

The separateness in theorem 1.2.2 was proven by Blum and Xu [84] and the existence of good moduli with the assumption that the automorphism group is reductive was proven by Alper, Blum, Haiper and Xu [85]. All is left to prove in the construction of the coarse moduli space is the following

Conjecture 1.2.1. *The good moduli space $\mathcal{M}_{n,V}^{K-ps}$ is proper and projective.*

Note that the smoothable case was proven analytically trough the machinery of the partial C^0-estimates developed by Chen, Donaldson, ad Sun [51], [52], [53].
A recent step, towards a reformulation of the above conjecture was proven by Codogni, Patakfalvi and by Xu-Zhang independently

Theorem 1.2.3. *([87, Theorem 1.1], [76, Theorem 1.2]) The CM line bundle is ample over a proper and closed subspace $M \subset \mathcal{M}_{n,V}^{K-ps}$ that intersects the uniformly $K-$stable locus, and is a quasi projective scheme.*

A big role in the last part of the proof of the above stated result is the Nakai-Moishezon criterion [9] and a result of Birkar [27]. The above stated conjecture could be reformulated by proving that all $\mathbb{Q}$-Fano admits such compact set Msuch that $M = \mathcal{M}_{n,V}^{K-ps}$.

Chapter 2

The CM line bundle

The CM line bundle, where CM stands for *Chow-Mumford* is a functorial line bundle defined on the base of a family of polarized varieties, and it is realised so that its weight is given by some multiple of the *Donaldson-Futaki invariant.* It was firstly introduced by Tian in [43]. In the following, we give the definition and we describe the main properties of the CM line bundles, that will be used during this work
In the following we fix $f : \mathcal{X} \to B$ to be a proper and flat morphism of schemes of finite type over $\mathbb{C}$. Let $\mathcal{L}$ be a relatively ample line bundle on $\mathcal{X}$. Assume that the fibers of f, $(\mathcal{X}_b, \mathcal{L}_b)$ have constant relative dimension n, and that $\mathcal{X}$ and B are normal and projective. The CM line bundle can be defined in terms of some *functorial* line bundles on B. These latter are defined in the following well known result.

Theorem 2.0.1 (Mumford-Knudsen [21])**.** *Let $f : \mathcal{X} \to B$, and $\mathcal{L}$ be as above. Then, there exist line bundles $\lambda_i = \lambda_i(\mathcal{X}, B)$ on B such that, if $k >> 0$ we have*

$$\det(f_!(\mathcal{L}^k)) \simeq \lambda_{n+1}^{\binom{k}{n+1}} \otimes \lambda_n^{\binom{k}{n}} \otimes ... \otimes \lambda_0,$$

Where $f_!$ denotes the Grothendieck's pull down map [4, Appendix I, 23.3].

Since f is flat, then at every point of the base B the Hilbert polynomial is fixed along the fibers of f, namely

$$p(k) = \chi(\mathcal{L}_b^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}), \quad \forall b \in B. \tag{2.1}$$

With the above data, we can calculate explicitly the coefficients of 2.1 using the *Hirzebruch-Riemann-Roch theorem* [4, Chapter 4,]. By characterization of ampleness studied by Cartan and Serre the sheaf cohomology groups $H^i(\mathcal{X}, \mathcal{L})$ vanishes for all $i > 0$, [9, Book I, Chapter 1, sec. 1.2] therefore the Euler characteristics of the family equals the Hilbert polynomial. Hence, by Hirzebruch-Riemann-Roch the coefficients are given by

$$a_0 = \frac{1}{n!}\int_{\mathcal{X}_b} c_1(\mathcal{L}_b)^n := \frac{1}{n!}\mathcal{L}_b^n \tag{2.2}$$

$$a_1 = \frac{1}{2(n-1)!}\int_{\mathcal{X}_b} c_1(\mathcal{L}_b)^{n-1}\cdot \mathrm{td}(\mathcal{X}_b) := \frac{-K_{\mathcal{X}_b}\cdot \mathcal{L}_b^{n-1}}{2(n-1)!} \tag{2.3}$$

Again, since $\mathcal{L}$ is relatively ample, then $\forall b \in B,\ \mathcal{L}_b$ is ample. Hence, by the *Nakai-Moishezon criterion* [5, Theorem 1.37], [9, Book I, Chapter 1, section 1.2], the coefficient a_0 is strictly positive. Hence, the following makes sense

$$\mu = \mu(\mathcal{X}, \mathcal{L}) = \frac{2a_1}{a_0}. \tag{2.4}$$

Definition 2.0.1. *(*Paul-Tian*, [47]) The CM line bundle associated to the family $(\mathcal{X}, \mathcal{L})$ is defined as*

$$\lambda_{CM} = \lambda_{CM}(\mathcal{X}, \mathcal{L}) = \lambda_{n+1}^{\mu+n(n+1)} \otimes \lambda_n^{-2(n+1)}. \tag{2.5}$$

2.0.1 Properties of the CM line bundle

As a first property, we calculate the degree of the CM line bundle, i.e. its first Chern class.

Proposition 2.0.1. *Let $f : \mathcal{X} \to B$ be a proper and flat family of normal projective schemes of relative dimension n, and let $\mathcal{L}$ be a f-ample line bundle. Suppose that the relative canonical bundle $K_{\mathcal{X}/B}$ of f makes sense. Then, the first Chern class of the CM line bundle on the base B is given by*

$$c_1(\lambda_{CM}) = f_*(\mu c_1(\mathcal{L})^{n+1} + (n+1)c_1(\mathcal{L})^n \cdot c_1(K_{\mathcal{X}/B}))) \tag{2.6}$$

Before proving the above result, we give an explicit example of such families

Example 2.0.1. *Fix two cubic surfaces c_1, and c_2 in $\mathbb{P}^3$. Consider the following family*

$$sc_1 + tc_2 = 0, \quad (s:t) \in \mathbb{P}^1$$

Assume that the above is a Lefschetz pencil, that is a pencil of cubics whose singularities are only of A_1-type, also known as nodal singularities. The base locus $B = c_1 \cap c_2$ is smooth and of codimension two. Consider the blow up of $\mathbb{P}^3$ in the base locus B, and denote it by $\mathcal{X} = \mathrm{Bl}_B(\mathbb{P}^3)$. From the blow up, we get fibration over $\mathbb{P}^1$, $\gamma\colon \mathcal{X} \to \mathbb{P}^1$.The latter is a proper and flat family of Fano varieties. As a relatively ample for such a family we can directly chose $\mathcal{L} = -K_{\mathcal{X}/\mathbb{P}^1}$. Which is given by

$$-K_{\mathcal{X}/\mathbb{P}^1} = -K_{\mathbb{P}^3} - E + \gamma^* K_{\mathbb{P}^{\nVdash}} = 2H - E.$$

Where E is the exceptional divisor and H is the hyperplane section.

Proof (of proposition 2.0.1). By the properties of the first Chern class we find

$$c_1(\lambda_{CM}) = (\mu + n(n+1))c_1(\lambda_{n+1}) - 2(n+1)c_1(\lambda_n). \tag{2.7}$$

By 2.0.1 and by definition of determinant bundle,

$$c_1(det(f_!\mathcal{L}^k)) = c_1(f_!\mathcal{L}^k) = \sum_{i=0}^{n+1} \binom{k}{n+1-i} c_1(\lambda_{n+1-i}).$$

Observe that the binomial coefficients can be expressed as

$$\binom{k}{n+1} = \frac{k^{n+1}}{(n+1)!} - \frac{n(n+1)}{2(n+1)!}k^n + O(k^{n-1}),$$

$$\binom{k}{n} = \frac{k^n}{n!} + O(k^{n-1}).$$

Therefore,

$$c_1(f_!\mathcal{L}^k) = \frac{k^{n+1}}{(n+1)!}c_1(\lambda_{n+1}) + \frac{k^n}{2n!}\left(2c_1(\lambda_n) - nc_1(\lambda_{n+1})\right).$$

By the Grothendieck-Riemann-Roch theorem, we have

$$f_!\mathcal{L}^k = f_*(ch(\mathcal{L}^k) \cdot td(K^*_{\mathcal{X}/B})).$$

Then taking the first Chern class of the LHS means taking the degree one terms in the RHS, moreover since $\mathcal{L}^k$ is a line bundle, then we have

$$c_1(f_!\mathcal{L}^k) = \left[\sum_{i=0}^{\infty} \frac{k^i}{i!} f_*(c_1(\mathcal{L}^i) \cdot td(K^*_{\mathcal{X}/B}))\right]_{(2)}.$$

Since

$$td(K_{\mathcal{X}/B}) = 1 + \frac{1}{2}c_1(K_{\mathcal{X}/B}),$$

then we find

$$c_1(f_!\mathcal{L}^k) = \frac{k^{n+1}}{n!} f_* c_1(\mathcal{L})^{(n+1)} - \frac{k^n}{2n!} f_*(c_1(\mathcal{L}^n) \cdot c_1(K_{\mathcal{X}/B})).$$

Hence,

$$c_1(\lambda_{n+1}) = f_* c_1(\mathcal{L}^{n+1}),$$

$$nc_1(\lambda_{n+1}) - 2c_1(\lambda_n)) = f_*(c_1(\mathcal{L}^n) \cdot c_1(K_{\mathcal{X}/B})).$$

Introducing these latter in the equation 2.6 the statement follows. □

As mentioned in the beginning of this chapter, the CM line bundle is realised so that its weight is the Donaldson-Futaki invariant. In order to see this we prove the following statement:

Proposition 2.0.2. *Let $f : \mathcal{X} \to B$ be a proper and flat family of normal schemes of relative dimension n, and let $\mathcal{L}$ be a $f-$ample line bundle. Suppose that $\mathbb{G}_m$ acts on $(\mathcal{X}, \mathcal{L})$ covering an action on B and $b \in B$ is a fixed point of this action. The weight function of the in induced action on $H^0(\mathcal{X}_b, \mathcal{L}_b^k)$ is a polynomial in k, namely*

$$w(k) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}).$$

Then the weight of the induced action on $\lambda_{CM}|_b$ is proportional to the Donaldson-Futaki invariant

$$w(\lambda_{CM}) = 2(n+1)! DF(\mathcal{X}, \mathcal{L}).$$

Proof By addictivity of weights we find

$$w(\lambda_{CM}) = (\mu + n(n+1))w(\lambda_{n+1}) - 2(n+1)w(\lambda_n).$$

And,

$$w(f_! \mathcal{L}^k) = \sum_{i=0}^{n+1} \binom{k}{n+1-i} w_k(\lambda_{n+1-i})$$

Using the binomial coefficient expansion, along the line of the proof of previous proposition, we immediately find

$$w(\lambda_{n+1}) = (n+1)! b_0, \quad w(\lambda_n) = 2(n+1)! \left(\frac{n}{2} b_0 + \frac{b_0}{n+1} \right).$$

Therefore,

$$w(\lambda_{CM}) = (n+1)! \left(\mu b_0 - 2b_1 \right) = (n+1)! \left(\frac{2a_1}{a_0} b_0 - 2b_1 \right)$$

the claim follows by the classical definition of DF invariant [46]. □

It is possible to realize the CM line bundle as a $\mathbb{Q}$-line bundle when $\mathcal{L}$ is the line bundle associated to some ample $\mathbb{Q}$-divisor. Indeed the CM line bundle is homogeneous of degree n, as the following shows:

Proposition 2.0.3. *If* $r \in \mathbb{Z}_+$ *then*

$$\lambda_{CM}(\mathcal{X}, \mathcal{L}^r) = r^n \lambda_{CM}(\mathcal{X}, \mathcal{L})$$

Proof Directly by 2.2, 2.3 and 2.4 we immediately get that

$$\mu(\mathcal{X}, \mathcal{L}^r) = \frac{1}{r}\frac{2a_1}{a_0}.$$

By the properties of the first Chern class and 2.6 the claim follows. □

Given a proper and flat family of relative dimension n of a polarized and normal variety (X, L) on the projective line $f : \mathcal{X} \to \mathbb{P}^1$ endowed with a relative ample line bundle $\mathcal{L}$ and a linearized equivariant $\mathbb{G}_m-$action, we have seen in 2.0.2 that the weight of the CM line bundle on the central fiber is proportional to the Donaldson Futaki invatiant. Along the line of 2.0.2 the following result shows an intersection theoretic formula for the DF invariant, thus a new definition of this latter.

Proposition 2.0.4. *([71], [70]). In the above setting we have that the DF invariant is given by*

$$DF(\mathcal{X}/\mathbb{P}^1, \mathcal{L}) = \frac{1}{2(n+1)\cdot \mathcal{L}_t^n}\left(n\mu(\mathcal{X}_t, \mathcal{L}_t)\mathcal{L}^{n+1} + (n+1)\mathcal{L}^n \cdot K_{\mathcal{X}/\mathbb{P}^1}\right) \quad (2.8)$$

Where $\mu(\mathcal{X}_t, \mathcal{L}_t) = \frac{\mathcal{L}_t^n\cdot(-K_{\mathcal{X}_t})}{\mathcal{L}_t^n}$

Proof By proposition 2.0.2

$$w(\lambda_{CM}) = 2(n+1)!\left(\frac{a_1}{a_0}b_0 - b_1\right)$$

Claim. $b_0 = \frac{\mathcal{L}^{n+1}}{(n+1)!}$, $b_1 = \frac{\mathcal{L}^n\cdot(-K_{\mathcal{X}})}{2n!} - a_0$.
For convenience, we will be denoting by (X, L) the fibers $(\mathcal{X}_t, \mathcal{L}_t)$. Assuming the claim true the above becomes

$$w(\lambda_{CM}) = \frac{1}{a_0}\left(\frac{a_1}{a_0}\mathcal{L}^{n+1} + \frac{(n+1)}{2}\mathcal{L}^n \cdot K_{\mathcal{X}}\right) + 1$$

Where we divided by a_0 since it is nonzero as previously shown. Observe that the coefficients a_i have the same expression on every fiber of the family and are computed via the Hirzebruch-Riemann-Roch theorem. In particular it is immediate to show that

$$\frac{a_1}{a_0} = n\mu(X, L).$$

Hence, by definition of relative canonical bundle the above becomes

$$w(\lambda_{CM}) = \frac{1}{a_0}\left(\frac{a_1}{a_0}\mathcal{L}^{n+1} + \frac{(n+1)}{2}\mathcal{L}^n \cdot K_{\mathcal{X}/\mathbb{P}^1}\right).$$

Using proposition 2.0.2 we see that

$$DF(\mathcal{X}, \mathcal{L}) = \frac{w(\lambda_{CM})}{2(n+1)!}.$$

And, the assertion of the proposition follows. We shall now prove the claim. By applying [5, Proposition 1.48] we can choose a large integer N such that the result of twisting the line bundle $\mathcal{L}$ by the pullback of another ample line bundle on $\mathbb{P}^1$ is still ample. Namely, the line bundle

$$\mathcal{E} = \mathcal{L} + N \cdot f^*\mathcal{O}_{\mathbb{P}^1}(\infty)$$

Is still f-ample (i.e. relatively ample). Let s_0 and s_∞ be sections of $\mathcal{E}$ that are the pullbacks of the divisors $\{0\}$, $\{\infty\}$, respectively. Assume that the $\mathbb{G}_m-$weight on those is -1 and 0 respectively. We have the following short exact sequences

$$0 \to \mathcal{E}(-\mathcal{X}_0) \to \mathcal{E} \to \mathcal{E}_{|\mathcal{X}_0} \to 0,$$
$$0 \to \mathcal{E}(-\mathcal{X}_\infty) \to \mathcal{E} \to \mathcal{E}_{|\mathcal{X}_\infty} \to 0.$$

Were the first maps are obtained by tensoring with the sections s_0 and s_∞ respectively. The exactness is preserved if we take tensor powers of the same lines. Moreover by taking a big enough positive tensor power k of $\mathcal{E}$, by the Cartan-Serre-Grothendieck *Vanishing theorem* [9, Chapter 1, Theorem 1.2.6], the exactness of the above will be preserved in cohomology. Namely,

$$0 \to H^0(\mathcal{X}, \mathcal{E}^k(-\mathcal{X}_0)) \to H^0(\mathcal{X}, \mathcal{E}^k) \to H^0(\mathcal{X}_0, \mathcal{E}^k_{\mathcal{X}_{\prime}}) \to 0$$
$$0 \to H^0(\mathcal{X}, \mathcal{E}^k(-\mathcal{X}_\infty)) \to H^0(\mathcal{X}, \mathcal{E}^k) \to H^0(\mathcal{X}_\infty, \mathcal{E}^k_{\mathcal{X}_\infty}) \to 0$$

Call $A = H^0(\mathcal{X}, \mathcal{E}^k(-\mathcal{X}_0)) = H^0(\mathcal{X}, \mathcal{E}^k(-\mathcal{X}_\infty))$, $B = H^0(\mathcal{X}, \mathcal{E}^k)$, $C = H^0(\mathcal{X}_0, \mathcal{E}^k_{\mathcal{X}_{\prime}})$ and $D = H^0(\mathcal{X}_\infty, \mathcal{E}^k_{\mathcal{X}_\infty})$. We call d_A the dimension of the vector space A and w_A the total weight of the $\mathbb{G}_m$-action on A, similarly we call d_B, d_C, d_D and w_B, w_C, w_D. From exactness we have the following equalities

$$w_B = w_A - d_A + w_C = w_A + w_D$$
$$d_B = d_A + d_C = d_A + d_B$$

Since on the $\mathcal{O}_{\mathbb{P}^1}(1)|_\infty$ the weight is -1 and the $\mathbb{G}_m-$action is trivial on $\mathcal{L}|_{\mathcal{X}_\infty}$ then, unraveling the definition of $\mathcal{E}$ we find

$$w_D = -kNdimH^0(\mathcal{X}_\infty, \mathcal{L}^k|_{\mathcal{X}_\infty})$$

Therefore,

$$w_C = d_B - (kN+1)d_C.$$

The $\mathbb{G}_m-$action is trivial on $\mathcal{O}_{\mathbb{P}^1}(1)|_0$ then, unraveling the definition of $\mathcal{E}$ we find that the total weight on $H^0(\mathcal{X}_0, \mathcal{E}^k_{\mathcal{X},})$ only depends on $\mathcal{L}$, namely

$$w(k) = dimH^0(\mathcal{X}, \mathcal{E}^k) - (kN+1)dimH^0(\mathcal{X}_0, \mathcal{L}^k|_{\mathcal{X}_0}).$$

At this point the claim is straightforward and it can be obtained unravelling the right hand side of the above equality via the Hirzebruch-Riemann-Roch theorem. □

In this work we are interested in the CM line bundle on the K-moduli space of Fano varieties. Therefore, it is reasonable to tailor the degree of the CM line bundle directly for this case.

Lemma 2.0.1. *Let X be a Fano variety of dimension n. Then $\mu = n$.*

Proof The Hilbert polynomial with respect to the canonical polarization $-K_X^m$ is by definition

$$p(m) = \chi(-K_X^m) = a_0 m^n + a_1 m^{n-1} + O(m^{n-2}).$$

Since X is a Fano variety, then $-K_X^m$ is ample. As before, by the *Nakai-Moishezon* criterion $a_0 > 0$, therefore 2.4, $\mu = \frac{2a_1}{a_0}$,makes sense. Recall that the Hilbert polynomial is the Euler characteristic of X with respect to the polarization $-K_X^m$. And by the Hirzebruch-Riemann-Roch theorem we have

$$\chi(-K_X^m) = \int_X ch(-K_X^m) \cdot Td(X) = a_0 m^n + a_1 m^{n-1} + O(m^{n-2})$$

Since $-K_X^m$ is a line bundle, we have

$$ch(-K_X^m) = \exp(ma), \text{ where } a = c_1(-K_X).$$

Since $\exp(ma) = \sum_{i=0}^n \frac{m^i a^i}{n!}$, we have

$$\chi(-K_X^m) = \sum_{i=0}^n \frac{m^i}{n!} \int_X [a^i]_{2i} \cdot [Td(X)]_{2(n-i)}.$$

Therefore, the coefficients a_0 and a_1 of the Hilbert polynomial are given by:

$$a_0 = \frac{1}{n!} \int_X a^n \cdot Td_0(X), \quad a_1 = \frac{1}{(n-1)!} \int_X a^{n-1} \cdot Td_2(X)$$

Moreover, we find that $Td_0(X) = 1$, and $Td_2(X) = \frac{1}{2}c_1(X)$.
Hence

$$a_0 = \frac{1}{n!}\int_X a^n \cdot 1 = \frac{1}{n!}\int_X c_1(X)^n,$$

$$a_1 = \frac{1}{(n-1)!}\int_X a^{n-1} \cdot \frac{1}{2}c_1(X) = \frac{1}{2(n-1)!}\int_X c_1(X)^n.$$

Introducing the above coefficients into 2.4 we obtain the claim. □

The following result shows how the degree of the CM line bundle is defined on families of Fano varieties in K-moduli spaces.

Corollary 2.0.1. *Let $\gamma : \mathcal{X} \to B$ be a proper and flat family of Fano varieties of relative dimension n. Assume that $-K_{\mathcal{X}/B}$ is well defined as γ-ample line bundle. Then*

$$c_1(\lambda_{CM}) = -\gamma_* c_1(-K_{\mathcal{X}/B})^{n+1} \tag{2.9}$$

Proof The claim follows immediately form 2.0.1 and introducing $\mathcal{L} = -K_{\mathcal{X}/B}$ into 2.6. □

2.0.2 The CM line bundle and the Weil-Petersson metric

Another picture of the CM line bundle for Fano families arises from a metric point of view, namely the degree of the CM line bundle for these families can be seen as the Chern curvature of some *Quillen* metric, and in particular equals the *Weil-Petersson* metric as a positive current. In this section we revisit some aspect of the work of [76] in order to illustrate the mentioned picture.

The Weil Petersson metric as the first Chern class of the CM line bundle

Let $f : \mathcal{X} \to B$ be a proper and flat family of K-polystastable Fano varieties over a complex space B, suppose that the fibers are of constant relative dimension n. By [46] we know that each fiber $\mathcal{X}_b$ posses a Kaehler-Einstein (KE) metric ω_b, for which there exist a *potential* φ, that solves the complex Monge Ampere equation

$$\omega_b^n = e^{-\varphi} dV_b,$$

where dV_b is some smooth volume form. The family of metrics $\{\omega_b^n\}$ defines an Hermitian metric on $K_{\mathcal{X}/B}$. Let A be a subset of B that parametrizes singularities of B. Denote by B^0 its complement and by $\mathcal{X}^0 \to B^0$ the corresponding smooth family. Denote by $\omega_{\mathcal{X}^0}$ the Chern curvature of $\{\omega_b^n\}$, namely

$$\omega_{\mathcal{X}^0} = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\{\omega_b^n\}.$$

A Theorem of Fujiki and Schumacher [59] states that on the induced smooth family $\mathcal{X}^0 \to B^0$ the *Weil-Petersson* metric can be represented in fiber integral

$$\omega_{WP}^0 = -\int_{\mathcal{X}^0/B^0} \omega_{\mathcal{X}^0}^{n+1}. \tag{2.10}$$

In this discussion, we take 2.10 as the main definition of the Weil-Petersson metric. We shall mention, for completeness, the *classical* Weil-Petersson metric in the following

Remark 2.0.1. *Let $f : \mathcal{X} \to B$ be as above. Starting with the induced smooth family $\mathcal{X}^0 \to B^0$, for all elements $t \in B^0$ we consider the $\mathcal{T}\mathcal{X}_t-$valued differential forms*

$$\Omega^{(0,k)} = \Omega^{(0,k)}(\mathcal{X}_t, \mathcal{T}\mathcal{X}_t).$$

Then for every couple of elements $\sigma_1, \sigma_2 \in \Omega^{(0,k)}$ there is a L^2-metric defined as follows

$$(\sigma_1, \sigma_2)_{L^2} := \int_{\mathcal{X}|B} < \sigma_1, \sigma_2 >_{\omega_t} \omega_t^n.$$

Where ω_t is a KE metric on $\mathcal{X}_t$. The scalar product $< -, - >_{\omega_t}$ is a Hermitian metric on $\mathcal{T}\mathcal{X}_t \otimes \mathcal{T}^{*(0,k)}\mathcal{X}_t$. Since we have a L^2 metric on each fiber and these latter are compact and Kaehler then for each class $[\sigma] \in H^1(\mathcal{X}_t, \mathcal{T}\mathcal{X}_t)$ there exists a unique harmonic representative σ_H. The chosen family can be interpreted as a complex deformation therefore we have a short exact sequence of locally free sheaves

$$0 \to \mathcal{T}\mathcal{X} \to \mathcal{T}\mathcal{X}_{|B} \to \mathcal{T}\mathcal{X} \otimes \mathcal{O}_{\mathcal{X}} \to 0$$

The boundary map, from the long exact sequence in sheaf cohomology, gives the Kodaira-Spencer map for all values $t \in B^0$

$$\kappa : T_t B \to H^1(\mathcal{X}_t, \mathcal{T}\mathcal{X}_t),$$

then we can define the Weil Petersson metric *for all elements $v, v' \in T_t B$ to be*

$$\omega_{WP}^0(v, v') := (\kappa_t(v), \kappa_t(v'))_{L_t^2}.$$

An infinitesimal direction $v \in T_t B$ is called effective if $\kappa_t(v) \neq 0$, therefore ω_{WP}^0 is positive-definite along effective directions. It was indeed proven in [59] that this definition of the Weil-Petersson metric coincides with the one given in Equation 2.10.

The relation with the CM line bundle will be extrapolated from the following

Theorem 2.0.2. *([59], Section 10). There is a Quillen metric h_{QM} on $\lambda_{CM}|_{B^0}$ such that*

$$-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h_{\mathrm{QM}} = \omega^0_{WP}$$

Quillen's metrics are in general difficult to calculate. In [17] Deligne proposed a machinery that allows to calculate such metrics, namely the so called *Deligne's Pairing* or *Deligne's Metric*. In the next result we collect two results that will be useful for later discussions.

Theorem 2.0.3. *Let $g : \mathcal{X} \to B$ be a flat projective morphism of integral schemes, of pure dimension n. Let $(\mathcal{L}_0, h_0), ..., (\mathcal{L}_n, h_n)$ be a $(n+1)-$tuple of hermitian line bundles on $\mathcal{L}$, denote the Deligne pairing associated to that tuples as the symbol $< \mathcal{L}_0, ..., \mathcal{L}_n >$. We have*

1. *([62]) Suppose h_i are smooth metrics $\forall i \in \{0, 1, ..., n\}$. Then the Deligne's metric h_{DP} is continuous on $< \mathcal{L}_0, ..., \mathcal{L}_n >$.*
2. *([17]) The following curvature formula holds for Deligne's metric*

$$c_1(< \mathcal{L}_0, ..., \mathcal{L}_n >) = \int_{\mathcal{X}/B} c_1(\mathcal{L}_0) \wedge ... \wedge c_1(\mathcal{L}_n).$$

Henceforth, we will denote the Deligne's Pairing of a line bundle $\mathcal{L}$ as

$$\mathcal{L}^{<n+1>} = \underbrace{< \mathcal{L}, ..., \mathcal{L} >}_{(n+1)-\text{times}}.$$

Theorem 2.0.4. *The CM line bundle can be interpreted with the Deligne's Pairing as*

$$\lambda_{\mathrm{CM},\mathcal{D}} = -\left(-K_{\mathcal{X}/\mathcal{B}}\right)^{<n+1>}$$

Proof Observe that 2.9 can be rephrased in the following way

$$c_1(\lambda_{\mathrm{CM},\mathcal{D}}) = -\int_{\mathcal{X}|B} \underbrace{c_1(-K_{\mathcal{X}|B}) \wedge ... \wedge c_1(-K_{\mathcal{X}|B})}_{(n+1)-\text{times}}.$$

By the second assertion of 2.0.3, we have

$$c_1(\lambda_{\mathrm{CM},\mathcal{D}}) = c_1\left((-K_{\mathcal{X}/B})^{<n+1>}\right)$$

Since tensoring with $\mathbb{Q}$ is an exact functor over $\mathbb{Z}$, the injectivity of the first Chern class is preserved, therefore the claim follows. □

The next result shows the mentioned picture at the beginning of this section.

Theorem 2.0.5. *[76] Let $f : \mathcal{X} \to B$ be a proper, flat and projective family of Fano varieties. Then, there exists a continuous Deligne's pairing metric h_{DP} on λ_{CM} such that*

1. $\omega_{WP} = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h_{DP}$ *is a positive (1,1)-current.*
2. $\omega_{WP|B^0} = \omega^0_{WP}$

It is proven that the CM line bundle descends to a line bundle on the proper moduli space $\bar{\mathcal{M}}$. Moreover, it is not hard to prove, using Theorem 2.0.5 of [76], that there is a canonically defined continuous Hermitian metric h_{DP} on the descending λ_{CM} whose curvature form is a positive current ω_{WP} on $\bar{\mathcal{M}}$ which extends the canonical Weil-Petersson current ω^0_{WP} on $\mathcal{M}$.

Chapter 3

The CM volume of del Pezzo Quartic Moduli space

A brief description of the intersection of two quadrics

Let $X = Q_1 \cap Q_2$ be the intersection of two quadrics Q_1 and Q_2, in $\mathbb{P}(V)$, where V is a complex vector space of dimension $n+1$. Consider a pencil of the quadrics Q_1 and Q_2, that is $\Phi = Span\{\phi_\lambda\}_{\lambda \in \mathbb{P}^1}$.Where every ϕ_λ denote the bilinear form associated to Q_λ.A pencil of quadrics Φ is said to be smooth, or non singular, if Q_λ is non singular or have at most simple cones as singularities $\forall \lambda \in \mathbb{P}^1$, [14]. For the λ's for which Q_λ is degenerate, if e_λ is a basis for $ker(\phi_\lambda)$, then $\phi_\mu(e_\lambda) \neq 0$ for all $\lambda \neq \mu$. If $e \in V$, we define the orthogonal complement of this latter with respect to Φ as follows

$$e^\perp = \cap_{\lambda \in \mathbb{P}^1} e^{\perp_{\phi_\lambda}} = e^{\perp_{\phi_1}} \cap e^{\perp_{\phi_2}}$$

Note that, for most of e, the above will be of codimension two. If the codimension of $e^\perp$ is less than two, then $e \in ker(\phi_\lambda)$ for some $\lambda \in \mathbb{P}^1$. Hence X is nonsingular and of codimension two in $\mathbb{P}(V)$ if and only if for all $x \in X$, $e \in V$ representing x, then e is not in the kernel of ϕ_λ for any $\lambda \in \mathbb{P}^1$. Smoothness of X can be characterized trough smoothness of Φ as the following proposition shows

Proposition 3.0.1. *([14]) Let X be as above, then the following conditions are equivalent*

1. *X is smooth and of codimension 2,*
2. *Φ is non singular,*
3. *The discriminant $det(\phi_\lambda) \neq 0$ is a polynomial in λ of degree the dimension of V and has $n+1$ distinct roots,*

4. there exists a unique basis of V $\{e_i\}_{i=1}^{dimV}$ such that

$$\begin{cases} \phi_1(\sum_i x_i e_i) = \sum_i x_i^2 \\ \phi_2(\sum_i x_i e_i) = \sum_i \lambda_i x_i^2, \ \lambda_i \neq \lambda_j, \ i \neq j \end{cases} \tag{3.1}$$

3.1

The moduli space of the intersection of Quadrics
Denote by M_Φ the moduli space of pencils of quadrics. This latter moduli space arises as a GIT quotient as shown by Avritzer and Lange [22] where the stability conditions are described in the following

Theorem 3.1.1. *([22])*

a. A pencil $\Phi \in Gr(2, n+1)$ is semistable if and only if its discriminant admits no roots of multiplicity $> n/2$.

b. A pencil $\Phi \in Gr(2, n+1)$ is stable if and only if its discriminant admits no multiple roots.

In the previous section we saw that if the discriminant admits no multiple root, then the pencil is smooth, smoothness of pencils are equivalent to the smoothness of the intersection of two quadrics, hence the above theorem says that all smooth intersections of two quadrics are smooth. With this in mind, we can easily see that stable intersection of quadrics are given by all simultaneously diagonalizable quadrics, therefore we can describe the moduli space of quartic del Pezzo's (intersection of two quadrics) by the roots $\lambda \in \mathbb{P}^1$. Hence, the desired GIT quotients is given by

$$M_{dP^4} = S^m\mathbb{P}^1 //_{\mathcal{L}} \mathrm{SL}(2).$$

Where, the linearization $\mathcal{L}$ is given by $\mathcal{O}(1)$. M_{dP^4} is a genuine K-moduli space, indeed the following result, which is based on the picture for K-stability described in Chapter one by Equation 1.3.

Theorem 3.1.2. *([67], Theorem 1.1) For integer $d \in \{1, 2, 3, 4\}$, there is a compact moduli algebraic space M_d, and a homeomorphism $\Phi : M_d^{GH} \to M_d$, such that $[X]$ and $\Phi([X])$ parametrize isomorphic log Del Pezzo surfaces for any $[X] \in M_d^{GH}$. Moreover, M_d contains a (Zariski) open dense subset that parametrizes all smooth degree d Del Pezzo surfaces.*

The case of degree $d = 4$ is our case, i.e. the case of quartic del Pezzo. To see that, in this case GIT stability is the same as K-stability. We should apply the following result

Theorem 3.1.3. *([66], Theorem 3.4) Let G be a reductive algebraic group without nontrivial characters. Let B be a projective scheme and $\pi : \mathcal{X} \to B$ be a equidimensional, flat and proper family of varieties. Let $\mathcal{L} \to \mathcal{X}$ be a relatively ample line bundle for π. Assume*

1. *The Picard rank of the base B is one, i.e. $\rho(B) = 1$.*
2. *There exists a test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ that degenerates in B, i.e. the test configuration is not of product type.*

Then $b \in B$ is GIT-(poly,semi)-stable if $(\mathcal{X}_b, \mathcal{L}_b)$ is K-(poly,semi)-stable.

In order to verify condition 2 of the above theorem, in the proof of Theorem 3.1.2 in the case of $d = 4$ in [66], it is proven that the chosen degenerations of Del Pezzo's does not give any almost trivial test configuration, in the sense of [49], [63] . The reason of that is rather technical: the central fibers of such pathological test configurations are not equidimensional, so they are especially not Cohen–Macaulay. However, the degenerations that are considered in [66] are all Cohen–Macaulay. This is because, in general, weighted projective spaces only have quotient singularities, and so are Cohen–Macaulay. Then the finite times cut by Cartier divisors (hypersurfaces) are inductively Cohen–Macaulay. The other conditions are true by construction at every point. Then M_{dP^4} is a K-moduli space.

The intersection number of the CM line bundle with a smooth curve in the moduli of quartic del Pezzo's

Let $\gamma \colon \mathcal{X} \to C$ be a flat and proper family of intersection of quadrics over a smooth curve C of genus g, assume that $-K_{\mathcal{X}/C}$ is γ-ample. A direct application of 2.6, and the Riemann-Roch theorem for curves, shows that the intersection number of the CM line bundle with the above family is given by

$$c_1(\lambda_{CM}) \cdot (\mathcal{X} \to C) = 8(n+1)(n-1)^n(1-g) - c_1(\mathcal{X})^{n+1} \tag{3.2}$$

The above family can be specified by fixing a quadric Q_0, and considering the following Lefschetz pencil

$$s(Q_0 \cap Q_1) + t(Q_0 \cap Q_2) = 0, \quad (s:t) \in \mathbb{P}^1.$$

This is by construction a smooth family of intersection of quadrics in $\mathbb{P}^{n+2}$ that has only nodal singularities. By definition of Lefschetz pencil [7], the base locus B is smooth and of codimension two in Q_0. The total space of the family γ can be specified by considering the blow up in the base locus of the quadric Q_0, i.e.

$$\mathcal{X} = Bl_B(Q_0).$$

Because of this construction the family γ becomes a fibration over $\mathbb{P}^1$, i.e.

$$\gamma\colon \mathcal{X} \to \mathbb{P}^1.$$

Hence, with respect to the above family, equation 3.2 becomes

$$c_1(\lambda_{CM}) \cdot (\chi \to C) = 8(n+1)(n-1)^n - c_1(\mathcal{X})^{n+1} \tag{3.3}$$

Hence, in order to calculate the above intersection number it sufficies to calculate the $(n+1)-$power of the first Chern class of $\mathcal{X}$. Denoting by p the blow up map, and using the fact that $\mathcal{X}$ is Fano we have

$$c_1(\chi)^{n+1} = -K_{\mathcal{X}}^{n+1} = ((n+1)H - E)^{n+1}.$$

In order to calculate the contributions of the intersection products of the above we need the following

Lemma 3.1.1.

$$c_1(N_{B|Q_0})_{|B} = 4c_1(\mathcal{O}_B(1))$$
$$c_2(N_{B|Q_0})_{|B} = 4c_1(\mathcal{O}_B(1))^2$$

Proof Consider the following chain of inclusions

$$B \hookrightarrow Q_0 \cap Q_1 \hookrightarrow Q_0,$$

from which follows the short exact sequence of normal bundles

$$0 \to N_{B|Q_0 \cap Q_2} \to N_{B|Q_0} \to N_{Q_0 \cap Q_2|Q_0} \to 0.$$

By the functorial properties of the total Chern class we have

$$c(N_{B|Q_0}) = c(N_{B|Q_0 \cap Q_1}) \cdot c(Q_0 \cap Q_2|Q_0).$$

Hence,

$$c_1(N_{B|Q_0}) = c_1(N_{B|Q_0 \cap Q_1}) + c_1(Q_0 \cap Q_2|Q_0).$$

$$c_2(N_{B|Q_0}) = c_1(N_{B|Q_0 \cap Q_1}) \cdot c_1(Q_0 \cap Q_2|Q_0).$$

Observe that B is a divisor in $Q_0 \cap Q_1$, and $Q_0 \cap Q_1$ is a divisor in Q_0, therefore

$$c_1(N_{B|Q_0}) = c_1(\mathcal{O}_{Q_0 \cap Q_1}(1)) + c_1(\mathcal{O}_{Q_0}(1)).$$
$$c_2(N_{B|Q_0}) = c_1(\mathcal{O}_{Q_0 \cap Q_1}(1)) \cdot c_1(\mathcal{O}_{Q_0}(1)).$$

By restricting the above equations to the base locus B the assertion follows immediately. □

By the general theory of blow ups, it follows that

$$c_1(E)^2 = -p^*c_1(N_{B|Q_0}) \cdot c_1(E) - p^*c_2(N_{B|Q_0}). \tag{3.4}$$

Set $x = c_1(E)^2$. By the above Lemma we have $p^*c_1(N_{B|Q_0}) = 4p^*c_1(\mathcal{O}_B(1)) = 4h$, hence

$$x^2 = -4hx - 4h^2.$$

Lemma 3.1.2.

$$x^k = (-1)^{k-1}(k2^{k-1}h^{k-1}x + (k-1)2^k h^k), \quad \forall k \in \mathbb{N}_+$$

Proof Set $x^k = A_k h^{k-1}x + B_k h^k$, where $A_k = (-1)^{k-1}k2^{k-1}$ and $B_k = (-1)^k(k-1)2^k$
For $k = 1$ the above is true. Therefore, we can inductively assume that the assertion holds for x^k, and we prove it for the next step

$$\begin{aligned} x^{k+1} &= x^k x = A_k h^{k-1}x^2 + B_k h^k x \\ &= A_k h^{k-1}(-4hx - 4h^2) + B_k h^k x \\ &= -4A_k h^k x - 4A_k h^{k+1} + B_k h^k x \\ &= (B_k - 4A_k)h^k x - 4A_k h^{k+1}. \end{aligned}$$

Therefore, the assertion holds if and only if

$$A_{k+1} = B_k - 4A_k \qquad B_{k+1} = -4A_k \tag{3.5}$$

Using the above definition of A_k and B_k, equations (5) hold. Hence, the assertion follows. □

With this in mind we consider the binomial expansion

$$\begin{aligned} c_1(\chi)^{n+1} &= -K_\chi^{n+1} = ((n+1)H - E)^{n+1} \\ &= (n+1)^{n+1}H^{n+1} - n(n+1)^n H^n E + \cdots \\ &+ \binom{n+1-i}{i}(n+1)^{n+1}H^{n+1-i}(-E)^i \\ &+ \cdots - E^{n+1}. \end{aligned}$$

Now, we calculate each contribution of the various intersection products. We note immediately that $H^{n+1} = 1$, and as a consequence of the moving lemma we see that $H^n E = 0$. By using the previous two lemmas we can compute the general term

$$\begin{aligned}
H^{n+1-i}E^i &= \int_{H^{n+1-i}E} c_1(E)^{i-1} \\
&= \int_{H^{n+1-i}E} (-1)^{i-2}[(i-1)2^{i-2}h^{i-2}x + (i-2)2^{i-1}h^{i-1}] \\
&= (-1)^{i-2} \int_{H^{n+1-i}E} (i-1)2^{i-2}p^*c_1(\mathcal{O}_B(1))^{i-2} \smile c_1(E) \\
&+ (-1)^{i-2} \int_{H^{n+1-i}E} (i-2)2^{i-1}p^*c_1(\mathcal{O}_B(1))^{i-1} \smile 1 \\
&= (-1)^{i-1}(i-1)2^{i-2} \int_{H^{n+1-i}B} c_1(\mathcal{O}_B(1))^{i-2} \\
&= (-1)^{i-1}(i-1)2^{i-2} H^{n+1-i} \cdot Q_0 \cdot Q_1 \cdot Q_2 \cdot H^{i-2} \\
&= (-1)^{i-1}(i-1)2^{i-2} \cdot 8 = (-1)^{i-1}(i-1)2^{i+1}.
\end{aligned}$$

Hence the desired intersection number is given by

$$\begin{aligned}
c_1(\lambda_{CM}) \cdot (\mathcal{X} \to C) &= 8(n+1)(n-1)^n \\
&+ \sum_{i=1}^{n} (-1)^{i-1} \binom{n+1}{i} (n+1)^{n+1-i}(i-1)2^{i+1} \qquad (3.6)
\end{aligned}$$

3.2 The CM volume of the moduli space of quartic del Pezzo

Theorem 3.2.1. *Let M_{dP^4} be the K-moduli space of quartic del Pezzo's of odd degree. The CM volume of M_{dP^4} is given by*

$$Vol(M_{dP^4}, \lambda_{CM}) = \left(\frac{c}{m}\right)^{m-3} \frac{1}{2^{m-1}} \left(\sum_{k:0<m-2k\leq m} \frac{(m-2k)^{m-3}}{k!(m-k)!} \right). \qquad (3.7)$$

Where c is the intersection number 3.6 and $m = n+2$.

Proof Observe that , since $S^m\mathbb{P}^1$ can be identified with a m-dimensional projective space, then $\mathrm{Pic}(M_{dP^4}) = \mathbb{Z}[h]$. Therefore, $c_1(\lambda_{CM}) = r[h]$, where $r \in \mathbb{Z}$ and h is the tautological class. This latter can be expressed as where $h = c_1(\tau)$, and τ is the tautological bundle of $S^m\mathbb{P}^1$, i.e. $\tau = \mathcal{O}(1)$.
The r coefficient can be easily calculated as follows

$$\begin{aligned} c &= c_1(\lambda_{CM}) \cdot (\mathcal{X} \to \mathbb{P}^1) \\ &= r \int_{(\mathcal{X} \to \mathbb{P}^1)} c_1(\mathcal{O}(1)) = r \cdot m. \end{aligned}$$

Therefore, $r = c/m$. The volume form of M_{dP^4} is given by

$$vol(M_{dP^4}, \lambda_{CM}) = \left(\frac{c}{m}\right)^{m-3} (c_1(\mathcal{O}(1))^{m-3}.$$

Where $m - 3 = dim(M_{dP^4})$. The volume will be given by

$$Vol(M_{dP^4}, \lambda_{CM}) = \int_{M_{dP^4}} vol(M_{dP^4}, \lambda_{CM}).$$

In order to calculate the above integral we will use the symplectic version of M_{dP^4}. We notice that the action of the compactification of SL(2), which is SU(2), is Hamiltonian with moment map μ . By assumption, the dimension of the moduli space is odd, therefore 0 is a regular value of the moment map and the symplectic reduction is a orbifold. In the GIT interpretation this means that the semistable locus is equal to the stable locus. Then, the Kirwan map $k : H^{\bullet}_{SU(2)}(S^m\mathbb{P}^1) \to H^{\bullet}(M_{dP^4})$ is surjective. Therefore, we can apply the nonabelian localization theoreom [19]. The latter, for the case of an Hamiltonian SU(2)-action is as follows:

$$k(\eta)e^{\omega_0}[M] = \frac{n_0}{2}\mathrm{Res}_{X=0}\left(4X^2 \sum_{F \in \mathcal{F}_+} \int_F \frac{i_F^* \eta e^{\bar{\omega}}}{e_F(X)} dX\right),$$

where $\mathcal{F}_+$ denotes the set of positive fixed cycles. Since the $\mathcal{F}_+$ is given by a finite number of points the integral on the right hand side reduces to a sum. If η has degree equal to the real dimension of the quotient, then we can omit the exponential in the right hand side. In our case $\eta = c_1^T(\tau)^{m-3}$, since τ is a *prequantum* line bundle then the pullback of the inclusion of η is a multiple of the moment map evaluated at the fixed point F, see [12, Lemma 9.31]. The action of the maximal torus $T = diag(t, t^{-1})$ on a generic element $p = p(u, v)$ of $S^m\mathbb{P}^1$ is given by

$$T \cdot p(u, v) = p(t^{-1}u, tv) = a_0 t^{-m} u^m + ... + a_k t^{2k-m} u^{m-k} v^k + ... + a_m v^m t^m.$$

The symmetric polynomial space $S^m\mathbb{P}^1$ can be canonically identified with $\mathbb{P}^m$, therefore the above action can be specified as

$$T \cdot (z_0, ..., z_m) = (t^{-m} z_0, ..., t^{2k-m} z_k, ..., t^m z_m)$$

With this prescription, the fixed point set for this latter action is given by the standard basis $\{e_i\}_{i=0}^{m}$ of $\mathbb{C}^{m+1}$. The moment map $\mu_T : \mathbb{P}^m \to i\mathfrak{t}^*$ for this action is given by

$$\mu_T(z_0, ..., z_m) = \sum_{j=1}^{m} (m-2j)\frac{|z_j|^2}{|z|^2}x$$

Where x denotes the basis of $i\mathfrak{t}^*$. The image of the fixed point e_k is given by

$$\mu_T(e_k) = (m-2k)x$$

Since all the fixed points are isolated and τ is a prequantum line bundle we conclude that, at any fixed point e_k the numerator of the meromorphic function into the localization formula is given by

$$i^*_{e_k}\eta = (-1)^{m-3}(m-2k)^{m-3}x^{m-3}$$

Since the maximal torus is identified with the one dimensional unitary group, the Euler class at the fixed point e_k to the normal bundle is simply given by the products of the weight, that is

$$e_{e_k}(X) = \prod_{j\neq k} 2(j-k)x^m = 2^m k!(m-k)!(-1)^{m-3}$$

Thus, the meromorphic form is given by

$$\frac{i^*_{e_k}\eta(X)}{e_{e_k}(X)} = \frac{(m-2k)^{m-3}}{2^m k!(m-k)!x^3}$$

Therefore,

$$\begin{aligned} Vol(M_{dP^4}, \lambda_{CM}) &= \left(\frac{c}{m}\right)^{m-3} Res_{x=0} 4x^2 \left(\sum_{k:0<m-2k\leq m} \frac{(m-2k)^{m-3}x^{m-3}}{2^m k!(m-k)!x^m} \right) \\ &= \left(\frac{c}{m}\right)^{m-3} Res_{x=0} \frac{1}{x} \left(\sum_{k:0<m-2k\leq m} \frac{(m-2k)^{m-3}}{2^{m-2} k!(m-k)!} \right) \end{aligned}$$

Hence, the volume of the moduli space of quartic del Pezzo's of odd dimension, with respect to the CM line bundle is given by

$$Vol(M_{dP^4}, \lambda_{CM}) = \left(\frac{c}{m}\right)^{m-3} \frac{1}{2^{m-1}} \left(\sum_{k:0<m-2k<m} \frac{(m-2l)^{m-3}}{k!(m-k)!} \right). \quad (3.8)$$

□

Remark 3.2.1. *In the even case, notice that the semistable locus does not equal the stable locus, therefore in order to apply the localization theorem we should use the Kirwan partial desingularization technique [16]. The semistable elements are fixed by nontrivial connected reductive subgroups of* $\mathrm{SL}(2)$ *and are those of the form* $(z_1, ..., z_m)$ *for which there exist distinct points* p *and* q *in* $\mathbb{P}^1$ *with exactly half of the points of the* m*-tuple equal to* p *and the rest equal to* q *. The partial desingularization it is achieved by blowing up, along the orbits corresponding to these latter points, the GIT quotient* $S^m\mathbb{P}^1//\mathrm{SL}(2)$*.It turns out that result 3.7 holds as well in the even case. The reason of that is encoded in the nature of the* $\mathrm{SL}(2)$ *action on the symmetric power of* $\mathbb{P}^1$*, namely the action is* weakly balanced *([23] definition 15, examples 2.4 and 2.5). In [23](Theorem 25, Example 26) is shown that under the hypothesis of* weakly balanced actions *then any intersection pairing on the partial desingularization can be localized in the undesingularized GIT quotient.*

Chapter 4

The log CM line bundle

We divide this Chapter in two parts, in the first one we will introduce the notion of CM line bundle for pairs, and in the second one we examine its relations with the log Weil-Petersson metric.

4.0.1 The CM line bundle for pairs

Fix $f : \mathcal{X} \to B$ to be a proper flat morphism of scheme of finite type over $\mathbb{C}$. Let $\mathcal{L}$ be a relatively ample on $\mathcal{X}$ and assume that f has relative dimension $n \geq 1$, namely $\forall b \in B,\ (\mathcal{X}_b, \mathcal{L}_b)$ has constant dimension n.

Definition 4.0.1. *[89] Let $\mathcal{D}_i, \forall i = 1, 2, .., k$ be a closed subscheme of $\mathcal{X}$ such that $f|_{\mathcal{D}_i} : \mathcal{D}_i \to B$ has relarive dimension $n-1$, and $f|_{\mathcal{D}_i}$ is proper and flat. Let $d_i \in [0,1] \cap \mathbb{Q}$ we define the log CM $\mathbb{Q}-$line bundle of the data $(f, \mathcal{D} := \sum_{i=1}^{k} d_i \mathcal{D}_i), \mathcal{L})$ to be*

$$\lambda_{CM,\mathcal{D}} := \lambda_{CM} - \frac{n\mathcal{L}_b^{n-1} \cdot \mathcal{D}_b}{(\mathcal{L}_b^n)} \lambda_{\mathrm{CH}} + (n+1)\lambda_{\mathrm{CH},\mathcal{D}},$$

Where λ_{CM} is the CM line bundle defined in Chapter two and λ_{CH} is the Chow line bundle, defined as the leading order term of the Hilbert Mumford expansion, namely $\lambda_{CH} = \lambda_{n+1}$, and $\lambda_{CH,\mathcal{D}} = \bigotimes_{i=1}^{k} \lambda_{CH}^{d_i}$.

Proposition 4.0.1. *Let $f : (\mathcal{X}, \mathcal{D}) \to B$ be a $\mathbb{Q}-$Gorenstein flat family of n dimensional pairs over a normal proper variety B. Then for any $\mathcal{L}$ relatively $f-$ample line bundle we have*

$$c_1(\lambda_{CM,\mathcal{D}}) = n\frac{(-K_{\mathcal{X}_b} - \mathcal{D}_b) \cdot \mathcal{L}_b^{n-1}}{(\mathcal{L}_b^n)} f_* c_1(\mathcal{L})^{n+1} - (n+1) f_*((-K_{\mathcal{X}/B} - \mathcal{D}) \cdot c_1(\mathcal{L})^n). \tag{4.1}$$

Proof From the definition we get

$$c_1(\lambda_{CM,\mathcal{D}}) = c_1(\lambda_{CM}) - \frac{n\mathcal{L}_b^{n-1} \cdot \mathcal{D}_b}{(\mathcal{L}_b^n)} c_1(\lambda_{\mathrm{CH}}) + (n+1)c_1(\lambda_{\mathrm{CH},\mathcal{D}}). \tag{4.2}$$

Again from the definition

$$c_1(\lambda_{CH,\mathcal{D}}) = \sum_{i=1}^{k} d_i c_1(\lambda_{CH})_{f_{|\mathcal{D}_i}} = \sum_{i=1}^{k} d_i c_1(\lambda_n) \cdot \mathcal{D}_i.$$

Recall from 2.0.1 that

$$c_1(\lambda_{CM}) = f_*(\mu c_1(\mathcal{L})^{n+1} + (n+1)c_1(\mathcal{L})^n \cdot c_1(K_{\mathcal{X}|B})))$$

Where the first Chern class of $\mathcal{L}$ was derived from the Hilbert-Mumford expansion using the Grothendieck-Riemann-Roch theorem, namely

$$c_1(\lambda_{n+1}) = f_* c_1(\mathcal{L}^{n+1})$$
$$nc_1(\lambda_{n+1}) - 2c_1(\lambda_n)) = f_*(c_1(\mathcal{L}^n) \cdot c_1(K_{\mathcal{X}/B})).$$

Recall that $\mu = \frac{2a_1}{a_0}$, and by the Hirzebruch-Riemann-Roch theorem we find

$$a_0 = \frac{1}{n!}\int_{\mathcal{X}_b} c_1(\mathcal{L}_b)^n = \frac{1}{n!}(\mathcal{L}_b)^n$$

$$a_1 = \frac{1}{2(n-1)!}\int_{\mathcal{X}_b} c_1(\mathcal{L}_b)^{n-1} \cdot Td(\mathcal{X}_b) = -\frac{K_{\mathcal{X}_b} \cdot \mathcal{L}_b^{n-1}}{2(n-1)!}.$$

Therefore $\mu = -n\frac{\mathcal{L}_b^{n-1} \cdot K_{\mathcal{X}_b}}{(\mathcal{L}^n)}$. Using 4.2 we have

$$\begin{aligned} c_1(\lambda_{CM}, \mathcal{D}) &= -n\frac{\mathcal{L}_b^{n-1} \cdot K_{\mathcal{X}_b}}{(\mathcal{L}^n)} f_* c_1(\mathcal{L})^{n+1} + (n+1) f_* c_1(\mathcal{L})^n \cdot c_1(K_{\mathcal{X}/B}) \\ &- n\frac{\mathcal{L}_b^{n-1} \cdot \mathcal{D}_b}{(\mathcal{L}^n)} + (n+1) f_* c_1(\mathcal{L})^n \cdot \mathcal{D} \\ &= n\frac{\mathcal{L}_b^{n-1} \cdot (-K_{\mathcal{X}_b} - \mathcal{D}_b)}{(\mathcal{L}^n)} f_* c_1(\mathcal{L}^{n+1} - (n+1) f_*(-K_{\mathcal{X}/B} - \mathcal{D}) \cdot c_1(\mathcal{L})^n. \end{aligned}$$

□

Remark 4.0.1. *In [88], there is another definition of the log CM line bundle for the case of one divisor with weight $1-\beta$. And in the same work (Theorem 2.7 first assertion) there is a calculation of the first Chern class of the defined log CM line bundle. We point out that the definition we gave in this section it* *is exactly the same for the case of one divisor. Indeed, for the first assertion of Theorem 2.7 we just require to let*

$$\mu(\mathcal{L}_b) := -\frac{-K_{\mathcal{X}_b} \cdot \mathcal{L}^{n-1}}{(\mathcal{L})^n}, \quad \mu(\mathcal{L}_b, \mathcal{D}_b) := \frac{\mathcal{D}_b \cdot \mathcal{L}_b^{n-1}}{(\mathcal{L}^n)}$$

and of course $\mathcal{D} = (1-\beta)D$. An easy calculation leads to

$$c_1(\lambda_{CM,\mathcal{D}}) = n\mu(\mathcal{L}_b)f_*c_1(\mathcal{L})^{n+1} + (n+1)f_*c_1(\mathcal{L})^n \cdot c_1(K_{\mathcal{X}/B}) + \\ +(1-\beta)((n+1)c_1(\mathcal{L}^n) \cdot \mathcal{D} - n\mu(L,\mathcal{D})c_1(\mathcal{L})^{n+1}).$$

The definition of the log CM line bundle given in the same work [88] definition 2.2 is also a functorial definition, namely

$$\Lambda_{CM,\mathcal{D}} := \lambda_{n+1}^{n(n+1)+\frac{2a_1-(1-\beta)\tilde{a}_0}{a_0}} \otimes \lambda_n^{-2(n+1)} \otimes \tilde{\lambda}_n^{(1-\beta)(n+1)}$$

The element $\tilde{\lambda}_n$ refers to the leading order term of the Hilbert-Mumford expansion for $f_{!|\mathcal{D}}\mathcal{L}$, and $\tilde{a}_0$ to the leading order term coefficient of the corresponding Hilbert polynomial, that is by the Hirzebruch-Riemann-Roch expansion

$$\tilde{a}_0 = \frac{1}{n!}\mathcal{L}_b^n \cdot \mathcal{D}_b.$$

Therefore, by switching back and forth from the additive and multiplicative notation, by definition we have

$$(n+1)\lambda_{CH,\mathcal{D}} = \tilde{\lambda}_n^{\otimes(1-\beta)(n+1)}.$$

As we previously calculated we have $\frac{2a_1}{a_0} = -n\frac{\mathcal{L}_b^{n-1} \cdot K_{\mathcal{X}_b}}{(\mathcal{L}^n)}$, and $\frac{\tilde{a}_0}{a_0} = -\frac{n\mathcal{L}_b^{n-1} \cdot \mathcal{D}_b}{(\mathcal{L}_b^n)}$. Hence,

$$\lambda_{CM} - \frac{n\mathcal{L}_b^{n-1} \cdot \mathcal{D}_b}{(\mathcal{L}_b^n)}\lambda_{\mathrm{CH}} = \lambda_{CM} - (1-\beta)\frac{\tilde{a}_0}{a_0}\lambda_{\mathrm{CH}} = \lambda_{n+1}^{n(n+1)+\frac{2a_1-(1-\beta)\tilde{a}_0}{a_0}} \otimes \lambda_n^{-2(n+1)},$$

Where the last equality follows from the functorial definition of the regular *CM line bundle given in Chapter 2.*

When computing the intersection number of the log CM line bundle on a $\mathbb{Q}-$Gorenstein flat family on the moduli of pairs, it is important to choose a ample line bundle on such family. In the state of the art of these possible choices, mainly we can make two of such. These choices are presented in the following

Corollary 4.0.1. *Let $f : (\mathcal{X}, \mathcal{D}) \to B$ be a $\mathbb{Q}-$Gorenstein flat family over a normal proper variety B. We have*

1. *[87] Given* $\mathcal{L} = -K_{\mathcal{X}/B} - \mathcal{D}$, *then*

$$c_1(\lambda_{CM,\mathcal{D}}) = -f_* c_1(-K_{\mathcal{X}/B} - \mathcal{D})^{n+1}.$$

2. *[88] Given* $\mathcal{L} = -K_{\mathcal{X}/B}$, *and* $\mathcal{D}_{\mathcal{X}_b} \in |-K_{\mathcal{X}_b}|$, $\forall b \in B$, *then*

$$c_1(\lambda_{CM,\mathcal{D}}) = f_* \left(c_1(-K_{\mathcal{X}/B})^n \cdot (-c_1(-K_{\mathcal{X}/B}) + \sum_{i=1}^{k} d_i((n+1)\mathcal{D}_i - nc_1(-K_{\mathcal{X}/B})) \right).$$

Proof Claim 1. follows from 4.0.1 by computing 4.2 in $\mathcal{L} = -K_{\mathcal{X}|B} - \mathcal{D}$, indeed noticing that fibers of $\mathcal{L}$ are $-K_{\mathcal{X}_b} - D_b$, the intersection number $\frac{(-K_{\mathcal{X}_b} - D_b)\cdot\mathcal{L}_b^{n-1}}{(\mathcal{L}_b)^n} = 1$, $\forall b \in B$. Then we have

$$\begin{aligned} c_1(\lambda_{CM,\mathcal{D}}) &= nf_* c_1(-K_{\mathcal{X}/B} - \mathcal{D})^{n+1} - (n+1) f_*(c_1(-K_{\mathcal{X}/B} - \mathcal{D})^{n+1} \\ &= -f_*(c_1(-K_{\mathcal{X}/B} - \mathcal{D})^{n+1}). \end{aligned}$$

Similarly, by introducing $\mathcal{L} = -K_{\mathcal{X}/B}$ in 4.2 and noticing that $\mu(\mathcal{L}, \mathcal{D}) = 1$, since, $\mathcal{D}_{\mathcal{X}_b} \in |-K_{\mathcal{X}_b}|$ the claim follows immediately. □

Of course, depending on the choice of $\mathcal{L}$ we will get different results in terms of intersection number on a generic curve into the moduli of pairs we want to study. In the next Chapter, as an example, we will point out this difference when calculating the intersection number of the log CM line bundle with a generic $\mathbb{P}^1$-*curve* in the moduli space of log Fano hyperplane arrangements.

4.0.2 On the log Weil-Petersson metric.

Consider a $\mathbb{Q}$-Gorenstein smoothable family $f : (\mathcal{X}, \mathcal{D}) \to B$ over a complex space B. Suppose f is of pure dimension n, $(\mathcal{X}, \mathcal{D})$ is a log Fano klt Pair. We assume, henceforth, that the divisor $\mathcal{D} = \sum_k (1 - \beta_k)[s_k = 0]$ is simple normal crossing. With this geometric data we have a Kähler metric ω on $X \setminus \bigcup_k D_k$ which is quasi isometric to the model cone metric with cone angles $2\pi\beta_k$ along $[z_k = 0]$

$$\omega_{\text{cone}} = \sum_{k=1}^{m} \frac{1}{|z_k|^{2(1-\beta_k)}} \sqrt{-1} dz_k \wedge d\bar{z}_k + \sum_{k=m+1}^{n} \sqrt{-1} dz_k \wedge d\bar{z}_k.$$

One might ask whether one can find a Kaehler-Einstein metric (KE) ω on $X \setminus \text{supp}(\mathcal{D})$ having conic singularities along $\mathcal{D}$. This metric will be referred henceforth as conic Kaehler metric (cKE). In [57, Theorem A] it is proven that the answer to that question is positive and it is sufficient to produce a weak solution $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ of the so called *global Monge-Ampere* equation

$$\omega^n = \frac{e^\varphi dV}{\prod_k |s_k|^{2(1-\beta_k)}},$$

where dV is a smooth volume form. Moreover it is proven [57, Theorem B] that any solution φ of the above equation is bounded and plurisubharmonic, more precisely $\varphi \in \mathcal{C}^{2,\alpha,\beta}$, where $C^{2,\alpha,\beta}$ is a *conic H ölder space* defined in [57, sec. 7.1 Definition 1] sec. 7.1 Definition 1.
The log K-stability was introduced in [50]. It is proved in [55] that if a log Fano pair (X, D) admits a cKE metric, then it is log K-polystable as defined in [50]. The converse is also true and it has been recently proven by [58]. Therefore we can assume that the above family $f : (\mathcal{X}, \mathcal{D}) \to B$ has log K-polystable fibers. Namely, $\forall b \in B$, $(\mathcal{X}_b, \mathcal{D}_b)$ is a log K-polystable Fano pair with cKE metric $\omega_b = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$, where $\varphi \in \mathcal{C}^{2,\alpha,\beta}$, is a solution of the global Monge Ampere equation, and $\omega_0 \in c_1(-K_{\mathcal{X}/B} - \mathcal{D})$. Let A be the subset of B parametrizing the singularities of B, denote its complement, i.e. the *smooth locus* by B^0 and by $f^0 : \mathcal{X}^0 \to B^0$ the corresponding smooth family. For all $b \in B^0$ the corresponding cKE metric varies in $\mathcal{X}_b^0$. At every $b \in B^0$ we obtain a Hermitian metric $\{\omega_b^n\}$ on $-K_{\mathcal{X}^0/B^0} - D$. At every $b \in B$, we can produce a map

$$\mathcal{E} : C^{2,\alpha,\beta} \times [0,1] \to C^{\alpha,\beta}$$

defined as

$$\mathcal{E}(\varphi, \epsilon) = \log \frac{\omega_{b,\varphi_\epsilon}}{\Omega_{b,\epsilon}} - (F_\epsilon + \phi_\epsilon).$$

Where ϕ_ϵ is a smooth approximation of the solution φ and

$$F_\epsilon = -\log\left(\frac{\prod_k (\epsilon^2 + |s_k|^2)^{1-\beta_k}}{\Omega_{b,\epsilon}}\right).$$

The derivative of $\mathcal{E}$ in φ leads to the *Laplacian operator*, which by [57], and [54], it has bounded estimates. Moreover, also in [57] it is proven that $\omega_{b,\varphi_\epsilon}$ converges smoothly to $\omega_{b,\varphi}$ outside the support of the divisor and along D it has only conic singularities. We denote by $\omega_{\mathcal{X}^0}$ its Chern curvature, namely

$$\omega_{\mathcal{X}^0} = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\{\omega_b^n\}.$$

Because of [57, Lemma 4,section 7.1.2.] we still have a control on the derivatives of such metrics, namely these latters belong to the class $C^{\alpha,\beta}$. In this setting, we define as in [59], and [76, Theorem 4.1].

Definition 4.0.2. *In the above setting, we define the* log Weil-Petersson *metric as the fiber integral*

$$\omega^0_{\mathrm{WP}} := -\int_{\mathcal{X}^0/B^0} \omega^{n+1}_{\mathcal{X}^0}. \tag{4.3}$$

In [59] it is proven in the non log case that there exists a Quillen metric h_{QM} on $\lambda_{\mathrm{CM}_{|B^0}}$ such that its Chern curvature is the Weil-Petersson metric. In [17] Deligne showed how to calculate such Quillen metric, via the so called *metrized Deligne Pairing.* In the next result we collect two results that will be useful for later discussions.

Theorem 4.0.1. *Let $g : \mathcal{X} \to B$ be a flat projective morphism of integral projective schemes of finite type, of relative dimension n. Let $(\mathcal{L}_0, h_0), ..., (\mathcal{L}_n, h_n)$ be a $(n+1)-$tuple of hermitian line bundles on $\mathcal{L}$, denote the Deligne pairing associated to that tuples as the symbol $< \mathcal{L}_0, ..., \mathcal{L}_n >$. We have*

1. *([62]) Suppose h_i are smooth metrics $\forall i \in \{0, 1, ..., n\}$. Then the Deligne's metric h_{DP} is continuous on $< \mathcal{L}_0, ..., \mathcal{L}_n >$.*
2. *([17]) The following curvature formula holds for Deligne's metric*

$$c_1(< \mathcal{L}_0, ..., \mathcal{L}_n >) = \int_{\mathcal{X}/B} c_1(\mathcal{L}_0) \wedge ... \wedge c_1(\mathcal{L}_n).$$

Henceforth, we will denote the Deligne's Pairing of a repeated line bundle $\mathcal{L}$ as

$$\mathcal{L}^{<n+1>} = \underbrace{< \mathcal{L}, ..., \mathcal{L} >}_{(n+1)-\text{times}}.$$

Theorem 4.0.2. *The previously defined log CM line bundle can be interpreted with the Deligne Pairing. Namely*

$$\lambda_{\mathrm{CM},\mathcal{D}} = -\left(-K_{\mathcal{X}/\mathcal{B}} - \mathcal{D}\right)^{<n+1>}$$

Proof Because of Corollary 4.0.1,for a $\mathbb{Q}$-Gorenstein family of log Fano pair $f : (\mathcal{X}, \mathcal{D}) \to B$ the first Chern class of the log CM line bundle is given by $c_1(\lambda_{\mathrm{CM},\mathcal{D}}) = -f_* c_1(-K_{\mathcal{X}/B} - \mathcal{D})^{n+1}$. It can be rephrased in the following way

$$c_1(\lambda_{\mathrm{CM},\mathcal{D}}) = -\int_{\mathcal{X}/B} \underbrace{c_1(-K_{\mathcal{X}/B} - \mathcal{D}) \wedge ... \wedge c_1(-K_{\mathcal{X}/B} - \mathcal{D})}_{(n+1)-\text{times}}.$$

By point 2 in 4.0.1, we have

$$c_1(\lambda_{\mathrm{CM},\mathcal{D}}) = c_1\left((-K_{\mathcal{X}/B} - \mathcal{D})^{<n+1>}\right)$$

Since the considered family is of projective schemes of finite type (e.g. projective varieties) then their Picard group is finitely generated. Thus, in this case tensoring with $\mathbb{Q}$ is an exact functor over $\mathbb{Z}$. Hence, the injectivity of the first Chern class is preserved, therefore the claim follows. □

Now we are ready to state the main result of this section

Theorem 4.0.3. *Let $f : (\mathcal{X}, \mathcal{D}) \to B$ as above. Then there exists a continuous metric h_{DP} on $\lambda_{\mathrm{CM},\mathcal{D}}$ such that*

1. $\omega_{\mathrm{WP}} := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h_{\mathrm{DP}}$ *is a positive current.*
2. $\omega_{\mathrm{WP}_{|B^0}} = \omega^0_{WP}$.

Proof It is proved in [58] that the partial C^0-estimates naturally extends in the log case. Therefore, as in [76] by embedding the total space into a suitable projective space, we can define two sets of metrics along fibers $\{\omega_b^n\}$, and $\{\tilde{\omega}_b^n\}$. By assertion 1 of 4.0.1 both sets, outside the support of the divisor, define Deligne's metrics on $\lambda_{\mathrm{CM},\mathcal{D}}$, which we call h_{DP} and $\tilde{h}_{\mathrm{DP}}$ respectively. Since, the Chern curvature depends fiberwise on the $\partial\bar{\partial}$ derivatives of ω_b^n, forall $b \in B$, then by [57, Lemma 4 in section 7.1.2.] we still have a control on the derivatives of such metrics. Therefore we can consider the Deligne's metrics h_{DP}, and $\tilde{h}_{\mathrm{DP}}$ to be well defined also along the Divisors.
By the *change of metric* formula of Deligne's pairing [76], [17], [62] we have

$$h_{\mathrm{DP}} = \widetilde{h_{\mathrm{DP}}} \cdot e^{-\Phi},$$

Where

$$\Phi = -\frac{1}{(n+1)!}\sum_{i=1}^{n}\int_{\mathcal{X}_b}\tilde{\varphi}\tilde{\omega}_b^i \wedge \omega_b^{n-i}.$$

It is proved in [56, Proposition 2.14] that Φ is continuous and uniformly bounded. Notice that over each fiber $(\mathcal{X}_b, \mathcal{D}_b)$ we have $\omega_b = \tilde{\omega}_b + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\tilde{\varphi}$. By looking at the curvature data we find

$$\omega_{\mathcal{X}} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\{\omega_b^n\} = \tilde{\omega}_{\mathcal{X}} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\tilde{\varphi}.$$

In the smooth part we have that the following identity holds:

$$\omega_{\mathcal{X}^0}^{n+1} = \tilde{\omega}^{n+1} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Phi,$$

where we denoted by $\tilde{\omega}$ the curvature data of $\{\tilde{\omega}_b^n\}$. Notice that both $\omega_{\mathcal{X}^0}$ and $\tilde{\omega}$ are smooth $(1,1)$-forms over $\mathcal{X}^0$, so we can take fiber integrals

$$\int_{\mathcal{X}^0/B^0} \omega_{\mathcal{X}^0}^{n+1} = \int_{\mathcal{X}^0/B^0} \tilde{\omega}^{n+1} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Phi,$$

we see that the above identity is nothing but

$$\omega_{\mathrm{WP}}^0 = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\tilde{h}_{\mathrm{DP}} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Phi = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h_{\mathrm{DP}|B^0}.$$

The claim follows from regularity results in pluripotential theory, see [76, Lemma 4.13]. □

The CM line bundle, since its weight coincides with [87], then it is zero on K-polystable points and descends as a $\mathbb{Q}$-line bundle on the K-proper moduli space and there is a well defined continuous Deligne's pairing metric whose curvature is a positive current and on families is like in 4.0.3. In conclusion, to compute the log Weil-Petersson volume we can simply compute the degree of the descended CM line bundle.

Chapter 5

The CM Volume of Log Fano Hyperplane arrangements

5.1 Some generalities on the log Fano hyperplane arrangements and their moduli.

In this section, we will give a brief description of the moduli space of log Fano hyperplane arrangements, which will be the main object of this entire chapter.

Definition 5.1.1. *Let (X, D) be a $n-$ dimensional hyperplane arrangement, that is $X = \mathbb{P}^n$ and D is a divisor of X of the form $\sum_{i=1}^{m} d_i H_i$, where $d_i \in \mathbb{Q}_{>0}$, and the hyperplanes H_i are mutually distinct $\forall i \in \{1, 2, ..., m\}$.*

1. *(X, D) is said to be a* log Fano hyperplane arrangement *if (X, D) is a log Fano pair, i.e. X is a normal projective variety over $\mathbb{C}$, D is an effective $\mathbb{Q}-$divisor on X such that, the pair (X, D) is klt and $-(K_X + D)$ is ample.*

2. *(X, D) is said to be a* log Calabi-Yau hyperplane arrangement*, i.e. X is a normal projective variety over $\mathbb{C}$, D is an effective $\mathbb{Q}-$divisor on X such that, the pair (X, D) is lc and $K_X + D$ is rationally equivalent to* 0.

3. *(X, D) is said to be a* log general type hyperplane arrangement*, i.e. X is a normal projective variety over $\mathbb{C}$, D is an effective $\mathbb{Q}-$divisor on X such that, the pair (X, D) is lc and $K_X + D$ is ample.*

Consider a (X, D) n-dimensional hyperplane arrangement like in 5.1.1. From the above conditions, the choices of the weights $d_i \in \mathbb{Q}_{>0}$ determine the *nature* of the hyperplane arrangement, namely one of the kind defined in 5.1.1, as the following shows

Proposition 5.1.1. *Let (X, D) be a n-dimensional hyperplane arrangement like in 5.1.1. Then*

- *(X, D) is a klt log Fano hyperplane arrangement if and only if*
$$\sum_{i=1}^{m} d_i < n+1.$$
- *(X, D) is a lc log Calabi-Yau hyperplane arrangement if and only if*
$$\sum_{i=1}^{m} d_i = n+1.$$
- *(X, D) is a lc log general type hyperplane arrangement if and only if*
$$\sum_{i=1}^{m} d_i > n+1.$$

Proof By definition we have $(X, D) = (\mathbb{P}^n, \sum_{i=1}^m d_i H_i)$. Then $K_X + D = -(n+1)H + \sum_{i=1}^m d_i H_i$. The hyperplanes have the same class H in the Chow ring, therefore we can factor H to get

$$L := K_X + D = \left(\sum_{i=1}^{m} d_i - (n+1) \right) H.$$

If (X, D) is log Calabi-Yau, then by definition 5.1.1 $L \sim_{\mathbb{Q}} 0$, that leads to $\sum_{i=1}^m d_i = n+1$. If (X, D) is of log general type, then L is ample, by the Kodaira theorem L is positive, and therefore $\sum_{i=1}^m d_i > n+1$. At end, by the same argument if (X, D) is a log Fano hyperplane arrangement then $-L$ is positive, and the claim follows. □

Let (X, D) be a klt log Fano hyperplane arrangement, then $(X, D) = (\mathbb{P}^n, \sum_{i=1}^m d_i H_i)$ with $\sum_{i=1}^m d_i < n+1$. An hyperplane $H_i \subset \mathbb{P}^n$ correspond to a point p_i in the dual projective space $\mathbb{P}^{n*}$, or equivalently in the Grassmannian $\mathrm{Gr}(n-1, n)$. Let $p := (p_1, ..., p_m) \in (\mathbb{P}^{n*})^m$, the group $\mathrm{SL}(n+1)$ acts naturally on $(\mathbb{P}^{n*})^m$ via its linear representation in $\mathbb{C}^{n+1}$ [8]. The line bundle $\mathcal{L}_d = \mathcal{O}(d_1, ..., d_m)$ admits a unique $\mathrm{SL}(n+1)-$linearization. The moduli space of log Fano hyperplane arrangement can be described in a GIT fashion. The notion of stability on the point $p \in (\mathbb{P}^{n*})^m$ with respect to $\mathcal{L}_d$ is the *classical* one, given in the standard literature [3], [8]. Namely, a point $p \in (\mathbb{P}^{n*})^m$ with respect to the $\mathrm{SL}(n+1)$-linearized line bundle $\mathcal{L}_d$ is called

- GIT *unstable*, if for the corresponding point in the open cone $\tilde{p} \in \operatorname{Spec}\left(\oplus_{d\geq 0} H^0((\mathbb{P}^{n*})^m, \mathcal{L}_d)\right)$ one has $0 \in \overline{\mathrm{SL}(n+1)\cdot \tilde{p}}$. Call the set of all unstable points as $(\mathbb{P}^{n*})^{m,us}$.

- GIT *semistable*, if $p \in (\mathbb{P}^{n*})^m \setminus (\mathbb{P}^{n*})^{m,us}$. Call the set of all semistable points as $(\mathbb{P}^{n*})^{m,ss}$.

- GIT *polystable*, if $p \in (\mathbb{P}^{n*})^{m,ss}$ and $\mathrm{SL}(n+1)\cdot\tilde{p}$ is closed in $(\mathbb{P}^{n*})^{m,ss}$.

- GIT *stable*, if it is GIT polystable and the isotropy group at p is finite.

It is proven by Fujita in [78, Theorem 1.5, Corollary 8.3] that the above GIT definitions matches with the K-stability conditions. Therefore, the K-moduli of log Fano hyperplane arrangements can be identified with the GIT quotient

$$M_d := (\mathbb{P}^{n*})^m //_{\mathcal{L}_d} \mathrm{SL}(n+1).$$

Although it is not object of this work, for completeness, we want to mention the construction of the moduli space of weighted stable hyperplane arrangements of Alexeev in [25], and [11]. This construction characterizes the moduli of log Calabi-Yau and log Fano hyperplane arrangements. The idea of Alexeev is to generalise the moduli space of n-pointed rational curves $\overline{\mathcal{M}_{0,n}}$ proposed by Hasset in [24]. First we recall the notion of stability given in [25] and [11, Theorem 5.2.1]. A hyperplane arrangement $(\mathbb{P}^n, \sum_{i=1}^m d_i H_i)$, is stable if and only if is log canonical and of log general type. This notion of stability is coherent with the one given in the general case of *stable pair*. The existence of a fine moduli space for such pair is proven in the following

Theorem 5.1.1. *([25] Theorem 1.1) For each n, m and $d := (d_1, ..., d_m) \in \mathbb{Q}^m_{i>0}$ with $\sum_{i=1}^m d_i > n+1$, there exists a projective scheme $\overline{M}_d(n,m)$ together with a locally free and flat family $f : (\mathcal{X}, \mathcal{D}) \to \overline{M}_d(n,m)$ such that:*

1. *Every geometric fiber of f is a $n-$ dimensional variety X together with n Weil divisors D_i such that the pair $(X, \sum_{i=1}^m d_i D_i)$ is stable.*

2. *For distinct geometric points of $\overline{M}_d(n,m)$, the fibers are non-isomorphic.*

3. *Over an open (but not dense in general) subset $M_d(n,m) \subset \overline{M}_d(n,m)$, f coincides with the universal family of weighted hyperplane arrangements.*

Moreover, for every $m \in \mathbb{Z}_{>0}$ such that all $md_i \in \mathbb{N}$, the sheaf $\mathcal{O}_{\mathcal{X}}(m(K_{\mathcal{X}} + \mathcal{D}))$ is relatively ample and free over $\overline{M}_d(n,m)$.

In order to explore the *geography* of $\overline{M}_d(n,m)$ one defines a *weight domain,* namely

$$\mathfrak{D}(n,m) := \{(d_1,...,d_m) \in \mathbb{Q}^n \ |0 < d_i \leq 1, \sum_{i=1}^{m} d_i > n+1\}.$$

The above open polytope could be sliced with hyperplanes into chambers

$$\mathrm{Ch}(d) := \{d \in \mathfrak{D}(n,m) \mid \sum_{i\in I} d_i = k, \ 1 \leq k \leq n, \ I \subset \{1,2,...,m\}\}.$$

We have the following result

Theorem 5.1.2. *([11, Theorem 5.5.2]) The weight domain $\mathfrak{D}(n,m)$ is divided by hyperplanes into finitely many chambers.*

1. *If d and d' are m-tuples of weights that belongs to the same chamber then $\overline{M}_d(n,m) = \overline{M}_{d'}(n,m)$.*
2. *If $d \in \overline{\mathrm{Ch}(d')}$, then there exists a contraction $\overline{M}_{d'}(n,m) \to \overline{M}_d(n,m)$ on the moduli spaces and on the corresponding families.*
3. *If $d \in \overline{\mathrm{Ch}(d')}$, and $d' \leq d$ then $\overline{M}_{d'}(n,m) = \overline{M}_d(n,m)$ and the morphism on families is birational on every irreducible component.*

The above theorem generalises the ideas of Hasset when constructing the moduli space of weighted stable curves. The fine moduli space, is however not complete, except for the case when the weights are *small*. In order to have a notion of "smallness" for weights we notice that the closure of the weight domain contain the regular simplex

$$\Delta(n,m) := \{d \in \mathbb{Q}^n | 0 < d_i \leq 1 \sum_{i=1}^{n+1} d_i = n+1\}.$$

Then, we have the following

Theorem 5.1.3. *([11, Theorem 5.5.2], [25, Theorem 1.5]) Let $a \in \Delta(n,m)$ be a generic element such that $a \in \overline{Ch(d)}$. Then*

$$\overline{M}_d(n,m) = \mathrm{Gr}(n,m)//_{\mathcal{L}_a}T \tag{5.1}$$

The above theorem says that for small perturbations of $a \in \Delta(n,m)$, namely $a-\epsilon, a+\epsilon$, for a positive and small ϵ, then the moduli description coincides with the GIT picture. Using the *Gelfand-MacPherson's correspondence* [11, sec. 2.6.1], [30]) one can prove that the GIT quotient in 5.1 is equivalent to

$$(\mathbb{P}^n)^m//_{\mathcal{L}_a}\mathrm{SL}(n+1).$$

For the details on the GIT correspondence see [18].

Consider the following GIT setting,

$$M_d = (\mathbb{P}^n)^m //_{\mathcal{L}_d} \mathrm{SL}(n+1),$$

Where $\mathcal{L}_d = \mathcal{O}(d_1, ..., d_m)$ is the linearization, and $d_i \in (0,1) \cap \mathbb{Q}, \forall i \in \{1, 2, ..., m\}$, such that $\sum_{i=1}^m d_i < n+1$. Suppose we choose some other linearization $\mathcal{L}_{d'} = \mathcal{O}(d'_1, ..., d'_m)$, where the d'_is have the same properties of the d'_is. Then, we get another moduli space $M_{d'}$like above.

Theorem 5.1.4. *([11, Theorem 5.3.6]) In the above setting, let $d' = d - \epsilon$, where ϵ is small. Then*

1. *The semistable locus of $M_{d'}$ equals the stable locus of M_d.*
2. *There exists a contraction of GIT quotients $\pi\colon M_{d'} \to M_d$ which is crepant with respect to d,i.e. $K_{M_{d'}} + \sum_{i=1}^m d'_i H'_i = \pi^*(K_{M_d} + \sum_{i=1}^m d_i H_i)$.*
3. *The morphism $\pi : M_{d'} \to M_d$ is birational (on every irreducible component), and it is an isomorphism on the support of the divisors.*

5.2 The Futaki Invariant of Log Fano Hyperplane arrangements

In this section we will prove a result about the K-stability of log Fano hyperplane arrangements by revisiting the results of Fujita in [78]. We begin by stating the main result of this section

Theorem 5.2.1. *Let (X, D) be a klt log Fano hyperplane arrangement. Then, for any integral test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})/\mathbb{A}^1$ coming from a dreamy prime divisor, the Futaki invariant for (X, D) is*

$$DF_D(\mathcal{X}, \mathcal{L}) = k \sum_{i=1}^m d_i - (n+1) \sum_{j=1}^k d_{i_j} \quad \forall 1 \leq k \leq n, \ 1 \leq i_1 < ... <\leq m. \tag{5.2}$$

The Futaki invariant for one dimensional log Fano hyperplane arrangements is a classical well known result, and it is expressed in the following

Theorem 5.2.2. *([50], [77]). Let (X, D) be a one dimensional klt hyperplane arrangement, namely $X = \mathbb{P}^1$ and $D = \sum_{i=1}^m d_i p_i$.Then for every integral test configuration, (X, D) is (uniformly) K-stable if and only if*

$$\sum_{j \neq i} d_j > d_i$$

The 5.2.2 condition are also called the *Troyanov condition.* In [50, Example 2], using a 1-parameter subgroup acting on $\mathbb{P}^1$ by multiplication, the author proved that the Donaldson-Futaki invariant for the one dimensional hyperplane is

$$DF(\mathcal{X}, \mathcal{L}) = \sum_{j \neq i} d_j - d_i, \tag{5.3}$$

where the test configuration $(\mathcal{X}, \mathcal{L})$ in the above notation, can be easily reconstructed from the chosen 1-parameter subgroup, which plays the role of a genuine $\mathbb{C}^*$-action. The central fiber of such test configuration is given by a point, and the weighted sum of two prime divisors $D_0 = \sum_{i \neq j} d_j\{\infty\} + d_i\{0\}$. Since $\mathbb{P}^1$ endowed with the action of the chosen 1-parameter subgroup is a toric variety, then the two prime divisors are also dreamy in the sense of Definition 1.1.6. Therefore, by the *Fujita-correspondence* between dreamy prime divisors and integral test configurations discussed in Chapter 1, we conclude that $(\mathcal{X}, \mathcal{D})/\mathbb{C}$ is an integral test configuration. As we can see the result of the author (equation 5.3), is a particular case of 5.2. Indeed, for $k = 1$ the second addend in the right hand side of 5.2 is determined by a single element. Therefore,

$$DF_D(\mathcal{X}, \mathcal{L}) = \sum_{i=1}^{m} d_i - 2d_j = DF(\mathcal{X}, \mathcal{L})$$

as wanted. Henceforth, we will always refer to compactified test configurations in the sense of [70], [71]. We recall the definition of the Donaldson-Futaki invariant in the case of log Fano log Pairs, given in Chapter 1

$$DF_D(\mathcal{X}, \mathcal{L}) = \frac{1}{L^n} \left(\frac{n}{n+1} \mathcal{L}^{n+1} + \mathcal{L}^n \cdot \left(K_{\mathcal{X}/\mathbb{P}^1} + D_{\mathcal{X}} \right) \right)$$

for any (compactified) test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ for the log Fano pair $(X, D, L := -(K_X + D))$.

Theorem 5.2.3. *([72], [77]). Let (X, D) be a log Fano klt pair. The K-stability condition defined through 5.2 on any compactified test configuration $(\mathcal{X}, \mathcal{L})$ are the same as the K-stability condition defined through 5.2 on any compactified test configuration $(\mathcal{X}, \mathcal{L})$ with integral central fiber.*

As we discussed in chapter one, in [78] the author translates the meaning of test configuration with integral central fiber within the language of dreamy prime divisors. With this in mind, we are ready to prove the main statement of this section.

Proof (5.2.1) We will first prove 5.2.2 with the tools presented until now and then we move back to 5.2.1. It will turn out that the technique of the proof is pretty much the same in both cases. We start with $(\mathbb{P}^1, \sum_{i=1}^{m} d_i p_i)$

where all $p_i, i \in \{1, 2, ..., m\}$ distinct points. Points in $\mathbb{P}^1$ are obviously dreamy prime divisors, so let $F = d_1 p_1$. In order to calculate the β-invariant we need a normalization of $\mathbb{P}^1$, we will take the blow up of $\mathbb{P}^1$ at F :

$$\tilde{\mathbb{P}^1} = Bl_F(\mathbb{P}^1) \simeq \mathbb{P}^1.$$

Then, for this case, the normalization map is just the identity. Set $d = \sum_{i=1}^{m} d_i$. We compute explicitly the $\beta-$ invariant for this case:

$$\beta(F) = 1 - \frac{\int_0^\infty vol_X(L - xF)dx}{A_{(X,D)}(F) \cdot L^1}$$

$L^1 = 2 - d$, $L = (2 - d)h$, where h is the class of a point in $\mathbb{P}^1$.

$$A_{(X,D)}(F) = ord_F(K_{\mathbb{P}^1} - id^*(K_{\mathbb{P}^1} + D)) + 1 = 1 - d_1$$

Because of the definition of vol_X there exists a $x_0 \in \mathbb{R}_{>0}$ such that vol_X is identically zero.

$$L - x_0 F = 0 \Rightarrow x_0 = 2 - d.$$

$$\int_0^\infty vol_X(L - xF) = \int_0^{2-d} (2 - d - x)hdx = \frac{1}{2}(2 - d)^2$$

$$\beta(F) = \frac{2(1 - d_1) - (2 - d)}{2(1 - d_1)} = \frac{d - 2d_1}{2(1 - d_1)}.$$

By applying the second assertion of 1.1.4, we get $DF_D(\mathcal{X}, \mathcal{L}) = \frac{1}{2}(\sum_{i=2}^{m} d_i - d_1)$ and hence the claim in the case of $\mathbb{P}^1$. Now, consider the case $(X, D) = (\mathbb{P}^n, \sum_{i=1}^{m} d_i)$, let $W = \bigcap_{i \in I} H_i$ the non empty intersection, where $I \subset \{1, ..., m\}$. Consider $Y = Bl_W(\mathbb{P}^n)$. Choose $F = \sigma^{-1}(W)$ where σ is the blow-up map to the base. Since Y is a toric variety, then F is dreamy prime. If $c = \text{codim}_X W$ then

$$Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^{c-1}}^{n+1-d} \oplus \mathcal{O}_{\mathbb{P}^{c-1}}(1)) = \tilde{X}.$$

Call with $\pi : \tilde{X} \to \mathbb{P}^{c-1}$ the fibration on $\mathbb{P}^{c-1}$. Denote by $\xi = \sigma^* \mathcal{O}_{\mathbb{P}^n}(1)$, $A = \pi^* \mathcal{O}_{\mathbb{P}^{c-1}}(1)$. Then F is rationally equivalent to $\xi - A$.

$$\begin{aligned} A_{(X,D)}(F) &= ord_F(K_{\tilde{X}} - \sigma^*(K_X + D)) + 1 \\ &= ord_F((c - d)\xi - F) + 1 \\ &= c - d^W. \end{aligned}$$

Where $d^W = d_{|W}$ is the sum of the weights of the hyperplanes of W.

$$L^n = ((n + 1)h - dh)^n = (n + 1 - d)^n h^n = (n + 1 - d).$$

The function vol_X is identically zero at $x_0 = n + 1 - d$. Therefore

$$\beta(F) = 1 - \frac{\int_0^{n+1-d} vol_X(L - xF)dx}{(c - d^W)(n + 1 - d)^n} = \frac{cd - (n + 1)d^W}{(n + 1)(c - d^W)}.$$

Using 1.1.4 we conclude that

$$DF_D(\mathcal{X}, \mathcal{L}) = \frac{1}{n + 1}(cd - d^W(n + 1)).$$

Calling $c = k$ and $d^W = \sum_{j=1}^{k} d_{i_j}$, we obtain the assertion. □

5.3 The intersection number of the log CM line bundle

In this section we compute the intersection number of the log CM line bundle with a family of log Fano hyperplane arrangements with base $\mathbb{P}^1$. Consider the product $\mathcal{X} := \mathbb{P}^n \times \mathbb{P}^1$, We define a divisor $\mathcal{D} \subset \mathcal{X}$, as

$$\mathcal{D} := \sum_{i=1}^{m-1} d_i \mathrm{pr}_2^* h + d_m(\mathrm{pr}_1^* l + \mathrm{pr}_2^* h)$$

Where the $\mathrm{pr}_i's$, $i = 1, 2$ denotes the projections onto the first and second factor of $\mathcal{X}$. We can think about $\mathcal{D}$ as $m-1$ fixed hyperplanes of $\mathbb{P}^n$ together with a line l with weight $d_m$free to move along the *diagonal* of $\mathcal{X}$. We assume that $d_i \in (0, 1) \cap \mathbb{Q}$, $\forall i \in \{1, 2, .., m\}$, $\sum_{i=1}^{m} d_i < n + 1$, and the chosen line and hyperplanes are in general position.

Proposition 5.3.1. *With the above data we have*

$$c_1(\lambda_{CM,\mathcal{D}}) = (n + 1)d_j \left(n + 1 - \sum_{i=1}^{m} d_i\right)^n, \ \forall j \in \{1, 2, ..., m\}.$$

Proof We see that the projection $\mathrm{pr}_2 : \mathcal{X} \to \mathbb{P}^1$ gives a proper and flat family of relative dimension n. Choose $\mathcal{L} = -K_{\mathcal{X}/B} - \mathcal{D}$and use assertion 1. of 4.0.1, we find that

$$\mathcal{L} = \left(n + 1 - \sum_{i=1}^{m} d_i\right) \mathrm{pr}_1^* h - d_m \mathrm{pr}_2^* l$$

Using the binomial expansion, the only term that survives is

$$c_1(\lambda_{CM,\mathcal{D}}) = -\mathrm{pr}_{2*}c_1(\mathcal{L})^{n+1} = (n+1)d_m\left(n+1-\sum_{i=1}^{m}d_i\right)^n \mathrm{pr}_{2*}\left(\mathrm{pr}_1^*h \cdot \mathrm{pr}_2^*l\right).$$

By using the projection formula and the arbitrariness of the choice of the weight on the line, the claim follows. □

Remark 5.3.1. *If we would have chosen $\mathcal{L} = -K_{\mathcal{X}/B}$, we can see that the hypothesis $\mathcal{D}_{|\mathcal{X}_b} \in |-K_{\mathcal{X}_b}|$ would not be satisfied. Indeed, it is true if and only if $\sum_{i=1}^m d_i = (n+1)$, i.e. a Calabi-Yau log hyperplane arrangement, but this is in contradiction with the log Fano assumption. However, $-K_{\mathcal{X}/B}$ it is still a relatively ample line bundle on $\mathrm{pr}_2 : \mathcal{X} \to \mathbb{P}^1$. We can compute 4.2 with this choice, and relaxing the hypothesis for which $\mathcal{D}_{|\mathcal{X}_b} \in |-K_{\mathcal{X}_b}|$ as the next result will show.*

Proposition 5.3.2. *With the above data and $\mathcal{L} = -K_{\mathcal{X}/B}$, we have*

$$c_1(\lambda_{CM,\mathcal{D}}) = (n+1)^2 d_j.$$

Proof The first summand is zero, since $\mathcal{X}/\mathbb{P}^1$ is a trivial fibration. From the second summand we get

$$\begin{aligned} c_1(\lambda_{CM,\mathcal{D}}) &= \\ &= -(n+1)\mathrm{pr}_{2*}\left[\left(n+1-\sum_{i=1}^{m}d_i\right)\mathrm{pr}_2^*h - d_m\mathrm{pr}_1^*l\right]\cdot(n+1)\mathrm{pr}_2^*h = \\ &= (n+1)^2 d_m. \end{aligned}$$

Because of the arbitrariness of choices the claim follows $\forall j \in \{1, 2, ..., m\}$. □

From the choice of the two ample line bundles on the given flat and proper family we see that they mainly differ by a factor which involves the volume of the fiber, that is

$$\mathrm{Vol}(\mathrm{pr}_{2,t}) := \left(n+1-\sum_{i=1}^{m}d_i\right)^n.$$

This difference might become important when calculating the volume of the moduli space of weighted hyperplane arrangements. To see that, we shall calculate the log CM line on to the mentioned moduli space. Recall that the moduli space of weighted log Fano hyperplane arrangement is the GIT quotient

$$M_d := (\mathbb{P}^n)^m //_{\mathcal{L}_d} \mathrm{SL}_{n+1}$$

where $\mathcal{L}_d = \mathcal{O}(d_1, ..., d_m)$, is the linearization and $d_i \in (0,1) \cap \mathbb{Q}, \forall i \in \{1, 2, ..., m\}$. It is known that,

$$\mathrm{Pic}(M_d) = \mathbb{Z}^m$$

see for example [8, Chapter 11, Lemma 11.1]. Therefore $c_1(\lambda_{CM,\mathcal{D}}) \in \mathrm{Pic}(M_d) \Rightarrow c_1(\lambda_{CM,\mathcal{D}}) = \mathcal{O}(r_1, ..., r_m)$, for some $(r_1, ..., r_m) \in \mathbb{Z}^m$. In order to compute the r_j's we can use the results of 5.3.1 and 5.3.2. Namely, in both cases we compute the following intersection number

$$c_1(\lambda_{CM,\mathcal{D}}) \cdot (\mathcal{X} \to \mathbb{P}^1) = k,$$

which means, $\forall j \in \{1, 2, ..., m\}$

$$\int_{(\mathbb{P}^1)_j} \mathcal{O}(r_1, ..., r_m) = r_j = k.$$

Therefore, from 5.3.1 we get $r_j = (n+1)d_j \mathrm{Vol}(\mathrm{pr}_{2,t})$, and from 5.3.2 we get $r_j = (n+1)^2 d_j$. Hence, we just proved the following result.

Proposition 5.3.3. *The log CM line bundle on the moduli space of log Fano hyperplane arrangement is given by*

- *When $\mathcal{L} = -K_{\mathcal{X}/B} - \mathcal{D}$, we have that*

$$c_1(\lambda_{CM,\mathcal{D}}) = \mathrm{Vol}(\mathrm{pr}_{2,t})\mathcal{O}((n+1)d_1, ..., (n+1)d_m)$$

- *When $\mathcal{L} = -K_{\mathcal{X}/B}$, we have*

$$c_1(\lambda_{CM,\mathcal{D}}) = (n+1)\mathcal{O}((n+1)d_1, ..., (n+1)d_m).$$

Remark 5.3.2. *In both cases of 5.3.3 we can see that the log CM line bundle is a multiple of the* weighted prequantum *line bundle on* $(\mathbb{P}^n)^m$*hence of its Kähler form, we recall that*

$$\left[\frac{\omega}{2\pi}\right] = c_1(-K_{(\mathbb{P}^n)^m}) = c_1(\mathcal{O}((n+1)d_1, .., (n+1)d_m)))$$

Where the Kähler form ω is given by the weighted sum of the Fubini Study metrics on each $\mathbb{P}^n$, namely $\omega = \sum_{i=1}^m d_i \omega_{FS,i}$.

5.4 A first study: the case of four points on the complex projective line.

As a first study, we consider the case of four points $\{p_i\}_{i=1}^4$ in $\mathbb{P}^1$ with rational weights $d_i \in (0,1) \cap \mathbb{Q}, i \in \{1,2,3,4\}$. Given the pair $(\mathbb{P}^1, \sum_{i=1}^4 d_i p_i)$, a choice of the rational weights correspond to three distinguished *geometries*

$$\begin{cases} \sum_{i=1}^4 d_i < 2 & \text{log Fano,} \\ \sum_{i=1}^4 d_i = 2 & \text{log Calabi-Yau,} \\ \sum_{i=1}^4 d_i > 2 & \text{log General type.} \end{cases}$$

We want to study how the *Volume* of the moduli space of four points in the projective line varies in the above geometries. We begin with the Calabi-Yau geometry. Applying Theorem 5.1.3, we can see that the moduli space of four points in $\mathbb{P}^1$ can be described as the following GIT quotient

$$M_d = (\mathbb{P}^1)^4 //_{\mathcal{L}_d} \mathrm{SL}_2\mathbb{C}, \tag{5.4}$$

with linearization $\mathcal{L}_d = \mathcal{O}(d_1, d_2, d_3, d_4)$. Furthermore, M_d is isomorphic to the moduli space of marked curves of genus 0, denoted by $\mathcal{M}_{0,4}$. A result of Thurston [31] shows that the general moduli space of marked curves of genus 0, with the additional condition that the weights sum up to 2, is a complex hyperbolic cone manifold. The latter, in general, have a natural *stratification*, and the strata of a cone manifold are connected and Riemannian. The *solid angle* of a cone manifold is constant on each strata. We know that for a closed complex hyperbolic manifold M of dimension n, its volume can be expressed by its Euler characteristic

$$\mathrm{Vol}(M) = \frac{(-4\pi)^n}{(n+1)!}\chi(M) \tag{5.5}$$

The dimension of M_d is 1, and it is homeomorphic to $\mathcal{M}_{0,4}$. By applying 5.5 we get

$$\mathrm{Vol}(M_d) = -2\pi \sum_{B \subset \mathcal{P} \subset \{1,2,3,4\}} (1 - \sum_{i \in B} d_i) \tag{5.6}$$

Where $\mathcal{P}$ is a partition of $\{1,2,3,4\}$ into blocks B for which $\sum_{i\in B} d_i < 1$. The above result is obtained by considering the analogy with the 2-sphere with four marked points with weights d_1, d_2, d_3, d_4, respectively. We can associate to these marked points a cell complex, where its skeleton consist of four distinct points and six triangles. These latter correspond to the four points partitions of size two and three. We will use the following theorem which will be proved in the next sections.

Theorem 5.4.1. *[37] The volume of M_d with $d_i \in (0,1) \cap \mathbb{Q}$, for all $i \in \{1, 2, ..., n\}$ is given by*

$$\mathrm{Vol}(M_d) = -\frac{(2\pi)^{n-3}}{2(n-3)!}\sum_{k=0}^{n-1}(-1)^k \sum_{I\subset\mathcal{F}_+,|I|=k}\left(D_I^+ - D_I^-\right)^{m-3}. \tag{5.7}$$

Where $D_I^+ = \sum_{i\notin I} d_i$, $D_I^- = \sum_{i\in I} d_i$, *and*

$$\mathcal{F}_+ = \{I \subset \{1,2,...,n\} \mid D_I^+ - D_I^- > 0\}.$$

In the case of four points in $\mathbb{P}^1$ 5.7 becomes

$$\frac{\mathrm{Vol}(M_d)}{\pi} = \sum_{k=0}^{3}(-1)^{k+1}\sum_{I\subset\mathcal{I}_+,|I|=k}\left(D_I^+ - D_I^-\right). \tag{5.8}$$

Now assume that $\sum_{i=1}^4 d_i = 2$. Unravelling the definition of D_I^+ and D_I^-, we have

$$\begin{aligned}\sum_{i\notin I} d_i - \sum_{i\in I} d_i &= 2 - \sum_{i\in I} d_i - \sum_{i\in I} d_i\\ &= 2(1-\sum_{i\in I} d_i).\end{aligned}$$

By substituting in Equation5.8, we find

$$\mathrm{Vol}(M_d) = 2\pi\sum_{k=1}^{4}(-1)^{k+1}\sum_{I\subset\mathcal{F}_+,|I|=k}(1-\sum_{i\in I} d_i) \tag{5.9}$$

In order to see that Equation 5.9 matches with Equation 5.6, we shall give an order to the weights. Firstly we assume that the *Troyanov conditions* [40] are satisfied. These latter are expressed by the following

$$d_i < \sum_{j\neq i} d_j.$$

A simple analysis shows that since $d_i \in (0,1)\cap\mathbb{Q}, \forall i$, and $\sum_{i=1}^4 d_i = 2$, then we can have three possible orders, by choosing d_4 to be the biggest among the $\{d_i\}_{i=1}^4$ then we have

$$\begin{aligned} d_1 + d_2 &< d_3 + d_4 \\ d_1 + d_3 &< d_2 + d_4 \\ d_2 + d_3 &< d_1 + d_4. \end{aligned} \tag{5.10}$$

This configuration is possible whenever we choose d_1 sufficiently close to zero and d_4 sufficiently close to 1. However, if we imagine to accumulate d_3, and d_4 sufficiently close to 1, then at least one of the above inequalities in 5.10 will be violated. Therefore, this leads to the next set of inequalities

$$\begin{aligned} d_1 + d_2 &< d_3 + d_4 \\ d_2 + d_4 &< d_1 + d_3 \\ d_1 + d_4 &< d_2 + d_3. \end{aligned} \tag{5.11}$$

When d_1, and d_2 accumulate sufficiently close to zero and d_4 is *nearly* one then we have the following set

$$\begin{aligned} d_1 + d_2 &< d_3 + d_4 \\ d_1 + d_3 &< d_2 + d_4 \\ d_1 + d_4 &< d_2 + d_3. \end{aligned} \tag{5.12}$$

We have fifteen partitions of the indices $\{1, 2, 3, 4\}$, but the only ones that contributes to the sum in 5.6 are those of length four and three (here the length is the size of B). The others will contribute zero. With this in mind, we have the following partitions of indexes

$$\begin{aligned} &\{1\}, \{2\}, \{3\}, \{4\}; \\ \{1\}, \{2\}, \{3, 4\}; &\{1\}, \{3\}, \{2, 4\}; \\ \{1\}, \{4\}, \{3, 2\}; &\{2\}, \{3\}, \{1, 4\}; \\ \{2\}, \{4\}, \{1, 3\}; &\{3\}, \{4\}, \{1, 2\}. \end{aligned}$$

By applying 5.6 to 5.10 we find

$$\mathrm{Vol}(\mathcal{M}_{0,4}) = 2\pi(2(d_1 + d_2 + d_3) - 2),$$

by applying 5.6 to 5.11 we find

$$\mathrm{Vol}(\mathcal{M}_{0,4}) = 2\pi(2(d_1 + d_2 + d_4) - 2),$$

and to 5.12 we find

$$\mathrm{Vol}(\mathcal{M}_{0,4}) = 4\pi d_1.$$

As before, we apply 5.9 in the above conditions:

- 5.10 gives$\mathrm{Vol}(M_d) = \pi \left(-\sum_{i=1}^{4} d_i + 2\sum_{i=1}^{4} d_i - (3d_4 - (d_1 + d_2 + d_3)) \right) = 2\pi(d_1 + d_2 + d_3) - 2\pi d_4$,
- 5.11 gives $\mathrm{Vol}(M_d) = \pi \left(\sum_{i=1}^{4} d_i - (3d_3 - (d_1 + d_2 + d_4)) \right) = 2\pi(d_1 + d_2 + d_3) - 2\pi d_3$,

- 5.12 gives $\mathrm{Vol}(M_d) = \pi\left(\sum_{i=1}^4 d_i - (d_2 + d_3 + d_4 - 3d_1)\right) = 4\pi d_1$.

We can see that 5.12 matches between 5.6 and 5.7. This is also the case for 5.10 and 5.11 whenever $\sum_{i=1}^4 d_i = 2$, indeed substituting, we find

$$\begin{aligned}\mathrm{Vol}(M_d) &= 2\pi(d_1 + d_2 + d_3) - 2\pi(2 - (d_1 + d_2 + d_3)) \\ &= 2\pi(2(d_1 + d_2 + d_3) - 2) \\ &= \mathrm{Vol}(\mathcal{M}_{0,4}).\end{aligned}$$

Similarly follows for the other cases. This proves, in the four point case, that the formula 5.7 has 5.6 as a limit whenever the sum of the weights equals 2. That means, the *volume function* is continuous when passing from the Fano geometry to the Calabi-Yau geometry. When $\sum_{i=1}^4 d_i > 2$ we may loose this behaviour unless we are still sufficiently close to 2, or we have some *control* on the weights. Alexeev, in [25, Theorem 1.1], proved that for small weights, i.e. in our case when $\sum_{i=1}^4 d_i = 2$, or small *perturbations* of this latter, the moduli space M_d can be identified with a GIT quotient. But, when the sum of the weights exceed significantly 2, we may not have a GIT quotient, therefore the previous techniques for calculating the volume can not be applied. As discussed in the first section of this chapter, the work of Alexeev [25] describes the moduli space of hyperplane arrangements by generalising the work of Hasset [24] so that the one dimensional case coincides with the *Hasset moduli space*. The dimension of the moduli space is 1 for any choice of weights, then we can calculate the volume in the general type geometry, by computing the degree of the log CM line bundle. To do so, we notice that M_d has a *universal family* obtained by considering the product $\mathbb{P}^1 \times \mathbb{P}^1$, fixing three points, $0, 1, \infty$, and allowing the last one to move in the diagonal $\Delta = \{(s,t) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid s = t\}$. We set

$$\mathcal{X} = \mathbb{P}^1 \times \mathbb{P}^1, \text{ and } \mathcal{D} = d_1 \cdot \mathbb{P}^1 \times \{0\} + d_2 \cdot \mathbb{P}^1 \times \{1\} + d_3 \cdot \mathbb{P}^1 \times \{\infty\} + d_4 \cdot \Delta.$$

Since we fixed three points, then as divisors we can assume that they have the same hyperplane class, that will be called h_2. Hence, we can rewrite $\mathcal{D}$ as follows

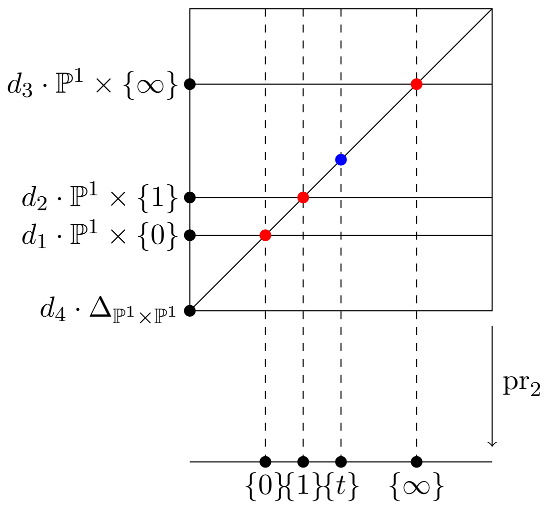
$$\mathcal{D} = \sum_{i=1}^{4} d_i \mathrm{pr}_2^* h_2 + d_4 \mathrm{pr}_1^* h_1.$$

Because of the construction of the Hasset's moduli space we must distinguish several situations. Heuristically, we have the following phenomena: when the point on the diagonal with weight d_4 meets one, two or three points, then a *wall* is crossed and therefore the universal family becomes singular at that point. This phenomena is encoded in the notion of *stability* given in [24] that we recall in the following

Definition 5.4.1. *Let $\pi : (C, s_1, ..., s_n) \to B$ be a proper and flat morphism of nodal curves of arithmetic genus g, and $s_1, ..., s_n$ are the sections of π corresponding to the marked points on C. A collection of data $(g, \mathcal{A}) = (g, a_1, ..., a_n)$ consist of an integer $g \geq 0$ and weights $(a_1, ..., a_n) \in \mathbb{Q}^n$, such that $0 < a_j \leq 1, \forall j \in \{1, 2, ..., n\}$ and $2g - 2 + a_1 + a_2 + \cdots + a_n > 0$. We say that π is* stable *if the following conditions are satisfied*

1. *The sections $s_1, ..., s_n$ are in the smooth locus of π, and for every subset $\{s_{i_1}, ...s_{i_r}\}$ with nonempty intersection we have $a_{i_1} + ... + a_{i_r} \leq 1$.*
2. *$K_\pi + \sum_{i=1}^n a_i s_i$ is relatively ample.*

Clearly, in our case $g = 0$, and $M_{0,4} \simeq \mathbb{P}^1$. The above definition, and the heuristic are rather *intuitive* as the following picture shows



In this picture, the dashed vertical lines represents the fibers of the map pr_2. To the red dots correspond singular points in the fibers, namely those for which the diagonal meets one or more divisors. The blue point is a smooth point for the fiber of $\{t\}$

When looking at the above picture, we shall assume that the weight d_4 is the smallest among the weights. So, while d_4 is free to move, and meet one (or more) points in the diagonal, then the rest of the points remain fixed.

- The following inequalities holds, when the diagonal does not meet any of the fixed points

$$\begin{cases} d_i + d_j > 1 & i, j \in \{1, 2, 3\} \\ d_k + d_4 < 1 & \text{for all } k \in \{1, 2, 3, 4\} \end{cases} \tag{5.13}$$

- If d_4 grows, then it meets one (or more) points along the diagonal, and a desingularization is needed

$$\begin{cases} d_i + d_4 > 1 & i \in \{1, 2, 3\} \\ d_j + d_4 \leq 1 & i \neq j. \end{cases} \tag{5.14}$$

As the above picture suggest we can take the projection onto the second factor of $\mathcal{X}$, namely $\mathrm{pr}_2 : (\mathcal{X}, \mathcal{D}) \to \mathbb{P}^1$, to get a $\mathbb{Q}$-Gorenstein family of log general type varieties. The stability conditions of Hasset suggest that we should choose as relatively ample line bundle for pr_2 the log canonical polarization $K_{\mathcal{X}/\mathbb{P}^1} + \mathcal{D}$. We also need some *adjustments* on the CM line bundle. As a consequence of Proposition 4.0.1, which is a general result, we get the following

Corollary 5.4.1. *Let $f\colon (\mathcal{X}, \mathcal{D}) \to B$ be a $\mathbb{Q}$-Gorenstein family of log general type varieties of relative dimension n, and with relatively ample line bundle $\mathcal{L} = K_{\mathcal{X}/B} + \mathcal{D}$. Then,*

$$c_1(\lambda_{\mathrm{CM}}) = f_* c_1(K_{\mathcal{X}/B} + \mathcal{D})^{n+1} \tag{5.15}$$

Proof By applying directly Proposition 4.0.1 we get

$$n \frac{(-K_{\mathcal{X}_b} - \mathcal{D}_b) \cdot (K_{\mathcal{X}_b} + \mathcal{D})^{n-1}}{(K_{\mathcal{X}_b + \mathcal{D}_b)^n})} = -n.$$

Thus,

$$\begin{aligned} c_1(\lambda_{\mathrm{CM},\mathcal{D}}) &= -n f_* c_1(K_{\mathcal{X}/B} + \mathcal{D})^{n+1} - (n+1) f_*(-(K_{\mathcal{X}/B} + \mathcal{D}) c_1(K_{\mathcal{X}/B} + \mathcal{D})^n) \\ &= -n f_* c_1(K_{\mathcal{X}/B} + \mathcal{D})^{n+1} + (n+1) f_* c_1(K_{\mathcal{X}/B} + \mathcal{D})^{n+1} \\ &= f_* c_1(K_{\mathcal{X}/B} + \mathcal{D})^{n+1}. \end{aligned}$$

□

Remark 5.4.1. *Consider (X, K_X) to be a general type normal variety, then the coefficient μ of the* absolute *CM line bundle equals $-n$. Indeed, consider a proper flat family $\pi : \mathcal{X} \to B$ of relative dimension n. Suppose that $\mathcal{L} \to \mathcal{X}$ is a relatively ample line bundle, and fibers of π $(\mathcal{X}_b, \mathcal{L}_b)$ are isomorphic to (X, K_X). Then, since π is flat, then the Hilbert polynomial doesn't change along its fibers, and it is like in Equation 2.1. by the Hirzebruch-Riemann-Roch Theorem we have $a_0 = \frac{\mathcal{L}_b^n}{n!}$, and $a_1 = \frac{-K_{\mathcal{X}_b} \cdot \mathcal{L}_b^{n-1}}{2(n-1)!}$. Replacing $\mathcal{L}_b$ with K_X we get that $a_1 = \frac{-K_X^n}{2(n-1)!}$. Since $\mu = \frac{2a_1}{a_0}$, then $\mu = -n$.*

We begin by studying the 5.13. We chose $K_{\mathcal{X}/\mathbb{P}^1} + \mathcal{D}$ as a relative polarization for the family $\mathrm{pr}_2\colon (\mathcal{X}, \mathcal{D}) \to \mathbb{P}^1$. By a direct application of Corollary 5.4.1 we have that

$$c_1(\lambda_{\mathrm{CM},\mathcal{D}}) = 2 d_4 \left(\sum_{i=1}^{4} d_i - 2 \right). \tag{5.16}$$

In 5.14, it is the situation when the diagonal meets one point. Suppose, without loss of generality, that the diagonal meets the zeroth fiber. The universal family becomes singular at that point, therefore we take the blowup at $\{0\}$ of the total space of the universal family. Set $\tilde{\mathcal{X}} = \mathrm{Bl}_0(\mathcal{X})$. The divisor $\mathcal{D}$, modifies as follows

$$\begin{aligned}\mathcal{D} &= d_1(\mathrm{pr}_2^* h_2 - E) + (d_2 + d_3)\mathrm{pr}_2^* h_2 + d_4(\mathrm{pr}_2^* h_2 + \mathrm{pr}_1^* h_1 - E) \\ &= \sum_{i=1}^{4} d_i \mathrm{pr}_2^* h_2 + d_4 \mathrm{pr}_1^* h_1 - (d_1 + d_4)E.\end{aligned}$$

Then,

$$K_{\tilde{\mathcal{X}}/\mathbb{P}^1} + \mathcal{D} = \left(\sum_{i=1}^{4} d_i - 2 \right) \mathrm{pr}_2^* h_2 + d_4 \mathrm{pr}_1^* h_1 - (d_1 + d_4 + 1)E.$$

By applying directly Corollary 5.4.1 we find Equation 5.16. Indeed the Chow ring of the blow up suggests that the intersection products $\mathrm{pr}_i^* h_i \cdot E = 0, i = 1, 2$. Moreover, the pushforward of the constant term $(d_1 + d_4 + 1)$ coming from $E^2 = -1$ term, is zero, therefore the only term that survives is only $\mathrm{pr}_{2*} 2d_4 \left(\sum_{i=1}^{4} d_i - 2 \right) \mathrm{pr}_2^* h_2 \cdot \mathrm{pr}_1^* h_1$, that yields to 5.16. The same holds if more than one desingularization is needed. The formula of the volume function for the log general type case is therefore constant with respect to the log canonical polarization. We notice also that when Equation 5.16 is evaluated in the Calabi-Yau zone then the volume function is zero. With respect to the anticanonical log polarization given in the Fano case we have a change of sign. This shows that the volume function has a discontinuity point in the Calabi-Yau zone. However, according to Theorem 5.1.3 if we choose weights whose sum is slightly greater than 2 the moduli space do not change, and it is described as the GIT quotient 5.4. Then, at least for 5.13, we can chose as relative polarization $-K_{\mathcal{X}/\mathbb{P}^1}$. Therefore, a fast computation proves that

$$c_1(\lambda_{\mathrm{CM},\mathcal{D}}) = 4d_4.$$

We can easily observe that up to a normalization factor, π, it coincides with the result obtained by applying 5.6, and 5.9. We resume all these results in the following

Proposition 5.4.1. *Let M_d be the moduli space of weighted hyperplane arrangements of dimension* 1. *Suppose that the weights $d_i \in (0,1) \cap \mathbb{Q}, i \in \{1, 2, 3, 4\}$ satisfy the following conditions*

- $d_j < \sum_{i \neq j} d_i, \forall j \in \{1, 2, 3, 4\}$;

•

$$d_1 + d_2 < d_3 + d_4$$
$$d_1 + d_3 < d_2 + d_4$$
$$d_1 + d_4 < d_2 + d_3.$$

Then, for small weights the volume of M_d changes continuously along the Fano, Calabi-Yau and general type geometry and it is given by $4\pi d_1$.

Remark 5.4.2. *In the condition of Proposition 5.4.1, when the sum of the weights largely exceed two, then we must chose another polarization, i.e. the log canonical polarization, as the Hasset compactification suggests. The calculations shows that the volume of M_d in this case is given by $2d_4\left(\sum_{i=1}^4 d_i - 2\right)$. Note that when approaching the Calabi-Yau geometry from the* far away *general type geometry, then the volume of M_d goes to zero.*

We wish to conclude this section with an observation. Consider the log Fano pair $(\mathbb{P}^1, \sum_{i=1}^4 d_i p_i)$. Chose some coordinate $z \in \mathbb{P}^1$, such that for a fixed $i \in \{1, 2, 3, 4\}$ we have $z(p_i) = 0$. Consider the vector field $v = z\partial_z$, and notice that it generates a one parameter subgroup $\lambda : \mathbb{C}^* \hookrightarrow \mathrm{GL}_1(\mathbb{C})$, that acts on $\mathbb{P}^1$ in the following fashion $(\lambda(t), z) := t \cdot z$. This actions easily translates on the divisor, $\mathcal{D}_t := t \cdot (\sum_{i=1}^4 d_i p_i)$, and on the anticanonical polarization of $\mathbb{P}^1$, namely $\mathcal{O}_{\mathbb{P}^1}(2)$. This data define a test configuration $\{(\mathbb{P}^1_t, \mathcal{D}_t, -K_{\mathbb{P}^1_t})\}_{t\in\mathbb{C}^*} = (\mathcal{X}, \mathcal{D}, K_{\mathcal{X}})$ whose central fiber $\mathcal{X}_0$ is just a point and the divisor, at the central fiber, behave as

$$\mathcal{D}_0 = d_i\{\infty\} + \sum_{i\neq j} d_j\{0\}. \tag{5.17}$$

Then, this latter defines an *integral test configuration.* As we mentioned in previous discussions and in [77] there exist a bijection between integral test configurations and *dreamy prime divisors.* We observe that since we are dealing with toric varieties, then every prime divisor is also dreamy, in the sense of definition 1.1.6. In particular in our central fiber 5.17 we have the sum of two dreamy prime divisor

$$W_1 = \{0\}, \text{ and } W_2 = \{\infty\}.$$

We have that

$$\mathrm{ord}_{W_1}(\mathcal{D}_0) = d_i, \text{ and } \mathrm{ord}_{W_2}(\mathcal{D}_0) = \sum_{i\neq j} d_j$$

The divisor W_1 was used to prove in 5.2.2 that $\mathrm{DF}_{\mathcal{D}}(\mathcal{X}, \mathcal{L}) = \sum_{i\neq j} d_j - d_i$. We will now show that the volume of $M_{0,4}$ can be obtained by summing all the Donaldson-Futaki invariant of the corresponding integral test configuration.

In 5.8 every single term $D_F^+ - D_F^-$, by the Jeffrey-Kirwan *residue theorem* [19] is associated to

$$\frac{i_B^* c_1(\mathcal{L}_d)(X)}{e_B(X)} = (-1)^{4-k}\mu(B)X^{-3} \tag{5.18}$$

where $B \in \mathcal{F} = \{(z^1, ..., z^4) \in (\mathbb{P}^1)^4 | \ z^j \in \{0, \infty\}\}$ is a fixed point for the maximal torus action. In 5.18 we used the fact that since $\mathcal{L}_d$ is a prequantum line bundle and the maximal torus is the unit circle then ([12], Lemma 9.31). The image of the moment map $\mu(B)$ is given by the weighted sum of the height function on each $\mathbb{P}^1$, namely

$$\mu(B) = \sum_{i=1}^{4} d_i \mu^i(z^i),$$

where $\mu^i(z^i)$ is the height function, i.e. the moment map for the maximal torus action on each $\mathbb{P}^1$. Clearly, the fixed points for this action are the standard basis element e_1, e_2 of $\mathbb{C}^2$ that corresponds to $0, \infty$ respectively. We have

$$\mu^i(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ -1 & \text{if } i = 2. \end{cases}$$

It follows in 5.8 that the index F is the one that tells how many e_2 we have in the fixed point B.

Example 5.4.1. $F = \{1\}$, *then*

$$\mu(B_F) = \sum_{i=2}^{4} d_i - d_1.$$

By theorem 5.2.2 the above is the Donaldson-Futaki invariant for the configurations of four points in $\mathbb{P}^1$.

If $|F| \geq 2$ we can always think of a integral test configuration for which two points came together into one point, where the weight of the latter is the sum of two or more weights. With this in mind, it is easy to convince ourselves that the size of the index set F tells how many points should come together into one point, and hence how many weights should be summed at that point. Therefore, the general term

$$\frac{i_B^* c_1(\mathcal{L})}{e_B(X)} = -\frac{(-1)^{4-k}}{x^3}\mathrm{DF}_{\mathcal{D}}(\mathcal{X}, \mathcal{L})(B),$$

as wanted. This observation, can be immediately generalised for a configurations of $n-$points in $\mathbb{P}^1$. The key point is that in the work of [78] it is shown that in the case of log Fano hyperplane arrangements the notion of

K-stability coincides with the notion of GIT-stability. Moreover by changing the weights, the corresponding moduli spaces are *well behaved* so that the Jeffrey-Kirwan localization formula of [19] can be applied for calculating the volume. The GIT quotient describing the hyperplane arrangements of dimension one is the following

$$M_d = (\mathbb{P}^1)^n//_{\mathcal{L}_d}\mathrm{SL}_2\mathbb{C}.$$

The dimension of M_d is $n-3$. We apply directly the Jeffrey-Kirwan theorem for the case of SU(2), on the top class namely

$$k(c_1(\mathcal{L}_d)^{n-3})e^{\omega_0}[M_d] = \frac{n_0}{2}\mathrm{Res}_{X=0}\left(4X^2\sum_{F\in\mathcal{F}_+}\int_F \frac{i_F^*c_1(\mathcal{L}_d)^{n-3}(X)}{e_F(X)}dX\right)$$

since $\mathcal{L}_d$ is a prequantum line bundle and the maximal torus is the unit circle then by Lemma 9.31 of [12] $i_f^*c_1(\mathcal{L}_d)^{n-3}(X) = (-1)^{n-3}\mu(F)^{n-3}(X)$. Hence,

$$\frac{i_F^*c_1(\mathcal{L}_d)^{n-3}}{e_F(X)} = -(-1)^{n-k}(\mu(F)(X))^{n-3}X^{-n-3}.$$

Because of the above observation, $\mu(F) = \mathrm{DF}_{\mathcal{D}}(\mathcal{X}_F, \mathcal{L}_F) = D_F^+ - D_F^-$. We have

$$4X^2\frac{i_F^*c_1(\mathcal{L}_d)^{n-3}}{e_F(X)} = -(-1)^{n-k}\mathrm{DF}_{\mathcal{D}}(\mathcal{X}_F, \mathcal{L}_F)^{n-3}X^{-1}. \tag{5.19}$$

The residue of X^{-1} in 5.19 is one. By taking the sum, we proved the following result

Theorem 5.4.2. *The volume of the moduli space of log Fano hyperplane arrangements of dimension one is the sum of the Donaldson-Futaki invariants associated to test configurations* $(\mathcal{X}_F, \mathcal{L}_F)$*. Where* $F \in \mathcal{F}_+$ *is like in 5.7, and* $(\mathcal{X}_F, \mathcal{L}_F)$ *is the test configuration for which* $|F|$ *points come together into one point.*

$$\mathrm{Vol}(M_d) = -\frac{(2\pi)^{n-3}}{2(n-3)!}\sum_{i=0}^{n-3}(-1)^k\sum_{F\in\mathcal{F}_+,\ |F|=k}\mathrm{DF}_{\mathcal{D}}(\mathcal{X}_F, \mathcal{L}_F)^{n-3}.$$

Remark 5.4.3. *It is natural to ask if more generally one can express the columes as* sums *of CM degrees of special unstable families, using the* Θ-*stratification [28] to generic* Kirwan's *conditions [28, Theorem 5.6].*

5.5 The volume of the moduli space of weighted hyperplane arrangements

Consider the GIT quotient

$$M_d = (\mathbb{P}^n)^m // _{\mathcal{L}_d} \mathrm{SL}(n+1)(\mathbb{C}) \tag{5.20}$$

with linearization $\mathcal{L}_d = \mathcal{O}(d_1, ..., d_m)$. A point in M_d corresponds to a log Fano hyperplane arrangement. $(\mathbb{P}^n, \sum_{i=1}^m d_i H_i)$. When a compact group K acts on a symplectic manifold X and $\mu\colon X \to \mathfrak{k}^*$ is a moment map for the action, the symplectic form on M induces a symplectic form on the quotient $\mu^{-1}(0)/K$, away from its singularities, which is the Marsden-Weinstein reduction or symplectic quotient of M by the action of K. In this setting, the symplectic quotient can be identified with a geometric invariant quotient $X//G$, where G is such that $K^{\mathbb{C}} = G$, where $K^{\mathbb{C}}$ is the complexification of K. The details of this correspondence can be found in [1, Remark 8.14].

Proposition 5.5.1. *Let X be a symplectic manifold endowed with a Hamiltonian action of a compact Lie group K. Let $\mu\colon X \to \mathfrak{k}^*$ be the moment map for this action. Then*

1. *$x \in X^{ss}$ if and only if $\mu^{-1}(0)$ meets the closure of the orbit Gx in M.*
2. *$x \in X^s$ if and only if $\mu^{-1}(0)$ meets the orbit Gx at a point whose stabilizer in K is finite.*
3. *The inclusion of $\mu^{-1}(0)$ in X^{ss} induces a homeomorphism $\mu^{-1}(0)/K \to X//G$.*

This construction applies immediately to Equation 5.20, indeed $(\mathbb{P}^n)^m$ is a symplectic manifold, and $\mathrm{SU}(n+1)^{\mathbb{C}} = G$. The moment map is well known in the literature, see for example [10, Example 5.5]. In this Chapter we wish to compute the volume of M_d using the Jeffrey-Kirwan non abelian localization theorem, also known as the residue theorem, which we briefly recall in the following

Theorem 5.5.1. *(Residue Theorem) [19], [20]Let (M, ω) be a symplectic manifold with a Hamiltonian action of a compact Lie group G with moment map μ. Assume that the center of $\mathfrak{g}$ is a regular value of the moment map.. Denote by M_{red} the corresponding symplectic reduction. Denote by K the Kirwan map. Then, given a formal equivariant class $\eta e^{\bar{\omega}} \in H^{\bullet}_G(M)$ is given by*

$$k(\eta e^{\bar{\omega}})[M_{red}] = n_0 C^G \mathrm{Res}\left(\mathcal{D}^2(X) \sum_{F \in \mathcal{F}} H^{\eta}_F(X)[dX]\right) \tag{5.21}$$

where n_0 denotes the order of the stabilizer in G of a generic point in $\mu^{-1}(0)$, the constant C^G depends on the volume of the maximal torus, on the Weyl factor and on the positive roots, namely

$$C^G = \frac{(-1)^{s+n_+}}{|W| vol(T)}$$

Where $s = \text{dim}G$*,* $n_+ = \frac{s-l}{2}$ *is the number of positive roots,* $l = \text{dim}T$ *and* W *is the Weyl group of* G*. The factor* $\mathcal{D}(X)$ *is called the* Weyl factor *and is defined as the product of the positive roots of* G*. The* $H_F^\eta(X)$ *are given by*

$$H_F^\eta(X) = e^{\mu(F)} \int_F \frac{i_F^* \eta(X) e^\omega}{e_F(X)}.$$

Here F *is a fixed point for the action of the maximal torus of* G*. The sum runs over the fixed* point *set* $\mathcal{F}$ *for the maximal torus action.*

Theorem 5.1.4 gives the necessary conditions for applying the Jeffrey-Kirwan residue theorem to M_d. For computing the volume, it is sufficient to set $\eta = 1$ in 5.21, so that the terms H_F^η becomes simply

$$H_F(X) := H_F^1(X) = \frac{e^{\mu(F)(X)}}{e_F(X)}$$

Indeed, the fixed point set for the maximal torus action consist of points F, hence $i_F^* \eta(X) e^\omega = (e^\omega)_0(F) = 1$. Furthermore, we chose a normalization for which $Vol(T) = 1$. Hence the formula to apply for calculating the volume will be the following

$$k(1.e^{\bar{\omega}})[M_{red}] = \frac{n_0(-1)^{s+n_+}}{|W|} \text{Res} \left(\mathcal{D}^2(X) \sum_{F \in \mathcal{F}} \frac{e^{\mu(F)(X)}}{e_F(X)} [dX] \right). \qquad (5.22)$$

The fixed points for the maximal torus action on each $\mathbb{P}^n$ are the standard basis elements of $\mathbb{C}^{n+1}$, which will be denoted as usual $\{e_i\}_{i=1}^{n+1}$. Therefore we have $(n+1)^m$ fixed points in $(\mathbb{P}^n)^m$, each of them is a string of elements of the standard basis. For the sake of clarity, we define a general notation that will be used henceforth.

- $d = (d_1, ..., d_m) \in \mathbb{R}_+^m$
- $\delta := \sum_i d_i$
- $[m] = \{1, 2, ..., m\}$
- If $I \subset [m]$, $D_I := \sum_{i \in I} d_i$
- $n := \text{dim}\mathbb{P}^n$
- $f := (f_1, ..., f_m) \in \mathcal{F} := [n+1]^m$, is identified with a fixed point in $(\mathbb{P}^n)^m$for the T-action.
- If $j \in [n+1]$, call $I_j(f) := \{i \in [n] : f_i = j\}$, $m_j(f) := |I_j(f)|$
- $\sum_{i=1}^{n+1} m_j(f) = m$

- $\delta_j(f) := D_{I_j(f)} = \sum_{i\in I_j(f)} d_i$, sometimes this latter will be just denoted by δ_j.

Set $h_f(X) := \mathcal{D}^2(X)\frac{e^{\mu(F)(X)}}{e_F(X)}$.

Proposition 5.5.2. *The function $h_f(X)$ with respect to the open cone Λ_+, is given by*

$$h_f(X) = (-1)^{m\cdot(n+1)-\sum_{j=1}^{n+1} m_j(f)\cdot j}\,\frac{e^{\sum_{i=1}^{n+1}\left(\sum_{k=1}^{i}(1-\frac{i}{n+1})\delta_k(f)-\sum_{k=i+1}^{n+1} i\frac{\delta_k(f)}{n+1}\right)\alpha_i\cdot}}{\prod_{1\le i<j\le n+1}\alpha_{ij}^{m_j+m_i-2}}.$$

Proof The *Weyl factor* is $\mathcal{D}^2(X) = \prod_{i<j}(\epsilon_i-\epsilon_j)^2$. The T-moment map on each $\mathbb{P}^n$ is simply given by

$$\begin{aligned}\mu_j(z_1,...,z_{n+1}) &= \sum_{i=1}^{n+1}\frac{|z_i|^2}{|z|^2}\epsilon_i\\ &= \sum_{i=1}^{n}\left(\sum_{k=1}^{i}\frac{|z_i|^2}{|z|^2}-\frac{k}{n+1}\right)\alpha_i.\end{aligned}$$

The *total* moment map is given by the weighted sum of the moment maps over each factor of $(\mathbb{P}^n)^m$, namely

$$\mu(z^1,...,z^j) = \sum_{j=1}^{m} d_j(f)\mu_j(z^j).$$

We recall that, via the Killing form, a basis for the dual of the lie algebra of the maximal torus of $SU(n+1)$ can be chosen such that it coincides with the dual basis $\{\epsilon_i\}_{i=1}^{n+1}$ of the Euclidean space $\mathbb{R}^{n+1*}$, with the condition that the sum of all the elements of such a basis is equal to zero. Using this basis, we can express the positive roots Δ_+ of $SU(n+1)$, as follows

$$\Delta_+ := \{\epsilon_i-\epsilon_j \mid 1\le i<j\le n+1\}.$$

Define $\alpha_{ij} = \epsilon_i-\epsilon_j$, and $\alpha_i = \alpha_{i,i+1}$.
On the fixed point f, we have

$$\mu(f) = \sum_{j=1}^{m} d_j(f)\mu_j(e_j) = \sum_{s=1}^{n+1}\delta_s(f)\mu(e_s). \tag{5.23}$$

Moreover,

$$\mu(e_s) = \sum_{i=1}^{k}\frac{|z_i|^2}{|z|^2}-\frac{k}{n+1} = \begin{cases}-\frac{k}{n+1} & \text{if } s>j\\ 1-\frac{k}{n+1} & \text{if } s\le j.\end{cases}$$

Then,

$$\mu(e_s) = \frac{1}{n+1}\big(-\alpha_1 - ... - (s-1)\alpha_{s-1} + \\ +(n+1-s)\alpha_s + (n-s)\alpha_{s+1} + ... + 1\cdot\alpha_n\big).$$

The coefficient α_i is $\frac{-i}{n+1}$, when $i < s$, and when $i \geq s$ is $\frac{n+1-i}{n}$. Therefore the coefficient of α_i is given by

$$\sum_{k=1}^{i}\left(1-\frac{i}{n+1}\right)\delta_k(f) - \sum_{k=i+1}^{n+1} i\frac{\delta_k}{n+1}. \tag{5.24}$$

Therefore, we get

$$\mu(f) = \sum_{i=1}^{n+1}\left(\sum_{k=1}^{i}\left(1-\frac{i}{n+1}\right)\delta_k(f) - \sum_{k=i+1}^{n+1} i\frac{\delta_k}{n+1}\right)\alpha_i.$$

Comparing with 5.23, we get

$$\begin{aligned}
\mu(f) &= \sum_{s=1}^{n+1}\delta_s\left(\sum_{i=s}^{n}\left(1-\frac{i}{n+1}\right)\alpha_i + \sum_{i\leq s-1}\frac{-i}{n+1}\alpha_i\right)\\
&= \sum_{s}\sum_{i\leq s-1}\delta_s\left(\frac{-i}{n+1}\right)\alpha_i + \sum_{s}\sum_{i\geq s}\delta_s\left(1-\frac{i}{n+1}\right)\alpha_i\\
&= \sum_{i=1}^{n}\left[\sum_{i\leq s-1}\left(\frac{-i}{n+1}\right)\delta_s\right]\alpha_i + \sum_{i=1}^{n}\left[\sum_{i\geq s}\left(1-\frac{i}{n+1}\right)\delta_s\right]\alpha_i\\
&= \sum_{i=1}^{n}\left[-\sum_{s>i}\frac{i\delta_s}{n+1} + \sum_{s\leq i}\left(1-\frac{i}{n+1}\right)\delta_s\right]\alpha_i.
\end{aligned}$$

Hence,

$$\mu(f) = \sum_{i=1}^{n}\left[\left(1-\frac{i}{n+1}\right)\cdot\sum_{k=1}^{i}\delta_k - \frac{i}{n+1}\cdot\left(\sum_{k=i+1}^{n+1}\delta_k\right)\right]\alpha_i.$$

Now we calculate the Euler class of the normal bundle at the generic fixed point f. The equivariant Euler class of the normal bundle to f is defined as:

$$e_f(X) = \prod_j (c_1(\nu_{f,j}) + \beta_{f,j}(X))$$

Where the $\nu_{f,j}$ are the line bundles of the formal splitting of the normal bundle to f given by the *splitting principle* and the $\beta_{f,j}$ are the weights of the torus action at the fixed point f. Observe that in our case the terms $c_1(\nu_{f,j})$ contributes to zero, and

$$\nu_f = \bigoplus_{m_1} T_{e_1}\mathbb{P}^n \oplus \bigoplus_{m_2} T_{e_2}\mathbb{P}^n \oplus \ ... \ \oplus \bigoplus_{m_{n+1}} T_{e_{n+1}}\mathbb{P}^n$$

In order to calculate the weights on the normal bundle we can take the chart on $\mathbb{P}^n$ such that $z_1 \neq 0$, then we define new coordinates $u_1 = \frac{z_2}{z_1}$ and $u_2 = \frac{z_3}{z_1}, ..., \frac{z_{n+1}}{z_1}$. Thus $T \cdot (u_1, u_2, ..., u_n) = \left(e^{t_2 - t_1} u_1, e^{t_3 - t_1} u_2, ..., e^{t_{n+1} - t_1} u_n\right)$. Identifying the t_i with the ϵ_i and noticing that

$$\sum_{j=k}^{i-1} \alpha_j = \epsilon_k - \epsilon_i, \text{ for } k = 1, ..., i-1,$$

$$-\sum_{j=i}^{k-1} \alpha_j = \epsilon_k - \epsilon_i, \text{ for } k = i+1, ..., n+1,$$

we find that the equivariant Euler class at e_1 is given by

$$e(T_{e_1}\mathbb{P}^n) = (-1)^n \prod_{k=1}^{n+1} \left(\sum_{j=1}^{k-1} \alpha_j \right).$$

Similarly, we find, for all $s \in \{1, 2, ..., n+1\}$

$$e(T_{e_s}\mathbb{P}^n) = (-1)^{n-s+1} \prod_{k=1}^{s-1} \left(\sum_{j=k}^{s-1} \alpha_j \right) \cdot \prod_{k=s+1}^{n+1} \left(\sum_{j=s}^{k-1} \alpha_j \right). \tag{5.25}$$

Since

$$\sum_{j=k}^{s-1} \alpha_j = \alpha_{ks}, \text{ if } k \leq s-1,$$

$$\sum_{j=s}^{k-1} \alpha_j = \alpha_{sk}, \text{ if } k \geq s+1,$$

5.25 becomes

$$e(T_{e_s}\mathbb{P}^n) = (-1)^{n-s+1} \prod_{k=1}^{s-1} \alpha_{ks} \cdot \prod_{k=s+1}^{n+1} \alpha_{sk}. \tag{5.26}$$

Let f be a fixed point, then the Euler class at that point is

$$e_f(X) = \prod_{s=1}^{n+1} e(T_{e_s}\mathbb{P}^n)^{m_s(f)}. \tag{5.27}$$

Using 5.26 we can compute 5.27. Firstly, notice that, since $\sum_{s=1}^{n+1} m_s(f) = m$ we have the following

$$\begin{aligned}\sum_{s=1}^{n+1} m_s(f)(n-s+1) &= (n+1)\sum_{s=1}^{n+1} m_s(f) - \sum_{i=1}^{n+1} m_s(f)\cdot s \\ &= (n+1)m - \sum_{s=1}^{n+1} m_s(f)\cdot s.\end{aligned}$$

For later use, we define

$$\epsilon = (-1)^{(n+1)m - \sum_{j=1}^{n+1} m_j(f)\cdot j}.$$

It remains to calculate the exponent of the remaining factors in 5.26. In order to do so, suppose we take the positive root $\alpha_{ij} := \epsilon_i - \epsilon_j$, with $i < j$. It appears in 5.26 exactly one time if $s = j$, and exactly one time if $i = s$. Hence, the exponent of α_{ij} is given by $m_j + m_i$. Thus,

$$e_f(\nu_f) = \epsilon \cdot \prod_{1\le i<j\le n+1} \alpha_{ij}^{m_j+m_i}.$$

Finally, noticing that the Weyl factor is the product of positive roots, by using the definition of $h_f(X)$ the claim follows. □

In the next two sections we calculate explicitly the GIT volume for $n = 1$, and $n = 2$.

5.6 The dimension one case

The volume formula for M_d, in the one dimensional case, is calculated in many works [37], [34], [36], the case of $d = 1$ was calculated in [33]. We will now provide a simple proof of the M_d's volume which is coherent with the cited works.

Theorem 5.6.1. *The volume of M_d with $d_i \in (0,1)\cap\mathbb{Q}$, for all $i \in \{1,2,....,m\}$, and $m \ge 4$ is given by*

$$\mathrm{vol}(M_d) = -\frac{1}{2(m-3)!}\sum_{f\in\mathcal{F}_+} (-1)^{m_1(f)}\left(\delta_1(f) - \delta_2(f)\right)^{m-3} \qquad (5.28)$$

Where

$$\mathcal{F}_+ = \{f \in \mathcal{F} = [2]^m | \delta_1(f) - \delta_2(f) > 0\}$$

Proof By a direct application of 5.5.2 we find that the general term of the residue $h(X)$ is given by

$$h(X) = \frac{e^{\mu(F)(X)}}{e_F(X)} = (-1)^{m_1(f)} \frac{e^{\lambda \alpha_1}}{\alpha_1^{m-2}}.$$

Where in the Euler class we used the identity $m_1(f) + m_2(f) = m$. .We see that the given meromorphic function $\frac{e^{\lambda \alpha_1}}{\alpha_1^{m-2}}$ as a pole in zero with mutiplicity $m-2$. Then the residue at zero is given by

$$\text{res}^+_{\alpha_1=0} \left(\frac{e^{\lambda \alpha_1}}{\alpha_1^{m-2}} \right) := \begin{cases} \frac{\lambda^{m-3}}{(m-3)!} & \text{if } \lambda > 0 \\ 0 & \text{else.} \end{cases} \tag{5.29}$$

The coefficient λ is easily calculated using 5.5.2 to find

$$\mu(f) = \frac{1}{2}(\delta_1(f) - \delta_2(f))\alpha_1, \tag{5.30}$$

therefore $\lambda = \frac{1}{2}(\delta_1(f) - \delta_2(f))$. The Weyl group W has dimension 2 and therefore the constant C^G is equal to $-1/2$. Since $n_0 = 2^{m-3}$, by taking the sum over the fixed points such that $2\lambda > 0$ the claim follows immediately.

□

Remark 5.6.1. *In the volume formula obtained in [37], differs just by a different choice of normalization. Indeed in [37] the Fubini study form is defined with integral weights. With this in mind the result is given by*

$$\text{vol}(M_d) = -\frac{(2\pi)^{m-3}}{2(m-3)!} \sum_{f \in \mathcal{F}_+} (-1)^{m_1(f)} \left(\delta_1(f) - \delta_2(f)\right)^{m-3} \tag{5.31}$$

A natural question is whether there is a continuity argument for the volume function also in the general case. A result from McMullen in [32] (Theorem 8.1), provides a general formula for the volume of $\mathcal{M}_{0,n}$.

Theorem 5.6.2. *([32], Theorem 8.1) Let $\mathcal{M}_{0,n}$ be the moduli space of n ordered points on the Riemann sphere. Then the complex hyperbolic volume of $\mathcal{M}_{0,n}$ is given by*

$$\text{Vol}(\mathcal{M}_{0,n}) = C_{n-3} \sum_{\mathcal{P}} (-1)^{|\mathcal{P}|+1} (|\mathcal{P}| - 3)! \prod_{B \in \mathcal{P}} \max\left(0, 1 - \sum_{i \in B} \mu_i \right)^{|B|-1}, \tag{5.32}$$

where $\sum_{i=1}^n \mu_i = 2$, $0 < \mu_i < 1$, $\forall i \in \{1, 2, ..., n\}$, and $C_n = \frac{(-4\pi)^n}{(n+1)!}$. Here $\mathcal{P}$ ranges over all partitions of the indices $(1, ..., n)$ into blocks B.

Equation 5.32 follows from the general Gauss-Bonnet theorem for cone manifolds

Theorem 5.6.3. *([32, Theorem 1.1]) Let M, be a cone manifold of dimension n. Denote by $M[n]$ the union of its strata M^σ. Then,*

$$\int_{M[n]} \psi(x)dV(x) = \sum_{\sigma} \chi(M^\sigma)\Theta^\sigma. \tag{5.33}$$

Where on the left hand side we have the curvature intgral, $\chi(M^\sigma)$ denotes the Euler characteristic of the strata, and Θ^σ is the solid angle at each stratum.

The solid angle for the moduli space of polygons was calculated by Thurston in [31, Proposition 3.6], and is given by

$$\Theta^{\mathcal{P}} = \prod_{B\in\mathcal{P}} (1 - \sum_{i\in B} \mu_i)^{|B|-1}.$$

The strata of M_d , $M^{\mathcal{P}}$, are indexed by some partition set $\mathcal{P} \subset [m]$, and they result from collisions between the vertices of the associated polyhedron which keep it locally convex. The solid angle at each strata, is nothing but the vertex angle. In the case of four points we have 3 connected strata, that are two tetrahedron, that correspond to the case when one out of four points come together with another point, and a square, that correspond to the case when we have two couple of two points coming together. The cardinality of the partition set $\mathcal{P}$ expresses the number of blocks $B \subset \mathcal{P}$. The size of those blocks correspond to the vertex of the polyhedron. In general, the Euler characteristic of $\mathcal{M}_{0,n}$ is simply given by $(-1)^{n-3}(n-3)!$, this can be proved inductively by using the map $\mathcal{M}_{0,n} \to \mathcal{M}_{0,n-1}$, which is a fibration with fiber $\mathbb{P}^1 \setminus \{$*(n-3)-points*$\}$. At each $\mathcal{P} \subset \{1, 2, ...n\}$, $M^{\mathcal{P}} \simeq \mathcal{M}_{0,|\mathcal{P}|}$. Thus, the Euler characteristic at each strata, is given by $\chi(M^{\mathcal{P}}) = \chi(\mathcal{M}_{0,|\mathcal{P}|}) = (-1)^{|\mathcal{P}|+1}(|\mathcal{P}| - 3)!$. Using Theorem 5.6.3, and Equation 5.5, together with all these observations, the volume formula 5.32 follows. Notice that the case of four points on the projective line follows from 5.32. Indeed, $C_1 = -2\pi$, the only partitions that make sense are those with four and three blocks. When $|\mathcal{P}| = 4$ then we have four blocks of size 1, and since there is only one partition with four blocks, then it contributes -1 in the sum. When $|\mathcal{P}| = 3$ then we have six different partitions each of which consist of a block of size 2 and two blocks of size 1. But, this is exactly like Formula 5.6. When the sum of all the weights is equal to 2, then Equation 5.30 becomes

$$\lambda = \mu(f) = (1 - \delta_2(f))\alpha_1.$$

If we allow Equation 5.29 to consider also negative value, then we have

$$\operatorname{res}_{\alpha_1=0}\left(\frac{e^{\lambda\alpha_1}}{\alpha_1^{m-2}}\right) = \frac{(2\pi)^{m-3}}{(m-3)!}\max\left(0, 1 - \delta_2(f)\right)^{m-3}.$$

The set $\mathcal{F}_+$ in this case is the set of all fixed points, indeed whenever $\delta_2(f) > 1$, $\max(0, 1 - \delta_2(f)) = 0$.
Thus, Equation 5.31 becomes

$$\mathrm{Vol}(M_d) = \frac{(4\pi)^{m-3}}{(m-3)!} \sum_{f \in \mathcal{F}} (-1)^{m_1(f)+1} \max(0, 1 - \delta_2(f))^{m-3} \tag{5.34}$$

Along the line of the case of four points, in order to see that Equation 5.32 and Equation 5.34 are equal, we shall firstly give an order to the weights d_i. We carry on explicitly the computation for the case $m = 5$. Suppose that the following inequalities hold

$$\begin{aligned} d_i &< \sum_{j \neq i} d_i, \ \forall i \in [5]; \\ \sum_{i \in \{3,j\}^c} d_i &< \sum_{i \in \{3,j\}} d_i, \forall j = 1,2,4,5, \ \{3,j\}^c = [5] \setminus \{3,j\}; \\ \sum_{i \in \{j,k,l\}^c} d_i &< \sum_{i \in \{j,k,l\}} d_i, \forall j,k,l = 1,2,3,4; \{j,k,l\}^c = [5] \setminus \{j,k,l\}. \end{aligned} \tag{5.35}$$

A set of weights that satisfies the above conditions together with the Calabi Yau condition, could be given by the following vector in $\mathbb{Q}^5$:

$$\frac{2}{17}(2, 1, 8, 2, 4). \tag{5.36}$$

By applying Equation 5.34 to the conditions 5.35, and to the vector 5.36, we get

$$\mathrm{Vol}(M_d) = 2\pi^2(2 - 2d_3) = \left(\frac{2}{17}\right)^2 2\pi^2. \tag{5.37}$$

Equation 5.32 is more laborious to apply. We carry on the computation as follows. The number of blocks, that is the cardinality of each permutation $\mathcal{P}$, that make sense in 5.32 is when $|\mathcal{P}| = 5, 4, 3$. When $|\mathcal{P}| = 5$, then there is only one permutation made of blocks whose size is 1. This latter permutation contributes 2 to the whole sum. When $|\mathcal{P}| = 4$, then we have 10 partitions with four blocks. These blocks, in each partition of this kind, have one element of size 2, and three elements of size 1. And are listed below

$$\{1,2\},\{3\},\{4\},\{5\};$$
$$\{1,3\},\{2\},\{4\},\{5\};$$
$$\{1,4\},\{2\},\{3\},\{5\};$$
$$\{1,5\},\{2\},\{3\},\{4\};$$
$$\{2,3\},\{1\},\{4\},\{5\};$$
$$\{2,4\},\{1\},\{3\},\{5\};$$
$$\{2,5\},\{1\},\{3\},\{4\};$$
$$\{3,4\},\{1\},\{2\},\{5\};$$
$$\{3,5\},\{1\},\{2\},\{4\};$$
$$\{4,5\},\{1\},\{2\},\{3\}.$$

Using the conditions 5.35 we see that not all of the above listed partition will contribute to the sum. Indeed, when the block of size two contains 3 then it contributes to zero. For example, consider the partition $\{1,3\},\{2\},\{4\},\{5\}$, then the solid angle term for this partition is given by

$$\max(0,1-d_1-d_3)\cdot\max(0,1-d_2)^0\cdot$$
$$\max(0,1-d_4)^0\cdot\max(0,1-d_5)^0=\max(0,1-d_1-d_3).$$

But, because of the conditions 5.35 and 5.36, d_3 is the biggest weight, therefore the sum of $d_1+d_3>1$, thus $\max(0,1-d_1-d_3)=0$. Similarly holds for the rest of the partitions whose block of size two contains 3. Hence, the contribution to the sum in 5.32 for the above set of partitions consist of 6 partitions. Therefore, the sum gives

$$\begin{aligned}&-(1-d_1-d_2+1-d_1-d_4+1-d_1-d_5\\&+1-d_2-d_4+1-d_2-d_5+1-d_4-d_5)\\=&-(6-3(d_1+d_2+d_4+d_5)=-(6-3(2-d_3))\\&=-3d_3=-\frac{48}{17}.\end{aligned}$$

When $|P|=3$ then we have 25 partitions. These latter can be subdivided into two groups depending on the size of their blocks. Indeed, we have

(a) a group of two blocks of size 2, and one block of size 1.

(b) a group of one block of size 3, and two blocks of size 1.

In the group itemized by the letter (a), most of the terms will contribute to zero in the sum, since most of them have 3 in most of the blocks of size 2. The partitions that survive in this case are listed below

$$\{1,2\},\{4,5\},\{3\};$$
$$\{1,4\},\{2,5\},\{3\};$$
$$\{1,5\},\{2,4\},\{3\}.$$

In the group itemized by the letter (b), the blocks of size three that contain 3 will contribute to zero in the sum. Therefore, the partitions that will contribute to the sum are the following

$$\{1,2,4\},\{3\},\{5\};$$
$$\{1,2,5\},\{3\},\{4\};$$
$$\{1,4,5\},\{2\},\{3\};$$
$$\{2,4,5\},\{1\},\{3\}.$$

Thus, for the partitions of 3 blocks, the sum gives

$$\begin{aligned}(1-d_1-d_2)(1-d_4-d_5)+(1-d_1-d_4)(1-d_2-d_5)\\ +(1-d_1-d_5)(1-d_2-d_4)+(1-d_1-d_2-d_4)^2\\ +(1-d_1-d_2-d_5)^2+(1-d_1-d_4-d_5)^2+\\ (1-d_2-d_4-d_5)^2=\frac{241}{17^2}.\end{aligned}$$

Now, we are ready to apply 5.32

$$\begin{aligned}\mathrm{Vol}(\mathcal{M}_{0,5}) &= \frac{(-4\pi)^2}{3!}\cdot\left(2-\frac{48}{17}+\frac{241}{17^2}\right)\\ &= \frac{2^4\pi^2}{1\cdot 2\cdot 3}\cdot\frac{3}{17^2}=2^3\pi^2\left(\frac{1}{17}\right)^2\\ &= \left(\frac{2}{17}\right)^2 2\pi^2.\end{aligned}$$

That means, $\mathrm{Vol}(\mathcal{M}_{0,5})=\mathrm{Vol}(M_d)$, as wanted. We have two explicit examples where these formulas coincide. That is, the case of four, and five points in $\mathbb{P}^1$. A general purely *combinatoric* proof of the two formulas, could be rather complicated. Indeed, it is hard to express 5.32 as a function of the number of points m, like in 5.34 and then argue by induction. But, the calculations on these explicit cases, suggest that when the sum of the weights equals two, then the two formulas match, and therefore the *volume function* is continuous when passing from the Fano geometry to the Calabi-Yau geometry.

5.7 The dimension two case

Consider the GIT quotient

$$M_d = (\mathbb{P}^2)^m //_{\mathcal{L}_d} \mathrm{SU}(3)$$

with linearizzation $\mathcal{L} = \mathcal{O}(d_1, ..., d_m)$. Where $d_i \in (0,1) \cap \mathbb{Q}, \forall i \in \{1, 2..., m\}$. Points in M_d are pairs (X, D) where X is a copy of $\mathbb{P}^2$ and D is a divisor in $\mathbb{P}^2$ identified as the weighted sum of the hyperplane classes H_i, that is $D = \sum_{i=1}^m d_i H_i$. Furthermore, we require for each pair (X, D) that the relative log anticanonical bundle $L = -(K_X + D)$ is ample. This latter leads to require that $\sum_{i=1}^m d_i < 3$ and therefore every pair is a log Fano pair. Fix the following notation. Set $\xi_i := \delta_i - \delta/3$ for $i = 1, 2, 3$. Then $\xi_1 + \xi_2 + \xi_3 = 0$ and

$$\begin{aligned} \lambda_1 &= \frac{2\delta_1 - \delta_2 - \delta_3}{3} = \delta_1 - \delta/3 = \xi_1, \\ \lambda_2 &= \frac{\delta_1 + \delta_2 - 2\delta_3}{3} = \delta/3 - \delta_3 = -\xi_3 = \xi_1 + \xi_2, \\ \lambda_1 - \lambda_2 &= \delta/3 - \delta_2 = -\xi_2. \end{aligned} \tag{5.38}$$

$$\begin{aligned} \mathcal{A}(d) &= \{f \in [3]^m \ | \lambda_1 > \lambda_2 > 0\} = \{f \in [3]^m : \xi_2 < 0, \xi_3 < 0\}, \\ \mathcal{B}(d) &= \{f \in [3]^m | \ \lambda_2 > \lambda_1 > 0\} = \{f \in [3]^m | \ \xi_1 > 0, \ \xi_2 > 0\}. \end{aligned} \tag{5.39}$$

This section is dedicated to the proof of the following result.

Theorem 5.7.1. *The GIT volume of the moduli space M_d of line arrangements in $\mathbb{P}^2$ is given by*

$$\mathrm{Vol}(M_d) =$$

$$= -\sum_{f \in \mathcal{A}(d)} \frac{(-1)^{m_2(f)}}{6(2m-8)!} \sum_{j=0}^{2m-8} \binom{2m-8}{j} \binom{m + m_2 - 6 - j}{m_2 + m_3 - 3} \xi_2^j \xi_3^{2m-8-j} +$$

$$- \sum_{f \in \mathcal{B}(d)} \frac{(-1)^{m_2(f)}}{6(2m-8)!} \sum_{j=0}^{2m-8} \binom{2m-8}{j} \binom{m + m_2 - 6 - j}{m_1 + m_2 - 3} \xi_1^j \xi_2^{2m-8-j}.$$

Proof We start recalling some elementary combinatorial identities. The *falling factorial powers* are defined as follows [13, p.47-48]: for $k \in \mathbb{Z}, k \geq 0$ and $z \in \mathbb{C}$ set

$$z^{\underline{k}} := \begin{cases} 1 & \text{if } k = 0 \\ \prod_{j=0}^{k-1} (z - j) & \text{otherwise.} \end{cases} \tag{5.40}$$

We set $0^0 := 1$, so the function x^0 is identically $= 1$ on the real line. Thus for any $\xi \in \mathbb{R}$ and $k \in \mathbb{Z}$, $k \geq 0$ we have

$$D^k(x^\xi) = \xi^{\underline{k}} x^{\xi - k}. \tag{5.41}$$

For $z \in \mathbb{C}$ and $k \in \mathbb{Z}$ set [13, p. 154]:

$$\binom{z}{k} := \begin{cases} \frac{z^{\underline{k}}}{k!} & \text{if } k \geq 0; \\ 0 & \text{if } k < 0. \end{cases} \tag{5.42}$$

We have

$$z^{\underline{k}} = (-1)^k (k - z - 1)^{\underline{k}}, \quad \binom{z}{k} = (-1)^k \binom{k - z - 1}{k}.$$

(See [13, p.164].)

$$\sum_j (-1)^j \binom{s+j}{n} \binom{l}{m+j} = (-1)^{l+m} \binom{s-m}{n-l}. \tag{5.43}$$

See formula (5.24) in [13, p. 169]. If $l \geq 0$, s, m are integers, then

$$\sum_j \binom{l}{m+j} \binom{s}{n+j} = \binom{l+s}{l-m+n}. \tag{5.44}$$

See (5.23) in [13, p. 169]. If $n > m \geq 0$, then

$$\binom{m}{n} = 0 \tag{5.45}$$

For integers $n, m \geq 0$ we have

$$(-1)^m \binom{-n-1}{m} = (-1)^n \binom{-m-1}{n} \tag{5.46}$$

See (5.15) of [13, p. 165].

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k} \tag{5.47}$$

This is (5.14) of [13, p. 164].
Since we are specializing to $n = 2$ we have $f \in [3]^m$ the formula in Proposition 5.5.2 becomes

$$h_f = (-1)^{m - \sum_{j=1}^{3} j \cdot m_j(f)} \frac{e^{\lambda_1 \alpha_1 + \lambda_2 \alpha_2}}{\alpha_1^{m_1+m_2-2} \alpha_2^{m_2+m_3-2} (\alpha_1 + \alpha_2)^{m_1+m_3-2}} \tag{5.48}$$

$$\lambda_1 = \lambda_1(f, d) := \frac{2\delta_1 - \delta_2 - \delta_3}{3}$$

$$\lambda_2 = \lambda_2(f, d) := \frac{\delta_1 + \delta_2 - 2\delta_3}{3}.$$

We use $\{x = \alpha_1, y = \alpha_2\}$ as a basis. For $\lambda_1, \lambda_2 \in \mathbb{R}$ and $a, b, c \in \mathbb{Z}$, set

$$\begin{aligned} R(\lambda_1, \lambda_2, a, b, c,) := \operatorname{res}^{\lambda}\left[\frac{e^{\lambda_1 x + \lambda_2 y}}{x^a y^b (x+y)^c}\right] = \\ = R(\lambda_1, \lambda_2, a, b, c,) = \operatorname{res}_x^+ \operatorname{res}_y^+ \left[\frac{e^{\lambda_1 x + \lambda_2 y}}{x^a y^b (x+y)^c}\right] = \operatorname{res}_x^+ \frac{e^{\lambda_1 x}}{x^a} \operatorname{res}_y^+ \left[\frac{e^{\lambda_2 y}}{y^b (x+y)^c}\right]. \end{aligned} \tag{5.49}$$

The following is well-known.

Lemma 5.7.1. *If f is a meromorphic function which is holomorphic at z_0, then*

$$\operatorname{res}_{z=z_0} \frac{g(z)}{(z-z_0)^k} = \frac{D^{k-1} g(z_0)}{(k-1)!} \quad \textit{if } k \geq 1 \tag{5.50}$$

and vanishes otherwise.

For $n \in \mathbb{Z}$ set

$$\phi_n(x) := \frac{e^x}{x^n} = x^{-n} e^x. \tag{5.51}$$

By Leibniz formula for higher derivatives we have

$$D^k \phi_n = D^k(x^{-n} e^x) = e^x \sum_{j=0}^{k} \binom{k}{j} (-n)^{\underline{j}} x^{-n-j}. \tag{5.52}$$

For $\lambda, a \in \mathbb{R}$ and $n \in \mathbb{Z}$ set

$$g_{\lambda,a,n}(z) := \frac{e^{\lambda z}}{(z-a)^n} = e^{\lambda a} \lambda^n \phi_n(\lambda z - \lambda a).$$

Observe that if f is a function, λ, b are constants and $h(x) := f(\lambda x + b)$, then for any k

$$D^k h(x) = \lambda^k \cdot (D^k f)(\lambda x + b).$$

Thus

$$\begin{aligned} \left(D^k g_{\lambda,a,n}\right)(z) = \lambda^{n+k} e^{\lambda a} \left(D^k \phi_n\right)(\lambda z - \lambda a) = \\ = \left(D^k g_{\lambda,a,n}\right)(z) = e^{\lambda z} \sum_{j=0}^{k} \binom{k}{j} (-n)^{\underline{j}} \lambda^{k-j} (z-a)^{-n-j}. \end{aligned} \tag{5.53}$$

Let us adopt the following convention: if P is an inequality, then $[P]$ is equal to 1 if P holds and to 0 otherwise. By Lemma 5.7.1 for any $p, q \in \mathbb{Z}$ we have

$$\begin{aligned}\operatorname{res}_{y=0}\left[\frac{e^{\lambda y}}{y^p(x+y)^q}\right] &= \operatorname{res}_{y=0}\left[\frac{g_{\lambda,-x,q}(y)}{y^p}\right] = \frac{[p \geq 1]}{(p-1)!} D^{p-1} g_{\lambda,-x,q}(0) = \\ &= \frac{1}{(p-1)!}\sum_{j=0}^{p-1}\binom{p-1}{j}\frac{(-q)^{\underline{j}}\lambda^{p-1-j}}{x^{q+j}} = \\ &= \sum_{j=0}^{p-1}\frac{(-q)^{\underline{j}}\lambda^{p-1-j}}{j!(p-1-j)!x^{q+j}}.\end{aligned} \tag{5.54}$$

In the same way

$$\operatorname{res}_{y=-x}\left[\frac{e^{\lambda y}}{y^p(x+y)^q}\right] = e^{-\lambda x}\sum_{j=0}^{q-1}\frac{(-1)^{p+j}(-p)^{\underline{j}}\lambda^{q-1-j}}{j!(q-1-j)!x^{p+j}}. \tag{5.55}$$

Hence

$$\operatorname{res}_y^+\left[\frac{e^{\lambda_2 y}}{y^b(x+y)^c}\right] =$$
$$= [\lambda_2 \geq 0]\cdot\sum_{j=0}^{b-1}\frac{(-c)^{\underline{j}}\lambda_2^{b-1-j}}{j!(b-1-j)!x^{c+j}} + [\lambda_2 \geq 0]\cdot\sum_{j=0}^{c-1}\frac{(-1)^{b+j}(-b)^{\underline{j}}\lambda_2^{c-1-j}e^{-\lambda_2 x}}{j!(c-1-j)!x^{b+j}}.$$

We plug this in (5.49):

$$\begin{aligned}R(\lambda_1,\lambda_2,a,b,c) &= [\lambda_2 \geq 0]\cdot\sum_{j=0}^{b-1}\frac{(-c)^{\underline{j}}\lambda_2^{b-1-j}}{j!(b-1-j)!}\cdot\operatorname{res}_x^+\frac{e^{\lambda_1 x}}{x^{a+c+j}} + \\ &+[\lambda_2 \geq 0]\cdot\sum_{j=0}^{c-1}\frac{(-1)^{b+j}(-b)^{\underline{j}}\lambda_2^{c-1-j}}{j!(c-1-j)!}\cdot\operatorname{res}_x^+\frac{e^{(\lambda_1-\lambda_2)x}}{x^{a+b+j}}.\end{aligned} \tag{5.56}$$

By Lemma 5.7.1

$$\operatorname{res}_{x=0}\frac{e^{\lambda x}}{x^k} = \frac{[k \geq 1]\cdot\lambda^{k-1}}{(k-1)!}, \qquad \operatorname{res}_x^+\frac{e^{\lambda x}}{x^k} = [\lambda \geq 0]\cdot[k \geq 1]\cdot\frac{\lambda^{k-1}}{(k-1)!}.$$

So in (5.56) only the terms with $a+b+j \geq 1$ survive:

$$R(\lambda_1,\lambda_2,a,b,c,) =$$
$$= [\lambda_1 \geq 0]\cdot[\lambda_2 \geq 0]\cdot\sum_{j=\max\{0,1-a-c\}}^{b-1}\frac{(-c)^{\underline{j}}\lambda_1^{a+c+j-1}\lambda_2^{b-j-1}}{j!(b-j-1)!(a+c+j-1)!} +$$
$$+[\lambda_1 \geq \lambda_2]\cdot[\lambda_2 \geq 0]\cdot\sum_{j=\max\{0,1-a-b\}}^{c-1}\frac{(-1)^{b+j}(-b)^{\underline{j}}\lambda_2^{c-j-1}(\lambda_1-\lambda_2)^{a+b+j-1}}{j!(c-j-1)!(a+b+j-1)!}.$$

We have $\{f \in [3]^m : \lambda_1 \geq \lambda_2 \geq 0\} \subset \{f \in [3]^m : \lambda_1 \geq 0, \lambda_2 \geq 0\}$. Moreover $\{f : \lambda_1 \geq 0, \lambda_2 \geq 0\} = \{f : \lambda_1 \geq \lambda_2 \geq 0\} \cup \{f : \lambda_2 \geq \lambda_1 \geq 0\}$ and this union is disjoint for generic d. So (at least for generic d) we have

$$R = [\lambda_1 \geq \lambda_2 \geq 0] \cdot R_1 + [\lambda_2 \geq \lambda_1 \geq 0] \cdot R_2$$

$$R_1 := \sum_{j=\max\{0,1-a-c\}}^{b-1} \frac{(-c)^{\underline{j}}\ \lambda_1^{a+c+j-1} \lambda_2^{b-j-1}}{j!(b-j-1)!(a+c+j-1)!} +$$

$$+ \sum_{j=\max\{0,1-a-b\}}^{c-1} \frac{(-1)^{b+j}(-b)^{\underline{j}}\ \lambda_2^{c-j-1}(\lambda_1-\lambda_2)^{a+b+j-1}}{j!(c-j-1)!(a+b+j-1)!}$$

$$R_2 := \sum_{j=\max\{0,1-a-c\}}^{b-1} \frac{(-c)^{\underline{j}}\ \lambda_1^{a+c+j-1} \lambda_2^{b-j-1}}{j!(b-j-1)!(a+c+j-1)!}.$$

By (5.48)

$$\operatorname{res}^{\Lambda} h_f = (-1)^{m_2} R(\lambda_1, \lambda_2, a, b, c) \tag{5.57}$$

where $m_j = m_j(f), \delta_j = \delta_j(f), \delta = \delta(f)$ and

$$a = m_1 + m_2 - 2, \quad b = m_2 + m_3 - 2, \quad c = m_1 + m_3 - 2$$
$$a + c - 1 = m + m_1 - 5 \quad a + b - 1 = m + m_2 - 5.$$

Moreover $m - \sum_{j=1}^{3} j \cdot m_j(f) \equiv_2 -2m_3 \mod 2$ and $\epsilon(f) := (-1)^{m - \sum_{j=1}^{3} j \cdot m_j(f)} = (-1)^{m_2}$. Notice that $1-a-c = 5-m-m_1 \leq 5-m \leq 0$. So $\max\{0, 1-a-c\} = 0$ and similarly $\max\{0, 1-a-b\} = 0$. Hence

$$R_1 = R_{11} + R_{12}$$

$$R_{11} := \sum_{j=0}^{b-1} \frac{(-c)^{\underline{j}}\ \lambda_1^{a+c+j-1} \lambda_2^{b-j-1}}{j!(b-j-1)!(a+c+j-1)!}$$

$$R_{12} := \sum_{j=0}^{c-1} \frac{(-1)^{b+j}(-b)^{\underline{j}}\ \lambda_2^{c-j-1}(\lambda_1-\lambda_2)^{a+b+j-1}}{j!(c-j-1)!(a+b+j-1)!}.$$

$$R_2 := \sum_{j=0}^{b-1} \frac{(-c)^{\underline{j}}\ \lambda_1^{a+c+j-1} \lambda_2^{b-j-1}}{j!(b-j-1)!(a+c+j-1)!}.$$

Let us start from R_{11}. We wish to expand it in powers of ξ_2 and ξ_3 instead of ξ_1 and ξ_2. Set for simplicity $\delta := a + c - 1$. Recall that $\lambda_1 = -(\xi_2 + \xi_3)$,

$\lambda_2 = -\xi_3$ and $(a+c+j-1)+(b-j-1) = b+\delta-1 = 2m-8$. So

$$R_{11} = \sum_{j=0}^{b-1} \binom{-c}{j} \frac{(\xi_2+\xi_3)^{\delta+j}\xi_3^{b-j-1}}{(b-j-1)!(\delta+j)!} =$$

$$= \sum_{j=0}^{b-1}\sum_{i=0}^{\delta+j} \binom{\delta+j}{i}\binom{-c}{j} \frac{1}{(b-j-1)!(\delta+j)!}\xi_2^i\xi_3^{2n-8-i}.$$

$$\binom{\delta+j}{i}\frac{1}{(b-j-1)!(\delta+j)!} = \frac{1}{i!(b+\delta-1-i)!}\binom{b+\delta-1-i}{\delta+j-i}$$

$$R_{11} = \frac{1}{(2m-8)!}\sum_{j=0}^{b-1}\sum_{i=0}^{\delta+j} z_{ij}\xi_2^i\xi_3^{2n-8-i}$$

$$\text{with } z_{ij} := \binom{-c}{j}\binom{2m-8}{i}\binom{2m-8-i}{\delta+j-i} \quad i,j\in\mathbb{Z}.$$

We rearrange by summing first on i next on j. The result is as follows:

$$\begin{aligned} R_{11} &= R_{111} + R_{112} \\ R_{111} &:= \frac{1}{(2m-8)!}\sum_{i=0}^{\delta}\sum_{j=0}^{b-1} z_{ij}\xi_2^i\xi_3^{2n-8-i} \\ R_{112} &:= \frac{1}{(2m-8)!}\sum_{i=\delta+1}^{2m-8}\sum_{j=i-\delta}^{b-1} z_{ij}\xi_2^i\xi_3^{2n-8-i}. \end{aligned} \tag{5.58}$$

Lemma 5.7.2. *$z_{ij} = 0$ if $j<0$ or $j>b-1$. If $j<i-\delta$, then $z_{ij}=0$.*

Proof. If $j<0$, then $\binom{-c}{j}=0$. If $j>b-1$, then $\delta+j-i>\delta+b-1-i = 2m-8-i$, so $\binom{2m-8-i}{\delta+j-i}=0$. Finally if $j<i-\delta$, then $\delta+j-i<0$, so again $\binom{2m-8-i}{\delta+j-i}=0$. □

It follows that both sums in (5.58) can be extended over arbitrary integer j. Hence

$$R_{111} = \frac{1}{(2m-8)!}\sum_{i=0}^{\delta}\mathscr{Z}_i\xi_2^i\xi_3^{2n-8-i} \quad R_{112} = \frac{1}{(2m-8)!}\sum_{i=\delta+1}^{2m-8}\mathscr{Z}_i\xi_2^i\xi_3^{2n-8-i}$$

$$R_{11} = \frac{1}{(2m-8)!}\sum_{i=0}^{2m-8}\mathscr{Z}_i\xi_2^i\xi_3^{2n-8-i} \quad \text{with } \mathscr{Z}_i := \sum_j z_{ij}.$$

Using (5.44)

$$\mathscr{Z}_i = \binom{2m-8}{i}\sum_j \binom{2m-8-i}{\delta+j-i}\binom{-c}{j} = \binom{2m-8}{i}\binom{2m-8-i-c}{2m-8-\delta} =$$
$$= \binom{2m-8}{i}\binom{m+m_2-6-i}{m_2+m_3-3}$$
$$R_{11} = \frac{1}{(2m-8)!}\sum_{i=0}^{2m-8}\binom{2m-8}{i}\binom{m+m_2-6-i}{m_2+m_3-3}\xi_2^i\xi_3^{2n-8-i}$$

Next we analyse the term R_{12}. Set $\gamma := c-1$, $\epsilon := a+b-1$. Notice that $\epsilon \geq 0$ and $\gamma+\epsilon = a+b+c-2 = 2m-8$. Substituting $\lambda_2 = -\xi_3$ and $\lambda_1 - \lambda_2 = -\xi_2$ we get

$$R_{12} = (-1)^b \sum_{j=0}^{\gamma}(-1)^j\binom{-b}{j}\frac{1}{(\gamma-j)!(\epsilon+j)!}\xi_2^{\epsilon+j}\xi_3^{\gamma-j}$$
$$= \frac{(-1)^{b+\epsilon}}{(2m-8)!}\sum_{i=\epsilon}^{2m-8}(-1)^i\binom{-b}{i-\epsilon}\binom{2m-8}{i}\xi_2^{\epsilon+j}\xi_3^{\gamma-j}.$$

Since $\binom{-b}{i-\epsilon} = 0$ for $i < \epsilon$, we have indeed

$$R_{12} = \frac{(-1)^{b+\epsilon}}{(2m-8)!}\sum_{i=0}^{2m-8}(-1)^i\binom{-b}{i-\epsilon}\binom{2m-8}{i}\xi_2^{\epsilon+j}\xi_3^{\gamma-j}.$$
$$R_1 = R_{11} + R_{12} = \frac{1}{(2m-8)!}\sum_{i=0}^{2m-8}\binom{2m-8}{i}\mathscr{W}_i\xi_2^i\xi_3^{2n-8-i}$$
$$\text{with } \mathscr{W}_i := \binom{m+m_2-6-i}{m_2+m_3-3} + (-1)^{b+\epsilon+i}\binom{-b}{i-\epsilon} =$$
$$= \binom{\epsilon-i-1}{b-1} + (-1)^{b+\epsilon+i}\binom{-b}{i-\epsilon}.$$

To compute $\mathscr{W}_i$ we distinguish three cases.

$$\mathscr{W}_i = \begin{cases} (-1)^{b+\epsilon+i}\binom{-b}{i-\epsilon} & \text{if } b < 1, \\ \binom{\epsilon-i-1}{b-1} & \text{if } i-\epsilon < 0. \end{cases}$$

If $b \geq 1$ and $\epsilon \leq i$, then we can use (5.46) to get

$$(-1)^{i-\epsilon}\binom{-(b-1)-1}{i-\epsilon} = (-1)^{b-1}\binom{-(i-\epsilon)-1}{b-1}$$
$$(-1)^{i+\epsilon}\binom{-b}{i-\epsilon} = -(-1)^b\binom{\epsilon-i-1}{b-1}$$
$$\mathscr{W}_i = 0.$$

Hence if $b \geq 1$

$$R_1 = \frac{1}{(2m-8)!} \sum_{i=0}^{\epsilon-1} \binom{2m-8}{i} \binom{\epsilon-i-1}{b-1} \xi_2^i \xi_3^{2n-8-i} =$$

$$= \frac{1}{(2m-8)!} \sum_{i=0}^{m+m_2-6} \binom{2m-8}{i} \binom{m+m_2-6-i}{m_2+m_3-3} \xi_2^i \xi_3^{2n-8-i}.$$

If instead $b < 1$, then $R_{11} = 0$, so

$$R_1 = R_{12} = \frac{(-1)^{b+\epsilon}}{(2m-8)!} \sum_{i=0}^{2m-8} (-1)^i \binom{-b}{i-\epsilon} \binom{2m-8}{i} \xi_2^{\epsilon+j} \xi_3^{\gamma-j}.$$

By (5.47)

$$\binom{-b}{i-\epsilon} = (-1)^{i-\epsilon} \binom{b+i-\epsilon-1}{i-\epsilon},$$

and this vanishes by (5.45) since $b+i-\epsilon-1 < i-\epsilon$. Hence for $b < 1$, $R_1 = 0$. But for $b < 1$ $\binom{\epsilon-i-1}{b-1} = 0$. So we have at any case

$$R_1 = \frac{1}{(2m-8)!} \sum_{i=0}^{m+m_2-6} \binom{2m-8}{i} \binom{m+m_2-6-i}{m_2+m_3-3} \xi_2^i \xi_3^{2n-8-i}. \quad (5.59)$$

Finally let us analyse R_2. Substituting the expressions for λ_1 and λ_2 from (5.38) we get

$$R_2 := \sum_{j=0}^{b-1} \frac{(-c)^{\underline{j}}\ \xi_1^{a+c+j-1} (\xi_1+\xi_2)^{b-j-1}}{j!(b-j-1)!(a+c+j-1)!} =$$

$$= \sum_{j=0}^{b-1} \sum_{i=0}^{b-j-1} \binom{b-j-1}{i} \frac{(-c)^{\underline{j}}}{j!(b-j-1)!(a+c+j-1)!} \xi_1^{a+c+j-1+b-j-1-i} \xi_2^i$$

Set

$$z_{ij} := \binom{b-j-1}{i} \frac{(-c)^{\underline{j}}}{j!(b-j-1)!(a+c+j-1)!} \qquad \mathscr{Z}_i := \sum_{j=0}^{b-i-1} z_{ij}.$$

Then

$$R_2 = \sum_{\substack{i,j\geq 0\\ i+j\leq b-1}} z_{ij}\, \xi_1^{2m-8-i} \xi_2^i = \sum_{i=0}^{b-1} \mathscr{Z}_i\, \xi_1^{2m-8-i} \xi_2^i.$$

Set for simplicity $\beta = b-1-i, \gamma = c-1, \delta = a+c-1$. Observe that $\beta+\delta = a+b+c-2-i = 2m-8-i$. Then

$$\begin{aligned}
z_{ij} &= \frac{(-c)^{\underline{j}}}{j!i!(b-j-i-1)!(a+c+j-1)!} = \binom{-c}{j}\frac{1}{i!}\frac{1}{(\beta-j)!(\delta+j)!} = \\
&= \binom{-c}{j}\frac{1}{i!}\binom{\beta+\delta}{\delta+j}\frac{1}{(\beta+\delta)!} = \frac{1}{i!(2m-8-i)!}\binom{-c}{j}\binom{\beta+\delta}{\delta+j} = \\
&= \frac{1}{(2m-8)!}\binom{2m-8}{i}\binom{-c}{j}\binom{\beta+\delta}{\delta+j} \\
\mathscr{Z}_i &= \frac{1}{(2m-8)!}\binom{2m-8}{i}\sum_{j=0}^{\beta}\binom{\beta+\delta}{\delta+j}\binom{-c}{j}.
\end{aligned}$$

By definition (5.42) $\binom{-c}{j}\binom{\beta+\delta}{\delta+j} = \binom{-c}{j}\binom{\beta+\delta}{\beta-j} = 0$ if $j < 0$ or $j > \beta$. Hence using (5.44) (since $\beta+\delta \geq 0$)

$$\sum_{j=0}^{\beta}\binom{\beta+\delta}{\delta+j}\binom{-c}{j} = \sum_{j}\binom{\beta+\delta}{\delta+j}\binom{-c}{j} = \binom{\beta+\delta-c}{\beta}$$

Now $\beta+\delta-c = m+m_2-6-i$ and $\delta-c = a-1 = m_1+m_2-3$. So

$$\begin{aligned}
\sum_{j=0}^{\beta}\binom{\beta+\delta}{\delta+j}\binom{-c}{j} &= \binom{m+m_2-6-i}{m_1+m_2-3} \\
\mathscr{Z}_i &= \frac{1}{(2m-8)!}\binom{2m-8}{i}\binom{m+m_2-6-i}{m_1+m_2-3}.
\end{aligned}$$

We get

$$R_2 = \frac{1}{(2m-8)!}\sum_{i=0}^{b-1}\binom{2m-8}{i}\binom{m+m_2-6-i}{m_1+m_2-3}\xi_1^{2m-8-i}\xi_2^{i}. \tag{5.60}$$

To conclude we apply the residue theorem, (5.57), (5.59) and (5.60)

$$\mathrm{Vol}(M_d) =$$
$$= \frac{n_0(-1)^{s+n_+}}{|W|} \sum_{f\in[3]^m} \mathrm{res}^{\Lambda} h_f(X) =$$
$$= -\frac{1}{6} \sum_{f\in[3]^m} (-1)^{m_2}\Big([\lambda_1 > \lambda_2 > 0]\cdot R_1 + [\lambda_2 > \lambda_1 > 0]\cdot R_2\Big) =$$
$$= -\frac{1}{6}\sum_{\mathcal{A}(d)}(-1)^{m_2}R_1 + -\frac{1}{6}\sum_{\mathcal{B}(d)}(-1)^{m_2}R_2 =$$
$$= -\sum_{f\in\mathcal{A}(d)} \frac{(-1)^{m_2(f)}}{6(2m-8)!} \sum_{j=0}^{2m-8} \binom{2m-8}{j}\binom{m+m_2-6-j}{m_2+m_3-3}\xi_2^j\xi_3^{2m-8-j} +$$
$$- \sum_{f\in\mathcal{B}(d)} \frac{(-1)^{m_2(f)}}{6(2m-8)!} \sum_{j=0}^{2m-8} \binom{2m-8}{j}\binom{m+m_2-6-j}{m_1+m_2-3}\xi_1^j\xi_2^{2m-8-j}.$$

This completes the proof of the Theorem. □

Remark 5.7.1. *This formula was found in 2008 by Suzuki and Takakura [35] with a different technique. To restore exactly their result of [35, Theorem 5.6], one needs to interchange the index* 1 *with the index* 3*.*

Final Remarks

The main results of this work were the extension in the log case of the Weil-Peterson metrics, and then provide some examples of the main philosophy of this work. That is, the study of the geometric properties when the K-moduli is proper, projective and explicit.

A GIT related problem

In [66] it is proven that the K-moduli space of del Pezzo of degree $d \in \{1, 2, 3, 4\}$ is proper and projective, and moreover when $d = 3, 4$ there is a nice GIT description of such moduli spaces. In degree 3 we have the moduli space of cubics

$$M_3 = \mathbb{P}(\mathrm{Sym}^3(\mathbb{C}^4)^*)//\mathrm{SL}(4).$$

We aim in a future work to calculate the volume of such moduli space. In this case, the GIT-stability conditions tells that the semistable locus do not match with the stable locus. Then, in this case the Kirwan map is no longer surjective and therefore the Jeffrey-Kirwan non abelian localization can not be applied. One way to proceed is considering the *Kirwan partial desingularization* [16], and calculate the volume of the desingularized GIT quotient. In [23] a picture of the Kirwan map is described via the *intersection cohomology ring*, and in this framework, when the action describing the GIT quotient has a special property, that is the action is *weakly balanced*, then the Jeffrey-Kirwan localization of the partial desingularization coincides with the Jeffrey-Kirwan localization of the base space (Proposition 23 in [23]). Unfortunately, the action of SL(4) on $\mathbb{P}(\mathrm{Sym}^3(\mathbb{C}^4)^*)$ is not weakly balanced.

CM Volumes as the sum of K-stability invariants

In Theorem 5.4.2 we have seen that the volume of the moduli space of log Fano hyperplane arrangements of dimension one can be expressed as the sum of the Futaki invariant of the all possible integral test configuration. This result was mainly based on the *Fujita-correspondence*. Of course, the one dimensional case is a *lucky case* since the points fixed by the S^1-action on $(\mathbb{P}^1)^m$ are just points. Already in dimension 2 the picture is much more complicated. Firstly, one should localize "from below", that is one should consider the smallest subtorus of the maximal torus in order to reproduce the analogy with $\mathbb{G}_m$actions of test configurations. Furthermore, there are some issues. Indeed, the S^1-action on $(\mathbb{P}^2)^m$ fixes points, and projective lines. But, points are not divisors in $(\mathbb{P}^2)^m$, therefore the Fujita-correspondece works only partially in this case, that is it works for the fixed lines. It is addressed to future works to see if these volumes can be expressed as sums of *more* K-stability invariants. Using the ideas behind Θ-stratifications [28], we hope to make sense of a *stacky oriented* Kirwan non abelian localization theorem tailored for calculating some geometric properties in the modern theory of K-moduli spaces.

The classical definition of the log Weil-Petersson metric

In Definition 4.0.2 we have defined the log Weil-Petersson metric via fiber integral. It would be interesting to restore the classical definition of the log Weil-Petersson metric given in Remark 2.0.1. That is, define the Weil-Petersson metric as an L^2-metric via the Kodaira-Spencer map. For general type geometries, Schumacher and Trapani in [64] gave a decription of the classical definition of Weil-Petersson metric. In a future work we aim to extend this work to the log Fano case.

Bibliography

[1] Kirwan, F., *Cohomology of quotients in symplectic and algebraic geometry.* ISBN-13: 9780691083704 (1984)

[2] Tian G., *Canonical Metrics in Kähler geometry*, Birkäuser 2000 ISBN 978-3-0348-8389-4

[3] Mumford, D, Fogarty, J., and Kirwan, F.,*Geometric invariant theory,* Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) Results in Mathematics and Related Areas (2), 34. Springer-Verlag, Berlin, 1994.

[4] Hirzebruch, F., *Topological Methods in Algebraic Geometry.* Reprint of the 1978 Edition, Springer ISBN 0-387-03525-7

[5] Kollar, J., Mori, M., *Birational Geometry of Algebraic Varieties,* Cambridge University Press 1998, ISBN 0 521 632773

[6] Burago D., Burago Y., Ivanov S., *A Course in Metric Geometry.* Graduate Studies in Mathematics, vol.33. A.M.S., Providence, RI, (2001)

[7] Voisin C., *complex algebraic geometry II.* Cambridge university press 2002

[8] Dolgachev, I., *Lectures on invariant theory,* London Mathematical Society Lecture Note Series, 296. Cambridge University Press, Cambridge, 2003.

[9] Lazarsfeld, R., *Positivity in Algebraic Geometry, I, II* Springer 2003

[10] Szekelyhidi G., *An introduction to Extremal Kähler metrics,* graduate text in mathematics Vol. 152, American Mathematical Society 2014

[11] Alexeev, V., *Moduli of weighted hyperplane arrangements,* Edited by Gilberto Bini, Marti Lahoz, Emanuele Macri and Paolo Stellari. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser/Springer, Basel, 2015.

[12] Jeffrey, L. van den Hurk, T., Herman, J., Dwivedi, S., *Hamiltonian group actions and equivariant cohomology* SpringerBriefs in Mathematics, (2019), ISBN 978-3-030-27226-5

[13] Graham, Knuth, Patashnik *Concrete mathematics. Second edition.* 1984, Addison-Wesley.

[14] M. Reid *The complete intersection of two or more quadrics* Phd Thesis, Trinity College, Cambridge, June 1972

[15] Kollar J., Shepherd-Barron N.,*Threefolds and deformations of surface sin-gularities,* Invent. Math. (1988).

[16] Kirwan F., *Partial Desingularisations of Quotients of Nonsingular Varieties and their Betti Numbers.* Annals of Mathematics Second Series, Vol. 122, No. 1 (Jul., 1985), pp. 41-85 (45 pages)

[17] Deligne, P. *Le Determinant de la Cohomologie,*Current Trends in Arithmetical Algebraic Geometry, Contemp. Math 67 (1987), 93-177

[18] Kapranov, M., *Chow quotients of Grassmannians.* I, I. M. Gel'fand Seminar, Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 29–110.

[19] Jeffrey L., Kirwan F., *Localizations for Nonabelian Group Actions*, Topology, 34 (1995) pp 219-268

[20] Jeffrey, L., Kirwan F., *Localization and the quantization conjecture*,Topology, 36 (1997), pp. 647-693

[21] Knudsen F., Mumford, D., *The projectivity of the moduli space of stable curves.* I. math.AG/0605278

[22] D. Avritzer and H. Lange *Pencils of Quadrics, Binary Forms, Hyperelliptic curves.* To Robin Hartshorne on his 60th birthday, Communication in Algebra Volume 28, 2000-issue 12

[23] Jeffrey, Kiem, Young-Hoon, Kirwan and Woolf, *Cohomology pairings on singular quotients in geometric invariant theory* Transformation Groups, 2001

[24] Hasset, B., *Moduli spaces of weighted pointed stable curves*, Advances in Mathematics 173 (2003) 316–352.

[25] Alexeev, V., *Weighted Grassmannians and stable hyperplane arrangements* arXiv:0806.0881, 2008

[26] Alper J., *Good moduli spaces for Artin stacks,* Ann. Inst. Fourier (Grenoble) 63 (2013), no. 6, 2349–2402.

[27] Birkar C, *The augmented base locus of real divisors over arbitrary fields,* arXiv:1312.0239 (2013)

[28] Halpern-Leistner D., *Theta-stratifications, Theta-reductive stacks, and applications.* arXiv:1608.04797 [math.AG]

[29] Guillemin, V., Sternberg, S.,*Convexity properties of the moment mapping I and II*, Invent. Math., 77 (1984), pp. 533-546

[30] Gelfand, I., Goresky, R., MacPherson, R., Serganova, V. *Combinatorial geometries, convex polyhedra, and Schubert cells,* Adv. in Math. 63 (1987), no. 3, 301–316.

[31] Thurston W., *Shapes of polyhedra and triangulations of the sphere.* In The Epstein Birthday Schrift, volume 1 of Geom. Topol. Monogr., pages 511–549. Geom. Topol. Publ., 1998.

[32] McMullen, C.T., *The Gauss-Bonnet theorem for cone manifolds and vol- umes of moduli spaces*, Amer. J. Math., 139(1):261–291, 2017.

[33] Martin, S., *Transversality theory, cobordisms, and invariants of symplectic quotients* from PhD thesis.

[34] Takakura, T., *Intersection theory on symplectic quotients of products of spheres*, Int. J. Math. 12(1) (2001), 97–111.

[35] Suzuki, T. , Takakura, T. *Symplectic Volumes of Certain Symplectic Quotients Associated with the Special Unitary Group of Degree Three*, Tokyo J. Math., Volume 31, Number 1 (2008), 1-26

[36] Khoi, V., *On the symplectic volume of the moduli space of spherical and Euclidean polygons*, Kodai Math. J. 28(1) (2005), 199–208

[37] Mandini, A., *A note on the moduli space of Polygons*, AIP Conference Proceedings 1023, 158 (2008); https://doi.org/10.1063/1.2958168

[38] Matsushima Y., *Sur la structure du groupe d'homeomorphismes analytiques d'une certaine variete kählerienne,* Nagoya Math. J. 11 (1957), 145–150.

[39] Futaki A., *An obstruction to the existence of Einstein Kähler metrics,* Invent. Math. 73 (1983), no. 3, 437–443.

[40] Troyanov, M., *Prescribing curvature on compact surfaces with conical singu-larities*, Trans. Amer. Math. Soc., 324(2):793–821, 1991.

[41] Bando S., Mabuchi S., *Uniqueness of Einstein Kähler metrics modulo connected group actions,* Algebraic geometry, Sendai, 1985, 11-40, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[42] Tian G., *On Calabi's conjecture for complex surfaces with positive first Chern class,* Invent. Math. 101 (1990), 101-172.

[43] Tian, G. *The K-energy on hypersurfaces and stability.* Comm. Anal. Geom., 2(2):239–265, 1994.

[44] Cheeger J., Colding T.H., *On the structure of spaces with Ricci curvature bounded below.* I. J. Diff. Geom. Volume 46,3 (1997), 406-480.

[45] Tian, G., *Kaehler-Einstein metrics with positive scalar curvature*, Invent math 130, 1–37 (1997). https://doi.org/10.1007/s002220050176

[46] Donaldson, S. K. *Scalar curvature and stability of toric varieties.* J. Differential Geom.,62(2):289–349, 2002.

[47] Paul, S.T., Tian, G. *CM stability and the generalized Futaki invariant I, II*, Asterisque 328 (2009), 339-354.

[48] Stoppa J.,*K-stability of constant scalar curvature Kaehler manifolds,* Advances in Mathematics 221, no. 4, (2009) 1397-1408.

[49] Stoppa J., *J. Stoppa, A note on the definition of K-stability,* arXiv:1111.5826.

[50] Li, C. *Remarks on logarithmic K-stability*, Commun. Contemp. Math., 17 (2014), no. 2, 1450020, 1-17.

[51] Chen X., Donaldson D., and Sun S., *Kaehler-Einstein metrics on Fano manifolds, I: approximation of metrics with cone singularities.* arXiv 1211.4566.

[52] Chen X., Donaldson S., Sun S., *Kaehler-Einstein metrics on Fano manifolds, II: limits with cone angle less than* 2π. arXiv 1212. 4714.

[53] Chen X., Donaldson S., Sun S.,*Kaehler-Einstein metrics on Fano manifolds, III: limits as cone angle approaches* 2π *and completion of the main proof.* arXiv 1302.0282.

[54] Donaldson, S., *Kähler metrics with cone singularities along a divisor, in Essays in mathematics and its applications,* pages 49–79. Springer, Heidelberg, 2012.

[55] Berman R., *K-polystability of Q-Fano varieties admitting Kähler-Einstein metrics,* Invent. Math. 203 (2016), 973-1025.

[56] Spotti C., Sun S., Yao *Explicit Gromov-Hausdorff compactifications of moduli spaces of Kaehler-Einstein Fano Manifolds.* Differential Geometry: Algebraic Geometry arXiv: 1705.00377 [math.DG]

[57] Guenancia, H., Paun, M. *Conic singularities metrics with prescribed Ricci curvature: General cone angles along normal crossing divisors* J. Differential Geom. Volume 103, Number 1 (2016), 15-57.

[58] Tian, G. Wang, F. *On the existence of conic Kaehler-Einstein metrics* arXiv:1903.12547v1 [math.DG].

[59] Fujiki, A., Schumacher, G., *the moduli space of extremal compact Kaehler manifolds and generalized Weil-Petersson metrics*, Publ. Res. Inst Math. 26 (1990), 101-183

[60] S. Mukai, *New developments in Fano manifolds theory related to the vector bundle method and moduli problems,* (Japanese) Sugaku 47 (1995), no.2, 125-144

[61] T. Mabuchi and S. Mukai *Stability and Einstein-Kaehler metrics of Quartic del Pezzo Surfaces* Documents to U (MNU/MNUG) University of Minnesota- Interlibrary Loan Lending

[62] Moriwaki, A., *The continuity of Deligne's pairing,*Internat. Math. Ann. 318 (2000), no. 4, 837-847

[63] Odaka Y, *On parametrization, optimization and triviality of test configurations,* arXiv:1201.0692.

[64] Schumacher G., Trapani S.,*Weil-Petersson geometry for families of hyperbolic conical Riemann surfaces.* Michigan Math. J., Volume 60, Issue 1 (2011), 3-33.

[65] Donaldson S., Sun S., *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry,* Acta Math. 213 (2014), 63-106.

[66] Odaka Y., Spotti C., Sun S. *Compact moduli spaces of del Pezzo surfaces and Kaehler-Einstein metrics,* Journal of Differential Geometry, Volume 102. Number 1 (2016), 127-172

[67] Spotti C., Sun S., *Explicit Gromov-Hausdorff compactifications of moduli spaces of Kähler-Einstein Fano manifolds.*Pure and Applied Mathematics Quarterly, Volume 13 (2017). Special Issue in Honor of Simon Donaldson

[68] Spotti C., *Kähler-Einstein metrics on Q-smoothable Fano varieties, their moduli and some applications*, arXiv:1705.00374 [math.AG] (2017)

[69] Fine, J., Ross, J. *A note on positivity of the CM line bundle.* International Mathematics Research Notices, Volume 2006, 1 January 2006, 095875

[70] Wang, X., *Height and GIT weight,* Math. Res. Lett. 19 (2012), 909–926. MR 3008424. Zbl 06165866

[71] Odaka, Y., *A generalization of the Ross-Thomas slope theory*, Osaka J. Math. 50 (2013), 171–185. MR 3080636. Zbl 1271.14067.

[72] Li, C, *special test configurations and K-stability of Fano varieties,* Annals of Mathematics 180 (2014), 197–232

[73] Li, C., Xu, C., *Stability of valuations and Kollár components*, to appear in J. Eur. Math. Soc. (JEMS), preprint, arXiv:1604.05398v5 [math.AG].

[74] Li, C., Wang, X., Xu, C. *Degeneration of Fano K ähler-Einstein manifolds*, arxiv:1411.0761 (2014).

[75] Li C, *K-semistability is equivariant volume minimization,* arXiv:1512.07205v3.

[76] Li, C., Wang, X., Xu, C *Quasi-Projectivity of the moduli space of smooth Kaehler-Einstein Fano manifolds* https://doi.org/10.24033/ASENS.236

[77] Fujita, K., *A valuative criterion for uniform K-stability of Q-Fano varieties*, J. Reine Angew. Math. 751 (2019), 309–338.

[78] Fujita, K. *K-stability of log Fano hyperplane arrangements*, Kyoto J. Math., Volume 59, Number 2 (2019), 399-418.

[79] Fujita K., *Examples of K-unstable Fano manifolds with the Picard number 1,* Proc. Edinb. Math. Soc. (2) 60 (2017), no. 4, 881–891.

[80] Fujita K., *Optimal bounds for the volumes of K ähler-Einstein Fano manifolds,* Amer. J. Math. 140 (2018), no. 2, 391–414.

[81] Fujita K., *Openness results for uniform K-stability,* Math. Ann. 373 (2019), no. 3-4, 1529–1548.

[82] Fujita K., *Uniform K-stability and plt blowups of log Fano pairs,* Kyoto J. Math. 59 (2019), no. 2, 399–418.

[83] Blum H., Liu Y., Xu C. *openess of K-semistability for Fano varieties*, arXiv:1907.02408 [math.AG] (2019)

[84] Blum H., Xu C., *Uniqueness of K-polystable degenerations of Fano varieties,* arXiv:1812.03538 [math.AG] (2019)

[85] Alper J., Blum H., Halpern-Leistner D., Xu C *Reductivity of the automorphism group of K-polystable Fano varieties*, Inventiones mathematicae, 2020

[86] Fujita K., Odaka Y., *On the K-stability of Fano varieties and anti-canonical divisors,* Tohoku Math. J. (2) 70 (2018), no. 4, 511–521.

[87] Codogni, G., Patakfalvi, Z., *Positivity of the CM line bundle for families of K-stable klt Fanos*, 2018. arXiv:1806.07180.

[88] Spotti, C., Gallardo, P., Martinez, J., *Application of the moduli continuity method to log K-stable pair*, arXiv:1811.00088 [math.AG]

[89] Ascher, K. , Devleming, K., Liu, Y., *Wall crossing for K-moduli spaces of plane curves* arXiv:1909.04576 [math.AG]

[90] Liu Y., Xu C., *K-stability of cubic threefolds.* Duke Math. J., Volume 168, Number 11 (2019), 2029-2073

[91] Liu Y., *K-stability of cubic fourfolds.* arXiv:2007.14320 [math.AG].