UNIVERSITY OF MILANO BICOCCA
DIPARTIMENTO DI MATEMATICA E APPLICAZIONI

JOINT PHD PROGRAM IN MATHEMATICS
MILANO BICOCCA - PAVIA - INDAM

UNDERSTANDING THE EVOLUTION OF TUMOURS, A PHASE-FIELD APPROACH: ANALYTIC RESULTS AND OPTIMAL CONTROL

Doctoral Dissertation of:
Andrea Signori

Supervisor:
Prof. Pierluigi Colli

Tutor:
Prof. Mauro Garavello

The Chair of the Doctoral Program:
Prof. Thomas Weigel

Cycle XXXIII – 2020

Contents

Declarations

I declare that this thesis contribution contains research entirely conducted by myself under the supervision of Prof. Pierluigi Colli, except where otherwise stated.

In this regards, it is worth mentioning that the material contained in Chapter 3 and Chapter 5 are based on the published papers [136] and [137, 140]. Furthermore, what presented in Chapter 4 is largely inspired by the results established in the submitted paper [134] which has been performed in collaboration with Luca Scarpa, whereas the results presented in Chapter 6 are based on the submitted paper [133] co-authored with Elisabetta Rocca and Luca Scarpa.

The author also declares that he has no conflict of interest. Moreover, this manuscript has not been submitted for other purposes at any other university.

List of publications

The results presented in this PhD thesis are largely inspired by the following published or submitted papers:

(1) E. Rocca, L. Scarpa and A. Signori, Parameter identification for nonlocal phase field models for tumor growth via optimal control and asymptotic analysis. Preprint arXiv:2009.11159 [math.AP], (2020), 1-39.

(2) L. Scarpa and A. Signori, On a class of non-local phase-field models for tumor growth with possibly singular potentials, chemotaxis, and active transport. Preprint arXiv:2002.12702 [math.AP], (2020), 1-40.

(3) A. Signori, Vanishing parameter for an optimal control problem modeling tumor growth. *Asymptot. Anal.* **117** (2020), 43-66.

(4) A. Signori, Optimal treatment for a phase field system of Cahn–Hilliard type modeling tumor growth by asymptotic scheme, *Math. Control Relat. Fields* **10** (2020), 305-331.

(5) A. Signori, Optimal distributed control of an extended model of tumor growth with logarithmic potential. *Appl. Math. Optim.* **82** (2020), 517-549.

Furthermore, let us mention that during the PhD period the author was also involved in the following research projects:

(6) H. Garcke, K.F. Lam and A. Signori, Sparse optimal control of a phase field tumour model with mechanical effects. Preprint arXiv:2010.03767 [math.OC], (2020), 1-26.

(7) P. Colli, A. Signori and J. Sprekels, Second-order analysis of an optimal control problem in a phase field tumor growth model with singular potentials and chemotaxis, Preprint arXiv:2009.07574 [math.AP], (2020), 1-52.

(8) S. Frigeri, K. F. Lam and A. Signori, Strong well-posedness and inverse identification problem of a non-local phase field tumor model with degenerate mobilities. Preprint arXiv:2004.04537 [math.AP], (2020), 1-41.

(9) P. Knopf and A. Signori, On the nonlocal Cahn–Hilliard equation with nonlocal dynamic boundary condition and boundary penalization. Preprint arXiv:2004.00093 [math.AP], (2020), 1-51.

(10) H. Garcke, K.F. Lam and A. Signori, On a phase field model of Cahn–Hilliard type for tumour growth with mechanical effects. *Nonlinear Anal. Real World Appl.* **57** (2021), 103192, https://doi.org/10.1016/j.nonrwa.2020.103192.

(11) P. Colli, A. Signori and J. Sprekels, Optimal control of a phase field system modelling tumor growth with chemotaxis and singular potentials, *Appl. Math. Optim.* (2019), https://doi.org/10.1007/s00245-019-09618-6.

(12) P. Colli and A. Signori, Boundary control problem and optimality conditions for the Cahn-Hilliard equation with dynamic boundary conditions, *Internat. J. Control* (2019), https://doi.org/10.1080/00207179.2019.1680870.

(13) A. Signori, Penalisation of long treatment time and optimal control of a tumour growth model of Cahn–Hilliard type with singular potential, *Discrete Contin. Dyn. Syst. Ser. A*, (2020), https://doi.org/10.3934/dcds.2020373.

(14) A. Signori, Optimality conditions for an extended tumor growth model with double obstacle potential via deep quench approach, *Evol. Equ. Control Theory* **9** (2020) 193-217.

Abstract

Over the last decades, great strides have been made by the mathematical and medical communities towards the understanding of tumor growth. The recently achieved novelties arise from two leading factors: on the one hand, the flourishing of mathematical models for biological systems, and on the other hand, the more and more accurate computational methods and numerical solvers rose in the last decades. Despite the deep and challenging aim of understanding the hidden mechanisms behind the disease, the scientists' factual goal is to provide robust methods that may help the practitioners suiting the best therapy for every single patient. In this sense, the mathematical approach to tumour growth models might bring new lymph and hope to this arduous journey.

This thesis aims at contributing to this common effort by providing some mathematical insights for two classes of tumor growth models capturing cell-to-cell adhesion effects of local and nonlocal nature, respectively. The common denominator of the two models under consideration is the assumption that the tumour cells are submerged in a nutrient-rich environment which is the primary source of nourishment for the tumorous cells: this is a reasonable assumption at least for young tumours (*avascular tumours*). This paradigm leads us to analyse four-species PDE systems (tumour cells, healthy cells, nutrient-rich concentration, nutrient-poor concentration) which couple a Cahn–Hilliard type equation with source term for the tumour with a reaction-diffusion equation for the surrounding nutrient.

For both models, we provide a rich spectrum of mathematical results. In the first part of the thesis, we discuss the weak well-posedness of the models under very general frameworks. Then, postulating further natural assumptions, we establish the strong well-posedness of the systems which lays the groundwork for possible further investigations. Lastly, we perform some singular limit analysis as some of the coefficients appearing in the systems approach zero.

In the second part, based on the analytic results presented in the first one, we discuss some optimal control problems in which the governing equations are ruled by the previously analysed systems. In this direction, we provide the existence of minimisers and first-order necessary conditions for optimality. Via suitable asymptotic

approaches, we then investigate the optimal control problems for the aforementioned systems as some parameters occurring in the systems go to zero.

CHAPTER 1

Introduction

ca. 460 BC – ca. 370 BC: those are the years in which Hippocrates of Kos, often referred to as "Father of Medicine", first employed the word *karkinos (carcinos)*, the Greek word for crab, to describe the cancer disease due to the picturesque finger-like shape created by the blood vessels sprouting around a solid tumour (see [127, 147]). Remarkable is both the long time that has elapsed since the discovery of the disease as well as the incessant fervor with which scientists have continued to investigate it.

The understanding of solid tumour growth, which is nowadays a leading cause of death worldwide, is one of the main scientists' challenges of this century. Although it is undeniable that considerable progress has been made towards more efficient therapies, the deep comprehension behind tumour's development still remains a challenging open problem. Although, it is now more than ever clear that only interdisciplinary collaborations may allow us to grasp more insights on cancer's evolution. In this scenario, the mathematics could play a crucial role as multiscale mathematical modelling provides a quantitative tool which may help in diagnostic and prognostic applications. Among others, mathematics has in fact two significant advantages: the first one is that of being able to select particular mechanisms which may be more relevant than others, while the second one is that of being able to foresee and make predictions that may be precious for medical practitioners without inflicting serious harm to the patients. Besides, the great flourishment of computational methods simulating nonlinear PDEs allows for the development of numerical solvers that may be implemented as a supporting tool in clinical therapies.

A major breakthrough in this field of research has sprouted from realising that the aggregation of tumorous cells, like any other material, have to obey physical laws: in this sense, from a modelling perspective, a tumour mass does not behave that different from other complex materials investigated by scientists. In this direction, many mathematical models of Cahn–Hilliard type have been proposed to capture the complexity of the underlying biological and chemical phenomena: see, e.g., the seminal works [8, 15, 22, 23, 50, 71, 121] and the references therein.

It is then the purpose of this thesis to present and provide some theoretical mathematical background concerning two families of phase-field models which are thermodynamically consistent and are suited to describe the evolution of young tumours. In fact, the key modelling assumption behind the models we are going to consider is that the tumour cells are embedded in a nutrient-rich environment from which the tumour draws nourishment to grow which is a reasonable assumption for young tumours as they do not possess yet their own vascular system (avascular tumours) and must, therefore, absorb growth factors (e.g., oxygen, glucose) from the surrounding region.

The two classes under consideration in the thesis consist of systems of partial differential equations coupling a Cahn–Hilliard type equation for the phase variable, tracking the tumour's evolution, with a reaction-diffusion equation for the nutrient. They differ from each other from considering short-ranged only and possibly also long-ranged interactions between the particles, respectively. For both the models we establish weak and strong well-posedness, regularity results, asymptotic analysis as some characteristic parameters approach zero, and some non-trivial application of optimal control.

1.1 The Abstract Models

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ denotes the spatial domain (the involved tissue) and let $T > 0$ be some final time. The two families of phase-field models of Cahn–Hilliard type whose physical context is that of tumor growth dynamics that will be analysed in this thesis are the following:

- The first family, that will be referred to as the *local model*, reads as

$$\alpha \partial_t \mu + \partial_t \varphi = \operatorname{div}(m(\varphi)\nabla \mu) + \mathcal{S}^1 \qquad \text{in } \Omega \times (0,T), \tag{1.1}$$
$$\mu = \beta \partial_t \varphi - \gamma\varepsilon \Delta \varphi + \tfrac{\gamma}{\varepsilon} F'(\varphi) - \chi \varphi \qquad \text{in } \Omega \times (0,T), \tag{1.2}$$
$$\partial_t \sigma = \operatorname{div}(n(\varphi)\nabla(\sigma - \chi\varphi)) - \mathcal{S}^2 \qquad \text{in } \Omega \times (0,T). \tag{1.3}$$

- The second family, that will be referred to as the *nonlocal model*, reads as

$$\alpha \partial_t \mu + \partial_t \varphi = \operatorname{div}(m(\varphi)\nabla \mu) + \mathcal{S}^1 \qquad \text{in } \Omega \times (0,T), \tag{1.4}$$
$$\mu = \beta \partial_t \varphi + \gamma\varepsilon (a\varphi - J * \varphi) + \tfrac{\gamma}{\varepsilon} F'(\varphi) - \chi \varphi \qquad \text{in } \Omega \times (0,T), \tag{1.5}$$
$$\partial_t \sigma = \operatorname{div}(n(\varphi)\nabla(\sigma - \chi\varphi)) - \mathcal{S}^2 \qquad \text{in } \Omega \times (0,T). \tag{1.6}$$

It is worth noticing that these models can be derived using balance and constitutive laws, such as mass and momentum balances, and thermodynamic principles; more insights in this direction will be provided in Section 1.4. Let us refer the reader, e.g., to the contributions [62, 92, 96], where the derivation of similar models can be found.

Here, we just give some general overview of the occurring symbols and refer to the following sections for a more in-depth discussion on the associated assumptions and on the choice of the boundary and initial conditions. The primary variables of the above systems are φ, μ, and σ. The variable φ, called phase variable, is an order parameter representing the difference of tumour cells and healthy cells volume fractions ranging between -1 and 1. It allows us to keep track of the evolution of the interface of the tumour since this latter can be recovered from the level sets of φ. Namely, the level sets $\{\varphi = 1\}$ and $\{\varphi = -1\}$ describe the region of pure phases: the tumourous phase and the healthy phase, respectively. Moreover, the diffuse interface approach postulates (see, e.g., [148]) that in most of the spatial region the solution takes values close to ± 1, the pure phases, and that there exists a narrow transition layer $\{-1 < \varphi < 1\}$ of thickness ε in which φ smoothly passes from one phase to the other. One of the major advantages of this modelling approach is that, unlike free boundary models, which formally correspond to the choice of the thickness parameter $\varepsilon = 0$, it takes into account also possible delicate behaviours such as topological changes in the tumorous regions, occurring for example during break-up and coalescence phenomena. For some free boundary problems modelling tumor growth we refer the reader to, e.g., [21,25,53,58,70,72,73] and the references therein. The second variable μ denotes the chemical potential related to φ as in the framework of the Cahn–Hilliard equation. As already mentioned, we postulate the growth and proliferation of the tumour to be driven by absorption and consumption of some nutrient σ with the convention that $\sigma \simeq 1$ represents a rich nutrient concentration, whereas $\sigma \simeq 0$ a poor one. Thus, the third variable σ captures the evolution of an unknown species nutrient, typically oxygen or glucose, in which the tumour is embedded.

The functions $m(\varphi)$ and $n(\varphi)$ are non-negative mobility functions related to the phase variable and to the nutrient variable, respectively. The physical constants ε and γ are related to the interfacial thickness, i.e., the Lebesgue measure of the level set $\{-1 < \varphi < 1\}$, and to the surface tension, respectively. It is worth noticing that, as they will not cover any role in our analysis, we will simply take $\gamma = \varepsilon = 1$ in the following chapters. Meanwhile, the non-negative constant χ represents the chemotactic sensitivity and it will cover a fundamental role in the investigation of the nonlocal models (1.4)–(1.6) (cf. Chapter 4). From a biological viewpoint the chemotaxis refers to the movement of the tumour cells towards regions of high nutrient concentration. Actually, the pure chemotaxis effect is solely captured by the term $-\chi\sigma$ in (1.2) ((1.5) respectively), while the second contribution $-\chi \operatorname{div}(n(\varphi)\nabla\varphi)$ in (1.3) ((1.6) respectively) is called *active transport* and is responsible of the active movement, in a biological sense, of the nutrient towards regions with high tumour concentration. From a modelling viewpoint, it is indeed possible to decouple these two mechanisms and take two different non-negative constants in front of them. Namely, we can keep $-\chi\sigma$ in (1.2) ((1.5) respectively) with the constant χ and consider the second contribution as $-\eta \operatorname{div}(n(\varphi)\nabla\varphi)$ with a possibly different non-negative constant η. We refer the reader to [89, 90, 96] for more details on chemotaxis and active transport effects as well as their decoupling.

As a common feature of phase-field models, F represents a double-well shaped nonlinearity. Typical examples are given by the regular, logarithmic, and double-obstacle potentials, which are defined, in this order, by

$$F_{reg}(s) := \frac{1}{4}(s^2-1)^2\,, \quad s \in \mathbb{R}\,, \tag{1.7}$$

$$F_{log}(s) := \begin{cases} \frac{\vartheta}{2}\left[(1+s)\ln(1+s) + (1-s)\ln(1-s)\right] - \frac{\vartheta_0}{2}s^2, & \text{if } s \in (-1,1), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.8}$$

$$F_{dob}(s) := \begin{cases} c(1-s^2) & \text{if } s \in [-1,1]\,, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.9}$$

for some positive constant c and $0 < \vartheta < \vartheta_0$. Observe that $F_{\log}$ is very relevant in the applications, where $F'_{\log}(s)$ approaches infinity as $s \to \pm 1$ and that, in the case of (1.9), the second equation (1.2) ((1.5) respectively) has to be modified into a differential inclusion, with $F'(\varphi)$ intended in the sense of subdifferentials.

In the nonlocal model (1.4)–(1.6), J stands for an even spatial convolution kernel and $a := J * 1$. The positive constants α and β may be regarded as relaxation parameters since $\alpha\partial_t\mu$ in (1.1) ((1.4) respectively) provides a parabolic structure to the first equation, whereas the term $\beta\partial_t\varphi$ in (1.2) ((1.5) respectively) represents the classical viscosity contribution as in the Cahn–Hilliard equation (see, e.g., [9,66]). The key idea behind these regularisations, which goes back to [36] in the context of tumour growth models (see also [38, 39]), stems from the fact that their regularising effects allow taking into account general potentials that may be singular, and possibly non-regular, including (1.8) and (1.9) as well as complex biological mechanisms like chemotaxis and active transport. Lastly, the functions $\mathcal{S}^1$ and $\mathcal{S}^2$ design some source/sink terms which somehow encapsulate the mutual interplay between the tumour cells and nutrient cells. For the discussion of some possible specific form for these source terms we refer to Section 1.4.

Let us mention that, in order to better emulate in-vivo tumour growth, numerous authors have proposed to include fluid motion by further coupling systems similar to the models above with a velocity law of Darcy's or Brinkmann's type: see, e.g., [60–62, 65, 82–84, 88–91, 96, 107, 149] and the references therein. In this regard, we stress that the second main assumption of our approach relies on neglecting the velocity effects, thus leading to simpler systems. However, the absence of velocity contribution will allow us to investigate the presented models under very general assumptions covering the case of singular potentials and further complex mechanisms like the active transport which, usually, can not be included in the corresponding mathematical analysis if a velocity field is prescribed.

Let us mention that the last assumption that has been tacitly considered, which is however in line with all the aforementioned models, is that we consider only single-species tumour growth models: for multi-species models, we refer to [54, 80, 82, 92].

From the modelling viewpoint, lots of additional non-trivial mechanisms can be incorporated such as apoptosis or necrosis as well as effects of elasticity or viscoelasticity; see [8, 24, 52, 92, 96, 106, 122, 129, 131, 132] and the references therein. As far as elasticity effects are concerned, we point out the recent work [95] (see also

[68, 116, 117]) wrote by the author in collaboration with H. Garcke and K. F. Lam. There, elasticity effects are taken into account as physical evidence showed that the presence of the extracellular matrix or a rigid bone can assert significant influences on tumour proliferation.

To conclude, a different choice for the potential has been proposed by A. Agosti et al. in [2], where they combine degenerate mobility with single-well potential of Lennard–Jones type. All this introduces several mathematical challenges as the degeneracy set of the mobility and of the singularity set of the potential do not coincide.

Lastly, we mention the contributions [2–5, 49, 51, 69, 150], where numerical simulations and comparison with clinical data can be found.

1.2 The Classical Cahn–Hilliard Equation

In this section, we temporarily set aside the tumour growth models and collect some well-known facts on the celebrated Cahn–Hilliard equation. This will help to comprehend the deep connection between phase segregation and the structure of the above tumour growth models.

The original Cahn–Hilliard equation dates back to 1958 by J. W. Cahn and J. E. Hilliard in [28] (see also [26]) and it was primarily introduced to model the spinodal decomposition in binary alloy mixtures. Since then it has found more and more applications in various fields of applied sciences of which we just mention material science, engineering, chemistry, ecology, natural sciences, mathematics, and imaging sciences. The Cahn–Hilliard equation is a semilinear parabolic equation of fourth-order which, in its classical form, reads as the system

$$\begin{cases} \partial_t \varphi = \operatorname{div}(m(\varphi)\nabla\mu) & \text{in } \Omega \times (0,T), \\ \mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi) & \text{in } \Omega \times (0,T), \end{cases} \tag{1.10}$$

or, equivalently, as the single equation

$$\partial_t \varphi - \operatorname{div}\big(m(\varphi)\nabla(-\varepsilon\Delta\varphi + \tfrac{1}{\varepsilon}F'(\varphi))\big) = 0 \quad \text{in } \Omega \times (0,T). \tag{1.11}$$

As before, φ denotes the phase variable, i.e., the rescaled concentration of one of the material's components which takes values between ± 1 so that the two extremes represent the pure phases, μ is the corresponding chemical potential, $m(\varphi)$ stands for a non-negative mobility, ε for the thickness of the transition layer, and F is a double-well nonlinearity. About it, we note that the most thermodynamically relevant example is the logarithmic nonlinearity (1.8), which can be derived from a mean-field model and it is deeply connected with the entropy of the binary mixture. In the contex of binary mixtures, the constants ϑ and ϑ_0 in (1.8) are proportional, in this order, to the absolute temperature and the critical temperature of the binary mixture we are dealing with. Notice that, the requirement $0 < \vartheta < \vartheta_0$ entails that $F_{\log}$ does possess a double-well structure so that the phase segregation takes place. It is extremely frequent in the literature to employ the regular quartic potential (1.7) as a reasonable polynomial approximation of the logarithmic potential which, from a phenomenological viewpoint, is accurate when the quench is shallow, i.e., when the absolute temperature ϑ is close to the critical value ϑ_0. As the phase mobility $m(\varphi)$ is concerned, there are essentially two main possible choices: regular mobility or degenerate mobility. The first approach consists in assuming a regular, concentration-dependent, non-negative and bounded mobility. Namely, it can be assumed that there exist some constants m_* and m^* such that

$$0 \le m_* \le m(s) \le m^*, \quad s \in \mathbb{R}.$$

Let us point out that the case of concentration dependent mobility already shows some mathematical challenges. In fact, for the classical Cahn–Hilliard equation (1.10) with non-constant and non-degenerate mobility uniqueness of weak solutions is not known for $d \ge 3$: see [13] for the case $d \in \{1, 2\}$. For this reason, the classical assumption is to postulate constant mobility that can be taken without loss of generality equal to one. The last popular choice is the so-called degenerate mobility (proposed in, e.g., [27, 29, 145]) consisting of a polynomial function defined on the physically relevant domain $[-1, 1]$ with degeneracy at the extremal points ± 1. Namely, it can be assumed that

$$m(s) = (1 - s^2)_+ := \max\{1 - s^2, 0\}, \quad s \in \mathbb{R},$$

or, more generally, that

$$m(s) = (1 - s^2)_+^\gamma, \quad \gamma \ge 1,\ s \in \mathbb{R}.$$

Usually, the above choices for the mobility are matched with the choice (1.8) for the potential. In this direction, let us refer to the well-known paper [67] which proves the existence of weak solutions for the case of degenerate mobility, in both two and three dimensions, by employing a suitable approximation argument: the uniqueness result is still an open problem due to the low regularity that can be established. Let us observe that, on the other hand, existence and uniqueness of weak solutions for the nonlocal problem are well-known (see, e.g., [85]). In the context of tumour growth models the degenerate mobility case was analysed in [79, 81].

More recent is instead the derivation of the corresponding nonlocal model performed by G. Giacomin and J. L. Lebowitz in [99–101] (see also the contributions [35, 74–77, 86] and the references therein). In its strong formulation the so-called nonlocal Cahn–Hilliard equation reads as

$$\begin{cases} \partial_t \varphi = \operatorname{div}(m(\varphi)\nabla\mu) & \text{in } \Omega \times (0, T), \\ \mu = \varepsilon(a\varphi - J * \varphi) + \frac{1}{\varepsilon} F'(\varphi) & \text{in } \Omega \times (0, T), \end{cases} \tag{1.12}$$

where $J : \mathbb{R}^d \to \mathbb{R}$ is an even sufficiently fast decaying kernel whose significant examples are given for instance by the Riesz, Bessel or Newtonian potentials, and

$$a(\mathbf{x}) := (J * 1)(\mathbf{x}) = \int_\Omega J(\mathbf{x} - \mathbf{y}) \mathrm{d}\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^d.$$

It is worth noticing that the nonlocal Cahn–Hilliard equation, in contrast to the local one, reduces to a second-order semilinear parabolic equation. In fact, the formal difference with respect to (1.10) is that the Laplace operator is substituted by a suitable spatial convolution kernel J. Even if this may appear as a simplification from a mathematical viewpoint, it is however balanced from the fact that the absence of the Laplace operator in the second equation results in a lack of spatial regularity for the phase variable. Besides, let us remark that by taking $\alpha = \beta = \chi = 0$, $\mathcal{S}^1 = 0$ and $\gamma = 1$ in the abstract system (1.1)–(1.3) we are limited to (1.10) (respectively (1.12) if we consider system (1.4)–(1.6) as a starting point).

Let us now focus our attention on the local model (1.10), even if similar considerations can be extended to the nonlocal case. Typically, the system (1.10) is complemented with homogeneous Neumann (i.e., no flux) boundary conditions

$$\partial_{\mathbf{n}}\varphi = 0, \quad \partial_{\mathbf{n}}\mu = 0 \quad \text{on } \partial\Omega \times (0,T),$$

and with an initial condition of the form

$$\varphi(0) = \varphi_0 \quad \text{in } \Omega,$$

for some given function φ_0. It is worth noticing that the no-flux condition $\partial_{\mathbf{n}}\mu = 0$ entails that the mass is conserved during the evolution as the integration of the first equation of (1.10) produces

$$\int_\Omega \varphi(t) = \int_\Omega \varphi_0 =: m_0 \quad \text{for every } t \in (0,T),$$

being m_0 a fixed quantity since φ_0 is prescribed. Besides, the condition $\partial_{\mathbf{n}}\varphi = 0$ has the physical interpretation that the interface intersects the boundary of Ω at a static contact angle of $\frac{\pi}{2}$. Concerning the nonlocal case let us just mention that we do not need any boundary condition on the phase variable φ due to its lower order.

For further notions on the Cahn–Hilliard equation, we refer to the book [125] and the references therein.

1.3 Energies in Comparison: Short-ranged and Long-ranged Interactions

From a variational perspective, the Cahn–Hilliard equation (1.10) can be derived as the conserved dynamic of the Ginzburg–Landau free energy functional $\mathcal{G}$, reading as

$$\mathcal{G}(\varphi) := \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}F(\varphi)\Big)\mathrm{d}\mathbf{x}. \tag{1.13}$$

Then, one assumes the chemical potential μ to be defined as the variational derivative of the above energy funcitonal, i.e., $\mu := \frac{\delta\mathcal{G}}{\delta\varphi}$. Let us comment on the two terms occurring in the energy: on the one hand the first one, encapsulating surface energy effects, penalises rapid spatial changes, i.e., too many interfaces, whereas on the other hand the second term vanishes in the pure phases and it is positive otherwise. Moreover, it can be shown that the Cahn–Hilliard equation is the H^{-1} type gradient flow of $\mathcal{G}$ in the sense that

$$(\partial_t \varphi, \eta)_* = -\frac{\delta \mathcal{G}}{\delta \varphi}(\varphi)[\eta] \quad \forall \eta \in \mathring{H}^1(\Omega), \tag{1.14}$$

where the above space $\mathring{H}^1(\Omega)$ and the scalar product $(\cdot, \cdot)_*$ are defined later on in Section 2.4.1 (cf. equations (2.7) and (2.8)), and where in this case we assume for simplicity that F is defined by the regular potential (1.7). Since it is not the focus of our study, let us proceed formally and just sketch some heuristic computations which justify our statements: for more details see [48]. In this direction, let us consider a smooth enough solution φ so that we can integrate by parts in (1.14). This leads to obtain that

$$\int_\Omega (-\Delta)^{-1} \partial_t \varphi v \,\mathrm{d}\mathbf{x} = \int_\Omega (\varepsilon \Delta \varphi - \tfrac{1}{\varepsilon} F'(\varphi)) v \mathrm{d}\mathbf{x} - \int_{\partial\Omega} \varepsilon \partial_{\mathbf{n}} \varphi v \mathrm{d}\mathbf{x}, \quad \forall v \in \mathring{H}^1(\Omega).$$

Thus, we invoke the fundamental lemma of calculus of variations which entails that, pointwise, we have

$$(-\Delta)^{-1} \partial_t \varphi = \varepsilon \Delta \varphi - \tfrac{1}{\varepsilon} F'(\varphi) \quad \text{in } \Omega, \quad \text{and} \quad \partial_{\mathbf{n}} \varphi = 0 \quad \text{on } \partial\Omega.$$

Then, upon defining $\mu := -\varepsilon \Delta \varphi + \frac{1}{\varepsilon} F'(\varphi)$, we infer that

$$\begin{cases} \Delta \mu = \partial_t \varphi & \text{in } \Omega, \\ \partial_{\mathbf{n}} \mu = 0 & \text{on } \partial\Omega, \end{cases}$$

and collecting the above terms we deduce that (1.14) produces the Cahn–Hilliard equation (1.10). Let us also mention that by using the same free energy functional it can be shown that the Allen–Cahn equation, which is strictly connected with the Cahn–Hilliard one, may be regarded as the L^2-gradient flow associated to the same free energy $\mathcal{G}$. For more details in this direction, we refer the reader to [16, 17, 48, 87].

The intrinsic limitation of the Ginzburg–Landau energy approach, and in turn of the classical Cahn–Hilliard equation (1.10), relies on the fact that it postulates the evolution to be driven by short-range interactions only, whereas for several physical situations, one has to take to include long-range interactions into account. For the corresponding derivation, we refer to [99–101], where the authors started from a discrete formulation on a lattice and deduce, by using a suitable asymptotic technique (hydrodynamic limit), a nonlocal macroscopic equation. There, instead of the classical (local) Ginzburg–Landau free energy functional, they propose to employ the nonlocal Helmholtz free energy

$$\mathcal{H}(\varphi) := -\frac{\varepsilon}{2} \int_\Omega \int_\Omega J(\mathbf{x} - \mathbf{y}) \varphi(\mathbf{x}) \varphi(\mathbf{y}) \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} + \frac{1}{\varepsilon} \int_\Omega \hat{F}(\varphi) \mathrm{d}\mathbf{x}, \tag{1.15}$$

where $\hat{F}$ denotes the convex part of the double-well potentials introduced above. However, as pointed out in [86], it is possible to rewrite (1.15) in the form

$$\mathcal{H}(\varphi) = \frac{\varepsilon}{4} \int_\Omega \int_\Omega J(\mathbf{x} - \mathbf{y}) |\varphi(\mathbf{x}) - \varphi(\mathbf{y})|^2 \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} + \frac{1}{\varepsilon} \int_\Omega F(\varphi) \mathrm{d}\mathbf{x}, \tag{1.16}$$

where F is a double-well nonlinearity so that (1.16) resembles the strucure of (1.13) and by following similar lines of argument as in the local case it can be proved that the first variation of the Helmholtz free energy leads to the nonlocal Cahn–Hilliard system (1.12). To be precise, this new F, which possess a double-well shape as happened for the classical Cahn–Hilliard equation, has been obtained as

$$F(s) = F(s, \mathbf{x}) = \hat{F}(s) - \frac{1}{2} a(\mathbf{x}) s^2. \tag{1.17}$$

Thus, due to the identity (1.17), the nonlocal Cahn–Hilliard system can be presented as (1.12) with the function $a(\cdot)$ and $F(s) = F(s, \mathbf{x})$ having a double-well shape or, equivalently, as

$$\begin{cases} \partial_t \varphi = \operatorname{div}(m(\varphi)\nabla\mu) & \text{in } \Omega \times (0,T), \\ \mu = -\varepsilon J * \varphi + \frac{1}{\varepsilon}\hat{F}'(\varphi) & \text{in } \Omega \times (0,T), \end{cases} \tag{1.18}$$

where $\hat{F}$ stands for the convex part of the aforementioned double-well potentials. Notice that in the literature, the usual mathematical framework postulates both the presence of the function a as well as the double-well shape for the potential F which is assumed to be independent of the space variable $\mathbf{x}$ (in analogy to (1.13)). It is then clear that all of these results are slightly more general than what described above and therefore in line with the correct nonlocal model.

Besides, (1.16) shows that its first approximation is formally given by $\frac{\varepsilon}{2}|\nabla\varphi|^2$ provided that the convolution kernel J is sufficiently peaked around the origin. In this regards, let us mention that, recently, it has been shown in a series of contributions [55–57, 123] how the asymptotic convergence of the solutions of the nonlocal Cahn–Hilliard equation to the respective solutions to the local one can be obtained when the kernel suitably peaks around zero. As the asymptotic limit performed in those works is performed in the framework of double-well shaped potentials F, which entails including $a(\cdot)$ in the second equation, we have decided to follow the same choice in the formulation of the nonlocal system (1.4)–(1.6) for possible future asymptotic investigations.

To conclude, let us point out that both the local and nonlocal Chan–Hilliard equations verify the energy identities

$$\frac{d}{dt}\mathcal{G}(\varphi(t)) + \int_\Omega |\sqrt{m(\varphi)}\nabla\mu(t)|^2 \,\mathrm{d}\mathbf{x} = 0 \quad \text{for all } t \in [0,T],$$

and

$$\frac{d}{dt}\mathcal{H}(\varphi(t)) + \int_\Omega |\sqrt{m(\varphi)}\nabla\mu(t)|^2 \,\mathrm{d}\mathbf{x} = 0 \quad \text{for all } t \in [0,T],$$

respectively. Lastly, we mention [111–113, 120] where more involved models based on the Cahn–Hilliard equation have been derived as flows of suitable free energies and scalar products.

1.4 Modelling and Biological Insights

Here, we present further modelling details behind the biological models (1.1)–(1.3) and (1.4)–(1.6) and specify the most common choices in the literature for the source/sink terms $\mathcal{S}^1$ and $\mathcal{S}^2$. First, let us notice that the above models considered with $\alpha = \beta = 0$ and $\gamma = 1$ for simplicity, are thermodynamically consistent and can be related to the following local free energy function

$$\mathcal{E}^{\mathrm{loc}}(\varphi, \sigma) = \mathcal{G}(\varphi) + \mathcal{N}(\varphi, \sigma). \tag{1.19}$$

This latter combines a Ginzburg–Landau energy, defined in (1.13), with a nutrient free energy $\mathcal{N}$ defined as

$$\mathcal{N}(\varphi, \sigma) := \int_\Omega \Big(\frac{1}{2}|\sigma|^2 + \chi\sigma(1-\varphi)\Big)\mathrm{d}\mathbf{x} \tag{1.20}$$

taking into account the nutrient diffusion and the chemotaxis effects. In the context of tumour growth, the energy $\mathcal{G}$ defined by (1.13) accounts for cell-to-cell adhesion, modelling the fact that tumour cells prefer to adhere to each other rather than to non-tumour cells. Let us remark that the chemotaxis term in (1.20) does not possess a positive sign in general as we may not be able to show that the phase and nutrient variables stay in the physical regions $[-1, 1]$ and $[0, 1]$, respectively.

While phase segregation described by utilising the local Cahn–Hilliard equation is widely accepted, it is not effective in capturing possible non-local competitions such as competition for space and degradation [144], spatial redistribution [18, 115], and cell-to-cell adhesion phenomena [7, 31, 83, 98]. Hence, in complete analogy of what has been proposed above for the local case, it is possible to define a free energy functional associated to the nonlocal model by simply substitute the Ginzburg–Landau part with the Helmholtz contribution (1.16) leading to the nonlocal free energy

$$\mathcal{E}^{\mathrm{nonloc}}(\varphi, \sigma) = \mathcal{H}(\varphi) + \mathcal{N}(\varphi, \sigma). \tag{1.21}$$

Next, as far as the source/sink terms are concerned, we point out that the two main choices in the literature are the following:

- On the one hand, accounting for linear phenomenological laws for chemical reactions A. Hawkins-Daarud et al. in [103] (see also [102, 104]) take

$$\mathcal{S}^1 = \mathcal{S}^2 = P(\varphi)(\sigma + \chi(1-\varphi) - \mu), \tag{1.22}$$

where $P(\cdot)$ stands for a proliferation function. For this latter they suggest the form

$$P(s) = \delta P_0(1+s)\,\chi_{\{s\geq -1\}}, \quad s \in \mathbb{R}, \tag{1.23}$$

where δ is a positive constant (very small in the applications), P_0 is a non-negative constant, whereas $\chi_E(\cdot)$ is the characteristic function of a set $E \subset \mathbb{R}$ defined by

$$\chi_E(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in E, \\ 0 & \text{if } \mathbf{x} \in \mathbb{R} \setminus E. \end{cases}$$

It is worth noticing that the expression (1.23) yields a proliferation function which is not bounded. We stress this fact since in some works concerning the mathematical analysis of similar models the boundedness of P is required. Another explicit choice has been postulated by D. Hilhorst et al. in [104] assuming $P(\cdot)$ to possess the form

$$P(s) = \frac{2}{\varepsilon} P_0 \sqrt{F(s)}\, \chi_{\{-1 \le s \le 1\}}, \quad s \in \mathbb{R},$$

where ε stands for the thickness parameter introduced before, F for the double-well potential, while P_0 is a non-negative constant. Let us point out that, despite these meaningful explicit expressions, from the mathematical point of view classical assumptions to infer the well-posedness results of the associated systems are that P is a non-negative, regular, bounded and Lipschitz continuous function. The form (1.22) has been the choice of the tumor growth models analysed in [30, 36, 38–40, 42, 43, 45, 78, 97, 136–140].

- On the other hand, using linear kinetics H. Garcke et al. in [88, 96] proposed to employ

$$\mathcal{S}^1 = (\mathcal{P}\sigma - \mathcal{A}) f(\varphi), \quad \mathcal{S}^2 = \mathcal{B}(\sigma - \sigma_S) + \mathcal{C}\sigma f(\varphi), \tag{1.24}$$

where $\mathcal{P}, \mathcal{A}, \mathcal{B}, \mathcal{C}$ are non-negative constants taking into account the proliferation rate of tumoural cells by consumption of nutrient, the apoptosis rate, the consumption rate of the nutrient with respect to a pre-existing concentration σ_S, and the nutrient consumption rate, respectively. Moreover, $f(\cdot)$ denotes an interpolation function between -1 and 1 such that $f(-1) = 0$ and $f(1) = 1$ so to weight the corresponding mechanisms compared to the amount of cancer located in that region and to switch off the associated mechanisms where the tumour cells are not present. This form was the choice of the models investigated in [46, 88–90, 95, 96, 126, 133, 134, 142].

In the next chapters, we will make use of both the paradigms as we will consider the first choice for the local family (1.1)–(1.3), and the second one for the nonlocal family (1.4)–(1.6). For more details on the modelling aspects, we refer to [96] and the references cited therein.

1.5 Structure of the Thesis

The rest of the thesis is organised as follows: Chapter 2 is primarily devoted to introduce and set, once for all, the employed conventions and notation. Moreover, we recollect there some selected topics and well-known results of functional analysis that will be, sometimes tacitly, utilised in the forthcoming chapters.

Then, we can subdivide the remaining part of the thesis in two main parts: the first one is dedicated to address several analytic aspects of the systems (1.1)–(1.3) and (1.4)–(1.6) including the weak and strong well-posedness and the vanishing viscosities analysis as the relaxation parameters α and β tend to zero. The second part takes advantage of some results obtained in those chapters and instead addresses non-trivial applications in terms of optimal control problems in which the governing equations are determined by the models previously examined.

More precisely, in Chapter 3, we discuss many mathematical aspects on the local model (1.1)–(1.3): we first address the weak well-posedness for the system covering the scenario of singular potentials, and we then moved to establish the strong well-posedness still including singular, while regular, potentials of which (1.8) is the prototypical example. It is worth noticing that Chapter 3 revisits some partial results obtained by the author in [136]. Lastly, we present the asymptotic behaviour of the local model (1.1)–(1.3) by discussing the singular limits of the system as $\alpha \to 0$ and $\beta \to 0$; these results are inspired by [36, 38, 39] and are propaedeutical to the asymptotic analysis as α and β go to zero performed in Chapter 5 at the level of the associated optimal control problems.

An analogous investigation has been performed in Chapter 6 for the nonlocal class (1.4)–(1.6). We remark that the results there established are reminiscent of the ones in [134].

Concerning the associated optimal control problems carried out in the second part of the thesis, we first analyse in Chapter 5 the optimal control for the local model with a classical tracking type cost functional. We prove the existence of minimisers and obtain the first-order necessary conditions for optimality. These results extend somehow previous contributions by including in the analysis also singular potentials. Then, we employ suitable asymptotic techniques to let $\alpha \to 0$ and $\beta \to 0$ separately at the level of the optimal control problem by exploiting the asymptotic results investigated in Chapter 3. We remark that the results there provided are largely inspired by [136] and [137, 140].

By following similar lines, in Chapter 6 we study the optimal control problem for the nonlocal model by revising the results achieved in [133]. Instead of considering the classical objective type cost functional, we postulate a specific form of the cost functional and of the control variables by taking inspiration from [109], which turns the minimisation problem into a parameter identification problem.

CHAPTER 2

Mathematical Preliminaries and Notation

In this chapter we present and fix our conventions and basic notation. Furthermore, for the reader's convenience, we collect some selected topics and fundamental tools that will be employed several times throughout the rest of the thesis.

2.1 Basic Notation

In what follows we assume $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ to be a bounded domain with smooth boundary $\Gamma := \partial\Omega$ and that $T > 0$ is a fixed final time horizon. Then, we set for convenience the parabolic cylinders

$$Q_t := \Omega \times (0, t), \quad \Sigma_t := \Gamma \times (0, t), \quad Q_t^T := \Omega \times (t, T), \quad t \in (0, T),$$

and, for brevity, we set

$$Q := Q_T, \qquad \Sigma := \Sigma_T.$$

For a given (real) Banach space X, we indicate with X^*, $\|\cdot\|_X$ and $\langle \cdot, \cdot \rangle_X$ its topological dual, its norm, and the duality pairing between X^* and X, respectively. Moreover, we employ the classical notation $L^p(\Omega)$ and $W^{k,p}(\Omega)$, with $p \in [1, \infty]$ and $k > 0$, for the standard Lebesgue and Sobolev spaces over Ω along with their norms $\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{k,p}(\Omega)}$, respectively. For the special case $p = 2$ we have Hilbert spaces and we convey to denote $H^k(\Omega) := W^{k,2}(\Omega)$ for every $k > 0$. Since the spaces $L^2(\Omega)$ and

$H^1(\Omega)$ will be intensively used, we introduce the following notation

$$H := L^2(\Omega), \qquad V := H^1(\Omega),$$

and also

$$W := \left\{ f \in H^2(\Omega) : \partial_{\mathbf{n}} f = 0 \ \text{ a.e. on } \Gamma \right\},$$

and we equip them with their natural norms $\|\cdot\| := \|\cdot\|_H$, $\|\cdot\|_V$, and $\|\cdot\|_W$, respectively. As usual, H is identified with its own dual H^* through its Riesz isomorphism, so that

$$W \subset\subset V \subset\subset H \approx H^* \subset\subset V^* \subset\subset W^*,$$

where all inclusions are dense, continuous, and compact. In this regards, for arbitrary Banach spaces A and B, we use $A \subset B$ to denote the continuous embedding of A into B and $A \subset\subset B$ for the continuous and compact embedding. The duality pairing between V^* and V, and the scalar product in H will be denoted by the symbols $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$, respectively.

For Bochner spaces with values in a generic Banach space X, we use the notation $L^p(0,T;X)$ with $p \in [1,\infty]$ and if $X = L^p(\Omega)$ we recall that it holds that $L^p(0,T;L^p(\Omega)) \approx L^p(Q) = L^p(\Omega \times (0,T))$. For two given Bochner spaces A and B we employ the symbol $\|\cdot\|_{A\cap B}$ to indicate $\|\cdot\|_A + \|\cdot\|_B$.

Besides, we define the generalised spatial mean value of a function v by

$$v_\Omega := \begin{cases} \frac{1}{|\Omega|} \int_\Omega v(\mathbf{x}) \mathrm{d}\mathbf{x} & \text{if } v \in L^1(\Omega), \\ \frac{1}{|\Omega|} \langle v, 1 \rangle & \text{if } v \in V^*, \end{cases} \tag{2.1}$$

where $|\Omega|$ stands for the d-dimensional Lebesgue measure of Ω.

As far as the constants are concerned, let us set once and for all our convention: the symbol C is used to indicate every constant that depends only on the structural data of the problem, such as T, Ω, α or β, the shape of the nonlinearities, and the norms of the involved functions. On the other hand, with specific capital letters and possible subscripts we specify particular constants. Therefore, notice that the meaning of the constant C may change from line to line.

2.2 Significant Inequalities

We recollect here some fundamental classical inequalities which will be employed later on.

- The Young inequality (see, e.g., [6]): let $a, b \geq 0$ and $\delta > 0$. Moreover, let $p, q \in (1,\infty)$ be two conjugate exponents, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Then, it holds that

$$ab \leq \frac{\delta}{p} a^p + \frac{(\delta)^{-q/p}}{q} b^q. \tag{2.2}$$

This estimate will be usually employed in a simplified form with the particular choice $p = q = 2$ so that (2.2) reduces to (with a slight abuse of notation taking as δ in the previous estimate 2δ)

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2. \tag{2.3}$$

For simplicity, we will simply refer to this latter as Young's inequality and to (2.2) as to generalised Young's inequality.

- The Hölder inequality: assume $\Omega \subset \mathbb{R}^d$ with $d \geq 1$ be a bounded domain. Let the exponents p, q, r be such that $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then it holds that

$$fg \in L^r(\Omega), \quad \text{and} \quad \|fg\|_r \leq \|f\|_p \|g\|_q.$$

- The Poincaré–Wirtinger inequality: suppose $\Omega \subset \mathbb{R}^d$ with $d \geq 2$ be a bounded domain with Lipschitz boundary Γ. Then,

$$\|v\|_V^2 \leq C_\Omega \big(\|\nabla v\|^2 + |v_\Omega|^2\big) \quad \text{for every } v \in V, \tag{2.4}$$

where the constant $C_\Omega > 0$ depends only on Ω.

2.2.1 The Gronwall Lemma

Let us collect here one version of the celebrated Gronwall's lemma in integral form. For the proof, we refer the reader to, e.g., [19].

Lemma 2.1. *Let $T > 0$, $g \in L^1(0, T; \mathbb{R})$ a.e. non-negative on $[0, T]$, and a be a non-negative constant. Moreover, let $f : [0, T] \to \mathbb{R}$ be a continuous function such that*

$$f(t) \leq a + \int_0^t g(s) f(s) \mathrm{ds} \quad \forall\, t \in [0, T].$$

Then, it holds that

$$f(t) \leq a \exp\Big(\int_0^t g(s) \mathrm{ds}\Big) \quad \forall t \in [0, T].$$

2.3 Continuous and Compact Embeddings

As far as some well-known results concerning continuous and compact inclusions of Sobolev spaces are concerned, we recall the following results which can be found, e.g., in [6, Thm. 10.9, Thm. 10.13] (see also [1, Thm. 4.12, Thm. 6.3] and [20]).

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^d$ with $d \geq 1$ be a bounded domain with Lipschitz boundary Γ. Moreover, suppose that $1 \leq p_1 \leq \infty$, $1 \leq p_2 < \infty$, and $k_1, k_2 \geq 0$ be some integers with the convention that $W^{0,p}(\Omega) := L^p(\Omega)$ for every $1 \leq p \leq \infty$. Then, the following holds:*

- *If* $k_1 \geq k_2$ *we have the continuous embedding*

$$W^{k_1,p_1}(\Omega) \subset W^{k_2,p_2}(\Omega) \quad \textit{if} \quad k_1 - \tfrac{d}{p_1} \geq k_2 - \tfrac{d}{p_2}.$$

  Moreover, there exists a positive contant K *which depends on* $\Omega, d, p_1, k_1, p_2, k_2$ *such that*

$$\|u\|_{W^{k_2,p_2}(\Omega)} \leq K\|u\|_{W^{k_1,p_1}(\Omega)}, \quad \textit{for every } u \in W^{k_1,p_1}(\Omega).$$

- *If* $k_1 > k_2$ *we have the compact embedding*

$$W^{k_1,p_1}(\Omega) \subset\subset W^{k_2,p_2}(\Omega) \quad \textit{if} \quad k_1 - \tfrac{d}{p_1} > k_2 - \tfrac{d}{p_2}.$$

- *If* $k_1 \geq 1$ *we have the continuous and compact embedding*

$$W^{k_1,p_1}(\Omega) \subset\subset C^{k_2}(\overline{\Omega}) \quad \textit{if} \quad k_1 - \tfrac{d}{p_1} > k_2.$$

Remark 2.3. *It is worth noticing that, as an easy consequence of the above results, in the case* $d \in \{2, 3\}$*, we have the following well-known continuous and compact embeddings:*

- *For every* $p \in [1, \infty], q \in [1, \infty)$ *if* $d = 2$ *and every* $p \in [1, 6], q \in [1, 6)$ *if* $d = 3$,

$$H^1(\Omega) \subset L^p(\Omega), \quad H^1(\Omega) \subset\subset L^q(\Omega).$$

- *If* $d = 3$ *we have the following continuous and compact inclusion (which also holds for* $d = 2$*)*

$$H^2(\Omega) \subset\subset C^0(\overline{\Omega}).$$

2.3.0.1 The Aubin–Lions Lemma

The Aubin–Lions lemma, also known as the Aubin–Lions–Simon lemma, is a fundamental tool in the theory of evolutionary partial differential equations since it provides an important compactness criterion. In fact, a typical way to establish the existence of a solution of an evolutionary PDE is to first construct approximate solutions (e.g., Galerkin solutions, see below) and then prove that there exists a subsequence of approximate solutions which converge, in a suitable sense, and whose limit yields a solution to the original equation. In many instances, the Aubin–Lions lemma provides sufficient compactness to ensure the existence of such a subsequence.

Lemma 2.4. *Let* X_0, X, X_1 *be Banach spaces such that*

$$X_0 \subset\subset X \subset X_1.$$

Namely, we assume that X_0 *is compactly embedded in* X *and that* X *is continuously embedded in* X_1*. Moreover, let* p, q *such that* $1 \leq p, q \leq \infty$*. Then:*

∘ *If $p < \infty$ it holds that*

$$\{v \in L^p(0,T;X_0) : \partial_t v \in L^q(0,T;X_1)\} \subset\subset L^p(0,T;X).$$

∘ *If $q > 1$ it holds that*

$$\{v \in L^\infty(0,T;X_0) : \partial_t v \in L^q(0,T;X_1)\} \subset\subset C^0([0,T];X).$$

Proof. See, e.g., [141, Sec. 8, Cor. 4] (see also [118, Thm. 5.1, p. 58]). □

Remark 2.5. *We will apply the Aubin–Lions lemma several times using the following compact inclusions:*

$$H^2(\Omega) \subset\subset H^1(\Omega) \subset\subset L^2(\Omega), \quad L^2(\Omega) \approx (L^2(\Omega))^* \subset\subset (H^1(\Omega))^* \subset\subset (H^2(\Omega))^*.$$

In particular, for $d = 3$, it follows that

$$\begin{aligned}
&H^1(0,T;(H^1(\Omega))^*) \cap L^\infty(0,T;H^1(\Omega)) \subset\subset C^0([0,T];L^2(\Omega)),\\
&H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \subset\subset L^2(0,T;L^r(\Omega)), \quad r \in [1,6),\\
&H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \subset\subset L^2(0,T;H^1(\Omega)),\\
&H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^2(\Omega)) \subset\subset C^0([0,T];W^{1,r}(\Omega)), \quad r \in [1,6).
\end{aligned}$$

Moreover, using $W^{1,r}(\Omega) \subset\subset C^0(\overline{\Omega})$, $r \in (3,6)$, we also deduce that

$$H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^2(\Omega)) \subset\subset C^0(\overline{Q}).$$

2.4 Important Tools for the Well-posedness

2.4.1 The Neumann–Laplace Problem

The homogeneous Neumann–Laplace problem with source term f (which has to possess zero mean value to be solved) reads as

$$\begin{cases} -\Delta u = f & \text{in } \Omega,\\ \partial_{\mathbf{n}} u = 0 & \text{on } \Gamma. \end{cases} \tag{2.5}$$

Moreover, we recall that the Neumann–Laplace operator may be seen as a variational operator by setting

$$-\Delta : V \to V^*, \quad \langle -\Delta v, w\rangle := \int_\Omega \nabla v \cdot \nabla w, \quad \forall\, v, w \in V. \tag{2.6}$$

For convenience, we set

$$\mathring{H}^1(\Omega) := \{v \in V \,:\, v_\Omega = 0\}, \quad (\mathring{H}^1(\Omega))^* := \{v^* \in V^* \,:\, v^*_\Omega = 0\}, \tag{2.7}$$

where v_Ω denotes the mean value of v as introduced by (2.1). Then, it is well known, as a consequence of the Poincaré–Wirtinger inequality (2.4), that the restriction of $-\Delta$ to $\mathring{H}^1(\Omega)$ is positive definite and self-adjoint with well-defined inverse

$$\mathcal{N} := (-\Delta)^{-1} : (\mathring{H}^1(\Omega))^* \to \mathring{H}^1(\Omega).$$

Namely, we have that for every $f \in (\mathring{H}^1(\Omega))^*$, $u = \mathcal{N} f \in \mathring{H}^1(\Omega)$ is the unique weak solution to (2.5) with zero mean value, meaning that

$$\langle -\Delta u, v\rangle = \int_\Omega \nabla u \cdot \nabla v = \langle f, v\rangle, \quad \forall\, v \in \mathring{H}^1(\Omega).$$

This allows us to define the inner product

$$(u^*, v^*)_* := (\nabla \mathcal{N} u^*, \nabla \mathcal{N} v^*), \quad \forall\, u^*, v^* \in (\mathring{H}^1(\Omega))^*, \tag{2.8}$$

and the associated norm $\|\cdot\|_* := (\cdot,\cdot)_*^{1/2}$. Moreover, it follows that

$$\langle -\Delta u, \mathcal{N} v^*\rangle = \langle v^*, u\rangle, \qquad \forall\, u \in \mathring{H}^1(\Omega), v^* \in (\mathring{H}^1(\Omega))^*, \tag{2.9}$$

$$\langle v^*, \mathcal{N} u^*\rangle = \langle u^*, \mathcal{N} v^*\rangle = (v^*, u^*)_*, \qquad \forall\, u^*, v^* \in (\mathring{H}^1(\Omega))^*. \tag{2.10}$$

Besides, it is a standard matter to show that in the larger space V^* we can simply introduce the norm, again indicated with $\|\cdot\|_*$, given by

$$v^* \mapsto \|v^*\|_* := \left(\|\nabla \mathcal{N}(v^* - v^*_\Omega)\|^2 + |v^*_\Omega|^2\right)^{1/2}, \qquad \forall\, v^* \in V^*,$$

which yields an equivalent norm on V^* (thanks, e.g., to the open mapping theorem). Then, due to (2.10) we infer that, for every $v^* \in H^1(0,T; (\mathring{H}^1(\Omega))^*)$ and for almost every $t \in (0,T)$, we have the identity

$$\langle \partial_t v^*(t), \mathcal{N} v^*(t)\rangle = \frac{1}{2}\frac{d}{dt}\|v^*(t)\|_*^2.$$

Furthermore, let us introduce the Riesz isomorphism of V given by

$$\mathcal{R} = I - \Delta : V \to V^*, \quad \langle \mathcal{R} u, v\rangle := \int_\Omega (\nabla u \cdot \nabla v + uv), \quad \forall\, u, v \in V.$$

It is well-known that $\mathcal{R}_{|W}$ yields an isomorphism from W to H with well-defined inverse $\mathcal{R}^{-1} : H \to W$. As above, for the inverse map $\mathcal{R}^{-1} : V^* \to V$, we have the identities

$$\begin{aligned} \langle \mathcal{R} u, \mathcal{R}^{-1} v^*\rangle &= \langle v^*, u\rangle, && \forall u \in V, \forall v^* \in V^*, \\ \langle u^*, \mathcal{R}^{-1} v^*\rangle = \langle v^*, \mathcal{R}^{-1} u^*\rangle &= (u^*, v^*)_{V^*}, && \forall u^*, v^* \in V^*, \end{aligned} \tag{2.11}$$

where the symbol $(\cdot,\cdot)_{V^*}$ denotes the dual inner product in V^*. Furthermore, for every $f \in V$ it holds that

$$\|f\|^2 = \langle f, f\rangle = \langle \mathcal{R} f, \mathcal{R}^{-1} f\rangle \le \|f\|_V \|\mathcal{R}^{-1} f\|_V \le \|f\|_V \|f\|_{V^*}. \tag{2.12}$$

Besides, if $v^* \in H^1(0,T; V^*)$ we have, for almost every $t \in (0,T)$, that

$$\langle \partial_t v^*(t), \mathcal{R}^{-1} v^*(t)\rangle = \frac{1}{2}\frac{d}{dt}\|v^*(t)\|_{V^*}^2. \tag{2.13}$$

2.4.2 The Faedo–Galerkin Scheme

Approximation schemes such as the Faedo–Galerkin one are powerful tools to prove the existence of solutions for evolutionary partial differential equations. The idea behind this procedure can be schematised in the following steps:

- Select a Schauder basis by means of eigenfunctions $\{w_j\}_j$ of a certain PDE which allows to introduce the *Galerkin approximated problem* which consists in a finite-dimensional ODE system in the finite subset $\mathcal{W}_n := \mathrm{span}\{w_1, ..., w_n\}$.
- Employ well-known results for systems of (non-linear) ODEs, like the Picard–Lindelöf theorem or the Carathéodory theorem, to establish the well-posedness of the approximated system.
- Show that the approximating solutions are independent of n, being n the dimension of the finite-dimensional subspaces $\mathcal{W}_n$. This is usually checked by showing some a priori estimates for the approximated solutions which are independent of n.
- Pass to the limit in the approximating system in a suitable sense as $n \to \infty$ and show that the limit yields a solution to the original problem. This usually requires the use of weak and weak star compactness arguments and the aforementioned Aubin–Lions lemma.

From spectral theory we obtain the classical result below: see, e.g., [108].

Theorem 2.6. *The eigenfunctions of the Neumann–Laplace operator* (2.6) *form an orthonormal Schauder basis in $L^2(\Omega)$ which is also orthogonal in $H^1(\Omega)$.*

Next, we recollect some fundamental classical tool for establishing the existence and uniqueness of an initial value problem. In this direction, let us refer, e.g., to [32] and the references therein for more details and the corresponding proofs.

The classical (forward) Cauchy problem can be formalised as follows:
Let $D \subset \mathbb{R} \times \mathbb{R}^d$ and let $f : D \to \mathbb{R}^d$ with $f = f(t, \mathbf{y})$. For a given initial datum $(t_0, \mathbf{y}_0) \in D$ and an interval $I \subset \mathbb{R}$ containing t_0, we consider the Cauchy problem:

$$\begin{cases} \mathbf{y}'(t) = f(t, \mathbf{y}(t)) & \text{in } I, \\ \mathbf{y}(t_0) = \mathbf{y}_0. \end{cases} \tag{2.14}$$

A fundamental result to solve (2.14) is the Picard–Lindelöf theorem, also known as Cauchy–Lipschitz theorem.

Theorem 2.7 (Picard–Lindelöf's Theorem)**.** *Let $D \subset \mathbb{R} \times \mathbb{R}^d$ and let $f : D \to \mathbb{R}^d$ with $f = f(t, \mathbf{y})$ be continuous and locally Lipschitz continuous with respect to its second variable. Hence, for every admissible initial datum $(t_0, \mathbf{y}_0) \in D$ there exists an interval $I \subset \mathbb{R}$ containing t_0 so that the initial value problem problem* (2.14) *admits a unique classical solution $\mathbf{y}$ in I in the sense that all the following conditions are fulfilled:*

$$\mathbf{y} \in C^1(I; \mathbb{R}^d), \qquad \mathbf{y}(t) \in D \quad \textit{for every } \ t \in I, \qquad \mathbf{y}(t_0) = \mathbf{y}_0.$$

Notice that the above result require f to be continuous on D. However, after integrating the first equation in (2.14) and using the initial condition, it is clear that we can consider a weaker notion of solution by just assuming that $\mathbf{y}$ satisfies the integral equation

$$\mathbf{y}(t) = \mathbf{y}_0 + \int_{t_0}^{t} f(s, \mathbf{y}(s))\mathrm{ds}, \quad \forall t \in I. \tag{2.15}$$

The existence result that can be established for this formulation is known as Carathéodory's existence theorem.

Theorem 2.8 (Carathéodory's Theorem)**.** *Let $D \subset \mathbb{R} \times \mathbb{R}^d$ and let $f : D \to \mathbb{R}^d$ with $f = f(t, \mathbf{y})$ satisfies the Carathéodory conditions on D. Hence, for every admissible initial datum $(t_0, \mathbf{y}_0) \in D$ there exists an interval $I \subset \mathbb{R}$ containing t_0 so that the initial value problem problem* (2.14) *admits a unique weak solution in I. Namely, there exists a unique absolutely continuous function $\mathbf{y} : I \to \mathbb{R}^d$ which fulfils* (2.15) *and solves* (2.14) *for almost every $t \in I$.*

2.4.3 Maximal Monotone Operators and the Yosida Approximation

Let us here briefly review some basic notions concerning maximal monotone operators in Hilbert spaces as well as the associated Moreau–Yosida approximation. Let us mention that lots of the following results can be extended to the framework of Banach spaces and that, as usual, we tacitly identified the dual H^* with H and denote the norm and the inner product of H by $\|\cdot\|$ and $(\cdot, \cdot)$, respectively. For further details and the corresponding proofs, we refer the reader, e.g., to [19, Chapter 3], [10], [135, Chapter 4].

Definition 2.9. *Let H be a (real) Hilbert space and let A be a possible multivalued operator $A : H \to 2^H$ with domain, range and graph defined, in this order, by*

$$D(A) := \{x \in H : Ax \neq \emptyset\}, \quad R(A) := \{y \in Ax : x \in D(A)\},$$
$$G(A) := \{(x, y) \in H \times H : x \in D(A), y \in Ax\}.$$

Then, the operator A is said to be monotone if

$$(f - g, x - y) \geq 0 \quad \forall x, y \in D(A), \quad f \in Ax, \ g \in Ay.$$

Moreover, the operator A is said to be maximal if

$$(f - g, x - y) \geq 0 \quad \forall (y, g) \in G(A) \quad \Rightarrow \quad x \in D(A), \ f \in Ax.$$

For a maximal monotone multivalued mapping it can be shown that, for any $\lambda > 0$, denoting by I the identity operator of H, the operator $I + \lambda A$ is surjective, i.e., $R(I + \lambda A) = H$. This entails that for every given datum $u \in H$, there exists a couple $(x, f) \in G(A)$ which solves the problem

$$x + \lambda f = u, \quad f \in Ax.$$

Hence, for every $\lambda > 0$, we can introduce the so-called resolvent operator $\mathcal{J}_\lambda$ and the associated Moreau–Yosida approximation A_λ by

$$\mathcal{J}_\lambda := (I + \lambda A)^{-1}, \quad A_\lambda := \frac{I - (I + \lambda A)^{-1}}{\lambda} = \frac{1}{\lambda}(I - \mathcal{J}_\lambda).$$

Let us recall that $\mathcal{J}_\lambda$ and A_λ are Lipschitz continuous operators with Lipschitz constant equal to 1 and $\frac{1}{\lambda}$, respectively. Moreover, we denote by $A^\circ x$ the element of minimum norm of Ax, i.e., for every $x \in D(A)$

$$\|A^\circ x\| := \inf\{\|f\| : f \in Ax\},$$

and recall that it holds

$$\|A_\lambda x\| \leq \|A^\circ x\|, \quad \lambda > 0,$$

and, as $\lambda \to 0$,

$$A_\lambda x \to A^\circ x \quad \text{weakly in } H, \forall x \in D(A).$$

As well-known, a fundamental class of maximal monotone operators, sometimes referred to as *subpotential maximal monotone operators*, is given by the subdifferentials of proper, convex and lower semicontinuous functions.

Theorem 2.10. *Let $\hat{\beta} : H \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be a proper (i.e., not identically $+\infty$), convex and lower semicontinuous function. Then, its subdifferential $\beta := \partial\hat{\beta} : H \to 2^H$, defined by*

$$\beta(x) = \{f \in H : \hat{\beta}(y) \geq \hat{\beta}(x) + (f, y - x) \quad \forall y \in H\}, \quad x \in D(\beta),$$

is a maximal monotone operator.

Lastly, for every $\lambda > 0$, we set the Yosida approximation of $\hat{\beta}$ by

$$\hat{\beta}_\lambda(x) := \inf_{y \in H} \left\{ \frac{1}{2\lambda}\|x - y\|^2 + \hat{\beta}(y) \right\}$$

and recall that $\hat{\beta}_\lambda$ is convex, Fréchet differentiable and it holds that

$$\hat{\beta}_\lambda(x) \leq \hat{\beta}(x) \quad \text{for every } x \in H, \lambda > 0, \quad \hat{\beta}_\lambda(x) \nearrow \hat{\beta}(x) \quad \text{as } \lambda \to 0.$$

Let us now present the following basic result which explains in which sense a proper, convex and lower semicontinuous function in H induces a proper, convex and lower semicontinuous function in $L^2(0, T; H)$ in a canonical fashion.

Theorem 2.11. *Assume that $\hat{\beta} : H \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous operator. Let $\hat{B} : L^2(0, T; H) \to \overline{\mathbb{R}}$ be the function given by*

$$\hat{B}(u) := \begin{cases} \int_0^T \hat{\beta}(u(t))\mathrm{dt} & \text{if } \hat{\beta}(u) \in L^1(0, T), \\ +\infty & \text{otherwise}. \end{cases}$$

Then, $\hat{B}$ is proper, convex and lower semicontinuous operator in $L^2(0,T;H)$ and it holds that

$$\xi \in \partial\hat{B}(u) \quad \Leftrightarrow \quad \xi(t) \in \partial\hat{\beta}(u(t)) \quad \textit{for } a.e.\, t \in (0,T). \tag{2.16}$$

Setting $\beta := \partial\hat{\beta}$ and $B := \partial\hat{B}$ this reads as

$$\xi \in B(u) \quad \Leftrightarrow \quad \xi(t) \in \beta(u(t)) \quad \textit{for } a.e.\, t \in (0,T)$$

and, from the above theorem, that they yield maximal monotone operator on H and on $L^2(0,T;H)$, respectively.

Proof. At first we note that $\hat{B}$ is proper as $\hat{\beta}$ is so. Moreover, being $\hat{\beta}$ convex, let us claim that $\hat{B}$ can not attain the value $-\infty$ anywhere. In fact, from the convexity of $\hat{\beta}$ we infer the existence of a positive constant c and $y \in H$ such that

$$\hat{\beta}(x) \geq c + (y,x) \quad \text{for every } x \in H.$$

Let now u_i, $i = 1,2$, be two element of the domain of

$$D(\hat{B}) := \{u \in L^2(0,T;H) : \hat{\beta}(u) \in L^1(0,T)\}$$

and, without loss of generality, we can assume that

$$u_i(t) \in D(\hat{\beta}) \quad \text{for } a.e.\, t \in (0,T), \quad i = 1,2.$$

Hence, it suffices to integrate over time the convexity condition of $\hat{\beta}$ given by

$$\hat{\beta}(\lambda u_1(t) + (1-\lambda)u_2(t)) \leq \lambda\hat{\beta}(u_1(t)) + (1-\lambda)\hat{\beta}(u_2(t)) \quad \forall\lambda \in [0,1].$$

As we will employ Fatou's lemma we can directly assume that $\hat{\beta}$ is non-negative. If that is not the case we can perturb $\hat{\beta}$ and define $\tilde{\beta}(u) := \hat{\beta}(u) + \tilde{c}(1 + \|u\|^2)$, where $\tilde{c}$ has to be chosen large enough so to have the non-negativity of $\tilde{\beta}$. Then the following lines will imply that $\tilde{\beta}$ is lower semicontinuous which in turn implies the same property for $\hat{\beta}$. Next, let $\{u_n\}_n \subset L^2(0,T;H)$ be a sequence such that

$$u_n \to u \quad \text{strongly in } L^2(0,T;H). \tag{2.17}$$

Assume by contradiction that $\hat{B}$ is not semicontinuous. Then, there exists a subsequence $\{n_k\}_k$ such that

$$\hat{B}(u) > \lim_{k\to\infty} \hat{B}(u_{n_k}).$$

Moreover, (2.17) entails the existence of a subsequence $\{n_{k_j}\}_j$ such that, as $j \to \infty$,

$$u_{n_{k_j}}(t) \to u(t) \quad \text{strongly in } H \quad \text{for } a.e.\, t \in (0,T)$$

and therefore that

$$\hat{\beta}(u(t)) \leq \liminf_{j\to\infty} \hat{\beta}(u_{n_{k_j}}(t)) \quad \text{for } a.e.\, t \in (0,T).$$

Thus, integrating over time the above expression and using the Fatou's lemma we obtain a contradiction.

We are then reduced to show that (2.16) holds. In this direction, let $\xi \in L^2(0,T;H)$, $\xi(t) \in \partial\hat{\beta}(u(t))$ for $a.e.\, t \in (0,T)$. Then, for $a.e.\, t \in (0,T)$ it holds that

$$(\xi(t), v(t) - u(t)) + \hat{\beta}(u(t)) \leq \hat{\beta}(v(t)) \quad \text{for every } v \in L^2(0,T;H).$$

Again, an integration over time produces $\xi \in \partial\hat{B}(u)$. Thus, the proof is completed. □

2.5 Minimisation Problems in Hilbert Spaces

In the second part of this thesis, we will deal with some optimal control problems for the systems (1.1)–(1.3) and (1.4)–(1.6). Since these problems are essentially constrained optimisation problems in Banach spaces, we recollect here some basic results related to constrained minimisation problems first in the Euclidian space $\mathbb{R}^d$ and then in a generic Banach and Hilbert space.

2.5.1 Minimisation Problems in $\mathbb{R}^d$

To begin with, let us state the natural extension of the celebrated Weierstrass' theorem.

Lemma 2.12. *Let K be a closed subset of $\mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ be a lower semicontinuous function over K. Then, if K is compact the minimisation problem*

$$\min_{\mathbf{x}\in K} f(\mathbf{x}) \tag{2.18}$$

admits a solution $\mathbf{x}_0$, meaning that

$$\exists\, \mathbf{x}_0 \in K \quad s.t. \quad f(\mathbf{x}_0) \leq f(\mathbf{x}) \quad \forall \mathbf{x} \in K.$$

Proof. Here, we just provide a sketch of the proof since the result is very standard and classical.

- First step: consider a minimising sequence $\{\mathbf{x}_k\}_k \subset \mathbb{R}^d$, i.e., a sequence such that

  $$\lim_{k\to\infty} f(\mathbf{x}_k) = \inf_{\mathbf{x}\in K} f(\mathbf{x}) =: \lambda$$

  and notice that $\lambda < +\infty$ or $\lambda = -\infty$.

- Second step: since K is compact, we are allowed to extract a convergent subsequence. Namely we deduce that there exists an element $\mathbf{x}_0 \in K$, since K is also closed, and a subsequence $\{k_j\}_j$ such that $\mathbf{x}_{kj} \to \mathbf{x}_0$, as $j \to \infty$.

◦ Third step: we now make use of the lower semicontinuity of f to infer that

$$f(\mathbf{x}_0) \leq \liminf_{j\to\infty} f(\mathbf{x}_{kj}) = \lim_{j\to\infty} f(\mathbf{x}_{kj}) = \lambda$$

so that $\mathbf{x}_0$ is the minimiser are looking for.

□

Lemma 2.13. *Assume that the assumptions of Lemma 2.12 hold. Moreover, suppose that K is convex and that f is strictly convex over K, i.e., it holds that*

$$f(t\mathbf{x} + (1-t)\mathbf{y}) < tf(\mathbf{x}) + (1-t)f(\mathbf{y}) \quad \forall t \in (0,1),\, \mathbf{x}, \mathbf{y} \in K, \mathbf{x} \neq \mathbf{y}. \tag{2.19}$$

Then, the solution to the minisation problem (2.18) *is unique.*

Once the existence of a minimiser has been established, a natural goal is to obtain some first-order necessary conditions for optimality. In this direction here is the main result.

Lemma 2.14. *Let K be a closed and convex subset of $\mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ be differentiable on K. Hence, every minimiser $\mathbf{x}_0$ satisfies the following variational inequality*

$$\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \geq 0 \quad \forall \mathbf{x} \in K. \tag{2.20}$$

Proof. Let $\mathbf{x}_0$ be a minimiser. Then, we have

$$f(\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0)) - f(\mathbf{x}_0) \geq 0 \quad \forall \mathbf{x} \in K, t \in (0,1).$$

Thus, we divide by t and pass to the limit as $t \to 0$ to infer that

$$\mathrm{d}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \geq 0 \quad \forall \mathbf{x} \in K$$

producing the claim. □

Combining the above lemmas, we have:

Corollary 2.15. *Let K be a bounded, closed and convex subset of $\mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ be a lower semicontinuous function (strictly convex) over K. Then, there exists a (unique) minimiser $\mathbf{x}_0 \in K$. Moreover, if f is differentiable, the minimiser $\mathbf{x}_0$ necessarily (and sufficiently) fulfils the variational inequality* (2.20).

2.5.2 Extension to Banach and Hilbert Spaces

The above results have been presented since they contain all the key ingredients to generalise the minimisation problem (2.18) in the framework of Banach and Hilbert spaces. Arguing in a similar fashion, let $\hat{\mathcal{J}} : \mathcal{U} \to \mathbb{R}$ be a functional and let $\mathcal{U}_{\mathrm{ad}}$ be a subset of a reflexive Banach space $\mathcal{U}$. Then, we are going to study the following optimisation problem:

$$\min_{v \in \mathcal{U}_{\mathrm{ad}}} \hat{\mathcal{J}}(v). \tag{2.21}$$

To prove the existence of minimisers, we aim at mimicking the strategy employed in the proof of Lemma 2.12. The first step can be repeated exactly in the same fashion. As for the second step, we would like to invoke some compactness principles. The difference is that in infinite dimensions, bounded and closed sets are not necessarily compact with respect to the strong topology. However, assuming $\mathcal{U}$ to be a reflexive Banach space we can restrict ourselves to work with the weak topology as it holds that from every bounded sequence we can extract a weakly convergent subsequence. It is then enough to ensure that every minimising sequence is bounded which is straightforwardly true if $\mathcal{U}_{\text{ad}}$ is bounded or if it satisfies some suitable weak coercivity property. Notice that we will also employ that a closed and convex set of a reflexive Banach space is weakly sequentially closed. Thus, we can repeat the proof of Lemma (2.12) to obtain the following extension:

Theorem 2.16. *Let $\mathcal{U}$ be a reflexive Banach space, $\mathcal{U}_{\text{ad}} \subset \mathcal{U}$ be closed and convex and let $\hat{\mathcal{J}} : \mathcal{U} \to \mathbb{R}$. Furthermore, we assume that*

- *Either $\mathcal{U}_{\text{ad}}$ is bounded or, if it is unbounded, it holds that*

$$\hat{\mathcal{J}}(v) \to +\infty \quad \text{as} \quad \|v\|_{\mathcal{U}} \to +\infty.$$

- *$\hat{\mathcal{J}}$ is weakly sequentially lower semicontinuous, i.e.,*

$$v_k \rightharpoonup v \quad \Rightarrow \quad \hat{\mathcal{J}}(v) \leq \liminf_{k\to\infty} \hat{\mathcal{J}}(v_k).$$

Then, there exists $\overline{u} \in \mathcal{U}_{\text{ad}}$ such that

$$\hat{\mathcal{J}}(\overline{u}) = \min_{v \in \mathcal{U}_{\text{ad}}} \hat{\mathcal{J}}(v).$$

Moreover, if $\hat{\mathcal{J}}$ is strictly convex, the minimiser $\overline{u}$ is unique.

Besides, we can generalise Lemma 2.14 and obtain the following variational inequality determining the first-order necessary conditions for optimality.

Theorem 2.17. *Let $\mathcal{U}$ be a reflexive Banach space, $\mathcal{U}_{\text{ad}} \subset \mathcal{U}$ be closed and convex and let $\hat{\mathcal{J}} : \mathcal{U} \to \mathbb{R}$ be Gâteaux differentiable. Then, every minimiser $\overline{u}$ necessarily fulfils*

$$D_G\hat{\mathcal{J}}(\overline{u})(u - \overline{u}) \geq 0 \quad \forall u \in \mathcal{U}_{\text{ad}}, \tag{2.22}$$

where $D_G\hat{\mathcal{J}}(\overline{u})$ stands for the Gâteaux derivative of $\hat{\mathcal{J}}$ at $\overline{u}$. In addition, if $\hat{\mathcal{J}}$ is convex, the minimiser $\overline{u}$ is unique and the above variational inequality yields also a sufficient condition for minimality.

2.5.3 The Abstract Control Problem

Optimal control theory has undergone a fast development in the past few decades becoming an independent and fundamental branch of applied mathematics on its own.

Roughly speaking, an optimal control problem aims at finding the smarter choice to address the solution of a problem by minimising suitable quantities which may represent, in a general sense, some costs. Here, we follow the classical notation u for the control variable which goes back to the first Russian's contributions in which u was employed in analogy with the Russian word "*upravlenie*" for control.

More precisely, if we should outline the mathematical structure of an optimal control problem, the following ingredients will be in order:

- A differential equation $F(y,u)=0$, called *state equation*, describing the governing physical phenomenon we have to deal with.

- Two variables: the control variable u, which belongs to a set of admissible controls $\mathcal{U}_{\text{ad}}$ (which is usually a closed and convex subset of a Banach space), and the associated state variable $y=y(u)$ which depends on u and solves the state equation $F(y(u),u)=0$.

- A cost functional $\mathcal{J}(y,u)$ to be minimised which depends both on the control variable u and on the solution y to the state equation (in which u appears).

- Possible further constraints on the state variable y or on the control variable u.

Thus, a generic optimal control problem consists of looking for the best admissible strategy (i.e., the choice of the control $u \in \mathcal{U}_{\text{ad}}$ with the associated state $y(u)$) which leads the cost functional $\mathcal{J}$ to be minimised. Namely, it can be formulated as the constrained minimisation problem

$$\min \mathcal{J}(y,u), \quad \text{subject} \quad F(y,u)=0, \quad u \in \mathcal{U}_{\text{ad}}. \tag{2.23}$$

Classical goals from the theoretical viewpoint are establishing the existence of a minimiser and providing first-order necessary conditions for optimality. As these latter are concerned, they usually read as a variational inequality that every optimal control has to verify.

Once a well-posedness result for the state equation $F(y,u)=0$ has been proved, it is possible to define the solution mapping, also known as *control-to-state operator*, which assigns to every admissible control $u \in \mathcal{U}_{\text{ad}}$ the unique corresponding solution to $F(y(u),u)=0$. Namely, we set

$$\mathcal{S}:\mathcal{U}_{\text{ad}} \to \mathcal{Y}, \quad \mathcal{S}: u \mapsto y(u),$$

where the space $\mathcal{Y}$ stands for the functional space in which the solution y belongs. Moreover, the operator $\mathcal{S}$ allows us define the so-called *reduced cost* functional

$$\mathcal{J}_{\text{red}}(u) := \mathcal{J}(\mathcal{S}(u),u),$$

which reduce (2.23) to

$$\min \mathcal{J}_{\text{red}}(u), \quad \text{subject} \quad u \in \mathcal{U}_{\text{ad}},$$

where the constraint "y yields a solution to the state equation" is implicitely captured by the definition of $\mathcal{J}_{\mathrm{red}}$. Notice that this last form of the optimal control problem (2.23) turns out to be a classical minimisation problem in Hilbert spaces as analysed in the previous section (with the choice $\hat{\mathcal{J}} = \mathcal{J}_{\mathrm{red}}$).

Let us refer the reader to the well-written book [146] for more details on basic notions of optimal control theory for linear and semilinear parabolic differential equations (see also, e.g., [105, 119, 128]).

Part I

Analytic Results

CHAPTER 3

Mathematical Analysis of a Family of Local Tumour Growth Models

This chapter is devoted to the investigation of the well-posedness of the local model (1.1)–(1.3), depending on α and β, with the following specifications:

- The thickness parameter ε and the surface tension parameter γ are set to one.
- We postulate constant mobilities m and n: without loss of generality we let $m = n = 1$.
- We neglect the chemotaxis contribution (as well as the active transport effect), i.e., $\chi = 0$.
- For the source/sink terms $\mathcal{S}^1$ and $\mathcal{S}^2$ we consider linear phenomenological laws for chemical reactions as introduced in (1.22).

Summing up, the family of models under consideration in this chapter reads as

$$\alpha\partial_t\mu + \partial_t\varphi - \Delta\mu = P(\varphi)(\sigma - \mu) \qquad \text{in } Q, \tag{3.1}$$

$$\mu = \beta\partial_t\varphi - \Delta\varphi + F'(\varphi) \qquad \text{in } Q, \tag{3.2}$$

$$\partial_t\sigma - \Delta\sigma = -P(\varphi)(\sigma - \mu) + g \qquad \text{in } Q, \tag{3.3}$$

with positive coefficients α and β. It is worth noticing that, compared to the abstract system (1.1)–(1.3), we add a generic source function g in equation (3.3) which will play the role of control variable later on in Chapter 5. Moreover, we complement the above system with classical homogeneous Neumann boundary conditions for all the involved variables and we prescribe some initial conditions. Namely, we endow (3.1)–(3.3) with

$$\partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}\sigma = 0 \qquad \text{on } \Sigma, \tag{3.4}$$

$$\varphi(0) = \varphi_0, \quad \mu(0) = \mu_0, \quad \sigma(0) = \sigma_0 \qquad \text{in } \Omega. \tag{3.5}$$

To begin with, we establish the weak and strong well-posedness of (3.1)–(3.5) under general assumptions for the potentials F assuming α and β to be fixed positive constants; secondly, we discuss the asymptotic behaviour of the system as these relaxation parameters approach zero. This family of systems have already been investigated as far as weak well-posedness is concerned in [36,38,39] for the case $\alpha, \beta > 0$ and in [78] for the case $\alpha = \beta = 0$, respectively. Moreover, P. Colli et al. established in [38, 39] how these relaxation parameters affect the behaviour of the solutions to system (3.1)–(3.5) specifying in which sense the solutions to the relaxed system (3.1)–(3.5) converge to the solutions to the limit system, obtained from (3.1)–(3.5) by formally setting α and/or β equal to zero, as the parameters α and β approach zero both separately and jointly. In this direction, let us anticipate that the results presented in Section 3.1 and Section 3.3 are not original but, however, we have decided to recollect there the statements of the obtained results for reader's convenience since they will play a fundamental role in the asymptotic analysis for the corresponding optimal control problems performed in Chapter 5.

Let us also mentioned the works by P. Colli et al. [42, 45], where the model (3.1)–(3.5) has been extended to the fractional case with $\chi = 0$ (and $g = 0$). Namely, instead of the Laplace operators, present in all the equations in (3.1)–(3.5), they consider some fractional powers of selfadjoint, monotone, unbounded, linear operators having compact resolvents. Moreover, they discuss the asymptotic behaviour of the system as the coefficients α and β tend to zero.

Concerning the long-time behaviour analysis for similar models, we point out the paper [36], where the omega-limit set is shown to be non-empty in the case $\alpha = \beta > 0, \chi = 0$ (and $g = 0$). We also mention the contribution [30] by C. Cavaterra et al., where the convergence of any global solution to a single equilibrium as time goes to infinity in the case $\alpha, \beta > 0, \chi = 0$ is proved by prescribing suitable decaying property for the source term g. Lastly, we refer to the recent work by H. Garcke et al. [97] which establishes the existence of the associated global attractor in a suitable phase-space defined via mass conservation, including the highly non-trivial chemotaxis effect in the case $\alpha = \beta = 0, \chi \geq 0$ (and $g = 0$). In this direction, see also [126], where A. Miranville et al. investigated the existence of the global attractor for a similar local model as (3.1)–(3.5) with $\alpha = \beta = \chi = 0$ (without any source term g) in which the constitutive assumption (1.24) has been employed instead of (1.22).

3.1 Weak Well-posedness

The weak well-posedness of the system (3.1)–(3.5) can be established under the following structural assumptions:

A1 $P : \mathbb{R} \to \mathbb{R}$ is non-negative, bounded and Lipschitz continuous.

A2 We assume

$$F := F_1 + F_2 \geq 0, \tag{3.6}$$

where

$$F_1 : \mathbb{R} \to [0, +\infty] \quad \text{is proper, convex, and lower semicontinuous,}$$

and

$$F_2 \in C^1(\mathbb{R}), \quad F_2' : \mathbb{R} \to \mathbb{R} \ \text{ is Lipschitz continuous,} \quad F_2'(0) = 0$$

with Lipschitz constant denoted by L. Let us notice that the subdifferential $\partial F_1 : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone operator (see, e.g., [19, Ex. 2.3.4, p. 25]) with domain denoted by $D(\partial F_1)$ and we postulate that $0 \in \partial F_1(0)$. For convenience, with a slight abuse of notation we set $F'(\cdot) := \partial F_1(\cdot) + F_2'(\cdot)$ and employ the same symbol to denote the operator induced on $L^2(Q)$ (cf. Section 2.4.3).

Remark 3.1. *It is worth noticing that all the classical choices for the potentials given by* (1.7)*,* (1.8) *and* (1.9) *comply with the requirement* **A2**. *Indeed, they enjoy the splitting required by* (3.6) *with the following prescriptions:*

$$\begin{aligned}
&F_{\mathrm{reg},1}(s) = \frac{1}{4}s^4, \quad F_{\mathrm{reg},2}(s) = \frac{1}{4}(1-2s^2),\\
&F_{\log,1}(s) = \begin{cases} \frac{\vartheta}{2}[(1+s)\ln(1+s) + (1-s)\ln(1-s)], & \text{if } s \in (-1,1),\\ 2\vartheta \log 2, & \text{if } s \in \{-1,1\},\\ +\infty, & \text{otherwise,} \end{cases}\\
&F_{\log,2}(s) = -\frac{\vartheta_0}{2}s^2,\\
&F_{\mathrm{dob},1}(s) = I_{[-1,1]}(s), \quad F_{\mathrm{dob},2}(s) = c(1-s^2),
\end{aligned}$$

where, for every subset $E \subset \mathbb{R}$, $I_E(\cdot)$ stands for the indicator function of E defined as

$$I_E(s) := \begin{cases} 0 & \text{if } s \in E,\\ +\infty & \text{otherwise.} \end{cases}$$

Furthermore, we stress that the subdifferential of the convex part of the above potentials may generate a multi-valued function (in the case (1.9)*) so that, the term $F'(\varphi)$ in* (3.2) *(cf.* (4.2)*) has to be treated using a selection $\xi \in \partial F_1(\varphi)$ or as a differential inclusion.*

We now present the weak well-posedness result for the system (3.1)–(3.5) with $\alpha, \beta > 0$ which was first established in [36] and then slightly improved in [38].

Theorem 3.2 (Existence of weak solutions: $\alpha, \beta > 0$, [36, Theorem 2.2], [38, Theorem 2.2])**.** *Assume* **A1–A2** *and let $\alpha, \beta \in (0,1)$. Let the initial data $(\varphi_0, \mu_0, \sigma_0)$ satisfy*

$$\varphi_0 \in H^1(\Omega), \quad \mu_0, \sigma_0 \in L^2(\Omega), \quad F(\varphi_0) \in L^1(\Omega), \tag{3.7}$$

and let the source term $g \in L^2(0,T;H)$. Then, the system (3.1)–(3.5) *admits a unique weak solution $(\varphi, \mu, \sigma, \xi)$ in the sense that*

$$\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{3.8}$$

$$\mu, \sigma \in H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V), \tag{3.9}$$

$$\xi \in L^2(0,T;H), \tag{3.10}$$

and $(\varphi, \mu, \sigma, \xi)$ verifies

$$\begin{aligned}
&\langle \partial_t(\alpha\mu + \varphi), v\rangle + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega P(\varphi)(\sigma - \mu)v, \\
&\mu = \beta\partial_t\varphi - \Delta\varphi + \xi + F_2'(\varphi), \qquad \xi \in \partial F_1(\varphi) \quad \textit{a.e. in } Q, \\
&\langle \partial_t\sigma, v\rangle + \int_\Omega \nabla\sigma \cdot \nabla v = -\int_\Omega P(\varphi)(\sigma - \mu)v + \int_\Omega gv,
\end{aligned}$$

for every $v \in V$ and almost everywhere in $(0,T)$, and the initial conditions

$$\varphi(0) = \varphi_0, \quad \mu(0) = \mu_0, \quad \sigma(0) = \sigma_0 \quad \textit{in } \Omega.$$

Proof of Theorem 3.2. For the proof we refer the reader to [36, 38]. □

Remark 3.3. *In the case $F = F_{\text{reg}}$ the last condition of* (3.7) *is a straightforward consequence of Sobolev's embedding theorem and the growth rate of the potential. In fact, we have that $F_{\text{reg}}(s) = \mathcal{O}(s^4)$ as well as $\varphi_0 \in V \subset L^4(\Omega)$ producing the claim.*

Theorem 3.4 (Continuous dependence: $\alpha, \beta > 0$)**.** *Assume* **A1–A2** *and let $\alpha, \beta \in (0,1)$. Then there exists a constant $K_1 > 0$ such that, for any triplet of initial data $\{(\varphi_0^i, \mu_0^i, \sigma_0^i)\}_i$, $i = 1, 2$, satisfying* (3.7) *and for any respective weak solutions $\{(\varphi_i, \mu_i, \sigma_i, \xi_i)\}_i$, $i = 1, 2$, obtained from Theorem 3.2, and source terms $g_i \in L^2(0,T;H)$ it holds that*

$$\begin{aligned}
&\|(\alpha\mu_1 + \varphi_1 + \sigma_1) - (\alpha\mu_2 + \varphi_2 + \sigma_2)\|_{L^\infty(0,T;V^*)} + \|\mu_1 - \mu_2\|_{L^2(0,T;H)} \\
&\quad + \|\varphi_1 - \varphi_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\sigma_1 - \sigma_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\
&\le K_1\Big(\|\alpha(\mu_0^1 - \mu_0^2) + (\varphi_0^1 - \varphi_0^2) + (\sigma_0^1 - \sigma_0^2)\|_{V^*}\Big) \\
&\quad + K_1\Big(\|g_1 - g_2\|_{L^2(0,T;H)} + \|\varphi_0^1 - \varphi_0^2\| + \|\sigma_0^1 - \sigma_0^2\|\Big).
\end{aligned} \tag{3.11}$$

Proof of Theorem 3.4. First of all, let us set for convenience the following notation

$$
\begin{aligned}
&\varphi := \varphi_1 - \varphi_2, \quad \mu := \mu_1 - \mu_2, \quad \sigma := \sigma_1 - \sigma_2, \quad \xi := \xi_1 - \xi_2,\\
&R_i := P(\varphi_i)(\sigma_i - \mu_i) \quad i \in \{1,2\}, \quad g := g_1 - g_2,\\
&\varphi_0 := \varphi_0^1 - \varphi_0^2, \quad \mu_0 := \mu_0^1 - \mu_0^2, \quad \sigma_0 := \sigma_0^1 - \sigma_0^2.
\end{aligned} \tag{3.12}
$$

We then write the system (3.1)–(3.5) for $\{(\varphi_i, \mu_i, \sigma_i, \xi_i)\}_i$, $i = 1, 2$, and take the difference of the equations to obtain that

$$\alpha\partial_t\mu + \partial_t\varphi - \Delta\mu = R_1 - R_2 \qquad \text{in } Q, \tag{3.13}$$

$$\mu = \beta\partial_t\varphi - \Delta\varphi + \xi + (F_2'(\varphi_1) - F_2'(\varphi_2)) \qquad \text{in } Q, \tag{3.14}$$

$$\partial_t\sigma - \Delta\sigma = -(R_1 - R_2) + g \qquad \text{in } Q, \tag{3.15}$$

$$\partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}\sigma = 0 \qquad \text{on } \Sigma, \tag{3.16}$$

$$\varphi(0) = \varphi_0, \quad \mu(0) = \mu_0, \quad \sigma(0) = \sigma_0 \qquad \text{in } \Omega. \tag{3.17}$$

Now, we add (3.13) with (3.15) and add to both sides of the new equation the term $\mu + \sigma$ to obtain that

$$\partial_t(\alpha\mu + \varphi + \sigma) + \mathcal{R}(\mu + \sigma) = g + \mu + \sigma \quad \text{in } Q, \tag{3.18}$$

where $\mathcal{R}$ is the Riesz operator associated to V defined in Subsection 2.4.1. Moreover, recalling the properties pointed out by (2.11)–(2.13), we multiply (3.18) by $\mathcal{R}^{-1}(\alpha\mu + \varphi + \sigma)$, (3.14) by $-\varphi$, and (3.15) by σ, add the resulting equalities, and integrate over Q_t to infer that

$$
\begin{aligned}
&\frac{1}{2}\|(\alpha\mu + \varphi + \sigma)(t)\|_{V^*}^2 + \int_{Q_t} (\mu + \sigma)(\alpha\mu + \varphi + \sigma) - \int_{Q_t} \varphi\mu\\
&\quad + \frac{\beta}{2}\|\varphi(t)\|^2 + \int_{Q_t} |\nabla\varphi|^2 + \int_{Q_t} \xi\varphi + \frac{1}{2}\|\sigma(t)\|^2 + \int_{Q_t} |\nabla\sigma|^2\\
&= \underbrace{\frac{1}{2}\|\alpha\mu_0 + \varphi_0 + \sigma_0\|_{V^*}^2 + \frac{\beta}{2}\|\varphi_0\|^2 + \frac{1}{2}\|\sigma_0\|^2}_{=:\mathbb{I}_1 = \mathbb{I}_1^1 + \mathbb{I}_1^2 + \mathbb{I}_1^3}\\
&\quad \underbrace{+ \int_0^t \langle g + \mu + \sigma, \mathcal{R}^{-1}(\alpha\mu + \varphi + \sigma)\rangle}_{=:\mathbb{I}_2}\\
&\quad \underbrace{- \int_{Q_t} (F_2'(\varphi_1) - F_2'(\varphi_2))\varphi - \int_{Q_t} \big(P(\varphi_1) - P(\varphi_2)\big)(\sigma_1 - \mu_1)\sigma}_{=:\mathbb{I}_3 = \mathbb{I}_3^1 + \mathbb{I}_3^2}\\
&\quad \underbrace{- \int_{Q_t} P(\varphi_2)(\sigma - \mu)\sigma + \int_{Q_t} g\sigma}_{=:\mathbb{I}_4 = \mathbb{I}_4^1 + \mathbb{I}_4^2}\,.
\end{aligned} \tag{3.19}
$$

From now onward, whenever an estimate will appear, let us convey to denote by $\mathbb{I}_i$ the i-th line on the right-hand side, whereas we specify with $\mathbb{I}_i^j$ the j-th term on the

i-th line. Notice that the sixth term on the left-hand side is non-negative due to the monotonicity of ∂F_1 and that the second term on the left-hand side can be developed as

$$\alpha \int_{Q_t} |\mu|^2 + \int_{Q_t} |\sigma|^2 + \int_{Q_t} \mu\varphi + (1+\alpha) \int_{Q_t} \mu\sigma + \int_{Q_t} \sigma\varphi,$$

so that we keep the first two terms on the left-hand side of (3.19), whereas the other terms are moved to the right-hand side and include them in $\mathbb{I}_4$. Using the assumptions on the initial data (3.7) we readily have that $\mathbb{I}_1$ is bounded above by a constant and from Young's inequality that

$$\begin{aligned}
|\mathbb{I}_2| &\leq \int_0^t \|g+\mu+\sigma\|_{V^*} \|\alpha\mu+\varphi+\sigma\|_{V^*} \\
&\leq \delta \int_0^t \|g+\mu+\sigma\|_{V^*}^2 + C(\delta) \int_0^t \|\alpha\mu+\varphi+\sigma\|_{V^*}^2,
\end{aligned}$$

for a positive δ yet to be chosen, and due to the continuous embedding $V^* \subset H$ and Young's inequality we also have that

$$\delta \int_0^t \|g+\mu+\sigma\|_{V^*}^2 \leq C\delta \int_0^t \|g+\mu \ +\sigma\|^2 \leq \frac{\alpha}{4} \int_{Q_t} |\mu|^2 + C \int_{Q_t} (|g|^2 + |\sigma|^2)$$

provided we choose δ sufficiently small. Moreover, combining the Hölder inequality with the Sobolev continuous embedding $V \subset L^4(\Omega)$, and using the Lipschitz continuity of P and F_2, we realise that

$$\begin{aligned}
|\mathbb{I}_3| &\leq C \int_0^t \|\varphi\|(\|\sigma_1\|_4 + \|\mu_1\|_4)\|\sigma\|_4 + C \int_{Q_t} |\varphi|^2 \\
&\leq C \int_0^t \|\varphi\|(\|\sigma_1\|_V + \|\mu_1\|_V)\|\sigma\|_V + C \int_{Q_t} |\varphi|^2 \\
&\leq \frac{1}{2} \int_{Q_t} \big(|\sigma|^2 + |\nabla\sigma|^2\big) + C \int_0^t (\|\sigma_1\|_V^2 + \|\mu_1\|_V^2)\|\varphi\|^2 + C \int_{Q_t} |\varphi|^2 \\
&\leq \frac{1}{2} \int_{Q_t} \big(|\sigma|^2 + |\nabla\sigma|^2\big) + C \int_{Q_t} |\varphi|^2,
\end{aligned}$$

where we also use the regularity of the solutions σ_1 and μ_1 expressed by (3.9). Furthermore, $\mathbb{I}_4$ can be easily bounded owing to Young's inequality and the boundedness of P as

$$|\mathbb{I}_4| \leq \frac{\alpha}{2} \int_{Q_t} |\mu|^2 + C \int_{Q_t} (|\sigma|^2 + |g|^2 + |\varphi|^2).$$

Hence, upon collecting the above estimates, we infer that, for a sufficiently small δ and for every $t \in [0, T]$, it holds that

$$\frac{1}{2}\|(\alpha\mu + \varphi + \sigma)(t)\|_{V^*}^2 + \frac{\alpha}{4}\int_{Q_t} |\mu|^2 + \frac{\beta}{2}\|\varphi(t)\|^2 + \int_{Q_t} |\nabla\varphi|^2$$
$$+ \int_{Q_t} |\sigma|^2 + \frac{1}{2}\int_{Q_t} |\nabla\sigma|^2 + \frac{1}{2}\|\sigma(t)\|^2$$
$$\leq \frac{1}{2}\|\alpha\mu_0 + \varphi_0 + \sigma_0\|_{V^*}^2 + \frac{\beta}{2}\|\varphi_0\|^2 + \frac{1}{2}\|\sigma_0\|^2$$
$$+ C\int_0^t \|(\alpha\mu + \varphi + \sigma)(s)\|_{V^*}^2 \mathrm{ds} + C\int_{Q_t} (|\sigma|^2 + |g|^2 + |\varphi|^2)$$

so that Gronwall's lemma produces the claim. □

3.2 Strong Well-posedness

This section is entirely devoted to discussing the strong well-posedness of the system (3.1)–(3.5) with $\alpha, \beta > 0$, where with *strong solutions* we mean that the equations of the system (3.1)–(3.3) are fulfilled pointwise in Q. Moreover, under natural additional assumptions on the phase variable at the initial time we are able to establish the so-called *separation property* which is of utmost importance to handle singular potentials like (1.8) and (1.9). For instance, for singular potentials, this property guarantees us that, as soon as the phase variable remains strictly detached from the boundary of the domain of the potential at the initial time, i.e., φ_0 it is not a pure phase, then the order parameter φ stays away from the pure states and assumes values only in a compact subset of the domain of the potential for the whole evolution. As a consequence, we derive that the solution is a physical one, meaning that φ just assumes physically admissible values between -1 and 1. Besides, the singular potentials may be regarded as Lipschitz nonlinearities opening the possibility for further mathematical investigations.

For the strong well-posedness result, in addition to **A1–A2**, we require that:

A3 $P \in C^2(\mathbb{R})$.

A4 Setting $(-\ell, \ell) := \mathrm{Int}\, D(\partial F_1)$, with $\ell \in [0, +\infty]$, we prescribe that

$$F \in C^3(-\ell, \ell), \qquad \lim_{s \to \pm(\ell)^{\mp}} F'(s) = \pm\infty.$$

It is worth underlying that due to the regularity of the potential required above we no longer need the selection ξ in equation (3.2) as the derivative of F can be now defined in the classical manner.

Theorem 3.5 (Existence of strong solutions and separation principle: $\alpha, \beta > 0$)**.** *Assume that* **A1–A4** *are fulfilled. Moreover, let* $\alpha, \beta \in (0, 1)$*,* $g \in L^2(0, T; H)$*, and let the initial data satisfy*

$$\varphi_0 \in W, \quad \mu_0 \in H^1(\Omega) \cap L^\infty(\Omega), \quad \sigma_0 \in H^1(\Omega), \tag{3.20}$$

as well as that

$$\big(\mu_0 + \Delta\varphi_0 - F'(\varphi_0)\big) \in L^2(\Omega). \tag{3.21}$$

Then, the system (3.1)–(3.5) *admits a unique strong solution* (φ, μ, σ) *which satisfies*

$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{3.22}$$
$$\mu, \sigma \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{3.23}$$
$$\mu \in L^\infty(Q), \tag{3.24}$$

and that

$$\varphi \in C^0([0,T]; C^0(\overline{\Omega})), \quad \mu, \sigma \in C^0([0,T]; V). \tag{3.25}$$

Moreover, there exists a constant $K_2 > 0$*, which depends only on structural data,* α *and* β *such that*

$$\begin{aligned} &\|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W)} + \|\mu\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)\cap L^\infty(Q)} \\ &\quad + \|\sigma\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le K_2. \end{aligned} \tag{3.26}$$

In addition, let us assume that

$$\exists\, s_0 \in (0,\ell) : \|\varphi_0\|_{L^\infty(\Omega)} \le s_0. \tag{3.27}$$

Then, there exist a constant $s^* \in (s_0, \ell)$ *such that*

$$\sup_{t\in[0,T]} \|\varphi(t)\|_{L^\infty(\Omega)} \le s^*, \tag{3.28}$$

which entails the existence of a constant $K_3 > 0$ *such that*

$$\|\varphi\|_{C^0(\overline{Q})} + \max_{0\le i\le 3} \|F^{(i)}(\varphi)\|_{L^\infty(Q)} + \max_{0\le j\le 2} \|P^{(j)}(\varphi)\|_{L^\infty(Q)} \le K_3. \tag{3.29}$$

Remark 3.6. *It is worth pointing out that* (3.21) *is automatically fulfilled in the case of polynomial growth type potentials. In fact, let for instance* F *be a* $q-$*polynomial growth type potential for some* $q > 1$*, i.e.,* $F(s) = \mathcal{O}(s^q)$*. Then, the continuous embedding* $W \subset L^p(\Omega)$*, which holds for every* p*, directly produces the claim.*

Proof of Theorem 3.5. The proof can be rigorously carried out within the framework of a Faedo–Galerkin scheme by constructing approximate solutions and then passing to the limit as the discretisation coefficient tends to infinity. For the sake of brevity, we limitate ourselves to present formal a priori estimates for the solutions which can be however reproduced rigorously employing a Galerkin scheme.

A Priori Estimates

First estimate: To begin with, we add to both sides of (3.2) the term φ. Next, we multiply (3.1) by μ, the new (3.2) by $-\partial_t\varphi$, and (3.3) by σ, add the resulting equations and integrate over Q_t and by parts to obtain

$$\begin{aligned}
&\frac{\alpha}{2}\|\mu(t)\|^2 + \int_{Q_t} |\nabla\mu|^2 + \beta\int_{Q_t} |\partial_t\varphi|^2 + \frac{1}{2}\|\varphi(t)\|_V^2 \\
&\quad + \int_\Omega F_1(\varphi(t)) + \frac{1}{2}\|\sigma(t)\|^2 + \int_{Q_t} |\nabla\sigma|^2 + \int_{Q_t} P(\varphi)(\sigma-\mu)^2 \\
&= \underbrace{\frac{\alpha}{2}\|\mu_0\|^2 + \frac{1}{2}\|\varphi_0\|_V^2 + \int_\Omega F_1(\varphi_0) + \frac{1}{2}\|\sigma_0\|^2}_{=:\mathbb{I}_1=\mathbb{I}_1^1+\mathbb{I}_1^2+\mathbb{I}_1^3+\mathbb{I}_1^4} \\
&\quad \underbrace{+ \int_{Q_t} g\sigma + \int_{Q_t} \varphi\,\partial_t\varphi - \int_{Q_t} F_2'(\varphi)\partial_t\varphi}_{=:\mathbb{I}_2=\mathbb{I}_2^1+\mathbb{I}_2^2+\mathbb{I}_2^3} = \mathbb{I}_1 + \mathbb{I}_2.
\end{aligned}$$

The boundedness of $\mathbb{I}_1$ readily follows from the assumptions on initial data (3.20), while $\mathbb{I}_2$ can be handled using Young's inequality to infer that, for every $\delta > 0$,

$$|\mathbb{I}_2| \le \frac{1}{2}\int_{Q_t} |g|^2 + \frac{1}{2}\int_{Q_t} |\sigma|^2 + 2\delta\int_{Q_t} |\partial_t\varphi|^2 + C(\delta)\int_{Q_t} (|\varphi|^2+1).$$

Hence, we choose $\delta < \frac{\beta}{2}$ so that Gronwall's lemma yields that

$$\begin{aligned}
&\|\varphi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\mu\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|F_1(\varphi)\|_{L^\infty(0,T;L^1(\Omega))}^{1/2} \\
&\quad + \|\sigma\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le C. \qquad (3.30)
\end{aligned}$$

Second estimate: We multiply (3.1) by $\partial_t\mu$ and (3.3) by $\partial_t\sigma$, add the resulting equations, integrate over Q_t and use (3.30) and the boundedness of the proliferation function P to deduce that

$$\|\mu\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\sigma\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le C. \qquad (3.31)$$

Third estimate: Equations (3.1) and (3.3) show an elliptic structure with respect to μ and σ, respectively. Furthermore, we are assuming the initial data $\mu_0, \sigma_0 \in V$ and both the forcing terms are bounded in $L^2(0,T;H)$ due to the above estimates. Therefore, straightforward computations along with the elliptic regularity theory produce that

$$\|\mu\|_{L^2(0,T;W)} + \|\sigma\|_{L^2(0,T;W)} \le C.$$

Fourth estimate: We rewrite equation (3.2) as the nonlinear elliptic equation

$$-\Delta\varphi + F_1'(\varphi) = f_\varphi := \mu - \beta\partial_t\varphi - F_2'(\varphi), \qquad (3.32)$$

where the forcing term f_φ is bounded in $L^2(0,T;H)$ due to the above estimates. Next, we multiply the above identity by $-\Delta\varphi$ and integrate over Ω. Again, this testing procedure can be rigorously justified within a Galerkin scheme. Then, using Young's inequality we have, for a.a. $t \in (0, T)$,

$$\|\Delta\varphi(t)\|^2 + \int_\Omega F_1''(\varphi(t))|\nabla\varphi(t)|^2 \le \int_\Omega |f(t)||\Delta\varphi(t)| \le \frac{1}{2}\|\Delta\varphi(t)\|^2 + \frac{1}{2}\|f(t)\|^2.$$

The last term on the right-hand side is bounded and the terms on the left-hand side are non-negative since F_1'' is so. Hence, we realise that

$$\|\Delta\varphi\|_{L^2(0,T;H)} \le C$$

so that elliptic regularity theory and then comparison in (3.32) leads us to obtain that

$$\|\varphi\|_{L^2(0,T;W)} + \|F_1'(\varphi)\|_{L^2(0,T;H)} \le C. \tag{3.33}$$

Fifth estimate: The following estimate can be carried out rigorously, e.g., within a time-discretisation scheme which we avoid by virtue of simplicity. Thus, we formally differentiate equation (3.2) with respect to time, multiply the resulting equation by $\partial_t\varphi$, and integrate over Q_t to get

$$\int_{Q_t} \partial_t\mu\,\partial_t\varphi = \beta\int_{Q_t} \partial_{tt}\varphi\,\partial_t\varphi - \int_{Q_t} (\Delta\partial_t\varphi)\,\partial_t\varphi + \int_{Q_t} (F_1''(\varphi) + F_2''(\varphi))\,|\partial_t\varphi|^2.$$

Using integration by parts, the boundary conditions (3.4) and the first of (3.20), we deduce that

$$\frac{\beta}{2}\|\partial_t\varphi(t)\|^2 + \int_{Q_t} |\nabla\partial_t\varphi|^2 + \int_{Q_t} F_1''(\varphi)|\partial_t\varphi|^2 = \frac{\beta}{2}\|\partial_t\varphi(0)\|^2$$
$$- \int_{Q_t} F_2''(\varphi)|\partial_t\varphi|^2 + \int_{Q_t} \partial_t\mu\,\partial_t\varphi,$$

where the terms on the left-hand side are non-negative. The first term of the right-hand side is bounded due to assumption (3.21), whereas the last two integrals can be estimate as follows by using the Young's inequality and the Lipschitz continuity of F_2''

$$- \int_{Q_t} F_2''(\varphi)|\partial_t\varphi|^2 + \int_{Q_t} \partial_t\mu\,\partial_t\varphi \le C\int_{Q_t} |\partial_t\varphi|^2 + \frac{1}{2}\int_{Q_t} |\partial_t\mu|^2.$$

Thus, recalling (3.30) and (3.31) we obtain that

$$\|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \le C.$$

Sixth estimate: We take again into account the form of equation (3.32). Due to the above estimate, we realise that the source term f_φ is now bounded in $L^\infty(0,T;H)$. Hence, by repeating the previous argument we end up with a $L^\infty(0,T;H)$-bound of $\Delta\varphi$ so that elliptic regularity and comparison in (3.32) leads us to improve (3.33) and deduce that

$$\|\varphi\|_{L^\infty(0,T;W)} + \|F_1'(\varphi)\|_{L^\infty(0,T;H)} \le C, \tag{3.34}$$

which, by the Sobolev embedding $W \subset L^\infty(\Omega)$, also entails

$$\|\varphi\|_{L^\infty(Q)} \leq C. \tag{3.35}$$

Moreover, let us notice that (3.25) are now immediate consequences of classical compact embedding results.

Seventh estimate: Before moving to proving the separation principle (3.28), we obtain a propedeutic uniform bound for the chemical potential μ in $L^\infty(Q)$. This can be achieved by applying [114, Thm. 7.1, p. 181] to the first equation (3.1). In fact, (3.1) can be rewritten as a parabolic equation as

$$\begin{cases} \alpha\partial_t\mu - \Delta\mu = f_\mu := P(\varphi)(\sigma - \mu) - \partial_t\varphi & \text{in } Q, \\ \partial_{\mathbf{n}}\mu = 0 & \text{on } \Sigma, \\ \mu(0) = \mu_0 & \text{in } \Omega. \end{cases}$$

Due to (3.20) and to the above estimates we readily deduce that $\mu_0 \in L^\infty(\Omega)$ and $f_\mu \in L^\infty(0,T;H)$ so that a direct application of [114, Thm. 7.1, p. 181] produces

$$\|\mu\|_{L^\infty(Q)} \leq C, \tag{3.36}$$

which concludes the proof of the first part of Theorem 3.5.

The Separation Property

With the above estimates, we are now ready to address the separation property for the phase variable given by condition (3.28). This property will be crucial to obtain (3.29), i.e., some uniform bounds on the phase variable and on the nonlinear functions $F(\varphi)$ and $P(\varphi)$ as well as on their higher-order derivatives. This bound will be fundamental in Chapter 5 to handle the optimal control problem since it will allow us considering singular, while regular, nonlinearities like the logarithmic potential $F_{\log}$. Moreover, notice that (3.29) straightforwardly follows once we show that the phase variable φ enjoys the separation property.

Eight estimate: Let us refer to [41, Proof of Thm. 2.6, pp. 992-994], where similar computations were performed to infer the separation result. To begin with, we rearrange equation (3.2) as the parabolic system

$$\begin{cases} \beta\partial_t\varphi - \Delta\varphi + F_1'(\varphi) = f := \mu - F_2'(\varphi) & \text{in } Q, \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on } \Sigma, \\ \varphi(0) = \varphi_0 & \text{in } \Omega. \end{cases} \tag{3.37}$$

Using the above estimates, we readily infer that $f \in L^\infty(Q)$. Then, we multiply the first equation of (3.37) by

$$|F_1'(\varphi)|^{p-1}\,\operatorname{sign}\varphi = |F_1'(\varphi)|^{p-2}F_1'(\varphi)$$

for a fixed $p > 2$, and integrate over Q_t and by parts to derive that, for every $t \in [0, T]$,

$$\beta \int_\Omega \mathcal{F}_p(\varphi(t)) + (p-1) \int_{Q_t} F_1''(\varphi)|F_1'(\varphi)|^{p-2}|\nabla\varphi|^2 + \int_{Q_t} |F_1'(\varphi)|^p$$
$$= \beta \int_\Omega \mathcal{F}_p(\varphi_0) + \int_{Q_t} f\,|F_1'(\varphi)|^{p-1} \operatorname{sign}\varphi, \tag{3.38}$$

where we put

$$\mathcal{F}_p(s) := \int_0^s |F_1'(s)|^{p-1} \operatorname{sign} s \,\mathrm{ds}.$$

Furthermore, notice that the terms on the left-hand side are non-negative since $p > 2$ and F_1'' is non-negative. As the right-hand side is concerned, the first term can be handled owing to (3.27) which entails that $|F_1'(\varphi_0)|$ is bounded by a positive constant. Namely, (3.27) implies the existence of a constant $M > 0$ such that

$$\beta \int_\Omega \mathcal{F}_p(\varphi_0) \le \beta M^{p-1} \int_\Omega |\varphi_0| \le C^p.$$

As for the last term of (3.38), by using the generalised Young inequality (2.2), we obtain that

$$\int_{Q_t} f\,|F_1'(\varphi)|^{p-1} \operatorname{sign}\varphi \le \frac{1}{p}\,C^p + \frac{1}{p'} \int_{Q_t} |F_1'(\varphi)|^{(p-1)p'} \le C^p + \frac{1}{p'} \int_{Q_t} |F_1'(\varphi)|^p,$$

where p' stands for the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Upon collecting the above estimates, we infer that (3.38) reduces to

$$\frac{1}{p} \int_{Q_t} |F_1'(\varphi)|^p \le C^p,$$

leading to

$$\|F_1'(\varphi)\|_{L^p(Q)} \le C, \tag{3.39}$$

for a positive constant C which is independent of p. Since the above procedure is independent of p, we can iterate the argument and deduce that (3.39) holds for every $p > 2$ so that it is a standard argument to pass to the limit and deduce

$$\|F_1'(\varphi)\|_{L^\infty(Q)} \le C$$

which in turn implies that

$$\|F'(\varphi)\|_{L^\infty(Q)} \le C$$

so that (3.28) follows. Lastly, from (3.28) we easily deduce (3.29) concluding the proof of Theorem 3.5. □

Exploiting the strong well-posedness established in Theorem 3.5, we are able to improve the stability estimate presented in Theorem 3.4. This new result will be crucial to handle the corresponding optimal control problem addressed in Chapter 5.

Theorem 3.7 (Refined continuous dependence: $\alpha, \beta > 0$). *Suppose that* **A1–A4** *hold and let* $\alpha, \beta \in (0,1)$. *Then there exists a constant* $K_4 > 0$ *such that, for any pair of initial data* $\{(\varphi_0^i, \mu_0^i, \sigma_0^i)\}_i$, $i = 1, 2$, *satisfying* (3.20)–(3.21) *and for any respective strong solutions* $\{(\varphi_i, \mu_i, \sigma_i)\}_i$, $i = 1, 2$, *obtained from Theorem 3.5, and source terms* $g_i \in L^2(0, T; H)$ *it holds that*

$$\begin{aligned}
&\|\varphi_1 - \varphi_2\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\mu_1 - \mu_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\
&\quad \le K_4\Big(\|g_1 - g_2\|_{L^2(0,T;H)} + \|\varphi_0^1 - \varphi_0^2\|_V + \|\mu_0^1 - \mu_0^2\| + \|\sigma_0^1 - \sigma_0^2\|\Big). \qquad (3.40)
\end{aligned}$$

Proof of Theorem 3.7. Here, we employ the same notation as in (3.12) and consider again the system of the differences (3.13)–(3.15). We then add to both sides of (3.14) the term $-\varphi$, test (3.13) by μ and this new second equation by $-\partial_t\varphi$. After adding the resulting equalities and integrating over Q_t and by parts we obtain that

$$\begin{aligned}
&\frac{\alpha}{2}\|\mu(t)\|^2 + \int_{Q_t} |\nabla\mu|^2 + \beta\int_{Q_t} |\partial_t\varphi|^2 + \frac{1}{2}\|\varphi(t)\|_V^2 \\
&\quad \le \frac{\alpha}{2}\|\mu_0\|^2 + \frac{1}{2}\|\varphi_0\|_V^2 + \int_{Q_t} (R_1 - R_2)\mu \\
&\qquad - \int_{Q_t} (F'(\varphi_1) - F'(\varphi_2))\partial_t\varphi + \int_{Q_t} \varphi\partial_t\varphi =: \mathbb{I}_1 + \mathbb{I}_2. \qquad (3.41)
\end{aligned}$$

The first two terms on the right-hand side are readily bounded due to the assumption on the initial conditions, whereas the third term on the right-hand side $\mathbb{I}_1^3$ can be bounded using (3.9), Young's and Hölder's inequalities, the boundedness and Lipschitz continuity of P and the Sobolev embedding $V \subset L^4(\Omega)$ to conclude that

$$\begin{aligned}
|\mathbb{I}_1^3| &\le \int_{Q_t} |P(\varphi_2)|(|\sigma| + |\mu|)|\mu| + \int_{Q_t} |P(\varphi_1) - P(\varphi_2)|(|\sigma_1| - |\mu_1|)|\mu| \\
&\le C\int_{Q_t} (|\sigma|^2 + |\mu|^2) + C\int_0^t \|\varphi\|_V(\|\sigma_1\|_4 + \|\mu_1\|_4)\|\mu\| \\
&\le C\int_{Q_t} (|\sigma|^2 + |\mu|^2) + C\int_0^t \|\varphi\|_V^2(\|\sigma_1\|_V^2 + \|\mu_1\|_V^2) \\
&\le C\Big(\|g_1 - g_2\|_{L^2(0,T;H)} + \|\varphi_0^1 - \varphi_0^2\| + \|\sigma_0^1 - \sigma_0^2\|\Big),
\end{aligned}$$

where we also used that σ_1 and μ_1 enjoy (3.9) and Theorem 3.4. Let us notice that we also tacitly employ that

$$\begin{aligned}
&\|P(\varphi_1) - P(\varphi_2)\|_4 \\
&\quad \le C\|P(\varphi_1) - P(\varphi_2)\|_V \\
&\quad \le C(\|P(\varphi_1) - P(\varphi_2)\| + \|P'(\varphi_1)\nabla\varphi_1 - P'(\varphi_2)\nabla\varphi_2\|) \\
&\quad \le C\|\varphi_1 - \varphi_2\| + C\|(P'(\varphi_1) - P'(\varphi_2))\nabla\varphi_1 + P'(\varphi_2)\nabla(\varphi_1 - \varphi_2)\| \\
&\quad \le C\big(\|\varphi_1 - \varphi_2\| + \|\varphi_1 - \varphi_2\| \, \|\nabla\varphi_1\| + \|\nabla(\varphi_1 - \varphi_2)\|\big) \\
&\quad \le C\|\varphi\|_V
\end{aligned}$$

which follows from the Cauchy–Schwarz inequality and the regularity (3.8) enjoyed by φ_1. On the other hand, the last two terms on the right-hand side can be handled by using Young's and Hölder's inequalities to infer that

$$\begin{aligned}
|\mathbb{I}_2| &\le 2\delta \int_{Q_t} |\partial_t\varphi|^2 + C(\delta)\int_{Q_t} |\varphi|^2 + C(\delta)\int_{Q_t} |F'(\varphi_1) - F'(\varphi_2)|^2 \\
&\le 2\delta \int_{Q_t} |\partial_t\varphi|^2 + C(\delta)\int_{Q_t} |\varphi|^2,
\end{aligned}$$

for $\delta > 0$ yet to be fixed and where in the last estimate we invoke the fact that F' turns out to be Lipschitz continuous in its domain due to the separation property (3.28). Hence, we take δ small enough and apply the Gronwall lemma to obtain that

$$\begin{aligned}
&\|\varphi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\mu\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\
&\qquad \le C(\|\varphi_0\|_V + \|\mu_0\| + \|\sigma_0\| + \|g\|_{L^2(0,T;H)})
\end{aligned} \tag{3.42}$$

concluding the proof of Theorem 3.7. □

3.3 Vanishing Viscosities Analysis

This section is completely devoted to addressing the asymptotic behaviour of the system (3.1)–(3.5) as the relaxation parameters α and β approach zero. This investigation has been the main goal of some papers by P. Colli et al. [36, 38, 39]. In those works it was addressed the asymptotic behaviour as $\alpha \to 0$ with $\beta > 0$ and fixed, the asymptotic behaviour with $\alpha > 0$ and fixed as $\beta \to 0$, as well as the joint asymptotic $\alpha \to 0$ and $\beta \to 0$. Moreover, the authors pointed out the specific mathematical framework under which the asymptotic can be performed and also proposed the corresponding error estimates which in turn allow them to derive the uniqueness of the solutions to the limiting systems. Before diving into the details by stating the results there established, let us briefly mention the mathematical challenges that they have to overcome in those passages.

Passage to the limit as $\boldsymbol{\alpha} \to \mathbf{0}$: In the first asymptotic scenario $\alpha \to 0$, the parabolic relaxation term $\alpha\partial_t\mu$ in (3.1) vanishes resulting in a lack of temporal regularity on the chemical potential μ. Hence, as usually happen for the classical Cahn–Hilliard equation, to bound μ in $L^2(0, T; V)$, a key argument is to derive some bounds on the spatial mean of the chemical potential which can be deduced from the second equation (3.2) as soon as we prescribe that the potential is defined on the whole real line and that it possesses some “controlled" growth. In this direction, a natural growth condition on the potential has to be assumed (c.f. (3.46)), allowing for any polynomial or first-order exponential potentials. As far as the error estimate is concerned, which in turn entails the uniqueness of the solution to the limit system, a rate of convergence of order $\alpha^{1/2}$ is obtained under the further restrictions that the proliferation function $P(\cdot)$ is a non-negative constant and that F has a polynomial growth of power four, still covering the case of the classical quartic potential F_{reg} (c.f. (3.53)).

Passage to the limit as $\boldsymbol{\beta \to 0}$: As for the vanishing of the viscosity term $\beta\partial_t\varphi$ in equation (3.2) we obtain, as expectable, a loss of temporal regularity on the phase variable. However, the presence of the relaxation term $\alpha\partial_t\mu$ with $\alpha > 0$ in (3.1) allows passing to the limit in a very general setting including singular potentials. In fact, we only requiring some smallness type assumptions on the relaxation parameter α (c.f. Theorem 3.13). Again, a corresponding error estimate is obtained with a convergence rate of order $\beta^{1/2}$, and therefore the uniqueness for the limit system is guaranteed.

Passage to the limit as $\boldsymbol{\alpha, \beta \to 0}$: The joint passage to the limit as both the parameters α and β go to zero has been investigated by [38] under a combination of the above restrictions.

3.3.1 Asymptotic Analysis as $\boldsymbol{\alpha \to 0}$

As mentioned, the discussion of the asymptotic analysis as $\alpha \to 0$ in (3.1)–(3.5) has been addressed in [39]. For the reader's convenience, we recollect here the main results there obtained which will be useful later on in Chapter 5. To begin with, let us specify the notion of weak solution to the system (3.1)–(3.5) with $\alpha = 0$.

Definition 3.8. *A quadruplet $(\varphi, \mu, \sigma, \xi)$ is said to be a weak solution to system* (3.1)–(3.5) *with $\alpha = 0$ if*

$$\begin{aligned}
&\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),\\
&\mu \in L^2(0,T;V),\\
&\sigma \in H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V),\\
&\xi \in L^2(0,T;H),
\end{aligned}$$

and $(\varphi, \mu, \sigma, \xi)$ verifies

$$\langle \partial_t\varphi, v\rangle + \int_\Omega \nabla\mu\cdot\nabla v = \int_\Omega P(\varphi)(\sigma-\mu)v, \tag{3.43}$$

$$\mu = \beta\partial_t\varphi - \Delta\varphi + \xi + F_2'(\varphi), \qquad \xi \in \partial F_1(\varphi) \quad \textit{a.e. in } Q, \tag{3.44}$$

$$\langle \partial_t\sigma, v\rangle + \int_\Omega \nabla\sigma\cdot\nabla v = -\int_\Omega P(\varphi)(\sigma-\mu)v + \int_\Omega gv, \tag{3.45}$$

for every $v \in V$, almost everywhere in $(0,T)$ and the initial conditions

$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 \quad \textit{in } \Omega.$$

The existence of a solution to (3.1)–(3.5) with $\alpha = 0$ in the sense of the above definition is obtained in [39] under the following restriction for the potentials:

A5 There exists a positive constant $\tilde{C}_F$ such that

$$D(F_1) = \mathbb{R}, \quad |\partial {F_1}^{\circ}(s)| \leq \tilde{C}_F(1 + F_1(s)) \quad \text{for every } s \in \mathbb{R}, \tag{3.46}$$

where $\partial F_1^{\circ}(\cdot)$ stands for the element of $\partial F_1(\cdot)$ having minimum norm as introduced in Section 2.4.3.

Let us remark that condition (3.46) is satisfied by (1.7) and by any other polynomial growth type potential (or first-order exponential potentials), whereas it is not verified by singular potentials like (1.8) and (1.9).

Theorem 3.9 (Asymptotics: $\alpha \to 0$, [39, Theorem 2.5])**.** *Suppose that* **A1–A2**, **A5** *hold and let $\alpha, \beta \in (0,1)$. Let the initial data $(\varphi_0, \mu_0, \sigma_0)$ satisfy* (3.7) *and let the source term $g_{\alpha,\beta} \in L^2(0,T;H)$ be such that $g_\beta \in L^2(0,T;H)$ and $g_{\alpha,\beta} \to g_\beta$ strongly in $L^2(0,T;H)$ as $\alpha \to 0$. Then, denoting by $(\varphi_{\alpha,\beta}, \mu_{\alpha,\beta}, \sigma_{\alpha,\beta}, \xi_{\alpha,\beta})$ the unique weak solution to* (3.1)–(3.5) *obtained by Theorem 3.2, it holds, as $\alpha \to 0$,*

$$\varphi_{\alpha,\beta} \to \varphi_\beta \quad \textit{weakly in } H^1(0,T;H) \cap L^2(0,T;W), \tag{3.47}$$
$$\mu_{\alpha,\beta} \to \mu_\beta \quad \textit{weakly in } L^2(0,T;V), \tag{3.48}$$
$$\sigma_{\alpha,\beta} \to \sigma_\beta \quad \textit{weakly in } H^1(0,T;V^*) \cap L^2(0,T;V), \tag{3.49}$$
$$\xi_{\alpha,\beta} \to \xi_\beta \quad \textit{weakly in } L^2(0,T;H), \tag{3.50}$$
$$\alpha\mu_{\alpha,\beta} \to 0 \quad \textit{weakly in } H^1(0,T;V^*)\textit{, and strongly in } L^2(0,T;V), \tag{3.51}$$

possibly for a subsequence and the limits $(\varphi_\beta, \mu_\beta, \sigma_\beta, \xi_\beta)$ and g_β solve system (3.1)–(3.5) *with $\alpha = 0$ in the sense of Definition 3.8. Moreover, up to a subsequence, we also have the strong convergence*

$$\varphi_{\alpha,\beta} \to \varphi_\beta \quad \textit{strongly in } L^2(0,T;H). \tag{3.52}$$

Proof of Theorem 3.9. For the proof we refer to [39]. □

The uniqueness of the limit system is proved as a consequence of a suitable error estimate between the solution to (3.1)–(3.5) with $\alpha > 0$ and with $\alpha = 0$, respectively. In this direction, two further restrictions have been postulated:

A6 P is a non-negative constant.

A7 $F \in C^2(\mathbb{R})$ and there exists a positive constant C_F such that

$$|F''(s)| \leq C_F(1 + |s|^2) \quad \text{for every } s \in \mathbb{R}. \tag{3.53}$$

Notice that (3.53) prevents the singular choices (1.8) and (1.9) to be considered, whereas (1.7) is still included.

Theorem 3.10 (Error estimate: $\alpha \to 0$, [39, Theorem 2.6])**.** *Assume* **A1–A2***,* **A5–A7***, and let $\alpha, \beta \in (0,1)$. Let the initial data $(\varphi_0, \mu_0, \sigma_0)$ satisfy* (3.7)*. Then, the solution $(\varphi_\beta, \mu_\beta, \sigma_\beta, \xi_\beta)$ to* (3.1)–(3.5) *with $\alpha = 0$ in the sense of Definition 3.8 obtained from Theorem 3.9 is unique.*

Moreover, denoting by $(\varphi_{\alpha,\beta}, \mu_{\alpha,\beta}, \sigma_{\alpha,\beta}, \xi_{\alpha,\beta})$ the unique weak solution obtained by Theorem 3.2 with $\alpha, \beta > 0$, it holds that

$$\begin{aligned} &\|\varphi_{\alpha,\beta} - \varphi_\beta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\mu_{\alpha,\beta} - \mu_\beta\|_{L^2(0,T;V)} \\ &\quad + \|\sigma_{\alpha,\beta} - \sigma_\beta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq K_\beta\big(\alpha^{1/2} + \|g_{\alpha,\beta} - g_\beta\|_{L^2(0,T;H)}\big) \end{aligned} \tag{3.54}$$

for a positive constant K_β which may depend on β but it is independent of α.

Proof of Theorem 3.10. For the proof we again refer to [39]. □

Combining the above theorems we end up with:

Corollary 3.11. *Assume* **A1–A2***,* **A5–A7***, and let $\beta \in (0,1)$. Then system* (3.1)–(3.5) *with $\alpha = 0$ admits a unique solution in the sense of Definition 3.8.*

3.3.2 Asymptotic Analysis as $\beta \to 0$

Here, we follow the same structure of the previous section by first introducing the notion of weak solution to (3.1)–(3.5) with $\beta = 0$ and then presenting the results related to the asymptotic as $\beta \to 0$ for the system (3.1)–(3.5).

Definition 3.12. *A quadruplet $(\varphi, \mu, \sigma, \xi)$ is said to be a weak solution to* (3.1)–(3.5) *with $\beta = 0$ if*

$$\begin{aligned} \varphi &\in L^\infty(0,T;V) \cap L^2(0,T;W), \\ \mu &\in L^\infty(0,T;H) \cap L^2(0,T;V), \\ \alpha\mu + \varphi &\in H^1(0,T;V^*), \\ \sigma &\in H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V), \\ \xi &\in L^2(0,T;H), \end{aligned}$$

and $(\varphi, \mu, \sigma, \xi)$ verifies

$$\langle \partial_t(\alpha\mu + \varphi), v\rangle + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega P(\varphi)(\sigma - \mu)v, \tag{3.55}$$

$$\mu = -\Delta\varphi + \xi + F_2'(\varphi), \qquad \xi \in \partial F_1(\varphi) \quad \text{a.e. in } Q, \tag{3.56}$$

$$\langle \partial_t\sigma, v\rangle + \int_\Omega \nabla\sigma \cdot \nabla v = -\int_\Omega P(\varphi)(\sigma - \mu)v + \int_\Omega gv, \tag{3.57}$$

for every $v \in V$, almost everywhere in $(0,T)$ and

$$(\alpha\mu + \varphi)(0) = \alpha\mu_0 + \varphi_0, \quad \sigma(0) = \sigma_0 \quad \text{in } \Omega. \tag{3.58}$$

Theorem 3.13 (Asymptotics: $\beta \to 0$, [39, Theorem 2.2])**.** *Assume* **A1–A2***. Let the initial data* $(\varphi_0, \mu_0, \sigma_0)$ *satisfy* (3.7) *and let the source term* $g_{\alpha,\beta} \in L^2(0,T;H)$ *be such that* $g_\alpha \in L^2(0,T;H)$ *and* $g_{\alpha,\beta} \to g_\alpha$ *strongly in* $L^2(0,T;H)$ *as* $\beta \to 0$*. Then, there exists* $\alpha_0 \in (0,1)$ *such that, denoting by* $(\varphi_{\alpha,\beta}, \mu_{\alpha,\beta}, \sigma_{\alpha,\beta}, \xi_{\alpha,\beta})$ *the unique weak solution to* (3.1)–(3.5) *obtained by Theorem 3.2 with* $\alpha \in (0,\alpha_0)$ *and* $\beta \in (0,1)$*, it holds, as* $\beta \to 0$*,*

$$\varphi_{\alpha,\beta} \to \varphi_\alpha \quad \textit{weakly star in } L^\infty(0,T;V) \cap L^2(0,T;W), \tag{3.59}$$

$$\mu_{\alpha,\beta} \to \mu_\alpha \quad \textit{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V), \tag{3.60}$$

$$\sigma_{\alpha,\beta} \to \sigma_\alpha \quad \textit{weakly in } H^1(0,T;V^*) \cap L^2(0,T;V), \tag{3.61}$$

$$\partial_t(\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta}) \to \partial_t(\alpha\mu_\alpha + \varphi_\alpha) \quad \textit{weakly in } L^2(0,T;V^*), \tag{3.62}$$

$$\xi_{\alpha,\beta} \to \xi_\alpha \quad \textit{weakly in } L^2(0,T;H), \tag{3.63}$$

$$\beta\varphi_{\alpha,\beta} \to 0 \quad \textit{strongly in } H^1(0,T;H) \cap L^2(0,T;W), \tag{3.64}$$

possibly for a subsequence and the limits $(\varphi_\alpha, \mu_\alpha, \sigma_\alpha, \xi_\alpha)$ *and* g_α *solve system* (3.1)–(3.5) *with* $\beta = 0$ *in the sense of Definition 3.12. Moreover, up to a subsequence, we also have the strong convergence*

$$\varphi_{\alpha,\beta} \to \varphi_\alpha \quad \textit{strongly in } L^2(0,T;H). \tag{3.65}$$

Proof of Theorem 3.13. For the proof we refer to [39]. □

Here, a suitable error estimate can be derived in a general framework under an additional smallness type assumption on the relaxation parameter α.

Theorem 3.14 (Error estimate: $\beta \to 0$, [39, Theorem 2.3])**.** *Suppose* **A1–A2***. Let the initial data* $(\varphi_0, \mu_0, \sigma_0)$ *satisfy* (3.7)*. Then, the solution* $(\varphi_\alpha, \mu_\alpha, \sigma_\alpha, \xi_\alpha)$ *to* (3.1)–(3.5) *with* $\beta = 0$ *in the sense of Definition 3.12 is unique.*

Moreover, there exists $\alpha_{00} \in (0,\alpha_0)$ *such that denoting by* $(\varphi_{\alpha,\beta}, \mu_{\alpha,\beta}, \sigma_{\alpha,\beta}, \xi_{\alpha,\beta})$ *the unique weak solution obtained by Theorem 3.2 with* $\alpha \in (0,\alpha_{00})$ *and* $\beta \in (0,1)$*, it holds, as* $\beta \to 0$*,*

$$\begin{aligned} &\|\varphi_{\alpha,\beta} - \varphi_\alpha\|_{L^2(0,T;V)} + \|\mu_{\alpha,\beta} - \mu_\alpha\|_{L^2(0,T;H)} + \|\sigma_{\alpha,\beta} - \sigma_\alpha\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\ &\quad + \|(\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta} + \sigma_{\alpha,\beta}) - (\alpha\mu_\alpha + \varphi_\alpha + \sigma_\alpha)\|_{L^\infty(0,T;V^*)} \\ &\le K_\alpha\big(\beta^{1/2} + \|g_{\alpha,\beta} - g_\alpha\|_{L^2(0,T;H)}\big) \end{aligned} \tag{3.66}$$

for a positive constant K_α *which may depend on* α *but it is independent of* β*.*

Proof of Theorem 3.14. For the proof we again refer to [39]. □

Combining the above results we conclude that:

Corollary 3.15. *Assume* **A1–A2** *and let* $\alpha \in (0,\alpha_{00})$*. Then system* (3.1)–(3.5) *with* $\beta = 0$ *admits a unique solution in the sense of Definition 3.12.*

CHAPTER 4

Mathematical Analysis of a Family of Nonlocal Tumour Growth Models

This chapter provides a unified mathematical treatment of the family of nonlocal diffuse interface models for tumour growth (1.4)–(1.6) which takes into account long-range interactions occurring in biological phenomena. In particular, we are now considering the nonlocal system (1.4)–(1.6) with the following specifications:

- The thickness parameter ε and the surface tension parameter γ are set to one.
- We prescribe constant mobilities m and n: without loss of generality we let $m = n = 1$.
- For the source/sink terms $\mathcal{S}^1$ and $\mathcal{S}^2$ we assume linear kinetics as expressed in (1.24).
- We consider two different non-negative constants χ and η for the chemotaxis parameter and the active transport rate, respectively. Let us refer to [96] (see also [90]), where the authors explained how these two mechanisms can be decoupled. This choice will be extremely useful since several mathematical results hold under the assumption $\chi \geq 0$ while $\eta = 0$.

As already pointed out, the model in consideration couples a nonlocal Cahn–Hilliard type equation for the tumour phase variable with a reaction-diffusion equation for the nutrient concentration, and allows for significant mechanisms such as chemotaxis and active transport effects. The first part of the chapter is devoted to the analysis of the system when both the regularisations $\alpha\partial_t\mu$ and $\beta\partial_t\varphi$ are present, i.e., $\alpha, \beta > 0$. Under this assumption, a rich spectrum of results is presented: weak well-posedness is first addressed, also including singular potentials like (1.8) and (1.9). Despite the generality of the existence result, we can provide a stability estimate, which in turn entails the uniqueness of weak solutions, just under the restricting assumption $\eta = 0$. Then, under suitable conditions on the potential setting, existence of strong solutions enjoying the separation property is proved. This allows also to obtain a refined stability estimate with respect to the data, including both chemotaxis and active transport. In the second part, we study the asymptotic behaviour of the system as the relaxation parameters α and β approach zero both separately and jointly, and exact error estimates are obtained. As a by-product, the well-posedness of the corresponding limit systems is established.

Summing up, the two-parameter class of nonlocal models we are going to deal with in this chapter reads as:

$$\alpha\partial_t\mu + \partial_t\varphi - \Delta\mu = (\mathcal{P}\sigma - \mathcal{A})f(\varphi) \qquad \text{in } Q, \tag{4.1}$$

$$\mu = \beta\partial_t\varphi + a\varphi - J * \varphi + F'(\varphi) - \chi\sigma \qquad \text{in } Q, \tag{4.2}$$

$$\partial_t\sigma - \Delta\sigma + \mathcal{B}(\sigma - \sigma_S) + \mathcal{C}\sigma f(\varphi) = -\eta\Delta\varphi \qquad \text{in } Q, \tag{4.3}$$

$$\partial_{\mathbf{n}}\mu = \eta\partial_{\mathbf{n}}\varphi - \partial_{\mathbf{n}}\sigma = 0 \qquad \text{on } \Sigma, \tag{4.4}$$

$$\mu(0) = \mu_0, \quad \varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 \qquad \text{in } \Omega. \tag{4.5}$$

It is worth pointing out that it corresponds to a nonlocal variant of the local model proposed by H. Garcke et al. in [96]. In fact, at least formally, by setting $\alpha = \beta = 0$ and by substituting the nonlocality $a\varphi - J * \varphi$ with the "corresponding local term" $-\Delta\varphi$, we obtain exactly a particular case of the system analysed in [96]. We refer to Section 1.4 for more details on the modeling aspects and the meaning of the occurring variables.

Let us immediately point out that the above nonlocal model is new in the literature so that all the mathematical results revisited in this chapter are original and established in [134] by the author in collaboration with L. Scarpa in contrast to the situation of the local model (3.1)–(3.5) for which lots of results were already known.

Up to the author's knowledge, there are still few contributions devoted to the mathematical analysis of nonlocal tumour growth models of phase-field type of which we mention [79, 81, 84, 133] (see also [59, 124], where some results for the standard nonlocal Cahn–Hilliard equation with source terms can be found).

4.1 Weak Well-posedness

This first section is devoted to the weak analysis of the system (4.1)–(4.5) when both the regularisations α and β are present. In this setting, we investigate existence of weak solutions, even when singular potentials as (1.8) or (1.9) are present, including chemotaxis and active transport. Secondly, we show that without active transport, i.e., $\eta = 0$, continuous dependence on the data (hence uniqueness) holds for weak solutions. This limitation will be overcome later on when dealing with strong solutions (cf. Theorem 4.8), which however restricts us to avoid considering the singular and non-regular double-obstacle potential (1.9).

The following structural assumptions on the data will be in order in this section.

B1 $\mathcal{P}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \chi, \eta$ are non-negative constants.

B2 $f : \mathbb{R} \to [0, +\infty)$ is bounded and Lipschitz continuous.

B3 $\sigma_S \in L^\infty(Q)$ and

$$0 \leq \sigma_S(\mathbf{x}, t) \leq 1 \quad \text{for a.e. } (\mathbf{x}, t) \in Q.$$

B4 $F := F_1 + F_2 \geq 0$, where

$$F_1 : \mathbb{R} \to [0, +\infty] \quad \text{is proper, convex, and lower semicontinuous,}$$

and

$$F_2 \in C^2(\mathbb{R}), \qquad F_2' : \mathbb{R} \to \mathbb{R} \quad \text{is Lipschitz continuous,} \qquad F_2'(0) = 0.$$

In particular, the subdifferential $\partial F_1 : \mathbb{R} \to 2^{\mathbb{R}}$ is well defined in the sense of convex analysis, and we require that $0 \in \partial F_1(0)$. The Moreau regularisation of F_1 and the Yosida approximation of ∂F_1 are defined, respectively, as

$$F_{1,\lambda} : \mathbb{R} \to [0, +\infty), \qquad F_{1,\lambda}(s) := F_1(0) + \int_0^s F_{1,\lambda}'(r)\, \mathrm{dr}, \quad \mathrm{s} \in \mathbb{R},$$

and

$$F_{1,\lambda}' : \mathbb{R} \to \mathbb{R}, \qquad F_{1,\lambda}' := \frac{I - (I + \lambda \partial F_1)^{-1}}{\lambda}, \quad \lambda > 0,$$

where I stands for the identity operator (see Section 2.4.3 for more details). We recall that $F_{1,\lambda}'$ is $\frac{1}{\lambda}$-Lipschitz continous and we set

$$F_\lambda := F_{1,\lambda} + F_2.$$

B5 The kernel $J \in W^{1,1}_{loc}(\mathbb{R}^d)$ is such that $J(\mathbf{x}) = J(-\mathbf{x})$ for a.e. $\mathbf{x} \in \mathbb{R}^d$, $d \in \{2, 3\}$. For any measurable $v : \Omega \to \mathbb{R}$ we use the notation

$$(J * v)(\mathbf{x}) := \int_\Omega J(\mathbf{x} - \mathbf{y}) v(\mathbf{y})\, \mathrm{d}\mathbf{y}, \quad \mathbf{x} \in \Omega,$$

and set $a := J * 1$. Moreover, we suppose that

$$a_* := \inf_{\mathbf{x} \in \Omega} \int_\Omega J(\mathbf{x} - \mathbf{y})\, \mathrm{d}\mathbf{y} = \inf_{\mathbf{x} \in \Omega} a(\mathbf{x}) \geq 0,$$

$$a^* := \sup_{\mathbf{x} \in \Omega} \int_\Omega |J(\mathbf{x} - \mathbf{y})|\, \mathrm{d}\mathbf{y} < +\infty, \qquad b^* := \sup_{\mathbf{x} \in \Omega} \int_\Omega |\nabla J(\mathbf{x} - \mathbf{y})|\, \mathrm{d}\mathbf{y} < +\infty,$$

and we set $c_a := \max\{a^* - a_*, 1\}$. Finally, we suppose that there exists a positive constant C_0 such that, for $i = 1, 2$, and $s_1 \neq s_2$, we have

$$a_* + \frac{w_1 - w_2}{s_1 - s_2} \geq C_0, \qquad \forall\, s_i \in D(\partial F_1), \quad \forall\, w_i \in \partial F_1(s_i) + F_2'(s_i).$$

Note that if F is of class C^2, the last condition is equivalent to the classical one

$$a_* + F''(s) \geq C_0 \qquad \forall\, s \in D(F').$$

For convenience, we introduce the following upper bounds for the coefficients α and β

$$\alpha_0 := \min\left\{\frac{1}{4c_a}, \frac{1}{\max\{1, a^* - \min\{a^*, C_0\}\}}, \frac{2C_0}{3(a^*+b^*)^2 K_0^2}\right\}, \quad \beta_0 := 1, \qquad (4.6)$$

where K_0 denotes the norm of the continuous inclusion $H \subset V^*$. This is only a technical requirement on the coefficients, which is clearly not restrictive as α and β have to be considered as small perturbations.

The first main result deals with existence of global weak solutions to the system (4.1)–(4.4) under very general assumptions on the data. In particular, we stress that any type of potential as in (1.7)–(1.9) is included in this first result.

Theorem 4.1 (Existence of weak solutions: $\alpha, \beta > 0$)**.** *Assume* **B1–B5***, and let* $\alpha \in (0, \alpha_0)$ *and* $\beta \in (0, \beta_0)$*. Moreover, let the triplet of initial data* $(\varphi_0, \mu_0, \sigma_0)$ *satisfy*

$$\varphi_0 \in V, \qquad \mu_0, \sigma_0 \in H, \qquad F(\varphi_0) \in L^1(\Omega). \qquad (4.7)$$

Then, there exists a quadruplet $(\varphi, \mu, \sigma, \xi)$ *such that*

$$\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V), \qquad (4.8)$$

$$\mu, \sigma \in H^1(0,T;V^*) \cap L^2(0,T;V), \qquad (4.9)$$

$$\xi \in L^2(0,T;H), \qquad (4.10)$$

where

$$\mu = \beta\partial_t\varphi + a\varphi - J*\varphi + \xi + F_2'(\varphi) - \chi\sigma, \qquad \xi \in \partial F_1(\varphi) \quad \textit{a.e. in } Q, \qquad (4.11)$$

with

$$\varphi(0) = \varphi_0, \quad \mu(0) = \mu_0, \quad \sigma(0) = \sigma_0 \quad \textit{in } \Omega,$$

and such that

$$\langle \partial_t(\alpha\mu + \varphi), v\rangle + \int_\Omega \nabla\mu\cdot\nabla v = \int_\Omega (\mathcal{P}\sigma - \mathcal{A}) f(\varphi) v, \qquad (4.12)$$

$$\langle \partial_t\sigma, v\rangle + \int_\Omega \nabla\sigma\cdot\nabla v + \int_\Omega (\mathcal{B}(\sigma - \sigma_S) + \mathcal{C}\sigma f(\varphi))v = \eta\int_\Omega \nabla\varphi\cdot\nabla v, \qquad (4.13)$$

for every $v \in V$*, almost everywhere in* $(0,T)$*.*

Furthermore, if $\eta = 0$ *and*

$$0 \le \sigma_0(\mathbf{x}) \le 1 \quad \textit{for a.e. } \mathbf{x} \in \Omega, \qquad (4.14)$$

then $\sigma(t) \in L^\infty(\Omega)$ *for all* $t \in [0,T]$ *and it holds the comparison principle*

$$0 \le \sigma(\mathbf{x}, t) \le 1 \quad \textit{for a.e. } \mathbf{x} \in \Omega, \quad \forall\, t \in [0,T]. \qquad (4.15)$$

It is worth mentioning that, in the case of singular potentials such as (1.8) and (1.9), the assumption $F(\varphi_0) \in L^1(\Omega)$ entails that $\varphi_0 \in L^\infty(\Omega)$ and that $|\varphi_0(\mathbf{x})| \le 1$ for almost every $\mathbf{x} \in \Omega$.

Proof of Theorem 4.1. To prove the existence of solutions we rely on an approximation procedure based on two parameters $n \in \mathbb{N}$ and $\lambda > 0$, involving a Faedo–Galerkin approximation on the functional space and the Yosida approximation on the potential, respectively.

The Approximation

Let $\{e_j\}_{j\in\mathbb{N}}$ and $\{l_j\}_{j\in\mathbb{N}}$ be the sequences of eigenfunctions and eigenvalues of the Laplace operator $-\Delta$ endowed with homogeneous Neumann conditions, renormalised in such a way that $\|e_j\| = 1$ for all $j \in \mathbb{N}$. Then it is well known that $\{e_j\}_j$ yields a complete orthonormal system in H, and orthogonal in V. For every $n \in \mathbb{N}$, let $\mathcal{W}_n := \operatorname{span}\{e_1, \ldots, e_n\}$, and define $\mathbb{P}_n : H \to \mathcal{W}_n$ as the orthogonal projection on $\mathcal{W}_n$ with respect to the scalar product of H. Then, as $n \to \infty$, it holds that $\mathbb{P}_n v \to v$ in H (resp. V or W) for every $v \in H$ (resp. V or W). We then consider the following approximated system: find a triplet $(\varphi_{\lambda,n}, \mu_{\lambda,n}, \sigma_{\lambda,n})$ such that

$$\alpha\partial_t\mu_{\lambda,n} + \partial_t\varphi_{\lambda,n} - \Delta\mu_{\lambda,n} = (\mathcal{P}\sigma_{\lambda,n} - \mathcal{A})f(\varphi_{\lambda,n}) \qquad \text{in } Q, \qquad (4.16)$$

$$\mu_{\lambda,n} = \beta\partial_t\varphi_{\lambda,n} + a\varphi_{\lambda,n} - J * \varphi_{\lambda,n} + F'_\lambda(\varphi_{\lambda,n}) - \chi\sigma_{\lambda,n} \qquad \text{in } Q, \qquad (4.17)$$

$$\partial_t\sigma_{\lambda,n} - \Delta\sigma_{\lambda,n} + \mathcal{B}(\sigma_{\lambda,n} - \sigma_{S,n}) + \mathcal{C}\sigma_{\lambda,n}f(\varphi_{\lambda,n}) = -\eta\Delta\varphi_{\lambda,n} \qquad \text{in } Q, \qquad (4.18)$$

$$\partial_{\mathbf{n}}\mu_{\lambda,n} = \eta\partial_{\mathbf{n}}\varphi_{\lambda,n} - \partial_{\mathbf{n}}\sigma_{\lambda,n} = 0 \qquad \text{on } \Sigma, \qquad (4.19)$$

$$\mu_{\lambda,n}(0) = \mathbb{P}_n\mu_0, \quad \varphi_{\lambda,n}(0) = \mathbb{P}_n\varphi_0, \quad \sigma_{\lambda,n}(0) = \mathbb{P}_n\sigma_0 \qquad \text{in } \Omega, \qquad (4.20)$$

where $\sigma_{S,n} := \mathbb{P}_n\sigma_S$, in the form

$$\varphi_{\lambda,n}(\mathbf{x}, t) := \sum_{j=1}^{n} \vartheta_j^{\lambda,n}(t)e_j(\mathbf{x}), \quad \mu_{\lambda,n}(\mathbf{x}, t) := \sum_{j=1}^{n} \omega_j^{\lambda,n}(t)e_j(\mathbf{x}),$$
$$\sigma_{\lambda,n}(\mathbf{x}, t) := \sum_{j=1}^{n} \gamma_j^{\lambda,n}(t)e_j(\mathbf{x}),$$

for $t \in [0, T]$, $\mathbf{x} \in \Omega$, and $j \in \{1, ..., n\}$. Moreover, we introduce the vectors

$$\boldsymbol{\vartheta}^{\lambda,n}, \boldsymbol{\omega}^{\lambda,n}, \boldsymbol{\gamma}^{\lambda,n} : [0, T] \to \mathbb{R}^n,$$

by setting

$$\boldsymbol{\vartheta}^{\lambda,n} := (\vartheta_1^{\lambda,n}, ..., \vartheta_n^{\lambda,n})^T, \quad \boldsymbol{\omega}^{\lambda,n} := (\omega_1^{\lambda,n}, ..., \omega_n^{\lambda,n})^T, \quad \boldsymbol{\gamma}^{\lambda,n} := (\gamma_1^{\lambda,n}, ..., \gamma_n^{\lambda,n})^T.$$

Plugging these expression in (4.16)–(4.20) and taking arbitrary $e_i \in \mathcal{W}_n$ as test functions, for $i = 1, \ldots, n$, we deduce that $(\varphi_{\lambda,n}, \mu_{\lambda,n}, \sigma_{\lambda,n})$ solves the approximated system above if and only if the triplet $(\boldsymbol{\vartheta}^{\lambda,n}, \boldsymbol{\omega}^{\lambda,n}, \boldsymbol{\gamma}^{\lambda,n})$ solves the following system of ODEs, for $i = 1, \ldots, n$:

$$\alpha\partial_t\omega_i^{\lambda,n} + \partial_t\vartheta_i^{\lambda,n} + l_i\omega_i^{\lambda,n} = \int_\Omega \Big(\mathcal{P}\sum_{j=1}^{n}\gamma_j^{\lambda,n}e_j - \mathcal{A}\Big)f\Big(\sum_{j=1}^{n}\vartheta_j^{\lambda,n}e_j\Big)e_i,$$

$$\omega_i^{\lambda,n} = \beta\partial_t\vartheta_i^{\lambda,n} + \sum_{j=1}^{n}\vartheta_j^{\lambda,n}\int_\Omega ae_je_i - \sum_{j=1}^{n}\vartheta_j^{\lambda,n}\int_\Omega (J * e_j)e_i$$
$$+ \int_\Omega F'_\lambda\Big(\sum_{j=1}^{n}\vartheta_j^{\lambda,n}e_j\Big)e_i - \chi\gamma_i^{\lambda,n},$$

$$\partial_t\gamma_i^{\lambda,n} + l_i\gamma_i^{\lambda,n} + \mathcal{B}\Big(\gamma_i^{\lambda,n} - \int_\Omega \sigma_{S,n}e_i\Big) + \mathcal{C}\sum_{j=1}^{n}\gamma_j^{\lambda,n}\int_\Omega f\Big(\sum_{m=1}^{n}\vartheta_m^{\lambda,n}e_m\Big)e_je_i = \eta l_i\vartheta_i^{\lambda,n},$$

$$\vartheta_i^{\lambda,n}(0) = (\varphi_0, e_i), \quad \omega_i^{\lambda,n}(0) = (\mu_0, e_i), \quad \gamma_i^{\lambda,n}(0) = (\sigma_0, e_i).$$

Since $f, F'_\lambda : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous and f is bounded, such initial value system can be written in the form

$$\begin{cases} \partial_t(\boldsymbol{\vartheta}^{\lambda,n}, \boldsymbol{\omega}^{\lambda,n}, \boldsymbol{\gamma}^{\lambda,n}) & = g_{\lambda,n}(\boldsymbol{\vartheta}^{\lambda,n}, \boldsymbol{\omega}^{\lambda,n}, \boldsymbol{\gamma}^{\lambda,n}), \\ (\boldsymbol{\vartheta}^{\lambda,n}, \boldsymbol{\omega}^{\lambda,n}, \boldsymbol{\gamma}^{\lambda,n})(0) & = ((\varphi_0, e_i), (\mu_0, e_i), (\sigma_0, e_i)), \end{cases}$$

where $g_{\lambda,n} : \mathbb{R}^{3n} \to \mathbb{R}^{3n}$ is locally Lipschitz continuous and linearly bounded. Hence, by the Cauchy–Peano theorem (cf. Section 2.4.2), the system above admits a unique global solution

$$\boldsymbol{\vartheta}^{\lambda,n}, \boldsymbol{\omega}^{\lambda,n}, \boldsymbol{\gamma}^{\lambda,n} \in C^1([0,T]; \mathbb{R}^n),$$

implying that

$$\varphi_{\lambda,n}, \mu_{\lambda,n}, \sigma_{\lambda,n} \in C^1([0,T]; \mathcal{W}_n)$$

are the unique solutions to the approximated problem (4.16)–(4.20).

Uniform Estimates

To justify the forthcoming passage to the limit, as $n \to \infty$ and $\lambda \to 0$, we show that the approximate solutions verify some energetic estimates which are uniform with respect to λ and n, still keeping α and $\beta > 0$ positive and fixed.

Testing (4.16) by $\mu_{\lambda,n}$, (4.17) by $-\partial_t\varphi_{\lambda,n}$, (4.18) by $\sigma_{\lambda,n}$, taking the sum and integrating over $(0,t)$, yields by symmetry of the kernel J, for every $t \in [0,T]$,

$$\begin{aligned} &\frac{\alpha}{2}\|\mu_{\lambda,n}(t)\|^2 + \int_{Q_t} |\nabla\mu_{\lambda,n}|^2 + \beta\int_{Q_t} |\partial_t\varphi_{\lambda,n}|^2 \\ &\quad + \frac{1}{4}\int_{\Omega\times\Omega} J(\mathbf{x}-\mathbf{y})|\varphi_{\lambda,n}(\mathbf{x},t) - \varphi_{\lambda,n}(\mathbf{y},t)|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} + \int_\Omega F_\lambda(\varphi_{\lambda,n}(t)) \\ &\quad + \frac{1}{2}\|\sigma_{\lambda,n}(t)\|^2 + \int_{Q_t} |\nabla\sigma_{\lambda,n}|^2 + \int_{Q_t} (\mathcal{B} + Ch(\varphi_{\lambda,n}))|\sigma_{\lambda,n}|^2 \\ &= \frac{\alpha}{2}\|\mathbb{P}_n\mu_0\|^2 + \frac{1}{4}\int_{\Omega\times\Omega} J(\mathbf{x}-\mathbf{y})|\mathbb{P}_n\varphi_0(\mathbf{x}) - \mathbb{P}_n\varphi_0(\mathbf{y})|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} + \int_\Omega F_\lambda(\mathbb{P}_n\varphi_0) \\ &\quad + \frac{1}{2}\|\mathbb{P}_n\sigma_0\|^2 + \int_{Q_t} (\mathcal{P}\sigma_{\lambda,n} - \mathcal{A})f(\varphi_{\lambda,n})\mu_{\lambda,n} + \chi\int_{Q_t} \sigma_{\lambda,n}\partial_t\varphi_{\lambda,n} \\ &\quad + \mathcal{B}\int_{Q_t} \sigma_{S,n}\sigma_{\lambda,n} + \eta\int_{Q_t} \nabla\varphi_{\lambda,n}\cdot\nabla\sigma_{\lambda,n}. \end{aligned}$$

Now, note that by assumption **B5** we have that

$$\begin{aligned} &\frac{1}{4}\int_{\Omega\times\Omega} J(\mathbf{x}-\mathbf{y})|\varphi_{\lambda,n}(\mathbf{x},t) - \varphi_{\lambda,n}(\mathbf{y},t)|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\ &\quad = \frac{1}{2}\int_\Omega [a(\mathbf{x})|\varphi_{\lambda,n}|^2 - (J * \varphi_{\lambda,n})\varphi_{\lambda,n}](\mathbf{x},t)\,\mathrm{d}\mathbf{x} \\ &\quad \geq \frac{a_*}{2}\|\varphi_{\lambda,n}(t)\|^2 - \frac{1}{2}\|J * \varphi_{\lambda,n}(t)\|\|\varphi_{\lambda,n}(t)\| \\ &\quad \geq \frac{a_* - a^*}{2}\|\varphi_{\lambda,n}(t)\|^2, \end{aligned} \tag{4.21}$$

and similarly that

$$
\begin{aligned}
&\frac{1}{4}\int_{\Omega\times\Omega} J(\mathbf{x}-\mathbf{y})|\mathbb{P}_n\varphi_0(\mathbf{x})-\mathbb{P}_n\varphi_0(\mathbf{y})|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\
&\quad=\frac{1}{2}\int_\Omega [a|\mathbb{P}_n\varphi_0|^2-(J*\mathbb{P}_n\varphi_0)\mathbb{P}_n\varphi_0](\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&\quad\le \frac{a^*+a^*}{2}\|\mathbb{P}_n\varphi_0\|^2\le a^*\|\varphi_0\|^2.
\end{aligned}
$$

Using that $F_\lambda \ge 0$, (4.21) along with the definition of c_a, recalling also that f is non-negative and bounded and that $\mathbb{P}_n$ is a contraction on H, owing to the Young inequality we infer that

$$
\begin{aligned}
&\frac{\alpha}{2}\|\mu_{\lambda,n}(t)\|^2+\int_{Q_t}|\nabla\mu_{\lambda,n}|^2+\beta\int_{Q_t}|\partial_t\varphi_{\lambda,n}|^2+\frac{1}{2}\|\sigma_{\lambda,n}(t)\|^2+\int_{Q_t}|\nabla\sigma_{\lambda,n}|^2\\
&\le \frac{\alpha}{2}\|\mu_0\|^2+a^*\|\varphi_0\|^2+\|F_\lambda(\mathbb{P}_n\varphi_0)\|_{L^1(\Omega)}+\frac{1}{2}\|\sigma_0\|^2+\frac{c_a}{2}\|\varphi_{\lambda,n}(t)\|^2\\
&\quad+\int_{Q_t}(\mathcal{P}\sigma_{\lambda,n}-\mathcal{A})f(\varphi_{\lambda,n})\mu_{\lambda,n}+\frac{1}{4}\int_{Q_t}|\sigma_{\lambda,n}|^2+|Q|\mathcal{B}^2\|\sigma_{S,n}\|^2_{L^\infty(Q)}\\
&\quad+\chi\int_{Q_t}\sigma_{\lambda,n}\partial_t\varphi_{\lambda,n}+\eta\int_{Q_t}\nabla\varphi_{\lambda,n}\cdot\nabla\sigma_{\lambda,n}. \qquad (4.22)
\end{aligned}
$$

Here, we recall that $a^*-a_*\ge 0$ which entails that $c_a=\max\{a^*-a_*,1\}>0$. Then, we test equation (4.16) by $4c_a(\alpha\mu_{\lambda,n}+\varphi_{\lambda,n})$ and (4.17) by $4c_a\Delta\varphi_{\lambda,n}$, add the resulting equalities and integrate over $(0,t)$ and by parts, getting, thanks to assumption **B5**,

$$
\begin{aligned}
&2c_a\|(\alpha\mu_{\lambda,n}+\varphi_{\lambda,n})(t)\|^2+4c_a\alpha\int_{Q_t}|\nabla\mu_{\lambda,n}|^2+2c_a\beta\|\nabla\varphi_{\lambda,n}(t)\|^2\\
&\quad+4c_aC_0\int_{Q_t}|\nabla\varphi_{\lambda,n}|^2\\
&\le 2c_a\|\mathbb{P}_n(\alpha\mu_0+\varphi_0)\|^2+2c_a\beta\|\nabla\mathbb{P}_n\varphi_0\|^2\\
&\quad+4c_a\int_{Q_t}(\mathcal{P}\sigma_{\lambda,n}-\mathcal{A})f(\varphi_{\lambda,n})(\alpha\mu_{\lambda,n}+\varphi_{\lambda,n})\\
&\quad+4c_a\chi\int_{Q_t}\nabla\sigma_{\lambda,n}\cdot\nabla\varphi_{\lambda,n}+8c_ab^*\|\varphi_{\lambda,n}\|_{L^2(Q_t)}\|\nabla\varphi_{\lambda,n}\|_{L^2(Q_t)},
\end{aligned}
$$

from which we infer, thanks to the Young inequality and the boundedness of f, that

$$
\begin{aligned}
&2c_a\|(\alpha\mu_{\lambda,n}+\varphi_{\lambda,n})(t)\|^2+4c_a\alpha\int_{Q_t}|\nabla\mu_{\lambda,n}|^2+2c_a\beta\|\nabla\varphi_{\lambda,n}(t)\|^2\\
&\quad+2c_aC_0\int_{Q_t}|\nabla\varphi_{\lambda,n}|^2\\
&\le 4c_a\alpha^2\|\mu_0\|^2+4c_a\|\varphi_0\|^2+2c_a\beta\|\nabla\varphi_0\|^2+4c_a\chi\int_{Q_t}\nabla\sigma_{\lambda,n}\cdot\nabla\varphi_{\lambda,n}\\
&\quad+C(1+\int_{Q_t}|\alpha\mu_{\lambda,n}+\varphi_{\lambda,n}|^2+\int_{Q_t}|\varphi_{\lambda,n}|^2+\int_{Q_t}|\sigma_{\lambda,n}|^2) \qquad (4.23)
\end{aligned}
$$

for a constant $C > 0$, independent of λ, n, α, and β. Summing (4.22) and (4.23), we infer that, possibly updating C,

$$\begin{aligned}
&\frac{\alpha}{2}\|\mu_{\lambda,n}(t)\|^2 + (1+4c_a\alpha)\int_{Q_t}|\nabla\mu_{\lambda,n}|^2 + \beta\int_{Q_t}|\partial_t\varphi_{\lambda,n}|^2 \\
&\quad + \frac{1}{2}\|\sigma_{\lambda,n}(t)\|^2 + \int_{Q_t}|\nabla\sigma_{\lambda,n}|^2 + 2c_a\|(\alpha\mu_{\lambda,n}+\varphi_{\lambda,n})(t)\|^2 + \\
&\quad 2c_a\beta\|\nabla\varphi_{\lambda,n}(t)\|^2 + 2c_aC_0\int_{Q_t}|\nabla\varphi_{\lambda,n}|^2 \\
&\quad \leq \big(\frac{\alpha}{2} + 4c_a\alpha^2\big)\|\mu_0\|^2 + (a^* + 4c_a)\|\varphi_0\|^2 + 2c_a\beta\|\nabla\varphi_0\|^2 + \|F_\lambda(\mathbb{P}_n\varphi_0)\|_{L^1(\Omega)} \\
&\quad + \frac{1}{2}\|\sigma_0\|^2 + \frac{c_a}{2}\|\varphi_{\lambda,n}(t)\|^2 + \chi\int_{Q_t}\sigma_{\lambda,n}\partial_t\varphi_{\lambda,n} + (\eta+4c_a\chi)\int_{Q_t}\nabla\sigma_{\lambda,n}\cdot\nabla\varphi_{\lambda,n} \\
&\quad + C(1 + \int_{Q_t}|\alpha\mu_{\lambda,n}+\varphi_{\lambda,n}|^2 + \int_{Q_t}|\varphi_{\lambda,n}|^2 + \int_{Q_t}|\sigma_{\lambda,n}|^2) \\
&\quad + \int_{Q_t}(\mathcal{P}\sigma_{\lambda,n} - \mathcal{A})f(\varphi_{\lambda,n})\mu_{\lambda,n}. \qquad (4.24)
\end{aligned}$$

Note that

$$\frac{c_a}{2}\|\varphi_{\lambda,n}(t)\|^2 \leq c_a\|(\alpha\mu_{\lambda,n}+\varphi_{\lambda,n})(t)\|^2 + c_a\alpha^2\|\mu_{\lambda,n}(t)\|^2,$$

where the two terms on the right-hand side can be incorporated in the left-hand side of (4.24) as $2c_a - c_a = c_a > 0$ and $\frac{\alpha}{2} - c_a\alpha^2 \geq \frac{\alpha}{4}$ since $\alpha \in (0,\alpha_0)$. Furthermore, using the Young inequality we have

$$\begin{aligned}
&\chi\int_{Q_t}\sigma_{\lambda,n}\partial_t\varphi_{\lambda,n} + (\eta+4c_a\chi)\int_{Q_t}\nabla\sigma_{\lambda,n}\cdot\nabla\varphi_{\lambda,n} \\
&\quad \leq \frac{\beta}{2}\int_{Q_t}|\partial_t\varphi_{\lambda,n}|^2 + \frac{\chi^2}{2\beta}\int_{Q_t}|\sigma_{\lambda,n}|^2 + \frac{1}{2}\int_{Q_t}|\nabla\sigma_{\lambda,n}|^2 + \frac{(\eta+4c_a\chi)^2}{2}\int_{Q_t}|\nabla\varphi_{\lambda,n}|^2.
\end{aligned}$$

Collecting the above estimates, we infer that

$$
\begin{aligned}
&\frac{\alpha}{4}\|\mu_{\lambda,n}(t)\|^2 + (1+4c_a\alpha)\int_{Q_t}|\nabla\mu_{\lambda,n}|^2 + \frac{\beta}{2}\int_{Q_t}|\partial_t\varphi_{\lambda,n}|^2 \\
&\quad + \frac{1}{2}\|\sigma_{\lambda,n}(t)\|^2 + \int_{Q_t}|\nabla\sigma_{\lambda,n}|^2 + c_a\|(\alpha\mu_{\lambda,n}+\varphi_{\lambda,n})(t)\|^2 \\
&\quad + 2c_a\beta\|\nabla\varphi_{\lambda,n}(t)\|^2 + 2c_aC_0\int_{Q_t}|\nabla\varphi_{\lambda,n}|^2 \\
&\leq \frac{3}{2}\alpha\|\mu_0\|^2 + (a^*+4c_a)\|\varphi_0\|^2 + 2c_a\beta\|\nabla\varphi_0\|^2 + \|F_\lambda(\mathbb{P}_n\varphi_0)\|_{L^1(\Omega)} \\
&\quad + \frac{1}{2}\|\sigma_0\|^2 + C(1+\int_{Q_t}|\alpha\mu_{\lambda,n}+\varphi_{\lambda,n}|^2 + \int_{Q_t}|\varphi_{\lambda,n}|^2 + \int_{Q_t}|\sigma_{\lambda,n}|^2) \\
&\quad + \frac{\chi^2}{2\beta}\int_{Q_t}|\sigma_{\lambda,n}|^2 + \frac{1}{2}\int_{Q_t}|\nabla\sigma_{\lambda,n}|^2 + \frac{(\eta+4c_a\chi)^2}{2}\int_{Q_t}|\nabla\varphi_{\lambda,n}|^2 \\
&\quad + \int_{Q_t}(\mathcal{P}\sigma_{\lambda,n}-\mathcal{A})f(\varphi_{\lambda,n})\mu_{\lambda,n}. \qquad (4.25)
\end{aligned}
$$

Moreover, the last term on the right-hand side can be easily bounded owing to Young's inequality.

Then, we fix $\lambda > 0$, and since F_λ has at most quadratic growth (depending on λ) and $\varphi_0 \in H$, we have that $\|F_\lambda(\mathbb{P}_n\varphi_0)\|_{L^1(\Omega)} \leq C_\lambda$ uniformly in $n \in \mathbb{N}$, for a certain $C_\lambda > 0$ independent of n. Therefore, Gronwall's lemma yields that

$$
\begin{aligned}
&\|\mu_{\lambda,n}\|^2_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\varphi_{\lambda,n}\|^2_{H^1(0,T;H)\cap L^\infty(0,T;V)} \\
&\quad + \|\sigma_{\lambda,n}\|^2_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq C_\lambda, \qquad (4.26)
\end{aligned}
$$

where the constant C_λ is independent of n (but not of α and β). Furthermore, by comparison in equations (4.16) and (4.18), we in turn deduce that

$$
\|\partial_t(\alpha\mu_{\lambda,n}+\varphi_{\lambda,n})\|^2_{L^2(0,T;V^*)} + \|\partial_t\mu_{\lambda,n}\|^2_{L^2(0,T;V^*)} + \|\partial_t\sigma_{\lambda,n}\|^2_{L^2(0,T;V^*)} \leq C_\lambda. \quad (4.27)
$$

Passage to the Limit

We pass now to the limit, keeping α and β fixed, first as $n \to \infty$ and then as $\lambda \to 0$. From the estimates (4.26)–(4.27) and Lemma 2.4, we deduce that the exists a triplet $(\varphi_\lambda, \mu_\lambda, \sigma_\lambda)$, with

$$
\varphi_\lambda \in H^1(0,T;H)\cap L^\infty(0,T;V), \qquad \mu_\lambda, \sigma_\lambda \in H^1(0,T;V^*)\cap L^2(0,T;V),
$$

such that, as $n \to \infty$,

$$
\begin{aligned}
\varphi_{\lambda,n} \to \varphi_\lambda \quad & \text{weakly star in } H^1(0,T;H)\cap L^\infty(0,T;V), \\
& \quad \text{and strongly in } C^0([0,T];H), \\
\mu_{\lambda,n} \to \mu_\lambda \quad & \text{weakly in } H^1(0,T;V^*)\cap L^2(0,T;V), \\
& \quad \text{and strongly in } C^0([0,T];V^*)\cap L^2(0,T;H), \\
\sigma_{\lambda,n} \to \sigma_\lambda \quad & \text{weakly in } H^1(0,T;V^*)\cap L^2(0,T;V), \\
& \quad \text{and strongly in } C^0([0,T];V^*)\cap L^2(0,T;H).
\end{aligned}
$$

Since F'_λ is Lipschitz continuous and f is Lipschitz continuous and bounded, it is a standard matter to pass the limit in the approximated problem (4.16)–(4.20) as $n \to \infty$ to obtain, for every test function $v \in V$,

$$\langle \partial_t(\alpha\mu_\lambda + \varphi_\lambda), v\rangle + \int_\Omega \nabla\mu_\lambda \cdot \nabla v = \int_\Omega (\mathcal{P}\sigma_\lambda - \mathcal{A}) f(\varphi_\lambda) v, \tag{4.28}$$

$$\mu_\lambda = \beta\partial_t\varphi_\lambda + a\varphi_\lambda - J * \varphi_\lambda + F'_\lambda(\varphi_\lambda) - \chi\sigma_\lambda, \tag{4.29}$$

$$\langle \partial_t\sigma_\lambda, v\rangle + \int_\Omega \nabla\sigma_\lambda \cdot \nabla v + \int_\Omega [\mathcal{B}(\sigma_\lambda - \sigma_S) + \mathcal{C}\sigma_\lambda f(\varphi_\lambda)] v = \eta \int_\Omega \nabla\varphi_\lambda \cdot \nabla v, \tag{4.30}$$

almost everywhere in $(0, T)$, and

$$\mu_\lambda(0) = \mu_0, \quad \varphi_\lambda(0) = \varphi_0, \quad \sigma_\lambda(0) = \sigma_0 \qquad \text{a.e. in } \Omega \tag{4.31}$$

meaning that the triplet $(\varphi_\lambda, \mu_\lambda, \sigma_\lambda)$ satisfies the analogous of conditions (4.11)–(4.13) at level λ.

Clearly, by weak lower semicontinuity of the norms and the convex integrands, passing to the inferior limit as $n \to \infty$ in the estimates (4.26) and (4.27), and recalling that $F_\lambda \leq F$, we infer that there exists $C > 0$, independent of λ (but not of α and β), such that

$$\begin{aligned} &\|\mu_\lambda\|_{H^1(0,T;V^*)\cap L^2(0,T;V)} + \|\varphi_\lambda\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \\ &\quad + \|\sigma_\lambda\|_{H^1(0,T;V^*)\cap L^2(0,T;V)} \leq C. \end{aligned} \tag{4.32}$$

Furthermore, estimate (4.32) readily implies, by comparison in equation (4.29), that

$$\|F'_{1,\lambda}(\varphi_\lambda)\|_{L^2(0,T;H)} \leq C. \tag{4.33}$$

Hence, there exists a quadruplet $(\varphi, \mu, \sigma, \xi)$, with

$$\begin{aligned} \varphi &\in H^1(0,T;H) \cap L^\infty(0,T;V), \quad \xi \in L^2(0,T;H), \\ \mu, \sigma &\in H^1(0,T;V^*) \cap L^2(0,T;V), \end{aligned}$$

such that, as $\lambda \to 0$,

$$\begin{aligned} \varphi_\lambda \to \varphi \quad &\text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V), \\ &\quad \text{and strongly in } C^0([0,T];H), \\ \mu_\lambda \to \mu \quad &\text{weakly in } H^1(0,T;V^*) \cap L^2(0,T;V), \\ &\quad \text{and strongly in } C^0([0,T];V^*) \cap L^2(0,T;H), \\ \sigma_\lambda \to \sigma \quad &\text{weakly in } H^1(0,T;V^*) \cap L^2(0,T;V), \\ &\quad \text{and strongly in } C^0([0,T];V^*) \cap L^2(0,T;H), \\ F'_{1,\lambda}(\varphi_\lambda) \to \xi \quad &\text{weakly in } L^2(0,T;H). \end{aligned}$$

The strong weak closure of the maximal monotone operator ∂F_1 implies that $\xi \in \partial F_1(\varphi)$ almost everywhere in Q. Moreover, by the Lipschitz continuity of F'_2 and f, and the boundedness of f, we have that

$$\begin{aligned} f(\varphi_\lambda) \to f(\varphi) \quad &\text{strongly in } L^p(Q) \quad \forall\, p \geq 1, \\ F'_2(\varphi_\lambda) \to F'_2(\varphi) \quad &\text{strongly in } L^2(0,T;H). \end{aligned}$$

Consequently, letting $\lambda \to 0$ in the variational formulation of (4.28)–(4.31), we obtain exactly (4.11)–(4.13) completing the proof concerning the existence of weak solutions in Theorem 4.1.

Comparison Principle for σ

We prove here the last assertion of Theorem 4.1, concerning a comparison principle for σ under the additional requirement $\eta = 0$. Testing equation (4.13) by $f_+(\sigma) := (\sigma - 1)_+$, we have

$$\frac{1}{2}\|f_+(\sigma(t))\|^2 + \int_{Q_t} f_+'(\sigma)|\nabla\sigma|^2 + \mathcal{B}\int_{Q_t} f_+(\sigma)(\sigma - \sigma_S) + \mathcal{C}\int_{Q_t} f_+(\sigma)\sigma f(\varphi) = 0,$$

where we have used the fact that $f_+(\sigma_0) = 0$, being $(\cdot)_+$ the positive part function defined by

$$(s)_+ := \max\{0, s\}, \quad s \in \mathbb{R}.$$

Since f_+ is non-decreasing and f is non-negative, we infer that the second and fourth terms on the left-hand side are non-negative so that

$$\frac{1}{2}\|f_+(\sigma(t))\|^2 + \mathcal{B}\int_{Q_t} f_+(\sigma)(\sigma - \sigma_S) \leq 0. \tag{4.34}$$

Moreover, since $\sigma_S \leq 1$ by assumption **B3**, we have that

$$\mathcal{B}\int_{Q_t} f_+(\sigma)(\sigma - \sigma_S) = \mathcal{B}\int_{Q_t\cap\{\sigma>1\}} (\sigma - 1)(\sigma - \sigma_S) \geq 0.$$

Therefore, coming back to (4.34), we realise that $f_+(\sigma(t)) = 0$ which gives us the upper bound $\sigma(t) \leq 1$ a.e in Ω, for every $t \in [0, T]$, as desired. The lower inequality follows by a similar argument testing by $f_-(\sigma) := -(\sigma)_-$, where

$$(s)_- := \max\{0, -s\}, \quad s \in \mathbb{R}.$$

□

The second result concerns the continuous dependence of the data for weak solutions. This result applies again to any choice of the potential F, but we are forced (so far) to restrict ourselves to the case without active transport (i.e., $\eta = 0$). As a consequence of the following result, we infer the uniqueness of the weak solution obtained in Theorem 4.1 under the only additional requirement that $\eta = 0$.

Theorem 4.2 (Continuous dependence: $\alpha, \beta > 0$)**.** *Assume* **B1–B5***, and let* $\eta = 0$, $\alpha \in (0, \alpha_0)$ *and* $\beta \in (0, \beta_0)$. *Then there exists a constant* $K > 0$ *independent of* β *such that, for any pair of initial data* $\{(\varphi_0^i, \mu_0^i, \sigma_0^i)\}_i$, $i = 1, 2$, *satisfying* (4.7) *and* (4.14), *and for any respective solutions* $\{(\varphi_i, \mu_i, \sigma_i, \xi_i)\}_i$, $i = 1, 2$, *obtained from Theorem 4.1, it holds that*

$$\begin{aligned}
&\|(\alpha\mu_1 + \varphi_1) - (\alpha\mu_2 + \varphi_2)\|_{L^\infty(0,T;V^*)} + \|\mu_1 - \mu_2\|_{L^2(0,T;H)} \\
&\quad + \beta^{1/2}\|\varphi_1 - \varphi_2\|_{C^0([0,T];H)} + \|\varphi_1 - \varphi_2\|_{L^2(0,T;H)} + \|\sigma_1 - \sigma_2\|_{C^0([0,T];H)\cap L^2(0,T;V)} \\
&\leq K(\|(\alpha\mu_0^1 + \varphi_0^1) - (\alpha\mu_0^2 + \varphi_0^2)\|_{V^*} + \beta^{1/2}\|\varphi_0^1 - \varphi_0^2\| + \|\sigma_0^1 - \sigma_0^2\|).
\end{aligned} \tag{4.35}$$

Proof of Theorem 4.2. To begin with, we set

$$\varphi := \varphi_1 - \varphi_2, \quad \mu := \mu_1 - \mu_2, \quad \sigma := \sigma_1 - \sigma_2, \quad \xi := \xi_1 - \xi_2,$$
$$\varphi_0 := \varphi_0^1 - \varphi_0^2, \quad \mu_0 := \mu_0^1 - \mu_0^2, \quad \sigma_0 := \sigma_0^1 - \sigma_0^2.$$

Then, we consider the difference of system (4.1)–(4.5) written for the two solutions to obtain

$$\alpha\partial_t\mu + \partial_t\varphi - \Delta\mu = \mathcal{P}\sigma f(\varphi_1) + (\mathcal{P}\sigma_2 - \mathcal{A})(f(\varphi_1) - f(\varphi_2)) \quad \text{in } Q, \tag{4.36}$$
$$\mu = \beta\partial_t\varphi + a\varphi - J*\varphi + \xi + F_2'(\varphi_1) - F_2'(\varphi_2) - \chi\sigma \quad \text{in } Q, \tag{4.37}$$
$$\partial_t\sigma - \Delta\sigma + \mathcal{B}\sigma + \mathcal{C}\sigma f(\varphi_1) = \mathcal{C}\sigma_2(f(\varphi_2) - f(\varphi_1)) - \eta\Delta\varphi \quad \text{in } Q, \tag{4.38}$$
$$\partial_{\mathbf{n}}\mu = \eta\partial_{\mathbf{n}}\varphi - \partial_{\mathbf{n}}\sigma = 0 \quad \text{on } \Sigma, \tag{4.39}$$
$$\mu(0) = \mu_0, \quad \varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 \quad \text{in } \Omega. \tag{4.40}$$

Next, we test the equation (4.36) by $\mathcal{R}^{-1}(\alpha\mu+\varphi)$, (4.37) by $-\varphi$, (4.38) by σ, and take the sum to get, after integration on $[0,t]$,

$$\begin{aligned}
&\frac{1}{2}\|(\alpha\mu+\varphi)(t)\|_{V^*}^2 + \alpha\int_{Q_t}|\mu|^2 + \frac{\beta}{2}\|\varphi(t)\|^2 + \int_{Q_t}[a|\varphi|^2 + \xi\varphi + (F_2'(\varphi_1) - F_2'(\varphi_2))\varphi] \\
&\quad + \frac{1}{2}\|\sigma(t)\|^2 + \int_{Q_t}|\nabla\sigma|^2 + \int_{Q_t}(\mathcal{B} + \mathcal{C}f(\varphi_1))|\sigma|^2 \\
&\quad = \frac{1}{2}\|\alpha\mu_0 + \varphi_0\|_{V^*}^2 + \frac{\beta}{2}\|\varphi_0\|^2 + \frac{1}{2}\|\sigma_0\|^2 \\
&\quad + \int_{Q_t}(\chi\sigma + J*\varphi)\varphi + \int_{Q_t}[\mathcal{C}\sigma_2(f(\varphi_2) - f(\varphi_1))]\sigma \\
&\quad + \int_{Q_t}[\mu + \mathcal{P}\sigma f(\varphi_1) + (\mathcal{P}\sigma_2 - \mathcal{A})(f(\varphi_1) - f(\varphi_2))]\mathcal{R}^{-1}(\alpha\mu+\varphi) \\
&\quad + \eta\int_{Q_t}\nabla\varphi\cdot\nabla\sigma.
\end{aligned} \tag{4.41}$$

Note that the last term on the left-hand side is non-negative due to the non-negativity of f. Hence, using the monotonicity of ∂F_1 and recalling assumption **B5**, we have

$$\int_{Q_t}[a|\varphi|^2 + \xi\varphi + (F_2'(\varphi_1) - F_2'(\varphi_2))\varphi] + \int_{Q_t}(\mathcal{B} + \mathcal{C}f(\varphi_1))|\sigma|^2 \geq C_0\int_{Q_t}|\varphi|^2.$$

Moreover, under the assumption $\eta = 0$, we have, owing to the comparison principle (4.15) that $\sigma_2 \in L^\infty(Q)$ with $\|\sigma_2\|_{L^\infty(Q)} \leq 1$ and that the last term on the right-hand side of (4.41) disappears. Let us estimate the remaining terms on the right-hand side. First of all, recalling that K_0 denotes the norm of the inclusion $H \subset V^*$, by the Young inequality we have that, for every $\delta_1, \delta_2 > 0$,

$$\int_{Q_t} \left[\mu + P\sigma f(\varphi_1) + (P\sigma_2 - A)(f(\varphi_1) - f(\varphi_2))\right]\mathcal{R}^{-1}(\alpha\mu + \varphi)$$
$$\leq \delta_1\alpha \int_{Q_t} |\mu|^2 + \delta_2 \int_{Q_t} |\varphi|^2 + \frac{P^2\|f\|^2_{L^\infty(\mathbb{R})}}{2} \int_{Q_t} |\sigma|^2$$
$$+ K_0^2 \left(\frac{1}{4\delta_1\alpha} + \frac{1}{2} + \frac{(P+A)^2\|f'\|^2_{L^\infty(\mathbb{R})}}{4\delta_2}\right) \int_0^t \|(\alpha\mu + \varphi)(s)\|^2_{V^*} \mathrm{ds}.$$

Secondly, analogous computations yield

$$\chi \int_{Q_t} \sigma\varphi + \int_{Q_t} [C\sigma_2(f(\varphi_2) - f(\varphi_1))]\sigma$$
$$\leq \delta_2 \int_{Q_t} |\varphi|^2 + \frac{\chi^2 + C^2\|f'\|^2_{L^\infty(\mathbb{R})}}{2\delta_2} \int_{Q_t} |\sigma|^2.$$

Finally, we have that

$$\int_{Q_t} (J * \varphi)\varphi \leq \int_0^t \|J * \varphi(s)\|_V \|\varphi(s)\|_{V^*} \mathrm{ds}$$
$$\leq (a^* + b^*) \int_0^t \|\varphi(s)\|\|\varphi(s)\|_{V^*} \mathrm{ds}$$
$$\leq \delta_2 \int_{Q_t} |\varphi|^2 + \frac{(a^* + b^*)^2}{4\delta_2} \int_0^t \|\varphi(s)\|^2_{V^*} \mathrm{ds}$$
$$\leq \delta_2 \int_{Q_t} |\varphi|^2 + \frac{(a^* + b^*)^2}{2\delta_2} \int_0^t \|(\alpha\mu + \varphi)(s)\|^2_{V^*} \mathrm{ds}$$
$$+ \frac{\alpha(a^* + b^*)^2 K_0^2}{2\delta_2} \left(\alpha \int_{Q_t} |\mu|^2\right).$$

Rearranging the terms we deduce that

$$\frac{1}{2}\|(\alpha\mu + \varphi)(t)\|^2_{V^*} + \alpha \int_{Q_t} |\mu|^2 + \frac{\beta}{2}\|\varphi(t)\|^2 + C_0 \int_{Q_t} |\varphi|^2$$
$$+ \frac{1}{2}\|\sigma(t)\|^2 + \int_{Q_t} |\nabla\sigma|^2$$
$$\leq \frac{1}{2}\|\alpha\mu_0 + \varphi_0\|^2_{V^*} + \frac{\beta}{2}\|\varphi_0\|^2 + \frac{1}{2}\|\sigma_0\|^2$$
$$+ C(\delta_1, \delta_2, \alpha) \int_0^t \left(\|\sigma(s)\|^2 + \|(\alpha\mu + \varphi)(s)\|^2_{V^*}\right) \mathrm{ds}$$
$$+ \left(\delta_1 + \frac{\alpha(a^* + b^*)^2 K_0^2}{2\delta_2}\right) \alpha \int_{Q_t} |\mu|^2 + 3\delta_2 \int_{Q_t} |\varphi|^2$$

for some positive constant $C(\delta_1, \delta_2, \alpha)$ depending on the data of the problem and α, but independent of β. Now, it clear that the last two terms on the right-hand side can be incorporated in the corresponding ones on the left provided to choose and fix $\delta_1, \delta_2 > 0$ such that

$$\delta_1 + \frac{\alpha(a^* + b^*)^2 K_0^2}{2\delta_2} < 1, \qquad 3\delta_2 < C_0.$$

An elementary computation shows that this is possible if and only if

$$\frac{\alpha(a^* + b^*)^2 K_0^2}{2} < \frac{C_0}{3},$$

which is indeed guaranteed since $\alpha < \alpha_0$ and by the smallness assumption on α_0. The thesis then follows by the Gronwall lemma. □

4.2 Strong Well-posedness

We then aim at investigating regularity properties of the solutions, and prove existence of strong solutions as well as the separation result from the potential barriers.

Theorem 4.3 (Regularity: $\alpha, \beta > 0$)**.** *Assume* **B1–B5***, let $\alpha \in (0, \alpha_0)$ and $\beta \in (0, \beta_0)$. Moreover, let the triplet of initial data $(\varphi_0, \mu_0, \sigma_0)$ satisfy* (4.7) *and also*

$$\exists\, \xi_0 \in H : \ \xi_0 \in \partial F_1(\varphi_0) \ \text{ a.e. in } \Omega, \qquad \mu_0, \sigma_0 \in V, \tag{4.42}$$

and suppose that $t = 0$ is a Lebesgue point for σ_S with

$$\sigma_S(0) \in H. \tag{4.43}$$

Then, the solution $(\varphi, \mu, \sigma, \xi)$ to (4.8)–(4.13) *given by Theorem 4.1 satisfies*

$$\varphi \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V), \tag{4.44}$$

$$\mu, \sigma - \eta\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{4.45}$$

$$\sigma \in H^1(0,T;H) \cap L^\infty(0,T;V). \tag{4.46}$$

Proof of Theorem 4.3. To begin with, we improve the regularity of φ and σ by showing that the approximate solutions $(\varphi_\lambda, \mu_\lambda, \sigma_\lambda)$ to the system (4.28)–(4.31) satisfy further estimates uniformly in λ. We proceed formally, to avoid a further regularisation on the system based on time discretisations. First, we analyse the system (4.28)–(4.31) at the initial time $t = 0$ and let us claim that there exists a unique pair $(\partial_t\varphi_\lambda(0), \partial_t\mu_\lambda(0), \partial_t\sigma_\lambda(0)) \in H \times V^* \times V^*$ such that, in Ω,

$$\begin{cases} \alpha\partial_t\mu_\lambda(0) + \partial_t\varphi_\lambda(0) - \Delta\mu_0 = (\mathcal{P}\sigma_0 - \mathcal{A})f(\varphi_0), \\ \mu_0 = \beta\partial_t\varphi_\lambda(0) + a\varphi_0 - J * \varphi_0 + F'_\lambda(\varphi_0) - \chi\sigma_0, \\ \partial_t\sigma_\lambda(0) - \Delta\sigma_0 + \mathcal{B}(\sigma_0 - \sigma_S(0)) + \mathcal{C}\sigma_0 f(\varphi_0) = -\eta\Delta\varphi_0. \end{cases}$$

Indeed, the existence and uniqueness of $\partial_t\sigma_\lambda(0)$ is given by the third equation and the assumptions (4.7), (4.42) and (4.43). It follows directly then from the second equation the unique definition for $\partial_t\varphi_\lambda(0)$, and finally from the first equation the one of

$\partial_t\mu_\lambda(0)$. Furthermore, from the second equation and assumption (4.42) it follows that $\{\partial_t\varphi_\lambda(0)\}_\lambda$ is uniformly bounded in H, which in turn yields that $\{\partial_t\mu_\lambda(0)\}_\lambda$ is uniformly bounded in V^*.

Bearing this in mind, we test (4.28) by $\partial_t\mu_\lambda$, the time derivative of (4.29) by $-\partial_t\varphi_\lambda$, (4.30) by $\partial_t(\sigma_\lambda - \eta\varphi_\lambda)$, and take the sum: after integrating in time we obtain

$$\begin{aligned}
\alpha \int_{Q_t} &|\partial_t\mu_\lambda|^2 + \frac{1}{2}\|\nabla\mu_\lambda(t)\|^2 + \frac{\beta}{2}\|\partial_t\varphi_\lambda(t)\|^2 + \int_{Q_t}(a + F''_\lambda(\varphi_\lambda))|\partial_t\varphi_\lambda|^2 \\
&+ \int_{Q_t}|\partial_t\sigma_\lambda|^2 + \frac{1}{2}\|\nabla(\sigma_\lambda - \eta\varphi_\lambda)(t)\|^2 \\
&= \frac{1}{2}\|\nabla\mu_0\|^2 + \frac{\beta}{2}\|\partial_t\varphi_\lambda(0)\|^2 + \frac{1}{2}\|\nabla(\sigma_0 - \eta\varphi_0)\|^2 + \int_{Q_t}(\mathcal{P}\sigma_\lambda - \mathcal{A})f(\varphi_\lambda)\partial_t\mu_\lambda \\
&+ \int_{Q_t}\big(J * (\partial_t\varphi_\lambda) + (\eta + \chi)\partial_t\sigma_\lambda\big)\partial_t\varphi_\lambda \\
&+ \int_{Q_t}\big(\mathcal{B}(\sigma_S - \sigma_\lambda) - Cf(\varphi_\lambda)\sigma_\lambda\big)(\partial_t\sigma_\lambda - \eta\partial_t\varphi_\lambda).
\end{aligned} \tag{4.47}$$

Now, the second term on the right-hand side is uniformly bounded in λ thanks to the remarks above, and so is the first one by assumption. Hence, recalling again **B5**, we infer that

$$\begin{aligned}
\alpha \int_{Q_t} &|\partial_t\mu_\lambda|^2 + \frac{1}{2}\|\nabla\mu_\lambda(t)\|^2 + \frac{\beta}{2}\|\partial_t\varphi_\lambda(t)\|^2 + C_0\int_{Q_t}|\partial_t\varphi|^2 \\
&+ \int_{Q_t}|\partial_t\sigma_\lambda|^2 + \frac{1}{2}\|\nabla(\sigma_\lambda - \eta\varphi_\lambda)(t)\|^2 \\
&\leq C + \frac{\alpha}{2}\int_{Q_t}|\partial_t\mu_\lambda|^2 + \frac{1}{2\alpha}\int_{Q_t}|(\mathcal{P}\sigma_\lambda - \mathcal{A})f(\varphi)|^2 + \frac{1}{2}\int_{Q_t}|\partial_t\sigma_\lambda|^2 \\
&+ (a^* + (\eta + \chi)^2 + \frac{\eta^2}{2})\int_{Q_t}|\partial_t\varphi_\lambda|^2 + \frac{3}{2}\int_{Q_t}|\mathcal{B}(\sigma_S - \sigma_\lambda) - Cf(\varphi_\lambda)\sigma_\lambda|^2.
\end{aligned}$$

Taking the estimate (4.32) into account and using the boundedness of f and σ_S we infer that

$$\begin{aligned}
&\|\varphi_\lambda\|^2_{W^{1,\infty}(0,T;H)} + \|\mu_\lambda\|^2_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\sigma_\lambda\|^2_{H^1(0,T;H)} \\
&\quad + \|\sigma_\lambda - \eta\varphi_\lambda\|^2_{L^\infty(0,T;V)} \leq C
\end{aligned}$$

for some $C > 0$ independent of λ. As we already know that $\{\varphi_\lambda\}_\lambda$ is uniformly bounded in $L^\infty(0,T;V)$ by (4.32), it is now a standard matter to pass to the limit as $\lambda \to 0$. Recalling then (4.8)–(4.9) and using a comparison argument for the linear combination $\sigma - \eta\varphi$, we have

$$\begin{aligned}
\varphi &\in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V), \\
\mu, \sigma - \eta\varphi &\in H^1(0,T;H) \cap L^\infty(0,T;V), \\
\sigma &\in H^1(0,T;H) \cap L^2(0,T;V).
\end{aligned}$$

Moreover, note that (4.1) and (4.3) can be rewritten as

$$\alpha\partial_t\mu - \Delta\mu = f_\mu := (\mathcal{P}\sigma - \mathcal{A})f(\varphi) - \partial_t\varphi \qquad \text{in } Q, \quad (4.48)$$
$$\partial_t(\sigma - \eta\varphi) - \Delta(\sigma - \eta\varphi) = f_\sigma := -\mathcal{B}(\sigma - \sigma_S) - \mathcal{C}\sigma f(\varphi) - \eta\partial_t\varphi \quad \text{in } Q, \quad (4.49)$$

endowed with homogeneus Neumann boundary conditions and initial data $\mu_0, \sigma_0 - \eta\varphi_0 \in V$. Since the forcing terms satisfy $f_\mu, f_\sigma \in L^2(0,T;H)$, the classical parabolic regularity theory yields

$$\mu, \sigma - \eta\varphi \in L^2(0,T;W),$$

completing the proof of Theorem 4.3. □

Our next result is concerned with the separation property, magnitude regularity, and existence of strong solutions. In this direction, we postulate the following assumptions for F and J.

B6 Setting $(-\ell, \ell) := \operatorname{Int} D(\partial F_1)$, with $\ell \in [0, +\infty]$, we assume that

$$F \in C^4(-\ell,\ell), \qquad \lim_{s\to(\pm\ell)^\mp}[F'(s) - \chi\eta s] = \pm\infty.$$

It is worth pointing out that **B6** excludes potentials F of double-obstacle type as in (1.9). Nevertheless, the logarithmic potential (1.8) and any polynomial super-quadratic potential as (1.7) is allowed.

As for the kernel, a natural requirement from the analytical point of view is to require

$$J \in W^{2,1}(\mathcal{B}_R), \quad \text{where } \mathcal{B}_R := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| < R := \operatorname{diam}(\Omega)\}, \quad R > 0. \quad (4.50)$$

However, this condition prevents some relevant cases of kernels such as the Newtonian or the Bessel potential from being considered if $d = 3$. Following the ideas of [75, 86] (see also [14, Def. 1]), it is possible to cover also these situations by replacing the above condition by assuming that J is *admissible* in the following sense.

Definition 4.4. *A convolution kernel $J \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ is admissible if it satisfies:*

- *$J \in C^3(\mathbb{R}^d \setminus \{0\})$.*
- *J is radially symmetric, i.e., $J(\cdot) = \widetilde{J}(|\cdot|)$ for a non-increasing $\widetilde{J} : \mathbb{R}_+ \to \mathbb{R}$.*
- *There exists $R_0 > 0$ such that $s \mapsto \widetilde{J}''(s)$ and $s \mapsto \widetilde{J}'(s)/s$ are monotone on $(0, R_0)$.*
- *There exists $C_d > 0$ such that $|D^3 J(\mathbf{x})| \le C_d|\mathbf{x}|^{-d-1}$ for every $\mathbf{x} \in \mathbb{R}^d \setminus \{0\}$.*

Let us point out that the above definition ensures the kernel J to be radially symmetric and non-repulsive and, moreover, both the Newtonian and Bessel potentials do verify the above conditions for $d \in \{2, 3\}$. Thus, we require:

B7 J satisfies (4.50) or it is admissible in the sense of Definition 4.4.

Theorem 4.5 (Existence of strong solutions, separation property: $\alpha, \beta > 0$)**.** *Assume conditions* **B1–B7***, and let* $\alpha \in (0, \alpha_0)$ *and* $\beta \in (0, \beta_0)$*. Let the initial data* $(\varphi_0, \mu_0, \sigma_0)$ *satisfy* (4.7)*,* (4.42)*, and also*

$$\varphi_0 \in H^2(\Omega), \quad \mu_0, \sigma_0 \in L^\infty(\Omega), \quad \exists\, s_0 \in (0, \ell) : \ \|\varphi_0\|_{L^\infty(\Omega)} \le s_0. \tag{4.51}$$

Then, the solution $(\varphi, \mu, \sigma, \xi)$ *to* (4.8)–(4.13) *given by Theorems 4.1 and 4.3 satisfies*

$$\varphi \in W^{1,\infty}(0, T; V) \cap H^1(0, T; H^2(\Omega)), \tag{4.52}$$
$$\partial_t \varphi \in L^\infty(Q), \quad \eta\varphi \in L^2(0, T; W), \tag{4.53}$$
$$\exists\, s^* \in (s_0, \ell) : \quad \sup_{t \in [0,T]} \|\varphi(t)\|_{L^\infty(\Omega)} \le s^*, \tag{4.54}$$
$$\mu, \sigma \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \cap L^\infty(Q). \tag{4.55}$$

In particular, equations (4.1)–(4.3) *hold almost everywhere in* Q*.*

Remark 4.6. (i) *Note that the equation* (4.2) *at time* 0 *reads as*

$$\mu_0 = \beta \partial_t \varphi(0) + a\varphi_0 - J * \varphi_0 + F'(\varphi_0) - \chi\sigma_0,$$

where $\partial_t \varphi(0)$ *"represents" the initial value of the time-derivative of* φ*. Under the assumptions* (4.7)*,* (4.42)*, and* (4.51) *we have that* $\partial_t\varphi(0) \in V \cap L^\infty(\Omega)$*, so that the improved regularities* $\partial_t\varphi \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega))$ *and* $\partial_t\varphi \in L^\infty(Q)$ *obtained in Theorem 4.5 are naturally expectable.*
(ii) *Let us point out that, in the case of* $F_{\log}$*,* (4.51) *prevents the initial tumour distribution to possess any region occupied by solely tumorous cells. In this setting the best one can do is to invoke some approximation argument.*

Proof of Theorem 4.5. Let us remark that the separation result will allow us to exploit the regularity of the linear combination $\sigma - \eta\varphi$ to derive further regularity for φ and σ separately.

First of all, by virtue of Theorem 4.3 we realise that (4.48) consists of a parabolic equation in the variable μ with source term $f_\mu \in L^\infty(0, T; H)$, and with initial datum $\mu_0 \in L^\infty(\Omega)$ by (4.51). Therefore, an application of [114, Thm. 7.1, p. 181] yields that

$$\mu \in L^\infty(Q).$$

In a similar fashion, we notice that in (4.49) we have initial datum $\sigma_0 - \eta\varphi_0 \in V \cap L^\infty(\Omega)$ and forcing term $f_\sigma \in L^\infty(0, T; H)$ by virtue of Theorem 4.3. Hence, again an application of [114, Thm. 7.1, p. 181] produces

$$\sigma - \eta\varphi \in L^\infty(Q).$$

Furthermore, we claim that owing to assumption **B7**, we can deduce further regularity also for the convolution term $J*\varphi$. Indeed, every kernel verifying Definition 4.4 satisfy the following result, whose proof can be found, e.g., in [14, Lemma 2].

Lemma 4.7. *Assume that the kernel J is admissible in the sense of the Definition 4.4. Then, for every $p \in (1, \infty)$, there exists a constant $C_p > 0$ such that*

$$\|\mathrm{div}(\nabla J * \psi)\|_{L^p(\Omega)^{d\times d}} \leq C_p \|\psi\|_{L^p(\Omega)} \quad \forall\, \psi \in L^p(\Omega). \tag{4.56}$$

As a consequence, by taking $p = 2$ in the formula (4.56), we deduce that

$$\|J * \varphi\|_{L^\infty(0,T;H^2(\Omega))} \leq C_2 \|\varphi\|_{L^\infty(0,T;H)},$$

which readily implies, thanks to the continuous inclusion $H^2(\Omega) \subset L^\infty(\Omega)$, that

$$J * \varphi \in L^\infty(0, T; H^2(\Omega)) \cap L^\infty(Q).$$

We are now ready to prove the separation property. To this end, note that, taking these remarks into account, under the assumption **B6** on F, we can rewrite equation (4.11) as

$$\beta \partial_t \varphi + a\varphi + F'(\varphi) - \chi\eta\varphi = f_\varphi := \mu + \chi(\sigma - \eta\varphi) + J * \varphi. \tag{4.57}$$

Besides, we have already proved that $f_\varphi \in L^2(0, T; H^2(\Omega)) \cap L^\infty(Q)$, so that there exists a constant $\overline{C} > 0$ such that

$$\|f_\varphi\|_{L^\infty(Q)} \leq \overline{C}.$$

Next, by **B6** and (4.51) we infer the existence of $s^* \in (s_0, \ell)$ such that

$$F'(s) - \chi\eta s \geq \overline{C} \quad \forall\, s \in (s^*, \ell), \qquad F'(s) - \chi\eta s \leq -\overline{C} \quad \forall\, s \in (-\ell, -s^*).$$

We claim that this choice entails $\varphi(t) \leq s^*$ almost everywhere in Ω, for all $t \in [0, T]$. In fact, by testing (4.57) by $(\varphi - s^*)_+$ and integrating on $[0, t]$, we immediately infer that

$$\frac{\beta}{2}\|(\varphi(t) - s^*)_+\|^2 + \int_{Q_t} a\varphi(\varphi - s^*)_+ = \frac{\beta}{2}\|(\varphi_0 - s^*)_+\|^2 + \int_{Q_t} [f_\varphi - (F'(\varphi) - \chi\eta\varphi)](\varphi - s^*)_+.$$

Now, since $s^* \in (s_0, \ell)$ and $\|\varphi_0\|_{L^\infty(\Omega)} \leq s_0$, the first term on the right-hand side vanishes. Moreover, by definition of $\overline{C}$ and s^* we have that

$$\int_{Q_t} [f_\varphi - (F'(\varphi) - \chi\eta\varphi)](\varphi - s^*)_+ = \int_{Q_t \cap \{\varphi > s^*\}} [f_\varphi - (F'(\varphi) - \chi\eta\varphi)](\varphi - s^*) \leq 0.$$

Recalling also **B5**, we infer that, for every $t \in [0, T]$,

$$\frac{\beta}{2}\|(\varphi(t) - s^*)_+\|^2 + a_* \int_{Q_t \cap \{\varphi > s^*\}} \varphi(\varphi - s^*) \leq 0.$$

Hence, since the second term on the left-hand side is non-negative, we deduce that

$$(\varphi(t) - s^*)_+ = 0 \quad \forall\, t \in [0, T], \quad \text{i.e.,} \quad \varphi(\mathbf{x}, t) \leq s^* \quad \text{for a.e. } \mathbf{x} \in \Omega \quad \forall\, t \in [0, T],$$

as required. The other inequality $\varphi \geq -s^*$ can be deduced analogously by testing by $-(\varphi+s^*)_-$ instead. Thus, we have shown that

$$\sup_{t\in[0,T]} \|\varphi(t)\|_{L^\infty(\Omega)} \leq s^*, \qquad \text{with } s^* \in (s_0, \ell).$$

Let us now show the $L^2(0,T;W)$-regularity for σ and $\eta\varphi$. To this end, we test the gradient of equation (4.57) by $|\nabla\varphi|^{p-2}\nabla\varphi$ and integrate over Q_t to obtain, by assumption **B5**, the Hölder inequality and the generalised Young inequality (2.2), that

$$\begin{aligned}
&\frac{\beta}{p}\sup_{s\in[0,t]} \|\nabla\varphi(s)\|^p_{L^p(\Omega)} + C_0\int_{Q_t} |\nabla\varphi|^p \\
&\quad = \frac{\beta}{p}\|\nabla\varphi_0\|^p_{L^p(\Omega)} + \chi\eta\int_{Q_t} |\nabla\varphi|^p - \int_{Q_t} (\nabla a)\varphi\,|\nabla\varphi|^{p-2}\nabla\varphi \\
&\qquad + \int_{Q_t} \nabla f_\varphi \cdot |\nabla\varphi|^{p-2}\nabla\varphi \\
&\quad \leq \frac{\beta}{p}\|\nabla\varphi_0\|^p_{L^p(\Omega)} + \chi\eta\int_{Q_t} |\nabla\varphi|^p + \frac{\beta}{2p}\sup_{s\in[0,t]} \|\nabla\varphi(s)\|^p_{L^p(\Omega)} \\
&\qquad + \frac{[4(p-1)]^{p-1}(b^*)^p}{p\beta^{p-1}}\|\varphi\|^p_{L^1(0,T;L^p(\Omega))} + \frac{[4(p-1)]^{p-1}}{p\beta^{p-1}}\|\nabla f_\varphi\|^p_{L^1(0,T;L^p(\Omega))}.
\end{aligned}$$

Owing to the already proved regularities $f_\varphi \in L^2(0,T;H^2(\Omega))$ and $\varphi \in L^2(0,T;V)$, we deduce in particular that $\nabla f_\varphi \in L^2(0,T;V)$ so that, using the continuous embedding $V \subset L^6(\Omega)$, also $\nabla f_\varphi, \varphi \in L^2(0,T;L^6(\Omega))$. Moreover, $\varphi_0 \in H^2(\Omega)$ also entails that $\nabla\varphi_0 \in L^6(\Omega)$. Choosing then $p = 6$ and using the Gronwall lemma yields

$$\varphi \in L^\infty(0,T;W^{1,6}(\Omega)). \tag{4.58}$$

For brevity we proceed formally: a rigorous argument can be reproduced on suitable approximations. Applying the second-order differential operator $\partial_{\mathbf{x}_i\mathbf{x}_j}$ $(i,j = 1,...,d)$ to equation (4.57), testing it by $\partial_{\mathbf{x}_i\mathbf{x}_j}\varphi$, and integrating on $[0,t]$ lead to

$$\begin{aligned}
&\frac{\beta}{2}\|\partial_{\mathbf{x}_i\mathbf{x}_j}\varphi(t)\|^2 + \int_{Q_t} (a + F''(\varphi))|\partial_{\mathbf{x}_i\mathbf{x}_j}\varphi|^2 \\
&\quad = \frac{\beta}{2}\|\partial_{\mathbf{x}_i\mathbf{x}_j}\varphi_0\|^2 + \int_{Q_t} \partial_{\mathbf{x}_i\mathbf{x}_j} f_\varphi \partial_{\mathbf{x}_i\mathbf{x}_j}\phi + \chi\eta\int_{Q_t} |\partial_{\mathbf{x}_i\mathbf{x}_j}\varphi|^2 \\
&\qquad - \int_{Q_t} [\partial_{\mathbf{x}_i} a\partial_{\mathbf{x}_j}\varphi + \partial_{\mathbf{x}_j} a\partial_{\mathbf{x}_i}\varphi + (\partial_{\mathbf{x}_i\mathbf{x}_j} a)\varphi + F^{(3)}(\varphi)\partial_{\mathbf{x}_i}\varphi\partial_{\mathbf{x}_j}\varphi]\partial_{\mathbf{x}_i\mathbf{x}_j}\varphi.
\end{aligned}$$

Now, due to the already proved separation property $\|\varphi\|_{L^\infty(Q)} \leq s^* < \ell$, and recalling that $F \in C^3(-\ell,\ell)$ by **B6**, we have that $F^{(3)}(\varphi) \in L^\infty(Q)$. Hence, exploiting **B5**, using the Young inequality, and summing on $i,j = 1,...,d$ we deduce, recalling that

$\varphi \in L^2(0,T;V)$, that

$$\begin{aligned}
&\frac{\beta}{2}\|\varphi(t)\|_{H^2(\Omega)}^2 + C_0\int_0^t \|\varphi(s)\|_{H^2(\Omega)}^2\,\mathrm{ds}\\
&\quad\le \frac{\beta}{2}\|\varphi_0\|_{H^2(\Omega)}^2 + (2+\chi\eta)\int_0^t \|\varphi(s)\|_{H^2(\Omega)}^2\,\mathrm{ds}\\
&\quad + \frac{1}{2}\|f_\varphi\|_{L^2(0,T;H^2(\Omega))}^2 + 2\int_{Q_t}|\nabla a|^2|\nabla\varphi|^2 + \frac{1}{2}\int_{Q_t}\sum_{i,j=1}^d |\partial_{\mathbf{x}_i\mathbf{x}_j}a|^2|\varphi|^2\\
&\quad + \frac{1}{2}\|F^{(3)}(\varphi)\|_{L^\infty(Q)}^2\int_{Q_t}|\nabla\varphi|^4.
\end{aligned}$$

Moreover, as $\|\nabla a\|_{L^\infty(\Omega)} \le b^*$ by **B5**, $\|a\|_{W^{2,p}(\Omega)} \le C_p$ for all $p \in (1,+\infty)$ by (4.56) and $\varphi \in L^\infty(0,T;V)$, the Hölder inequality yields

$$\int_{Q_t}|\nabla a|^2|\nabla\varphi|^2 \le (b^*)^2\|\varphi\|_{L^2(0,T;V)}^2 \le C$$

and, by the continuous embedding $V \subset L^4(\Omega)$, also that

$$\int_{Q_t}\sum_{i,j=1}^d |\partial_{\mathbf{x}_i\mathbf{x}_j}a|^2|\varphi|^2 \le \|a\|_{W^{2,4}(\Omega)}^2\|\varphi\|_{L^4(0,T;L^4(\Omega))}^2 \le C\|\varphi\|_{L^4(0,T;V)}^2 \le C.$$

Using then (4.58), we are left with

$$\frac{\beta}{2}\|\varphi(t)\|_{H^2(\Omega)}^2 + C_0\int_0^t\|\varphi(s)\|_{H^2(\Omega)}^2\,\mathrm{ds} \le \frac{\beta}{2}\|\varphi_0\|_{H^2(\Omega)}^2 + C(1+\int_0^t\|\varphi(s)\|_{H^2(\Omega)}^2\,\mathrm{ds})$$

so that a Gronwall argument produces

$$\varphi \in L^\infty(0,T;H^2(\Omega)).$$

At this point, the equation for σ can be written also as

$$\partial_t\sigma - \Delta\sigma = f_\sigma := -\mathcal{B}(\sigma-\sigma_S) - \mathcal{C}\sigma f(\varphi) - \eta\Delta\varphi \in L^\infty(0,T;H),$$

with initial datum $\sigma_0 \in V \cap L^\infty(\Omega)$. Hence, by parabolic regularity theory and again [114, Thm. 7.1], we deduce that

$$\sigma \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap L^\infty(Q).$$

Since we already know that $\sigma - \eta\varphi \in L^2(0,T;W)$, by comparison, we also infer

$$\eta\varphi \in L^2(0,T;W).$$

To conclude, we go back to equation (4.57) and note that, by difference, we have also the regularity

$$\partial_t\varphi \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(Q)$$

which completes the proof of Theorem 4.5. □

Relying on the extra-regularity and the separation property, we can now show a refined continuous dependence result for strong solutions, where the stability estimates are verified in stronger topologies. Let us stress that in this case, we can include in the analysis both the chemotaxis and the active transport mechanisms, complementing thus the previous Theorem 4.2.

Theorem 4.8 (Refined continuous dependence: $\alpha, \beta > 0$)**.** *Assume* **B1–B7***, and let* $\alpha \in (0, \alpha_0)$ *and* $\beta \in (0, \beta_0)$*. Then for any pair of initial data* $\{(\varphi_0^i, \mu_0^i, \sigma_0^i)\}_i$*,* $i = 1, 2$*, satisfying* (4.7)*,* (4.42)*, and* (4.51)*, there exists a constant* $K > 0$ *such that, for any respective strong solutions* $\{(\varphi_i, \mu_i, \sigma_i)\}_i$ *obtained from Theorem 4.5,* $i = 1, 2$*, it holds that*

$$\begin{aligned}
&\|\mu_1 - \mu_2\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} + \|\varphi_1 - \varphi_2\|_{W^{1,\infty}(0,T;V)\cap H^1(0,T;H^2(\Omega))} \\
&\qquad + \|\sigma_1 - \sigma_2\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \\
&\quad \le K(\|\mu_0^1 - \mu_0^2\|_V + \|\varphi_0^1 - \varphi_0^2\|_{H^2(\Omega)} + \|\sigma_0^1 - \sigma_0^2\|_V),
\end{aligned} \tag{4.59}$$

where K *only depends on* $\Omega, T, \alpha, \beta, \mathcal{P}, \mathcal{A}, \mathcal{B}, \mathcal{C}, C_0, a_*, a^*, b^*, s^*, \|F\|_{C^4([-s^*,s^*])}$ *and the initial data* $\{(\varphi_0^i, \mu_0^i, \sigma_0^i)\}_{i=1,2}$.

In turn, the uniqueness of strong solutions in the sense of Theorem 4.5 holds.

Proof of Theorem 4.8. Employing the same notation of the proof of Theorem 4.2, we consider the system (4.36)–(4.40) and test (4.36) by $\partial_t\mu$, the time-derivative of (4.37) by $-\partial_t\varphi$, (4.38) by $\partial_t(\sigma - \eta\varphi)$, and integrate over $[0, t]$, to obtain

$$\begin{aligned}
&\alpha \int_{Q_t} |\partial_t\mu|^2 + \frac{1}{2}\|\nabla\mu(t)\|^2 + \frac{\beta}{2}\|\partial_t\varphi(t)\|^2 + \int_{Q_t} (a + F''(\varphi_1))|\partial_t\varphi|^2 \\
&\qquad + \int_{Q_t} |\partial_t\sigma|^2 + \frac{1}{2}\|\nabla(\sigma - \eta\varphi)(t)\|^2 \\
&\quad = \frac{1}{2}\|\nabla\mu_0\|^2 + \frac{\beta}{2}\|\partial_t\varphi(0)\|^2 + \frac{1}{2}\|\nabla(\sigma_0 - \eta\varphi_0)\|^2 \\
&\qquad + \int_{Q_t} [\mathcal{P}\sigma f(\varphi_1) + (\mathcal{P}\sigma_2 - \mathcal{A})(f(\varphi_1) - f(\varphi_2))]\partial_t\mu \\
&\qquad + \int_{Q_t} [(F''(\varphi_2) - F''(\varphi_1))\partial_t\varphi_2 + \chi\partial_t\sigma + J * \partial_t\varphi]\partial_t\varphi + \eta\int_{Q_t} \partial_t\sigma\partial_t\varphi \\
&\qquad + \int_{Q_t} [\mathcal{C}\sigma_2(f(\varphi_2) - f(\varphi_1)) - \mathcal{C}\sigma f(\varphi_1) - \mathcal{B}\sigma](\partial_t\sigma - \eta\partial_t\varphi).
\end{aligned}$$

First of all, notice that $\partial_t\varphi(0)$ is such that

$$\mu_0 = \beta\partial_t\varphi(0) + a\varphi_0 - J * \varphi_0 + F'(\varphi_0^1) - F'(\varphi_0^2) - \chi\sigma_0.$$

Since the initial data satisfy (4.7), (4.42), and (4.51), for $i = 1, 2$ we have that $\partial_t\varphi(0) \in V \cap L^\infty(\Omega)$. Now, recalling that $F \in C^3([-s_0, s_0])$, we have

$$\|\partial_t\varphi(0)\| \le \frac{1}{\beta}(\|\mu_0\| + 2a^*\|\varphi_0\| + \|F''\|_{C^0([-s_0,s_0])}\|\varphi_0\| + \chi\|\sigma_0\|).$$

Secondly, by the separation property for φ_1 and φ_2, we have $\|\varphi_i\|_{L^\infty(Q)} \leq s^* < \ell$ for $i = 1, 2$ and combined with $F \in C^3([-s^*, s^*])$ we have $F'' \in W^{1,\infty}(-s^*, s^*)$, so that

$$|F''(\varphi_1) - F''(\varphi_2)| \leq \|F^{(3)}\|_{C^0([-s^*,s^*])}|\varphi_1 - \varphi_2| \quad \text{a.e. in } Q.$$

Taking this information into account, using **B5**, and exploiting the regularities $f \in W^{1,\infty}(\mathbb{R})$, $\sigma_2 \in L^\infty(Q)$, and $\partial_t\varphi_2 \in L^\infty(Q)$, we invoke the Young inequality to infer

$$\begin{aligned}\int_{Q_t} |\partial_t\mu|^2 + \|\nabla\mu(t)\|^2 + \|\partial_t\varphi(t)\|^2 + \int_{Q_t} |\partial_t\varphi|^2 + \int_{Q_t} |\partial_t\sigma|^2 + \|\nabla(\sigma - \eta\varphi)(t)\|^2 \\ \leq C(\|\mu_0\|_V^2 + \|\varphi_0\|^2 + \|\sigma_0\|^2 + \|\nabla(\sigma_0 - \eta\varphi_0)\|^2 \\ + \int_{Q_t} (|\sigma|^2 + |\varphi|^2 + |\partial_t\varphi|^2)),\end{aligned} \tag{4.60}$$

where the constant $C > 0$ may depend on α, β and on structural data. Now, we take the gradient of (4.37) and test it by $\nabla\varphi$, getting

$$\begin{aligned}&\frac{\beta}{2}\|\nabla\varphi(t)\|^2 + \int_{Q_t} (a + F''(\varphi_1))|\nabla\varphi|^2 \\ &= \frac{\beta}{2}\|\nabla\varphi_0\|^2 + \int_{Q_t} (F''(\varphi_2) - F''(\varphi_1))\nabla\varphi_2 \cdot \nabla\varphi \\ &\quad + \int_{Q_t} (\nabla\mu + \chi\nabla\sigma + (\nabla J) * \varphi - (\nabla a)\varphi) \cdot \nabla\varphi.\end{aligned}$$

Using **B5**, along with the Lipschitz continuity of F'' on $[-s^*, s^*]$, the identity

$$\chi\nabla\sigma \cdot \nabla\varphi = \chi(\nabla(\sigma - \eta\varphi) + \eta\nabla\varphi) \cdot \nabla\varphi,$$

and the Young inequality lead to

$$\begin{aligned}\frac{\beta}{2}\|\nabla\varphi(t)\|^2 + C_0\int_{Q_t} |\nabla\varphi|^2 \leq \frac{\beta}{2}\|\nabla\varphi_0\|^2 + \|F^{(3)}\|_{C^0([-s^*,s^*])}\int_{Q_t} |\varphi||\nabla\varphi_2||\nabla\varphi| \\ + \int_{Q_t} |\nabla\mu|^2 + \chi^2\int_{Q_t} |\nabla(\sigma - \eta\varphi)|^2 \\ + (1 + \chi\eta)\int_{Q_t} |\nabla\varphi|^2 + 2(b^*)^2\int_{Q_t} |\varphi|^2.\end{aligned}$$

From the embedding $V \subset L^4(\Omega)$, Hölder's inequality and the $\varphi_2 \in L^\infty(0, T; H^2(\Omega))$, we find

$$\int_{Q_t} |\varphi||\nabla\varphi_2||\nabla\varphi| \leq C\int_0^t \|\varphi(s)\|_V\|\varphi_2(s)\|_{H^2(\Omega)}\|\nabla\varphi(s)\| \,\mathrm{ds} \leq C\int_0^t \|\varphi(s)\|_V^2 \,\mathrm{ds}.$$

We deduce then that, for every $t \in [0, T]$,

$$\|\nabla\varphi(t)\|^2 \leq C(\|\nabla\varphi_0\|^2 + \int_{Q_t} |\nabla\mu|^2 + \int_{Q_t} |\nabla(\sigma - \eta\varphi)|^2 + \int_0^t \|\varphi(s)\|_V^2 \,\mathrm{ds}). \tag{4.61}$$

Collecting (4.60) and (4.61), we infer that, for all $t \in [0,T]$,

$$\int_{Q_t} |\partial_t \mu|^2 + \|\nabla \mu(t)\|^2 + \|\partial_t \varphi(t)\|^2 + \|\nabla \varphi(t)\|^2 + \int_{Q_t} |\partial_t \sigma|^2 + \|\nabla \sigma(t)\|^2$$
$$\leq C(\|\mu_0\|_V^2 + \|\varphi_0\|_V^2 + \|\sigma_0\|_V^2)$$
$$+ \int_0^t (\|\nabla \mu(s)\|^2 + \|\sigma(s)\|_V^2 + \|\varphi(s)\|_V^2 + \|\partial_t \varphi(s)\|^2)\,\mathrm{ds}.$$

Since the quantities $\|\sigma_2\|_{L^\infty(Q)}$, $\|\partial_t \varphi_2\|_{L^\infty(Q)}$, and $\|\varphi_2\|_{L^\infty(0,T;H^2(\Omega))}$ appearing implicitly in the constant C can be in turn handled in terms on the norms of the initial data appearing in (4.7), (4.42), and (4.51), we can close the estimate by the Gronwall lemma. Moreover, comparison in equation (4.36) produces

$$\|\Delta \mu\|_{L^2(0,T;H)} \leq C(\|\varphi\|_{H^1(0,T;H)} + \|\partial_t \mu\|_{L^2(0,T;H)} + \|\sigma\|_{L^2(0,T;H)}),$$

where all the terms on the right-hand side have already been estimated. Similarly, from (4.37) we get

$$\|\partial_t \varphi\|_{L^\infty(0,T;V)} \leq C(\|\mu\|_{L^\infty(0,T;V)} + \|\varphi\|_{L^\infty(0,T;V)} + \|\sigma\|_{L^\infty(0,T;V)}),$$

while from (4.38) we get

$$\|\Delta(\sigma - \eta\varphi)\|_{L^2(0,T;H)} \leq C(\|\sigma\|_{H^1(0,T;H)} + \|\varphi\|_{L^2(0,T;H)}).$$

Collecting the above estimates, along with elliptic regularity theory, we deduce that

$$\|\mu\|^2_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} + \|\varphi\|^2_{W^{1,\infty}(0,T;V)} + \|\sigma\|^2_{H^1(0,T;H)\cap L^\infty(0,T;V)}$$
$$+ \|\sigma - \eta\varphi\|^2_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq C(\|\mu_0\|_V^2 + \|\varphi_0\|_V^2 + \|\sigma_0\|_V^2). \quad (4.62)$$

To complete the proof, we need to show a stability estimate for $\partial_t \varphi$ and σ also in $L^2(0,T;H^2(\Omega))$ and $L^2(0,T;W)$, respectively. In this direction, for any $i,j = 1,...,d$, we apply the differential operator $\partial_{\mathbf{x}_i \mathbf{x}_j}$ to (4.37) and test the obtained equation by $\partial_{\mathbf{x}_i \mathbf{x}_j}\varphi$, getting

$$\frac{\beta}{2}\|\partial_{\mathbf{x}_i \mathbf{x}_j}\varphi(t)\|^2 + \int_{Q_t} (a + F''(\varphi_1))|\partial_{\mathbf{x}_i \mathbf{x}_j}\varphi|^2$$
$$= \frac{\beta}{2}\|\partial_{\mathbf{x}_i \mathbf{x}_j}\varphi_0\|^2 + \int_{Q_t} \partial_{\mathbf{x}_i \mathbf{x}_j}(\mu + \chi(\sigma - \eta\varphi) + J * \varphi)\partial_{\mathbf{x}_i \mathbf{x}_j}\varphi$$
$$+ \chi\eta \int_{Q_t} |\partial_{\mathbf{x}_i \mathbf{x}_j}\varphi|^2 - \int_{Q_t} (\partial_{\mathbf{x}_i} a \partial_{\mathbf{x}_j}\varphi + \partial_{\mathbf{x}_j} a \partial_{\mathbf{x}_i}\varphi + (\partial_{\mathbf{x}_i \mathbf{x}_j} a)\varphi)\partial_{\mathbf{x}_i \mathbf{x}_j}\varphi$$
$$+ \int_{Q_t} [(F''(\varphi_2) - F''(\varphi_1))\partial_{\mathbf{x}_i \mathbf{x}_j}\varphi_2 + (F^{(3)}(\varphi_2) - F^{(3)}(\varphi_1))\partial_{\mathbf{x}_i}\varphi_1 \partial_{\mathbf{x}_j}\varphi_2]\partial_{\mathbf{x}_i \mathbf{x}_j}\varphi$$
$$- \int_{Q_t} [F^{(3)}(\varphi_1)\partial_{\mathbf{x}_i}\varphi_1 \partial_{\mathbf{x}_j}\varphi + F^{(3)}(\varphi_2)\partial_{\mathbf{x}_i}\varphi \partial_{\mathbf{x}_j}\varphi_2]\partial_{\mathbf{x}_i \mathbf{x}_j}\varphi.$$

We recall that, due to **B6**, $F \in C^4([-s^*, s^*])$, so that $F^{(3)}$ is Lipschitz continuous on $[-s^*, s^*]$, and as a consequence of the separation result, also $F^{(3)}(\varphi_i) \in L^\infty(Q)$, for

$i = 1, 2$. Now, we use the Hölder and Young inequalities and sum on $i, j = 1, ..., d$: proceeding as in the proof of Theorem 4.5 and exploiting assumptions **B5** and **B7**, we get

$$\begin{aligned}
&\frac{\beta}{2}\|\varphi(t)\|^2_{H^2(\Omega)} + C_0\int_0^t \|\varphi(s)\|^2_{H^2(\Omega)}\,\mathrm{ds} \\
&\quad\le \frac{\beta}{2}\|\varphi_0\|^2_{H^2(\Omega)} + C(\|\mu\|^2_{L^2(0,T;W)} + \|\sigma-\eta\varphi\|^2_{L^2(0,T;W)} + \int_0^t \|\varphi(s)\|^2_{H^2(\Omega)}\,\mathrm{ds}) \\
&\quad\quad + C\sum_{i,j=1}^d (\int_{Q_t} |\varphi|^2(|\partial_{\mathbf{x}_i\mathbf{x}_j}\varphi_2|^2 + |\partial_{\mathbf{x}_i}\varphi_1|^2|\partial_{\mathbf{x}_j}\varphi_2|^2) \\
&\quad\quad + \int_{Q_t}(|\partial_{\mathbf{x}_i}\varphi_1|^2|\partial_{\mathbf{x}_j}\varphi|^2 + |\partial_{\mathbf{x}_i}\varphi|^2|\partial_{\mathbf{x}_j}\varphi_2|^2)).
\end{aligned}$$

The first bracket on the right-hand side can be controlled using (4.62) and the Gronwall lemma, while the sum-term can be estimated using the Hölder inequality and the continuous inclusions $V \subset L^4(\Omega)$ and $H^2(\Omega) \subset L^\infty(\Omega)$ by

$$\begin{aligned}
&\int_0^t \|\varphi(s)\|^2_{L^\infty(\Omega)}(\|\varphi_2(s)\|^2_{H^2(\Omega)} + \|\nabla\varphi_1(s)\|^2_V\|\nabla\varphi_2(s)\|^2_V)\,\mathrm{ds} \\
&\quad + \int_0^t \|\nabla\varphi(s)\|^2_{L^4(\Omega)}(\|\nabla\varphi_1(s)\|^2_{L^4(\Omega)} + \|\nabla\varphi_2(s)\|^2_{L^4(\Omega)})\,\mathrm{ds} \\
&\le C(\|\varphi_2\|^2_{L^\infty(0,T;H^2(\Omega))} + \|\varphi_1\|^2_{L^\infty(0,T;H^2(\Omega))}\|\varphi_2\|^2_{L^\infty(0,T;H^2(\Omega))})\int_0^t \|\varphi(s)\|^2_{H^2(\Omega)}\,\mathrm{ds} \\
&\quad + C(\|\varphi_1\|^2_{L^\infty(0,T;H^2(\Omega))} + \|\varphi_2\|^2_{L^\infty(0,T;H^2(\Omega))})\int_0^t \|\varphi(s)\|^2_{H^2(\Omega)}\,\mathrm{ds}.
\end{aligned}$$

Taking these estimates into account, and recalling $\varphi_1, \varphi_2 \in L^\infty(0, T; H^2(\Omega))$, we conclude that

$$\begin{aligned}
&\|\varphi(t)\|^2_{H^2(\Omega)} \\
&\quad\le \|\varphi_0\|^2_{H^2(\Omega)} + C\Big(\|\mu\|^2_{L^2(0,T;W)} + \|\sigma-\eta\varphi\|^2_{L^2(0,T;W)} + \int_0^t \|\varphi(s)\|^2_{H^2(\Omega)}\,\mathrm{ds}\Big)
\end{aligned}$$

so that Gronwall's lemma, along with the above estimates, produces

$$\|\varphi\|^2_{L^\infty(0,T;H^2(\Omega))} \le C(\|\mu_0\|^2_V + \|\varphi_0\|^2_{H^2(\Omega)} + \|\sigma_0\|^2_V).$$

The stability estimate for σ in $L^2(0, T; W)$ follows by comparison in (4.38) and elliptic regularity theory. Finally, by comparison in equation (4.37) we also infer the stability estimate for $\partial_t\varphi$ in $L^2(0, T; H^2(\Omega))$, concluding the proof of Theorem 4.8. □

4.3 Vanishing Viscosities Analysis

The second part of the chapter is focused on the study of the asymptotic behaviour of the system (4.1)–(4.5) as $\alpha \to 0$ and/or $\beta \to 0$. These investigations are performed both separately (i.e., $\alpha \to 0$ with $\beta > 0$, and $\beta \to 0$ with $\alpha > 0$) and jointly (i.e., $\alpha, \beta \to 0$). In each of these cases, under suitable conditions, we are able to show convergence of the system to the respective limit problem, hence also the corresponding well-posedness. Also, we give the exact rates of convergence through precise error estimates. Before entering the details, let us briefly mention here the mathematical challenges that we have to overcome.

Passage to the limit as $\boldsymbol{\alpha \to 0}$: In this first asymptotic study, the parabolic regularisation on μ is "removed", resulting in a lack of regularity on the chemical potential. As a consequence, due to the presence of proliferation term in the Cahn–Hilliard equation, a very natural growth condition on the potential has to be required (c.f. (4.63)), allowing for any polynomial or first-order exponential potentials. The passage to the limit, hence the existence for the limit problem with $\alpha = 0$, is proved in the setting of no active transport term, i.e., $\eta = 0$, due to the need of a comparison principle argument for σ (cf. Theorem 4.1). As for the error estimate, and therefore the uniqueness for the limit system, a rate of convergence of order $\alpha^{1/4}$ is obtained by showing refined estimates on the solutions and exploiting a locally-Lipschitz assumption on the potential, still including the classical quartic case (1.7).

Passage to the limit as $\boldsymbol{\beta \to 0}$: In the second passage to the limit, the viscosity of the Cahn–Hilliard equation vanishes, and this results in a loss of temporal regularity on the phase variable φ. Anyhow, the presence of $\alpha > 0$ still allows passing to the limit in very general settings, such as singular potentials, chemotaxis, and active transport, only requiring some compatibility conditions (smallness type assumptions) on the involved constants. The separation from the potential barriers is not preserved though, as it is naturally expectable. Moreover, a corresponding error estimate showing a convergence rate of order $\beta^{1/2}$ is obtained, and therefore the uniqueness for the limit system is guaranteed.

Passage to the limit as $\boldsymbol{\alpha, \beta \to 0}$: In the last passage to the limit, the parameters α and β vanish simultaneously. Here, the convergence is proved by showing some refined estimates on the solutions, depending on both parameters, and combining the assumptions above on the potential and the coefficients. Moreover, the error estimate (and the resulting well-posedness of the limit problem) is obtained with a rate of convergence of $\alpha^{1/4} + \beta^{1/2}$, under a suitable scaling on the two parameters.

4.3.1 Asymptotics Analysis as $\boldsymbol{\alpha \to 0}$

We now will present the results concerning the asymptotic analysis of (4.1)–(4.5) with respect to the parameter α, assuming $\beta > 0$ to be fixed. In this direction, we need to enforce some additional conditions on the potential F. In fact, proceeding with classical estimates, just a bound of $\nabla\mu$ in $L^2(0, T; H)$ can be proved, having no information on the behaviour of μ in $L^2(0, T; H)$. This gap is usually bridged via the application of a Poincaré type inequality, which yields the control of μ in the full space $L^2(0, T; V)$. To this end, some control on the spatial mean of μ is in order: if $\alpha > 0$ is fixed, this automatically follows from the estimates, whereas in the limit $\alpha \to 0$ it has to be obtained from a suitable prescription on potential growth. Namely, the assumption

$$D(\partial F_1) = \mathbb{R}, \qquad \exists\, C_F > 0: \quad |\partial F_1^\circ(s)| \le C_F(F_1(s)+1) \quad \forall\, s \in \mathbb{R}, \tag{4.63}$$

has to be prescribed for F, where $\partial F_1^\circ(s)$ stands for the element of $\partial F_1(s)$ having minimum modulus introduced in Section 2.4.3. This implies that for every $z \in H$ and $w \in \partial F_1(z)$ it holds

$$\int_\Omega |w| \le C_F \int_\Omega (F_1(z)+1).$$

Let us point out that the above requirement is met by all the regular potentials everywhere defined on the real line with polynomial or first-order exponential growth-rate.

Theorem 4.9 (Asymptotics: $\alpha \to 0$)**.** *Assume that* **B1–B5**, *and* (4.63) *hold, and let* $\beta \in (0, \beta_0)$ *and* $\eta = 0$. *Suppose also that*

$$\varphi_{0,\beta} \in V, \qquad \sigma_{0,\beta} \in H, \qquad F(\varphi_{0,\beta}) \in L^1(\Omega). \tag{4.64}$$

For every $\alpha \in (0, \alpha_0)$, *let the initial data* $(\varphi_{0,\alpha,\beta}, \mu_{0,\alpha,\beta}, \sigma_{0,\alpha,\beta})$ *satisfy assumptions* (4.7) *and* (4.14), *and denote by* $(\varphi_{\alpha,\beta}, \mu_{\alpha,\beta}, \sigma_{\alpha,\beta}, \xi_{\alpha,\beta})$ *the respective unique weak solution to the system* (4.1)-(4.5) *obtained from Theorem 4.1. In addition, we assume that, as* $\alpha \to 0$,

$$\varphi_{0,\alpha,\beta} \to \varphi_{0,\beta} \quad \textit{weakly in } V, \qquad \sigma_{0,\alpha,\beta} \to \sigma_{0,\beta} \quad \textit{strongly in } H, \tag{4.65}$$

and

$$\exists\, M_0 > 0: \quad \alpha^{1/2}\|\mu_{0,\alpha,\beta}\| + \|F(\varphi_{0,\alpha,\beta})\|_{L^1(\Omega)} \le M_0 \quad \forall\, \alpha \in (0, \alpha_0). \tag{4.66}$$

Then, there exists a quadruplet $(\varphi_\beta, \mu_\beta, \sigma_\beta, \xi_\beta)$, *with*

$$\begin{aligned}
&\varphi_\beta \in H^1(0,T;H) \cap L^\infty(0,T;V),\\
&\mu_\beta \in L^2(0,T;V),\\
&\sigma_\beta \in H^1(0,T;V^*) \cap L^2(0,T;V) \cap L^\infty(Q),\\
&0 \le \sigma_\beta(\mathbf{x},t) \le 1 \quad \textit{for a.e. } \mathbf{x} \in \Omega, \quad \forall\, t \in [0,T],\\
&\xi_\beta \in L^2(0,T;H),
\end{aligned}$$

such that

$$\begin{aligned}
&\langle \partial_t \varphi_\beta, v\rangle + \int_\Omega \nabla \mu_\beta \cdot \nabla v = \int_\Omega (\mathcal{P}\sigma_\beta - \mathcal{A}) f(\varphi_\beta) v,\\
&\langle \partial_t \sigma_\beta, v\rangle + \int_\Omega \nabla \sigma_\beta \cdot \nabla v + \mathcal{B}\int_\Omega (\sigma_\beta - \sigma_S) v + \mathcal{C}\int_\Omega \sigma_\beta f(\varphi_\beta) v = 0,
\end{aligned}$$

for every $v \in V$, *almost everywhere in* $(0,T)$, *and*

$$\begin{aligned}
&\mu_\beta = \beta \partial_t \varphi_\beta + a\varphi_\beta - J * \varphi_\beta + \xi_\beta + F_2'(\varphi_\beta) - \chi\sigma_\beta, \quad \xi_\beta \in \partial F_1(\varphi_\beta) \quad \textit{a.e. in } Q,\\
&\varphi_\beta(0) = \varphi_{0,\beta}, \quad \sigma_\beta(0) = \sigma_{0,\beta} \qquad \textit{a.e. in } \Omega.
\end{aligned}$$

Moreover, as $\alpha \to 0$, along a non-relabelled subsequence it holds that

$$\varphi_{\alpha,\beta} \to \varphi_\beta \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V), \tag{4.67}$$
$$\mu_{\alpha,\beta} \to \mu_\beta \quad \text{weakly in } L^2(0,T;V), \tag{4.68}$$
$$\sigma_{\alpha,\beta} \to \sigma_\beta \quad \text{weakly star in } H^1(0,T;V^*) \cap L^2(0,T;V) \cap L^\infty(Q), \tag{4.69}$$
$$\xi_{\alpha,\beta} \to \xi_\beta \quad \text{weakly in } L^2(0,T;H), \tag{4.70}$$
$$\alpha\mu_{\alpha,\beta} \to 0 \quad \text{strongly in } C^0([0,T];H) \cap L^2(0,T;V), \tag{4.71}$$

hence in particular that

$$\varphi_{\alpha,\beta} \to \varphi_\beta \quad \text{strongly in } C^0([0,T];H), \tag{4.72}$$
$$\sigma_{\alpha,\beta} \to \sigma_\beta \quad \text{strongly in } C^0([0,T];V^*) \cap L^2(0,T;H). \tag{4.73}$$

Proof of Theorem 4.9. Proceeding as in the proof of Theorem 4.1, we perform the analogous estimates that we used to deduce (4.25). In particular, since the implicit constant C in (4.25) is independent of α and β, recalling that we are assuming $\eta = 0$, we realise that

$$\begin{aligned}
&\frac{\alpha}{4}\|\mu_{\alpha,\beta}(t)\|^2 + (1+4c_a\alpha)\int_{Q_t} |\nabla\mu_{\alpha,\beta}|^2 + \frac{\beta}{2}\int_{Q_t} |\partial_t\varphi_{\alpha,\beta}|^2 + \int_\Omega F(\varphi_{\alpha,\beta}(t)) \\
&\quad + \frac{1}{2}\|\sigma_{\alpha,\beta}(t)\|^2 + \int_{Q_t} |\nabla\sigma_{\alpha,\beta}|^2 + c_a\|(\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta})(t)\|^2 \\
&\quad + 2c_a\beta\|\nabla\varphi_{\alpha,\beta}(t)\|^2 + 2c_aC_0\int_{Q_t} |\nabla\varphi_{\alpha,\beta}|^2 \\
&\le \frac{3}{2}\alpha\|\mu_{0,\alpha,\beta}\|^2 + (a^* + 4c_a)\|\varphi_{0,\alpha,\beta}\|^2 + 2c_a\beta\|\nabla\varphi_{0,\alpha,\beta}\|^2 + \|F(\varphi_{0,\alpha,\beta})\|_{L^1(\Omega)} \\
&\quad + \frac{1}{2}\|\sigma_{0,\alpha,\beta}\|^2 + C(1 + \int_{Q_t} |\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta}|^2 + \int_{Q_t} |\varphi_{\alpha,\beta}|^2 + \int_{Q_t} |\sigma_{\alpha,\beta}|^2) \\
&\quad + \frac{\chi^2}{2\beta}\int_{Q_t} |\sigma_{\alpha,\beta}|^2 + \frac{1}{2}\int_{Q_t} |\nabla\sigma_{\alpha,\beta}|^2 + 8c_a^2\chi^2\int_{Q_t} |\nabla\varphi_{\alpha,\beta}|^2 \\
&\quad + \int_{Q_t} (\mathcal{P}\sigma_{\alpha,\beta} - \mathcal{A})f(\varphi_{\alpha,\beta})\mu_{\alpha,\beta}.
\end{aligned} \tag{4.74}$$

All the terms referring to the initial data on the right-hand side are uniformly bounded in α by virtue of assumptions (4.65)–(4.66). Moreover, all the remaining terms can be handled using the Gronwall lemma, except for the last one. To this end, note that by the Poincaré–Wirtinger inequality (2.4), using the fact that f is bounded, and the uniform bound $\|\sigma_{\alpha,\beta}\|_{L^\infty(Q)} \le 1$, recall that now $\eta = 0$, we have

$$\begin{aligned}
&\int_{Q_t} (\mathcal{P}\sigma_{\alpha,\beta} - \mathcal{A})f(\varphi_{\alpha,\beta})\mu_{\alpha,\beta} \\
&\quad \le \int_{Q_t} (\mathcal{P}\sigma_{\alpha,\beta} - \mathcal{A})f(\varphi_{\alpha,\beta})(\mu_{\alpha,\beta} - (\mu_{\alpha,\beta})_\Omega) + \int_{Q_t} (\mathcal{P}\sigma_{\alpha,\beta} - \mathcal{A})f(\varphi_{\alpha,\beta})(\mu_{\alpha,\beta})_\Omega \\
&\quad \le \frac{1}{2}\int_{Q_t} |\nabla\mu_{\alpha,\beta}|^2 + C + (\mathcal{P} + \mathcal{A})\|f\|_{L^\infty(\mathbb{R})} t^{1/2}\|(\mu_{\alpha,\beta})_\Omega\|_{L^2(0,t)}.
\end{aligned}$$

Furthermore, noting that $(a\varphi_{\alpha,\beta} - J * \varphi_{\alpha,\beta})_\Omega = 0$, by comparison in equation (4.2) we get

$$(\mu_{\alpha,\beta})_\Omega = \beta(\partial_t\varphi_{\alpha,\beta})_\Omega + (\xi_{\alpha,\beta} + F_2'(\varphi_{\alpha,\beta}))_\Omega - \chi(\sigma_{\alpha,\beta})_\Omega,$$

so that thanks to assumption (4.63) implies that

$$\begin{aligned}\|(\mu_{\alpha,\beta})_\Omega\|_{L^2(0,t)} &\le \beta\|\partial_t\varphi_{\alpha,\beta}\|_{L^2(Q_t)} + \|\xi_{\alpha,\beta} + F_2'(\varphi_{\alpha,\beta})\|_{L^2(0,t;L^1(\Omega))} + \chi\|\sigma_{\alpha,\beta}\|_{L^2(0,t;H)} \\ &\le C(1 + \beta^2 \int_{Q_t} |\partial_t\varphi_{\alpha,\beta}|^2 + \sup_{s\in[0,t]} \int_\Omega F(\varphi_{\alpha,\beta}(s)) + \sup_{s\in[0,t]} \|\sigma_{\alpha,\beta}(s)\|^2),\end{aligned}$$

for a certain constant $C > 0$, independent of α. Putting this information together, we first choose $t \in [0, T_0]$, where $T_0 \in (0, T]$ is fixed sufficiently small so that the term corresponding to $t^{1/2}$ can be incorporated on the left-hand side, for example by picking a T_0 such that

$$(\mathcal{P} + \mathcal{A})\|f\|_{L^\infty(\mathbb{R})}T_0^{1/2} < \frac{1}{C}.$$

We then take supremum in $t \in [0, T_0]$ on the left-hand side of the inequality (4.74) and rearrange the terms: the estimate can be closed on the time interval $[0, T_0]$ using the Gronwall lemma. As the choice of T_0 is independent of α, β, and of the initial data (it only depends on $\mathcal{A}$, $\mathcal{P}$, C_F, f, and χ), repeating the same argument we can close the estimate also on $[T_0, 2T_0]$, and so on, so that a classical patching argument guarantees the existence of a constant $C > 0$, independent of α, such that

$$\|\varphi_{\alpha,\beta}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\sigma_{\alpha,\beta}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le C, \tag{4.75}$$

$$\|(\mu_{\alpha,\beta})_\Omega\|_{L^2(0,T)} + \|\nabla\mu_{\alpha,\beta}\|_{L^2(0,T;H)} + \alpha^{1/2}\|\mu_{\alpha,\beta}\|_{L^\infty(0,T;H)} \le C. \tag{4.76}$$

From estimate (4.76), the Poincaré–Wirtinger inequality (2.4) yields

$$\|\mu_{\alpha,\beta}\|_{L^2(0,T;V)} \le C. \tag{4.77}$$

Lastly, by comparison in (4.3), we also deduce that

$$\|\sigma_{\alpha,\beta}\|_{H^1(0,T;V^*)} \le C, \tag{4.78}$$

while by comparison in (4.2) we have that

$$\|\xi_{\alpha,\beta}\|_{L^2(0,T;H)} \le C. \tag{4.79}$$

Passage to the Limit

From the estimates (4.75)–(4.79) and classical compactness arguments, we infer the existence of a quadruplet $(\varphi_\beta, \mu_\beta, \sigma_\beta, \xi_\beta)$ with

$$\begin{aligned}&\varphi_\beta \in H^1(0,T;H) \cap L^\infty(0,T;V), &&\mu_\beta \in L^2(0,T;V),\\ &\sigma_\beta \in H^1(0,T;V^*) \cap L^2(0,T;V), &&\xi_\beta \in L^2(0,T;H),\end{aligned}$$

such that, as $\alpha \to 0$, along a non-relabelled subsequence, it holds that the weak, weak star and strong convergences (4.67)–(4.71) and (4.72)–(4.73) are fulfilled. We are then left to show that $(\varphi_\beta, \mu_\beta, \sigma_\beta, \xi_\beta)$ yields a solution to (4.1)–(4.5) with $\alpha = 0$ in the sense of Theorem 4.9. In this direction, let us exploit the strong convergence of the phase variable (4.72) along with the continuity and boundedness of f, and Lebesgue convergence theorem, to deduce that, as $\alpha \to 0$,

$$\begin{aligned} f(\varphi_{\alpha,\beta}) \to f(\varphi_\beta) \quad & \text{strongly in } L^p(Q) \quad \forall\, p \geq 1, \\ F_2'(\varphi_{\alpha,\beta}) \to F_2'(\varphi_\beta) \quad & \text{strongly in } C^0([0,T]; H). \end{aligned}$$

Moreover, the strong-weak closure of ∂F_1 (see, e.g., [12, Cor. 2.4, p. 41]) entails that $\xi_\beta \in \partial F_1(\varphi_\beta)$ almost everywhere in Q. Lastly, it is not difficult to pass to the limit in the weak formulation of (4.1)–(4.5) to conclude that $(\mu_\beta, \varphi_\beta, \sigma_\beta, \xi_\beta)$ solves (4.1)–(4.5) with $\alpha = 0$, as we claimed. The comparison principle for σ_β can be then obtained repeating the argument of the proof of Theorem 4.1 leading to $\sigma_\beta \in L^\infty(Q)$. This concludes the proof of Theorem 4.9. $\square$

Theorem 4.10 (Error estimate: $\alpha \to 0$)**.** *In the setting of Theorem 4.9, suppose also that*

$$F \in C^1(\mathbb{R}), \quad |F'(r) - F'(s)| \leq C_F(1 + |r|^2 + |s|^2)|r - s|, \quad r, s \in \mathbb{R}, \tag{4.80}$$

and that there exists $M_0 > 0$ such that

$$\alpha^{1/4}(\|\mu_{0,\alpha,\beta}\|_V + \|\sigma_{0,\alpha,\beta}\|_V + \|F'(\varphi_{0,\alpha,\beta})\|) \leq M_0 \quad \forall\, \alpha \in (0, \alpha_0). \tag{4.81}$$

Then the solution $(\varphi_\beta, \mu_\beta, \sigma_\beta, \xi_\beta)$ to the system (4.1)–(4.5) *with $\alpha = 0$ is unique, the convergences obtained in Theorem 4.9 hold along the entire sequence $\alpha \to 0$, and there exists $K_\beta > 0$, independent of α, such that the following error estimate holds:*

$$\begin{aligned} & \|\varphi_{\alpha,\beta} - \varphi_\beta\|_{L^\infty(0,T;H)} + \|\mu_{\alpha,\beta} - \mu_\beta\|_{L^2(0,T;V)} + \|\sigma_{\alpha,\beta} - \sigma_\beta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\ & \quad \leq K_\beta(\alpha^{1/4} + \|\varphi_{0,\alpha,\beta} - \varphi_{0,\beta}\| + \|\sigma_{0,\alpha,\beta} - \sigma_{0,\beta}\|). \end{aligned}$$

Remark 4.11. *Note that given $(\varphi_{0,\beta}, \sigma_{0,\beta})$ satisfying* (4.64)*, a natural choice for the approximating sequence of initial data $(\varphi_{0,\alpha,\beta}, \sigma_{0,\alpha,\beta})$ satisfying* (4.65)–(4.66) *and* (4.81) *is given by the solutions to the elliptic problems*

$$\varphi_{0,\alpha,\beta} + \alpha^{1/2}\mathcal{R}\varphi_{0,\alpha,\beta} = \varphi_{0,\beta}, \quad \sigma_{0,\alpha,\beta} + \alpha^{1/2}\mathcal{R}\sigma_{0,\alpha,\beta} = \sigma_{0,\beta},$$

being $\mathcal{R}$ the Riesz isomorphism introduced in Section 2.4.1. In this case, if for example $\sigma_{0,\beta} \in V$, it is immediate to check that

$$\|\varphi_{0,\alpha,\beta} - \varphi_{0,\beta}\| + \|\sigma_{0,\alpha,\beta} - \sigma_{0,\beta}\| \leq M_0\alpha^{1/4}$$

for a certain $M_0 > 0$, so that the rate of convergence given by Theorem 4.10 is exactly $1/4$.

Proof of Theorem 4.10. First of all, we need to deduce an additional estimate of $\partial_t\mu_{\alpha,\beta}$. Arguing as in the proof of Theorem 4.3, by considering (4.47) and multiplying it by

$\alpha^{1/2}$ (recall that $\eta = 0$), we obtain

$$
\begin{aligned}
&\alpha^{3/2}\int_{Q_t}|\partial_t\mu_{\alpha,\beta}|^2 + \frac{\alpha^{1/2}}{2}\|\nabla\mu_{\alpha,\beta}(t)\|^2 + \frac{\beta\alpha^{1/2}}{2}\|\partial_t\varphi_{\alpha,\beta}(t)\|^2 + C_0\alpha^{1/2}\int_{Q_t}|\partial_t\varphi_{\alpha,\beta}|^2 \\
&\quad + \alpha^{1/2}\int_{Q_t}|\partial_t\sigma_{\alpha,\beta}|^2 + \frac{\alpha^{1/2}}{2}\|\nabla\sigma_{\alpha,\beta}(t)\|^2 \\
&\quad \le \frac{\alpha^{1/2}}{2}\|\nabla\mu_{0,\alpha,\beta}\|^2 + \frac{\beta\alpha^{1/2}}{2}\|\partial_t\varphi_{\alpha,\beta}(0)\|^2 + \frac{\alpha^{1/2}}{2}\|\nabla\sigma_{0,\alpha,\beta}\|^2 \\
&\quad + \alpha^{1/2}\int_{Q_t}(\mathcal{P}\sigma_{\alpha,\beta} - \mathcal{A})f(\varphi_{\alpha,\beta})\partial_t\mu_{\alpha,\beta} \\
&\quad + \alpha^{1/2}\int_{Q_t}(J*(\partial_t\varphi_{\alpha,\beta}) + \chi\partial_t\sigma_{\alpha,\beta})\partial_t\varphi_{\alpha,\beta} \\
&\quad + \alpha^{1/2}\int_{Q_t}\big(\mathcal{B}(\sigma_S - \sigma_{\alpha,\beta}) - Ch(\varphi_{\alpha,\beta})\sigma_{\alpha,\beta}\big)\partial_t\sigma_{\alpha,\beta}.
\end{aligned}
$$

The last two terms on the right-hand side can be easily handled as in the proof of Theorem 4.3, using the averaged Young inequality. Moreover, since $\partial_t\varphi_{\alpha,\beta}(0)$ satisfies

$$\mu_{0,\alpha,\beta} = \beta\partial_t\varphi_{\alpha,\beta}(0) + a\varphi_{0,\alpha,\beta} - J*\varphi_{0,\alpha,\beta} + F'(\varphi_{0,\alpha,\beta}) - \chi\sigma_{0,\alpha,\beta},$$

the first three terms on the right-hand side of the inequality above are uniformly bounded in α thanks to the assumptions (4.65)–(4.66) and (4.81). As for the fourth term, this can be treated using integration by parts in time and the boundedness of $\sigma_{\alpha,\beta}$ in (4.15) as

$$
\begin{aligned}
&-\alpha^{1/2}\mathcal{P}\int_{Q_t}\partial_t\sigma_{\alpha,\beta}f(\varphi_{\alpha,\beta})\mu_{\alpha,\beta} - \alpha^{1/2}\int_{Q_t}(\mathcal{P}\sigma_{\alpha,\beta} - \mathcal{A})f'(\varphi_{\alpha,\beta})\partial_t\varphi_{\alpha,\beta}\mu_{\alpha,\beta} \\
&\quad + \alpha^{1/2}\int_\Omega(\mathcal{P}\sigma_{\alpha,\beta}(t) - \mathcal{A})f(\varphi_{\alpha,\beta}(t))\mu_{\alpha,\beta}(t) \\
&\quad - \alpha^{1/2}\int_\Omega(\mathcal{P}\sigma_{0,\alpha,\beta} - \mathcal{A})f(\varphi_{0,\alpha,\beta})\mu_{0,\alpha,\beta} \\
&\quad \le \frac{\alpha^{1/2}}{4}\int_{Q_t}|\partial_t\sigma_{\alpha,\beta}|^2 + \alpha^{1/2}\mathcal{P}^2\|f\|^2_{L^\infty(\mathbb{R})}\|\mu_{\alpha,\beta}\|^2_{L^2(0,T;H)} \\
&\quad + \alpha^{1/2}(\mathcal{P}+\mathcal{A})\|f\|_{W^{1,\infty}(\mathbb{R})}\big(\|\partial_t\varphi_{\alpha,\beta}\|_{L^2(0,T;H)}\|\mu_{\alpha,\beta}\|_{L^2(0,T;H)} + 2\|\mu_{\alpha,\beta}\|_{C^0([0,T];H)}\big),
\end{aligned}
$$

where the right-hand side is uniformly bounded in α thanks to (4.75)–(4.79). Putting this information together, we deduce that

$$\alpha^{3/4}\|\mu_{\alpha,\beta}\|_{H^1(0,T;H)} + \alpha^{1/4}\|\mu_{\alpha,\beta}\|_{L^\infty(0,T;V)} \le C, \tag{4.82}$$

$$\alpha^{1/4}\|\varphi_{\alpha,\beta}\|_{W^{1,\infty}(0,T;H)} + \alpha^{1/4}\|\sigma_{\alpha,\beta}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le C. \tag{4.83}$$

We are now ready to show the error estimate. Taking the difference between the unique solution $(\mu_{\alpha,\beta}, \varphi_{\alpha,\beta}, \sigma_{\alpha,\beta}, \xi_{\alpha,\beta})$ to (4.1)–(4.5) with $\alpha, \beta > 0$ and $\eta = 0$ and the solution $(\mu_\beta, \varphi_\beta, \sigma_\beta, \xi_\beta)$ to (4.1)–(4.5) with $\alpha = \eta = 0$ obtained in Theorem 4.9 leads us to

$$\alpha\partial_t\mu_{\alpha,\beta} + \partial_t\varphi - \Delta\mu = \mathcal{P}\sigma f(\varphi_{\alpha,\beta}) + (\mathcal{P}\sigma_\beta - \mathcal{A})(f(\varphi_{\alpha,\beta}) - f(\varphi_\beta)) \quad \text{in } Q, \tag{4.84}$$
$$\mu = \beta\partial_t\varphi + a\varphi - J * \varphi + F'(\varphi_{\alpha,\beta}) - F'(\varphi_\beta) - \chi\sigma \quad \text{in } Q, \tag{4.85}$$
$$\partial_t\sigma - \Delta\sigma + \mathcal{B}\sigma + \mathcal{C}\sigma f(\varphi_{\alpha,\beta}) = \mathcal{C}\sigma_\beta(f(\varphi_\beta) - f(\varphi_{\alpha,\beta})) \quad \text{in } Q, \tag{4.86}$$
$$\partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}\sigma = 0 \quad \text{on } \Sigma, \tag{4.87}$$
$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 \quad \text{in } \Omega\,, \tag{4.88}$$

where the equations are intended in the usual variational setting, and where we have set

$$\varphi := \varphi_{\alpha,\beta} - \varphi_\beta, \quad \mu := \mu_{\alpha,\beta} - \mu_\beta, \quad \sigma := \sigma_{\alpha,\beta} - \sigma_\beta,$$
$$\varphi_0 := \varphi_{0,\alpha,\beta} - \varphi_{0,\beta}, \quad \sigma_0 := \sigma_{0,\alpha,\beta} - \sigma_{0,\beta}.$$

Next, we multiply (4.84) by $\beta\mu$, (4.85) by $\mu-\varphi$, (4.86) by σ, add the resulting equality and integrate over Q_t to obtain, thanks to assumption **B5**,

$$\begin{aligned}
&\int_{Q_t} |\mu|^2 + \beta\int_{Q_t} |\nabla\mu|^2 + \frac{\beta}{2}\|\varphi(t)\|^2 + C_0\int_{Q_t} |\varphi|^2 + \frac{1}{2}\|\sigma(t)\|^2 \\
&\quad + \int_{Q_t} |\nabla\sigma|^2 + \int_{Q_t} (\mathcal{B} + Ch(\varphi_{\alpha,\beta}))|\sigma|^2 \\
&\le \frac{\beta}{2}\|\varphi_0\|^2 + \frac{1}{2}\|\sigma_0\|^2 - \int_{Q_t} \alpha\partial_t\mu_{\alpha,\beta}\beta\mu \\
&\quad + \int_{Q_t} (\mu + \chi\sigma)\varphi + \int_{Q_t} \mathcal{C}\sigma_\beta(f(\varphi_\beta) - f(\varphi_{\alpha,\beta}))\sigma \\
&\quad + \beta\int_{Q_t} \big[\mathcal{P}\sigma f(\varphi_{\alpha,\beta}) + (\mathcal{P}\sigma_\beta - \mathcal{A})(f(\varphi_{\alpha,\beta}) - f(\varphi_\beta))\big]\mu \\
&\quad + \int_{Q_t} (a\varphi - J * \varphi + F'(\varphi_{\alpha,\beta}) - F'(\varphi_\beta) - \chi\sigma)\mu =: \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4.
\end{aligned}$$

Let us estimate the terms on the right-hand side separately. The third and fourth ones yield, thanks to the Young inequality and the refined estimate (4.82),

$$\begin{aligned}
|\mathbb{I}_1^3| + |\mathbb{I}_2^1| &\le \frac{1}{2}\int_{Q_t} |\mu|^2 + \beta^2\alpha^2\|\partial_t\mu_{\alpha,\beta}\|^2_{L^2(0,T;H)} + 2\int_{Q_t} |\varphi|^2 + \frac{\chi^2}{4}\int_{Q_t} |\sigma|^2 \\
&\le C\alpha^{1/2} + \frac{1}{2}\int_{Q_t} |\mu|^2 + 2\int_{Q_t} |\varphi|^2 + \frac{\chi^2}{4}\int_{Q_t} |\sigma|^2,
\end{aligned}$$

for a certain constant C independent of α. The fifth and sixth terms can be easily handled using the Young inequality, the Lipschitz continuity and boundedness of f, and the uniform bound $\|\sigma_\beta\|_{L^\infty(Q)} \le 1$, as

$$\begin{aligned}
|\mathbb{I}_2^2| + |\mathbb{I}_3| &\le \frac{1}{4}\int_{Q_t} |\mu|^2 + \|f\|^2_{W^{1,\infty}(\mathbb{R})}\Big((2\beta^2\mathcal{P}^2 + \mathcal{C}^2)\int_{Q_t} |\sigma|^2 \\
&\quad + \big(\frac{1}{4} + 2\beta^2(\mathcal{P} + \mathcal{A})^2\big)\int_{Q_t} |\varphi|^2\Big).
\end{aligned}$$

Moreover, the last term satisfies, thanks to the Young inequality and the growth assumption (4.80),

$$
\begin{aligned}
|\mathbb{I}_4| \leq \frac{1}{8}\int_{Q_t} |\mu|^2 + 12(a^*)^2 \int_{Q_t} |\varphi|^2 + 6\chi^2 \int_{Q_t} |\sigma|^2 \\
+ C_F \int_{Q_t} (1 + |\varphi_{\alpha,\beta}|^2 + |\varphi_\beta|^2)|\varphi||\mu|,
\end{aligned}
$$

where, thanks to the inclusion $V \subset L^6(\Omega)$ and the Hölder inequality,

$$
\begin{aligned}
&\int_{Q_t} (1 + |\varphi_{\alpha,\beta}|^2 + |\varphi_\beta|^2)|\varphi||\mu| \\
&\quad \leq \int_0^t (|\Omega|^{1/3} + \|\varphi_{\alpha,\beta}\|_6^2 + \|\varphi_\beta\|_6^2)\|\varphi(s)\|\|\mu(s)\|_6 \,\mathrm{ds} \\
&\quad \leq C(1 + \|\varphi_{\alpha,\beta}\|^2_{L^\infty(0,T;V)} + \|\varphi_\beta\|^2_{L^\infty(0,T;V)}) \int_0^t \|\varphi(s)\|\|\mu(s)\|_V \,\mathrm{ds},
\end{aligned}
$$

which yields, thanks to the estimate (4.75) and again the Young inequality, that

$$
\int_{Q_t} (1 + |\varphi_{\alpha,\beta}|^2 + |\varphi_\beta|^2)|\varphi||\mu| \leq \min\{1/16, \beta/2\}\|\mu\|^2_{L^2(0,t;V)} + C_\beta \int_{Q_t} |\varphi|^2
$$

for a certain constant $C_\beta > 0$ independent of α. Hence, collecting the above estimates we obtain

$$
\begin{aligned}
\min\{1/16, \beta/2\}\|\mu\|^2_{L^2(0,t;V)} + \frac{\beta}{2}\|\varphi(t)\|^2 + \frac{1}{2}\|\sigma(t)\|^2 + \int_{Q_t} |\nabla\sigma|^2 \\
\leq C\Big(\alpha^{1/2} + \frac{\beta}{2}\|\varphi_0\|^2 + \frac{1}{2}\|\sigma_0\|^2 + \int_{Q_t} |\varphi|^2 + \int_{Q_t} |\sigma|^2\Big),
\end{aligned}
$$

where the updated constant C depends on β, and the initial data $(\varphi_{0,\beta}, \sigma_{0,\beta})$. The error estimate follows then by the Gronwall lemma.

Finally, it is not difficult to check that the error estimate performed here yields uniqueness of the solution $(\varphi_\beta, \mu_\beta, \sigma_\beta, \xi_\beta)$ for the system (4.1)–(4.5) at $\alpha = 0$. This reality implies then that the convergences as $\alpha \to 0$ hold along the entire sequence α which completes the proof of Theorem 4.10. □

4.3.2 Asymptotics Analysis as $\beta \to 0$

The second asymptotic study that we are going to address is the one as $\beta \to 0$, when $\alpha \in (0, \alpha_0)$ is fixed. In this case, the presence of the parabolic regularisation on μ provided by $\alpha\partial_t\mu$ with $\alpha > 0$ allows considering also very general potentials and to avoid limiting assumptions as (4.63). Let us mention that the limit as $\beta \to 0$ corresponds to a vanishing viscosity argument on the system in consideration. We expect then to lose, at the limit $\beta = 0$, temporal regularity on the phase variable, as well as the separation principle.

Theorem 4.12 (Asymptotics: $\beta \to 0$)**.** *Assume* **B1–B5***,* $\alpha \in (0, \alpha_0)$*, and*

$$0 \le \chi < \sqrt{c_a}, \qquad (\chi + \eta + 4c_a\chi)^2 < 8c_a C_0 + 4\chi\eta. \tag{4.89}$$

Moreover, let us suppose that

$$\varphi_{0,\alpha}, \mu_{0,\alpha}, \sigma_{0,\alpha} \in H, \qquad F(\varphi_{0,\alpha}) \in L^1(\Omega). \tag{4.90}$$

For every $\beta \in (0, \beta_0)$*, let the initial data* $(\varphi_{0,\alpha,\beta}, \mu_{0,\alpha,\beta}, \sigma_{0,\alpha,\beta})$ *satisfy* (4.7)*, and denote by* $(\varphi_{\alpha,\beta}, \mu_{\alpha,\beta}, \sigma_{\alpha,\beta}, \xi_{\alpha,\beta})$ *the corresponding weak solution to* (4.1)-(4.5) *obtained from Theorem 4.1. Suppose also that, as* $\beta \to 0$*,*

$$\begin{aligned} &\varphi_{0,\alpha,\beta} \to \varphi_{0,\alpha} \quad \textit{strongly in } H, \qquad \mu_{0,\alpha,\beta} \to \mu_{0,\alpha} \quad \textit{strongly in } H, \\ &\sigma_{0,\alpha,\beta} \to \sigma_{0,\alpha} \quad \textit{strongly in } H, \end{aligned} \tag{4.91}$$

and

$$\exists\, M_0 > 0 : \quad \beta^{1/2}\|\varphi_{0,\alpha,\beta}\|_V + \|F(\varphi_{0,\alpha,\beta})\|_{L^1(\Omega)} \le M_0 \quad \forall\, \beta \in (0, \beta_0). \tag{4.92}$$

Then, there exists a quadruplet $(\varphi_\alpha, \mu_\alpha, \sigma_\alpha, \xi_\alpha)$*, with*

$$\begin{aligned} \varphi_\alpha, \mu_\alpha &\in L^\infty(0,T;H) \cap L^2(0,T;V), \\ \alpha\mu_\alpha + \varphi_\alpha &\in H^1(0,T;V^*) \cap L^2(0,T;V), \\ \sigma_\alpha &\in H^1(0,T;V^*) \cap L^2(0,T;V), \\ \xi_\alpha &\in L^2(0,T;V), \end{aligned}$$

such that

$$\begin{aligned} &\langle \partial_t(\alpha\mu_\alpha + \varphi_\alpha), v\rangle + \int_\Omega \nabla\mu_\alpha \cdot \nabla v = \int_\Omega (\mathcal{P}\sigma_\alpha - \mathcal{A}) f(\varphi_\alpha) v, \\ &\langle \partial_t\sigma_\alpha, v\rangle + \int_\Omega \nabla\sigma_\alpha \cdot \nabla v + \mathcal{B}\int_\Omega (\sigma_\alpha - \sigma_S) v + \mathcal{C}\int_\Omega \sigma_\alpha f(\varphi_\alpha) v \\ &\quad = \eta \int_\Omega \nabla\varphi_\alpha \cdot \nabla v, \end{aligned}$$

for every $v \in V$*, almost everywhere in* $(0,T)$*, and*

$$\begin{aligned} &\mu_\alpha = a\varphi_\alpha - J * \varphi_\alpha + \xi_\alpha + F_2'(\varphi_\alpha) - \chi\sigma_\alpha, \quad \xi_\alpha \in \partial F_1(\varphi_\alpha) \qquad \textit{a.e. in } Q, \\ &\varphi_\alpha(0) = \varphi_{0,\alpha}, \quad \sigma_\alpha(0) = \sigma_{0,\alpha} \qquad \textit{a.e. in } \Omega. \end{aligned}$$

Moreover, as $\beta \to 0$*, along a non-relabelled subsequence it holds that*

$$\varphi_{\alpha,\beta} \to \varphi_\alpha \quad \textit{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V), \tag{4.93}$$

$$\mu_{\alpha,\beta} \to \mu_\alpha \quad \textit{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V), \tag{4.94}$$

$$\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta} \to \alpha\mu_\alpha + \varphi_\alpha \quad \textit{weakly in } H^1(0,T;V^*) \cap L^2(0,T;V), \tag{4.95}$$

$$\sigma_{\alpha,\beta} \to \sigma_\alpha \quad \textit{weakly in } H^1(0,T;V^*) \cap L^2(0,T;V), \tag{4.96}$$

$$\xi_{\alpha,\beta} \to \xi_\alpha \quad \textit{weakly in } L^2(0,T;H), \tag{4.97}$$

$$\beta\varphi_{\alpha,\beta} \to 0 \quad \textit{strongly in } H^1(0,T;H) \cap L^\infty(0,T;V), \tag{4.98}$$

hence in particular that

$$\varphi_{\alpha,\beta} \to \varphi_\alpha \text{ strongly in } L^2(0,T;H), \quad \mu_{\alpha,\beta} \to \mu_\alpha \text{ strongly in } L^2(0,T;H), \tag{4.99}$$
$$\sigma_{\alpha,\beta} \to \sigma_\alpha \quad \text{strongly in } C^0([0,T];V^*) \cap L^2(0,T;H). \tag{4.100}$$

Furthermore, if $\eta = 0$ *and* $\sigma_{0,\alpha,\beta}$ *satisfies* (4.14) *for all* $\beta > 0$*, then the limit* σ_α *satisfies* (4.15) *as well, and*

$$\sigma_{\alpha,\beta} \to \sigma_\alpha \quad \text{weakly star in } L^\infty(Q).$$

Proof of Theorem 4.12. To begin with, we present some a priori estimates.

Uniform Estimates

Performing the same estimates as the proof of Theorem 4.1, and noting that the constant C in (4.24) is independent of α and β, we infer that

$$\begin{aligned}
&\frac{\alpha}{2}\|\mu_{\alpha,\beta}(t)\|^2 + (1+4c_a\alpha)\int_{Q_t} |\nabla \mu_{\alpha,\beta}|^2 + \beta \int_{Q_t} |\partial_t \varphi_{\alpha,\beta}|^2 \\
&\quad + \frac{1}{2}\|\sigma_{\alpha,\beta}(t)\|^2 + \int_{Q_t} |\nabla \sigma_{\alpha,\beta}|^2 + 2c_a\|(\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta})(t)\|^2 \\
&\quad + 2c_a\beta\|\nabla\varphi_{\alpha,\beta}(t)\|^2 + 2c_aC_0\int_{Q_t} |\nabla\varphi_{\alpha,\beta}|^2 \\
&\le \frac{3}{2}\alpha\|\mu_{0,\alpha,\beta}\|^2 + (a^* + 4c_a)\|\varphi_{0,\alpha,\beta}\|^2 + 2c_a\beta\|\nabla\varphi_{0,\alpha,\beta}\|^2 \\
&\quad + \|F(\varphi_{0,\alpha,\beta})\|_{L^1(\Omega)} + \frac{1}{2}\|\sigma_{0,\alpha,\beta}\|^2 + \frac{c_a}{2}\|\varphi_{\alpha,\beta}(t)\|^2 + \chi\int_{Q_t} \sigma_{\alpha,\beta}\partial_t\varphi_{\alpha,\beta} \\
&\quad + (\eta + 4c_a\chi)\int_{Q_t} \nabla\sigma_{\alpha,\beta}\cdot\nabla\varphi_{\alpha,\beta} \\
&\quad + C(1 + \int_{Q_t} |\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta}|^2 + \int_{Q_t} |\varphi_{\alpha,\beta}|^2 + \int_{Q_t} |\sigma_{\alpha,\beta}|^2) \\
&\quad + \int_{Q_t} (\mathcal{P}\sigma_{\alpha,\beta} - \mathcal{A})f(\varphi_{\alpha,\beta})\mu_{\alpha,\beta}.
\end{aligned} \tag{4.101}$$

First of all, note that all the terms on the right-hand side referring to the initial data are uniformly bounded in β due to assumptions (4.91)–(4.92). Moreover, since $\alpha \in (0, \frac{1}{4c_a})$ we have a bound from below on the left-hand side in the form

$$\begin{aligned}
&2c_a\|(\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta})(t)\|^2 + \frac{\alpha}{2}\|\mu_{\alpha,\beta}(t)\|^2 \\
&\quad \ge 2c_a\|(\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta})(t)\|^2 + 2c_a\alpha^2\|\mu_{\alpha,\beta}(t)\|^2 \\
&\quad \ge (c_a - \rho)\|\varphi_{\alpha,\beta}(t)\|^2 + 2\rho\alpha^2\|\mu_{\alpha,\beta}(t)\|^2
\end{aligned} \tag{4.102}$$

for every $\rho \in (0, c_a)$. Hence the corresponding term $\frac{c_a}{2}\|\varphi_{\alpha,\beta}(t)\|^2$ on the right-hand side can be incorporated on the left-hand side of (4.101), provided we choose $\rho < c_a/2$. Furthermore, from the boundedness of f the last term in (4.101) can be easily handled using the Young inequality and the Gronwall lemma. Hence, we only need to estimate the terms involving χ and η. To this end, we first use integration by parts and the equation (4.13) to deduce, thanks to the Young inequality and the boundedness of f, that

$$\begin{aligned}
&\chi \int_{Q_t} \sigma_{\alpha,\beta} \partial_t \varphi_{\alpha,\beta} \\
&\quad = -\chi \int_0^t \langle \partial_t \sigma_{\alpha,\beta}(s), \varphi_{\alpha,\beta}(s) \rangle \,\mathrm{ds} + \chi \int_\Omega \sigma_{\alpha,\beta}(t)\varphi_{\alpha,\beta}(t) - \chi \int_\Omega \sigma_{0,\alpha,\beta}\varphi_{0,\alpha,\beta} \\
&\quad = \chi \int_{Q_t} \nabla \sigma_{\alpha,\beta} \cdot \nabla \varphi_{\alpha,\beta} + \chi \int_{Q_t} (\mathcal{B}(\sigma_{\alpha,\beta} - \sigma_S) + \mathcal{C}\sigma_{\alpha,\beta} f(\varphi_{\alpha,\beta}))\varphi_{\alpha,\beta} \\
&\qquad - \chi\eta \int_{Q_t} |\nabla \varphi_{\alpha,\beta}|^2 + \chi \int_\Omega \sigma_{\alpha,\beta}(t)\varphi_{\alpha,\beta}(t) - \chi \int_\Omega \sigma_{0,\alpha,\beta}\varphi_{0,\alpha,\beta} \\
&\quad \leq \chi \int_{Q_t} \nabla \sigma_{\alpha,\beta} \cdot \nabla \varphi_{\alpha,\beta} - \chi\eta \int_{Q_t} |\nabla \varphi_{\alpha,\beta}|^2 + C(1 + \int_{Q_t} |\varphi_{\alpha,\beta}|^2 + \int_{Q_t} |\sigma_{\alpha,\beta}|^2) \\
&\qquad + \delta\chi^2 \|\varphi_{\alpha,\beta}(t)\|^2 + \frac{1}{4\delta}\|\sigma_{\alpha,\beta}(t)\|^2,
\end{aligned} \tag{4.103}$$

for every $\delta > 0$. Now, it is immediate to check that assumption (4.89) yields $\frac{1}{2} < \frac{c_a}{2\chi^2}$ (with the convention that $\frac{1}{\chi} = +\infty$ if $\chi = 0$): hence

$$\exists\, \bar\delta \in \Big(\frac{1}{2}, \frac{c_a}{2\chi^2}\Big) \quad \text{such that} \quad \bar\delta\chi^2 < \frac{c_a}{2}, \qquad \frac{1}{4\bar\delta} < \frac{1}{2}, \tag{4.104}$$

so that we can incorporate the last two terms on the right-hand side of (4.103) on the left-hand side of (4.101). Taking these remarks into account, we are left with

$$\begin{aligned}
&2\rho\alpha^2 \|\mu_{\alpha,\beta}(t)\|^2 + (\frac{c_a}{2} - \rho - \bar\delta\chi^2)\|\varphi_{\alpha,\beta}(t)\|^2 + \big(1 + 4c_a\alpha\big) \int_{Q_t} |\nabla \mu_{\alpha,\beta}|^2 \\
&\quad + \beta \int_{Q_t} |\partial_t \varphi_{\alpha,\beta}|^2 + (\frac{1}{2} - \frac{1}{4\bar\delta})\|\sigma_{\alpha,\beta}(t)\|^2 + \int_{Q_t} |\nabla \sigma_{\alpha,\beta}|^2 \\
&\quad + 2c_a\beta \|\nabla \varphi_{\alpha,\beta}(t)\|^2 + (2c_a C_0 + \chi\eta) \int_{Q_t} |\nabla \varphi_{\alpha,\beta}|^2 \\
&\leq C(1 + \int_{Q_t} |\mu_{\alpha,\beta}|^2 + \int_{Q_t} |\varphi_{\alpha,\beta}|^2 + \int_{Q_t} |\sigma_{\alpha,\beta}|^2) \\
&\quad + (\chi + \eta + 4c_a\chi) \int_{Q_t} \nabla \sigma_{\alpha,\beta} \cdot \nabla \varphi_{\alpha,\beta},
\end{aligned} \tag{4.105}$$

which holds for every $\rho \in (0, c_a/2)$. By choosing $\bar\delta$ such that (4.104) are fulfilled, it is also possible to choose and fix $\bar\rho \in (0, c_a/2)$ such that

$$\frac{c_a}{2} - \bar\rho - \bar\delta\chi^2 > 0.$$

Next, we use again the averaged Young inequality to obtain, for every $\kappa > 0$,

$$(\chi + \eta + 4c_a\chi) \int_{Q_t} \nabla \sigma_{\alpha,\beta} \cdot \nabla \varphi_{\alpha,\beta} \leq \kappa \int_{Q_t} |\nabla \sigma_{\alpha,\beta}|^2 + \frac{(\chi + \eta + 4c_a\chi)^2}{4\kappa} \int_{Q_t} |\nabla \varphi_{\alpha,\beta}|^2$$

where the two terms on the right-hand side can be incorporated on the left-hand side of (4.105) provided to choose κ such that

$$\kappa < 1, \qquad \frac{(\chi + \eta + 4c_a\chi)^2}{4\kappa} < 2c_a C_0 + \chi\eta.$$

Easy computations show that this is possible if and only if

$$\frac{(\chi + \eta + 4c_a\chi)^2}{4(2c_a C_0 + \chi\eta)} < 1,$$

which is verified owing to (4.89).

Therefore, after rearranging the terms and using the Gronwall lemma, we infer that there exists a constant $C > 0$, which may depend on α, but it is independent of β, such that

$$\begin{aligned} &\|\varphi_{\alpha,\beta}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\mu_{\alpha,\beta}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\ &\quad + \|\sigma_{\alpha,\beta}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq C, \qquad (4.106) \\ &\beta^{1/2}\|\varphi_{\alpha,\beta}\|_{H^1(0,T;H)} + \beta^{1/2}\|\varphi_{\alpha,\beta}\|_{L^\infty(0,T;V)} \leq C, \qquad (4.107) \end{aligned}$$

yielding in turn, by comparison in equations (4.1) and (4.3),

$$\|\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta}\|_{H^1(0,T;V^*)} + \|\sigma_{\alpha,\beta}\|_{H^1(0,T;V^*)} \leq C. \qquad (4.108)$$

Testing equation (4.2) by $\xi_{\alpha,\beta}$ and using the estimate (4.106), its is a standard matter to deduce also that

$$\|\xi_{\alpha,\beta}\|_{L^2(0,T;H)} \leq C. \qquad (4.109)$$

Passage to the Limit

The estimates (4.106)–(4.109) and Lemma 2.4 ensure that there exists a quadruplet $(\varphi_\alpha, \mu_\alpha, \sigma_\alpha, \xi_\alpha)$ with

$$\begin{aligned} &\varphi_\alpha, \mu_\alpha \in L^\infty(0,T;H) \cap L^2(0,T;V), \\ &\lambda_\alpha := \alpha\mu_\alpha + \varphi_\alpha \in H^1(0,T;V^*) \cap L^2(0,T;V), \\ &\sigma_\alpha \in H^1(0,T;V^*) \cap L^2(0,T;V), \\ &\xi_\alpha \in L^2(0,T;H), \end{aligned}$$

such that, as $\beta \to 0$ (on a subsequence) it holds that (4.93)–(4.98) and (4.99)–(4.100) are satisfied, and also that

$$\begin{aligned} &\lambda_{\alpha,\beta} \to \lambda_\alpha \quad \text{weakly in } H^1(0,T;V^*) \cap L^2(0,T;V), \\ &\lambda_{\alpha,\beta} \to \lambda_\alpha \quad \text{strongly in } C^0([0,T];V^*) \cap L^2(0,T;H). \end{aligned}$$

Moreover, let us claim that the above strong convergences imply the strong convergences

$$\mu_{\alpha,\beta} \to \mu_\alpha \text{ strongly in } L^2(0,T;H), \quad \varphi_{\alpha,\beta} \to \varphi_\alpha \text{ strongly in } L^2(0,T;H). \quad (4.110)$$

To this end, we argue as in [39, Sec. 3], checking that the sequence $\{\lambda_{\alpha,\beta}\}_\beta$ is a Cauchy sequence in $L^2(0,T;H)$. In this direction, let us pick two arbitrary $\beta, \beta' > 0$ and take the difference of the corresponding equation (4.2) written for β and β', respectively. Next, we multiply the resulting equation by α, add to both sides the term $\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}$, test the resulting equation by $\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}$, and integrate over Q_t to obtain

$$\begin{aligned}
&\int_{Q_t} |\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}|^2 \\
&\quad + \alpha \int_{Q_t} \Big(a(\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}) + \xi_{\alpha,\beta} - \xi_{\alpha,\beta'} + F_2'(\varphi_{\alpha,\beta}) - F_2'(\varphi_{\alpha,\beta'})\Big)(\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}) \\
&\leq \int_{Q_t} \big((\lambda_{\alpha,\beta} - \lambda_{\alpha,\beta'}) - \alpha(\beta\partial_t\varphi_{\alpha,\beta} - \beta'\partial_t\varphi_{\alpha,\beta'}) + \alpha\chi(\sigma_{\alpha,\beta} - \sigma_{\alpha,\beta'})\big)(\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}) \\
&\quad + \alpha \int_{Q_t} J * (\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'})(\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}).
\end{aligned}$$

Owing to (4.93)–(4.98) and (4.99)–(4.100) we easily infer that the first term on the right-hand side goes to zero as $\beta, \beta' \to 0$. Moreover, on the left-hand side we have, thanks to assumption **B5**,

$$\begin{aligned}
&\int_{Q_t} (a(\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}) + \xi_{\alpha,\beta} - \xi_{\alpha,\beta'} + F_2'(\varphi_{\alpha,\beta}) - F_2'(\varphi_{\alpha,\beta'}))(\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}) \\
&\quad \geq C_0 \int_{Q_t} |\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}|^2,
\end{aligned}$$

while the last term on the right-hand side satisfies

$$\int_{Q_t} J * (\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'})(\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}) \leq a^* \int_{Q_t} |\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}|^2.$$

Rearranging the terms leads us to

$$\begin{aligned}
&(1 + (C_0 - a^*)\alpha) \int_{Q_t} |\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}|^2 \\
&\leq \int_{Q_t} \big((\lambda_{\alpha,\beta} - \lambda_{\alpha,\beta'}) - \alpha(\beta\partial_t\varphi_{\alpha,\beta} - \beta'\partial_t\varphi_{\alpha,\beta'}) + \chi(\sigma_{\alpha,\beta} - \sigma_{\alpha,\beta'})\big)(\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'})
\end{aligned}$$

where the right-hand side converges to 0 as $\beta \to 0$. Since $\alpha a^* < \alpha C_0 + 1$ as a consequence of the smallness assumption on α_0, this yields the second of (4.110) and by comparison also the first one follows, as we claimed.

With the strong convergence of the phase variable at disposal it is now straightforward to infer by combining the boundedness of f and the Lebesgue convergence theorem that, as $\beta \to 0$,

$$\begin{aligned}
f(\varphi_{\alpha,\beta}) &\to f(\varphi_\alpha) \quad \text{strongly in } L^p(Q) \quad \forall\, p \geq 1, \\
F_2'(\varphi_{\alpha,\beta}) &\to F_2'(\varphi_\alpha) \quad \text{strongly in } L^2(0,T;H).
\end{aligned}$$

Hence, since $\xi_\alpha \in \partial F_1(\varphi_\alpha)$ by the strong-weak closure of ∂F_1, it is a standard matter to pass to the limit as $\beta \to 0$ in the weak formulation of (4.1)–(4.5) and deduce that the limit $(\mu_\alpha, \varphi_\alpha, \sigma_\alpha, \xi_\alpha)$ yields a solution to (4.1)–(4.5) with $\beta = 0$. Notice in particular that by difference in the limit equation (4.2) we deduce the further regularity $\xi_\alpha \in L^2(0,T;V)$, while the last assertion of Theorem 4.12 follows as before by repeating the computations of the proof of Theorem 4.1 completing the proof of Theorem 4.12. □

Theorem 4.13 (Error estimate: $\beta \to 0$)**.** *In the setting of Theorem 4.12, suppose that $\eta = 0$. Then the solution $(\varphi_\alpha, \mu_\alpha, \sigma_\alpha, \xi_\alpha)$ to the system* (4.1)–(4.5) *with $\beta = 0$ is unique, the convergences obtained in Theorem 4.12 hold along the entire sequence $\beta \to 0$, and there exists $K_\alpha > 0$, independent of β, such that the following error estimate holds:*

$$\begin{aligned} &\|(\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta}) - (\alpha\mu_\alpha + \varphi_\alpha)\|_{L^\infty(0,T;V^*)} + \|\varphi_{\alpha,\beta} - \varphi_\alpha\|_{L^2(0,T;H)} \\ &\quad + \|\mu_{\alpha,\beta} - \mu_\alpha\|_{L^2(0,T;H)} + \|\sigma_{\alpha,\beta} - \sigma_\alpha\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\ &\quad \le K_\alpha \left(\beta^{1/2} + \|(\alpha\mu_{0,\alpha,\beta} + \varphi_{0,\alpha,\beta}) - (\alpha\mu_{0,\alpha} + \varphi_{0,\alpha})\|_{V^*} + \|\sigma_{0,\alpha,\beta} - \sigma_{0,\alpha}\|\right). \end{aligned}$$

Remark 4.14. *Note that given $(\varphi_{0,\alpha}, \mu_{0,\alpha}, \sigma_{0,\alpha})$ satisfying* (4.90)*, a natural choice for the approximating sequence $(\varphi_{0,\alpha,\beta}, \mu_{0,\alpha,\beta}, \sigma_{0,\alpha,\beta})$ is given by the solutions to the elliptic problems*

$$\varphi_{0,\alpha,\beta} + \beta\mathcal{R}\varphi_{0,\alpha,\beta} = \varphi_{0,\alpha}, \quad \mu_{0,\alpha,\beta} + \beta\mathcal{R}\mu_{0,\alpha,\beta} = \mu_{0,\alpha}, \quad \sigma_{0,\alpha,\beta} + \beta\mathcal{R}\sigma_{0,\alpha,\beta} = \sigma_{0,\alpha}.$$

In such a case, hypotheses (4.91)–(4.92) *are readily satisfied. Moreover, if for example $\varphi_{0,\alpha}, \mu_{0,\alpha}, \sigma_{0,\alpha} \in V$, it is immediate to check that, there is $M_0 > 0$, independent of β, such that*

$$\|\varphi_{0,\alpha,\beta} - \varphi_{0,\alpha}\| + \|\mu_{0,\alpha,\beta} - \mu_{0,\alpha}\| + \|\sigma_{0,\alpha,\beta} - \sigma_{0,\alpha}\| \le M_0\beta^{1/2},$$

so that the rate of convergence given by Theorem 4.13 is exactly $1/2$*.*

Proof of Theorem 4.13. The last result of this section follows with few changes from the proof of the continuous dependence estimate (4.35) established in Theorem 4.2.

Indeed, we can repeat almost the same computations performed the proof of Theorem 4.2 with the choices

$$(\varphi_1, \mu_1, \sigma_1, \xi_1) := (\varphi_{\alpha,\beta}, \mu_{\alpha,\beta}, \sigma_{\alpha,\beta}, \xi_{\alpha,\beta}), \quad (\varphi_2, \mu_2, \sigma_2, \xi_2) := (\varphi_\alpha, \mu_\alpha, \sigma_\alpha, \xi_\alpha).$$

Moreover, by setting

$$\begin{aligned} &\varphi := \varphi_{\alpha,\beta} - \varphi_\alpha, \quad \mu := \mu_{\alpha,\beta} - \mu_\alpha, \quad \sigma := \sigma_{\alpha,\beta} - \sigma_\alpha, \\ &\varphi_0 := \varphi_{0,\alpha,\beta} - \varphi_{0,\alpha}, \quad \mu_0 := \mu_{0,\alpha,\beta} - \mu_{0,\alpha}, \quad \sigma_0 := \sigma_{0,\alpha,\beta} - \sigma_{0,\alpha}, \end{aligned}$$

and recalling that we are assuming $\eta = 0$, we infer from (4.41) that

$$\begin{aligned}
&\frac{1}{2}\|(\alpha\mu+\varphi)(t)\|_{V^*}^2 + \alpha\int_{Q_t}|\mu|^2 + C_0\int_{Q_t}|\varphi|^2 + \frac{1}{2}\|\sigma(t)\|^2 + \int_{Q_t}|\nabla\sigma|^2 \\
&\leq -\beta\int_{Q_t}\partial_t\varphi_{\alpha,\beta}\varphi + \frac{1}{2}\|\alpha\mu_0+\varphi_0\|_{V^*}^2 + \frac{1}{2}\|\sigma_0\|^2 + \int_{Q_t}(\chi\sigma + J*\varphi)\varphi \\
&\quad + \int_{Q_t}[C\sigma_\alpha(f(\varphi_\alpha) - f(\varphi_{\alpha,\beta}))]\sigma \\
&\quad + \int_{Q_t}\big[\mu + P\sigma f(\varphi_{\alpha,\beta}) + (P\sigma_\alpha - A)(f(\varphi_{\alpha,\beta}) - f(\varphi_\alpha))\big]\mathcal{R}^{-1}(\alpha\mu+\varphi).
\end{aligned}$$

All the terms on the right-hand side, except for the first one, can be handled in exactly the same way as in the proof of Theorem 4.2. As for the first one, we use the Young inequality and estimate (4.107) to infer, for every $\delta > 0$,

$$-\beta\int_{Q_t}\partial_t\varphi_{\alpha,\beta}\varphi \leq \delta\int_{Q_t}|\varphi|^2 + \frac{\beta^2}{4\delta}\int_{Q_t}|\partial_t\varphi_{\alpha,\beta}|^2 \leq \delta\int_{Q_t}|\varphi|^2 + C_\delta\beta,$$

so that the first term on the right-hand side can be absorbed on the left provided to choose again δ small enough, which is indeed possible as we noted in the proof of Theorem 4.2. We can now argue as before and conclude using Gronwall's lemma. Moreover, the same argument on the limit problem yields uniqueness of solution for the system with $\beta = 0$, hence also that the convergences hold along the entire sequence and the proof of Theorem 4.13 is concluded.

□

4.3.3 Asymptotics Analysis as $\alpha, \beta \to 0$

The last two results we present deal with the asymptotic study of the system (4.1)–(4.5) as the parameters α and β go to 0 simultaneously.

Theorem 4.15 (Asymptotics: $\alpha, \beta \to 0$)**.** *Assume* **B1–B5***,* (4.63)*,* (4.89)*,* $\eta = 0$*, and suppose that*

$$\varphi_0, \sigma_0 \in H, \quad F(\varphi_0) \in L^1(\Omega). \tag{4.111}$$

For every $\alpha \in (0, \alpha_0)$ *and* $\beta \in (0, \beta_0)$*, let the initial data* $(\varphi_{0,\alpha,\beta}, \mu_{0,\alpha,\beta}, \sigma_{0,\alpha,\beta})$ *satisfy* (4.7) *and* (4.14)*, and denote by* $(\varphi_{\alpha,\beta}, \mu_{\alpha,\beta}, \sigma_{\alpha,\beta}, \xi_{\alpha,\beta})$ *the respective unique weak solution to the system* (4.1)-(4.5) *obtained from Theorem 4.1. Suppose also that, as* $(\alpha, \beta) \to (0, 0)$*,*

$$\varphi_{0,\alpha,\beta} \to \varphi_0 \quad \textit{strongly in } H, \qquad \sigma_{0,\alpha,\beta} \to \sigma_0 \quad \textit{strongly in } H, \tag{4.112}$$

and that there exists $M_0 > 0$ *such that, for every* $(\alpha, \beta) \in (0, \alpha_0) \times (0, \beta_0)$*,*

$$\beta^{1/2}\|\varphi_{0,\alpha,\beta}\|_V + \alpha^{1/2}\|\mu_{0,\alpha,\beta}\| + \|F(\varphi_{0,\alpha,\beta})\|_{L^1(\Omega)} \leq M_0. \tag{4.113}$$

Then, there exists a quadruplet $(\varphi, \mu, \sigma, \xi)$*, with*

$$\begin{aligned}
&\varphi \in H^1(0,T;V^*) \cap L^2(0,T;V),\\
&\mu = a\varphi - J * \varphi + \xi + F_2'(\varphi) - \chi\sigma \in L^2(0,T;V),\\
&\sigma \in H^1(0,T;V^*) \cap L^2(0,T;V) \cap L^\infty(Q),\\
&0 \leq \sigma(\mathbf{x},t) \leq 1 \quad \textit{for a.e. } \mathbf{x} \in \Omega, \quad \forall\, t \in [0,T],\\
&\xi \in L^2(0,T;V), \qquad \xi \in \partial F_1(\varphi) \qquad \textit{a.e. in } Q,\\
&\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 \qquad \textit{a.e. in } \Omega,
\end{aligned}$$

such that, for every $v \in V$*, almost everywhere in* $(0,T)$*, it holds*

$$\begin{aligned}
&\langle \partial_t \varphi, v\rangle + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega (\mathcal{P}\sigma - \mathcal{A}) f(\varphi) v,\\
&\langle \partial_t \sigma, v\rangle + \int_\Omega \nabla\sigma \cdot \nabla v + \mathcal{B}\int_\Omega (\sigma - \sigma_S) v + \mathcal{C}\int_\Omega \sigma f(\varphi) v = 0.
\end{aligned}$$

Moreover, as $(\alpha,\beta) \to (0,0)$*, along a non-relabelled subsequence it holds that*

$$\varphi_{\alpha,\beta} \to \varphi \quad \textit{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V), \tag{4.114}$$

$$\mu_{\alpha,\beta} \to \mu \quad \textit{weakly in } L^2(0,T;V), \tag{4.115}$$

$$\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta} \to \varphi \quad \textit{weakly in } H^1(0,T;V^*) \cap L^2(0,T;V), \tag{4.116}$$

$$\sigma_{\alpha,\beta} \to \sigma \quad \textit{weakly star in } H^1(0,T;V^*) \cap L^2(0,T;V) \cap L^\infty(Q), \tag{4.117}$$

$$\alpha\mu_{\alpha,\beta} \to 0 \quad \textit{strongly in } C^0([0,T];H) \cap L^2(0,T;V), \tag{4.118}$$

$$\beta\varphi_{\alpha,\beta} \to 0 \quad \textit{strongly in } H^1(0,T;H) \cap L^\infty(0,T;V), \tag{4.119}$$

hence in particular that

$$\begin{aligned}
&\varphi_{\alpha,\beta} \to \varphi \quad \textit{strongly in } L^2(0,T;H),\\
&\sigma_{\alpha,\beta} \to \sigma \quad \textit{strongly in } C^0([0,T];V^*) \cap L^2(0,T;H).
\end{aligned} \tag{4.120}$$

Proof of Theorem 4.15. Let us come back to estimate (4.24) with $\eta = 0$. We have

$$\begin{aligned}
&\frac{\alpha}{2}\|\mu_{\alpha,\beta}(t)\|^2 + (1+4c_a\alpha)\int_{Q_t} |\nabla\mu_{\alpha,\beta}|^2 + \beta\int_{Q_t} |\partial_t\varphi_{\alpha,\beta}|^2 + \int_\Omega F(\varphi_{\alpha,\beta}(t))\\
&\quad + \frac{1}{2}\|\sigma_{\alpha,\beta}(t)\|^2 + \int_{Q_t} |\nabla\sigma_{\alpha,\beta}|^2 + 2c_a\|(\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta})(t)\|^2\\
&\quad + 2c_a\beta\|\nabla\varphi_{\alpha,\beta}(t)\|^2 + 2c_aC_0\int_{Q_t} |\nabla\varphi_{\alpha,\beta}|^2\\
&\leq \frac{3}{2}\alpha\|\mu_{0,\alpha,\beta}\|^2 + (a^* + 4c_a)\|\varphi_{0,\alpha,\beta}\|^2 + 2c_a\beta\|\nabla\varphi_{0,\alpha,\beta}\|^2 + \|F(\varphi_{0,\alpha,\beta})\|_1\\
&\quad + \frac{1}{2}\|\sigma_{0,\alpha,\beta}\|^2 + \frac{c_a}{2}\|\varphi_{\alpha,\beta}(t)\|^2 + \chi\int_{Q_t} \sigma_{\alpha,\beta}\partial_t\varphi_{\alpha,\beta} + 4c_a\chi\int_{Q_t} \nabla\sigma_{\alpha,\beta}\cdot\nabla\varphi_{\alpha,\beta}\\
&\quad + C(1 + \int_{Q_t} |\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta}|^2 + \int_{Q_t} |\varphi_{\alpha,\beta}|^2 + \int_{Q_t} |\sigma_{\alpha,\beta}|^2)\\
&\quad + \int_{Q_t} (\mathcal{P}\sigma_{\alpha,\beta} - \mathcal{A}) f(\varphi_{\alpha,\beta})\mu_{\alpha,\beta},
\end{aligned} \tag{4.121}$$

where the constant $C > 0$ is independent of both α and β. Now, all the terms on the right-hand side referring to the initial data are uniformly bounded in both α and β thanks to assumptions (4.112)–(4.113). Moreover, as done in (4.102), on the left-hand side we have

$$2c_a\|(\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta})(t)\|^2 + \frac{\alpha}{2}\|\mu_{\alpha,\beta}(t)\|^2 \geq (c_a - \rho)\|\varphi_{\alpha,\beta}(t)\|^2 + 2\rho\alpha^2\|\mu_{\alpha,\beta}(t)\|^2$$

for every $\rho \in (0, c_a/2)$, so that the term on the right-hand side of the above inequality can be absorbed on the left-hand side of (4.121). Furthermore, proceeding again as in the proof of Theorem 4.12 and recalling that here $\eta = 0$, we have

$$\begin{aligned} \chi \int_{Q_t} \sigma_{\alpha,\beta}\partial_t\varphi_{\alpha,\beta} &= -\chi \int_0^t \langle \partial_t\sigma_{\alpha,\beta}(s), \varphi_{\alpha,\beta}(s)\rangle \,\mathrm{ds} \\ &\quad + \chi \int_\Omega \sigma_{\alpha,\beta}(t)\varphi_{\alpha,\beta}(t) - \chi \int_\Omega \sigma_{0,\alpha,\beta}\varphi_{0,\alpha,\beta} \\ &\leq \chi \int_{Q_t} \nabla\sigma_{\alpha,\beta}\cdot\nabla\varphi_{\alpha,\beta} + C(1 + \int_{Q_t} |\varphi_{\alpha,\beta}|^2 + \int_{Q_t} |\sigma_{\alpha,\beta}|^2) \\ &\quad + \delta\chi^2\|\varphi_{\alpha,\beta}(t)\|^2 + \frac{1}{4\delta}\|\sigma_{\alpha,\beta}(t)\|^2, \end{aligned}$$

for every $\delta > 0$. Moreover, we can choose $\bar{\delta}$ such that (4.104) are satisfied, so that the corresponding two terms on the right-hand side can be incorporated on the left. The remaining terms on the right-hand side of (4.121) containing χ can be handled as, for every $\kappa > 0$,

$$(\chi + 4c_a\chi) \int_{Q_t} \nabla\sigma_{\alpha,\beta}\cdot\nabla\varphi_{\alpha,\beta} \leq \kappa \int_{Q_t} |\nabla\sigma_{\alpha,\beta}|^2 + \frac{(\chi + 4c_a\chi)^2}{4\kappa} \int_{Q_t} |\nabla\varphi_{\alpha,\beta}|^2.$$

Again, the two terms on the right can be incorporated on the left-hand side of (4.121) provided that we choose κ such that

$$\kappa < 1, \qquad \frac{(\chi + 4c_a\chi)^2}{4\kappa} < 2c_aC_0,$$

which is indeed possible since (4.89) and the fact that $\eta = 0$ yield $\frac{(\chi+4c_a\chi)^2}{8c_aC_0} < 1$. To close the estimate, we only need to handle the last term on the right-hand side of (4.121): this can be done exactly in the same way as in the proof of Theorem 4.9. Indeed, on the right-hand side we have, thanks to the boundedness of f and the fact that $\|\sigma_{\alpha,\beta}\|_{L^\infty(Q)} \leq 1$,

$$\int_{Q_t} (\mathcal{P}\sigma_{\alpha,\beta} - \mathcal{A})f(\varphi_{\alpha,\beta})\mu_{\alpha,\beta} \leq \frac{1}{2}\int_{Q_t} |\nabla\mu_{\alpha,\beta}|^2 + C(1 + T_0^{1/2}\|(\mu_{\alpha,\beta})_\Omega\|_{L^2(0,t)})$$

for every $t \in [0, T_0]$ and $T_0 < T$. Furthermore, by comparison in equation (4.2) and thanks to (4.63), since $\beta \in (0, 1)$, we have

$$|(\mu_{\alpha,\beta}(t))_\Omega| \leq C\Big(1 + \beta\int_{Q_t} |\partial_t\varphi_{\alpha,\beta}(t)|^2 + \sup_{s\in[0,t]}\int_\Omega F(\varphi_{\alpha,\beta}(s)) + \sup_{s\in[0,t]}\|\sigma_{\alpha,\beta}(s)\|^2\Big).$$

Hence, using a patching argument as in the proof of Theorem 4.9, we deduce the following uniform estimates

$$
\begin{aligned}
&\|\varphi_{\alpha,\beta}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\mu_{\alpha,\beta}\|_{L^2(0,T;V)} \\
&\quad + \|\sigma_{\alpha,\beta}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq C, && (4.122)\\
&\|F(\varphi_{\alpha,\beta})\|_{L^\infty(0,T;L^1(\Omega))} \leq C, && (4.123)\\
&\alpha^{1/2}\|\mu_{\alpha,\beta}\|_{L^\infty(0,T;H)} + \beta^{1/2}\|\varphi_{\alpha,\beta}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq C. && (4.124)
\end{aligned}
$$

Comparison then in the system gives us in particular that

$$
\|\xi_{\alpha,\beta}\|_{L^2(0,T;H)} + \|\sigma_{\alpha,\beta}\|_{H^1(0,T;V^*)} + \|\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta}\|_{H^1(0,T;V^*)} \leq C, \qquad (4.125)
$$

as well as

$$
\alpha\beta^{1/2}\|\mu_{\alpha,\beta}\|_{H^1(0,T;V^*)} \leq C. \qquad (4.126)
$$

The uniform bound for $\sigma_{\alpha,\beta}$ in $L^\infty(Q)$ can be obtained as before using the same lines of argument employed in the proof of Theorem 4.1.

Passage to the Limit

The estimates (4.122)–(4.126) ensure, thanks to the classical compactness results, that there exists a quadruplet $(\varphi, \mu, \sigma, \xi)$, with

$$
\begin{aligned}
&\varphi \in H^1(0,T;V^*) \cap L^2(0,T;V), \qquad \mu \in L^2(0,T;V),\\
&\sigma \in H^1(0,T;V^*) \cap L^2(0,T;V) \cap L^\infty(Q),\\
&0 \leq \sigma(\mathbf{x},t) \leq 1 \quad \text{for a.e. } \mathbf{x} \in \Omega, \quad \forall\, t \in [0,T],\\
&\xi \in L^2(0,T;H),
\end{aligned}
$$

such that, as $(\alpha,\beta) \to 0$ it holds that, along a non-relabelled subsequence, (4.114)–(4.119) and (4.120) are fulfilled. In addition, setting $\lambda_{\alpha,\beta} := \alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta}$, we have

$$
\begin{aligned}
\lambda_{\alpha,\beta} \to \varphi \quad & \text{weakly star in } H^1(0,T;V^*) \cap L^2(0,T;V),\\
& \qquad \text{and strongly in } C^0([0,T];V^*) \cap L^2(0,T;H),\\
\xi_{\alpha,\beta} \to \xi \quad & \text{weakly in } L^2(0,T;H).
\end{aligned}
$$

In particular, by difference we deduce that

$$
\varphi_{\alpha,\beta} = \lambda_{\alpha,\beta} - \alpha\mu_{\alpha,\beta} \to \varphi \quad \text{strongly in } L^2(0,T;H)
$$

which readily implies that $\xi \in \partial F_1(\varphi)$ almost everywhere in Q, and that

$$
\begin{aligned}
f(\varphi_{\alpha,\beta}) \to f(\varphi) \quad & \text{strongly in } L^p(Q) \quad \forall\, p \geq 1,\\
F_2'(\varphi_{\alpha,\beta}) \to F_2'(\varphi) \quad & \text{strongly in } L^2(0,T;H).
\end{aligned}
$$

It is then a standard matter to let $\alpha, \beta \to 0$ in the weak formulation of (4.1)–(4.5) to conclude. Note in particular that by difference in the limit equation (4.2) we deduce the further regularity $\xi \in L^2(0,T;V)$, which concludes the proof of Theorem 4.15. □

Theorem 4.16 (Error estimate: $\alpha, \beta \to 0$)**.** *In the setting of Theorem 4.15, assume* (4.80)*. Suppose also that there exist a positive constant c_F such that*

$$F(s) \geq c_F|s|^4 - c_F^{-1} \quad \forall\, s \in \mathbb{R}, \tag{4.127}$$

and there exists $M_0 > 0$ such that, for every $(\alpha, \beta) \in (0, \alpha_0) \times (0, \beta_0)$,

$$\frac{\alpha^{1/4}}{\beta^{1/2}}(\|\mu_{0,\alpha,\beta}\| + \|F'(\varphi_{0,\alpha,\beta})\|) + \alpha^{1/4}(\|\mu_{0,\alpha,\beta}\|_V + \|\sigma_{0,\alpha,\beta}\|_V) \leq M_0. \tag{4.128}$$

Then the solution $(\varphi, \mu, \sigma, \xi)$ to the system (4.1)–(4.5) *with $\alpha = \beta = 0$ is unique. Moreover, the convergences obtained in Theorem 4.15 hold along every zero subsequence $\{(\alpha_k, \beta_k)\}_k$ satisfying*

$$\limsup_{k\to\infty} \frac{\alpha_k^{1/2}}{\beta_k} < +\infty, \tag{4.129}$$

and in this case, there exists $K > 0$, independent of k, such that the following error estimate holds:

$$\begin{aligned}
&\|\varphi_{\alpha_k,\beta_k} - \varphi\|_{C^0([0,T];V^*)\cap L^2(0,T;H)} + \|\sigma_{\alpha_k,\beta_k} - \sigma\|_{C^0([0,T];H)\cap L^2(0,T;V)} \\
&\quad \leq K(\alpha_k^{1/4} + \beta_k^{1/2} + \|\varphi_{0,\alpha_k,\beta_k} - \varphi_0\|_{V^*} + \|\sigma_{0,\alpha_k,\beta_k} - \sigma_0\|).
\end{aligned}$$

Proof of Theorem 4.16. The idea is to adapt the argument presented in the proof of Theorem 4.10. First of all, we need to prove a refined estimate. Proceeding as in the proof of Theorem 4.10, we know that

$$\begin{aligned}
&\alpha^{3/2}\int_{Q_t} |\partial_t\mu_{\alpha,\beta}|^2 + \frac{\alpha^{1/2}}{2}\|\nabla\mu_{\alpha,\beta}(t)\|^2 + \frac{\beta\alpha^{1/2}}{2}\|\partial_t\varphi_{\alpha,\beta}(t)\|^2 + C_0\alpha^{1/2}\int_{Q_t} |\partial_t\varphi_{\alpha,\beta}|^2 \\
&\quad + \alpha^{1/2}\int_{Q_t} |\partial_t\sigma_{\alpha,\beta}|^2 + \frac{\alpha^{1/2}}{2}\|\nabla\sigma_{\alpha,\beta}(t)\|^2 \\
&\leq \frac{\alpha^{1/2}}{2}\|\nabla\mu_{0,\alpha,\beta}\|^2 + \frac{\beta\alpha^{1/2}}{2}\|\partial_t\varphi_{\alpha,\beta}(0)\|^2 + \frac{\alpha^{1/2}}{2}\|\nabla\sigma_{0,\alpha,\beta}\|^2 \\
&\quad + \alpha^{1/2}\int_{Q_t} (\mathcal{P}\sigma_{\alpha,\beta} - \mathcal{A})f(\varphi_{\alpha,\beta})\partial_t\mu_{\alpha,\beta} \\
&\quad + \alpha^{1/2}\int_{Q_t} (J * (\partial_t\varphi_{\alpha,\beta}) + \chi\partial_t\sigma_{\alpha,\beta})\partial_t\varphi_{\alpha,\beta} \\
&\quad + \alpha^{1/2}\int_{Q_t} (\mathcal{B}(\sigma_S - \sigma_{\alpha,\beta}) - Ch(\varphi_{\alpha,\beta})\sigma_{\alpha,\beta})\partial_t\sigma_{\alpha,\beta}.
\end{aligned} \tag{4.130}$$

The first and third terms on the right-hand side are uniformly bounded in α and β due to assumptions (4.113) and (4.128). As for the second term on the right-hand side, using (4.2) we realise that

$$\mu_{0,\alpha,\beta} = \beta\partial_t\varphi_{\alpha,\beta}(0) + a\varphi_{0,\alpha,\beta} - J * \varphi_{0,\alpha,\beta} + F'(\varphi_{0,\alpha,\beta}) - \chi\sigma_{0,\alpha,\beta},$$

so that, multiplying both sides by $\alpha^{1/4}/\beta^{1/2}$ and squaring,

$$\beta\alpha^{1/2}\|\partial_t\varphi_{\alpha,\beta}(0)\|^2 \leq 5\frac{\alpha^{1/2}}{\beta}(\|\mu_{0,\alpha,\beta}\|^2+2(a^*)^2\|\varphi_{0,\alpha,\beta}\|^2+\|F'(\varphi_{0,\alpha,\beta})\|^2+\chi^2\|\sigma_{0,\alpha,\beta}\|^2),$$

from which we deduce by (4.128) that the second term on the right-hand side of (4.130) is uniformly bounded in α and β. Let us focus on the fourth term on the right-hand side: proceeding as in the proof of Theorem 4.10, this can be bounded using integration by parts and the Young inequality by the quantity

$$\frac{\alpha^{1/2}}{4}\int_{Q_t}|\partial_t\sigma_{\alpha,\beta}|^2 + C\alpha^{1/2}(\|\mu_{\alpha,\beta}\|^2_{L^2(0,T;H)} + \|\partial_t\varphi_{\alpha,\beta}\|^2_{L^2(0,T;H)} + \alpha^{1/2}\|\mu_{\alpha,\beta}\|_{C^0([0,T];H)})$$

for a positive constant C independent of α and β. The first term can be then incorporated on the left-hand side, and the remaining others are uniformly bounded in α and β thanks to the estimates (4.122), (4.124), and condition (4.129) on (α,β). Finally, noting that

$$\begin{aligned}\alpha^{1/2}\int_{Q_t}(J*\partial_t\varphi_{\alpha,\beta})\partial_t\varphi_{\alpha,\beta} &\leq (a^*+b^*)\alpha^{1/2}\int_0^t\|\partial_t\varphi_{\alpha,\beta}(s)\|\|\partial_t\varphi_{\alpha,\beta}(s)\|_{V^*}\mathrm{ds}\\ &\leq C\alpha^{1/2}\|\partial_t\varphi_{\alpha,\beta}\|^2_{L^2(0,T;H)}.\end{aligned}$$

The remaining terms on the right-hand side of (4.130) can be handled similarly, using the averaged Young inequality, estimate (4.122)–(4.124), and condition (4.129). Thus, there exists $C>0$, independent of both α and β, such that

$$\alpha^{3/4}\|\mu_{\alpha,\beta}\|_{H^1(0,T;H)} + \alpha^{1/4}\|\mu_{\alpha,\beta}\|_{L^\infty(0,T;V)} \leq C, \tag{4.131}$$
$$\beta^{1/2}\alpha^{1/4}\|\varphi_{\alpha,\beta}\|_{W^{1,\infty}(0,T;H)} + \alpha^{1/4}\|\sigma_{\alpha,\beta}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq C. \tag{4.132}$$

We are now ready to show the error estimate. Setting

$$\begin{aligned}&\overline{\varphi} := \varphi_{\alpha,\beta}-\varphi, \quad \overline{\mu} := \mu_{\alpha,\beta}-\mu, \quad \overline{\sigma} := \sigma_{\alpha,\beta}-\sigma,\\ &\overline{\varphi}_0 := \varphi_{0,\alpha,\beta}-\varphi_0, \quad \overline{\sigma}_0 := \sigma_{0,\alpha,\beta}-\sigma_0,\end{aligned}$$

we write the difference of the system (4.1)–(4.5) with $\eta=0$ at $\alpha,\beta>0$ and $\alpha=\beta=0$ to find that

$$\alpha\partial_t\mu_{\alpha,\beta} + \partial_t\overline{\varphi} - \Delta\overline{\mu} = \mathcal{P}\overline{\sigma}f(\varphi_{\alpha,\beta}) + (\mathcal{P}\sigma-\mathcal{A})(f(\varphi_{\alpha,\beta})-f(\varphi)) \quad \text{in } Q, \tag{4.133}$$
$$\overline{\mu} = \beta\partial_t\varphi_{\alpha,\beta} + a\overline{\varphi} - J*\overline{\varphi} + F'(\varphi_{\alpha,\beta}) - F'(\varphi) - \chi\overline{\sigma} \quad \text{in } Q, \tag{4.134}$$
$$\partial_t\overline{\sigma} - \Delta\overline{\sigma} + \mathcal{B}\overline{\sigma} + \mathcal{C}\overline{\sigma}f(\varphi_{\alpha,\beta}) = \mathcal{C}\sigma(f(\varphi)-f(\varphi_{\alpha,\beta})) \quad \text{in } Q, \tag{4.135}$$
$$\partial_{\mathbf{n}}\overline{\mu} = \partial_{\mathbf{n}}\overline{\sigma} = 0 \quad \text{on } \Sigma, \tag{4.136}$$
$$\overline{\varphi}(0) = \overline{\varphi}_0, \quad \overline{\sigma}(0) = \overline{\sigma}_0 \quad \text{in } \Omega\,, \tag{4.137}$$

where the equations have to be intended in the usual variational framework. We then test (4.133) by $\mathcal{N}(\overline{\varphi} - (\overline{\varphi})_\Omega)$, (4.134) by $\overline{\varphi} - (\overline{\varphi})_\Omega$, (4.135) by $\overline{\sigma}$, integrate over Q_t, add the resulting equalities and use **B5** to get

$$\begin{aligned}
&\frac{1}{2}\|(\overline{\varphi} - (\overline{\varphi})_\Omega)(t)\|_*^2 + C_0 \int_{Q_t} |\overline{\varphi}|^2 + \frac{1}{2}\|\overline{\sigma}(t)\|^2 + \int_{Q_t} |\nabla\overline{\sigma}|^2 + \int_{Q_t} (\mathcal{B} + Cf(\varphi_{\alpha,\beta}))|\overline{\sigma}|^2 \\
&\quad = \frac{1}{2}\|\overline{\varphi}_0 - (\overline{\varphi}_0)_\Omega\|_*^2 + \frac{1}{2}\|\overline{\sigma}_0\|^2 - \alpha \int_{Q_t} \partial_t\mu_{\alpha,\beta}\mathcal{N}(\overline{\varphi} - (\overline{\varphi})_\Omega) \\
&\qquad + \int_{Q_t} \overline{\varphi}(\chi\overline{\sigma} - \beta\partial_t\varphi_{\alpha,\beta}) + \int_{Q_t} \overline{\mu}(\overline{\varphi})_\Omega + \int_{Q_t} (J * \overline{\varphi})\overline{\varphi} \\
&\qquad + \mathcal{C} \int_{Q_t} \sigma(f(\varphi) - f(\varphi_{\alpha,\beta}))\overline{\sigma} \\
&\qquad + \int_{Q_t} \Big(\mathcal{P}\overline{\sigma}f(\varphi_{\alpha,\beta}) + (\mathcal{P}\sigma - \mathcal{A})(f(\varphi_{\alpha,\beta}) - f(\varphi))\Big)\mathcal{N}(\overline{\varphi} - (\overline{\varphi})_\Omega). \qquad (4.138)
\end{aligned}$$

Now, note that the Young inequality and the estimates (4.124) and (4.131) yield

$$\begin{aligned}
&- \alpha \int_{Q_t} \partial_t\mu_{\alpha,\beta}\mathcal{N}(\overline{\varphi} - (\overline{\varphi})_\Omega) + \int_{Q_t} \overline{\varphi}(\chi\overline{\sigma} - \beta\partial_t\varphi_{\alpha,\beta}) \\
&\quad \leq \alpha^2\|\partial_t\mu_{\alpha,\beta}\|_{L^2(0,T;H)}^2 + \frac{1}{4}\int_0^t \|\mathcal{N}(\overline{\varphi} - (\overline{\varphi})_\Omega)(s)\|_H^2 \,\mathrm{ds} + \frac{C_0}{4}\int_{Q_t} |\overline{\varphi}|^2 \\
&\qquad + \frac{2}{C_0}(\beta^2\|\partial_t\varphi_{\alpha,\beta}\|_{L^2(0,T;H)}^2 + \chi^2 \int_{Q_t} |\overline{\sigma}|^2) \\
&\quad \leq \frac{C_0}{4}\int_{Q_t} |\overline{\varphi}|^2 + C(\alpha^{1/2} + \beta + \int_0^t \|(\overline{\varphi} - (\overline{\varphi})_\Omega)(s)\|_*^2 \,\mathrm{ds} + \int_{Q_t} |\overline{\sigma}|^2),
\end{aligned}$$

for a certain constant $C > 0$ independent of α and β. Furthermore, using the boundedness and Lipschitz continuity of f, and the fact that $\|\sigma\|_{L^\infty(Q)} \leq 1$, the last two terms in (4.138) can be handled again by the Young inequality as

$$\begin{aligned}
&\mathcal{C} \int_{Q_t} \sigma(f(\varphi) - f(\varphi_{\alpha,\beta}))\overline{\sigma} \\
&\qquad + \int_{Q_t} \Big(\mathcal{P}\overline{\sigma}f(\varphi_{\alpha,\beta}) + (\mathcal{P}\sigma - \mathcal{A})(f(\varphi_{\alpha,\beta}) - f(\varphi))\Big)\mathcal{N}(\overline{\varphi} - (\overline{\varphi})_\Omega) \\
&\quad \leq \frac{C_0}{4}\int_{Q_t} |\overline{\varphi}|^2 + C(\int_{Q_t} |\overline{\sigma}|^2 + \int_0^t \|(\overline{\varphi} - (\overline{\varphi})_\Omega)(s)\|_*^2 \,\mathrm{ds})
\end{aligned}$$

for a certain $C > 0$ independent of α and β, and similarly we have the estimate

$$\int_{Q_t} (J * \overline{\varphi})\overline{\varphi} \leq (a^* + b^*) \int_0^t \|\overline{\varphi}(s)\|\|\overline{\varphi}(s)\|_* \mathrm{ds} \leq \frac{C_0}{8}\int_{Q_t} |\overline{\varphi}|^2 + C\int_0^t \|\overline{\varphi}(s)\|_*^2 \mathrm{ds}.$$

Finally, as for the fifth term on the right-hand side of (4.138) we have, for a positive δ yet to be chosen,

$$\int_{Q_t} \overline{\mu}(\overline{\varphi})_\Omega = |\Omega| \int_0^t (\overline{\varphi}(s))_\Omega(\overline{\mu}(s))_\Omega \,\mathrm{ds} \leq \delta\int_0^t |(\overline{\mu}(s))_\Omega|^2 \,\mathrm{ds} + \frac{|\Omega|}{4\delta}\int_0^t |(\overline{\varphi}(s))_\Omega|^2 \,\mathrm{ds},$$

where, by comparison in equation (4.134) and by using the estimate (4.124),

$$\int_0^t |(\overline{\mu}(s))_\Omega|^2 \,\mathrm{ds} \le C_*(\beta + \int_0^t \|F'(\varphi_{\alpha,\beta}(s)) - F'(\varphi(s))\|_1^2 \,\mathrm{ds} + \chi^2 \int_{Q_t} |\overline{\sigma}|^2)$$

for some $C_* > 0$ independent of both α and β. Next, owing to (4.80) and the Hölder inequality, we infer that

$$\int_0^t \|F'(\varphi_{\alpha,\beta}(s)) - F'(\varphi(s))\|_1^2 \,\mathrm{ds} \le C_F^2 \int_0^t \|(1 + |\varphi_{\alpha,\beta}|^2 + |\varphi|^2)\overline{\varphi}\|_1^2 \,\mathrm{ds}$$
$$\le C(1 + \|\varphi_{\alpha,\beta}\|^2_{L^\infty(0,T;L^4(\Omega))} + \|\varphi\|^2_{L^\infty(0,T;L^4(\Omega))}) \int_0^t \|\overline{\varphi}(s)\|^2 \,\mathrm{ds},$$

which yields in turn, due to assumption (4.127) and to the previous estimates,

$$\int_0^t \|F'(\varphi_{\alpha,\beta}(s)) - F'(\varphi(s))\|_1^2 \,\mathrm{ds} \le C^* \int_{Q_t} |\overline{\varphi}|^2$$

for a constant $C^* > 0$ independent of α and β. Thus, collecting the above estimates and rearranging the terms, we see that choosing $\delta > 0$ sufficiently small, for example $\delta = \frac{C_0}{4C_*C^*}$, we are left with

$$\frac{1}{2}\|(\overline{\varphi} - (\overline{\varphi})_\Omega)(t)\|_*^2 + \frac{C_0}{8}\int_{Q_t} |\overline{\varphi}|^2 + \frac{1}{2}\|\overline{\sigma}(t)\|^2 + \int_{Q_t} |\nabla\overline{\sigma}|^2$$
$$\le \frac{1}{2}\|\overline{\varphi}_0\|_*^2 + \frac{1}{2}\|\overline{\sigma}_0\|^2 + C\Big(\alpha^{1/2} + \beta + \int_0^t \|(\overline{\varphi} - (\overline{\varphi})_\Omega)(s)\|_*^2 \,\mathrm{ds}$$
$$+ \int_{Q_t} |\overline{\sigma}|^2 + \int_0^t |(\overline{\varphi}(s))_\Omega|^2 \,\mathrm{ds}\Big). \tag{4.139}$$

To conclude, we only need to handle the last term on the right-hand side of (4.139). To this end, note that integrating equation (4.133) on Ω and testing by $(\overline{\varphi})_\Omega$ yields, using the estimate (4.131), the Young inequality, the boundedness of σ and f, and the Lipschitz continuity of f,

$$\frac{1}{2}|(\overline{\varphi}(t))_\Omega|^2$$
$$= \frac{1}{2}|(\overline{\varphi}_0)_\Omega|^2 - \alpha \int_0^t (\overline{\varphi}(s))_\Omega (\partial_t \mu_{\alpha,\beta}(s))_\Omega \,\mathrm{ds}$$
$$+ \int_0^t \Big(\mathcal{P}\overline{\sigma}(s) f(\varphi_{\alpha,\beta}(s)) + (\mathcal{P}\sigma(s) - \mathcal{A})(f(\varphi_{\alpha,\beta}(s)) - f(\varphi(s)))\Big)_\Omega (\overline{\varphi}(s))_\Omega \,\mathrm{ds}$$
$$\le \frac{1}{2}\|\overline{\varphi}_0\|_*^2 + C(\alpha^{1/2} + \int_0^t |(\overline{\varphi}(s))_\Omega|^2 \,\mathrm{ds} + \int_{Q_t} |\overline{\sigma}|^2) + \frac{C_0}{16}\int_{Q_t} |\overline{\varphi}|^2. \tag{4.140}$$

Summing then (4.139) and (4.140), we infer that

$$\|\overline{\varphi}(t)\|_*^2 + \int_{Q_t} |\overline{\varphi}|^2 + \|\overline{\sigma}(t)\|^2 + \int_{Q_t} |\nabla\overline{\sigma}|^2$$
$$\le C(\|\overline{\varphi}_0\|_*^2 + \|\overline{\sigma}_0\|^2 + \alpha^{1/2} + \beta + \int_0^t \|\overline{\varphi}(s)\|_*^2 \,\mathrm{ds} + \int_{Q_t} |\overline{\sigma}|^2)$$

for a certain constant C, independent of α and β. Therefore, we invoke the Gronwall lemma to complete the proof of Theorem 4.16. $\square$

Part II

Optimal Control Theory

CHAPTER 5

Optimal Control Theory of the Local Model

The goal of this chapter is to investigate an optimal control problem whose governing equation is given by the local system (3.1)–(3.5) analysed in Chapter 3, with standard tracking type cost functional, and where the role of control variable is played by the source term g appearing in equation (3.3).

First, we assume the presence of both the relaxation parameters α and β and we establish the existence of a minimiser and the related first-order necessary conditions for optimality. Then, by using asymptotic approaches, we let the parameters α and β go to zero to solve similar minimisation problems associated with (3.1)–(3.5) with $\alpha = 0$ and $\beta > 0$ and $\alpha > 0$ and $\beta = 0$, respectively.

5.1 Optimal Control Theory of the Local Relaxed Model

In this first part of the chapter, we aim at minimising the *cost functional* of standard tracking type form

$$\begin{aligned}\mathcal{J}(\varphi, \sigma, u) = \frac{b_1}{2}\|\varphi - \varphi_Q\|^2_{L^2(Q)} + \frac{b_2}{2}\|\varphi(T) - \varphi_\Omega\|^2_{L^2(\Omega)} + \frac{b_3}{2}\|\sigma - \sigma_Q\|^2_{L^2(Q)} \\ + \frac{b_4}{2}\|\sigma(T) - \sigma_\Omega\|^2_{L^2(\Omega)} + \frac{b_0}{2}\|u\|^2_{L^2(Q)}, \qquad (5.1)\end{aligned}$$

subject to (3.1)–(3.5) with $g = u$ and the control constraint

$$u \in \mathcal{U}_{\rm ad},$$

where b_0, b_1, b_2, b_3, b_4 denote some non-negative constants (not all zero), and $\varphi_Q, \sigma_Q : Q \to \mathbb{R}$, $\varphi_\Omega, \sigma_\Omega : \Omega \to \mathbb{R}$ some prescribed target functions. Moreover, we assume that the space of admissible controls is defined as

$$\mathcal{U}_{\rm ad} := \{u \in L^\infty(Q) : u_* \leq u \leq u^* \text{ a.e. in } Q\}, \tag{5.2}$$

for fixed functions $u_*, u^* \in L^\infty(Q)$. Notice that assuming $u_* \leq u^*$ a.e. in Q implies that $\mathcal{U}_{\rm ad}$ is a non-empty, closed and convex subset of $L^2(Q)$. In addition, since in the following it will be sometimes necessary to work with an open superset of $\mathcal{U}_{\rm ad}$, we fix some constant $R > 0$ such that the open ball

$$\mathcal{U}_R := \{u \in L^2(Q) : \|u\|_{L^2(Q)} < R\} \quad \text{contains} \quad \mathcal{U}_{\rm ad}. \tag{5.3}$$

Thus, the control u may represent a supply of a nutrient or a medication in chemotherapy.

Before moving on, let us spend some words on the classical tracking type structure of the cost functional $\mathcal{J}$ of which (5.1) is an example. As the functions $\varphi_Q, \sigma_Q, \varphi_\Omega, \sigma_\Omega$ are fixed, they represent some targets we want to approach. In fact, for instance, the first term in the cost functional $\frac{b_1}{2}\|\varphi - \varphi_Q\|^2_{L^2(Q)}$ is mininimised when the state variable φ is as close as possible to the given φ_Q in the sense of $L^2(Q)$-norm. In a similar fashion, it goes for the other variables. Thus, in the context of tumor growth, the functions $\varphi_Q, \sigma_Q, \varphi_\Omega, \sigma_\Omega$ should be chosen as stable configurations of the system or as some desirable objective configurations which, e.g., are meaningful for surgery. Differently, the last term $\frac{b_0}{2}\|u\|^2_{L^2(Q)}$ is a classical L^2-penalisation on large values of the control variable designing the side-effect that the dispensation of too many drugs to the patient may cause. Moreover, let us notice that the constants b_0, b_1, b_2, b_3, b_4 can be chosen accordingly to the therapeutic goal we are interested in. Other terms of interest that can be possibly included in the cost functional are the initial data, in the context of parameter estimation, as in [81], the time treatment as in [30, 93, 139], sparsity effects as in [94, 142], and the estimation of the physical parameters appearing in the model as in [109, 133].

Summarising, the optimal control problem we are going to deal with in this chapter is:

$(CP)_{\alpha,\beta}$ Minimise $\mathcal{J}(\varphi, \sigma, u)$ subject to:
(i) (φ, μ, σ) yields a solution to (3.1)–(3.5) obtained from Theorem 3.5 with $g = u$;
(ii) $u \in \mathcal{U}_{\rm ad}$.

In the second part of the chapter, after the minimisation problem $(CP)_{\alpha,\beta}$ has been treated, by employing the asymptotic results established in Chapter 3, we show that the obtained results for $(CP)_{\alpha,\beta}$ allow us to investigate the asymptotic behaviour of

$(CP)_{\alpha,\beta}$ as α and β approach zero in the state system (3.1)–(3.5). We will employ the symbols $(CP)_\beta$ and $(CP)_\alpha$ to denote the corresponding optimal control problems in which $\alpha = 0$, $\beta > 0$ and $\beta = 0$, $\alpha > 0$, respectively. Namely, we set

$(CP)_\beta$ Minimise $\mathcal{J}(\varphi, \sigma, u)$ subject to:
(i) (φ, μ, σ) yields a solution to (3.1)–(3.5) obtained from Theorem 3.9 with $g = u$, $\alpha = 0$;
(ii) $u \in \mathcal{U}_{\mathrm{ad}}$.

The optimal control problem $(CP)_\alpha$ can be defined analogously by using Theorem 3.13. Throughout this section, we employ the notation u instead of g (cf. Chapter 3) since here the source term g in (3.1)–(3.5) plays the role of control variable. For the optimal control problem, we postulate the following structural assumptions:

C1 b_0, b_1, b_2, b_3, b_4 are non-negative constants, not all zero.

C2 $\varphi_Q, \sigma_Q : Q \to \mathbb{R}$, $\varphi_\Omega, \sigma_\Omega : \Omega \to \mathbb{R}$ and $\varphi_Q, \sigma_Q \in L^2(Q)$, $\varphi_\Omega, \sigma_\Omega \in L^2(\Omega)$.

C3 $u_*, u^* \in L^\infty(Q)$ with $u_* \le u^*$ a.e. in Q.

By virtue of the well-posedness result established by Theorem 3.5, we already know that the solution mapping, also referred to as the *control-to-state operator*, is well defined between the spaces specified in the statement. Namely, we have

$$\mathcal{S} : \mathcal{U}_R \to \mathcal{Y}, \quad u \mapsto \mathcal{S}(u) = (\varphi, \mu, \sigma),$$

where the triplet (φ, μ, σ) is the unique strong solution to (3.1)–(3.5) obtained from Theorem 3.5, and the solution space $\mathcal{Y}$ is identified by the regularity requirements (3.22)–(3.24). Moreover, we use $\mathcal{S}_i$ to denote the i-th component of the solution operator $\mathcal{S}$.

Besides, a wide number of results concerning optimal controllability for similar tumor growth models have been performed. Up to the author's knowledge, the first optimal control problem for system (3.1)–(3.5) is the work by P. Colli et al. [40]. There, the authors investigate the classical control problem with tracking type cost functional, with the choices $\alpha = \beta = \chi = 0$ and where the source term g, there called u, acts as a control. Moreover, they were forced to restrict the investigation to the case of polynomial-growth type potentials. The inclusion of singular, while regular, potentials has been tackled by the author in [136] for (3.1)–(3.5) with the choices $\alpha, \beta \ge 0$, $\chi = 0$. As appears in the strong well-posedness result presented in Chapter 3 (cf. Theorem 3.5), the relaxation terms $\alpha\partial_t\mu$ and $\beta\partial_t\varphi$ allow establishing the separation property so that the optimal control problem treated in [40] can be extended to the case of the logarithmic potential (1.8): this extension will be presented in this first part of this chapter. Next, the same author, using the so-called deep-quench asymptotic technique, proved in [138] how non-smooth potentials like the double-obstacle potential can also be admitted. The second part of this chapter is devoted to the asymptotic analysis performed in [137, 140]. There the author showed that it is possible to let α and β approach zero separately in order to recover the existence of optimal controls and to characterise the corresponding first-order necessary conditions for optimality. A control problem similar to [136] has been addressed recently by C. Cavaterra et al. in [143] with a different choice of the cost functional which includes long-time treatment penalisation with the choices $\alpha = \beta = \chi = 0$ taking inspiration from the previous contribution [93] (see also [139]). For optimal control problems with different choices for the source terms like (1.24) and for different phase-field models of Cahn–Hilliard type which may also include velocity flow effects, we mention [37, 43, 46, 47, 63, 64]. We also refer to [109] (see also [110]) by C. Kahle et al., where a different choice of the cost functional and control variables turn the optimal control problem in a parameter identification problem: this idea will be used in Chapter 6 for the nonlocal models (4.1)–(4.5) studied in Chapter 4.

To conclude the overview, let us mention the work by C. Orrieri et al. [130], where a phase-field model for tumor growth is analysed also taking into account possible stochastic perturbations of the system when two Wiener type noises act on the proliferation of tumor cells and the evolution of nutrient.

5.1.1 Existence of a Minimiser

The first issue we are going to address concerns the existence of minimisers of the minimisation problem $(CP)_{\alpha,\beta}$. In this direction, let us point out that the proof easily follows from combining the direct method of calculus of variations presented by Theorem 2.16 with the strong well-posedness result established by Theorem 3.5.

Theorem 5.1. *Assume* **A1–A4**, **C1–C3***, and* $\alpha, \beta \in (0, 1)$. *Then, the optimal control problem* $(CP)_{\alpha,\beta}$ *admits at least a minimiser.*

Proof of Theorem 5.1. To begin with, let us note that the cost functional $\mathcal{J}$ is nonnegative so that we can consider a minimising sequence $\{u_n\}_n$ of elements of $\mathcal{U}_{\text{ad}}$ with the corresponding sequence of states $\{(\varphi_n, \mu_n, \sigma_n)\}_n$. Namely, we have

$$\begin{aligned}
&(\varphi_n, \mu_n, \sigma_n) = \mathcal{S}(u_n) \quad \text{for every } n \in \mathbb{N}, \\
&\lim_{n\to\infty} \mathcal{J}(\varphi_n, \sigma_n, u_n) = \inf \big\{ \mathcal{J}(\varphi, \sigma, u) : u \in \mathcal{U}_{\text{ad}},\ (\varphi, \sigma) = (\mathcal{S}_1(u), \mathcal{S}_3(u)) \big\} =: \lambda \geq 0.
\end{aligned}$$

Furthermore, from the estimate (3.26), which is uniform in n, we infer the existence of limits $\overline{u} \in \mathcal{U}_{\text{ad}}$ and $(\overline{\varphi}, \overline{\mu}, \overline{\sigma})$ such that along a non-relabelled subsequence, as $n \to \infty$,

$$\begin{aligned}
&u_n \to \overline{u} \ \text{ weakly star in } L^\infty(Q), \\
&\mu_n \to \overline{\mu} \ \text{ weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap L^\infty(Q), \\
&\varphi_n \to \overline{\varphi} \ \text{ weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \\
&\sigma_n \to \overline{\sigma} \ \text{ weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W).
\end{aligned}$$

Moreover, owing to standard compactness results (cf. Lemma 2.4) we have that, possible up to a further extraction and as $n \to \infty$,

$$\varphi_n \to \overline{\varphi} \quad \text{strongly in } C^0([0,T]; C^0(\overline{\Omega})) \ \text{ and a.e. in } Q.$$

Using then the Lebesgue dominated convergence theorem and the continuity of the nonlinear functions F' and P it is a standard matter to deduce that

$$F'(\varphi_n) \to F'(\overline{\varphi}) \text{ and } P(\varphi_n) \to P(\overline{\varphi}) \quad \text{strongly in } C^0([0,T]; C^0(\overline{\Omega})).$$

These considerations allow us passing to the limit as $n \to \infty$ in the variational formulation of (3.1)–(3.5) written for $(\varphi_n, \mu_n, \sigma_n)$ so that $\mathcal{S}(\overline{u}) = (\overline{\varphi}, \overline{\mu}, \overline{\sigma})$. Lastly, due to the weak sequential lower semicontinuity of $\mathcal{J}$ we eventually realise that $\overline{u}$ and $(\overline{\varphi}, \overline{\mu}, \overline{\sigma})$ is indeed a minimiser for $(CP)_{\alpha,\beta}$. This conclude the proof of Theorem 5.1. □

5.1.2 Linearised System

In this section, we start investigating the first-order necessary conditions for optimality. The first step consists in establishing the Fréchet differentiability of the solution operator $\mathcal{S}$. It can be shown (cf. Theorem 5.3) that the control-to-state operator $\mathcal{S}$ is Fréchet differentiable between suitable Banach spaces and that its directional derivative is captured by the unique solution of the linearised system of (3.1)–(3.5) along a suitable direction. Thus, let us set some preliminary notation: with $\overline{u}$ we denote a fixed admissible control, not necessarily optimal, with the corresponding state $(\overline{\varphi}, \overline{\mu}, \overline{\sigma}) := \mathcal{S}(\overline{u})$. Hence, for every $h \in L^2(Q)$, the linearised system to (3.1)–(3.5), expressed in the variables (ϑ, ν, ρ), reads as

$$\alpha\partial_t\nu + \partial_t\vartheta - \Delta\nu = P'(\overline{\varphi})(\overline{\sigma} - \overline{\mu})\vartheta + P(\overline{\varphi})(\rho - \nu) \quad \text{in } Q, \tag{5.4}$$
$$\nu = \beta\partial_t\vartheta - \Delta\vartheta + F''(\overline{\varphi})\vartheta \quad \text{in } Q, \tag{5.5}$$
$$\partial_t\rho - \Delta\rho = -P'(\overline{\varphi})(\overline{\sigma} - \overline{\mu})\vartheta - P(\overline{\varphi})(\rho - \nu) + h \quad \text{in } Q, \tag{5.6}$$
$$\partial_{\mathbf{n}}\vartheta = \partial_{\mathbf{n}}\nu = \partial_{\mathbf{n}}\rho = 0 \quad \text{on } \Sigma, \tag{5.7}$$
$$\vartheta(0) = \nu(0) = \rho(0) = 0 \quad \text{in } \Omega. \tag{5.8}$$

Here, is the corresponding well-posedness result.

Theorem 5.2 (Well-posedness of the linearised system: $\alpha, \beta > 0$)**.** *Assume that* **A1–A4**, **C1–C3** *hold, and* $\alpha, \beta \in (0,1)$. *Then, for every* $h \in L^2(Q)$, *the linarised system* (5.4)–(5.8) *admits a unique solution* (ϑ, ν, ρ) *such that*

$$\vartheta, \nu, \rho \in H^1(0,T; H) \cap L^\infty(0,T; V) \cap L^2(0,T; W). \tag{5.9}$$

Proof of Theorem 5.2. The idea of the proof consists in applying a Faedo–Galerkin scheme (cf. Section 2.4.2). Due to the well-known spectral property of the operator $-\Delta + I$, we consider the family $\{w_j\}_j$ of eigenfunctions for the eigenvalue problem

$$\begin{cases} -\Delta w_j + w_j = \lambda_j w_j & \text{in } \Omega, \\ \partial_{\mathbf{n}} w_j = 0 & \text{on } \Gamma, \end{cases} \tag{5.10}$$

which produces an orthonormal Schauder basis in H. Moreover, $\{w_j\}_j$ can be renormalised to have a complete orthonormal system in $(H, (\cdot,\cdot))$ which is also orthogonal in $(V, (\cdot,\cdot)_V)$. For every given n, we denote by $\mathcal{W}_n := \operatorname{span}\{w_1, ..., w_n\}$, and with $\mathbb{P}_n$ the corresponding projection. We then aim at finding functions $(\vartheta_n, \nu_n, \rho_n)$ of the form

$$\vartheta_n(\mathbf{x}, t) = \sum_{k=1}^{n} a_k^n(t) w_k(\mathbf{x}), \quad \nu_n(\mathbf{x}, t) = \sum_{k=1}^{n} b_k^n(t) w_k(\mathbf{x}),$$
$$\rho_n(\mathbf{x}, t) = \sum_{k=1}^{n} c_k^n(t) w_k(\mathbf{x}),$$

for suitable unknown sequences a_k^n, b_k^n, c_k^n such that, for every $v \in \mathcal{W}_n$,

$$\int_\Omega \alpha \partial_t \nu_n v + \int_\Omega \partial_t \vartheta_n v + \int_\Omega \nabla \nu_n \cdot \nabla v = \int_\Omega P'(\overline{\varphi}_n)(\overline{\sigma}_n - \overline{\mu}_n)\vartheta_n v$$
$$+ \int_\Omega P(\overline{\varphi}_n)(\rho_n - \nu_n) v, \tag{5.11}$$
$$\int_\Omega \nu_n v = \int_\Omega \beta \partial_t \vartheta_n v + \int_\Omega \nabla \vartheta_n \cdot \nabla v + \int_\Omega F''(\overline{\varphi}_n)\vartheta_n v, \tag{5.12}$$
$$\int_\Omega \partial_t \rho_n v + \int_\Omega \nabla \rho_n \cdot \nabla v = -\int_\Omega P'(\overline{\varphi}_n)(\overline{\sigma}_n - \overline{\mu}_n)\vartheta_n v$$
$$- \int_\Omega P(\overline{\varphi}_n)(\rho_n - \nu_n) v + \int_\Omega h_n v, \tag{5.13}$$
$$\vartheta_n(0) = \nu_n(0) = \rho(0) = 0, \tag{5.14}$$

where

$$\overline{\varphi}_n := \mathbb{P}_n(\overline{\varphi}), \quad \overline{\mu}_n := \mathbb{P}_n(\overline{\mu}), \quad \overline{\sigma}_n := \mathbb{P}_n(\overline{\sigma}), \quad h_n := \mathbb{P}_n(h).$$

Using (5.12), we infer that the above system can be expressed in terms of a_i^n and c_i^n, $1 \le i \le n$ so that it can be reformulated as a Cauchy problem for a nonlinear system of $2n$ first-order ODE in the unknowns a_i^n, c_i^n. Due to the Lipschitz continuity and regularity of the nonlinear terms, we obtain by the Cauchy–Lipschitz theorem the existence of a unique solution such that $(a_1^n, ..., a_n^n, c_1^n, ..., c_n^n) \in (C^1(0,T))^{2n}$ which entails that $(\vartheta_n, \nu_n, \rho_n) \in C^1([0,T]; \mathcal{W}_n)^3$.

A Priori Estimates

Here, we point out some uniform estimates with respect to n which allow us to rigorously pass to the limit as $n \to \infty$ and show that the limit of the approximated solutions $(\vartheta_n, \nu_n, \rho_n)$ produces a solution to the original problem (5.4)–(5.8). To avoid a heavy notation, in the following estimates we avoid writing the subscript n, while we will reintroduce the correct notation at the end of each computation. Notice that all the employed test functions are admissible within the approximation scheme.

First estimate: Firstly, we add to both sides of (5.5) the term ϑ. Then, we test (5.4) by ν, the new second equation by $-\partial_t \vartheta$, and (5.6) by ρ, add the resulting equalities and integrate over Q_t and by parts to obtain

$$\begin{aligned}
&\frac{\alpha}{2}\|\nu(t)\|^2 + \int_{Q_t} |\nabla \nu|^2 + \beta \int_{Q_t} |\partial_t \vartheta|^2 + \frac{1}{2}\|\vartheta\|_V^2 + \frac{1}{2}\|\rho(t)\|^2 \\
&\quad + \int_{Q_t} |\nabla \rho|^2 + \int_{Q_t} P(\overline{\varphi})(\rho - \nu)^2 \\
&\quad = \underbrace{\int_{Q_t} h\rho - \int_{Q_t} F''(\overline{\varphi})\, \vartheta \partial_t \vartheta + \int_{Q_t} P'(\overline{\varphi})(\overline{\sigma} - \overline{\mu})\vartheta(\nu - \rho) + \int_{Q_t} \vartheta \partial_t \vartheta}_{=:\mathbb{I}_1 = \mathbb{I}_1^1 + \mathbb{I}_1^2 + \mathbb{I}_1^3 + \mathbb{I}_1^4},
\end{aligned}$$

where, as above, we convey to indicate by $\mathbb{I}_i$ the i-th line on the right-hand side, whereas with $\mathbb{I}_i^j$ the j-th term on the i-th line. It is worth noting that all the terms of the left-hand side are non-negative since they all are squares and P attains non-negative values by **A1**. Using Young's inequality along with the estimate (3.29), we easily infer that, for every $\delta > 0$,

$$|\mathbb{I}_1^1| + |\mathbb{I}_1^2| + |\mathbb{I}_1^4| \le \frac{1}{2} \int_{Q_t} (|h|^2 + |\rho|^2) + 2\delta \int_{Q_t} |\partial_t \vartheta|^2 + C(\delta) \int_{Q_t} |\vartheta|^2.$$

Moreover, using Hölder's and Young's inequalities, the bounds for $\overline{\mu}$ and $\overline{\sigma}$ pointed out by (3.26) and (3.29), and the Sobolev embedding $V \subset L^6(\Omega)$, we find that

$$\begin{aligned}
|\mathbb{I}_1^3| &\le C \int_0^t (\|\overline{\sigma}\|_6 + \|\overline{\mu}\|_6)\|\vartheta\|_3(\|\nu\| + \|\rho\|) \\
&\le C \int_0^t (\|\overline{\sigma}\|_V^2 + \|\overline{\mu}\|_V^2)\|\vartheta\|_V^2 + C \int_{Q_t} (|\nu|^2 + |\rho|^2) \\
&\le C \int_0^t \|\vartheta\|_V^2 + C \int_{Q_t} (|\nu|^2 + |\rho|^2).
\end{aligned}$$

Thus, we adjust $\delta \in (0,1)$ sufficiently small so that Gronwall's lemma yields

$$\begin{aligned}
&\|\vartheta_n\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} + \|\nu_n\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \\
&\quad + \|\rho_n\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \le C
\end{aligned} \tag{5.15}$$

for a positive constant C independent of n.
Second estimate: Multiplying (5.5) by $\Delta\vartheta$ and using (3.29) and the previous estimate lead us to deduce that

$$\|\Delta \vartheta\|_{L^2(0,T;H)} \le C$$

so that elliptic regularity theory entails that

$$\|\vartheta_n\|_{L^2(0,T;W)} \le C. \tag{5.16}$$

Third estimate: Next, we multiply (5.4) by $\partial_t \nu$, (5.6) by $\partial_t \rho$, integrate over Q_t and by parts to obtain that

$$
\begin{aligned}
&\alpha \int_{Q_t} |\partial_t \nu|^2 + \frac{1}{2}\|\nabla \nu(t)\|^2 + \int_{Q_t} |\partial_t \rho|^2 + \frac{1}{2}\|\nabla \rho(t)\|^2 \\
&\quad = - \int_{Q_t} \partial_t \vartheta \, \partial_t \nu + \int_{Q_t} P'(\overline{\varphi})(\overline{\sigma} - \overline{\mu})\vartheta \, \partial_t \nu + \int_{Q_t} P(\overline{\varphi})(\rho - \nu)\, \partial_t \nu \\
&\qquad - \int_{Q_t} P'(\overline{\varphi})(\overline{\sigma} - \overline{\mu})\vartheta \, \partial_t \rho - \int_{Q_t} P(\overline{\varphi})(\rho - \nu)\, \partial_t \rho + \int_{Q_t} h \, \partial_t \rho =: \mathbb{I}_1 + \mathbb{I}_2.
\end{aligned}
$$

As $\mathbb{I}_1^2$ and $\mathbb{I}_2^1$ are concerned, using (3.26), (3.29), Hölder's inequality along with the continuous embedding $V \subset L^6(\Omega)$, we infer that

$$
\begin{aligned}
|\mathbb{I}_1^2| + |\mathbb{I}_2^1| &\le C \int_{Q_t} (|\overline{\sigma}| + |\overline{\mu}|)|\vartheta||\partial_t \nu| + C \int_{Q_t} (|\overline{\sigma}| + |\overline{\mu}|)|\vartheta||\partial_t \rho| \\
&\le C \int_0^t (\|\overline{\sigma}\|_6 + \|\overline{\mu}\|_6)\|\vartheta\|_3 \|\partial_t \nu\| + C \int_0^t (\|\overline{\sigma}\|_6 + \|\overline{\mu}\|_6)\|\vartheta\|_3 \|\partial_t \rho\| \\
&\le \delta \int_{Q_t} (|\partial_t \nu|^2 + |\partial_t \rho|^2) + C(\delta) \int_0^t (\|\overline{\sigma}\|_V^2 + \|\overline{\mu}\|_V^2)\|\vartheta\|_V^2,
\end{aligned}
$$

for a positive δ yet to be chosen. Moreover, in a similar fashion we can bound the other terms as

$$
\begin{aligned}
|\mathbb{I}_1^1| + |\mathbb{I}_1^3| + |\mathbb{I}_2^2| + |\mathbb{I}_2^3| &\le 2\delta \int_{Q_t} (|\partial_t \nu|^2 + |\partial_t \rho|^2) \\
&\quad + C(\delta) \int_{Q_t} (|\partial_t \vartheta|^2 + |\rho|^2 + |\nu|^2 + |h|^2)
\end{aligned}
$$

and, upon choosing δ small enough, Gronwall's lemma produces

$$
\|\nu_n\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} + \|\rho_n\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \le C. \tag{5.17}
$$

Fourth estimate: Next, a comparison argument in (5.4), and (5.6) easily produce that

$$
\|\Delta \nu\|_{L^2(0,T;H)} + \|\Delta \rho\|_{L^2(0,T;H)} \le C
$$

so that, from the elliptic regularity theory, we infer that

$$
\|\nu_n\|_{L^2(0,T;W)} + \|\rho_n\|_{L^2(0,T;W)} \le C. \tag{5.18}
$$

Passage to the Limit as $n \to \infty$

From these a priori estimates independent of n it is now a standard matter by the Banach–Alaoglu theorem to infer the existence of limits (ν, ϑ, ρ) such that, as $n \to \infty$,

$$
\begin{aligned}
\vartheta_n &\to \vartheta \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \\
\nu_n &\to \nu \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \\
\rho_n &\to \rho \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W).
\end{aligned}
$$

Furthermore, classical compactness embedding results (see, e.g. [141]) allow us to deduce that, as $n \to \infty$,

$$\vartheta_n \to \vartheta, \quad \nu_n \to \nu, \quad \rho_n \to \rho \quad \text{strongly in } C^0([0,T];H) \cap L^2(0,T;V),$$

and

$$\vartheta_n \to \vartheta, \quad \nu_n \to \nu, \quad \rho_n \to \rho \quad \text{a.e. in } Q.$$

Hence, using the Lebesgue dominated convergence theorem and the aforementioned structural assumptions, it is easy to pass to the limit in the variational formulation (5.11)–(5.13) as $n \to \infty$ and infer that the limit (ϑ, ν, ρ) yields a solution to the linearised system (5.4)–(5.8).

As far as the uniqueness is concerned, it readily follows from the linearity of the system along with the above estimates. In fact, taking two solutions $\{(\vartheta_i, \nu_i, \rho_i)\}_i$, $i = 1, 2$, and denoting the differences by $\vartheta := \vartheta_1 - \vartheta_2,\ \nu := \nu_1 - \nu_2,\ \rho := \rho_1 - \rho_2$, we deduce that the above estimates are verified with $C = 0$ which directly implies that $\vartheta = \nu = \rho = 0$ proving the uniqueness. □

5.1.3 Fréchet Differentiability of $\mathcal{S}$

Here, we show that the control-to-state operator $\mathcal{S}$ is Fréchet differentiable between suitable Banach spaces. As a consequence, using the chain rule and the definition of the cost functional $\mathcal{J}$, we will express the abstract optimality condition of (2.22) in an explicit form in terms of the linerised variables. To begin with, we introduce some notation: let $\overline{u} \in \mathcal{U}_{\text{ad}}$ be fixed and denote by $(\overline{\varphi}, \overline{\mu}, \overline{\sigma}) = \mathcal{S}(\overline{u})$ the corresponding state. Then, we consider any $h \in L^2(Q)$ such that $\overline{u} + h$ belongs to $\mathcal{U}_R$. This is fulfilled as soon as h is chosen sufficiently small since $\mathcal{U}_R$ is open: from now on, we tacitly assume that h verifies this condition. Moreover, we denote by

$$(\overline{\varphi}^h, \overline{\mu}^h, \overline{\sigma}^h) := \mathcal{S}(\overline{u} + h),$$

and set

$$\psi := \overline{\varphi}^h - \overline{\varphi} - \vartheta, \quad \zeta := \overline{\mu}^h - \overline{\mu} - \nu, \quad \omega := \overline{\sigma}^h - \overline{\sigma} - \rho\,. \tag{5.19}$$

Thus, our goal is to prove that there exists a linear operator $D\mathcal{S}(\overline{u})$ such that

$$\mathcal{S}(\overline{u} + h) = \mathcal{S}(\overline{u}) + [D\mathcal{S}(\overline{u})](h) + o(\|h\|_{L^2(0,T;H)}) \quad \text{as} \quad \|h\|_{L^2(0,T;H)} \to 0.$$

The expectation is that $D\mathcal{S}(\overline{u})[h] = (\vartheta, \nu, \rho)$ for every $h \in L^2(Q)$, where (ϑ, ν, ρ) is the unique solution to the linearised system associated with h. In light of the aforementioned notation, we realise that it is enough to check that

$$\|(\psi, \zeta, \omega)\|_{\mathcal{X}} \leq C\|h\|^2_{L^2(0,T;H)} \quad \text{as} \quad \|h\|_{L^2(0,T;H)} \to 0, \tag{5.20}$$

for a suitable Bancach space $\mathcal{X}$ yet to be defined. It is worth noting that due to Theorem 3.5 and Theorem 5.2, we already know that

$$\psi, \zeta, \omega \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W).$$

Theorem 5.3 (Fréchet differentiability of $\mathcal{S}$: $\alpha, \beta > 0$)**.** *Suppose that* **A1–A4**, **C1–C3** *hold, and $\alpha, \beta \in (0,1)$. Let $\overline{u} \in \mathcal{U}_R$ be a fixed control with the corresponding state $(\overline{\varphi}, \overline{\mu}, \overline{\sigma}) = \mathcal{S}(\overline{u})$. Then the control-to-state operator $\mathcal{S}$ is Fréchet differentiable at $\overline{u}$ as a mapping from $\mathcal{U}_R$ into the Banach space $\mathcal{X}$ defined by*

$$\mathcal{X} := \big(H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)\big)^3. \tag{5.21}$$

Moreover, for every $h \in L^2(Q)$, $D\mathcal{S}(\overline{u})[h] = (\vartheta, \nu, \rho)$, where (ϑ, ν, ρ) is the unique solution to the linearised system associated to h obtained from Theorem 5.2.

Proof of Theorem 5.3. In view of the above comments, we are reduced to show that (5.20) is satisfied for the Banach space $\mathcal{X}$ specified by (5.21). In this direction, we consider the system (3.1)–(3.5) written for $\overline{u} + h$ and for $\overline{u}$ as well as the linearised system with respect to $\overline{u}$. Taking the difference of the equations and employing the notation (5.19), we obtain that (ψ, ζ, ω) solves the following system

$$\alpha\partial_t\zeta + \partial_t\psi - \Delta\zeta = \Theta \qquad \text{in } Q, \tag{5.22}$$
$$\zeta = \beta\partial_t\psi - \Delta\psi + \Xi \qquad \text{in } Q, \tag{5.23}$$
$$\partial_t\omega - \Delta\omega = -\Theta \qquad \text{in } Q, \tag{5.24}$$
$$\partial_{\mathbf{n}}\psi = \partial_{\mathbf{n}}\zeta = \partial_{\mathbf{n}}\omega = 0 \qquad \text{on } \Sigma, \tag{5.25}$$
$$\psi(0) = \zeta(0) = \omega(0) = 0 \qquad \text{in } \Omega, \tag{5.26}$$

where the source terms Ξ and Θ are defined as follows

$$\begin{aligned}
&\Xi := F'(\overline{\varphi}^h) - F'(\overline{\varphi}) - F''(\overline{\varphi})\,\vartheta,\\
&\Theta := P(\overline{\varphi}^h)(\overline{\sigma}^h - \overline{\mu}^h) - P(\overline{\varphi})(\overline{\sigma} - \overline{\mu}) - P'(\overline{\varphi})(\overline{\sigma} - \overline{\mu})\,\vartheta - P(\overline{\varphi})(\rho - \nu).
\end{aligned}$$

Rearranging the terms and using Taylor's theorem with integral remainder, we deduce that

$$\begin{aligned}
\Xi &= F''(\overline{\varphi})\psi + R_1^h(\overline{\varphi}^h - \overline{\varphi})^2,\\
\Theta &= P(\overline{\varphi})(\omega - \zeta) + \big(P(\overline{\varphi}^h) - P(\overline{\varphi})\big)\big((\overline{\sigma}^h - \overline{\sigma}) - (\overline{\mu}^h - \overline{\mu})\big)\\
&\quad + P'(\overline{\varphi})(\overline{\sigma} - \overline{\mu})\psi + (\overline{\sigma} - \overline{\mu})R_2^h(\overline{\varphi}^h - \overline{\varphi})^2,
\end{aligned}$$

where the remainders R_1^h and R_2^h are defined by

$$\begin{aligned}
R_1^h &:= \int_0^1 (1-s)\,F^{(3)}(\overline{\varphi} + s\,(\overline{\varphi}^h - \overline{\varphi}))\mathrm{d}s,\\
R_2^h &:= \int_0^1 (1-s)\,P''(\overline{\varphi} + s\,(\overline{\varphi}^h - \overline{\varphi}))\mathrm{d}s,
\end{aligned}$$

respectively.

Preliminary Estimates

Let us premise here some useful estimates on the above source terms. First of all, thanks to (3.29) and the regularity assumptions of the nonlinear terms F and P, we have the uniform bound

$$\|R_1^h\|_{L^\infty(Q)} + \|R_2^h\|_{L^\infty(Q)} \le C. \tag{5.27}$$

Using the above estimates, the continuous embedding $V \subset L^6(\Omega)$ and the Young and the Hölder inequalties, we derive that

$$\begin{aligned}\int_0^t \left\|R_1^h(s)(\overline{\varphi}^h(s) - \overline{\varphi}(s))^2\right\|^2 \mathrm{ds} &\le C \int_{Q_t} |\overline{\varphi}^h - \overline{\varphi}|^4 \\ &\le C\|\overline{\varphi}^h - \overline{\varphi}\|^4_{L^\infty(0,T;V)} \\ &\le C\|h\|^4_{L^2(0,T;H)}. \end{aligned}\tag{5.28}$$

Moreover, similar arguments lead us to infer that

$$\begin{aligned}&\int_{Q_t} \left|\big(P(\overline{\varphi}^h) - P(\overline{\varphi})\big)\big((\overline{\sigma}^h - \overline{\sigma}) - (\overline{\mu}^h - \overline{\mu})\big)\right|^2 \qquad (5.29)\\ &\le C \int_{Q_t} |\overline{\varphi}^h - \overline{\varphi}|^2(|\overline{\sigma}^h - \overline{\sigma}|^2 + |\overline{\mu}^h - \overline{\mu}|^2) \\ &\le C \int_0^t \|\overline{\varphi}^h(s) - \overline{\varphi}(s)\|_4^2(\|\overline{\sigma}^h(s) - \overline{\sigma}(s)\|_4^2 + \|\overline{\mu}^h(s) - \overline{\mu}(s)\|_4^2)\,\mathrm{ds} \\ &\le C \int_0^t (\|\overline{\varphi}^h - \overline{\varphi}\|_V^2 \|\overline{\sigma}^h - \overline{\sigma}\|_V^2 + \|\overline{\mu}^h - \overline{\mu}\|_V^2) \le C\,\|h\|^4_{L^2(0,T;H)},\end{aligned}$$

and, from (3.26) and (3.29), also that

$$\begin{aligned}\int_{Q_t} \left|P'(\overline{\varphi})(\overline{\sigma} - \overline{\mu})\psi\right|^2 &\le C \int_{Q_t} (|\overline{\sigma}|^2 + |\overline{\mu}|^2)|\psi|^2 \\ &\le C \int_0^t (\|\overline{\sigma}\|_V^2 + \|\overline{\mu}\|_V^2)\|\psi\|_V^2 \\ &\le C \int_0^t \|\psi\|_V^2. \end{aligned}\tag{5.30}$$

Lastly, thanks to (5.27), Hölder's and Young's inequalities, (3.26), (3.11), (3.40), and the continuous inclusion $V \subset L^6(\Omega)$, we obtain that

$$\begin{aligned}&\int_{Q_t} \left|(\overline{\sigma} - \overline{\mu})R_2^h(\overline{\varphi}^h - \overline{\varphi})^2\right|^2 \\ &\le C \int_{Q_t} (|\overline{\sigma}|^2 + |\overline{\mu}|^2)|\overline{\varphi}^h - \overline{\varphi}|^4 \\ &\le C \int_0^t (\|\overline{\sigma}(s)\|_6^2 + \|\overline{\mu}(s)\|_6^2)\|\overline{\varphi}^h(s) - \overline{\varphi}(s)\|_6^4\,\mathrm{ds} \\ &\le C \int_0^t (\|\overline{\sigma}\|_V^2 + \|\overline{\mu}\|_V^2)\|\overline{\varphi}^h - \overline{\varphi}\|_V^4 \le C\|h\|^4_{L^2(0,T;H)}. \end{aligned}\tag{5.31}$$

With these preliminary we are now ready to address the Fréchet differentiability of $\mathcal{S}$ by checking (5.20).

Uniform Estimates

First estimate: First, we add to both sides of (5.23) the term ψ. Then, we multiply (5.22) by ζ, this new second equation by $-\partial_t\psi$, (5.24) by ω, add the resulting equalities and integrate over Q_t and by parts to get that

$$\begin{aligned}
&\frac{\alpha}{2}\|\zeta(t)\|^2 + \int_{Q_t}|\nabla\zeta|^2 + \frac{1}{2}\|\psi(t)\|_V^2 + \beta\int_{Q_t}|\partial_t\psi|^2 + \|\omega(t)\|^2 + \int_{Q_t}|\nabla\omega|^2 \\
&\quad = \int_{Q_t}\Theta\zeta - \int_{Q_t}F''(\overline{\varphi})\psi\,\partial_t\psi - \int_{Q_t}R_1^h(\overline{\varphi}^h - \overline{\varphi})^2\,\partial_t\psi \\
&\qquad + \int_{Q_t}\psi\,\partial_t\psi - \int_{Q_t}\Theta\omega =: \mathbb{I}_1 + \mathbb{I}_2.
\end{aligned}$$

Using (3.29), Young's inequality, and (5.28), we deduce that

$$\begin{aligned}
|\mathbb{I}_1^2| + |\mathbb{I}_1^2| + |\mathbb{I}_2^1| &\le 3\delta\int_{Q_t}|\partial_t\psi|^2 + C(\delta)\int_{Q_t}|\psi|^2 + C(\delta)\int_{Q_t}|R_1^h(\overline{\varphi}^h - \overline{\varphi})^2|^2 \\
&\le 3\delta\int_{Q_t}|\partial_t\psi|^2 + C(\delta)\int_{Q_t}|\psi|^2 + C(\delta)\|h\|_{L^2(0,T;H)}^4
\end{aligned}$$

for a constant $\delta > 0$ yet to be chosen. Moreover, the first term on the right-hand side can be bounded above by

$$\begin{aligned}
|\mathbb{I}_1^1| &\le \int_{Q_t}\big|P(\overline{\varphi})(\omega - \zeta)\,\zeta + P(\overline{\varphi}^h) - P(\overline{\varphi})(\overline{\sigma}^h - \overline{\sigma}) - (\overline{\mu}^h - \mu)\zeta\big| \\
&\quad + \int_{Q_t}\big|P'(\overline{\varphi})(\overline{\sigma} - \overline{\mu})\,\psi\,\zeta + (\overline{\sigma} - \overline{\mu})R_2^h(\overline{\varphi}^h - \overline{\varphi})^2\,\zeta\big| \\
&\le C\int_{Q_t}(|\omega|^2 + |\zeta|^2) + C\int_0^t\|\psi\|_V^2 + C\|h\|_{L^2(0,T;H)}^4.
\end{aligned}$$

The last term $\mathbb{I}_2^2$ can be treated in the same way, while it is related to the variable ω instead of ζ. Therefore, we pick δ small enough so that the Gronwall lemma produces

$$\begin{aligned}
&\|\psi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\zeta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\
&\quad + \|\omega\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le C\|h\|_{L^2(0,T;H)}^2.
\end{aligned} \tag{5.32}$$

Second estimate: Accounting for the previous estimate, from a comparison argument in (5.23), we readily obtain that

$$\|\Delta\psi\|_{L^2(0,T;H)} \le C\|h\|_{L^2(0,T;H)}^2$$

so that elliptic regularity theory produces

$$\|\psi\|_{L^2(0,T;W)} \le C\|h\|_{L^2(0,T;H)}^2.$$

Third estimate: Next, let us rewrite the equations (5.22) and (5.24) as

$$\alpha\partial_t\zeta - \Delta\zeta = \Theta - \partial_t\varphi := g_1, \quad \partial_t\omega - \Delta\omega = \Theta := g_2.$$

Owing to (5.32), we have that

$$\|g_1\|_{L^2(0,T;H)} + \|g_2\|_{L^2(0,T;H)} \le C\|h\|^2_{L^2(0,T;H)}$$

so that parabolic regularity theory gives

$$\|\zeta\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} + \|\omega\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le C\|h\|^2_{L^2(0,T;H)}$$

concluding the proof. □

Then, by virtue of the above theorem and to the abstract necessary condition for optimality pointed out by the variational inequality (2.22), we obtain the first-order necessary conditions for optimality.

Theorem 5.4. *Suppose that* **A1–A4**, **C1–C3** *hold, and* $\alpha, \beta \in (0,1)$. *Let* $\overline{u} \in \mathcal{U}_{\rm ad}$ *be an optimal control for* $(CP)_{\alpha,\beta}$ *with the corresponding state* $(\overline{\varphi}, \overline{\mu}, \overline{\sigma}) = \mathcal{S}(\overline{u})$. *Then* $\overline{u}$ *necessarily satisfies*

$$\begin{aligned} b_1 \int_Q (\overline{\varphi} - \varphi_Q)\vartheta + b_2 \int_\Omega (\overline{\varphi}(T) - \varphi_\Omega)\vartheta(T) + b_3 \int_Q (\overline{\sigma} - \sigma_Q)\rho \\ + b_4 \int_\Omega (\overline{\sigma}(T) - \sigma_\Omega)\rho(T) + b_0 \int_Q \overline{u}(u - \overline{u}) \ge 0 \quad \forall u \in \mathcal{U}_{\rm ad}, \end{aligned} \tag{5.33}$$

where ϑ *and* ρ *are two components of the unique solution* (ϑ, ν, ρ) *to the linearised system* (5.4)–(5.8) *associated with* $h = u - \overline{u}$ *as given by Theorem 5.2.*

5.1.4 Adjoint System

Once first-order necessary conditions for optimality for problem $(CP)_{\alpha,\beta}$ has been obtained as the variation inequality (5.33), we aim at simplyfing that condition. In this direction, a classical tool is to introduce the so-called *adjoint system* which is a backward-in-time system in the variables (p, q, r) and reads as

$$- \partial_t(p + \beta q) - \Delta q + F''(\overline{\varphi})q + P'(\overline{\varphi})(\overline{\sigma} - \overline{\mu})(r - p) \tag{5.34}$$
$$= b_1(\overline{\varphi} - \varphi_Q) \quad \text{in } Q, \tag{5.35}$$
$$- \alpha\partial_t p - q - \Delta p + P(\overline{\varphi})(p - r) = 0 \quad \text{in } Q, \tag{5.36}$$
$$- \partial_t r - \Delta r + P(\overline{\varphi})(r - p) = b_3(\overline{\sigma} - \sigma_Q) \quad \text{in } Q, \tag{5.37}$$
$$\partial_{\mathbf{n}} p = \partial_{\mathbf{n}} q = \partial_{\mathbf{n}} r = 0 \quad \text{on } \Sigma, \tag{5.38}$$
$$(p + \beta q)(T) = b_2(\overline{\varphi}(T) - \varphi_\Omega), \quad \alpha p(T) = 0,$$
$$r(T) = b_4(\overline{\sigma}(T) - \sigma_\Omega) \quad \text{in } \Omega. \tag{5.39}$$

Here, the corresponding well-posedness result follows.

Theorem 5.5 (Well-posedness of the adjoint system: $\alpha, \beta > 0$). *Suppose that* **A1–A4**, **C1–C3**, *and* $\alpha, \beta \in (0,1)$. *Then, the adjoint system* (5.35)–(5.39) *admits a unique solution* (p, q, r) *such that*

$$p, q, r \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W). \tag{5.40}$$

Proof of Theorem 5.5. A rigorous proof should involve an approximation argument such as the Faedo–Galerkin scheme. However, since the system is linear and the arguments are standard, we just point out the corresponding formal a priori estimates, leaving the details to the reader.

First estimate: First, we add to both sides of (5.36) the term p. Then, we test (5.35) by $-q$, this new (5.36) by $-\partial_t p$, (5.37) by r, add the resulting equalities and integrate over $[t,T]$ and by parts to obtain that

$$\begin{aligned}
&\frac{\beta}{2}\|q(t)\|^2 + \int_{Q_t^T} |\nabla q|^2 + \alpha \int_{Q_t^T} |\partial_t p|^2 + \frac{1}{2}\|p(t)\|_V^2 + \frac{1}{2}\|r(t)\|^2 + \int_{Q_t^T} |\nabla r|^2 \\
&\quad = \frac{1}{2}\|r(T)\|^2 + \frac{\beta}{2}\|q(T)\|^2 + \frac{1}{2}\|p(T)\|_V^2 - \int_{Q_t^T} b_1(\overline{\varphi} - \varphi_Q) q + \int_{Q_t^T} b_3(\overline{\sigma} - \sigma_Q) r \\
&\qquad + \int_{Q_t^T} P'(\overline{\varphi})(\overline{\sigma} - \overline{\mu})(r-p)q - \int_{Q_t^T} F''(\overline{\varphi})|q|^2 \\
&\qquad - \int_{Q_t^T} P(\overline{\varphi})(r-p)(r + \partial_t p) - \int_{Q_t^T} p\partial_t p =: \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3.
\end{aligned}$$

Using assumption **C1–C2** and Young's inequality we readily infer that

$$|\mathbb{I}_1| \le \int_{Q_t^T} (|q|^2 + |r|^2 + 1).$$

Due to (3.29) and again to the Young inequality, we get

$$|\mathbb{I}_2^2| + |\mathbb{I}_3| \le 2\delta \int_{Q_t^T} |\partial_t p|^2 + C(\delta) \int_{Q_t^T} (|p|^2 + |r|^2)$$

for a positive δ yet to be chosen. Lastly, (3.26), (3.29), the continuous embedding $V \subset L^6(\Omega)$, Young's and Hölder's inequalities produce

$$\begin{aligned}
|\mathbb{I}_2^1| &\le C \int_{Q_t^T} |P'(\overline{\varphi})(\overline{\sigma} - \overline{\mu})q|^2 + C \int_{Q_t^T} |r-p|^2 \\
&\le C \int_t^T (\|\overline{\sigma}\|_6^2 + \|\overline{\mu}\|_6^2)\|q\|_6\|q\| + C \int_{Q_t^T} (|p|^2 + |r|^2) \\
&\le \frac{1}{2} \int_{Q_t^T} (|q|^2 + |\nabla q|^2) + C \int_t^T (\|\overline{\sigma}\|_V^4 + \|\overline{\mu}\|_V^4)\|q\|^2 + C \int_{Q_t^T} (|p|^2 + |r|^2) \\
&\le \frac{1}{2} \int_{Q_t^T} |\nabla q|^2 + C \int_{Q_t^T} (|p|^2 + |q|^2 + |r|^2).
\end{aligned}$$

Hence, we fix δ small enough and apply the backward-in-time Gronwall lemma to infer that

$$\|p\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|q\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|r\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq C.$$

Second estimate: By testing (5.36) by Δp and using similar argument as above we deduce that

$$\|\Delta p\|_{L^2(0,T;H)} \leq C$$

so that elliptic regularity theory entails that

$$\|p\|_{L^2(0,T;W)} \leq C.$$

Third estimate: We now multiply (5.35) by $\partial_t q$ and integrate over Q_t^T and by parts to obtain that

$$\beta \int_{Q_t^T} |\partial_t q|^2 + \frac{1}{2}\|\nabla q(t)\|^2 = \frac{1}{2}\|\nabla q(T)\|^2 + \int_{Q_t^T} \partial_t p \partial_t q$$
$$- \int_{Q_t^T} P'(\overline{\varphi})(\overline{\sigma} - \overline{\mu})(r-p)\partial_t q + \int_{Q_t^T} F''(\overline{\varphi}) q\, \partial_t q + \int_{Q_t^T} b_1(\overline{\varphi} - \varphi_Q)\partial_t q.$$

Except for the second term on the right-hand side the other terms can be bounded above by

$$3\delta \int_{Q_t^T} |\partial_t q|^2 + C(\delta) \int_{Q_t^T} (|\partial_t p|^2 + |q|^2 + 1) \tag{5.41}$$

by using Young's inequality for a positive δ yet to be chosen. Furthermore, owing to the continuous inclusion $V \subset L^6(\Omega)$, the boundedness of P' and Hölder's inequality we have

$$\begin{aligned}
&\int_{Q_t^T} P'(\overline{\varphi})(\overline{\sigma} - \overline{\mu})(r-p)\partial_t q \\
&\leq C \int_t^T (\|\overline{\sigma}\|_6 + \|\overline{\mu}\|_6)(\|r\|_3 + \|p\|_3)\|\partial_t q\| \\
&\leq \delta \int_{Q_t^T} |\partial_t q|^2 + C(\delta) \int_t^T (\|\overline{\sigma}\|_V^2 + \|\overline{\mu}\|_V^2)(\|r\|_V^2 + \|p\|_V^2),
\end{aligned}$$

where all the terms on the right-hand side have already been estimated. Therefore, fixing δ sufficientely small we conclude that

$$\|q\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq C.$$

Fourth estimate: Testing (5.37) by $-\partial_t r$ and using similar aguments as above we deduce that

$$\|r\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq C.$$

Fifth estimate: A comparison argument in (5.35) and (5.37) along with elliptic regularity theory imply

$$\|q\|_{L^2(0,T;W)} + \|r\|_{L^2(0,T;W)} \leq C$$

which concludes the existence part of the proof.

As far as the uniqueness is concerned, it is enough to consider two solutions $\{(p_i, q_i, r_i)\}_i$, $i = 1, 2$ to (5.35)–(5.39) and after setting $p := p_1 - p_2$, $q := q_1 - q_2$, $r := r_1 - r_2$, realise that all the above estimates are verified with $C = 0$ so that $p = q = r = 0$. □

5.1.5 First-order Optimality Conditions

Using the adjoint variables we are now in the position to eliminate the linearised variables from the variational inequality (5.33) and obtain:

Theorem 5.6 (First-order necessary optimality condition: $\alpha, \beta > 0$)**.** *Suppose* **A1–A4**, **C1–C3***, and* $\alpha, \beta \in (0, 1)$*. Then, every optimal control* $\overline{u}$ *necessarily satifies*

$$\int_Q (r + b_0\overline{u})(u - \overline{u}) \geq 0 \quad \forall u \in \mathcal{U}_{\mathrm{ad}}, \tag{5.42}$$

where r *is the third component of the adjoint system obtained from Theorem 5.5. Moreover, if* $b_0 > 0$*,* $\overline{u}$ *is the* $L^2(0, T; H)$*-orthogonal projection of* $-{b_0}^{-1}r$ *onto the closed subspace* $\mathcal{U}_{\mathrm{ad}}$ *and we have the following pointwise characterisation for the minimiser*

$$\overline{u}(\mathbf{x}, t) = \max\left\{u_*(\mathbf{x}, t), \min\{u^*(\mathbf{x}, t), -b_0^{-1}r(\mathbf{x}, t)\}\right\} \quad \textit{for a.a. } (\mathbf{x}, t) \in Q.$$

It is worth noticing that, as a consequence of (5.42), we deduce via Riesz's representation theorem that the gradient of the reduced cost functional $\mathcal{J}_{\mathrm{red}}$ can be identified as $\nabla\mathcal{J}_{\mathrm{red}}(\overline{u}) = r + b_0\overline{u}$. From a numerical perspective this plays an important role since the optimal control problem reduces to minimise a function $\mathcal{J}_{red}$ whose gradient is known.

Proof of Theorem 5.6. Comparing the inequalities (5.33) with (5.42), we realise that it suffices to check the identity

$$\begin{aligned} \int_Q rh = b_1 \int_Q (\overline{\varphi} - \varphi_Q)\vartheta + b_2 \int_\Omega (\overline{\varphi}(T) - \varphi_\Omega)\vartheta(T) \\ + b_3 \int_Q (\overline{\sigma} - \sigma_Q)\rho + b_4 \int_\Omega (\overline{\sigma}(T) - \sigma_\Omega)\rho(T), \end{aligned} \tag{5.43}$$

with $h = v - \overline{u}$, where ϑ and ρ are the corresponding linearised variables obtained from Theorem 5.2. In this direction, we multiply (5.4)–(5.6) by $p, q, r,$ in this order and add the equalities to obtain that

$$
\begin{aligned}
0 &= \int_Q p[\alpha\,\partial_t\nu + \partial_t\vartheta - \Delta\nu - P'(\overline{\varphi})(\overline{\sigma}-\overline{\mu})\,\vartheta - P(\overline{\varphi})(\rho-\nu)] \\
&\quad + \int_Q q[-\nu + \beta\partial_t\vartheta - \Delta\vartheta + F''(\overline{\varphi})\,\vartheta] \\
&\quad + \int_Q r[\partial_t\rho - \Delta\rho + P'(\overline{\varphi})(\overline{\sigma}-\overline{\mu})\,\vartheta + P(\overline{\varphi})(\rho-\nu) - h].
\end{aligned}
$$

Integration by parts with respect to time, using the initial conditions (5.8), and rearranging the terms leads us to

$$
\begin{aligned}
\int_Q rh &= \int_Q \nu[q - \alpha\partial_t p - \Delta p + P(\overline{\varphi})(p-r)] \\
&\quad + \int_Q \vartheta[\beta\partial_t q - \partial_t p + \Delta q - F''(\overline{\varphi})q + P'(\overline{\varphi})(\overline{\sigma}-\overline{\mu})(r-p)] \\
&\quad + \int_Q \rho[-\partial_t r - \Delta r + P(\overline{\varphi})(r-p)] \\
&\quad + \int_\Omega [(p+\beta q)(T)\vartheta(T) + \alpha\nu(T)p(T) + r(T)\rho(T)].
\end{aligned}
$$

Thus, using the terminal conditions (5.38) and recalling the definition of the adjoint variables (5.35)–(5.39), we realise that the most part of the above terms simplify and the remaining equality is exactly (5.43), as we claimed. □

5.2 Asymptotic Analysis

In this section, we aim at exploiting the results established so far for the optimal control problem $(CP)_{\alpha,\beta}$ to address the minimisation problems $(CP)_\alpha$ and $(CP)_\beta$ via asymptotic approach. In particular, the main goal is to pass to the limit as α and β go to zero in the variational inequality (5.42) to characterise the corresponding first-order necessary conditions for optimality. Notice that these passages to the limit are rather involved as we need to keep track the behaviour of the state and adjoint systems as well as the behaviour of the control as the parameters approach zero. In this direction, we recall that the asymptotic analysis on the state system has already been addressed in Chapter 3: see Theorems 3.9, 3.10, 3.13, and 3.14.

Then, the first novelty addressed in this section consists in understanding the asymptotic behaviour of the adjoint system. To this aim, the first step is to point out some a priori estimates which will allow us to let the parameters α and β go to zero by using classical weak and weak star compactness arguments.

As a second step, we prove how to approximate optimal controls of $(CP)_\alpha$ and $(CP)_\beta$ through sequences of optimal controls of the relaxed problem $(CP)_{\alpha,\beta}$. Then, we combine the results to rigorously pass to the limit in the variational inequality (5.42) obtaining the corresponding optimality conditions for the limit problems $(CP)_\alpha$ and $(CP)_\beta$, respectively. Moreover, let us recall that the well-posedness of the state system

(3.1)–(3.5) with $\alpha = \beta = 0$ has been already addressed in [78] and the associated optimal control problem was investigated in [40] for polynomial growth type potentials.

5.2.1 The Optimisation Problem $(CP)_{\alpha,\beta}$ as $\alpha \to 0$

From now onward, we set the following notation: for every $\alpha \in (0, \alpha_{00})$ and $\beta \in (0, \beta_0)$, let $\overline{u}_{\alpha,\beta} \in \mathcal{U}_{\text{ad}}$ be a fixed admissible control, and let $(\overline{\varphi}_{\alpha,\beta}, \overline{\mu}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta})$ and $(p_{\alpha,\beta}, q_{\alpha,\beta}, r_{\alpha,\beta})$ denote the associated unique solutions to the state system (3.1)–(3.5) and the adjoint system (5.35)–(5.39) with $\alpha, \beta > 0$ as obtained from Theorem 3.2 and Theorem 5.5, respectively. The first result concerns the existence of minimisers for the optimisation problem $(CP)_\beta$ which can be deduced by adapting the lines of argument in the proof of Theorem 5.1.

5.2.1.1 Existence of minimisers

Theorem 5.7 (Existence of a minimiser: $\alpha \to 0$)**.** *Assume that the assumption of Theorem 3.10 are fulfilled and suppose that* **C1–C3** *hold. Then the optimisation problem* $(CP)_\beta$ *admits a minimiser.*

Proof of Theorem 5.7. The proof readily follows as an application of the direct methods of calculus of variations along with the asymptotic behaviour results pointed out by Theorem 3.9. To begin with, notice that the cost functional $\mathcal{J}$ is non-negative so that we can consider a minimising sequence of optimal controls for $(CP)_{\alpha,\beta}$. Hence, we take a sequence $\{\alpha_n\}_n \subset (0, 1]$ which goes to zero as $n \to \infty$, and consider a minimising sequence $\{u_{n,\beta}\}_n := \{u_{\alpha_n,\beta}\}_n \subset \mathcal{U}_{\text{ad}}$ such that, for every n, $u_{n,\beta}$ is optimal for $(CP)_{\alpha_n,\beta}$, and we denote by $\{(\varphi_{n,\beta}, \mu_{n,\beta}, \sigma_{n,\beta})\}_n$ the sequence of the corresponding states. Namely, we have

$$
\begin{aligned}
&(\varphi_{n,\beta}, \mu_{n,\beta}, \sigma_{n,\beta}) = \mathcal{S}(u_{n,\beta}) \quad \text{for every } n \in \mathbb{N}, \\
&\lim_{n\to\infty} \mathcal{J}(\varphi_{n,\beta}, \sigma_{n,\beta}, u_{n,\beta}) \\
&\qquad = \inf \left\{ \mathcal{J}(\varphi, \sigma, u) : u \in \mathcal{U}_{\text{ad}}, \ (\varphi, \sigma) = \big(\mathcal{S}_1(u), \mathcal{S}_3(u)\big) \right\} =: \lambda \geq 0.
\end{aligned}
$$

Moreover, notice that the convergences (3.47)–(3.51) are independent of n (i.e., on α) and from the assumptions on $\mathcal{U}_{\text{ad}}$, it follows from classical weak and weak star compactness results the existence of a non-relabelled subsequence and limits $\overline{u}_\beta \in \mathcal{U}_{\text{ad}}$ and $(\overline{\varphi}_\beta, \overline{\mu}_\beta, \overline{\sigma}_\beta)$ such that, as $n \to \infty$,

$$
\begin{aligned}
&u_{n,\beta} \to \overline{u}_\beta \ \text{ weakly star in } L^\infty(Q), \\
&\varphi_{n,\beta} \to \overline{\varphi}_\beta \ \text{ weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \\
&\mu_{n,\beta} \to \overline{\mu}_\beta \ \text{ weakly in } L^2(0,T;V), \\
&\sigma_{n,\beta} \to \overline{\sigma}_\beta \ \text{ weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V).
\end{aligned}
$$

Moreover, a compactness argument (see Lemma 2.4) yields that

$$
\varphi_{n,\beta} \to \overline{\varphi}_\beta \quad \text{strongly in } C^0([0,T];H) \cap L^2(0,T;V).
$$

This, along with the growth assumption of the potential postulated by (3.53) allows us to infer that

$$F'(\varphi_{n,\beta}) \to F'(\overline{\varphi}_\beta) \quad \text{strongly in } L^2(0,T;H).$$

Then, it suffices to take into account the variational formulation of system (3.1)–(3.5) written for the solution triplet $(\varphi_{n,\beta}, \mu_{n,\beta}, \sigma_{n,\beta})$, and the control $u_{n,\beta}$ and pass to the limit as $n \to \infty$ to conclude that $(\overline{\varphi}_\beta, \overline{\mu}_\beta, \overline{\sigma}_\beta)$ and $\overline{u}_\beta$ yield a minimiser for $(CP)_\beta$. □

5.2.1.2 A Priori Estimates on the Adjoint Variables

Then, we present some a priori estimates on the adjoint variables of (5.35)–(5.39) which are independent of α.

Theorem 5.8. *Suppose that the assumption of Theorem 3.10 are fulfilled and that* **C1–C3** *hold. Let $\overline{u}_\beta$ be an optimal control of $(CP)_\beta$ and $\{u_{\alpha,\beta}\}_\alpha \subset \mathcal{U}_{\mathrm{ad}}$ be such that $u_{\alpha,\beta} \to \overline{u}_\beta$ strongly in $L^2(0,T;H)$ as $\alpha \to 0$. Let $(\varphi_{\alpha,\beta}, \mu_{\alpha,\beta}, \sigma_{\alpha,\beta})$ and $(p_{\alpha,\beta}, q_{\alpha,\beta}, r_{\alpha,\beta})$ denote the state associated to $u_{\alpha,\beta}$ and the unique solution to the adjoint system* (5.35)–(5.39) *as given by Theorem 3.5 and Theorem 5.5, respectively. Let $(\overline{\varphi}_\beta, \overline{\mu}_\beta, \overline{\sigma}_\beta)$ be the unique solution to* (3.1)–(3.5) *with $\alpha = 0$ obtained from Theorem 3.10 associated to $\overline{u}_\beta$. Then, there exists a triplet $(p_\beta, q_\beta, r_\beta)$ with*

$$\begin{aligned} p_\beta &\in L^2(0,T;W), \\ q_\beta &\in L^2(0,T;V), \\ p_\beta + \beta q_\beta &\in H^1(0,T;V^*), \\ r_\beta &\in H^1(0,T;V^*) \cap C^0([0,T];H) \cap L^2(0,T;V), \end{aligned}$$

such that it holds, as $\alpha \to 0$,

$$p_{\alpha,\beta} \to p_\beta \quad \textit{weakly in } L^2(0,T;W), \tag{5.44}$$
$$q_{\alpha,\beta} \to q_\beta \quad \textit{weakly in } L^2(0,T;V), \tag{5.45}$$
$$r_{\alpha,\beta} \to r_\beta \quad \textit{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V), \quad \textit{and strongly in } L^2(0,T;H), \tag{5.46}$$
$$p_{\alpha,\beta} + \beta q_{\alpha,\beta} \to p_\beta + \beta q_\beta \quad \textit{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V), \quad \textit{and strongly in } L^2(0,T;H), \tag{5.47}$$
$$\alpha p_{\alpha,\beta} \to 0 \quad \textit{strongly in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W). \tag{5.48}$$

Moreover, there exists a positive constant K_β, which may depend on β but it is independent of α, such that

$$\begin{aligned} &\|p_{\alpha,\beta} - \beta q_{\alpha,\beta}\|_{H^1(0,T;V^*)\cap L^\infty(0,T;H)\cap L^2(0,T;V)} + \|q_{\alpha,\beta}\|_{L^2(0,T;V)} \\ &\quad + \alpha \|p_{\alpha,\beta}\|_{H^1(0,T;H)} + \alpha^{1/2}\|p_{\alpha,\beta}\|_{L^\infty(0,T;V)} \\ &\quad + \|p_{\alpha,\beta}\|_{L^2(0,T;W)} + \|r_{\alpha,\beta}\|_{H^1(0,T;V^*)\cap L^\infty(0,T;H)\cap L^2(0,T;V)} \le K_\beta. \end{aligned} \tag{5.49}$$

In addition, the limit triplet $(p_\beta, q_\beta, r_\beta)$ *is the unique weak solution to the adjoint system* (5.35)–(5.39) *with* $\alpha = 0$ *in the sense that it fulfils*

$$
\begin{aligned}
&- \langle \partial_t (p_\beta + \beta q_\beta)(t), v\rangle + \int_\Omega \nabla q_\beta(t)\cdot\nabla v + \int_\Omega F''(\overline{\varphi}_\beta(t)) q_\beta(t) v \\
&\qquad = \int_\Omega b_1(\overline{\varphi}_\beta(t) - \varphi_Q(t)) v, \\
&- \int_\Omega q_\beta(t) v + \int_\Omega \nabla p_\beta(t)\cdot\nabla v + P\int_\Omega (p_\beta(t) - r_\beta(t)) v = 0, \\
&- \langle \partial_t r_\beta(t), v\rangle + \int_\Omega \nabla r_\beta(t)\cdot\nabla v + P\int_\Omega (r_\beta(t) - p_\beta(t)) v = \int_\Omega b_3(\overline{\sigma}_\beta(t) - \sigma_Q(t)) v,
\end{aligned}
$$

for every $v \in V$ *and almost everywhere in* $(0,T)$*, and*

$$
(p_\beta + \beta q_\beta)(T) = b_2(\overline{\varphi}_\beta(T) - \varphi_\Omega), \quad r_\beta(T) = \int_\Omega b_4(\overline{\sigma}_\beta(T) - \sigma_\Omega).
$$

Proof of Theorem 5.8. For brevity, the estimates presented below are formal, but they can be easily justified by introducing a Faedo–Galerkin scheme. Moreover, in what follows, the symbol C will denote a positive constant which may depend on β but it is independent of α.

First estimate: To begin with, we set

$$
w_\alpha := p_\alpha + \beta q_\alpha, \tag{5.50}
$$

which in turn implies the identities

$$
q_\alpha = \frac{w_\alpha - p_\alpha}{\beta}, \qquad p_\alpha = w_\alpha - \beta q_\alpha.
$$

Hence, we rewrite the system (5.35)–(5.39) in terms of p_α, w_α, and r_α to obtain that

$$
\begin{aligned}
&- \partial_t w_{\alpha,\beta} + \tfrac{1}{\beta}\Delta(p_{\alpha,\beta} - w_{\alpha,\beta}) + \tfrac{1}{\beta}F''(\overline{\varphi}_{\alpha,\beta})(w_{\alpha,\beta} - p_{\alpha,\beta}) \\
&\qquad = b_1(\overline{\varphi}_{\alpha,\beta} - \varphi_Q) && \text{in } Q, && (5.51)\\
&- \alpha\partial_t p_{\alpha,\beta} + \tfrac{1}{\beta}(p_{\alpha,\beta} - w_{\alpha,\beta}) - \Delta p_{\alpha,\beta} + P(p_{\alpha,\beta} - r_{\alpha,\beta}) = 0 && \text{in } Q, && (5.52)\\
&- \partial_t r_{\alpha,\beta} - \Delta r_{\alpha,\beta} + P(r_{\alpha,\beta} - p_{\alpha,\beta}) = b_3(\overline{\sigma}_{\alpha,\beta} - \sigma_Q) && \text{in } Q, && (5.53)\\
&\partial_{\mathbf{n}} w_{\alpha,\beta} = \partial_{\mathbf{n}} p_{\alpha,\beta} = \partial_{\mathbf{n}} r_{\alpha,\beta} = 0 && \text{on } \Sigma, && (5.54)\\
&w_{\alpha,\beta}(T) = b_2(\overline{\varphi}_{\alpha,\beta}(T) - \varphi_\Omega), \ \ \alpha p_{\alpha,\beta}(T) = 0, \\
&r_{\alpha,\beta}(T) = b_4(\overline{\sigma}_{\alpha,\beta}(T) - \sigma_\Omega) && \text{in } \Omega. && (5.55)
\end{aligned}
$$

Then, we multiply (5.51) by $w_{\alpha,\beta}$, (5.52) by $p_{\alpha,\beta} - \Delta p_{\alpha,\beta}$, (5.53) by $r_{\alpha,\beta}$ add the resulting equalities and integrate over Q_t^T and by parts to obtain, upon rearranging the terms, that

$$
\begin{aligned}
&\frac{1}{2}\|w_{\alpha,\beta}(t)\|^2 + \frac{1}{\beta}\int_{Q_t^T} |\nabla w_{\alpha,\beta}|^2 + \frac{\alpha}{2}\|p_{\alpha,\beta}(t)\|_V^2 + \big(\frac{1}{\beta} + P\big)\int_{Q_t^T} |p_{\alpha,\beta}|^2 \\
&\quad + \big(\frac{1}{\beta} + P + 1\big)\int_{Q_t^T} |\nabla p_{\alpha,\beta}|^2 + \int_{Q_t^T} |\Delta p_{\alpha,\beta}|^2 + \frac{1}{2}\|r_{\alpha,\beta}(t)\|^2 \\
&\quad + \int_{Q_t^T} |\nabla r_{\alpha,\beta}|^2 + P\int_{Q_t^T} |r_{\alpha,\beta}|^2 \\
&= \frac{1}{2}\|b_2(\overline{\varphi}_{\alpha,\beta}(T) - \varphi_\Omega)\|^2 + \frac{1}{2}\|b_4(\overline{\sigma}_{\alpha,\beta}(T) - \sigma_\Omega)\|^2 + \int_{Q_t^T} b_1(\overline{\varphi}_{\alpha,\beta} - \varphi_Q) w_{\alpha,\beta} \\
&\quad + \int_{Q_t^T} b_3(\overline{\sigma}_{\alpha,\beta} - \sigma_Q) r_{\alpha,\beta} + \frac{1}{\beta}\int_{Q_t^T} F''(\overline{\varphi}_{\alpha,\beta}) p_{\alpha,\beta} w_{\alpha,\beta} - \frac{1}{\beta}\int_{Q_t^T} F''(\overline{\varphi}_{\alpha,\beta})|w_{\alpha,\beta}|^2 \\
&\quad - \frac{2}{\beta}\int_{Q_t^T} \Delta p_{\alpha,\beta} w_{\alpha,\beta} + P\int_{Q_t^T} r_{\alpha,\beta}(p_{\alpha,\beta} - \Delta p_{\alpha,\beta}) \\
&\quad + \frac{1}{\beta}\int_{Q_t^T} w_{\alpha,\beta} p_{\alpha,\beta} + P\int_{Q_t^T} p_{\alpha,\beta} r_{\alpha,\beta} =: \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4.
\end{aligned}
$$

Using the terminal conditions (5.55), assumptions **C1–C2**, we infer by the Young inequality that

$$
|\mathbb{I}_1| + |\mathbb{I}_2^1| \le C\int_{Q_t^T} (|w_{\alpha,\beta}|^2 + |r_{\alpha,\beta}|^2 + 1).
$$

As for $\mathbb{I}_2^2$ and $\mathbb{I}_2^3$, we recall the growth assumption (3.53) and the fact that $\overline{\varphi}_{\alpha,\beta}$ is bounded in $H^1(0,T;H) \cap L^2(0,T;W)$, uniformely with respect to α, due to (3.47). Thus, Hölder's and Young's inequalities along with the continuous injection $V \subset L^6(\Omega)$ lead us to infer that

$$
\begin{aligned}
|\mathbb{I}_2^2| + |\mathbb{I}_2^3| &\le C\int_t^T (1 + \||\overline{\varphi}_{\alpha,\beta}|^2\|_3)\|p_{\alpha,\beta}\|_6 \|w_{\alpha,\beta}\| \\
&\quad + C\int_t^T (1 + \||\overline{\varphi}_{\alpha,\beta}|^2\|_3)\|w_{\alpha,\beta}\|_6 \|w_{\alpha,\beta}\| \\
&\le C\int_t^T (1 + \|\overline{\varphi}_{\alpha,\beta}\|_V^2)\|p_{\alpha,\beta}\|_V \|w_{\alpha,\beta}\| \\
&\quad + C\int_t^T (1 + \|\overline{\varphi}_{\alpha,\beta}\|_V^2)\|w_{\alpha,\beta}\|_V \|w_{\alpha,\beta}\| \\
&\le \delta\int_t^T \|p_{\alpha,\beta}\|_V^2 + \delta\int_{Q_t^T} |\nabla w_{\alpha,\beta}|^2 + C(\delta)\int_{Q_t^T} |w_{\alpha,\beta}|^2,
\end{aligned}
$$

for a positive constant δ yet to be determined. Next, similar arguments lead us to obtain that

$$
|\mathbb{I}_3^1| \le \frac{1}{4}\int_{Q_t^T} |\Delta p_{\alpha,\beta}|^2 + \frac{2}{\beta}\int_{Q_t^T} |w_{\alpha,\beta}|^2,
$$

as well as that

$$|\mathbb{I}_3^2| + |\mathbb{I}_4| \leq 3\delta \int_{Q_t^T} |p_{\alpha,\beta}|^2 + \frac{1}{4} \int_{Q_t^T} |\Delta p_{\alpha,\beta}|^2 + C(\delta) \int_{Q_t^T} (|w_{\alpha,\beta}|^2 + |r_{\alpha,\beta}|^2).$$

Hence, upon collecting all these terms, we realise that it suffices to fix δ small enough so that Gronwall's lemma along with elliptic regularity theory to deduce that

$$\begin{aligned}&\|w_{\alpha,\beta}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \alpha^{1/2}\|p_{\alpha,\beta}\|_{L^\infty(0,T;V)} + \|p_{\alpha,\beta}\|_{L^2(0,T;W)} \\ &\quad + \|r_{\alpha,\beta}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq C\end{aligned}$$

for a suitable positive constant C independent of α. Moreover, let us note that in turn

$$\|\alpha p_{\alpha,\beta}\|_{L^\infty(0,T;V)} \leq C\alpha^{1/2}.$$

Second estimate: We multiply (5.51) by an arbitrary $v \in L^2(0,T;V)$, integrate over Q and by parts, and make use of the above bounds to infer that

$$\begin{aligned}\Big|\int_Q \partial_t w_{\alpha,\beta}\, v\Big| &\leq C\|\nabla p_{\alpha,\beta}\|_{L^2(0,T;H)}\|\nabla v\|_{L^2(0,T;H)} + C\|\nabla w_{\alpha,\beta}\|_{L^2(0,T;H)}\|\nabla v\|_{L^2(0,T;H)} \\ &\quad + C\|p_{\alpha,\beta}\|_{L^2(0,T;H)}\|v\|_{L^2(0,T;V)} \\ &\quad + C\|w_{\alpha,\beta}\|_{L^2(0,T;H)}\|v\|_{L^2(0,T;V)} + C\|v\|_{L^2(0,T;H)} \\ &\leq C\|v\|_{L^2(0,T;V)}.\end{aligned}$$

Thus, dividing both sides by $\|v\|_{L^2(0,T;V)}$ and passing to the superior limit yields

$$\|\partial_t w_{\alpha,\beta}\|_{L^2(0,T;V^*)} \leq C.$$

Third estimate: Arguing in a similar fashion, we readily infer from comparison in equation (5.53) that

$$\|\partial_t r_{\alpha,\beta}\|_{L^2(0,T;V^*)} \leq C,$$

and from a comparison in equation (5.52) that

$$\|\alpha\partial_t p_{\alpha,\beta}\|_{L^2(0,T;H)} \leq C.$$

Passage to the Limit

Here, we draw some consequences from the aforementioned estimates. Owing to standard weak and weak star compactness arguments it follows that there exist limits $w_\beta, p_\beta, r_\beta$ such that, as $\alpha \to 0$,

$$\begin{aligned}&w_{\alpha,\beta} \to w_\beta \quad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V),\\ &p_{\alpha,\beta} \to p_\beta \quad \text{weakly in } L^2(0,T;W),\\ &r_{\alpha,\beta} \to r_\beta \quad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V).\end{aligned}$$

Moreover, the compact embedding of $H^1(0,T;V^*) \cap L^2(0,T;V)$ into $C^0([0,T];H)$ guarantees that the final data are meaningful and that

$$\begin{aligned} w_{\alpha,\beta} \to w_\beta \quad &\text{strongly in } L^2(0,T;H), \\ r_{\alpha,\beta} \to r_\beta \quad &\text{strongly in } L^2(0,T;H), \end{aligned}$$

and we also have that

$$\alpha p_{\alpha,\beta} \to 0 \quad \text{strongly in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W).$$

Hence, by combining the above convergences above with the definition of the auxiliary variable $w_{\alpha,\beta}$, we also deduce that

$$q_{\alpha,\beta} \to q_\beta \quad \text{weakly in } L^2(0,T;V). \tag{5.56}$$

Therefore, the above convergences implies that the weak limit of $w_{\alpha,\beta}$ can be identified with $w_\beta = p_\beta - \beta q_\beta$.

Next, we take into account the variational formulation of system (5.35)–(5.39) written in the variables $(w_{\alpha,\beta}, p_{\alpha,\beta}, r_{\alpha,\beta})$:

$$\begin{aligned} &-\langle \partial_t w_{\alpha,\beta}(t), v\rangle - \int_\Omega \nabla q_{\alpha,\beta}(t) \cdot \nabla v - \int_\Omega F''(\overline{\varphi}_{\alpha,\beta}(t)) q_{\alpha,\beta}(t) v \\ &\quad = \int_\Omega b_1(\overline{\varphi}_{\alpha,\beta}(t) - \varphi_Q(t)) v, \\ &\int_\Omega q_{\alpha,\beta}(t) v - \alpha, \beta \int_\Omega \partial_t p_{\alpha,\beta}(t) v + \int_\Omega \nabla p_{\alpha,\beta}(t) \cdot \nabla v \\ &\quad + P \int_\Omega (p_{\alpha,\beta}(t) - r_{\alpha,\beta}(t)) v = 0, \\ &-\langle \partial_t r_{\alpha,\beta}(t), v\rangle + \int_\Omega \nabla r_{\alpha,\beta}(t) \cdot \nabla v + P \int_\Omega (r_{\alpha,\beta}(t) - p_{\alpha,\beta}(t)) v \\ &\quad = \int_\Omega b_3(\overline{\sigma}_{\alpha,\beta}(t) - \sigma_Q(t)) v, \end{aligned}$$

for every $v \in V$ and almost everywhere in $(0,T)$, and the terminal conditions

$$w_{\alpha,\beta}(T) = \int_\Omega b_2(\overline{\varphi}_{\alpha,\beta}(T) - \varphi_\Omega), \quad r_{\alpha,\beta}(T) = \int_\Omega b_4(\overline{\sigma}_{\alpha,\beta}(T) - \sigma_\Omega). \tag{5.57}$$

By virtue of the above convergences it follows that, as $\alpha \to 0$, the above system converges to

$$\begin{aligned}
&- \langle \partial_t (p_\beta + \beta q_\beta)(t), v \rangle + \int_\Omega \nabla q_\beta(t) \cdot \nabla v + \int_\Omega F''(\overline{\varphi}_\beta(t)) q_\beta(t) v \\
&\quad = \int_\Omega b_1(\overline{\varphi}_\beta(t) - \varphi_Q(t)) v, \\
&- \int_\Omega q_\beta(t) v + \int_\Omega \nabla p_\beta(t) \cdot \nabla v + P \int_\Omega (p_\beta(t) - r_\beta(t)) v = 0, \\
&- \langle \partial_t r_\beta(t), v \rangle + \int_\Omega \nabla r_\beta(t) \cdot \nabla v + P \int_\Omega (r_\beta(t) - p_\beta(t)) v \\
&\quad = \int_\Omega b_3(\overline{\sigma}_\beta(t) - \sigma_Q(t)) v,
\end{aligned}$$

for every $v \in V$ and almost everyewhere in $(0, T)$, and

$$(p_\beta + \beta q_\beta)(T) = b_2(\overline{\varphi}_\beta(T) - \varphi_\Omega), \quad r_\beta(T) = \int_\Omega b_4(\overline{\sigma}_\beta(T) - \sigma_\Omega).$$

In fact, almost all the terms pass to the limit straightforwardly by using classical reasoning and the above estimates. The only term which deserve to be commented is the nonlinear term $F''(\overline{\varphi}_{\alpha,\beta}) q_{\alpha,\beta}$. However, by combining the growth assumptions of the potential F (3.53) with the strong convergence (3.52), it follows that

$$F''(\overline{\varphi}_{\alpha,\beta}) \to F''(\overline{\varphi}_\beta) \quad \text{strongly in } L^2(0, T; H) \tag{5.58}$$

so that, by the weak-strong principle we readily infer that

$$F''(\overline{\varphi}_{\alpha,\beta}) q_{\alpha,\beta} \to F''(\overline{\varphi}_\beta) q_\beta \quad \text{weakly in } L^2(0, T; H),$$

and the proof is concluded. □

5.2.1.3 Approximation of Optimal Controls and Optimality Conditions

Once the existence of minimisers for $(CP)_\beta$ has been established, we aim at deriving the corresponding first-order necessary conditions for optimality. From a formal perspective, it could seem reasonable to let $\alpha \to 0$ in the necessary condition for $(CP)_{\alpha,\beta}$ expressed by the variational inequality (5.42). However, this would be possible, without any additional restrictions, if we can guarantee that every optimal control for $(CP)_\beta$ can be recovered as limit of a sequence of optimal controls of $(CP)_{\alpha,\beta}$. Unfortunately, we can not prove this general fact and to overcome this issue we follow the same line of argument of [11] (see also [33, 34, 44, 138], where an application of such a technique can be found). Namely, we define a new cost functional, called *adapted cost functional*, depending on a fixed minimiser $\overline{u}_\beta$ of $(CP)_\beta$, which is defined as

$$\mathcal{J}_{\text{ad}}(\varphi, \sigma, u) := \mathcal{J}(\varphi, \sigma, u) + \frac{1}{2} \|u - \overline{u}_\beta\|^2_{L^2(Q)}. \tag{5.59}$$

Notice that $\mathcal{J}_{\text{ad}}$ turns out to be a local perturbation of the cost functional $\mathcal{J}$ since it reduces to $\mathcal{J}$ when acts on the minimisers of $(CP)_\beta$. The main idea behind this definition concerns the fact that for the associated optimal control problem we can obtain a compactness type property in the sense that every arbitrary minimiser $\overline{u}_\beta$ of $(CP)_\beta$ can be recovered as limit of a sequence of minimisers of $(CP)^{\text{ad}}_{\alpha,\beta}$, as $\alpha \to 0$. Hence, the auxiliary minimisation problem we are going to consider, which will be referred to as adapted, reads as

$(CP)^{\text{ad}}_{\alpha,\beta}$ Minimise $\mathcal{J}_{\text{ad}}(\varphi, \sigma, u)$ subject to:

(i) (φ, μ, σ) yields a strong solution to (3.1)–(3.5) obtained from Theorem 3.5 with $g = u$;

(ii) $u \in \mathcal{U}_{\text{ad}}$. (5.60)

In a sense to yet to be specified, we will prove that $(CP)^{\text{ad}}_{\alpha,\beta}$ approximates $(CP)_\beta$ as $\alpha \to 0$ so that the passage to the limit as $\alpha \to 0$ in the variational inequality (5.42) can be rigorously performed producing in turn the optimality condition of $(CP)_\beta$.

It is straightforward to infer that the adapted control problem $(CP)^{\text{ad}}_{\alpha,\beta}$ perfectly complies with the framework of $(CP)_{\alpha,\beta}$ so that we directly deduce the existence of minimisers and the optimality condition for optimality (in analogy with Theorem 5.1 and Theorem 5.6, respectively).

Lemma 5.9. *Assume that the assumption of Theorem 3.10 hold and let* **C1–C3** *be verified. Then, for every optimal control $\overline{u}_\beta$ of $(CP)_\beta$, the adapted optimisation problem $(CP)^{\text{ad}}_{\alpha,\beta}$ defined by* (5.60) *admits a minimiser.*

In a similar fashion as in Theorem 5.6 we derive the first-order necessary condition for optimality of $(CP)^{\text{ad}}_{\alpha,\beta}$ as follows.

Theorem 5.10. *Assume that the assumption of Theorem 3.10 hold and let* **C1–C3** *are in force. Let $\overline{u}_\beta$ be an optimal control for $(CP)_\beta$ and let $\mathcal{J}_{\text{ad}}$ be defined by* (5.59). *Then, if $\overline{u}_{\alpha,\beta} \in \mathcal{U}_{\text{ad}}$ is an optimal control for $(CP)^{\text{ad}}_{\alpha,\beta}$ it necessarily fulfils*

$$\int_Q \big(r_{\alpha,\beta} + b_0 \overline{u}_{\alpha,\beta} + (\overline{u}_\beta - \overline{u}_{\alpha,\beta})\big)(u - \overline{u}_{\alpha,\beta}) \geq 0 \qquad \forall u \in \mathcal{U}_{\text{ad}}, \tag{5.61}$$

where $r_{\alpha,\beta}$ is the third component of the unique solution $(p_{\alpha,\beta}, q_{\alpha,\beta}, r_{\alpha,\beta})$ to the adjoint system (5.35)–(5.39) *obtained from Theorem 5.5.*

The sense in which the minimisers of $(CP)^{\text{ad}}_{\alpha,\beta}$ approximate the ones of $(CP)_\beta$ as $\alpha \to 0$ is finally specified in the following statement.

Theorem 5.11. *Suppose the assumption of Theorem 3.10 and let* **C1–C3** *hold. Let $\overline{u}_\beta$ be an optimal control for $(CP)_\beta$ with the corresponding state $(\overline{\varphi}_\beta, \overline{\mu}_\beta, \overline{\sigma}_\beta)$ obtained from Theorem 3.14. Then, for every family $\{\overline{u}_{\alpha,\beta}\}_\alpha$ of optimal controls for $(CP)^{\text{ad}}_{\alpha,\beta}$ with the corresponding states $\{(\overline{\varphi}_{\alpha,\beta}, \overline{\mu}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta})\}_\alpha$ it holds that, as $\alpha \to 0$,*

$$\overline{u}_{\alpha,\beta} \to \overline{u}_\beta \quad \textit{strongly in } L^2(Q), \tag{5.62}$$

$$\overline{\varphi}_{\alpha,\beta} \to \overline{\varphi}_\beta \quad \textit{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{5.63}$$

$$\overline{\sigma}_{\alpha,\beta} \to \overline{\sigma}_\beta \quad \textit{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V), \tag{5.64}$$

$$\mathcal{J}_{\text{ad}}(\overline{\varphi}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta}, \overline{u}_{\alpha,\beta}) \to \mathcal{J}(\overline{\varphi}_\beta, \overline{\sigma}_\beta, \overline{u}_\beta). \tag{5.65}$$

Proof of Theorem 5.11. Using the uniform estimates (3.47)–(3.51) along with the structure of $\mathcal{U}_{\rm ad}$ we deduce the existence of limits $u^*_\beta, \varphi^*_\beta, \sigma^*_\beta$ such that, along a non-relabelled subsequence $\{\alpha_n\}_n$ which goes to zero as $n \to \infty$, it holds, as $n \to \infty$,

$$\begin{aligned}
\overline{u}_{\alpha_n,\beta} &\to u^*_\beta \quad \text{weakly star in } L^\infty(Q),\\
\overline{\varphi}_{\alpha_n,\beta} &\to \varphi^*_\beta \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),\\
\overline{\sigma}_{\alpha_n,\beta} &\to \sigma^*_\beta \quad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V).
\end{aligned}$$

As far as (5.65) is concerned, let us note that on the one hand the minimality of $(\overline{\varphi}_n, \overline{\sigma}_n, \overline{u}_n)$ for $(CP)^{\rm ad}_{\alpha,\beta}$ entails that

$$\mathcal{J}_{\rm ad}(\overline{\varphi}_{\alpha_n,\beta}, \overline{\sigma}_{\alpha_n,\beta}, \overline{u}_{\alpha_n,\beta}) \leq \mathcal{J}_{\rm ad}(\overline{\varphi}_\beta, \overline{\sigma}_\beta, \overline{u}_\beta) \quad \text{for every } n \in \mathbb{N}$$

so that passing to the superior limit in both sides and exploiting the definition of $\mathcal{J}_{\rm ad}$, we get

$$\limsup_{n\to\infty} \mathcal{J}_{\rm ad}(\overline{\varphi}_{\alpha_n,\beta}, \overline{\sigma}_{\alpha_n,\beta}, \overline{u}_{\alpha_n,\beta}) \leq \mathcal{J}_{\rm ad}(\overline{\varphi}_\beta, \overline{\sigma}_\beta, \overline{u}_\beta) = \mathcal{J}(\overline{\varphi}_\beta, \overline{\sigma}_\beta, \overline{u}_\beta). \tag{5.66}$$

On the other hand, the weak sequential lower semicontinuity of $\mathcal{J}_{\rm ad}$ implies that

$$\begin{aligned}
\liminf_{n\to\infty} \mathcal{J}_{\rm ad}(\overline{\varphi}_{\alpha_n,\beta}, \overline{\sigma}_{\alpha_n,\beta}, \overline{u}_{\alpha_n,\beta}) &\geq \mathcal{J}_{\rm ad}(\varphi^*_\beta, \sigma^*_\beta, u^*_\beta)\\
&= \mathcal{J}(\varphi^*_\beta, \sigma^*_\beta, u^*_\beta) + \frac{1}{2}\|u^*_\beta - \overline{u}_\beta\|^2_{L^2(Q)}\\
&\geq \mathcal{J}(\overline{\varphi}_\beta, \overline{\sigma}_\beta, \overline{u}_\beta) + \frac{1}{2}\|u^*_\beta - \overline{u}_\beta\|^2_{L^2(Q)}.
\end{aligned} \tag{5.67}$$

Since φ^*_β and σ^*_β are solution to (3.1)–(3.5) with $\alpha = 0$ associated to u^*_β, combining (5.66) and (5.67) with the optimality of $\overline{u}_\beta$ for $(CP)_\beta$ yields

$$u^*_\beta = \overline{u}_\beta$$

so that the uniqueness of (3.1)–(3.5) with $\alpha = 0$ also implies that $\varphi^*_\beta = \overline{\varphi}_\beta$ and $\sigma^*_\beta = \overline{\sigma}_\beta$. Hence, we realise that the following chain of equality has been shown:

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{J}_{\rm ad}(\overline{\varphi}_{\alpha_n,\beta}, \overline{\sigma}_{\alpha_n,\beta}, \overline{u}_{\alpha_n,\beta}) &= \liminf_{n\to\infty} \mathcal{J}_{\rm ad}(\overline{\varphi}_{\alpha_n,\beta}, \overline{\sigma}_{\alpha_n,\beta}, \overline{u}_{\alpha_n,\beta})\\
= \limsup_{n\to\infty} \mathcal{J}_{\rm ad}(\overline{\varphi}_{\alpha_n,\beta}, \overline{\sigma}_{\alpha_n,\beta}, \overline{u}_{\alpha_n,\beta}) &= \mathcal{J}(\overline{\varphi}_\beta, \overline{\sigma}_\beta, \overline{u}_\beta)
\end{aligned}$$

which proves (5.65). We are now reduced to show the strong convergence (5.62), but a careful look of the above estimates leads us to

$$\frac{1}{2}\|\overline{u}_{\alpha_n,\beta} - \overline{u}_\beta\|^2_{L^2(Q)} \to 0,$$

concluding the proof. □

With the approximation results presented in Theorem 5.11, we are now in a position to let $\alpha \to 0$ in the variational inequality (5.61) to derive the optimality conditions of $(CP)_\beta$ in a rigorous fashion and obtain the optimality conditions for the limit problem.

Theorem 5.12. *Suppose the assumption of Theorem 3.10 and let* **C1–C3** *hold. Let* $\overline{u}_\beta$ *be a minimiser of* $(CP)_\beta$*. Then,* $\overline{u}_\beta$ *necessarily satisfies*

$$\int_Q (r_\beta + b_0\overline{u}_\beta)(u - \overline{u}_\beta) \geq 0 \qquad \forall u \in \mathcal{U}_{\mathrm{ad}},$$

where r_β *is the third component of the unique solution* $(p_\beta, q_\beta, r_\beta)$ *of the adjoint system* (5.35)–(5.39) *with* $\alpha = 0$ *obtained from Theorem 5.8.*

Proof of Theorem 5.12. As already mention we aim at passing as $\alpha \to 0$ in (5.61). In this direction, we first consider the approximating sequence of controls $\{\overline{u}_{\alpha_n,\beta}\}_n \subset \mathcal{U}_{\mathrm{ad}}$ obtained from Theorem 5.11 with the corresponding states $\{(\overline{\varphi}_{\alpha_n,\beta}, \overline{\mu}_{\alpha_n,\beta}, \overline{\sigma}_{\alpha_n,\beta})\}_n$ and adjoint variables $\{(p_{\alpha_n,\beta}, q_{\alpha_n,\beta}, r_{\alpha_n,\beta})\}_n$. Then, by virtue of the convergences pointed out by Theorem 3.9 and Theorem 5.8 the thesis readily follows by letting $n \to \infty$ in (5.61) using the Lebesgue dominated convergence theorem. □

Finally, due to the structure of the control-box $\mathcal{U}_{\mathrm{ad}}$, in the case $b_0 > 0$, we can provide a pointwise characterisation of the minimiser (see, e.g., [146]).

Corollary 5.13. *Suppose that the assumption of Theorem 5.12 hold and let* $b_0 > 0$ *and* $\overline{u}_\beta$ *be an optimal control for* $(CP)_\beta$*. Then, the minimiser* $\overline{u}_\beta$ *is the* $L^2(0,T;H)$*-orthogonal projection of* $-{b_0}^{-1}r_\beta$ *onto the closed subspace* $\mathcal{U}_{\mathrm{ad}}$ *and*

$$\overline{u}_\beta(\mathbf{x},t) = \max\left\{u_*(\mathbf{x},t), \min\{u^*(\mathbf{x},t), -{b_0}^{-1}r_\beta(\mathbf{x},t)\}\right\} \quad \textit{for a.a. } (\mathbf{x},t) \in Q,$$

being r_β *the third component of the unique solution of the adjoint system* (5.35)–(5.39) *with* $\alpha = 0$ *obtained from Theorem 5.8.*

Proof of Corollary 5.13. The claim directly follows from the Hilbert projection theorem, since $\mathcal{U}_{\mathrm{ad}}$ is a non-empty, closed and convex subset of $L^2(0,T;H)$, whereas the pointwise characterisation follows from the box-structure of $\mathcal{U}_{\mathrm{ad}}$. □

5.2.2 The Optimisation Problem $(CP)_{\alpha,\beta}$ as $\beta \to 0$

5.2.2.1 Existence of Minimisers

As a first step we prove the existence of a minimiser for $(CP)_\alpha$.

Theorem 5.14 (Existence of a minimiser: $\beta \to 0$)**.** *Suppose the assumption of Theorem 3.14 and let* **C1–C3** *hold. Then the optimisation problem* $(CP)_\alpha$ *admits a minimiser.*

Proof of Theorem 5.14. The result directly follows by adapting the lines of arguments in the proof of Theorem 5.7 (see also the proof of Theorem 5.1) along with uniform in β convergences obtained in Theorem 3.13. □

5.2.2.2 A Priori Estimates on the Adjoint Variables

Here, we are going to address the asymptotic behaviour of the adjoint system (5.35)–(5.39) as $\beta \to 0$.

Theorem 5.15. *Assume that the assumption of Theorem 3.14 hold, let* **C1–C3** *and* $b_2 = b_4 = 0$. *Let* $\overline{u}_\alpha$ *be an optimal control of* $(CP)_\alpha$ *and* $\{u_{\alpha,\beta}\}_\alpha \subset \mathcal{U}_{\text{ad}}$ *be such that* $u_{\alpha,\beta} \to \overline{u}_\alpha$ *strongly in* $L^2(0,T;H)$ *as* $\beta \to 0$. *Let* $(\varphi_{\alpha,\beta}, \mu_{\alpha,\beta}, \sigma_{\alpha,\beta})$ *and* $(p_{\alpha,\beta}, q_{\alpha,\beta}, r_{\alpha,\beta})$ *be the state associated to* $u_{\alpha,\beta}$ *and the unique solution to the adjoint system* (5.35)–(5.39) *as given by Theorem 3.5 and Theorem 5.5, respectively and let* $(\overline{\varphi}_\alpha, \overline{\mu}_\alpha, \overline{\sigma}_\alpha)$ *denote the unique solution to* (3.1)–(3.5) *with* $\beta = 0$ *obtained from Theorem 3.14 associated to* $\overline{u}_\alpha$. *Then, there exists a triplet* $(p_\alpha, q_\alpha, r_\alpha)$ *with*

$$
\begin{aligned}
p_\alpha, r_\alpha &\in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),\\
q_\alpha &\in L^2(0,T;W),
\end{aligned}
$$

such that, as $\beta \to 0$,

$$
\begin{aligned}
p_{\alpha,\beta} &\to p_\alpha \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), && (5.68)\\
q_{\alpha,\beta} &\to q_\alpha \quad \text{weakly in } L^2(0,T;W), && (5.69)\\
r_{\alpha,\beta} &\to r_\alpha \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), && (5.70)\\
\beta q_{\alpha,\beta} &\to 0 \quad \text{strongly in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W). && (5.71)
\end{aligned}
$$

Moreover, there exists a positive constant K_α, *which may depend on* α *but it is independent of* β, *such that*

$$
\begin{aligned}
&\|p_{\alpha,\beta}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} + \beta\|q_{\alpha,\beta}\|_{H^1(0,T;H)} + \beta^{1/2}\|q_{\alpha,\beta}\|_{L^\infty(0,T;V)}\\
&\quad + \|q_{\alpha,\beta}\|_{L^2(0,T;W)} + \|r_{\alpha,\beta}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le K_\alpha. \qquad (5.72)
\end{aligned}
$$

In addition, the limit triplet $(p_\alpha, q_\alpha, r_\alpha)$ *is the unique weak solution to the adjoint system* (5.35)–(5.39) *with* $\beta = 0$ *in the sense that it verifies*

$$
\begin{aligned}
&- \langle \partial_t p_\alpha(t), v\rangle + \int_\Omega \nabla q_\alpha(t) \cdot \nabla v + \int_\Omega F''(\overline{\varphi}_\alpha(t)) q_\alpha(t) v\\
&\qquad + \int_\Omega P'(\overline{\varphi}_\alpha(t))(\overline{\sigma}_\alpha(t) - \overline{\mu}_\alpha(t))(r_\alpha(t) - p_\alpha(t))v = \int_\Omega b_1(\overline{\varphi}_\alpha(t) - \varphi_Q(t))v,\\
&- \langle a\partial_t p_\alpha(t), v\rangle - \int_\Omega q_\alpha(t) v + \int_\Omega \nabla p_\alpha(t) \cdot \nabla v + P\int_\Omega (p_\alpha(t) - r_\alpha(t))v = 0,\\
&- \langle \partial_t r_\alpha(t), v\rangle + \int_\Omega \nabla r_\alpha(t) \cdot \nabla v + P\int_\Omega (r_\alpha(t) - p_\alpha(t))v\\
&= \int_\Omega b_3(\overline{\sigma}_\alpha(t) - \sigma_Q(t))v,
\end{aligned}
$$

for every $v \in V$ *and almost everywhere in* $(0,T)$, *and*

$$
\alpha p_\alpha(T) = 0, \quad r_\alpha(T) = 0.
$$

Proof of Theorem 5.15. As before, since the adjoint system (5.35)–(5.39) is linear and the arguments are standard, we proceed formally for convenience even though the same computation can be made rigorous within a Galerkin scheme. Notice that in what follows, the symbol C will denote a positive constant which may depend on α but it is independent of β.

First estimate: To begin with, we add to both sides of (5.36) the term $p_{\alpha,\beta}$. Then, we test (5.35) by $-q_{\alpha,\beta}$, the new second equation by $-\partial_t p_{\alpha,\beta}$, and (5.37) by $r_{\alpha,\beta}$. Summing the resulting equalities and integrating over Q_t^T leads us to obtain that

$$\begin{aligned}
&\frac{\beta}{2}\|q_{\alpha,\beta}(t)\|^2 + \int_{Q_t^T} |\nabla q_{\alpha,\beta}|^2 + \int_{Q_t^T} F''(\overline{\varphi}_{\alpha,\beta})|q_{\alpha,\beta}|^2 + \frac{1}{2}\|p_{\alpha,\beta}\|_V^2 \\
&\quad + \alpha \int_{Q_t^T} |\partial_t p_{\alpha,\beta}|^2 + \frac{1}{2}\|r_{\alpha,\beta}(t)\|^2 + \int_{Q_t^T} |\nabla r_{\alpha,\beta}|^2 \\
&= -b_1 \int_{Q_t^T} (\overline{\varphi}_{\alpha,\beta} - \varphi_Q) q_{\alpha,\beta} + b_3 \int_{Q_t^T} (\overline{\sigma}_{\alpha,\beta} - \sigma_Q) r_{\alpha,\beta} \\
&\quad + \int_{Q_t^T} P'(\overline{\varphi}_{\alpha,\beta})(\overline{\sigma}_{\alpha,\beta} - \overline{\mu}_{\alpha,\beta})(r_{\alpha,\beta} - p_{\alpha,\beta}) q_{\alpha,\beta} \\
&\quad + \int_{Q_t^T} P(\overline{\varphi}_{\alpha,\beta})(p_{\alpha,\beta} - r_{\alpha,\beta})\partial_t p_{\alpha,\beta} - \int_{Q_t^T} p_{\alpha,\beta}\, \partial_t p_{\alpha,\beta} \\
&\quad - \int_{Q_t^T} P(\overline{\varphi}_{\alpha,\beta})(r_{\alpha,\beta} - p_{\alpha,\beta}) r_{\alpha,\beta}.
\end{aligned} \tag{5.73}$$

As the third term on the left-hand side is concerned, let us recall that F'' is bounded below in terms of the Lipschitz constant of F_2' (see **A3**) so that we have

$$\int_{Q_t^T} F''(\overline{\varphi}_{\alpha,\beta})|q_{\alpha,\beta}|^2 \geq -L \int_{Q_t^T} |q_{\alpha,\beta}|^2.$$

Next, we test (5.36) by $Kq_{\alpha,\beta}$, for a positive constant K yet to be determined, and integrate over Q_t^T to obtain that

$$\begin{aligned}
K \int_{Q_t^T} |q_{\alpha,\beta}|^2 &= \alpha K \int_{Q_t^T} \partial_t p_{\alpha,\beta} q_{\alpha,\beta} - K \int_{Q_t^T} \nabla p_{\alpha,\beta} \cdot \nabla q_{\alpha,\beta} \\
&\quad - K \int_{Q_t^T} P(\overline{\varphi}_{\alpha,\beta})(p_{\alpha,\beta} - r_{\alpha,\beta}) q_{\alpha,\beta}.
\end{aligned} \tag{5.74}$$

Then, we add (5.73) to (5.74) to infer that

$$\begin{aligned}
&\frac{\beta}{2}\|q_{\alpha,\beta}(t)\|^2 + (K-L)\int_{Q_t^T}|q_{\alpha,\beta}|^2 + \int_{Q_t^T}|\nabla q_{\alpha,\beta}|^2 + \alpha\int_{Q_t^T}|\partial_t p_{\alpha,\beta}|^2\\
&\quad + \frac{1}{2}\|p_{\alpha,\beta}\|_V^2 + \frac{1}{2}\|r_{\alpha,\beta}(t)\|^2 + \int_{Q_t^T}|\nabla r_{\alpha,\beta}|^2\\
&\le -b_1\int_{Q_t^T}(\overline{\varphi}_{\alpha,\beta}-\varphi_Q)q_{\alpha,\beta} + b_3\int_{Q_t^T}(\overline{\sigma}_{\alpha,\beta}-\sigma_Q)r_{\alpha,\beta}\\
&\quad + \int_{Q_t^T}P'(\overline{\varphi}_{\alpha,\beta})(\overline{\sigma}_{\alpha,\beta}-\overline{\mu}_{\alpha,\beta})(r_{\alpha,\beta}-p_{\alpha,\beta})q_{\alpha,\beta}\\
&\quad + \int_{Q_t^T}P(\overline{\varphi}_{\alpha,\beta})(p_{\alpha,\beta}-r_{\alpha,\beta})\partial_t p_{\alpha,\beta} - \int_{Q_t^T}p_{\alpha,\beta}\,\partial_t p_{\alpha,\beta}\\
&\quad - \int_{Q_t^T}P(\overline{\varphi}_{\alpha,\beta})(r_{\alpha,\beta}-p_{\alpha,\beta})r_{\alpha,\beta} + \alpha K\int_{Q_t^T}\partial_t p_{\alpha,\beta}\,q_{\alpha,\beta} - K\int_{Q_t^T}\nabla p_{\alpha,\beta}\cdot\nabla q_{\alpha,\beta}\\
&\quad - K\int_{Q_t^T}P(\overline{\varphi}_{\alpha,\beta})(p_{\alpha,\beta}-r_{\alpha,\beta})q_{\alpha,\beta} = \mathbb{I}_1+\mathbb{I}_2+\mathbb{I}_3+\mathbb{I}_4+\mathbb{I}_5.
\end{aligned}$$

Owing to assumptions **C1–C2**, Young's inequality, and the fact that $\overline{\varphi}_{\alpha,\beta}$ and $\overline{\sigma}_{\alpha,\beta}$ satisfy (3.59)–(3.64) uniformely in β, we have that

$$|\mathbb{I}_1| \le \delta\int_{Q_t^T}|q_{\alpha,\beta}|^2 + C(\delta)\int_{Q_t^T}(|r_{\alpha,\beta}|^2+1),$$

for a positive δ yet to be determined. From the Hölder and Young inequalities, the continuous embedding $V\subset L^6(\Omega)$, the boundedness of the proliferation function P, and (3.59)–(3.64), we infer that

$$\begin{aligned}
|\mathbb{I}_2| &\le C\int_t^T\|\overline{\sigma}_{\alpha,\beta}-\overline{\mu}_{\alpha,\beta}\|_6\|r_{\alpha,\beta}-p_{\alpha,\beta}\|\|q_{\alpha,\beta}\|_3\\
&\le \delta\int_t^T\|q_{\alpha,\beta}\|_V^2 + C(\delta)\int_t^T(\|\overline{\sigma}_{\alpha,\beta}\|_V^2+\|\overline{\mu}_{\alpha,\beta}\|_V^2)(\|r_{\alpha,\beta}\|^2+\|p_{\alpha,\beta}\|^2)\\
&\le \delta\int_{Q_t^T}(|q_{\alpha,\beta}|^2+|\nabla q_{\alpha,\beta}|^2) + C(\delta)\int_{Q_t^T}(|r_{\alpha,\beta}|^2+|p_{\alpha,\beta}|^2).
\end{aligned}$$

In a similar fashion we have

$$|\mathbb{I}_3^1| \le C\int_{Q_t^T}|p_{\alpha,\beta}-r_{\alpha,\beta}||\partial_t p_{\alpha,\beta}| \le \delta\int_{Q_t^T}|\partial_t p_{\alpha,\beta}|^2 + C(\delta)\int_{Q_t^T}(|p_{\alpha,\beta}|^2+|r_{\alpha,\beta}|^2),$$

and by using the Young inequality once more that

$$\begin{aligned}
&|\mathbb{I}_3^2| + |\mathbb{I}_4^1| + |\mathbb{I}_4^2|\\
&\quad\le \delta\int_{Q_t^T}|\partial_t p_{\alpha,\beta}|^2 + C(\delta)\int_{Q_t^T}(|p_{\alpha,\beta}|^2+|r_{\alpha,\beta}|^2) + \Big|\alpha K\int_{Q_t^T}\partial_t p_{\alpha,\beta}\,q_{\alpha,\beta}\Big|\\
&\quad\le \Big(\frac{\alpha^2K}{2}+\delta\Big)\int_{Q_t^T}|\partial_t p_{\alpha,\beta}|^2 + C(\delta)\int_{Q_t^T}(|p_{\alpha,\beta}|^2+|r_{\alpha,\beta}|^2) + \frac{K}{2}\int_{Q_t^T}|q_{\alpha,\beta}|^2
\end{aligned}$$

and also that

$$|\mathbb{I}_4^3| + |\mathbb{I}_5| \leq \delta \int_{Q_t^T} (|\nabla q_{\alpha,\beta}|^2 + |q_{\alpha,\beta}|^2) + C(\delta) \int_{Q_t^T} (|\nabla p_{\alpha,\beta}|^2 + |p_{\alpha,\beta}|^2 + |r_{\alpha,\beta}|^2).$$

Upon collecting the previous estimates, we realise that the backward-in-time Gronwall lemma yields the estimate we are looking for, provided that K and δ satisfy the following requirements:

$$\min \left\{ K - \tfrac{K}{2} - L - 3\delta, 1 - 2\delta, \alpha - \tfrac{\alpha^2 K}{2} - 2\delta \right\} > 0.$$

Since δ can be taken arbitrarily small, we are reduced to check that

$$\min \left\{ \tfrac{K}{2} - L, \alpha - \tfrac{\alpha^2 K}{2} \right\} > 0$$

which is fulfilled as soon as α is small enough. Lastly, we pick δ small enough and apply Gronwall's lemma to conclude that

$$\begin{aligned} &\|p_{\alpha,\beta}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \beta^{1/2}\|q_{\alpha,\beta}\|_{L^\infty(0,T;H)} + \|q_{\alpha,\beta}\|_{L^2(0,T;V)} \\ &\quad + \|r_{\alpha,\beta}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq C. \end{aligned} \tag{5.75}$$

Second estimate: We test (5.36) by $-\Delta p_{\alpha,\beta}$ and integrate over (t,T) to infer that

$$\begin{aligned} &\frac{\alpha}{2} \int_\Omega |\nabla p_{\alpha,\beta}(t)|^2 + \int_{Q_t^T} |\Delta p_{\alpha,\beta}|^2 \\ &\quad = \int_{Q_t^T} q_{\alpha,\beta}\, \Delta p_{\alpha,\beta} + \int_{Q_t^T} P(\overline{\varphi}_{\alpha,\beta})(p_{\alpha,\beta} - r_{\alpha,\beta})\Delta p_{\alpha,\beta} =: \mathbb{I}_1. \end{aligned} \tag{5.76}$$

Using Young's inequality, along with the boundedness of P, we get that

$$|\mathbb{I}_1| \leq \frac{1}{2} \int_{Q_t^T} |\Delta p_{\alpha,\beta}|^2 + \int_{Q_t^T} |q_{\alpha,\beta}|^2 + C \int_{Q_t^T} (|p_{\alpha,\beta}|^2 + |r_{\alpha,\beta}|^2)$$

so that the previous estimate produces

$$\|\nabla p_{\alpha,\beta}\|_{L^\infty(0,T;H)} + \|\Delta p_{\alpha,\beta}\|_{L^2(0,T;H)} \leq C. \tag{5.77}$$

Hence, from the elliptic regularity theory we deduce that

$$\|p_{\alpha,\beta}\|_{L^\infty(0,T;V)\cap L^2(0,T;W)} \leq C. \tag{5.78}$$

Third estimate: A simple analysis of equation (5.37) (e.g., by first multiplying by $-\partial_t r_{\alpha,\beta}$ and then by $-\Delta r_{\alpha,\beta}$) allows us to infer the parabolic regularity

$$\|r_{\alpha,\beta}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq C. \tag{5.79}$$

Fourth estimate: Next, by testing (5.35) by $\Delta q_{\alpha,\beta}$ and integrating over (t,T), we find that

$$\begin{aligned}\frac{\beta}{2}\|\nabla q_{\alpha,\beta}(t)\|^2 + \int_{Q_t^T} |\Delta q_{\alpha,\beta}|^2 &= \frac{\beta}{2}\|\nabla q_{\alpha,\beta}(T)\|^2 + \int_{Q_t^T} \partial_t p_{\alpha,\beta}\, \Delta q_{\alpha,\beta} \\ &+ \int_{Q_t^T} F''(\overline{\varphi}_{\alpha,\beta}) q_{\alpha,\beta}\, \Delta q_{\alpha,\beta} - \int_{Q_t^T} P'(\overline{\varphi}_{\alpha,\beta})(\overline{\sigma}_{\alpha,\beta} - \overline{\mu}_{\alpha,\beta})(r_{\alpha,\beta} - p_{\alpha,\beta})\Delta q_{\alpha,\beta} \\ &+ \int_{Q_t^T} b_1(\overline{\varphi}_{\alpha,\beta} - \varphi_Q)\Delta q_{\alpha,\beta} =: \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3.\end{aligned}$$

Similar arguments as in the previous estimates allow us to obtain that

$$\begin{aligned}&|\mathbb{I}_1| + |\mathbb{I}_2| + |\mathbb{I}_3| \\ &\quad \le \frac{4}{5}\int_{Q_t^T} |\Delta q_{\alpha,\beta}|^2 + C\int_{Q_t^T} (|\partial_t p_{\alpha,\beta}|^2 + |q_{\alpha,\beta}|^2 + |p_{\alpha,\beta}|^2 + |r_{\alpha,\beta}|^2 + 1),\end{aligned}$$

where the bounds producing (3.60) and (3.61) for the solutions $\overline{\mu}_{\alpha,\beta}$ and $\overline{\sigma}_{\alpha,\beta}$, are also taken into account. Hence, we find that

$$\beta^{1/2}\|\nabla q_{\alpha,\beta}\|_{L^\infty(0,T;H)} + \|\Delta q_{\alpha,\beta}\|_{L^2(0,T;H)} \le C,$$

and from elliptic regularity that

$$\beta^{1/2}\|\nabla q_{\alpha,\beta}\|_{L^\infty(0,T;H)} + \|q_{\alpha,\beta}\|_{L^2(0,T;W)} \le C.$$

Fifth estimate: Lastly, by comparison in equation (5.35) we also realise that

$$\beta\|\partial_t q_{\alpha,\beta}\|_{L^2(0,T;H)} \le C. \tag{5.80}$$

Passage to the Limit

Here, let us draw some consequence of the a priori estimates obtained so far. Let us recall that the constant C appearing at the end of every estimate is independent of β. Thus, from Banach–Alaoglu theorem we infer the existence of variables p_α, q_α and r_α such that, as $\beta \to 0$,

$$\begin{aligned} p_{\alpha,\beta} &\to p_\alpha \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \\ q_{\alpha,\beta} &\to q_\alpha \quad \text{weakly in } L^2(0,T;W), \\ r_{\alpha,\beta} &\to r_\alpha \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \\ \beta q_{\alpha,\beta} &\to 0 \quad \text{strongly in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W). \end{aligned}$$

Moreover, up to a non-relabelled subsequence, we also have by compactness arguments that

$$p_{\alpha,\beta} \to p_\alpha \quad \text{strongly in } C^0([0,T];H) \cap L^2(0,T;V), \tag{5.81}$$

$$r_{\alpha,\beta} \to r_\alpha \quad \text{strongly in } C^0([0,T];H) \cap L^2(0,T;V). \tag{5.82}$$

Besides, by (3.59) and (3.65), along a non-relabelled subsequence, for every $\kappa \geq 1$ if $d = 2$ and $1 \leq \kappa < 6$ if $d = 3$, we have that

$$\overline{\varphi}_{\alpha,\beta} \to \overline{\varphi}_\alpha \quad \text{a.e. in } Q, \quad \overline{\varphi}_{\alpha,\beta} \to \overline{\varphi}_\alpha, \quad \text{strongly in } L^2(0,T;L^\kappa(\Omega)).$$

Hence, we consider the weak formulation of (5.35)-(5.39) which reads as

$$\begin{aligned}
&- \langle \partial_t(p_{\alpha,\beta}(t) + \beta q_{\alpha,\beta}(t), v\rangle + \int_\Omega \nabla q_{\alpha,\beta}(t) \cdot \nabla v + \int_\Omega F''(\overline{\varphi}_{\alpha,\beta}(t)) q_{\alpha,\beta}(t)\, v \\
&\qquad + \int_\Omega P'(\overline{\varphi}_{\alpha,\beta}(t))(\overline{\sigma}_{\alpha,\beta}(t) - \overline{\mu}_{\alpha,\beta}(t))(r_{\alpha,\beta}(t) - p_{\alpha,\beta}(t))v \\
&\qquad = \int_\Omega b_1(\overline{\varphi}_{\alpha,\beta}(t) - \varphi_Q(t))v, \\
&-\alpha\langle \partial_t p_{\alpha,\beta}(t), w\rangle - \int_\Omega q_{\alpha,\beta}(t) w + \int_\Omega \nabla p_{\alpha,\beta}(t) \cdot \nabla w \\
&\qquad + \int_\Omega P(\overline{\varphi}_{\alpha,\beta}(t))(p_{\alpha,\beta}(t) - r_{\alpha,\beta}(t))w = 0, \\
&- \langle \partial_t r_{\alpha,\beta}(t), z\rangle + \int_\Omega \nabla r_{\alpha,\beta}(t) \cdot \nabla z + \int_\Omega P(\overline{\varphi}_{\alpha,\beta}(t))(r_{\alpha,\beta}(t) - p_{\alpha,\beta}(t))z \\
&\qquad = \int_\Omega b_3(\overline{\sigma}_{\alpha,\beta}(t) - \sigma_Q(t))z,
\end{aligned}$$

for every $v, w, z \in V$ and almost everywhere in $(0,T)$ and the terminal conditions

$$(p_{\alpha,\beta} + \beta q_{\alpha,\beta})(T) = 0, \quad \alpha p_{\alpha,\beta}(T) = 0, \quad r_{\alpha,\beta}(T) = 0.$$

Let us just comment how to handle the passage as $\beta \to 0$ of the nonlinear term involving F'' as the other terms easily pass to the weak limit as $\beta \to 0$ due to the regularity of P and to the aforementioned convergences. By continuity of F'' and from the above remark, it follows that, as $\beta \to 0$,

$$F''(\overline{\varphi}_{\alpha,\beta}) \to F''(\overline{\varphi}_\alpha) \quad \text{a.e. in } Q.$$

Now, since $\{F(\overline{\varphi}_{\alpha,\beta})\}_\beta$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$, by the growth condition (3.53) we know that $\{F''(\overline{\varphi}_{\alpha,\beta})\}_\beta$ is uniformly bounded in $L^\infty(0,T;H)$. Furthermore, the boundedness of $\{\varphi_{\alpha,\beta}\}_\beta$ in $L^2(0,T;L^6(\Omega))$ and again (3.53) ensure also that $\{F''(\overline{\varphi}_{\alpha,\beta})\}_\beta$ is uniformly bounded in $L^1(0,T;L^3(\Omega))$. For any $\vartheta \in (0,1)$, setting $\kappa_\vartheta \in (2,3)$ such that $\frac{1}{\kappa_\vartheta} := \frac{\vartheta}{2} + \frac{1-\vartheta}{3}$, by interpolation we have that

$$\|F''(\overline{\varphi}_{\alpha,\beta})\|_{\kappa_\vartheta} \leq C\|F''(\overline{\varphi}_{\alpha,\beta})\|^\vartheta \|F''(\overline{\varphi}_{\alpha,\beta})\|_3^{1-\vartheta} \quad \text{a.e. in } (0,T),$$

from which it follows that

$$\|F''(\overline{\varphi}_{\alpha,\beta})\|_{L^{\frac{1}{1-\vartheta}}(0,T;L^{\kappa_\vartheta}(\Omega))} \leq C_\alpha.$$

In particular, there exists $\overline{\vartheta} \in (0,1)$ such that $\overline{\kappa} := \kappa_{\overline{\vartheta}} = \frac{1}{1-\overline{\vartheta}} \in (2,3)$: an easy computation yields $\overline{\vartheta} = \frac{4}{7}$ and $\overline{\kappa} = \frac{7}{3}$. This implies that

$$\|F''(\overline{\varphi}_{\alpha,\beta})\|_{L^{7/3}(Q)} \leq C_\alpha.$$

By the Severini–Egorov theorem we infer that, for all $\kappa \in [1, \frac{7}{3})$,

$$F''(\overline{\varphi}_{\alpha,\beta}) \to F''(\overline{\varphi}_\alpha) \quad \text{weakly in } L^{7/3}(Q) \text{ and strongly in } L^\kappa(Q).$$

In particular, since $\frac{7}{3} > 2$ and (5.69), this implies that, as $\beta \to 0$,

$$F''(\overline{\varphi}_{\alpha,\beta})q_{\alpha,\beta} \to F''(\overline{\varphi}_\alpha)q_\alpha \quad \text{weakly in } L^1(Q).$$

Since $W \subset L^\infty(\Omega)$, this allows to pass to the limit as $\beta \to 0$ in the first line of the above variational formulation for every test function $v \in W$. Since $F''(\overline{\varphi}_\alpha) \in L^1(0,T; L^3(\Omega))$, at the limit we find that $F''(\overline{\varphi}_\alpha)q_\alpha \in L^{6/5}(\Omega) \subset V^*$ almost everywhere in $(0,T)$, and the variational formulation holds also for all $v \in V$ by the density of W in V. Thus, after passing to the limit as $\beta \to 0$ we find that the limit triplet $(p_\alpha, q_\alpha, r_a)$ verifies

$$\begin{aligned}
&- \langle \partial_t p_\alpha(t), v\rangle + \int_\Omega \nabla q_\alpha(t) \cdot \nabla v + \int_\Omega F''(\overline{\varphi}_\alpha(t))q_\alpha(t)\, v \\
&\quad + \int_\Omega P'(\overline{\varphi}_\alpha(t))(\overline{\sigma}_\alpha(t) - \overline{\mu}_\alpha(t))(r_\alpha(t) - p_\alpha(t))v = \int_\Omega b_1(\overline{\varphi}_\alpha(t) - \varphi_Q(t))v, \\
&-\alpha\langle \partial_t p_\alpha(t), v\rangle - \int_\Omega q_\alpha(t)v + \int_\Omega \nabla p_\alpha(t) \cdot \nabla v + \int_\Omega P(\overline{\varphi}_\alpha(t))(p_\alpha(t) - r_\alpha(t))v \\
&\quad = 0, \\
&- \langle \partial_t r_\alpha(t), v\rangle + \int_\Omega \nabla r_\alpha(t) \cdot \nabla v + \int_\Omega P(\overline{\varphi}_\alpha(t))(r_\alpha(t) - p_\alpha(t))v \\
&\quad = \int_\Omega b_3(\overline{\sigma}_\alpha(t) - \sigma_Q(t))v,
\end{aligned}$$

for every $v \in V$ and almost everywhere in $(0,T)$ and the terminal conditions

$$p_\alpha(T) = 0, \quad r_\alpha(T) = 0$$

which concludes the proof. □

5.2.2.3 Approximation of Optimal Controls and Optimality Conditions

We are now ready to derive the first-order necessary conditions for optimality of $(CP)_\alpha$ by adapting the lines of argument in Subsection 5.2.1.3 for the case $\beta \to 0$ instead of $\alpha \to 0$. Namely, for a given minimiser $\overline{u}_\alpha$ of $(CP)_\alpha$ we define the adapted cost functional by

$$\mathcal{J}_{\text{ad}}(\varphi,\sigma,u) := \mathcal{J}(\varphi,\sigma,u) + \frac{1}{2}\|u - \overline{u}_\alpha\|^2_{L^2(Q)}$$

along with the corresponding adapted optimisation problem

$(CP)^{\text{ad}}_{\alpha,\beta}$ Minimise $\mathcal{J}_{\text{ad}}(\varphi,\sigma,u)$ subject to:

(i) (φ,μ,σ) yields a solution to (3.1)–(3.5) obtained from Theorem 3.5 with $g = u$;

(ii) $u \in \mathcal{U}_{\text{ad}}$.

This will allow us to show that $(CP)^{\mathrm{ad}}_{\alpha,\beta} \searrow (CP)_\alpha$ in a sense to be precised later on so to pass rigorously to the limit as $\beta \to 0$ in the variational inequality (5.42). First of all, we obtain the corresponding result to Lemma 5.9 which again straightforwardly follows from the previous investigation on $(CP)_{\alpha,\beta}$.

Lemma 5.16. *Suppose the assumption of Theorem 3.14 and let* **C1–C3** *hold and* $b_2 = 0$*. Then, for every* $\beta \in (0,1)$ *and every optimal control* $\overline{u}_\alpha$ *of* $(CP)_\alpha$*, the adapted optimisation problem* $(CP)^{\mathrm{ad}}_{\alpha,\beta}$ *admits a minimiser.*

Theorem 5.17. *Assume that the assumption of Theorem 3.14 are in force and let* **C1–C3** *hold and* $b_2 = 0$*. Let* $\overline{u}_\alpha$ *be an optimal control for* $(CP)_\alpha$*. Then, for every* $\beta \in (0,1)$*, if* $\overline{u}_{\alpha,\beta} \in \mathcal{U}_{\mathrm{ad}}$ *is an optimal control for* $(CP)^{\mathrm{ad}}_{\alpha,\beta}$ *it necessarily fulfils*

$$\int_Q \big(r_{\alpha,\beta} + b_0\overline{u}_{\alpha,\beta} + (\overline{u}_\alpha - \overline{u}_{\alpha,\beta})\big)(u - \overline{u}_{\alpha,\beta}) \geq 0 \qquad \forall u \in \mathcal{U}_{\mathrm{ad}}, \tag{5.83}$$

where $r_{\alpha,\beta}$ *is the third component of the unique solution* $(p_{\alpha,\beta}, q_{\alpha,\beta}, r_{\alpha,\beta})$ *to the adjoint system* (5.35)–(5.39) *obtained from Theorem 5.5.*

Hence, it holds that $(CP)^{\mathrm{ad}}_{\alpha,\beta} \searrow (CP)_\alpha$ as $\beta \to 0$ in the following sense:

Theorem 5.18. *Suppose the assumption of Theorem 3.14 and let* **C1–C3** *hold and* $b_2 = 0$*. Let* $\overline{u}_\alpha$ *be an optimal control for* $(CP)_\alpha$ *with the corresponding state* $(\overline{\varphi}_\alpha, \overline{\mu}_\alpha, \overline{\sigma}_\alpha)$ *obtained from Theorem 3.10. Then, for every family* $\{\overline{u}_{\alpha,\beta}\}_\beta$ *of optimal controls for* $(CP)^{\mathrm{ad}}_{\alpha,\beta}$ *with the corresponding states* $\{(\overline{\varphi}_{\alpha,\beta}, \overline{\mu}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta})\}_\alpha$ *it holds that, as* $\beta \to 0$*,*

$$\overline{u}_{\alpha,\beta} \to \overline{u}_\alpha \quad \text{strongly in } L^2(Q), \tag{5.84}$$

$$\overline{\varphi}_{\alpha,\beta} \to \overline{\varphi}_\alpha \quad \text{weakly star in } H^1(0,T;V) \cap L^\infty(0,T;W), \tag{5.85}$$

$$\overline{\sigma}_{\alpha,\beta} \to \overline{\sigma}_\alpha \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{5.86}$$

$$\mathcal{J}_{\mathrm{ad}}(\overline{\varphi}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta}, \overline{u}_{\alpha,\beta}) \to \mathcal{J}(\overline{\varphi}_\alpha, \overline{\sigma}_\alpha, \overline{u}_\alpha). \tag{5.87}$$

Proof of Theorem 5.18. The proof follows the same lines of argument of the proof of Theorem 5.11 with the help of the asymptotic analysis performed by Theorem 3.13. □

Lastly, we combine Theorem 5.18 with Theorem 5.6 to derive the optimality condition of $(CP)_\alpha$ by letting $\beta \to 0$ in the variational inequality (5.42). Here, the obtained result follows.

Theorem 5.19. *Suppose the assumption of Theorem 3.14 and* **C1–C3** *hold. Moreover, let* $b_2 = b_4 = 0$ *and let* $\overline{u}_\alpha$ *be a minimiser of* $(CP)_\alpha$*. Then, it necessarily satisfies*

$$\int_Q (r_\alpha + b_0\overline{u}_\alpha)(u - \overline{u}_\alpha) \geq 0 \qquad \forall u \in \mathcal{U}_{\mathrm{ad}},$$

where r_α *is the third component of the unique solution* $(p_\alpha, q_\alpha, r_\alpha)$ *of the adjoint system* (5.35)–(5.39) *with* $\beta = 0$ *obtained from Theorem 5.15.*

Proof of Theorem 5.19. The proof can be obtained by adapting the line of arguments in Theorem 5.12 by using the approximation results pointed out by Theorem 5.15 and Theorem 5.18. □

Lastly, we obtain the corresponding of Corollary 5.13.

Corollary 5.20. *Suppose that the assumption of Theorem 5.19 are satisfied. Moreover, let $b_0 > 0$ and $\overline{u}_\alpha$ be an optimal control for $(CP)_\alpha$. Then, the minimiser $\overline{u}_\alpha$ is the $L^2(0,T;H)$-orthogonal projection of $-{b_0}^{-1}r_\alpha$ onto the closed subspace $\mathcal{U}_{\mathrm{ad}}$ and*

$$\overline{u}_\alpha(\mathbf{x},t) = \max\left\{u_*(\mathbf{x},t), \min\{u^*(\mathbf{x},t), -{b_0}^{-1}r_\alpha(\mathbf{x},t)\}\right\} \quad \text{for a.a. } (\mathbf{x},t) \in Q,$$

where r_α is the third component of the unique solution of the adjoint system (5.35)–(5.39) *with $\beta = 0$ obtained from Theorem 5.15.*

CHAPTER 6

Optimal Control Theory of the Nonlocal Model

This chapter is completely devoted to studying an optimal control problem associated with the nonlocal system (4.1)–(4.5) analysed in Chapter 4. To differentiate the presentation with respect to Chapter 5, we postulate a different form for the cost functional and for the control variables. In fact, inspired from the work by C. Kahle et al. [109], we formulate the problem of parameters identification in the framework of optimal control theory. In particular, the physical parameters we aim to identify are the constants $\mathcal{P}$, χ, η, and $\mathcal{C}$ occurring in the system (4.1)–(4.5) and the role of controls is now played by these parameters.

As done in the previous chapter, we first start considering the optimal control problem with both the relaxation parameters α and β positive and fixed in the system (4.1)–(4.5), and then we let them to zero via asymptotic approaches.

6.1 Optimal Control Theory of the Nonlocal Relaxed Model

Motivated by the above comments, we consider the objective type *cost functional* of the following form:

$$\mathcal{J}(\varphi,\mathcal{P},\chi,\eta,\mathcal{C}) := \frac{b_1}{2}\|\varphi-\varphi_Q\|^2_{L^2(Q)} + \frac{b_2}{2}\|\varphi(T)-\varphi_\Omega\|^2_{L^2(\Omega)} + \frac{\alpha_P}{2}|\mathcal{P}-\mathcal{P}_*|^2 + \frac{\alpha_\chi}{2}|\chi-\chi_*|^2 + \frac{\alpha_\eta}{2}|\eta-\eta_*|^2 + \frac{\alpha_C}{2}|\mathcal{C}-\mathcal{C}_*|^2, \tag{6.1}$$

for some prescribed target functions $\varphi_\Omega : \Omega \to \mathbb{R}$, $\varphi_Q : Q \to \mathbb{R}$, non-negative constants $b_1, b_2, \alpha_P, \alpha_\chi, \alpha_\eta, \alpha_C$ (not all zero), and non-negative constants $\mathcal{P}_*, \chi_*, \eta_*, \mathcal{C}_*$ representing some a priori information for the parameters. Moreover, the set of *admissible controls* is defined as

$$\begin{aligned}\mathcal{U}_{\mathrm{ad}} :=\{&(\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathbb{R}^4 : \\ &0 \le \mathcal{P} \le \mathcal{P}_{\max},\ 0 \le \chi \le \chi_{\max},\ 0 \le \eta \le \eta_{\max},\ 0 \le \mathcal{C} \le \mathcal{C}_{\max}\}, \quad (6.2)\end{aligned}$$

for some prescribed non-negative constants $\mathcal{P}_{\max}, \chi_{\max}, \eta_{\max}, \mathcal{C}_{\max}$. It is worth underlying that (6.2) is a non-empty and compact (closed and bounded) subset of $\mathbb{R}^4$.

Hence, the identification problem we are going to address in this first part of the chapter can be summarised as

$(CP)_{\alpha,\beta}$ Minimise $\mathcal{J}(\varphi, \mathcal{P}, \chi, \eta, \mathcal{C})$ subject to:
(i) (φ, μ, σ) yields a solution to (4.1)–(4.5) obtained from Theorem 4.5;
(ii) $(\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathcal{U}_{\mathrm{ad}}$.

Thus, we would like to tackle the following problem: given a set of data $\mathcal{P}_*, \chi_*, \eta_*, \mathcal{C}_*$, identify the optimal parameter values $\mathcal{P}, \chi, \eta, \mathcal{C}$ so that the resulting model predictions and the data are close in some proper sense. An alternative approach for parameter identification relies on the Bayeasian calibration which has been recently used by C. Kahle et al. in [110] in the framework of local tumour growth models.

Throughout this first part of the chapter we work with $\alpha \in (0, \alpha_0)$ and $\beta \in (0, \beta_0)$ fixed, being α_0 and β_0 the bounds defined by (4.6). For this reason and for the sake of simplicity, we avoid writing the unnecessary subscripts α, β under the involved variables.

As far as the assumptions are concerned, in addition to **B1–B7,** we postulate the following assumptions:

D1 $\mathcal{P}_{\max}, \chi_{\max}, \eta_{\max}, \mathcal{C}_{\max}$ are non-negative constants.

D2 $F : (-\ell, \ell) \to [0, +\infty)$ is of class C^4, where $\ell \in (0, +\infty]$, and

$$F'(0) = 0, \qquad \lim_{r \to (\pm\ell)^{\mp}} [F'(r) - \chi_{\max}\eta_{\max} r] = \pm\infty.$$

This latter condition allows both for the logarithmic potential and for any polynomial super-quadratic potential. Nonetheless, potentials of double-obstacle type like (1.9) are excluded. Let us note also that the limiting condition at $\pm\ell$ is satisfied in particular by any $\eta \in [0, \eta_{\max}]$ and $\chi \in [0, \chi_{\max}]$.

When dealing with the aforementioned optimal control problem, we postulate that the cost functional $\mathcal{J}$ and the space of admissible controls $\mathcal{U}_{\mathrm{ad}}$ are defined by (6.1), and (6.2) and that the following are fulfilled.

D3 The target functions $\varphi_\Omega : \Omega \to \mathbb{R}$ and $\varphi_Q : Q \to \mathbb{R}$ verify $\varphi_\Omega \in L^2(\Omega)$ and $\varphi_Q \in L^2(Q)$.

D4 $b_1, b_2, \alpha_P, \alpha_\chi, \alpha_\eta, \alpha_C$ are non-negative constants, not all zero.

D5 $\mathcal{P}_*, \chi_*, \eta_*, \mathcal{C}_*$ are non-negative constants.

D6 $f \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$.

Let us recall that from Theorem 4.1 and Theorem 4.2 we already deduce the existence and uniqueness of a weak solution to system (4.1)–(4.5) under a rather general framework. However, since we are interested in solving the optimal control problem $(CP)_{\alpha,\beta}$, we are forced to work with strong solutions instead, which possess better stability properties with respect to the involved parameters. In particular, unlike weak solutions, strong solutions allow to consider also non-negative chemotaxis and active transport coefficients (cf. Theorem 4.5 and Theorem 4.8). Moreover, let us notice that it is straightforward to complement (4.59) and obtain the following continuous dependence result.

Lemma 6.1. *Let the assumptions of Theorem 4.8 be fulfilled. Moreover, for any pair of initial data* $\{(\varphi_0^i, \mu_0^i, \sigma_0^i)\}_{i=1,2}$ *satisfying* (4.7), (4.42), *and* (4.51), *there exists a constant* $K > 0$, *depending on the structural data,* α, *and* β, *such that, for every pair of parameters* $\{(\mathcal{P}^i, \chi^i, \eta^i, \mathcal{C}^i)\}_{i=1,2} \in \mathcal{U}_{\text{ad}}$, *and for any respective strong solutions* $\{(\varphi_i, \mu_i, \sigma_i)\}_{i=1,2}$ *satisfying* (4.52)–(4.55), *it holds that*

$$
\begin{aligned}
&\|\varphi_1 - \varphi_2\|_{W^{1,\infty}(0,T;V)\cap H^1(0,T;H^2(\Omega))} + \|\mu_1 - \mu_2\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \\
&\quad + \|\sigma_1 - \sigma_2\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \\
&\le K\left(\|\varphi_0^1 - \varphi_0^2\|_{H^2(\Omega)} + \|\mu_0^1 - \mu_0^2\|_V + \|\sigma_0^1 - \sigma_0^2\|_V\right) \\
&\quad + K\left(|\mathcal{P}^1 - \mathcal{P}^2| + |\chi^1 - \chi^2| + |\eta^1 - \eta^2| + |\mathcal{C}^1 - \mathcal{C}^2|\right).
\end{aligned}
$$

The strong well-posedness of the state system (4.1)–(4.5) in Theorem 4.5 and Theorem 4.8 allows us to define the *control-to-state* operator $\mathcal{S}$, assigning to every given admissible control $(\mathcal{P}, \chi, \eta, \mathcal{C})$ the unique corresponding state (φ, μ, σ). Namely, we have

$$
\mathcal{S} : (\mathcal{P}, \chi, \eta, \mathcal{C}) \mapsto (\varphi, \mu, \sigma),
$$

where (φ, μ, σ) is the unique solution to (4.1)–(4.5) obtained from Theorem 4.5. Moreover, let us draw a straightforward consequence of the separation property (4.54).

Corollary 6.2. *In addition to the assumptions of Theorem 4.5, suppose that* $F \in C^k(-\ell, \ell)$ *for some* $k \ge 4$. *Then, there exists a positive constant* K, *depending only on the structural data, on the initial data, and possibly on* α *and* β, *such that*

$$
\|\varphi\|_{L^\infty(Q)} + \max_{i=0,\dots,k} \|F^{(i)}(\varphi)\|_{L^\infty(Q)} \le K \quad \forall\, (\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathcal{U}_{\text{ad}}. \tag{6.3}
$$

6.1.1 Existence of a Minimiser

The first problem that we address concerns the existence of a minimiser of the optimal control problem $(CP)_{\alpha,\beta}$, with $\alpha, \beta > 0$ being fixed. Its proof is rather standard and follows as a consequence of the direct method of calculus of variations.

Theorem 6.3 (Existence of a minimiser)**.** *Suppose that* **B1–B7** *and* **D1–D6** *are fulfilled. Then, the optimisation problem* $(CP)_{\alpha,\beta}$ *admits a minimiser.*

Proof of Theorem 6.3. Without loss of generality, we assume that all the constants $\alpha_P, \alpha_\chi, \alpha_\eta, \alpha_C$ are positive. In fact, if this is not the case, we can consider the corresponding control $\mathcal{P}, \chi, \eta$ and/or $\mathcal{C}$ as a prescribed constant, redefine $\mathcal{U}_{\rm ad}$ accordingly and argue in the same way.

To begin with, notice that the cost functional is non-negative so that we can consider a minimising sequence of elements of $\mathcal{U}_{\rm ad}$. Namely, we take the minimising sequence $\{(\mathcal{P}_n, \chi_n, \eta_n, \mathcal{C}_n)\}_n \subset \mathcal{U}_{\rm ad}$ and the corresponding sequence of states $\{(\varphi_n, \mu_n, \sigma_n)\}_n = \{\mathcal{S}(\mathcal{P}_n, \chi_n, \eta_n, \mathcal{C}_n)\}_n$ all related to the same initial data $(\varphi_0, \mu_0, \sigma_0)$. Namely, we have

$$0 \le \overline{\lambda} := \inf \Big\{ \mathcal{J}(\varphi, \mathcal{P}, \chi, \eta, \mathcal{C}) \big| \, (\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathcal{U}_{\rm ad}, \, \varphi = \mathcal{S}_1(\mathcal{P}, \chi, \eta, \mathcal{C}) \Big\},$$

and, as $n \to \infty$,

$$\mathcal{J}(\varphi_n, \mathcal{P}_n, \chi_n, \eta_n, \mathcal{C}_n) \to \overline{\lambda}.$$

Since the bounds (4.52)–(4.55) are uniform in n thanks to the structure of $\mathcal{U}_{\rm ad}$, we invoke standard compactness results (cf., Lemma 2.4) to obtain the existence of limits $\overline{\varphi}$, $(\overline{P}, \overline{\chi}, \overline{\eta}, \overline{C}) \in \mathcal{U}_{\rm ad}$, and a non-relabelled subsequence such that, as $n \to \infty$,

$$\begin{aligned}
&\varphi_n \to \overline{\varphi} \quad \text{weakly star in } W^{1,\infty}(0,T;V) \cap H^1(0,T;H^2(\Omega)), \\
&\qquad\qquad \text{and strongly in } C^0([0,T];C^0(\overline{\Omega})), \\
&\mu_n \to \overline{\mu} \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap L^\infty(Q), \\
&\sigma_n \to \overline{\sigma} \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap L^\infty(Q), \\
&\mathcal{P}_n \to \overline{\mathcal{P}}, \; \chi_n \to \overline{\chi}, \; \eta_n \to \overline{\eta}, \; \mathcal{C}_n \to \overline{\mathcal{C}}.
\end{aligned}$$

It is then a standard matter to pass to the limit in the variational formulation of (4.1)–(4.5) written for $\{(\varphi_n, \mu_n, \sigma_n)\}_n$ to deduce that $(\overline{\varphi}, \overline{\mu}, \overline{\sigma}) = \mathcal{S}(\overline{P}, \overline{\chi}, \overline{\eta}, \overline{C})$. Lastly, the weak sequential lower semincontinuity of $\mathcal{J}$ entails that $(\overline{P}, \overline{\chi}, \overline{\eta}, \overline{C})$ is a minimiser for $(CP)_{\alpha,\beta}$ together with its corresponding state $(\overline{\varphi}, \overline{\mu}, \overline{\sigma})$. ☐

6.1.2 Linearised System

We study here the linearised system, which can be formally obtained by differentiating the state system (4.1)–(4.5) with respect to the control in a certain direction. First, we fix some preliminary notation: let $(\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathcal{U}_{\rm ad}$ be fixed, set

$$(\overline{\varphi}, \overline{\mu}, \overline{\sigma}) := \mathcal{S}(\mathcal{P}, \chi, \eta, \mathcal{C}),$$

and consider an arbitrary increment

$$\mathbf{h} := (h_{\mathcal{P}}, h_\chi, h_\eta, h_{\mathcal{C}}) \in \mathbb{R}^4 \quad \text{such that} \quad (\mathcal{P}, \chi, \eta, \mathcal{C}) + \mathbf{h} \in \mathcal{U}_{\rm ad}.$$

The variables of the linearised system are denoted by (ξ, ν, ζ): of course, they depend on the increment $\mathbf{h}$, but we avoid keeping track of this explicitly for the brevity of notation.

The linearised system reads:

$$\partial_t(\alpha\nu + \xi) - \Delta\nu = (\mathcal{P}\overline{\sigma} - \mathcal{A})f'(\overline{\varphi})\xi + \mathcal{P}\zeta f(\overline{\varphi}) + h_{\mathcal{P}}\overline{\sigma} f(\overline{\varphi}) \qquad \text{in } Q, \tag{6.4}$$
$$\nu = \beta\partial_t\xi + a\xi - J * \xi + F''(\overline{\varphi})\xi - \chi\zeta - h_\chi\overline{\sigma} \qquad \text{in } Q, \tag{6.5}$$
$$\partial_t\zeta - \Delta\zeta + \mathcal{B}\zeta + \mathcal{C}(\zeta f(\overline{\varphi}) + \overline{\sigma} f'(\overline{\varphi})\xi) + h_{\mathcal{C}}\overline{\sigma} f(\overline{\varphi}) = -\Delta(\eta\xi + h_\eta\overline{\varphi}) \qquad \text{in } Q, \tag{6.6}$$
$$\partial_{\mathbf{n}}\nu = \partial_{\mathbf{n}}(\eta\xi + h_\eta\overline{\varphi}) = \partial_{\mathbf{n}}\zeta = 0 \qquad \text{on } \Sigma, \tag{6.7}$$
$$\xi(0) = \nu(0) = \zeta(0) = 0 \qquad \text{in } \Omega. \tag{6.8}$$

Here is the corresponding well-posedness result.

Theorem 6.4 (Well-posedness of the linearised system: $\alpha, \beta > 0$)**.** *Assume* **B1–B7** *and* **D1–D6***. Then, the linearised system* (6.4)–(6.8) *admits a unique solution* (ξ, ν, ζ) *satisfying*

$$\xi \in H^1(0,T;H) \cap L^\infty(0,T;V), \qquad \nu, \zeta \in H^1(0,T;V^*) \cap L^2(0,T;V).$$

Proof of Theorem 6.4. In what follows we proceed formally by pointing out some a priori estimates. Anyway, it is a standard matter to perform the same computations within a Galerkin scheme (cf. Section 2.4.2) and then pass to the limit as the discretisation parameter approach infinity to deduce the same results at the continuous level. Moreover, let us notice that the symbols C and $C(\delta)$ will denote generic constants depending only on structural data and possibly on an additional positive constant δ and may change from line to line.

First estimate: To begin with we add to both sides of (6.5) the term $(c_a + 2)\xi$. Next, we multiply (6.4) by ν, the new (6.5) by $-\partial_t\xi$, the gradient of (6.5) by $-\nabla\xi$, (6.6) by ζ, integrate over Q_t and by parts. Adding the resulting equalities we obtain that

$$\begin{aligned}
&\frac{\alpha}{2}\|\nu(t)\|^2 + \int_{Q_t} |\nabla\nu|^2 + \beta\int_{Q_t} |\partial_t\xi|^2 + \frac{\beta}{2}\|\nabla\xi(t)\|^2 + (\tfrac{c_a}{2} + 1)\|\xi(t)\|^2 \\
&\quad + \int_{Q_t} (a + F''(\overline{\varphi}))|\nabla\xi|^2 + \int_{Q_t} (a\xi - J * \xi)\partial_t\xi + \frac{1}{2}\|\zeta(t)\|^2 \\
&\quad + \mathcal{B}\int_{Q_t} |\zeta|^2 + \int_{Q_t} |\nabla\zeta|^2 \\
&= \int_{Q_t} (\mathcal{P}\overline{\sigma} - \mathcal{A})f'(\overline{\varphi})\xi\nu + \int_{Q_t} \mathcal{P}\zeta f(\overline{\varphi})\nu + \int_{Q_t} h_{\mathcal{P}}\overline{\sigma} f(\overline{\varphi})\nu - \int_{Q_t} F''(\overline{\varphi})\xi\partial_t\xi \\
&\quad + \int_{Q_t} \chi\zeta\partial_t\xi + \int_{Q_t} h_\chi\overline{\sigma}\partial_t\xi - \int_{Q_t} (c_a + 2)\xi\partial_t\xi \\
&\quad + \int_{Q_t} \nabla\nu\cdot\nabla\xi - \int_{Q_t} (\nabla a)\xi\cdot\nabla\xi + \int_{Q_t} (\nabla J * \xi)\cdot\nabla\xi - \int_{Q_t} F^{(3)}(\overline{\varphi})\xi\nabla\overline{\varphi}\cdot\nabla\xi \\
&\quad + \int_{Q_t} \chi\nabla\zeta\cdot\nabla\xi + \int_{Q_t} h_\chi\nabla\overline{\sigma}\cdot\nabla\xi - \int_{Q_t} \mathcal{C}(\zeta f(\overline{\varphi}) + \overline{\sigma} f'(\overline{\varphi})\xi)\zeta - \int_{Q_t} h_{\mathcal{C}}\overline{\sigma} f(\overline{\varphi})\zeta \\
&\quad + \int_{Q_t} \eta\nabla\xi\cdot\nabla\zeta - \int_{Q_t} h_\eta\Delta\overline{\varphi}\,\zeta =: \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4 + \mathbb{I}_5.
\end{aligned}$$

The first two terms of the second line can be easily estimated by using **B5**, which produces

$$\int_{Q_t} (a + F''(\overline{\varphi}))|\nabla \xi|^2 \geq C_0 \int_{Q_t} |\nabla \xi|^2, \tag{6.9}$$

and similarly for the next term we have

$$\int_{Q_t} (a\xi - J * \xi)\partial_t \xi \geq \frac{a_* - a^*}{2} \|\xi(t)\|^2 \geq -\frac{c_a}{2} \|\xi(t)\|^2. \tag{6.10}$$

Moreover, the terms on the right-hand side can be easily estimated using Young and Hölder inequalities, the regularity of the state variables $\overline{\varphi}, \overline{\sigma}$ expressed in (4.52)–(4.55), the boundedness of f and f', and Corollary 6.2. In fact, for every $\delta > 0$, we obtain that

$$\begin{aligned}
|\mathbb{I}_1| &\leq C \int_{Q_t} (|\overline{\sigma}||\xi| + |\zeta| + |\overline{\sigma}|)|\nu| + C\|F''(\overline{\varphi})\|_{L^\infty(Q)} \int_{Q_t} |\xi||\partial_t \xi| \\
&\leq \delta \int_{Q_t} |\partial_t \xi|^2 + C(\delta) \int_{Q_t} (|\xi|^2 + |\nu|^2 + |\zeta|^2 + 1), \\
|\mathbb{I}_2| &\leq C \int_{Q_t} (|\zeta| + |\overline{\sigma}| + |\xi|)|\partial_t \xi| \leq \delta \int_{Q_t} |\partial_t \xi|^2 + C(\delta) \int_{Q_t} (|\zeta|^2 + |\xi|^2 + 1).
\end{aligned}$$

Similarly, using the continuous embedding $V \subset L^6(\Omega)$, Hölder's inequality, the regularity of $\overline{\varphi}$ and again Corollary 6.2, we find that

$$\begin{aligned}
|\mathbb{I}_3| &\leq C \int_{Q_t} (|\nabla \nu| + |\xi|)|\nabla \xi| + C\|F^{(3)}(\overline{\varphi})\|_{L^\infty(Q)} \int_{Q_t} |\xi||\nabla \overline{\varphi}||\nabla \xi| \\
&\leq \delta \int_{Q_t} |\nabla \nu|^2 + C(\delta) \int_{Q_t} (|\xi|^2 + |\nabla \xi|^2) + C \int_0^t \|\overline{\varphi}(s)\|_{H^2(\Omega)} \|\xi(s)\|_V \|\nabla \xi(s)\| \, ds \\
&\leq \delta \int_{Q_t} |\nabla \nu|^2 + C(\delta) \int_{Q_t} (|\xi|^2 + |\nabla \xi|^2).
\end{aligned}$$

By analogous computations, we have that

$$\begin{aligned}
|\mathbb{I}_4| &\leq C \int_{Q_t} (|\nabla \zeta| + |\nabla \overline{\sigma}|)|\nabla \xi| + C \int_{Q_t} (|\zeta| + |\overline{\sigma}||\xi| + |\overline{\sigma}|)|\zeta| \\
&\leq \delta \int_{Q_t} |\nabla \zeta|^2 + C(\delta) \int_{Q_t} (|\nabla \xi|^2 + |\xi|^2 + |\zeta|^2 + 1), \\
|\mathbb{I}_5| &\leq C \int_{Q_t} |\nabla \xi||\nabla \zeta| + C \int_{Q_t} |\Delta \overline{\varphi}||\zeta| \leq \delta \int_{Q_t} |\nabla \zeta|^2 + C(\delta) \int_{Q_t} (|\zeta|^2 + |\nabla \xi|^2 + 1).
\end{aligned}$$

Upon collecting the above estimates, we infer that

$$\begin{aligned}
&\frac{\alpha}{2} \|\nu(t)\|^2 + (1 - \delta) \int_{Q_t} |\nabla \nu|^2 + (\beta - 2\delta) \int_{Q_t} |\partial_t \xi|^2 + \|\xi(t)\|^2 + \frac{\beta}{2} \|\nabla \xi(t)\|^2 \\
&\quad + C_0 \int_{Q_t} |\nabla \xi|^2 + \frac{1}{2} \|\zeta(t)\|^2 + \mathcal{B} \int_{Q_t} |\zeta|^2 + (1 - 2\delta) \int_{Q_t} |\nabla \zeta|^2 \\
&\leq C(\delta) \int_{Q_t} (|\xi|^2 + |\nabla \xi|^2 + |\zeta|^2 + 1).
\end{aligned}$$

Then, we choose $\delta := \min\{\frac{\beta}{4}, \frac{1}{4}\}$ so that Gronwall's lemma yields that

$$\|\xi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\nu\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\zeta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le C.$$

Second estimate: In light of the above estimate, a comparison argument in (6.4) and (6.6) produces

$$\|\partial_t(\alpha\nu + \xi)\|_{L^2(0,T;V^*)} + \|\partial_t\zeta\|_{L^2(0,T;V^*)} \le C,$$

hence also

$$\|\partial_t\nu\|_{L^2(0,T;V^*)} \le C.$$

Conclusion: It is clear that these estimates are enough to pass to the limit in the linearised system. Furthermore, it is worth noting that the uniqueness directly follows from the linearity of the system and the estimates above. The proof of Theorem 6.4 is then concluded. □

6.1.3 Fréchet Differentiability of $\mathcal{S}$

Theorem 6.5 (Fréchet differentiability: $\alpha, \beta > 0$)**.** *Assume* **B1–B7** *and* **D1–D6** *and let* $(\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathcal{U}_{\mathrm{ad}}$ *be fixed. Then, the control-to-state operator* $\mathcal{S} : \mathbb{R}^4 \to \mathcal{X}$ *is Fréchet differentiable at* $(\mathcal{P}, \chi, \eta, \mathcal{C})$*, where*

$$\mathcal{X} := \big(H^1(0,T;H) \cap L^\infty(0,T;V)\big) \times \big(L^\infty(0,T;H) \cap L^2(0,T;V)\big)^2.$$

Moreover, for every increment $\mathbf{h} \in \mathbb{R}^4$*, we have* $D\mathcal{S}(\mathcal{P}, \chi, \eta, \mathcal{C})[\mathbf{h}] = (\xi, \nu, \zeta)$*, where* (ξ, ν, ζ) *is the unique solution to* (6.4)–(6.8) *associated to* $\mathbf{h}$*, obtained by Theorem 6.4.*

Proof of Theorem 6.5. To begin with, let us recall that $\mathbf{h} = (h_{\mathcal{P}}, h_\chi, h_\eta, h_{\mathcal{C}})$ and let us set the corresponding control

$$(\mathcal{P}^{\mathbf{h}}, \chi^{\mathbf{h}}, \eta^{\mathbf{h}}, \mathcal{C}^{\mathbf{h}}) := (\mathcal{P} + h_{\mathcal{P}}, \chi + h_\chi, \eta + h_\eta, \mathcal{C} + h_{\mathcal{C}}).$$

Next, we denote by

$$\begin{aligned}
(\overline{\varphi}, \overline{\mu}, \overline{\sigma}) &:= \mathcal{S}(\mathcal{P}, \chi, \eta, \mathcal{C}),\\
(\overline{\varphi}^{\mathbf{h}}, \mu^{\mathbf{h}}, \sigma^{\mathbf{h}}) &:= \mathcal{S}(\mathcal{P}^{\mathbf{h}}, \chi^{\mathbf{h}}, \eta^{\mathbf{h}}, \mathcal{C}^{\mathbf{h}}),\\
(\xi, \nu, \zeta) &:= \text{Solution to the linearised system associated to } \mathbf{h},
\end{aligned}$$

and set

$$\phi := \overline{\varphi}^{\mathbf{h}} - \overline{\varphi} - \xi, \quad \rho := \mu^{\mathbf{h}} - \overline{\mu} - \nu, \quad \omega := \sigma^{\mathbf{h}} - \overline{\sigma} - \zeta.$$

By comparing Theorem 4.5 with Theorem 6.4, we deduce that the triplet (ϕ, ρ, ω) satisfies

$$\phi \in H^1(0,T;H) \cap L^\infty(0,T;V), \qquad \rho, \omega \in H^1(0,T;V^*) \cap L^2(0,T;V).$$

To prove the assertion, it is then enough to show that

$$\frac{\|\mathcal{S}(\mathcal{P}^{\mathbf{h}}, \chi^{\mathbf{h}}, \eta^{\mathbf{h}}, \mathcal{C}^{\mathbf{h}}) - \mathcal{S}(\mathcal{P}, \chi, \eta, \mathcal{C}) - (\xi, \nu, \zeta)\|_{\mathcal{X}}}{|\mathbf{h}|} \to 0 \quad \text{as } |\mathbf{h}| \to 0,$$

where $|\mathbf{h}| := |\mathcal{P}^{\mathbf{h}}| + |\chi^{\mathbf{h}}| + |\eta^{\mathbf{h}}| + |\mathcal{C}^{\mathbf{h}}|$. By using the above notation, this amounts to show that

$$\frac{\|(\phi, \rho, \omega)\|_{\mathcal{X}}}{|\mathbf{h}|} \to 0 \quad \text{as } |\mathbf{h}| \to 0, \tag{6.11}$$

so that it suffices to check that there exist two constants $C > 0$ and $\gamma > 1$, independent of $\mathbf{h}$, such that

$$\|(\phi, \rho, \omega)\|_{\mathcal{X}} \le C|\mathbf{h}|^{\gamma}. \tag{6.12}$$

Taking the difference of the corresponding systems, we infer that the triplet (ϕ, ρ, ω) solves the following system

$$\partial_t(\alpha\rho + \phi) - \Delta\rho = L_{\mathcal{P}} \qquad \text{in } Q, \tag{6.13}$$
$$\rho = \beta\partial_t\phi + a\phi - J * \phi + F''(\overline{\varphi})\phi + L_{\chi} \qquad \text{in } Q, \tag{6.14}$$
$$\partial_t\omega - \Delta\omega + \mathcal{B}\omega + L_{\mathcal{C}} = L_{\eta} \qquad \text{in } Q, \tag{6.15}$$
$$\eta\partial_{\mathbf{n}}\phi - \partial_{\mathbf{n}}\omega = \partial_{\mathbf{n}}\rho = 0 \qquad \text{on } \Sigma, \tag{6.16}$$
$$\phi(0) = \rho(0) = \omega(0) = 0 \qquad \text{in } \Omega, \tag{6.17}$$

where

$$\begin{aligned}
L_{\mathcal{P}} &:= \mathcal{P}f(\overline{\varphi})\omega + \mathcal{P}\big(f(\overline{\varphi}^{\mathbf{h}}) - f(\overline{\varphi})\big)\big(\sigma^{\mathbf{h}} - \overline{\sigma}\big) + (\mathcal{P}\overline{\sigma} - \mathcal{A})\big(f(\overline{\varphi}^{\mathbf{h}}) - f(\overline{\varphi}) - f'(\overline{\varphi})\xi\big) \\
&\quad + h_{\mathcal{P}}\big[\big(f(\overline{\varphi}^{\mathbf{h}}) - f(\overline{\varphi})\big)\big(\sigma^{\mathbf{h}} - \overline{\sigma}\big) + (f(\overline{\varphi}^{\mathbf{h}}) - f(\overline{\varphi}))\overline{\sigma} + (\sigma^{\mathbf{h}} - \overline{\sigma})f(\overline{\varphi})\big], \\
L_{\chi} &:= -\chi\omega - h_{\chi}(\sigma^{\mathbf{h}} - \overline{\sigma}) + F'(\overline{\varphi}^{\mathbf{h}}) - F'(\overline{\varphi}) - F''(\overline{\varphi})(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}), \\
L_{\eta} &:= -\eta\Delta\phi - h_{\eta}\Delta(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}), \\
L_{\mathcal{C}} &:= \mathcal{C}f(\overline{\varphi})\omega + \mathcal{C}\big(f(\overline{\varphi}^{\mathbf{h}}) - f(\overline{\varphi})\big)\big(\sigma^{\mathbf{h}} - \overline{\sigma}\big) + \mathcal{C}\big(f(\overline{\varphi}^{\mathbf{h}}) - f(\overline{\varphi}) - f'(\overline{\varphi})\xi\big)\overline{\sigma} \\
&\quad + h_{\mathcal{C}}\big[\big(f(\overline{\varphi}^{\mathbf{h}}) - f(\overline{\varphi})\big)\big(\sigma^{\mathbf{h}} - \overline{\sigma}\big) + (f(\overline{\varphi}^{\mathbf{h}}) - f(\overline{\varphi}))\overline{\sigma} + (\sigma^{\mathbf{h}} - \overline{\sigma})f(\overline{\varphi})\big].
\end{aligned}$$

As an easy consequence of Theorem 4.5 and Lemma 6.1, the following estimates hold:

$$\begin{aligned}
&\|\overline{\varphi}\|_{W^{1,\infty}(0,T;V)\cap H^1(0,T;H^2(\Omega))} + \|\eta\overline{\varphi}\|_{L^2(0,T;W)} + \|\partial_t\overline{\varphi}\|_{L^\infty(Q)} \\
&\quad + \|\overline{\mu}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)\cap L^\infty(Q)} \\
&\quad + \|\overline{\sigma}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)\cap L^\infty(Q)} \le K,
\end{aligned}$$

and

$$\begin{aligned}
&\|\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}\|_{W^{1,\infty}(0,T;V)\cap H^1(0,T;H^2(\Omega))} + \|\mu^{\mathbf{h}} - \overline{\mu}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \\
&\quad + \|\sigma^{\mathbf{h}} - \overline{\sigma}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le K|\mathbf{h}|,
\end{aligned} \tag{6.18}$$

where the constant $K > 0$ is independent of $\mathbf{h}$. In order to prove (6.11), we multiply (6.13) by ρ, (6.14) to which we add to both sides $(c_a + 2)\phi$ by $-\partial_t\phi$, the gradient of (6.14) by $\nabla\phi$, and (6.15) by ω. Using the same argument employed in (6.9)–(6.10) in the proof of the linearised system, we deduce that, upon integration over Q_t and addition of the resulting equalities,

$$\begin{aligned}
&\frac{\alpha}{2}\|\rho(t)\|^2 + \int_{Q_t} |\nabla\rho|^2 + \beta\int_{Q_t} |\partial_t\phi|^2 + \|\phi(t)\|^2 + C_0\int_{Q_t} |\nabla\phi|^2 + \frac{\beta}{2}\|\nabla\phi(t)\|^2 \\
&\quad + \frac{1}{2}\|\omega(t)\|^2 + \mathcal{B}\int_{Q_t} |\omega|^2 + \int_{Q_t} |\nabla\omega|^2 \\
&\quad \le \int_{Q_t} L_{\mathcal{P}}\rho - \int_{Q_t} F''(\overline{\varphi})\phi\partial_t\phi - \int_{Q_t} L_\chi\partial_t\phi - \int_{Q_t} (c_a+2)\phi\partial_t\phi - \int_{Q_t} \nabla L_\chi\cdot\nabla\phi \\
&\quad - \int_{Q_t} \phi\nabla a\cdot\nabla\phi + \int_{Q_t} (\nabla J)*\phi\cdot\nabla\phi - \int_{Q_t} F^{(3)}(\overline{\varphi})\phi\nabla\overline{\varphi}\cdot\nabla\phi \\
&\quad + \int_{Q_t} (L_\eta - L_{\mathcal{C}})\omega.
\end{aligned} \tag{6.19}$$

The second term on the right-hand side can be estimated by using the separation property and Young's inequality which lead to

$$\int_{Q_t} F''(\overline{\varphi})\phi\partial_t\phi \le C\int_{Q_t} |\phi||\partial_t\phi| \le \delta\int_{Q_t} |\partial_t\phi|^2 + C(\delta)\int_{Q_t} |\phi|^2,$$

for a positive constant δ yet to be chosen. Let us recall Taylor's formula with integral remainder for f, which is well-defined owing to the required regularity:

$$f(\overline{\varphi}^{\mathbf{h}}) - f(\overline{\varphi}) - f'(\overline{\varphi})\xi \;=\; f'(\overline{\varphi})\phi + R_f^{\mathbf{h}}(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi})^2, \tag{6.20}$$

where the remainder $R_f^{\mathbf{h}}$ is defined as

$$R_f^{\mathbf{h}} := \int_0^1 f''(\overline{\varphi} + s(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}))(1-s)\,\mathrm{ds}$$

and it is uniformly bounded due to **D6**. Using the Young and Hölder inequalities, the boundedness and the Lipschitz continuity of f and f', the Taylor's formula (6.20), the estimate (6.18), as well as the regularity of $\overline{\varphi}$ and $\overline{\sigma}$, we have

$$
\begin{aligned}
&\int_{Q_t} L_{\mathcal{P}}\rho - \int_{Q_t} L_{\mathcal{C}}\omega \\
&\quad \leq C \int_{Q_t} (|\omega| + |\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}||\sigma^{\mathbf{h}} - \overline{\sigma}| + |\phi| + |\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}|^2)(|\rho| + |\omega|) \\
&\qquad + C|\mathbf{h}| \int_{Q_t} (|\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}||\sigma^{\mathbf{h}} - \overline{\sigma}| + |\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}| + |\sigma^{\mathbf{h}} - \overline{\sigma}|)(|\rho| + |\omega|) \\
&\quad \leq C \int_{Q_t} (|\omega|^2 + |\rho|^2 + |\phi|^2) + C \int_0^t \|\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}\|_4^2 (\|\rho\| + \|\omega\|) \\
&\qquad + C(1 + |\mathbf{h}|) \int_0^t \|\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}\|_4 \|\sigma^{\mathbf{h}} - \overline{\sigma}\|_4 (\|\rho\| + \|\omega\|) \\
&\qquad + C|\mathbf{h}| \int_0^t (\|\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}\| + \|\sigma^{\mathbf{h}} - \overline{\sigma}\|)(\|\rho\| + \|\omega\|) \\
&\quad \leq C \int_{Q_t} (|\omega|^2 + |\rho|^2 + |\phi|^2) + C(|\mathbf{h}|^6 + |\mathbf{h}|^4).
\end{aligned}
$$

Now, arguing again by Taylor's formula with integral remainder, we have that

$$
F'(\overline{\varphi}^{\mathbf{h}}) - F'(\overline{\varphi}) - F''(\overline{\varphi})(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}) = R^{\mathbf{h}}_{F'}(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi})^2,
$$

with remainder

$$
R^{\mathbf{h}}_{F'} := \int_0^1 F^{(3)}(\overline{\varphi} + s(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}))(1 - s)\,\mathrm{ds}.
$$

Besides, this latter is uniformly bounded by Corollary 6.2. Taking these remarks into account, we infer that

$$
\begin{aligned}
&- \int_{Q_t} L_{\chi}\partial_t\phi + \int_{Q_t} L_{\eta}\omega \\
&\quad \leq C \int_{Q_t} |\omega||\partial_t\phi| + C|\mathbf{h}| \int_0^t \|\sigma^{\mathbf{h}} - \overline{\sigma}\| \|\partial_t\phi\| + C \int_{Q_t} |\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}|^2 |\partial_t\phi| \\
&\qquad + C \int_{Q_t} |\nabla\phi||\nabla\omega| + C|\mathbf{h}| \int_0^t \|\Delta(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi})\| \|\omega\| \\
&\quad \leq \delta \int_{Q_t} (|\partial_t\phi|^2 + |\nabla\omega|^2) \\
&\qquad + C(\delta) \int_{Q_t} (|\omega|^2 + |\mathbf{h}|^2 |\sigma^{\mathbf{h}} - \overline{\sigma}|^2 + |\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}|^4 + |\nabla\phi|^2 + |\mathbf{h}|^2 |\Delta(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi})|^2) \\
&\quad \leq \delta \int_{Q_t} (|\partial_t\phi|^2 + |\nabla\omega|^2) + C(\delta) \int_{Q_t} (|\omega|^2 + |\nabla\phi|^2) + C(\delta)|\mathbf{h}|^4.
\end{aligned}
$$

Furthermore, we have that

$$-\int_{Q_t} \nabla L\chi \cdot \nabla\phi = \chi \int_{Q_t} \nabla\omega \cdot \nabla\phi + h\chi \int_{Q_t} \nabla(\sigma^{\mathbf{h}} - \overline{\sigma}) \cdot \nabla\phi \\ - \int_{Q_t} \nabla(F'(\overline{\varphi}^{\mathbf{h}}) - F'(\overline{\varphi}) - F''(\overline{\varphi})(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi})) \cdot \nabla\phi,$$

where, by the Young inequality,

$$\begin{aligned} &\chi \int_{Q_t} \nabla\omega \cdot \nabla\phi + h\chi \int_{Q_t} \nabla(\sigma^{\mathbf{h}} - \overline{\sigma}) \cdot \nabla\phi \\ &\quad \leq \delta \int_{Q_t} |\nabla\omega|^2 + C(\delta) \int_{Q_t} (|\nabla\phi|^2 + |\mathbf{h}|^2 |\nabla(\sigma^{\mathbf{h}} - \overline{\sigma})|^2) \\ &\quad \leq \delta \int_{Q_t} |\nabla\omega|^2 + C(\delta) \int_{Q_t} |\nabla\phi|^2 + C(\delta)|\mathbf{h}|^4, \end{aligned}$$

and

$$\begin{aligned} &\int_{Q_t} \nabla(F'(\overline{\varphi}^{\mathbf{h}}) - F'(\overline{\varphi}) - F''(\overline{\varphi})(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi})) \cdot \nabla\phi \\ &\quad = \int_{Q_t} (F''(\overline{\varphi}^{\mathbf{h}})\nabla\overline{\varphi}^{\mathbf{h}} - F''(\overline{\varphi})\nabla\overline{\varphi} - F^{(3)}(\overline{\varphi})\nabla\overline{\varphi}(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}) - F''(\overline{\varphi})\nabla(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi})) \cdot \nabla\phi \\ &\quad = \int_{Q_t} (F''(\overline{\varphi}^{\mathbf{h}}) - F''(\overline{\varphi}) - F^{(3)}(\overline{\varphi})(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}))\nabla\overline{\varphi} \cdot \nabla\phi \\ &\qquad + \int_{Q_t} (F''(\overline{\varphi}^{\mathbf{h}}) - F''(\overline{\varphi}))\nabla(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}) \cdot \nabla\phi. \end{aligned}$$

Hence, using again the Taylor formula with integral remainder for F'', Corollary 6.2, and the estimate (6.18), it is straightforward to see that

$$-\int_{Q_t} \nabla(F'(\overline{\varphi}^{\mathbf{h}}) - F'(\overline{\varphi}) - F''(\overline{\varphi})(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi})) \cdot \nabla\phi \leq C \int_{Q_t} |\nabla\phi|^2 + C|\mathbf{h}|^4.$$

Finally, the last line of (6.19) can be bounded from above using the Hölder inequality, the continuous embedding $V \subset L^4(\Omega)$, and the regularity of $\overline{\varphi}$ as

$$\begin{aligned} &\frac{\beta}{2} \int_{Q_t} |\partial_t\phi|^2 + C \int_{Q_t} |\phi|^2 + C \int_{Q_t} (|\phi|^2 + |\nabla\phi|^2) + C \int_0^t \|\nabla\overline{\varphi}\|_4^2 \|\phi\|_4^2 \\ &\quad \leq \frac{\beta}{2} \int_{Q_t} |\partial_t\phi|^2 + C \int_0^t \|\phi\|_V^2. \end{aligned}$$

Therefore, upon collecting the above estimates, picking δ small enough, and invoking Gronwall's lemma, we conclude the proof since (6.12) has been shown with $\gamma = 2$. □

6.1.4 Adjoint System

In order to study first-order necessary conditions for optimality for problem $(CP)_{\alpha,\beta}$, for a fixed admissible control $(\overline{\mathcal{P}}, \overline{\chi}, \overline{\eta}, \overline{\mathcal{C}}) \in \mathcal{U}_{\rm ad}$ with corresponding state $(\overline{\varphi}, \overline{\mu}, \overline{\sigma})$, we introduce and solve the auxiliary backward-in-time problem called *adjoint system*, in the new variables (p, q, r). This system is formally obtained by taking the adjoint of the linearised system (6.4)–(6.8), and reads

$$
\begin{aligned}
&- \partial_t(p+\beta q) + aq - J * q + \overline{\eta}\Delta r + F''(\overline{\varphi})q \\
&\quad + \overline{\mathcal{C}}\overline{\sigma} f'(\overline{\varphi})r - (\overline{\mathcal{P}}\overline{\sigma} - \mathcal{A})f'(\overline{\varphi})p \,=\, b_1(\overline{\varphi} - \varphi_Q) && \text{in } Q, && (6.21)\\
&- \alpha\partial_t p - \Delta p - q \,=\, 0 && \text{in } Q, && (6.22)\\
&- \partial_t r - \Delta r + (\mathcal{B} + \overline{\mathcal{C}} f(\overline{\varphi}))r - \overline{\mathcal{P}} f(\overline{\varphi})p - \overline{\chi} q \,=0 && \text{in } Q, && (6.23)\\
&\partial_{\mathbf{n}} p \,=\, \partial_{\mathbf{n}} r \,=\, 0 && \text{on } \Sigma, && (6.24)\\
&\alpha p(T) \,=\, 0, \quad (p+\beta q)(T) \,=\, b_2(\overline{\varphi}(T) - \varphi_\Omega), \quad r(T) \,=0 && \text{in } \Omega. && (6.25)
\end{aligned}
$$

Here is the corresponding well-posedness result.

Theorem 6.6 (Well-posedness of the adjoint system: $\alpha, \beta > 0$)**.** *Assume* **B1–B7***,* **D1–D6***, and let* $(\overline{\mathcal{P}}, \overline{\chi}, \overline{\eta}, \overline{\mathcal{C}}) \in \mathcal{U}_{\rm ad}$ *be an admissible control, with corresponding state* $(\overline{\varphi}, \overline{\mu}, \overline{\sigma})$*. Then, the adjoint system* (6.21)–(6.25) *admits a unique solution* (p, q, r) *such that*

$$
\begin{aligned}
p, r &\in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W),\\
q &\in H^1(0,T;H) \cap L^\infty(0,T;H).
\end{aligned}
$$

Proof of Theorem 6.6. A rigorous proof has to be addressed within an approximation scheme. Anyhow, since the system is linear and the arguments are standard we just point out the formal a priori estimates, leaving the details to the reader.
First estimate: We multiply (6.21) by q, (6.22) by $-\partial_t p + p$, (6.23) to which we add to both sides the term r by $-\partial_t r$ and (6.23) by $-\Delta r$. After integrating over Q_t^T and adding the resulting equalities, we obtain

$$
\begin{aligned}
&\frac{\beta}{2}\|q(t)\|^2 + C_0 \int_{Q_t^T} |q|^2 + \alpha \int_{Q_t^T} |\partial_t p|^2 + \frac{\alpha}{2}\|p(t)\|^2 + \frac{1}{2}\|\nabla p(t)\|^2 + \int_{Q_t^T} |\nabla p|^2 \\
&\quad + \frac{\mathcal{B}+1}{2}\|r(t)\|^2 + \|\nabla r(t)\|^2 + \mathcal{B}\int_{Q_t^T} |\nabla r|^2 + \int_{Q_t^T} |\partial_t r|^2 + \int_{Q_t^T} |\Delta r|^2 \\
&\le \frac{1}{2\beta}\|b_2(\overline{\varphi}(T) - \varphi_\Omega)\|^2 + \int_{Q_t^T} b_1(\overline{\varphi} - \varphi_Q)q - \int_{Q_t^T} \overline{\eta}\Delta r q + \int_{Q_t^T} (J * q)q \\
&\quad - \int_{Q_t^T} \overline{\mathcal{C}}\overline{\sigma} f'(\overline{\varphi}) r q + \int_{Q_t^T} (\overline{\mathcal{P}}\overline{\sigma} - \mathcal{A}) f'(\overline{\varphi}) p q + \int_{Q_t^T} qp \\
&\quad + \int_{Q_t^T} \overline{\mathcal{C}} f(\overline{\varphi}) r(\partial_t r + \Delta r) - \int_{Q_t^T} \mathcal{P} f(\overline{\varphi}) p(\partial_t r + \Delta r) \\
&\quad - \int_{Q_t^T} \overline{\chi} q(\partial_t r + \Delta r) - \int_{Q_t^T} r\partial_t r =: \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4.
\end{aligned}
$$

By virtue of the regularity of $\overline{\sigma}$, the boundedness of f and f' and Young's inequality, we easily infer that

$$|\mathbb{I}_1| \le \delta \int_{Q_t} |\Delta r|^2 + C(\delta) \int_{Q_t^T} (|q|^2 + 1),$$
$$|\mathbb{I}_2| + |\mathbb{I}_3| + |\mathbb{I}_4| \le \delta \int_{Q_t^T} (|\partial_t r|^2 + |\Delta r|^2) + C(\delta) \int_{Q_t^T} (|p|^2 + |q|^2 + |r|^2),$$

for a positive constant δ yet to be chosen. Hence, we take δ small enough so that Gronwall's lemma along with elliptic regularity, produces

$$\|p\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|q\|_{L^\infty(0,T;H)} + \|r\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le C.$$

Second estimate: A comparison argument in (6.22) easily produces an $L^2(0,T;H)$ bound for Δp. Hence, using elliptic regularity theory we easily infer that

$$\|p\|_{L^2(0,T;W)} \le C.$$

Third estimate: From the above estimate, a comparison argument in (6.21) leads us to obtain

$$\|\partial_t q\|_{L^2(0,T;H)} \le C.$$

Fourth estimate: Notice that (6.22) and (6.23) have a parabolic structure in p and r with zero final condition and source term bounded in $L^\infty(0,T;H)$. Therefore, it easily follows from classical parabolic regularity theory that

$$\|p\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W)} + \|r\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W)} \le C.$$

Arguing in a similar fashion as for the linearised system, due to the linearity of the adjoint system (6.21)–(6.25) the uniqueness directly follows from the above estimates and the proof is concluded. □

6.1.5 First-order Optimality Conditions

This section is devoted to the study of necessary conditions for optimality for the optmisation problem $(CP)_{\alpha,\beta}$. First of all, we employ a classical tool to derive first-order necessary conditions for $(CP)_{\alpha,\beta}$. In fact, provided that $\mathcal{J}$ is sufficiently smooth and recalling the structure of $\mathcal{U}_{\text{ad}}$, a first-order necessary condition for $(\overline{P}, \overline{\chi}, \overline{\eta}, \overline{C}) \in \mathcal{U}_{\text{ad}}$ to be optimal is to verify the following variational inequality

$$\langle D\mathcal{J}_{\text{red}}(\overline{P}, \overline{\chi}, \overline{\eta}, \overline{C}), (\mathcal{P}, \chi, \eta, \mathcal{C}) - (\overline{P}, \overline{\chi}, \overline{\eta}, \overline{C}) \rangle \ge 0 \quad \forall (\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathcal{U}_{\text{ad}}, \tag{6.26}$$

where $D\mathcal{J}_{\text{red}}$ denotes the Gâteaux derivative of the *reduced* cost functional defined as

$$\mathcal{J}_{\text{red}}(\mathcal{P}, \chi, \eta, \mathcal{C}) := \mathcal{J}(\mathcal{S}_1(\mathcal{P}, \chi, \eta, \mathcal{C}), \mathcal{P}, \chi, \eta, \mathcal{C}), \quad (\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathcal{U}_{\text{ad}}.$$

Theorem 6.5 allows us to exploit this result to obtain an explicit expression in terms of the linearised variables.

Theorem 6.7 (First-order necessary condition for optimality: $\alpha, \beta > 0$)**.** *Assume* **B1–B7** *and* **D1–D6***, and let* $(\overline{P}, \overline{\chi}, \overline{\eta}, \overline{C})$ *be an optimal control for problem* $(CP)_{\alpha,\beta}$*, with corresponding state* $(\overline{\varphi}, \overline{\mu}, \overline{\sigma})$*. Then,* $(\overline{P}, \overline{\chi}, \overline{\eta}, \overline{C})$ *necessarily satisfies*

$$\begin{aligned}\int_\Omega b_2(\overline{\varphi}(T) - \varphi_\Omega)\xi(T) + \int_Q b_1(\overline{\varphi} - \varphi_Q)\xi + \alpha_P(\overline{\mathcal{P}} - \mathcal{P}_*)(\mathcal{P} - \overline{\mathcal{P}}\,) \\ + \alpha_\chi(\overline{\chi} - \chi_*)(\chi - \overline{\chi}) + \alpha_\eta(\overline{\eta} - \eta_*)(\eta - \overline{\eta}) \\ + \alpha_C(\overline{\mathcal{C}} - \mathcal{C}_*)(\mathcal{C} - \overline{\mathcal{C}}) \geq 0 \qquad \forall(\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathcal{U}_{\text{ad}},\end{aligned} \tag{6.27}$$

where ξ *is the first component of the unique solution* (ξ, ν, ζ) *to the linearised system obtained by Theorem 6.4 associated to* $\mathbf{h} = (\mathcal{P} - \overline{\mathcal{P}},\ \chi - \overline{\chi},\ \eta - \overline{\eta},\ \mathcal{C} - \overline{\mathcal{C}})$*.*

Proof of Theorem 6.7. By Theorem 6.5 and the usual chain rule for Fréchet-differentiable functions, it follows immediately that the reduced cost functional $\mathcal{J}_{\text{red}} : \mathcal{U}_{\text{ad}} \to \mathbb{R}$ is Fréchet-differentiable at $(\overline{P}, \overline{\chi}, \overline{\eta}, \overline{C})$. Hence, the optimality of $(\overline{P}, \overline{\chi}, \overline{\eta}, \overline{C})$ yields directly (6.26), which in turn reads as (6.27). □

The next step consists in simplifying the necessary conditions for the minimiser presented above, by using the adjoint system.

Theorem 6.8 (Final first-order necessary conditions for optimality: $\alpha, \beta > 0$)**.** *Assume* **B1–B7** *and* **D1–D6***, let* $(\overline{\mathcal{P}}, \overline{\chi}, \overline{\eta}, \overline{\mathcal{C}}) \in \mathcal{U}_{\text{ad}}$ *be an optimal control for* $(CP)_{\alpha,\beta}$*, and let* $(\overline{\varphi}, \overline{\mu}, \overline{\sigma})$ *and* (p, q, r) *be the corresponding state and adjoint variables, respectively. Then,* $(\overline{P}, \overline{\chi}, \overline{\eta}, \overline{C})$ *necessarily verifies*

$$\begin{aligned}\int_Q (\mathcal{P} - \overline{\mathcal{P}}\,)\overline{\sigma} f(\overline{\varphi})p + \int_Q (\chi - \overline{\chi})\overline{\sigma} q - \int_Q (\eta - \overline{\eta})\Delta\overline{\varphi}\, r - \int_Q (\mathcal{C} - \overline{\mathcal{C}})\overline{\sigma} f(\overline{\varphi})\, r \\ + \alpha_P(\overline{\mathcal{P}} - \mathcal{P}_*)(\mathcal{P} - \overline{\mathcal{P}}\,) + \alpha_\chi(\overline{\chi} - \chi_*)(\chi - \overline{\chi}\,) \\ + \alpha_\eta(\overline{\eta} - \eta_*)(\eta - \overline{\eta}\,) + \alpha_C(\overline{\mathcal{C}} - \mathcal{C}_*)(\mathcal{C} - \overline{\mathcal{C}}\,) \geq 0 \quad \forall(\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathcal{U}_{\text{ad}}.\end{aligned} \tag{6.28}$$

Proof of Theorem 6.8. We note that (6.28) directly follows from (6.27), provided to show the identity

$$\begin{aligned}\int_Q b_1(\overline{\varphi} - \varphi_Q)\xi + \int_\Omega b_2(\overline{\varphi}(T) - \varphi_\Omega)\xi(T) \\ = \int_Q h_{\mathcal{P}}\overline{\sigma} f(\overline{\varphi})p + \int_Q h_\chi \overline{\sigma} q - \int_Q h_\eta \Delta\overline{\varphi}\, r - \int_Q h_{\mathcal{C}}\overline{\sigma} f(\overline{\varphi})\, r\end{aligned} \tag{6.29}$$

with $\mathbf{h} = (\mathcal{P} - \overline{\mathcal{P}}, \chi - \overline{\chi}, \eta - \overline{\eta}, \mathcal{C} - \overline{\mathcal{C}})$. To this end, we multiply (6.4)–(6.6) by $p, q,$ and r in the order, integrate over Q, and sum the equalities to obtain

$$\begin{aligned}0 = \int_Q p[\partial_t(\alpha\nu + \xi) - \Delta\nu - (\overline{\mathcal{P}}\overline{\sigma} - \mathcal{A})f'(\overline{\varphi})\xi - \overline{\mathcal{P}}\zeta f(\overline{\varphi}) - h_{\mathcal{P}}\overline{\sigma} f(\overline{\varphi})] \\ + \int_Q q[-\nu + \beta\partial_t\xi + a\xi - J * \xi + F''(\overline{\varphi})\xi - \overline{\chi}\zeta - h_\chi\overline{\sigma}] \\ + \int_Q r[\partial_t\zeta - \Delta\zeta + \mathcal{B}\zeta + \overline{\mathcal{C}}(\zeta f(\overline{\varphi}) + \overline{\sigma} f'(\overline{\varphi})\xi) + h_{\mathcal{C}}\overline{\sigma} f(\overline{\varphi}) + \overline{\eta}\Delta\xi + h_\eta\Delta\overline{\varphi}].\end{aligned}$$

The terms involving the time derivatives can be easily handled by integrating by parts and using the initial conditions (6.8) and the terminal conditions (6.25) to obtain that

$$\begin{aligned}
&\int_Q p\partial_t(\alpha\nu+\xi)+\int_Q \beta q\partial_t\xi+\int_Q \partial_t\zeta r\\
&\quad=-\int_Q \alpha\partial_t p\nu-\int_Q \partial_t p\xi-\int_Q \beta\partial_t q\xi-\int_Q \partial_t r\zeta+\int_\Omega (p+\beta q)(T)\xi(T)\\
&\quad=-\int_Q \alpha\partial_t p\nu-\int_Q \partial_t(p+\beta q)\xi-\int_Q \partial_t r\zeta+\int_\Omega b_2(\overline{\varphi}(T)-\varphi_\Omega)\xi(T).
\end{aligned}$$

Moreover, integrating by parts and rearranging the terms we get

$$\begin{aligned}
0=&\int_Q \xi[-\partial_t(p+\beta q)+aq-J*q+\overline{\eta}\Delta r+F''(\overline{\varphi})q+\overline{\mathcal{C}}\overline{\sigma}f'(\overline{\varphi})r\\
&\quad-(\overline{\mathcal{P}}\overline{\sigma}-\mathcal{A})f'(\overline{\varphi})p]\\
&+\int_Q \nu[-\alpha\partial_t p-\Delta p-q]\\
&+\int_Q \zeta[-\partial_t r-\Delta r+(\mathcal{B}+\overline{\mathcal{C}}f(\overline{\varphi}))r-\overline{\mathcal{P}}f(\overline{\varphi})p-\overline{\chi}q]\\
&-\int_Q h_{\mathcal{P}}\overline{\sigma}f(\overline{\varphi})p-\int_Q h_\chi\overline{\sigma}q+\int_Q h_\eta\Delta\overline{\varphi}\,r+\int_Q h_{\mathcal{C}}\overline{\sigma}f(\overline{\varphi})\,r\\
&+\int_\Omega b_2(\overline{\varphi}(T)-\varphi_\Omega)\xi(T).
\end{aligned}$$

Hence, we recall the definition of the adjoint variables (6.21)–(6.23) to realise that the most part of the above terms simplify and the remaining equality is (6.29), as we claimed. □

6.2 Asymptotic Analysis

This second part of the chapter concerns the asymptotic behaviour of $(CP)_{\alpha,\beta}$ as α and/or β approach zero in the state system above which is made possible by virtue of the asymptotic investigation performed in Chapter 4: see Theorems 4.9, 4.12 and 4.15. Before moving on, let us point out an important fact concerning these results. Due to the nature of the identification problem, in the minimisation problem $(CP)_{\alpha,\beta}$ we consider as control variable the quadruplet $(\mathcal{P}_{\alpha,\beta},\chi_{\alpha,\beta},\eta_{\alpha,\beta},\mathcal{C}_{\alpha,\beta})$. This forces us to adjust the aforementioned theorems by including in the discussion the asymptotic behaviours of the controls $(\mathcal{P}_{\alpha,\beta},\chi_{\alpha,\beta},\eta_{\alpha,\beta},\mathcal{C}_{\alpha,\beta})$, which was fixed at the level of the asymptotic analysis results performed in Chapter 4. Thus, Theorems 4.9, 4.12 and 4.15 have to be straightforwardly modified by assuming that every solution triplet $(\varphi_{\alpha,\beta},\mu_{\alpha,\beta},\sigma_{\alpha,\beta})$ is also referred to the associated parameters $(\mathcal{P}_{\alpha,\beta},\chi_{\alpha,\beta},\eta_{\alpha,\beta},\mathcal{C}_{\alpha,\beta})$, and that the set of parameters $(\mathcal{P}_{\alpha,\beta},\chi_{\alpha,\beta},\eta_{\alpha,\beta},\mathcal{C}_{\alpha,\beta})$ converges to the corresponding limit. For instance, concerning Theorem 4.9, in addition to (4.64), we assume that

$$(\mathcal{P}_\beta,\chi_\beta,\eta_\beta,\mathcal{C}_\beta)\in\mathcal{U}_{\mathrm{ad}},\quad \eta_\beta=0,$$

and that, as $\alpha \to 0$,

$$(\mathcal{P}_{\alpha,\beta}, \chi_{\alpha,\beta}, \eta_{\alpha,\beta}, \mathcal{C}_{\alpha,\beta}) \to (\mathcal{P}_{\beta}, \chi_{\beta}, \eta_{\beta}, \mathcal{C}_{\beta}).$$

Theorems 4.12 and 4.15 can be extended analogously. Therefore, in what follows, when we will refer to those results, we tacitly assume that these minor corrections are in order.

Besides, we will employ the symbols $(CP)_\alpha$, $(CP)_\beta$, and $(\overline{CP})$ to denote the corresponding optimal control problems in which $\beta = 0$, $\alpha = 0$ and $\alpha = \beta = 0$ in the order. For instance, we have

$(CP)_\beta$ Minimise $\mathcal{J}(\varphi, \mathcal{P}, \chi, \mathcal{C})$ subject to:
(i) (φ, μ, σ) yields a solution to (4.1)–(4.5) with $\alpha = 0$ obtained from Theorem 4.9;
(ii) $(\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathcal{U}_{\mathrm{ad}}$.

The problems $(CP)_\alpha$ and $(\overline{CP})$ are defined analogously (cf. Theorems 4.12 and 4.15). Different requirements on the structural data are in order, depending on the asymptotic study under consideration. In particular, we aim at passing to the limit as α and β to zero, both separately and jointly, in the optimality condition (6.28).

As the asymptotic analysis for the state system has already been addressed in Theorems 4.9, 4.12, and 4.15, the first novelty here consists in understanding the asymptotic behaviour of the adjoint system (6.21)–(6.25). In this direction, we show that the adjoint variables, depending on $\alpha, \beta > 0$, converge in some topology. To this aim, we begin with obtaining some uniform estimates with respect to α and β so to pass to the limit using classical weak and weak star compactness arguments.

The second step deals with the approximation of the optimal controls of $(CP)_\alpha$, $(CP)_\beta$, $(\overline{CP})$ by means of sequences of optimal controls of $(CP)_{\alpha,\beta}$. A combination of these steps will allow us to rigorously pass to the limit in the optimality conditions (6.28), recovering thus the corresponding ones for $(CP)_\alpha$, $(CP)_\beta$, and $(\overline{CP})$.

Despite part of the following results works also under more general assumptions on the potential (cf. Chapter 4), from now on we will assume that in addition to **D2**, the potential F fulfils the following:

D7 There exist two positive constants c_F and C_F such that

$$|F''(r)| \leq C_F(1 + |r|^2), \quad F(r) \geq c_F|r|^4 - C_F \qquad \forall\, r \in \mathbb{R}.$$

It is worth noting that these conditions are met by the classical regular potential (1.7), whereas prevent the singular choices (1.8) and (1.9) to be considered.

6.2.1 Uniform Estimates on the Adjoint Problem

Now, let us assume to be in the setting of either Theorem 4.9, 4.12, or 4.15. For every $\alpha \in (0, \alpha_0)$ and $\beta \in (0, \beta_0)$, let $(\mathcal{P}_{\alpha,\beta}, \chi_{\alpha,\beta}, \eta_{\alpha,\beta}, \mathcal{C}_{\alpha,\beta}) \in \mathcal{U}_{\mathrm{ad}}$ be an admissible control, and let $(\overline{\varphi}_{\alpha,\beta}, \overline{\mu}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta})$ and $(p_{\alpha,\beta}, q_{\alpha,\beta}, r_{\alpha,\beta})$ denote the unique solutions to the state system (4.1)–(4.5) and the adjoint system (6.21)–(6.25) with $\alpha, \beta > 0$, respectively.

First of all, performing the same estimate as in the proof of Theorem 6.6, noting that $\{\overline{\varphi}_{\alpha,\beta}\}_{\alpha,\beta}$ is always bounded in $C^0([0,T];H)$ uniformly in both α and β thanks to Theorems 4.9, 4.12 and 4.15, we have that

$$\begin{aligned}
&\frac{\beta}{2}\|q_{\alpha,\beta}(t)\|^2 + C_0\int_{Q_t^T}|q_{\alpha,\beta}|^2 + \alpha\int_{Q_t^T}|\partial_t p_{\alpha,\beta}|^2 + \frac{\alpha}{2}\|p_{\alpha,\beta}(t)\|^2 + \frac{1}{2}\|\nabla p_{\alpha,\beta}(t)\|^2 \\
&\quad + \int_{Q_t^T}|\nabla p_{\alpha,\beta}|^2 + \frac{\mathcal{B}+1}{2}\|r_{\alpha,\beta}(t)\|^2 + \|\nabla r_{\alpha,\beta}(t)\|^2 \\
&\quad + \mathcal{B}\int_{Q_t^T}|\nabla r_{\alpha,\beta}|^2 + \int_{Q_t^T}|\partial_t r_{\alpha,\beta}|^2 + \int_{Q_t^T}|\Delta r_{\alpha,\beta}|^2 \\
&\le C\left(\frac{b_2^2}{2\beta}+1\right) + \delta\int_{Q_t^T}|q_{\alpha,\beta}|^2 - \int_{Q_t^T}\eta_{\alpha,\beta}\Delta r_{\alpha,\beta}q_{\alpha,\beta} \\
&\quad + C(\delta)\int_{Q_t^T}(|\overline{\sigma}_{\alpha,\beta}p_{\alpha,\beta}|^2 + |\overline{\sigma}_{\alpha,\beta}r_{\alpha,\beta}|^2) + \int_{Q_t^T}(J*q_{\alpha,\beta})q_{\alpha,\beta} \\
&\quad + \delta'\int_{Q_t^T}(|\partial_t r_{\alpha,\beta}|^2 + |\Delta r_{\alpha,\beta}|^2) + C(\delta,\delta')\int_{Q_t^T}(|r_{\alpha,\beta}|^2 + |p_{\alpha,\beta}|^2) \\
&\quad - \int_{Q_t^T}\chi_{\alpha,\beta}q_{\alpha,\beta}(\partial_t r_{\alpha,\beta} + \Delta r_{\alpha,\beta}),
\end{aligned} \tag{6.30}$$

where the constants C, δ, δ', $C(\delta)$, and $C(\delta,\delta')$ are independent of α and β. The Hölder inequality and the continuous inclusion $V \subset L^4(\Omega)$ yield also

$$C(\delta)\int_{Q_t^T}(|\overline{\sigma}_{\alpha,\beta}p_{\alpha,\beta}|^2 + |\overline{\sigma}_{\alpha,\beta}r_{\alpha,\beta}|^2) \le C(\delta)\int_t^T \|\overline{\sigma}_{\alpha,\beta}\|_V^2(\|p_{\alpha,\beta}\|_V^2 + \|r_{\alpha,\beta}\|_V^2).$$

Secondly, by taking the mean value of (6.21) we have that

$$\begin{aligned}
&-\partial_t((p_{\alpha,\beta})_\Omega + \beta(q_{\alpha,\beta})_\Omega) + (F''(\overline{\varphi}_{\alpha,\beta})q_{\alpha,\beta})_\Omega + \mathcal{C}_{\alpha,\beta}(\overline{\sigma}_{\alpha,\beta}f'(\overline{\varphi}_{\alpha,\beta})r_{\alpha,\beta})_\Omega \\
&\qquad = ((\mathcal{P}_{\alpha,\beta}\overline{\sigma}_{\alpha,\beta} - \mathcal{A})f'(\overline{\varphi}_{\alpha,\beta})p_{\alpha,\beta})_\Omega + b_1((\overline{\varphi}_{\alpha,\beta} - \varphi_Q))_\Omega,
\end{aligned}$$

so that, testing this latter by $(p_{\alpha,\beta})_\Omega + \beta(q_{\alpha,\beta})_\Omega$, we get

$$\begin{aligned}
&\frac{1}{2}|(p_{\alpha,\beta}(t))_\Omega + \beta(q_{\alpha,\beta}(t))_\Omega|^2 \\
&\quad \le \frac{b_2^2}{2}|(\overline{\varphi}_{\alpha,\beta}(T) - \varphi_\Omega)_\Omega|^2 + \int_t^T b_1^2\|\overline{\varphi}_{\alpha,\beta} - \varphi_Q\|_1^2 \\
&\qquad + C(\delta)\int_t^T |(p_{\alpha,\beta})_\Omega + \beta(q_{\alpha,\beta})_\Omega|^2(1 + \|F''(\overline{\varphi}_{\alpha,\beta})\|^2) \\
&\qquad + \delta\int_t^T \|q_{\alpha,\beta}\|^2 + C\int_t^T(\|\overline{\sigma}_{\alpha,\beta}r_{\alpha,\beta}\|_1^2 + \|\overline{\sigma}_{\alpha,\beta}p_{\alpha,\beta}\|_1^2 + \|p_{\alpha,\beta}\|_1^2)
\end{aligned} \tag{6.31}$$

where again the constant C and $C(\delta)$ are independent of α and β. Now, since $\beta \leq 1$, by the Jensen inequality we have

$$\frac{1}{8}|(p_{\alpha,\beta}(t))_\Omega|^2 \leq \frac{1}{4}|(p_{\alpha,\beta}(t))_\Omega + \beta(q_{\alpha,\beta}(t))_\Omega|^2 + \frac{\beta}{4|\Omega|}\|q_{\alpha,\beta}(t)\|^2,$$

so that summing (6.30) and (6.31), using again the fact that $\{\overline{\varphi}_{\alpha,\beta}\}_{\alpha,\beta}$ is always bounded in $C^0([0,T];H)$ uniformly in both α and β, and rearranging the terms, by the Poincaré-Wirtinger inequality we infer that

$$\begin{aligned}
&\frac{\beta}{2}\|q_{\alpha,\beta}(t)\|^2 + C_0\int_{Q_t^T}|q_{\alpha,\beta}|^2 + \alpha\int_{Q_t^T}|\partial_t p_{\alpha,\beta}|^2 + \frac{\alpha}{2}\|p_{\alpha,\beta}(t)\|^2 + m\|p_{\alpha,\beta}(t)\|_V^2 \\
&\quad + \int_{Q_t^T}|\nabla p_{\alpha,\beta}|^2 + \frac{1}{2}\|r_{\alpha,\beta}(t)\|_V^2 + \mathcal{B}\int_{Q_t^T}|\nabla r_{\alpha,\beta}|^2 + \int_{Q_t^T}|\partial_t r_{\alpha,\beta}|^2 + \int_{Q_t^T}|\Delta r_{\alpha,\beta}|^2 \\
&\leq C\left(\frac{b_2^2}{2\beta}+1\right) + \delta\int_{Q_t^T}|q_{\alpha,\beta}|^2 + \delta'\int_{Q_t^T}(|\partial_t r_{\alpha,\beta}|^2 + |\Delta r_{\alpha,\beta}|^2) + \int_{Q_t^T}(J * q_{\alpha,\beta})q_{\alpha,\beta} \\
&\quad + C(\delta,\delta')\int_t^T(\|r_{\alpha,\beta}\|^2 + \|p_{\alpha,\beta}\|^2) + C(\delta)\int_t^T\|F''(\overline{\varphi}_{\alpha,\beta})\|^2(\|p_{\alpha,\beta}\|^2 + \beta\|q_{\alpha,\beta}\|^2) \\
&\quad + C(\delta)\int_t^T\|\overline{\sigma}_{\alpha,\beta}\|_V^2(\|p_{\alpha,\beta}\|_V^2 + \|r_{\alpha,\beta}\|_V^2) - \int_{Q_t^T}\eta_{\alpha,\beta}\Delta r_{\alpha,\beta}q_{\alpha,\beta} \\
&\quad - \int_{Q_t^T}\chi_{\alpha,\beta}q_{\alpha,\beta}(\partial_t r_{\alpha,\beta} + \Delta r_{\alpha,\beta}),
\end{aligned} \tag{6.32}$$

where $\delta, \delta' > 0$ are arbitrary, and $m, C, C(\delta), C(\delta,\delta') > 0$ are independent of α and β.

6.2.2 The Optimisation Problem $(CP)_{\alpha,\beta}$ as $\alpha \to 0$

Here, we solve $(CP)_\beta$ through an asymptotic approach by exploiting the proved results for $(CP)_{\alpha,\beta}$ by letting $\alpha \to 0$. Throughout the whole Section 6.2.2, we assume the following framework:

$$\beta \in (0,\beta_0) \text{ fixed}, \qquad \eta_{\max} = \alpha_\eta = 0, \qquad \text{(4.64)–(4.66), (4.81)}.$$

This means that we neglect the term in η in the cost functional and in the state system, implying that all the admissible controls are in the form $(\mathcal{P}, \chi, 0, \mathcal{C})$: with a slight abuse of notation, we look at $\mathcal{U}_{\text{ad}}$ as a compact set in $\mathbb{R}^3$, and use the symbol $(\mathcal{P}, \chi, \mathcal{C})$ for the generic admissible control in $\mathcal{U}_{\text{ad}}$.

The first result that we present concerns existence of optimal controls for $(CP)_\beta$.

Theorem 6.9. *Assume* **B1–B7**, **D1–D7**. *Then, the optimisation problem* $(CP)_\beta$ *admits a solution.*

Proof of Theorem 6.9. This result follows directly by adapting the direct method used in the proof of Theorem 6.3, taking into account the compactness of $\mathcal{U}_{\text{ad}}$ and the convergence pointed out in Theorem 4.9. □

After the existence is established, our main goal is to provide some necessary conditions for optimality by letting $\alpha \to 0$ in (6.28) written with the subscripts α, β. Let then $(\overline{P}_\beta, \overline{\chi}_\beta, \overline{C}_\beta) \in \mathcal{U}_{\text{ad}}$ be an optimal control for problem $(CP)_\beta$, and let $(\overline{\varphi}_\beta, \overline{\mu}_\beta, \overline{\sigma}_\beta)$ be the corresponding state variables solving (4.1)–(4.5) with $\alpha = 0$, in the sense of Theorem 4.9. Formally we expect that, as $\alpha \to 0$, the optimality condition reads

$$\begin{aligned}
&\int_Q (\mathcal{P} - \overline{\mathcal{P}}_\beta)\overline{\sigma}_\beta f(\overline{\varphi}_\beta) p_\beta - \int_Q (\chi - \overline{\chi}_\beta)\overline{\sigma}_\beta q_\beta - \int_Q (\mathcal{C} - \overline{\mathcal{C}}_\beta)\overline{\sigma}_\beta f(\overline{\varphi}_\beta)\, r_\beta \\
&\quad + \alpha_P(\overline{\mathcal{P}}_\beta - \mathcal{P}_*)(\mathcal{P} - \overline{\mathcal{P}}_\beta) + \alpha_\chi(\overline{\chi}_\beta - \chi_*)(\chi - \overline{\chi}_\beta) \\
&\quad + \alpha_C(\overline{\mathcal{C}}_\beta - \mathcal{C}_*)(\mathcal{C} - \overline{\mathcal{C}}_\beta) \geq 0 \quad \forall (\mathcal{P}, \chi, \mathcal{C}) \in \mathcal{U}_{\text{ad}},
\end{aligned}$$

where $(p_\beta, q_\beta, r_\beta)$ stands for some adjoint variables solving (6.21)–(6.25) with $\alpha = 0$, whose meaning is yet to be defined. Unfortunately, the situation is slightly more delicate. In fact, even if we prove that the adjoint variables $(p_{\alpha,\beta}, q_{\alpha,\beta}, r_{\alpha,\beta})$ converge to some limit $(p_\beta, q_\beta, r_\beta)$ in a suitable sense as $\alpha \to 0$, it is not obvious that every optimal control $(\overline{P}_\beta, \overline{\chi}_\beta, \overline{C}_\beta)$ can be recovered as the limit of a sequence of optimal controls $\{(\overline{P}_{\alpha,\beta}, \overline{\chi}_{\alpha,\beta}, \overline{C}_{\alpha,\beta})\}_\alpha$ of $(CP)_{\alpha,\beta}$.

To overcome this issue, as did in the previous chapter, we follow the same line of argument of [11] (see also [137, 138, 140] in the context of tumor growth models) and introduce the adapted cost functional, depending on the fixed minimiser $(\overline{P}_\beta, \overline{\chi}_\beta, \overline{C}_\beta)$ of $(CP)_\beta$, which is defined as

$$\mathcal{J}_{\text{ad}}(\varphi, \mathcal{P}, \chi, \mathcal{C}) := \mathcal{J}(\varphi, \mathcal{P}, \chi, \mathcal{C}) + \tfrac{1}{2}|\mathcal{P} - \overline{\mathcal{P}}_\beta|^2 + \tfrac{1}{2}|\chi - \overline{\chi}_\beta|^2 + \tfrac{1}{2}|\mathcal{C} - \overline{\mathcal{C}}_\beta|^2.$$

Keeping the optimal control $(\overline{P}_\beta, \overline{\chi}_\beta, \overline{C}_\beta)$ of $(CP)_\beta$ fixed, note that $\mathcal{J}_{\text{ad}} \equiv \mathcal{J}$ on the minimisers of $(CP)_\beta$. The main idea behind this local perturbation concerns the fact that for the associated optimal control problem, which will be referred to as adapted, we can obtain a compactness type property. Namely, we prove that every arbitrary minimiser $(\overline{P}_\beta, \overline{\chi}_\beta, \overline{C}_\beta)$ of $(CP)_\beta$ can be recovered as limit of a sequence of minimisers of $(CP)^{\text{ad}}_{\alpha,\beta}$, as $\alpha \to 0$. The just mentioned adapted optimal control problem associated with α, β reads as

$(CP)^{\text{ad}}_{\alpha,\beta}$ Minimise $\mathcal{J}_{\text{ad}}(\varphi, \mathcal{P}, \chi, \mathcal{C})$ subject to:

(i) (φ, μ, σ) yields a solution to (4.1)–(4.5) obtained from Theorem 4.5;

(ii) $(\mathcal{P}, \chi, \mathcal{C}) \in \mathcal{U}_{\text{ad}}$. (6.33)

In a sense to be made rigorous later, we will prove that $(CP)^{\text{ad}}_{\alpha,\beta} \searrow (CP)_\beta$ so that the passage to the limit as $\alpha \to 0$ in the variational inequality (6.28) can be rigorously performed producing in turn the optimality condition of $(CP)_\beta$. Since $(CP)^{\text{ad}}_{\alpha,\beta}$ fulfils the same assumptions of $(CP)_{\alpha,\beta}$, for what we already proved in Section 6.1 we readily infer the following.

Lemma 6.10. *Assume* **B1–B7** *and* **D1–D5**. *Then, for every $\alpha \in (0, \alpha_0)$ and for every optimal control $(\overline{\mathcal{P}}_\beta, \overline{\chi}_\beta, \overline{\mathcal{C}}_\beta) \in \mathcal{U}_{\text{ad}}$ of $(CP)_\beta$, the optimisation problem $(CP)^{\text{ad}}_{\alpha,\beta}$ admits a minimiser.*

Lemma 6.11. *Assume* **B1–B7** *and* **D1–D6***, and let* $(\overline{\mathcal{P}}_\beta, \overline{\chi}_\beta, \overline{\mathcal{C}}_\beta) \in \mathcal{U}_{\rm ad}$ *be an optimal control for* $(CP)_\beta$*. For every* $\alpha \in (0, \alpha_0)$*, if* $(\overline{\mathcal{P}}_{\alpha,\beta}, \overline{\chi}_{\alpha,\beta}, \overline{\mathcal{C}}_{\alpha,\beta}) \in \mathcal{U}_{\rm ad}$ *is an optimal control for* $(CP)^{\rm ad}_{\alpha,\beta}$*, then the following first-order necessary condition holds*

$$\begin{aligned}
&\int_Q (\mathcal{P} - \overline{\mathcal{P}}_{\alpha,\beta})\overline{\sigma}_{\alpha,\beta} f(\overline{\varphi}_{\alpha,\beta}) p_{\alpha,\beta} - \int_Q (\chi - \overline{\chi}_{\alpha,\beta})\overline{\sigma}_{\alpha,\beta} q_{\alpha,\beta} \\
&\quad - \int_Q (\mathcal{C} - \overline{\mathcal{C}}_{\alpha,\beta})\overline{\sigma}_{\alpha,\beta} f(\overline{\varphi}_{\alpha,\beta})\, r_{\alpha,\beta} \\
&\quad + (\mathcal{P} - \overline{\mathcal{P}}_{\alpha,\beta})\big(\alpha_P(\overline{\mathcal{P}}_{\alpha,\beta} - \mathcal{P}_*) + (\overline{\mathcal{P}}_{\alpha,\beta} - \overline{\mathcal{P}}_\beta)\big) \\
&\quad + (\chi - \overline{\chi}_{\alpha,\beta})\big(\alpha_\chi(\overline{\chi}_{\alpha,\beta} - \chi_*) + (\overline{\chi}_{\alpha,\beta} - \overline{\chi}_\beta)\big) \\
&\quad + (\mathcal{C} - \overline{\mathcal{C}}_{\alpha,\beta})\big(\alpha_C(\overline{\mathcal{C}}_{\alpha,\beta} - \mathcal{C}_*) + (\overline{\mathcal{C}}_{\alpha,\beta} - \overline{\mathcal{C}}_\beta)\big) \geq 0 \quad \forall (\mathcal{P}, \chi, \mathcal{C}) \in \mathcal{U}_{\rm ad}, \qquad (6.34)
\end{aligned}$$

where $(\overline{\varphi}_{\alpha,\beta}, \overline{\mu}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta})$ *and* $(p_{\alpha,\beta}, q_{\alpha,\beta}, r_{\alpha,\beta})$ *denote the corresponding unique solutions to* (4.1)–(4.5) *and* (6.21)–(6.25) *with* $\alpha, \beta > 0$.

The sense in which the minimisers of $(CP)^{\rm ad}_{\alpha,\beta}$ approximate the ones of $(CP)_\beta$ as $\alpha \to 0$ is specified in the following theorem.

Theorem 6.12. *Assume* **B1–B7**, **D1–D7**. *Let* $(\overline{P}_\beta, \overline{\chi}_\beta, \overline{C}_\beta) \in \mathcal{U}_{\rm ad}$ *be an optimal control for* $(CP)_\beta$*, with corresponding state* $(\overline{\varphi}_\beta, \overline{\mu}_\beta, \overline{\sigma}_\beta)$*. Then, for every family* $\{(\overline{\mathcal{P}}_{\alpha,\beta}, \overline{\chi}_{\alpha,\beta}, \overline{\mathcal{C}}_{\alpha,\beta})\}_\alpha$ *of optimal controls for* $(CP)^{\rm ad}_{\alpha,\beta}$*, with corresponding states* $\{(\overline{\varphi}_{\alpha,\beta}, \overline{\mu}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta})\}_\alpha$ *it holds that, as* $\alpha \to 0$,

$$\begin{aligned}
&\overline{\varphi}_{\alpha,\beta} \to \overline{\varphi}_\beta \quad \textit{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V), \\
&\qquad\qquad\qquad \textit{and strongly in } C^0([0,T];H), && (6.35) \\
&\overline{\mathcal{P}}_{\alpha,\beta} \to \overline{\mathcal{P}}_\beta, \quad \overline{\chi}_{\alpha,\beta} \to \overline{\chi}_\beta, \quad \overline{\mathcal{C}}_{\alpha,\beta} \to \overline{\mathcal{C}}_\beta, && (6.36) \\
&\mathcal{J}_{\rm ad}(\overline{\varphi}_{\alpha,\beta}, \overline{\mathcal{P}}_{\alpha,\beta}, \overline{\chi}_{\alpha,\beta}, \overline{\mathcal{C}}_{\alpha,\beta}) \to \mathcal{J}(\overline{\varphi}_\beta, \overline{\mathcal{P}}_\beta, \overline{\chi}_\beta, \overline{\mathcal{C}}_\beta). && (6.37)
\end{aligned}$$

Proof of Theorem 6.12. Since $\mathcal{U}_{\rm ad}$ is a compact subset of $\mathbb{R}^3$, by virtue of (4.67)–(4.69), (4.72)–(4.73), and the Bolzano–Weierstrass theorem, we infer the existence of limits $\hat{\varphi}, \hat{\mu}, \hat{\sigma}$, and $(\hat{\mathcal{P}}, \hat{\chi}, \hat{\mathcal{C}}) \in \mathcal{U}_{\rm ad}$ such that, along a non-relabelled zero subsequence α_k, as $k \to \infty$,

$$\begin{aligned}
&\overline{\varphi}_{\alpha_k,\beta} \to \hat{\varphi} \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V), \\
&\qquad\qquad \text{and strongly in } C^0([0,T];H), \\
&\overline{\mu}_{\alpha_k,\beta} \to \hat{\mu} \quad \text{weakly in } L^2(0,T;V), \\
&\overline{\sigma}_{\alpha_k,\beta} \to \hat{\sigma} \quad \text{weakly star in } H^1(0,T;V^*) \cap L^2(0,T;V) \cap L^\infty(Q), \\
&\qquad\qquad \text{and strongly in } C^0([0,T];V^*) \cap L^2(0,T;H), \\
&\overline{\mathcal{P}}_{\alpha_k,\beta} \to \hat{\mathcal{P}}, \quad \overline{\chi}_{\alpha_k,\beta} \to \hat{\chi}, \quad \overline{\mathcal{C}}_{\alpha_k,\beta} \to \hat{\mathcal{C}}.
\end{aligned}$$

Arguing in a similar fashion as in Theorem 4.9, by employing the above uniform convergences, we can pass to the limit as $k \to \infty$ in the variational formulation of

(4.1)–(4.5) to infer that $\hat{\varphi}$ is actually the first component of the state system (4.1)–(4.5) with $\alpha = 0$ and parameters $(\hat{\mathcal{P}}, \hat{\chi}, \hat{\mathcal{C}})$. Now, on the one hand the minimality of $(\overline{\mathcal{P}}_{\alpha_k,\beta}, \overline{\chi}_{\alpha_k,\beta}, \overline{\mathcal{C}}_{\alpha_k,\beta})$ for $(CP)^{\text{ad}}_{\alpha_k,\beta}$ entails that

$$\mathcal{J}_{\text{ad}}(\overline{\varphi}_{\alpha_k,\beta}, \overline{\mathcal{P}}_{\alpha_k,\beta}, \overline{\chi}_{\alpha_k,\beta}, \overline{\mathcal{C}}_{\alpha_k,\beta}) \leq \mathcal{J}_{\text{ad}}(\overline{\varphi}_\beta, \overline{\mathcal{P}}_\beta, \overline{\chi}_\beta, \overline{\mathcal{C}}_\beta) = \mathcal{J}(\overline{\varphi}_\beta, \overline{\mathcal{P}}_\beta, \overline{\chi}_\beta, \overline{\mathcal{C}}_\beta),$$

so that passing to the superior limit to both sides leads to

$$\limsup_{k\to\infty} \mathcal{J}_{\text{ad}}(\overline{\varphi}_{\alpha_k,\beta}, \overline{\mathcal{P}}_{\alpha_k,\beta}, \overline{\chi}_{\alpha_k,\beta}, \overline{\mathcal{C}}_{\alpha_k,\beta}) \leq \mathcal{J}(\overline{\varphi}_\beta, \overline{\mathcal{P}}_\beta, \overline{\chi}_\beta, \overline{\mathcal{C}}_\beta).$$

On the other hand, by the lower semincontinuity of $\mathcal{J}_{\text{ad}}$, we also have that

$$\begin{aligned}\liminf_{k\to\infty} \mathcal{J}_{\text{ad}}(\overline{\varphi}_{\alpha_k,\beta}, \overline{\mathcal{P}}_{\alpha_k,\beta}, \overline{\chi}_{\alpha_k,\beta}, \overline{\mathcal{C}}_{\alpha_k,\beta}) &\geq \mathcal{J}_{\text{ad}}(\hat{\varphi}, \hat{\mathcal{P}}, \hat{\chi}, \hat{\mathcal{C}}) \\ &= \mathcal{J}(\hat{\varphi}, \hat{\mathcal{P}}, \hat{\chi}, \hat{\mathcal{C}}) + \tfrac{1}{2}(|\hat{\mathcal{P}} - \overline{\mathcal{P}}_\beta|^2 + |\hat{\chi} - \overline{\chi}_\beta|^2 + |\hat{\mathcal{C}} - \overline{\mathcal{C}}_\beta|^2).\end{aligned}$$

Since $\hat{\varphi}$ is the first state component of the system with $\alpha = 0$ and coefficients $(\hat{\mathcal{P}}, \hat{\chi}, \hat{\mathcal{C}})$, combining the above inequalities with the optimality of $(\overline{\varphi}_\beta, \overline{\mathcal{P}}_\beta, \overline{\chi}_\beta, \overline{\mathcal{C}}_\beta)$ for $(CP)_\beta$ yields directly

$$\hat{\mathcal{P}} = \overline{\mathcal{P}}_\beta, \quad \hat{\chi} = \overline{\chi}_\beta, \quad \hat{\mathcal{C}} = \overline{\mathcal{C}}_\beta,$$

from which also $\hat{\varphi} = \overline{\varphi}_\beta$ by uniqueness of the state system (4.1)–(4.5) with $\alpha = 0$. Also, we have the chain of equalities

$$\begin{aligned}\lim_{k\to\infty} \mathcal{J}_{\text{ad}}(\overline{\varphi}_{\alpha_k,\beta}, \overline{\mathcal{P}}_{\alpha_k,\beta}, \overline{\chi}_{\alpha_k,\beta}, \overline{\mathcal{C}}_{\alpha_k,\beta}) &= \liminf_{k\to\infty} \mathcal{J}_{\text{ad}}(\overline{\varphi}_{\alpha_k,\beta}, \overline{\mathcal{P}}_{\alpha_k,\beta}, \overline{\chi}_{\alpha_k,\beta}, \overline{\mathcal{C}}_{\alpha_k,\beta}) \\ = \limsup_{k\to\infty} \mathcal{J}_{\text{ad}}(\overline{\varphi}_{\alpha_k,\beta}, \overline{\mathcal{P}}_{\alpha_k,\beta}, \overline{\chi}_{\alpha_k,\beta}, \overline{\mathcal{C}}_{\alpha_k,\beta}) &= \mathcal{J}(\overline{\varphi}_\beta, \overline{\mathcal{P}}_\beta, \overline{\chi}_\beta, \overline{\mathcal{C}}_\beta).\end{aligned}$$

As the same argument holds along every arbitrary subsequence $\{\alpha_k\}_k$, by uniqueness of the limits the convergences actually hold along the whole sequence α, and the proof is concluded. □

6.2.2.1 Letting $\alpha \to 0$ in the Adjoint System

This section is devoted to discuss and analyse the asymptotic behaviour of the adjoint system (6.21)–(6.25) as $\alpha \to 0$, which will be a key ingredient to derive the optimality conditions of $(CP)_\beta$. To begin with, let us state the established result.

Theorem 6.13. *Assume* **B1–B7**, **D1–D7**. *Let* $(\mathcal{P}_\beta, \chi_\beta, \mathcal{C}_\beta) \in \mathcal{U}_{\text{ad}}$, $\{(\mathcal{P}_{\alpha,\beta}, \chi_{\alpha,\beta}, \mathcal{C}_{\alpha,\beta})\}_\alpha \subset \mathcal{U}_{\text{ad}}$ *be such that* $(\mathcal{P}_{\alpha,\beta}, \chi_{\alpha,\beta}, \mathcal{C}_{\alpha,\beta}) \to (\mathcal{P}_\beta, \chi_\beta, \mathcal{C}_\beta)$ *as* $\alpha \to 0$. *Let* $(\overline{\varphi}_\beta, \overline{\mu}_\beta, \overline{\sigma}_\beta)$ *and* $(\overline{\varphi}_{\alpha,\beta}, \overline{\mu}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta})$ *be the unique solutions to the state system* (4.1)–(4.5) *in the cases* $\alpha = 0$ *with coefficients* $(\mathcal{P}_\beta, \chi_\beta, \mathcal{C}_\beta)$ *and* $\alpha \in (0, \alpha_0)$ *with coefficients* $(\mathcal{P}_{\alpha,\beta}, \chi_{\alpha,\beta}, \mathcal{C}_{\alpha,\beta})$, *as given by Theorems 4.9 and 4.5, respectively. Let also* $(p_{\alpha,\beta}, q_{\alpha,\beta}, r_{\alpha,\beta})$ *be the unique solution to the adjoint system* (6.21)–(6.25) *with* $\alpha \in (0, \alpha_0)$ *and coefficients* $(\mathcal{P}_{\alpha,\beta}, \chi_{\alpha,\beta}, \mathcal{C}_{\alpha,\beta})$, *as given by Theorem 6.6. Then, there exists a triplet* $(p_\beta, q_\beta, r_\beta)$, *with*

$$\begin{aligned} p_\beta &\in L^\infty(0,T;V) \cap L^2(0,T;W), \\ q_\beta &\in L^\infty(0,T;H), \\ p_\beta + \beta q_\beta &\in H^1(0,T;V^*), \\ r_\beta &\in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \end{aligned}$$

such that, for every $\kappa \geq 1$ if $d = 2$ and $1 \leq \kappa < 6$ if $d = 3$, as $\alpha \to 0$ it holds

$$\begin{aligned}
p_{\alpha,\beta} &\to p_\beta && \text{weakly star in } L^\infty(0,T;V) \cap L^2(0,T;W), \\
q_{\alpha,\beta} &\to q_\beta && \text{weakly star in } L^\infty(0,T;H), \\
r_{\alpha,\beta} &\to r_\beta && \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \\
& && \quad \text{strongly in } C^0([0,T];L^\kappa(\Omega)) \cap L^2(0,T;V), \\
p_{\alpha,\beta} + \beta q_{\alpha,\beta} &\to p_\beta + \beta q_\beta && \text{weakly in } H^1(0,T;V^*), \\
\alpha p_{\alpha,\beta} &\to 0 && \text{strongly in } H^1(0,T;H).
\end{aligned}$$

Moreover, $(p_\beta, q_\beta, r_\beta)$ is the unique weak solution to the adjoint system (6.21)–(6.25) *with $\alpha = 0$ and coefficients $(\mathcal{P}_\beta, \chi_\beta, \mathcal{C}_\beta)$, in the sense that*

$$\begin{aligned}
& - \langle \partial_t (p_\beta + \beta q_\beta), v \rangle_V + \int_\Omega (a q_\beta - J * q_\beta) v + \int_\Omega F''(\overline{\varphi}_\beta) q_\beta v \\
& \qquad + \int_\Omega \mathcal{C}_\beta \overline{\sigma}_\beta f'(\overline{\varphi}_\beta) r_\beta v - \int_\Omega \mathcal{P}_\beta \overline{\sigma}_\beta f'(\overline{\varphi}_\beta) p_\beta v = \int_\Omega b_1 (\overline{\varphi}_\beta - \varphi_Q) v, \\
& \int_\Omega \nabla p_\beta \cdot \nabla v - \int_\Omega q_\beta v \; = \; 0, \\
& - \int_\Omega \partial_t r_\beta v + \int_\Omega \nabla r_\beta \cdot \nabla v + \int_\Omega \mathcal{C}_\beta f(\overline{\varphi}_\beta) r_\beta v - \int_\Omega \mathcal{P}_\beta f(\overline{\varphi}_\beta) p_\beta v - \int_\Omega \chi_\beta q_\beta v \; = 0,
\end{aligned}$$

for every $v \in V$, almost everywhere in $(0,T)$, and

$$(p_\beta + \beta q_\beta)(T) \; = \; b_2(\overline{\varphi}_\beta(T) - \varphi_\Omega), \qquad r_\beta(T) \; = 0.$$

Proof of Theorem 6.13. We use the estimate (6.32). First of all, by **D7** and Theorem 4.9, we have that $\{F''(\overline{\varphi}_{\alpha,\beta})\}_\alpha$ is uniformly bounded in $L^\infty(0,T;L^3(\Omega))$ and $\{\overline{\sigma}_{\alpha,\beta}\}_\alpha$ is uniformly bounded in $L^2(0,T;V)$: hence, recalling that β is fixed and $\eta = 0$, by the Gronwall lemma along with elliptic regularity theory, there exists a positive constant C_β, which may depend on β but is independent of α, such that

$$\begin{aligned}
\alpha^{1/2} \|p_{\alpha,\beta}\|_{H^1(0,T;H)} + \|p_{\alpha,\beta}\|_{L^\infty(0,T;V)} + \|q_{\alpha,\beta}\|_{L^\infty(0,T;H)} \\
+ \|r_{\alpha,\beta}\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} \leq C_\beta.
\end{aligned}$$

In particular, by the Hölder inequality it follows that

$$\|F''(\overline{\varphi}_{\alpha,\beta}) q_{\alpha,\beta}\|_{L^\infty(0,T;L^{6/5}(\Omega))} \leq C_\beta.$$

Secondly, elliptic regularity theory and equation (6.22) entail that

$$\|p_{\alpha,\beta}\|_{L^2(0,T;W)} \leq C_\beta.$$

Moreover, since $L^{6/5}(\Omega) \subset V^*$, a comparison argument in (6.21) yields, by the boundedness of $\{\overline{\sigma}_{\alpha,\beta}\}_\alpha$ in $L^\infty(Q)$ and the estimates above, that

$$\|p_{\alpha,\beta} + \beta q_{\alpha,\beta}\|_{H^1(0,T;V^*)} \leq C_\beta.$$

Banach–Alaoglu theorem and classical compact embedding results (see, e.g., [141]) allow us to obtain from the above a priori estimates the existence of functions $(p_\beta, q_\beta, r_\beta)$ such that as $\alpha \to 0$ it holds, for every $\kappa \geq 1$ if $d = 2$ and $1 \leq \kappa < 6$ if $d = 3$, and along a non-relabelled subsequence,

$$\begin{aligned}
p_{\alpha,\beta} &\to p_\beta && \text{weakly star in } L^\infty(0,T;V) \cap L^2(0,T;W),\\
q_{\alpha,\beta} &\to q_\beta && \text{weakly star in } L^\infty(0,T;H),\\
r_{\alpha,\beta} &\to r_\beta && \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),\\
& && \quad \text{strongly in } C^0([0,T];L^\kappa(\Omega)) \cap L^2(0,T;V),\\
p_{\alpha,\beta} + \beta q_{\alpha,\beta} &\to p_\beta + \beta q_\beta && \text{weakly in } H^1(0,T;V^*),\\
\alpha p_{\alpha,\beta} &\to 0 && \text{strongly in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)\,.
\end{aligned}$$

We claim that these limit variables yield a weak solution to the adjoint system (6.21)–(6.25) in which we formally set $\alpha = 0$. In this direction, we just need to justify the passage to the limit as $\alpha \to 0$ in the variational formulation for system (6.21)–(6.25) written for the triplet $(p_{\alpha,\beta}, q_{\alpha,\beta}, r_{\alpha,\beta})$, which reads

$$\begin{aligned}
&-\int_\Omega \partial_t(p_{\alpha,\beta} + \beta q_{\alpha,\beta})v + \int_\Omega (aq_{\alpha,\beta} - J * q_{\alpha,\beta})v + \int_\Omega F''(\overline{\varphi}_{\alpha,\beta})q_{\alpha,\beta}v\\
&\quad + \int_\Omega \mathcal{C}_{\alpha,\beta}\overline{\sigma}_{\alpha,\beta}f'(\overline{\varphi}_{\alpha,\beta})r_{\alpha,\beta}v - \int_\Omega \mathcal{P}_{\alpha,\beta}\overline{\sigma}_{\alpha,\beta}f'(\overline{\varphi}_{\alpha,\beta})p_{\alpha,\beta}v\\
&= \int_\Omega b_1(\overline{\varphi}_{\alpha,\beta} - \varphi_Q)v, && (6.38)\\
&-\int_\Omega \alpha\partial_t p_{\alpha,\beta}w + \int_\Omega \nabla p_{\alpha,\beta}\cdot\nabla w - \int_\Omega q_{\alpha,\beta}w \;=\; 0, && (6.39)\\
&-\int_\Omega \partial_t r_{\alpha,\beta}z + \int_\Omega \nabla r_{\alpha,\beta}\cdot\nabla z + \int_\Omega \mathcal{C}_{\alpha,\beta}f(\overline{\varphi}_{\alpha,\beta})r_{\alpha,\beta}z - \int_\Omega \mathcal{P}_{\alpha,\beta}f(\overline{\varphi}_{\alpha,\beta})p_{\alpha,\beta}z\\
&\quad - \int_\Omega \chi_{\alpha,\beta}q_{\alpha,\beta}z \;= 0, && (6.40)
\end{aligned}$$

for every $v, w, z \in V$ and almost every $t \in (0,T)$ and also in the terminal conditions

$$\alpha p_{\alpha,\beta}(T) \;=\; 0, \quad (p_{\alpha,\beta} + \beta q_{\alpha,\beta})(T) \;=\; b_2(\overline{\varphi}_{\alpha,\beta}(T) - \varphi_\Omega), \quad r_{\alpha,\beta}(T) = 0. \quad (6.41)$$

It is worth noting that since $\alpha > 0$ the second condition of (6.41) reduces to $\beta q_{\alpha,\beta}(T) = b_2(\overline{\varphi}_{\alpha,\beta}(T) - \varphi_\Omega)$. Moreover, from the convergences (4.72)–(4.73), we also have that, possibly after another extraction,

$$\begin{aligned}
\overline{\varphi}_{\alpha,\beta} &\to \overline{\varphi}_\beta && \text{strongly in } C^0([0,T];L^\kappa(\Omega)), \text{ and } a.e. \text{ in } Q, && (6.42)\\
\overline{\sigma}_{\alpha,\beta} &\to \overline{\sigma}_\beta && \text{strongly in } C^0([0,T];V^*) \cap L^2(0,T;L^\kappa(\Omega)), && (6.43)
\end{aligned}$$

for every $\kappa \geq 1$ if $d = 2$ and $1 \leq \kappa < 6$ if $d = 3$. Therefore, most part of the above limits are easy consequence of (4.72)–(4.73), the above estimates and Lebesgue's dominated convergence theorem as well. For instance, due to the boundedness and continuity of f' we have, e.g., that $f'(\overline{\varphi}_{\alpha,\beta}) \to f'(\overline{\varphi}_\beta)$ $a.e.$ in Q so that we easily infer that

$$\int_\Omega \mathcal{C}_{\alpha,\beta}\overline{\sigma}_{\alpha,\beta} f'(\overline{\varphi}_{\alpha,\beta}) r_{\alpha,\beta} v \to \int_\Omega \mathcal{C}_\beta \overline{\sigma}_\beta f'(\overline{\varphi}_\beta) r_\beta v \quad \text{for every } v \in V,$$

and the other terms can be handled in a similar fashion. The only term which has to be treated differently is the one involving the potential. It can be dealt with invoking the almost everywhere convergence (6.42), the weak star convergence (4.67), the continuous embedding $V \subset L^6(\Omega)$, and the Severini–Egorov theorem. In fact, these properties imply in particular that, as $\alpha \to 0$,

$$F''(\overline{\varphi}_{\alpha,\beta}) \to F''(\overline{\varphi}_\beta) \quad \text{strongly in } L^\kappa(Q) \quad \text{for all } \kappa \in [1,3)$$

so that from the weak-strong convergence principle, we get

$$F''(\overline{\varphi}_{\alpha,\beta}) q_{\alpha,\beta} \to F''(\overline{\varphi}_\beta) q_\beta \quad \text{weakly in } L^2(0,T;L^\gamma(\Omega)) \text{ for all } \gamma \in [1,6/5).$$

It follows then that

$$\int_\Omega F''(\overline{\varphi}_{\alpha,\beta}) q_{\alpha,\beta} v \to \int_\Omega F''(\overline{\varphi}_\beta) q_\beta v \qquad \forall\, v \in W.$$

This is enough to pass to the limit in the variational formulation (6.38)–(6.40) as $\alpha \to 0$ for every $v \in W$, $w, z \in V$, and to obtain the required terminal conditions. Since at the limit $F''(\overline{\varphi}_\beta) q_\beta \in L^\infty(0,T;L^{6/5}(\Omega))$, by the density of W in V the variational formulation holds also for all $v \in V$. Thus, we realise that the limit variables obtained above yield a weak solution to (6.21)–(6.25) in which α is set to zero.

By linearity and the estimate (6.32), we deduce that $(p_\beta, q_\beta, r_\beta)$ is the unique weak solution to (6.21)–(6.25) with $\alpha = 0$, hence also that the convergences above hold along the entire sequence $\alpha \to 0$, and the proof is concluded. □

6.2.2.2 Letting $\alpha \to 0$ in the Optimality Condition

In this last step, we draw some consequences from the approximation of controls presented in Theorem 6.12 and Subsection 6.2.2.1 by passing to the limit in the variational inequality (6.34) as $\alpha \to 0$. This allows us to prove the optimality conditions of $(CP)_\beta$ as follows:

Theorem 6.14. *Assume* **B1–B7**, **D1–D7**. *Then, every optimal control* $(\overline{\mathcal{P}}_\beta, \overline{\chi}_\beta, \overline{\mathcal{C}}_\beta)$ *of* $(CP)_\beta$ *necessarily verifies*

$$\begin{aligned}\int_Q (\mathcal{P} - \overline{\mathcal{P}}_\beta)\overline{\sigma}_\beta f(\overline{\varphi}_\beta) p_\beta - \int_Q (\chi - \overline{\chi}_\beta)\overline{\sigma}_\beta q_\beta - \int_Q (\mathcal{C} - \overline{\mathcal{C}}_\beta)\overline{\sigma}_\beta f(\overline{\varphi}_\beta) r_\beta \\ + \alpha_P(\overline{\mathcal{P}}_\beta - \mathcal{P}_*)(\mathcal{P} - \overline{\mathcal{P}}_\beta) + \alpha_\chi(\overline{\chi}_\beta - \chi_*)(\chi - \overline{\chi}_\beta) \\ + \alpha_C(\overline{\mathcal{C}}_\beta - \mathcal{C}_*)(\mathcal{C} - \overline{\mathcal{C}}_\beta) \geq 0 \quad \forall (\mathcal{P}, \chi, \mathcal{C}) \in \mathcal{U}_{\text{ad}},\end{aligned}$$

where $(\overline{\varphi}_\beta, \overline{\mu}_\beta, \overline{\sigma}_\beta)$ *and* $(p_\beta, q_\beta, r_\beta)$ *are the unique solutions to* (4.1)–(4.5) *and* (6.21)–(6.25) *with* $\alpha = 0$ *in the sense of Theorems 4.9 and 6.13, respectively.*

Proof of Theorem 6.14. Let $\{(\overline{\mathcal{P}}_{\alpha,\beta}, \overline{\chi}_{\alpha,\beta}, \overline{\mathcal{C}}_{\alpha,\beta})\}_\alpha$ be an approximating sequence of minimisers for $(CP)^{\mathrm{ad}}_{\alpha,\beta}$, as given by Lemma 6.10. Then, by Lemma 6.11 the corresponding states $\{(\overline{\varphi}_{\alpha,\beta}, \overline{\mu}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta})\}_\alpha$ and adjoint variables $\{(p_{\alpha,\beta}, q_{\alpha,\beta}, r_{\alpha,\beta})\}_\alpha$ satisfy (6.34). By the convergences in Theorems 4.9, 6.12, and 6.13, the thesis follows letting $\alpha \to 0$ in (6.34) using the dominated convergence theorem. □

6.2.3 The Optimisation Problem $(CP)_{\alpha,\beta}$ as $\beta \to 0$

In this section, we continue the asymptotic analysis of the optimisation problem $(CP)_{\alpha,\beta}$, focusing on the case $\beta \to 0$, and keeping α fixed instead. Namely, throughout the whole Section 6.2.3 we assume the following framework:

$$\alpha \in (0, \alpha_0)\,, \quad b_2 = 0, \quad \text{(4.90)–(4.92)}$$
$$\chi_{\max} < \sqrt{c_a}, \quad (\chi_{\max} + \eta_{\max} + 4c_a\chi_{\max})^2 < 8c_a C_0, \quad \eta_{\max}^2 + \chi_{\max}^2 < \frac{4}{9}C_0.$$

It is worth underlying that, from (6.25), it is clear that the compatibility condition $b_2 = 0$ has to be imposed in the scenario $\alpha > 0$ and $\beta = 0$.

Existence of optimal controls for $(CP)_\alpha$ is given in the following result.

Theorem 6.15. *Assume* **A1–A7** *and* **C1–C3**. *Then, the optimisation problem* $(CP)_\alpha$ *admits a solution.*

Proof of Theorem 6.15. This result follows directly by adapting the direct method used in the proof of Theorem 6.3, taking into account the compactness of $\mathcal{U}_{\mathrm{ad}}$ and the convergence presented in Theorem 4.12. □

The main goal is to obtain now necessary conditions for optimality of $(CP)_\alpha$. The idea is to proceed on the same line of Section 6.2.2, by passing to the limit as $\beta \to 0$ in the adjoint problem and in the first-order conditions for optimality for the adapted optimisation problem $(CP)^{\mathrm{ad}}_{\alpha,\beta}$. The main difference with respect to the case $\alpha \to 0$ is that the state system (4.1)–(4.5) with $\beta = 0$ does not have a unique solution, as stated in Theorem 4.12. Notice that also uniqueness has been established in Theorem 4.13 as a consequence of a suitable error estimate between the solutions to the α, β and α problems. However, that result oblige us to assume $\eta = 0$. This means that under no additional requirements on the data (in particular, if $\eta_{\max} > 0$), the control-to-state operator $\mathcal{S}$ is not even well-defined when $\beta = 0$. The main problem is that, despite Theorem 4.12, for a given minimiser $(\varphi_\alpha, \mathcal{P}_\alpha, \chi_\alpha, \eta_\alpha, \mathcal{C}_\alpha)$ of $(CP)_\alpha$, it is not necessarily true that the corresponding state φ_α can be approximated by some corresponding state solutions $\{\varphi_{\alpha,\beta}\}_\beta$ of the state system with $\alpha, \beta > 0$ as $\beta \to 0$. For this reason, it is important that the adapted cost functional is modified accordingly, accounting also for the phase variable. Namely, in this section, given a certain minimiser $(\overline{\varphi}_\alpha, \overline{\mathcal{P}}_\alpha, \overline{\chi}_\alpha, \overline{\eta}_\alpha, \overline{\mathcal{C}}_\alpha)$ for $(CP)_\alpha$, we consider the following adapted cost functional:

$$\mathcal{J}_{\mathrm{ad}}(\varphi, \mathcal{P}, \chi, \eta, \mathcal{C}) := \mathcal{J}(\varphi, \mathcal{P}, \chi, \eta, \mathcal{C}) + \tfrac{1}{2}\|\varphi - \overline{\varphi}_\alpha\|^2_{L^2(Q)} + \tfrac{1}{2}|\mathcal{P} - \overline{\mathcal{P}}_\alpha|^2$$
$$+ \tfrac{1}{2}|\chi - \overline{\chi}_\alpha|^2 + \tfrac{1}{2}|\eta - \overline{\eta}_\alpha|^2 + \tfrac{1}{2}|\mathcal{C} - \overline{\mathcal{C}}_\alpha|^2.$$

The adapted optimisation problem $(CP)^{\text{ad}}_{\alpha,\beta}$ is then defined exactly as in (6.33) with this new definition for the cost functional.

Arguing as in Section 6.2.2, we straightforwardly infer the corresponding of Lemmas 6.10, 6.11 and Theorem 6.12, whose proofs are omitted since can be reproduced in the same fashion. These results concern solvability of $(CP)^{\text{ad}}_{\alpha,\beta}$, necessary conditions for $(CP)^{\text{ad}}_{\alpha,\beta}$, and approximation $(CP)^{\text{ad}}_{\alpha,\beta} \searrow (CP)_\alpha$ as $\beta \to 0$. Let us just point out that since the cost functional is corrected also with respect to the state variable, the forcing term of the corresponding adjoint system has to be corrected too, with no additional effort though.

Lemma 6.16. *Assume* **B1–B7** *and* **D1–D5**. *Then, for every $\beta \in (0, \beta_0)$ and for every minimiser $(\overline{\varphi}_\alpha, \overline{\mathcal{P}}_\alpha, \overline{\chi}_\alpha, \overline{\eta}_\alpha, \overline{\mathcal{C}}_\alpha)$ of $(CP)_\alpha$, the optimisation problem $(CP)^{\text{ad}}_{\alpha,\beta}$ admits a minimiser.*

Lemma 6.17. *Assume* **B1–B7** *and* **D1–D5**, *and let $(\overline{\varphi}_\alpha, \overline{\mathcal{P}}_\alpha, \overline{\chi}_\alpha, \overline{\eta}_\alpha, \overline{\mathcal{C}}_\alpha)$ be a minimiser for $(CP)_\alpha$. For every $\beta \in (0, \beta_0)$, if $(\overline{\mathcal{P}}_{\alpha,\beta}, \overline{\chi}_{\alpha,\beta}, \overline{\eta}_{\alpha,\beta}, \overline{\mathcal{C}}_{\alpha,\beta})$ is an optimal control for $(CP)^{\text{ad}}_{\alpha,\beta}$, then the following first-order necessary condition holds*

$$
\begin{aligned}
&\int_Q (\mathcal{P} - \overline{\mathcal{P}}_{\alpha,\beta})\overline{\sigma}_{\alpha,\beta} f(\overline{\varphi}_{\alpha,\beta}) p_{\alpha,\beta} - \int_Q (\chi - \overline{\chi}_{\alpha,\beta})\overline{\sigma}_{\alpha,\beta} q_{\alpha,\beta} \\
&\quad - \int_Q (\mathcal{C} - \overline{\mathcal{C}}_{\alpha,\beta})\overline{\sigma}_{\alpha,\beta} f(\overline{\varphi}_{\alpha,\beta}) r_{\alpha,\beta} \\
&\quad + (\mathcal{P} - \overline{\mathcal{P}}_{\alpha,\beta})\big(\alpha_P(\overline{\mathcal{P}}_{\alpha,\beta} - \mathcal{P}_*) + (\overline{\mathcal{P}}_{\alpha,\beta} - \overline{\mathcal{P}}_\alpha)\big) \\
&\quad + (\chi - \overline{\chi}_{\alpha,\beta})\big(\alpha_\chi(\overline{\chi}_{\alpha,\beta} - \chi_*) + (\overline{\chi}_{\alpha,\beta} - \overline{\chi}_\alpha)\big) \\
&\quad + (\eta - \overline{\eta}_{\alpha,\beta})\big(\alpha_\eta(\overline{\eta}_{\alpha,\beta} - \eta_*) + (\overline{\eta}_{\alpha,\beta} - \overline{\eta}_\alpha)\big) \\
&\quad + (\mathcal{C} - \overline{\mathcal{C}}_{\alpha,\beta})\big(\alpha_C(\overline{\mathcal{C}}_{\alpha,\beta} - \mathcal{C}_*) + (\overline{\mathcal{C}}_{\alpha,\beta} - \overline{\mathcal{C}}_\alpha)\big) \geq 0 \quad \forall (\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathcal{U}_{\text{ad}}, \qquad (6.44)
\end{aligned}
$$

where $(\overline{\varphi}_{\alpha,\beta}, \overline{\mu}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta})$ and $(p_{\alpha,\beta}, q_{\alpha,\beta}, r_{\alpha,\beta})$ are the corresponding unique solutions to the state system (4.1)–(4.5) *and the adjoint system* (6.21)–(6.25) *with $\alpha, \beta > 0$ and with respect to the coefficients $(\overline{\mathcal{P}}_{\alpha,\beta}, \overline{\chi}_{\alpha,\beta}, \overline{\eta}_{\alpha,\beta}, \overline{\mathcal{C}}_{\alpha,\beta})$, the right-hand side of* (6.21) *being modified as $b_1(\overline{\varphi}_{\alpha,\beta} - \varphi_Q) + (\overline{\varphi}_{\alpha,\beta} - \overline{\varphi}_\alpha)$.*

Theorem 6.18. *Assume* **B1–B7**, **D1–D6**, *and let $(\overline{\varphi}_\alpha, \overline{\mathcal{P}}_\alpha, \overline{\chi}_\alpha, \overline{\eta}_\alpha, \overline{\mathcal{C}}_\alpha)$ be a minimiser for $(CP)_\alpha$. Then, for every family of optimal controls $\{(\overline{\mathcal{P}}_{\alpha,\beta}, \overline{\chi}_{\alpha,\beta}, \overline{\eta}_{\alpha,\beta}, \overline{\mathcal{C}}_{\alpha,\beta})\}_\beta$ for $(CP)^{\text{ad}}_{\alpha,\beta}$, with corresponding states $\{(\overline{\varphi}_{\alpha,\beta}, \overline{\mu}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta})\}_\beta$, as $\beta \to 0$ it holds that, for every $\kappa \geq 1$ if $d = 2$ and $1 \leq \kappa < 6$ if $d = 3$,*

$$
\begin{aligned}
&\overline{\varphi}_{\alpha,\beta} \to \overline{\varphi}_\alpha \quad \textit{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V), \\
&\qquad\qquad\qquad \textit{and strongly in } L^2(0,T;L^\kappa(\Omega)), \\
&\overline{\mathcal{P}}_{\alpha,\beta} \to \overline{\mathcal{P}}_\alpha, \quad \overline{\chi}_{\alpha,\beta} \to \overline{\chi}_\alpha, \quad \overline{\eta}_{\alpha,\beta} \to \overline{\eta}_\alpha, \quad \overline{\mathcal{C}}_{\alpha,\beta} \to \overline{\mathcal{C}}_\alpha, \\
&\mathcal{J}_{\text{ad}}(\overline{\varphi}_{\alpha,\beta}, \overline{\mathcal{P}}_{\alpha,\beta}, \overline{\chi}_{\alpha,\beta}, \overline{\eta}_{\alpha,\beta}, \overline{\mathcal{C}}_{\alpha,\beta}) \to \mathcal{J}(\overline{\varphi}_\alpha, \overline{\mathcal{P}}_\alpha, \overline{\chi}_\alpha, \overline{\eta}_\alpha, \overline{\mathcal{C}}_\alpha).
\end{aligned}
$$

Proof of Theorem 6.18. The proof is analogous to the one of Theorem 6.12, the only difference being that the identification of the limit $\hat{\varphi} = \overline{\varphi}_\alpha$ follows from the additional correction in the cost functional and not from the uniqueness of the state system with $\beta = 0$, which is indeed not true. The strong convergence of $\{\overline{\varphi}_{\alpha,\beta}\}_\beta$ is a consequence of the convergences in Theorem 4.12 and the compact inclusion $V \subset L^\kappa(\Omega)$. □

6.2.3.1 Letting $\beta \to 0$ in the Adjoint System

In this section we study the passage to the limit as $\beta \to 0$ in the adjoint system (6.21)–(6.25), where the forcing term of (6.21) is modified as stated in Lemma 6.17.

Theorem 6.19. *Assume* **B1–B7**, **D1–D7**. *Let the parameters* $(\mathcal{P}_\alpha, \chi_\alpha, \eta_\alpha, \mathcal{C}_\alpha) \in \mathcal{U}_{\mathrm{ad}}$ *and* $\{(\mathcal{P}_{\alpha,\beta}, \chi_{\alpha,\beta}, \eta_{\alpha,\beta}, \mathcal{C}_{\alpha,\beta})\}_\beta \subset \mathcal{U}_{\mathrm{ad}}$ *be such that* $(\mathcal{P}_{\alpha,\beta}, \chi_{\alpha,\beta}, \eta_{\alpha,\beta}, \mathcal{C}_{\alpha,\beta}) \to (\mathcal{P}_\alpha, \chi_\alpha, \eta_\alpha, \mathcal{C}_\alpha)$ *as* $\beta \to 0$. *Let* $(\overline{\varphi}_\alpha, \overline{\mu}_\alpha, \overline{\sigma}_\alpha)$ *be a solution to the state system* (4.1)–(4.5) *with* $\beta = 0$ *and coefficients* $(\mathcal{P}_\alpha, \chi_\alpha, \eta_\alpha, \mathcal{C}_\alpha)$ *as given by Theorem 4.12, and let* $(\overline{\varphi}_{\alpha,\beta}, \overline{\mu}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta})$ *be the solution to the state system* (4.1)–(4.5) *with* $\alpha, \beta > 0$ *and coefficients* $(\mathcal{P}_{\alpha,\beta}, \chi_{\alpha,\beta}, \eta_{\alpha,\beta}, \mathcal{C}_{\alpha,\beta})$ *as given by Theorem 4.5. Let also* $(p_{\alpha,\beta}, q_{\alpha,\beta}, r_{\alpha,\beta})$ *be the unique solution to the adjoint system* (6.21)–(6.25) *with* $\alpha, \beta > 0$*, coefficients* $(\mathcal{P}_{\alpha,\beta}, \chi_{\alpha,\beta}, \eta_{\alpha,\beta}, \mathcal{C}_{\alpha,\beta})$*, and forcing term in* (6.21) *given by* $b_1(\overline{\varphi}_{\alpha,\beta} - \varphi_Q) + (\overline{\varphi}_{\alpha,\beta} - \overline{\varphi}_\alpha)$*. Then, there exist a triplet* $(p_\alpha, q_\alpha, r_\alpha)$*, with*

$$p_\alpha, r_\alpha \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \quad q_\alpha \in L^2(0,T;H),$$

such that, for every $\kappa \geq 1$ *if* $d = 2$ *and* $1 \leq \kappa < 6$ *if* $d = 3$*, it holds that, as* $\beta \to 0$,

$$\begin{aligned}
p_{\alpha,\beta} \to p_\alpha \qquad & \textit{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \\
& \quad \textit{strongly in } C^0([0,T];L^\kappa(\Omega)) \cap L^2(0,T;V), \\
q_{\alpha,\beta} \to q_\alpha \qquad & \textit{weakly in } L^2(0,T;H), \\
r_{\alpha,\beta} \to r_\alpha \qquad & \textit{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \\
& \quad \textit{strongly in } C^0([0,T];L^\kappa(\Omega)) \cap L^2(0,T;V), \\
\beta q_{\alpha,\beta} \to 0 \qquad & \textit{strongly in } H^1(0,T;L^1(\Omega)) \cap L^2(0,T;H).
\end{aligned}$$

Moreover, $(p_\alpha, q_\alpha, r_\alpha)$ *is the unique weak solution to the adjoint system* (6.21)–(6.25) *with* $\beta = 0$ *and coefficients* $(\mathcal{P}_\alpha, \chi_\alpha, \eta_\alpha, \mathcal{C}_\alpha)$*, in the sense that*

$$\begin{aligned}
&- \int_\Omega \partial_t p_\alpha v + \int_\Omega (a q_\alpha - J * q_\alpha) v + \eta_\alpha \int_\Omega \Delta r_\alpha v + \int_\Omega F''(\overline{\varphi}_\alpha) q_\alpha v \\
&\qquad + \int_\Omega \mathcal{C}_\alpha \overline{\sigma}_\alpha f'(\overline{\varphi}_\alpha) r_\alpha v - \int_\Omega \mathcal{P}_\alpha \overline{\sigma}_\alpha f'(\overline{\varphi}_\alpha) p_\alpha v = \int_\Omega b_1(\overline{\varphi}_\alpha - \varphi_Q) v, \\
&- \alpha \int_\Omega \partial_t p_\alpha w + \int_\Omega \nabla p_\alpha \cdot \nabla w - \int_\Omega q_\alpha w = 0, \\
&- \int_\Omega \partial_t r_\alpha z + \int_\Omega \nabla r_\alpha \cdot \nabla z + \int_\Omega \mathcal{C}_\alpha f(\overline{\varphi}_\alpha) r_\alpha z - \int_\Omega \mathcal{P}_\alpha f(\overline{\varphi}_\alpha) p_\alpha z - \int_\Omega \chi_\alpha q_\alpha z = 0,
\end{aligned}$$

for every $v, w, z \in V$*, almost everywhere in* $(0,T)$*, and*

$$p_\alpha(T) = 0, \qquad r_\alpha(T) = 0.$$

Proof of Theorem 6.19. We proceed by pointing out some a priori estimates on the adjoint variables uniformly with respect to β, using again the estimate (6.32) as a starting point. First of all, by Theorem 4.12 we have that $\{F(\overline{\varphi}_{\alpha,\beta})\}_\beta$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$, so that by **D7** we have that $\{F''(\overline{\varphi}_{\alpha,\beta})\}_\beta$ is uniformly bounded in $L^\infty(0,T;H)$. Secondly, by the assumption on the kernel J, the Young inequality, and by comparison in equation (6.22) we have

$$\begin{aligned}
&\int_{Q_t^T} (J * q_{\alpha,\beta}) q_{\alpha,\beta} \\
&\quad \le \int_t^T \|J * q_{\alpha,\beta}\|_V \|q_{\alpha,\beta}\|_{V^*} \le (a^* + b^*) \int_t^T \|q_{\alpha,\beta}\| \|q_{\alpha,\beta}\|_{V^*} \\
&\quad \le \delta \int_{Q_t^T} |q_{\alpha,\beta}|^2 + \frac{(a^*+b^*)^2}{2\delta} \int_t^T \|p_{\alpha,\beta}\|_V^2 + \frac{(a^*+b^*)^2}{2\delta} \alpha^2 \int_{Q_t^T} |\partial_t p_{\alpha,\beta}|^2
\end{aligned}$$

while the last two terms on the right-hand side of (6.32) can be bounded as

$$\begin{aligned}
&- \int_{Q_t^T} \eta_{\alpha,\beta} \Delta r_{\alpha,\beta} q_{\alpha,\beta} - \int_{Q_t^T} \chi_{\alpha,\beta} q_{\alpha,\beta} (\partial_t r_{\alpha,\beta} + \Delta r_{\alpha,\beta}) \\
&\quad \le \delta \int_{Q_t^T} |q_{\alpha,\beta}|^2 + \frac{3({\eta_{\alpha,\beta}}^2 + {\chi_{\alpha,\beta}}^2)}{4\delta} \int_{Q_t^T} |\Delta r_{\alpha,\beta}|^2 + \frac{3{\chi_{\alpha,\beta}}^2}{4\delta} \int_{Q_t^T} |\partial_t r_{\alpha,\beta}|^2.
\end{aligned}$$

Hence, recalling that $\{\overline{\sigma}_{\alpha,\beta}\}_\beta$ is uniformly bounded in $L^2(0,T;V)$, all the terms in (6.32) can be rearranged and treated by the Gronwall lemma, provided to fix $\delta, \delta' > 0$ such that

$$3\delta < C_0, \quad \frac{(a^*+b^*)^2}{2\delta} \alpha < 1, \quad \frac{3({\eta_{\alpha,\beta}}^2 + {\chi_{\alpha,\beta}}^2)}{4\delta} < 1, \quad \delta' < 1 - \frac{3({\eta_{\alpha,\beta}}^2 + {\chi_{\alpha,\beta}}^2)}{4\delta}.$$

An easy computation shows that this is possible if and only if

$$\max \left\{ \frac{(a^*+b^*)^2}{2} \alpha, \frac{3({\eta_{\alpha,\beta}}^2 + {\chi_{\alpha,\beta}}^2)}{4} \right\} < \frac{C_0}{3}$$

which is indeed true by (4.6) and the fact that $\eta_{\max}^2 + \chi_{\max}^2 < \frac{4}{9} C_0$. Hence, (6.32) can be closed uniformly in β, and we obtain

$$\begin{aligned}
&\|p_{\alpha,\beta}\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} + \beta^{1/2} \|q_{\alpha,\beta}\|_{L^\infty(0,T;H)} \\
&\quad + \|q_{\alpha,\beta}\|_{L^2(0,T;H)} + \|r_{\alpha,\beta}\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} \le C_\alpha
\end{aligned}$$

for a positive constant C_α which may depend on α but it is independent of β. Then, elliptic regularity theory and (6.22) lead us to infer that

$$\|p_{\alpha,\beta}\|_{L^2(0,T;W)} \le C_\alpha.$$

Moreover, noting that by the Hölder inequality we have that

$$\|F''(\overline{\varphi}_{\alpha,\beta}) q_{\alpha,\beta}\|_{L^2(0,T;L^1(\Omega))} \le C_\alpha,$$

arguing as in the proof of Theorem 6.13, by a comparison argument in (6.21) we deduce that

$$\beta \|q_{\alpha,\beta}\|_{H^1(0,T;L^1(\Omega))} \le C_\alpha.$$

Recalling the continuous embedding $W \subset L^\infty(\Omega)$, Banach–Alaoglu theorem and standard compactness results allow us to obtain from the above a priori estimates that there exist functions $(p_\alpha, q_\alpha, r_\alpha)$ such that it holds, along a non-relabeled subsequence, as $\beta \to 0$,

$$\begin{aligned}
p_{\alpha,\beta} \to p_\alpha &\quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),\\
q_{\alpha,\beta} \to q_\alpha &\quad \text{weakly in } L^2(0,T;H),\\
r_{\alpha,\beta} \to r_\alpha &\quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),\\
&\quad\quad \text{strongly in } C^0([0,T];L^\kappa(\Omega)) \cap L^2(0,T;V),\\
\beta q_{\alpha,\beta} \to 0 &\quad \text{weakly in } H^1(0,T;W^*) \text{ and strongly in } L^2(0,T;H),
\end{aligned}$$

for every $\kappa \geq 1$ if $d = 2$ and $1 \leq \kappa < 6$ if $d = 3$. Arguing as in the proof of Theorem 6.13, we exploit the above convergences to pass to the limit in the variational formulation of the adjoint system given by (6.38)–(6.40) and in the terminal conditions (6.41). As a by-product, we obtain that the above limits are a weak solution to (6.21)–(6.25) with $\beta = 0$. To this end, note that by (4.93) and (4.99)–(4.100), along a non-relabelled subsequence, for every $\kappa \geq 1$ if $d = 2$ and $1 \leq \kappa < 6$ if $d = 3$, we have that

$$\overline{\varphi}_{\alpha,\beta} \to \overline{\varphi}_\alpha \quad \text{a.e. in } Q, \quad \overline{\varphi}_{\alpha,\beta} \to \overline{\varphi}_\alpha, \quad \overline{\sigma}_{\alpha,\beta} \to \overline{\sigma}_\alpha \quad \text{strongly in } L^2(0,T;L^\kappa(\Omega)).$$

Consequently, all terms in (6.38)–(6.40) and (6.41) pass to the weak limit as $\beta \to 0$. The only delicate term to treat, as usual, is the one containing F'': let us spend a few words on this. By continuity of F'' it follows that, as $\beta \to 0$,

$$F''(\overline{\varphi}_{\alpha,\beta}) \to F''(\overline{\varphi}_\alpha) \quad \text{a.e. in } Q.$$

Now, since $\{F(\overline{\varphi}_{\alpha,\beta})\}_\beta$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$, by **D7** we know that $\{F''(\overline{\varphi}_{\alpha,\beta})\}_\beta$ is uniformly bounded in $L^\infty(0,T;H)$. Furthermore, the boundedness of $\{\varphi_{\alpha,\beta}\}_\beta$ in $L^2(0,T;L^6(\Omega))$ and again **D7** ensure also that $\{F''(\overline{\varphi}_{\alpha,\beta})\}_\beta$ is uniformly bounded in $L^1(0,T;L^3(\Omega))$. For any $\vartheta \in (0,1)$, setting $\kappa_\vartheta \in (2,3)$ such that $\frac{1}{\kappa_\vartheta} := \frac{\vartheta}{2} + \frac{1-\vartheta}{3}$, by interpolation we have that

$$\|F''(\overline{\varphi}_{\alpha,\beta})\|_{\kappa_\vartheta} \leq C\|F''(\overline{\varphi}_{\alpha,\beta})\|^\vartheta \|F''(\overline{\varphi}_{\alpha,\beta})\|_3^{1-\vartheta} \quad \text{a.e. in } (0,T),$$

from which it follows that

$$\|F''(\overline{\varphi}_{\alpha,\beta})\|_{L^{\frac{1}{1-\vartheta}}(0,T;L^{\kappa_\vartheta}(\Omega))} \leq C_\alpha.$$

In particular, there exists $\overline{\vartheta} \in (0,1)$ such that $\overline{\kappa} := \kappa_{\overline{\vartheta}} = \frac{1}{1-\overline{\vartheta}} \in (2,3)$: an easy computation yields $\overline{\vartheta} = \frac{4}{7}$ and $\overline{\kappa} = \frac{7}{3}$. This implies that

$$\|F''(\overline{\varphi}_{\alpha,\beta})\|_{L^{7/3}(Q)} \leq C_\alpha.$$

By the Severini–Egorov theorem we infer that, for all $\kappa \in [1, \frac{7}{3})$,

$$F''(\overline{\varphi}_{\alpha,\beta}) \to F''(\overline{\varphi}_\alpha) \quad \text{weakly in } L^{7/3}(Q) \text{ and strongly in } L^\kappa(Q).$$

In particular, since $\frac{7}{3} > 2$, this implies that, as $\beta \to 0$,

$$F''(\overline{\varphi}_{\alpha,\beta})q_{\alpha,\beta} \to F''(\overline{\varphi}_\alpha)q_\alpha \quad \text{weakly in } L^1(Q).$$

Since $W \subset L^\infty(\Omega)$, this allows to pass to the limit as $\beta \to 0$ in (6.38) for every test function $v \in W$. Since $F''(\overline{\varphi}_\alpha) \in L^1(0,T;L^3(\Omega))$, at the limit we have that $F''(\overline{\varphi}_\alpha)q_\alpha \in L^{6/5}(\Omega) \subset V^*$ almost everywhere in $(0,T)$, and the variational formulation holds also for all $v \in V$ by the density of W in V.

Finally, by linearity of the system, the same estimates yield also uniqueness of $(p_\alpha, q_\alpha, r_\alpha)$, and the convergences hold along the entire sequence $\beta \to 0$, as desired. □

6.2.3.2 Letting $\beta \to 0$ in the Optimality Condition

Lastly, we argue as in Theorem 6.14 to pass to the limit as $\beta \to 0$ to establish the necessary conditions for optimality of $(CP)_\alpha$.

Theorem 6.20. *Assume* **B1–B7**, **D1–D7**. *Then, every minimiser* $(\overline{\varphi}_\alpha, \overline{\mathcal{P}}_\alpha, \overline{\chi}_\alpha, \overline{\eta}_\alpha, \overline{\mathcal{C}}_\alpha)$ *of* $(CP)_\alpha$ *necessarily satisfies*

$$\begin{aligned}&\int_Q (\mathcal{P} - \overline{\mathcal{P}}_\alpha)\overline{\sigma}_\alpha f(\overline{\varphi}_\alpha)p_\alpha - \int_Q (\chi - \overline{\chi}_\alpha)\overline{\sigma}_\alpha q_\alpha - \int_Q (\mathcal{C} - \overline{\mathcal{C}}_\alpha)\overline{\sigma}_\alpha f(\overline{\varphi}_\alpha)r_\alpha \\ &\quad + \alpha_P(\overline{\mathcal{P}}_\alpha - \mathcal{P}_*)(\mathcal{P} - \overline{\mathcal{P}}_\alpha) + \alpha_\chi(\overline{\chi}_\alpha - \chi_*)(\chi - \overline{\chi}_\alpha) \\ &\quad + \alpha_\eta(\overline{\eta}_\alpha - \eta_*)(\eta - \overline{\eta}_\alpha) + \alpha_C(\overline{\mathcal{C}}_\alpha - \mathcal{C}_*)(\mathcal{C} - \overline{\mathcal{C}}_\alpha) \geq 0 \quad \forall (\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathcal{U}_{\text{ad}},\end{aligned}$$

where $(\overline{\varphi}_\alpha, \overline{\mu}_\alpha, \overline{\sigma}_\alpha)$ *is a solution to the state system* (4.1)–(4.5) *and and* $(p_\alpha, q_\alpha, r_\alpha)$ *is the unique weak solution to the adjoint system* (6.21)–(6.25) *with* $\beta = 0$ *and coefficients* $(\overline{\mathcal{P}}_\alpha, \overline{\chi}_\alpha, \overline{\eta}_\alpha, \overline{\mathcal{C}}_\alpha)$*, in the sense of Theorems 4.12 and Theorem 6.19, respectively.*

Proof of Theorem 6.20. The proof is analogous to the one of Theorem 6.14, using Lemma 6.16 and 6.17, and Theorems 4.12, 6.18, and 6.19 instead. □

6.2.4 The Optimisation Problem $(CP)_{\alpha,\beta}$ as $\alpha, \beta \to 0$

In this last section we deal with the optimisation problem $(\overline{CP})$, by letting $\alpha, \beta \to 0$ jointly. Since most of the ideas have already been explained and motivated in detail in the previous Sections 6.2.2 and 6.2.3, we proceed here more quickly, avoiding technical details for brevity. Throughout the whole Section 6.2.4, we assume the following framework:

$$\begin{aligned}&\eta_{\max} = \alpha_{\max} = 0, \quad \varphi_\Omega \in V, \quad (4.111)\text{–}(4.113), \\ &\chi_{\max} < \sqrt{c_a}, \quad (\chi_{\max} + \eta_{\max} + 4c_a\chi_{\max})^2 < 8c_aC_0, \quad \eta_{\max}^2 + \chi_{\max}^2 < \frac{4}{9}C_0.\end{aligned}$$

As in Section 6.2.2, since $\eta_{\max} = 0$, we shall consider $\mathcal{U}_{\text{ad}}$ as a subset of $\mathbb{R}^3$ instead, and write $(\mathcal{P}, \chi, \mathcal{C}) \in \mathcal{U}_{\text{ad}}$ for the generic admissible control. As usual, the first result concerns existence of optimal controls for $(\overline{CP})$ which is given in the following result.

Theorem 6.21. *Assume* **B1–B7***,* **D1–D7***. Then, the optimisation problem* $(\overline{CP})$ *admits a solution.*

Proof of Theorem 6.21. This result follows directly by adapting the direct method used in the proof of Theorem 6.3, taking into account the compactness of $\mathcal{U}_{\rm ad}$ and the convergence presented in Theorem 4.15. □

Now, we investigate the necessary conditions for optimality. First of all, note that for every admissible control $(\mathcal{P}, \chi, \mathcal{C})$, the state system (4.1)–(4.5) with $\alpha = \beta = 0$ admits a unique solution by Theorem 4.15. Consequently, when introducing the adapted cost functional, by contrast with Section 6.2.3, it is not necessary here to use a perturbation with respect to the phase variable.

Nevertheless, looking at the final condition (6.25) and taking formally $\alpha = \beta = 0$, we immediately see that $p_{\alpha,\beta}(T) = 0$ for all $\alpha \in (0, \alpha_0)$ while at the limit $p(T) = b_2(\overline{\varphi}(T) - \varphi_\Omega)$. This immediately suggests that if $b_2 > 0$, we can not pass to the joint limit $\alpha, \beta \to 0$ in the adjoint problem (6.21)–(6.25) as it is. At an intuitive level for the moment, the limit adjoint problem (6.21)–(6.25) with $\alpha = \beta = 0$ is still well-posed also when $b_2 > 0$. These heuristic considerations suggest that the right assumption is to keep a generic $b_2 \geq 0$, but to modify the final condition for $p_{\alpha,\beta}$ at the approximate level in a smart way, in order to recover the compatibility condition $p_{\alpha,\beta}(T) = b_2(\overline{\varphi}(T) - \varphi_\Omega)$ also when $\alpha, \beta > 0$. In order to do this, we introduce a correction in the adapted cost functional, depending on the terminal values of both the variables φ and μ.

For a given optimal control $(\overline{\mathcal{P}}, \overline{\chi}, \overline{\mathcal{C}})$ of $(\overline{CP})$ and for all $\alpha \in (0, \alpha_0)$ and $\beta \in (0, \beta_0)$, the idea is to set

$$\begin{aligned} \mathcal{J}_{\rm ad}(\varphi, \mu, \mathcal{P}, \chi, \mathcal{C}) := \mathcal{J}(\varphi, \mathcal{P}, \chi, \mathcal{C}) &+ (\alpha\mu(T), b_2(\varphi(T) - \varphi_\Omega)) \\ &+ \tfrac{1}{2}|\mathcal{P} - \overline{\mathcal{P}}|^2 + \tfrac{1}{2}|\chi - \overline{\chi}|^2 + \tfrac{1}{2}|\mathcal{C} - \overline{\mathcal{C}}|^2 \end{aligned}$$

and define the adapted optimisation problem $(CP)^{\rm ad}_{\alpha,\beta}$ as in (6.33).

It is clear that the optimality condition for $(\overline{CP})$ follows similar lines of the previous sections, by firstly obtaining the approximating sequence of optimal controls of minimisers of $(CP)^{\rm ad}_{\alpha,\beta}$ and then pass to the limit as $\alpha, \beta \to 0$. The major difference is the nature of the correction in the adapted cost functional, which yields a correction in the terminal values of the adapted adjoint system at $\alpha, \beta > 0$: for this reason, we recall such corrected adjoint system explicitly in Lemma 6.23 below. The proof of first-order conditions for optimality at the level $\alpha, \beta > 0$ follows, *mutatis mutandis*, the proof of Theorem 6.7, and is omitted for brevity.

The following results concern existence of optimal controls and necessary conditions for $(CP)^{\rm ad}_{\alpha,\beta}$, and the convergence $(CP)^{\rm ad}_{\alpha,\beta} \searrow (\overline{CP})$ as $\alpha, \beta \to 0$.

Lemma 6.22. *Assume* **B1–B7***,* **D1–D7***. Then, for every optimal control* $(\overline{\mathcal{P}}, \overline{\chi}, \overline{\mathcal{C}})$ *for* $(\overline{CP})$*, for every* $\alpha \in (0, \alpha_0)$ *and* $\beta \in (0, \beta_0)$ *the optimisation problem* $(CP)^{\rm ad}_{\alpha,\beta}$ *admits a solution.*

Lemma 6.23. *Assume* **B1–B7**, **D1–D7**, *and let* $(\overline{\mathcal{P}}, \overline{\chi}, \overline{\mathcal{C}})$ *be an optimal control for* $(\overline{CP})$. *For every* $\alpha \in (0, \alpha_0)$ *and* $\beta \in (0, \beta_0)$, *if* $(\overline{\mathcal{P}}_{\alpha,\beta}, \overline{\chi}_{\alpha,\beta}, \overline{\mathcal{C}}_{\alpha,\beta})$ *is an optimal control of* $(CP)^{\mathrm{ad}}_{\alpha,\beta}$*, then the following first-order necessary condition holds*

$$\begin{aligned}
&\int_Q (\mathcal{P} - \overline{\mathcal{P}}_{\alpha,\beta})\overline{\sigma}_{\alpha,\beta} f(\overline{\varphi}_{\alpha,\beta}) p_{\alpha,\beta} - \int_Q (\chi - \overline{\chi}_{\alpha,\beta})\overline{\sigma}_{\alpha,\beta} q_{\alpha,\beta} \\
&\quad - \int_Q (\mathcal{C} - \overline{\mathcal{C}}_{\alpha,\beta})\overline{\sigma}_{\alpha,\beta} f(\overline{\varphi}_{\alpha,\beta}) r_{\alpha,\beta} \\
&\quad + (\mathcal{P} - \overline{\mathcal{P}}_{\alpha,\beta})\big(\alpha_P(\overline{\mathcal{P}}_{\alpha,\beta} - \mathcal{P}_*) + (\overline{\mathcal{P}}_{\alpha,\beta} - \overline{\mathcal{P}})\big) \\
&\quad + (\chi - \overline{\chi}_{\alpha,\beta})\big(\alpha_\chi(\overline{\chi}_{\alpha,\beta} - \chi_*) + (\overline{\chi}_{\alpha,\beta} - \overline{\chi})\big) \\
&\quad + (\mathcal{C} - \overline{\mathcal{C}}_{\alpha,\beta})\big(\alpha_C(\overline{\mathcal{C}}_{\alpha,\beta} - \mathcal{C}_*) + (\overline{\mathcal{C}}_{\alpha,\beta} - \overline{\mathcal{C}})\big) \geq 0 \quad \forall (\mathcal{P}, \chi, \mathcal{C}) \in \mathcal{U}_{\mathrm{ad}},
\end{aligned} \tag{6.45}$$

where $(\overline{\varphi}_{\alpha,\beta}, \overline{\mu}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta})$ *is the unique solution to* (4.1)–(4.5) *with* $\alpha, \beta > 0$ *and parameters* $(\overline{\mathcal{P}}_{\alpha,\beta}, \overline{\chi}_{\alpha,\beta}, \overline{\mathcal{C}}_{\alpha,\beta})$, *and* $(p_{\alpha,\beta}, q_{\alpha,\beta}, r_{\alpha,\beta})$ *is the unique solution to the following adapted adjoint system*

$$\begin{aligned}
&- \partial_t(p_{\alpha,\beta} + \beta q_{\alpha,\beta}) + a q_{\alpha,\beta} - J * q_{\alpha,\beta} + F''(\overline{\varphi}_{\alpha,\beta}) q_{\alpha,\beta} \\
&\qquad + \overline{\mathcal{C}}\overline{\sigma} f'(\overline{\varphi}_{\alpha,\beta}) r_{\alpha,\beta} - (\overline{\mathcal{P}}\overline{\sigma}_{\alpha,\beta} - \mathcal{A}) f'(\overline{\varphi}_{\alpha,\beta}) p_{\alpha,\beta} \\
&\quad = b_1(\overline{\varphi}_{\alpha,\beta} - \varphi_Q) && \text{in } Q, \qquad (6.46) \\
&- \alpha \partial_t p_{\alpha,\beta} - \Delta p_{\alpha,\beta} - q_{\alpha,\beta} = 0 && \text{in } Q, \qquad (6.47) \\
&- \partial_t r_{\alpha,\beta} - \Delta r_{\alpha,\beta} + (\mathcal{B} + \overline{\mathcal{C}} f(\overline{\varphi}_{\alpha,\beta})) r_{\alpha,\beta} - \overline{\mathcal{P}} f(\overline{\varphi}_{\alpha,\beta}) p_{\alpha,\beta} \\
&\quad - \overline{\chi} q_{\alpha,\beta} = 0 && \text{in } Q, \qquad (6.48) \\
&\partial_{\mathbf{n}} p_{\alpha,\beta} = \partial_{\mathbf{n}} r_{\alpha,\beta} = 0 && \text{on } \Sigma, \qquad (6.49) \\
&\alpha p_{\alpha,\beta}(T) = \alpha, \beta_2(\overline{\varphi}_{\alpha,\beta}(T) - \varphi_\Omega), \\
&\quad (p_{\alpha,\beta} + \beta q_{\alpha,\beta})(T) = b_2(\overline{\varphi}_{\alpha,\beta}(T) - \varphi_\Omega) + \alpha, \beta_2 \overline{\mu}_{\alpha,\beta}(T), \\
&\quad r_{\alpha,\beta}(T) = 0 && \text{in } \Omega. \qquad (6.50)
\end{aligned}$$

Theorem 6.24. *Assume* **B1–B7**, **D1–D7**. *Let* $(\overline{\mathcal{P}}, \overline{\chi}, \overline{\mathcal{C}})$ *be an optimal control for* $(\overline{CP})$, *with corresponding state* $(\overline{\varphi}, \overline{\mu}, \overline{\sigma})$. *Then, for every family* $\{(\overline{\mathcal{P}}_{\alpha,\beta}, \overline{\chi}_{\alpha,\beta}, \overline{\mathcal{C}}_{\alpha,\beta})\}_{\alpha,\beta}$ *of optimal controls for* $(CP)^{\mathrm{ad}}_{\alpha,\beta}$, *with corresponding states* $\{(\overline{\varphi}_{\alpha,\beta}, \overline{\mu}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta})\}_{\alpha,\beta}$, *as* $\alpha, \beta \to 0$ *it holds that, for every* $\kappa \geq 1$ *if* $d = 2$ *and* $1 \leq \kappa < 6$ *if* $d = 3$,

$$\begin{aligned}
&\overline{\varphi}_{\alpha,\beta} \to \overline{\varphi} \quad \text{weakly}^* \text{ in } L^\infty(0,T;H) \cap L^2(0,T;V), \\
&\qquad\qquad \text{strongly in } C^0([0,T];V^*) \cap L^2(0,T;L^\kappa(\Omega)), \\
&\overline{\mathcal{P}}_{\alpha,\beta} \to \overline{\mathcal{P}}_\beta, \quad \overline{\chi}_{\alpha,\beta} \to \overline{\chi}_\beta, \quad \overline{\mathcal{C}}_{\alpha,\beta} \to \overline{\mathcal{C}}_\beta, \\
&\mathcal{J}_{\mathrm{ad}}(\overline{\varphi}_{\alpha,\beta}, \overline{\mathcal{P}}_{\alpha,\beta}, \overline{\chi}_{\alpha,\beta}, \overline{\mathcal{C}}_{\alpha,\beta}) \to \mathcal{J}(\overline{\varphi}, \overline{\mathcal{P}}, \overline{\chi}, \overline{\mathcal{C}}).
\end{aligned}$$

Proof of Theorem 6.24. The proof is analogous to Theorem 6.12, by using the convergences of Theorem 4.15. □

6.2.4.1 Letting $\alpha, \beta \to 0$ in the Adjoint System

We focus here on the passage to the limit as $\alpha, \beta \to 0$ in the adjoint system (6.21)–(6.25).

Theorem 6.25. *Assume* **B1–B7**, **D1–D7**. *Let* $(\mathcal{P}, \chi, \mathcal{C}) \in \mathcal{U}_{\text{ad}}$, $\{(\mathcal{P}_{\alpha,\beta}, \chi_{\alpha,\beta}, \mathcal{C}_{\alpha,\beta})\}_{\alpha,\beta} \subset \mathcal{U}_{\text{ad}}$ *be such that* $(\mathcal{P}_{\alpha,\beta}, \chi_{\alpha,\beta}, \mathcal{C}_{\alpha,\beta}) \to (\mathcal{P}, \chi, \mathcal{C})$ *as* $\alpha, \beta \to 0$. *Let* $(\overline{\varphi}, \overline{\mu}, \overline{\sigma})$ *and* $(\overline{\varphi}_{\alpha,\beta}, \overline{\mu}_{\alpha,\beta}, \overline{\sigma}_{\alpha,\beta})$ *be the unique solutions to the state system* (4.1)–(4.5) *in the cases* $\alpha = \beta = 0$ *with coefficients* $(\mathcal{P}, \chi, \mathcal{C})$, *and* $\alpha \in (0, \alpha_0)$ *and* $\beta \in (0, \beta_0)$ *with coefficients* $(\mathcal{P}_{\alpha,\beta}, \chi_{\alpha,\beta}, \mathcal{C}_{\alpha,\beta})$, *as given by Theorems 4.15 and 4.5, respectively. Let also* $(p_{\alpha,\beta}, q_{\alpha,\beta}, r_{\alpha,\beta})$ *be the unique solution to the adapted adjoint system* (6.46)–(6.50) *with* $\alpha \in (0, \alpha_0)$, $\beta \in (0, \beta_0)$, *and coefficents* $(\mathcal{P}_{\alpha,\beta}, \chi_{\alpha,\beta}, \mathcal{C}_{\alpha,\beta})$, *as given by Theorem 6.6. Then, there exists a triplet* (p, q, r), *with*

$$
\begin{aligned}
&p \in H^1(0,T;V^*) \cap L^2(0,T;W),\\
&q \in L^2(0,T;H),\\
&r \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),
\end{aligned}
$$

such that, for every $\kappa \geq 1$ *if* $d = 2$ *and* $1 \leq \kappa < 6$ *if* $d = 3$, *along any sequence*

$$(\alpha, \beta) \to (0,0) \quad \textit{such that} \quad \limsup_{(\alpha,\beta)\to(0,0)} \frac{\alpha}{\beta} < +\infty,$$

it holds

$$
\begin{aligned}
p_{\alpha,\beta} &\to p && \textit{weakly in } H^1(0,T;V^*) \cap L^2(0,T;W),\\
q_{\alpha,\beta} &\to q && \textit{weakly in } L^2(0,T;H),\\
r_{\alpha,\beta} &\to r && \textit{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),\\
& && \quad \textit{strongly in } C^0([0,T];L^\kappa(\Omega)) \cap L^2(0,T;H),\\
\alpha p_{\alpha,\beta} &\to 0 && \textit{strongly in } H^1(0,T;H) \cap L^\infty(0,T;V^*),\\
\beta q_{\alpha,\beta} &\to 0 && \textit{strongly in } L^\infty(0,T;V).
\end{aligned}
$$

Moreover, (p, q, r) *is the unique weak solution to the adjoint system* (6.21)–(6.25) *with* $\alpha = \beta = 0$ *and coefficients* $(\mathcal{P}, \eta, \mathcal{C})$, *in the sense that*

$$
\begin{aligned}
&-\langle \partial_t p, v\rangle_V + \int_\Omega (aq - J*q + F''(\overline{\varphi}))v + \int_\Omega \mathcal{C}\overline{\sigma} f'(\overline{\varphi}) r v\\
&\qquad - \int_\Omega \mathcal{P}\overline{\sigma} f'(\overline{\varphi}) p v = \int_\Omega b_1(\overline{\varphi} - \varphi_Q) v,\\
&\int_\Omega \nabla p \cdot \nabla v - \int_\Omega q v = 0,\\
&-\int_\Omega \partial_t r v + \int_\Omega \nabla r \cdot \nabla v + \int_\Omega \mathcal{C} f(\overline{\varphi}) r v - \int_\Omega \mathcal{P} f(\overline{\varphi}) p v - \int_\Omega \chi q v = 0,
\end{aligned}
$$

for every $v \in V$, *almost everywhere in* $(0,T)$, *and*

$$p(T) = b_2(\overline{\varphi}(T) - \varphi_\Omega), \qquad r(T) = 0.$$

Proof of Theorem 6.25. We perform on the adapted adjoint system (6.46)–(6.50) the same first estimate of the proof of Theorem 6.6, getting

$$
\begin{aligned}
&\frac{\beta}{2}\|q_{\alpha,\beta}(t)\|^2 + C_0\int_{Q_t^T}|q_{\alpha,\beta}|^2 + \alpha\int_{Q_t^T}|\partial_t p_{\alpha,\beta}|^2 + \frac{\alpha}{2}\|p_{\alpha,\beta}(t)\|^2 + \frac{1}{2}\|\nabla p_{\alpha,\beta}(t)\|^2\\
&\quad + \int_{Q_t^T}|\nabla p_{\alpha,\beta}|^2 + \frac{\mathcal{B}+1}{2}\|r_{\alpha,\beta}(t)\|^2 + \|\nabla r_{\alpha,\beta}(t)\|^2 + \mathcal{B}\int_{Q_t^T}|\nabla r_{\alpha,\beta}|^2\\
&\quad + \int_{Q_t^T}|\partial_t r_{\alpha,\beta}|^2 + \int_{Q_t^T}|\Delta r_{\alpha,\beta}|^2\\
&\leq \frac{\beta}{2}\|q_{\alpha,\beta}(T)\|^2 + \frac{\alpha}{2}\|p_{\alpha,\beta}(T)\|^2 + \frac{1}{2}\|\nabla p_{\alpha,\beta}(T)\|^2 + \int_{Q_t^T} b_1(\overline{\varphi}_{\alpha,\beta} - \varphi_Q)q_{\alpha,\beta}\\
&\quad + \int_{Q_t^T}(J * q_{\alpha,\beta})q_{\alpha,\beta} - \int_{Q_t^T}\overline{\mathcal{C}}_{\alpha,\beta}\overline{\sigma} f'(\overline{\varphi}_{\alpha,\beta})r_{\alpha,\beta}q_{\alpha,\beta}\\
&\quad + \int_{Q_t^T}(\overline{\mathcal{P}}_{\alpha,\beta}\overline{\sigma}_{\alpha,\beta} - \mathcal{A})f'(\overline{\varphi}_{\alpha,\beta})p_{\alpha,\beta}q_{\alpha,\beta} + \int_{Q_t^T} q_{\alpha,\beta}p_{\alpha,\beta}\\
&\quad + \int_{Q_t^T}\overline{\mathcal{C}}_{\alpha,\beta} f(\overline{\varphi}_{\alpha,\beta})r_{\alpha,\beta}(\partial_t r_{\alpha,\beta} + \Delta r_{\alpha,\beta})\\
&\quad - \int_{Q_t^T}\overline{\mathcal{P}}_{\alpha,\beta} f(\overline{\varphi}_{\alpha,\beta})p_{\alpha,\beta}(\partial_t r_{\alpha,\beta} + \Delta r_{\alpha,\beta})\\
&\quad - \int_{Q_t^T}\overline{\chi}_{\alpha,\beta}q_{\alpha,\beta}(\partial_t r_{\alpha,\beta} + \Delta r_{\alpha,\beta}) - \int_{Q_t^T} r_{\alpha,\beta}\partial_t r_{\alpha,\beta}.
\end{aligned}
$$

We show only how to bound the first three terms on the right-hand side, as all the other terms on the right-hand side can be treated exactly in the same way as in Section 6.2.1 and in the proof of Theorem 6.19, using Theorem 4.15. To this end, taking into account the modified terminal conditions (6.50) we have

$$
p_{\alpha,\beta}(T) = b_2(\overline{\varphi}_{\alpha,\beta}(T) - \varphi_\Omega) \in V, \qquad q_{\alpha,\beta}(T) = \frac{\alpha}{\beta} b_2\overline{\mu}_{\alpha,\beta}(T).
$$

Then, it follows that

$$
\begin{aligned}
&\frac{\beta}{2}\|q_{\alpha,\beta}(T)\|^2 + \frac{\alpha}{2}\|p_{\alpha,\beta}(T)\|^2 + \frac{1}{2}\|\nabla p_{\alpha,\beta}(T)\|^2\\
&\quad = \frac{b_2^2}{2}\frac{\alpha^2}{\beta}\|\overline{\mu}_{\alpha,\beta}(T)\|^2 + \frac{\alpha}{2}\|b_2(\overline{\varphi}_{\alpha,\beta}(T) - \varphi_\Omega)\|^2 + \frac{1}{2}\|\nabla b_2(\overline{\varphi}_{\alpha,\beta}(T) - \varphi_\Omega)\|^2.
\end{aligned}
$$

Now, by Theorem 4.15 we have that $\{\alpha^{1/2}\overline{\mu}_{\alpha,\beta}\}_{\alpha,\beta}$ is uniformly bounded in $C^0([0,T];H)$: hence, thanks also to the regularity $\varphi_\Omega \in V$, we deduce that there exists $C > 0$, independent of α, β, such that

$$
\frac{\beta}{2}\|q_{\alpha,\beta}(T)\|^2 + \frac{\alpha}{2}\|p_{\alpha,\beta}(T)\|^2 + \frac{1}{2}\|\nabla p_{\alpha,\beta}(T)\|^2 \leq C(1 + \alpha + \tfrac{\alpha}{\beta}).
$$

Consequently, the scaling $\limsup \frac{\alpha}{\beta} < +\infty$ on (α, β) yields that

$$
\frac{\beta}{2}\|q_{\alpha,\beta}(T)\|^2 + \frac{\alpha}{2}\|p_{\alpha,\beta}(T)\|^2 + \frac{1}{2}\|\nabla p_{\alpha,\beta}(T)\|^2 \leq C.
$$

As we anticipated above, all the remaining terms on the right-hand side can be treated exactly as in Section 6.2.1 and in the proof of Theorem 6.19, so that we infer that there exists a positive constant C, independent of both α and β, such that

$$\alpha^{1/2}\|p_{\alpha,\beta}\|_{H^1(0,T;H)} + \|p_{\alpha,\beta}\|_{L^\infty(0,T;V)} + \beta^{1/2}\|q_{\alpha,\beta}\|_{L^\infty(0,T;H)} + \|q_{\alpha,\beta}\|_{L^2(0,T;H)} + \|r_{\alpha,\beta}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq C.$$

Moreover, elliptic regularity theory and (6.22)–(6.23) leads to

$$\|p_{\alpha,\beta}\|_{L^2(0,T;W)} \leq C,$$

while by comparison in (6.21), as in the proof of Theorem 6.19, we infer that

$$\|p_{\alpha,\beta} + \beta q_{\alpha,\beta}\|_{H^1(0,T;L^1(\Omega))} \leq C.$$

By the usual weak compactness criteria, we infer the existence of functions (p, q, r) such that as $\alpha, \beta \to 0$ it holds, for every $\kappa \geq 1$ if $d = 2$ and $1 \leq \kappa < 6$ if $d = 3$, and along a non-relabelled subsequence,

$$\begin{aligned}
p_{\alpha,\beta} &\to p && \text{weakly in } L^2(0,T;W),\\
q_{\alpha,\beta} &\to q && \text{weakly in } L^2(0,T;H),\\
p_{\alpha,\beta} + \beta q_{\alpha,\beta} &\to p && \text{weakly in } H^1(0,T;W^*),\\
r_{\alpha,\beta} &\to r && \text{weakly star in } H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W),\\
&&& \quad \text{strongly in } C^0([0,T];L^\kappa(\Omega))\cap L^2(0,T;V),\\
\alpha p_{\alpha,\beta} &\to 0 && \text{strongly in } H^1(0,T;H)\cap L^2(0,T;W),\\
\beta q_{\alpha,\beta} &\to 0 && \text{strongly in } L^\infty(0,T;H).
\end{aligned}$$

Moreover, since by Theorem 4.15 we have that, for every $\kappa \geq 1$ if $d = 2$ and $1 \leq \kappa < 6$ if $d = 3$,

$$\overline{\varphi}_{\alpha,\beta} \to \overline{\varphi} \quad \text{strongly in } C^0([0,T];V^*)\cap L^2(0,T;L^\kappa(\Omega)),$$

we can pass to the limit as $\alpha, \beta \to 0$ in the variational formulation (6.38)–(6.41), treating the term with F'' as in the proof of Theorem 6.19, and conclude. The uniqueness of the weak solution (p, q, r) follows from linearity and the estimates already performed. □

6.2.4.2 Letting $\alpha, \beta \to 0$ in the Optimality Condition

Here, we conclude the asymptotic analysis by letting $\alpha, \beta \to 0$ in the optimality condition for $(CP)^{\text{ad}}_{\alpha,\beta}$, and proving the corresponding necessary conditions for $(\overline{CP})$.

Theorem 6.26. *Assume* **B1–B7**, **D1–D7**. *Then, every optimal control* $(\overline{\mathcal{P}}, \overline{\chi}, \overline{\mathcal{C}})$ *of* $(\overline{CP})$ *necessarily satisfies*

$$\begin{aligned}
&\int_Q (\mathcal{P} - \overline{\mathcal{P}})\overline{\sigma} f(\overline{\varphi}) p - \int_Q (\chi - \overline{\chi})\overline{\sigma} q - \int_Q (\mathcal{C} - \overline{\mathcal{C}})\overline{\sigma} f(\overline{\varphi}) r\\
&\quad + \alpha_P(\overline{\mathcal{P}} - \mathcal{P}_*)(\mathcal{P} - \overline{\mathcal{P}}) + \alpha_\chi(\overline{\chi} - \chi_*)(\chi - \overline{\chi})\\
&\quad + \alpha_C(\overline{\mathcal{C}} - \mathcal{C}_*)(\mathcal{C} - \overline{\mathcal{C}}) \geq 0 \quad \forall (\mathcal{P}, \chi, \mathcal{C}) \in \mathcal{U}_{\text{ad}},
\end{aligned}$$

where $(\overline{\varphi}, \overline{\mu}, \overline{\sigma})$ *and* (p, q, r) *are the unique solutions to* (4.1)–(4.5) *and* (6.21)–(6.25) *with* $\alpha = \beta = 0$ *in the sense of Theorems 4.15 and 6.25, respectively.*

Bibliography

[1] R. A. Adams and J. J. F. Fournier. Sobolev spaces, volume 140 of Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] A. Agosti, P. F. Antonietti, P. Ciarletta, M. Grasselli, and M. Verani. A Cahn–Hilliard-type equation with application to tumor growth dynamics. Mathematical Methods in the Applied Sciences, 40(18):7598–7626, 2017.

[3] A. Agosti, C. Cattaneo, C. Giverso, D. Ambrosi, and P. Ciarletta. A computational framework for the personalized clinical treatment of glioblastoma multiforme. ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik, 98(12):2307–2327, 2018.

[4] A. Agosti, P. Ciarletta, H. Garcke, and M. Hinze. Learning patient-specific parameters for a diffuse interface glioblastoma model from neuroimaging data. arXiv preprint arXiv:1912.08036, 2019.

[5] A. Agosti, C. Giverso, E. Faggiano, A. Stamm, and P. Ciarletta. A personalized mathematical tool for neuro-oncology: A clinical case study. International Journal of Non-Linear Mechanics, 107:170–181, 2018.

[6] H. W. Alt. Linear functional analysis. Springer, 1992.

[7] N. J. Armstrong, K. J. Painter, and J. A. Sherratt. A continuum approach to modelling cell-cell adhesion. Journal of Theoretical Biology, 243(1):98 – 113, 2006.

[8] S. Astanin and L. Preziosi. Multiphase Models of Tumour Growth, pages 1–31. Birkhäuser Boston, Boston, 2008.

[9] F. Bai, C. M. Elliott, A. Gardiner, A. Spence, and A. M. Stuart. The viscous Cahn–Hilliard equation. i. computations. Nonlinearity, 8(2):131–160, 1995.

[10] V. Barbu. Nonlinear semigroups and differential equations in Banach spaces. Springer, 1976.

[11] V. Barbu. Necessary conditions for nonconvex distributed control problems governed by elliptic variational inequalities. Journal of Mathematical Analysis and Applications, 80(2):566–597, 1981.

[12] V. Barbu. Nonlinear differential equations of monotone types in Banach spaces. Springer Monographs in Mathematics. Springer, New York, 2010.

[13] J. Barrett and J. Blowey. Finite element approximation of the Cahn–Hilliard equation with concentration dependent mobility. Mathematics of Computation, 68(226):487–517, 1999.

[14] J. Bedrossian, N. Rodríguez, and A. L. Bertozzi. Local and global well-posedness for aggregation equations and Patlak-Keller-Segel models with degenerate diffusion. Nonlinearity, 24(6):1683–1714, 2011.

[15] N. Bellomo, N. K. Li, and P. K. Maini. On the foundations of cancer modelling: Selected topics, speculations, and perspectives. Mathematical Models and Methods in Applied Sciences, 18(04):593–646, 2008.

[16] J. F. Blowey and C. M. Elliott. The Cahn–Hilliard gradient theory for phase separation with non-smooth free energy part I: Mathematical analysis. European Journal of Applied Mathematics, 2(3):233–280, 1991.

[17] J. F. Blowey and C. M. Elliott. The Cahn–Hilliard gradient theory for phase separation with non-smooth free energy part II: numerical analysis. European Journal of Applied Mathematics, 3(2):147–179, 1992.

[18] I. Borsi, A. Fasano, M. Primicerio, and T. Hillen. A non-local model for cancer stem cells and the tumour growth paradox. Mathematical Medicine and Biology: A Journal of the IMA, 34(1):59–75, 2017.

[19] H. Brézis. Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[20] H. Brezis. Functional analysis, Sobolev spaces and partial differential equations. Springer Science & Business Media, 2010.

[21] H. Byrne and M. A. Chaplain. Free boundary value problems associated with the growth and development of multicellular spheroids. European Journal of Applied Mathematics, 8(6):639–658, 1997.

[22] H. Byrne et al. Using mathematics to study solid tumour growth. In Proceedings of the 9th General Meetings of European Women in Mathematics, pages 81–107. New York: Hindawi Publishing, 1999.

[23] H. Byrne and L. Preziosi. Modelling solid tumour growth using the theory of mixtures. Mathematical Medicine and Biology: A Journal of the IMA, 20(4):341–366, 2003.

[24] H. M. Byrne and M. Chaplain. Growth of necrotic tumors in the presence and absence of inhibitors. Mathematical biosciences, 135(2):187–216, 1996.

[25] H. M. Byrne, J. R. King, D. S. McElwain, and L. Preziosi. A two-phase model of solid tumour growth. Applied Mathematics Letters, 16(4):567–573, 2003.

[26] J. W. Cahn. On spinodal decomposition. Acta metallurgica, 9(9):795–801, 1961.

[27] J. W. Cahn, C. M. Elliott, and A. Novick-Cohen. The Cahn–Hilliard equation with a concentration dependent mobility: motion by minus the laplacian of the mean curvature. European journal of applied mathematics, 7(3):287–302, 1996.

[28] J. W. Cahn and J. E. Hilliard. Free energy of a nonuniform system. i. interfacial free energy. The Journal of chemical physics, 28(2):258–267, 1958.

[29] J. W. Cahn and J. E. Taylor. Overview no. 113 surface motion by surface diffusion. Acta metallurgica et materialia, 42(4):1045–1063, 1994.

[30] C. Cavaterra, E. Rocca, and H. Wu. Long-time dynamics and optimal control of a diffuse interface model for tumor growth. Appl. Math. Optim., pages 1–49, 2019.

[31] M. A. J. Chaplain, M. Lachowicz, Z. Szymańska, and D. Wrzosek. Mathematical modelling of cancer invasion: The importance of cell-cell adhesion and cell-matrix adhesion. Mathematical Models and Methods in Applied Sciences, 21(04):719–743, 2011.

[32] E. A. Coddington and N. Levinson. Theory of ordinary differential equations. Tata McGraw-Hill Education, 1955.

[33] P. Colli, M. H. Farshbaf-Shaker, G. Gilardi, and J. Sprekels. Optimal boundary control of a viscous Cahn–Hilliard system with dynamic boundary condition and double obstacle potentials. SIAM J. Control Optim., 53(4):2696–2721, 2015.

[34] P. Colli, M. H. Farshbaf-Shaker, and J. Sprekels. A deep quench approach to the optimal control of an Allen–Cahn equation with dynamic boundary conditions and double obstacles. Appl. Math. Optim., 71(1):1–24, 2015.

[35] P. Colli, S. Frigeri, and M. Grasselli. Global existence of weak solutions to a nonlocal Cahn–Hilliard–Navier–Stokes system. Journal of Mathematical Analysis and Applications, 386(1):428 – 444, 2012.

[36] P. Colli, G. Gilardi, and D. Hilhorst. On a Cahn–Hilliard type phase field system related to tumor growth. Discrete Contin. Dyn. Syst., 35(6):2423–2442, 2015.

[37] P. Colli, G. Gilardi, G. Marinoschi, and E. Rocca. Sliding mode control for a phase field system related to tumor growth. Appl. Math. Optim., 79:647–670, 2019.

[38] P. Colli, G. Gilardi, E. Rocca, and J. Sprekels. Vanishing viscosities and error estimate for a Cahn–Hilliard type phase field system related to tumor growth. Nonlinear Anal. Real World Appl., 26:93–108, 2015.

[39] P. Colli, G. Gilardi, E. Rocca, and J. Sprekels. Asymptotic analyses and error estimates for a Cahn–Hilliard type phase field system modelling tumor growth. Discrete Contin. Dyn. Syst. Ser. S, 10(1):37–54, 2017.

[40] P. Colli, G. Gilardi, E. Rocca, and J. Sprekels. Optimal distributed control of a diffuse interface model of tumor growth. Nonlinearity, 30(6):2518–2546, 2017.

[41] P. Colli, G. Gilardi, and J. Sprekels. On the Cahn–Hilliard equation with dynamic boundary conditions and a dominating boundary potential. J. Math. Anal. Appl., 419(2):972–994, 2014.

[42] P. Colli, G. Gilardi, and J. Sprekels. Asymptotic analysis of a tumor growth model with fractional operators. Asymptotic Analysis, pages 1–32, 2019.

[43] P. Colli, G. Gilardi, and J. Sprekels. A distributed control problem for a fractional tumor growth model. Mathematics, 7(9):792, 2019.

[44] P. Colli, G. Gilardi, and J. Sprekels. Optimal velocity control of a convective Cahn–Hilliard system with double obstacles and dynamic boundary conditions: a deep quench approach. J. Convex Anal., 26(2):485–514, 2019.

[45] P. Colli, G. Gilardi, and J. Sprekels. Well-posedness and regularity for a fractional tumor growth model. Adv. Math. Sci. Appl., 28:343–375, 2019.

[46] P. Colli, A. Signori, and J. Sprekels. Optimal control of a phase field system modelling tumor growth with chemotaxis and singular potentials. Appl. Math. Optim., pages 1–33, 2019.

[47] P. Colli, A. Signori, and J. Sprekels. Second-order analysis of an optimal control problem in a phase field tumor growth model with singular potentials and chemotaxis. arXiv preprint arXiv:2009.07574, 2020.

[48] C. Cowan. The Cahn–Hilliard equation as a gradient flow. PhD thesis, Department of Mathematics-Simon Fraser University, 2005.

[49] V. Cristini, X. Li, J. S. Lowengrub, and S. M. Wise. Nonlinear simulations of solid tumor growth using a mixture model: invasion and branching. Journal of Mathematical Biology, 58(4):723, 2008.

[50] V. Cristini and J. Lowengrub. Multiscale Modeling of Cancer: An Integrated Experimental and Mathematical Modeling Approach. Cambridge University Press, 2010.

[51] V. Cristini, J. Lowengrub, and Q. Nie. Nonlinear simulation of tumor growth. Journal of mathematical biology, 46(3):191–224, 2003.

[52] S. Cui and A. Friedman. Analysis of a mathematical model of the growth of necrotic tumors. Journal of Mathematical Analysis and Applications, 255(2):636–677, 2001.

[53] S. B. Cui. Analysis of a free boundary problem modeling tumor growth. Acta Mathematica Sinica, 21(5):1071–1082, 2005.

[54] M. Dai, E. Feireisl, E. Rocca, G. Schimperna, and M. E. Schonbek. Analysis of a diffuse interface model of multispecies tumor growth. Nonlinearity, 30(4):1639–1658, 2017.

[55] E. Davoli, H. Ranetbauer, L. Scarpa, and L. Trussardi. Degenerate nonlocal Cahn–Hilliard equations: well-posedness, regularity and local asymptotics. Ann. Inst. H. Poincaré Anal. Non Linéaire, DOI:10.1016/j.anihpc.2019.10.002, 2019.

[56] E. Davoli, L. Scarpa, and L. Trussardi. Local asymptotics for nonlocal convective Cahn–Hilliard equations with $W^{\{1,1\}}$ kernel and singular potential. arXiv preprint arXiv:1911.12770, 2019.

[57] E. Davoli, L. Scarpa, and L. Trussardi. Nonlocal-to-local convergence of Cahn–Hilliard equations: Neumann boundary conditions and viscosity terms. arXiv preprint arXiv:1908.00945, 2019.

[58] E. De Angelis and L. Preziosi. Advection-diffusion models for solid tumour evolution in vivo and related free boundary problem. Mathematical Models and Methods in Applied Sciences, 10(03):379–407, 2000.

[59] F. Della Porta and M. Grasselli. Convective nonlocal Cahn–Hilliard equations with reaction terms. Discrete Contin. Dyn. Sys. B, 20(5):1529–1553, 2015.

[60] M. Ebenbeck and H. Garcke. Analysis of a Cahn–Hilliard–Brinkman model for tumour growth with chemotaxis. Journal of Differential Equations, 266(9):5998 – 6036, 2019.

[61] M. Ebenbeck and H. Garcke. On a Cahn–Hilliard–Brinkman model for tumor growth and its singular limits. SIAM Journal on Mathematical Analysis, 51(3):1868–1912, 2019.

[62] M. Ebenbeck, H. Garcke, and R. Nürnberg. Cahn–Hilliard–Brinkman systems for tumour growth. arXiv preprint arXiv:2003.08314, 2020.

[63] M. Ebenbeck and P. Knopf. Optimal medication for tumors modeled by a Cahn–Hilliard–Brinkman equation. Calculus of Variations and Partial Differential Equations, 58(4), 2019.

[64] M. Ebenbeck and P. Knopf. Optimal control theory and advanced optimality conditions for a diffuse interface model of tumor growth. ESAIM: COCV, 2020.

[65] M. Ebenbeck and K. F. Lam. Weak and stationary solutions to a Cahn–Hilliard–Brinkman model with singular potentials and source terms. Advances in Nonlinear Analysis, 10(1):24 – 65, 2021.

[66] C. Elliott and A. Stuart. Viscous Cahn–Hilliard equation II. analysis. Journal of Differential Equations, 128(2):387 – 414, 1996.

[67] C. M. Elliott and H. Garcke. On the Cahn–Hilliard equation with degenerate mobility. SIAM J. Math. Anal., 27(2):404–423, 1996.

[68] D. Faghihi, X. Feng, E. A. Lima, J. T. Oden, and T. E. Yankeelov. A coupled mass transport and deformation theory of multi-constituent tumor growth. Journal of the Mechanics and Physics of Solids, page 103936, 2020.

[69] H. B. Frieboes, J. S. Lowengrub, S. Wise, X. Zheng, P. Macklin, E. L. Bearer, and V. Cristini. Computer simulation of glioma growth and morphology. Neuroimage, 37:S59–S70, 2007.

[70] A. Friedman. A free boundary problem for a coupled system of elliptic, hyperbolic, and stokes equations modeling tumor growth. Interfaces and Free boundaries, 8(2):247–261, 2006.

[71] A. Friedman. Mathematical analysis and challenges arising from models of tumor growth. Mathematical Models and Methods in Applied Sciences, 17:1751–1772, 2007.

[72] A. Friedman. Free boundary problems associated with multiscale tumor models. Mathematical Modelling of Natural Phenomena, 4(3):134–155, 2009.

[73] A. Friedman and F. Reitich. Analysis of a mathematical model for the growth of tumors. Journal of mathematical biology, 38(3):262–284, 1999.

[74] S. Frigeri, C. G. Gal, and M. Grasselli. On nonlocal Cahn-Hilliard-Navier-Stokes systems in two dimensions. J. Nonlinear Sci., 26(4):847–893, 2016.

[75] S. Frigeri, C. G. Gal, and M. Grasselli. On nonlocal Cahn–Hilliard–Navier–Stokes systems in two dimensions. Journal of Nonlinear Science, 26(4):847–893, 2016.

[76] S. Frigeri, C. G. Gal, M. Grasselli, and J. Sprekels. Two-dimensional nonlocal Cahn–Hilliard–Navier–Stokes systems with variable viscosity, degenerate mobility and singular potential. Nonlinearity, 32(2):678–727, 2019.

[77] S. Frigeri, M. Grasselli, and E. Rocca. A diffuse interface model for two-phase incompressible flows with non-local interactions and non-constant mobility. Nonlinearity, 28(5):1257–1293, 2015.

[78] S. Frigeri, M. Grasselli, and E. Rocca. On a diffuse interface model of tumour growth. European J. Appl. Math., 26(2):215–243, 2015.

[79] S. Frigeri, K. F. Lam, and E. Rocca. On a diffuse interface model for tumour growth with non-local interactions and degenerate mobilities. In Solvability, regularity, and optimal control of boundary value problems for PDEs, volume 22 of Springer INdAM Ser., pages 217–254. Springer, Cham, 2017.

[80] S. Frigeri, K. F. Lam, E. Rocca, and G. Schimperna. On a multi-species Cahn-Hilliard-Darcy tumor growth model with singular potentials. Comm. in Math. Sci., 16:821–856, 2018.

[81] S. Frigeri, K. F. Lam, and A. Signori. Strong well-posedness and inverse identification problem of a non-local phase field tumor model with degenerate mobilities. arXiv preprint arXiv:2004.04537, 2020.

[82] M. Fritz, P. K. Jha, T. Köppl, J. T. Oden, and B. Wohlmuth. Analysis of a new multi-species tumor growth model coupling 3d phase-fields with a 1d vascular network. arXiv preprint arXiv:2006.10477, 2020.

[83] M. Fritz, E. A. Lima, V. Nikolić, J. T. Oden, and B. Wohlmuth. Local and nonlocal phase-field models of tumor growth and invasion due to ECM degradation. arXiv preprint arXiv:1906.07788, 2019.

[84] M. Fritz, E. A. B. F. Lima, J. Tinsley Oden, and B. Wohlmuth. On the unsteady Darcy–Forchheimer–Brinkman equation in local and nonlocal tumor growth models. Mathematical Models and Methods in Applied Sciences, 29(09):1691–1731, 2019.

[85] H. Gajewski and K. Zacharias. On a nonlocal phase separation model. Journal of mathematical analysis and applications, 286(1):11–31, 2003.

[86] C. G. Gal, A. Giorgini, and M. Grasselli. The nonlocal Cahn–Hilliard equation with singular potential: well-posedness, regularity and strict separation property. J. Differential Equations, 263(9):5253–5297, 2017.

[87] H. Garcke and P. Knopf. Weak solutions of the Cahn–Hilliard equation with dynamic boundary conditions: A gradient flow approach. SIAM J. Math. Anal., 2019.

[88] H. Garcke and K. F. Lam. Global weak solutions and asymptotic limits of a Cahn–Hilliard–Darcy system modelling tumour growth. AIMS Mathematics, 1(Math-01-00318):318, 2016.

[89] H. Garcke and K. F. Lam. Analysis of a Cahn–Hilliard system with non-zero Dirichlet conditions modeling tumor growth with chemotaxis. Discrete Contin. Dyn. Syst., 37(8):4277–4308, 2017.

[90] H. Garcke and K. F. Lam. Well-posedness of a Cahn–Hilliard system modelling tumour growth with chemotaxis and active transport. European J. Appl. Math., 28(2):284–316, 2017.

[91] H. Garcke and K. F. Lam. On a Cahn–Hilliard–Darcy system for tumour growth with solution dependent source terms. In E. Rocca, U. Stefanelli, L. Truskinovsky, and A. Visintin, editors, Trends in Applications of Mathematics to Mechanics, pages 243–264, Cham, 2018. Springer International Publishing.

[92] H. Garcke, K. F. Lam, R. Nürnberg, and E. Sitka. A multiphase Cahn–Hilliard–Darcy model for tumour growth with necrosis. Mathematical Models and Methods in Applied Sciences, 28(03):525–577, 2018.

[93] H. Garcke, K. F. Lam, and E. Rocca. Optimal control of treatment time in a diffuse interface model of tumor growth. Appl. Math. Optim., 2017.

[94] H. Garcke, K. F. Lam, and A. Signori. Sparse optimal control of a phase field tumour model with mechanical effects. arXiv preprint arXiv:2010.03767, 2020.

[95] H. Garcke, K. F. Lam, and A. Signori. On a phase field model of Cahn–Hilliard type for tumour growth with mechanical effects. Nonlinear Analysis: Real World Applications, 57:103192, 2021.

[96] H. Garcke, K. F. Lam, E. Sitka, and V. Styles. A Cahn–Hilliard–Darcy model for tumour growth with chemotaxis and active transport. Math. Models Methods in Appl. Sci., 26(06):1095–1148, 2016.

[97] H. Garcke and S. Yayla. Long-time dynamics for a Cahn–Hilliard tumor growth model with chemotaxis. arXiv preprint arXiv:2005.02803, 2020.

[98] A. Gerisch and M. Chaplain. Mathematical modelling of cancer cell invasion of tissue: Local and non-local models and the effect of adhesion. Journal of Theoretical Biology, 250(4):684 – 704, 2008.

[99] G. Giacomin and J. L. Lebowitz. Exact macroscopic description of phase segregation in model alloys with long range interactions. Phys. Rev. Lett., 76:1094–1097, 1996.

[100] G. Giacomin and J. L. Lebowitz. Phase segregation dynamics in particle systems with long range interactions. I. macroscopic limits. Journal of Statistical Physics, 87:37–61, 1997.

[101] G. Giacomin and J. L. Lebowitz. Phase segregation dynamics in particle systems with long range interactions II: Interface motion. SIAM Journal on Applied Mathematics, 58(6):1707–1729, 1998.

[102] A. Hawkins-Daarud, S. Prudhomme, K. G. van der Zee, and J. T. Oden. Bayesian calibration, validation, and uncertainty quantification of diffuse interface models of tumor growth. J. Math. Biol., 67(6):1457–1485, 2013.

[103] A. Hawkins-Daarud, K. G. van der Zee, and J. Tinsley Oden. Numerical simulation of a thermodynamically consistent four-species tumor growth model. Int. J. Numer. Methods Biomed. Eng., 28(1):3–24, 2012.

[104] D. Hilhorst, J. Kampmann, T. N. Nguyen, and K. G. Van Der Zee. Formal asymptotic limit of a diffuse-interface tumor-growth model. Mathematical Models and Methods in Applied Sciences, 25(06):1011–1043, 2015.

[105] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich. Optimization with PDE constraints, volume 23. Springer Science & Business Media, 2008.

[106] M. Holmes and B. Sleeman. A mathematical model of tumour angiogenesis incorporating cellular traction and viscoelastic effects. Journal of theoretical biology, 202(2):95–112, 2000.

[107] J. Jiang, H. Wu, and S. Zheng. Well-posedness and long-time behavior of a non-autonomous Cahn–Hilliard–Darcy system with mass source modeling tumor growth. Journal of Differential Equations, 259(7):3032 – 3077, 2015.

[108] J. Jost. Partial Differential Equations. Springer, third edition, 2013.

[109] C. Kahle and K. F. Lam. Parameter identification via optimal control for a Cahn–Hilliard-chemotaxis system with a variable mobility. Applied Mathematics & Optimization, pages 1–42, 2018.

[110] C. Kahle, K. F. Lam, J. Latz, and E. Ullmann. Bayesian parameter identification in Cahn–Hilliard models for biological growth. SIAM/ASA Journal on Uncertainty Quantification, 7(2):526–552, 2019.

[111] P. Knopf and K. F. Lam. Convergence of a robin boundary approximation for a Cahn–Hilliard system with dynamic boundary conditions. Nonlinearity, 2019.

[112] P. Knopf, K. F. Lam, C. Liu, and S. Metzger. Phase-field dynamics with transfer of materials: the Cahn–Hillard equation with reaction rate dependent dynamic boundary conditions. arXiv preprint arXiv:2003.12983, 2020.

[113] P. Knopf and A. Signori. On the nonlocal Cahn–Hilliard equation with nonlocal dynamic boundary condition and boundary penalization. arXiv preprint arXiv:2004.00093, 2020.

[114] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva. Linear and quasilinear equations of parabolic type. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[115] C. Lee, M. Hoopes, J. Diehl, W. Gilliland, G. Huxel, E. Leaver, K. McCann, J. Umbanhowar, and A. Mogilner. Non-local concepts and models in biology. J. Theor. Biol., 210:201–219, 2001.

[116] E. Lima, J. Oden, B. Wohlmuth, A. Shahmoradi, D. Hormuth, T. Yankeelov, L. Scarabosio, and T. Horger. Selection and validation of predictive models of radiation effects on tumor growth based on noninvasive imaging data. Computer Methods in Applied Mechanics and Engineering, 327:277 – 305, 2017. Advances in Computational Mechanics and Scientific Computation-the Cutting Edge.

[117] E. A. B. F. Lima, J. T. Oden, D. A. Hormuth, T. E. Yankeelov, and R. C. Almeida. Selection, calibration, and validation of models of tumor growth. Mathematical Models and Methods in Applied Sciences, 26(12):2341–2368, 2016.

[118] J. L. Lions. Quelques méthodes de résolution des problemes aux limites non linéaires. Dunod, 1969.

[119] J. L. Lions. Optimal control of systems governed by partial differential equations problèmes aux limites. Springer, 1971.

[120] S. Lisini, D. Matthes, and G. Savaré. Cahn–Hilliard and thin film equations with nonlinear mobility as gradient flows in weighted-Wasserstein metrics. Journal of Differential Equations, 253(2):814–850, 2012.

[121] J. S. Lowengrub, H. B. Frieboes, F. Jin, Y.-L. Chuang, X. Li, P. Macklin, S. M. Wise, and V. Cristini. Nonlinear modelling of cancer: bridging the gap between cells and tumours. Nonlinearity, 23(1):R1, 2009.

[122] D. McElwain and L. Morris. Apoptosis as a volume loss mechanism in mathematical models of solid tumor growth. Mathematical Biosciences, 39(1-2):147–157, 1978.

[123] S. Melchionna, H. Ranetbauer, L. Scarpa, and L. Trussardi. From nonlocal to local Cahn–Hilliard equation. Adv. Math. Sci. Appl., 28(2):197–211, 2019.

[124] S. Melchionna and E. Rocca. On a nonlocal Cahn–Hilliard equation with a reaction term. Adv. Math. Sci. Appl., 24(2):461–497, 2014.

[125] A. Miranville. The Cahn–Hilliard Equation: Recent Advances and Applications. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2019.

[126] A. Miranville, E. Rocca, and G. Schimperna. On the long time behavior of a tumor growth model. Journal of Differential Equations, 267(4):2616 – 2642, 2019.

[127] L. T. Morton and R. J. Moore. A chronology of medicine and related sciences. Scolar Press, 1997.

[128] P. Neittaanmaki, J. Sprekels, and D. Tiba. Optimization of elliptic systems: theory and applications. Springer Science & Business Media, 2007.

[129] J. T. Oden, A. Hawkins, and S. Prudhomme. General diffuse-interface theories and an approach to predictive tumor growth modeling. Math. Models Methods Appl. Sci., 20(03):477–517, 2010.

[130] C. Orrieri, E. Rocca, and L. Scarpa. Optimal control of stochastic phase-field models related to tumor growth. ESAIM Control Optim. Calc. Var., 2020.

[131] B. Perthame and N. Vauchelet. Incompressible limit of a mechanical model of tumour growth with viscosity. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 373(2050):20140283, 2015.

[132] L. Preziosi and A. Tosin. Multiphase modelling of tumour growth and extracellular matrix interaction: mathematical tools and applications. Journal of mathematical biology, 58(4-5):625, 2009.

[133] E. Rocca, L. Scarpa, and A. Signori. Parameter identification for nonlocal phase field models for tumor growth via optimal control and asymptotic analysis. arXiv preprint arXiv:2009.11159, 2020.

[134] L. Scarpa and A. Signori. On a class of non-local phase-field models for tumor growth with possibly singular potentials, chemotaxis, and active transport. arXiv preprint arXiv:2002.12702, 2020.

[135] R. E. Showalter. Monotone operators in Banach space and nonlinear partial differential equations, volume 49. American Mathematical Soc., 2013.

[136] A. Signori. Optimal distributed control of an extended model of tumor growth with logarithmic potential. Appl. Math. Optim., 82:517–549, 2020.

[137] A. Signori. Optimal treatment for a phase field system of Cahn–Hilliard type modeling tumor growth by asymptotic scheme. Math. Control Relat. Fields, 10:305–331, 2020.

[138] A. Signori. Optimality conditions for an extended tumor growth model with double obstacle potential via deep quench approach. Evol. Equ. Control Theory, 9:193–217, 2020.

[139] A. Signori. Penalisation of long treatment time and optimal control of a tumour growth model of Cahn–Hilliard type with singular potential. Discrete Contin. Dyn. Syst. Ser. A, 2020.

[140] A. Signori. Vanishing parameter for an optimal control problem modeling tumor growth. Asymptot. Anal., 117:46–66, 2020.

[141] J. Simon. Compact sets in the space $L^p(0,T;B)$. Ann. Mat. Pura Appl. (4), 146:65–96, 1987.

[142] J. Sprekels and F. Tröltzsch. Sparse optimal control of a phase field system with singular potentials arising in the modeling of tumor growth. arXiv preprint arXiv:2005.02784, 2020.

[143] J. Sprekels and H. Wu. Optimal distributed control of a Cahn–Hilliard–Darcy system with mass sources. App. Math. Opt., 2019.

[144] Z. Szymańska, C. Rodrigo, M. Lachowicz, and M. Chaplain. Mathematical modelling of cancer invasion of tissue: The role and effect of nonlocal interactions. Math. Models Methods Appl. Sci., 19(2):257–281, 2009.

[145] J. E. Taylor and J. W. Cahn. Linking anisotropic sharp and diffuse surface motion laws via gradient flows. Journal of Statistical Physics, 77(1-2):183–197, 1994.

[146] F. Tröltzsch. Optimal control of partial differential equations: theory, methods, and applications, volume 112. American Mathematical Soc., 2010.

[147] F. E. Udwadia. Man and medicine: A history. Oxford University Press, USA, 2000.

[148] J. D. van der Waals. The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density. Journal of Statistical Physics, 20(2):200–244, 1979.

[149] S. Wise, J. Lowengrub, H. Frieboes, and V. Cristini. Three-dimensional multispecies nonlinear tumor growth-I: Model and numerical method. Journal of Theoretical Biology, 253(3):524 – 543, 2008.

[150] X. Zheng, S. Wise, and V. Cristini. Nonlinear simulation of tumor necrosis, neo-vascularization and tissue invasion via an adaptive finite-element/level-set method. Bulletin of mathematical biology, 67(2):211, 2005.