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Convergence of Solutions of a Set Optimization Problem in the Image Space

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Abstract The present work is devoted to the study of stability in set optimization. In particular, a sequence of perturbed set optimization problems, with a fixed objective map, is studied under suitable continuity assumptions. A formulation of external and internal stability of the solutions is considered in the image space, in such a way that the convergence of a sequence of solutions of perturbed problems to a solution of the original problem is studied

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under appropriate compactness assumptions. Our results can also be seen as an extension to the set-valued framework of known stability results in vector optimization.

Keywords Set optimization · Set relations · Minimal solutions · Stability · Set-convergence

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1 Introduction

In the last decades, set-valued optimization problems, where the objective function is a set-valued map, have been extensively studied. By generalizing the notions of solution already well established within the framework of vector optimization, two distinct approaches to this class of problems have been developed. The first one concerns the study of the minimal boundary of the union of the images of the feasible region through the set-valued objective map (see, e.g., [1–3]). More recently, an alternative approach, usually called set optimization, was introduced by Kuroiwa [4,5]. It considers an order relation between sets and studies a minimality notion induced by this order.

The second approach seems to be more suitable for applications. Indeed, it can be related to the study of robust multiobjective optimization (see, e.g., [6, 7]). In particular, the lower order relation can be used to obtain an "optimistic" notion of robust solution for multiobjective optimization (see [8]).

A number of specific topics related to set optimization has already been investigated. Indeed, besides the seminal work of Kuroiwa, Tanaka and Ha [9] on cone convexity of set-valued maps, we recall the papers [10–16].

Until now, only a few efforts have been devoted to study the stability properties of the solutions of a set optimization problem. Moreover, even if the behavior of minimal solutions under perturbations plays an important role in the special case of vector optimization problems (see, e.g., [17–21]), only in [22] the continuity of the solution map of a parametric set optimization problem is investigated.

In the present work, we consider a sequence of perturbed problems with a fixed set-valued objective map converging to a given set optimization problem. We limit our study to the convergence of the solution sets in the image space. The image space analysis of optimization problems provides an interesting insight (see [23] and the references therein). If we aim at a direct extension of stability results developed in the special framework of vector optimization, then we need to define the convergence of a sequence of collections of sets. Our approach avoids a direct definition of the convergence of sequences of collections of sets. Indeed, we obtain "lower convergence" or "internal stability" results by requiring that each set in the solution of the original set optimization problem is the limit of a converging sequence of solutions of perturbed problems. Moreover, we consider a formulation of "upper convergence" or "external stability" by requiring that the limit of a converging sequence of solutions of perturbed problems belongs to the solution of the original problem. Under suitable continuity requirements on the set-valued objective map, we obtain both internal and external stability results in the image space. We underline that our approach is completely different from the one developed in [22]. Moreover, even in the special case of vector optimization, our results are new, since they are not comparable with the existing ones.

In vector optimization, the notion of weak minimality plays a fundamental role whenever the study of upper convergence of minimal solutions is involved [18, 24]. Hence, it is not surprising that, in order to prove our external stability results, we refer to a notion of weak minimality in set optimization. Indeed, we prove that the limit, in the image space, of a sequence of weakly minimal sets of perturbed set optimization problems is a weakly minimal solution of the original problem.

The internal stability of the solutions in the image space is studied under appropriate compactness assumptions, involving the whole collection of solutions of the perturbed problems in the image space. Such a strong assumption can be avoided if we strengthen the lower continuity property of the set-valued objective map by requiring cone lower Hausdorff continuity instead of the classical lower semicontinuity. As a consequence, the convergence result works for the conical extension in the image space of the solution of the original problem.

The paper is organized as follows. In Section 2, some preliminary notions and results are collected. In Section 3, the external stability of the solutions of a set optimization problem in the image space is considered. In Section 4, the internal stability of the solutions of a set optimization problem in the image space is investigated under appropriate compactness assumptions.

2 Preliminaries and Notations

Let X be a metric space and let ρ denote the metric of X. Moreover, let $B(x, r)$ denote the open ball centred at x with radius $r > 0$, let $\text{int}B$ be the topological interior of a set $B \subset X$ and let $\mathbb{R}^p_+$ denote the non-negative orthant of $\mathbb{R}^p$.

2.1 Set-Convergences

We briefly recall the definitions of two classical notions of set-convergence.

Let $\{A_n\}$ be a sequence of nonempty subsets of X. The *Kuratowski-Painlevé lower* and *upper limits* are defined, respectively, by:

$$\text{Li}A_n := \left\{x \in X : x = \lim_{n\to\infty} x_n,\, x_n \in A_n \,\text{eventually}\right\},$$

$$\text{Ls}A_n := \left\{x \in X : x = \lim_{s\to\infty} x_s,\, x_s \in A_{n_s},\, \{n_s\} \text{ subsequence of } \{n\}\right\}.$$

If a subset A of X satisfies the conditions

$$\text{Ls}A_n \subset A \subset \text{Li}A_n,$$

then we say that the sequence $\{A_n\}$ converges to A in the sense of Kuratowski-Painlevé and we denote it by $A_n \xrightarrow{K} A$. The inclusion $\text{Ls}A_n \subset A$ is known as the upper part of Kuratowski-Painlevé convergence (and we denote it by $A_n \xrightarrow{K} A$), while $A \subset \mathrm{Li} A_n$ is the lower part of Kuratowski-Painlevé convergence (denoted by $A_n \stackrel{K}{\rightharpoondown} A$).

Now we recall a stronger notion of set-convergence, the so called Hausdorff convergence. Let $x \in X$ and let $\emptyset \neq A, B \subset X$, we have

$$d(x, A) := \inf_{a \in A} \rho(x, a) \quad (d(x, \emptyset) := \infty),$$

$$e(A, B) := \sup_{a \in A} d(a, B) \quad (e(\emptyset, A) := 0,\ e(\emptyset, \emptyset) := 0,\ e(A, \emptyset) := \infty).$$

Let $\{A_n\}$ be a sequence of nonempty subsets of X. The sequence $\{A_n\}$ converges to $A \subset X$ in the sense of Hausdorff iff

$$e(A_n, A) \to 0, \qquad e(A, A_n) \to 0,$$

and we denote it by $A_n \stackrel{H}{\to} A$. Condition $e(A_n, A) \to 0$ is the upper part of Hausdorff convergence (denoted by $A_n \stackrel{H}{\rightharpoonup} A$), while condition $e(A, A_n) \to 0$ is the lower part of Hausdorff convergence (denoted by $A_n \stackrel{H}{\rightharpoondown} A$). Clearly, $A_n \stackrel{H}{\rightharpoondown} A$ if and only if there exists a sequence $\{\varepsilon_n\}$ of positive real numbers such that $\varepsilon_n \to 0$ and $A \subset A_n + B(0, \varepsilon_n)$, for all n. Moreover, we remark that $A_n \stackrel{H}{\to} A$ implies $A_n \stackrel{K}{\to} A$, whenever A is closed.

2.2 Continuity Notions for Set-Valued Functions

Let Y be a normed vector space, let $F : X \rightrightarrows Y$ be a set-valued map and let $K \subset Y$ be a closed, convex and proper ($\{0\} \neq K \neq Y$) cone. Let us recall some well-known continuity notions for F [3,17,25].

1. F is *upper* (respectively *lower*) *semicontinuous* at $x_0 \in X$ iff for every open set Q such that $F(x_0) \subset Q$ (respectively $F(x_0) \cap Q \neq \emptyset$) there exists a neighbourhood U of x_0 such that $F(x) \subset Q$ (respectively $F(x) \cap Q \neq \emptyset$) for every $x \in U$.

   F is *continuous* at a point $x_0 \in X$ iff F is upper and lower semicontinuous at x_0.

   F is *continuous* (respectively lower semicontinuous) on X iff F is upper and lower semicontinuous (respectively lower semicontinuous) at every point $x \in X$.
2. F is *upper K-semicontinuous* at $x_0 \in X$ iff for every open set Q such that $F(x_0) \subset Q$ there exists a neighbourhood U of x_0 such that $F(x) \subset Q + K$ for every $x \in U$.
3. F is *upper* (respectively *lower*) *Hausdorff continuous* at $x_0 \in X$ iff for every neighbourhood W of 0 there exists a neighbourhood U of x_0 such that

   $F(x) \subset F(x_0) + W$ (respectively $F(x_0) \subset F(x) + W$) for every $x \in U$.
4. F is *K-lower Hausdorff continuous* at $x_0 \in X$ iff for every neighbourhood W of 0 there exists a neighbourhood U of x_0 such that $F(x_0) \subset F(x) + W + K$ for every $x \in U$.

Remark 2.1 If F is upper semicontinuous at x_0 and $F(x_0)$ is closed, then it is easy to see that $\mathrm{Ls}F(x_n) \subset F(x_0)$, for each sequence $\{x_n\} \subset X$ converging to x_0.

Now we prove a useful lemma, that investigates the behavior of a continuous set-valued map with respect to set-convergences. Let

$$\mathcal{F}_A := \{F(a) : a \in A \cap \mathrm{dom} F\},$$

where $\emptyset \neq A \subset X$ and $\mathrm{dom} F := \{x \in X : F(x) \neq \emptyset\}$. Moreover, we say that F is closed-valued on X iff $F(x)$ is closed, for all $x \in \mathrm{dom} F$.

Lemma 2.1 *Let $\emptyset \neq A \subset X$ and let $\{A_n\}$ be a sequence of nonempty subsets of X such that $A_n \xrightarrow{K} A$.*

1. *If F is lower semicontinuous on X, then for every $H \in \mathcal{F}_A$ there exists a sequence $\{H_n\}$ of nonempty subsets of Y such that $H_n \in \mathcal{F}_{A_n}$ for all n and $H_n \xrightarrow{K} H$.*
2. *If F is continuous and closed-valued on X, then for every $H \in \mathcal{F}_A$ there exists a sequence $\{H_n\}$ of nonempty subsets of Y such that $H_n \in \mathcal{F}_{A_n}$ for all n and $H_n \xrightarrow{K} H$.*
3. *If F is lower (respectively K-lower) Hausdorff continuous on X, then for every $H \in \mathcal{F}_A$ there exists a sequence $\{H_n\}$ of nonempty subsets of Y such that $H_n \in \mathcal{F}_{A_n}$ for all n and $H_n \xrightarrow{H} H$ (respectively $H_n + K \xrightarrow{H} H$).*

Proof 1. If $H \in \mathcal{F}_A$, then there exists $a \in A$ such that $F(a) = H$. Since $A_n \xrightarrow{K} A$ then there exists a sequence $\{a_n\}$ such that $a_n \in A_n$ for all n and $a_n \to a$. By the lower semicontinuity of F at a, we obtain the inclusion $F(a) \subset \mathrm{Li} F(a_n)$.

2. By the first part we see that there exists a sequence $\{a_n\}$ such that $a_n \in A_n$ for all n, $a_n \to a$ and $F(a) \subset \mathrm{Li}F(a_n)$. On the other hand, since F is upper semicontinuous at a and $F(a)$ is closed, by Remark 2.1 we obtain the inclusion $\mathrm{Ls}F(a_n) \subset F(a)$.

3. Let us suppose that F is lower Hausdorff continuous on X (the proof of the case where F is K-lower Hausdorff continuous on X is similar, hence it is omitted). Since $H \in \mathcal{F}_A$, there exists $a \in A$ such that $F(a) = H$. The lower Kuratowski-Painlevé convergence of A_n to A implies that there exists a sequence $\{a_n\}$ converging to a, where $a_n \in A_n$ for every n. By the lower Hausdorff continuity of the map F we have that $F(a_n) \xrightarrow{H} F(a)$, and the assertion is proved. □

2.3 l-Minimality Notions in Set Optimization

In the family of all the nonempty subsets of Y (in the sequel denoted by $\mathscr{P}_0(Y)$) we consider the quasi order $\lesssim^l$ (see [12] and references therein) defined by

$$A \lesssim^l B \quad \text{if and only if} \quad B \subset A + K,$$

where $A, B \in \mathscr{P}_0(Y)$. Let $\mathcal{G}$ be a subcollection of $\mathscr{P}_0(Y)$; if $G \in \mathcal{G}$ then we call the collection $\mathscr{S}_{\mathcal{G}}(G) := \left\{C \in \mathcal{G} : C \lesssim^l G\right\}$ the *l-section* of $\mathcal{G}$ at G.

If $A \lesssim^l B$ and $B \lesssim^l A$ then we say that $A \sim^l B$. We remark that $A \sim^l B$ if and only if $A + K = B + K$. Moreover, whenever $\mathrm{int}K \neq \emptyset$, we can consider also a strict order relation, defined as follows:

$$A <^l B \quad \text{if and only if} \quad B \subset A + \text{int}K.$$

We say that $A \in \mathcal{G}$ is an l-minimal element of $\mathcal{G}$ iff

$$B \in \mathcal{G} \quad \text{and} \quad B \lesssim^l A \quad \text{imply} \quad A \lesssim^l B.$$

Let l-Min$(\mathcal{G})$ be the family of all l-minimal elements of $\mathcal{G}$. If $A \in l$-Min$(\mathcal{G})$ then $A + K \in l$-Min$(\mathcal{G})$ whenever $A + K \in \mathcal{G}$. On the other hand, clearly,

$$\mathscr{S}_{\mathcal{G}}(A) = \{B \in \mathcal{G} : A \sim^l B\} \subset l\text{-Min}(\mathcal{G}), \quad \forall A \in l\text{-Min}(\mathcal{G}). \tag{1}$$

We can also consider a stronger notion of l-minimality called strict l-minimality (see, e.g., [12]): $A \in \mathcal{G}$ is a strictly l-minimal element of $\mathcal{G}$ iff

$$B \in \mathcal{G} \quad \text{and} \quad B \lesssim^l A \quad \text{imply} \quad A = B.$$

Each strictly l-minimal element in $\mathcal{G}$ is also a l-minimal element but, in general, the reverse implication does not hold. Finally, following the definition introduced in [15], if int$K \neq \emptyset$, then we say that $A \in \mathcal{G}$ is a weakly l-minimal element of $\mathcal{G}$ iff

$$B \in \mathcal{G} \quad \text{and} \quad B <^l A \quad \text{imply} \quad A <^l B.$$

We denote by l-WMin$(\mathcal{G})$ the family of all weakly l-minimal elements of $\mathcal{G}$.

It can be proved that l-Min$(\mathcal{G}) \subset l$-WMin$(\mathcal{G})$ (see Proposition 2.7 in [15]).

Remark 2.2 If $\mathcal{G}$ is a collection of singletons, then the notions of l-minimality and weak l-minimality reduce to the standard definitions of minimality and weak minimality (with respect to the ordering cone K) in vector optimization. Recall that the minimal and weak minimal sets of a nonempty set $A \subset Y$ are, respectively,

$$\mathrm{Min}(A) := \{a \in A : A \cap (a - K) \subseteq a + (K \cap (-K))\},$$
$$\mathrm{WMin}(A) := \{a \in A : A \cap (a - \mathrm{int}K) = \emptyset\}.$$

If, additionally, the ordering cone K is pointed (i.e., $K \cap (-K) = \{0\}$), then the two notions of l-minimality and strict l-minimality coincide.

It is trivial to see that, if it does not exist an element $B \in \mathcal{G}$ such that $B <^l A$, then $A \in l$-WMin$(\mathcal{G})$. Under mild assumptions, also the reverse implication holds.

Proposition 2.1 *(see* [15, Lemma 2.6]*) Let $A \in \mathcal{G}$. If* WMin$(A) \neq \emptyset$*, then* $A \in l$-WMin$(\mathcal{G})$ *if and only if it does not exist an element $B \in \mathcal{G}$ such that* $B <^l A$.

We remark that, if A is a compact set, then the property WMin$(A) \neq \emptyset$ holds true (see, e.g., [3]). Moreover, the following example shows that assumption WMin$(A) \neq \emptyset$ cannot be dropped in Proposition 2.1.

Example 2.1 Let $Y := \mathbb{R}^2$, $K := \mathbb{R}^2_+$ and $\mathcal{G} := \{A, B\}$, where

$$A := \left\{(x_1, x_2) \in \mathbb{R}^2 : \, x_2 = 2^{x_1}, \, x_1 < 0\right\},$$

$$B := \left\{(x_1, x_2) \in \mathbb{R}^2 : \, x_2 = 3^{x_1}, \, x_1 < 0\right\}.$$

It is easy to see that $A, B \in l\text{-WMin}(\mathcal{G})$, $\text{WMin}(A) = \text{WMin}(B) = \emptyset$ and it holds $B <^l A$ and $A <^l B$.

Finally, we recall the l-domination property introduced in [14].

Definition 2.1 We say that $\mathcal{G} \subset \mathscr{P}_0(Y)$ has the l-domination property iff for each $A \in \mathcal{G}$ there exists $B \in l\text{-Min}(\mathcal{G})$ such that $B \lesssim^l A$.

3 External Stability

In this section, we study the external stability of the solutions of a set optimization problem in the image space, i.e., the fact that the limit of a converging sequence of solutions of perturbed problems is a solution of the original set optimization problem. We recall that in vector optimization there are straightforward examples of sequences of minimal solution sets of perturbed problems that do not converge to a minimal solution set of the original problem (see Example 4.1 in [20]). Indeed, even in the special case of a vector optimization problem, the upper convergence of the solutions in the image space can only be obtained for weak minimality, or in the case where minima and weak minima coincide (see, e.g., Proposition 3.1 in [19], Proposition 2.3 in [24] and part (a) of Theorem 3.1(iii) in [26]) . Hence, it is not surprising that we study the external stability of the solutions of a sequence of perturbed set optimization problems only in the sense of weak l-minimality.

Theorem 3.1 *Let $\{A_n\}$ be a sequence of nonempty subsets of X. Let us suppose that the following assumptions hold:*

1. *F is lower Hausdorff continuous on X;*
2. *$A_n \xrightarrow{K} A$.*

Let $\{B_n\}$ be a sequence of sets of Y such that $\mathrm{WMin}(B_n) \neq \emptyset$*, for all n. If $B_n \in l$-*$\mathrm{WMin}(\mathcal{F}_{A_n})$ *for every n and $B_n \xrightarrow{H} B$, where $B \in \mathcal{F}_A$ is a compact set, then $B \in l$-*$\mathrm{WMin}(\mathcal{F}_A)$.

Proof By contradiction, let $B \notin l\text{-WMin}(\mathcal{F}_A)$. Then, there exists $C \in \mathcal{F}_A$ such that $B \subset C + \mathrm{int}K$. Since B is a compact set and $C + \mathrm{int}K$ is an open set, there exists a real number $\delta > 0$ such that

$$\inf_{b \in B} d\left(b, (C + \mathrm{int}K)^C\right) = 2\delta \tag{2}$$

(where given $E \subset Y$, $E^C := Y \setminus E$). Therefore $B + B(0,\delta) \subset C + \mathrm{int}K$. Since $B_n \xrightarrow{H} B$, eventually it holds

$$B_n \subset B + B(0,\delta) \subset C + \mathrm{int}K. \tag{3}$$

Moreover, by Lemma 2.1, there exists a sequence $\{C_n\}$ such that $C_n \in \mathcal{F}_{A_n}$ and $C_n \xrightarrow{H} C$. Therefore, there exists a sequence $\{\eta_n\}$ of positive real numbers such that $\eta_n \to 0$ and $C \subset C_n + B(0, \eta_n)$. Hence, by (3) we have

$$B + B(0, \delta) \subset C_n + B(0, \eta_n) + \text{int}K,$$

for every n. Now, let us suppose that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $h_k \in B_{n_k} \cap (C_{n_k} + \text{int}K)^C$, for every $k \in \mathbb{N}$. By (3) there exist two sequences $\{y_k\} \subset B$ and $\{z_k\} \subset B(0, \delta)$ such that $h_k = y_k + z_k$, for all k. It holds

$$\begin{aligned} \inf_{b \in B} d\left(b, (C + \text{int}K)^C\right) &\leq \inf_k d\left(y_k, (C + \text{int}K)^C\right) \\ &\leq \inf_k d\left(h_k, (C + \text{int}K)^C\right) + \delta \\ &\leq \inf_k d\left(h_k, (C_{n_k} + B(0, \eta_{n_k}) + \text{int}K)^C\right) + \delta \\ &\leq \eta_{n_k} + \delta, \quad \forall\, n_k, \end{aligned}$$

and so $\inf_{b \in B} d\left(b, (C + \text{int}K)^C\right) \leq \delta$.

Equation (2) shows that the last assertion is a contradiction. Therefore, it holds

$$B_n \subset C_n + \text{int}K,$$

for any n large enough. Since $\text{WMin}(B_n) \neq \emptyset$ for all n, by recalling Proposition 2.1, the last inclusion gives a contradiction against $B_n \in l\text{-WMin}(\mathcal{F}_{A_n})$. □

In the previous theorem, the assumption that B is a compact set cannot be avoided, as it is shown by the following example.

Example 3.1 Let $X := \mathbb{R}$, $Y := \mathbb{R}^2$, $K := \mathbb{R}^2_+$ and $S := A \cup (\cup_{n=2}^{\infty} A_n) \subset X$, where $A := \{0, 1\}$ and $A_n := \{1/n, 1 + (1/n)\}$, for all $n \geq 2$. Moreover, let $F : X \rightrightarrows Y$, $\{B_n\}_{n\geq 2}$ and B be defined as follows: $F(x) := \mathbb{R}^2$, for all $x \notin S$, $F(0) := \{(0, 0)\}$, $F(1/n) := \{(1/n, 1/n)\}$, for all $n \geq 2$,

$$B = F(1) := \left\{(x_1, x_2) \in \mathbb{R}^2 : \, x_2 = \frac{1}{x_1}, \, x_1 > 0\right\},$$
$$B_n = F(1 + (1/n)) := \left\{(x_1, x_2) \in \mathbb{R}^2 : \, x_2 = \frac{1 + (1/n)}{x_1}, \, x_1 > 0\right\}, \quad \forall\, n \geq 2.$$

It is easy to see that F is lower Hausdorff continuous on X, $A_n \xrightarrow{K} A$, $\mathrm{WMin}(B_n) \neq \emptyset$ and $B_n \in l\text{-}\mathrm{WMin}(\mathcal{F}_{A_n})$, $\forall\, n \geq 2$, $B \in \mathcal{F}_A$, $B_n \xrightarrow{H} B$, but $B \notin l\text{-}\mathrm{WMin}(\mathcal{F}_A)$.

Remark 3.1 In order to compare Theorem 3.1 with the well-known result on the upper convergence of the weak minimal solutions of a vector optimization problem in the image space (see, e.g., Proposition 3.1 in [19]), we should reformulate it in the special case, where $X = Y$ and the set-valued map F reduces to the single-valued identity map. Then, it is proved that, if a subsequence b_{n_k} such that $b_{n_k} \in \mathrm{WMin}(A_{n_k})$ converges to an element of A, then the limit of b_{n_k} is also in $\mathrm{WMin}(A)$. Therefore, for arbitrary sets $\{A_n\}$ (not necessarily closed) and under the slightly weaker assumptions $A_n \xrightarrow{K} A$ and $\mathrm{LsWMin}(A_n) \subseteq A$, one obtains the same thesis as in Proposition 3.1 in [19].

On the other hand, these assumptions are not comparable with the assumptions of Proposition 2.3 in [24]. For example, if $X := \mathbb{R}^2$, $K := \mathbb{R}^2_+$, $A_n := \{\alpha(1, 1/n) : \alpha \in [0, 1[\}$ and $A = \{(0, 0), (1/2, 0)\}$, then the assumptions of Theorem 3.1 are fulfilled and Proposition 3.1 in [19] and Proposition 2.3 in [24] cannot be applied.

An application of the previous result concerns the approximate solutions of a vector optimization problem.

Remark 3.2 By using Theorem 3.1, we can prove that the set of weak approximate solutions of a vector optimization problem in the Kutateladze sense (see [27]) is an external approximation of the weak minimal set in the image space. This property has been used to define well-posedness notions in vector optimization (see [28,29]).

Let $M \subset Y$ be a nonempty and closed set, $q \in \text{int}K$ and $\varepsilon \geq 0$. Recall that $y_0 \in M$ is said to be a (q, ε)-approximate point of M in the Kutateladze sense, denoted by $y_0 \in \text{WMin}(M, q, \varepsilon)$, iff $(y_0 - \varepsilon q - \text{int}K) \cap M = \emptyset$. Consider $X = Y$, $F(y) = \{y\}$, for all $y \in Y$ and the sets $M_\varepsilon := M + \varepsilon q$, for all $\varepsilon \geq 0$. It is easy to check that $l\text{-WMin}(\mathcal{F}_{M_\varepsilon}) = \text{WMin}(M_\varepsilon)$, for all $\varepsilon \geq 0$ and

$$\text{WMin}(M, q, \varepsilon) \subset \bigcup_{\delta \in [0,\varepsilon]} \text{WMin}(M_\delta), \quad \forall \varepsilon \geq 0. \tag{4}$$

Let y_0, $\{y_n\}$ and $\{\varepsilon_n\}$ be such that $y_n \in \text{WMin}(M, q, \varepsilon_n)$, $y_n \to y_0$ and $\varepsilon_n \to 0$. By (4) we deduce that there exists a sequence $\{\delta_n\}$ such that $y_n \in \text{WMin}(M_{\delta_n})$ for all n and $\delta_n \to 0$.

Clearly, F is lower Hausdorff continuous on Y and $M_{\delta_n} \overset{K}{\to} M$. By applying Theorem 3.1, it follows $y_0 \in l\text{-WMin}(\mathcal{F}_M) = \text{WMin}(M)$. Hence, the sets $\mathrm{WMin}(M, q, \varepsilon)$, $\varepsilon \geq 0$, can be seen as external approximations of the set $\mathrm{WMin}(M)$. Further results in this vein can be found in [24].

4 Internal Stability

In this section, we prove some internal stability results on the solution sets in the image space, i.e., we obtain that a given solution of a set optimization problem can be expressed as a limit of solutions of a sequence of perturbed problems. In the special case of vector optimization, this kind of results follows two different approaches. The first one relies on convexity assumptions (see, e.g., [19,20]), avoiding the compactness assumptions that play a crucial role in the second approach (see, e.g., [18,21]). As a first attempt to study lower stability properties in set optimization, we prove some results based on compactness properties globally involving the whole collection of minimal sets in the image space of a sequence of perturbed set optimization problems.

Theorem 4.1 *Let $\{A_n\}$ be a sequence of nonempty subsets of X. Let us suppose that the following assumptions hold:*

1. *F is lower semicontinuous on X;*
2. *$A_n \xrightarrow{K} A$;*
3. *$\mathcal{F}_{A_n}$ has the l-domination property for every n;*
4. *let $\{H_n\}$ such that $H_n \in \mathcal{F}_{A_n}$ for every n and $H_n \xrightarrow{K} H \in \mathcal{F}_A$, then every sequence $\{W_n\}$ such that $W_n \in l\text{-}\mathrm{Min}(\mathcal{F}_{A_n}) \cap \mathscr{S}_{\mathcal{F}_{A_n}}(H_n)$ has a subsequence $\{W_{n_j}\}$ such that $W_{n_j} \xrightarrow{K} W \in \mathcal{F}_A$, $W \neq \emptyset$ and $\mathrm{cl}\left(\cup_j W_{n_j}\right)$ is a compact set.*

If $B \in l\text{-Min}(\mathcal{F}_A)$*, then there exist a sequence* $\{B_j\}$ *and a set* $C \in l\text{-Min}(\mathcal{F}_A)$ *such that* $C \sim B$*,* $B_j \in l\text{-Min}(\mathcal{F}_{A_j})$ *and* $B_j \xrightarrow{K} C$*.*

Proof Since $B \in \mathcal{F}_A$, by Lemma 2.1, there exists a sequence $\{H_n\}$ such that $H_n \in \mathcal{F}_{A_n}$ for every n and $H_n \xrightarrow{K} B$. By Assumption 3 in the present theorem, there exists a sequence $\{W_n\}$ such that

$$W_n \in l\text{-Min}(\mathcal{F}_{A_n}) \cap \mathscr{S}_{\mathcal{F}_{A_n}}(H_n), \quad \forall\, n.$$

By Assumption 4 in the present theorem, we obtain that $\{W_n\}$ has a subsequence $\left\{W_{n_j}\right\}$ such that $W_{n_j} \xrightarrow{K} W \in \mathcal{F}_A$, $W \neq \emptyset$. Since $W_{n_j} \in \mathscr{S}_{\mathcal{F}_{A_{n_j}}}(H_{n_j})$, it holds that

$$W_{n_j} \lesssim^l H_{n_j}, \quad \forall\, n_j. \tag{5}$$

Now let us consider a generic element $b \in B$. Since $H_n \xrightarrow{K} B$, there exists a sequence $\{h_j\}$ such that $h_j \in H_{n_j}$ and $h_j \to b$. Relation (5) implies that there exist two sequences $\{w_j\}$ and $\{k_j\}$ such that $w_j \in W_{n_j}$, $k_j \in K$ and $h_j = w_j + k_j$. By Assumption 4 in this theorem, there exist a subsequence $\{w_{j_s}\}$ of $\{w_j\}$ and an element $w \in W$ such that $w_{j_s} \to w$. Therefore, there exists $k \in K$ such that $k_{j_s} \to k$ and $b = w + k$. Hence, we conclude that $W \lesssim^l B$, and by (1) it holds that $B \sim^l W \in l\text{-Min}(\mathcal{F}_A)$. The thesis holds by taking $C = W$ and $B_j = W_{n_j}$. □

Remark 4.1

1. If F is continuous on X, $A \subset \mathrm{dom}F$, $A_n \overset{K}{\rightharpoondown} A$ and two compact sets $G_X \subset X$ and $G_Y \subset Y$ exist such that

$$\bigcup_n A_n \subset G_X, \qquad \bigcup_n \left(\bigcup_{W \in l\text{-}\mathrm{Min}(\mathcal{F}_{A_n})} W \right) \subset G_Y,$$

then Assumption 4 in Theorem 4.1 holds. Indeed, let $\{W_n\}$ be such that $W_n \in l$-Min$(\mathcal{F}_{A_n})$. Therefore there exists $\{a_n\}$ such that $a_n \in A_n$ and $W_n = F(a_n)$. Since $\{a_n\} \subset G_X$, a subsequence $\{a_{n_j}\}$ such that $a_{n_j} \to a \in A$ can be found. As F is continuous on X, it holds

$$F(a_{n_j}) = W_{n_j} \overset{K}{\to} F(a) = W$$

and $W \neq \emptyset$. Finally, cl $\left(\cup_j W_{n_j}\right)$ is a compact set, since it is a closed subset of G_Y.

2. Assumption 3 is satisfied whenever F is upper K-semicontinuous on X and A_n is compact, for all n (see Corollary 5.6 in [14]).

3. If we suppose that B is a strictly l-minimal element of $\mathcal{F}_A$, then we can directly prove, under the same assumptions as in Theorem 4.1, that there exists a sequence $\{B_j\}$, where $B_j \in l$-Min$(\mathcal{F}_{A_j})$, such that $B_j \overset{K}{\to} B$.

4. If F is continuous and closed-valued on X, then Assumption 4 of Theorem 4.1 can be weakened by referring it to sequences $H_n \overset{K}{\to} H \in \mathcal{F}_A$ instead of $H_n \overset{K}{\rightharpoondown} H \in \mathcal{F}_A$.

5. As in Remark 3.1, in order to compare Theorem 4.1 with some known stability results in vector optimization, we reformulate it in the special

case where $X = Y$ and the set-valued map F reduces to the single-valued identity map. Among various known results related to internal stability properties (see, e.g., [20] and the references therein), Theorem 3.1 in [18] is directly related to the approach developed here. A detailed comparison shows that, in the special framework of vector optimization, Assumption 4 in Theorem 4.1 is implied by Assumptions (i) (only upper part) and (iii) in Theorem 3.1 in [18]. Hence, Theorem 4.1 slightly weakens the upper part of the convergence requirement in (i) in Theorem 3.1 in [18] (with a fixed order structure). A similar, but non comparable, result can be found also in Theorem 2.2 in [24]. Indeed, here the lower convergence assumption is weaker than Assumption 2 in Theorem 4.1, whereas the compactness assumption partially implies Assumption 4 in the same theorem, as shown in Item 1 of this remark. Moreover, in Theorem 4.1 no upper convergence of A_n to A is required. Next, we show an example where the assumptions of Theorem 4.1 are fulfilled, whereas Theorem 3.1 in [18] and Theorem 2.2 in [24] cannot be applied.

Example 4.1 Let $X := \mathbb{R}^2$, $K := \mathbb{R}^2_+$, $A = \{(0,0)\}$ and the sequence $\{A_n\}$ defined by

$$A_n := \left\{(0,0)\,, \left(-\frac{n}{\sqrt{n^2+1}}, \frac{1}{\sqrt{n^2+1}}\right)\right\}.$$

It is easy to show that $\mathrm{Min}(A_n) = A_n$, $\mathrm{Min}(A) = A$ and $A_n \xrightarrow{K} A$. Moreover, Assumption 4 of Theorem 4.1 is satisfied. Thus, this theorem can be applied.

However, the upper convergence of A_n to A or $A+K$ is not fulfilled, and so Theorem 3.1 in [18] and Theorem 2.2 in [24] cannot be applied.

If we strengthen the continuity properties of the set-valued objective map F by requiring K-lower Hausdorff continuity, then we can avoid the global compactness Assumption 4 in Theorem 4.1 in order to obtain a lower Hausdorff convergence result for the conical extension of the l-minimal solution sets in the image space. We say that a family $\mathcal{G} \subset \mathscr{P}_0(Y)$ is closed for the conical extension by K (K-conical closed, for short) iff $A+K \in \mathcal{G}$, for all $A \in \mathcal{G}$.

Theorem 4.2 *Let $\{A_n\}$ be a sequence of nonempty subsets of X. Let us suppose that the following assumptions hold:*

1. *F is K-lower Hausdorff continuous on X;*
2. *$A_n \xrightarrow{K} A$;*
3. *$\mathcal{F}_{A_n}$ has the l-domination property for every n;*
4. *The families $\mathcal{F}_A$ and $\mathcal{F}_{A_n}$, for all n, are K-conical closed.*

If $B \in l\text{-Min}(\mathcal{F}_A)$, then there exists a sequence $\{B_n\}$ and a set $C \in l\text{-Min}(\mathcal{F}_A)$ such that $C \sim B$, $B_n \in l\text{-Min}(\mathcal{F}_{A_n})$ and $B_n \xrightarrow{H} C$.

Proof Since $B \in \mathcal{F}_A$, by Lemma 2.1, there exists a sequence $\{H_n\}$ such that $H_n \in \mathcal{F}_{A_n}$ and $H_n + K \xrightarrow{H} B$. By Assumption 3, there exists a sequence $\{W_n\}$ such that

$$W_n \in l\text{-Min}(\mathcal{F}_{A_n}) \cap \mathscr{S}_{\mathcal{F}_{A_n}}(H_n), \quad \forall\, n. \tag{6}$$

Since $H_n+K \overset{H}{\to} B$, there exist a sequence of positive real numbers ε_n, $\varepsilon_n \to 0$, and an integer n_0 such that

$$B \subset H_n + B(0,\varepsilon_n) + K, \quad \forall\, n \geq n_0. \tag{7}$$

If we combine relations (6) and (7), then we obtain

$$B \subset W_n + B(0,\varepsilon_n) + K, \quad \forall\, n \geq n_0$$

and hence

$$B + K \subset W_n + B(0,\varepsilon_n) + K, \quad \forall\, n \geq n_0.$$

Therefore, $W_n + K \overset{H}{\to} B + K$. Now, let $B_n := W_n + K$ and $C := B + K$. By Assumption 4 it holds $B_n \in \mathcal{F}_{A_n}$ and $C \in \mathcal{F}_A$. As $B \in l\text{-Min}(\mathcal{F}_A)$ we see that $C \sim B$. Moreover, since $W_n \in l\text{-Min}(\mathcal{F}_{A_n})$, for all n, it holds $B_n \in l\text{-Min}(\mathcal{F}_{A_n})$ and the thesis is proved. □

We underline that Assumption 4 in the previous theorem is not unduly restrictive. Indeed, if a family $\mathcal{F}_A$ is not K-conical closed, then one can always enlarge it by considering the family

$$\mathcal{F}_A^K := \mathcal{F}_A \cup \{B + K : B \in \mathcal{F}_A\}.$$

It is easy to see that the following implications hold:

$$B \in l\text{-Min}(\mathcal{F}_A) \Rightarrow B \in l\text{-Min}(\mathcal{F}_A^K) \Leftrightarrow B + K \in l\text{-Min}(\mathcal{F}_A^K).$$

Moreover, it is worth to underline that, to the best of our knowledge, the previous theorem provides a new result, even in the special case of a vector optimization problem.

5 Conclusions

To the best of our knowledge, the results proved in this work represent the first attempt in the literature to study stability properties of a set optimization problem by using set-convergence notions. Our approach can be considered as an extension of known results on upper and lower convergence of the solution sets of a vector optimization problem in the image space to the more general framework of set optimization. In particular, we developed lower convergence results under compactness assumptions involving the whole family of solutions of the perturbed problems in the image space. In vector optimization, an alternative approach based on convexity was developed in [19, 20]. It is still an open question if the existing notions of convexity for set-valued maps (see [9]) allow us to obtain some alternative internal stability results.

We underline that the analysis of stability in the present work is carried out only in the image space. It may be interesting in the future to consider also some stability properties of the solutions in the decision space.

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