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C^*-algebras associated to monoids and groupoids, and Bass-Serre theory for groupoids

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Abstract

In this thesis we establish some connections between the theory of self-similar fractals in the sense of J. E. Hutchinson (cf. [20]), and the theory of boundary quotients of C^*-algebras associated to monoids. We show that the existence of self-similar M-fractals for a given monoid M, gives rise to examples of C^*-algebras generalizing the boundary quotients discussed by X. Li in [27, §7, p. 71]. The starting point for our investigations is the observation that the universal boundary of a finitely 1-generated monoid carries naturally two topologies: the cone topology and the fine topology. The first is used to define canonical measures on the attractor of an M-fractal for a finitely 1-generated monoid M, while the latter plays a prominent role in the construction of the boundary quotients mentioned above.

Moreover, we construct a Bass-Serre theory (cf. [35]) in the groupoid setting and prove a structure theorem. Any groupoid action without inversion of edges on a forest induces a graph of groupoids, while any graph of groupoids satisfying certain hypothesis has a canonical associated groupoid, called the fundamental groupoid, and a forest, called the Bass–Serre forest, such that the fundamental groupoid acts on the Bass–Serre forest. The structure theorem says that these processes are mutually inverse, so that graphs of groupoids "encode" groupoid actions on forests.

Finally, we prove a groupoid C^*-algebraic Bass-Serre theorem following the ideas in [8], where such a theorem is proved in the group setting. To this end, we need to consider a different groupoid and forest as the ones defined above: we associate to any graph of groupoids a groupoid, called the universal fundamental groupoid, and a forest, called the universal forest, on which the universal fundamental groupoid acts. To a large class of graph of groupoids we associate a C^*-algebra universal for generators and relations and show that it is isomorphic to the action groupoid C^*-algebra induced by the action of the universal fundamental groupoid on the boundary of the universal forest.

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Thanks to Filippo, who happened to be in the library when I was only looking for a topology book, accompanied me worldwide throughout the following years, and forever will.

Ora corollari dar già, ma mai gradir alloro caro

Questa è la strada incalcolabile
di chi nasce in calcol abile
e prova con tentativo maldestro
a curar il proprio mal d'estro.

Con tratto fine firmai il contratto:
Matematica fu decisa.
Ma tematica incerta, titolo confuso,
con fuso ampio mi spostai.

Nota azione coraggiosa,
cambiar la notazione!
Ma la mente si abitua,
seppur reagisca malamente.

O pacata inversa calotta,
hai placato l'opaca, cieca lotta!
Sedotta da nuove vie, poi edotta,
ti lasciai con lemmi e ritrovai dilemmi.

Rime di ogni tipo furon rimedi
adatti seppur mai messi ad atti.
Integrata non fui mai
e riposi fiducia in te grata.

D'altronde il conto è assai diverso
da chi ama andar di verso,
e alquanto bassa è la probabilità
di viver della mia proba abilità.

Non fu mai tutto fiori e rose
questa scelta che mi erose:
invero ha causato varia bile
al mio umore così variabile.

Conferita l'aurea laurea con ferita,
Perversi miei desideri per versi
Con dotti metodi via condotti,
Certa me ne sto col mio certame.

Contents

Chapter 1

Introduction

This thesis contains a number of results concerning two main topics: C^*-algebras associated to the boundary of monoids and C^*-algebras associated to graphs of groupoids. When studying the latter, it is predictable to encounter the so-called Bass-Serre theory (cf. [35]) and to adapt its techniques and results to the groupoid context; hence, this thesis also contains the construction of a Bass-Serre theory for groupoids.

In the first part of the thesis (cf. Chapter 3) we establish some connections between the theory of self-similar fractals in the sense of J. E. Hutchinson (cf. [20]), and the theory of boundary quotients of C^*-algebras associated to monoids. The motivation for this work came from attempting to use monoids as partially ordered sets to generalize the boundary quotient discussed by X. Li in [27, §7]. On a monoid M there is naturally defined a reflexive and transitive relation "$\preceq$", i.e., for $\omega, \tau \in M$ one defines $\omega \preceq \tau$ if, and only if, there exists $\sigma \in M$ satisfying $\omega = \tau \cdot \sigma$. In particular, one may consider $(M, \preceq)$ as a partially ordered set. Moreover, if M is $\mathbb{N}_0$-graded (cf. Definition 2.1.24) and finitely 1-generated (cf. Definition 2.1.28), then $(M, \preceq)$ is a *Noetherian* partially ordered set (see Corollary 2.1.34). Such a poset has a poset completion $i_M \colon M \longrightarrow \bar{M}$ (see Section 2.1.2), and one defines the *universal boundary* ∂M of M by

$$\partial M = (\bar{M} \setminus \mathrm{im}(i_M))/\approx, \tag{1.0.1}$$

where $\approx$ is the equivalence relation induced by "$\preceq$" on $\bar{M} \setminus \mathrm{im}(i_M)$ (see Section 2.1.2). For several reasons (cf. Theorem A, Theorem B, Theorem C) one may consider ∂M as the natural boundary associated to the monoid M. However, it is less clear what topology one should consider. Apart from the cone topology $\mathcal{T}_c(\bar{M})$ there is another finer topology $\mathcal{T}_f(\bar{M})$ which will be called the *fine topology* on ∂M (cf. Section 2.1.2), i.e., the identity

$$\mathrm{id}_{\partial M} \colon (\partial M, \mathcal{T}_f(\bar{M})) \longrightarrow (\partial M, \mathcal{T}_c(\bar{M})) \tag{1.0.2}$$

is a continuous map. The monoid M will be said to be $\mathcal{T}$*-regular* if $\mathrm{id}_{\partial M}$ is a homeomorphism. The universal boundary $\eth M = (\partial M, \mathcal{T}_f(\bar{M}))$ with the fine topology can be identified with the Laca boundary $\widehat{E}(M)$ of the monoid M. This topological space plays an essential role for defining boundary quotients of C^*-algebras associated to monoids (cf. [27, § 7], [28]). Indeed one has the following (cf. Theorem 3.1.6).

Theorem A. *The map* $\overline{\chi}_. \colon (\partial M, \mathcal{T}_f(\bar{M})) \longrightarrow \widehat{E}(M)$ *defined by* (3.1.13) *is a homeomorphism.*

By Theorem A, the topological space $(\partial M, \mathcal{T}_f(\bar{M}))$ is totally-disconnected and compact and thus it has the nicest topological regularity property that one can wish for.

On the contrary, in general one can only show that $(\partial M, \mathcal{T}_c(\bar{M}))$ is a T_0-space, which is a low level regularity property. Indeed, if $(\partial M, \mathcal{T}_c(\bar{M}))$ happens to be Hausdorff, then (1.0.2) is necessarily a homeomorphism and M is $\mathcal{T}$-regular (cf. Proposition 3.1.7).

If $\phi\colon Q \to M$ is a surjective graded homomorphism of finitely 1-generated monoids, then, by construction, ϕ induces a surjective, continuous and open map

$$\partial\bar{\phi}\colon (\partial Q, \mathcal{T}_c(\bar{Q})) \longrightarrow (\partial M, \mathcal{T}_c(\bar{M})) \tag{1.0.3}$$

(cf. Proposition 2.1.26). This property can be used to establish the following.

Theorem B. *Let M be a finitely 1-generated $\mathbb{N}_0$-graded monoid. Then ∂M carries naturally a Borel probability meausure*

$$\mu_M\colon \mathrm{Bor}(\partial M) \longrightarrow \mathbb{R}^+ \cup \{\infty\} \tag{1.0.4}$$

induced by the canonical homomorphism of monoids $\phi_M\colon \mathscr{F}\langle M_1\rangle \longrightarrow M$ (cf. (2.1.24)).

On the other hand, the induced mapping $\phi_{\widehat{E}}$ is given by a map

$$\phi_{\widehat{E}}\colon \widehat{E}(M) \longrightarrow \widehat{E}(Q) \tag{1.0.5}$$

(cf. Proposition 3.1.8). Hence for the purpose of constructing Borel measures the fine topology seems to be inappropriate.

Theorem B can be used to define the C^*-algebra

$$C^*(M, \mu_M) = C^*\big(\{\, \beta_\omega, \mid \omega \in M\,\}\big) \subseteq \mathcal{B}(L^2(\partial M, \mathbb{C}, \mu_M))$$

for every finitely 1-generated $\mathbb{N}_0$-graded monoid M, where β_ω is the mapping induced by left multiplication with ω (cf. § 3.1.5). We will show by explicit calculation that for the monoid $\mathscr{F}_n$, freely generated by a set of cardinality n, the C^*-algebra $C^*(\mathscr{F}_n, \mu_{\mathscr{F}_n})$ coincides with the Cuntz algebra $\mathcal{O}_n$ (cf. Proposition 3.1.21), while for the right-angled Artin monoid M^Γ associated to the finite graph Γ, $C^*(M^\Gamma, \mu_M^\Gamma)$ coincides with the boundary quotients introduced by Crisp and Laca in [9] (cf. § 3.1.6).

Let $M = \bigcup_{k\in\mathbb{N}_0} M_k$ be a $\mathbb{N}_0$-graded finitely 1-generated monoid. In the context of self-similar fractals in the sense of J. E. Hutchinson (cf. [20]) it will turn out to be convenient to endow ∂M with the cone topology $\mathcal{T}_c(\bar{M})$. Let (X, d) be a complete metric space with a left M-action $\alpha\colon M \longrightarrow \mathcal{C}(X, X)$ by continuous maps. Such a presentation is said to be *contracting*, if there exists a positive real number $\delta < 1$ such that

$$d(\alpha(s)(x), \alpha(s)(y)) \leq \delta \cdot d(x, y), \tag{1.0.6}$$

for all $x, y \in X$, $s \in M_1$ (cf. [20, § 2.2]). For a contrating metric space (X, d) there exists a unique compact subset $K \subseteq X$ such that

(1) $K = \bigcup_{s\in M_1} \alpha(s)(K)$,

(2) $K = \mathrm{cl}(\{\, \mathrm{Fix}(\alpha(t)) \mid t \in M\,\} \subseteq X$.

Obviously, by definition every map $\alpha(t) \in \mathcal{C}(X, X)$ is contracting, and thus has a unique fixed point $x_t \in X$. For short we call $K = K(\alpha) \subset X$ the *attractor* of the representation α. One has the following (cf. Proposition 3.2.5).

Theorem C. *Let $M = \bigcup_{k \in \mathbb{N}_0} M_k$ be a finitely 1-generated $\mathbb{N}_0$-graded monoid, let (X, d) be a compact metric space and let $\alpha \colon M \to \mathcal{C}(X, X)$ be a contracting representation of M. Then for any point $x \in X$, α induces a continuous map*

$$\kappa_x \colon \partial M \longrightarrow K(\alpha).$$

Moreover, if M is $\mathcal{T}$-regular, then κ_x is surjective.

Under the general hypothesis of Theorem C we do not know whether the topological space $(\partial M, \mathcal{T}_c(\bar{M}))$ is necessarily compact. However, in case that it is compact, we call $(\partial M, \mathcal{T}_c(\bar{M}))$ the *universal attractor of the finitely 1-generated $\mathbb{N}_0$-graded $\mathcal{T}$-regular monoid M.*

For a finitely 1-generated monoid M, ∂M carries canonically a probability measure μ_M (cf. § 3.1.5). Thus, by Theorem C, the attractor of the M-fractal $((X, d), \alpha)$ carries the *contact probability measure* $\mu_x = \mu_\circ^{\kappa_x}$ for every point $x \in X$, which is given by $\mu_x(B) = \mu_M(\kappa_x^{-1}(B))$, for $B \in \mathrm{Bor}(K)$.

Since the monoid M is acting on K, it also acts on $L^2(K, \mathbb{C}, \mu_x)$ by bounded linear operators γ_ω, $\omega \in M$ (cf. § 3.2.2) This defines a C^*-algebra (cf. § 3.2.2)

$$C^*(M, X, d, \mu_x) = C^*\big(\{\gamma_\omega, \mid \omega \in M\}\big) \subseteq \mathcal{B}(L^2(K, \mathbb{C}, \mu)). \tag{1.0.7}$$

In case that the equivalence relation $\hat{\sim}$ generated by $\preceq$ on ∂M is different from $\approx$ (cf. (1.0.1)) the canonical map $\tilde{j} \colon \partial M \to \partial M/\hat{\sim}$ is not the identity.

In the second part of the thesis (cf. Chapters 4,5), we deal with graphs of groupoids and their C^*-algebras. The motivation for our investigation came from the observation that group actions on trees can be generalized by groupoid actions on forests, as well as the Bass-Serre theory can be extended to the groupoid setting. Moreover, following the ideas in [8], it became natural to aim to build a Bass-Serre theory in the groupoid C^*-algebraic setting. The well-known Bass-Serre theory (cf. §2.2.3) gives a complete and satisfactory description of groups acting on trees via the structure theorem. A graph of groups consists of a connected graph Γ together with a group for each vertex and edge of Γ, and group monomorphisms from each edge group to the adjacent vertex groups. Any group action (*without inversion*) on a tree induces a graph of groups, while any graph of groups has a canonical associated group, called the *fundamental group*, and a tree, called the *Bass–Serre tree*, such that the fundamental group acts on the Bass–Serre tree. The structure theorem (cf. Theorem 2.2.24) says that these processes are mutually inverse, so that graphs of groups “encode” group actions on trees. Following these ideas, we build the appropriate analogue of Bass-Serre theory for groupoids.

Groupoids (cf. §2.3) are algebraic objects that behave like a group (i.e., they satisfy conditions of associativity, left and right identities and inverses) except that the multiplication operation is only partially defined. The collection $\mathcal{G}^{(0)}$ of idempotent elements in a groupoid $\mathcal{G}$ is called its *unit space*, since these are precisely the elements x that satisfy $x\alpha = \alpha$ and $\beta x = \beta$ whenever these products are defined. When considering an action of $\mathcal{G}$ on a graph Γ, this leads to a fibred structure of Γ over $\mathcal{G}^{(0)}$. As for group actions, one associates to a groupoid action on a set a certain groupoid, called the *action groupoid* (cf. Remark 2.4.5), which will be widely used in Chapter 5.

A graph of groupoids $\mathcal{G}(\Gamma)$ is given by a connected graph Γ together with a groupoid for each vertex and edge of Γ, and monomorphisms from each edge groupoid to the adjacent vertex groupoid. We will only work with graph of groupoids having discrete vertex and edge groupoids. In Chapter 4 we associate to any graph of groupoids $\mathcal{G}(\Gamma)$ a groupoid $\pi_1(\mathcal{G}(\Gamma))$, called the *fundamental groupoid*, and a forest $X_{\mathcal{G}(\Gamma)}$, called the *Bass-Serre forest*, such that $\pi_1(\mathcal{G}(\Gamma))$ acts on $X_{\mathcal{G}(\Gamma)}$. Hence, we prove a structure theorem (cf. Theorem 4.3.22) in this setting.

Theorem D. *Let $\mathcal{G}$ be a groupoid acting without inversion of edges on a forest F. Then $\mathcal{G}$ is isomorphic to the fundamental groupoid of the graph of groupoids defined by the desingularization of the action of $\mathcal{G}$ on F.*

One of the main differences between the two settings is the following: in the classical setting, given a group action without inversion on a graph, one of the ingredients used to build a graph of groups is the quotient graph given by such action; in the groupoid context, there is not a canonical graph associated to the action of a groupoid on a graph. Hence, we need to resort to the difficult notion of *desingularization* (cf. Definition 4.2.13) of a groupoid action on a graph. In case we consider groupoids whose unit space is a singleton, we recover the classical Bass-Serre theory.

In Chapter 5 we associate a *different* groupoid and forest to a graph of groupoids as those defined in Chapter 4. In particular, for any graph of groupoids $\mathcal{G}(\Gamma)$, we define a *universal fundamental groupoid* $\Pi_1(\mathcal{G}(\Gamma))$ which is more general than the fundamental groupoid $\pi_1(\mathcal{G}(\Gamma))$. Indeed, the two groupoids $\pi_1(\mathcal{G}(\Gamma))$ and $\Pi_1(\mathcal{G}(\Gamma))$ have the same generating set, and one obtains $\pi_1(\mathcal{G}(\Gamma))$ from $\Pi_1(\mathcal{G}(\Gamma))$ by adding two relations to the defining relations of $\Pi_1(\mathcal{G}(\Gamma))$ (cf. Remark 5.1.6). Moreover, we associate to $\mathcal{G}(\Gamma)$ a forest, called the *universal forest* and denoted by $Y_{\mathcal{G}(\Gamma)}$, on which the universal fundamental groupoid acts. Such an action induces an action of $\Pi_1(\mathcal{G}(\Gamma))$ on the boundary $\partial Y_{\mathcal{G}(\Gamma)}$ of the universal forest $Y_{\mathcal{G}(\Gamma)}$.

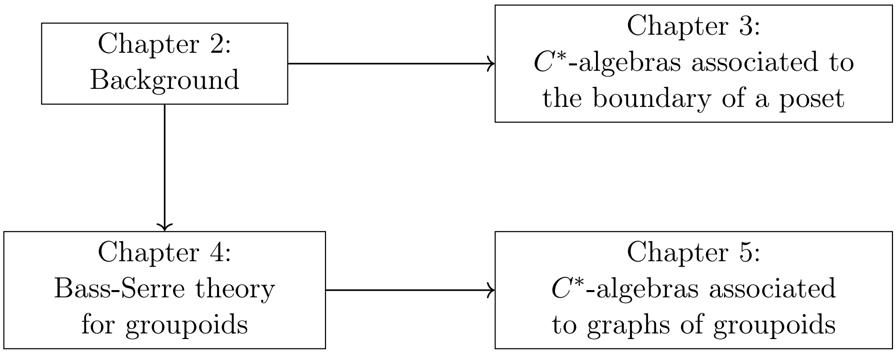
The motivation for our interest in the universal fundamental groupoid and in the universal forest comes from the fact that we want to prove a groupoid C^*-algebraic Bass-Serre theorem following the ideas in [8], where the authors prove such a theorem in the group setting: given a graph of groups, they prove that the graph of groups C^*-algebra, which is universal for generators and relations, is stably isomorphic to the boundary action crossed product C^*-algebra induced by the action of the fundamental group on the boundary of the Bass-Serre tree.

Given a locally-finite nonsingular graph of groupoids $\mathcal{G}(\Gamma)$, we work with two C^*-algebras: the graph of groupoids C^*-algebra $C^*(\mathcal{G})$ (cf. Definition 5.4.4), which is universal for generators and relations, and the *action groupoid* C^*-algebra $C^*(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$ induced by the action of $\Pi_1(\mathcal{G}(\Gamma))$ on $\partial Y_{\mathcal{G}(\Gamma)}$. We prove the following theorem (cf. Theorem 5.5.3).

Theorem E. *Let $\mathcal{G}(\Gamma)$ be a locally finite nonsingular graph of groupoids. Then there is an isomorphism $\Phi\colon C^*(\mathcal{G}) \longrightarrow C^*\big(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}\big)$.*

For the proof of this theorem we first use the universality of $C^*(\mathcal{G})$ to find a *-homomorphism $\Phi\colon C^*(\mathcal{G}) \to C^*(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$. Then, we use the fact that the space of continuous function with compact support $\mathcal{C}_c(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$ is dense in $C^*(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$ to build a representation $\pi\colon \mathcal{C}_c(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}) \to C^*(\mathcal{G})$ which induces a *-homomorphism $\Psi\colon C^*(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}) \to C^*(\mathcal{G})$. Finally, we prove that such Ψ is the inverse of Φ.

The logical structure of the thesis may be outlined in the following diagram.



In Chapter 2 we recall background material and establish our notation. We begin in Section 2.1 with the definitions and topological properties of boundaries of posets and we recall some basic notions in monoid theory. In Section 2.2 we recall the necessary concepts from graph theory and the main definitions and results of Bass-Serre theory. In Section 2.3 we give some basic definitions and properties in the groupoid setting, then in Section 2.4 we define groupoid actions on topological space and, in particular, on graphs. Finally, in Section 2.5 we describe the construction of groupoid C^*-algebras.

In Chapter 3 we show that the existence of self-similar M-fractals for a given monoid M, gives rise to example of C^*-algebras generalizing the boundary quotients of X. Li. In Section 3.1 we study the boundaries of certain monoids and their C^*-algebras. In Section 3.2 we associate a C^*-algebra to the action of a finitely 1-generated monoid on a contracting metric space, which we will call an *M-fractal*.

In Chapter 4 we build a Bass-Serre theory in the groupoid setting. We begin with the definitions of a graph of groupoid and its representation on a forest in Section 4.1. Then, in Section 4.2 we define the desingularization of a groupoid action on a forest and associate a graph of groupoids to such action. In Section 4.3 we define the fundamental groupoid and the Bass-Serre forest associated to a graph of groupoids, and then we prove the structure theorem. Finally, in Section 4.4 we give an example of the constructions developed in the previous sections.

In Chapter 5 we prove a groupoid C^*-algebraic Bass-Serre theorem. We begin with the definitions of the universal fundamental groupoid and the universal forest of a graph of groupoids in Section 5.1. Then, we define the boundary of the universal forest in Section 5.2. In Section 5.3 we consider the action groupoid defined by the action of the universal fundamental groupoid on the boundary of the universal forest and we define a topology which makes it a Hausdorff *ample* groupoid. In Section 5.4 we define the graph of groupoids C^*-algebra generated by a $\mathcal{G}$-family and prove some properties of such a family. Finally, in Section 5.5 we prove that the graph of groupoids C^*-algebra is isomorphic to the action groupoid C^*-algebra.

The prerequisites for understanding the content of this thesis include a basic knowledge of measure theory, C^*-algebras and functional analysis. For a comprehensive overview of such topics we refer the reader to

- W. Arveson, *An Invitation to C^*-algebras*, [3];
- B. Blackadar, *Operator Algebras - Theory of C^*-algebras and von Neumann algebras*, [6];
- W. Rudin, *Real and Complex Analysis*, [33].

Notation. By $\mathcal{B}(\mathcal{H})$ we will denote the C^*-algebra of bounded linear operators on a Hilbert space $\mathcal{H}$.

Given a set of operators $\{ s_v \mid v \in V \}$, we use the notation $C^*(\{ s_v \mid v \in V \})$ for the C^*-algebra generated by the operators in $\{ s_v \mid v \in V \}$.

Given a set of elements V we use the notation $\langle x \in V \mid \mathcal{R} \rangle^+$ and $\langle x \in V \mid \mathcal{R} \rangle$ to mean, respectively, the monoid and the group generated by the set V subject to the relations in $\mathcal{R}$.

Chapter 2

Background

In this chapter we present basic background material. In Section 2.1 we define the boundary of a partially ordered set and recall some definitions concerning monoids. In Section 2.2 we give some background of graph theory and then we recall the main definitions and results of the so-called Bass-Serre theory. In Section 2.3 we give the definition a groupoid and outline some of the elementary facts. In Section 2.4 we recall some basic constructions concerning groupoid actions on topological spaces. Section 2.5 contains the basic constructions in the theory of groupoids C^*-algebras.

2.1 Posets, monoids and their boundaries

All the definitions and results in this section have been pubished in [11].

2.1.1 Posets

Definition 2.1.1. A *poset* (or *partially ordered set*) is a set X together with a reflexive and transitive relation $\preceq\colon X \times X \to \{t, f\}$ such that for all $x, y \in X$ satisfying $x \preceq y$ and $y \preceq x$ follows that $x = y$.

Definition 2.1.2. For a poset $(X, \preceq)$ and $\tau, \omega \in X$ the set

$$\mathrm{C}_\omega = \{\, x \in X \mid x \preceq \omega \,\} \tag{2.1.1}$$

will be called the *cone* defined by ω, and

$$\mho_\tau = \{\, y \in X \mid y \succeq \tau \,\} \tag{2.1.2}$$

the *cocone* defined by τ. For $\tau \preceq \omega$ the set

$$[\tau, \omega] = \mho_\tau \cap \mathrm{C}_\omega = \{\, x \in X \mid \tau \preceq x \preceq \omega \,\} \tag{2.1.3}$$

is called the *closed interval* from τ to ω, i.e., $[\omega, \omega] = \{\omega\}$. The poset $(X, \preceq)$ is said to be *noetherian*, if $\mathrm{card}(\mho_\tau) < \infty$ for all $\tau \in X$.

Definition 2.1.3. Let X be a countable set. For a poset $(X, \preceq)$ let

$$\mathscr{D}(\mathbb{N}, X, \preceq) = \{\, f \in \mathscr{F}(\mathbb{N}, X) \mid \forall n, m \in \mathbb{N}:\ n \leq m \implies f(n) \succeq f(m) \,\} \tag{2.1.4}$$

denote the set of decreasing functions which we will - if necessary - identify with the set of decreasing sequences.
A poset $(X, \preceq)$ is said to be *complete*, if for all $f \in \mathscr{D}(\mathbb{N}, X, \preceq)$ there exists an element $z \in X$ such that

(CP$_1$) $f(n) \succeq z$ for all $n \in \mathbb{N}$, and

(CP$_2$) if $y \in X$ satisfies $f(n) \succeq y$ for all $n \in \mathbb{N}$, then $z \succeq y$.

Note that - if it exists - $z \in X$ is the unique element satisfying (CP$_1$) and (CP$_2$) for $f \in \mathscr{D}(\mathbb{N}, X, \preceq)$. As usual, $z = \min(f)$ is called the *minimum* of $f \in \mathscr{D}(\mathbb{N}, X, \preceq)$.

Notation 2.1.4. Let $(X, \preceq)$ be a poset. For $u, v \in \mathscr{D}(\mathbb{N}, X, \preceq)$ we put

$$u \preceq v \iff \forall n \in \mathbb{N} \quad \exists k_n \in \mathbb{N}\colon\ u(k_n) \preceq v(n), \tag{2.1.5}$$

and

$$u \sim v \iff \big[\,(u \preceq v \wedge v \preceq u) \vee \big(v \preceq u \wedge v = c_m, m = min(u)\big)\,\big], \tag{2.1.6}$$

where $c_z \in \mathscr{D}(\mathbb{N}, X, \preceq)$, $z \in X$, is given by $c_z(n) = z$ for all $n \in \mathbb{N}$.

Let $\approx$ be the equivalence relation generated by $\sim$ and put

$$\overline{X} = \mathscr{D}(\mathbb{N}, X, \preceq)/\approx\ . \tag{2.1.7}$$

Then the following properties hold for $(\overline{X}, \preceq)$.

Proposition 2.1.5. *Let $(X, \preceq)$ be a poset.*

(a) The relation $\preceq$ defined in (2.1.5) is reflexive and transitive.

(b) For any strictly increasing function $\alpha\colon \mathbb{N} \to \mathbb{N}$ and $u \in \mathscr{D}(\mathbb{N}, X, \preceq)$ one has $u \approx u \circ \alpha$.

(c) Define for $[u], [v] \in \overline{X}$ that $[u] \preceq [v]$ if, and only if, $u \preceq v$. Then $(\overline{X}, \preceq)$ is a poset.

(d) $(\overline{X}, \preceq)$ is complete.

Proof. (a) The relation $\preceq$ is obviously reflexive. Let $u, v, w \in \mathscr{D}(\mathbb{N}, X, \preceq)$, $u \preceq v$, $v \preceq w$. Then for all $n \in \mathbb{N}$ there exists $h_n, k_n \in \mathbb{N}$ such that $u(h_n) \preceq v(k_n) \preceq w(n)$. Thus, $u \preceq w$.
(b) Let $u \in \mathscr{D}(\mathbb{N}, X, \preceq)$ and let $\alpha\colon \mathbb{N} \to \mathbb{N}$ be a strictly increasing function. Let $m < n$, $m, n \in \mathbb{N}$. Since α is strictly increasing, $\alpha(m) < \alpha(n)$. Then there exist $m_0, n_0 \in \mathbb{N}$ such that $m_0 \leq \alpha(m) < \alpha(n) \leq n_0$. Then one has $u(m_0) \succeq u(\alpha(m)) \succeq u(\alpha(n)) \succeq u(n_0)$. Thus $u \preceq u \circ \alpha$ and $u \circ \alpha \preceq u$, proving that $u \approx u \circ \alpha$.
(c) Let $[u], [v] \in \overline{X}$, $[u] \preceq [v]$ and $[v] \preceq [u]$. Then, by definition, $u \preceq v$ and $v \preceq u$, and thus $u \approx v$, i.e., $[u] = [v]$.
(d) Let $\{u_k\}_{k\in\mathbb{N}} \in \mathscr{D}(\mathbb{N}, \overline{X}, \preceq)$, i.e., $u_k \in \overline{X}$ for all $k \in \mathbb{N}$. Then one has $u_1 \succeq u_2 \succeq \ldots$ by definition. Since each $u_k \in \mathscr{D}(\mathbb{N}, X, \preceq)$, one has $u_k(n) \succeq u_k(m)$ for all $n \leq m$, $m, n \in \mathbb{N}$. We define $v \in \mathscr{D}(\mathbb{N}, X, \preceq)$ by $v(n) = u_n(n)$, $n \in \mathbb{N}$. Then $[v] \in \overline{X}$ is the minimum of $\{u_k\}_{k\in\mathbb{N}}$. This yields the claim. □

Assigning every element $x \in X$ the equivalence class containing the constant function $c_x \in \mathscr{D}(\mathbb{N}, X, \preceq)$ yields a strictly decreasing mapping of posets $\iota_X\colon X \to \overline{X}$. From now on $(X, \preceq)$ will be considered as a sub-poset of $(\overline{X}, \preceq)$.

Definition 2.1.6. The poset $(\overline{X}, \preceq)$ will be called the *poset completion of* $(X, \preceq)$.

The following proposition is straightforward.

Proposition 2.1.7. *The map ι_X is a bijection if, and only if, $(X, \preceq)$ is complete.*

Example 1. Let $X = \mathbb{N} \sqcup \{\infty\}$ and define $n \preceq m$ if, and only if, $n \geq m$, where "$\geq$" denotes the natural order relation. Then the poset $(X, \preceq)$ is complete and $\overline{X} = X$.

2.1.2 Boundaries of posets

Definition 2.1.8. For a poset $(X, \preceq)$ the poset $\partial X = \overline{X} \setminus \mathrm{im}(i_X)$ will be called the *universal boundary* of the poset $(X, \preceq)$.

Notation 2.1.9. From now on we use the notation $x \succ y$ as a short form for $x \succeq y$ and $x \neq y$.

Definition 2.1.10. A function $f \colon \mathbb{N} \to X$ will be said to be *strictly decreasing* if $f(n+1) \prec f(n)$ for all $n \in \mathbb{N}$.

The following Proposition will turn out to be useful.

Proposition 2.1.11. *Let $f \in \mathscr{D}(\mathbb{N}, X, \preceq)$ be a decreasing function such that $[f] \in \partial X$. Then there exists a strictly decreasing function $h \in \mathscr{D}(\mathbb{N}, X, \preceq)$ such that $f \approx h$, i.e., $[f] = [h]$.*

Proof. By hypothesis, $J = \mathrm{im}(f)$ is an infinite set. In particular, the set

$$\Omega = \{\, \min(f^{-1}(\{j\}) \mid j \in J \,\}$$

is an infinite and unbounded subset of $\mathbb{N}$. Let $e \colon \mathbb{N} \to \Omega$ be the enumeration function of Ω, i.e., $e(1) = \min(\Omega)$, and recursively one has $e(k+1) = \min(\Omega \setminus \{e(1), \dots, e_k\})$. Then, by construction, $h = f \circ e$ is strictly decreasing, and, by Proposition 2.1.5(b), one has $f \approx h$, and hence the claim. □

Proposition 2.1.12. *Let $(X, \preceq)$ be a noetherian poset, and let $(\overline{X}, \preceq)$ be its completion. Then for all $\tau \in X$ one has $\mho_\tau(\overline{X}) \subseteq X$. In particular, $\mho_\tau(\overline{X}) = \mho_\tau(X)$, where the cocones are taken in the respective posets.*

Example 2. Let $X = A \sqcup B$, where $A, B = \mathbb{Z}$ and define

$$n \preceq m \iff \Big(((n, m \in A \vee n, m \in B) \wedge n \leq m) \vee (n \in A \wedge m \in B)\Big), \tag{2.1.8}$$

where "$\leq$" denotes the natural order relation on $\mathbb{Z}$. Then $(X, \preceq)$ is a poset and its completion is given by $\overline{X} = \mathbb{Z} \sqcup \{-\infty\} \bigsqcup \mathbb{Z} \sqcup \{-\infty\}$. For $n \in A$, one has $\mho_n(X) \neq \mho_n(\overline{X})$, since $-\infty \in B$ is in $\mho_n(\overline{X})$, but not in $\mho_n(X)$.

We now introduce two different topologies on the poset completion of a poset.

Definition 2.1.13. Let $(X, \preceq)$ be a poset, and let $(\overline{X}, \preceq)$ denote its completion. For $\tau, \omega \in X$ let

$$S(\tau, \omega) = \{\, x \in X \mid x \preceq \tau \ \wedge \ x \preceq \omega \,\}. \tag{2.1.9}$$

By transitivity,

$$\mathrm{C}_\tau(\overline{X}) \cap \mathrm{C}_\omega(\overline{X}) = \bigcup_{z \in S(\tau,\omega)} \mathrm{C}_z(\overline{X}). \tag{2.1.10}$$

In particular,

$$\mathcal{B}_c(\overline{X}) = \big\{\, \{x\} \mid x \in X \,\big\} \cup \big\{\, \mathrm{C}_\omega(\overline{X}) \mid \omega \in X \,\big\} \tag{2.1.11}$$

is a base of a topology $\mathcal{T}_c(\overline{X})$ - the *cone topology* - on $\overline{X}$.

By construction, the subspace X is discrete and open, and the subspace ∂X is closed.

Notation 2.1.14. For $\omega \in \overline{X}$ let $\mathcal{N}_c(\omega)$ denote the set of all open neighborhoods of ω with respect to the cone-topology, and put $\mathcal{S}(\omega) = \bigcap_{U \in \mathcal{N}_c(\omega)} U$.

Then, by construction, one has $\mathcal{S}(\omega) = \{\omega\}$ for $\omega \in X$, and $\mathcal{S}(\omega) = \mathrm{C}_\omega(\overline{X})$ for $\omega \in \partial X$. This implies the following.

Proposition 2.1.15. *Let $(X, \preceq)$ be a poset, and let $(\overline{X}, \preceq)$ denote its completion. Then $(\overline{X}, \mathcal{T}_c(\overline{X}))$ is a T_0-space (or Kolmogorov space).*

Proof. Let $\tau, \omega \in \overline{X}$, $\tau \neq \omega$. If either $\tau \in X$ or $\omega \in X$, then either $\{\tau\}$ or $\{\omega\}$ is an open set. So we may assume that $\tau, \omega \in \partial X$. As $\mathcal{S}(\omega) = \mathrm{C}_\omega(\overline{X})$, either there exists $U \in \mathcal{N}_c(\omega)$, $\tau \notin U$, or $\tau \preceq \omega$. By changing the role of ω and τ, either there exists $V \in \mathcal{N}_c(\tau)$, $\omega \notin V$, or $\omega \preceq \tau$. Since $\tau \preceq \omega$ and $\omega \preceq \tau$ is impossible, this yields the claim. □

Definition 2.1.16. For a partially ordered set $(X, \preceq)$ let

$$\mathscr{S} = \{ \{\tau\}, C_\tau(\overline{X}), C_\tau(\overline{X})^C \mid \tau \in X \} \tag{2.1.12}$$

denote the set of all subsets of X of cardinality 1, all cones and their complements in X. Then $\mathscr{S}$ is a subbasis of a topology $\mathcal{T}_f(\overline{X})$ on $\overline{X}$ which we will call the *fine topology* on $\overline{X}$. In particular, the set $\Omega = \{ X = \bigcap_{1 \le j \le r} X_j \mid X_1, \dots, X_r \in \mathscr{S} \}$ is a base of the topology $\mathcal{T}_f(\overline{X})$.

Note that the identity

$$\mathrm{id}_{\partial M} \colon (\partial M, \mathcal{T}_f(\bar{M})) \longrightarrow (\partial M, \mathcal{T}_c(\bar{M})) \tag{2.1.13}$$

is a continuous map.

Definition 2.1.17. The monoid M will be said to be *$\mathcal{T}$-regular* if $\mathrm{id}_{\partial M}$ is a homeomorphism.

By definition, the fine topology has the following properties.

Proposition 2.1.18. *Let $(X, \preceq)$ be a partially ordered set. Then*

(a) $(\overline{X}, \mathcal{T}_f(\overline{X}))$ is a T_2-space (or Hausdorff space).

(b) $\mathcal{T}_c(\overline{X}) \subseteq \mathcal{T}_f(\overline{X})$.

There is another type of boundary for a poset, the $\sim$-boundary, which we will use later on in this thesis. Before defining it, we need some more notation.

Notation 2.1.19. Let $(X, \preceq)$ be a noetherian poset, and let $(\overline{X}, \preceq)$ denote its completion. We put

$$\Omega = \Delta(X) \sqcup \{ (\varepsilon, \eta) \in \partial X \times \partial X \mid \varepsilon \preceq \eta \}, \tag{2.1.14}$$

where $\Delta(X) = \{ (x, x) \mid x \in X \}$, and let $\hat{\sim}$ denote the equivalence relation on $\overline{X}$ generated by the relation Ω. Then one has a canonical map

$$\pi \colon \overline{X} \to \widetilde{X}, \tag{2.1.15}$$

where $\widetilde{X} = \overline{X}/\hat{\sim}$. By construction, $\pi|_X$ is injective.

Definition 2.1.20. The set $\widetilde{\partial} X = \widetilde{X} \setminus \pi(X)$ will be called the $\sim$-*boundary* of the poset $(X, \preceq)$.

Example 3. Let X be the monoid $X = \langle\, x, y, z \mid xy = yx \,\rangle^+$. For $\sigma, \tau \in X$ put $\sigma \preceq \tau$ if and only if $\sigma = \tau\eta$, for $\eta \in X$. Then one has that $(X, \preceq)$ is a poset and $x^\infty y^\infty \in \widetilde{\partial} X$ is such that $x^\infty y^\infty \preceq y^i x^\infty$ and $x^\infty y^\infty \preceq x^j y^\infty$ for all $i, j \geq 0$. That is, $y^i x^\infty, x^j y^\infty \in [x^\infty y^\infty]^{\hat{\sim}}$ for all $i, j \geq 0$, where $[x^\infty y^\infty]^{\hat{\sim}}$ denotes the equivalence class of $x^\infty y^\infty$ with respect to the relation $\hat{\sim}$. However, whenever a generator z appears in an infinite word ω, one has that $[\omega]^{\hat{\sim}} = \{\omega\}$.

Remark 2.1.21. We put

$$I(\sim) = \{\, (\omega, \tau) \in \overline{X} \times \overline{X} \mid \omega \mathrel{\hat{\sim}} \tau \,\} \subseteq \overline{X} \times \overline{X} \tag{2.1.16}$$

The set $\widetilde{X}$ carries the *quotient topology* $\mathcal{T}_q(\widetilde{X})$ with respect to the mapping π and the topological space $(\overline{X}, \mathcal{T}_c(\overline{X}))$. In particular, the subspace $\pi(X) \subseteq \widetilde{X}$ is discrete and open, and $\widetilde{\partial} X \subseteq \widetilde{X}$ is closed. For $\omega \in \overline{X}$ we put $\widetilde{\mathrm{C}}_\omega = \pi(\mathrm{C}_\omega(\overline{X}))$.

The space $\widetilde{X}$ will be considered merely as topological space. It has the following property.

Proposition 2.1.22. *The topological space $(\widetilde{X}, \mathcal{T}_q(\widetilde{X}))$ is a T_1-space.*

Proof. For $\omega \in \overline{X}$ one has

$$\mathcal{S}(\pi(\omega)) = \pi\Big(\bigcap_{\tau \hat{\sim} \omega} \mathcal{S}(\tau)\Big) = \pi\Big(\bigcap_{\tau \hat{\sim} \omega} \mathrm{C}_\tau(\overline{X})\Big) = \{\pi(\omega)\}. \tag{2.1.17}$$

This yields the claim. □

2.1.3 Monoids

Definition 2.1.23. A *monoid* (or semigroup with unit) M is a set with an associative multiplication $\_ \cdot \_ \colon M \times M \to M$ and a distinguished element $1 \in M$ satisfying $1 \cdot x = x \cdot 1 = x$ for all $x \in M$. For a monoid M we denote by

$$M^\times = \{\, x \in M \mid \exists y \in M \colon x \cdot y = y \cdot x = 1 \,\} \tag{2.1.18}$$

the *maximal subgroup* contained in M. Elements in $M^\sharp = M \setminus M^\times$ will be called *non-invertible.*

Example 4. The set of non-negative integers $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ together with addition is a monoid.

Definition 2.1.24. A *homomorphism* of monoids is a map $\phi \colon Q \to M$ between two monoids Q and M such that $\phi(q \cdot r) = \phi(q) \cdot \phi(r)$ for all $q, r \in Q$ and $\phi(1) = 1$.
A monoid M together with a homomorphism of monoids $|\_| \colon M \to \mathbb{N}_0$ is called an $\mathbb{N}_0$-*graded monoid.*
For $k \in \mathbb{N}_0$ one defines $M_k = \{\, x \in M \mid |x| = k \,\}$. The $\mathbb{N}_0$-graded monoid M is called *connected*, if $M_0 = \{1\}$.
If Q and M are $\mathbb{N}_0$-graded monoids, a homomorphism $\phi \colon Q \to M$ of monoids is a *homomorphism of* $\mathbb{N}_0$*-graded monoids*, if $\phi(Q_k) \subseteq M_k$ for all $k \in \mathbb{N}_0$.

Notation 2.1.25. Let M be a monoid. For $x \in M$, put

$$Mx = \{\, yx \mid y \in M \,\}; \tag{2.1.19}$$
$$xM = \{\, xy \mid y \in M \,\}. \tag{2.1.20}$$

For $x, y \in M$ we define

$$x \preceq y \iff xM \subseteq yM, \tag{2.1.21}$$

i.e., $x \preceq y$ if, and only if, there exists $z \in M$ such that $x = yz$.

The following property is straightforward.

Proposition 2.1.26. *Let $\phi\colon Q \to M$ be a homomorphism of $\mathbb{N}_0$-graded monoids. Then ϕ is monotone, i.e., $x, y \in Q$, $x \preceq y$ implies $\phi(x) \preceq \phi(y)$, and thus induces a monotone map*

$$\mathscr{D}\phi\colon \mathscr{D}(\mathbb{N}, Q, \preceq) \longrightarrow \mathscr{D}(\mathbb{N}, M, \preceq). \tag{2.1.22}$$

Let $\bar{\phi}\colon \bar{Q} \to \bar{M}$ denote the induced map. Let $\partial\bar{\phi}\colon \partial Q \longrightarrow \partial M$ be the map induced by $\bar{\phi}$. Then $\partial\bar{\phi}$ is continuous with repect to the cone topology.

Proof. Let $\tau \in M$. Then the monotony of $\mathscr{D}\phi$ implies that

$$\bar{\phi}^{-1}(C_\tau(\bar{M})) = \bigcup_{y \in \mathscr{S}} C_y(\bar{Q}), \tag{2.1.23}$$

where $\mathscr{S} = \{\, q \in \bar{Q} \mid \bar{\phi}(q) \in C_\tau(\bar{M}) \,\}$. Thus $\bar{\phi}$ and $\partial\phi$ are continuous. □

Definition 2.1.27. Let Y be a set. The *free monoid on* Y is the monoid $F\langle Y \rangle$ whose elements are all words in the alphabet Y, where multiplication is given by concatenation of words and the unit of the monoid is the empty word.

It is straightforward to see that $F\langle Y \rangle$ is naturally $\mathbb{N}_0$-graded. Moreover, $F\langle Y \rangle$ is connected and $F\langle Y \rangle_1 = Y$.

Definition 2.1.28. For an $\mathbb{N}_0$-graded monoid M there exists a canonical homomorphism of $\mathbb{N}_0$-graded monoids

$$\phi_M\colon F\langle M_1 \rangle \longrightarrow M \tag{2.1.24}$$

satisfying $\phi_{M_1} = \mathrm{id}_{M_1}$. The $\mathbb{N}_0$-graded monoid M is said to be 1-*generated*, if ϕ_M is surjective. In particular, such a monoid is connected. By definition, free monoids are 1-generated. Moreover, M is said to be *finitely* 1-*generated*, if it is 1-generated and M_1 is a finite set.

Definition 2.1.29. A monoid M is *left cancellative* if $xy = xz$ implies $y = z$ for all $x, y, z \in M$; it is *right cancellative* if $yx = zx$ implies $y = z$ for all $x, y, z \in M$.

Proposition 2.1.30. *Let M be a left-cancellative monoid. For $x, y \in M$ one has $xM = yM$ if, and only if, there exists $z \in M^\times$ such that $y = xz$.*

Proof. For $z \in M^\times$ one has $zM = M$. If $y = xz$, then one has $yM = xM$, for $x, y \in M$. Viceversa, suppose $xM = yM$ for x, y in M. Then there exist $z, w \in M$ such that $y = xz$ and $x = yw$, so $y = ywz$ and $x = xzw$. Thus left cancellation implies $zw = 1 = wz$, showing that $z, w \in M^\times$. □

Corollary 2.1.31. *Let M be a left-cancellative monoid. Then $(M/M^\times, \preceq)$ is a poset.*

Remark 2.1.32. If left cancellation is replaced by right cancellation, then one has $Mx = My$ if, and only if, there exists $z \in M^\times$ such that $y = zx$.

Proposition 2.1.33. *Let M be a connected $\mathbb{N}_0$-graded monoid. For $x, y \in M$ one has $xM = yM$ if, and only if, $x = y$.*

Proof. Suppose $xM = yM$, for $x, y \in M$. Then there exist $z, w \in M$ such that $x = yz$ and $y = xw$, so $|x| = |y| + |z|$ and $|y| = |x| + |w|$. Thus $|z| = 0 = |w|$. Since M is connected, this implies $z = 1 = w$. □

As a consequence one obtains the following.

Corollary 2.1.34. *Let M be a 1-generated $\mathbb{N}_0$-graded connected monoid. Then $(M, \preceq)$ is a poset. If M is finitely 1-generated, then $(M, \preceq)$ is a noetherian poset.*

Remark 2.1.35. The following example shows that the universal boundary ∂M is in general different from the $\sim$-boundary $\widetilde{\partial} M$. Let $M = \langle x, y, z \mid xz = zx \rangle$. Consider

$$\begin{aligned} &f_1 \colon \mathbb{N} \to M, && f_1(n) = (xz)^n, \\ &f_2 \colon \mathbb{N} \to M, && f_2(n) = x^n, \\ &f_3 \colon \mathbb{N} \to M, && f_3(n) = z^n. \end{aligned} \tag{2.1.25}$$

Then $f_2 \succeq f_1 \preceq f_3$. Hence $\pi(f_1) = \pi(f_2) = \pi(f_3) \in \widetilde{\partial} M$, and $\pi \colon \partial M \to \widetilde{\partial} M$ is not injective.

2.2 Graphs, Trees and the Bass-Serre theory

2.2.1 Graphs

The notion of graph used in this thesis comes from J-P. Serre (cf. [35]).

Definition 2.2.1. A *graph* Γ (in the sense of J-P. Serre) consists of a non-empty set of vertices Γ^0, a set of edges Γ^1, a terminus map $t \colon \Gamma^1 \to \Gamma^0$, an origin map $o \colon \Gamma^1 \to \Gamma^0$ and an edge-reversing map $^- \colon \Gamma^1 \to \Gamma^1$ satisfying

$$\bar{e} \neq e, \quad \bar{\bar{e}} = e, \quad t(\bar{e}) = o(e).$$

Such a graph Γ can be viewed as an *unoriented* (or *undirected*) graph in which each geometric edge is replaced by a pair of edges e and $\bar{e}$.

A *subgraph* Λ of Γ consists of a non-empty subset $\Lambda^0 \subseteq \Gamma^0$ and a subset $\Lambda^1 \subseteq \Gamma^1$ such that $\overline{\Lambda^1} \subseteq \Lambda^1$, $t(\Lambda^1) \subseteq \Lambda^0$ and $o(\Lambda^1) \subseteq \Lambda^0$.

An *orientation* $\Gamma^1_+ \subseteq \Gamma^1$ of Γ is a set of edges containing exactly one edge from each pair $\{e, \bar{e}\}$, $e \in \Gamma^1$

By definition, every subgraph of a graph is a graph. We will use graphs in the sense of Serre in Chapter 4 to construct a Bass-Serre theory for groupoids and undirected graphs in Chapter 3.

Definition 2.2.2. Let Γ and Λ be two graphs. A *graph homomorphism* is a pair of mappings $\psi = (\psi^0, \psi^1)$, where $\psi^0 \colon \Gamma^0 \to \Lambda^0$ and $\psi^1 \colon \Gamma^1 \to \Lambda^1$, which commutes with t, o and $^-$, i.e.,

$$\psi^0(o(e)) = o(\psi^0(e)), \quad \psi^0(t(e)) = t(\psi^0(e)), \quad \psi^1(\bar{e}) = \overline{\psi^1(e)}.$$

A homomorphism of graphs is called an *isomorphism* if ψ^0 and ψ^1 are bijective.

By definition, any subgraph Λ of a graph Γ defines a canonical injective homomorphism of graphs $\iota\colon \Lambda \to \Gamma$.

Definition 2.2.3. Let Γ be a graph. For $v \in \Gamma^0$, we define the *star* of v in Γ by

$$\mathrm{st}_\Gamma(v) = \{e \in \Gamma^1 \mid t(e) = v\}$$

and we call the *valence* of v the cardinality of $\mathrm{st}_\Gamma(v)$. We say that Γ is *locally finite* if each vertex has finite valence, i.e., if $|t^{-1}(v)| < \infty$ for all $v \in \Gamma^0$. A locally-finite graph such that each vertex has valence k will be called k-regular.
A vertex of Γ is said to be *singular* if it has valence one, i.e., if it is the terminus (equivalently, origin) of a unique edge. We say that Γ is *nonsingular* if it has no singular vertices.

Definition 2.2.4. A *path* of length n in Γ is either a vertex $v \in \Gamma^0$ (when $n = 0$), or a sequence of edges $e_1 \dots e_n$ with $o(e_i) = t(e_{i+1})$ for all $i = 1, \dots, n-1$. For a path $p = e_1 \dots e_n$ with $t(e_1) = v$ and $o(e_n) = w$ we say that p is a path from v to w. We put $t(p) = t(e_1)$ and $o(p) = o(e_n)$ and denote by $\ell(p) = n$ the length of the path p. If $e_{i+1} \neq \bar{e}_i$ for all $i = 1, \dots, n-1$, then we say that p is *without backtracking*. A path of length n is said to be *reduced* if either $n = 0$ or if there is no backtracking. We denote by $\mathcal{P}_{v,w}$ the set of all paths from v to w without backtracking. A *cycle* is a path with origin equal to its terminus. We say that Γ is *connected* if $\mathcal{P}_{v,w} \neq \emptyset$ for all $v, w \in \Gamma^0$.
For a path $p = e_1 \dots e_n$, we define the *reversal* path $\bar{p} = \bar{e}_n \cdots \bar{e}_1$.

In a connected graph Γ one defines a distance function $\mathrm{dist}_\Gamma\colon \Gamma^0 \times \Gamma^0 \to \mathbb{N}_0$ by

$$\mathrm{dist}_\Gamma(v, w) = \min\{\, \ell(p) \mid p \in \mathcal{P}_{v,w} \,\}, \quad v, w, \in \Gamma^0, \tag{2.2.1}$$

which satisfies $\mathrm{dist}_\Gamma(v, w) = 0$ if and only if $v = w$.

Note that we use the "Australian" convention for paths in a graph, which is suitable for the operator-algebraic methods used in this thesis.

Definition 2.2.5. A graph Γ is said to be a *tree* if it is connected and for every $v \in \Gamma^0$, the only reduced path which starts and ends at v is the path of length 0 at v. Equivalently, Γ is a tree if $|\mathcal{P}_{v,w}| = 1$ for all $v, w \in \Gamma^0$.

Remark 2.2.6. Let Γ be a connected graph. The set

$$\mathrm{SubTr}(\Gamma) = \{\, \Lambda \subseteq \Gamma \mid \Lambda \text{ is a tree} \,\}$$

with the relation "$\subseteq$" is a non-trivial poset. Moreover, if $\Xi \subseteq \mathrm{SubTr}(\Gamma)$ is a totally-ordered subset of $\mathrm{SubTr}(\Gamma)$, then the graph given by

$$\Gamma(\Xi)^0 = \bigcup_{\Lambda \in \Xi} \Lambda^0, \quad \Gamma(\Xi)^1 = \bigcup_{\Lambda \in \Xi} \Lambda^1, \tag{2.2.2}$$

is a connected subgraph of Γ. Let $u, v \in \Gamma(\Xi)^0$ and suppose that there exist two distinct paths $p, q \in \mathcal{P}_{u,v}(\Gamma(\Xi))$, where $p = e_1 \cdots e_n$ and $q = f_1 \cdots f_m$. As Ξ is totally ordered, there exists a tree $\Lambda \in \Xi$ such that $e_i, f_j \in \Lambda$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$. Hence $p, q \in \mathcal{P}_{u,v}(\Lambda)$, a contradiction. This shows that $\Gamma(\Xi)$ is a tree. Thus, by Zorn's lemma, $\mathrm{SubTr}(\Gamma)$ contains maximal elements.

Definition 2.2.7. Let Γ be a connected graph. The maximal elements in $\mathrm{SubTr}(\Gamma)$ are called *maximal subtrees*.

Remark 2.2.8. Note that for a maximal subtree $T \subseteq \Gamma$ one has that $T^0 = \Gamma^0$ (see [35, Proposition I.11]).

2.2.2 The boundary of a tree

Definition 2.2.9. Let $T = (T^0, T^1)$ be a tree and let $v \in T^0$. A *ray* $\rho = (e_i)_{i\in\mathbb{N}}$ is an infinite reduced path, i.e., $e_i \in T^1$, $o(e_i) = t(e_{i+1})$ and $e_{i+1} \neq \bar{e}_i$ for all $i \in \mathbb{N}$. For any $m > 0$ we define the m-shift of ρ by $\rho[m] = (e_{j+m})_{j\in\mathbb{N}}$. We define a relation $\sim$, called the *shift relation*, on the set of infinite reduced paths of T by $\rho \sim \eta$ if and only if there exist $m, n \geq 0$ such that $\rho[m] = \eta[n]$.

Lemma 2.2.10. *The relation $\sim$ defined above is an equivalence relation.*

Proof. Clearly, $\sim$ is symmetric. It is also reflexive, since $\rho = \rho[0]$ for any infinite reduced path ρ. Finally, it is transitive: suppose that $\rho \sim \eta$ and $\eta \sim \zeta$. Then there exist $m, n, k, l \geq 0$ such that $\rho[m] = \eta[n]$ and $\eta[k] = \zeta[l]$. Without loss of generality, suppose that $m \geq n \geq k \geq l$. Then one has that $\rho[m - n + k] = \zeta[l]$. □

Definition 2.2.11. Let T be a tree and let ρ be a ray in T. We denote by $[\rho]$ the equivalence class of ρ with respect to the shift relation. The *boundary* ∂T of T is given by

$$\partial T = \{\, [\rho] \mid \rho \text{ is a ray in } T \,\}.$$

Equivalently, fixed a base vertex $x_0 \in T$, one has that

$$\partial T = \{\, p = e_1 e_2 \cdots \mid p \text{ is a reduced infinite path, } t(p) = x_0 \,\}.$$

Definition 2.2.12. Let $T = (T^0, T^1)$ be a tree and let $e \in T^1$. Put $T - e = (T^0, T^1 \setminus \{e, \bar{e}\})$. Then $T - e$ is a subgraph containing two connected components. We denote by T_e the connected component of $T - e$ containing the vertex $o(e)$. Then the connected component of $T - e$ containing $t(e)$ concides with $T_{\bar{e}}$. We put $B_e = \partial T_e \subseteq \partial T$.

Proposition 2.2.13. *Let $T = (T^0, T^1)$ be a tree. Then the set $\{\, B_e \mid e \in T^1 \,\}$ is a basis for a topology on ∂T making it a totally-disconnected Hausdorff space. Moreover, if T is locally-finite, ∂T is compact.*

Proof. We first note that $\partial T = B_e \cup B_{\bar{e}}$ for all $e \in T^1$. Fix $e, f \in T^1$. If $e = f$, one has that $B_e = B_f$. Suppose that $e \neq f$. If $f \in T^1_e$, one has two cases: either $\mathrm{dist}_\Gamma((o(e), t(f)) < \mathrm{dist}_\Gamma((o(e), o(f))$, or $\mathrm{dist}_\Gamma((o(e), t(f)) > \mathrm{dist}_\Gamma((o(e), o(f))$. In the first case, one has that $B_f \subseteq B_e$. In the second case, one may express the intersection $B_e \cap B_f$ as follows. Let $p = e_1 \cdots e_n \in \mathcal{P}_{o(e), o(f)}$ and put $\Omega = \{o(e_i), t(e_i) \mid i = 1, \ldots, n\}$. Put

$$E_{e,f} = \Big(\bigcup_{v\in\Omega} \mathrm{st}_T(v) \Big) \setminus \big(\{\bar{e}, \bar{f}\} \cup \{e_i, \bar{e}_i \mid 1 \leq i \leq n\} \big).$$

Then one has

$$B_e \cap B_f = \bigcup_{d\in E(e,f)} B_d.$$

Now suppose that $f \in T^1_{\bar{e}}$. If $\mathrm{dist}_\Gamma((t(e), o(f)) < \mathrm{dist}_\Gamma((t(e), t(f))$, then $B_e \subseteq B_f$. If $\mathrm{dist}_\Gamma((t(e), o(f)) > \mathrm{dist}_\Gamma((t(e), t(f))$, then $B_e \cap B_f = \emptyset$.

Let $[\rho], [\sigma] \in \partial T$, $[\rho] \neq [\sigma]$. Then for any $\rho = (e_i)_{i\in\mathbb{N}} \in [\rho]$ and for any $\sigma = (f_i)_{i\in\mathbb{N}} \in [\sigma]$, there exists $k \in \mathbb{N}$ such that $e_i \neq f_k$ for all $i \in \mathbb{N}$. Thus one has that $[\rho] \in B_{\bar{f}_k}$, $[\sigma] \in B_{f_k}$ and $B_{\bar{f}_k} \cap B_{f_k} = \emptyset$. Hence, ∂T is a Hausdorff space. Moreover, since for any $e \in T^1$ one has that $\partial T = B_e \sqcup B_{\bar{e}}$, where "$\sqcup$"denotes disjoint union, each B_e is also closed. Hence, the component of a point x contains only the point x. This shows that ∂T is totally disconnected.

Finally, suppose that T is locally-finite and fix $x_0 \in T^0$. For $r \geq 0$, let $V_r = \{ v \in T^0 \mid \mathrm{dist}_T(v, x_0) = r \}$. Since T is locally-finite, V_k is finite for all $k \geq 0$ and hence it is also a compact discrete topological space. Let $\pi_r \colon \partial T \to V_r$ be the map defined by $\pi_r(\rho) = t(e_r)$, where $\rho = (e_i)_{i\in\mathbb{N}}$ and e_r is such that $\mathrm{dist}_T(t(e_r), x_0) = r$. We define $\pi_{r,r-1} \colon V_r \to V_{r-1}$ by $\pi_{r,r-1}(v) = v_{r-1}$, where v_{r-1} is the unique vertex in the path from v to x_0 such that $\mathrm{dist}_T(v_{r-1}, x_0) = r-1$. Similarly, we define the map $\pi_{r,k} \colon V_r \to V_k$ for any $k \leq r$. Finally, let $\pi \colon \partial T \to \prod_{k\in\mathbb{N}} V_k$ be definded by $\pi(\rho) = (\pi_k(\rho))_{k\in\mathbb{N}}$. Then, π is continuous and $\mathrm{im}(\partial T) = \varprojlim(V_k, \pi_{r,k})$. Note that $\prod_{k\in\mathbb{N}} V_k$ is compact by Tychonoff's theorem. Since any subspace of $\prod_{k\in\mathbb{N}} V_k$ is Hausdorff and hence $\mathrm{im}(\partial T)$ is a closed subset of $\prod_{k\in\mathbb{N}} V_k$, it is compact. Moreover, $\pi_* \colon \partial T \to \varprojlim(V_k, \pi_{r,k})$ is bijective, and also a homeomorphism. Thus, ∂T is compact. $\square$

2.2.3 Classical Bass-Serre theory

In this section we recall the theory of graphs of groups, also known as Bass-Serre theory. It gives a complete and satisfactory description of groups acting on trees via the structure theorem. Any group action on a tree (satisfying some mild hypotheses) induces a graph of groups, while any graph of groups has a canonical associated group, called the *fundamental group* (cf. Definition 2.2.20), and a tree, called the *Bass–Serre tree* (cf. Definition 2.2.21), such that the fundamental group acts on the Bass–Serre tree. The structure theorem (cf. Theorem 2.2.24) says that these processes are mutually inverse.

Definition 2.2.14. Let Γ be a graph and let G be a group. We say that G *is acting on the graph* Γ (on the left) if Γ^0 and Γ^1 carry the structure of (left) G-sets and if t, o, and $\bar{}$ are morphisms of (left) G-sets. We say that G acts on Γ *without inversion* of edges if $ge \neq \bar{e}$ for all $g \in G$ and $e \in \Gamma^1$.

Definition 2.2.15. Let G be a group acting without inversion of edges on a graph Γ. There is a well-defined *quotient graph* $G\backslash\backslash\Gamma$ given by

$$(G\backslash\backslash\Gamma)^0 = \{ G.v \mid v \in \Gamma^0 \}, \quad (G\backslash\backslash\Gamma)^1 = \{ G.e \mid e \in \Gamma^e \}, \tag{2.2.3}$$

and mappings

$$o(G.e) = G.o(e), \quad t(G.e) = G.t(e), \quad \overline{G.e} = G.\bar{e}. \tag{2.2.4}$$

Since G is acting without reversing any edge, one has that $\overline{G.e} \neq G.e$. Thus, there is a natural projection of graphs $\pi_G = (\pi_G^0, \pi_G^1) \colon \Gamma \to G\backslash\backslash\Gamma$.

Definition 2.2.16. Let Γ be a connected graph. A *graph of groups* $G(\Gamma)$ based on Γ consists of the following data:

(i) a *vertex group* G_v for every vertex $v \in \Gamma^0$;

(ii) an *edge group* G_e for every edge $e \in \Gamma^1$ satisfying $G_e = G_{\bar{e}}$;

(iii) a monomorphism $\alpha_e \colon G_e \to G_{t(e)}$ for every edge $e \in \Gamma^1$.

Definition 2.2.17. Let Γ be a connected graph, and let $G(\Gamma)$ be a graph of groups based on Γ. We define the group $F(G, \Gamma)$ to be the group with generating set

$$\Gamma^1 \sqcup \left(\bigsqcup_{v\in\Gamma^0} G_v \right), \tag{2.2.5}$$

that is, the edge set of the graph Γ together with the elements of the vertex groups of $G(\Gamma)$, and defining relations the relations in the vertex groups, together with:

(R1) $\bar{e} = e^{-1}$ for all $e \in \Gamma^1$;

(R2) $e\alpha_{\bar{e}}(g)\bar{e} = \alpha_e(g)$ for all $e \in \Gamma^1$ and $g \in G_e$.

Definition 2.2.18. A sequence of elements $p = g_1e_1g_2e_2\cdots g_ne_ng_{n+1}$ is called a *generalized path* if $e_1\cdots e_n$ is a path in Γ, $g_i \in G_{t(e_i)}$ for all $1 \le i \le n$ and $g_{n+1} \in G_{o(e_n)}$. As before we put $t(p) = t(e_1)$ and $o(p) = o(e_n)$. We denote by $[p] \in F(G,\Gamma)$ the associated element in $F(G,\Gamma)$. A generalized path $p = g_1e_1g_2e_2\cdots g_ne_ng_{n+1}$ is said to be *reduced* if $n = 0$ and $g_1 \neq 1$, or $n \geq 1$ and $g_i \notin \mathrm{im}(\alpha_{\bar{e}_i})$ for all $i \in \{1,\dots,n\}$ with $e_{i+1} = \bar{e}_i$.

In particular, every generalized path $p = g_1e_1g_2e_2\cdots g_ne_ng_{n+1}$ such that $e_1\cdots e_n$ is a path without backtracking of length $n \geq 1$ in Γ is reduced.

Remark 2.2.19. One has that if p is a reduced generalized path, then $[p] \neq 1$ in (see [35, Theorem I.11]).

Definition 2.2.20. Let $G(\Gamma)$ be a graph of groups and let $v \in \Gamma^0$. Then we define the *fundamental group* of $G(\Gamma)$ based at v by

$$\pi_1(G,\Gamma,v) = \{\, [p] \mid p \text{ is a generalized path with } o(p) = t(p) = v \,\} \subseteq F(G,\Gamma). \tag{2.2.6}$$

Let $T \subseteq \Gamma$ be a maximal subtree of Γ. Then a second version of the *fundamental group* is given by

$$\pi_1(G,\Gamma,T) = F(G,\Gamma)/\langle\, e \in T^1 \,\rangle, \tag{2.2.7}$$

where $\langle e \in T^1\rangle$ denotes the normal subgroup generated by the elements of T^1. Equivalently, $\pi_1(G,\Gamma,T)$ has generating set (2.2.5) as the group $F(G,\Gamma)$ and defining relations the relations in the vertex groups of $G(\Gamma)$ together with (R1), (R2) and the additional relation

(R3) $e = 1$ for all $e \in T^1$.

For each choice of base vertex v and maximal subtree $T \subseteq \Gamma$, the induced projection $F(G,\Gamma) \to \pi_1(G,\Gamma,T)$ restricts to an isomorphism of fundamental groups $\pi_1(G,\Gamma,v) \to \pi_1(G,\Gamma,T)$ (see [35, Proposition I.20]). Thus, up to isomorphism the fundamental group of a graph of groups is independent of the choice of base vertex or of maximal subtree.

Definition 2.2.21. Let $G(\Gamma)$ be a graph of groups based on Γ, let $T \subseteq \Gamma$ be a maximal subtree of Γ and let $\Gamma^1_+ \subseteq \Gamma^1$ be an orientation of Γ such that $o(e) \in T^0$ for all $e \in \Gamma^1_+$. For $e \in \Gamma^1$ we define the map

$$\varepsilon(e) = \begin{cases} 0 & \text{if } e \in \Gamma^1_+, \\ 1 & \text{if } e \in \Gamma^1_- = \Gamma^1 \setminus \Gamma^1_+. \end{cases} \tag{2.2.8}$$

and put $|e| = \Gamma^1_- \cap \{e,\bar{e}\}$. For $e \in \Gamma^1$, we put $H_e = \mathrm{im}(\alpha_e)$. Then the graph $X_{G(\Gamma)}$ given by

$$X^0_{G(\Gamma)} = \bigsqcup_{v\in\Gamma^0} \pi_1(G,\Gamma,T)/G_v[v], \tag{2.2.9}$$

$$X^1_{G(\Gamma)} = \bigsqcup_{e\in\Gamma^1} \pi_1(G,\Gamma,T)/H_{|e|}[e] \tag{2.2.10}$$

with mappings given by

$$o(gH_{|e|}[e]) = ge^{-\varepsilon(e)}G_{o(e)}[o(e)], \tag{2.2.11}$$

$$t(gH_{|e|}[e]) = ge^{1-\varepsilon(e)}G_{t(e)}[t(e)], \tag{2.2.12}$$

$$\overline{gH_{|e|}[e]} = gH_{|e|}[\bar{e}], \tag{2.2.13}$$

for $g \in \pi_1(G, \Gamma, T)$, is called the *Bass-Serre tree* (or *universal cover*) of the graph of groups $G(\Gamma)$. Note that there is a natural left action of $\pi_1(G, \Gamma, T)$ on $X_{G(\Gamma)}$.

Note that o and t are well-defined (see [35, Section I.5.3]). Moreover, the graph $X_{G(\Gamma)}$ is a tree (see [35, Theorem I.12]), so that the terminology used in Definition 2.2.21 is justified.

Definition 2.2.22. Let G be a group acting on a graph Γ without inversion of edges and let $\Lambda = G\backslash\!\backslash\Gamma$ be the quotient graph with canonical projection $\pi_G = (\pi_G^0, \pi_G^1) \colon \Gamma \to \Lambda$. A *fundamental domain* for the G-action on Γ consists of

(i) a subtree $\mathcal{T} \subseteq \Gamma$ such that $\pi_G^0|_{\mathcal{T}^0} \colon \mathcal{T}^0 \to \Lambda^0$ is bijective;

(ii) a subset $\Omega^+ \subseteq \Gamma^1 \setminus \mathcal{T}^1$ such that for $\Omega^- = \{\, \bar{e} \mid e \in \Omega^+ \,\}$ one has that

  (a) $\Omega^+ \cap \Omega^- = \emptyset$;

  (b) for $\Omega = \Omega^+ \sqcup \Omega^-$ and $\Delta^1 = \mathcal{T}^1 \sqcup \Omega$ one has that $\pi_G^1|_{\Delta^1} \colon \Delta^1 \to \Lambda^1$ is bijective;

  (c) $o(e) \in \mathcal{T}^0$ for all $e \in \Omega^+$;

(iii) let $s = (s^0, s^1) \colon \Lambda \to \Gamma$ be the pair of section associated to $(\mathcal{T}^0, \Delta^1)$ and let $\Upsilon^+ = \pi_G^1(\Omega^+)$, i.e.,

$$s^0 = \pi_G^0|_{\mathcal{T}^0}^{-1}, \quad s^1 = \pi_G^1|_{\Delta^1}^{-1}.$$

Then for $e \in \Upsilon^+$ there exists an element $g_e \in G$ satisfying

$$g_e \cdot s^0(t(e)) = t(s^1(e)). \tag{2.2.14}$$

Remark 2.2.23. One has that fundamental domains always exist (see [35, Section I.3.1]). By definition, a fundamental domain defines a maximal subtree $T = \pi_G(\mathcal{T}) \subseteq \Lambda$ of Λ. Moreover, the assignment

(i) $G_v = \mathrm{Stab}_G(s^0(v))$ for $v \in \Lambda^0$;

(ii) $G_e = \mathrm{Stab}_G(s^1(e))$ for $e \in \Lambda^1$;

(iii) for $e \in \Lambda^1 \setminus \Upsilon^+$, the map $\alpha_e \colon G_e \to G_{t(e)}$ is just the canonical inclusion; while for $e \in \Upsilon^+$ one puts

$$\alpha_e \colon G_e \hookrightarrow \mathrm{Stab}_G\big(t(s^1(e))\big) \xrightarrow{i_{g_e^{-1}}} \mathrm{Stab}_G\big(s^0(t(e))\big) = G_{t(e)} \tag{2.2.15}$$

defines a graph of groups, where i_{g_e} denotes left conjugation by the element $g_e \in G$. For simplicity, one puts $g_e = 1$ if $e \in T^1$ and $g_{\bar{e}} = g_e^{-1}$ if $e \in \Upsilon^- = \pi_G^1(\Omega^-)$.

Theorem 2.2.24 (Structure theorem [35, Theorem I.13])**.** *Let G be a group acting on a graph Γ without inversion of edges and let $\mathcal{T} \subseteq \Gamma$ be a maximal subtree of Γ. Let Λ, $T \subseteq \Lambda$ and $(g_e)_{e \in \Lambda^1}$ be given as described in Definition 2.2.22, and let $s \colon \Lambda \to \Gamma$ be the section associated to a fundamental domain. Then one has the following.*

(a) The map $\psi_o\colon \pi_1(G,\Lambda,T) \to G$ *given by*

$$\psi_o|_{G_v} = \mathrm{id}_{G_v}, \quad v \in \Lambda^0, \tag{2.2.16}$$
$$\psi_o(e) = g_e, \quad e \in \Lambda^1, \tag{2.2.17}$$

extends to a homomorphism of groups

$$\psi\colon \pi_1(G,\Lambda,T) \to G. \tag{2.2.18}$$

(b) The map $\Psi = (\Psi^0,\Psi^1)\colon X_{G(\Gamma)} \to \Gamma$ *given by*

$$\Psi^0(gG_v[v]) = \psi(g)\cdot s^0(v), \tag{2.2.19}$$
$$\Psi^1(gH_{|e|}[e]) = \psi(g)\cdot s^1(e), \tag{2.2.20}$$

is a ψ*-equivariant homomorphism of graphs, i.e., for* $w \in X^0_{G(\Gamma)}$ *and* $f \in X^1_{G(\Gamma)}$ *one has*

$$\Psi^0(y\cdot w) = \psi(y)\cdot\Psi^0(w), \tag{2.2.21}$$
$$\Psi^1(y\cdot f) = \psi(y)\cdot\Psi^1(f), \tag{2.2.22}$$

for all $y \in \pi_1(G,\Gamma,\mathcal{T})$*.*

(c) If Γ *is a tree, then* ψ *is an isomorphism of group and* Ψ *is an isomorphism of graphs.*

2.3 Categories and groupoids

We will use two approaches to category theory, categories of structures and a category as an algebraic object. The mathematical structures that we will study are groupoids and topological spaces. We refer the reader to [18] for a full account on category theory.

Definition 2.3.1. A *category* $\mathcal{C}$ consists of the following data:

(i) a class of objects $\mathrm{Ob}(\mathcal{C})$;

(ii) a set of arrows $\mathrm{Arr}(\mathcal{C})$;

(iii) an underlying graph $\Gamma_{\mathcal{C}}$ given by $\Gamma^0_{\mathcal{C}} = \mathrm{Ob}(\mathcal{C})$ and $\Gamma^1_{\mathcal{C}} = \mathrm{Arr}(\mathcal{C})$;

(iv) a family of multiplications

$$\begin{aligned}\mathrm{Arr}(x,y)\times\mathrm{Arr}(y,z) &\to \mathrm{Arr}(x,z)\\ (g,h) &\mapsto gh\end{aligned}$$

satisfying

(1) if $g \in \mathrm{Arr}(x,y)$, $h \in \mathrm{Arr}(y,z)$ and $k \in \mathrm{Arr}(z,w)$, then $(gh)k = g(hk)$;

(2) for all $x \in \mathrm{Ob}(\mathcal{C})$ there exists an element $1_x \in \mathrm{Arr}(x,x)$ such that $1_x x = x$ and $x 1_x = x$ whenever these multiplications are defined.

A small category $\mathcal{C}$ is category where the objects $\mathrm{Ob}(\mathcal{C})$ of $\mathcal{C}$ form a set. A set is a class which is a member of some other class.

Definition 2.3.2. A *groupoid* $\mathcal{G}$ is a small category in which every arrow has an inverse, i.e., for all $x, y \in \text{Ob}(\mathcal{G})$ and $g \in \text{Arr}(x, y)$ there exists an element $g^{-1} \in \text{Arr}(y, x)$ such that $gg^{-1} = 1_x$ and $g^{-1}g = 1_y$.

We denote by $\mathcal{C}(x, y)$ the set of arrows $\text{Arr}(x, y)$ from y to x in $\mathcal{C}$.

Definition 2.3.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *covariant functor* $F: \mathcal{C} \to \mathcal{D}$ assigns to each object $x \in \mathcal{C}$ an object $F(x) \in \mathcal{D}$ and to each arrow $g \in \mathcal{C}(x, y)$ an arrow $F(g) \in \mathcal{D}(F(x), F(y))$ such that $F(1_x) = 1_{F(x)}$ for each $u \in \text{Ob}(\mathcal{C})$ and $F(gh) = F(g)F(h)$ whenever gh is defined.

A *contravariant functor* $F: \mathcal{C} \to \mathcal{D}$ assigns to each object $x \in \mathcal{C}$ an object $F(x) \in \mathcal{D}$ and to each arrow $g \in \mathcal{C}(x, y)$ an arrow $F(g) \in \mathcal{D}(F(y), F(x))$ such that $F(1_x) = 1_{F(x)}$ for each $u \in \text{Ob}(\mathcal{C})$ and $F(gh) = F(h)F(g)$ whenever gh is defined.

A covariant functor $F: \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there exists a a covariant functor $G: \mathcal{D} \to \mathcal{C}$ such that GF and FG are isomorphic to the identity functors $1_{\mathcal{C}}$ and $1_{\mathcal{D}}$, respectively. If such an equivalence exists, we say that te categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*.

Remark 2.3.4. The identity functor $1_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$ is defined to be the identity map on objects and arrows.
An important class of functors are *forgetful functors*. We obtain forgetful functors from a groupoid to a category to a graph, $\mathcal{G} \to \mathcal{C} \to \Gamma_{\mathcal{C}}$, by thinking of a groupoid as a category and a category as a graph.

Examples 5. The category of groups $\mathcal{G}p$ has as objects all groups and as arrows all homomorphisms of groups. We also note that a group G is a category with one object, the identity $1_G \in G$, arrows given by the elements of G, and composition given by the multiplication in G.

The category of graphs $\mathcal{G}ph$ has as objects all graphs and arrows all graph homomorphisms. Composition is given by composing the object and arrow maps in the obvious way. The identity arrow for each object Γ is the identity graph map on $\text{Ob}(\Gamma)$ and $\text{Arr}(\Gamma)$.

Similarly, one defines the categories $\mathcal{S}et$, $\mathcal{T}op$ and $\mathcal{G}pd$ of sets, topological spaces and groupoids respectively.

2.3.1 Groupoids

As we have seen in Definition 2.3.2, a groupoid is a small category with inverses. We now give an equivalent definition which we will use later on in this thesis. All the results in this section are taken from [36].

Definition 2.3.5. A *groupoid* is a set $\mathcal{G}$ together with a multiplication map $(\alpha, \beta) \mapsto \alpha\beta$ from $\mathcal{G}^{(2)}$ to $\mathcal{G}$, where $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ is a distinguished subset of $\mathcal{G} \times \mathcal{G}$ called the set of *composable pairs*, and an inverse map $\gamma \mapsto \gamma^{-1}$ from $\mathcal{G}$ to $\mathcal{G}$ such that the following relations are satisfied:

(i) $(\gamma^{-1})^{-1} = \gamma$ for all $\gamma \in \mathcal{G}$;

(ii) if (α, β) and (β, γ) belong to $\mathcal{G}^{(2)}$, then $(\alpha\beta, \gamma)$ and $(\alpha, \beta\gamma)$ belong to $\mathcal{G}^{(2)}$, and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$;

(iii) $(\gamma^{-1}, \gamma) \in \mathcal{G}^{(2)}$ for all $\gamma \in \mathcal{G}$, and for all $(\gamma, \eta) \in \mathcal{G}^{(2)}$, one has $\gamma^{-1}(\gamma\eta) = \eta$ and $(\gamma\eta)\eta^{-1} = \gamma$.

Definition 2.3.6. Given a groupoid $\mathcal{G}$, we call the set

$$\mathcal{G}^{(0)} = \{ \gamma^{-1}\gamma \mid \gamma \in \mathcal{G} \} \tag{2.3.1}$$

the *unit space* of $\mathcal{G}$ and refer to elements of $\mathcal{G}^{(0)}$ as *units*. We define the maps $r, s \colon \mathcal{G} \to \mathcal{G}^{(0)}$ by

$$r(\gamma) = \gamma\gamma^{-1} \quad \text{and} \quad s(\gamma) = \gamma^{-1}\gamma \tag{2.3.2}$$

for all $\gamma \in \mathcal{G}$, and we call them the *range* map and the *source* map respectively.

The following results show that the Definitions 2.3.5 and 2.3.2 are equivalent.

Lemma 2.3.7 ([36, Lemma 2.1.2])**.** *Let $\mathcal{G}$ be a groupoid. If $\gamma \in \mathcal{G}$, then $\big(r(\gamma), \gamma\big)$, $\big(\gamma, s(\gamma)\big) \in \mathcal{G}^{(2)}$, and*

$$r(\gamma)\gamma = \gamma s(\gamma).$$

One has that $r(\gamma^{-1}) = s(\gamma)$ and $s(\gamma^{-1}) = r(\gamma)$. Moreover, γ^{-1} is the unique element satisfying $(\gamma, \gamma^{-1}) \in \mathcal{G}^{(2)}$ and $\gamma\gamma^{-1} = r(\gamma)$, and also the unique element satisfying $(\gamma^{-1}, \gamma) \in \mathcal{G}^{(2)}$ and $\gamma^{-1}\gamma = s(\gamma)$.

Proof. Let $\gamma \in \mathcal{G}$. Then by (2.3.2) and by (iii) of Defintion 2.3.5 one has that

$$r(\gamma)\gamma = (\gamma\gamma^{-1})\gamma = (\gamma\gamma^{-1})(\gamma^{-1})^{-1} = \gamma = \gamma(\gamma^{-1}\gamma) = \gamma s(\gamma).$$

By definition, $r(\gamma^{-1}) = \gamma^{-1}(\gamma^{-1})^{-1} = \gamma^{-1}\gamma = s(\gamma)$. Suppose that there exists $\eta \in \mathcal{G}$ such that $(\gamma, \eta) \in \mathcal{G}^{(2)}$ and $\gamma\eta = r(\gamma) = \gamma\gamma^{-1}$. Then by (ii) of Defintion 2.3.5 one has that $(\gamma^{-1}\gamma, \eta) \in \mathcal{G}^{(2)}$ $\eta = \gamma^{-1}\gamma\eta = \gamma^{-1}r(\gamma) = \gamma^{-1}s(\gamma^{-1}) = \gamma^{-1}$. Similarly, one shows that if there exists $\beta \in \mathcal{G}$ such that $(\beta, \gamma) \in \mathcal{G}^{(2)}$ and $\beta\gamma = s(\gamma)$, then it must be $\beta = \gamma^{-1}$. □

Lemma 2.3.8 ([36, Lemma 2.1.3])**.** *Let $\mathcal{G}$ be a groupoid. Suppose that $(\alpha, \gamma), (\beta, \gamma) \in \mathcal{G}^{(2)}$ and that $\alpha\gamma = \beta\gamma$. Then $\alpha = \beta$. Similarly, if $(\gamma, \alpha), (\gamma, \beta) \in \mathcal{G}^{(2)}$ and $\gamma\alpha = \gamma\beta$, then $\alpha = \beta$.*

Proof. Let $\alpha, \beta, \gamma \in \mathcal{G}$ be such that $\alpha\gamma = \beta\gamma$. Then combining (ii) and (iii) of Defintion 2.3.5 one has that $\alpha = \alpha\gamma\gamma^{-1} = \beta\gamma\gamma^{-1} = \beta$. □

Lemma 2.3.9 ([36, Lemma 2.1.4])**.** *Let $\mathcal{G}$ be a groupoid. Then $(\alpha, \beta) \in \mathcal{G}^{(2)}$ if and only if $s(\alpha) = r(\beta)$. Moreover, the range, source and inverse maps satisfy the following:*

(1) $r(\alpha\beta) = r(\alpha)$ and $s(\alpha\beta) = s(\beta)$ for all $(\alpha, \beta) \in \mathcal{G}^{(2)}$;

(2) $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$ for all $(\alpha, \beta) \in \mathcal{G}^{(2)}$;

(3) $r(x) = x = s(x)$ for all $x \in \mathcal{G}^{(0)}$.

Proof. Suppose that $(\alpha, \beta) \in \mathcal{G}^{(2)}$. Then by (ii) and (iii) of Definition 2.3.5 one has $(\alpha^{-1}, \alpha\beta) \in \mathcal{G}^{(2)}$ and $\alpha^{-1}\alpha\beta = \beta$. Since $\beta = r(\beta)\beta$ by Lemma 2.3.7, one has that $s(\alpha)\beta = \alpha^{-1}\alpha\beta = r(\beta)\beta$. Thus, $s(\alpha) = r(\beta)$ by Lemma 2.3.8.

Now suppose that $s(\alpha) = r(\beta)$. By definition, one has that $\alpha^{-1}\alpha = \beta\beta^{-1}$. Hence one has that $(\alpha, \beta\beta^{-1}) = (\alpha, \alpha^{-1}\alpha) \in \mathcal{G}^{(2)}$ and $(\alpha^{-1}\alpha, \beta) = (\beta\beta^{-1}, \beta) \in \mathcal{G}^{(2)}$. By (ii) of Definition 2.3.5 one has that $(\alpha, \beta\beta^{-1}\beta) \in \mathcal{G}^{(2)}$, and thus $(\alpha, \beta) \in \mathcal{G}^{(2)}$ since $\beta\beta^{-1}\beta = r(\beta)\beta = \beta$.

We now prove (1) – (3). Let $(\alpha, \beta) \in \mathcal{G}^{(2)}$ By (ii) of Definition 2.3.5 one has that $(r(\alpha), \alpha\beta) \in \mathcal{G}^{(2)}$ and $r(\alpha)(\alpha\beta) = (r(\alpha)\alpha)\beta = \alpha\beta = r(\alpha\beta)\alpha\beta$. Thus, $r(\alpha) = r(\alpha\beta)$ by Lemma 2.3.8. Similarly, one proves that $s(\alpha\beta) = s(\beta)$. Thus, (1) is proved.

Since $s(\alpha) = r(\beta)$, one has that $s(\beta^{-1}) = r(\beta) = s(\alpha) = r(\alpha^{-1})$ and hence $(\beta^{-1}, \alpha^{-1}) \in \mathcal{G}^{(2)}$. Moreover, $s(\alpha\beta) = s(\beta) = r(\beta^{-1}) = r(\beta^{-1}\alpha^{-1})$ by (1), and hence $(\alpha\beta, \beta^{-1}\alpha^{-1}) \in \mathcal{G}^{(2)}$. Again, by (1) one has that $s(\alpha\beta\beta^{-1}) = s(\beta^{-1}) = r(\beta) = s(\alpha) = r(\alpha^{-1})$ and thus $(\alpha\beta\beta^{-1}, \alpha^{-1}) \in \mathcal{G}^{(2)}$. Thus, $(\alpha\beta)(\beta^{-1}\alpha^{-1}) = (\alpha\beta\beta^{-1})\alpha^{-1} = \alpha\alpha^{-1} = r(\alpha) = r(\alpha\beta)$. Hence one has that $\beta^{-1}\alpha^{-1} = (\alpha\beta)^{-1}$ by Lemma 2.3.7, which proves (2).

Finally, let $x = \gamma^{-1}\gamma \in \mathcal{G}^{(0)}$. Then one has $r(x) = r(\gamma^{-1}\gamma) = r(\gamma^{-1}) = s(\gamma) = \gamma^{-1}\gamma = x$ and $s(x) = s(\gamma^{-1}\gamma) = s(\gamma) = \gamma^{-1}\gamma = x$. ☐

Remark 2.3.10. For a groupoid $\mathcal{G}$, one has that

$$\mathcal{G}^{(2)} = \{\, (\alpha, \beta) \in \mathcal{G} \times \mathcal{G} \mid s(\alpha) = r(\beta) \,\} \tag{2.3.3}$$

by Lemma 2.3.9.

Lemma 2.3.11. *Let $\mathcal{G}$ be a groupoid. Then one has that $\mathcal{G}^{(0)} = \{\, \gamma \in \mathcal{G} \mid (\gamma, \gamma) \in \mathcal{G}^{(2)}$ and $\gamma^2 = \gamma \,\}$.*

Proof. For $x \in \mathcal{G}^{(0)}$, one has that $r(x) = x = s(x)$ and hence $x = x\,s(x) = x\,x = x^2$. Thus, the first inclusion is proved. Viceversa, let $\gamma \in \mathcal{G}$ be such that $(\gamma, \gamma) \in \mathcal{G}^{(2)}$ and $\gamma^2 = \gamma$. Then one has that $r(\gamma)\gamma = \gamma = \gamma\gamma$ and thus $\gamma = r(\gamma) \in \mathcal{G}^{(0)}$ by Lemma 2.3.8, which proves the lemma. ☐

Sometimes it is easier to work with the following definition of a groupoid.

Definition 2.3.12. A *groupoid* is a set $\mathcal{G}$ with a distinguished subset $\mathcal{G}^{(0)}$, maps $r, s\colon \mathcal{G} \to \mathcal{G}^{(0)}$, a map $(\alpha, \beta) \mapsto \alpha\beta$ from $\{(\alpha, \beta) \in \mathcal{G} \times \mathcal{G} \mid s(\alpha) = r(\beta)\}$ to $\mathcal{G}$ and an inverse map $\mathcal{G} \to \mathcal{G}$ given by $\gamma \mapsto \gamma^{-1}$ satisfying

(1) $r(x) = x = s(x)$ for all $x \in \mathcal{G}^{(0)}$;
(2) $r(\gamma)\gamma = \gamma = \gamma s(\gamma)$ for all $\gamma \in \mathcal{G}$;
(3) $r(\gamma^{-1}) = s(\gamma)$ and $s(\gamma^{-1}) = r(\gamma)$ for all $\gamma \in \mathcal{G}$;
(4) $\gamma^{-1}\gamma = s(\gamma)$ and $\gamma\gamma^{-1} = r(\gamma)$ for all $\gamma \in \mathcal{G}$;
(5) $r(\alpha\beta) = r(\alpha)$, $s(\alpha\beta) = s(\beta)$ whenever $s(\alpha) = r(\beta)$;
(6) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ whenever $s(\alpha) = r(\beta)$ and $s(\beta) = r(\gamma)$.

Remark 2.3.13. One has that Definition 2.3.5 and Definition 2.3.12 are equivalent: every groupoid in the sense of Definition 2.3.5 satisfies (1)-(6) by Lemmas 2.3.7 and 2.3.9. One the other hand, given the structure above, by putting $\mathcal{G}^{(2)} = \{(\alpha, \beta) \mid s(\alpha) = r(\beta)\}$ one has a groupoid according to Definition 2.3.5: we only have to show that (1)-(6) implies that $(\gamma^{-1})^{-1} = \gamma$. By (2),(4) and (6) one has cancellativity: if $\alpha, \beta, \gamma \in \mathcal{G}$ are such that $s(\alpha) = s(\beta) = r(\gamma)$ and $\alpha\gamma = \beta\gamma$, then one has that

$$\alpha = \alpha\, s(\alpha) = \alpha\, r(\gamma) = \alpha\gamma\gamma^{-1} = \beta\,\gamma\gamma^{-1} = \beta\, r(\gamma) = \beta\, s(\beta) = \beta. \tag{2.3.4}$$

By (3) and (4) one has that $(\gamma^{-1})^{-1}\gamma^{-1} = s(\gamma^{-1}) = r(\gamma) = \gamma\gamma^{-1}$. Then, one concludes that $(\gamma^{-1})^{-1} = \gamma$ by (2.3.4).

Example 6. Every group G can be viewed as a groupoid, with $G^{(0)} = \{1\}$, multiplication given by the group operation, and inversion the usual group inverse. A groupoid is a group if and only if its unit space is a singleton.

Examples 7. (a) Fix $n \geq 1$. Define $R_n = \{1, \ldots, n\} \times \{1, \ldots, n\}$ and put $R_n^{(0)} = \{(i,i) \mid 1 \leq i \leq n\}$, $r(i,j) = (i,i)$, $s(i,j) = (j,j)$ and $(i,j)(j,k) = (i,k)$. Then R_n is a groupoid, and $(i,j)^{-1} = (j,i)$ for all i,j. We usually identify $R_n^{(0)}$ with $\{1, \ldots, n\}$ in the obvious way.
(b) For any set X, the set $R_X := X \times X$ is a groupoid with operations analogous to above. Again, we identify $R_X^{(0)}$ with X.

Example 8. If R is an equivalence relation on a set X, then $R^{(0)} := \{(x,x) \mid x \in X\}$ is contained in R by reflexivity; we identify $R^{(0)}$ with X again. Then R is a groupoid with $r(x,y) = x$, $s(x,y) = y$, $(x,y)(y,z) = (x,z)$ and $(x,y)^{-1} = (y,x)$.

Example 9. Let X be a set and let G be a group acting on X by bijections. Let $\mathcal{G} := G \times X$ and put $\mathcal{G}^{(0)} = \{1\} \times X$. Define $r(g,x) = g \cdot x$, $s(g,x) = x$, $(g, h \cdot x)(h,x) = (gh, x)$ and $(g,x)^{-1} = (g^{-1}, g \cdot x)$. Then $\mathcal{G}$ is a groupoid, called the *transformation groupoid.*

Notation 2.3.14. For $x \in \mathcal{G}^{(0)}$, we write $\mathcal{G}_x = \mathcal{G}x := \{\gamma \in \mathcal{G} \mid s(\gamma) = x\}$ and $\mathcal{G}^x = x\mathcal{G} := \{\gamma \in \mathcal{G} \mid r(\gamma) = x\}$. We write $\mathcal{G}_x^y := \mathcal{G}_x \cap \mathcal{G}^y$.

Definition 2.3.15. A *subgroupoid* $\mathcal{H}$ of a groupoid $\mathcal{G}$, denoted by $\mathcal{H} \leq \mathcal{G}$ is a groupoid with $\mathcal{H}^{(0)} \subseteq \mathcal{G}^{(0)}$, $\mathcal{H} \subseteq \mathcal{G}$ and induced multiplication on $\mathcal{H}$. A subgroupoid is *full* if for any two objects $u, v \in \mathcal{H}^{(0)}$ one has $\mathcal{H}_u^v = \mathcal{G}_u^v$. A subgroupoid $\mathcal{H}$ of $\mathcal{G}$ is said to be *wide* if $\mathcal{H}^{(0)} = \mathcal{G}^{(0)}$. A subgroupoid $\mathcal{N}$ of a groupoid $\mathcal{G}$, denoted by $\mathcal{H} \trianglelefteq \mathcal{G}$ is said to be *normal* if $\mathcal{N}$ is wide in $\mathcal{G}$, i.e., $\mathcal{N}^{(0)} = \mathcal{G}^{(0)}$, and for all $\gamma \in \mathcal{G}_x^y$, $x, y \in \mathcal{G}^{(0)}$, one has $\gamma \mathcal{N}_x^x \gamma^{-1} \subseteq \mathcal{N}_y^y$. A groupoid $\mathcal{G}$ is *connected* if for any objects $x, y \in \mathcal{G}^{(0)}$ there is a path in $\mathcal{G}$ from x to y or, equivalently, $\mathcal{G}_x^y \neq \emptyset$.

The *components* of a groupoid $\mathcal{G}$ are the full connected subgroupoids of $\mathcal{G}$. If the components are all vertex groups, then $\mathcal{G}$ is totally disconnected.

A groupoid $\mathcal{G}$ is *discrete* if $\mathcal{G}$ is totally disconnected and for each object $x \in \mathcal{G}^{(0)}$, the vertex group $\mathcal{G}_x^x$ is the identity group.

Definition 2.3.16. We say that a groupoid $\mathcal{G}$ is *principal* if the map $\gamma \mapsto (r(\gamma), s(\gamma))$ is injective.

Definition 2.3.17. Let $\mathcal{G}$ be a groupoid. The full subgroupoid $\mathcal{G}_x^x$, $x \in \mathcal{G}^{(0)}$ is a group called the *vertex group* at x. We call the *isotropy subgroupoid of* $\mathcal{G}$, or just the isotropy of $\mathcal{G}$, the subset

$$\mathrm{Iso}(\mathcal{G}) = \bigcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}_x^x = \{\, \gamma \in \mathcal{G} \mid r(\gamma) = s(\gamma) \,\}. \tag{2.3.5}$$

It is straightforward to see that the isotropy subgroupoid really is a subgroupoid. Clearly $\mathcal{G}^{(0)} \subseteq \mathrm{Iso}(\mathcal{G})$.

Lemma 2.3.18 ([36, Lemma 2.2.1]). *A groupoid $\mathcal{G}$ is principal if and only if* $\mathrm{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$.

Proof. Suppose that $\mathcal{G}$ is principal and let $\gamma \in \mathrm{Iso}(\mathcal{G})$. Put $x = r(\gamma) = s(\gamma)$. Then one has $(r(\gamma), s(\gamma) = (r(x), s(x))$ and since $\mathcal{G}$ is principal it follows that $\gamma = x \in \mathcal{G}^{(0)}$.

Now suppose that $\mathrm{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$ and let $\gamma, \alpha \in \mathcal{G}$ be such that $(r(\gamma), s(\gamma)) = (r(\alpha), s(\alpha))$. Then $\alpha\gamma^{-1} \in \mathrm{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$ and hence $\alpha\gamma^{-1} = r(\alpha)$. Thus, one concludes that $\alpha = \gamma$ by Lemma 2.3.7. $\square$

2.3.2 Topological groupoids

Since we will work with C^*-algebras, we will want to endow our groupoids with a topology.

Definition 2.3.19. A *topological groupoid* is a groupoid $\mathcal{G}$ endowed with a locally compact topology under which $\mathcal{G}^{(0)} \subseteq \mathcal{G}$ is Hausdorff in the relative topology, the maps r, s and $\gamma \mapsto \gamma^{-1}$ are continuous, and the map $(g, h) \mapsto gh$ is continuous with respect to the relative topology on $\mathcal{G}^{(2)}$ as a subset of $\mathcal{G} \times \mathcal{G}$.

Note the unit space $\mathcal{G}^{(0)}$ of a topological groupoid $\mathcal{G}$ is closed in $\mathcal{G}$ only when $\mathcal{G}$ is Hausdorff.

Lemma 2.3.20 ([36, Lemma 2.3.2])**.** *If $\mathcal{G}$ is a topological groupoid, then $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$ if and only if $\mathcal{G}$ is Hausdorff.*

Proof. Suppose that $\mathcal{G}$ is Hausdorff and let $(x_i)_{i \in I}$ be a net in $\mathcal{G}^{(0)}$ such that $x_i \to \gamma \in \mathcal{G}$. Since r is continuous, one has that $x_i = r(x_i) \to r(\gamma) \in \mathcal{G}^{(0)}$. Since $\mathcal{G}$ is Hausdorff, this limit point is unique, and hence one has that $\gamma = r(\gamma)$.

Now suppose that $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$. To prove that $\mathcal{G}$ is Hausdorff, it suffices to show that any convergent net has a unique limit point. Let $(\gamma_i)_{i \in I}$ be a net in $\mathcal{G}$ and suppose that there exist $\alpha, \beta \in \mathcal{G}$ such that $\gamma_i \to \alpha$ and $\gamma_i \to \beta$. Since both multiplication and inversion are continuous, one has that $\gamma_i^{-1}\gamma_i \to \alpha^{-1}\beta$. Since $\gamma_i^{-1}\gamma_i = s(\gamma_i) \in \mathcal{G}^{(0)}$ and $\mathcal{G}^{(0)}$ is closed, one has that $\alpha^{-1}\beta \in \mathcal{G}^{(0)}$ and hence $\alpha = \beta$. ☐

Example 10. Every groupoid is a topological groupoid in the discrete topology.

Example 11. If X is a second-countable Hausdorff space and R is an equivalence relation on X, then R is a topological groupoid in the relative topology inherited from $X \times X$.

Throughout this thesis, we will only consider second-countable and Hausdorff topological groupoids. In particular, we will focus on étale groupoids. These are the analogue, in the groupoid context, of discrete groups.

Definition 2.3.21. A topological groupoid $\mathcal{G}$ is *étale* if the range map $r\colon \mathcal{G} \to \mathcal{G}$ is a local homeomorphism.

We recall thst a *local homeomorphism* between two topological spaces X and Y is a continuous map $h\colon X \to Y$ such that every $x \in X$ has an open neighbourhood U_x such that $h(U_x) \subseteq Y$ is open and $h\colon U_x \to h(U_x)$ is a homeomorphism.

Note that r is a local homeomorphism as a map from $\mathcal{G}$ to $\mathcal{G}$; not just from $\mathcal{G}$ to $\mathcal{G}^{(0)}$ in the relative topology. One fas the following Lemma.

Lemma 2.3.22 ([36, Lemma 2.4.2])**.** *If $\mathcal{G}$ is an étale groupoid, then $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$.*

Proof. For each $\gamma \in \mathcal{G}$, choose an open set U_γ containing γ such that $r\colon U_\gamma \to r(U_\gamma)$ is a local homeomorphism. Then $\mathcal{G}^{(0)} = \bigcup_{\gamma \in \mathcal{G}} r(U_\gamma)$ is open. ☐

Note that if $\mathcal{G}$ is étale, since $\gamma \mapsto \gamma^{-1}$ is continuous and self-inverse, then the source map $s\colon \mathcal{G} \to \mathcal{G}^{(0)}$ is also a local homeomorphism. So there exist open sets of $\mathcal{G}$ on which r, s are both homeomorphisms.

Definition 2.3.23. A *bisection* of an étale groupoid $\mathcal{G}$ is a subset B such that there exists an open set U containing B such that $r\colon U \to r(U)$ and $s\colon U \to s(U)$ are both homeomorphisms onto open subsets of $\mathcal{G}^{(0)}$.

Lemma 2.3.24 ([36, Lemma 2.4.9]). *Let $\mathcal{G}$ be a second-countable Hausdorff étale groupoid. Then $\mathcal{G}$ has a countable base of open bisections.*

Proof. Let $\{\gamma_n\}$ be a countable dense subset of $\mathcal{G}$. For each γ_n, let $\{U_{n_i}\}_i$ and $\{Vn, i\}_i$ be countable neighbourhood bases such that r is a homeomorphism of each $U_{n,i}$ onto an open set, and s is a homeomorphism of each $V_{n,i}$ onto an open set. Then $\{U_{n,i} \cap V_{n,i} \mid n, i \in \mathbb{N}\}$ is a countable base of open bisections. □

2.3.3 Groupoid homomorphisms and quotient groupoids

The theory of quotient groupoids is modelled on that of quotient groups, but differs from it in important respects. In particular, the First Isomorphism Theorem of group theory (that every surjective morphism of groups is obtained essentially by factoring out its kernel) is no longer true for groupoids, so we need to characterise those groupoid morphisms for which this isomorphism theorem holds.

Definition 2.3.25. Let $\mathcal{G}$ be a groupoid and let $\mathcal{N} \subseteq \mathcal{G}$ be a normal subgroupoid. We define an equivalence relation $\sim$ on $\mathcal{G}$ by

$$\alpha \sim \beta \Leftrightarrow s(\alpha) = s(\beta) \text{ and } \alpha\beta^{-1} \in \mathcal{N}. \tag{2.3.6}$$

Clearly, $\sim$ is reflexive and symmetric. Moreover, it is transitive since $\alpha \sim \beta$ and $\beta \sim \gamma$ implies that $s(\alpha) = s(\beta) = s(\gamma)$ and $\alpha\gamma^{-1} = \alpha\beta^{-1}\beta\gamma^{-1} \in \mathcal{N}$. Hence, $\sim$ is an equivalence relation. The equivalence class of $\sim$ containing α is the set $\alpha\mathcal{N} = \{\, \alpha\beta \mid \beta \in \mathcal{N},\, s(\alpha) = r(\beta)\,\}$ and is called the *left coset* of $\mathcal{N}$ in $\mathcal{G}$. We denote by $\mathcal{G}/\mathcal{N}$ the quotient set $\mathcal{G}/\sim$, i.e., the set of all left cosets of $\mathcal{N}$ in $\mathcal{G}$. Then one has a quotient map $\pi\colon \mathcal{G} \to \mathcal{G}/\mathcal{N}$.

Proposition 2.3.26. *Let $\mathcal{G}$ be a groupoid, let $\mathcal{N} \subseteq \mathcal{G}$ be a normal subgroupoid and let $\pi\colon \mathcal{G} \to \mathcal{G}/\mathcal{N}$ be the quotient map. Then $\mathcal{G}/\mathcal{N}$ is a groupoid with unit space $\pi(\mathcal{G}^{(0)})$, multiplication given by*

$$(\alpha\mathcal{N})(\beta\mathcal{N}) = \alpha\beta\mathcal{N}, \tag{2.3.7}$$

for $\alpha, \beta \in \mathcal{G}$ such that $s(\alpha) = r(\beta)$, and source, range and inverse map given by

$$s(\alpha\mathcal{N}) = s(\alpha)\mathcal{N}, \quad r(\alpha\mathcal{N}) = r(\alpha)\mathcal{N}, \quad (\alpha\mathcal{N})^{-1} = \alpha^{-1}\mathcal{N}, \tag{2.3.8}$$

for $\alpha \in \mathcal{G}$.

Proof. We need to prove that the maps defined above are well defined. Suppose that α, $\beta \in \mathcal{G}$ satisfy $\alpha \sim \beta$. Then, by definition, one has that $s(\alpha) = s(\beta)$. Moreover, since $\alpha\beta^{-1} \in \mathcal{N}$, one has that $r(\alpha) = r(\alpha\beta^{-1}) = s(\alpha\beta^{-1}) = s(\beta^{-1}) = r(\beta)$. Thus, r and s are well defined.

Let $\alpha, \alpha', \beta, \beta' \in \mathcal{G}$ satisfy $\alpha \sim \beta$, $\alpha' \sim \beta'$ and $s(\alpha) = r(\alpha')$. Then one has $s(\alpha) = s(\beta)$ and $r(\alpha') = r(\beta')$ by the argument above. Hence, $s(\beta) = s(\alpha) = r(\alpha') = r(\beta')$. Thus, one has

$$s(\alpha\alpha') = s(\alpha') = s(\beta') = s(\beta\beta')$$

and

$$\alpha\alpha'(\beta\beta')^{-1} = \alpha\alpha'\beta'^{-1}\beta^{-1} = \alpha\alpha'\beta'^{-1}\alpha^{-1}\alpha\beta^{-1} = (\alpha\alpha'\beta'^{-1}\alpha^{-1})(\alpha\beta^{-1}).$$

Since $\mathcal{N}$ is normal, one has that $\alpha\alpha'\beta'^{-1}\alpha^{-1} \in \mathcal{N}$ and since $\alpha \sim \beta$, one has that $\alpha\beta^{-1} \in \mathcal{N}$. Thus, one has that $\alpha\alpha'(\beta\beta')^{-1} \in \mathcal{N}$ and hence $\alpha\alpha' \sim \beta\beta'$. Finally, one has that $(\alpha\mathcal{N})^{-1} = \alpha^{-1}\mathcal{N}$ since $(\alpha\mathcal{N})(\alpha^{-1}\mathcal{N}) = \alpha\alpha^{-1}\mathcal{N} = r(\alpha)\mathcal{N}$ and $(\alpha^{-1}\mathcal{N})(\alpha\mathcal{N}) = \alpha^{-1}\alpha\mathcal{N} = s(\alpha)\mathcal{N}$. □

If $\mathcal{G}$ is a topological groupoid, then we consider the quotient topology as a topology of $\mathcal{G}/\mathcal{N}$. If $\mathcal{N}$ is open, then the projection π is an open map. Moreover, it is a local homeomorphism (see [22, Lemma 2.1.7]).

Definition 2.3.27. Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. A map $\phi\colon \mathcal{G} \to \mathcal{H}$ is a *groupoid homomorphism* if whenever $(\alpha, \beta) \in \mathcal{G}^{(2)}$, then $(\phi(\alpha), \phi(\beta)) \in \mathcal{H}^{(2)}$ and in this case $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$. If ϕ is also bijective then it is called a *groupoid isomorphism.*

Lemma 2.3.28 ([36, Lemma 2.1.12])**.** *If $\phi\colon \mathcal{G} \to \mathcal{H}$ is a groupoid homomorphism, then $\phi(\mathcal{G}^{(0)}) \subseteq \mathcal{H}^{(0)}$. We have $\phi(r(\gamma)) = r(\phi(\gamma))$, $\phi(s(\gamma)) = s(\phi(\gamma))$ and $\phi(\gamma^{-1}) = \phi(\gamma)^{-1}$ for all $\gamma \in \mathcal{G}$.*

Proof. For $x \in \mathcal{G}^{(0)}$, one has that $\phi(x)^2 = \phi(x^2) = \phi(x)$. So, $\phi(x) \in \mathcal{H}^{(0)}$ by Lemma 2.3.11. This proves that $\phi(\mathcal{G}^{(0)}) \subseteq \mathcal{H}^{(0)}$. Let $\gamma \in \mathcal{G}$. Then one has that

$$\phi(r(\gamma))\,\phi(\gamma) = \phi(r(\gamma)\gamma) = \phi(\gamma) = r(\phi(\gamma))\,\phi(\gamma)$$
$$\phi(\gamma)\,\phi(s(\gamma)) = \phi(\gamma s(\gamma)) = \phi(\gamma) = \phi(\gamma)\,s(\phi(\gamma))$$

and thus $\phi(r(\gamma)) = r(\phi(\gamma))$ and $\phi(s\gamma) = s(\phi(\gamma))$ by Lemma 2.3.7. Finally, one has that $\phi(\gamma)\phi(\gamma^{-1}) = \phi(\gamma\gamma^{-1}) = \phi(r(\gamma)) = r(\phi(\gamma))$ and hence $\phi(\gamma^{-1}) = \phi(\gamma)^{-1}$ by Lemma 2.3.7. □

Definition 2.3.29. For a groupoid homomorphism $\phi\colon \mathcal{G} \to \mathcal{H}$, we define the *kernel* and the *image* of ϕ to be the subsets

$$\ker\phi = \{\, g \in \mathcal{G} \mid \phi(g) \in \mathcal{H}^{(0)} \,\},$$
$$\operatorname{im}\phi = \{\, \phi(g) \mid g \in \mathcal{G} \,\},$$

respectively.

One has that $\ker\phi \trianglelefteq \mathcal{G}$ is a normal subgroupoid of $\mathcal{G}$, but, in general, $\operatorname{im}\phi$ is not a subgroupoid of $\mathcal{H}$ (see [4, Proposition 3.11]). Thus, we need to resort to *strong* homomorphisms, which have the same properties as group homomorphisms.

Definition 2.3.30. A groupoid homomorphism $\phi\colon \mathcal{G} \to \mathcal{H}$ is said to be *strong* if for every $(\phi(\alpha), \phi(\beta)) \in \mathcal{H}^{(2)}$ one has that $(\alpha, \beta) \in \mathcal{G}^{(2)}$.

We will need the following result. We omit its proof.

Proposition 2.3.31 ([4, Proposition 3.14])**.** *Let $\phi\colon \mathcal{G} \to \mathcal{H}$ be strong groupoid homomorphism. Then the following hold:*

(i) if $\mathcal{G}' \leq \mathcal{G}$ is a subgroupoid of $\mathcal{G}$, then $\phi(\mathcal{G}') \leq \mathcal{H}$ is a subgroupoid of $\mathcal{H}$;

(ii) if $\mathcal{G}' \trianglelefteq \mathcal{G}$ is a normal subgroupoid of $\mathcal{G}$, then $\phi(\mathcal{G}') \trianglelefteq \mathcal{H}$ is a normal subgroupoid of $\phi(\mathcal{G})$;

(iii) ϕ is injective if and only if $\ker\phi = \mathcal{G}^{(0)}$;

(iv) (The Correspondence Theorem for Groupoids) There exists a one-to-one correspondence between the sets $\mathfrak{A} = \{\, \mathcal{G}' \leq \mathcal{G} \mid \ker\phi \subseteq \mathcal{G}' \,\}$ and $\mathfrak{B} = \{\, \mathcal{H}' \leq \phi(\mathcal{G}) \,\}$. Moreover, this correspondence preserves normal subgroupoids.

2.4 Groupoid actions on topological spaces

The notion of a groupoid action on a space is a straightforward generalization of group actions. Much of this section is inspired by [23] and [12].

Definition 2.4.1. A groupoid $\mathcal{G}$ is said to *act (on the left)* on a set X if there are given a surjective map $\varphi\colon X \to \mathcal{G}^{(0)}$, called the *momentum*, and a map $\mathcal{G} * X \to X$, $(g, x) \mapsto gx$, where

$$\mathcal{G} * X = \{(g, x) \in \mathcal{G} \times X \mid s(g) = \varphi(x)\}, \tag{2.4.1}$$

that satisfy

(A1) $\varphi(gx) = r(g)$ for all $(g, x) \in \mathcal{G} * X$;

(A2) $(g_1, g_2) \in \mathcal{G}^{(2)}, (g_2, x) \in \mathcal{G} * X$ implies $(g_1g_2, x), (g_1, g_2x) \in \mathcal{G} * X$ and

$$g_1(g_2x) = (g_1g_2)x;$$

(A3) $\varphi(x)x = x$ for all $x \in X$.

We say that X is a *left $\mathcal{G}$-set.* The action is said to be *free* if $gx = x$ for some x implies $g = \varphi(x) \in \mathcal{G}^{(0)}$.

Right actions and right $\mathcal{G}$-spaces are defined similarly except that the action is defined on the set $X * \mathcal{G} = \{(x, g) \in X \times \mathcal{G} \mid r(g) = \varphi(x)\}$.

Remark 2.4.2. Let X be a left $\mathcal{G}$-set. Then one defines a relation $\sim$ on X defined by $x \sim y$ if and only if there exists $g \in \mathcal{G}$ such that $y = gx$. Then, $\sim$ is reflexive since $x = \varphi(x)x$ for all $x \in X$. It is also symmetric, since $y = gx$ implies that $x = g^{-1}y$. Finally, it is transitive: suppose that $x \sim y$ and $y \sim z$. Then there exist $g, h \in \mathcal{G}$ such that $y = gx$ and $z = hy$. Since $r(g) = r(y) = s(h)$, one has that $(h, g) \in \mathcal{G}^{(2)}$. Hence one has that $z = hgx$, which implies that $x \sim z$. Thus, $\sim$ is an equivalence relation.

When X is a right $\mathcal{H}$-set the orbit relation is defined similarly and one uses the notation $X/\mathcal{H}$. If X is both a left $\mathcal{G}$-space and a right $\mathcal{H}$-space, one denotes it by $\mathcal{G}\backslash X/\mathcal{H}$.

Definition 2.4.3. Let X be a left $\mathcal{G}$-set. We call the equivalence relation $\sim$ defined above the *orbit relation.* The quotient space of X with respect to this relation is denoted by $\mathcal{G}\backslash X$ and the equivalence classes, i.e., the elements of $\mathcal{G}\backslash X$, are denoted by

$$\mathcal{G}x = \{\, gx \mid g \in \mathcal{G}, s(g) = \varphi(x)\,\}, \quad x \in X. \tag{2.4.2}$$

We define the *stabilizer* of x to be the subgroupoid

$$\mathrm{Stab}_{\mathcal{G}}(x) = \{\, g \in \mathcal{G} \mid gx = x\,\}.$$

Note that $\mathrm{Stab}_{\mathcal{G}}(x)$ is a subgroup of $\mathcal{G}_u^u$, where $u = \varphi(x)$.

Definition 2.4.4. A topological groupoid $\mathcal{G}$ is said to *act (on the left)* on a locally compact space X, if there are given a continuous, open surjection $\varphi\colon X \to \mathcal{G}^{(0)}$, called the *momentum*, and a continuous map $\mathcal{G} * X \to X$, $(g, x) \mapsto gx$, where

$$\mathcal{G} * X = \{(g, x) \in \mathcal{G} \times X \mid s(g) = \varphi(x)\}, \tag{2.4.3}$$

that satisfy (A1)-(A3) of Definition 2.4.1.

Remark 2.4.5. The fibered product $\mathcal{G} * X$ has a natural structure of groupoid, called the *action groupoid* or *semi-direct product* and is denoted by $\mathcal{G} \ltimes X$, where

$$(\mathcal{G} \ltimes X)^{(2)} = \{ \big((g_1, x_1), (g_2, x_2)\big) \mid x_1 = g_2 x_2 \}, \tag{2.4.4}$$

with operations

$$\begin{aligned} (g_1, g_2x_2)(g_2, x_2) &= (g_1g_2, x_2), \\ (g, x)^{-1} &= (g^{-1}, gx), \end{aligned}$$

source and range maps given by

$$\begin{aligned} s(g, x) &= (s(g), x) = (\varphi(x), x) \\ r(g, x) &= (r(g), g \cdot x) = (\varphi(gx), gx), \end{aligned}$$

and the unit space $(\mathcal{G} \ltimes X)^{(0)}$ may be identified with X via the map

$$i \colon X \to \mathcal{G} \ltimes X, \quad i(x) = (\varphi(x), x).$$

Definition 2.4.6. Let $\mathcal{G}$ be a groupoid and let X and X' be left $\mathcal{G}$-sets. A *morphism* of left $\mathcal{G}$-sets is a map $F \colon X \to X'$ such that the following diagrams commute

Definition 2.4.7. A left $\mathcal{G}$-set X is said to be *transitive* if given $x, y \in X$ there exists $g \in \mathcal{G}$ such that $gx = y$.

We now clarify the notions of groupoid cosets and of conjugation between subgroupoids.

Definition 2.4.8. Let $\mathcal{H}$ be a wide subgroupoid of a groupoid $\mathcal{G}$. We define a relation $\sim_{\mathcal{H}}$ on $\mathcal{G}$ by

$$g_1 \sim_{\mathcal{H}} g_2 \iff \text{there exists } h \in \mathcal{H} : \ g_1 = g_2 h. \tag{2.4.5}$$

Remark 2.4.9. One has that $\sim_{\mathcal{H}}$ is an equivalence relation. It is reflexive: $g = gs(g)$ and $s(g) \in \mathcal{H}$ for any $g \in \mathcal{G}$, since $\mathcal{H}$ is wide in $\mathcal{G}$. It is symmetric: if $g_1 = g_2 h$ for $g_1, g_2 \in \mathcal{G}$ and $h \in \mathcal{H}$, then $g_2 = g_1 h^{-1}$ and $h^{-1}\mathcal{H}$ since $\mathcal{H}$ is closed under inversion. Finally, it is transitive: if $g_1 = g_2 h$ and $g_2 = g_3 k$ for $g_1, g_2, g_3 \in \mathcal{G}$ and $h, k \in \mathcal{H}$, then $g_1 = g_3 kh$ and $kh \in \mathcal{H}$ since $s(k) = s(g_2) = r(h)$ and $\mathcal{H}$ is closed under multiplication.

Definition 2.4.10. The equivalence classes of $\sim_{\mathcal{H}}$ are called *cosets* and denoted by

$$g\mathcal{H} = \{ gh \mid h \in \mathcal{H}, s(g) = r(h) \},$$

and the set of cosets is denoted by $\mathcal{G} \backslash \mathcal{H} = \{ g\mathcal{H} \mid g \in \mathcal{G} \}$. As in group theory, one may choose a set of coset representatives called a *transversal*.

Proposition 2.4.11. *Let $\mathcal{H}$ be a wide subgroupoid of a groupoid $\mathcal{G}$. For $g_1, g_2 \in \mathcal{G}$, one has that $g_1 \in g_2\mathcal{H}$ if and only if $g_2 \in g_1\mathcal{H}$. In this case, $g_1\mathcal{H} = g_2\mathcal{H}$.*

Proof. Let $g_1 \in g_2\mathcal{H}$. Then there exists $h \in \mathcal{H}$ such that $g_1 = g_2h$. Thus, $g_2 = g_1h^{-1} \in g_1\mathcal{H}$. Viceversa, if $g_2 \in g_1\mathcal{H}$, then there exists $k \in \mathcal{H}$ such that $g_2 = g_1k$. Then one has that $g_1 = g_2k^{-1} \in g_2\mathcal{H}$. Let $a = g_1l \in g_1\mathcal{H}$. Then $a = g_2hl \in g_2\mathcal{H}$, since $r(l) = s(g_1) = s(h)$. Hence, one has that $g_1\mathcal{H} \subseteq g_2\mathcal{H}$. Similarly, for $b = g_2m \in g_2\mathcal{H}$ one has that $b = g_1km \in g_1\mathcal{H}$, since $r(m) = s(g_2) = s(k)$. Thus, $g_1\mathcal{H} = g_2\mathcal{H}$. □

One has the following straightforward proposition.

Proposition 2.4.12. *Let $\mathcal{H}$ be a wide subgroupoid of a groupoid $\mathcal{G}$. Then $\mathcal{G}\backslash\mathcal{H}$ is a left $\mathcal{G}$-set with momentum map $\varsigma\colon \mathcal{G}\backslash\mathcal{H} \to \mathcal{G}^{(0)}$ given by*

$$\varsigma(g\mathcal{H}) = r(g),$$

and the action $\mathcal{G} * \mathcal{G}\backslash\mathcal{H} \to \mathcal{G}\backslash\mathcal{H}$ defined by

$$(g_1, g_2\mathcal{H}) \mapsto g_1g_2\mathcal{H}.$$

Proof. We have to prove that (A1)-(A3) of Definition 2.4.1 are satisfied. One has that $\mathcal{G} * \mathcal{G}\backslash\mathcal{H} = \{\, (g_1, g_2\mathcal{H}) \in \mathcal{G} \times \mathcal{G}\backslash\mathcal{H} \mid s(g_1) = \varsigma(g_2\mathcal{H}) = r(g_2) \,\}$. Let $(g_1, g_2\mathcal{H}) \in \mathcal{G} * \mathcal{G}\backslash\mathcal{H}$. Then one has that $\varsigma(g_1g_2\mathcal{H}) = r(g_1g_2) = r(g_1)$, which proves (A1). For $g' \in \mathcal{G}$ with $s(g') = r(g_1)$, one has that $(g'g_1, g_2\mathcal{H}), (g', g_1g_2\mathcal{H}) \in \mathcal{G} * \mathcal{G}\backslash\mathcal{H}$ and $(g'g_1)g_2\mathcal{H} = g'g_1g_2\mathcal{H} = g'(g_1g_2\mathcal{H})$, which proves (A2). Finally, one has that $\varsigma(g\mathcal{H})g\mathcal{H} = r(g)g\mathcal{H} = g\mathcal{H}$ for all $g \in \mathcal{G}$, which yields (A3). □

The following proposition from [13] characterizes, as in the classical case of groups, the right cosets by the stabilizer subgroupoid. We omit its proof.

Proposition 2.4.13 ([13, Proposition 3.10])**.** *Let $\mathcal{G}$ be a groupoid acting on the left on a set X. For $x \in X$, let $\mathcal{H} = \mathrm{Stab}_{\mathcal{G}}(x)$. Then one has an isomorphism of left $\mathcal{G}$-sets*

$$\begin{aligned} \mathcal{G}\backslash\mathcal{H} &\to \mathcal{G}x \\ g\mathcal{H} &\mapsto gx. \end{aligned}$$

In Chapter 4 we will use conjugacy in the groupoid context. The concept of conjugation for subgroupoids of a given groupoid is rather recent and unexplored. The following definition is taken from [14, Definition 4.2].

Definition 2.4.14. Let $\mathcal{G}$ be a groupoid and let $\mathcal{K}$, $\mathcal{H}$ be subgroupoids of $\mathcal{G}$. We say that $\mathcal{K}$ and $\mathcal{H}$ are *conjugated* (or *conjugally equivalent*) if there exists a functor $F\colon \mathcal{K} \to \mathcal{H}$ which is an equivalence of categories (cf. Definition 2.3.3) and if there exists a family $\{g_x\}_{x\in\mathcal{K}^{(0)}} \subseteq \mathcal{G}$ such that

(i) $g_x \in \mathcal{G}(F(x), x)$ for all $x \in \mathcal{K}^{(0)}$;

(ii) for all $k \in \mathcal{K}(x_2, x_1)$ one has that $F(k) = g_{x_2}\, k\, g_{x_1}^{-1} \in \mathcal{H}\big(F(x_2), F(x_1)\big)$.

It is shown in [14] that the conjugacy relation is reflexive, symmetric and also transitive, i.e., it is an equivalence relation on the set of all subgroupoids of a given groupoid. In contrast with the classical group setting, conjugated subgroupoids are not necessarily isomorphic. In fact, we only know that conjugated subgroupoids have equivalent underlying categories.

We will use conjugated subgroupoids such that the functor F is an injective equivalence of categories. In particular, one has the following lemma.

Lemma 2.4.15. *Let $\mathcal{G}$ be a groupoid and let $\mathcal{K}$, $\mathcal{H}$ be conjugated subgroupoids of $\mathcal{G}$. Let $i_g\colon \mathcal{K} \to \mathcal{H}$ denote the conjugation map, i.e., $i_g(k) = g_{r(\gamma)}\, k\, g_{s(\gamma)}^{-1}$. If the equivalence of categories $F\colon \mathcal{K} \to \mathcal{H}$ is injective, then i_g is an injective homomorphism of groupoids.*

Proof. We first prove that i_g is a groupoid homomorphism. For $x \in \mathcal{K}^{(0)}$, one has that

$$i_g(x) = g_x x g_x^{-1} = g_x g_x^{-1} = F(x) \in \mathcal{H}^{(0)}. \tag{2.4.6}$$

Moreover, for $k, l \in \mathcal{K}$ such that $s(k) = r(l)$, one has

$$i_g(kl) = g_{r(k)}\, k\, l\, g_{s(l)}^{-1} = g_{r(k)}\, k\, g_{s(k)}^{-1}\, g_{r(l)}\, l\, g_{s(l)}^{-1} = i_g(k)\, i_g(l). \tag{2.4.7}$$

Hence, i_g is a groupoid homomorphism. Suppose that there exist $h, h' \in \mathcal{H}$ such that $i_g(h) = i_g(h')$. Then one has

$$g_{r(h)}\, h\, g_{s(h)}^{-1} = g_{r(h')}\, h'\, g_{s(h')}^{-1}, \tag{2.4.8}$$

which implies that

$$r(g_{r(h)}) = r(g_{r(h')}) \quad \text{and} \quad s(g_{s(h)}^{-1}) = s(g_{s(h')}^{-1}). \tag{2.4.9}$$

Hence, by (i) of Definition 2.4.14, one has that

$$F(r(h)) = F(r(h')) \quad \text{and} \quad F(s(h)) = F(s(h')). \tag{2.4.10}$$

Since F is injective, this implies that $r(h) = r(h')$ and $s(h) = s(h')$. Then one has that

$$\begin{aligned} h &= g_{r(h)}^{-1}\, g_{r(h)}\, h\, g_{s(h)}^{-1}\, g_{s(h)} \\ &= g_{r(h)}^{-1}\, g_{r(h')}\, h'\, g_{s(h')}^{-1}\, g_{s(h)} \\ &= g_{r(h')}^{-1}\, g_{r(h')}\, h'\, g_{s(h')}^{-1}\, g_{s(h')} \\ &= h'. \end{aligned} \tag{2.4.11}$$

That is, i_g is injective. □

2.4.1 Groupoids actions on graphs

Definition 2.4.16. Let $\Gamma = (\Gamma^0, \Gamma^1, o, t)$ be a topological graph, i.e. Γ^0 and Γ^1 are locally compact spaces, $o, t\colon \Gamma^1 \to \Gamma^0$ are continuous maps and o is a local homeomorphism. We say that a topological groupoid $\mathcal{G}$ *acts on* Γ if there exist a continuous open surjection $\varphi\colon \Gamma^0 \to \mathcal{G}^{(0)}$, called the *momentum*, such that

$$\varphi \circ o = \varphi \circ t\colon \Gamma^1 \to \mathcal{G}^{(0)} \tag{2.4.12}$$

and continuous maps

$$\mu^0\colon \mathcal{G} * \Gamma^0 \to \Gamma^0 \quad \text{and} \quad \mu^1\colon \mathcal{G} * \Gamma^1 \to \Gamma^1 \tag{2.4.13}$$

which satisfy (A1) – (A3) and

$$o\big(\mu^1(\gamma, e)\big) = \mu^0(\gamma, o(e)), \tag{2.4.14}$$

$$t\big(\mu^1(\gamma, e)\big) = \mu^0(\gamma \cdot t(e)) \tag{2.4.15}$$

for all $(\gamma, e) \in \mathcal{G} * \Gamma^1$.

Note that since φ and o are open maps, $\varphi \circ o$ is open.

Remark 2.4.17. Since $\varphi \circ o = \varphi \circ t$, if $\mathcal{G}^{(0)}$ is discrete one has that Γ is a union of graphs $x\Gamma$ for $x \in \mathcal{G}^{(0)}$, where

$$x\Gamma^0 = \varphi^{-1}(x), \tag{2.4.16}$$

$$x\Gamma^1 = t^{-1}\big(\varphi^{-1}(x)\big). \tag{2.4.17}$$

Definition 2.4.18. Let $\mathcal{G}$ be a groupoid acting on a graph $\Gamma = (\Gamma^0, \Gamma^1)$. Then we define $\mathcal{G}\backslash\backslash\Gamma$ by

$$(\mathcal{G}\backslash\backslash\Gamma)^0 = \{\, \mu^0(\mathcal{G} * x) \mid x \in \Gamma^0 \,\}, \tag{2.4.18}$$

$$(\mathcal{G}\backslash\backslash\Gamma)^1 = \{\, \mu^1(\mathcal{G} * e) \mid e \in \Gamma^1 \,\}. \tag{2.4.19}$$

Then one has maps

$$o\colon (\mathcal{G}\backslash\backslash\Gamma)^1 \to (\mathcal{G}\backslash\backslash\Gamma)^0, \tag{2.4.20}$$

$$t\colon (\mathcal{G}\backslash\backslash\Gamma)^1 \to (\mathcal{G}\backslash\backslash\Gamma)^0, \tag{2.4.21}$$

$$^{-}\colon (\mathcal{G}\backslash\backslash\Gamma)^1 \to (\mathcal{G}\backslash\backslash\Gamma)^1, \tag{2.4.22}$$

given by

$$o(\mu^1(\mathcal{G} * e)) = \mu^0(\mathcal{G} * o(e)), \tag{2.4.23}$$

$$t(\mu^1(\mathcal{G} * e)) = \mu^0(\mathcal{G} * t(e)), \tag{2.4.24}$$

$$\overline{\mu^1(\mathcal{G} * e)} = \mu^1(\mathcal{G} * \bar{e}), \tag{2.4.25}$$

satisfying for all $e \in \Gamma^1$

(i) $\overline{\overline{\mu^1(\mathcal{G} * e)}} = \mu^1(\mathcal{G} * e)$;

(ii) $o(\overline{\mu^1(\mathcal{G} * e)}) = t(\mu^1(\mathcal{G} * e))$ and $t(\overline{\mu^1(\mathcal{G} * e)}) = o(\mu^1(\mathcal{G} * e))$.

However, $\overline{\mu^1(\mathcal{G} * e)} \neq \mu^1(\mathcal{G} * e)$ is not necessarily satisfied.

Definition 2.4.19. Let $\mathcal{G}$ be a groupoid acting on a graph Γ. We say that $\mathcal{G}$ *is acting without inversion of edges* if for all $(\gamma, e) \in \mathcal{G} * \Gamma^1$ one has

$$\mu^1(\gamma, e) \neq \bar{e}.$$

Then, for a groupoid $\mathcal{G}$ acting on a graph Γ, one has that $\mathcal{G}\backslash\backslash\Gamma$ is a graph if and only if $\mathcal{G}$ is acting on Γ without inversion of edges. Thus, one has a projection of graphs $\pi_G\colon \Gamma \to \mathcal{G}\backslash\backslash\Gamma$.

Definition 2.4.20. Let $\mathcal{G}$ be a groupoid acting without inversion on a graph Γ. We say that Γ is a *$\mathcal{G}$-forest* if Γ_x is a tree for every $x \in \mathcal{G}^{(0)}$.

2.4.2 Cayley graphs

Definition 2.4.21. Let $\mathcal{G}$ be a groupoid. A subset $\mathcal{S}$ of $\mathcal{G}$ is said to be an *admissible system* for $\mathcal{G}$ if it does not contain any unit of $\mathcal{G}$ and if $\mathcal{S} = \mathcal{S}^{-1}$, where $\mathcal{S}^{-1} = \{s^{-1} \mid s \in \mathcal{S}\}$. We say that an admissible system $\mathcal{S}$ is a *generating system* if the groupoid generated by $\mathcal{S}$ coincides with $\mathcal{G}$.

Definition 2.4.22. Given a groupoid $\mathcal{G}$ and a generating system $\mathcal{S}$ of $\mathcal{G}$, the *Cayley graph* $\Gamma(\mathcal{G}, \mathcal{S})$ of $(\mathcal{G}, \mathcal{S})$ is the graph defined by

(i) $$\Gamma(\mathcal{G}, \mathcal{S})^0 = \mathcal{G}$$

(ii) $$\Gamma(\mathcal{G}, \mathcal{S})^1 = \{\, (gs, g) \mid (g, s) \in \mathcal{G} * \mathcal{S} \,\}$$

where the maps t, o and $^-$ are given by

$$t\big((gs, g)\big) = gs, \quad o\big((gs, g)\big) = g, \quad \overline{(gs, g)} = (g, gs).$$

Since for any unit $x \in \mathcal{G}^{(0)}$ one has $x \notin \mathcal{S}$, one has that $\Gamma(\mathcal{G}, \mathcal{S})^1 \subseteq \mathcal{G} * \mathcal{G} \setminus \Delta(\mathcal{G})$, where $\Delta(\mathcal{G}) = \{\, (g, g) \mid g \in \mathcal{G} \,\}$.

Remark 2.4.23. Note that the Cayley graph $\Gamma(\mathcal{G}, \mathcal{S})$ is fibered on $\mathcal{G}^{(0)}$ via the map $\varphi = r \colon \Gamma(\mathcal{G}, \mathcal{S})^0 = \mathcal{G} \to \mathcal{G}^{(0)}$. For x in $\mathcal{G}^{(0)}$ we denote by $\Gamma(\mathcal{G}, \mathcal{S})_x$ the fiber of x, i.e.,

$$\Gamma(\mathcal{G}, \mathcal{S})^0_x = \varphi^{-1}(x),$$
$$\Gamma(\mathcal{G}, \mathcal{S})^1_x = o^{-1}(\varphi^{-1}(x)).$$

Since for $(gs, s) \in \Gamma(\mathcal{G}, \mathcal{S})^1_x$ and $(hr, r) \in \Gamma(\mathcal{G}, \mathcal{S})^1_y$ one has that $\varphi(o(gs, s)) = \varphi(t(gs, s)) = x$ and $\varphi(o(hr, r)) = \varphi(t(hr, r)) = y$, one has that $\Gamma(\mathcal{G}, \mathcal{S})_x$ and $\Gamma(\mathcal{G}, \mathcal{S})_y$ are disjoint for any $x \neq y$.

Proposition 2.4.24. *Let $\mathcal{G}$ be a groupoid and $\mathcal{S} \subseteq \mathcal{G}$ be an admissible system. Then the following are equivalent*

(i) S is a generating system;

(ii) $\Gamma(\mathcal{G}, \mathcal{S})_x$ is connected for any $x \in \mathcal{G}^{(0)}$.

Proof. Fix $x \in \mathcal{G}^0$ Suppose that (i) holds and let $g, h \in x\mathcal{G} = \varphi^{-1}(x)$ such that $g \neq h$. Since $\mathcal{S}$ generates $\mathcal{G}$, there exist elements $a_1, \dots, a_n \in \mathcal{S}$ such that $s(a_i) = r(a_{i+1})$, $s(h) = r(a_1)$, $s(g) = s(a_n)$ and $h^{-1}g = a_1 \cdots a_n$. Hence one has $g = ha_1 \cdots a_n$, and thus

$$(h, ha_1)\,(ha_1, ha_1a_2) \cdots (ha_1a_2 \cdots a_{n-1}, ha_1a_2 \cdots a_n)$$

is a path from h to g. Thus, $\Gamma(\mathcal{G}, \mathcal{S})_x$ is connected.

Now suppose that (ii) holds and let $g \in \mathcal{G}$ with $r(g) = x$. Then there exists a path from x to g given by

$$(x, b_1)\,(b_1, b_2) \cdots (b_{m-1}, b_m)$$

where $r(b_i) = x$ for all $i = 1, \dots, m$ and $b_m = g$. By definition, $b_1 \in \mathcal{S}$ and $b_{j+1}^{-1} b_j \in \mathcal{S}$. So, by induction, $g = b_m \in \langle \mathcal{S} \rangle$. Thus $\mathcal{S}$ generates $\mathcal{G}$. □

2.5 Groupoids C^*-algebras

In this section we define the universal C^*-algebra of an étale groupoid. The definitions and proofs in this section are taken from [36, Chapter 3]. We refer the reader to [32] for a full account on groupoids C^*-algebras.

Let

$$\mathcal{C}_c(\mathcal{G}) = \{\, f \colon \mathcal{G} \to \mathbb{C} \mid f \text{ is continuous and } \operatorname{supp}(f) \text{ is compact} \,\}.$$

Then $\mathcal{C}_c(\mathcal{G})$ is a complex vector space with the obvious linear structure. We first discuss the convolution product on $\mathcal{C}_c(\mathcal{G})$ and then its C^*-completion $C^*(\mathcal{G})$.

Proposition 2.5.1 ([36, Proposition 3.1.1])**.** *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. For $f, g \in \mathcal{C}_c(\mathcal{G})$ and $\gamma \in \mathcal{G}$, the set*

$$\{ (\alpha, \beta) \in \mathcal{G}^{(2)} \mid \alpha\beta = \gamma \text{ and } f(\alpha)g(\beta) \neq 0 \}$$

is finite. Then $\mathcal{C}_c(\mathcal{G})$ is a *-algebra with multiplication and involution given by

$$(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta), \tag{2.5.1}$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}, \tag{2.5.2}$$

respectively. For $f, g \in \mathcal{C}_c(\mathcal{G})$ one has that $\mathrm{supp}(f * g) \subseteq \mathrm{supp}(f)\mathrm{supp}(g)$.

Proof. Let $\gamma \in \mathcal{G}$ and let $\alpha, \beta \in \mathcal{G}$ be such that $\alpha\beta = \gamma$. Then $\alpha \in \mathcal{G}^{r(\gamma)}$ and $\beta \in \mathcal{G}_{s(\gamma)}$. Since $\mathcal{G}$ is étale, both $\mathcal{G}^{r(\gamma)}$ and $\mathcal{G}_{s(\gamma)}$ are discrete sets. Hence, their intersections with the compact sets $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ are finite. It remains to prove that $\mathcal{C}_c(\mathcal{G})$ is a *-algebra with multiplication and involution defined as above. For $f \in \mathcal{C}_c(\mathcal{G})$ and $\gamma \in \mathcal{G}$, one has that

$$(f^*)^*(\gamma) = \overline{f^*(\gamma^{-1})} = \overline{\overline{f(\gamma)}} = f(\gamma). \tag{2.5.3}$$

Let $g \in \mathcal{C}_c(\mathcal{G})$. Then one has

$$\begin{aligned}
(f * g)^*(\gamma) &= \overline{(f * g)(\gamma^{-1})} \\
&= \sum_{\alpha\beta=\gamma^{-1}} \overline{f(\alpha)g(\beta)} \\
&= \sum_{\alpha\beta=\gamma^{-1}} \overline{f(\alpha)}\,\overline{g(\beta)} \\
&= \sum_{\beta^{-1}\alpha^{-1}=\gamma} \overline{g(\beta)}\,\overline{f(\alpha)} \\
&= \sum_{\beta^{-1}\alpha^{-1}=\gamma} g^*(\beta^{-1})\, f^*(\alpha^{-1}) \\
&= (g^* * f^*)(\gamma).
\end{aligned} \tag{2.5.4}$$

Finally, for $c \in \mathbb{C}$ one has that

$$\begin{aligned}
(f + cg)^*(\gamma) &= \overline{(f + cg)(\gamma^{-1})} \\
&= \overline{f(\gamma^{-1}) + cg(\gamma^{-1})} \\
&= \overline{f(\gamma^{-1})} + \bar{c}\overline{g(\gamma^{-1})} \\
&= f^*(\gamma) + \bar{c}g^*(\gamma) \\
&= (f^* + \bar{c}g^*)(\gamma).
\end{aligned} \tag{2.5.5}$$

Thus, $\mathcal{C}_c(\mathcal{G})$ is a *-algebra. □

It will be helpful for the calculations in the next sections to know that the convolution algebra $\mathcal{C}_c(\mathcal{G})$ is spanned by those functions in $\mathcal{C}_c(\mathcal{G})$ whose supports are contained in a bisection.

Lemma 2.5.2 ([36, Lemma 3.1.3])**.** *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. Then*

$$\mathcal{C}_c(\mathcal{G}) = \mathrm{span}\{ f \in \mathcal{C}_c(\mathcal{G}) \mid \mathrm{supp}(f) \text{ is a bisection} \}.$$

Proof. Let $f \in \mathcal{C}_c(\mathcal{G})$. Since $\mathcal{G}$ is étale, it has a countable base of open bisections. Thus, $\operatorname{supp}(f)$ can be covered by open bisections $\{V_i\}_{i\in\mathbb{N}}$, i.e., $\operatorname{supp}(f) \subseteq \cup_{i\in\mathbb{N}} V_i$. Since $\mathcal{G}$ is locally compact, there exists a finite subcover $\{U_i\}_{i=1}^n$ such that $\operatorname{supp}(f) \subseteq \bigcup_{i=1}^n U_i$. Let $\{h_i\}$ be a partition of unity on $\bigcup_{i=1}^n U_i$ subordinate to the U_i, i.e., $\operatorname{supp}(h_i) \subseteq U_i$ for all $1 \le i \le n$ and $\sum_i h_i(x) = 1$ for all $x \in \mathcal{G}$. Let $f_i = f \cdot h_i$ be the pointwise product of f with h_i, $i = 1, \dots, n$. Then one has that $f = \sum_{i=1}^n f_i$ and $\operatorname{supp}(f_i) \subseteq U_i$ for all $1 \le i \le n$, and hence $f_i \in \mathcal{C}_c(\mathcal{G})$. ☐

Lemma 2.5.3 ([36, Lemma 3.1.3])**.** *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. If $U, V \subseteq \mathcal{G}$ are open bisections and $f, g \in \mathcal{C}_c(\mathcal{G})$ are such that $\operatorname{supp}(f) \subseteq U$ and $\operatorname{supp}(g) \subseteq V$, then $\operatorname{supp}(f * g) \subseteq UV$ and for $\gamma = \alpha\beta \in UV$, one has*

$$(f * g)(\gamma) = f(\alpha)g(\beta). \tag{2.5.6}$$

One has $\mathcal{C}_c(\mathcal{G}^{(0)}) \subseteq \mathcal{C}_c(\mathcal{G})$. If $f \in \mathcal{C}_c(\mathcal{G})$ is such that $\operatorname{supp}(f)$ is a bisection, then $f^* * f \in \mathcal{C}_c(\mathcal{G}^{(0)})$ is supported on $s(\operatorname{supp}(f))$ and $(f^* * f)(s(\gamma)) = |f(\gamma)|^2$ for any $\gamma \in \operatorname{supp}(f)$. Similarly, $f * f^* \in \mathcal{C}_c(\mathcal{G}^{(0)})$ is supported on $r(\operatorname{supp}(f))$ and $(f * f^*)(\gamma) = |f(\gamma)|^2$ for $\gamma \in \operatorname{supp}(f)$.

Proof. Let $\gamma = \alpha\beta \in UV$. One has that $(f * g)(\gamma) = \sum_{\eta\zeta=\gamma} f(\eta)g(\zeta)$ by definition. For any η, ζ appearing in the sum, one has that $r(\eta) = r(\gamma)$ and $s(\zeta) = s(\gamma)$. Since f and g are supported on bisections and since $\alpha \in \mathcal{G}^{r(\gamma)} \subseteq \operatorname{supp}(f)$ and $\beta \in \mathcal{G}_{s(\gamma)} \subseteq \operatorname{supp}(g)$, the only nonzero term in the sum is $f(\alpha)g(\beta)$.

Since $\mathcal{G}$ is étale, $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$. Then, one may regard $\mathcal{C}_c(\mathcal{G}^{(0)})$ as a subalgebra of $\mathcal{C}_c(\mathcal{G})$ as follows: $f \in \mathcal{C}_c(\mathcal{G}^{(0)})$ can be extended to a function in $\mathcal{C}_c(\mathcal{G})$ which agrees with f on $\mathcal{C}_c(\mathcal{G}^{(0)})$ and vanishes on its complement. Let $f \in \mathcal{C}_c(\mathcal{G})$ be supported on a bisection. Then clearly $f^* * f$ is supported on $s(\operatorname{supp}(f))$ by (2.5.6). For $\gamma \in \operatorname{supp}(f)$, one has

$$(f^* * f)(s(\gamma)) = f^*(\gamma^{-1})\, f(\gamma) = \overline{f(\gamma)}\, f(\gamma) = |f(\gamma)|^2.$$

Similarly, one proves the remaining statements. ☐

The following results from [36] allows us to define the universal C^*-algebra of an étale groupoid.

Proposition 2.5.4 ([36, Proposition 3.2.1])**.** *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. For each $f \in \mathcal{C}_c(\mathcal{G})$, there is a constant $K_f \ge 0$ such that $\|\pi(f)\| \le K_f$ for every *-representation $\pi\colon \mathcal{C}_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ of $\mathcal{C}_c(\mathcal{G})$. If f is supported on a bisection, we can take $K_f = \|f\|_\infty$.*

Proof. Let $f \in \mathcal{C}_c(\mathcal{G})$. By Lemma 2.5.2 one has $f = \sum_{i=1}^n f_i$, where f_i is supported on a bisection, $i = 1, \dots, n$. Put $K_f = \sum_{i=1}^n \|f_i\|_\infty$. Let π be a *-representation of $\mathcal{C}_c(\mathcal{G})$. Then the restriction $\pi|_{\mathcal{C}_c(\mathcal{G}^{(0)})}$ is a *-representation of the commutative algebra $\mathcal{C}_c(\mathcal{G}^{(0)})$. Hence, $\|\pi(h)\| \le \|h\|_\infty$ for every $h \in \mathcal{C}_c(\mathcal{G}^{(0)})$. For any $i = 1 \dots n$ one has that $f_i^* * f_i$ is supported on $\mathcal{G}^{(0)}$ and $\|f_i^* * f_i\|_\infty = \|f_i\|_\infty$ by Lemma 2.5.3. Thus, one has

$$\|\pi(f_i)\|^2 = \|\pi(f_i)^*\pi(f_i)\| = \|\pi(f_i^* * f_i)\| \le \|f_i^* * f_i\|_\infty = \|f_i\|_\infty^2, \tag{2.5.7}$$

which implies that $\|\pi(f_i)\| \le \|f_i\|_\infty$ for eachf_i. By the triangle inequality one has that

$$\|\pi(f)\| = \left\| \sum_{i=1}^n \pi(f_i) \right\| \le \sum_{i=1}^n \|\pi(f_i)\| \le \sum_{i=1}^n \|f_i\|_\infty = K_f, \tag{2.5.8}$$

that is, $\|\pi(f)\| \le K_f$. Finally, if f is supported on a bisection, then there is just one term in the sum $f = \sum_{i=1}^n f_i$, so $K_f = \|f\|_\infty$. ☐

Theorem 2.5.5 ([36, Theorem 3.2.2])**.** *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. Then there exists a C^*-algebra $C^*(\mathcal{G})$ and a *-homomorphism $\pi_{\max}\colon \mathcal{C}_c(\mathcal{G}) \to C^*(\mathcal{G})$ such that $\pi_{\max}(\mathcal{C}_c(\mathcal{G}))$ is dense in $C^*(\mathcal{G})$ and such that for every *-representation $\pi\colon C_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ there is a representation $\psi\colon C^*(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ such that $\psi \circ \pi_{\max} = \pi$. The norm on $C^*(\mathcal{G})$ satisfies*

$$\|\pi_{\max}(f)\| = \sup \left\{ \|\pi(f)\| \mid \pi \text{ is a } *\text{-representation of } \mathcal{C}_c(\mathcal{G}) \right\}$$

for all $f \in \mathcal{C}_c(\mathcal{G})$.

Proof. For each $f \in \mathcal{C}_c(\mathcal{G})$, the set $\{\pi(f) \mid \pi$ is a *-representation of $\mathcal{C}_c(\mathcal{G})\}$ is bounded above by Proposition 2.5.4. Moreover, it is nonempty because it contains $\pi_0(f)$, where π_0 is the zero representation. Hence we define $\rho\colon \mathcal{C}_c(\mathcal{G}) \to [0,\infty)$ by

$$\rho(f) = \sup\{ \|\pi(f)\| \mid \pi\colon \mathcal{C}_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H}) \text{ is a } ^*\text{-representation} \}. \tag{2.5.9}$$

Clearly, $\rho(f) \geq 0$ for all $f \in \mathcal{C}_c(\mathcal{G})$. For $f, g \in \mathcal{C}_c(\mathcal{G})$ and $\lambda \in \mathbb{C}$ one has

$$\begin{aligned} \rho(\lambda f) &= \sup\{ \|\pi(\lambda f)\| \mid \pi\colon \mathcal{C}_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H}) \text{ is a } ^*\text{-representation} \} \\ &= \sup\{ |\lambda|\, \|\pi(f)\| \mid \pi\colon \mathcal{C}_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H}) \text{ is a } ^*\text{-representation} \} \\ &= |\lambda| \sup\{ \|\pi(f)\| \mid \pi\colon \mathcal{C}_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H}) \text{ is a } ^*\text{-representation} \} \\ &= |\lambda|\, \rho(f), \end{aligned} \tag{2.5.10}$$

and

$$\begin{aligned} \rho(f+g) &= \sup\{ \|\pi(f+g)\| \mid \pi\colon \mathcal{C}_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H}) \text{ is a } ^*\text{-representation} \} \\ &= \sup\{ \|\pi(f)+\pi(g)\| \mid \pi\colon \mathcal{C}_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H}) \text{ is a } ^*\text{-representation} \} \\ &\leq \sup\{ \|\pi(f)\| + \|\pi(g)\| \mid \pi\colon \mathcal{C}_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H}) \text{ is a } ^*\text{-representation} \} \\ &\leq \rho(f) + \rho(g). \end{aligned} \tag{2.5.11}$$

Moreover, one has

$$\begin{aligned} \rho(f^* * f) &= \sup\{ \|\pi(f^* * f)\| \mid \pi\colon \mathcal{C}_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H}) \text{ is a } ^*\text{-representation} \} \\ &= \sup\{ \|\pi(f)^*\pi(f)\| \mid \pi\colon \mathcal{C}_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H}) \text{ is a } ^*\text{-representation} \} \\ &= \sup\{ \|\pi(f)\|^2 \mid \pi\colon \mathcal{C}_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H}) \text{ is a } ^*\text{-representation} \} \\ &= \rho(f)^2. \end{aligned} \tag{2.5.12}$$

Thus, ρ is a C^*-seminorm. So one defines the C^*-algebra $C^*(\mathcal{G})$ to be the completion of the quotient of $\mathcal{C}_c(\mathcal{G})$ by $N = \{f \in \mathcal{C}_c(\mathcal{G}) \mid \|f\| = 0\}$ with respect to the pre-C^*-norm $\|\cdot\|$ induced by ρ.

We define $\pi_{\max}\colon \mathcal{C}_c(\mathcal{G}) \to C^*(\mathcal{G})$ by $\pi_{\max}(f) = f + N$. For any *-representation π of $\mathcal{C}_c(\mathcal{G})$, one has that $\|\pi(f)\| \leq \rho(f) = \|\pi_{\max}(f)\|$ for any $f \in \mathcal{C}_c(\mathcal{G})$ by construction. Then there is a well-defined norm-decreasing linear map $\psi\colon C^*(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ such that $\psi \circ \pi_{\max} = \pi$. Finally, ψ is a C^*-homomorphism by continuity. □

One has that $\pi_{\max}$ is injective (see [36, Corollary 3.3.4]).

Remark 2.5.6. For a groupoid $\mathcal{G}$, one may consider two different norms on $\mathcal{C}_c(\mathcal{G})$: the uniform norm and the I-norm. If $\mathcal{G}$ is an étale groupoid, then the *I-norm* on $\mathcal{C}_c(\mathcal{G})$ is given by

$$\|f\|_I = \sup_{x \in G^{(0)}} \max \left\{ \sum_{\gamma \in G_x} |f(\gamma)|, \sum_{\gamma \in G^x} |f(\gamma)| \right\}. \tag{2.5.13}$$

Note that in [32] Renault defines the universal norm not as the supremum over all *-representations of $C_c(\mathcal{G})$, but as the supremum only over *-representations of $C_c(\mathcal{G})$ that are bounded with respect to the *I-norm* on $\mathcal{C}_c(\mathcal{G})$. In [32, Proposition II.1.4] it is proved that boundedness in the I-norm is equivalent to continuity in the inductive-limit topology on $\mathcal{C}_c(\mathcal{G})$, i.e., the topology obtained by regarding $\mathcal{C}_c(\mathcal{G})$ as the inductive limit of the subspaces $X_K := \big(\{ f \in \mathcal{C}(\mathcal{G}) \mid \operatorname{supp}(f) \subseteq K \}, \|\cdot\|_\infty\big)$, for K compact subspace of $\mathcal{G}$. In the general non-étale setting, this equivalence is nontrivial. It has been proved in [36, Lemma 3.2.2] that every *-representation of $\mathcal{C}_c(\mathcal{G})$ is continuous in the inductive limit topology when $\mathcal{G}$ is étale and bounded with respect to the I-norm. This guarantees that the universal C^*-algebra defined in Theorem 2.5.5 coincides with the one defined by Renault.

The following proposition will be useful to establish surjectivity of a homomorphism into $C^*(\mathcal{G})$.

Proposition 2.5.7 ([36, Proposition 3.3.5])**.** *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff etale groupoid. Let A be a C^*-algebra and suppose that $\pi\colon A \to C^*(\mathcal{G})$ is a homomorphism. Suppose that for each open bisection $U \subseteq \mathcal{G}$ and each pair of distinct points $\beta, \gamma \in U$ there exists $a \in A$ such that $\pi(a) \in \mathcal{C}_0(U)$, $\pi(a)(\beta) = 0$ and $\pi(a)(\gamma) = 1$. Then π is surjective.*

Proof. We prove that $C^*(\mathcal{G}) \subseteq \pi(A)$. By the Stone-Weierstrass theorem, one has that $\pi(A)$ contains $\mathcal{C}_0(\mathcal{G}^{(0)})$. Thus, by Lemma 2.5.3, for any open bisection $U \subseteq \mathcal{G}$ the set $\pi(A) \cap \mathcal{C}_0(U)$ is closed under pointwise multiplication. Let $f, g \in \pi(A) \cap \mathcal{C}_0(r(U))$. Then $f \circ r^{-1} \in \mathcal{C}_0(r(U)) \subseteq \mathcal{C}_0(\mathcal{G}^{(0)}) \subseteq \pi(A)$ and $f \cdot g = (f \circ r^{-1}) * g$. By combining the Stone-Weierstrass theorem with the fact that $\|\cdot\|_{C^*(\mathcal{G})}$ agrees with $\|\cdot\|_\infty$ on $\mathcal{C}_c(U)$, one has that $\mathcal{C}_c(U) \subseteq \pi(A)$. By Lemma 2.5.2, one has that $\mathcal{C}_c(\mathcal{G}) \subseteq \pi(A)$. Since π is a C^*-homomorphism, it has closed range, and hence $C^*(\mathcal{G}) \subseteq \pi(A)$. This proves that $C^*(\mathcal{G}) = \pi(A)$, i.e., π is surjective. □

Chapter 3

C^*-algebras associated to the boundary of a poset

In this chapter we show that, for a monoid M, the existence of self-similar M*-fractals* gives rise to example of C^*-algebras generalizing the boundary quotients of X. Li (cf. [27]). In Section 3.1 we study the boundaries of a certain class of monoids and their C^*-algebras. In Section 3.2 we associate a C^*-algebra to the action of a finitely 1-generated monoid on a contracting metric space, which we will call an M*-fractal*.

All the results contained in this chapter have been published in [11].

3.1 Boundaries of monoids and their C^*-algebras

3.1.1 Abelian semigroups generated by idempotents

Definition 3.1.1. Let E be an abelian semigroup with unit, i.e., a monoid, generated by a set of elements $\Sigma \subseteq E$ satisfying $\sigma^2 = \sigma$ for all $\sigma \in \Sigma$, i.e., all non-trivial elements of Σ are idempotents. Then every element $u \in E$ is an idempotent, and one may define a partial order "$\preceq$" on E by

$$u \preceq v \qquad \Longleftrightarrow \qquad u \cdot v = v, \tag{3.1.1}$$

for $u, v \in E$. Let $\mathcal{R} = \{\, (u, v) \in \Sigma \times \Sigma \mid u \preceq v \,\}$. By definition, one has

$$E = \{\, u = \sigma_1 \cdots \sigma_r \mid \sigma_i \in \Sigma \,\}. \tag{3.1.2}$$

Hence

$$E \simeq \mathscr{F}^{\mathrm{ab}}(\Sigma)/R, \tag{3.1.3}$$

where $\mathscr{F}^{\mathrm{ab}}(\Sigma)$ is the free abelian semigroup over the set Σ, and R is the relation

$$R = \{\, (uv, v) \mid (u, v) \in \mathcal{R} \,\} \subseteq \mathscr{F}^{\mathrm{ab}}(\Sigma) \times \mathscr{F}^{\mathrm{ab}}(\Sigma), \tag{3.1.4}$$

i.e., $E = \mathscr{F}^{\mathrm{ab}}(\Sigma)/R^{\sim}$, where $R^{\sim}$ is the equivalence relation on $\mathscr{F}^{\mathrm{ab}}(\Sigma)$ generated by the set R. Let

$$\widehat{E} = \{\, \chi \colon E \to \{0, 1\} \mid \chi \text{ a semigroup homomorphism, } \chi \not\equiv 0 \,\}. \tag{3.1.5}$$

Note that $\chi(0) = 0$ by definition.

Example 12. Let $\Sigma = \{1, e, f\}$, where e, f are idempotents (in a commutative algebra) such that $e, f, e - ef, f - ef$ are all nonzero. Then one has

$$\begin{aligned}
E &= \{ 1, e, f, ef \}, \\
\mathcal{R} &= \{ (1,1), (1,e), (1,f), (1,ef), (e,e), (f,f), (e,ef), (f,ef) \}, \\
R &= \{ (1,1), (e,e), (f,f), (ef,ef), (e^2,e), (f^2,f), (e^2f,ef), (ef^2,ef) \}, \\
R^{\sim} &= \{ (1,1), (e^k,e), (e,e^k), (f^k,f), (f,f^k), (e^if^j,ef), (ef,e^if^j) \mid i,j,k > 0\}, \\
\mathscr{F}^{\mathrm{ab}}(\Sigma) &= \{ e^if^j \mid i+j > 0 \} \sqcup \{1\}, \\
\mathscr{F}^{\mathrm{ab}}(\Sigma)/R^{\sim} &= \{ [1], [e], [f], [ef] \}.
\end{aligned}$$

Remark 3.1.2. One has that $\widehat{E}$ coincides with the set of characters of the C^*-algebra $C^*(E)$ generated by E (satisfying $e^* = e$ for all $e \in E$), and hence carries naturally the structure of a compact topological space (cf. [27, Corollary 6.25]). By construction, $\widehat{E}$ can be identified with a subset of $\mathcal{F}(\Sigma, \{0,1\})$ - the set of functions from Σ to $\{0,1\}$. In more detail,

$$\widehat{E} = \{ \phi \in \mathcal{F}(\Sigma, \{0,1\}) \mid \forall (u,v) \in \mathcal{R}: \ \phi(v) = \phi(u) \cdot \phi(v) \}. \tag{3.1.6}$$

Thus, identifying $\mathcal{F}(\Sigma, \{0,1\})$ with $\{0,1\}^{\Sigma}$, one obtains that

$$\widehat{E} = \{ (\eta_\sigma)_{\sigma \in \Sigma} \in \{0,1\}^{\Sigma} \mid \forall (u,v) \in \mathcal{R} : \sigma_v = \sigma_u \cdot \sigma_v \}. \tag{3.1.7}$$

Definition 3.1.3. Let X be a set, and let $S \subseteq \mathscr{P}(X)$ be a set of subsets of X. Then S generates an algebra of sets $\mathcal{A}(S) \subseteq \mathscr{P}(X)$, i.e., the sets of $\mathcal{A}(S)$ consist of the finite intersections and finite unions of sets in S. Then

$$E(S) = \langle I_A \mid A \in \mathcal{A}(S) \rangle \subseteq \mathcal{F}(X, \{0,1\}) \tag{3.1.8}$$

is an abelian semigroup being generated by the set of idempotents

$$\Sigma = \{ I_Y \mid Y \in S \}. \tag{3.1.9}$$

Moreover, by (3.1.6), one has

$$\widehat{E}(S) = \{ \phi \in \mathcal{F}(S, \{0,1\}) \mid \forall U, V \in S, \ V \subseteq U : \phi(V) = \phi(U) \cdot \phi(V) \}. \tag{3.1.10}$$

3.1.2 The Laca-boundary of a monoid

Definition 3.1.4. Let M be a 1-generated monoid. Then one chooses

$$S = \{ \omega \cdot M \mid \omega \in M \} \tag{3.1.11}$$

to consist of all principal right ideals. For short we call the compact set $\eth M = \widehat{E}(S)$ for S as in (3.1.11) the *Laca boundary* of M. For an infinite word $\underline{\omega} = (\omega_k) \in \mathscr{D}(\mathbb{N}, M, \succeq)$ and for $\tau \in M$ one defines the element $\chi_{\underline{\omega}} \in \widehat{E}(S)$ by $\chi_{\underline{\omega}}(\tau M) = 1$ if, and only if, there exists $k \in \mathbb{N}$ such that $\omega_k \in \tau M$, i.e., $\tau \succeq \omega_k$, and thus $\tau \succeq \underline{\omega}$. This yields a map

$$\chi_. : \mathscr{D}(\mathbb{N}, M, \preceq) \longrightarrow \widehat{E}(S) \tag{3.1.12}$$

(cf. [28, § 2.2]).

By definition, it has the following property:

Proposition 3.1.5. *For* $\underline{\omega} = (\omega_k) \in \mathscr{D}(\mathbb{N}, M, \preceq), \tau \in M$*, one has* $\chi_{\underline{\omega}}(\tau M) = 1$ *if, and only if,* $\tau \succeq \underline{\omega}$*. In particular, one has* $\chi_{\underline{\eta}} = \chi_{\underline{\omega}}$ *if, and only if,* $\underline{\eta} \approx \underline{\omega}$*, and hence* $\chi_{\cdot}$ *induces an injective map*

$$\overline{\chi}_{\cdot} \colon \partial M \longrightarrow \eth M. \tag{3.1.13}$$

Proof. The first part has already been established before. Let $\underline{\eta} = (\eta_k)$. Then by the first part, $\underline{\omega} \succeq \underline{\eta}$ implies that for all $\tau \in M$ one has

$$\chi_{\underline{\omega}}(\tau M) = 1 \Longrightarrow \chi_{\underline{\eta}}(\tau M) = 1. \tag{3.1.14}$$

Thus as $\mathrm{im}(\chi_\omega) \subseteq \{0, 1\}$ one concludes that $\underline{\omega} \succeq \underline{\eta}$ and $\underline{\omega} \preceq \underline{\eta}$ implies that $\chi_{\underline{\omega}} = \chi_{\underline{\eta}}$. On the other hand $\chi_{\underline{\eta}} = \chi_{\underline{\omega}}$ implies that $1 = \chi_{\underline{\eta}}(\eta_k M) = \chi_{\underline{\omega}}(\eta_k M)$ for all $k \in \mathbb{N}$. In particular, $\underline{\eta} \succeq \underline{\omega}$. Interchanging the roles of $\underline{\eta}$ and $\underline{\omega}$ yields $\underline{\omega} \succeq \underline{\eta}$, and thus $\underline{\eta} \approx \underline{\omega}$ (cf. Section 2.1). The last part is a direct consequence of the definition of ∂M. □

The following theorem shows that for a 1-generated $\mathbb{N}_0$-graded left-cancellative monoid M its universal boundary ∂M with the fine topology is a totally-disconnected compact space.

Theorem 3.1.6. *The map* $\overline{\chi}_{\cdot} \colon (\partial M, \mathcal{T}_f(\bar{M})) \longrightarrow \eth M$ *is a homeomorphism.*

Proof. It is well known that χ is surjective (see [28, Lemma 2.3]), and thus $\overline{\chi}$ is surjective. By Proposition 3.1.5, $\overline{\chi}$ is injective, and hence $\overline{\chi}$ is a bijection. The sets

$$U_\tau^\varepsilon = \{\, \eta \in \widehat{E}(M) \mid \eta(\tau M) = \varepsilon \,\}, \qquad \tau \in M, \qquad \varepsilon \in \{0, 1\} \tag{3.1.15}$$

form a subbasis of the topology of $\widehat{E}(M)$, and

$$U_\tau^1 = \overline{\chi}(C_\tau(\bar{M}) \cap \partial M) \tag{3.1.16}$$

by (3.1.14). Hence $U_\tau^0 = \overline{\chi}(C_\tau(\bar{M})^C \cap \partial M)$ and this shows that $\overline{\chi}^{-1}$ is continuous, which yields the claim. □

The proof of Theorem 3.1.6 has also shown that

$$\overline{\chi}^{-1} \colon \eth M \longrightarrow (\partial M, \mathcal{T}_c(\bar{M})) \tag{3.1.17}$$

is a bijective and continuous map. Thus, if $(\partial M, \mathcal{T}_c(\bar{M}))$ is Hausdorff, then $\overline{\chi}^{-1}$ is a homeomorphism (see [7, § 9.4, Corollary 2]). This has the following consequence.

Proposition 3.1.7. *Let* M *be a left-cancellative* 1*-generated* $\mathbb{N}_0$*-graded monoid such that* $(\partial M, \mathcal{T}_c(\bar{M}))$ *is Hausdorff. Then* M *is* $\mathcal{T}$*-regular.*

In contrast to Proposition 2.1.26 one has the following property for the Laca boundary of monoids.

Proposition 3.1.8. *Let* $\phi \colon Q \to M$ *be a surjective homomorphism of connected* $\mathbb{N}_0$*-graded monoids. Then* ϕ *induces an injective continuous map* $\phi_\eth \colon \eth M \longrightarrow \eth Q$*.*

Proof. By Proposition 2.1.33, ϕ induces a map $\phi_\Sigma \colon \Sigma(Q) \to \Sigma(M)$ given by

$$\phi_\Sigma(\omega Q) = \phi(\omega) M. \tag{3.1.18}$$

Moreover, for $x, y \in Q$ one has $x \preceq y$, if and only if, $xQ \subseteq yQ$, if and only if there exists $z \in Q$ such that $x = y \cdot z$. From the last statement one concludes that $\phi_\Sigma(xQ) \subseteq \phi_\Sigma(zQ)$. Thus, by (3.1.3), ϕ_Σ induces a homomorphism of semigroups

$$\phi_E \colon E(Q) \longrightarrow E(M), \tag{3.1.19}$$

and thus a map

$$\phi^\circ_{\widehat{E}} \colon \widehat{E}(M) \cup \{0\} \longrightarrow \widehat{E}(Q) \cup \{0\}. \tag{3.1.20}$$

If ϕ is surjective, then ϕ_E is surjective, and $\phi^\circ_{\widehat{E}}$ restricts to a map

$$\phi_{\widehat{E}} \colon \widehat{E}(M) \longrightarrow \widehat{E}(Q). \tag{3.1.21}$$

It is straightforward to verify that $\phi_{\widehat{E}}$ is continuous and injective. □

Example 13. Let M be the monoid being freely generated by a set $\Sigma = \{s_k \mid k \in \mathbb{N}\}$ of countably many generators. Then $\bar{M}$ with the fine topology is a compact Hausdorff space, but it is not sequentially compact, as $(s_i)_{i\in\mathbb{N}}$ does not have a convergent subsequence. Similarly, $\bar{M}$ with the cone topology is Hausdorff, but not compact, as $\{s_i^\infty \mid i \in \mathbb{N}\}$ is a discrete infinite subset of $\bar{M}$.

3.1.3 The poset completion of free monoids

Definition 3.1.9. Let $\mathscr{F}_n = \mathscr{F}\langle x_1, \dots, x_n\rangle$ be the free monoid on n generators. Let $S = \{x_1, \dots, x_n\}$ be the set of generators, and let $|\_| \colon \mathscr{F}_n \to \mathbb{N}_0$ be the grading morphisms, i.e., $|y| = 1$ if and only if $y \in S$. The *Cayley graph* $\Gamma(\mathscr{F}_n, S)$ of $\mathscr{F}_n$ with respect to S is the graph defined by

$$V = \{\, x \mid x \in \mathscr{F}_n \,\} \tag{3.1.22}$$
$$E = \big\{\, (x, xx_i) \in V \times V \mid x \in \mathscr{F}_n, x_i \in S \,\big\}. \tag{3.1.23}$$

The terminus and origin maps $t, o \colon E \to V$ are given by the projection onto the first and second coordinate, respectively.

Remark 3.1.10. One has that $\Gamma(\mathscr{F}_n, S)$ is an n-regular tree with root 1 and all edges pointing away from 1. The graph $\Gamma(\mathscr{F}_n, S)$ coincides with an orientation of the n-regular tree T_n.

Definition 3.1.11. Let T_n be the n-regular tree. The boundary ∂T_n of T_n is the set of equivalence classes of infinite reduced paths under the shift relation $\sim$. We denote by $[v, \rho)$ the unique path starting at v in the class $[\rho]$ and define

$$I_v = \{\, \omega \in \partial T_n \mid v \in [1, \omega) \,\} \tag{3.1.24}$$

the *interval* of ∂T_n starting at v.

Then ∂T_n is compact with respect to the topology $\mathcal{T}_I$ generated by $\{I_v\}_{v\in V}$.

Remark 3.1.12. For any $[\rho] \in \partial T_n$ there exists a unique ray $\rho = (e_k)_{k\in\mathbb{N}}$, $t(\rho) = t(e_1) = 1$. One can assign to ρ the decreasing function $\omega_\rho \in \mathscr{D}(\mathbb{N}, \mathscr{F}_n, \preceq)$ given by $\omega_\rho(k) = o(e_k)$. The map $\varphi \colon \partial T_n \to \partial \mathscr{F}_n$ given by

$$\varphi([\rho]) = [\omega_\rho] \tag{3.1.25}$$

is a bijection. Hence one can identify ∂T_n with $\partial \mathscr{F}_n$.

Let $\bar{\mathscr{F}}_n$ be the poset completion of $\mathscr{F}_n$ and consider the cone topology on $(\bar{\mathscr{F}}_n)$.

Proposition 3.1.13. *The topological space $(\bar{\mathscr{F}}_n, \mathcal{T}_c(\bar{\mathscr{F}}_n))$ is compact.*

Proof. Every cone $C_\tau(\bar{\mathscr{F}}_n)$ defines a rooted subtree T_τ of T_n satisfying $\partial T_\tau = \partial \mathscr{F}_n \cap C_\tau(\bar{\mathscr{F}}_n)$. Thus every covering $\bigcup_{\tau \in U} C_\tau(\bar{\mathscr{F}}_n) \cap \partial \mathscr{F}_n$ of the boundary of $\partial \mathscr{F}_n$ by cones defines a forest $F = \bigcup_{\tau \in U} T_\tau$. Let $F = \bigcup_{i \in I} F_i$ be the decomposition of F in connected components. Then $\partial T_n = \partial F = \bigsqcup_{i \in I} \partial F_i$, where $\sqcup$ denotes disjoint union. Hence the compactness of ∂T_n implies $|I| < \infty$.

As $\partial F_i \subseteq \partial T_n$ is closed, and hence compact, a similar argument shows that there exist finitely many cones $C_{\tau_{i,j}}$, $1 \leq j \leq r_i$, such that $F_i = \bigcup_{1 \leq j \leq r_i} T_{\tau_{i,j}}$. Thus, if $\bigcup V$ is an open covering of $\bar{\mathscr{F}}_n$ by open sets, it can be refined to a covering $\bigcup U$, where U consists either of a cone $C_\tau(\bar{\mathscr{F}}_n)$ or of a singleton set $\{\omega\}$, $\omega \in \mathscr{F}_n$. Let $\Lambda \subseteq T_n$ be the subtree being generated by the vertices $\tau_{i,j}$. Then Λ is a finite subtree, and the only vertices of T_n not being covered by $\bigcup_{i,j} C_{\tau_{i,j}}(\bar{\mathscr{F}}_n))$ are contained in $V(\Lambda)$. This shows that $(\bar{\mathscr{F}}_n, \mathcal{T}_c(\bar{\mathscr{F}}_n))$ is a compact space. □

Proposition 3.1.14. *Let M be a finitely 1-generated $\mathbb{N}_0$-graded $\mathcal{T}$-regular monoid. Then $(\bar{M}, \mathcal{T}_c(\bar{M}))$ is a compact space.*

Proof. By definition, $(\partial M, \mathcal{T}_c(\bar{M}))$ is a Hausdorff space, and hence $(\bar{M}, \mathcal{T}_c(\bar{M}))$ is a Hausdorff space. By Proposition 2.1.26, the canonical mapping $\phi_M \colon \mathscr{F} \to M$ (cf. (2.1.24)) induces a continuous surjective map $\bar{\phi}_M \colon \bar{\mathscr{F}} \to \bar{M}$. This yields to the thesis. □

3.1.4 The canonical probability measure on the boundary of a regular tree

Let $\mathscr{F}_n$ be the free monoid on n generators and let T_n be the n-regular tree. By Carathéodory's extension theorem the assignment

$$\mu(I_v) = n^{-|v|}, \tag{3.1.26}$$

$v \in T^0$, defines a unique probability measure $\mu \colon \mathrm{Bor}(\partial T_n) \to \mathbb{R}_0^+$. Hence the corresponding probability measure $\mu \colon \mathrm{Bor}(\partial \mathscr{F}_n) \to \mathbb{R}_0^+$ satisfies

$$\mu(\partial \mathscr{F}_n \cap C_\tau(\bar{\mathscr{F}}_n)) = n^{-|\tau|} \text{ for } \tau \in \mathscr{F}_n. \tag{3.1.27}$$

Definition 3.1.15. Let $\_ \cdot \_ \colon \mathscr{F}_n \times \partial \mathscr{F}_n \to \partial \mathscr{F}_n$ be the map given by

$$x \cdot [\omega] = [x\omega], \tag{3.1.28}$$

where $x\omega \colon \mathbb{N} \to \mathscr{F}_n$ is given by $(x\omega)(n) = x\omega(n)$

Note that this action is well defined, since $\omega \sim \omega'$ implies that $x\omega \sim x\omega'$.

Definition 3.1.16. Let $\_ \cdot \_ \colon L^2(\partial \mathscr{F}_n, \mathbb{C}, \mu) \times \mathscr{F}_n \to L^2(\partial \mathscr{F}_n, \mathbb{C}, \mu)$ be the map given by

$$f.x = {}^x f, \tag{3.1.29}$$

where

$$({}^x f)([\omega]) = f([x\omega]). \tag{3.1.30}$$

Remark 3.1.17. Note that for $f \in L^2(\partial\mathscr{F}_n, \mathbb{C}, \mu)$ one has ${}^x f \in L^2(\partial\mathscr{F}_n, \mathbb{C}, \mu)$, since

$$\| {}^x f\|_2^2 = \int_{\partial T_n} | {}^x f|^2 \, d\mu \tag{3.1.31}$$

$$= \int_{x\partial T_n} |f|^2 \, d\mu \tag{3.1.32}$$

$$\leq \int_{\partial T_n} |f|^2 \, d\mu \tag{3.1.33}$$

$$= \|f\|_2^2, \tag{3.1.34}$$

where (3.1.33) follows since $x\partial\mathscr{F}_n \subseteq \partial\mathscr{F}_n$.

Definition 3.1.18. For $z \in \mathscr{F}_n$ we define the map $T_z \colon L^2(\partial\mathscr{F}_n, \mathbb{C}, \mu) \to L^2(\partial\mathscr{F}_n, \mathbb{C}, \mu)$ by

$$T_z(f) = {}^z f. \tag{3.1.35}$$

Proposition 3.1.19. *$\mathscr{F}_n$ acts via $T_\cdot$ on $L^2(\partial\mathscr{F}_n, \mathbb{C}, \mu)$ by bounded linear operators.*

Proof. Let $z \in \mathscr{F}_n$. For $f, g \in L^2(\partial\mathscr{F}_n, \mathbb{C}, \mu)$, $[\omega] \in \partial\mathscr{F}_n$, one has

$$\big(T_z(f+g)\big)([\omega]) = \big(T_z(f)\big)([\omega]) + \big(T_z(g)\big)([\omega])$$

by definition. Thus T_z is linear. It is also bounded, since

$$\|T_z\|_\infty = \sup_{\|f\|_2=1} \|T_z(f)\|_2 \leq \sup_{\|f\|_2=1} \|f\|_2 \leq 1. \tag{3.1.36}$$

□

Hence $T_z \in \mathcal{B}\big(L^2(\partial\mathscr{F}_n, \mathbb{C}, \mu)\big)$ for all $z \in \mathscr{F}_n$. As $\mathcal{B}\big(L^2(\partial\mathscr{F}_n, \mathbb{C}, \mu)\big)$ is a C^*-algebra, T_z has an adjoint operator T_z^*, which is the bounded operator satisfying

$$\langle T_z f, g \rangle = \langle f, T_z^* g \rangle, \tag{3.1.37}$$

for all $f, g \in L^2(\partial\mathscr{F}_n, \mathbb{C}, \mu)$.

Proposition 3.1.20. *The bounded operator T_z^*, for $z \in \mathscr{F}_n$, is given by*

$$(T_z^* f)([\omega]) = \begin{cases} 0 & \text{if } [\omega] \notin z\partial\mathscr{F}_n \\ f([\omega']) & \text{if } [\omega] = z[\omega']. \end{cases} \tag{3.1.38}$$

Proof. Note that $T_z^* f \in L^2(\partial\mathscr{F}_n, \mathbb{C}, \mu)$, since

$$\|T_z^* f\|_2^2 = \int_{\partial\mathscr{F}_n} |T_z^* f|^2 \, d\mu \tag{3.1.39}$$

$$= \int_{z\partial\mathscr{F}_n} |T_z^* f|^2 \, d\mu \tag{3.1.40}$$

$$\leq \int_{\partial T_n} |f|^2 \, d\mu. \tag{3.1.41}$$

Let $f, g \in L^2(\partial\mathscr{F}_n, \mathbb{C}, \mu)$. Then one has

$$\langle f, T_z^* g \rangle = \int_{\partial T_n} f(\overline{T_z^* g})\, d\mu \tag{3.1.42}$$

$$= \int_{z\partial T_n} f(\overline{T_z^* g})\, d\mu \tag{3.1.43}$$

$$= \int_{z\partial T_n} (T_z f)\overline{g}\, d\mu \tag{3.1.44}$$

$$\leq \int_{\partial T_n} (T_z f)\overline{g}\, d\mu \tag{3.1.45}$$

$$= \langle T_z f, g \rangle. \tag{3.1.46}$$

where equality (3.1.44) holds by

$$f([z\omega'])\, \overline{T_z^* g}([z\omega']) = (T_z f)([\omega'])\, \overline{g}([\omega']). \tag{3.1.47}$$

□

Proposition 3.1.21. *The following identities hold for all $x, y \in S \subseteq \mathscr{F}_n$*

$$T_x^* T_y = \delta_{xy}; \tag{3.1.48}$$

$$\sum_{i=1}^{n} T_{x_i} T_{x_i}^* = 1. \tag{3.1.49}$$

In particular, the C^*-algebra $C^*(\mathscr{F}_n, \mu) \subseteq \mathcal{B}(L^2(\partial\mathscr{F}_n, \mathbb{C}, \mu))$ generated by $\mathscr{F}_n$ is isomorphic to the Cuntz algebra $\mathcal{O}_n$.

Proof. Let $x, y \in S \subseteq \mathscr{F}_n$ and let $f \in L^2(\partial\mathscr{F}_n, \mathbb{C}, \mu)$. For any $[\omega] \in \partial\mathscr{F}_n$ one has

$$T_x^* T_y(f)([\omega]) = \delta_{xy} f([\omega]) \tag{3.1.50}$$

by Proposition 3.1.20. This proves identity (3.1.48).

Let $[\omega] \in \partial\mathscr{F}_n$. Then there exists $x_j \in S$ such that $[\omega] \in x_j\partial\mathscr{F}_n$. Hence one has

$$T_{x_i} T_{x_i}^* f([\omega]) = \delta_{ij} f([\omega]) \tag{3.1.51}$$

for any $f \in L^2(\partial\mathscr{F}_n, \mathbb{C}, \mu)$. This yields the identity (3.1.49). □

3.1.5 Finitely 1-generated monoids

Let M be a finitely 1-generated $\mathbb{N}_0$-graded monoid. Then one has a canonical surjective graded homomorphism $\phi_M : \mathscr{F} \to M$, where $\mathscr{F}$ is a finitely generated free monoid (cf. (2.1.24)), which induces a continuous map $\partial\phi : \partial\mathscr{F} \longrightarrow \partial M$ (cf. Proposition 2.1.26). In particular,

$$\mu_M : \mathrm{Bor}(\partial M) \longrightarrow \mathbb{R}_0^+ \tag{3.1.52}$$

given by $\mu_M(A) = \mu((\partial\phi_M)^{-1}(A))$ is a Borel probability measure on ∂M.

Definition 3.1.22. For $s \in M$, define the map $\beta_s : \partial M \to \partial M$ by

$$\beta_s([f]) = [sf], \quad [f] \in \partial M, \tag{3.1.53}$$

where $(sf)(n) = s \cdot f(n)$ for all $n \in \mathbb{N}$, $f \in \mathscr{D}(\mathbb{N}, M, \preceq)$.

Remark 3.1.23. As β_s is mapping cones to cones, β_s is continuous. Hence one has a representation

$$\beta\colon M \to \mathcal{C}(\partial M, \partial M). \tag{3.1.54}$$

Proposition 3.1.24. *There exists an antirepresentation*

$$\beta_*\colon M \to \mathcal{B}\big(L^2(\partial M, \mathbb{C}, \mu)\big) \tag{3.1.55}$$

given by

$$\begin{aligned}\beta_{*,s}\colon L^2(\partial M, \mathbb{C}, \mu) &\to L^2(\partial M, \mathbb{C}, \mu)\\ \beta_{*,s}(g)([f]) &= g(\beta_s([f])) = g([sf]),\end{aligned} \tag{3.1.56}$$

for $g \in L^2(\partial M, \mu)$, $[f] \in \partial M$, $s \in M$.

Proof. One has

$$\begin{aligned}\|\beta_{*,s}(g)\|_2^2 &= \int_{\partial M} \big|g\big(\beta_s([f]\big)\big|^2\, d\mu_M && (3.1.57)\\ &= \int_{\partial M} \big|g\big([sf]\big)\big|^2\, d\mu_M && (3.1.58)\\ &= \int_{s\partial M} \big|g\big([f]\big)\big|^2\, d\mu_M && (3.1.59)\\ &\le \int_{\partial M} \big|g\big([f]\big)\big|^2\, d\mu_M && (3.1.60)\\ &= \|g\|_2^2, && (3.1.61)\end{aligned}$$

for all $g \in L^2(\partial M, \mathbb{C}, \mu)$, $s \in M$. Thus,

$$\|\beta_{*,s}\| = \sup_{\|g\|_2=1} \|\beta_{*,s}(g)\|_2 \le 1 \tag{3.1.62}$$

for all $s \in M$, i.e., $\beta_{*,s}$ is a bounded operator on $L^2(\partial M, \mathbb{C}, \mu)$. By an argument similar to the one used in the proof of Proposition 3.1.19 one can show that it is also linear. □

Then one can define the following C^* algebra for every finitely 1-generated $\mathbb{N}_0$-graded monoid M.

Definition 3.1.25. Let M be a finitely 1-generated $\mathbb{N}_0$-graded monoid. We define the C^*-algebra

$$C^*(M, \mu_M) = C^*\big(\{\, \beta_\omega \mid \omega \in M \,\}\big) \subseteq \mathcal{B}(L^2(\partial M, \mathbb{C}, \mu_M)) \tag{3.1.63}$$

where β_ω is the mapping induced by left multiplication with ω.

3.1.6 Right-angled Artin monoids

Definition 3.1.26. Let $\Gamma = (V, E)$ be a finite undirected graph, i.e. $|V| = n < \infty$ and $E \subseteq \mathscr{P}_2(V)$, where $\mathscr{P}_2(V)$ denotes the set of subsets of cardinality 2 of V. The *right-angled Artin monoid* associated to Γ is the monoid M^Γ defined by

$$M^\Gamma = \langle\, x \in V \mid xy = yx \text{ if } \{x, y\} \in E \,\rangle^+. \tag{3.1.64}$$

Remark 3.1.27. Clearly, M^Γ is $\mathbb{N}_0$-graded and finitely 1-generated. By Luis Paris theorem (cf. [31]), M^Γ embeds into the right-angled Artin group

$$G_\Gamma = \langle\, x \in V \mid xy = yx \text{ if } \{x,y\} \in E \,\rangle.$$

Thus M^Γ has the left-cancellation property as well as the right-cancellation property. The canonical homomorphism $\phi_\Gamma \colon \mathscr{F}\langle V\rangle \longrightarrow M^\Gamma$ is surjetcive and induces a continuous surjective map

$$\partial\phi_\Gamma \colon \partial\mathscr{F}\langle V\rangle \longrightarrow \partial M^\Gamma. \tag{3.1.65}$$

(cf. Proposition 2.1.26). We denote by $\mu_\Gamma \colon \mathrm{Bor}(\partial M^\Gamma) \longrightarrow \mathbb{R}_0^+$ the Borel probability measure induced by $\partial\phi_\Gamma$, i.e., for $A \in \mathrm{Bor}(\partial M^\Gamma)$ one has

$$\mu_\Gamma(A) = \mu(\partial\phi_\Gamma^{-1}(A)), \tag{3.1.66}$$

where μ is the measure defined on $\partial\mathscr{F}\langle V\rangle$ by (3.1.27).

Definition 3.1.28. Let $\Gamma = (V,E)$ be a graph, and let $\Gamma_1 = (V_1,E_1)$ and $\Gamma_2 = (V_2,E_2)$ be subgraphs of Γ. We say that Γ is *bipartitly decomposed* by Γ_1 and Γ_2, if $V = V_1 \sqcup V_2$ and

$$E = E_1 \sqcup E_2 \sqcup \{\, \{v_1,v_2\} \mid v_1 \in V_1,\ v_2 \in V_2 \,\}. \tag{3.1.67}$$

In this case we will write $\Gamma = \Gamma_1 \vee \Gamma_2$. If no such decomposition exists, Γ will be called *coconnected*.

Any graph Γ can be decomposed into *connected components* Γ_i, i.e. $\Gamma = \sqcup_{i\in I}\Gamma_i$. In a similar fashion one may define a decomposition in *coconnected components*.

Definition 3.1.29. Let $\Gamma = (V,E)$ be a graph and let $\Gamma^{\mathrm{op}} = \bigsqcup_{i\in I}\Lambda_i$ be the decomposition of Γ^{op} in its connected components. We will call

$$\Gamma = \bigvee_{i\in I}\Lambda_i^{\mathrm{op}}, \tag{3.1.68}$$

the *decomposition of* Γ *in coconnected components*.

One has the following property.

Proposition 3.1.30. *Let $\Gamma = (V,E)$ be an undirected graph. Then Γ is coconnected if, and only if, Γ^{op} is connected. In particular, if $\Gamma^{\mathrm{op}} = \bigsqcup_{i\in I}\Lambda_i$ is the decomposition of Γ^{op} in its connected components, then one has*

$$\Gamma = \bigvee_{i\in I}\Lambda_i^{\mathrm{op}}, \tag{3.1.69}$$

where Λ_i^{op} are coconnected subgraphs of Γ.

Proof. Obviously, the graph $\Gamma = \Gamma_1 \vee \Gamma_2$ is bipartitly decomposed if, and only if, $\Gamma^{\mathrm{op}} = \Gamma_1^{\mathrm{op}} \sqcup \Gamma_2^{\mathrm{op}}$ is not connected. This yields to the claim. □

Note that the decomposition in coconnected components implies that ant two vertices in different components must be connected by an edge. From this property one concludes the following straightforward fact.

Proposition 3.1.31. *Let $\Gamma = (V, E)$ be a finite graph with unoriented edges, and let $\Gamma = \bigvee_{1 \leq i \leq r} \Gamma_i$ be its decomposition in coconnected components, $\Gamma_i = (V_i, E_i)$. Then*

$$M^\Gamma = M^{\Gamma_1} \times \cdots \times M^{\Gamma_r}, \tag{3.1.70}$$

where $M^{\Gamma_i} = \langle v \in V_i \rangle$. In particular, $\partial M^\Gamma = \times_{1 \leq r} \partial M^{\Gamma_i}$ and

$$L^2(\partial M^\Gamma, \mathbb{C}, \mu_\Gamma) = L^2(\partial M^{\Gamma_r}, \mathbb{C}, \mu_{\Gamma_1}) \widehat{\otimes} \cdots \widehat{\otimes} L^2(\partial M^{\Gamma_r}, \mathbb{C}, \mu_{\Gamma_r}). \tag{3.1.71}$$

In [9], J. Crisp and M.Laca has shown the following.

Theorem 3.1.32 ([9], Theorem 6.7)**.** *Let $\Gamma = (V, E)$ be a finite unoriented graph such that Γ^{op} has no isolated vertices, and let $\Gamma = \bigvee_{i=1}^r \Gamma_i$ be the decomposition of Γ in coconnected components, $\Gamma_i = (V_i, E_i)$. Then the universal C^*-algebra with generators $\{S_x \mid x \in V\}$ subject to the relations*

(i) $S_x^* S_x = 1$ for each $x \in V$;

(ii) $S_x S_y = S_y S_x$ and $S_x^* S_y = S_y S_x^*$ if x and y are adjacent in Γ;

(iii) $S_x^* S_y = 0$ if x and y are distinct and not adjacent in Γ;

(iv) $\prod_{x \in V_i} (1 - S_x S_x^*) = 0$ for each $i \in \{1, \ldots, r\}$;

is canonically isomorphic to the boundary quotient $\partial C_\lambda(M^\Gamma)$ for M^Γ and it is a simple C^*-algebra.

Hence, one has the following proposition.

Proposition 3.1.33. *The C^*-algebra $C^*(M^\Gamma, \mu_\Gamma)$ (cf. (3.1.63)) of a right-angled Artin monoid M^Γ is isomorphic to the boundary quotient $\partial C_\lambda(M^\Gamma)$ of Theorem 3.1.32.*

Proof. Let $\Gamma = (V, E)$ be a finite unoriented graph such that $|V| = n$ and let $\Gamma = \bigvee_{i=1}^r \Gamma_i$ be its decomposition in coconnected components. It is straightforward to verify (i)-(iii) for the set of operators $\{ T_x \mid x \in V \}$, where the operator $T_x \in \mathcal{B}(L^2(\partial M^\Gamma, \mathbb{C}, \mu_\Gamma))$, μ_Γ as in (3.1.66), is defined by

$$T_x(f)([\omega]) = f([x\omega]), \tag{3.1.72}$$

and the adjoint operators are given by

$$(T_x^* f)([u]) = \begin{cases} 0 & \text{if } [u] \notin x\partial M^\Gamma \\ f([u']) & \text{if } [u] = x[u'], \end{cases} \tag{3.1.73}$$

where $f \in L^2(\partial M^\Gamma, \mathbb{C}, \mu_\Gamma)$. It remains to prove that it also satisfies (iv). Let

$$\mathbf{e}_i = \prod_{x \in V_i} (1 - T_x T_x^*). \tag{3.1.74}$$

In order to show that $\mathbf{e}_i(f) = 0$ for all $f \in L^2(\partial M^\Gamma, \mathbb{C}, \mu_\Gamma)$ it suffices to show that $\mathbf{e}_i(f) = 0$ for $f = f_1 \otimes \cdots \otimes f_r$, $f_i \in L^2(\partial M^{\Gamma_i}, \mathbb{C}, \mu_{\Gamma_i})$ (cf. (3.1.71)). Note that

$$(1 - T_x T_x^*)(f)([u]) = \begin{cases} 0 & \text{if } [u] \in x\partial M^\Gamma, \\ f([u]) & \text{otherwise.} \end{cases} \tag{3.1.75}$$

Let $[u] = [u_1] \cdots [u_r]$, $[u_j] \in \partial M^{\Gamma_j}$. Then there exists $y \in V_i$ such that $[u_i] \in y\partial M^{\Gamma_i}$. Hence, by (3.1.75)

$$(1 - T_y T_y^*)(f)([u]) = 0. \tag{3.1.76}$$

Hence $\mathbf{e}_i(f) = 0$ and this yields the claim. ∎

3.2 M-fractals

Definition 3.2.1. Let M be a finitely 1-generated monoid. By an *M-fractal* we will understand a compact metric space (X, d) with a contracting left M-action $\alpha\colon M \longrightarrow \mathcal{C}(X, X)$, i.e., there exists a real number $\delta < 1$ such that for all $x, y \in X$ and all $\omega \in M \setminus \{1\}$ one has

$$d\big(\alpha(\omega)(x), \alpha(\omega)(y)\big) < \delta \cdot d(x, y). \tag{3.2.1}$$

The real number δ will be called the *contraction constant*.

Example 14. Let $s_1, s_2\colon I \to I$, $I = [0, 1]$, be defined by $s_1(x) = \frac{1}{3}x$, $s_2(x) = \frac{2}{3} + s_1(x)$. Then $\langle s_1, s_2\rangle \subseteq \mathcal{C}(I, I)$ is isomorphic to the free monoid $\mathscr{F}_2$ on 2 generators. The $\mathscr{F}_2$-fractal (I, d, α), where d is the standard metric and α is the action described above, has as attractor the Cantor set (see [20], Ex. 3.3).

3.2.1 The action of the universal boundary on an M-fractal

Let M be a finitely 1-generated monoid with grading $|\_|\colon M \to \mathbb{N}_0$. For a strictly decreasing sequence $f \in \mathscr{D}(\mathbb{N}, M, \preceq)$ and for $n, m \in \mathbb{N}_0$, $m > n$, there exists $\tau_{m,n} \in M \setminus \{1\}$ such that $f(m) = f(n) \cdot \tau_{m,n}$. By induction, one concludes that $|f(n)| \geq n$. If $[f] \in \partial M$, then f can be represented by a strictly decreasing sequence (cf. Proposition 2.1.11).

As α is contracting, one concludes that $\big(\alpha(f(n))(x)\big)$ is a Cauchy sequence for every strictly decreasing sequence $f \in \mathscr{D}(\mathbb{N}, M, \preceq)$ and thus has a limit point $\alpha(f)(x) = \lim_{n\to\infty} \big(\alpha(f(n))(x)\big)$. In more detail, if α has contracting constant $\delta < 1$, one has for $n, m \in \mathbb{N}$, $m > n$, that

$$d(\alpha(f(m))(x), \alpha(f(n))(x)) < \delta^{|f(n)|} \cdot d(\alpha(\tau_{m,n})(x), x) \leq \delta^{|f(n)|} \cdot \mathrm{diam}(X), \tag{3.2.2}$$

where $\mathrm{diam}(X) = \max\{\, d(y, z) \mid y, z \in X \,\}$. Thus one has a map

$$\_ \cdot \_\colon \mathscr{D}(\mathbb{N}, M, \prec) \times X \longrightarrow X \tag{3.2.3}$$

given by $[f] \cdot x = \alpha(f)(x)$. This map has the following property.

Remark 3.2.2. (a) Let (X, d) be a compact metric space. For $A, B \subseteq X$ the Hausdorff metric $\mathfrak{d}\colon \mathscr{P}(X) \times \mathscr{P}(X) \to \mathbb{R}_0^+$, where $\mathscr{P}(X)$ denotes the set of subsets of X, is given by

$$\mathfrak{d}(A, B) = \sup\{\, d(a, B), d(b, A) \mid a \in A, b \in B \,\},$$

where $d(a, B) = \inf\{\, d(a, b) \mid b \in B \,\}$ (cf. [20, (2.4)]).
(b) Let M be a finitely 1-generated monoid, and let $((X, d), \alpha)$ be an M-fractal with attractor $K \subseteq X$. For $\mathscr{S}\colon \mathscr{P}(X) \to \mathscr{P}(X)$ given by

$$\mathscr{S}(A) = \bigcup_{\sigma\in M_1} \alpha(\sigma)(A),$$

it is well known that $(\mathscr{S}^k(A))_{k\in\mathbb{N}}$, where $\mathscr{S}^k(A) = \mathscr{S}(\mathscr{S}^{k-1}(A))$, converges to K in the Hausdorff metric (cf. [20, Statement (1)]).

Proposition 3.2.3. *Let M be a finitely 1-generated monoid, and let $((X, d), \alpha)$ be an M-fractal with attractor $K \subseteq X$. Then the map (3.2.3) is continuous and $[f] \cdot x \in K$ for all $f \in \mathscr{D}(\mathbb{N}, M, \prec)$ and $x \in X$.*

Proof. Let $f \in \mathscr{D}(\mathbb{N}, M, \prec)$ be a strictly decreasing function. For $A = \{x\}$, and $\mathscr{S}$ as above, the sequence $(\mathscr{S}^k(A))_{k\in\mathbb{N}}$ converges to K in the Hausdorff metric. Thus for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for all $n > N(\varepsilon)$ one has $\mathfrak{d}(\mathscr{S}^n(A), K) < \varepsilon$. Hence $d(\alpha(f(n))(x), K) < \varepsilon$ for all $n > N(\varepsilon)$, and $\alpha(f)(x)$ is a clusterpoint of K. As K is closed this implies $\alpha(f)(x) \in K$.

The map (3.2.3) is obviously continuous in the second argument. Moreover, let $f, h \in \mathscr{D}(\mathbb{N}, M, \prec)$, $f, h \prec \tau$, $\tau \in M$. Then

$$d(\alpha(f)(x)), \alpha(h)(x)) \leq 2 \cdot \delta^{|\tau|} \cdot \mathrm{diam}(X). \tag{3.2.4}$$

Thus (3.2.3) is continuous. □

Proposition 3.2.4. *Let $f, h \in \mathscr{D}(\mathbb{N}, M, \prec)$ satisfying $f \preceq h$. Then, $\alpha(f)(x) = \alpha(h)(x)$.*

Proof. We may assume that $f(n) \preceq h(n)$ for all $n \in \mathbb{N}$, i.e., there exists $y_n \in M$ such that $f(n) = h(n) \cdot y_n$. Then, by the same argument which was used for (3.2.2), one concludes that

$$d(\alpha(f(n))(x), \alpha(h(n)(x)) \leq \delta^{|h(n)|}\mathrm{diam}(X) \leq \delta^n \mathrm{diam}(X). \tag{3.2.5}$$

This yields the claim. □

From Proposition 3.2.4 one concludes that the map (3.2.3) induces a map

$$\_ \cdot \_ \colon \widetilde{\partial} M \times X \longrightarrow X \tag{3.2.6}$$

given by $\pi([f]) \cdot x = \alpha(f)(x)$ (cf. (2.1.15)), and thus an action of $\widetilde{\partial} M$ on X.

The following property suggest to think of $(\widetilde{\partial} M, \mathcal{T}_c)$ as the universal attractor of an M-fractal.

Proposition 3.2.5. *Let $x \in X$, and let $K \subset X$ be the attractor of the M-fractal $((X, d), \alpha)$. Then the induced map*

$$\kappa_x \colon \partial M \longrightarrow K \tag{3.2.7}$$

given by $\kappa_x([f]) = \alpha(f)(x)$ is surjective.

Proof. Let $z \in K$, and $A = \{x\}$. By (cf. [20, (2.4)]), for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for all $n > N(\varepsilon)$ one has $\mathfrak{d}(\mathscr{S}^n(A), z) < \varepsilon$, i.e., there exists a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n \in M_n$, $f_{n+1} \in \bigcup_{\sigma \in M_1}\{\sigma \cdot f_n\}$, such that $d(\alpha(f_n)(x), z) < \varepsilon$.

If M is $\mathcal{T}$-regular, then $(\bar{M}, \mathcal{T}_c(\bar{M}))$ is compact (cf. Proposition 3.1.14). Hence $(f_n)_{n\in\mathbb{N}}$ has a cluster point $f \in \bar{M}$. As $|f_n| = n$, one has $f \notin M$ and thus $f \in \partial M$. It is straightforward to verify that $[f] \cdot x = z$, showing that κ_x is surjective. □

Note that, in general, we do not know whether the topological space $(\partial M, \mathcal{T}_c(\bar{M}))$ is compact.

Definition 3.2.6. If the topological space $(\partial M, \mathcal{T}_c(\bar{M}))$ is compact, we call it the *universal attractor of the finitely 1-generated $\mathbb{N}_0$-graded $\mathcal{T}$-regular monoid M.*

Remark 3.2.7. Let M be a finitely 1-generated monoid. Then M carries canonically a probability measure μ_M (cf. § 3.1.5). Thus, by Proposition 3.2.5, the attractor of the M-fractal $((X, d), \alpha)$ carries the *contact probability measure* $\mu_x = \mu_\circ^{\kappa_x}$ for every point $x \in X$, which is given by

$$\mu_x(B) = \mu_M(\kappa_x^{-1}(B)), \qquad B \in \mathrm{Bor}(K). \tag{3.2.8}$$

3.2.2 The C^*-algebra associated to an M-fractals for a finitely 1-generated monoid M

Let M be a finitely 1-generated monoid, and let $((X,d),\alpha)$ be an M-fractal with attractor K. For $x \in X$, there exists a continuous mapping $\kappa_x\colon \partial M \to K$ by Proposition 3.2.5. Let $\mu_x\colon \mathrm{Bor}(K) \to \mathbb{R}_0^+$ be the probability measure given by (3.2.8). Then the action of M on K defines an action of M on $L^2(K, \mathbb{C}, \mu_x)$.

Proposition 3.2.8. *The monoid M acts on the Hilbert space $L^2(K, \mathbb{C}, \mu_x)$ by bounded linear operators*

$$\begin{aligned} \gamma_t\colon L^2(K,\mathbb{C},\mu_x) &\to L^2(K,\mathbb{C},\mu_x) \\ \gamma_t(g)(x) &= g(\alpha_t(x)), \quad g \in L^2(K,\mathbb{C},\mu_x) \end{aligned} \tag{3.2.9}$$

for any $t \in M$.

Proof. Let $t \in M$. Cleary, γ_t is linear. For $g \in L^2(K,\mathbb{C},\mu_x)$ one has

$$\|\gamma_t(g)\|_2^2 = \int_K |\gamma_t(g(z))|^2 \, d\mu_x = \int_K |g(\alpha_t(z))|^2 \, d\mu_x \le \|g\|_2^2.$$

Thus, γ_t is bounded. □

Definition 3.2.9. We define the C^*-algebra generated by the M-fractal $((X,d),\alpha)$ by

$$C^*(M,X,d,\mu_x) = C^*\big(\{\, \gamma_t \mid t \in M \,\}\big) \subseteq \mathcal{B}(L^2(K,\mathbb{C},\mu_x)). \tag{3.2.10}$$

Chapter 4

Bass-Serre theory for groupoids

In this chapter we develop the analogue of Bass-Serre theory in the groupoid context. We start by defining a graph of groupoids, then we construct a desingularization of a groupoid action on a graph. In particular, we define a graph of groupoids associated to a groupoid action on a graph. Then, given a graph of groupoids, we associate to it a groupoid, called the *fundamental groupoid*, and a forest, called the *Bass-Serre forest*, such that the fundamental groupoid acts on the Bass-Serre forest. Finally, we prove the structure theorem.

4.1 Graphs of groupoids

Definition 4.1.1. A *graph of groupoids* $\mathcal{G}(\Gamma)$ consists of a connected combinatorial graph $\Gamma = (\Gamma^0, \Gamma^1)$ together with the following data:

(i) a *vertex* groupoid $\mathcal{G}_v$ for every vertex $v \in \Gamma^0$;

(ii) an *edge* groupoid $\mathcal{G}_e$ for every edge $e \in \Gamma^1$ satisfying $\mathcal{G}_e = \mathcal{G}_{\bar{e}}$;

(iii) an injective homomorphism of groupoids $\alpha_e \colon \mathcal{G}_e \to \mathcal{G}_{t(e)}$ for every $e \in \Gamma^1$.

If Γ is a tree, we call $\mathcal{G}(\Gamma)$ a *tree of groupoids*.

Standing Assumption. We suppose that $\alpha_e(\mathcal{G}_e)$ is a wide subgroupoid of $\mathcal{G}_{t(e)}$ for all $e \in \Gamma^1$, i.e., it has the same unit space as $\mathcal{G}_{t(e)}$, and that $\mathcal{G}_v^{(0)}$ is discrete for all $v \in \Gamma^0$.

Notation 4.1.2. For $e \in \Gamma^1$, we put $\mathcal{H}_e = \alpha_e(\mathcal{G}_e)$. We denote by

$$\phi_e \colon \mathcal{H}_{\bar{e}} \subseteq \mathcal{G}_{o(e)} \to \mathcal{H}_e \subseteq \mathcal{G}_{t(e)}$$

the map given by

$$\phi_e(g) = (\alpha_e \circ \alpha_{\bar{e}}^{-1})(g), \quad g \in \mathcal{H}_{\bar{e}}.$$

Hence, ϕ_e is an isomorphisms between the groupoids $\mathcal{H}_{\bar{e}}$ and $\mathcal{H}_e$ and one has $\phi_{\bar{e}} = \phi_e^{-1}$.

Definition 4.1.3. A *graph of partial isomorphisms* Γ^{pi} is given by a graph $\Gamma = (\Gamma^0, \Gamma^1)$ together with

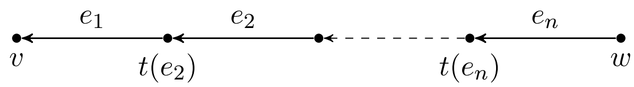
(i) a set Γ_v for each vertex $v \in \Gamma^0$;

(ii) a partial isomorphism ϕ_e^0 for each edge $e \in \Gamma^1$, i.e., an isomorphism such that the domain and image of ϕ_e^0 are subspaces $\mathrm{dom}(\phi_e^0) \subseteq \Gamma_{o(e)}$ and $\mathrm{im}(\phi_e^0) \subseteq \Gamma_{t(e)}$ respectively, and such that $\phi_{\bar{e}}^0 = (\phi_e^0)^{-1}$.

If Γ is a tree, we call Γ^{pi} a *tree of partial isomorphisms*.

Remark 4.1.4. Let Γ^{pi} be a tree of partial isomorphisms.

(a) Since Γ is a tree, for $v, w \in \Gamma^0$ there exists a unique reduced path $p \in \mathcal{P}_{v,w}$, i.e., $p = e_1 \cdots e_n$, $n \in \mathbb{N}$, $o(p) = o(e_n) = w$ and $t(p) = t(e_1) = v$.



Each edge e_i, $i = 1, \ldots, n$, carries a partial isomorphism

$$\phi_{e_i} \colon D_i \to C_i,$$

where $D_i = \mathrm{dom}(\phi_{e_i}) \subseteq \Gamma_{o(e_i)}$ and $C_i = \mathrm{im}\phi_{e_i} \subseteq \Gamma_{t(e_i)}$. Let $A_n = \mathrm{dom}(\phi_{e_n})$ and $A_i = D_i \cap C_{i+1}$ for $i = 1, \ldots, n-1$. Suppose that $A_i \neq \emptyset$ for all $i = 1, \ldots, n$. Then

$$\Phi_p := \phi_{e_1}|_{A_1} \circ \phi_{e_2}|_{A_2} \circ \cdots \circ \phi_{e_n}|_{A_n} \tag{4.1.1}$$

defines a partial isomorphism between Γ_w and Γ_v.

(b) Let $U = \bigsqcup_{v \in \Gamma^0} \Gamma_v$. Then the partial isomorphisms ϕ_e, $e \in \Gamma^1$, associated to the edges of Γ induce an equivalence relation on U given by

$$x \sim y \iff \text{there exists } p \in \mathcal{P}_{v,w} \text{ such that } \Phi_p(y) = x,\ x \in \Gamma_v,\ y \in \Gamma_w. \tag{4.1.2}$$

Clearly, $\sim$ is reflexive. It is also symmetric, since $\Phi_p(x) = y$ implies that $\Phi_{\bar{p}}(y) = x$, where $\bar{p}$ is the reversal path of p. Finally, it is transitive, since $\Phi_p(x) = y$ and $\Phi_q(y) = z$ implies that $o(p) = t(q)$ and $\Phi_{pq}(x) = z$.

Therefore one has that for each $x \in \mathrm{im}(\Phi_p) \subseteq \Gamma_v$ there exists a unique $y \in \mathrm{dom}(\Phi_p) \subseteq \Gamma_w$ such that $x \sim y$.

Definition 4.1.5. Let A be a set, and let $\mathcal{R}$ be an equivalence relation on A. A subset B of A is said to be *saturated* with respect to $\mathcal{R}$ if for all $x, y \in A$, $x \in B$ and $x\mathcal{R}y$ imply $y \in B$. Equivalently, B is saturated if it is the union of a family of equivalence classes with respect to $\mathcal{R}$. For a subset C of A, the *saturation* of C with respect to $\mathcal{R}$ is the least saturated subset $S(C)$ of A that contains C.

Definition 4.1.6. A tree of partial isomorphisms Γ^{pi} is said to be *rooted* if there exists a vertex $v \in \Gamma^0$, called the *root*, such that Γ_v is a system of representatives for the relation $\sim$ defined in (4.1.2).

Remark 4.1.7. Via the forgetful functor (see Remark 2.3.4), one may associate to any graph of groupoids $\mathcal{G}(\Lambda)$ a tree of partial isomorphisms Γ^{pi} where $\Gamma \subseteq \Lambda$ is a maximal subtree of Λ. The sets associated to the vertices of Γ are given by $\Gamma_v = \mathcal{G}_v^{(0)}$, $v \in \Gamma^0$, and the partial isomorphisms ϕ_e^0, $e \in \Gamma^1$, are given by

$$\phi_e^0 = \phi_e|_{\mathcal{G}^{(0)}_{o(e)}} \colon \mathcal{G}^{(0)}_{o(e)} \to \mathcal{G}^{(0)}_{t(e)}.$$

Since we will only consider graphs of groupoids where the edge groupoids are wide subgroupoids of the adjacent vertex groupoids, i.e., $\alpha_e(\mathcal{G}_e) = \mathcal{H}_e \subseteq \mathcal{G}_e$ is a wide subgroupoid of $\mathcal{G}_{t(e)}$ for all $e \in \Gamma^1$, we will only deal with tree of partial isomorphisms Γ^{pi} such that each $\phi_e^0 \colon \mathcal{G}^{(0)}_{o(e)} \to \mathcal{G}^{(0)}_{t(e)}$ is bijective, $e \in \Gamma^1$. Therefore, each Γ_v is a system of representative for the equivalence relation $\sim$ defined in (4.1.2).

Thus, under our assumptions, we will deal with rooted trees of full isomorphisms.

Remark 4.1.8. Let Γ^{pi} be a rooted tree of partial isomorphisms with root $r \in \Gamma^0$. Then Γ^{pi} defines naturally a forest $\mathcal{F}^{\mathrm{pi}}(\Gamma)$ given as follows.

$$\mathcal{F}^{\mathrm{pi}}(\Gamma)^0 = \bigsqcup_{v \in \Gamma^0} \Gamma_v$$
$$\mathcal{F}^{\mathrm{pi}}(\Gamma)^1 = \{ (x, y) \in \mathcal{F}^{\mathrm{pi}^0} \times \mathcal{F}^{\mathrm{pi}^0} \mid x \in \Gamma_{o(e)}, y \in \Gamma_{t(e)}, \phi_e(x) = y, e \in \Gamma^1 \}.$$

Clearly, $\mathcal{F}^{\mathrm{pi}}(\Gamma)$ is a graph with terminus and origin maps given by the projection on the first and second component, respectively, and inversion given by the interchange of the components. Moreover, it is a disjoint union of graphs since each connected component is given by an equivalence class of the relation $\sim$ defined in (4.1.2). Finally, since Γ is a tree, for each pair of vertices $x \neq y$, x and y in the same connected component of $\mathcal{F}^{\mathrm{pi}}(\Gamma)$ there is at most one path from x to y. Thus, each connected component of $\mathcal{F}^{\mathrm{pi}}(\Gamma)$ is a tree and hence $\mathcal{F}^{\mathrm{pi}}(\Gamma)$ is a forest.

Definition 4.1.9. Let Γ^{pi} be a rooted tree of partial isomorphisms and let F be a forest. A *representation* of Γ^{pi} on F is a graph homomorphism $\chi\colon \mathcal{F}^{\mathrm{pi}}(\Gamma) \to F$ from the forest $\mathcal{F}^{\mathrm{pi}}(\Gamma)$ defined by Γ^{pi} on its root to F. We say that $F' = \chi(\mathcal{F}^{\mathrm{pi}}(\Gamma)) \subseteq F$ *is the representation* of Γ^{pi} on F.

Given a tree of partial isomorphisms Γ^{pi}, we can associate to each edge $e \in \Gamma^1$ a set Γ_e with injections $\alpha_{\bar{e}}\colon \Gamma_e \to \Gamma_{o(e)}$ and $\alpha_e\colon \Gamma_e \to \Gamma_{t(e)}$ in such a way that $\alpha_{\bar{e}}(\Gamma_e) = \mathrm{dom}(\phi_e)$ and $\alpha_e(\Gamma_e) = \mathrm{im}(\phi_e)$.

Definition 4.1.10. Let $F' = \chi(\mathcal{F}^{\mathrm{pi}}(\Gamma)) \subseteq F$ be a representation of a tree of partial isomorphisms Γ^{pi} on F.
For $v \in \Gamma^0$, we denote by $\chi(\Gamma_v)$ the set

$$\chi(\Gamma_v) = \{ \chi(x) \mid x \in \Gamma_v \subseteq \mathcal{F}^{\mathrm{pi}}(\Gamma)^0 \} \subseteq F'^0$$

and we call it the *representation of* Γ_v *on* F. For $e \in \Gamma^1$, we denote by

$$\chi(\Gamma_e) = \{ \varepsilon \in F'^1 \mid o(\varepsilon) \in \chi(\Gamma_{o(e)}),\, t(\varepsilon) \in \chi(\Gamma_{t(e)}) \} \subseteq F'^1$$

and we call it the *representation of* Γ_e *on* F.

Remark 4.1.11. Let $\mathcal{G}$ be a groupoid acting on a forest F with momentum map $\varphi\colon F^0 \to \mathcal{G}^{(0)}$ and let $F' \subseteq F$ be any representation of a tree of partial isomorphisms Γ^{pi} on F as in Defintion 4.1.9. The action of $\mathcal{G}$ on F induces an action of $\mathcal{G}$ on F' which gives rise to a tree of groupoids $\mathcal{G}_\chi(\Gamma)$ based on Γ as follows. We put

$$\mathcal{G}^{(0)}_{\chi,v} := \Gamma_v, \quad v \in \Gamma^0,$$
$$\mathcal{G}^{(0)}_{\chi,e} := \Gamma_e, \quad e \in \Gamma^1.$$

Then we define $\mathcal{G}_{\chi,v}$ to be the groupoid on $\mathcal{G}^{(0)}_{\chi,v}$ whose morphisms are given by the action of $\mathcal{G}$ on $\chi(\Gamma_v) \subseteq F^0$, i.e.,

$$\mathcal{G}_{\chi,v} = \{ g \in \mathcal{G} \mid s(g) = \varphi(\nu),\, \nu \in \chi(\Gamma_v) \text{ and } \mu(g, \nu) \in \chi(\Gamma_v) \}. \tag{4.1.3}$$

Similarly, we define $\mathcal{G}_{\chi,e}$ to be the groupoid on $\mathcal{G}^{(0)}_{\chi,e}$ whose morphisms are given by the action of $\mathcal{G}$ on $\chi(\Gamma_e) \subseteq F^1$, i.e.,

$$\mathcal{G}_{\chi,e} = \{ g \in \mathcal{G} \mid s(g) = \varphi(o(\varepsilon)),\, \varepsilon \in \chi(\Gamma_e) \text{ and } \mu(g, \varepsilon) \in \chi(\Gamma_e) \}. \tag{4.1.4}$$

Definition 4.1.12. Let $\mathcal{G}(T)$ be a tree of groupoids and let F be a forest. A *representation of* $\mathcal{G}(T)$ *on* F is a representation χ of the tree of partial isomorphisms T^{pi} underlying $\mathcal{G}(T)$ on F (see Remark 4.1.7) such that there are groupoid isomorphisms between the vertex and edge groupoids of $\mathcal{G}(T)$ and the groupoids defined by the action of $\mathcal{G}$ on the representations of the T_v's and T_e's on F, i.e., groupoid isomorphisms

$$\mathcal{G}_v \to \mathcal{G}_{\chi,v}, \quad \mathcal{G}_e \to \mathcal{G}_{\chi,e},$$

for $v \in T^0$ and $e \in T^1$.

Remark 4.1.13. Let $\mathcal{G}$ be a groupoid acting on a forest F. The action of $\mathcal{G}$ on F induces an equivalence relation on F^0 and F^1, which we denote by $\mathcal{R}_{\mathcal{G}}$, defined by

$$\begin{aligned} v\mathcal{R}_{\mathcal{G}}w &\iff \text{there exists } g \in \mathcal{G} : \mu(g,v) = w \\ e\mathcal{R}_{\mathcal{G}}f &\iff \text{there exists } h \in \mathcal{G} : \mu(h,e) = f \end{aligned} \tag{4.1.5}$$

(see Remark 2.4.2).

Definition 4.1.14. Let $\mathcal{G}$ be a groupoid acting without inversion on a forest F. A *tree of representatives* $(\mathcal{G}(T),\chi)$ of the action of $\mathcal{G}$ on F is given by a rooted tree of groupoids $\mathcal{G}(T)$, based on a rooted tree T, with vertex groupoids $\mathcal{G}_x$, $x \in T^0$, and a representation χ of $\mathcal{G}(T)$ on F such that the saturations of the $\chi(\mathcal{G}_x^{(0)})$'s, $x \in T^0$, with respect to the equivalence relation $\mathcal{R}_{\mathcal{G}}$ give a partition of F^0.

4.2 The graph of groupoids associated to a groupoid action on a forest

Definition 4.2.1. A topological space F is said to be *fibered* on a topological space X if there exists a continuous and surjective map $\pi\colon F \to X$, called the *projection*, such that $\pi^{-1}(x)$ is countable for each $x \in X$. We call $F_x = \pi^{-1}(x)$ the *fiber* of x, $x \in X$. A *section* of F is a map $\sigma\colon X \to F$ such that $\pi \circ \sigma = \mathrm{id}_X$. If $A \subseteq X$ and $\sigma_A\colon A \to F$ is such that $\pi \circ \sigma_A = \mathrm{id}_A$, we say that σ_A is a *partial section* of F.

The following theorem (The Selection Theorem) guarantees the existence of a section for a fibered space F on X.

Theorem 4.2.2 ([25, Theorem 12.13])**.** *Let F be a fibered space on X. Then there exists a countable family of partial sections of F such that their images form a partition of F.*

In particular, we will use the following corollary.

Corollary 4.2.3 (Lusin-Novikov, [21, Theorem 18.10])**.** *Let F be a fibered space on X. Then there exists an enumeration of the fibers of F, i.e., there exists a map $N\colon F \to \mathbb{N}$ such that $N|_{F_x}\colon F_x \to \mathbb{N}$ is injective for all $x \in X$. Moreover, one may suppose that the enumeration in each fiber of F starts from 1 and follows the natural enumeration of $\mathbb{N}$.*

We will deal with groupoid actions on fibered spaces.

Definition 4.2.4. Let $\mathcal{G}$ be a groupoid acting on a fibered space (F,π) on $\mathcal{G}^{(0)}$. If $\sigma\colon A \subseteq \mathcal{G}^{(0)} \to F$ is a partial section of F, we call the *stabilizer* of σ the subgroupoid

$$\mathrm{Stab}_{\mathcal{G}}(\sigma) = \{\, g \in \mathcal{G} \mid g \cdot \sigma(s(g)) = \sigma(r(g)), s(g), r(g) \in A \,\} \subseteq \mathcal{G}. \tag{4.2.1}$$

Remark 4.2.5. Let $\mathcal{G}$ be a groupoid acting on a graph $\Gamma = (\Gamma^0, \Gamma^1)$ (cf. Definition 2.4.16). Then Γ^0 is fibered on $\mathcal{G}^0$ via the momentum map $\varphi\colon \Gamma^0 \to \mathcal{G}^{(0)}$.

Theorem 4.2.6. *Let $\mathcal{G}$ be a groupoid acting without inversion on a forest $F = (F^0, F^1)$. Then there exists a countable or finite family $\Sigma = \{\sigma_i\}_{i\in I}$, $I \subseteq \mathbb{N}$, of partial sections of F^0 with domains $X_i \subseteq \mathcal{G}^{(0)}$, i.e., $\sigma_i\colon X_i \to F^0$, $i \in I$, such that*

(i) $\tilde{U}_i \cap \tilde{U}_k = \emptyset$ for all $i \neq k$, where $U_i = \sigma_i(X_i)$ and $\tilde{U}_i = S(U_i)$ denotes the saturation of U_i with respect to $\mathcal{R}_{\mathcal{G}}$;

(ii) for $V = \sqcup_{i\in I} U_i$, one has that $V \cap \varphi_0^{-1}(x) \neq \emptyset$ for all $x \in \mathcal{G}^{(0)}$;

(iii) $\tilde{V} = F^0$.

Proof. We construct recursively a sequence of partial sections $\{\sigma_i\}_{i\in I}$, $I \subseteq \mathbb{N}$, satisfying (i) and (ii). Let $X = \mathcal{G}^{(0)}$ be the unit space of the groupoid $\mathcal{G}$. Since we only consider discrete and thus countable groupoids, we may suppose that $X = \{x_1, x_2, \cdots\}$. The graph $F = (F^0, F^1)$ is fibered on X via the map $\varphi = (\varphi_0, \varphi_1)$, where $\varphi_0\colon F^0 \to X$ and $\varphi_1 = \varphi_0 \circ t\colon F^1 \to X$. For each i, we enumerate the elements in the fiber $\varphi_0^{-1}(x_i)$ and choose the one corresponding to the smallest number, say y_i is such an element in $\varphi_0^{-1}(x_i)$. Let $\sigma_1\colon X \to F^0$ be the partial section defined by

$$\sigma_1(x_i) = y_i. \tag{4.2.2}$$

Let $U_1 = \mathrm{im}(\sigma_1)$ and let $\tilde{U}_1 = S(U_1)$ be the saturation of U_1 with respect to $\mathcal{R}_{\mathcal{G}}$. Put $C_1 = F^0 \setminus \tilde{U}_1$. If $C_1 = \emptyset$, then U_1 is a complete domain for the action of $\mathcal{G}$ on F^0 and the thesis follows by choosing $n = 1$ and $\Sigma = \{\sigma_1\}$. Otherwise, let $X_2 = \varphi_0(C_1)$. For each $x_i \in X_2$, let $z_i \in C_1 \cap \varphi_0^{-1}(x_i)$ be the element corresponding to the smallest number in the enumeration of $C_1 \cap \varphi_0^{-1}(x_i)$. Let $\sigma_2\colon X_2 \to F^0$ be the partial section defined by

$$\sigma_2(x_i) = z_i. \tag{4.2.3}$$

Let $U_2 = \mathrm{im}(\sigma_2)$, $\tilde{U}_2 = S(U_2)$ and $C_2 = F^0 \setminus (\tilde{U}_1 \sqcup \tilde{U}_2)$. If $C_2 = \emptyset$, then $U_1 \sqcup U_2$ is a complete domain for the action of $\mathcal{G}$ on F^0 and the thesis follows by choosing $n = 2$ and $\Sigma = \{\sigma_1, \sigma_2\}$. Otherwise, we use the same argument as above to construct the partial sections σ_k, $k > 2$, recursively.

Let $n \geq 1$. Suppose that we have constructed n partial sections $\{\sigma_i\}_{1\leq i\leq n}$ satisfying conditions (i) and (ii). Let $V_n = \sqcup_{i=1}^n U_i$ and let $C_n = F^0 \setminus \tilde{V}_n$. Then either $C_n = \emptyset$ and thus (iii) holds for $I = \{1, \ldots, n\}$, or $C_n \neq \emptyset$ and we construct a partial section σ_{n+1} as follows. Let $C'_n = \{\, x \in C_n \mid \mathrm{dist}(x, V_n) = 1 \,\}$ and put $X_{n+1} = \{\, x \in X \mid \varphi_0^{-1}(x) \cap C'_n \neq \emptyset \,\}$. We define $\sigma_{n+1}\colon X_{n+1} \to C'_n$ by $\sigma_{n+1}(x) = \min\{\, \varphi_0^{-1}(x) \cap C'_n \,\}$, where the minimum is taken over the enumeration of the elements of each fiber $\varphi_0^{-1}(x)$.

Suppose that $I = \mathbb{N}$ and $F^0 \neq \tilde{V}$, where $V = \sqcup_{k=1}^\infty U_k$. Then the set $C = \{\, x \in F^0 \setminus \tilde{V} \mid \mathrm{dist}(x, V) = 1 \,\}$ is nonempty. Let $a \in C$. Then there exists $v \in \tilde{U}_k$, for some $k \in \mathbb{N}$, such that $\mathrm{dist}(a, v) = 1$. Hence, $\mathrm{dist}(a, \tilde{V}_l) = 1$ for all $l \geq k$, and one has that if $a \in \varphi^{-1}(\varphi(a)) \cap C_l$ is the m-th element in the enumeration, $m \in \mathbb{N}$, then

$$a \in \bigcup_{1\leq j\leq m} \mathrm{im}(\sigma_{k+j}) \subseteq V, \tag{4.2.4}$$

a contradiction. Hence, one has that $V = F^0$. □

Remark 4.2.7. By the construction of the σ_i's used above, one has that $\mathrm{im}(\sigma_i)$ contains the smallest elements which have distance 1 from $V_{i-1} = \bigcup_{k=1}^{i-1} \mathrm{im}(\sigma_k)$ in each fiber $\varphi_0^{-1}(x)$, $x \in \mathcal{G}^{(0)}$. Then for each $i \in \{1, \dots, n\}$ one has a map

$$\begin{aligned} m_i \colon F^0 &\to \{1, \dots, n\} \\ a &\mapsto m_i(a), \end{aligned} \tag{4.2.5}$$

where $m_i(a)$ denotes the integer $m_i(a) < i$ such that the vertex a is adjacent to $\mathrm{im}(\sigma_{m_i(a)})$ in the fiber of $\varphi(a)$. That is, there exists $b \in \mathrm{im}(\sigma_{m_i(a)}) \cap \varphi_0^{-1}(y)$, where $y = \varphi(a)$, such that a and b are adjacent in F_y^0.

Notation 4.2.8. For a graph $\Gamma = (\Gamma^0, \Gamma^1)$, $|\Gamma^0| \geq 2$, and $v \in \Gamma^0$, we denote by $\Gamma - v$ the subgraph of Γ given by

$$\begin{aligned} (\Gamma - v)^0 &= \Gamma^0 \setminus \{v\} \\ (\Gamma - v)^1 &= \Gamma^1 \setminus \{\, e \in \Gamma^1 \mid t(e) = v \text{ or } o(e) = v \,\}. \end{aligned} \tag{4.2.6}$$

The following property will turn out to be useful for our purpose.

Lemma 4.2.9 ([35, Proposition 9])**.** *Let $T = (T^0, T^1)$ be a connected graph, $|T^0| \geq 2$, and let $v \in T^0$ be a terminal vertex. Then T is a tree if and only if $T - v$ is a tree.*

Proof. If T is a tree, then $T - v$ is a tree since it is a connected subgraph of T. Suppose that $T - v$ is a tree. Since v is a terminal vertex, one has that $T^1 = (T - v)^1 \cup \{e, \bar{e}\}$, where $e \in T^1$ is the unique edge such that $o(e) = t(\bar{e}) = v$. Let $u, w \in (T - v)^0$. By hypothesis, one has that $|\mathcal{P}_{u,w}| = 1$. If $q \in \mathcal{P}_{v,w}$, then $q = p\, e$, where $p \in \mathcal{P}_{o(e),w}$. Hence, $\mathcal{P}_{v,w}| = 1$ and thus T is a tree. □

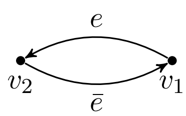
We are now ready to show the existance of a tree of representatives for the action of a groupoid on a forest (see Definition 4.1.14).

Proposition 4.2.10. *Let $\mathcal{G}$ be a groupoid acting without inversion on a forest F. Then there exists a tree of representatives $(\mathcal{G}(T), \chi)$ of the action of $\mathcal{G}$ on F such that $\varphi\big(\chi(\mathcal{G}_r^{(0)})\big) = \mathcal{G}^{(0)}$, where $r \in T^0$ is the root of T.*

Proof. By Theorem 4.2.6, there exists a family $\Sigma = \{\sigma_i\}_{i \in I}$, $I \subseteq \mathbb{N}$, of partial sections of F^0 satisfying (i), (ii) and (iii) of the theorem. If $I = \{1\}$, the thesis follows by choosing T to be the tree with one vertex v and no edges. Then $\mathcal{G}_v = \mathrm{Stab}_{\mathcal{G}}(\sigma_1)$ (cf. Definition 4.2.4) is the groupoid associated to the vertex v. Moreover, the graph $F' = (F'^0, F'^1)$ given by

$$F'^0 = U_1, \quad F'^1 = \emptyset$$

is the representation of $\mathcal{G}(T)$ on F. If $I = \{1, 2\}$, the thesis follows by choosing $T = (T^0, T^1)$ to be the segment tree, i.e., $T^0 = \{v_1, v_2\}$ and $T^1 = \{e, \bar{e}\}$ as follows



with vertex groupoids $\mathcal{G}_{v_1} = \mathrm{Stab}_{\mathcal{G}}(\sigma_1)$ and $\mathcal{G}_{v_2} = \mathrm{Stab}_{\mathcal{G}}(\sigma_2)$. Moreover, the graph $F' = (F'^0, F'^1)$ given by

$$F'^0 = U_1 \sqcup U_2, \quad F'^1 = \{\, e \in F^1 \mid o(e), t(e) \in F'^0 \,\}$$

is the representation of $\mathcal{G}(T)$ on F. For $k \geq 2$, $I = \{1, \dots, k\}$, we define the graph with k vertices $T_k = (T_k^0, T_k^1)$ by

$$(4.2.7) \qquad \begin{aligned} T_k^0 &= \{ v_1, \dots, v_k \} \\ T_k^1 &= \{ \{v_i, v_j\} \in T^0 \times T^0 \mid v_i \in \operatorname{im}(\sigma_{m_j(v_j)}),\, i \neq j, i, j = 1, \dots, k \}, \end{aligned}$$

where $v_i \in \operatorname{im}(\sigma_i)$ is adjacent to $v_j \in \operatorname{im}(\sigma_{m_i(v_i)})$ and m_i is the function defined in Remark 4.2.7. Then for each $k \geq 2$ one has that

$$(4.2.8) \qquad \begin{aligned} T_k^0 &= T_{k-1}^0 \sqcup \{v_k\} \\ T_k^1 &= T_{k-1}^1 \sqcup \{ e \in T^1 \mid o(e) = v_k \text{ or } t(e) = v_k \}. \end{aligned}$$

Thus, for $k \geq 1$, one has that T_k is a tree by Lemma 4.2.9. Finally, we put

$$(4.2.9) \qquad T = \bigcup_{i \in I} T_i.$$

We associate to each $v_i \in T^0$ the groupoid

$$(4.2.10) \qquad G_{v_i} = \operatorname{Stab}_{\mathcal{G}}(\sigma_i)$$

Moreover, for $e \in T^1$ we define the partial section $\sigma_e \colon A \subseteq X \to F^1$ by putting $A = \operatorname{dom}(\sigma_{o(e)}) \cap \operatorname{dom}(\sigma_{t(e)})$ and

$$(4.2.11) \qquad \sigma_e(x) = f,$$

where f is the unique $f \in F^1$ such that $o(f) \in \sigma_{o(e)}(x)$, $t(f) \in \sigma_{t(e)}(x)$, $x \in A$. Thus, we associate to each $e \in T^1$ the groupoid

$$(4.2.12) \qquad G_e = \operatorname{Stab}_{\mathcal{G}}(\sigma_e).$$

Finally, we put

$$F'^0 = V, \quad F'^1 = \{ e \in F^1 \mid o(e), t(e) \in F'^0 \}.$$

Since $\tilde{V} = F^0$, one has that the graph $F' = (F'^0, F'^1)$ is the representation of a tree of representatives of the action of $\mathcal{G}$ on F. By construction, if $v, w \in F^0$ are in the same orbit, then either $v \in F'^0$ or $w \in F'^0$. □

Remark 4.2.11. Note that by definition of T, one has an orientation $T_+^1 \subseteq T^1$ of T.

Remark 4.2.12. Let $\mathcal{G}$ be a groupoid acting on a forest F. Then there exists a subforest F' of F such that the vertices of F' are a fundamental domain for the action of $\mathcal{G}$ on F^0. In particular, the vertices of F' are a fundamental domain for the equivalence relation $\mathcal{R}_{\mathcal{G}}$ generated by the action of $\mathcal{G}$ on F^0. Then the restriction $\pi|_{F'} \colon F' \to \mathcal{G}\backslash\backslash F$ is injective on the vertex set and its image $\pi|_{F'}(F')$ is a maximal subforest of $\mathcal{G}\backslash\backslash F$, i.e., $\pi|_{F'^0}(F'^0) = (\mathcal{G}\backslash\backslash F)^0$.

In classical Bass-Serre theory, given a group acting without inversion on a tree, one defines a tree of groups by considering a maximal subtree of the quotient graph and the stabilizers of its vertices and edges. A tree of representatives has the same role in this context. However, the tree of representatives $(\mathcal{G}(T), \chi)$ associated to a groupoid action on a forest F does not contain all the information we need: in general, the $\mathcal{G}$-orbits of the $\chi(G_e)$'s, $e \in T^1$, do not cover F^1. Thus, we need the following definition.

Definition 4.2.13. Let $\mathcal{G}$ be a groupoid acting on a forest F. A *desingularization* $\mathfrak{D}(\mathcal{G}, F)$ of the action of $\mathcal{G}$ on F (or a desingularization of $\mathcal{G}\backslash\!\backslash F$) consists of

(D1) a connected graph Γ with orientation Γ^1_+;

(D2) a graph of groupoids $\mathcal{G}(\Gamma)$ based on Γ;

(D3) a maximal rooted subtree $T \subseteq \Gamma$;

(D4) a tree of representatives $(\mathcal{G}(T), \chi)$;

(D5) a partial section $\sigma_e \colon X_e \subseteq \mathcal{G}^{(0)} \to F^1$ together with a family of elements $g_e = \{\, g_{e,x} \,\}_{x\in\varphi(\mathrm{im}(\sigma_e))} \subseteq \mathcal{G}$ for any $e \in \Gamma^1_+ \setminus T^1$;

satisfying the following properties:

(i) the tree of groupoids $(\mathcal{G}(T), \chi)$ induced on T is a tree of representatives for the action of $\mathcal{G}$ on F where $\varphi\big(\,\chi\big(T_r^{(0)}\big)\,\big) = \mathcal{G}^{(0)}$ and $r \in T^0$ is the root of T;

(ii) for each $e \in \Gamma^1_+ \setminus T^1$ and for $\varepsilon \in \mathrm{im}(\sigma_e)$ one has that $o(\varepsilon) \in \chi\big(\mathcal{G}^{(0)}_{o(e)}\big)$ and $g_{e,\varphi(t(\varepsilon))} \cdot t(\varepsilon) \in \chi\big(\,\mathcal{G}^{(0)}_{t(e)}\,\big)$;

(iii) for each $e \in \Gamma^1_+ \setminus T^1$, the maps

$$\alpha_{\bar{e}} \colon \mathcal{G}_e \to \mathcal{G}_{o(e)},$$
$$\alpha_e \colon \mathcal{G}_e \to \mathcal{G}_{t(e)}$$

are given by inclusion and conjugation by g_e (cf. Definition 2.4.14) composed with inclusion, respectively;

(iv) the saturations with respect to $\mathcal{R}_\mathcal{G}$ of the $\chi(\mathcal{G}_e)$'s, $e \in \Gamma^1$, form a partition of F^1.

We prove that there exists a desingularization for any groupoid action without inversion on a forest. This result is analogous to the construction of a graph of groups associated to a group action without inversion on a tree. The main difference is that in the classical Bass-Serre theory the underlying graph Λ of the graph of groups $G(\Lambda)$ associated to a group action without inversion on a tree T is the quotient graph $\Lambda = G\backslash\!\backslash T$, while in this context there is no canonical graph underlying the desingularization.

Theorem 4.2.14. *Let $\mathcal{G}$ be a groupoid acting on a forest F. Then there exists a desingularization of the action of $\mathcal{G}$ on F.*

Proof. By Proposition 4.2.10, there exists a tree T (see (4.2.9)) and a tree of representatives $(\mathcal{G}(T), \chi)$ such that $\varphi\big(\chi(\mathcal{G}_r^{(0)})\big) = \mathcal{G}^{(0)}$, where $r \in T^0$ is the root of T. Let $F' = (F'^0, F'^1) \subseteq F$ be the representation of T^{pi} on F. Then one has

$$F^0 = \bigsqcup_{v\in T^0} \mu^0\big(\,\mathcal{G},\, \chi(\mathcal{G}_v^{(0)})\,\big), \tag{4.2.13}$$

i.e., the $\mathcal{G}$-orbits of the $\chi(\mathcal{G}_v^{(0)})$'s, $v \in T^0$, form a partition of F^0. Let

$$S = \bigsqcup_{e\in T^1} \mu^1\big(\,\mathcal{G},\, \chi(\mathcal{G}_e^{(0)})\,\big) \tag{4.2.14}$$

i.e., S is the union of the $\mathcal{G}$-orbits of the $\chi(\mathcal{G}_e)$'s, $e \in T^1$. Let $C = F^1 \setminus S$. If $C = \emptyset$, then we put $\Gamma = T$ and $\mathcal{G}(T)$ is the graph of groupoids required (cf. Proposition 4.2.10). Suppose that $C \neq \emptyset$ and let

$$C' = \{\, e \in C \mid o(e) \in F'^0 \,\}. \tag{4.2.15}$$

Then $C' \neq \emptyset$ since the fibers of F are connected graphs and since the action of $\mathcal{G}$ preserves the distances on F^0. Thus, we construct a family of partial sections of C such that the saturations of their images form a partition of C. Let $v \in T^0$ and let σ_v be the partial section associated to v. For all $w \in T^0$, let

$$C_{v,w} = \{\, e \in C \mid o(e) \in \mathrm{im}(\sigma_v),\, t(e) \in S\big(\chi(\mathcal{G}_w)\big) \,\}. \tag{4.2.16}$$

where $S(\chi(\mathcal{G}_w))$ denotes the saturation of $\chi(\mathcal{G}_w)$ with respect to $\mathcal{R}_{\mathcal{G}}$. If $C_{v,w} \neq \emptyset$, then it is fibered on $\varphi \circ o\,(C_{v,w}) \subseteq \mathcal{G}^{(0)}$, where φ is the momentum map. Then by Theorem 4.2.6 there exists a family $\{\sigma_i\}_{i\in I}$, $I \subseteq \mathbb{N}$, of partial sections such that

$$C_{v,w} = \bigsqcup_{i\in I} S\big(\mathrm{im}(\sigma_i)\big) \tag{4.2.17}$$

gives a partition of $C_{v,w}$. For each i in the decomposition of $C_{v,w}$ we define a new edge $f_i(v,w) \in \Gamma^1$ such that $o(f_i(v,w)) = v$, $t(f_i(v,w)) = w$. Thus, $\Gamma = (\Gamma^0, \Gamma^1)$ is the graph given by

$$\begin{aligned} \Gamma^0 &= T^0, \\ \Gamma^1 &= T^1 \sqcup \{\, f_i(v,w),\ \bar{f}_i(v,w) \mid i \in I,\, v,w \in T^0 \,\}. \end{aligned} \tag{4.2.18}$$

We put

$$\Upsilon_+ = \{\, f_i(v,w) \mid i \in I,\, v,w \in T^0 \,\}, \quad \Upsilon_- = \{\, \bar{f} \mid f \in \Upsilon_+ \,\}, \tag{4.2.19}$$

and $\Gamma^1_+ = T^1_+ \sqcup \Upsilon_+$. Thus, we have (D1) and (D3). It remains to construct (D2), (D4), (D5). For $v \in T^0$ and $e \in T^1$, we put $\mathcal{G}_v = \mathrm{Stab}_{\mathcal{G}}(\sigma_v)$ and $\mathcal{G}_e = \mathrm{Stab}_{\mathcal{G}}(\sigma_e)$ as in (4.2.10) and (4.2.12), respectively. For $f_i \in \Upsilon_+$, we associate to f_i the groupoid

$$\mathcal{G}_{f_i} = \mathcal{G}_{\bar{f}_i} = \mathrm{Stab}_{\mathcal{G}}(\sigma_i). \tag{4.2.20}$$

Then $\mathcal{G}_{f_i}$ injects naturally into $\mathcal{G}_v$, i.e.,

$$\alpha_{\bar{f}_i} \colon \mathcal{G}_{f_i} = \mathrm{Stab}_{\mathcal{G}}(\sigma_i) \hookrightarrow \mathcal{G}_v = \mathrm{Stab}_{\mathcal{G}}(\sigma_v) \tag{4.2.21}$$

is given by inclusion. Moreover, by construction one has that for $\varepsilon_i \in \mathrm{im}(\sigma_i)$, there exists $g_{f_i,\varphi(t(\varepsilon_i))} \in \mathcal{G}$ such that

$$\mu^0\big(\, g_{f_i,\varphi(t(\varepsilon_i))}, t(\varepsilon_i) \,\big) \in F'^0. \tag{4.2.22}$$

Let $\mathcal{G}_w = \mathrm{Stab}_{\mathcal{G}}(\sigma_w)$ as in (4.2.10). Then the groupoid $\mathcal{G}_{f_i}$ injects into $\mathcal{G}_w$ via the conjugation by $g_{f_i} = \{\, g_{f_i,\varphi(t(\varepsilon_i))} \,\}_{\varepsilon_i \in \mathrm{im}(\sigma_i)}$ as follows. Put $\sigma_\tau := t \circ \sigma_i \colon \mathrm{dom}(\sigma_i) \to F^0$. Then $\mathcal{G}_{f_i} = \mathrm{Stab}_{\mathcal{G}}(\sigma_i)$ injects naturally into $\mathrm{Stab}_{\mathcal{G}}(\sigma_\tau)$, i.e., the map $\iota \colon \mathcal{G}_{f_i} \hookrightarrow \mathrm{Stab}_{\mathcal{G}}(\sigma_\tau)$ is given by inclusion. Since the subgroupoid $\mathrm{Stab}_{\mathcal{G}}(\sigma_\tau) \subseteq \mathcal{G}$ and $\mathcal{G}_w \subseteq \mathcal{G}$ have the same unit space, the equivalence of categories $\mathrm{Stab}_{\mathcal{G}}(\sigma_\tau) \to \mathcal{G}_w$ in Definition 2.4.14 is an identity on the unit space $\mathrm{Stab}_{\mathcal{G}}(\sigma_\tau)^{(0)}$, and the conjugation map is given by

$$\begin{aligned} i_{g_{f_i}} \colon \mathrm{Stab}(\sigma_\tau) &\to \mathcal{G}_w \\ \gamma &\mapsto g_{f_i,r(\gamma)}\, \gamma\, g^{-1}_{f_i,s(\gamma)}. \end{aligned} \tag{4.2.23}$$

Since the identity functor is an injective equivalence of categories, by Lemma 2.4.15 one has that $i_{g_{f_i}}$ is injective. Hence, the composition

$$\alpha_{f_i}\colon \mathcal{G}_{f_i} \xrightarrow{\iota} \mathrm{Stab}_{\mathcal{G}}(\sigma_\tau) \xrightarrow{i_{g_{f_i}}} \mathcal{G}_w, \tag{4.2.24}$$

where the first map is given by inclusion and the second map by conjugation by g_{f_i}, is an injective homomorphism of groupoids. Thus, we have (D2), (D4) and (D5). □

4.3 The fundamental groupoid of a graph of groupoids

In the previous section we have proved that given a groupoid $\mathcal{G}$ acting without inversion of edges on a forest F, there exists a desingularization $\mathfrak{D}(\mathcal{G}, F)$ associated to the action of $\mathcal{G}$ on F (cf. Theorem 4.2.14), and hence a graph of groupoids associated to the action of $\mathcal{G}$ on F. We now want to prove that it is possible to reconstruct $\mathcal{G}$ in terms of the vertex end edge groupoids of the graph of groupoids $\mathcal{G}(\Gamma)$. In particular, we will prove that $\mathcal{G}$ is isomorphic to the fundamental groupoid of the graph of groupoids $\mathcal{G}(\Gamma)$.

4.3.1 The fundamental groupoid of a graph of groupoids

Let $\mathcal{G}(\Gamma)$ be a graph of groupoids and let $T \subseteq \Gamma$ be a maximal rooted subtree. Let $\mathcal{H}_e = \alpha_e(\mathcal{G}_e)$ and let $\phi_e\colon \mathcal{H}_{\bar{e}} \to \mathcal{H}_e$ be the map given by $\phi_e(g) = \alpha_e(\alpha_{\bar{e}}^{-1}(g))$. Since the groupoids $\mathcal{H}_e$ are wide subgroupoids of $\mathcal{G}_{t(e)}$ for all $e \in \Gamma^1$, the map

$$\phi_e^0 = \phi_e|_{\mathcal{G}_{o(e)}^{(0)}}\colon \mathcal{G}_{o(e)}^{(0)} \to \mathcal{G}_{t(e)}^{(0)} \tag{4.3.1}$$

is a bijection for all $e \in \Gamma^1$. Then the maps ϕ_e^0's induce a rooted tree of partial isomorphisms T^{pi} which defines an equivalence relation $\sim$ (see (4.1.2)) on $Y = \sqcup_{v\in\Gamma^0} \mathcal{G}_v^{(0)}$. Let $\tilde{Y} = Y/\sim$ and $\pi\colon Y \to \tilde{Y}$ be the canonical projection. Let $r \in T^0$ be the root of the tree T. Then

$$X = \mathcal{G}_r^{(0)} \tag{4.3.2}$$

is a fundamental domain for the equivalence relation $\sim$ and the map

$$\tau_0 := \pi|_X^{-1}\colon \tilde{Y} \to X \tag{4.3.3}$$

is bijective. Let

$$\tau = \tau_0 \circ \pi\colon Y \to X \tag{4.3.4}$$

the map assigning to each $y \in Y$ the representative $x \in X$ such that $y \sim x$.

Definition 4.3.1. The *fundamental groupoid* $\pi_1(\mathcal{G}(\Gamma))$ of the graph of groupoids $\mathcal{G}(\Gamma)$ is the groupoid defined as follows (see [17, §3] for presentation of groupoids). It has unit space X defined in (4.3.2) and generators

$$\Big(\bigsqcup_{v\in\Gamma^0} \mathcal{G}_v\Big) \sqcup \Big(\bigsqcup_{\substack{e\in\Gamma^1,\\ x\in\mathcal{G}_{o(e)}^{(0)}}} (\phi_e(x), x)\Big). \tag{4.3.5}$$

Note that for $z \in \mathcal{G}_e^{(0)}$ one has that $(\alpha_e(z), \alpha_{\bar{e}}(z))$ is a generator, since $\phi_e(\alpha_{\bar{e}}(z)) = \alpha_e \circ \alpha_{\bar{e}}^{-1}(\alpha_{\bar{e}}(z)) = \alpha_e(z)$. The range and source maps $r, s\colon \pi_1(\mathcal{G}(\Gamma)) \to X$ are defined by

$$r(g) = \tau(r'(g)), \quad r((\phi_e(x), x)) = \tau(\phi_e(x)), \tag{4.3.6}$$

$$s(g) = \tau(s'(g)), \quad s((\phi_e(x), x)) = \tau(x), \tag{4.3.7}$$

where r' and s' denotes the range and source maps in $\mathcal{G}_v$, $v \in \Gamma^0$, and multiplication is given by concatenation. The inverse map is given by the inverse map in $\mathcal{G}_v$ for elements $g \in \mathcal{G}_v$, $v \in \Gamma^0$, and by $(\phi_e(x), x)^{-1} = (x, \phi_e(x))$ for $e \in \Gamma^1$, $x \in \mathcal{G}^{(0)}_{o(e)}$. Moreover, the generators are subject to the relations in the vertex groupoids together with relations

(R1) $(\phi_e(x), x) = \mathrm{id}_{\tau(x)}$ for $e \in T^1$, $x \in \mathcal{G}^{(0)}_{o(e)}$;

(R2) $\big(\alpha_e(r(g)), \alpha_{\bar{e}}(r(g))\big) \cdot \alpha_{\bar{e}}(g) \cdot \big(\alpha_{\bar{e}}(s(g)), \alpha_e(s(g))\big) = \alpha_e(g)$ for $g \in \mathcal{G}_e, e \in \Gamma^1$.

Definition 4.3.2. We call the *free groupoid* $\mathcal{F}(\mathcal{G}(\Gamma))$ the groupoid with the same generators as $\pi_1(\mathcal{G}(\Gamma))$, i.e., the generators in (4.3.5), subject to the relations in the vertex groupoids together with the relation (R2).

Notation 4.3.3. In order to simplify calculations with fundamental groupoids elements, we will use the following notation:

$$\phi_e(x)e = ex := (\phi_e(x), x),$$

for $x \in \mathcal{G}_{o(e)}$, $e \in \Gamma^1$. Thus, we may rewrite relations (R1) and (R2) as follows:

(R1) $\quad \phi_e(x)e = ex = \mathrm{id}_{\tau(x)}, \quad$ for $x \in \mathcal{G}_{o(e)}$, $e \in T^1$.

(R2) $\quad e\,\alpha_{\bar{e}}(r(g)) \cdot \alpha_{\bar{e}}(g) \cdot \alpha_{\bar{e}}(s(g))\,\bar{e} = \alpha_e(g) \quad$ for $g \in \mathcal{G}_e$, $e \in \Gamma^1$.

Since for any groupoid G and any morphism $\gamma \in G$ one has $r(\gamma)\gamma = \gamma = \gamma s(\gamma)$, we put

$$g_1 e g_2 = g_1 \cdot s(g_1)\, e\, r(g_2) \cdot g_2 \tag{4.3.8}$$

for $g_1 \in \mathcal{G}_{t(e)}$, $g_2 \in \mathcal{G}_{o(e)}$, $s(g_1) = \phi_e(r(g_2))$. With this notation, (R2) becomes

(R2) $$e\,\alpha_{\bar{e}}(g)\,\bar{e} = \alpha_e(g),$$

for $g \in \mathcal{G}_e, e \in \Gamma^1$.

We can extend our definition of a path in Γ as follows.

Definition 4.3.4. A *graph of groupoids word* w in $\mathcal{G}(\Gamma)$ is a sequence of elements

$$w = g_1 e_1 g_2 e_2 \cdots g_n e_n g_{n+1}, \tag{4.3.9}$$

where

- $e_i \in \Gamma^1$ for all $i = 1, \dots, n$;
- $o(e_i) = t(e_{i+1})$ for all $i = 1, \dots, n-1$;
- $g_i \in \mathcal{G}_{t(e_i)}$ for all $i = 1, \dots, n$ and $g_{n+1} \in \mathcal{G}_{o(e_n)}$;
- for all $i = 1, \dots, n$ one has $\phi_{\bar{e}_i}\big(s(g_i)\big) = r(g_{i+1})$.

We say that w has *length* n. A *subword* of w is a sequence $g_i e_i \cdots g_{i+k} e_{i+k}$ such that $i \geq 1$, $k \geq 0$ and $i + k \leq n$.

Remark 4.3.5. For all $e \in \Gamma^1$ we choose a *left transversal* $\mathcal{T}_e$ of $\alpha_e(\mathcal{G}_e)$ in $\mathcal{G}_{t(e)}$ containing the identity elements of $\mathcal{G}_{t(e)}$, i.e., for all $g \in \mathcal{G}_{t(e)}$ there exist $\tau = \tau(g, e) \in \mathcal{T}_e$ and $h = h(g, e) \in \mathcal{G}_e$ such that

$$g = \tau \alpha_e(h). \tag{4.3.10}$$

Then we may represent a graph of groupoids word w by

$$w = g_1 e_1 g_2 e_2 \cdots g_n e_n g_{n+1}, \tag{4.3.11}$$

where

- $e_i \in \Gamma^1$ for all $i = 1, \ldots, n$;
- $t(e_i) = o(e_{i-1})$ for all $i = 2, \ldots, n$;
- $g_i \in \mathcal{G}_{t(e_i)}$ for all $i = 1, \ldots, n$ and $g_{n+1} \in \mathcal{G}_{o(e_n)}$;
- for all $i = 1, \ldots, n$, subwords $g_i e_i g_{i+1} = \tau_i \alpha_{e_i}(h_i)\, e_i\, \tau_{i+1} \alpha_{e_{i+1}}(h_{i+1})$ must satisfy
$$s(\alpha_{\bar{e}_i}(h_i)) = r(\tau_{i+1}).$$

Definition 4.3.6. A graph of groupoids word $w = g_1 e_1 \cdots g_n e_n g_{n+1}$ is said to be *reduced* if

- $g_i \in \mathcal{T}_{e_i}$ for all $i = 1, \ldots, n$ and $g_{n+1} \in \mathcal{G}_{o(e_n)}$;
- if $\bar{e}_i = e_{i-1}$ for some $i = 2, \ldots, n$, then $g_i \notin \mathrm{im}(\alpha_{e_i})$.

In particular, if $e_1 \cdots e_n$ is a path in Γ without backtracking, then w is reduced.

We define the *reduction* of words by using the relations (R2). There are two types of reduction: coset and length.

Definition 4.3.7. Let $u = geg'$ be a subword of a graph of groupoids word w, where $g = \tau \alpha_e(h)$ as in (4.3.10). The *coset reduction* w' is the graph of groupoids word obtained from w by replacing u by $v = \tau e g''$, where $g'' = \alpha_{\bar{e}}(h) g'$.

Definition 4.3.8. Let $u = eg\bar{e}$, $g \in \mathcal{H}_{\bar{e}}$, be a subword of a graph of groupoids word w. The *length reduction* w' is the graph of groupoids word obtained from w by replacing u by $\phi_e(g)$. Note that when length reduction is applied to a graph of groupoids word, its length is decreased by two.

Two graph of groupoids words w and w' are said to be *equivalent* if one can get one from the other by using coset and length reductions.

Hence a reduced graph of groupoids word is a word in which no coset and no length reduction can be applied. The reduction process removes subwords $eg\bar{e}$, where $g \in \mathcal{H}_{\bar{e}}$, and moves $\alpha_e(h)$ in the decomposition $\tau \alpha_e(h)$ to the right. We prove that $p \in \pi_1(\mathcal{G}(\Gamma))$ is represented uniquely by a reduced graph of groupoids word $p = g_1 e_1 g_2 e_2 \cdots g_n e_n g_{n+1}$. We adapt the proofs of [19, §3] and [29, Theorem 2.1.7].

Theorem 4.3.9. *Every morphism of $\mathcal{F}(\mathcal{G}(\Gamma))$ is represented by a unique reduced graph of groupoids word.*

Proof. Clearly, each $p \in \pi_1(\mathcal{G}(\Gamma))$ is represented by a graph of groupoids word by definition. Let $w = g_1e_1g_2e_2 \cdots g_ne_ng_{n+1}$ be a graph of groupoids word.

If $n = 0$, then $w = g_1 \in \mathcal{G}_v$ for some $v \in \Gamma^0$ is already reduced. Hence, suppose that $n \geq 1$. For all $i = 1, \dots, n$, we choose a left transversal $\mathcal{T}_{e_i}$ of $\alpha_{e_i}(\mathcal{G}_{e_i})$ in $\mathcal{G}_{t(e_i)}$ as in Remark 4.3.5. Thus, for all $i = 1, \dots, n$, we can write $g \in \mathcal{G}_{t(e_i)}$ as $g = \tau\alpha_{e_i}(h)$, where $\tau = \tau(g, e_i) \in \mathcal{T}_{e_i}$ and $h = h(g, e_i) \in \mathcal{G}_{e_i}$. Then one has that $g_1e_1g_2 = \tau_1\alpha_{e_1}(h_1)e_1g_2 = \tau_1e_1\alpha_{\bar{e}_1}(h_1)g_2$, for $\tau_1 \in \mathcal{T}_{e_1}$ and $h_1 \in \mathcal{G}_{e_1}$. We put $g_2' = \alpha_{\bar{e}_1}(h_1)g_2$. We repeat this process with $g_2'e_2g_3$ to obtain $g_1e_1g_2e_2g_3 = \tau_1e_1\tau_2e_2\alpha_{\bar{e}_2}(h_2)g_3$, where $\tau_2 \in \mathcal{T}_{e_2}$ and $h_2 \in \mathcal{G}_{e_2}$. By repeating this argument, one obtains the graph of groupoids word

$$w' = \tau_1e_1\tau_2e_2 \cdots \tau_ne_ng'_{n+1}$$

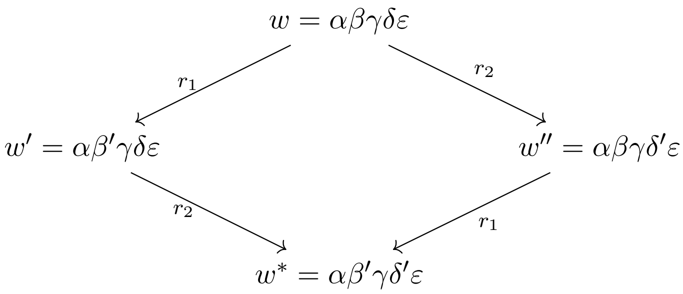
with $\tau_i \in \mathcal{T}_{e_i}$, $h_i \in \mathcal{G}_{e_i}$ for $i = 1, \dots, n$ and $g'_{n+1} \in \mathcal{G}_{o(e_n)}$. If w' is not reduced, there exists $i \in 1, \dots, n-1$ such that $e_{i+1} = \bar{e}_i$ and τ_{i+1} is a unit of $\mathcal{G}_{o(e_i)}$, i.e., $\tau_{i+1} = x_{i+1} \in \mathcal{G}^{(0)}_{o(e_i)}$. Then, we apply length reduction to $e_i\tau_ie_{i+1}$ and obtain $e_i\tau_ie_{i+1} = e_ix_{i+1}\bar{e}_i = x_i \in \mathcal{G}_{t(e_{i+2})} = \mathcal{G}_{o(e_{i-1})}$. Thus, we have a graph of groupoids word

$$w'' = \tau_1e_1 \cdots \tau_{i-1}e_{i-1}\tau_i\tau_{i+2}e_{i+2} \cdots e_ng'_{n+1}$$

of length $n-2$. By induction on n, one has that $w'' \in \pi_1(\mathcal{G}(\Gamma))$ is a reduced graph of groupoids word.

To prove uniqueness of the reduced graph of groupoids word, we use the diamond lemma method. We look at two different sequences of reductions for a graph of groupoids word w and we assume that the first steps in the two sequences are different: in the first one, the two (coset and length) reductions occur to disjoint subwords of w; in the second one, the two reductions occur to the same subwword of w.

Case I. We suppose that reductions occur to disjoint subwords of w. Let $w = \alpha\beta\gamma\delta\varepsilon$, where $\alpha, \beta, \gamma, \delta, \varepsilon$ are subwords such that both β and δ can be reduced. We denote by r_1 the reduction taking β to β' and r_2 the reduction taking δ to δ'. Then by applying r_1 and r_2 to w we will obtain to different graph of groupoids words w' and w'' that can be reduced to a common graph of groupoids word by a sequence of reductions. That is, we have the following commutative diamond:



Case II. We suppose that two reductions in the first steps of the sequence occur to the same subword of w. We have four cases.

Case (a). Let $w = \alpha g_1e_1x\bar{e}_1\tau he_2g_2\varepsilon$. If we apply length reduction first, we get

$$w' = \alpha\, g_1\, \tau\, h\, e_2\, g_2\, \varepsilon = \alpha\, g_1\, \tau'\, h'\, h\, e_2\, g_2\, \varepsilon.$$

We now apply coset reduction to w' to obtain

$$w^* = \alpha\, g_1\, \tau'\, e_2\, \phi_{\bar{e}_2}(h'h)\, g_2\, \varepsilon.$$

Otherwise, by applying coset reduction to w first, we get

$$w'' = \alpha\, g_1\, e_1\, x\, \bar{e}_1\, \tau\, e_2\, \phi_{\bar{e}_2}(h)\, g_2\, \varepsilon.$$

We now apply length reduction to w'' and get

$$\alpha\, g_1\, \tau\, e_2\, \phi_{\bar{e}_2}(h)\, g_2\, \varepsilon = \alpha\, \tau'\, h'\, e_2\, \phi_{\bar{e}_2}(h)\, g_2\, \varepsilon.$$

Again, we apply coset reduction and get

$$\alpha\, \tau'\, e_2\, \phi_{\bar{e}_2}(h')\, \phi_{\bar{e}_2}(h)\, g_2\, \varepsilon.$$

Finally, since $\phi_{\bar{e}_2}$ is an isomorphism, we obtain the graph of groupoids word

$$w^* = \alpha\, g_1\, \tau'\, e_2\, \phi_{\bar{e}_2}(h'h)\, g_2\, \varepsilon.$$

Case (b). Let $w = \alpha\, \tau\, h\, e\, x\, \bar{e}\, g\, \varepsilon$. If we apply length reduction first, we get

$$w' = \alpha\, \tau\, h\, g\, \varepsilon.$$

Otherwise, we apply coset reduction to w first and get

$$w'' = \alpha\, \tau\, e\, \phi_{\bar{e}}(h)\, \bar{e}\, g\, \varepsilon$$

Again, by applying coset reduction to w'' we get

$$\alpha\, \tau\, e\, r(\phi_{\bar{e}}(h))\bar{e}\, \phi_e(\phi_{\bar{e}}(h))\, g\, \varepsilon = \alpha\, \tau\, e\, r(\phi_{\bar{e}}(h))\, \bar{e}\, h\, g\, \varepsilon.$$

Finally, by applying length reduction we get

$$w^* = w' = \alpha\, \tau\, h\, g\, \varepsilon.$$

Case (c). Let $w = \alpha\, \tau_1\, h_1\, e_1 \tau_2\, h_2\, e_2\, g\, \varepsilon$. Then we can apply two different coset reductions to w. Let

$$w' = \alpha\, \tau_1\, e_1 \phi_{\bar{e}_1}(h_1) \tau_2\, h_2\, e_2\, g\, \varepsilon = \alpha\, \tau_1\, e_1 \tau_2'\, h_2'\, e_2\, g\, \varepsilon.$$

Then, we apply the second coset reduction to w' and obtain

$$w^* = \alpha\, \tau_1\, e_1 \tau_2'\, e_2 \phi_{\bar{e}_2}(h_2')\, g\, \varepsilon.$$

On the other hand, let

$$w'' = \alpha\, \tau_1\, h_1\, e_1 \tau_2\, e_2 \phi_{\bar{e}_2}(h_2)\, g\, \varepsilon.$$

Then, by applying the first coset reduction to w'' we get

$$\alpha\, \tau_1\, e_1 \phi_{\bar{e}_1}(h_1)\, \tau_2\, e_2 \phi_{\bar{e}_2}(h_2)\, g\, \varepsilon = \alpha\, \tau_1\, e_1 \tau_2'' h_2''\, e_2 \phi_{\bar{e}_2}(h_2)\, g\, \varepsilon.$$

and, by applying coset reduction, we have reduced w'' to

$$\hat{w} = \alpha\, \tau_1\, e_1 \tau_2''\, e_2\, \phi_{\bar{e}_2}(h_2'')\, \phi_{\bar{e}_2}(h_2)\, g\, \varepsilon = \alpha\, \tau_1\, e_1 \tau_2''\, e_2\, \phi_{\bar{e}_2}(h_2''\, h_2)\, g\, \varepsilon.$$

Note that $\phi_{\bar{e}_1}(h_1)\, \tau_2\, h_2 = \tau_2'\, h_2'$ and $\phi_{\bar{e}_1}(h_1)\, \tau_2 = \tau_2''\, h_2''$ by construction. Hence, one has that $\tau_2'\, h_2' = \phi_{\bar{e}_1}(h_1)\, \tau_2\, h_2 = \tau_2''\, h_2'' h_2$, which implies that $\tau_2' = \tau_2''$ and $h_2' = h_2'' h_2$. This proves that $w^* = \hat{w}$.

Case (d). Let $w = \alpha\, g\, e\, x\, \bar{e}\, z\, e\, g'\, \varepsilon$. Then we apply two different length reductions to w to obtain

$$w' = \alpha\, g\, y\, z\, e\, g'\, \varepsilon = \alpha\, g\, e\, g'\, \varepsilon,$$
$$w'' = \alpha\, g\, e\, x\, u\, g'\, \varepsilon = \alpha\, g\, e\, g'\, \varepsilon,$$

where $y = z = s(g) \in \mathcal{G}^{(0)}_{t(e)}$ and $x = u = r(g') \in \mathcal{G}^{(0)}_{o(e)}$. Thus, we have reduced both w' and w'' to the same graph of groupoids word $w^* = \alpha\, g\, e\, g'\, \varepsilon$.

Now, we consider two equivalent graph of groupoids words w and w' such that there exists a family $\{w_i\}_{i=1}^{n}$ of graph of groupoids words satisfying

(1) $w_0 = w$ and $w_n = w'$;

(2) for each $i = 1, \ldots, n$, either w_i is given by a simple reduction of w_{i+1} or w_{i+1} is given by a simple reduction of w_i.

Suppose that there exists k such that w_{i+1} is given by a simple reduction of w_i for $i < k$ and w_i is given by a simple reduction of w_{i+1} for $i \geq k$. Then one has both w and w' reduce to w_k. If no such k exists, then there exists l such that both w_{l-1} and w_{l+1} are given by a simple reduction of w_l. Then, by deleting w_l and w_{l+1} we obtain a family $\{w_i\}_{1\leq i\leq n,\, i\neq l,\, i\neq l+1}$ of graph of groupoids words satifying (1) and (2). Since by this process the number of elements of the family $\{w_i\}$ decreases, by repeating the process one finds a word w^* given by the reduction of both w and w'. □

One has the following analogue of Britton's Lemma.

Definition 4.3.10. A graph of groupoids word $g = g_1 e_1 g_2 e_2 \cdots g_n e_n g_{n+1}$ is said to be reduced *in the sense of Serre* if it satisfies the following: if $n = 0$, then g_1 is not a unit; if $n > 0$ and $e_{i+1} = \bar{e}_i$, then $g_{i+1} \notin \mathcal{H}_{\bar{e}_i}$.

Lemma 4.3.11 (Britton's Lemma)**.** *Let $\mathcal{G}(\Gamma)$ be a graph of groupoids based on Γ and let $p = g_1 e_1 g_2 \cdots g_n e_n g_{n+1}$ be a graph of groupoids word which is reduced in the sense of Serre. Then p is not a unit.*

Proof. The case $n = 0$ is trivial. Thus, let $n > 0$. For $i = 1, \ldots, n$ let $h_i \in \mathcal{H}_{e_i}$ and $\tau_i \in \mathcal{T}_{e_i}$ be such that

$$g_i = \tau_i h_i,$$
$$\phi_{\bar{e}_i}(h_i)\, g_{i+1} = \tau_{i+1} h_{i+1}.$$

Then by pulling the h_i's to the right one has that $p = \tau_1 e_1 \tau_2 \cdots \tau_n e_n a$, where $\tau_i \in \mathcal{T}_{e_i}$ and $a \in \mathcal{G}_{o(e_n)}$. If there exists i such that $e_{i+1} = \bar{e}_i$, then one has that $\phi_{\bar{e}_i}(h_i)$ and h_{i+1} are both in $\mathcal{H}_{\bar{e}_i}$ and hence it must be $\tau_{i+1} \neq x$, with $x = r(h_{i+1})$. Thus, $\tau_1 e_1 \tau_2 \cdots \tau_n e_n a$ is not an unit. □

4.3.2 The universal cover of a graph of groupoids

Let $\mathcal{G}(\Gamma)$ be a graph of groupoids and let $T \subseteq \Gamma$ be a maximal subtree of Γ which induces an equivalence relation as in (4.1.2). For $v \in \Gamma^0$ and $e \in \Gamma^1$, the groupoids $\mathcal{G}_v$ and $\mathcal{G}_e$ induce subgroupoids of $\pi_1(\mathcal{G}(\Gamma))$. This allows us to define a *universal cover* for $\mathcal{G}(\Gamma)$.

Definition 4.3.12. Let $\mathcal{G}(\Gamma)$ be a graph of groupoids and let $\mathcal{H}_e = \alpha_e(\mathcal{G}_e) \subseteq \mathcal{G}_{t(e)}$, $e \in \Gamma^1$. We fix an orientation Γ^1_+ of Γ and put

$$\varepsilon(e) = \begin{cases} 0 & \text{if } e \in \Gamma^1_+ \\ 1 & \text{if } e \in \Gamma^1 \setminus \Gamma^1_+. \end{cases} \tag{4.3.12}$$

In particular, one has that

$$\varepsilon(\bar{e}) = 1 - \varepsilon(e)$$

for all $e \in \Gamma^1$. We denote by $|e|$ the edge satisfying $\{|e|\} = \{e, \bar{e}\} \cap \Gamma^1_+$.

Note that by definition, one has that $\mathcal{H}_{|e|} = \mathcal{H}_{|\bar{e}|}$ for all $e \in \Gamma^1$.

Notation 4.3.13. For $e \in \Gamma^1$, we put

(i) $e\,\mathcal{G}_{o(e)} = \{\, \phi_e(r(g))eg \mid g \in \mathcal{G}_{o(e)} \,\}$,

(ii) $e\,\mathcal{G}_{o(e)}\,e^{-1} = \{\, \phi_e(r(g))ege^{-1}\phi_e(s(g)) \mid g \in \mathcal{G}_{o(e)} \,\}$.

Remark 4.3.14. Let $e \in \Gamma^1$. With the notations above, one has the following

(a) $$\mathcal{H}_e = \mathcal{G}_{t(e)} \cap e\,\mathcal{G}_{o(e)}\,e^{-1};$$

(b) $$\mathcal{H}_{|e|} = e^{1-\varepsilon(e)}\,\mathcal{H}_{\bar{e}}\,e^{\varepsilon(e)-1};$$

(c) $$\begin{aligned}\mathcal{H}_{|e|} &= e^{1-\varepsilon(e)}\,\mathcal{H}_{\bar{e}}\,e^{\varepsilon(e)-1} \\ &= e^{1-\varepsilon(e)}\,(\mathcal{G}_{o(e)} \cap e^{-1}\,\mathcal{G}_{t(e)}\,e)\,e^{\varepsilon(e)-1} \\ &= e^{1-\varepsilon(e)}\,\mathcal{G}_{o(e)}\,e^{\varepsilon(e)-1} \,\cap\, e^{-\varepsilon(e)}\,\mathcal{G}_{t(e)}\,e^{\varepsilon(e)}.\end{aligned}$$

We remind that for $v \in \Gamma^0$ and $p \in \pi_1(\mathcal{G}(\Gamma))$, the set $p\mathcal{G}_v$ is the set $p\mathcal{G}_v = \{\, pg \mid g \in \mathcal{G}_v, s(p) = r(g) \,\}$. In what follows we simplify the groupoid notation and write $\pi_1(\mathcal{G}(\Gamma))/\mathcal{G}_v$ to indicate the set $\big\{\, p\mathcal{G}_v \mid p \in \pi_1(\mathcal{G}(\Gamma)),\ s(p) \in \tau(\mathcal{G}_v^{(0)}) \,\big\}$, for $v \in \Gamma^0$.

Definition 4.3.15. The *Bass-Serre forest* $X_{\mathcal{G}(\Gamma)}$ of the graph of groupoids $\mathcal{G}(\Gamma)$, also known as its *universal cover*, is the graph defined by

$$X^0_{\mathcal{G}(\Gamma)} = \bigsqcup_{v \in \Gamma^0} \pi_1(\mathcal{G}(\Gamma))/\mathcal{G}_v[v], \tag{4.3.13}$$

$$X^1_{\mathcal{G}(\Gamma)} = \bigsqcup_{e \in \Gamma^1} \pi_1(\mathcal{G}(\Gamma))/\mathcal{H}_{|e|}[e]. \tag{4.3.14}$$

The maps $o, t\colon X^1_{\mathcal{G}(\Gamma)} \to \Lambda^0$ and $^-\colon X^1_{\mathcal{G}(\Gamma)} \to X^1_{\mathcal{G}(\Gamma)}$ are given by

$$o(p\mathcal{H}_{|e|}) = pe^{1-\varepsilon(e)}\mathcal{G}_{o(e)}[o(e)] \in \pi_1(\mathcal{G}(\Gamma))/\mathcal{G}_{o(e)}, \tag{4.3.15}$$

$$t(p\mathcal{H}_{|e|}) = pe^{-\varepsilon(e)}\mathcal{G}_{t(e)}[t(e)] \in \pi_1(\mathcal{G}(\Gamma))/\mathcal{G}_{t(e)}, \tag{4.3.16}$$

and

$$\overline{p\mathcal{H}_{|e|}[e]} = p\mathcal{H}_{|\bar{e}|}[\bar{e}]. \tag{4.3.17}$$

Note that t and o are well defined by Remark 4.3.14. In fact, suppose that for $p, q \in \pi_1(\mathcal{G}(\Gamma))$ and $e \in \Gamma^1$ one has that $p \in q\mathcal{H}_{|e|}$. Then there exists $h \in \mathcal{H}_{|e|}$ such that

$p = qh$. By Remark 4.3.14 (c), there exists $g \in \mathcal{G}_{o(e)}$ such that $h = e^{1-\varepsilon(e)}\, g\, e^{\varepsilon(e)-1}$. Then one has

$$\begin{aligned} o(p\mathcal{H}_{|e|}) &= p\, e^{1-\varepsilon(e)}\, \mathcal{G}_{o(e)}[o(e)] \\ &= q\, h\, e^{1-\varepsilon(e)}\, \mathcal{G}_{o(e)}[o(e)] \\ &= q\, e^{1-\varepsilon(e)}\, g\, e^{\varepsilon(e)-1}\, e^{1-\varepsilon(e)}\, \mathcal{G}_{o(e)}[o(e)] \\ &= q\, e^{1-\varepsilon(e)}\, g\, \mathcal{G}_{o(e)}[o(e)] \\ &= q\, e^{1-\varepsilon(e)}\, r(g)\, \mathcal{G}_{o(e)}[o(e)] \\ &= o(q\mathcal{H}_{|e|}). \end{aligned} \tag{4.3.18}$$

Moreover, it is easy to see that

$$\overline{p\mathcal{H}_{|e|}[e]} \neq p\mathcal{H}_{|e|}[e] \tag{4.3.19}$$

$$\overline{\overline{p\mathcal{H}_{|e|}[e]}} = p\mathcal{H}_{|e|}[e], \tag{4.3.20}$$

and, by (4.3.12)

$$\begin{aligned} o(\overline{p\mathcal{H}_{|e|}[e]}) &= o(p\mathcal{H}_{|\bar{e}|}[\bar{e}]) \\ &= p\bar{e}^{1-\varepsilon(\bar{e})}\mathcal{G}_{o(\bar{e})}[o(\bar{e})], \\ &= pe^{-\varepsilon(e)}\mathcal{G}_{t(e)}[t(e)], \\ &= t(p\mathcal{H}_{|e|}[e]). \end{aligned} \tag{4.3.21}$$

Hence, $X_{\mathcal{G}(\Gamma)}$ is a graph.

Remark 4.3.16. The fundamental groupoid $\pi_1(\mathcal{G}(\Gamma))$ acts on $X_{\mathcal{G}(\Gamma)}$ by left multiplication with momentum map

$$\begin{aligned} \tilde{\varphi}\colon X^0_{\mathcal{G}} &\to \pi_1(\mathcal{G}(\Gamma))^{(0)} \\ p\mathcal{G}_v &\mapsto r(p) \end{aligned} \tag{4.3.22}$$

(see Definition 2.4.16). That is, for $p, p' \in \pi_1(\mathcal{G}(\Gamma))$, $q\mathcal{G}_v[v] \in X^0_{\mathcal{G}(\Gamma)}$, $q'\mathcal{H}_e[e] \in X^1_{\mathcal{G}(\Gamma)}$ with $s(p) = \tilde{\varphi}(q\mathcal{G}_v) = r(q)$ and $s(p') = \tilde{\varphi} \circ o(q'\mathcal{H}_e) = r(q')$ one has that

$$\mu^0(p, q\mathcal{G}_v[v]) = pq\mathcal{G}_v[v], \tag{4.3.23}$$

$$\mu^1(p', q'\mathcal{H}_{|e|}[e]) = p'q'\mathcal{H}_{|e|}[e], \tag{4.3.24}$$

where $pq \in \pi_1(\mathcal{G}(\Gamma))$ is the reduced graph of groupoids word obtained from the concatenation of p and q.

Note further that for $x \in \pi_1(\mathcal{G}(\Gamma))^{(0)}$ and for all $e \in T^1$ one has that

$$o(x\mathcal{H}_{|e|}[e]) = x\mathcal{G}_{o(e)}[o(e)], \quad t(x\mathcal{H}_{|e|}[e]) = x\mathcal{G}_{t(e)}[t(e)], \tag{4.3.25}$$

and for all $e \in \Gamma^1_+ \setminus T^1$ one has that

$$o(x\mathcal{H}_{|e|}[e]) = xe\mathcal{G}_{o(e)}[o(e)], \quad t(x\mathcal{H}_{|e|}[e]) = x\mathcal{G}_{t(e)}[t(e)]. \tag{4.3.26}$$

Remark 4.3.17. The Bass-Serre forest $X_{\mathcal{G}(\Gamma)}$ is fibered on $\pi_1(\mathcal{G}(\Gamma))^{(0)}$ via the map $\tilde{\varphi}$ defined in (4.3.22). For each unit $x \in \pi_1(\mathcal{G}(\Gamma))^{(0)}$, we denote by $xX_{\mathcal{G}(\Gamma)}$ the subgraph of $X_{\mathcal{G}(\Gamma)}$ which is fibered on $\{x\}$, i.e.,

$$
\begin{aligned}
xX^0_{\mathcal{G}(\Gamma)} &= \tilde{\varphi}^{-1}(x),\\
xX^1_{\mathcal{G}(\Gamma)} &= o^{-1}(\tilde{\varphi}^{-1}(x)).
\end{aligned}
\tag{4.3.27}
$$

Clearly $xX_{\mathcal{G}(\Gamma)}$ and $yX_{\mathcal{G}(\Gamma)}$ are disconnected for all $x \neq y$, $x, y \in \pi_1(\mathcal{G}(\Gamma))^{(0)}$. Since each vertex in $xX_{\mathcal{G}(\Gamma)}$ starts with x and each vertex in $yX_{\mathcal{G}(\Gamma)}$ starts with y, i.e., for $p\mathcal{G}_v \in X_x$ and $q\mathcal{G}_w \in X_y$ one has $r(p) = x$ and $r(q) = y$, there is no edge connecting $xX_{\mathcal{G}(\Gamma)}$ and $yX_{\mathcal{G}(\Gamma)}$. In fact, the map $\tilde{\varphi}$ is such that $\tilde{\varphi} \circ o = \tilde{\varphi} \circ t$ by definition. Moreover, $xX_{\mathcal{G}(\Gamma)}$ is a tree (cf. Proposition 4.3.18). It follows that $X_{\mathcal{G}(\Gamma)}$ is a forest, so that the terminology used in Definition 4.3.15 is justified.

Proposition 4.3.18. *$xX_{\mathcal{G}(\Gamma)}$ is a tree for all $x \in \pi_1(\mathcal{G}(\Gamma))^{(0)}$.*

Proof. Fix $x \in \pi_1(\mathcal{G}(\Gamma))^{(0)}$. First we need to prove that $xX_{\mathcal{G}(\Gamma)}$ is a connected graph. Let Ξ be the smallest subgraph of $xX_{\mathcal{G}(\Gamma)}$ containing $\{\, x\mathcal{H}_{|e|}[e] \mid e \in \Gamma^1 \,\}$. Then Ξ is connected by (4.3.25). and (4.3.26). Moreover, $xX_{\mathcal{G}(\Gamma)} = x\pi_1(\mathcal{G}(\Gamma))x * \Xi$, where $x\pi_1(\mathcal{G}(\Gamma))x$ is the isotropy group of $\pi_1(\mathcal{G}(\Gamma))$ at x, i.e.,

$$x\pi_1(\mathcal{G}(\Gamma))x = \{p \in \pi_1(\mathcal{G}(\Gamma)) \mid s(p) = x = r(p)\}.$$

By Proposition 2.4.24, it suffices to show that there exists a generating system $\mathcal{S} \subseteq \pi_1(\mathcal{G}(\Gamma))$ such that $\Xi \cup a * \Xi$ is connected for all $a \in x\mathcal{S}x$. Then one has that the graph

$$\Xi \,\cup\, a_1 * \Xi \,\cup\, a_1a_2\Xi \,\cup\, \cdots \cup a_1a_2 \cdots a_n * \Xi \tag{4.3.28}$$

is connected for all $a_1, \ldots, a_n \in x\mathcal{S}x$ by induction on n. It suffices to prove the claim for $a \in x\mathcal{S}x$. Choosing

$$\mathcal{S} = \left(\bigsqcup_{v \in \Gamma^0} \mathcal{G}_v \right) \sqcup \left(\bigsqcup_{e \in \Gamma^1_+,\, y \in \mathcal{G}^{(0)}_{o(e)}} \phi_e(y)ey \right). \tag{4.3.29}$$

one verifies the claim for $a \in x\mathcal{S}x$. Since Ξ is connected by construction and $a * \Xi$ is connected because the action of $\pi_1(\mathcal{G}(\Gamma))$ preserves the distances, it is sufficient to prove that there exists a vertex lying in both Ξ and $a * \Xi$, i.e., $\Xi^0 \cap a * \Xi^0 \neq \emptyset$. By definition, for any $a \in \mathcal{S}$ there exists a vertex $p\mathcal{G}_v$, $v \in \Gamma^0$, such that $a * p\mathcal{G}_v$ is defined and $a * p\mathcal{G}_v = p\mathcal{G}_v$. Fix $a \in x\mathcal{S}x$, $v \in \Gamma^0$ such that $a * x\mathcal{G}_v = x\mathcal{G}_v$. Then one has that $x\mathcal{G}_v \in \Xi^0 \cap a * \Xi^0 \neq \emptyset$, which proves that $\Xi \cup a * \Xi$ is connected. It remains to prove that $xX_{\mathcal{G}(\Gamma)}$ does not contain non-trivial reduced paths from v to v, $v \in xX^0_{\mathcal{G}(\Gamma)}$. Let

$$\mathfrak{p} = (p_1\mathcal{H}_{|e_1|}[e_1])\,(p_2\mathcal{H}_{|e_2|}[e_2]) \cdots (p_n\mathcal{H}_{|e_n|}[e_n]), \quad n \geq 1 \tag{4.3.30}$$

be such a path. Put $v_i = o(e_i)$. In particular, one has $v_n = o(e_n) = t(e_1)$. Then one has

$$
\begin{aligned}
o(p_n\mathcal{H}_{|e_n|}[e_n]) &= p_ne_n^{1-\varepsilon(e_n)}\mathcal{G}_{v_n}[v_n] = p_1e_1^{-\varepsilon(e_1)}\mathcal{G}_{v_n}[v_1] = t(p_1\mathcal{H}_{|e_1|}[e_1]),\\
o(p_1\mathcal{H}_{|e_1|}[e_1]) &= p_1e_1^{1-\varepsilon(e_1)}\mathcal{G}_{v_1}[v_1] = p_2e_2^{-\varepsilon(e_2)}\mathcal{G}_{v_1}[v_2] = t(p_2\mathcal{H}_{|e_2|}[e_2]),\\
&\vdots\\
o(p_{n-1}\mathcal{H}_{|e_{n-1}|}[e_{n-1}]) &= p_{n-1}e_{n-1}^{1-\varepsilon(e_{n-1})}\mathcal{G}_{v_{n-1}}[v_{n-1}] = p_ne_n^{-\varepsilon(e_n)}\mathcal{G}_{v_{n-1}}[v_n] = t(p_n\mathcal{H}_{|e_n|}[e_n]).
\end{aligned}
$$

In particular, putting $q_i = p_i e_i^{-\varepsilon(e_i)}$, there exist elements $a_i \in \mathcal{G}_{v_i}$ with $r(a_i) = \phi_{\bar{e}_1}(s(q_i))$ such that

$$\begin{aligned} q_n e_n a_n &= q_1 \\ q_1 e_1 a_1 &= q_2 \\ &\vdots \\ q_{n-1} e_{n-1} a_{n-1} &= q_n. \end{aligned}$$

In particular, one has

$$\begin{aligned} s(q_n) e_n a_n &= q_n^{-1} q_1 \\ s(q_1) e_1 a_1 &= q_1^{-1} q_2 \\ &\vdots \\ s(q_{n-1}) e_{n-1} a_{n-1} &= q_{n-1}^{-1} q_n. \end{aligned}$$

Then one has

$$s(q_1) e_1 a_1 e_2 a_2 \cdots e_n a_n = s(q_1). \tag{4.3.31}$$

Thus, by Lemma 4.3.11, $\mathfrak{q} := s(q_1) e_1 a_1 e_2 a_2 \cdots e_n a_n$ is not reduced. Hence there exists $i \in \{1, \dots, n-1\}$ such that $e_{i+1} = \bar{e}_i$ and $a_i \in \mathrm{im}(\alpha_{\bar{e}_i}) = \mathcal{H}_{\bar{e}_i}$, i.e., there exists $b \in \mathcal{G}_{e_i}$ such that $a_i = \alpha_{\bar{e}_i}(b)$ and

$$e_i a_i e_{i+1} = e_i \alpha_{\bar{e}_i}(b) \bar{e}_i = \alpha_e(b) \in \mathcal{G}_{t(e_i)} = \mathcal{G}_{v_{i-1}} \tag{4.3.32}$$

Here we put $v_0 = v_n$. Then one has

$$\begin{aligned} p_{i+1} \mathcal{H}_{|e_{i+1}|}[e_{i+1}] &= q_{i+1} e_{i+1}^{\varepsilon(e_{i+1})} \mathcal{H}_{|e_i|}[\bar{e}_i] \\ &= q_i e_i a_i \bar{e}_i^{\varepsilon(\bar{e}_i)} \mathcal{H}_{|e_i|}[\bar{e}_i] \\ &= p_i e_i^{1-\varepsilon(e_i)} a_i \bar{e}_i^{1-\varepsilon(e_i)} \mathcal{H}_{|e_i|}[\bar{e}_i] \\ &= p_i e_i^{1-\varepsilon(e_i)} a_i e_i^{\varepsilon(e_i)-1} \mathcal{H}_{|e_i|}[\bar{e}_i]. \end{aligned} \tag{4.3.33}$$

Note that

$$\begin{aligned} e_i^{1-\varepsilon(e_i)} a_i e_i^{\varepsilon(e_i)-1} &= \begin{cases} e_i a_i e_i^{-1} & \text{if } |e_i| = e_i, \\ a_i & \text{if } |e_i| = \bar{e}_i \end{cases} \\ &= \begin{cases} \alpha_{e_i}(b) & \text{if } |e_i| = e_i, \\ a_i & \text{if } |e_i| = \bar{e}_i. \end{cases} \end{aligned} \tag{4.3.34}$$

Since $\alpha_{e_i}(b) \in \mathcal{H}_{e_i}$ and $a_i = \alpha_{\bar{e}_i}(b) \in \mathcal{H}_{\bar{e}_i}$, one has that $e_i^{1-\varepsilon(e_i)} a_i e_i^{\varepsilon(e_i)-1} \in \mathcal{H}_{|e_i|}$ by Remark 4.3.14 (b). Hence one has

$$p_{i+1} \mathcal{H}_{|e_{i+1}|}[e_{i+1}] = p_i \mathcal{H}_{|e_i|}[\bar{e}_i] = \overline{p_i \mathcal{H}_{|e_i|}[e_i]}. \tag{4.3.35}$$

Thus, $\mathfrak{p}$ is not reduced, a contradiction, and this yields the claim. □

4.3.3 The structure theorem

In this subsection we collect all the results of the previous sections. Let $\mathcal{G}$ be a groupoid acting on a forest F without inversion of edges. Then we have seen in Section 4.2 that one has a desingularization $\mathfrak{D}(\mathcal{G}, F)$, i.e., a graph of groupoids $\mathcal{G}(\Gamma)$ based on a graph $\Gamma = T \sqcup \Upsilon_+ \sqcup \Upsilon_-$, where $T \subseteq \Gamma$ is a maximal subtree such that $\mathcal{G}(T)$ is a tree of representatives for the action of $\mathcal{G}$ on F and Υ_+ and Υ_- are as in (4.2.19), and a family of groupoid elements $g_e = \{g_{e,x}\}_{x \in \varphi(t(\mathrm{im}(\sigma_e)))}$ for each $e \in \Gamma^1$, such that

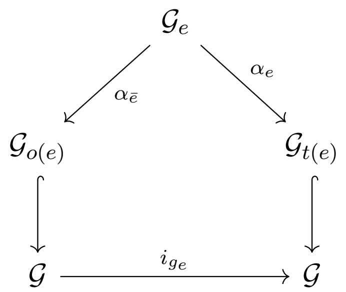
(i) $\mathcal{G}_v = \mathrm{Stab}_{\mathcal{G}}(\sigma_v)$ for $v \in \Gamma^0$;

(ii) $\mathcal{G}_e = \mathcal{G}_{\bar{e}} = \mathrm{Stab}_{\mathcal{G}}(\sigma_e)$ for $e \in \Gamma^1$;

(iii) $\alpha_e \colon \mathcal{G}_e \to \mathcal{G}_{t(e)}$ is given by inclusion for $e \in T^1 \sqcup \Upsilon_+$;

(iv) $\alpha_e \colon \mathcal{G}_e \to \mathcal{G}_{t(e)}$ is given by conjugation by g_e for $e \in \Upsilon_-$ (see (4.2.24)).

We put $g_e = \{\mathrm{id}_x\}_{x \in \varphi(\mathrm{im}(\sigma_e))}$ for $e \in T^1$ and $g_e = g_{\bar{e}}^{-1}$ for $e \in \Upsilon_+$. So by definition for all $e \in \Gamma^1_+$ one has a commutative diagram



Proposition 4.3.19. *For $\mathcal{G}$, F, $\mathfrak{D}(\mathcal{G}, F)$ and $\{g_e\}_{e \in \Gamma^1}$ as above, the assignment*

$$\psi_\circ \colon \bigsqcup_{\substack{e \in \Gamma^1_+ \\ x \in \mathcal{G}^{(0)}_{o(e)}}} (\phi_e(x), x) \sqcup \bigsqcup_{v \in \Gamma^0} \mathcal{G}_v \longrightarrow \mathcal{G} \tag{4.3.36}$$

given by

$$\begin{aligned} \psi_\circ(g) &= g, & g &\in \mathcal{G}_v, \\ \psi_\circ((\phi_e(x), x)) &= g_{e,x}, & e &\in \Gamma^1,\ x \in \mathcal{G}^{(0)}_{o(e)}, \end{aligned} \tag{4.3.37}$$

defines a groupoid homomorphism

$$\psi \colon \pi_1(\mathcal{G}(\Gamma)) \longrightarrow \mathcal{G}. \tag{4.3.38}$$

Proof. By definition, $\psi_\circ$ satisfies the relation

$$\psi_\circ\big((\phi_e(x), x)\big)\, \psi_\circ\big((x, \phi_e(x))\big) = \mathrm{id}_{\phi_e(x)} \tag{4.3.39}$$

for all $e \in \Gamma^1$, $x \in \mathcal{G}^{(0)}_{o(e)}$. Moreover, by definition one has

$$\begin{aligned} \psi_\circ\big((\alpha_e(x), \alpha_{\bar{e}}(x)\big)\, \psi_\circ(\alpha_{\bar{e}}(g))\, \psi_\circ\big((\alpha_{\bar{e}}(y), \alpha_e(y))\big) &= g_{e,\alpha_{\bar{e}}(x)}\, \alpha_{\bar{e}}(g)\, g^{-1}_{e,\alpha_{\bar{e}}(y)} \\ &= \alpha_e(g) \\ &= \psi_\circ(\alpha_e(g)) \end{aligned} \tag{4.3.40}$$

for all $e \in \Gamma^1$ and $g \in \mathcal{G}_e$ with $r(g) = x$ and $s(g) = y$. $\square$

Corollary 4.3.20. *One has that ψ is a strong groupoid homomorphism.*

Proof. Suppose that for $p, q \in \pi_1(\mathcal{G}(\Gamma))$ one has that $(\psi(p), \psi(q)) \in \mathcal{G}^{(2)}$, i.e., $s(\psi(p)) = r(\psi(q))$. Since ψ is a homomorphism, one has that $\psi(s(p)) = \psi(r(q))$. By (4.3.37), one has that $s(p) = r(q)$, i.e., $(p, q) \in \pi_1(\mathcal{G}(\Gamma))^{(2)}$. □

Since $\mathcal{G}(\Gamma)$ in the desingularization of the action of $\mathcal{G}$ on F is a graph of groupoids, one has that the fundamental groupoid $\pi_1(\mathcal{G}(\Gamma))$ of $\mathcal{G}(\Gamma)$ acts on the Bass-Serre forest $X_{\mathcal{G}(\Gamma)}$ associated to $\mathcal{G}(\Gamma)$ as we have seen is Subsection 4.3.1.

Proposition 4.3.21. *Let $\mathcal{G}(\Gamma)$, $\pi_1(\mathcal{G}(\Gamma))$ and $X_{\mathcal{G}(\Gamma)}$ be as above. Let*

$$\Psi = (\Psi^0, \Psi^1) \colon X_{\mathcal{G}(\Gamma)} \longrightarrow F \tag{4.3.41}$$

be the mapping defined by

$$\begin{aligned} \Psi^0(p\mathcal{G}_v[v]) &= \mu^0\big(\,\psi(p),\, \sigma_v(\psi(s(p)))\,\big), \\ \Psi^1(p\mathcal{H}_{|e|}[e]) &= \mu^1\big(\,\psi(p),\, \sigma_e(\psi(s(p)))\,\big), \end{aligned} \tag{4.3.42}$$

where σ_v and σ_e are the partial section associated to $v \in \Gamma^0$ and $e \in \Gamma^1$, respectively. Then Ψ is a ψ-equivariant homomorphism of graphs, i.e.,

$$\begin{aligned} \Psi^0(\mu^0(p, w)) &= \mu^0\big(\psi(p), \Psi^0(w)\big), \\ \Psi^1(\mu^1(p, f)) &= \mu^1\big(\psi(p), \Psi^1(f)\big) \end{aligned} \tag{4.3.43}$$

for all $p \in \pi_1(\mathcal{G}(\Gamma))$, $w \in X^0_{\mathcal{G}(\Gamma)}$ and $f \in X^1_{\mathcal{G}(\Gamma)}$.

Proof. We first prove that Ψ is a homomorphism of graphs. Note that for $e \in \Upsilon_-$ and $p\mathcal{H}_{|e|}[e] \in X^1_{\mathcal{G}(\Gamma)}$, one has

$$t\big(\sigma_e(\psi(s(p)))\big) = \mu^0\big(\psi(g_{e,s(p)}), \sigma_{t(e)}(\psi(s(p)))\big), \tag{4.3.44}$$

$$o\big(\sigma_e(\psi(s(p)))\big) = \sigma_{o(e)}\big(\psi(s(p))\big). \tag{4.3.45}$$

For $e \in T^1 \sqcup \Upsilon_+$ and $p\mathcal{H}_{|e|}[e] \in X^1_{\mathcal{G}(\Gamma)}$, one has

$$\begin{aligned} &t\big(\sigma_e(\psi(s(p)))\big) = o\big(\sigma_{\bar{e}}(\psi(s(p)))\big) = \sigma_{o(\bar{e})}(\psi(s(p))) = \sigma_{t(e)}(\psi(s(p))), \\ &o\big(\sigma_e(\psi(s(p)))\big) = t\big(\sigma_{\bar{e}}(\psi(s(p)))\big) = \mu^0\big(\psi(g_{\bar{e},s(p)}), \sigma_{t(\bar{e})}(\psi(s(p)))\big) = \mu^0\big(\psi(g_{e,s(p)})^{-1}, \sigma_{o(e)}(\psi(s(p)))\big). \end{aligned}$$

Thus with the notation as in (4.3.12), one obtains

$$\overline{\sigma_e(x)} = \sigma_{\bar{e}}(x), \tag{4.3.46}$$

$$t(\sigma_e(x)) = \mu^0\big(\psi(g_{e,x})^{1-\varepsilon(e)}, \sigma_{t(e)}(x)\big), \tag{4.3.47}$$

$$o(\sigma_e(x)) = \mu^0\big(\psi(g_e, x)^{-\varepsilon(e)}, \sigma_{o(e)}(x)\big), \tag{4.3.48}$$

for all $e \in \Gamma^1$, $x \in \operatorname{dom}\sigma_e$. This shows that Ψ is a homomorphism of graphs which commutes with the action of $\pi_1(\mathcal{G}(\Gamma))$ on $X_{\mathcal{G}(\Gamma)}$. □

One has the following structure theorem.

Theorem 4.3.22. *With the notation as above, ψ is an isomorphism of groupoids and Ψ is an isomorphism of graphs.*

We need the following lemma for the proof of Theorem 4.3.22.

Lemma 4.3.23. *Let $\Psi = (\Psi^0, \Psi^1)\colon \Lambda \to \Delta$ be a homomorphism of graphs such that*

(i) Λ is connected;

(ii) Δ is a tree;

(iii) for all $u \in \Lambda^0$ the map $\Psi^1|_{\mathrm{st}_\Lambda(u)}\colon \mathrm{st}_\Lambda(u) \to \mathrm{st}_\Delta(\Psi^0(u))$ is injective.

Then Ψ is injective.

Proof. Since Ψ^1 is injective by hypothesis, it suffices to show that Ψ^0 is injective. Suppose that there exist $v, w \in \Lambda^0$, $v \neq w$, such that $\Psi^0(v) = \Psi^0(w)$. Since Λ is connected, there exists a path $p \in \mathcal{P}_{v,w}(\Lambda)$ from v to w. Since Δ is a tree, by (iii) one has that $\Psi_e(p)$ is a reduced path from $\Psi^0(v)$ to $\Psi^0(v)$, which is a contradiction. Thus, Ψ^0 is injective. □

Remark 4.3.24. We recall that by the construction of the desingularization $\mathfrak{D}(\mathcal{G}, F)$, one has that Γ is a fundamental domain for the action of $\mathcal{G}$ on F. That is, for any $w \in F^0$ there exists $v \in \Gamma^0$ such that w is in the saturation of $\operatorname{im}\sigma_v$, i.e., there exists $u \in \operatorname{im}\sigma_v \subseteq F^0$ such that w is in the orbit of u. Similarly, for any $\varepsilon \in F^1$ there exists $e \in \Gamma^1$ such that ε is in the saturation of $\operatorname{im}\sigma_e$, i.e., there exists $f \in \operatorname{im}\sigma_e \subseteq F^1$ such that ε is in the orbit of f.

We are now ready to prove the structure theorem.

Proof of Theorem 4.3.22. Let $\mathfrak{D}\big(\pi_1(\mathcal{G}(\Gamma)), X_{\mathcal{G}(\Gamma)}\big)$ be the desingularization of the action of $\pi_1(\mathcal{G}(\Gamma))$ on $X_{\mathcal{G}(\Gamma)}$, given by a graph of groupoids $\mathfrak{G}(\Delta)$ based on a graph Δ and a maximal subtree $\Lambda \subseteq \Delta$ such that $(\mathfrak{G}(\Lambda), \rho)$ is a tree of representatives of the action of $\pi_1(\mathcal{G}(\Gamma))$ on $X_{\mathcal{G}(\Gamma)}$. For $v \in \Gamma^0$, consider the partial section σ_v associated to v and the stabilizer

$$\mathrm{Stab}_{\mathcal{G}}(\sigma_v) = \big\{\, g \in \mathcal{G} \mid \mu^0(g, \sigma_v(s(g))) = \sigma_v(r(g)),\ s(g), r(g) \in \mathrm{dom}(\sigma_v) \,\big\}.$$

Fix a unit $x \in \pi_1(\mathcal{G}(\Gamma))^{(0)}$ and consider the vertex $x\mathcal{G}_v[v] \in X^0_{\mathcal{G}(\Gamma)}$. Then there exists $w \in \Delta^0$ such that $x\mathcal{G}_v[v]$ is in the saturation of $\operatorname{im}\sigma_w$. Consider the stabilizer

$$\mathrm{Stab}_{\pi_1(\mathcal{G}(\Gamma))}(\sigma_w) = \big\{\, p \in \pi_1(\mathcal{G}(\Gamma)) \mid \mu^0(p, \sigma_w(s(p))) = \sigma_w(r(p)),\ s(p), r(p) \in \mathrm{dom}(\sigma_w) \,\big\}.$$

Then the map

$$\psi|_{\mathrm{Stab}_{\pi_1(\mathcal{G}(\Gamma))}(\sigma_w)}\colon \mathrm{Stab}_{\pi_1(\mathcal{G}(\Gamma))}(\sigma_w) \to \mathrm{Stab}_{\mathcal{G}}(\sigma_v) \tag{4.3.49}$$

is an isomorphism by construction. Then one has that the following diagram of graphs commutes:

$$\begin{array}{ccc} X_{\mathcal{G}(\Gamma)} & \xrightarrow{\ \Psi\ } & F \\ \rho\uparrow & & \uparrow\chi \\ \mathcal{F}^{\mathrm{pi}}(\Lambda) & \xrightarrow{\ \widehat{\Psi}\ } & \mathcal{F}^{\mathrm{pi}}(T) \end{array}$$

where $\mathcal{F}^{\mathrm{pi}}(\Lambda)$ and $\mathcal{F}^{\mathrm{pi}}(T)$ are the forests of partial isomorphisms induced by $\mathfrak{G}(\Lambda)$ and $\mathcal{G}(T)$, respectively (cf. Remark 4.1.8). In particular, $\widehat{\Psi}$ is an isomorphism of graphs. Hence one has that

$$\mu^0\Big(\,\psi\big(\pi_1(\mathcal{G}(\Gamma))\big),\ \operatorname{im}\sigma_v\Big) = \mu^0\big(\,\mathcal{G},\ \operatorname{im}\sigma_v\,\big) \tag{4.3.50}$$

and, by (4.3.49), the groupoid homomorphism $\psi\colon \pi_1(\mathcal{G}(\Gamma)) \to \mathcal{G}$ is surjective. Thus, since

$$
\begin{aligned}
F^0 &= \bigsqcup_{v\in\Gamma^0} \mu^0(\mathcal{G}, \operatorname{im}\sigma_v), \\
F^1 &= \bigsqcup_{e\in\Gamma^1} \mu^1(\mathcal{G}, \operatorname{im}\sigma_e),
\end{aligned}
\tag{4.3.51}
$$

by Remark 4.3.24, one has that Ψ is surjective.

Note that $X_{\mathcal{G}(\Gamma)}$ is fibered on $\pi_1(\mathcal{G}(\Gamma))$ and each fiber corresponds to a connected component of $X_{\mathcal{G}(\Gamma)}$, i.e.

$$X_{\mathcal{G}(\Gamma)} = \bigsqcup_{y\in\pi_1(\mathcal{G}(\Gamma))^{(0)}} yX_{\mathcal{G}(\Gamma)},$$

where $yX_{\mathcal{G}(\Gamma)}$ denotes the fiber of $y \in \pi_1(\mathcal{G}(\Gamma))^{(0)}$ (cf. Remark 4.3.17). Similarly, the forest F is fibered on $\mathcal{G}^{(0)}$ and hence one has

$$F = \bigsqcup_{a\in\mathcal{G}^{(0)}} aF.$$

Since Ψ is a ψ-equivariant homomorphism of graphs, one has that $\Psi(yX_{\mathcal{G}(\Gamma)}) \subseteq F_{\psi(y)}$ for all $y \in \pi_1(\mathcal{G}(\Gamma))^{(0)}$. Then for $e \in \mathrm{st}_\Gamma(v)$ one has canonical bijections

$$
\begin{array}{ccc}
 & \bigsqcup_{e\in\mathrm{st}_\Gamma(v)} x\mathcal{G}_v/\mathrm{im}(\alpha_{\bar{e}}) & \\
\swarrow^{\bigsqcup_{e\in\mathrm{st}_\Gamma(v)} a_e} & & \searrow^{\bigsqcup_{e\in\mathrm{st}_\Gamma(v)} b_e} \\
\mathrm{st}_{X_{\mathcal{G}(\Gamma)}}(x\mathcal{G}_v[v]) & \xrightarrow{\Psi^1|_{\mathrm{st}_{X_{\mathcal{G}(\Gamma)}}(x\mathcal{G}_v[v])}} & \mathrm{st}_F(\sigma_v(x))
\end{array}
$$

given by

$$a_e(g) = g\mathcal{H}_{|e|}[e], \tag{4.3.52}$$

$$b_e(g) = \sigma_e(s(g)), \tag{4.3.53}$$

for $g \in \mathcal{G}_v$ with $r(g) = x$. Hence $\Psi^1|_{\mathrm{st}_{X_{\mathcal{G}(\Gamma)}}(x\mathcal{G}_v[v])}$ is bijective for every $x\mathcal{G}_v[v] \in X^0_{\mathcal{G}(\Gamma)}$. Thus, for any $y \in \pi_1(\mathcal{G}(\Gamma))^{(0)}$ the restriction

$$\Psi|_{yX_{\mathcal{G}(\Gamma)}}\colon yX_{\mathcal{G}(\Gamma)} \to F_{\psi(y)} \tag{4.3.54}$$

is an injective homomorphism of graphs by Lemma 4.3.23. Therefore, Ψ is injective. Thus, Ψ is an isomorphism of graphs.

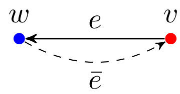
It remains to show that ψ is injective. Let $N := ker(\psi) = \{p \in \pi_1(\mathcal{G}(\Gamma)) \mid \psi(p) \in \mathcal{G}^{(0)}\}$. For $w \in \Lambda^0$, let $\pi_w = \mathrm{Stab}_{\pi_1(\mathcal{G}(\Gamma))}(\sigma_w)$. One has that $N \cap \pi_w = \pi_w^{(0)}$ by (4.3.49). On the other hand, for $n \in N$ and $v \in X^{(0)}_{\mathcal{G}(\Gamma)}$ such that $s(n) = \tilde{\varphi}(v)$ and v is in the saturation of $\operatorname{im}\sigma_w$, one has

$$\Psi^0\big(\mu^0(n, v)\big) = \mu^0\big(\psi(n), \Psi^0(v)\big) = \Psi^0(v) \tag{4.3.55}$$

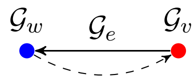
since Ψ is a ψ-equivariant homomorphism of graphs. Since Ψ is injective, one has that $\mu^0(n, v) = v$, i.e., $n = \tilde{\varphi}(v) \in N \cap \pi_w = \pi_w^{(0)}$. Thus, $N = \pi_1(\mathcal{G}(\Gamma))^{(0)}$ and since ψ is a strong groupoid homomorphism (see Corollary 4.3.20), it is injective by Proposition 2.3.31. This proves that ψ is an isomorphism of groupoids. $\square$

4.4 Example

Example 15. Let Γ be the graph



and let $\mathcal{G}(\Gamma)$ be the graph of groupoids



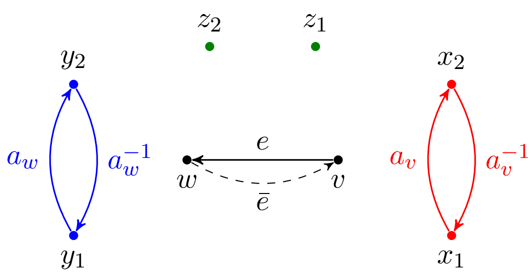
where

$$\mathcal{G}_v = \{x_1, x_2, a_v, a_v^{-1}\}, \quad s(a_v) = x_1,\, r(a_v) = x_2,$$
$$\mathcal{G}_w = \{y_1, y_2, a_w, a_w^{-1}\}, \quad s(a_w) = y_1,\, r(a_w) = y_2,$$

and

$$\mathcal{G}_e = \mathcal{G}_{\bar{e}} = \{z_1, z_2\}.$$

That is, $\mathcal{G}(\Gamma)$ is the graph of groupoids



The monomorphisms $\alpha_e\colon \mathcal{G}_e \to \mathcal{G}_w$ and $\alpha_{\bar{e}}\colon \mathcal{G}_e \to \mathcal{G}_v$ are given by

$$\alpha_e(z_i) = y_i,$$
$$\alpha_{\bar{e}}(z_i) = x_i,$$

for $i = 1, 2$. Then one has

$$\mathcal{H}_e = \operatorname{im}(\alpha_e) = \mathcal{G}_w^{(0)},$$
$$\mathcal{H}_{\bar{e}} = \operatorname{im}(\alpha_{\bar{e}}) = \mathcal{G}_v^{(0)},$$

and trasversals

$$\mathcal{T}_e = \mathcal{G}_w,$$
$$\mathcal{T}_{\bar{e}} = \mathcal{G}_v.$$

We identify x_i with y_i for $i = 1, 2$ via the relation (4.1.2). Then the fundamental groupoid $\pi_1(\mathcal{G}(\Gamma))$ has unit space $\pi_1(\mathcal{G}(\Gamma))^{(0)} = \{u_1, u_2\}$, generators

$$\{\, a_v,\, a_v^{-1},\, a_w,\, a_w^{-1},\, y_1 e x_1,\, y_2 e x_2,\, x_1 \bar{e} y_1,\, x_2 \bar{e} y_2 \,\}$$

and defining relations $a_v^{-1}a_v = u_1$, $a_v a_v^{-1} = u_2$, $a_w^{-1}a_w = u_1$, $a_w a_w^{-1} = u_2$, together with relations

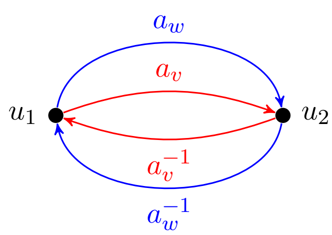
(R1) $\qquad x_1\bar{e}y_1 = \mathrm{id}_{u_1}, \quad x_2\bar{e}y_2 = \mathrm{id}_{u_2}, \quad y_1 e x_1 = \mathrm{id}_{u_1}, \quad y_2 e x_2 = \mathrm{id}_{u_2};$

(R2) $\qquad e x_1 \bar{e} = y_1, \quad e x_2 \bar{e} = y_2, \quad \bar{e} y_1 e = x_1, \quad \bar{e} y_2 e = x_2.$

Hence one has

$$\begin{aligned}\pi_1(\mathcal{G}(\Gamma)) = \{\, & u_1, u_2, \\ & a_v, a_v^{-1}, a_w, a_w^{-1}, \\ & a_v a_w^{-1},\ a_v^{-1}a_w,\ a_w a_v^{-1}, a_w^{-1}a_v \\ & a_v a_w^{-1}a_v,\ a_v^{-1}a_w a_v^{-1},\ a_w a_v^{-1}a_w, a_w^{-1}a_v a_w^{-1} \\ & \cdots \}.\end{aligned}$$

That is, the morphisms of $\pi_1(\mathcal{G}(\Gamma))$ are the finite words given by the alternation of composable red and blue arrows where no two successive arrows of the same color occur, as shown in the subsequent diagram:

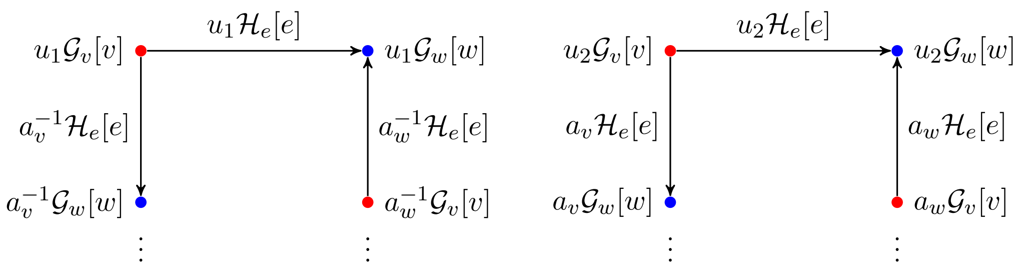


The Bass-Serre forest $X_{\mathcal{G}(\Gamma)}$ is defined as follows:

$$\begin{aligned} X^0_{\mathcal{G}(\Gamma)} &= \{\, p\mathcal{G}_v[v] \mid p \in \pi_1(\mathcal{G}(\Gamma)) \,\} \sqcup \{\, p\mathcal{G}_w[w] \mid p \in \pi_1(\mathcal{G}(\Gamma)) \,\}, \\ X^1_{\mathcal{G}(\Gamma)} &= \{\, p\mathcal{H}_{|e|}[e] \mid p \in \pi_1(\mathcal{G}(\Gamma)) \,\}, \end{aligned}$$

where $p\mathcal{G}_v = \{pg \mid g \in \mathcal{G}_v, s(p) = r(g)\}$ (see Definition 4.3.15). Then $X_{\mathcal{G}(\Gamma)}$ is a bipartite graph, since the vertices of $X_{\mathcal{G}(\Gamma)}$ are naturally partitioned into two disjoint sets and each edge has initial vertex in one of these sets and terminal vertex in the other set. The Bass-Serre forest $X_{\mathcal{G}(\Gamma)}$ is the union of the graphs $uX_{\mathcal{G}(\Gamma)}$, for $u \in \pi_1(\mathcal{G})^{(0)}$, i.e.,

$$X_{\mathcal{G}(\Gamma)} = u_1 X_{\mathcal{G}(\Gamma)} \sqcup u_2 X_{\mathcal{G}(\Gamma)} :$$



The fundamental groupoid $\pi_1(\mathcal{G}(\Gamma))$ acts on $X_{\mathcal{G}(\Gamma)}$ by left multiplication with momentum map $\varphi\colon X^0_{\mathcal{G}(\Gamma)} \to \pi_1(\mathcal{G})^{(0)}$ given by $\varphi(q\mathcal{G}_v) = r(q)$. Hence one has that $X_{\mathcal{G}(\Gamma)} = \varphi^{-1}(u_1) \sqcup \varphi^{-1}(u_2)$. There are two orbits of the action of $\pi_1(\mathcal{G}(\Gamma))$ on $X^0_{\mathcal{G}(\Gamma)}$:

one is given by the collection of the red vertices and the other is given by the collection of the blue vertices, i.e.,

$$X^0_{\mathcal{G}(\Gamma)} = \mu^0\big(\pi_1(\mathcal{G}(\Gamma)), u_1\mathcal{G}_v\big) \sqcup \mu^0\big(\pi_1(\mathcal{G}(\Gamma)), u_1\mathcal{G}_w\big).$$

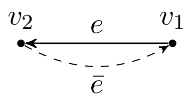
We enumerate the fibers of u_1 and u_2 (see the proof of Theorem 4.2.14), i.e., we enumerate the vertices of $u_1 X_{\mathcal{G}(\Gamma)}$ and $u_2 X_{\mathcal{G}}(\Gamma)$ as follows:

Such enumeration gives two partial sections of vertices $\sigma_1, \sigma_2 \colon \pi_1(\mathcal{G}(\Gamma))^{(0)} \to X^0_{\mathcal{G}(\Gamma)}$ and a partial section of edges $\sigma_{\mathbf{e}} \colon \pi_1(\mathcal{G}(\Gamma))^{(0)} \to X^1_{\mathcal{G}(\Gamma)}$ defined by

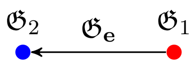
$$\begin{aligned} \sigma_1(u_i) &= u_i\mathcal{G}_v[v], \\ \sigma_2(u_i) &= u_i\mathcal{G}_w[w], \\ \sigma_{\mathbf{e}}(u_i) &= u_i\mathcal{H}_e[e], \end{aligned}$$

for $i = 1, 2$. Hence, we have a tree of representatives $\mathfrak{G}(T)$ based on the segment tree $T = \big(\{v_1, v_2\}, \{e, \bar{e}\}\big)$ as follows

T



$\mathfrak{G}(T)$



where $\mathfrak{G}_1 = \mathrm{Stab}_{\pi_1(\mathcal{G}(\Gamma))}(\sigma_1)$, $\mathfrak{G}_2 = \mathrm{Stab}_{\pi_1(\mathcal{G}(\Gamma))}(\sigma_2)$, and $\mathcal{G}_{\mathbf{e}} = \mathrm{Stab}_{\pi_1(\mathcal{G}(\Gamma))}(\sigma_{\mathbf{e}})$. In particular, $\mathfrak{G}(T)$ is a desingularization of the action of $\pi_1(\mathcal{G}(\Gamma))$ on $X_{\mathcal{G}(\Gamma)}$.

Chapter 5

A groupoid C^*-algebraic Bass-Serre theorem

In Chapter 4 we have constructed a fundamental groupoid $\pi_1(\mathcal{G}(\Gamma))$ associated to a graph of groupoids $\mathcal{G}(\Gamma)$ which generalizes the notion of the fundamental group of a graph of groups. That is, if we consider the vertex and edge groupoids of $\mathcal{G}(\Gamma)$ to be groups, then $\pi_1(\mathcal{G}(\Gamma))$ coincides with the fundamental group of the graph of groups $\pi_1(\mathcal{G}(\Gamma))$ as described in [35].

In [19] the author defines the fundamental groupoid $\mathcal{F}(G(\Gamma))$ of a graph of groups $G(\Gamma)$ as follows. The unit space of $\mathcal{F}(G(\Gamma))$ is the vertex set Γ^0, the generating set is the same as for the fundamental group $\pi_1(G(\Gamma))$ as defined in Definition 2.2.20, i.e., it is given by the edge set Γ^1 together with the elements of the vertex groups G_v, $v \in \Gamma^0$. For $e \in \Gamma^1$, the source and range map coincide with the origin and terminus map; for $g \in \mathcal{G}_v$, $v \in \Gamma^0$, the source and range map are given by $s(g) = r(g) = v$. The defining relations are given by the relations in the vertex groups together with the relation $e\alpha_{\bar{e}}(g)\bar{e} = \alpha_e(g)$ for each $g \in G_e$, $e \in \Gamma^1$. This relation implies that $e\bar{e} = 1_{r(e)}$, and hence e and $\bar{e}$ are inverse elements in the sense of groupoids. The groupoid approach is used in [19] to simplify calculations when establishing a normal form for elements of the fundamental group.

In this chapter to a graph of groupoids we associate a groupoid, and we call it *univeral fundamental groupoid*, which generalizes the fundamental groupoid of a graph of groups, and then we associate to it an action groupoid.

5.1 The universal fundamental groupoid of a graph of groupoids

Standing Assumption. From now on, $\mathcal{G}(\Gamma)$ will denote a locally-finite nonsingular graph of groupoids such that $\mathcal{G}_v$ is a discrete groupoid for every $v \in \Gamma^0$ and $\alpha_e(\mathcal{G}_e)$ is a wide subgroupoid of $\mathcal{G}_{t(e)}$ for each $e \in \Gamma^1$.

Definition 5.1.1. The *universal fundamental groupoid* of the graph of groupoids $\mathcal{G}(\Gamma)$ is the groupoid $\Pi_1(\mathcal{G}(\Gamma))$ defined as follows (see [17, §3] for presentation of groupoids). It is generated by the set

$$\left(\bigsqcup_{v\in\Gamma^0} \mathcal{G}_v \times \{v\} \right) \sqcup \left(\bigsqcup_{e\in\Gamma^1} \mathcal{G}^{(0)}_{t(e)} \times \{e\} \right) \tag{5.1.1}$$

with unit space

$$(5.1.2)\qquad \Pi_1(\mathcal{G}(\Gamma))^{(0)} = \bigsqcup_{v\in\Gamma^0} \mathcal{G}_v^{(0)} \times \{v\}$$

and range and source maps defined by

$$(5.1.3)\qquad r\big((x,e)\big) = (x, t(e)) \quad\text{and}\quad r\big((g,v)\big) = (r(g), v),$$

$$(5.1.4)\qquad s\big((x,e)\big) = (\phi_{\bar{e}}(x), o(e)) \quad\text{and}\quad s\big((g,v)\big) = (s(g), v),$$

for all $x \in \mathcal{G}_{t(e)}^{(0)}$, $e \in \Gamma^1$ and for all $g \in \mathcal{G}_v$, $v \in \Gamma^0$. Multiplication in $\Pi_1(\mathcal{G}(\Gamma))$ is defined by concatenation. Moreover, groupoid elements are subject to relations

$$\text{(R)}\qquad (g_1, v)\cdot(g_2, v) = (g_1 g_2, v)$$

$$\text{(R2')}\qquad \big(r(\alpha_{\bar{e}}(g)), \bar{e}\big)\cdot\big(\alpha_e(g), t(e)\big)\cdot\big(s(\alpha_e(g)), e\big) = \big(\alpha_{\bar{e}}(g), o(e)\big)$$

for all $g_1, g_2 \in \mathcal{G}_v$, $v \in \Gamma^0$ and $g \in \mathcal{G}_e$, $e \in \Gamma^1$. Note that

$$(5.1.5)\qquad (\phi_{\bar{e}}(x), \bar{e})\cdot(x, e) = (\phi_{\bar{e}}(x), \bar{e})\cdot(x, t(e))\cdot(x, e) = (\phi_{\bar{e}}(x), o(e))$$

by (R2'), that is, $(x,e)^{-1} = (\phi_{\bar{e}}(x), \bar{e})$, and that $(g,v)^{-1} = (g^{-1}, v)$ by (R).

Remark 5.1.2. Unlike the unit space of the fundamental groupoid $\pi_1(\mathcal{G}(\Gamma))$, where we identify some units of different vertex groupoids via the relation (4.1.2), we consider here all the units in the disjoint union of the vertex groupoids.

As we have done in Chapter 4, we now define graph of groupoids Π_1-words, a forest $Y_{\mathcal{G}(\Gamma)}$ on which $\Pi_1(\mathcal{G}(\Gamma))$ acts without inversion of edges, and then we define the action groupoid $\Pi_1(\mathcal{G}(\Gamma)) \ltimes Y_{\mathcal{G}(\Gamma)}$.

Definition 5.1.3. We call a *graph of groupoids* Π_1*-word* a sequence

$$w = (g_1, t(e_1))\,(s(g_1), e_1)\,(g_2, t(e_2))\,(s(g_2), e_2)\cdots(g_n, t(e_n))\,(s(g_n), e_n)\,(g_{n+1}, o(e_n)),$$

where

- $e_i \in \Gamma^1$ for all $i = 1, \ldots, n$;
- $o(e_i) = t(e_{i+1})$ for all $i = 1, \ldots, n-1$;
- $g_i \in \mathcal{G}_{t(e_i)}$ for all $i = 1, \ldots, n$ and $g_{n+1} \in \mathcal{G}_{o(e_n)}$;
- for all $i = 1, \ldots, n$ one has $\phi_{\bar{e}_i}\big(s(g_i)\big) = r(g_{i+1})$.

Notation 5.1.4. We denote a graph of groupoids Π_1-word $w = (g_1, t(e_1))\,(s(g_1), e_1)\cdots (g_n, t(e_n))(s(g_n), e_n)\,(g_{n+1}, o(e_n))$ by $w = g_1 e_1 \cdots g_n e_n g_{n+1}$, since it is clear that the concatenation $g_i e_i$ is possible if and only if $g_i \in \mathcal{G}_{t(e_i)}$.

For all $e \in \Gamma^1$ we choose a left transversal $\mathcal{T}_e$ of $\alpha_e(\mathcal{G}_e)$ in $\mathcal{G}_{t(e)}$ containing the identity elements of $\mathcal{G}_{t(e)}$, i.e., for all $g \in \mathcal{G}_{t(e)}$ there exist $\tau = \tau(g,e) \in \mathcal{T}_e$ and $h = h(g,e) \in \mathcal{G}_e$ such that $g = \tau\alpha_e(h)$.

Definition 5.1.5. A graph of groupoids Π_1-word $w = g_1 e_1 g_2 e_2 \cdots g_n e_n g_{n+1}$ is said to be *reduced* if it satisfies

(i) $g_i \in \mathcal{T}_{e_i}$ for all $i = 1, \ldots, n$ and $g_{n+1} \in \mathcal{G}_{o(e_n)}$;

(ii) if $e_{i-1} = \bar{e}_i$ for some $2 \leq i \leq n$, then g_i is not a unit of $\mathcal{G}_{t(e_i)}$.

Again, one has that each element of $\Pi_1(\mathcal{G}(\Gamma))$ is represented uniquely by a reduced graph of groupoids Π_1-word. We omit the proof of this statement, since it is identical to the one of Theorem 4.3.9.

Since we work with two different groupoids associated to a graph of groupoids $\mathcal{G}(\Gamma)$, i.e., the fundamental groupoid $\pi_1(\mathcal{G}(\Gamma))$ and the universal fundamental groupoid $\Pi_1(\mathcal{G}(\Gamma))$, we establish a connection between them.

Remark 5.1.6. The fundamental groupoid $\pi_1(\mathcal{G}(\Gamma))$ has the same generators as $\Pi_1(\mathcal{G}(\Gamma))$. In fact, for a generator $(x,e) \in \Pi_1(\mathcal{G}(\Gamma))$, where $e \in \Gamma^1$, $x \in \mathcal{G}^{(0)}_{t(e)}$, one has that $r((x,e)) = (x, t(e))$ and $s((x,e)) = (\phi_{\bar{e}}(x), o(e))$. Put $\phi_{\bar{e}}(x) = y \in \mathcal{G}^{(0)}_{o(e)}$. Then, (x,e) may be seen as an arrow from y to x. On the other hand, one has that $(x,y) = (\phi_e(y), y) \in \pi_1(\mathcal{G}(\Gamma))$. That is, for $x \in \mathcal{G}^{(0)}_{t(e)}$, $e \in \Gamma^1$, $(x, \phi_{\bar{e}}(x))$ is a generator of $\pi_1(\mathcal{G}(\Gamma))$. Hence, one obtains the fundamental groupoid $\pi_1(\mathcal{G}(\Gamma))$ by adding the relations $\sim$ (see (4.1.2)) and (R1) (see (R1)) to $\Pi_1(\mathcal{G}(\Gamma))$.

Furthermore, note that one may associate to a graph of groupoids $\mathcal{G}(\Gamma)$ a more general groupoid, i.e., the groupoid generated by the same generators as $\Pi_1(\mathcal{G}(\Gamma))$ (see (5.1.1)), and no relations.

As we have done in Chapter 4, we define a forest, which we call the *universal forest*, on which the universal fundamental groupoid acts.

Definition 5.1.7. The *universal forest* $Y_{\mathcal{G}(\Gamma)}$ of the graph of groupoids $\mathcal{G}(\Gamma)$ is the graph defined by

$$Y^0_{\mathcal{G}(\Gamma)} = \bigsqcup_{v \in \Gamma^0} \Pi_1(\mathcal{G}(\Gamma))/\mathcal{G}_v[v], \tag{5.1.6}$$

$$Y^1_{\mathcal{G}(\Gamma)} = \bigsqcup_{e \in \Gamma^1} \Pi_1(\mathcal{G}(\Gamma))/\mathcal{H}_{|e|}[e]. \tag{5.1.7}$$

The maps $o, t \colon Y^1_{\mathcal{G}(\Gamma)} \to Y^0_{\mathcal{G}(\Gamma)}$ and $^{-} \colon Y^1_{\mathcal{G}(\Gamma)} \to Y^1_{\mathcal{G}(\Gamma)}$ are given by

$$o(p\mathcal{H}_{|e|}) = pe^{1-\varepsilon(e)}\mathcal{G}_{o(e)}[o(e)] \in \Pi_1(\mathcal{G}(\Gamma))/\mathcal{G}_{o(e)}, \tag{5.1.8}$$

$$t(p\mathcal{H}_{|e|}) = pe^{-\varepsilon(e)}\mathcal{G}_{t(e)}[t(e)] \in \Pi_1(\mathcal{G}(\Gamma))/\mathcal{G}_{t(e)}, \tag{5.1.9}$$

$$\overline{p\mathcal{H}_{|e|}[e]} = p\mathcal{H}_{|\bar{e}|}[\bar{e}], \tag{5.1.10}$$

where $\varepsilon \colon \Gamma^1 \to \{0,1\}$ is the map defined in (4.3.12).

Remark 5.1.8. Note that the maps t and o are well defined and $Y_{\mathcal{G}(\Gamma)}$ is a graph (cf. Definition 4.3.15). The universal fundamental groupoid $\Pi_1(\mathcal{G}(\Gamma))$ acts on $Y_{\mathcal{G}(\Gamma)}$ by left multiplication with momentum map

$$\begin{aligned} \varphi \colon Y^0_{\mathcal{G}} &\to \Pi_1(\mathcal{G}(\Gamma))^{(0)} \\ p\mathcal{G}_v &\mapsto r(p). \end{aligned} \tag{5.1.11}$$

That is, for $p, p' \in \Pi_1(\mathcal{G}(\Gamma))$, $q\mathcal{G}_v[v] \in Y^0_{\mathcal{G}(\Gamma)}$, $q'\mathcal{H}_e[e] \in Y^1_{\mathcal{G}(\Gamma)}$ with $s(p) = \varphi(q\mathcal{G}_v) = r(q)$ and $s(p') = \varphi \circ o(q'\mathcal{H}_e) = r(q')$ one has that

$$\mu^0(p, q\mathcal{G}_v[v]) = pq\mathcal{G}_v[v], \tag{5.1.12}$$

$$\mu^1(p', q'\mathcal{H}_{|e|}[e]) = p'q'\mathcal{H}_{|e|}[e], \tag{5.1.13}$$

where $pq \in \Pi_1(\mathcal{G}(\Gamma))$ is the reduced graph of groupoids Π_1-word obtained from the concatenation of p and q. Moreover, $Y_{\mathcal{G}(\Gamma)}$ is fibered on $\Pi_1(\mathcal{G}(\Gamma))^{(0)}$ via the map φ and one has that

$$Y_{\mathcal{G}(\Gamma)}^{(0)} = \bigsqcup_{x \in \Pi_1(\mathcal{G}(\Gamma))^{(0)}} xY_{\mathcal{G}(\Gamma)},$$

where $xY_{\mathcal{G}(\Gamma)} = \varphi^{-1}(x)$ (see 4.3.17). In particular, $Y_{\mathcal{G}(\Gamma)}$ is a forest (see 4.3.18).

5.2 The boundary of the universal forest

Let $X = (X^0, X^1)$ be a locally finite, nonsingular tree, i.e., each vertex has finite valence bigger than 1, and choose a base vertex $x \in X^0$.

For $n \geq 0$ we write xX^n for the set of reduced paths of length n with terminus x. Since X is locally finite, the set xX^n is finite for each n. We denote by xX^* the set of all finite reduced paths in X with range x, that is,

$$xX^* = \bigcup_{n \in \mathbb{N}} xX^n. \tag{5.2.1}$$

Definition 5.2.1. The boundary (from x) $x\partial X$ of the tree X is the set of infinite reduced paths with range x, that is

$$x\partial X = \{\, e_1 e_2 \cdots \mid t(e_1) = x, e_i \in X^1, o(e_i) = t(e_{i+1}), i \in \mathbb{N} \,\}. \tag{5.2.2}$$

Definition 5.2.2. For a finite reduced path $\mu \in xX^*$, we define the *cylinder set* $Z(\mu)$ to be the elements of $x\partial X$ that extend μ, that is

$$Z(\mu) = \{\, \omega \in x\partial X \mid \omega = \mu\omega', \omega' \in o(\mu)\partial X \}. \tag{5.2.3}$$

Since X is nonsingular, the set $Z(\mu)$ is nonempty for all such μ. The collection $\{Z(\mu) \mid \mu \in xX^*\}$ is a base for a totally disconnected compact Hausdorff topology on $x\partial X$, coinciding with the topology described in Section 2.2.2.

We now consider the universal forest $Y_{\mathcal{G}(\Gamma)}$ for a locally finite, nonsingular graph of groupoids $\mathcal{G}(\Gamma)$. Let $\Pi_1(\mathcal{G}(\Gamma))$ be the universal fundamental groupoid of $\mathcal{G}(\Gamma)$. Then

$$Y_{\mathcal{G}(\Gamma)} = \bigsqcup_{x \in \pi_1(\mathcal{G})^{(0)}} xY_{\mathcal{G}(\Gamma)} \tag{5.2.4}$$

by Remark 2.4.17. Thus, we define the boundary $\partial Y_{\mathcal{G}(\Gamma)}$ of the universal forest $Y_{\mathcal{G}(\Gamma)}$ as the union of the boundaries of its subtrees $xY_{\mathcal{G}(\Gamma)}$, that is

$$\partial Y_{\mathcal{G}(\Gamma)} = \bigsqcup_{x \in \Pi_1(\mathcal{G})^{(0)}} x\partial Y_{\mathcal{G}(\Gamma)}. \tag{5.2.5}$$

Then $\partial Y_{\mathcal{G}(\Gamma)}$ is a totally disconnected compact Hausdorff space. We can then identify the boundary $\partial Y_{\mathcal{G}(\Gamma)}$ with the set of infinite reduced graph of groupoids Π_1-words, which is the set of all infinite sequences $g_1 e_1 g_2 e_2 \ldots$ such that each initial finite subsequence of the form $g_1 e_1 \cdots g_n e_n$ is a reduced graph of groupoids Π_1-word, i.e., an element of $\Pi_1(\mathcal{G}(\Gamma))$. The range map r extends to infinite reduced words in the obvious way:

$$r(g_1 e_1 g_2 e_2 \ldots) := r(g_1). \tag{5.2.6}$$

There is an induced action of $\Pi_1(\mathcal{G}(\Gamma))$ on the boundary $\partial Y_{\mathcal{G}(\Gamma)}$. Let $p \in \Pi_1(\mathcal{G}(\Gamma))$ and let $\xi \in \partial Y_{\mathcal{G}(\Gamma)}$ and suppose that $s(p) = r(\xi)$. The infinite word consisting of the concatenation of the reduced Π_1-word p with the reduced infinite Π_1-word ξ will not in general be reduced. However, by possibly infinitely many applications of the relations in the vertex groupoids and relations (R2'), this concatenation can be transformed into an infinite reduced graph of groupoids Π_1-word ξ'. This procedure of concatenation and then reduction taking ξ to ξ' defines an action of the groupoid $\Pi_1(\mathcal{G}(\Gamma))$ on the set $\partial Y_{\mathcal{G}(\Gamma)}$. The image of a cylinder set $Z(\mu)$ under this action is a union of cylinder sets, and so the fundamental groupoid $\Pi_1(\mathcal{G}(\Gamma))$ acts on the boundary $\partial Y_{\mathcal{G}(\Gamma)}$ by homeomorphisms.

5.3 The action groupoid $\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$

We have seen in Remark 2.4.5 that the action of a topological groupoid $\mathcal{G}$ on a locally compact space X defines naturally a groupoid $\mathcal{G} \ltimes X$, called the action groupoid. Thus, given a graph of groupoids $\mathcal{G}(\Gamma)$, we consider the action groupoid

$$\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$$

defined by the action of the fundamental groupoid $\Pi_1(\mathcal{G}(\Gamma))$ on the boundary $\partial Y_{\mathcal{G}(\Gamma)}$ of the universal forest $Y_{\mathcal{G}(\Gamma)}$. We want to make $\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$ into a locally compact étale groupoid. Much of this section is inspired by [24]. The idea is to use the sets

$$\{\, \{\mu\} \times Z(\nu) \mid \mu, \nu \in \Pi_1(\mathcal{G}(\Gamma)), s(\mu) = r(\nu) \,\}$$

as a basis for a topology on $\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$.

Since $Y_{\mathcal{G}(\Gamma)}$ is a forest, two cylinder sets $Z(\nu)$ and $Z(\zeta)$ in $Y_{\mathcal{G}(\Gamma)}$ only intersect if either ν is a subword of ζ, and in that case $Z(\zeta) \subseteq Z(\nu)$, or ζ is a subword of ν, in which case $Z(\nu) \subseteq Z(\zeta)$. Thus, the following lemma is straightforward.

Lemma 5.3.1. *For $\mu, \nu, \gamma, \zeta \in \Pi_1(\mathcal{G}(\Gamma))$ with $s(\mu) = r(\nu)$ and $s(\gamma) = r(\zeta)$, one has*

$$\{\mu\} \times Z(\nu) \cap \{\gamma\} \times Z(\zeta) = \begin{cases} \{\mu\} \times Z(\nu) & \text{if } \mu = \gamma,\ \nu = \zeta\nu' \text{ for } \nu' \in \Pi_1(\mathcal{G}(\Gamma)), \\ \{\gamma\} \times Z(\zeta) & \text{if } \mu = \gamma,\ \zeta = \nu\zeta' \text{ for } \zeta' \in \Pi_1(\mathcal{G}(\Gamma)), \\ \emptyset & \text{otherwise.} \end{cases}$$

Proposition 5.3.2. *Let $\mathcal{G}(\Gamma)$ be a locally finite nonsingular graph of groupoids. Then the sets*

$$\{\, \{\mu\} \times Z(\nu) \mid \mu, \nu \in \Pi_1(\mathcal{G}(\Gamma)), s(\mu) = r(\nu) \,\}.$$

form a basis for a locally compact Hausdorff topology on $\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$. Moreover, with this topology $\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$ is an ample groupoid in which each $\{\mu\} \times Z(\nu)$ is a compact open set.

Proof. By Lemma 5.3.1, each finite intersection of $\{\mu\} \times Z(\nu)$'s is another set of the type $\{\mu\} \times Z(\nu)$, and hence the sets $\{\mu\} \times Z(\nu)$ form a basis for a topology on $\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$. Moreover, we now prove that it is a Hausdorff topology. Let (μ, ξ), $(\mu', \xi') \in \Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$, $(\mu, \xi) \neq (\mu', \xi')$. If $\mu \neq \mu'$, then it suffices to choose ν and ν' in $\Pi_1(\mathcal{G}(\Gamma))$ such that $\xi = \nu\eta$, $\xi' = \nu'\eta'$ for $\eta, \eta' \in \partial Y_{\mathcal{G}(\Gamma)}$ to have $\{\mu\} \times Z(\nu) \cap \{\mu'\} \times Z(\nu') = \emptyset$, $(\mu, \xi) \in \{\mu\} \times Z(\nu)$ and $(\mu', \xi') \in \{\mu'\} \times Z(\nu')$. If $\mu = \mu'$, then it must be

$\xi \neq \xi'$. Suppose that $\xi = g_1 e_1 g_2 e_2 \cdots$ and $\xi' = g_1' e_1' g_2' e_2' \cdots$. Then there exists $n \in \mathbb{N}$ such that

$$g_{n+1} e_{n+1} g_{n+2} e_{n+2} \cdots \neq g_{n+1}' e_{n+1}' g_{n+2}' e_{n+2}' \cdots .$$

Thus, by putting $\zeta = g_1 e_1 g_2 e_2 \cdots g_{n+1} e_{n+1}$ and $\zeta' = g_1' e_1' g_2' e_2' \cdots g_{n+1}' e_{n+1}'$ we have that

$$\{\mu\} \times Z(\zeta) \cap \{\mu'\} \times Z(\zeta') = \emptyset,$$

$(\mu, \xi) \in \{\mu\} \times Z(\zeta)$ and $(\mu', \xi') \in \{\mu'\} \times Z(\zeta')$. Furthermore, inversion and product are continuous, since

$$\big(\{\mu\} \times Z(\nu)\big)^{-1} = \{\mu^{-1}\} \times Z(\mu\nu)$$

and

$$\_ \cdot \_^{-1}\big(\{\mu\} \times Z(\nu)\big) = \bigcup_{\gamma\zeta=\mu} \big(\{\gamma\} \times Z(\zeta\nu)\big) \times \big(\{\zeta\} \times Z(\nu)\big).$$

Next note that the range map r is a homeomorphism from $\{\mu\} \times Z(\nu)$ to $\{r(\mu)\} \times Z(\mu\nu)$ and the source map s is a homeomorphism from $\{\mu\} \times Z(\nu)$ to $\{s(\mu)\} \times Z(\nu)$. Thus, each $\{\mu\} \times Z(\nu)$ is an open bisection. Finally, since the $Z(\nu)$'s form a basis of compact open sets of $\partial Y_{\mathcal{G}(\Gamma)}$, one has that $\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$ is Hausdorff ample. □

5.4 The graph of groupoids C^*-algebra

In this section we build the groupoid analogous of the graph of groups C^*-algebra defined in [8, §3]. We begin with the definition of a unitary representation of a discrete groupoid, then we give the definition of a $\mathcal{G}$-family and define the graph of groupoid C^*-algebra $C^*(\mathcal{G})$.

Definition 5.4.1. Let $\mathcal{G}$ be a discrete groupoid. A function $U\colon \mathcal{G} \to \mathcal{B}(\mathcal{H})$ is a *unitary representation* of $\mathcal{G}$ if

- (i) for $x \in \mathcal{G}^{(0)}$, U_x is the orthogonal projection on a closed subspace of $\mathcal{H}$;
- (ii) for all $g \in \mathcal{G}$, U_g is a partial isometry with initial projection $U_{s(g)}$ and final projection $U_{r(g)}$, that is, $U_g^* U_g = U_{s(g)}$ and $U_g U_g^* = U_{r(g)}$;
- (iii) for $g, h \in \mathcal{G}$, one has

$$U_g U_h = \begin{cases} U_{gh} & \text{if } s(g) = r(h) \\ 0 & \text{otherwise.} \end{cases} \tag{5.4.1}$$

The multiplication formula (5.4.1) applies when g and h are units, and then says that the projections U_x, for $x \in \mathcal{G}^{(0)}$, are mutually orthogonal.

Definition 5.4.2. For each $e \in \Gamma^1$, choose a left transversal $\mathcal{T}_e$ of $\alpha_e(\mathcal{G}_e)$ in $\mathcal{G}_{t(e)}$ containing the identity elements of $\mathcal{G}_{t(e)}$. A $(\mathcal{G}, \mathcal{T})$*-family* is a collection of partial isometries S_e for each $e \in \Gamma^1$ and unitary representations $U_{\cdot,v}$ of $\mathcal{G}_v$ for each $v \in \Gamma^0$ satisfying the relations:

(G1) $U_{x,v} U_{y,w} = 0$ for all $v \neq w$, $x \in \mathcal{G}_v^{(0)}$, $y \in \mathcal{G}_w^{(0)}$;

(G2) $U_{\alpha_e(g),t(e)}S_e = S_e U_{\alpha_{\bar{e}}(g),o(e)}$ for each $e \in \Gamma^1$ and $g \in \mathcal{G}_e$;

(G3)
$$\sum_{x \in \mathcal{G}^{(0)}_{o(e)}} U_{x,o(e)} = S_e^* S_e + S_{\bar{e}} S_{\bar{e}}^*$$

for each $e \in \Gamma^1$;

(G4)
$$S_e^* S_e = \sum_{\substack{t(f)=o(e),\, h \in \mathcal{T}_f \\ hf \neq x\bar{e},\, x \in \mathcal{G}^{(0)}_{o(e)}}} U_{h,o(e)} S_f S_f^* U^*_{h,o(e)}$$

for each $e \in \Gamma^1$.

Remark 5.4.3. Relation (G4) is independent of the choice of transversals. Given a second choice of transversals $\{ \mathcal{T}'_e \mid e \in \Gamma^1 \}$, edges $e, f \in \Gamma^1$ with $t(f) = o(e)$, and $h' \in \mathcal{T}'_f$, we write $h' = h\alpha_f(g)$ for some $h \in \mathcal{T}_f$ and $g \in \mathcal{G}_f$. Then (G2) gives

$$\begin{aligned} U_{h',o(e)} S_f S_f^* U^*_{h',o(e)} &= U_{h,o(e)} U_{\alpha_f(g),o(e)}\, S_f\, S_f^*\, U^*_{\alpha_f(g),o(e)}\, U^*_{h,o(e)} \\ &= U_{h,o(e)}\, S_f\, U_{\alpha_{\bar{f}}(g),o(f)}\, U^*_{\alpha_{\bar{f}}(g),o(f)}\, S_f^*\, U^*_{h,o(e)} \\ &= U_{h,o(e)}\, S_f\, S_f^*\, U^*_{h,o(e)}. \end{aligned} \tag{5.4.2}$$

Given Remark 5.4.3, from now on we call a family of partial isometries and partial unitaries as in Definition 5.4.2 a $\mathcal{G}$*-family*. We can now define the graph of groupoids C^*-algebra.

Definition 5.4.4. Let $\mathcal{G}(\Gamma)$ be a locally finite nonsingular graph of groupoids as above. The *graph of groupoids algebra* $C^*(\mathcal{G})$ is the universal C^*-algebra generated by a $\mathcal{G}$-family (cf. [26, Proposition 4.1] for the existence and uniqueness of $C^*(\mathcal{G})$), in the sense that $C^*(\mathcal{G})$ is generated by a $\mathcal{G}$-family $\{ u_{\cdot,v}, s_e \mid v \in \Gamma^0, e \in \Gamma^1 \}$ such that if B is a C^*-algebra, and if $\{ U_{\cdot,v}, S_e \mid v \in \Gamma^0, e \in \Gamma^1 \}$ is a $\mathcal{G}$-family in B, there is a unique $*$-homomorphism from $C^*(\mathcal{G})$ to B such that $u_{\cdot,v} \mapsto U_{\cdot,v}$ and $s_e \mapsto S_e$.

Remark 5.4.5. We build a concrete $\mathcal{G}$-family using regular representations of the transformation groupoid $\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$. Let $\xi = g_1 e_1 g_2 e_2 \cdots \in \partial Y_{\mathcal{G}(\Gamma)}$. Put $\mathcal{H}_\xi = \ell^2\big(\Pi_1(\mathcal{G}(\Gamma))\xi\big)$, where $\Pi_1(\mathcal{G}(\Gamma))\xi = \{p\xi \mid p \in \Pi_1(\mathcal{G}(\Gamma)), s(p) = r(\xi)\}$. For $\gamma \in \Pi_1(\mathcal{G})$, let $\delta_{\gamma\xi}$ be the point mass function. For $e \in \Gamma^1$, let $\phi_e \colon \mathcal{H}_{\bar{e}} \to \mathcal{H}_e$ be the map defined by

$$\phi_e(h) = \alpha_e\big(\alpha_{\bar{e}}^{-1}(h)\big). \tag{5.4.3}$$

Since α_e is a monomorphism and $\mathcal{G}_e$ is a wide subgroupoid of $\mathcal{G}_{t(e)}$ for all $e \in \Gamma^1$, we have that ϕ_e is an isomorphism of groupoids for all $e \in \Gamma^1$. We define a $\mathcal{G}$-family in $\mathcal{B}(\mathcal{H}_\xi)$ by

$$S_e \delta_{\gamma\xi} = \begin{cases} \delta_{\phi_e(r(\gamma)) e \gamma\xi} & \text{if } r(\gamma) \in \mathcal{G}_{o(e)}, \gamma\xi \in \partial Y_{\mathcal{G}(\Gamma)} \setminus Z(r(\gamma)\bar{e}), \\ 0 & \text{otherwise.} \end{cases} \tag{5.4.4}$$

$$U_{g,v} \delta_{\gamma\xi} = \begin{cases} \delta_{g\gamma\xi} & \text{if } r(\gamma) = s(g), \\ 0 & \text{otherwise.} \end{cases} \tag{5.4.5}$$

for each $e \in \Gamma^1$, $v \in \Gamma^0$, $\gamma \in \Pi_1(\mathcal{G}(\gamma))$ and $g \in \mathcal{G}_v$.

We verify that this is indeed a $\mathcal{G}$-family. First, it is easy to see that (G1) follows from the above formula for $U_{g,v}$. Next note that

$$S_e^*\delta_{\gamma\xi} = \begin{cases} \delta_{\xi'} & \text{if } \gamma\xi = \phi_e(r(\xi'))e\xi' \text{ for some } \xi' \in \partial Y_{\mathcal{G}(\Gamma)} \setminus Z\big(r(\xi')\,\bar{e}\big), \\ 0 & \text{otherwise.} \end{cases} \tag{5.4.6}$$

It follows that $S_eS_e^*$ is the projection onto $\overline{\operatorname{span}}\{\,\delta_{\gamma\xi} \mid \gamma\xi \in Z(xe), x \in \mathcal{G}_{t(e)}^{(0)}\,\}$. Observe that $U_{x,v}$ is the projection onto $\overline{\operatorname{span}}\{\,\delta_{\gamma\xi} \mid r(\gamma) = x\,\}$, and hence $\sum_{x\in\mathcal{G}_v^{(0)}} U_{x,v}$ is the projection onto $\overline{\operatorname{span}}\{\,\delta_{\gamma\xi} \mid r(\gamma) \in \mathcal{G}_v^{(0)}\,\}$. It follows that

$$S_e^*S_e = \sum_{x\in\mathcal{G}_{o(e)}^{(0)}} U_{x,o(e)} - S_{\bar{e}}S_{\bar{e}}^*.$$

Therefore, (G3) holds.

Next we note that for $g \in \mathcal{G}_e$, $U_{\alpha_e(g),t(e)}S_e\delta_{\gamma\xi} \neq 0$ if and only if $s(\alpha_e(g)) = \phi_e(r(\gamma))$ and $\gamma\xi \in \partial Y_{\mathcal{G}(\Gamma)} \setminus Z(r(\gamma)\bar{e})$. In this case, we have

$$U_{\alpha_e(g),t(e)}S_e\delta_{\gamma\xi} = \delta_{\alpha_e(g)e\gamma\xi} = \delta_{r(\alpha_e(g))e\alpha_{\bar{e}}(g)\gamma\xi}.$$

We also note that $U_{\alpha_{\bar{e}}(g),o(e)}\delta_{\gamma\xi} = \delta_{\alpha_{\bar{e}}(g)\gamma\xi}$ if and only if $r(\gamma) = s(\alpha_{\bar{e}}(g))$. If moreover $\alpha_{\bar{e}}(g)\gamma\xi \in \partial Y_{\mathcal{G}(\Gamma)} \setminus Z\big(r(\alpha_{\bar{e}}(g))\bar{e}\big)$, we have

$$S_eU_{\alpha_{\bar{e}}(g),o(e)}\delta_{\gamma\xi} = S_e\delta_{\alpha_{\bar{e}}(g)\gamma\xi} = \delta_{r(\alpha_e(g))e\alpha_{\bar{e}}(g)\gamma\xi}. \tag{5.4.7}$$

Lemma 5.4.6. *Let $e \in \Gamma^1$, $g \in \mathcal{G}_e$, $\gamma \in \Pi_1(\mathcal{G})$ such that $r(\gamma) = s(\alpha_{\bar{e}}(g))$ and $\xi \in \partial Y_{\mathcal{G}(\Gamma)}$. Then $\alpha_{\bar{e}}(g)\gamma\xi \in Z\big(r(\alpha_{\bar{e}}(g))\bar{e}\big)$ if and only if $\gamma\xi \in Z\big(r(\gamma)\bar{e}\big)$.*

Proof. Let $\gamma = g_1e_1\dots g_ne_ng_{n+1}$. Then $r(\gamma) = r(g_1)$. Suppose that $\alpha_{\bar{e}}(g)\gamma\xi \in Z\big(r(\alpha_{\bar{e}}(g))\bar{e}\big)$. Then there exists $\xi' \in \partial X$ such that

$$\alpha_{\bar{e}}(g)g_1e_1\dots g_ne_ng_{n+1}\xi = r(\alpha_{\bar{e}}(g))\bar{e}\xi'. \tag{5.4.8}$$

This implies that $e_1 = \bar{e}$ and $\alpha_{\bar{e}}(g)g_1 = r(\alpha_{\bar{e}}(g))$, i.e., $\alpha_{\bar{e}}(g)g_1$ must be a unit. Then either $g_1 = \alpha_{\bar{e}}(g)^{-1}$ or g_1 is a unit. Since γ is reduced, one has that $g_1 \in \mathcal{T}_{\bar{e}}$ and hence $g_1 \neq \alpha_{\bar{e}}(g)^{-1}$. This implies that g_1 is a unit, i.e. $g_1 = r(g_1)$ and hence $\gamma\xi \in Z\big(r(g_1)\bar{e}\big)$.

Viceversa, suppose that $\gamma\xi \in Z(r(g_1)\bar{e})$. Then there exists $\xi' \in \partial Y_{\mathcal{G}(\Gamma)}$ such that $\gamma\xi = r(g_1)\bar{e}\xi'$. Thus, one has

$$\alpha_{\bar{e}}(g)\gamma\xi = \alpha_{\bar{e}}(g)r(g_1)\bar{e}\xi' = \alpha_{\bar{e}}(g)\bar{e}\xi' = r(\alpha_{\bar{e}}(g))\bar{e}\alpha_e(g)\xi'.$$

That is, $\alpha_{\bar{e}}(g)\gamma\xi \in Z\big(r(\alpha_{\bar{e}}(g))\bar{e}\big)$. □

By Lemma 5.4.6, one has that $S_eU_{\alpha_{\bar{e}}(g),o(e)}\delta_{\gamma\xi} = 0$ if and only if $U_{\alpha_e(g),t(e)}S_e\delta_{\gamma\xi} = 0$, and if non zero, they are equal. Hence (G2) holds.

Finally, one has that $\big(U_{h,t(f)}S_f\big)\big(U_{h,t(f)}S_f\big)^*$ is the projection onto $\overline{\operatorname{span}}\{\,\delta_{\gamma\xi} \mid \gamma\xi \in Z(hf)\,\}$ by the same argument as above. This observation together with (G3) yield to (G4).

There are some important consequences of the relations (G1)-(G4) in Definition 5.4.2.

Lemma 5.4.7. *Let $\mathcal{G}(\Gamma)$ be a graph of groupoids and let $\{\,U_{\cdot,v}, S_e \mid v \in \Gamma^0, e \in \Gamma^1\,\}$ be a $\mathcal{G}$-family. For $e \in \Gamma^1$ one has*

(i) $S_e S_{\bar{e}} = S_{\bar{e}} S_e = 0$;

(ii) $\sum_{x\in\mathcal{G}^{(0)}_{t(e)}} U_{x,t(e)}\, S_e = \sum_{y\in\mathcal{G}^{(0)}_{o(e)}} S_e\, U_{y,o(e)} = S_e$;

(iii) $S_e S_f = 0$ *for* $f \in \Gamma^1$ *with* $t(f) \neq o(e)$;

(iv) $U_{g_1,v_1} S_e U_{g_2,v_2} = 0$ *for* $v_1 \neq t(e)$ *or* $v_2 \neq o(e)$.

Proof. Since S_e and $S_{\bar{e}}$ are partial isometries and since $S_e^* S_e$ and $S_{\bar{e}} S_{\bar{e}}^*$ are orthogonal projections, one has

$$S_e S_{\bar{e}} = (S_e S_e^* S_e)(S_{\bar{e}} S_{\bar{e}}^* S_{\bar{e}}) = S_e (S_e^* S_e S_{\bar{e}} S_{\bar{e}}^*) S_{\bar{e}} = 0; \tag{5.4.9}$$

$$S_{\bar{e}} S_e = (S_{\bar{e}} S_{\bar{e}}^* S_{\bar{e}})(S_e S_e^* S_e) = S_{\bar{e}} (S_{\bar{e}}^* S_{\bar{e}} S_e S_e^*) S_e = 0. \tag{5.4.10}$$

This proves (i). It follows that

$$\sum_{x\in\mathcal{G}^{(0)}_{o(\bar{e})}} U_{x,o(\bar{e})}\, S_e = (S_{\bar{e}}^* S_{\bar{e}} + S_e S_e^*) S_e = S_{\bar{e}}^* S_{\bar{e}} S_e + S_e S_e^* S_e = S_e; \tag{5.4.11}$$

$$\sum_{y\in\mathcal{G}^{(0)}_{o(e)}} S_e\, U_{y,o(e)} = S_e (S_e^* S_e + S_{\bar{e}} S_{\bar{e}}^*) = S_e S_e^* S_e + S_e S_{\bar{e}} S_{\bar{e}}^* = S_e, \tag{5.4.12}$$

which proves (ii). Moreover, one has that

$$S_e S_f = \Big(\sum_{y\in\mathcal{G}^{(0)}_{o(e)}} S_e U_{y,o(e)}\Big)\Big(\sum_{x\in\mathcal{G}^{(0)}_{t(f)}} U_{x,t(f)} S_f\Big) = 0 \tag{5.4.13}$$

by (ii) and relation (G1), which proves (iii). Finally, (iv) follows by the same argument used to prove (iii). □

Lemma 5.4.8. *Let* $e, f \in \Gamma^1$ *and put* $v = t(f)$. *Then for* $g \in \mathcal{T}_f$ *one has*

(i) $S_e^* U_{g,v} S_f = \begin{cases} S_e^* S_e U_{y,o(e)} & \text{if } gf = xe,\, x \in \mathcal{G}^{(0)}_{t(e)},\, y = \phi_{\bar{e}}(x), \\ 0 & \text{otherwise.} \end{cases}$

(ii) $S_e^* S_e U_{g,v} S_f = \begin{cases} U_{g,v} S_f & \text{if } v = o(e),\, gf \neq y\bar{e} \text{ and } y \in \mathcal{G}^{(0)}_{o(e)}, \\ 0 & \text{otherwise.} \end{cases}$

Proof. If $gf = xe$ for some $x \in \mathcal{G}^{(0)}_{t(e)}$, then one has

$$S_e^* U_{g,v} S_f = S_e^* U_{x,t(e)} S_e = S_e^* S_e u_{y,o(e)} \tag{5.4.14}$$

by (G2), where $y = \phi_{\bar{e}}(x)$. If $t(f) \neq t(e)$, then one has

$$S_e^* U_{g,v} S_f = S_e^* \Big(\sum_{x\in\mathcal{G}^{(0)}_{t(e)}} u_{x,t(e)}\Big) U_{g,v} S_f = 0 \tag{5.4.15}$$

by Lemma 5.4.7 (ii). If $t(f) = t(e)$ and $gf \neq xe$, $x \in \mathcal{G}^{(0)}_{t(e)}$, one has

$$\begin{aligned} S_e^* U_{g,v} S_f &= (S_e^* S_e S_e^*) U_{g,v} (S_f S_f^* S_f) \\ &= S_e^* \Big(S_e S_e^* U_{g,v} S_f (S_f^* U_{g,v}^*) U_{g,v} S_f\Big) \\ &= S_e^* \Big(S_e S_e^* U_{g,v} S_f (U_{g,v} S_f)^*\Big) U_{g,v} S_f \\ &= 0, \end{aligned} \tag{5.4.16}$$

since $U_{g,v}S_f(U_{g,v}S_f)^*$ is a subprojection of $S_{\bar{e}}^*S_{\bar{e}}$ by (G4) and since $S_eS_e^*$ and $S_{\bar{e}}^*S_{\bar{e}}$ are orthogonal by (G3). Thus, (i) is proved.
It remains to prove (ii). If $o(e) \neq v$, then $S_e^*S_eU_{g,v}S_f = 0$ by Lemma 5.4.7 (iv). If $gf = y\bar{e}$ for some $y \in \mathcal{G}_{o(e)}^{(0)}$, $y = \phi_{\bar{e}}(x)$, then one has

$$S_e^*S_eU_{g,v}S_f = S_e^*S_eU_{y,o(e)}S_{\bar{e}} = S_e^*S_eS_{\bar{e}}U_{x,t(e)} = 0, \tag{5.4.17}$$

since $S_es_{\bar{e}} = 0$. Finally, if $o(e) = v$ and $gf \neq y\bar{e}$, $y \in \mathcal{G}_{o(e)}^{(0)}$, then one has

$$\begin{aligned} S_e^*S_eU_{g,v}S_f &= S_e^*S_eU_{g,v}S_f(U_{g,v}S_f)^*U_{g,v}S_f \\ &= U_{g,v}S_f(U_{g,v}S_f)^*U_{g,v}S_f \\ &= U_{g,v}S_f, \end{aligned} \tag{5.4.18}$$

since $U_{g,v}S_f$ is a partial isometry and $U_{g,v}S_f(U_{g,v}S_f)^*$ is a subprojection of $S_e^*S_e$. This proves (ii). □

Notation 5.4.9. For $v \in \Gamma^0$ and $\mu = g_1e_1 \cdots g_ne_n \in \pi_1(\mathcal{G}(\Gamma))$, we put

(a) $$p_v = \sum_{x \in \mathcal{G}_v^{(0)}} U_{x,v};$$

(b) $$s_\mu = U_{g_1,t(e_1)}S_{e_1}U_{g_2,t(e_2)}S_{e_2} \cdots U_{g_n,t(e_n)}S_{e_n};$$

(c) $$o(\mu) = o(e_n), \quad t(\mu) = t(e_1),$$

(d) $$s(\mu) = \phi_{\bar{e}_n}(s(g_n)), \quad r(\mu) = r(g_1).$$

Lemma 5.4.10. *Let $e \in \Gamma^1$. Then one has*

$$S_eS_e^* = \sum_{x \in \mathcal{G}_{t(e)}^{(0)}} S_{xe}S_{xe}^*.$$

Proof. By (G2) one has that $U_{x,t(e)}S_e = S_eU_{\phi_{\bar{e}}(x),o(e)}$. Then

$$\begin{aligned} \sum_{x \in \mathcal{G}_{t(e)}^{(0)}} S_{xe}S_{xe}^* &= \sum_{x \in \mathcal{G}_{t(e)}^{(0)}} U_{x,t(e)}\, S_e\, S_e^*U_{x,t(e)}^* \\ &= \sum_{x \in \mathcal{G}_{t(e)}^{(0)}} S_e\, U_{\phi_{\bar{e}}(x),o(e)}\, U_{\phi_{\bar{e}}(x),o(e)}^*\, S_e^* \\ &= S_e \Big(\sum_{y \in \mathcal{G}_{o(e)}^{(0)}} U_{y,o(e)}U_{y,o(e)}^* \Big) S_e^* \\ &= S_e \Big(\sum_{y \in \mathcal{G}_{o(e)}^{(0)}} U_{y,o(e)} \Big) S_e^* \\ &= S_e\, p_{o(e)}\, S_e^* \\ &= S_eS_e^*. \end{aligned}$$

□

Lemma 5.4.11. *Let $v \in \Gamma^0$ Then for any $k \geq 1$ one has that*

$$p_v = \sum_{\substack{\alpha \in \pi_1(\mathcal{G}(\Gamma)),\\ t(\alpha)=v,\, |\alpha|=k}} S_\alpha S_\alpha^*.$$

Proof. We prove it by induction on k.

Let $k = 1$. Then the right-hand side of the equation in the statement becomes

$$\sum_{\substack{\alpha\in\pi_1(\mathcal{G}(\Gamma)),\\ t(\alpha)=v,\,|\alpha|=1}} S_\alpha S_\alpha^* = \sum_{\substack{f\in\Gamma^1, t(f)=v,\\ h\in\mathcal{T}_f}} S_{hf} S_{hf}^*. \tag{5.4.19}$$

On the other hand, for $e \in \Gamma^1$ such that $o(e) = v$, one has that

$$\begin{aligned}
p_v &= S_e^* S_e + S_{\bar{e}} S_{\bar{e}}^* \\
&= \Big(\sum_{\substack{f\in\Gamma^1,\, t(f)=v,\\ h\in\mathcal{T}_f,\, hf\neq x\bar{e}}} S_{hf} S_{hf}^* \Big) + S_{\bar{e}} S_{\bar{e}}^* \\
&= \Big(\sum_{\substack{f\in\Gamma^1,\, t(f)=v,\\ h\in\mathcal{T}_f,\, hf\neq x\bar{e}}} S_{hf} S_{hf}^* \Big) + \sum_{x\in\mathcal{G}_v^{(0)}} S_{x\bar{e}} S_{x\bar{e}}^* \\
&= \sum_{\substack{f\in\Gamma^1, t(f)=v,\\ h\in\mathcal{T}_f}} S_{hf} S_{hf}^*,
\end{aligned}$$

where we have used (G3) for the first equality, (G4) for the second one and Lemma 5.4.10 for the third one.

Suppose the thesis is true for $k = n$, $n > 1$, and consider $k = n + 1$. Then

$$\begin{aligned}
\sum_{\substack{\alpha\in\pi_1(\mathcal{G}(\Gamma)),\\ t(\alpha)=v,\,|\alpha|=n+1}} S_\alpha S_\alpha^* &= \sum_{\substack{\beta\in\pi_1(\mathcal{G}(\Gamma)),\\ t(\beta)=v,\,|\beta|=n}} \sum_{\substack{\beta'\in\pi_1(\mathcal{G}(\Gamma)),\\ r(\beta')=s(\beta),\,|\beta'|=1}} S_{\beta\beta'} S_{\beta\beta'}^* \\
&= \sum_{\substack{\beta\in\pi_1(\mathcal{G}(\Gamma)),\\ t(\beta)=v,\,|\beta|=n}} \sum_{\substack{\beta'\in\pi_1(\mathcal{G}(\Gamma)),\\ r(\beta')=s(\beta),\,|\beta'|=1}} S_\beta S_{\beta'} S_{\beta'}^* S_\beta^* \\
&= \sum_{\substack{\beta\in\pi_1(\mathcal{G}(\Gamma)),\\ t(\beta)=v,\,|\beta|=n}} S_\beta \left(\sum_{\substack{\beta'\in\pi_1(\mathcal{G}(\Gamma)),\\ r(\beta')=s(\beta),\,|\beta'|=1}} S_{\beta'} S_{\beta'}^* \right) S_\beta^* \\
&= \sum_{\substack{\beta\in\pi_1(\mathcal{G}(\Gamma)),\\ t(\beta)=v,\,|\beta|=n}} S_\beta\, p_{s(\beta)}\, S_\beta^* \\
&= \sum_{\substack{\beta\in\pi_1(\mathcal{G}(\Gamma)),\\ t(\beta)=v,\,|\beta|=n}} S_\beta\, S_\beta^* \\
&= p_v,
\end{aligned}$$

where the last equality is given by the induction hypothesis. □

5.5 A groupoid C^*-algebraic Bass-Serre theorem

The action of the universal fundamental groupoid $\Pi_1(\mathcal{G}(\Gamma))$ of a graph of groupoids $\mathcal{G}(\Gamma)$ on the boundary $\partial Y_{\mathcal{G}(\Gamma)}$ of the universal forest $Y_{\mathcal{G}(\Gamma)}$ induces a groupoid, the action groupoid, to which is associated the C^*-algebra $C^*(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$. In this section we prove our main theorem, which shows that the graph of groupoids C^*-algebra $C^*(\mathcal{G})$ is isomorphic to the action groupoid C^*-algebra $C^*(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$.

Notation 5.5.1. (a) From here on we denote the graph of groupoid words $x\,e\,\phi_{\bar{e}}(x)$ by xe and $g\,e\,\phi_{\bar{e}}(s(g))$ by ge, where $e \in \Gamma_1$, $x \in \mathcal{G}^{(0)}_{t(e)}$ and $g \in \mathcal{G}_{t(e)}$.

(b) Note that multiplication and inversion in $C^*(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$ are defined as in 2.5.1. For $S, T \in C^*(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$, we wil denote the convolution $S * T$ by $S\,T$.

Proposition 5.5.2. *Let $\mathcal{G}(\Gamma)$ be a locally finite nonsingular graph of groupoids. For each $v \in \Gamma^0$, $g \in \mathcal{G}_v$, $e \in \Gamma^1$ define*

$$U_{g,v} = \chi_{\{g\}\times Z(s(g))}, \tag{5.5.1}$$

$$S_e = \chi_{Z_e} \tag{5.5.2}$$

where

$$Z_e = \bigsqcup_{x \in \mathcal{G}^{(0)}_{t(e)}} \{xe\} \times \Xi_{xe}, \tag{5.5.3}$$

and

$$\Xi_{xe} = Z\big(\phi_{\bar{e}}(x)\big) \setminus Z\big(\phi_{\bar{e}}(x)\bar{e}\big). \tag{5.5.4}$$

Then the collection $\{\, U_{\cdot,v}, S_e \mid v \in \Gamma^0, e \in \Gamma^1 \,\}$ is a $\mathcal{G}$-family in $C^*\big(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}\big)$.

Proof. Firstly, we prove that $U_{\cdot,v}$ is a unitary representation of $\mathcal{G}_v$ for each $v \in \Gamma^0$. Let $v \in \Gamma^0$ and $g \in \mathcal{G}_v$. For $(\gamma, \xi) \in \Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}}$ we have

$$\begin{aligned}
(U_{g,v}\,U^*_{g,v})(\gamma,\xi) &= \sum_{\substack{\gamma_1,\gamma_2 \in \Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} U_{g,v}(\gamma_1, \gamma_2\xi) \cdot U^*_{g,v}(\gamma_2, \xi) \\
&= \sum_{\substack{\gamma_1,\gamma_2 \in \Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} U_{g,v}(\gamma_1, \gamma_2\xi) \cdot \overline{U_{g,v}((\gamma_2, \xi)^{-1})} \\
&= \sum_{\substack{\gamma_1,\gamma_2 \in \Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} U_{g,v}(\gamma_1, \gamma_2\xi) \cdot U_{g,v}((\gamma_2^{-1}, \gamma_2\xi)) \\
&= \sum_{\substack{\gamma_1,\gamma_2 \in \Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} \chi_{\{g\}\times Z(s(g))}(\gamma_1, \gamma_2\xi) \cdot \chi_{\{g\}\times Z(s(g))}(\gamma_2^{-1}, \gamma_2\xi),
\end{aligned}$$

and $\chi_{\{g\}\times Z(s(g))}(\gamma_1, \gamma_2\xi) \cdot \chi_{\{g\}\times Z(s(g))}(\gamma_2^{-1}, \gamma_2\xi) \neq 0$ if and only if $\gamma_1 = g$, $\gamma_2^{-1} = g$ and $\gamma_2\xi \in Z(s(g))$, i.e., if and only if $\gamma = gg^{-1} = r(g)$ and $\xi \in Z(r(g))$. Hence, we have

$$(U_{g,v}\,U^*_{g,v})(\gamma,\xi) = \chi_{\{r(g)\}\times Z(r(g))} = U_{r(g),v}(\gamma,\xi). \tag{5.5.5}$$

Moreover, we have

$$\begin{aligned}
(U^*_{g,v}\,U_{g,v})(\gamma,\xi) &= \sum_{\substack{\gamma_1,\gamma_2 \in \Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} U^*_{g,v}(\gamma_1, \gamma_2\xi) \cdot U_{g,v}(\gamma_2, \xi) \\
&= \sum_{\substack{\gamma_1,\gamma_2 \in \Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} \overline{U_{g,v}((\gamma_1, \gamma_2\xi)^{-1})} \cdot U_{g,v}(\gamma_2, \xi) \\
&= \sum_{\substack{\gamma_1,\gamma_2 \in \Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} U_{g,v}(\gamma_1^{-1}, \gamma_1\gamma_2\xi) \cdot U_{g,v}(\gamma_2, \xi),
\end{aligned} \tag{5.5.6}$$

and $U_{g,v}(\gamma^{-1}, \gamma_1\gamma_2\xi) \cdot U_{g,v}(\gamma_2, \xi) \neq 0$ if and only if $\gamma_1^{-1} = g$, $\gamma_2 = g$, $\gamma\xi \in Z(s(g))$ and $\xi \in Z(s(g))$, i.e., if and only if $\gamma = g^{-1}g = s(g)$ and $\xi \in Z(s(g))$. Hence, we have

$$(U_{g,v}^* U_{g,v})(\gamma, \xi) = \chi_{\{s(g)\} \times Z(s(g))} = U_{s(g),v}(\gamma, \xi). \tag{5.5.7}$$

Thus, $U_{g,v} U_{g,v}^* = U_{r(g),v}$ and $U_{g,v}^* U_{g,v} = U_{s(g),v}$.
Finally, for $g, h \in \mathcal{G}_v$, $s(g) = r(h)$, we have

$$(U_{g,v} U_{h,v})(\gamma, \xi) = \sum_{\substack{\gamma_1, \gamma_2 \in \Pi_1(\mathcal{G}(\Gamma)), \\ \gamma_1\gamma_2 = \gamma}} U_{g,v}(\gamma_1, \gamma_2\xi) \cdot U_{h,v}(\gamma_2, \xi) \tag{5.5.8}$$

and $U_{g,v}(\gamma_1, \gamma_2\xi) \cdot U_{h,v}(\gamma_2, \xi) \neq 0$ if and only if $\gamma_1 = g$, $\gamma_2\xi \in Z(s(g))$ $\gamma_2 = h$ and $\xi \in Z(s(h))$, i.e., if and only if $\gamma = gh$ and $\xi \in Z(s(h)) = Z(s(gh))$. Hence, we have

$$(U_{g,v} U_{h,v})(\gamma, \xi) = \chi_{\{gh\} \times Z(s(h))} = U_{gh,v}(\gamma, \xi). \tag{5.5.9}$$

Thus, $U_{g,v} U_{h,v} = U_{gh,v}$. Hence, $U_{\cdot,v}$ is a unitary representation of $\mathcal{G}_v$.

Then, we prove that S_e is a partial isometry for all $e \in \Gamma^1$. Let $(\gamma, \xi) \in \Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$. Note that

$$S_e^*((\gamma, \xi)) = \overline{S_e((\gamma, \xi)^{-1})} = \chi_{Z_e}((\gamma, \xi)^{-1}) = S_e((\gamma, \xi)^{-1}). \tag{5.5.10}$$

Thus, one has

$$\begin{aligned} (S_e^* S_e)(\gamma, \xi) &= \sum_{\substack{\gamma_1, \gamma_2 \in \Pi_1(\mathcal{G}(\Gamma)), \\ \gamma_1\gamma_2 = \gamma}} S_e^*(\gamma_1, \gamma_2\xi) \cdot S_e(\gamma_2, \xi) \\ &= \sum_{\substack{\gamma_1, \gamma_2 \in \Pi_1(\mathcal{G}(\Gamma)), \\ \gamma_1\gamma_2 = \gamma}} S_e(\gamma_1^{-1}, \gamma_1\gamma_2\xi) \cdot S_e(\gamma_2, \xi) \end{aligned} \tag{5.5.11}$$

Note that $S_e(\gamma_1^{-1}, \gamma_1\gamma_2\xi) \cdot S_e(\gamma_2, \xi) \neq 0$ if and only if $\gamma_1^{-1} = xe$ for some $x \in \mathcal{G}_{t(e)}^{(0)}$, $\gamma_2 = ye$ for some $y \in \mathcal{G}_{t(e)}^{(0)}$, $\gamma\xi \in \Xi_{xe}$ and $\xi \in \Xi_{ye}$. That is, $S_e(\gamma_1^{-1}, \gamma_1\gamma_2\xi) \cdot S_e(\gamma_2, \xi) \neq 0$ if and only if $\gamma_1 = \phi_{\bar{e}}(x)\bar{e}$, $x = y$, $\gamma = \gamma_1\gamma_2 = \phi_{\bar{e}}(x)$ and $\xi \in \Xi_{xe}$. Thus, one has

$$(S_e^* S_e)(\gamma, \xi) = \begin{cases} 1 & \text{if } (\gamma, \xi) \in \bigsqcup_{x \in \mathcal{G}_{t(e)}^{(0)}} \{\phi_{\bar{e}}(x)\} \times \Xi_{xe}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.5.12}$$

Hence one has

$$(S_e S_e^* S_e)(\gamma, \xi) = \sum_{\substack{\gamma_1, \gamma_2 \in \Pi_1(\mathcal{G}(\Gamma)), \\ \gamma_1\gamma_2 = \gamma}} S_e(\gamma_1, \gamma_2\xi) \cdot (S_e^* S_e)(\gamma_2, \xi). \tag{5.5.13}$$

Note that $S_e(\gamma_1, \gamma_2\xi) \cdot (S_e^* S_e)(\gamma_2, \xi) \neq 0$ if and only if $\gamma_1 = xe$ for some $x \in \mathcal{G}_{t(e)}^{(0)}$, $\gamma_2\xi \in \Xi_{xe}$, $\gamma_2 = \phi_{\bar{e}}(y)$ for some $y \in \mathcal{G}_{t(e)}^{(0)}$ and $\xi \in \Xi_{ye}$. That is, $S_e(\gamma_1, \gamma_2\xi) \cdot (S_e^* S_e)(\gamma_2, \xi) \neq 0$ if and only if $x = y$, $\gamma = \gamma_1\gamma_2 = xe$ for some $x \in \mathcal{G}_{t(e)}^{(0)}$ and $\xi \in \Xi_{xe}$. Hence, one has

$$\begin{aligned} (S_e S_e^* S_e)(\gamma, \xi) &= \begin{cases} 1 & \text{if } (\gamma, \xi) \in \bigsqcup_{x \in \mathcal{G}_{t(e)}^{(0)}} \{xe\} \times \Xi_{xe} \\ 0 & \text{otherwise} \end{cases} \\ &= S_e(\gamma, \xi), \end{aligned} \tag{5.5.14}$$

which proves that S_e is a partial isometry.

Finally, we prove that the family $\{ U_{\cdot,v}, S_e \mid v \in \Gamma^0, e \in \Gamma^1 \}$ satisfies (G1)-(G4) in Definition 5.4.2.

Let $x \in \mathcal{G}_v$, $y \in \mathcal{G}_w$, $v, w \in \Gamma^0$ with $v \neq w$. For $(\gamma, \xi) \in \Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$, we have

$$(U_{x,v}\, U_{y,w})(\gamma,\xi) = \sum_{\substack{\gamma_1,\gamma_2 \in \Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} U_{x,v}(\gamma_1, \gamma_2\xi) \cdot U_{y,w}(\gamma_2,\xi) \tag{5.5.15}$$

In order to have $U_{x,v}(\gamma_1,\gamma_2\xi)\cdot U_{y,w}(\gamma_2,\xi) \neq 0$ it must be $\gamma_1 = x$ and $\gamma_2 = y$, with $\gamma_1\gamma_2 = \gamma$. Since $x \in \mathcal{G}_v$ and $y \in \mathcal{G}_w$, they are not composable. It follows that $(U_{x,v}\, U_{y,w})(\gamma,\xi) = 0$. This proves (G1).

Let $e \in \Gamma^1$, $g \in \mathcal{G}_e$ and $(\gamma,\xi) \in \Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$. Then one has

$$(U_{\alpha_e(g),t(e)}\, S_e)(\gamma,\xi) = \sum_{\substack{\gamma_1,\gamma_2 \in \Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} U_{\alpha_e(g),t(e)}(\gamma_1,\gamma_2\xi) \cdot S_e(\gamma_2,\xi). \tag{5.5.16}$$

Note that $U_{\alpha_e(g),t(e)}(\gamma_1,\gamma_2\xi)\cdot S_e(\gamma_2,\xi) \neq 0$ if and only if $\gamma_1 = \alpha_e(g)$, $\gamma_2\xi \in Z(s(\alpha_e(g)))$, $\gamma_2 = xe$ for some $x \in \mathcal{G}^{(0)}_{t(e)}$ and $\xi \in \Xi_{xe}$, i.e., if and only if $\gamma_1 = \alpha_e(g)$, $\gamma_2 = \alpha_e(s(g))\, e$ and $\xi \in \Xi_{\alpha_e(s(g))e}$. Hence one has

$$(U_{\alpha_e(g),t(e)}\, S_e)(\gamma,\xi) = \begin{cases} 1 & \text{if } (\gamma,\xi) \in \{\alpha_e(g)\, e\} \times \Xi_{\alpha_e(s(g))e}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.5.17}$$

Similarly, one has

$$(S_e\, U_{\alpha_{\bar{e}}(g),o(e)})(\gamma,\xi) = \sum_{\substack{\gamma_1,\gamma_2 \in \pi_1(\mathcal{G}),\\ \gamma_1\gamma_2=\gamma}} S_e(\gamma_1,\gamma_2\xi) \cdot U_{\alpha_{\bar{e}}(g),o(e)}(\gamma_2,\xi). \tag{5.5.18}$$

Note that $S_e(\gamma_1,\gamma_2\xi) \cdot U_{\alpha_{\bar{e}}(g),o(e)}(\gamma_2,\xi) \neq 0$ if and only if $\gamma_1 = xe$ for some $x \in \mathcal{G}^{(0)}_{t(e)}$, $\gamma_2\xi \in \Xi_{xe}$, $\gamma_2 = \alpha_{\bar{e}}(g)$ and $\xi \in Z\big(s(\alpha_{\bar{e}}(g))\big)$, that is, if and only if $\gamma_1 = \alpha_e(r(g))\, e$, $\gamma_2 = \alpha_{\bar{e}}(g)$ and $\xi \in \Xi_{\alpha_e(s(g))\, e}$. Hence one has that $S_e(\gamma_1,\gamma_2\xi) \cdot U_{\alpha_{\bar{e}}(g),o(e)}(\gamma_2,\xi) \neq 0$ if and only if $\gamma = \alpha_e(r(g))\, e\, \alpha_{\bar{e}}(g) = \alpha_e(g)\, e\, \alpha_{\bar{e}}(s(g))$ and $\xi \in \Xi_{\alpha_e(s(g))\, e}$.

Thus, one has

$$\begin{aligned}(S_e\, U_{\alpha_{\bar{e}}(g),o(e)})(\gamma,\xi) &= \begin{cases} 1 & \text{if } (\gamma,\xi) \in \{\alpha_e(g)\, e\} \times \Xi_{\alpha_e(s(g))\, e}, \\ 0 & \text{otherwise.} \end{cases} \\ &= (U_{\alpha_e(g),t(e)}\, S_e)(\gamma,\xi).\end{aligned} \tag{5.5.19}$$

Thus, (G2) is proved.

It remains to prove (G3) and (G4). For $e \in \Gamma^1$, note that

$$\sum_{x \in \mathcal{G}^{(0)}_{o(e)}} U_{x,o(e)} = \sum_{x \in \mathcal{G}^{(0)}_{o(e)}} \chi_{\{x\}\times Z(x)} = \chi_{Z_x}, \tag{5.5.20}$$

where

$$Z_x := \bigsqcup_{x \in \mathcal{G}^{(0)}_{o(e)}} \{x\} \times Z(x). \tag{5.5.21}$$

For $(\gamma,\xi)\in \Pi_1(\mathcal{G}(\Gamma))\ltimes \partial Y_{\mathcal{G}(\Gamma)}$, one has

$$\sum_{x\in\mathcal{G}^{(0)}_{o(e)}} U_{x,o(e)}(\gamma,\xi)=\begin{cases}1 & \text{if } (\gamma,\xi)\in Z_x\\ 0 & \text{otherwise.}\end{cases} \tag{5.5.22}$$

We also note that

$$\begin{aligned}(S_e\,S_e^*)(\gamma,\xi) &= \sum_{\substack{\gamma_1,\gamma_2\in\Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} S_e(\gamma_1,\gamma_2\xi)\cdot S_e^*(\gamma_2,\xi)\\ &= \sum_{\substack{\gamma_1,\gamma_2\in\Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} S_e(\gamma_1,\gamma_2\xi)\cdot S_e(\gamma_2^{-1},\gamma_2\xi).\end{aligned} \tag{5.5.23}$$

Note that $S_e(\gamma_1,\gamma_2\xi)\cdot S_e(\gamma_2^{-1},\gamma_2\xi)\neq 0$ if and only if $\gamma_1=xe$ for some $x\in\mathcal{G}^{(0)}_{t(e)}$, $\gamma_2^{-1}=ye$ for some $y\in\mathcal{G}^{(0)}_{t(e)}$, i.e., $\gamma_2=\phi_{\bar{e}}(y)\bar{e}$ for some $y\in\mathcal{G}^{(0)}_{t(e)}$, $\gamma_2\xi\in\Xi_{xe}\cap\Xi_{ye}$, that is, if and only if $x=y$, $\gamma=xe\phi_{\bar{e}}(x)\bar{e}=x$ and $\xi\in Z(xe)$ for some $x\in\mathcal{G}^{(0)}_{t(e)}$. Thus, one has

$$(S_e\,S_e^*)(\gamma,\xi)=\begin{cases}1 & \text{if } (\gamma,\xi)\in\bigsqcup_{x\in\mathcal{G}^{(0)}_{t(e)}}\{x\}\times Z(xe),\\ 0 & \text{otherwise.}\end{cases} \tag{5.5.24}$$

In particular,

$$(S_{\bar{e}}\,S_{\bar{e}}^*)(\gamma,\xi)=\begin{cases}1 & \text{if } (\gamma,\xi)\in\bigsqcup_{x\in\mathcal{G}^{(0)}_{o(e)}}\{x\}\times Z(x\bar{e}),\\ 0 & \text{otherwise}\end{cases}. \tag{5.5.25}$$

Then (G3) follows by the combination of (5.5.22), (5.5.12) and (5.5.25).

Finally, we prove (G4). Let $f\in\Gamma_1$ and $h\in\mathcal{T}_f$. We put

$$S_{hf}:=U_{h,t(f)}\,S_f. \tag{5.5.26}$$

and prove that

$$S_e^*\,S_e=\sum_{\substack{t(f)=o(e),\,h\in\mathcal{T}_f\\ hf\neq x\bar{e},\,x\in\mathcal{G}^{(0)}_{o(e)}}} S_{hf}\,S_{hf}^*. \tag{5.5.27}$$

Let $(\gamma,\xi)\in\Pi_1(\mathcal{G}(\Gamma))\ltimes\partial Y_{\mathcal{G}(\Gamma)}$.

Step 1. One has

$$S_{hf}(\gamma,\xi)=\sum_{\substack{\gamma_1,\gamma_2\in\Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} U_{h,t(f)}(\gamma_1,\gamma_2\xi)\cdot S_f(\gamma_2,\xi). \tag{5.5.28}$$

Note that $U_{h,t(f)}(\gamma_1,\gamma_2\xi)\cdot S_f(\gamma_2,\xi)\neq 0$ if and only if $\gamma_1=h$, $\gamma_2\xi\in Z(s(h))$, $\gamma_2=xf$ for some $x\in\mathcal{G}^{(0)}_{t(f)}$ and $\xi\in\Xi_{xf}$; that is, if and only if $x=s(h)$, $\gamma=\gamma_1\gamma_2=hf$ and $\xi\in\Xi_{s(h)f}$. Thus, one has

$$\begin{aligned}S_{hf}(\gamma,\xi) &= \chi_{\{hf\}\times\Xi_{s(h)f}}\\ &= \begin{cases}1 & \text{if } (\gamma,\xi)\in\{hf\}\times\Xi_{s(h)f},\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{5.5.29}$$

Step 2. One has

$$S^*_{hf} = (S^*_f\, U^*_{h,t(f)})(\gamma,\xi)$$
$$= \sum_{\substack{\gamma_1,\gamma_2\in\pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} S^*_f(\gamma_1,\gamma_2\xi)\cdot U^*_{h,t(f)}(\gamma_2,\xi) \tag{5.5.30}$$
$$= \sum_{\substack{\gamma_1,\gamma_2\in\Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} S_f(\gamma_1^{-1},\gamma_1\gamma_2\xi)\cdot U_{h,t(f)}(\gamma_2^{-1},\gamma_2\xi).$$

Note that $S_f(\gamma_1^{-1},\gamma_1\gamma_2\xi)\cdot U_{h,t(f)}(\gamma_2^{-1},\gamma_2\xi)\neq 0$ if and only if $\gamma_1^{-1}=xf$ for some $x\in\mathcal{G}^{(0)}_{t(f)}$, $\gamma_1\gamma_2\xi\in\Xi_{xf}$, $\gamma_2^{-1}=h$ and $\gamma_2\xi\in Z(s(h))$. That is, if and only if $x=s(h)$, $\gamma_1=\phi_{\bar f}(s(h))\bar f s(h)$, $\gamma\xi\in\Xi_{s(h)f}$, $\gamma_2=h^{-1}$ and $\xi\in Z(r(h))$. Thus, $S_f(\gamma_1^{-1},\gamma_1\gamma_2\xi)\cdot U_{h,t(f)}(\gamma_2^{-1},\gamma_2\xi)\neq 0$ if and only if $\gamma=\phi_{\bar f}(s(h))\bar f h^{-1}$ and $\xi\in Z(hf)$.
Thus, one has

$$S^*_{hf}(\gamma,\xi) = \chi_{\{\phi_{\bar f}(s(h))\,\bar f\,h^{-1}\}\times Z(hf)} = \begin{cases}1 & \text{if } (\gamma,\xi)\in\{\phi_{\bar f}(s(h))\,\bar f\,h^{-1}\}\times Z(hf)\\ 0 & \text{otherwise.}\end{cases} \tag{5.5.31}$$

Step 3. One has

$$(S_{hf}\,S^*_{hf})(\gamma,\xi) = \sum_{\substack{\gamma_1,\gamma_2\in\Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} S_{hf}(\gamma_1,\gamma_2\xi)\cdot S^*_{hf}(\gamma_2,\xi). \tag{5.5.32}$$

Note that $S_{hf}(\gamma_1,\gamma_2\xi)\cdot S^*_{hf}(\gamma_2,\xi)\neq 0$ if and only if $\gamma_1=hf$, $\gamma_2\xi\in Z(\tau l)$ for some $(\tau,l)\in\Xi_{s(h)}$, $\gamma_2=\phi_{\bar f}(s(h))\,\bar f\,h^{-1}$, $\xi\in Z(hf)$. That is, $S_{hf}(\gamma_1,\gamma_2\xi)\cdot S^*_{hf}(\gamma_2,\xi)\neq 0$ if and only if $\gamma=\gamma_1\gamma_2=r(h)$ and $\xi\in Z(hf)$.
Thus, one has

$$(S_{hf}\,S^*_{hf})(\gamma,\xi) = \begin{cases}1 & \text{if } (\gamma,\xi)\in\{r(h)\}\times Z(hf)\\ 0 & \text{otherwise.}\end{cases} \tag{5.5.33}$$

Step 4. Finally, one has that

$$S^*_e\,S_e = \sum_{\substack{t(f)=o(e),\,h\in\mathcal{T}_f\\ hf\neq x\bar e,\,x\in\mathcal{G}^{(0)}_{o(e)}}} S_{hf}\,S^*_{hf} \tag{5.5.34}$$

by combining (5.5.12) and (5.5.33).

We conclude that the collection $\{\,U_{\cdot,v}, S_e \mid v\in\Gamma^0, e\in\Gamma^1\,\}$ satisfies (G1)-(G4) in Definition 5.4.2 and hence it is a $\mathcal{G}$-family in $C^*\big(\Pi_1(\mathcal{G}(\Gamma))\ltimes\partial Y_{\mathcal{G}(\Gamma)}\big)$.

□

The main purpose of this section is to prove the following theorem.

Theorem 5.5.3. *Let $\mathcal{G}(\Gamma)$ be a locally finite nonsingular graph of groupoids. Then there is an isomorphism*

$$\Phi\colon C^*(\mathcal{G})\longrightarrow C^*\big(\Pi_1(\mathcal{G}(\Gamma))\ltimes\partial Y_{\mathcal{G}(\Gamma)}\big) \tag{5.5.35}$$

satisfying

$$u_{\cdot,v} \longmapsto U_{\cdot,v} \tag{5.5.36}$$
$$s_e \longmapsto S_e \tag{5.5.37}$$

for all $v \in \Gamma^0$, $e \in \Gamma^1$.

By Proposition 5.5.2 one has that $\{U_{\cdot,v}, S_e \mid v \in \Gamma^0, e \in \Gamma^1\}$ is a $\mathcal{G}$-family in $C^*\big(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}\big)$. Thus, using the universal property of $C^*(\mathcal{G})$ one has that there exists a unique *-homomorphism $\Phi\colon C^*(\mathcal{G}) \to C^*\big(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}\big)$ satisfying conditions (5.5.36) and (5.5.37). We need to prove that such Φ is an isomorphism.

Notation 5.5.4. For $\mu = g_1e_1\cdots g_ne_ng_{n+1} \in \Pi_1(\mathcal{G}(\Gamma))$, we put

$$S_\mu = U_{g_1,t(e_1)}\, S_{e_1}\, U_{g_2,t(e_2)}\, S_{e_2}\, \cdots\, U_{g_n,t(e_n)}\, S_{e_n}\, U_{g_{n+1},o(e_n)}.$$

Each S_μ is a partial isometry, because for each $i \in \{2,\dots,n\}$, the final projection of $U_{g_i,t(e_i)}\, S_{e_i}$ is a subprojection of the initial projection of $U_{g_{i-1},t(e_{i-1})}\, S_{e_{i-1}}$ by (G4).

The following lemma allows us to pull partial unitary representations of a vertex group along finite paths whose range or source is that vertex in a similar way to how (G2) works on an edge.

Lemma 5.5.5. *Let* $\mu = g_1e_1g_2\cdots g_ne_ng_{n+1}$ *and let* $a \in \mathcal{G}_{t(e_1)}$ *such that* $s(a) = r(g_1)$. *Then there exist a unique* $b \in \mathcal{G}_{o(e_n)}$ *and* $\mu' = g_1'e_1g_2'\cdots g_n'e_ng_{n+1}'$ *such that*

$$u_{a,t(e_1)}\, s_\mu = s_{\mu'}\, u_{b,o(e_n)}. \tag{5.5.38}$$

Proof. If $n = 1$, i.e., $\mu = g_1e_1g_2$, then one has

$$u_{a,t(e_1)}\, s_\mu = u_{a,t(e_1)}u_{g_1,t(e_1)}s_{e_1}u_{g_2,o(e_1)} = u_{ag_1,t(e_1)}s_{e_1}u_{g_2,o(e_1)}. \tag{5.5.39}$$

Since $ag_1 \in \mathcal{G}_{t(e_1)}$, there exist a unique $g_1' \in \mathcal{T}_{e_1}$ and $h_1 \in \alpha_{e_1}(\mathcal{G}_{e_1})$ such that $ag_1 = g_1'h_1$. Put $h_1' = \phi_{\bar{e}_1}(h_1)$. Then using (G2) and (5.5.39) one has

$$u_{a,t(e_1)}\, s_\mu = u_{g_1'h_1,t(e_1)}s_{e_1}u_{g_2,o(e_1)} = u_{g_1',t(e_1)}s_{e_1}u_{h_1'g_2,o(e_1)}. \tag{5.5.40}$$

If $n > 1$, then repeating this process we get the desired b and μ'. □

Lemma 5.5.6. *For* $\mu, \nu \in \Pi_1(\mathcal{G}(\Gamma))$, *one has*

$$s_\mu^* s_\nu = \begin{cases} s_{\mu'}^* & \text{if } \mu = \nu\mu', \\ s_{\nu'} & \text{if } \nu = \mu\nu', \\ u_{g_{n+1},o(e_n)}^* s_{e_n}^* s_{e_n} u_{g_{n+1},o(e_n)} & \text{if } \nu = \mu, \\ 0 & \text{otherwise}. \end{cases} \tag{5.5.41}$$

Proof. Let $\mu = g_1e_1\cdots g_ne_ng_{n+1}$ and $\nu = g_1'e_1'\cdots g_m'e_m'g_{m+1}'$. Then one has

$$\begin{aligned} s_\mu^* s_\nu &= \big(u_{g_1,t(e_1)}s_{e_1}\cdots s_{e_n}u_{g_{n+1},o(e_n)}\big)^* u_{g_1',t(e_1')}s_{e_1'}\cdots s_{e_m'}u_{g_{m+1}',o(e_m')} \\ &= u_{g_{n+1},o(e_n)}^* s_{e_n}^* \cdots s_{e_1}^* u_{g_1,t(e_1)}^* u_{g_1',t(e_1')}s_{e_1'}\cdots s_{e_m'}u_{g_{m+1}',o(e_m')}. \end{aligned} \tag{5.5.42}$$

Note that $u_{g_1,t(e_1)}^* u_{g_1',t(e_1')} \neq 0$ if and only if $t(e_1') = t(e_1)$ and $r(g_1) = r(g_1')$. In that case, we put $g_1^{-1}g_1' = \gamma \in \mathcal{G}_{t(e_1)}$ so that

$$u_{g_1,t(e_1)}^*\, u_{g_1',t(e_1')} = u_{g_1^{-1},t(e_1)}\, u_{g_1',t(e_1')} = u_{\gamma,t(e_1)}. \tag{5.5.43}$$

Hence, we have

$$s_\mu^* s_\nu = u_{g_{n+1},o(e_n)}^* s_{e_n}^* \cdots s_{e_1}^* u_{\gamma,t(e_1)} s_{e_1'} u_{g_2',t(e_2')} \cdots s_{e_m'} u_{g_{m+1}',o(e_m')}. \tag{5.5.44}$$

By Lemma 5.4.8 (i), we have that $s_{e_1}^* u_{\gamma,t(e_1)} s_{e_1'} \neq 0$ if and only if $\gamma e_1' = x e_1$ for some $x \in \mathcal{G}_{t(e_1)}^{(0)}$, i.e., if $g_1 = g_1'$. In that case, one has $s_{e_1}^* u_{x,t(e_1)} s_{e_1} = s_{e_1}^* s_{e_1} u_{y,o(e_1)}$, where $y = \phi_{\bar{e}_1}(x)$. Moreover, $u_{y,o(e_1)} u_{g_2',t(e_2')} \neq 0$ if and only if $o(e_1) = t(e_2')$ and $y = r(g_2')$. In that case one has $u_{y,o(e_1)} u_{g_2',t(e_2')} = u_{g_2',t(e_2')}$. Then one has

$$s_\mu^* s_\nu = u_{g_{n+1},o(e_n)}^* s_{e_n}^* \cdots s_{e_1}^* s_{e_1} u_{g_2',t(e_2')} s_{e_2'} \cdots s_{e_m'} u_{g_{m+1}',o(e_m')}. \tag{5.5.45}$$

By Lemma 5.4.8 (ii), we have that $s_{e_1}^* s_{e_1} u_{g_2',t(e_2')} s_{e_2'} \neq 0$ if and only if $o(e_1) = t(e_2')$ and $g_2' e_2' \neq y \bar{e}_1$, $y \in \mathcal{G}_{o(e)}^{(0)}$. In that case, one has that $s_{e_1}^* s_{e_1} u_{g_2',t(e_2')} s_{e_2'} = u_{g_2',t(e_2')} s_{e_2'}$. Hence one has

$$s_\mu^* s_\nu = u_{g_{n+1},o(e_n)}^* s_{e_n}^* \cdots u_{g_2,t(e_2)}^* u_{g_2',t(e_2')} s_{e_2'} \cdots s_{e_m'} u_{g_{m+1}',o(e_m')}. \tag{5.5.46}$$

This string has the same form as the one in (5.5.42), and so again we are forced to have $t(e_2') = t(e_2)$ and $r(g_2) = r(g_2')$ for this string to be non-zero. Assuming that $n < m$ and repeating this process $n-1$ times, μ will be forced to be a subpath of ν in order to get a non-zero string. In this case, we write $\nu = \mu\nu'$, where $\nu' \in \Pi_1(\mathcal{G}(\Gamma))$ is the extension of μ, and get

$$s_\mu^* s_\nu = s_\mu^* s_\mu s_{\nu'} = s_{\nu'}. \tag{5.5.47}$$

If $m < n$ and $\mu = \nu\mu'$, with a similar argument we obtain

$$s_\mu^* s_\nu = s_{\mu'}^* s_\nu^* s_\nu = s_{\mu'}^*. \tag{5.5.48}$$

If $\mu = \nu$, then one has

$$\begin{aligned}
s_\mu^* s_\mu &= u_{g_{n+1},o(e_n)}^* s_{e_n}^* \cdots u_{g_2,t(e_2)}^* s_{e_1}^* u_{g_1,t(e_1)}^* u_{g_1,t(e_1)} s_{e_1} u_{g_2,t(e_2)} \cdots s_{e_n} u_{g_{n+1},o(e_n)} \\
&= u_{g_{n+1},o(e_n)}^* s_{e_n}^* \cdots u_{g_2,t(e_2)}^* s_{e_1}^* u_{s(g_1),t(e_1)} s_{e_1} u_{g_2,t(e_2)} \cdots s_{e_n} u_{g_{n+1},o(e_n)} \\
&= u_{g_{n+1},o(e_n)}^* s_{e_n}^* \cdots u_{g_2,t(e_2)}^* s_{e_1}^* s_{e_1} u_{g_2,t(e_2)} s_{e_2} \cdots s_{e_n} u_{g_{n+1},o(e_n)} \\
&= u_{g_{n+1},o(e_n)}^* s_{e_n}^* \cdots s_{e_2}^* u_{g_2,t(e_2)}^* u_{g_2,t(e_2)} s_{e_2} \cdots s_{e_n} u_{g_{n+1},o(e_n)} \\
&= u_{g_{n+1},o(e_n)}^* s_{e_n}^* \cdots u_{g_3,t(e_3)}^* s_{e_2}^* s_{e_2} u_{g_3,t(e_3)} \cdots s_{e_n} u_{g_{n+1},o(e_n)} \\
&\vdots \\
&= u_{g_{n+1},o(e_n)}^* s_{e_n}^* s_{e_n} u_{g_{n+1},o(e_n)}.
\end{aligned}$$

□

Proposition 5.5.7. *Let $\{u,s\}$ be a $\mathcal{G}$-family in a C^*-algebra A and let $\mathcal{T}$ be a transversal for $\mathcal{G}(\Gamma)$. We put*

$$C^*(\{u,s\}) := C^*\Big(\{\, s_e, u_{g,t(e)} \mid e \in \Gamma^1, g \in \mathcal{T}_e \,\}\Big).$$

and

$$\begin{aligned}
B = \overline{\operatorname{span}}\{\, s_\mu\, u_{g,v}\, s_\nu^* \quad | \quad & \mu,\nu \in \Pi_1(\mathcal{G}(\Gamma)),\ g \in \mathcal{G}_v, \\
& o(\mu) = v = o(\nu),\ s(\mu) = r(g),\ s(g) = s(\nu)\,\}.
\end{aligned}$$

Then one has

$$C^*(\{u,s\}) = B. \tag{5.5.49}$$

Proof. Clearly, $s_\mu u_{g,v} s_\nu^* \in C^*(\{u,s\})$ for each μ, ν, g and v satisfing the conditions in (5.5.49). Since B is in particular a closed subspace of $C^*(\{u,s\})$ it follows that $B \subseteq C^*(\{u,s\})$. For the reverse inclusion, we first observe that

$$u_{g,v} = u_{r(g),v}\, u_{g,v}\, u^*_{s(g),v} \tag{5.5.50}$$

and

$$s_e = \sum_{x \in \mathcal{G}^{(0)}_{o(e)}} s_e u_{x,o(e)} = \sum_{x \in \mathcal{G}^{(0)}_{o(e)}} u_{\phi_e(x),t(e)} s_e u^*_{x,o(e)}, \tag{5.5.51}$$

where the second equality is given by Lemma 5.4.7(ii). Hence, it suffices to show that B is a C^*-algebra. Since A is a C^*-algebra, for any subset Y of A which is closed under multiplication and involution we have $C^*(Y) = \overline{\operatorname{span}}(Y)$, it suffices to show that B is closed under multiplication and involution. Clearly, B is closed under involution. It remains to prove that it is closed under multiplication. Let $\eta, \mu, \nu, \zeta \in \Pi_1(\mathcal{G}(\Gamma))$, $\mu = g_1 e_1 \cdots g_n e_n g_{n+1}$, $\nu = g'_1 e'_1 \cdots g'_m e'_m g'_{m+1}$ satisfying

$$o(\eta) = o(\mu), \quad o(\nu) = o(\zeta). \tag{5.5.52}$$

Let $a \in \mathcal{G}_{o(\mu)}$, $b \in \mathcal{G}_{o(\nu)}$ and suppose that $\nu = \mu\nu'$. Then one has

$$s_\eta\, u_{a,o(\mu)}\, s_\mu^*\, s_\nu\, u_{b,o(\nu)}\, s_\zeta^* = s_\eta\, u_{a,t(\nu')}\, s_{\nu'}\, u_{b,o(\nu)}\, s_\zeta^* \tag{5.5.53}$$

by Lemma 5.5.6. Moreover, by Lemma 5.5.5 one has that $u_{a,t(\nu')}\, s_{\nu'} = s_{\nu''}\, u_{c,o(\nu)}$ for some $c \in \mathcal{G}_{o(\nu)}$ and $\nu'' = c_1 e'_1 \cdots c_m e'_m c_{m+1}$. Then one has

$$\begin{aligned} s_\eta\, u_{a,o(e_n)}\, s_\mu^*\, s_\nu\, u_{b,o(\nu)}\, s_\zeta^* &= s_\eta\, s_{\nu''}\, u_{c,o(\nu)}\, u_{b,o(\nu)}\, s_\zeta^* \\ &= s_{\eta\nu''}\, u_{cb,o(\nu)}\, s_\zeta^*. \end{aligned} \tag{5.5.54}$$

Since $cb \in \mathcal{G}_{o(\nu)} = \mathcal{G}_{o(e'_m)}$, there exist $d \in \mathcal{T}_{e'_m}$ and $h \in \operatorname{im}(\alpha_{\bar{e}'_m})$ such that $cb = dh$. Thus, one has $u_{cb,o(\nu)}\, s_\zeta^* = u_{d,o(\nu)} u_{h,o(\nu)}\, s_\zeta^*$. Since $u_{h,o(\nu)} = u^*_{h^{-1},o(\nu)}$ and $o(\zeta) = o(\nu)$ by hypothesis, one has that $u_{h,o(\nu)}\, s_\zeta^* = u^*_{h^{-1},o(\zeta)} s_\zeta^* = (s_\zeta u_{h^{-1},o(\zeta)})^* = s^*_{\zeta'}$. Thus one has

$$s_\eta\, u_{a,o(e_n)}\, s_\mu^*\, s_\nu\, u_{b,o(\nu)}\, s_\zeta^* = s_{\eta\nu''}\, u_{d,o(\nu)}\, s^*_{\zeta'}, \tag{5.5.55}$$

which is in B. The argument is almost identical if $\mu = \nu\mu'$.

Finally, suppose that $\nu = \mu$. Then one has

$$s_\eta\, u_{a,o(\mu)}\, s_\mu^*\, s_\nu\, u_{b,o(\nu)}\, s_\zeta^* = s_\eta\, u_{a,o(\mu)} u^*_{g_{n+1},o(\mu)} s^*_{e_n} s_{e_n} u_{g_{n+1},o(\mu)} u_{b,o(\mu)}\, s_\zeta^* \tag{5.5.56}$$

by Lemma 5.5.6. Since $u^*_{g_{n+1},o(\mu)} = u_{g^{-1}_{n+1},o(\mu)}$, one has that (5.5.56) is non-zero if and only if $s(a) = s(g_{n+1}) = r(b)$. In this case, we put $c = a g^{-1}_{n+1}$ and $d = g_{n+1} b$. Then one has

$$\begin{aligned} s_\eta\, u_{a,o(\mu)}\, s_\mu^*\, s_\nu\, u_{b,o(\nu)}\, s_\zeta^* &= s_\eta\, u_{c,o(\mu)} s^*_{e_n} s_{e_n} u_{d,o(\mu)}\, s_\zeta^* \\ &= s_\eta\, u_{c,o(\mu)} \Big(\sum_{x \in \mathcal{G}_{o(e_n)}} u_{x,o(e_n)} - s_{\bar{e}_n} s^*_{\bar{e}_n} \Big) u_{d,o(\mu)}\, s_\zeta^* \\ &= \sum_{x \in \mathcal{G}_{o(e_n)}} s_\eta\, u_{c,o(\mu)}\, u_{x,o(e_n)}\, u_{d,o(\mu)}\, s_\zeta^* - s_\eta\, u_{c,o(\mu)}\, s_{\bar{e}_n} s^*_{\bar{e}_n}\, u_{d,o(\mu)}\, s_\zeta^* \\ &= s_\eta\, u_{cd,o(e_n)}\, s_\zeta^* - s_\eta u_{c,o(\mu)}\, s_{\bar{e}_n} u_{\phi_{e_n}(s(c)),t(e_n)}\, s^*_{\bar{e}_n}\, u^*_{d^{-1},o(\mu)}\, s_\zeta^* \\ &= s_\eta\, u_{h,o(e_n)}\, s_\zeta^* - s_{\eta'} u_{y,t(e_n)} s^*_{\zeta'}, \end{aligned}$$

where we put $h = cd$, $y = \phi_{e_n}(s(c))$, $\eta' = \eta\, c\, \bar{e}_n y$ and $\zeta' = \zeta d^{-1}\bar{e}_n y$. Again, we can pull $u_{h,o(\mu)} = u_{h,o(e_n)}$ through s^*_ζ to get $s_\eta\, u_{h,o(e_n)}\, s^*_\zeta = s_\eta\, u_{h',o(e_n)}\, s^*_{\zeta''} \in B$, with $h' \in \mathcal{T}_{\bar{e}_n}$. Similarly, we can get $s_{\eta'} u_{y,t(e_n)} s^*_{\zeta'} = s_{\eta'} u_{y',t(e_n)} s^*_{\zeta'''} \in B$. Hence, B is closed under multiplication and so $C^*(\{u, s\}) = B$. □

The following calculations will be useful for our purpose.

Lemma 5.5.8. *For $\{\mu\} \times Z(\nu)$, $\{\alpha\} \times Z(\beta) \in \Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$, one has*

$$\chi_{\{\mu\}\times Z(\nu)} \cdot \chi_{\{\alpha\}\times Z(\beta)} = \chi_{\{\mu\alpha\}\times \alpha^{-1}Z(\nu)\cap Z(\beta)}.$$

Proof. For $(\gamma, \xi) \in \Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$ one has

$$\chi_{\{\mu\}\times Z(\nu)} \cdot \chi_{\{\alpha\}\times Z(\beta)}(\gamma, \xi) = \sum_{\substack{\gamma_1,\gamma_2 \in \Pi_1(\mathcal{G}), \\ \gamma = \gamma_1\gamma_2}} \chi_{\{\mu\}\times Z(\nu)}(\gamma_1, \gamma_2\xi) \cdot \chi_{\{\alpha\}\times Z(\beta)}(\gamma_2, \xi).$$

Note that $\chi_{\{\mu\}\times Z(\nu)}(\gamma_1, \gamma_2\xi) \cdot \chi_{\{\alpha\}\times Z(\beta)}(\gamma_2, \xi) \neq 0$ if and only if $\gamma_1 = \mu$, $\gamma_2\xi \in Z(\nu)$, $\gamma_2 = \alpha$ and $\xi \in Z(\beta)$, i.e., if and only if $\gamma = \mu\alpha$, $s(\mu) = r(\alpha)$, and $\xi \in \alpha^{-1}Z(\nu) \cap Z(\beta)$. □

Lemma 5.5.9. *For $\mu = g_1e_1g_2e_2 \dots g_ne_n \in \Pi_1(\mathcal{G}(\Gamma))$, one has*

$$S_\mu = \chi_{\{\mu\}\times \Xi_{s(g_n)e_n}}. \tag{5.5.57}$$

Proof. Let $g \in \mathcal{G}_v$, $v \in \Gamma^0$, and let $e \in \Gamma^1$ such that $t(e) = v$. Then for $(\gamma, \xi) \in \Pi_1(\mathcal{G}(\Gamma)) \rtimes \partial Y_{\mathcal{G}}$ one has

$$U_{g,v}\, S_e\, (\gamma, \xi) = \sum_{\substack{\gamma_1,\gamma_2 \in \Pi_1(\mathcal{G}), \\ \gamma_1\gamma_2 = \gamma}} U_{g,v}(\gamma_1, \gamma_2\xi) \cdot S_e(\gamma_2, \xi). \tag{5.5.58}$$

Note that $U_{g,v}(\gamma_1, \gamma_2\xi) \cdot S_e(\gamma_2, \xi) \neq 0$ if and only if $\gamma_1 = g$, $\gamma_2\xi \in Z(s(g))$, $\gamma_2 = s(g)e$ and $\xi \in \Xi_{s(g)e}$. That is, $U_{g,v}(\gamma_1, \gamma_2\xi) \cdot S_e(\gamma_2, \xi) \neq 0$ if and only if $\gamma = ge$ and $\xi \in \Xi_{s(g)e}$. Thus, one has

$$U_{g,v}\, S_e = \chi_{\{ge\}\times \Xi_{s(g)e}}. \tag{5.5.59}$$

Moreover, for $(\gamma, \xi) \in \Pi_1(\mathcal{G}(\Gamma)) \rtimes \partial Y_{\mathcal{G}(\Gamma)}$ one has

$$\begin{aligned} U_{g_1,t(e_1)} S_{e_1} U_{g_2,t(e_2)} S_{e_2}\, (\gamma, \xi) &= (U_{g_1,t(e_1)}\, S_{e_1}) \cdot (U_{g_2,t(e_2)}\, S_{e_2})\, (\gamma, \xi) \\ &= \sum_{\substack{\gamma_1,\gamma_2 \in \Pi_1(\mathcal{G}(\Gamma)), \\ \gamma_1\gamma_2 = \gamma}} \chi_{\{g_1e_1\}\times \Xi_{s(g_1)e_1}}(\gamma_1, \gamma_2\xi) \cdot \chi_{\{g_2e_2\}\times \Xi_{s(g_2)e_2}}(\gamma_2, \xi). \end{aligned}$$

Note that $\chi_{\{g_1e_1\}\times \Xi_{s(g_1)e_1}}(\gamma_1, \gamma_2\xi) \cdot \chi_{\{g_2e_2\}\times \Xi_{s(g_2)e_2}}(\gamma_2, \xi) \neq 0$ if and only if $\gamma_1 = g_1e_1$, $\gamma_2\xi \in \Xi_{s(g_1)e_1}$, $\gamma_2 = g_2e_2$ and $\xi \in \Xi_{s(g_2)e_2}$, i.e., if and only if $\gamma = g_1e_1g_2e_2$ and $\xi \in \Xi_{s(g_2)e_2}$. Thus, one has

$$U_{g_1,t(e_1)} S_{e_1} U_{g_2,t(e_2)} S_{e_2} = \chi_{\{g_1e_1g_2e_2\}\times \Xi_{s(g_2)e_2}}. \tag{5.5.60}$$

By iterating this argument, one proves the statement. □

Lemma 5.5.10. *For $\mu = g_1e_1g_2e_2 \dots g_ne_n \in \Pi_1(\mathcal{G}(\Gamma))$, one has*

$$S_\mu S_\mu^* = \chi_{\{r(\mu)\}\times Z(\mu)}. \tag{5.5.61}$$

Proof. For $(\gamma,\xi) \in \Pi_1(\mathcal{G}(\Gamma)) \rtimes \partial Y_{\mathcal{G}(\Gamma)}$ one has

$$\begin{aligned} S_\mu S_\mu^*(\gamma,\xi) &= \sum_{\substack{\gamma_1,\gamma_2\in\Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} S_\mu(\gamma_1,\gamma_2\xi)\cdot S_\mu^*(\gamma_2,\xi) \\ &= \sum_{\substack{\gamma_1,\gamma_2\in\Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} S_\mu(\gamma_1,\gamma_2\xi)\cdot S_\mu(\gamma_2^{-1},\gamma_2\xi) \end{aligned}$$

Note that $S_\mu(\gamma_1,\gamma_2\xi)\cdot S_\mu(\gamma_2^{-1},\gamma_2\xi) \neq 0$ if and only if $\gamma_1 = \mu$, $\gamma_2\xi \in \Xi_{s(g_n)e_n}$, $\gamma_2 = \mu^{-1}$ and $\gamma_2\xi \in \Xi_{s(g_n)e_n}$. That is, $S_\mu(\gamma_1,\gamma_2\xi)\cdot S_\mu(\gamma_2^{-1},\gamma_2\xi) \neq 0$ if and only if $\gamma = \mu\mu^{-1}$ and $\xi \in \mu\Xi_{s(g_n)e_n}$, i.e., if and only if $\gamma = r(\mu)$ and $\xi \in Z(\mu)$. Hence one has

(5.5.62) $$S_\mu S_\mu^*(\gamma,\xi) = \chi_{\{r(\mu)\}\times Z(\mu)}(\gamma,\xi),$$

which proves the statement. □

Lemma 5.5.11. *For $\mu = g_1e_1\dots g_ne_n$, $\nu = h_1f_1\dots h_mf_m \in \Pi_1(\mathcal{G}(\Gamma))$ such that $o(e_n) = o(f_m) = v$, and $g \in \mathcal{G}_v$ with $s(g) = \phi_{\bar{f}_m}(s(h_m))$ and $r(g) = \phi_{\bar{e}_n}(s(g_n))$, one has*

(5.5.63) $$S_\mu\, U_{g,v}\, S_\nu^* = \chi_{\{\mu g\nu^{-1}\}\times\nu g^{-1}\Xi_{s(g_n)e_n}}.$$

Proof. Let $(\gamma,\xi) \in \Pi_1(\mathcal{G}(\Gamma)) \rtimes \partial Y_{\mathcal{G}(\Gamma)}$. Then one has

(5.5.64) $$S_\mu U_{g,v}(\gamma,\xi) = \sum_{\substack{\gamma_1,\gamma_2\in\Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} \chi_{\{\mu\}\times\Xi_{s(g_n)e_n}}(\gamma_1,\gamma_2\xi)\cdot\chi_{\{g\}\times Z(s(g))}(\gamma_2,\xi).$$

Note that $\chi_{\{\mu\}\times\Xi_{s(g_n)e_n}}(\gamma_1,\gamma_2\xi)\cdot\chi_{\{g\}\times Z(s(g))}(\gamma_2,\xi) \neq 0$ if and only if $\gamma_1 = \mu$, $\gamma_2 = g$, $\gamma_2\xi \in \Xi_{s(g_n)e_n}$ and $\xi \in Z(s(g))$. That is, $\chi_{\{\mu\}\times\Xi_{s(g_n)e_n}}(\gamma_1,\gamma_2\xi)\cdot\chi_{\{g\}\times Z(s(g))}(\gamma_2,\xi) \neq 0$ if and only if $\gamma = \mu g$ and $\xi \in g^{-1}\Xi_{s(g_n)e_n}$. Thus one has

(5.5.65) $$S_\mu U_{g,v} = \chi_{\{\mu g\}\times g^{-1}\Xi_{s(g_n)e_n}}.$$

Hence one has

$$S_\mu U_{g,v} S_\nu^*(\gamma,\xi) = \sum_{\substack{\gamma_1,\gamma_2\in\Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2=\gamma}} \chi_{\{\mu g\}\times g^{-1}\Xi_{s(g_n)e_n}}(\gamma_1,\gamma_2\xi)\cdot\chi_{\{\nu\}\times\Xi_{s(h_m)f_m}}(\gamma_2^{-1},\gamma_2\xi).$$

Note that $\chi_{\{\mu g\}\times g^{-1}\Xi_{s(g_n)e_n}}(\gamma_1,\gamma_2\xi)\cdot\chi_{\{\nu\}\times\Xi_{s(h_m)f_m}}(\gamma_2^{-1},\gamma_2\xi) \neq 0$ if and only if $\gamma_1 = \mu g$, $\gamma_2\xi \in g^{-1}\Xi_{s(g_n)e_n}$, $\gamma_2^-1 = \nu$, $\gamma_2\xi \in \Xi_{s(h_m)f_m}$. That is, $\chi_{\{\mu g\}\times g^{-1}\Xi_{s(g_n)e_n}}(\gamma_1,\gamma_2\xi)\cdot \chi_{\{\nu\}\times\Xi_{s(h_m)f_m}}(\gamma_2^{-1},\gamma_2\xi) \neq 0$ if and only if $\gamma = \mu g\nu^{-1}$, $\xi \in Z(\nu)$ and $\nu^{-1}\xi \in g^{-1}\Xi_{s(g_n)e_n}$, i.e., if and only if $\gamma = \mu g\nu^{-1}$ and $\xi \in \nu g^{-1}\Xi_{s(g_n)e_n}$. Thus, one has

(5.5.66) $$S_\mu\, U_{g,v}\, S_\nu^* = \chi_{\{\mu g\nu^{-1}\}\times\nu g^{-1}\Xi_{s(g_n)e_n}}$$

□

One has the following property.

Proposition 5.5.12. *Let $\mu = g_1e_1\cdots g_ne_n \in \Pi_1(\mathcal{G}(\Gamma))$. For $\nu \in \Pi_1(\mathcal{G}(\Gamma))$ such that $o(\mu) = o(\nu) = v$ and for $g \in \mathcal{G}_v$, $v \in \Gamma^0$, one has that the subset*

(5.5.67) $$\{\mu g\nu^{-1}\}\times\nu g^{-1}\Xi_{s(g_n)e_n} \subseteq \Pi_1(\mathcal{G}(\Gamma)) \rtimes \partial Y_{\mathcal{G}(\Gamma)}$$

is a bisection.

Proof. Note that

$$\{\mu g\nu^{-1}\} \times \nu g^{-1}\Xi_{s(g_n)e_n} = \{\, (\mu g\nu^{-1}, \nu g^{-1}\xi) \mid r(\xi) = s(g_n), \xi \in \Xi_{s(g_n)e_n} \}.$$

For $\xi, \xi' \in \Xi_{s(g_n)e_n}$, one has that

$$\begin{aligned} & r(\mu g\nu^{-1}, \nu g^{-1}\xi) = r(\mu g\nu^{-1}, \nu g^{-1}\xi') \\ \Leftrightarrow\ & (r(\mu g\nu^{-1}), \mu g\nu^{-1} \cdot \nu g^{-1}\xi) = (r(\mu g\nu^{-1}), \mu g\nu^{-1} \cdot \nu g^{-1}\xi') \\ \Leftrightarrow\ & (r(\mu), \mu\xi) = (r(\mu), \mu\xi') \\ \Leftrightarrow\ & \xi = \xi', \end{aligned}$$

since $\mu\xi$ and $\mu\xi'$ are reduced. Thus, the range map r is injective on the subset $\{\mu g\nu^{-1}\} \times \nu g^{-1}\Xi_{s(g_n)e_n}$. Similarly, one has that the source map s is also injective on $\{\mu g\nu^{-1}\} \times \nu g^{-1}\Xi_{s(g_n)e_n}$. Hence, $\{\mu g\nu^{-1}\} \times \nu g^{-1}\Xi_{s(g_n)e_n}$ is a bisection. □

Proposition 5.5.13. *For $\mu = g_1e_1 \dots g_ne_n$, $\nu = h_1f_1 \dots h_mf_m \in \Pi_1(\mathcal{G}(\Gamma))$ such that $r(\nu) = s(\mu)$, one has*

$$S_\mu\, S_\nu\, S_\nu^* = \chi_{\{\mu\}\times Z(\nu)}.$$

Proof. Let $(\gamma, \xi) \in \Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$. Then one has

$$S_\mu S_\nu S_\nu^*(\gamma, \xi) = \sum_{\substack{\gamma_1,\gamma_2 \in \Pi_1(\mathcal{G}(\Gamma)),\\ \gamma_1\gamma_2 = \gamma}} \chi_{\{\mu\}\times \Xi_{s(g_n)e_n}}(\gamma_1, \gamma_2\xi) \cdot \chi_{\{r(\nu)\}\times Z(\nu)}(\gamma_2, \xi). \tag{5.5.68}$$

Note that $\chi_{\{\mu\}\times \Xi_{s(g_n)e_n}}(\gamma_1, \gamma_2\xi) \cdot \chi_{\{r(\nu)\}\times Z(\nu)}(\gamma_2, \xi) \neq 0$ if and only if $\gamma_1 = \mu$, $\gamma_2\xi \in \Xi_{s(g_n)e_n}$, $\gamma_2 = r(\nu) = s(\mu)$ and $\xi \in Z(\nu)$, i.e., if and only if $\gamma = \mu$ and $\xi \in Z(\nu)$. Hence one has

$$S_\mu\, S_\nu\, S_\nu^*(\gamma, \xi) = \chi_{\{\mu\}\times Z(\nu)}(\gamma, \xi). \tag{5.5.69}$$

□

To prove that the map $\Phi\colon C^*(\mathcal{G}) \to C^*(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$ defined in Theorem 5.5.3 is an isomorphism, we construct a homomorphism Ψ from $C^*(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$ to $C^*(\mathcal{G})$, which we will prove to be the inverse of Φ. In order to do this, we note that $\mathcal{C}_c(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$ is dense in $C^*(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$ and hence we construct a representation $\pi\colon \mathcal{C}_c(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}) \to C^*(\mathcal{G})$ and use Theorem 2.5.5 to obtain Ψ. We follow the ideas and techniques in [24, §4].

Remark 5.5.14. The support of any fixed $f \in \mathcal{C}_c(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$ is contained in a union of compact open bisections of the form $\{\mu\} \times Z(\nu)$, $\mu, \nu \in \Pi_1(\mathcal{G}(\Gamma))$, by Proposition 5.3.2. Moreover, these bisections are disjoint by Lemma 5.3.1. Thus, f is a finite sum of functions with support in some $\{\mu\} \times Z(\nu)$. Since each $\{\mu\} \times Z(\nu)$ is a bisection, i.e., r and s are both homeomorphisms on $\{\mu\} \times Z(\nu)$, the uniform norm on $\mathcal{C}\big(\{\mu\} \times Z(\nu)\big)$ dominates the I-norm (cf. (2.5.13)), and thus it is enough to approximate f in the uniform norm. Note that $\mathcal{C}\big(\{\mu\} \times Z(\nu)\big)$ is a C^*-algebra with the uniform norm and pointwise operations. Hence, we can use the Stone-Weierstrass Theorem to see that the *-subalgebra

$$\begin{aligned} \mathcal{C}\big(\{\mu\} \times Z(\nu)\big) \cap \operatorname{span}\{\, \chi_{\{\alpha\}\times Z(\beta)} \mid \alpha, \beta \in \Pi_1(\mathcal{G}(\Gamma)), s(\alpha) = r(\beta) \,\} \\ = \operatorname{span}\{\, \chi_{\{\mu\}\times Z(\nu\zeta)} \mid \zeta \in \Pi_1(\mathcal{G}(\Gamma)), r(\zeta) = s(\nu) \,\} \end{aligned}$$

is dense in $\mathcal{C}\big(\{\mu\} \times Z(\nu)\big)$ with respect to the uniform norm. In fact, it separates points in $\{\mu\} \times Z(\nu)$: let $(\mu, \nu\zeta)$, $(\mu, \nu\eta) \in \{\mu\} \times Z(\nu)$ such that $\zeta \neq \eta$. Since $\zeta, \eta \in \partial Y_{\mathcal{G}(\Gamma)}$, they are of the form

$$\begin{aligned} \zeta &= g_1 e_1 g_2 e_2 \cdots, \\ \eta &= h_1 f_1 h_2 f_2 \cdots. \end{aligned}$$

Since $\zeta \neq \eta$, there exists $n \in \mathbb{N}$ such that $g_1 e_1 \cdots g_n e_n \neq h_1 f_1 \cdots h_n f_n$. Put $\xi = g_1 e_1 \cdots g_n$. Then $\chi_{\{\mu\}\times Z(\nu\xi)} \in \operatorname{span}\{\, \chi_{\{\mu\}\times Z(\nu\zeta)} \mid \zeta \in \Pi_1(\mathcal{G}(\Gamma)), r(\zeta) = s(\nu) \,\}$ is such that

$$\begin{aligned} \chi_{\{\mu\}\times Z(\nu\xi)}(\mu, \nu\zeta) &= 1 \\ \chi_{\{\mu\}\times Z(\nu\xi)}(\mu, \nu\eta) &= 0. \end{aligned}$$

Lemma 5.5.15. *For any $\mu, \nu \in \Pi_1(\mathcal{G}(\Gamma))$, the map*

$$\begin{aligned} h_{\mu,\nu} \colon Z(\nu) &\to \{\mu\} \times Z(\nu) \\ h_{\mu,\nu}(\nu x) &= (\mu, \nu x) \end{aligned} \tag{5.5.70}$$

is a homeomorphism.

Proof. Let $\mu, \nu \in \Pi_1(\mathcal{G}(\Gamma))$. Clearly, $h_{\mu,\nu}$ is bijective. Since both its domain and codomain are compact, it remains to prove that it is continuous. One has that $h_{\mu,\nu}$ is continuous since the preimage of any open set $\{\mu\} \times Z(\nu\nu') \subseteq \{\mu\} \times Z(\nu)$ is the set $Z(\nu\nu')$, which is open in $Z(\nu)$. □

Remark 5.5.16. By the Gelfand-Neumark theorem, the homeomorphism $h_{\mu,\nu}$ induces an isomorphism of C^*-algebras

$$\begin{aligned} \phi_{\mu,\nu} \colon \mathcal{C}(Z(\nu)) &\to \mathcal{C}(\{\mu\} \times Z(\nu)) \\ \phi_{\mu,\nu}(f) &= f \circ h_{\mu,\nu}^{-1}. \end{aligned} \tag{5.5.71}$$

We need the following lemma to construct a representation from $\mathcal{C}(Z(\nu))$ to $C^*(\mathcal{G})$. The proof is analogous to the proof of Lemma 4.4 in [8].

Lemma 5.5.17. *Let $\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$ be as above and let $\nu \in \Pi_1(\mathcal{G}(\Gamma))$. Then $\mathcal{C}(Z(\nu))$ is the universal C^*-algebra generated by a family*

$$\{\, p_\zeta \mid \zeta = \nu\nu', \nu' \in \Pi_1(\mathcal{G}(\Gamma)),\ r(\nu') = s(\nu) \,\}$$

satisfying the relations

(1) the p_ζ are communting projections;

(2) for all $\zeta = \nu\nu'$ one has

$$p_\zeta = \sum_{\substack{f \in \Gamma^1,\, t(f) = o(\zeta) \\ \zeta f \text{ reduced}}} p_{\zeta f}.$$

Proof. Let A denote the universal C^*-algebra in the statement. Note that the characteristic functions

$$\{\,\chi_{Z(\zeta)} \mid \zeta = \nu\nu',\, \nu' \in \Pi_1(\mathcal{G}(\Gamma)),\, s(\nu) = r(\nu')\,\}$$

satisfy the relations (1) and (2). Then there exists a *-homomorphism $A \to \mathcal{C}(Z(\nu))$ such that $p_\zeta \mapsto \chi_{Z(\zeta)}$. Let V be te complex vector space with basis

$$\{\,w_\zeta \mid \zeta = \nu\nu',\, \nu' \in \Pi_1(\mathcal{G}(\Gamma)),\, s(\nu) = r(\nu')\,\}. \tag{5.5.72}$$

Define a linear map $L\colon V \to \mathcal{C}(Z(\nu))$ by $L(w_\zeta) = \chi_{Z(\zeta)}$. Then the image of L is a dense *-subalgebra in $\mathcal{C}(Z(\nu))$. Let

$$E = \Big\{\, w_\zeta - \sum_{\zeta f \text{ reduced}} w_{\zeta f} \mid \zeta = \nu\nu',\, \nu' \in \Pi_1(\mathcal{G}(\Gamma)),\, s(\nu) = r(\nu')\,\Big\}$$

and let $M = \operatorname{span} E$. Clearly, $M \subseteq \ker L$. We claim that $M = \ker L$. Let $z \in \ker L$,

$$z = \sum_{\zeta=\nu\nu'} c_\zeta\, w_\zeta,$$

where only finitely many c_ζ are nonzero. For $\mu = g_1e_1\cdots g_ke_k \in \Pi_1(\mathcal{G}(\Gamma))$, we denote the length of μ by $|\mu|$, i.e., $|\mu| = k$. Let $n = \max\{\,|\zeta| \mid c_\zeta \neq 0\,\}$. Let $\zeta = \nu\nu'$ with $|\zeta| < n$. Then one has

$$w_\zeta = \sum_{\zeta f \text{ reduced}} w_{\zeta f} + \Big(w_\zeta - \sum_{\zeta f \text{ reduced}} w_{\zeta f}\Big) \in \Bigg(\sum_{\zeta f \text{ reduced}} w_{\zeta f}\Bigg) + M. \tag{5.5.73}$$

Applying this argument inductively, we find that for $\zeta = \nu\nu'$ with $|\zeta| < n$, one has

$$w_\zeta \in \Bigg(\sum_{\substack{\beta=\zeta\eta,\\ |\beta|=n}} w_\beta\Bigg) + M.$$

Then one has

$$z = \sum_{\zeta=\nu\nu'} c_\zeta\, w_\zeta \in \Bigg(\sum_{\zeta=\nu\nu'} c_\zeta \sum_{\substack{\beta=\zeta\eta,\\ |\beta|=n}} w_\beta\Bigg) + M = \Bigg(\sum_{\substack{\beta=\nu\alpha,\\ |\beta|=n}} \Big(\sum_{\substack{\zeta=\nu\nu',\\ \beta=\zeta\zeta'}} c_\zeta\Big) w_\beta\Bigg) + M.$$

In particular, since $L(z) = 0$, one has that

$$L\Bigg(\sum_{\substack{\beta=\nu\alpha,\\ |\beta|=n}} \Big(\sum_{\substack{\zeta=\nu\nu',\\ \beta=\zeta\zeta'}} c_\zeta\Big) w_\beta\Bigg) = \sum_{\substack{\beta=\nu\alpha,\\ |\beta|=n}} \Big(\sum_{\substack{\zeta=\nu\nu',\\ \beta=\zeta\zeta'}} c_\zeta\Big) \chi_{Z(\beta)} = 0.$$

Since the sets $Z(\beta)$ for $|\beta| = n$ are pairwise disjoint, it follows that for each $\beta = \nu\alpha$ one has

$$\sum_{\zeta=\nu\nu',\, \beta=\zeta\zeta'} c_\zeta = 0.$$

Thus, $z \in M$, which proves that $M = \ker L$. Hence there is a linear isomorphism

$$L_0\colon V/M \to \operatorname{span}\{\,\chi_{Z(\zeta)} \mid \zeta = \nu\nu'\,\}.$$

By the universal property of V, there exists a linear map $K\colon V/M \to A$ defined by

$$K(w_\zeta + M) = p_\zeta.$$

Then one has that $\varphi_0 := K \circ L_0^{-1}\colon \operatorname{span}\{\, \chi_{Z(\zeta)} \mid \zeta = \nu\nu' \,\} \to A$ is a linear map. Moreover, it is a *-homomorphism since these characteristic functions have the same multiplication relations as the corresponding generators of A. Since $\operatorname{span}\{\, \chi_{Z(\zeta)} \mid \zeta = \nu\nu' \,\}$ is an increasing union of finite-dimensional C^*-algebras, φ_0 extends to a *-homomorphism $\varphi\colon \mathcal{C}(Z(\nu)) \to A$, inverse to the canonical map of A onto $\mathcal{C}(Z(\nu))$. □

Remark 5.5.18. By the universal property of $\mathcal{C}(Z(\nu))$, there exists a representation

$$\begin{aligned} \pi_\nu\colon \mathcal{C}(Z(\nu)) &\to C^*(\mathcal{G}) \\ \pi_\nu(\chi_{Z(\alpha)}) &= s_\alpha s_\alpha^*, \end{aligned} \tag{5.5.74}$$

where $\alpha = \nu\nu'$ for some $\nu' \in \partial Y_{\mathcal{G}(\Gamma)}$ with $r(\nu') = s(\nu)$. The representation π_ν immediately gives a map

$$\begin{aligned} \pi_{\mu,\nu}\colon \mathcal{C}(\{\mu\} \times Z(\nu)) &\to C^*(\mathcal{G}) \\ \pi_{\mu,\nu}(f) &= s_\mu\, \pi_\nu\big(\phi_{\mu,\nu}^{-1}(f)\big) \end{aligned} \tag{5.5.75}$$

for any $\mu \in \Pi_1(\mathcal{G}(\Gamma))$. Note that

$$\begin{aligned} \pi_{\mu,\nu}(\chi_{\{\mu\}\times Z(\nu)}) &= s_\mu\, \pi_\nu\big(\phi_{\mu,\nu}^{-1}(\chi_{\{\mu\}\times Z(\nu)})\big) \\ &= s_\mu\, \pi_\nu(\chi_{Z(\nu)}) \\ &= s_\mu s_\nu s_\nu^*. \end{aligned} \tag{5.5.76}$$

We aim to define $\Psi_0\colon \mathcal{C}_c(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}) \to C^*(\mathcal{G})$ as follows. For any $f \in \mathcal{C}_c(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$, one has that $\operatorname{supp}(f)$ is contained in a disjoint union of basic bisections, i.e., $\operatorname{supp}(f) \subseteq \bigsqcup_{i=1}^n \{\mu_i\} \times Z(\nu_i)$. Then we put

$$\Psi_0(f) = \sum_{i=1}^n \pi_{\mu_i,\nu_i}\big(\, f|_{\{\mu_i\}\times Z(\nu_i)} \,\big). \tag{5.5.77}$$

Since there is more than one way of writing a compact set as the union of bisections, we need to check that the representation Ψ_0 is consistent. We need the following technical lemma. The proof is similar to the one of Lemma 4.4 in [24].

Lemma 5.5.19. *Let $f \in \mathcal{C}(\{\mu\} \times Z(\nu))$. For any $k \geq 1$, one has*

$$\pi_\nu\big(\phi_{\mu,\nu}^{-1}(f)\big) = \sum_{\substack{\zeta\in\pi_1(\mathcal{G}(\Gamma)),\, |\zeta|=k, \\ \zeta=\nu\nu'}} \pi_\zeta\big(\phi_{\mu,\zeta}^{-1}(f|_{\{\mu\}\times Z(\zeta)})\big) \tag{5.5.78}$$

Proof. Both sides of (5.5.78) are continuous and linear in f. Thus, we may choose $f = \phi_{\mu,\nu}(\chi_{Z(\alpha)}) = \chi_{\{\mu\}\times Z(\alpha)}$, for some $\alpha \in \Pi_1(\mathcal{G}(\Gamma))$ such that $\alpha = \nu\nu'$. Then the left-hand side of (5.5.78) is

$$\pi_\nu\big(\phi_{\mu,\nu}^{-1}(\chi_{\{\mu\}\times Z(\alpha)})\big) = \pi_\nu(\chi_{Z(\alpha)}) = s_\alpha s_\alpha^*. \tag{5.5.79}$$

If $k < |\alpha|$, then the only non-zero summand on the right-hand side of (5.5.78) occurs when ζ is a subword of α, i.e., when $\alpha = \zeta\zeta'$. In this case, the right-hand side of (5.5.78) becomes

$$\begin{aligned}\pi_\zeta\big(\phi_{\mu,\zeta}^{-1}(\chi_{\{\mu\}\times Z(\zeta\zeta')})\big) &= \pi_\zeta(\chi_{Z(\zeta\zeta')}) \\ &= s_{\zeta\zeta'}\, s_{\zeta\zeta'}^* \\ &= s_\alpha\, s_\alpha^*.\end{aligned}$$

If $k \geq |\alpha|$, then the non-zero summands occur when $\zeta = \alpha\alpha'$ for some $\alpha' \in \Pi_1(\mathcal{G}(\Gamma))$ with $r(\alpha') = s(\alpha)$. In this case, the right-hand side of (5.5.78) becomes

$$\begin{aligned}\sum_{\substack{\alpha'\in\pi_1(\mathcal{G}(\Gamma)),r(\alpha')=s(\alpha)\\|\alpha'|=k-|\alpha|}} &\pi_{\alpha\alpha'}\big(\phi_{\mu,\alpha\alpha'}^{-1}(\chi_{\{\mu\}\times Z(\alpha\alpha')})\big) \\ &= \sum_{\substack{\alpha'\in\pi_1(\mathcal{G}(\Gamma)),r(\alpha')=s(\alpha)\\|\alpha'|=k-|\alpha|}} \pi_{\alpha\alpha'}(\chi_{Z(\alpha\alpha')}) \\ &= \sum_{\substack{\alpha'\in\pi_1(\mathcal{G}(\Gamma)),r(\alpha')=s(\alpha)\\|\alpha'|=k-|\alpha|}} s_{\alpha\alpha'}\, s_{\alpha\alpha'}^* \\ &= \sum_{\substack{\alpha'\in\pi_1(\mathcal{G}(\Gamma)),r(\alpha')=s(\alpha)\\|\alpha'|=k-|\alpha|}} s_\alpha\, s_{\alpha'}\, s_{\alpha'}^* s_\alpha^* \\ &= s_\alpha\left(\sum_{\substack{\alpha'\in\pi_1(\mathcal{G}(\Gamma)),r(\alpha')=s(\alpha)\\|\alpha'|=k-|\alpha|}} s_{\alpha'}\, s_{\alpha'}^*\right) s_\alpha^* \\ &= s_\alpha\, s_\alpha^*,\end{aligned}$$

where the last equality is given by Lemma 5.4.11. Thus, one has that in both cases the equality in (5.5.78) holds, which proves the thesis. □

The proof of the following proposition is similar to the one of Lemma 4.5 in [24].

Proposition 5.5.20. *Let $\mathcal{G}$ and $\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$ be as above. Let $f \in \mathcal{C}_c(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$, $\operatorname{supp}(f) \subseteq \bigsqcup_{i=1}^n \{\mu_i\} \times Z(\nu_i)$ for $\mu_i, \nu_i \in \Pi_1(\mathcal{G}(\Gamma))$, $i = 1, \ldots, n$ and $n \in \mathbb{N}$. Then the map*

$$\begin{aligned}\Psi_0 \colon \mathcal{C}_c(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}) &\to C^*(\mathcal{G}) \\ \Psi_0(f) &= \sum_{i=1}^n \pi_{\mu_i,\nu_i}\big(f|_{\{\mu_i\}\times Z(\nu_i)}\big)\end{aligned} \tag{5.5.80}$$

is a well-defined *-homomorphism.

Proof. We first prove that Ψ_0 is well-defined. Thus, it remains to prove that Ψ_0 is well-defined. Suppose that there exist $m \in \mathbb{N}$ and $\alpha_j, \beta_j \in \Pi_1(\mathcal{G}(\Gamma))$ such that $\operatorname{supp}(f) \subseteq \bigsqcup_{j=1}^m \{\alpha_j\} \times Z(\beta_j)$. Since $\{\mu_i\} \times Z(\nu_i) \cap \{\alpha_j\} \times Z(\beta_j) \neq \emptyset$ only when one set is contained in the other by Lemma 5.3.1, one has that either each $\{\mu_i\} \times Z(\nu_i)$ is contained in a disjoint union of $\{\alpha_j\} \times Z(\beta_j)$'s, or each $\{\alpha_j\} \times Z(\beta_j)$ is contained in a disjoint union of $\{\mu_i\} \times Z(\nu_i)$'s. Without loss of generality suppose that

$$\{\mu_i\} \times Z(\nu_i) \subseteq \bigsqcup_{k=1}^l \{\alpha_k\} \times Z(\beta_k), \tag{5.5.81}$$

$i \in \{1, \dots, n\}$. Then each $\{\alpha_k\} \times Z(\beta_k)$, $k \in \{1, \dots, l\}$, has the form

$$\{\alpha_k\} \times Z(\beta_k) = \{\mu_i\} \times Z(\nu_i \nu'_k), \tag{5.5.82}$$

for some $\nu'_k \in \Pi_1(\mathcal{G}(\Gamma))$, $r(\nu'_k) = s(\nu_i)$. Let $\ell = \max_{k=1,\dots,n} |\nu'_k|$. Consider the decomposition

$$\{\mu_i\} \times Z(\nu_i) = \bigsqcup_{\substack{\zeta \in \Pi_1(\mathcal{G}(\Gamma)), |\zeta| = \ell \\ s(\nu_i) = r(\zeta)}} \{\mu_i\} \times Z(\nu_i \zeta). \tag{5.5.83}$$

Then the ζ in (5.5.83) must group together in subsets

$$F_k = \{\, \nu'_k \eta \mid |\eta| = \ell - |\nu'_k| \,\},$$

$k \in \{1, \dots, l\}$, to form decompositions of $\{\alpha_k\} \times Z(\beta_k)$. Thus we can rewrite (5.5.83) as

$$\{\mu_i\} \times Z(\nu_i) = \bigsqcup_{k=1}^{l} \left(\bigsqcup_{\nu'_k \eta \in F_k} \{\mu_i\} \times Z(\nu_i \nu'_k \eta) \right) \tag{5.5.84}$$

$$= \bigsqcup_{k=1}^{l} \left(\bigsqcup_{\nu'_k \eta \in F_k} \{\mu_i\} \times Z(\beta_k \eta) \right). \tag{5.5.85}$$

Now we apply recursively Lemma 5.5.19 to get

$$\begin{aligned}
\pi_{\mu_i,\nu_i}(f|_{\{\mu_i\} \times Z(\nu_i)}) &= s_{\mu_i}\, \pi_{\nu_i}\big(\phi^{-1}_{\mu_i,\nu_i}(f|_{\{\mu_i\} \times Z(\nu_i)})\big) \\
&= \sum_{\substack{\zeta \in \Pi_1(\mathcal{G}(\Gamma)), |\zeta| = \ell \\ \zeta = \nu_i \nu'_i}} s_{\mu_i}\, \pi_\zeta\big(\phi^{-1}_{\mu_i,\zeta}(f|_{\{\mu_i\} \times Z(\zeta)})\big) \\
&= \sum_{k=1}^{l} \sum_{\nu'_k \eta \in F_k} s_{\alpha_k}\, \pi_{\beta_k \eta}\big(\phi^{-1}_{\alpha_k, \beta_k \eta}(f|_{\{\alpha_k\} \times Z(\beta_k \eta)})\big) \\
&= \sum_{k=1}^{l} s_{\alpha_k}\, \pi_{\beta_k}\big(\phi^{-1}_{\alpha_k,\beta_k}(f|_{\{\alpha_k\} \times Z(\beta_k)})\big) \\
&= \sum_{k=1}^{l} \pi_{\alpha_k,\beta_k}(f|_{\{\alpha_k\} \times Z(\beta_k)}).
\end{aligned}$$

By repeating the argument above to the decomposition of each $\{\mu_i\} \times Z(\nu_i)$, $i = 1, \dots, n$, in terms of the $\{\alpha_j\} \times Z(\beta_j)'s$, one shows that $\Psi_0(f)$ is independent on the choice of the cover of $\operatorname{supp} f$.

It remains to prove that Ψ_0 is a *-homomorphism. We first check it on characteristic functions. For $\mu, \nu \in \Pi_1(\mathcal{G}(\Gamma))$, one has that

$$\begin{aligned}
\Psi_0(\chi_{\{\mu\} \times Z(\nu)})^* &= \pi_{\mu,\nu}(\chi_{\{\mu\} \times Z(\nu)})^* \\
&= (s_\mu\, s_\nu\, s_\nu^*)^* \\
&= s_\nu\, s_\nu^*\, s_\mu^*
\end{aligned}$$

and

$$\begin{aligned}\Psi_0(\chi^*_{\{\mu\}\times Z(\nu)}) &= \Psi_0(\chi_{\{\mu^{-1}\}\times Z(\mu\nu)})\\ &= \pi_{\mu^{-1},\mu\nu}(\chi_{\{\mu^{-1}\}\times Z(\mu\nu)})\\ &= s_{\mu^{-1}}\, s_{\mu\nu}\, s^*_{\mu\nu}\\ &= s_{\mu^{-1}}\, s_\mu s_\nu s^*_\nu s^*_\mu\\ &= s_\nu s^*_\nu s^*_\mu.\end{aligned}$$

Thus, one has that $\Psi_0(\chi_{\{\mu\}\times Z(\nu)})^* = \Psi_0(\chi^*_{\{\mu\}\times Z(\nu)})$.

Let $\mu,\nu,\alpha,\beta \in \Pi_1(\mathcal{G}(\Gamma))$ such that $s(\mu) = r(\nu)$, $s(\alpha) = r(\beta)$ and $s(\mu) = r(\alpha)$. Note that $\chi_{\{\mu\}\times Z(\nu)} \cdot \chi_{\{\alpha\}\times Z(\beta)} = \chi_{\{\mu\alpha\}\times Z(\beta)\cap\alpha^{-1}Z(\nu)}$ by Lemma 5.5.8. Moreover, one has that $Z(\beta)\cap\alpha^{-1}Z(\nu) \neq \emptyset$ if and only if either $\nu = \alpha\beta\beta'$ or $\nu = \alpha\beta'$ and $\beta = \beta'\beta''$. Then one has that

$$\begin{aligned}\Psi_0\big(\chi_{\{\mu\}\times Z(\nu)} \cdot \chi_{\{\alpha\}\times Z(\beta)}\big) &= \Psi_0\big(\chi_{\{\mu\alpha\}\times Z(\beta)\cap\alpha^{-1}Z(\nu)}\big)\\ &= \begin{cases} s_{\mu\alpha}\, s_{\beta\beta'}\, s^*_{\beta\beta'} & \text{if } \nu = \alpha\beta\beta',\\ s_{\mu\alpha}\, s_\beta\, s^*_\beta & \text{if } \nu = \alpha\beta' \text{ and } \beta = \beta'\beta'',\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases} s_\mu\, s_\alpha\, s_\beta\, s_{\beta'}\, s^*_{\beta'}\, s^*_\beta & \text{if } \nu = \alpha\beta\beta',\\ s_\mu\, s_\alpha\, s_\beta\, s^*_\beta & \text{if } \nu = \alpha\beta' \text{ and } \beta = \beta'\beta'',\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

and

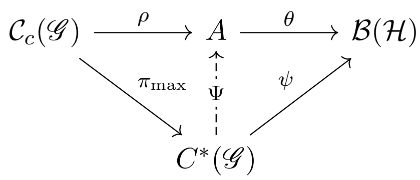
$$\begin{aligned}\Psi_0\big(\chi_{\{\mu\}\times Z(\nu)}\big)\cdot\Psi_0\big(\chi_{\{\alpha\}\times Z(\beta)}\big) &= s_\mu\, s_\nu\, s^*_\nu\, s_\alpha\, s_\beta\, s^*_\beta\\ &= \begin{cases} s_\mu\, s_{\alpha\beta\beta'}\, s^*_{\alpha\beta\beta'}\, s_\alpha\, s_\beta\, s^*_\beta & \text{if } \nu = \alpha\beta\beta',\\ s_\mu\, s_{\alpha\beta'}\, s^*_{\alpha\beta'}\, s_\alpha\, s_{\beta'\beta''}\, s^*_{\beta'\beta''} & \text{if } \nu = \alpha\beta' \text{ and } \beta = \beta'\beta'',\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases} s_\mu\, s_\alpha\, s_\beta\, s_{\beta'}\, s^*_{\beta'}\, s^*_\beta\, s^*_\alpha\, s_\alpha\, s_\beta\, s^*_\beta & \text{if } \nu = \alpha\beta\beta',\\ s_\mu\, s_\alpha\, s_{\beta'}\, s^*_{\beta'} s^*_\alpha\, s_\alpha\, s_{\beta'}\, s_{\beta''}\, s^*_{\beta''}\, s^*_{\beta'} & \text{if } \nu = \alpha\beta' \text{ and } \beta = \beta'\beta'',\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases} s_\mu\, s_\alpha\, s_\beta\, s_{\beta'}\, s^*_{\beta'}\, s^*_\beta & \text{if } \nu = \alpha\beta\beta',\\ s_\mu\, s_\alpha\, s_{\beta'}\, s_{\beta''}\, s^*_{\beta''}\, s^*_{\beta'} & \text{if } \nu = \alpha\beta' \text{ and } \beta = \beta'\beta'',\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases} s_\mu\, s_\alpha\, s_\beta\, s_{\beta'}\, s^*_{\beta'}\, s^*_\beta & \text{if } \nu = \alpha\beta\beta',\\ s_\mu\, s_\alpha\, s_\beta\, s^*_\beta & \text{if } \nu = \alpha\beta' \text{ and } \beta = \beta'\beta'',\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Thus, one has that $\Psi_0\big(\chi_{\{\mu\}\times Z(\nu)}\cdot\chi_{\{\alpha\}\times Z(\beta)}\big) = \Psi_0\big(\chi_{\{\mu\}\times Z(\nu)}\big)\cdot\Psi_0\big(\chi_{\{\alpha\}\times Z(\beta)}\big)$. Since Ψ_0 is linear and preserves multiplication and the adjoint on the characterstic functions, one concludes that Ψ_0 is a $*$-homomorphism by Remark 5.5.14. □

We now use Theorem 2.5.5 to get a *-homomorphism $\Psi\colon C^*(\Pi_1(\mathcal{G}(\Gamma))\ltimes\partial Y_{\mathcal{G}}) \to C^*(\mathcal{G})$.

Proposition 5.5.21. *Let $\mathscr{G}$ be a second-countable locally compact Hausdorff etale groupoid. Then there exists a C^*-algebra $C^*(\mathscr{G})$ and a *-homomorphism $\pi_{\max}\colon \mathcal{C}_c(\mathscr{G}) \to C^*(\mathscr{G})$ such that $\pi_{\max}(\mathcal{C}_c(\mathscr{G}))$ is dense in $C^*(\mathscr{G})$ and such that for every *-homomorphism $\rho\colon \mathcal{C}_c(\mathscr{G}) \to A$, where A is a C^*-algebra, there is a *-homomorphism $\Psi\colon C^*(\mathscr{G}) \to A$ such that $\Psi \circ \pi_{\max} = \rho$.*

Proof. Let A be a C^*-algebra and let $\rho\colon \mathcal{C}_c(\mathscr{G}) \to A$ be a *-homomorphism. By the Gelfand-Neumark theorem, there exists a Hilbert space $\mathcal{H}$ and a faithful *-homomorphism $\theta\colon A \to \mathcal{B}(\mathcal{H})$. Then one has that $\pi := \theta\circ\rho$ is a *-representation of $\mathcal{C}_c(\mathscr{G})$ on $\mathcal{H}$. Thus, by Theorem 2.5.5 there is a *-homomorphism $\psi\colon C^*(\mathscr{G}) \to \mathcal{B}(\mathcal{H})$ such that $\psi \circ \pi_{\max} = \pi$, where $\pi_{\max}$ is the *-homomorphism in Theorem 2.5.5. Hence one has the following diagram:



In order to define $\Psi := \theta^{-1} \circ \psi$, we must check that $\operatorname{im}\psi \subseteq \operatorname{im}(\theta)$. One has that

$$\operatorname{im}\psi = \psi\big(C^*(\mathscr{G})\big) = \psi\big(\overline{\pi_{\max}(\mathcal{C}_c(\mathscr{G}))}\big) \subseteq \overline{\psi\big(\pi_{\max}(\mathcal{C}_c(\mathscr{G}))\big)} \subseteq \overline{\theta\big(\rho(\mathcal{C}_c(\mathscr{G}))\big)} \subseteq \operatorname{im}\theta,$$

where the last equality is given by the fact that $\operatorname{im}\theta$ is closed in $\mathcal{B}(\mathcal{H})$. Thus, we put $\Psi := \theta^{-1} \circ \psi$. Then one has

$$\Psi \circ \pi_{\max} = \theta^{-1} \circ \psi \circ \pi_{\max} = \theta^{-1} \circ \pi = \theta^{-1} \circ \theta \circ \rho = \rho,$$

which concludes the proof. □

Remark 5.5.22. Consider $\mathscr{G}$ to be the action groupoid $\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$ and the C^*-algebra A to be the graph of groupoids C^*-algebra $C^*(\mathcal{G})$. Then, by Proposition 5.5.21 one has that the *-homomorphism $\Psi_0\colon \mathcal{C}_c(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}) \to C^*(\mathcal{G})$ defined in Proposition 5.5.20 induces a *-homomorphism $\Psi\colon C^*(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}) \to C^*(\mathcal{G})$ such that $\Psi \circ \pi_{\max} = \Psi_0$.

Finally, we prove that the homomorphism Φ defined in Theorem 5.5.3 is an isomorphism of C^*-algebras by proving that it is surjective and the homomorphism Ψ is its inverse. We begin by proving surjectivity.

Proposition 5.5.23. *Let $\mathcal{G}(\Gamma)$ be a locally finite nonsingular graph of groupoids. Then the map $\Phi\colon C^*(\mathcal{G}) \to C^*\big(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}\big)$ defined in Theorem 5.5.3 is surjective.*

Proof. We use Proposition 2.5.7 with $A = C^*(\mathcal{G})$ and $\mathcal{G} = \Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}$. Let $\{\mu\} \times Z(\nu) \subseteq C^*\big(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}\big)$ be an open bisection and let $\beta = (\mu, \nu\zeta)$, $\gamma = (\mu, \nu\eta) \in \{\mu\} \times Z(\nu)$, $\zeta, \eta \in \partial Y_{\mathcal{G}(\Gamma)}$, such that $\beta \neq \gamma$. Since $\zeta, \eta \in \partial Y_{\mathcal{G}(\Gamma)}$, they are of the form

$$\zeta = g_1 e_1 g_2 e_2 \cdots$$
$$\eta = h_1 f_1 h_2 f_2 \cdots$$

Since $\beta \neq \gamma$, it must be $\zeta \neq \eta$. Thus, there exists $n \in \mathbb{N}$ such that $g_1 e_1 \cdots g_n e_n \neq h_1 f_1 \cdots h_n f_n$. Put

$$\xi = g_1 e_1 \cdots g_n e_n \in \Pi_1(\mathcal{G}(\Gamma))$$
$$\xi' = h_1 e_1 \cdots h_n f_n \in \Pi_1(\mathcal{G}(\Gamma)).$$

Then one has $\beta = (\mu, \nu\xi\zeta')$ and $\gamma = (\mu, \nu\xi'\eta')$, where $\zeta', \eta' \in \partial Y_{\mathcal{G}(\Gamma)}$. Let $a = s_{\mu\nu\xi}\, u_{s(\xi),o(\xi)}\, s^*_{\mu\nu\xi'} \in C^*(\mathcal{G})$. Then one has

$$\Phi(a) = \chi_{\{\mu\nu\xi(\nu\xi)^{-1}\}\times Z(\nu\xi)} = \chi_{\{\mu\}\times Z(\nu\xi)}.$$

Hence one has

$$\begin{aligned}\Phi(a)(\beta) &= \chi_{\{\mu\}\times Z(\nu\xi)}(\mu, \nu\xi\zeta') = 1,\\ \Phi(a)(\gamma) &= \chi_{\{\mu\}\times Z(\nu\xi)}(\mu, \nu\xi'\eta') = 0.\end{aligned}$$

Thus, Φ is surjective by Proposition 2.5.7. □

It remains to show that Ψ is the inverse of Φ.

Lemma 5.5.24. *Let $\Phi\colon C^*(\mathcal{G}) \to C^*(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)})$ and $\Psi\colon C^*(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}) \to C^*(\mathcal{G})$ be the homomorphisms of C^*-algebras defined above. Then one has that $\Psi \circ \Phi = \mathrm{id}$.*

Proof. Since $C^*(\mathcal{G})$ is generated by the s_e's and $u_{g,v}$'s, $e \in \Gamma^1$, $v \in \Gamma^0$, $g \in \mathcal{G}_v$, it suffices to show that the statement holds on the generators. For $e \in \Gamma^1$, one has

$$\begin{aligned}
\Psi(\Phi(s_e)) &= \Psi(S_e)\\
&= \Psi_0(\chi_{Z_e})\\
&= \sum_{x\in\mathcal{G}^{(0)}_{t(e)}} \sum_{\substack{f\in\Gamma^1,\, t(f)=o(e),\\ g\in\mathcal{G}_{o(e)},\, r(g)=\phi_{\bar e}(x),\\ gf\neq\phi_{\bar e}(x)\bar e}} \Psi_0(\chi_{\{xe\}\times Z(gf)})\\
&= \sum_{x\in\mathcal{G}^{(0)}_{t(e)}} \sum_{\substack{f\in\Gamma^1,\, t(f)=o(e),\\ g\in\mathcal{G}_{o(e)},\, r(g)=\phi_{\bar e}(x),\\ gf\neq\phi_{\bar e}(x)\bar e}} \pi_{xe,gf}(\chi_{\{xe\}\times Z(gf)})\\
&= \sum_{x\in\mathcal{G}^{(0)}_{t(e)}} \sum_{\substack{f\in\Gamma^1,\, t(f)=o(e),\\ g\in\mathcal{G}_{o(e)},\, r(g)=\phi_{\bar e}(x),\\ gf\neq\phi_{\bar e}(x)\bar e}} s_{xe}\, s_{gf}\, s^*_{gf}\\
&= \sum_{x\in\mathcal{G}^{(0)}_{t(e)}} \sum_{\substack{f\in\Gamma^1,\, t(f)=o(e),\\ g\in\mathcal{G}_{o(e)},\, r(g)=\phi_{\bar e}(x),\\ gf\neq\phi_{\bar e}(x)\bar e}} u_{x,t(e)}\, s_e\, s_{gf}\, s^*_{gf}\\
&\overset{(G2)}{=} \sum_{x\in\mathcal{G}^{(0)}_{t(e)}} \sum_{\substack{f\in\Gamma^1,\, t(f)=o(e),\\ g\in\mathcal{G}_{o(e)},\, r(g)=\phi_{\bar e}(x),\\ gf\neq\phi_{\bar e}(x)\bar e}} s_e\, u_{\phi_{\bar e}(x),o(e)}\, s_{gf}\, s^*_{gf}\\
&= \sum_{x\in\mathcal{G}^{(0)}_{t(e)}} s_e \sum_{\substack{f\in\Gamma^1,\, t(f)=o(e),\\ g\in\mathcal{G}_{o(e)},\, r(g)=\phi_{\bar e}(x),\\ gf\neq\phi_{\bar e}(x)\bar e}} s_{gf}\, s^*_{gf}\\
&\overset{(G4)}{=} s_e s_e^* s_e\\
&= s_e.
\end{aligned}$$

Let $v \in \Gamma^0$, $g \in \mathcal{G}_v$. Then one has

$$\Psi(\Phi(u_{g,v})) = \Psi(U_{g,v})$$

$$
\begin{aligned}
&= \Psi_0\big(\chi_{\{g\}\times Z(s(g))}\big)\\
&= \Psi_0\left(\sum_{\substack{f\in\Gamma^1,\, t(f)=v,\\ h\in\mathcal{T}_f,\, r(h)=s(g)}} \chi_{\{g\}\times Z(hf)}\right)\\
&= \sum_{\substack{f\in\Gamma^1,\, t(f)=v,\\ h\in\mathcal{T}_f,\, r(h)=s(g)}} \pi_{g,hf}\big(\chi_{\{g\}\times Z(hf)}\big)\\
&= \sum_{\substack{f\in\Gamma^1,\, t(f)=v,\\ h\in\mathcal{T}_f,\, r(h)=s(g)}} u_{g,v}\, s_{hf}\, s_{hf}^*\\
&= u_{g,v} \sum_{\substack{f\in\Gamma^1,\, t(f)=v,\\ h\in\mathcal{T}_f,\, r(h)=s(g)}} s_{hf}\, s_{hf}^*\\
&\overset{\text{(G4)}}{=} u_{g,v}\left(s_e^*s_e + \sum_{x\in\mathcal{G}_{o(e)}^{(0)}} u_{x,o(e)}\, s_{\bar{e}}\, s_{\bar{e}}^* u_{x,o(e)}^*\right)\\
&\overset{\star}{=} u_{g,v}\,(s_e^*s_e + s_{\bar{e}}s_{\bar{e}}^*)\\
&\overset{\text{(G3)}}{=} u_{g,v} \sum_{x\in\mathcal{G}_v^{(0)}} u_{x,v}\\
&= u_{g,v}.
\end{aligned}
$$

where the equality $\star$ is given by Lemma 5.4.7 (ii). Hence, one has that $\Psi = \Phi^{-1}$. □

Remark 5.5.25. Proposition 5.5.23 together with Lemma 5.5.24 prove Theorem 5.5.3, i.e., one has that

$$C^*(\mathcal{G}) \cong C^*\big(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}\big).$$

Moreover, the action of the universal fundamental groupoid $\Pi_1(\mathcal{G}(\Gamma))$ on the locally compact space $\partial Y_{\mathcal{G}(\Gamma)}$ induces an action of $\Pi_1(\mathcal{G}(\Gamma))$ on $\mathcal{C}(\partial Y_{\mathcal{G}(\Gamma)})$ by

$$(p\cdot f)(\xi) = f(p^{-1}\cdot \xi), \quad p\in \Pi_1(\mathcal{G}(\Gamma)),\, \xi\in \partial Y_{\mathcal{G}(\Gamma)}$$

such that

$$\mathcal{C}(\partial Y_{\mathcal{G}(\Gamma)}) \rtimes \Pi_1(\mathcal{G}(\Gamma)) \cong C^*\big(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}\big),$$

where $\mathcal{C}(\partial Y_{\mathcal{G}(\Gamma)}) \rtimes \Pi_1(\mathcal{G}(\Gamma))$ is the crossed product given by the action of $\Pi_1(\mathcal{G}(\Gamma))$ on $\mathcal{C}(\partial Y_{\mathcal{G}(\Gamma)})$ (see [12, Example 5.4]). We refer the reader to [30] for a full account on groupoid crossed products. Hence one has the following isomorphisms of C^*-algebras:

$$C^*(\mathcal{G}) \;\cong\; C^*\big(\Pi_1(\mathcal{G}(\Gamma)) \ltimes \partial Y_{\mathcal{G}(\Gamma)}\big) \;\cong\; \mathcal{C}(\partial Y_{\mathcal{G}(\Gamma)}) \rtimes \Pi_1(\mathcal{G}(\Gamma)).$$

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