Università degli studi di Milano-Bicocca
Università degli studi di Pavia
Istituto Nazionale di Alta Matematica

Joint Ph.D. Program in Mathematics
XXXIII Cycle

Monotonicity formulas and blow-up methods for the study of spectral stability and fractional obstacle problems

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ABSTRACT

The present dissertation is essentially divided into two parts.

In the first part, we investigate questions of spectral stability for eigenvalue problems driven by the Laplace operator, under certain specific kinds of singular perturbations. More precisely, we start by considering the spectrum of the Laplacian on a fixed, bounded domain with prescribed homogeneous boundary conditions (of pure Dirichlet or Neumann type); then, we introduce a singular perturbation of the problem, which gives rise to a perturbed sequence of eigenvalues. Our goal is to understand the asymptotic behavior of the perturbed spectrum as long as the perturbation tends to disappear. In particular, we consider two different types of singular perturbation. On one hand, in the case of homogeneous Dirichlet boundary conditions, we consider a perturbation of the domain, which consists in attaching a thin cylindrical tube to the fixed limit domain and let its section shrink to a point. In this framework, we combine energy estimates coming from a tailor-made Almgren type monotonicity formula with the Courant-Fischer min-max characterization and then we perform a careful blow-up analysis for scaled eigenfunctions; with these ingredients, we identify the sharp rate of convergence of a perturbed eigenvalue in the case in which it is approaching a simple eigenvalue of the limit problem. On the other hand, we deal with a perturbation of the boundary conditions. More specifically, we start with the homogeneous Neumann eigenvalue problem for the Laplacian and we perturb it by prescribing zero Dirichlet boundary conditions on a small subset of the boundary. In this context, we describe the sharp asymptotic behavior of a perturbed eigenvalue when it is converging to a simple eigenvalue of the limit Neumann problem. In particular, the first term in the asymptotic expansion turns out to depend on the Sobolev capacity of the subset where the perturbed eigenfunction is vanishing. Then we focus on the case of Dirichlet boundary conditions imposed on a subset which is scaling to a point; through a blow-up analysis for suitable rescalings of capacitary potentials, we are able to detect the exact vanishing order of the Sobolev capacity of such shrinking Dirichlet boundary portion and consequently obtain sharp estimates for the eigenvalue variation. We point out that, in both these cases, the rate of spectral convergence strongly depends on the local behavior of the limit eigenfunction near the region where the perturbation is applied.

In the second part of this thesis, we deal with two problems, both governed by the fractional Laplace operator, i.e. the power of order between 0 and 1 of the classical

(negative) Laplacian. First, we address the question of positivity of a nonlocal Schrödinger operator, driven by the fractional Laplacian and with singular multipolar Hardy-type potentials. Namely, we provide necessary and sufficient conditions on the coefficients of the potential for the existence of a configuration of poles that ensures the positivity of the corresponding Schrödinger operator. In particular, the threshold is given by the best constant in the fractional Hardy inequality in the whole space. This result is based, in turn, on a criterion in the spirit of the Agmon-Allegretto-Piepenbrink principle, which establishes the equivalence between positivity of quadratic forms associated with fractional Schrödinger operators and the existence of positive supersolutions to a corresponding perturbed Schrödinger equation. Finally, another element in the proof of our main theorem is a result fitting in the theory of localization of binding. The second topic we investigate in this part concerns geometric properties of the free boundary of solutions of a two-phase penalized obstacle-type problem for the fractional Laplacian. In view of the Caffarelli-Silvestre extension, we can interpret it as a thin obstacle-type problem driven by a second-order differential operator living in one dimension more and with a Muckenhoupt weight, that can be either singular or degenerate on the thin space. Working in this framework, by means of Almgren and Monneau type monotonicity formulas and blow-up analysis, we first prove a classification of the possible vanishing orders on the thin space and, as a consequence, the boundary strong unique continuation principle. We finally establish a stratification result for the nodal set (which coincides with the free boundary) on the thin space and we provide sharp estimates on the Hausdorff dimension of its regular and singular part. The main tools we exploit here come from geometric measure theory; in particular, we rely on Whitney's Extension Theorem and Federer's Reduction Principle.

In the end things must be as they are and have always been:
the great things remain for the great, the abysses for the profound,
the delicacies and thrills for the refined, and, to sum up shortly,
everything rare for the rare.

Infine i fatti devono stare come stanno e sono sempre stati:
le cose grandi sono riservate ai grandi, gli abissi ai profondi,
le finezze e i brividi ai sottili, e per esprimerci sinteticamente,
ai rari le cose rare.

F. W. Nietzsche

ACKNOWLEDGEMENTS

This is the end of my Ph.D. course. The first word instinctively coming up in my mind right after is "unfortunately". Indeed, Milan - and in particular buildings U5 and U22 - have been a new, welcoming home to me and now it's difficult to separate from them and to express how grateful I am for the amazing time I spent here. From the very first moment I strongly wanted to pursue this Ph.D. programme and, even if this obviously had a cost, all the energies I invested returned to me in the enhanced form of deep, true happiness and gratification. I am sure of this: I would make the same choice again and again.

During this journey I met a great number of wonderful people. I am really in debt with all of them, and I sincerely apologize with those I am not thanking in these pages.

The first person I would like to thank is Veronica Felli, who has been my Ph.D. supervisor. Few lines are really too short to sum up the contribution she gave to my life and to my academic career: I owe her a lot, for everything she taught me, for her incredible patience and for her constant and insightful guidance. I am really glad, and also proud, of having been a student of her. Moreover, I am thankful to Benedetta Noris and Donatella Danielli, for their warm hospitality during my visit in Amiens and West Lafayette, for their precious advising and for the fruitful collaborations born in these occasions. I am also grateful to Laura Abatangelo for all the valuable and stimulating discussions we had and for her kindness. I really hope this is only the starting point of long-standing collaborations with all of you.

Another huge thank goes to all the other Ph.D. students, wonderful and reliable fellows with whom I shared the best moments, made of coffees, hard work, enjoyable conversations (scientific and not), social lunches and delightful nights out. You all surely played the foremost role for my happiness during these years in Milan. At this point some special thanks are required. I truly believe each of you would deserve a dedicated mention, but for sake of brevity I am forced to restrict: my apologies to the people whose names are not explicitly written here, I swear you are in my mind and you will receive my thanks personally. Thank you Jessica and Alberto for your kind welcoming at my arrival in 2112 and for the countless pleasant moments spent together, inside and outside the university. I also would like to thank my companions which I started the Ph.D. and shared all the ups and downs with, till the end: Anderson, Giulia, Andrea, Davide and in particular Michele, one of the most generous and trustful persons I know. Many thanks also to Stefano, first known as a colleague, but then become a true friend. Thank you Alessandra for your sincere and unconditional friendship, which means a lot to me.

Furthermore, I would like to thank all the math Ph.D. students in Amiens: I couldn't really imagine a better, warmer hospitality than the one you gave me. I spent amazing days there and this is mostly because of your *fraternité*.

I am immensely thankful to my friends from Modena: we started by studying mathematics, but then mathematics became just a "collateral effect". In fact, it's impossible to describe how much I owe you. So much time spent together, so many discussions and adventures cannot be thanked with so few words.

Thanks to Comitato a Favore degli Ecomostri, the punk band I am a member of, and thanks to the other components Eugenio, Gabriele, Michele.

Almost needless to say, as always I am grateful to all my friends from my hometown: there is no need for complicated words here, it suffices to say that you are essentially a family for me.

Thanks to Morgana, my mother. It's always difficult to properly express the gratitude one has for his/her mother, but I am lucky there is no real need for this, since she already knows. Thanks for your support, despite all the problems, the hard times, the difficulties.

To conclude, I would like to truly thank Alice, the person who first (and most) believed in me during these wonderful years and who has always been by my side. Because happiness is only real when shared.

Roberto

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Prologue

CHAPTER 1

INTRODUCTION

The purpose of the present dissertation is essentially twofold. On one hand, this thesis addresses the question of quantitative spectral stability for a couple of singularly perturbed eigenvalue problems on bounded domains, driven by the Laplace operator; this is treated in Part I and describes the results achieved in [FO20, FNO21]. On the other hand, the second part of this thesis deals with positivity principles and obstacle-type problems for nonlocal operators; these topics are carried out in Part II, which collects the results obtained in [FMO20] and [DO]. In this introduction we would like to outline the main themes touched throughout this exposition, to give a taste of what are the major issues that we had to face and to summarize the foremost techniques we employed in order to overcome them. We refer to the introductions of the single chapters for a more exhaustive description of the frameworks and of the particular problems we take into consideration.

Part I For what concerns Part I, we start by fixing a bounded domain $\Omega \subseteq \mathbb{R}^N$, with sufficiently smooth boundary. Then, in Ω we consider the eigenvalue problem for the Laplace operator, coupled with some kind of boundary conditions. Namely,

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ \text{B.C.}, & \text{on } \partial\Omega. \end{cases} \tag{EP}$$

In the particular cases we cover in this work, the boundary conditions we prescribe at this stage are of homogeneous Dirichlet or Neumann type, that is

$$u = 0, \quad \text{on } \partial\Omega \qquad \text{or} \qquad \frac{\partial u}{\partial \boldsymbol{\nu}} = 0, \quad \text{on } \partial\Omega,$$

where $\boldsymbol{\nu}$ denotes the outer unit normal. For both these choices, we refer to (EP) as the *unperturbed* (or *limit*) problem. If $u \not\equiv 0$ is a solution of (EP) for some $\lambda \in \mathbb{R}$, we say that λ is an *eigenvalue* of the Laplacian and that u is a corresponding *eigenfunction*. By classical spectral theory (see Chapter 3 for the details), it is known that problem (EP) admits a countable, diverging sequence of nonnegative eigenvalues, which we denote by

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots \to +\infty.$$

Analogously, we call the λ_ns the *unperturbed* (or *limit*) eigenvalues.

Now we consider a small, singular perturbation of (EP). In order to do this, we first introduce a perturbation parameter $\varepsilon > 0$, which plays the role of the index of the perturbation itself. Thence, we consider an ε-dependent family of eigenvalue problems. In particular, we focus on two classes of perturbations: perturbation of the domain or perturbation of the boundary conditions, that is

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega_{\varepsilon}, \\ \quad \text{B.C.}, & \text{on } \partial\Omega_{\varepsilon}, \end{cases} \qquad \begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ \quad \text{B.C.}^{\varepsilon}, & \text{on } \partial\Omega. \end{cases} \tag{EP$_\varepsilon$}$$

In the former case, we are dealing with a fixed differential equation and fixed boundary conditions, though prescribed in a varying domain Ω_ε; on the other hand, in the latter case, the domain is fixed, while the boundary conditions are changing with ε. In both these instances, we call (EP$_\varepsilon$) the *perturbed* problem. One should think that the problem (EP$_\varepsilon$) is more and more similar (in some sense) to (EP) as ε becomes smaller and smaller, but still (EP$_\varepsilon$) is different in "nature" from (EP): in this perspective, the perturbation may be regarded as *singular*. Let us assume that the perturbed problem admits a countable sequence of nonnegative eigenvalues, which we denote by

$$0 \le \lambda_1^\varepsilon \le \lambda_2^\varepsilon \le \cdots \le \lambda_n^\varepsilon \le \cdots \to +\infty,$$

emphasizing the dependence on ε; this actually happens in the situations we investigate in Part I. We refer to the eigenvalues λ_n^ε as the *perturbed* eigenvalues. The first question one can ask concerns the **stability** of the unperturbed eigenvalues λ_n with respect to the perturbation; in other words,

$$\text{is it true that} \quad \lambda_n^\varepsilon \to \lambda_n \quad \text{as } \varepsilon \to 0 \text{ ?}$$

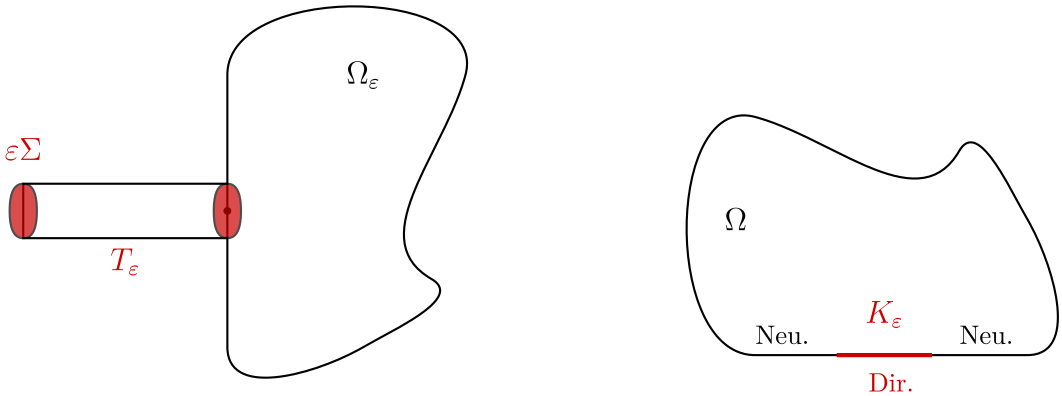
Again, in the cases under consideration in the present thesis, this question has a positive response for any index n; therefore it makes sense to research for the rate of convergence of λ_n^ε to λ_n, in terms of the parameter ε. In particular, the main goal of this first part of the thesis is to determine the sharp asymptotic behavior of the perturbed eigenvalues, as $\varepsilon \to 0$. Namely, we look for expansions of the type

$$\lambda_n^\varepsilon = \lambda_n + h(\varepsilon) + o(h(\varepsilon)), \quad \text{as } \varepsilon \to 0, \tag{1.0.1}$$

for some function $h\colon (0,+\infty) \to \mathbb{R}$, which vanishes at 0.

We now try to give the idea, in a broad sense, of what are the perturbations that we treat in Part I. Concerning the case of varying domains, we define the perturbed domain Ω_ε as the limit (unperturbed) domain Ω attached to a thin tube T_ε of fixed length and shrinking section with radius of order ε, i.e.

$$\Omega_\varepsilon := \Omega \cup T_\varepsilon, \tag{1.0.2}$$



(a) Perturbation of the domain (b) Perturbation of the B.C.

Figure 1.1: Types of perturbation

see Figure 1.1 (a). In this case, we prescribe homogeneous Dirichlet boundary conditions on $\partial\Omega$ and $\partial\Omega_\varepsilon$. We observe that Ω_ε does not converge to Ω in the sense of Hausdorff distance, even if convergence of the Lebesgue measures holds: in this sense we can label the perturbation as singular. Nevertheless, Ω_ε converges to Ω, as $\varepsilon \to 0$, in the sense of Mosco (see Definition 4.1.1) and this ensures that $\lambda_n^\varepsilon \to \lambda_n$, as proved in [Dan03]. We refer to [Hal05, BLLdK06] for thorough expositions of the topic of spectral stability for differential operators under domain perturbations (see also Section 4.1 of the present thesis). On the other side, concerning the perturbation of the boundary conditions, we start from an unperturbed problem with assigned

$$\frac{\partial u}{\partial \boldsymbol{\nu}} = 0, \quad \text{on } \partial\Omega$$

and we perturb it by prescribing homogeneous Dirichlet boundary conditions on a small region $K_\varepsilon \subseteq \partial\Omega$, which is "disappearing" when $\varepsilon \to 0$, see Figure 1.1 (b). Here the singular nature of the perturbation lies in the fact that in the unperturbed problem the solution is assumed to satisfy a single condition on the whole $\partial\Omega$, while in the perturbed setting we are dealing with a mixed Dirichlet-Neumann problem, for any ε. The question of stability of the spectrum for mixed problem has been initiated, up to our knowledge, in [Gad92] and [Gad93] for the two-dimensional case.

Actually, we work in a more general scenario, but for sake of simplicity in the exposition, we rather focus on these prototypical cases; for all the details, we refer to chapters 4 and 5, respectively. In both these cases we are able to provide the sharp asymptotic behavior of a perturbed eigenvalue when it is converging to a simple eigenvalue of the limit problem; this is what basically constitutes our main results, i.e. theorems 4.2.1 and 5.2.5. The rate of this spectral convergence turns out to strongly rely on the behavior of the (unique, up to a multiplicative constant) eigenfunction, corresponding to the limit (simple) eigenvalue, near the point where the perturbation is applied. This is due to the local character of the perturbations we investigate, being them circumscribed to a small, vanishing region. Let us quickly enter into the details. First, let us denote by φ_n an eigenfunction associated to the limit (simple) eigenvalue. Then, the function h describing the first order term in the asymptotic expansion of the eigenvalue variation, as in (1.0.1), in the case of the attachment of a thin tube, behaves like

$$h(\varepsilon) = -\varepsilon^{N+2k-2},$$

up to multiplication by a positive constant, where $k \in \mathbb{N}$ is the vanishing order of φ_n at the point of $\partial\Omega$ where the tube is shrinking; see Theorem 4.2.1 for the exact statement. The basic tool we exploit in order to prove this result is the Courant-Fischer min-max variational characterization of eigenvalues, that allows us to obtain lower and upper bounds for the difference $\lambda_n^\varepsilon - \lambda_n$. These bounds are based, in turn, on energy estimates for the perturbed eigenfunctions near the junction, that come as a consequence of a suitable perturbed Almgren-type monotonicity formula. When dealing with a boundary value problem with zero Dirichlet boundary conditions on a Lipschitz domain of the type (1.0.2), one faces lack of regularity of the solutions and the proof of the monotonicity formula gets stuck. In order to overcome this difficulty, we develop an approximation procedure for the perturbed problem, that basically smooths away the corners of the perturbed domain; in view also of a geometric assumption on the tube section, it is then possible to rule out a remainder term in the derivative of the Almgren frequency function and to pass to the limit, see Section 4.3.2. Finally, the sharpness of the energy estimates is in close connection with the identification of a nontrivial limit profile for the blow-up sequence of a proper rescaling of the perturbed eigenfunction, see Theorem 4.2.2. In order to uniquely recognize this limit profile, the monotonicity formula is not sufficient by itself, since it only provides an upper bound for its frequency at infinity. Therefore, in order to obtain a univocal identification, we employ an argument based on a local inversion result, that yields an energy control for the difference between the blow-up eigenfunction and a k-homogeneous profile.

On the other hand, as far as the case of perturbation of the boundary conditions is concerned, the proper parameter that quantifies the perturbation is a certain notion of capacity of a set, tailored for our purposes. In particular, we have that $\lambda_n^\varepsilon \to \lambda_n$ as $\varepsilon \to 0$, for any $n \in \mathbb{N}$, if and only if

$$\mathrm{Cap}_{\bar{\Omega}}(K_\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0,$$

where $\mathrm{Cap}_{\overline{\Omega}}(\cdot)$ stands for the Sobolev capacity of a set, relative to $\overline{\Omega}$, see Definition 5.2.1. Furthermore, we proved that also the rate of convergence of the perturbed eigenvalue is sharply measured by a notion of capacity, that takes into account the values of the limit eigenfunction φ_n on the perturbing set K_ε. Namely, we have that

$$h(\varepsilon) = \mathrm{Cap}_{\bar{\Omega}}(K_\varepsilon, \varphi_n),$$

see Definition 5.2.4 and Theorem 5.2.5. In addition, in the particular case in which K_ε is obtained by rescaling a fixed, compact shape Σ by ε, we have that

$$\mathrm{Cap}_{\bar{\Omega}}(K_\varepsilon, \varphi_n) = C_{\Sigma,k}\, \varepsilon^{N+2k-2} + o(\varepsilon^{N+2k-2}), \quad \text{as } \varepsilon \to 0, \tag{1.0.3}$$

where $k \in \mathbb{N}$ denotes the vanishing order of φ_n at the point where $K_\varepsilon \approx \varepsilon\Sigma$ is concentrating and $C_{\Sigma,k} > 0$ is again a notion of capacity, relative to the half space, see Theorem 5.2.12. Here the main difficulty comes from the fact that the region K_ε is vanishing on a (possibly) curved portion of the boundary of Ω. In order to overcome this issue, we had first to perform a straightening of this region of the boundary, through a particular diffemorphism that allows us to carry out a reflection argument. In this light, our initial problem with bent boundary is equivalent to another problem, driven by a second order elliptic operator in divergence form, but with flat boundary. At this point we implemented a careful blow-up analysis for a suitable rescaling of the capacitary potential. This, together with ad-hoc Poincaré and Hardy type inequalities and appropriate capacitary estimates, led us to the proof of (1.0.3).

Part II Let us now introduce the content of Part II. Here the protagonist is interpreted by the so called *fractional Laplacian*, that is the power of order $s \in (0,1)$ of the standard Laplace operator. This can be defined, for functions $u \in C_c^\infty(\mathbb{R}^N)$, as

$$\begin{aligned}(-\Delta)^s u(x) :&= \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y, \qquad x \in \mathbb{R}^N \\ &= \lim_{\rho\to 0^+} \int_{|x-y|>\rho} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y,\end{aligned}$$

up to a positive multiplicative constant, where P.V. means that the integral has to be seen in the principal value sense. We refer to Chapter 6 for a more detailed presentation of this nonlocal operator.

The first problem we investigate tackles the question of positivity of a Schrödinger operator, driven by the fractional Laplacian and with a multipolar Hardy-type potential. More precisely, for some fixed $m \in \mathbb{N}$, let us pick m coefficients $\mu_1, \dots, \mu_m \in \mathbb{R}$ and m poles $a_1, \dots, a_m \in \mathbb{R}^N$ and let us consider the following fractional Schrödinger operator

$$\mathcal{L} := (-\Delta)^s - \sum_{i=1}^m \frac{\mu_i}{|x-a_i|^{2s}}.$$

Loosely speaking, the aim of our investigation is to understand how the values μ_is and the position of the poles a_is influence the coercivity of this operator. In order to better describe the problem, let us first briefly introduce the functional setting. We work in the fractional Beppo Levi space $\mathcal{D}^{s,2}(\mathbb{R}^N)$, defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the following norm

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} := \left(\frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{1/2},$$

that amounts to the square root of the quadratic form naturally associated with $(-\Delta)^s$. Let us now consider the quadratic form corresponding to $\mathcal{L}$

$$Q_{\mathcal{L}}(u) :={}_{(\mathcal{D}^{s,2}(\mathbb{R}^N))^*} \langle \mathcal{L}u, u\rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^m \mu_i \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x-a_i|^{2s}}\,\mathrm{d}x.$$

We say that the operator $\mathcal{L}$ is *positive* or, equivalently, that the form $Q_{\mathcal{L}}$ is *positive definite* if there exists a constant $\Lambda > 0$ such that

$$Q_{\mathcal{L}}(u) \geq \Lambda \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}, \quad \text{for all } u \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

We point out that the form $Q_{\mathcal{L}}$ is well defined by virtue of the fractional Hardy inequality (proved in [Her77]), which reads as follows

$$\gamma_H \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} \, \mathrm{d}x, \quad \text{for all } u \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

where $\gamma_H > 0$ denotes the best constant possible. Our main result consists in a necessary and sufficient condition on the coefficients $\mu_1, \dots, \mu_m$ for the existence of a configuration of poles that makes the form $Q_{\mathcal{L}}$ positive definite. Namely, we have that

$$\mu_i < \gamma_H, \quad \text{for all } i = 1, \dots, m, \quad \text{and} \quad \sum_{i=1}^m \mu_i < \gamma_H$$

if and only if there exists a configuration of poles $a_1, \dots, a_m$ such that $Q_{\mathcal{L}}$ is positive definite, see Theorem 7.2.4. This result is based, in turn, on a positivity principle reminiscent of the theory of Agmon-Allegretto-Piepenbrink and that possesses its own interest: it establishes a relation between positivity of quadratic forms and existence of positive supersolutions to certain (possibly perturbed) Schrödinger equations. The precise statement can be found in Lemma 7.2.2. We refer to Section 7.1 for a more detailed description of the classical Agmon-Allegetto-Piepenbrink positivity principle, whose first version was proved in 1974, independently, in [All74] and [Pie74], and then refined by Agmon, see e.g. [Agm83]. Another key ingredient is a result that fits into the theory of localization of binding, see Theorem 7.2.3. Widely speaking, it says that, given two *positive* fractional Schrödinger operators

$$(-\Delta)^s - V_1 \quad \text{and} \quad (-\Delta)^s - V_2,$$

then the operator

$$(-\Delta)^s - (V_1 + V_2(\cdot - y))$$

is also positive if the translation vector $y \in \mathbb{R}^N$ is sufficiently large. This result was pointed out for the first time in [OS79], in a prototypical situation with the standard Laplacian, in the course of the proof of the Efimov's effect. In the subsequent years, it has then been deepened as a stand-alone matter of investigation, see e.g. [KS79, Sim80] and [Pin95].

We observe that this problem displays two opposite, discordant faces. Indeed, on one hand it seems reasonable to examine, locally near the singularities of the potential a_is, the behavior of solutions of PDEs driven by $\mathcal{L}$. This necessity of purely local arguments is actually crucial in many steps. On the other hand, the operator $\mathcal{L}$, in its principal part $(-\Delta)^s$, exhibits a nonlocal nature, that prevents us from easily understanding what happens near the poles. A strategy to bypass this issue is provided by the so called Caffarelli-Silvestre extension for functions in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, established in [CS07], which allows to recover a local framework, paying the price of working in one dimension more. Namely, this approach provides a way to translate a nonlocal problem governed by the fractional Laplacian in $\mathbb{R}^N$ into a boundary value problem in

$$\mathbb{R}^{N+1}_+ := \{(x,t)\colon x \in \mathbb{R}^N,\ t > 0\},$$

led by a second order elliptic *differential* operator, with a weight that can be either singular or degenerate at $\{t = 0\}$. To be more precise, we have that, for any $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ there exists $U\colon \mathbb{R}^{N+1}_+ \to \mathbb{R}$ belonging to a suitable functional space, such that

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla U) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ \qquad\qquad U = u, & \text{on } \partial\mathbb{R}^{N+1}_+ \cong \mathbb{R}^N, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\dfrac{\partial U}{\partial t} = \kappa_s(-\Delta)^s u, & \text{on } \partial\mathbb{R}^{N+1}_+ \cong \mathbb{R}^N, \end{cases} \tag{1.0.4}$$

with κ_s denoting a positive (explicit) constant. From this perspective, the fractional Laplacian may be seen as a Dirichlet-to-Neumann operator in $\mathbb{R}^{N+1}_+$. This procedure decisively enters in our proofs and provides a fundamental tool in our work.

The second (and last) topic we investigate in this part concerns the study of the nodal set of solutions to an obstacle-type problem. In particular, for $s \in (0,1)$, let us consider the following equation

$$(-\Delta)^s u = \lambda_-(u^-)^{p-1} - \lambda_+(u^+)^{p-1}, \quad \text{in } B_1', \tag{1.0.5}$$

where λ_- and λ_+ are positive constant, $p \geq 2$ and B_1' is the unit ball in $\mathbb{R}^N$. Here, u^- and u^+ denote, respectively, the positive and negative part of u. We notice that (1.0.5) is a nonlocal, semilinear problem exhibiting two phases and that, in view of the Caffarelli-Silvestre extension (1.0.4), it can be understood as follows

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla u) = 0, & \text{in } B_1^+, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\dfrac{\partial u}{\partial t} = \lambda_-(u^-)^{p-1} - \lambda_+(u^+)^{p-1} & \text{on } B_1', \end{cases} \tag{1.0.6}$$

up to a multiplicative constant, where

$$B_1^+ := \{z \in \mathbb{R}^{N+1}_+ \colon\ |z| < 1\}$$

stands for the positive half ball. With an abuse of notation, we denote by u both the solution of (1.0.6) defined on B_1^+ and its trace on B_1'. The obstacle is here assumed to be zero. Problem (1.0.6) emerges from physical motivations, such as boundary temperature control and osmosis through semi-permeable membranes, as explained, for instance, in [DL76] and [DJ21]. The object of our interest are the geometric properties of the free boundary of solutions to (1.0.6) on the thin ball B_1', defined as

$$\mathcal{Z}(u) := B_1' \cap \partial\{x \in B_1' \colon u(x) \neq 0\}.$$

In this direction, our first result Theorem 8.1.1, yields a classification of the admissible vanishing orders, which implies the strong unique continuation principle, see Corollary

8.1.2. In particular, this tells us that the nodal set on B'_1 has empty interior in the $\mathbb{R}^N$ topology; as a consequence we derive that the free boundary coincides with the nodal set of the solution u on the thin ball B'_1. The proofs of these facts heavily rely upon a weighted Almgren monotonicity formula, together with a fine blow-up analysis for suitable rescalings of the solution. Let us briefly explain this. For any $r \in (0,1)$ and any $x_0 \in B'_1$, let

$$B_r^+(x_0) := \{z \in \mathbb{R}^{N+1}_+ \colon\ |z - (x_0, 0)| < r\} \quad \text{and} \quad S_r^+(x_0) := \partial B_r^+(x_0) \cap \mathbb{R}^{N+1}_+$$

and let us consider the weighted Almgren frequency function of u at x_0

$$\mathcal{N}_{x_0}(u,r) := \frac{r \displaystyle\int_{B_r^+(x_0)} t^{1-2s} |\nabla u|^2 \,\mathrm{d}x\mathrm{d}t}{\displaystyle\int_{S_r^+(x_0)} t^{1-2s} u^2 \,\mathrm{d}S}.$$

We start by proving the monotonicity in r of a slight perturbation of $\mathcal{N}_{x_0}(u,r)$, which takes into account the lower order terms coming from the nonlinearities. Consequently, we obtain the existence of the limit at 0^+ of $\mathcal{N}_{x_0}(u,r)$,

$$\mathcal{N}_{x_0}(u,0^+) := \lim_{r\to 0^+} \mathcal{N}_{x_0}(u,r)$$

for any $x_0 \in B'_1$: we call it *frequency* of u at x_0. Moreover, this limit is a nonnegative, finite and integer number k and it coincides with the vanishing order of u at the point x_0. More specifically, the blow-up sequence

$$\frac{u(x_0 + rx, 0)}{r^k} \tag{1.0.7}$$

converges, as $r \to 0$, to a homogeneous polynomial of degree k, in the variable x. Basically, the study of the blow-up sequence is a procedure that enables to zoom in the graph of the function u closer and closer to the point x_0.

We finally investigate regularity and structural properties of the nodal set. As a first step, we distinguish between the regular and the singular part of the free boundary, defined respectively as

$$\begin{aligned} \mathcal{R}(u) &:= \mathcal{Z}_1(u), \\ \mathcal{S}(u) &:= \bigcup_{k\geq 2} \mathcal{Z}_k(u), \end{aligned}$$

where, for any integer $k \geq 1$,

$$\mathcal{Z}_k(u) := \{x_0 \in \mathcal{Z}(u) \colon \mathcal{N}_{x_0}(u,0^+) = k\}.$$

While we showed that $\mathcal{R}(u)$ is a regular submanifold of B'_1 (see Proposition 8.1.3), this does not hold true for the singular part. Nevertheless, we were able to prove a stratification result for $\mathcal{S}(u)$, which basically tells us that the singular part is contained in a countable union of lower dimensional C^1-manifolds, see Theorem 8.1.5. The proof is essentially based on the continuity of the function that maps any point $x_0 \in B_1'$ to the homogeneous blow-up limit of u at x_0, as in (1.0.7), and on Whitney's Extension Theorem, as stated in [Whi34]. The last properties of the free boundary we analyze are estimates on the Hausdorff dimension of $\mathcal{Z}(u)$, $\mathcal{R}(u)$ and $\mathcal{S}(u)$. In particular, we first proved that the nodal set and its regular part are $(N-1)$-dimensional. Finally we point out how the weight t^{1-2s} affects the analysis. Indeed, as a result of an explicit counterexample (provided in [STT20]) that exploits the presence of the extra variable $t > 0$, we have that $N-1$ is also the optimal bound for the Hausdorff dimension of $\mathcal{S}(u)$, contrariwise to what happens in other similar, unweighted, situations. These estimates are carried out with the aid of another tool coming from geometric measure theory, that is Federer's Reduction Principle, see [Sim83, Appendix A].

CHAPTER 2

THE ALMGREN MONOTONICITY FORMULA

In the proof of the main results contained in this manuscript, at many stages, an Almgren-type monotonicity formula plays a pivotal role, see sections 4.3.4 and 8.3. Therefore, in this chapter we give a friendly and self-contained exposition of the celebrated Almgren monotonicity formula in its classical version, first introduced in [Alm83] for harmonic functions. In particular, we first present the main concepts and we sketch some of the proofs; then we try to give a flavor of the importance of this idea, we exhibit some of its applications in the world of PDEs and we show some of its possible generalizations.

For any $r > 0$ let B_r be the ball of $\mathbb{R}^N$ centered at the origin and with radius r and let $S_r := \partial B_r$. Let us begin by considering a function $u \in H^1(B_1)$ such that

$$-\Delta u = 0 \quad \text{in } B_1, \tag{2.0.1}$$

in a weak sense, i.e.

$$\int_{B_r} \nabla u \cdot \nabla v \, \mathrm{d}x = 0 \quad \text{for all } v \in H_0^1(B_r).$$

In relation with u, we introduce the following functions:

$$\text{the } \textit{energy} \text{ function} \quad r \mapsto E(u,r) := \frac{1}{r^{N-2}} \int_{B_r} |\nabla u|^2 \, \mathrm{d}x,$$

$$\text{the } \textit{height} \text{ function} \quad r \mapsto H(u,r) := \frac{1}{r^{N-1}} \int_{S_r} u^2 \, \mathrm{d}S$$

and finally

$$\text{the } \textit{frequency} \text{ function} \quad r \mapsto \mathcal{N}(u,r) := \frac{E(u,r)}{H(u,r)} = \frac{r \displaystyle\int_{B_r} |\nabla u|^2 \, \mathrm{d}x}{\displaystyle\int_{S_r} u^2 \, \mathrm{d}S}.$$

First of all, we have the following preliminary properties of the functions just defined.

Lemma 2.0.1. *The following hold:*

1. $E(u,\cdot), H(u,\cdot) \in C^\infty(0,1)$;
2. $E(u,r) = o(1)$ *and* $H(u,r) = O(1)$ *as* $r \to 0^+$.

Proof. Point 1. immediately comes from classical regularity theory for harmonic functions.
Since $u, \nabla u \in L^\infty_{\mathrm{loc}}(B_1)$, we have that

$$E(u,r) \leq C_N \left\|\nabla u\right\|^2_{L^\infty(B_{1/2})} r^2 \quad \text{and} \quad H(u,r) \leq C_N \left\|u\right\|^2_{L^\infty(B_{1/2})}$$

for all $r \in (0,1/2)$; then letting $r \to 0^+$ proves 2.

□

The name *frequency* is justified by the following lemma (see also Figure 2.1).

Lemma 2.0.2. *If* $u \in H^1(B_1)$, $u \not\equiv 0$ *is a harmonic function, homogeneous of degree* $\gamma \geq 0$, *then* $\mathcal{N}(u,r) = \gamma$. *In particular, by classical regularity theory* $\gamma \in \mathbb{N}$.

Proof. Let us assume that u is a harmonic polynomial, homogeneous of degree γ, i.e.

$$u(\alpha x) = \alpha^\gamma u(x) \tag{2.0.2}$$

$$-\Delta u = 0, \quad \text{in } B_1. \tag{2.0.3}$$

From (2.0.2) we have that

$$\nabla u(x) \cdot x = \gamma u(x)$$

while multiplying (2.0.3) by u and integrating (by parts) on B_r we deduce that

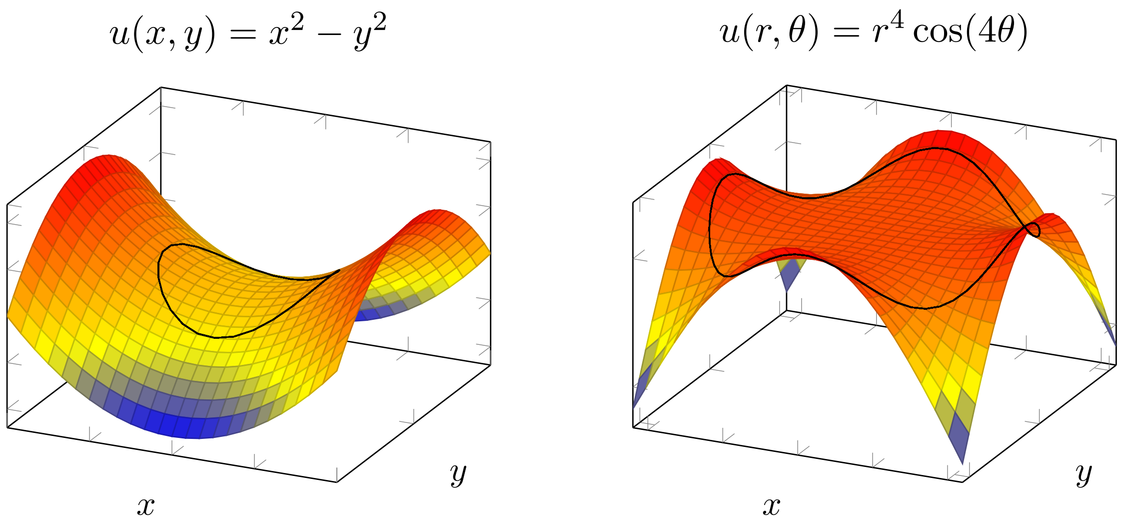
$$\int_{B_r} \left|\nabla u\right|^2 \mathrm{d}x = \int_{S_r} u \nabla u \cdot \boldsymbol{\nu} \, \mathrm{d}S = \frac{1}{r} \int_{S_r} u \nabla u \cdot x \, \mathrm{d}S,$$

since $\boldsymbol{\nu}(x) = x/r$ on S_r. Combining these two facts we have that

$$\mathcal{N}(u,r) = \frac{r \int_{B_r} \left|\nabla u\right|^2 \mathrm{d}x}{\int_{S_r} u^2 \, \mathrm{d}S} = \gamma.$$

□

Broadly speaking, the function $\mathcal{N}(u,r)$ counts (making an average) how many positive peaks and negative wells the function u has when restricted to S_r. In [Alm83] the author proved that, if u is harmonic in B_1, then the function $\mathcal{N}(u,r)$ is nondecreasing with respect to $r \in (0,1)$. Therefore the number of "ups and downs" of the function u on S_r can only increase when r goes away from zero.



(a) (b)

Figure 2.1: Frequencies of harmonic polynomials.

Proposition 2.0.3. *Let $u \in H^1(B_1)$ be such that $u \not\equiv 0$ and*

$$-\Delta u = 0 \quad \text{in } B_1$$

in the sense of distributions. If $H(u, r_0) > 0$ for some $r_0 \in (0,1)$, then there exists $r_1, r_2 \in (0,1)$ satisfying $r_1 < r_0 < r_2$, such that $\mathcal{N}(u,\cdot)$ is smooth in (r_1, r_2) and

$$\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}r}(u,r) = \frac{2r}{\left(\int_{S_r} u^2 \,\mathrm{d}S\right)^2} \left[\int_{S_r} u^2 \,\mathrm{d}S \int_{S_r} \left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)^2 \mathrm{d}S - \left(\int_{S_r} u \frac{\partial u}{\partial \boldsymbol{\nu}} \,\mathrm{d}S \right)^2 \right] \geq 0. \tag{2.0.4}$$

Proof. Thanks to Lemma 2.0.1 we know that the functions $E(u,\cdot)$ and $H(u,\cdot)$ are smooth in a neighborhood of $r \in (0,1)$. Since $H(u, r_0) > 0$, by continuity there exists a (possibly smaller) neighborhood $(r_1, r_2) \subseteq (0,1)$ of r_0, such that

$$H(u,r) > 0 \quad \text{for all } r \in (r_1, r_2).$$

First, let us compute the derivative with respect to r of the height function. By rewriting, with a change of variable,

$$H(u,r) = \frac{1}{r^{N-1}} \int_{S_r} u^2 \,\mathrm{d}S = \int_{S_1} u^2(rx) \,\mathrm{d}S(x)$$

and by dominated convergence theorem we have that

$$\frac{\mathrm{d}H}{\mathrm{d}r}(u,r) = \frac{2}{r^{N-1}} \int_{S_r} u \frac{\partial u}{\partial \boldsymbol{\nu}} \,\mathrm{d}S. \tag{2.0.5}$$

On the other hand, thanks to coarea formula we can compute

$$\frac{\mathrm{d}E}{\mathrm{d}r}(u,r)=\frac{1}{r^{N-2}}\int_{S_r}|\nabla u|^2\,\mathrm{d}S-\frac{N-2}{r^{N-1}}\int_{B_r}|\nabla u|^2\,\mathrm{d}x.$$

With the help of the standard Pohovaev identity

$$\int_{S_r}|\nabla u|^2\,\mathrm{d}S=2\int_{S_r}\left(\frac{\partial u}{\partial\boldsymbol{\nu}}\right)^2\mathrm{d}S+\frac{N-2}{r}\int_{B_r}|\nabla u|^2\,\mathrm{d}x$$

we obtain the following expression

$$\frac{\mathrm{d}E}{\mathrm{d}r}(u,r)=\frac{2}{r^{N-2}}\int_{S_r}\left(\frac{\partial u}{\partial\boldsymbol{\nu}}\right)^2\mathrm{d}S.$$

Finally, we observe that multiplying (2.0.1) by u and integrating by parts on B_r yields

$$E(u,r)=\frac{1}{r^{N-2}}\int_{S_r}u\frac{\partial u}{\partial\boldsymbol{\nu}}\,\mathrm{d}S. \tag{2.0.6}$$

Now, plugging (2.0.5) and (2.0.6) into

$$\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}r}(u,r)=\frac{H\frac{\mathrm{d}E}{\mathrm{d}r}-E\frac{\mathrm{d}H}{\mathrm{d}r}}{H^2}$$

leads to the desired expression (2.0.4) for the derivative of $\mathcal{N}(u,r)$, with Cauchy-Schwartz inequality telling us that it is nonnegative.

□

The present monotonicity formula has several applications in the framework of elliptic PDEs, in particular in relation with unique continuation principle. The first time in which this approach has been pursued was by Garofalo and Lin in [GL86]. The authors took into account solutions to a more general class of elliptic equations and were able to prove the validity of the unique continuation principle, exploiting the monotonicity of a suitably modified version of the frequency function. A key step in their investigation is the proof of the so called *doubling property*, which is a straightforward consequence of the boundedness of the frequency function near 0 (which is, in turn, a consequence of its monotonicity). In the following we present it in the model case of harmonic functions.

Lemma 2.0.4. *If $H(u,r_0)>0$ for some $r_0\in(0,1)$, then there exist $r_1,r_2\in(0,1)$, satisfying $r_1<r_0<r_2$, and a constant $C=C(u)>0$ such that*

$$H(u,R)\le\left(\frac{R}{r}\right)^{2C}H(u,r),\quad \textit{for all } r_1<r\le R<r_2. \tag{2.0.7}$$

Proof. From (2.0.5) and (2.0.6) we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}r}\log H(u,r)=\frac{2}{r}\mathcal{N}(u,r).$$

From Proposition 2.0.3 we know that $\mathcal{N}(u,\cdot)$ is differentiable in (r_1,r_2) and $\mathcal{N}(u,r) \le \mathcal{N}(u,r_2) =: C$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}r}\log H(u,r) \le \frac{\mathrm{d}}{\mathrm{d}r}\log r^{2C}.$$

By integration of this inequality in $(r,R) \subseteq (r_1,r_2)$ we obtain the thesis. □

As a consequence we obtain that the function $H(u,\cdot)$ is actually always positive near 0, if u does not vanishes identically.

Corollary 2.0.5. *If $H(u,r_0) > 0$ for some $r_0 \in (0,1)$, then $H(u,r) > 0$ for all $r \in (0,r_0]$.*

Proof. From (2.0.7) we deduce that

$$\lim_{r\to r_1^+} H(u,r) = H(u,r_1) > 0,$$

therefore $H(u,\cdot) > 0$ in a neighborhood of r_1. Iteratively applying Lemma 2.0.4 we can deduce the thesis. □

In addition, we can derive that the frequency function is well defined in a neighborhood of 0.

Corollary 2.0.6. *If $H(u,r_0) > 0$ for some $r_0 \in (0,1)$, then*

$$\mathcal{N}(u,r) = \frac{E(u,r)}{H(u,r)}$$

is well defined and smooth in $(0,r_0)$. Moreover (2.0.4) holds in $(0,r_0)$ and there exists

$$\lim_{r\to 0}\mathcal{N}(u,r) =: \gamma \in [0,+\infty).$$

Finally $\mathcal{N}(u,\cdot)$ is constant if and only if u is homogeneous of degree γ.

Proof. The first part is trivial by virtue of Proposition 2.0.3 and Corollary 2.0.5. Concerning the last statement, we observe that the Cauchy-Schwartz inequality (2.0.4) holds with equality if and only if $\frac{\partial u}{\partial \boldsymbol{\nu}} = \lambda u$ on S_r for some $\lambda \ge 0$ and for any $r \in (0,r_0)$, i.e. if and only if u is homogeneous of degree γ. □

This fact allows us to infer the doubling condition for the function u, which reads as follows.

Lemma 2.0.7 (Doubling property)**.** *If $H(u,r_0) > 0$ for some $r_0 \in (0,1)$, then there exists a constant $C = C(N,u) > 0$ such that*

$$\int_{B_{2r}} u^2\,\mathrm{d}x \le C\int_{B_r} u^2\,\mathrm{d}x$$

for all $r \in (0,r_0/2]$.

Proof. Reasoning analogously to the proof of Lemma 2.0.4, in view of corollaries 2.0.5 and (2.0.6), we obtain that

$$H(u, 2r) \leq 2^{2C} H(u, r)$$

for all $r \in (0, r_0/2]$, thus implying that

$$\int_{S_{2r}} u^2 \, \mathrm{d}S \leq 2^{N-1+2C} \int_{S_r} u^2 \, \mathrm{d}S.$$

Integrating the inequality above in $(0, r)$ allows us to conclude the proof. □

The doubling condition immediately implies the validity of the weak unique continuation principle.

Corollary 2.0.8 (WUCP)**.** *If $u \equiv 0$ in an open set $\omega \subseteq B_1$ then $u \equiv 0$ in B_1.*

Proof. Since the Laplace operator is translation invariant, without loss of generality we can assume that $0 \in \omega$. Then there exists $r \in (0, 1/2]$ such that $B_r \subseteq \omega$. Since $u \equiv 0$ in B_r, from Lemma 2.0.7 we have that $u \equiv 0$ in B_{2r}. Iterating this argument we may conclude the proof. □

Actually, it is possible to prove the strong unique continuation property, which says that a harmonic function cannot vanish with infinite order in a point, unless it is constantly equal to 0.

In fact, it is possible to push this approach further and obtain more accurate, quantitative results concerning the local behavior of solutions to elliptic PDEs (and derive the unique continuation principle as a direct consequence). More precisely, combining the monotonicity formula with a careful blow-up analysis for suitable rescalings of solutions and separation of variables, it is possible to describe the asymptotics of the solution itself in terms of some angular eigenvalue problems.

Several results have been produced in this spirit in the recent years. For instance, in [FFT11] the authors classified the behavior of solutions to a Schrödinger equation near the singularity of the potential (see also [FFT12]), while [FF13] focused on the boundary local asymptotics near a corner point.

To conclude the section, it is worth to spend a couple of words about a possible generalization of the classical monotonicity formula here described. In the last years, the so called *fractional Laplace operator* has attracted a lot of attention among mathematicians, also thanks to the new point of view provided by the celebrated work by Caffarelli and Silvestre [CS07]. We now try to give an idea of how to define an Almgren frequency function in the fractional context, avoiding all the details; we refer to Chapter 7 for the precise description of the functional framework and to [CS07] for what concerns the monotonicity formula. The fractional Laplacian of order $s \in (0, 1)$ is defined, for functions $u \in C_c^\infty(\mathbb{R}^N)$ as follows

$$\begin{aligned}(-\Delta)^s u(x) :&= \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, \mathrm{d}y \\ &= \lim_{\delta \to 0} \int_{\mathbb{R}^N \setminus B_\delta(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, \mathrm{d}y,\end{aligned}$$

up to a positive, multiplicative constant. It is clear from its definition that the fractional Laplacian is a nonlocal operator, in the sense that in order to compute $(-\Delta)^s u$ in a point x you need to know the values of u in the whole $\mathbb{R}^N$ and not only in a neighborhood of x, like for the standard Laplacian. Hence, at a first glance, it is not clear how to define the analogous of the functions E, H and $\mathcal{N}$, in view of their purely local nature. Here the work [CS07] enters: indeed, the authors proved that it is possible to see $(-\Delta)^s$ as a Dirichlet-to-Neumann operator for an elliptic problem in $\mathbb{R}^{N+1}_+ := \{(x,t)\colon t>0,\ x\in\mathbb{R}^N\}$, driven by a local, weighted operator. More precisely, for any $u\colon \mathbb{R}^N \to \mathbb{R}$ (in a suitable function space) there exists a function $U\colon \mathbb{R}^{N+1}_+ \to \mathbb{R}$ weakly satisfying

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla U) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ U = u, & \text{on } \partial\mathbb{R}^{N+1}_+, \end{cases}$$

and such that

$$(-\Delta)^s u(x) = -C_{N,s} \lim_{t\to 0^+} t^{1-2s}\frac{\partial U}{\partial t}(x,t).$$

In this sense, the operator

$$u \mapsto -C_{N,s} \lim_{t\to 0^+} t^{1-2s}\frac{\partial U}{\partial t}$$

is local and makes us able to define the frequency function as follows. For any $U\colon \mathbb{R}^{N+1}_+ \to \mathbb{R}$ in a suitable (weighted with t^{1-2s}) functional Sobolev space that ensure the following integrals to be well defined, we let

$$E_s(U,r) := \frac{1}{r^{N-2s}}\int_{B_r^+} t^{1-2s}\left|\nabla U\right|^2 \,\mathrm{d}x\mathrm{d}t, \quad H_s(U,r) := \frac{1}{r^{N-2s+1}}\int_{S_r^+} t^{1-2s} U^2 \,\mathrm{d}S,$$

where

$$B_r^+ := \{z \in \mathbb{R}^{N+1}_+\colon\ |z|<r\}, \quad S_r^+ := \partial B_r^+ \cap \mathbb{R}^{N+1}_+.$$

Then we can define the frequency function in the expected way

$$\mathcal{N}_s(U,r) := \frac{E_s(U,r)}{H_s(U,r)}.$$

In [CS07] it has been proved that, if $u\colon \mathbb{R}^N \to \mathbb{R}$ is such that

$$(-\Delta)^s u = 0, \quad \text{in } B_1,$$

then

$$\frac{\mathrm{d}\mathcal{N}_s}{\mathrm{d}r}(U,r) \geq 0,$$

where $U\colon \mathbb{R}^{N+1}_+ \to \mathbb{R}$ is the Caffarelli-Silvestre extension of u.

Making use of this powerful tool, in [FF14] the authors provided the local asymptotics of solutions to semilinear equations driven by the fractional Laplacian, with singular and critical potentials. Moreover they proved, as a consequence, the strong unique continuation principle. In the same spirit, similar results have been obtained in [FF15a] concerning a relativistic Schrödinger operator and in [FF20] for an application to fractional Laplacian of higher order $s = 3/2$. Finally, we adopted this tool in Chapter 8, where we define a tailor-made frequency function for solutions of a boundary weighted obstacle problem. This allows us to prove classification of the possible vanishing orders, unique continuation principles and geometric properties of the free boundary.

Aside from the study of the local behavior of solutions to elliptic PDEs, the Almgren monotonicity formula is of considerable help also in other situations. For example, the monotonicity of suitably modified frequency functions may be exploited in the area of spectral stability of singularly perturbed eigenvalue problem. Up to our knowledge, the first attempt in tackling such a problem making use of an Almgren-type monotonicity formula was made in [FFT12] and in the subsequent papers [AFT14b, AFT14a]. In those works, the picture is similar to the one presented in Section 4.2: they consider two disjoint domain connected by a thin tube of radius proportional to a small parameter $\varepsilon > 0$; therefore we now refer to the notation of Section 4.2. Since we are dealing with a parameter-dependent domain, as one may expect, the frequency function depends on the parameter as well. Namely, we define

$$E(\varphi_i^\varepsilon, r, \lambda_i^\varepsilon, \varepsilon) := \frac{1}{r^{N-2}} \int_{B_r^+ \cup T_\varepsilon} (|\nabla \varphi_i^\varepsilon|^2 - \lambda_i^\varepsilon p \, |\varphi_i^\varepsilon|^2) \, \mathrm{d}x,$$
$$H(\varphi_i^\varepsilon, r) := \frac{1}{r^{N-1}} \int_{S_r^+} |\varphi_i^\varepsilon|^2 \, \mathrm{d}S \quad \text{and} \quad \mathcal{N}(\varphi_i^\varepsilon, r, \lambda_i^\varepsilon, \varepsilon) := \frac{E(\varphi_i^\varepsilon, r, \lambda_i^\varepsilon, \varepsilon)}{H(\varphi_i^\varepsilon, r)}$$

where φ_i^ε is an eigenfunction of the perturbed problem and λ_i^ε the corresponding eigenvalue. We also observe that, being φ_i^ε an eigenfunction, the term $\lambda_i^\varepsilon \int_{B_r^+ \cup T_\varepsilon} p \, |\varphi_i^\varepsilon|^2 \, \mathrm{d}x$ naturally appears. A similar approach has been pursued in [AF15] (see also [AF16, AFNN17]), where the authors made a significant step forward in understanding the power of this tool. In [AF15] the problem under consideration is the eigenvalue problem for the Aharonov-Bohm operator with one pole $a \in \Omega \subseteq \mathbb{R}^2$ and homogeneous Dirichlet boundary conditions and the aim is the detection of the asymptotic behavior of an eigenvalue λ_i^a as $a \to (0,0) \in \Omega$. The perturbed Almgren functions are defined as follows

$$E(\varphi_i^a, r, \lambda_i^a) := \int_{B_r} (|(i\nabla + A_a)\varphi_i^a|^2 - \lambda_i^a \, |\varphi_i^a|^2) \, \mathrm{d}x,$$
$$H(\varphi_i^a, r) := \frac{1}{r} \int_{\partial B_r} |\varphi_i^a|^2 \, \mathrm{d}S \quad \text{and} \quad \mathcal{N}(\varphi_i^a, r, \lambda_i^a) := \frac{E(\varphi_i^a, r, \lambda_i^a)}{H(\varphi_i^a, r)},$$

with A_a denoting the Aharonov-Bohm potential. We remark that the above definitions have sense only under geometrical constraints: when attaching a thin tube, if $r > \varepsilon$, i.e. if the upper half ball B_r^+ is larger than the tube; concerning the Aharonov-Bohm operator, the frequency function is useful only if the ball B_r contains the pole a, that is if $r > |a|$. Hence we immediately observe that it is not allowed to let $r \to 0$. Nevertheless, the (almost) monotonicity of the frequency, in these frameworks, is a key tool in establishing estimates of the energy of appropriately modified eigenfunctions, adapted to be admissible test functions in the min-max characterization of eigenvalues, see for instance Lemma 4.3.34.

Part I

Singularly perturbed eigenvalue problems

CHAPTER 3

THE EIGENVALUES OF THE LAPLACE OPERATOR

We begin this part by introducing the main characters of our investigation, i.e. the eigenvalues of the Laplace operator, in the cases of homogeneous Dirichlet and Neumann boundary conditions as well as for mixed type problems. We only give the main ideas of the procedure that leads to the definition of eigenvalues and eigenfunctions for the Laplace operator and we refer to [Bre11, Section 9.8] and [Jos13, Section 11.5] for a comprehensive exposition. We also point out that what we present in this chapter is a classical matter, nevertheless we believe it is useful for the reader to recall the principal concepts.

Let $N \geq 2$ and let $\Omega \subseteq \mathbb{R}^N$ be an open and connected set. The eigenvalue problem for the Laplace operator consists in finding a nonzero function $u\colon \Omega \to \mathbb{R}$ and a real number $\lambda \in \mathbb{R}$ such that

$$-\Delta u = \lambda u, \quad \text{in } \Omega. \tag{3.0.1}$$

The number λ is called *eigenvalue* and u a corresponding *eigenfunction*. It is natural to couple equation (3.0.1) together with the prescription of boundary conditions on u. In this thesis, we treat homogeneous Dirichlet boundary conditions, that is

$$u = 0, \quad \text{on } \partial\Omega,$$

homogeneous Neumann boundary conditions, i.e.

$$\frac{\partial u}{\partial \boldsymbol{\nu}} = 0, \quad \text{on } \partial\Omega,$$

where $\boldsymbol{\nu}$ denotes the exterior normal vector to $\partial\Omega$, and mixed Dirichlet-Neumann homogeneous boundary conditions, namely

$$u = 0, \quad \text{on } \Gamma_D \qquad \text{and} \qquad \frac{\partial u}{\partial \boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N,$$

being $\Gamma_D, \Gamma_N \subseteq \partial\Omega$ such that $\Gamma_D \cap \Gamma_N = \emptyset$ and $\Gamma_D \cup \Gamma_N = \partial\Omega$, with Γ_D relatively closed in $\partial\Omega$. We approach these eigenvalue problems through a weak formulation in a general abstract setting, which embraces the specific cases we consider. More precisely, we shall work in a separable Hilbert space $\mathcal{H} \subseteq L^2(\Omega)$: in particular, we choose

$$\mathcal{H} := H_0^1(\Omega)$$

for the Dirichlet case and

$$\mathcal{H} := H^1(\Omega)$$

for the Neumann case, where $H^1(\Omega)$ and $H_0^1(\Omega)$ denote the classical Sobolev spaces $W^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$, respectively. For what concerns the case of mixed boundary conditions, we let $\mathcal{H} := H^1_{0,\Gamma_D}(\Omega)$ be the closure of $C_c^\infty(\overline{\Omega} \setminus \Gamma_D)$ with respect to the $H^1(\Omega)$ norm. We recall that, when one deals with a regular domain (and regular interfaces Γ_D and Γ_N), it is possible to characterize the aforementioned Sobolev spaces making use of the notion of trace. Namely, we have that

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \colon u = 0, \text{ on } \partial\Omega\} \tag{3.0.2}$$

and

$$H^1_{0,\Gamma_D}(\Omega) = \{u \in H^1(\Omega) \colon u = 0, \text{ on } \Gamma_D\}, \tag{3.0.3}$$

where the values of u on $\partial\Omega$ has to be understood in the sense of traces. We assume Ω to be such that the embedding

$$i \colon \mathcal{H} \hookrightarrow L^2(\Omega)$$

is compact. As an example, $\mathcal{H} = H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ if Ω is bounded; if, in addition, Ω has Lipschitz boundary, then also $\mathcal{H} = H^1(\Omega)$ satisfies the assumption. Moreover, concerning the space $\mathcal{H} = H^1_{0,\Gamma_D}(\Omega)$, the compact embedding into $L^2(\Omega)$ holds if either $\partial\Omega$ is Lipschitz regular or if a Poincaré type inequality is available. We equip the Hilbert space with the following scalar products:

$$(u, v)_{\mathcal{H}} := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \quad \text{if} \quad \mathcal{H} = H_0^1(\Omega)$$

and

$$(u, v)_{\mathcal{H}} := \int_\Omega (\nabla u \cdot \nabla v + uv) \,\mathrm{d}x, \quad \text{if} \quad \mathcal{H} = H^1(\Omega) \quad \text{or} \quad \mathcal{H} = H^1_{0,\Gamma_D}(\Omega).$$

We now introduce the abstract (weak) formulation of the eigenvalue problems. We say that $u \in \mathcal{H}$ is an *eigenfunction* for the scalar product $(\cdot, \cdot)_{\mathcal{H}}$ with respect to $L^2(\Omega)$ and $\lambda \in \mathbb{R}$ is the corresponding *eigenvalue* if $u \neq 0$ and

$$(u, v)_{\mathcal{H}} = \lambda \int_\Omega uv \,\mathrm{d}x \quad \text{for all } v \in \mathcal{H}. \tag{3.0.4}$$

In order to prove the existence of such eigenvalues and eigenfunctions we appeal to the well known Hilbert-Schmidt (spectral) theorem.

Theorem 3.0.1 (Hilbert-Schmidt). *Let $(\mathcal{E}, (\cdot,\cdot)_{\mathcal{E}})$ be a separable Hilbert space and let*

$$T\colon \mathcal{E} \to \mathcal{E}$$

be a linear, self-adjoint, compact operator. Then there exists a sequence of real numbers $(\mu_n)_n$ and a sequence of vectors $(\varphi_n)_n \subseteq \mathcal{E}$ that are, respectively, eigenvalues and eigenvectors of T, that is

$$T\varphi_n = \mu_n \varphi_n, \quad \text{for all } n \in \mathbb{N}.$$

Moreover $(\varphi_n)_n$ is an orthonormal basis of $\mathcal{E}$ and, if $\mathcal{E}$ is infinite dimensional, then

$$\lim_{n\to\infty} \mu_n = 0.$$

Let us now apply the spectral theorem by properly choosing the space $\mathcal{E}$ and the operator T. Let us consider a function $f \in L^2(\Omega)$. Since $\mathcal{H} \hookrightarrow L^2(\Omega)$, then $L^2(\Omega) \hookrightarrow \mathcal{H}^*$, where $\mathcal{H}^*$ denotes the dual space of $\mathcal{H}$, and so, from the Riesz representation theorem we know that there exists a unique $u_f \in \mathcal{H}$ such that

$$(u_f, v)_{\mathcal{H}} = \int_\Omega f v \,\mathrm{d}x \quad \text{for all } v \in \mathcal{H}. \tag{3.0.5}$$

We can now define the so called *Green operator* as follows

$$\begin{aligned} \mathcal{G}\colon L^2 &\to \mathcal{H} \\ f &\mapsto u_f. \end{aligned}$$

We notice that, again by the Riesz representation theorem, $\|\mathcal{G}u\|_{\mathcal{H}} = \|u_f\|_{\mathcal{H}} = \|f\|_{L^2(\Omega)}$, which implies the continuity of the Green operator $\mathcal{G}$. Now we let $\mathcal{E} := L^2(\Omega)$ and

$$T := i \circ \mathcal{G}\colon \mathcal{E} \xrightarrow{\mathcal{G}} \mathcal{H} \xrightarrow{i} \mathcal{E}.$$

It is easy to see that the operator T satisfies all the assumptions of Theorem 3.0.1; in particular T is compact due to compactness of the embedding $\mathcal{H} \hookrightarrow \mathcal{E}$. Therefore there exists a sequence of eigenvalues $(\mu_n)_n$ and eigenvectors $(\varphi_n)_n \subseteq \mathcal{E}$ of T and by definition of the operator we know that $T\varphi_n \in \mathcal{H}$, and then $\varphi_n \in \mathcal{H}$ for all n. From (3.0.5) we can say that, for any $v \in \mathcal{H}$

$$(T\varphi_n, v)_{\mathcal{H}} = (\mathcal{G}\varphi_n, v)_{\mathcal{H}} = (\varphi_n, v)_{\mathcal{E}},$$

while, on the other hand, being φ_n an eigenvector

$$(T\varphi_n, v)_{\mathcal{H}} = \mu_n (\varphi_n, v)_{\mathcal{H}}.$$

Coupling these two facts we obtain that

$$(\varphi_n, v)_{\mathcal{H}} = \frac{1}{\mu_n} (\varphi_n, v)_{\mathcal{E}} \quad \text{for all } v \in \mathcal{H}, \tag{3.0.6}$$

thus meaning that $\varphi_n \in \mathcal{H}$ is an eigenfunction with respect to the scalar product $(\cdot,\cdot)_{\mathcal{H}}$ and $\lambda_n := 1/\mu_n$ is the corresponding eigenvalue. Moreover, letting $v = \varphi_n$ in the equation above we obtain that

$$\|\varphi_n\|_{\mathcal{H}}^2 = \lambda_n \|\varphi_n\|_{\mathcal{E}}^2 = \lambda_n,$$

which implies that $\lambda_n > 0$, and therefore, in view of Theorem 3.0.1, we have that $\lambda_n \to +\infty$. We also observe that, since

$$(\varphi_n, \varphi_m)_{\mathcal{E}} = \delta_n^m,$$

where δ_n^m denotes the usual Kronecker's delta, from (3.0.6) we have

$$(\varphi_n, \varphi_m)_{\mathcal{H}} = \lambda_n \delta_n^m.$$

Next, we show that, under suitable assumptions on the domain Ω, the λ_ns are actually the eigenvalues of the Laplace operator in the classical sense (3.0.1). By elliptic regularity theory (see e.g. [GT83, Corollary 8.11]) we know that $\varphi_n \in C^\infty(\Omega)$ and hence, by the divergence theorem

$$(\varphi_n, v)_{\mathcal{H}} = -\int_\Omega v\Delta\varphi_n \,\mathrm{d}x, \quad \text{for all } v \in C_c^\infty(\Omega)$$

in the case $\mathcal{H} = H_0^1(\Omega)$ and

$$(\varphi_n, v)_{\mathcal{H}} = \int_\Omega v(-\Delta\varphi_n + \varphi_n)\,\mathrm{d}x, \quad \text{for all } v \in C_c^\infty(\Omega)$$

if $\mathcal{H} = H^1(\Omega)$ or $\mathcal{H} = H^1_{0,\Gamma_D}(\Omega)$. In view of (3.0.4), these two facts imply that, respectively, there holds

$$-\Delta\varphi_n = \lambda_n\varphi_n, \quad \text{and} \quad -\Delta\varphi_n + \varphi_n = \lambda_n\varphi_n$$

in Ω. Therefore we have that the λ_n and $\lambda_n - 1$ are indeed eigenvalues of Dirichlet and Neumann or mixed Laplacian, respectively.

One can observe that the boundary conditions, for the Dirichlet parts, can be recovered (in the trace sense) by characterization of the spaces $H_0^1(\Omega)$ and $H^1_{0,\Gamma_D}(\Omega)$ (given in (3.0.2) and (3.0.3), respectively), whereas the Neumann boundary condition naturally arises by integration by parts.

We finally describe a few remarkable features of the eigenelements of the Laplacian. We again refer to [Jos13, Section 11.5] for a complete exposition on this topic. Here we additionally assume that Ω is bounded; moreover, we label the eigenvalues in nondecreasing order, i.e.

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

and we fix the orthonormal family of eigenfunctions $(\varphi_n)_n \subseteq \mathcal{H}$. We begin by describing the features of the first eigenelements.

Proposition 3.0.2. *The following properties hold true:*

1. *the first eigenvalue is strictly positive, i.e.* $\lambda_1 > 0$*;*
2. *the first eigenvalue is simple, i.e. the corresponding eigenspace is one dimensional;*
3. *the first eigenfunction does not change sign in* Ω*, i.e.* $\varphi_1 > 0$ *or* $\varphi_1 < 0$ *in* Ω*.*

Remark 3.0.3. In fact, in the Neumann case $\mathcal{H} = H^1(\Omega)$, we know something more. In particular, one can easily see that $\lambda_1 = 1$ and φ_1 is a constant function, which can be explicitly found due to normalization, i.e.

$$\varphi_1 = \pm |\Omega|_N^{-1/2},$$

with $|\Omega|_N$ denoting the N-dimensional Lebesgue measure of Ω.

We now present a variational, iterative characterization of the eigenvalues, which basically describes the procedure used to derive the sequences $(\lambda_n)_n$ and $(\varphi_n)_n$ in [Jos13, Section 11.5]. We recall the notation $\mathcal{E} = L^2(\Omega)$.

Proposition 3.0.4. *There holds*

$$\lambda_n = \min \left\{ \frac{\|u\|_{\mathcal{H}}^2}{\|u\|_{\mathcal{E}}^2} : u \in \mathcal{H} \text{ such that } (u, \varphi_i)_{\mathcal{E}} = 0 \text{ for all } i \leq n-1 \right\}.$$

We finally recall a second variational, non-iterative characterization of the eigenvalues, known in the literature as the Courant-Fischer Min-Max principle. It turns out to be a fundamental tool in our work.

Proposition 3.0.5. *There holds*

$$\lambda_n = \min \left\{ \max_{u \in V_n} \frac{\|u\|_{\mathcal{H}}^2}{\|u\|_{\mathcal{E}}^2} : V_n \text{ is a } n\text{-dimensional subspace of } \mathcal{H} \right\}.$$

Notation We gather here the notation we adopt throughout this part:

- $\mathbb{N}_* := \mathbb{N} \setminus \{0\}$;
- $B_R := \{x \in \mathbb{R}^N : |x| < R\}$ and $S_R = \partial B_R$ for, respectively, balls and spheres centered at the origin;
- $\mathbb{R}^N_+ := \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_N > 0\}$ for the upper half space;
- we may identify $\mathbb{R}^{N-1} := \partial \mathbb{R}^N_+$;
- $B_R^+ := B_R \cap \mathbb{R}^N_+$ and $S_R^+ := \partial B_R^+ \cap \mathbb{R}^N_+$ for half balls and half spheres;
- $\mathbb{S}^{N-1} := S_1$ and $\mathbb{S}^{N-1}_+ := S_1^+$ denote respectively the unitary sphere and upper unitary half-sphere;
- $B_R' := B_R \cap \partial \mathbb{R}^N_+$ denote the balls in $\mathbb{R}^N$.

CHAPTER 4

PERTURBATION OF THE DOMAIN

4.1 An introduction to spectral stability for the Dirichlet eigenvalues

In this chapter we consider a perturbation of the eigenvalue problem for the Laplacian with homogeneous Dirichlet boundary conditions. This perturbation amounts to a small, localized singular modification of the domain.

First of all, in this section we discuss the general problem of spectral stability for the Dirichlet-Laplacian. We let $N \geq 2$ and we let $\Omega \subseteq \mathbb{R}^N$ be a bounded, open and connected set.

First of all, we consider the unperturbed problem, which consists in

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.1.1}$$

This boundary value problem must be thought in a weak sense. In particular, we say that $\lambda \in \mathbb{R}$ is an *eigenvalue* and $\varphi \not\equiv 0$ is a corresponding *eigenfunction* if $\varphi \in H^1_0(\Omega)$ and

$$\int_\Omega \nabla\varphi \cdot \nabla v \,\mathrm{d}x = \lambda \int_\Omega \varphi v \,\mathrm{d}x \quad \text{for all } v \in H^1_0(\Omega). \tag{4.1.2}$$

With reference to Chapter 3, we know that there exists a diverging sequence of eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

and a corresponding sequence of eigenfunctions, which we denote by $(\varphi_n)_n$, that forms an orthonormal basis of $H^1_0(\Omega)$.

Now let us introduce a perturbation of problem (4.1.1). We consider a family of bounded connected domains Ω^ε, indexed by a real parameter $\varepsilon \in (0,1)$. One must think Ω^ε to be closer and closer to Ω as ε approaches 0 (more details about "convergence" of domains will be given later). We consider, on each Ω^ε, the corresponding eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & \text{in } \Omega^\varepsilon, \\ \varphi = 0, & \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{4.1.3}$$

meant in a weak sense. Reasoning as before, we have that for any $\varepsilon \in (0,1)$ there exists a diverging sequence of positive eigenvalues of (4.1.3)

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon \le \lambda_3^\varepsilon \le \cdots \to +\infty,$$

repeated as many times as their multiplicity. Similarly, for every $\varepsilon \in (0,1)$, there exists a sequence of eigenfunctions $\{\varphi_n^\varepsilon\}_n \subseteq H_0^1(\Omega^\varepsilon)$ which forms an orthonormal basis of $L^2(\Omega^\varepsilon)$.

Now the following natural questions raise:

If Ω^ε is *close* to Ω, then is λ_n^ε close to λ_n, fo any n?

If yes, how close is λ_n^ε to λ_n in terms of ε?

Before trying to answer these questions, it is important to specify in which sense Ω^ε should be close to Ω. The suitable notion for our purposes turns out to be the convergence of sets in the sense of Mosco, introduced for the first time in 1969 in [Mos69].

Definition 4.1.1. Let $\varepsilon \in (0,1)$ and let $U_\varepsilon, U \subseteq \mathbb{R}^N$ be open sets. We say that U_ε *is converging to* U *in the sense of Mosco* as $\varepsilon \to 0$ if the following two properties hold:

(i) the weak limit points (as $\varepsilon \to 0$) in $H^1(\mathbb{R}^N)$ of every family of functions $\{u_\varepsilon\}_\varepsilon \subseteq H^1(\mathbb{R}^N)$, such that $u_\varepsilon \in H_0^1(U_\varepsilon)$ for every $\varepsilon > 0$, belong to $H_0^1(U)$;

(ii) for every $u \in H_0^1(U)$ there exists a family $\{u_\varepsilon\}_\varepsilon \subseteq H^1(\mathbb{R}^N)$ such that $u_\varepsilon \in H_0^1(U_\varepsilon)$ for every $\varepsilon > 0$ and $u_\varepsilon \to u$ in $H^1(\mathbb{R}^N)$, as $\varepsilon \to 0$.

We may also say that $H_0^1(U_\varepsilon)$ *is converging to* $H_0^1(U)$ *in the sense of Mosco.*

Hereafter, we refer to the domain Ω as the *unperturbed* (or *limit*) domain and to the domains Ω^ε as the *perturbed domains*. An analogous nomenclature is used for the eigenvalues $(\lambda_n)_n$ and $(\lambda_n^\varepsilon)_n$. From [Dan03, Section 4] we derive a sufficient condition for the stability with respect to ε of the Dirichlet eigenvalues (see also [BZ98]).

Theorem 4.1.2. *Let $\Omega, \Omega^\varepsilon \subseteq \mathbb{R}^N$ be open, bounded and connected for every $\varepsilon \in (0,1)$ and assume Ω^ε converges to Ω in the sense of Mosco as $\varepsilon \to 0$. Then, for any $n \in \mathbb{N}$,*

$$\lambda_n^\varepsilon \to \lambda_n \quad \textit{as } \varepsilon \to 0.$$

If the question of stability of the eigenvalues with respect to a perturbation has been solved in a pretty general framework, the problem of quantitatively estimating the difference $|\lambda_n^\varepsilon - \lambda_n|$ in terms of ε is much more subtle. Therefore, if investigating the asymptotic behavior of the perturbed eigenvalues λ_n^ε as $\varepsilon \to 0$ is, in its general formulation, an ambitious task, focusing on some particular classes of perturbation makes us able to give partial answers.

The first example of a perturbation one may think of, is a *regular perturbation* of the domain. By this we mean that the perturbed domain Ω^ε is the image of Ω through a sufficiently smooth diffeomorphism, i.e. $\Omega^\varepsilon = h_\varepsilon(\Omega)$ for some $h_\varepsilon \in C^r(\Omega;\mathbb{R}^N)$, being r a positive integer. We denote by $\mathcal{K}_r(\Omega)$ the collection of all regions which are C^r-diffeomorphic to Ω. The first stability result in this framework has been achieved in the early 20th century by Courant and Hilbert: in [CH53] they proved the continuity of the eigenvalues with respect to the perturbing diffeomorphism, taking into account various types of boundary conditions (see Theorem 10, Chapter VI, § 2). Some years later, in [Mic72], Micheletti gave consistency to this approach and proved that $\mathcal{K}_r(\Omega)$ is metrizable with respect to a suitable topology and may be considered as a separable complete metric space. We also mention [BV65], where the authors proved continuity of eigenvalues for general higher order differential operators. Another remarkable result in this direction is the celebrated Hadamard formula for the derivative of a simple eigenvalue with respect to the perturbation parameter. In spite of its name, this formula was first formally computed by Lord Rayleigh in some special cases in late 1800 (see [Ray45, eq. 11 pag. 338]) and then for two dimensional domains by Hadamard in 1908, see [Had08]. However, the fist rigorous proof was made in [GS53]. We briefly recall it and we refer to [Hen05, Section 3.6] for a comprehensive description. Let Ω be a fixed bounded domain with smooth boundary. Let $h(\cdot,\varepsilon)$ be a family of diffeomorphisms in $\mathcal{K}_2(\Omega)$, of class C^1 with respect to the variable $\varepsilon \in (0,1)$, and assume that $h(x,0) = i_\Omega(x)$, where i_Ω denotes the identity over Ω. Let us assume that $\lambda_n = \lambda_n(\Omega)$ is a simple eigenvalue (i.e. with a one dimensional eigenspace) of the Dirichlet-Laplacian over Ω and let λ_n^ε be the n-th eigenvalue of the Dirichlet-Laplacian on the perturbed domain $\Omega^\varepsilon = h(\Omega,\varepsilon)$. If we denote by φ_n the unique (up to a sign) eigenfunction corresponding to λ_n such that

$$\int_\Omega \varphi_n^2 \,\mathrm{d}x = 1,$$

then the following formula holds

$$\frac{\partial \lambda_n^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} = -\int_{\partial\Omega} \left(\frac{\partial \varphi_n}{\partial \boldsymbol{\nu}}(x)\right)^2 \frac{\partial h}{\partial \varepsilon}(x,0)\cdot \boldsymbol{\nu}(x)\,\mathrm{d}S(x).$$

Since then, a quantity of formulas of this kind has been proved for different types of problems and for multiple limit eigenvalues (see, among many others, [Hen06, Section 2.5] for a complete survey). We remark that, recently, an analogous formula has been derived for the eigenvalues of the fractional Laplacian, see [DFW20]. Again concerning regular perturbations of the domain it is worth mentioning the result obtained in [BL08] (see also [BL05] and [LLdC05]), where the authors proved estimates on the eigenvalue variation in a pretty general framework. Namely, as a particular case, they proved that if Ω and Ω^ε are domains of class $C^{1,1}$, then

$$|\lambda_n - \lambda_n^\varepsilon| \le C_n \,|\Omega \vartriangle \Omega^\varepsilon|_N\,,$$

where $|\cdot|_N$ denotes the N-dimensional Lebesgue measure of a set and C_n is a positive constant depending on the index n. We also cite the seminal paper [RT75], where the authors investigate some singular perturbations of the domain. We finally refer the interested readers to [BLLdK06], [Hal05] and [Hen05] (see also [Hen06]) for fairly complete surveys on regular domain perturbation theory in connection with the spectrum of differential operators.

4.2 A singular perturbation

Description of the perturbation In the rest of present chapter, we consider a particular singular perturbation of the domain, which does not fit in any of the previous results. In broad terms, the perturbed domain Ω^ε is made by attaching a tube of finite length and section with radius of order ε to the fixed, "limit" domain Ω. In order to be more precise, let us first set up the geometrical framework and little notation. We denote by

$$\mathbb{R}^N_+ := \{(x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N : x_N > 0\}$$

the upper half space and, for any $r > 0$,

$$B_r^+ := \{x \in \mathbb{R}^N_+ \colon |x| < r\}, \quad S_r^+ := \partial B_r^+ \cap \mathbb{R}^N_+, \quad B_r' := \{(x', 0) \in \partial\mathbb{R}^N_+ \colon |x'| < r\},$$

the upper half balls, half spheres and thin balls, respectively (centered at the origin). Let us assume that $0 \in \partial\Omega$ and that $\partial\Omega$ is flat in a neighborhood of the origin, i.e.

$$\text{there exists } R_{\max} > 0 \text{ such that } B'_{R_{\max}} \subseteq \partial\Omega. \tag{4.2.1}$$

Without loss of generality we can assume that

$$R_{\max} > 1 \quad \text{and} \quad B^+_{R_{\max}} \subseteq \Omega \subseteq \mathbb{R}^N_+,$$

see Figure 4.1 (a). Now let us rigorously introduce the perturbation. Let $\Sigma \subseteq B'_{R_{\max}}$ be open in the $\mathbb{R}^{N-1}$ topology, containing the origin and with boundary $\partial\Sigma$ (always thought in $\mathbb{R}^{N-1}$) of class $C^{1,1}$; nevertheless, this last regularity assumption can be relaxed, see Remark 4.3.14. Moreover we assume, for sake of simplicity, that Σ is contained in the unitary thin ball, i.e.

$$\Sigma \subseteq B_1'.$$

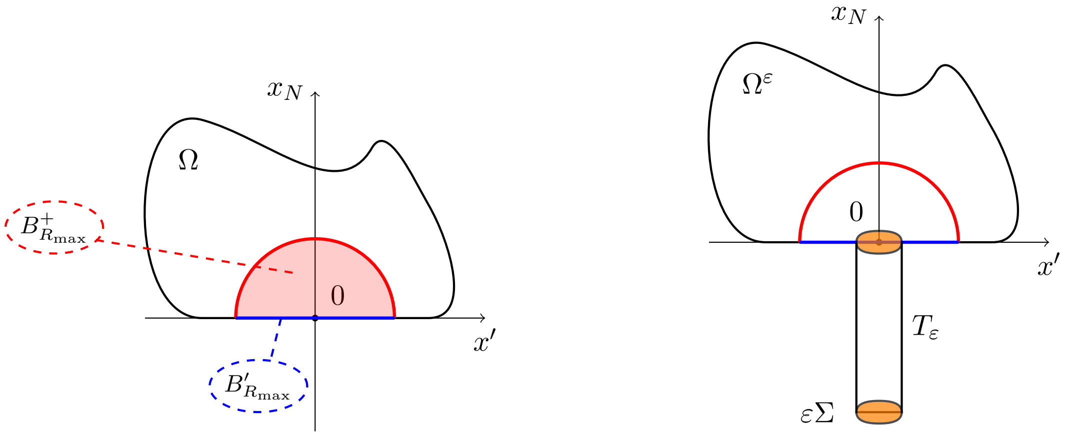
Finally we assume that Σ is **starshaped** with respect to 0, i.e.

$$x \cdot \boldsymbol{\nu}_\Sigma \geq 0 \text{ for all } x \in \partial\Sigma, \tag{4.2.2}$$

where $\boldsymbol{\nu}_\Sigma$ denotes the exterior unit normal vector to $\partial\Sigma$. Let $\varepsilon \in (0, 1]$ and let $T_\varepsilon := \varepsilon\Sigma \times (-1, 0]$ be a cylindrical tube with section $\varepsilon\Sigma := \{\varepsilon x \colon x \in \Sigma\}$. We point out that the length of the tube T_ε is not relevant for our purposes, therefore we fixed it to 1. Let us denote by

$$\Omega^\varepsilon = \Omega \cup T_\varepsilon \tag{4.2.3}$$

the *perturbed* domain (see Figure 4.1 (b)). We remark that this perturbation cannot be



(a) *Unperturbed Domain* (b) *Perturbed Domain*

Figure 4.1

regarded as a regular one, in the sense of the previous section, since there does not exists a regular family of diffeomorphisms that map Ω in Ω^ε. Moreover, Ω^ε is not even converging to Ω in the sense of Hausdorff distance.

We can actually treat eigenvalue problems slightly more general than (4.1.1). More precisely, let $p \in L^\infty(\mathbb{R}^N)$ be a weight function such that $p \geq 0$ a.e. and $p \not\equiv 0$ in Ω. For any open, bounded set $\omega \subseteq \mathbb{R}^N$, we consider the weighted Dirichlet eigenvalue problem for the Laplacian on ω

$$\begin{cases} -\Delta\varphi = \lambda p \varphi, & \text{in } \omega, \\ \varphi = 0, & \text{on } \partial\omega, \end{cases} \tag{E_ω}$$

thought in a weak sense, that is $\varphi \in H^1_0(\omega)$ and

$$\int_\omega \nabla\varphi \cdot \nabla v \,\mathrm{d}x = \lambda \int_\omega p\varphi v \,\mathrm{d}x \quad \text{for all} \quad v \in H^1_0(\omega).$$

By slight modifications of the arguments of Chapter 3, one can prove that, if $p \not\equiv 0$ in ω, there exists a sequence of positive eigenvalues of (E_ω)

$$0 < \lambda_1(\omega) < \lambda_2(\omega) \leq \lambda_3(\omega) \leq \dots$$

repeated according to their multiplicity. We denote by $(\lambda_n)_n := (\lambda_n(\Omega))_n$ the sequence of eigenvalues of the unperturbed problem (E_Ω), and by $(\varphi_n)_n$ a corresponding sequence of eigenfunctions such that

$$\int_\Omega p\,|\varphi_n|^2 \,dx = 1 \quad \text{and} \quad \int_\Omega p\varphi_n\varphi_m \,dx = 0 \quad \text{if} \quad n \neq m.$$

Similarly, we denote by $(\lambda^\varepsilon_n)_n := (\lambda_n(\Omega^\varepsilon))_n$ and $(\varphi^\varepsilon_n)_n$ the sequences of eigenvalues and eigenfunctions of the perturbed problem (E_{Ω^ε}), such that

$$\int_{\Omega^\varepsilon} p\,|\varphi^\varepsilon_n|^2 \,dx = 1 \quad \text{and} \quad \int_{\Omega^\varepsilon} p\varphi^\varepsilon_n\varphi^\varepsilon_m \,dx = 0 \quad \text{if} \quad n \neq m.$$

Let $n_0 \in \mathbb{N}$ be such that

$$\lambda_0 := \lambda_{n_0} \quad \text{is } \textbf{simple}. \tag{4.2.4}$$

Assumption (4.2.4) is not so restrictive: indeed, the simplicity of all eigenvalues is a generic property with respect to perturbations of the domain , see [Mic72, Uhl76]. Hereafter, we use the following notation for the perturbed n_0-th eigenvalue and eigenfunction, respectively

$$\lambda_\varepsilon := \lambda_{n_0}^\varepsilon \quad \text{and} \quad \varphi_\varepsilon := \varphi_{n_0}^\varepsilon.$$

As we said in Section 4.1, classical results (see for instance [BZ98, Dan03]) ensure the continuity with respect to our domain perturbation, i.e. λ_ε is simple for ε small and

$$\lambda_\varepsilon \longrightarrow \lambda_0 \quad \text{as } \varepsilon \to 0. \tag{4.2.5}$$

Furthermore, for every ε we can choose the eigenfunction φ_ε in such a way that

$$\varphi_\varepsilon \longrightarrow \varphi_0 \quad \text{in } H_0^1(\Omega^1) \quad \text{as } \varepsilon \to 0, \tag{4.2.6}$$

where the functions are trivially extended in Ω^1 outside their domains.

The main goal of this part of the thesis is to find the leading term in the asymptotic expansion of the difference $\lambda_0 - \lambda_\varepsilon$ as $\varepsilon \to 0$.

The behavior of the spectrum of the Dirichlet-Laplacian when the domain is subject to a singular perturbation has been widely studied in the past. In particular, concerning the perturbation through the attachment of a thin tube, in [Gad03] the author proved a power asymptotic expansion in ε for the difference $\lambda_0 - \lambda_\varepsilon$. However, Theorem 4.1 in [Gad03] provides the exact vanishing rate of the eigenvalue variation just in the case in which $\nabla\varphi_0(0) \neq 0$, but it does not says what happens when the limit eigenfuction has, in the origin, a zero of order higher than one. We also mention the work [ACG10] concerning a related two dimensional perturbed problem with thin shoots. In [Tay13] the author considers, in a framework similar to ours, an eigenvalue problem for a vector valued elliptic operator and gave any estimate of the type $|\lambda_0 - \lambda_\varepsilon| = O(\varepsilon^a)$, with the exponent a depending on the distance of λ_0 from its neighbors. The starting points of the present work are [Gad03] and [AFT14a, AFT14b, FT13]. On one hand, we aim at providing a criterion for selecting the leading term in the asymptotic expansion given in [Gad03], based on the vanishing order of the limit eigenfunction at the junction; on the other hand, we improve and generalize some results of [AFT14a]. We note that [AFT14a] (as well as many of the aforementioned articles) deals with dumbbell domains in which the tubular handle is vanishing. However, from the point of view of both the expected results and the technical approach, our method does not require substantial adaptions to treat also the dumbbell case; hence for the sake of simplicity of exposition, in the present part we consider only perturbations of type (4.2.3).

A motivation for the interest in studying the spectral behavior of the Laplacian on thin branching domains comes from physics: for instance, it occurs in the theory of quantum graphs, which models the propagation of waves in quasi one-dimensional structures, like quantum wires, narrow waveguides, photonic crystals, blood vessels, etc. (see e.g.

[BK13, Kuc05] and reference therein). Moreover, this topic is also related with engineering problems, such as elasticity and multi-structure problems, as well explained in surveys [Cia90, Mov06]. We also point out that our results may be seen in the framework of scattering theory: indeed, in [HM91] the authors proved that, if we consider a resonator made of an unbounded domain Ω^{ex} such that $\Omega \subseteq \mathbb{R}^N \setminus \Omega^{\mathrm{ex}}$ and connected with Ω through a thin channel T_ε, the squares of the scattering frequencies of the resonator $\Omega \cup T_\varepsilon \cup \Omega_{\mathrm{ex}}$ coincide with the Dirichlet eigenvalues on $\Omega \cup T_\varepsilon$ up to an exponentially small (in ε) error.

Statement of the main results In order to state our main results, we first need to recall some known facts. Let us consider the eigenvalue problem for the standard Laplacian on the $(N-1)$-dimensional unit sphere

$$-\Delta_{\mathbb{S}^{N-1}}\Psi = \mu\Psi \quad \text{in } \mathbb{S}^{N-1}. \tag{4.2.7}$$

It is well known that the eigenvalues of (4.2.7) are $\mu_n = n(n+N-2)$, for $n = 0, 1, \dots$ and that their multiplicities are (see [BGM71])

$$\mathfrak{m}_n = \binom{n+N-2}{n} + \binom{n+N-3}{n-1}.$$

We denote by E_n the eigenspace corresponding to the eigenvalue μ_n. We recall that the elements of E_n are spherical harmonics, i.e. the restriction to $\mathbb{S}^{N-1}$ of harmonic homogeneous polynomials (of N variables) of degree n. It is known that the local behavior of solutions to (E_Ω) near $0 \in \partial\Omega$ can be describe in terms of spherical harmonics vanishing on $\{x_N = 0\}$, see e.g. [Ber55] and [FFT11, Theorem 1.3]. Therefore, if we denote

$$E_n^0 := \{\psi \in E_n \colon \psi(\theta_1, \dots, \theta_{N-1}, 0) = 0\},$$

we have that there exist $k \in \mathbb{N}$, $k \geq 1$ and $\Psi_k \in E_k^0$ such that

$$r^{-k}\varphi_0(r\theta) \to \Psi_k(\theta) \quad \text{in } C^{1,\tau}(S_1^+), \quad \text{as } r \to 0^+, \tag{4.2.8}$$

$$r^{1-k}\nabla\varphi_0(r\theta) \to \nabla_{\mathbb{S}^{N-1}}\Psi_k(\theta) \quad \text{in } C^{0,\tau}(S_1^+, \mathbb{R}^N), \quad \text{as } r \to 0^+, \tag{4.2.9}$$

for all $\tau \in (0,1)$. Furthermore the asymptotic homogeneity order k can be characterized as the limit of an Almgren frequency function (see [Alm83]), i.e.

$$\lim_{r\to 0^+} \frac{r\int_{B_r^+}\left(|\nabla\varphi_0|^2 - \lambda_0|\varphi_0|^2\right)\,\mathrm{d}x}{\int_{S_r^+}|\varphi_0|^2\,\mathrm{d}S} = k.$$

Hereafter we will denote

$$\psi_k(r\theta) := r^k\Psi_k(\theta), \quad r \geq 0,\ \theta \in S_1^+. \tag{4.2.10}$$

The sharp asymptotic estimate of the eigenvalue variation we are going to prove involves a nonzero constant $m_k(\Sigma)$ which admits the following variational characterization. Let us consider the functional

$$\begin{aligned} J \colon \mathcal{D}^{1,2}(\Pi) &\longrightarrow \mathbb{R}, \\ J(u) :=& \frac{1}{2}\int_{\Pi} |\nabla u|^2 \,\mathrm{d}x - \int_{\Sigma} u \frac{\partial \psi_k}{\partial x_N} \,\mathrm{d}x', \end{aligned}$$

where

$$T_1^- := \Sigma \times (-\infty, 0], \qquad \Pi := T_1^- \cup \mathbb{R}^N_+,$$

and, for any open set $\omega \subseteq \mathbb{R}^N$, $\mathcal{D}^{1,2}(\omega)$ denotes the completion of the space $C_c^\infty(\omega)$ with respect to the L^2 norm of the gradient (see Section 4.3.1 for further details). In dimension 2, we will always deal with spaces $\mathcal{D}^{1,2}(\omega)$ with ω such that $\mathbb{R}^N \setminus \omega$ contains a half-line; in this case $\mathcal{D}^{1,2}(\omega)$ can be characterized as a concrete functional space thanks to the validity of a Hardy inequality also in dimension 2, see Theorem 4.3.2.

By standard minimization methods, one can prove that J is bounded from below and that the infimum

$$m_k(\Sigma) := \inf_{u \in \mathcal{D}^{1,2}(\Pi)} J(u) \tag{4.2.11}$$

is attained by some w_k. Moreover

$$m_k(\Sigma) = -\frac{1}{2}\int_{\Pi} |\nabla w_k|^2 \,\mathrm{d}x = -\frac{1}{2}\int_{\Sigma} w_k \frac{\partial \psi_k}{\partial x_N} \,\mathrm{d}x' < 0, \tag{4.2.12}$$

see [FT13]. With this framework in mind we are able to state the first (and main) result of this chapter.

Theorem 4.2.1. *Under assumptions (4.2.1), (4.2.2) and (4.2.4), let k denote the vanishing order of the unperturbed eigenfunction φ_0 as in (4.2.8)–(4.2.9). Then*

$$\lambda_\varepsilon = \lambda_0 - C_k(\Sigma)\varepsilon^{N+2k-2} + o(\varepsilon^{N+2k-2}),$$

as $\varepsilon \to 0$, where

$$C_k(\Sigma) = -2m_k(\Sigma) > 0 \tag{4.2.13}$$

and $m_k(\Sigma)$ is defined in (4.2.11).

We recall that, for $N \geq 3$, an asymptotic expansion for the eigenvalue variation is constructed using the *concordance method* in [Gad03, Theorem 4.1], but explicit formulas are given only for the first perturbed coefficient, which turns out to be a multiple of $|\nabla \varphi_j(0)|^2$; in dimension $N = 2$, [Gad03, Theorem 10.1] performs a more detailed asymptotic analysis with the computation of all the coefficients and takes into account also multiple limit eigenvalues. Hence, for $N \geq 3$, [Gad03] finds outs what is the leading term in the asymptotic expansion only when $\nabla \varphi_j(0) \neq 0$. We emphasize that, differently from [Gad03], Theorem 4.2.1 detects the exact vanishing rate of $\lambda_0 - \lambda_\varepsilon$ also when $\nabla \varphi_0(0) = 0$ and $N \geq 3$; more precisely it establishes a direct correspondence between the order of the infinitesimal $\lambda_0 - \lambda_\varepsilon$ and the number k, which is the order of vanishing of φ_0 at the junction point 0.

The proof of Theorem 4.2.1 is based on lower and upper bounds for the difference $\lambda_0 - \lambda_\varepsilon$ carried out using the Min-Max Courant-Fischer characterization of the eigenvalues, see Section 4.3.5. To obtain the exact asymptotics for the eigenvalue variation it is crucial to sharply control the energy of perturbed eigenfunctions in neighborhoods of the junction with radius of order ε. The sharpness of our energy estimates is related to the identification of a nontrivial limit profile for blow-up of scaled eigenfunctions, as stated in the following theorem.

Theorem 4.2.2. *Under the same assumptions of Theorem 4.2.1, let φ_ε be chosen as in* (4.2.6)*. Then*

$$\varepsilon^{-k}\varphi_\varepsilon(\varepsilon x) \to \Phi(x) \quad \text{as } \varepsilon \to 0,$$

in $H^1(T_1^- \cup B_R^+)$ for all $R > 1$, where $\Phi := w_k + \psi_k$, being w_k the minimizer for (4.2.11) *and ψ_k the homogeneous function defined in* (4.2.10)*.*

As mentioned before, Theorem 4.2.1 generalizes and improves [AFT14a, Th. 1.1]: indeed, in [AFT14a] the weight p was assumed to vanish in a neighborhood of the junction Σ and only the case of vanishing order $k = 1$ for the unperturbed eigenfunction φ_0 was considered. Furthermore the dimension $N = 2$ was not included in [AFT14a]. As in [AFT14a], a fundamental tool for the proof of the energy estimates needed to study the local behavior of eigenfunctions is an Almgren-type monotonicity formula, which was first introduced by Almgren [Alm83] and then used by Garofalo and Lin [GL86] to study unique continuation properties for elliptic partial differential equations.

In the particular case treated in [AFT14a, FT13], precise pointwise estimates from above and from below for the perturbed eigenfunction and its gradient were directly obtained via comparison and maximum principles: indeed, if the limit eigenfunction has minimal vanishing order at the origin and the weight vanishes around the junction, then such eigenfunction has a fixed sign and is harmonic in a neighborhood of 0. These estimates were used in [FT13] to get rid of a remainder term in the derivative of the Almgren quotient for the perturbed problem, however they are not available in the more general framework of the present chapter. Nevertheless, under the geometric assumption (4.2.2) on the tube section we succeed in proving that the remainder term has a positive sign, thus obtaining the monotonicity formula, see Proposition 4.3.27. We also point out that the 2-dimensional case requires the proof of an *ad hoc* Hardy type inequality for functions vanishing on a fixed half-line, see (4.3.5).

We observe that in [AFT14a, FT13] the limit of the blow-up family $\varepsilon^{-k}\varphi_\varepsilon(\varepsilon x)$ was recognized by its frequency at infinity, which must be necessarily equal to the minimal one, i.e. 1, in the particular case $k = 1$. In the general case $k \geq 1$, the monotonicity argument implies that the frequency of the limit profile is less than or equal to k, and this seems to be not enough for a univocal identification. To overcome this difficulty, we use here an argument inspired by [AF15] and based on a local inversion result giving an energy control for the difference between the blow-up eigenfunction and a k-homogeneous profile, see Corollary 4.3.48.

4.3 Proof of the asymptotic expansion

In this section we are going to prove our main results Theorem 4.2.1 and Theorem 4.2.2. We organize it as follows. After some preliminary results in Subsection 4.3.1, in Subsection 4.3.2 we prove a Pohozaev-type identity, which is combined with the Poincaré inequalities of Subsection 4.3.3 to develop a monotonicity argument in Subsection 4.3.4. From the monotonicity formula established in Corollary 4.3.28, we derive some local energy estimates which allow us to deduce sharp upper and lower bounds for the eigenvalue variation in Subsection 4.3.5. In Subsection 4.3.6 we perform a blow-up analysis for scaled eigenfunctions from which we deduce first Theorem 4.2.2 and then, in Subsection 4.3.7, our main result Theorem 4.2.1.

4.3.1 Preliminaries and notation

In this subsection we introduce some basic definitions and notation and we state some introductory results which will be useful in the rest of this chapter. We start fixing some notation:

$$\begin{aligned} &\Omega_r^\varepsilon := T_\varepsilon \cup B_r^+, \quad \varepsilon \in (0,1),\ r \in (\varepsilon, R_{\max}),\\ &\Pi_r := T_1^- \cup B_r^+, \quad r > 1. \end{aligned}$$

For any measurable set $\omega \subseteq \mathbb{R}^N$, we denote as $|\omega|$ its N-dimensional Lebesgue measure and, if ω is a regular domain and there is no ambiguity, we denote by $\boldsymbol{\nu}(x)$ the exterior unit normal to $\partial\omega$ at the point $x \in \partial\omega$.

For any $R \geq 2$, we will denote as η_R a cut-off function satisfying

$$\begin{aligned} &\eta_R \in C^\infty(\overline{\Pi}), \quad \eta_R(x) = \begin{cases} 1, & \text{for } x \in \Pi \setminus \Pi_R, \\ 0, & \text{for } x \in \Pi_{R/2}, \end{cases} \\ &|\eta_R(x)| \leq 1, \quad |\nabla \eta_R(x)| \leq 4/R \qquad \text{for all } x \in \Pi. \end{aligned} \tag{4.3.1}$$

We now recall a well known quantitative result about the first eigenvalue of the Dirichlet-Laplacian on bounded domains.

Theorem 4.3.1 (Faber-Krahn Inequality)**.** *Let $\omega \subseteq \mathbb{R}^N$ be open and bounded and let $\lambda_1^D(\omega)$ denote the first eigenvalue of the Dirichlet-Laplacian on ω. Then*

$$\lambda_1^D(\omega) \geq \frac{\lambda_1^D(B_1)\,|B_1|^{2/N}}{|\omega|^{2/N}},$$

where B_1 denotes the N-dimensional ball centered at the origin and with radius 1.

Moreover, if we denote

$$C_N := \frac{1}{|B_1|^{2/N}\,\lambda_1^D(B_1)}, \tag{4.3.2}$$

and we combine the previous Theorem with the usual Poincaré Inequality, we have that

$$\int_\omega |u|^2 \, \mathrm{d}x \leq C_N \, |\omega|^{2/N} \int_\omega |\nabla u|^2 \, \mathrm{d}x \quad \text{for all } u \in H_0^1(\omega). \tag{4.3.3}$$

It is well known that the classical Hardy's Inequality

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, \mathrm{d}x \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x, \quad u \in C_c^\infty(\mathbb{R}^N), \quad N \geq 3,$$

fails in dimension 2. However we observe, in the following theorem, that, under a vanishing condition on part of the domain (at least on a half-line), it is possible to recover a Hardy-type Inequality even in dimension 2.

Let $\boldsymbol{p} = (x_{\boldsymbol{p}}, 0) \in \mathbb{R}^2$ with $x_{\boldsymbol{p}} > 0$ and let $s_{\boldsymbol{p}} := \{(x,0) \colon x \geq x_{\boldsymbol{p}}\}$. Let $\mathcal{D}^{1,2}(\mathbb{R}^N \setminus s_{\boldsymbol{p}})$ denote the completion of the space $C_c^\infty(\mathbb{R}^2 \setminus s_{\boldsymbol{p}})$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N \setminus s_{\boldsymbol{p}})} := \left(\int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x\right)^{1/2}.$$

Let us consider the function

$$\theta_{\boldsymbol{p}} \colon \mathbb{R}^2 \setminus s_{\boldsymbol{p}} \longrightarrow (0, 2\pi), \quad \theta_{\boldsymbol{p}}(x_{\boldsymbol{p}} + r\cos t, r \sin t) = t.$$

We have that $\theta_{\boldsymbol{p}} \in C^\infty(\mathbb{R}^2 \setminus s_{\boldsymbol{p}})$.

Theorem 4.3.2. *For all $\varphi \in C_c^\infty(\mathbb{R}^2 \setminus s_{\boldsymbol{p}})$*

$$\frac{1}{4} \int_{\mathbb{R}^2} \frac{|\varphi(z)|^2}{|z - \boldsymbol{p}|^2} \, \mathrm{d}z \leq \int_{\mathbb{R}^2} |\nabla \varphi(z)|^2 \, \mathrm{d}z. \tag{4.3.4}$$

Moreover the space $\mathcal{D}^{1,2}(\mathbb{R}^N \setminus s_{\boldsymbol{p}})$ can be characterized as

$$\mathcal{D}^{1,2}(\mathbb{R}^N \setminus s_{\boldsymbol{p}}) = \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^2) : \nabla u \in L^2(\mathbb{R}^2), \ \tfrac{u}{|z - \boldsymbol{p}|} \in L^2(\mathbb{R}^2), \ \text{and } u = 0 \text{ on } s_{\boldsymbol{p}}\right\}$$

and inequality (4.3.4) holds for every $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus s_{\boldsymbol{p}})$.

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^2 \setminus s_{\boldsymbol{p}})$ and let $\tilde{\varphi}(z) := \varphi(z) e^{i\frac{\theta_{\boldsymbol{p}}(z)}{2}} \in C_c^\infty(\mathbb{R}^2 \setminus s_{\boldsymbol{p}}, \mathbb{C})$. By direct calculations, we have that

$$i\nabla\varphi(z) = e^{-i\frac{\theta_{\boldsymbol{p}}(z)}{2}} (i\nabla + \mathrm{A}_{\boldsymbol{p}})\tilde{\varphi}(z),$$

where

$$\mathrm{A}_{\boldsymbol{p}}(x,y) := \frac{1}{2}\left(\frac{-y}{(x - x_{\boldsymbol{p}})^2 + y^2}, \frac{x - x_{\boldsymbol{p}}}{(x - x_{\boldsymbol{p}})^2 + y^2}\right)$$

is the Aharonov-Bohm vector potential with pole $\boldsymbol{p}$ and circulation $1/2$. Now let us compute the L^2-norm of $|i\nabla\varphi|$ and use the Hardy's Inequality for Aharonov-Bohm operators (see [LW99]):

$$\int_{\mathbb{R}^2} |\nabla \varphi|^2 \, \mathrm{d}z = \int_{\mathbb{R}^2} |(i\nabla + \mathrm{A}_{\boldsymbol{p}})\tilde{\varphi}|^2 \, \mathrm{d}z \geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{|\tilde{\varphi}|^2}{|z - \boldsymbol{p}|^2} \, \mathrm{d}z = \frac{1}{4} \int_{\mathbb{R}^2} \frac{|\varphi|^2}{|z - \boldsymbol{p}|^2} \, \mathrm{d}z.$$

The second part of the statement follows from (4.3.4) by classical completion and density arguments. $\square$

Corollary 4.3.3. *There exists $C = C(\boldsymbol{p}) > 0$ such that*

$$\int_{\mathbb{R}^2} \frac{|\varphi(z)|^2}{1+|z|^2}\,\mathrm{d}z \le C \int_{\mathbb{R}^2} |\nabla\varphi(z)|^2\,\mathrm{d}z \quad \text{for all } \varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus s_{\boldsymbol{p}}). \tag{4.3.5}$$

Proof. We observe that there exists $K = K(|\boldsymbol{p}|) > 0$ such that

$$|z-\boldsymbol{p}|^2 \le K(|\boldsymbol{p}|)(1+|z|^2) \quad \text{for all } z \in \mathbb{R}^2.$$

Therefore the claim easily follows from Theorem 4.3.2 with $C(\boldsymbol{p}) = 4K(|\boldsymbol{p}|)$. □

The Space $\mathcal{H}_R$ For $R > 1$ let us define the function space $\mathcal{H}_R$ as the completion of $C_c^\infty(\Pi_R \cup S_R^+)$ with respect to the norm induced by the scalar product

$$(u,v)_{\mathcal{H}_R} := \int_{\Pi_R} \nabla u \cdot \nabla v\,\mathrm{d}x.$$

Since Π_R is bounded in at least 1 direction, the Poincaré Inequality holds. Hence $\mathcal{H}_R \hookrightarrow H^1(\Pi_R)$ continuously and we have the following characterization

$$\mathcal{H}_R := \left\{u \in H^1(\Pi_R) \colon u = 0 \text{ on } \partial\Pi_R \setminus S_R^+\right\}.$$

Moreover, when $N \ge 3$, the classical Sobolev inequality implies that $\mathcal{H}_R \hookrightarrow L^{2^*}(\Pi_R)$ continuously, where $2^* = \frac{2N}{N-2}$.

Limit Profiles In this section we introduce some limit profiles that will appear in the blow-up analysis of scaled eigenfunctions. We recall the following result from [FT13, Lemma 2.4].

Proposition 4.3.4. *For every $\psi \in C^2(\mathbb{R}^N_+) \cap C^1(\overline{\mathbb{R}^N_+})$ such that*

$$\begin{cases} -\Delta\psi = 0, & \text{in } \mathbb{R}^N_+, \\ \psi = 0, & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

there exists a unique $\Phi = \Phi(\psi) : \Pi \to \mathbb{R}$ such that

$$\Phi \in \mathcal{H}_R, \qquad \text{for all } R > 1, \tag{4.3.6}$$

$$\begin{cases} -\Delta\Phi = 0, & \text{in } \Pi, \\ \Phi = 0, & \text{on } \partial\Pi, \end{cases} \tag{4.3.7}$$

$$\int_\Pi |\nabla(\psi - \Phi)|^2\,\mathrm{d}x < +\infty. \tag{4.3.8}$$

Hereafter we will denote

$$\Phi := \Phi(\psi_k) \tag{4.3.9}$$

where ψ_k is the function defined in (4.2.10). As observed in [FT13] we have that

$$\Phi = \begin{cases} \psi_k + w_k, & \text{in } \mathbb{R}^N_+, \\ w_k, & \text{in } \Pi \setminus \mathbb{R}^N_+, \end{cases} \tag{4.3.10}$$

where w_k is the function realizing the minimum $m_k(\Sigma)$ in (4.2.11). We observe that, in the particular case $N = 2$, the function Φ corresponds to the function X_k introduced in [Gad03, Sections 10-11]. By a classical Dirichlet principle, one can easily obtain the following result.

Lemma 4.3.5. *For every $R > 1$ there exists a unique function $U_R \in \mathcal{H}_R$ solution to the following minimization problem*

$$\min_{u \in \mathcal{H}_R} \left\{ \int_{\Pi_R} |\nabla u|^2 \, dx \colon u = \psi_k \text{ on } S_R^+ \right\}.$$

Moreover it weakly solves

$$\begin{cases} -\Delta U_R = 0, & \text{in } \Pi_R, \\ U_R = 0, & \text{on } \partial\Pi_R \setminus S_R^+, \\ U_R = \psi_k, & \text{on } S_R^+. \end{cases}$$

The function U_R just defined turns out to be a good approximation of the limit profile Φ, as explained in the following lemma.

Lemma 4.3.6. *For every $r > 1$ we have*

$$U_R \longrightarrow \Phi \qquad \text{in } \mathcal{H}_r, \qquad \text{as } R \to +\infty.$$

Proof. We can assume $R > \max\{r, 2\}$. The function $U_R - \Phi$ satisfies, in a weak sense,

$$\begin{cases} -\Delta(U_R - \Phi) = 0, & \text{in } \Pi_R, \\ U_R - \Phi = 0, & \text{on } \partial\Pi_R \setminus S_R^+, \\ U_R - \Phi = \psi_k - \Phi, & \text{on } S_R^+, \end{cases}$$

and then it is the least energy function among those having these boundary conditions. Let $\eta = \eta_R \in C^\infty(\overline{\Pi})$ be the cut-off function defined in (4.3.1). Then

$$\begin{aligned} \int_{\Pi_r} |\nabla(U_R - \Phi)|^2 \, dx &\le \int_{\Pi_R} |\nabla(U_R - \Phi)|^2 \, dx \le \int_{\Pi_R} |\nabla(\eta(\psi_k - \Phi))|^2 \, dx \le \\ &\le 2 \int_{\Pi_R} |\nabla\eta|^2 \, |\psi_k - \Phi|^2 \, dx + 2 \int_{\Pi} |\eta|^2 \, |\nabla(\psi_k - \Phi)|^2 \, dx \le \\ &\le \frac{32}{R^2} \int_{\Pi_R \setminus \Pi_{R/2}} |\psi_k - \Phi|^2 \, dx + 2 \int_{\Pi - \Pi_{R/2}} |\nabla(\psi_k - \Phi)|^2 \, dx \le \\ &\le 32 \int_{\Pi \setminus \Pi_{R/2}} \frac{|\psi_k - \Phi|^2}{|x|^2} \, dx + 2 \int_{\Pi - \Pi_{R/2}} |\nabla(\psi_k - \Phi)|^2 \, dx \longrightarrow 0 \end{aligned}$$

thanks to (4.3.8) and Hardy's Inequality. In the case $N = 2$ we use the fact that $1 + |x|^2 \le 2\,|x|^2$ for $|x| \ge 1$ and the 2-dimensional Hardy inequality (4.3.5). □

Using again the Dirichlet principle, we construct also the limit profile Z_R as follows.

Lemma 4.3.7. *For every $R > 1$ there exists a unique function $Z_R \in H^1(B_R^+)$ solution to the following minimization problem*

$$\min_{u \in H^1(B_R^+)} \left\{ \int_{B_R^+} |\nabla u|^2 \, \mathrm{d}x \colon u = 0 \text{ on } B_R', \; u = \Phi \text{ on } S_R^+ \right\}.$$

Moreover it weakly solves

$$\begin{cases} -\Delta Z_R = 0, & \text{in } B_R^+, \\ Z_R = 0, & \text{on } B_R', \\ Z_R = \Phi, & \text{on } S_R^+. \end{cases}$$

4.3.2 A Pohozaev-type inequality

The purpose of this subsection is to prove the following inequality.

Proposition 4.3.8. *There exists $\tilde{\varepsilon}, \tilde{r} > 0$, with $0 < \tilde{\varepsilon} < \tilde{r} \le R_{\max}$, such that, for $\varepsilon \in (0, \tilde{\varepsilon}]$, $r \in (\varepsilon, \tilde{r}]$ and $i \in \{1, \dots, n_0\}$, we have*

$$\int_{S_r^+} |\nabla \varphi_i^\varepsilon|^2 \, \mathrm{d}S - \frac{N-2}{r} \int_{\Omega_r^\varepsilon} |\nabla \varphi_i^\varepsilon|^2 \, \mathrm{d}x \ge 2 \int_{S_r^+} \left(\frac{\partial \varphi_i^\varepsilon}{\partial \boldsymbol{\nu}} \right)^2 \mathrm{d}S + \frac{2\lambda_i^\varepsilon}{r} \int_{\Omega_r^\varepsilon} p \varphi_i^\varepsilon \nabla \varphi_i^\varepsilon \cdot x \, \mathrm{d}x. \tag{4.3.11}$$

We observe that solutions to problems of type

$$\begin{cases} -\Delta u = f, & \text{in } \Omega_r^\varepsilon, \\ u = 0, & \text{on } \partial \Omega_r^\varepsilon, \end{cases}$$

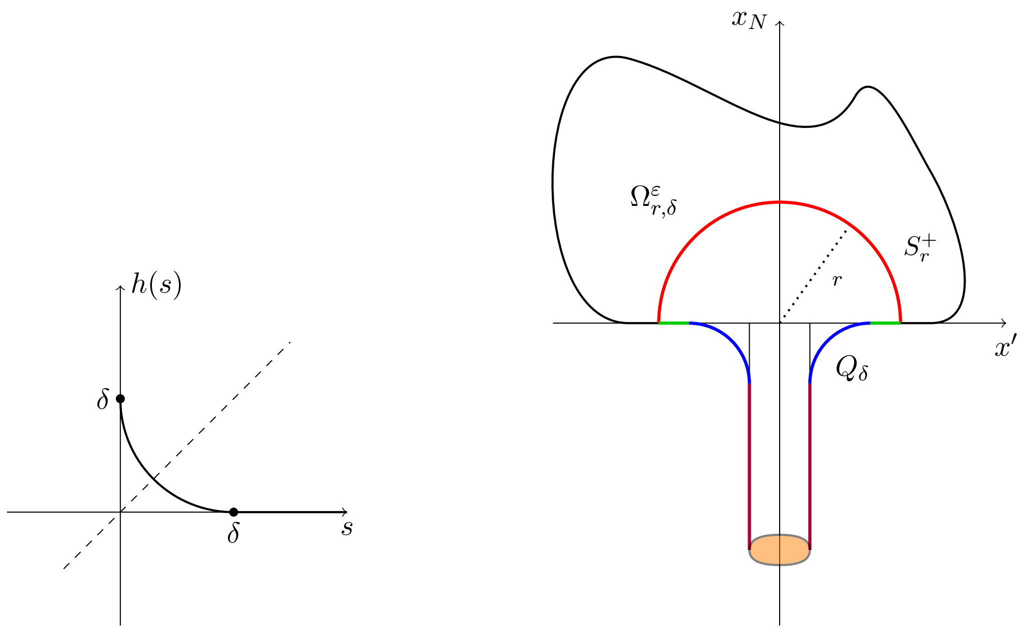
with $f \in L^2(\Omega_r^\varepsilon)$, in general do not belong to $H^2(\Omega_r^\varepsilon)$ since $\partial \Omega_r^\varepsilon$ is only Lipschitz continuous and doesn't verify a uniform exterior ball condition (which ensures L^2-integrability of second order derivatives, see [Ado92]). But, along a proof of a Pohozaev Identity, one tests the equation with the function $\nabla \varphi_i^\varepsilon \cdot x$, which could fail to be H^1 in our case. To overcome this difficulty we implement an approximation process.

Approximating domains Let $\varepsilon \in (0, 1]$ and $r \in (\varepsilon, R_{\max}]$: in order to remove the concave edge $\Gamma_\varepsilon := \partial(\varepsilon \Sigma)$, we will approximate the domain Ω_r^ε with a family of starshaped domains $\Omega_{r,\delta}^\varepsilon$ (with $0 \le \delta < r - \varepsilon$) such that

$$\Omega_{r,0}^\varepsilon = \Omega_r^\varepsilon, \quad \Omega_{r,\delta_1}^\varepsilon \subset \Omega_{r,\delta_2}^\varepsilon \quad \text{for all } 0 \le \delta_1 \le \delta_2 < r - \varepsilon,$$

and such that every $\Omega_{r,\delta}^\varepsilon$ verifies the uniform exterior ball condition. In particular we will define Q_δ such that

$$\Omega_{r,\delta}^\varepsilon = \Omega_r^\varepsilon \cup Q_\delta.$$



(a) *The function h*

(b) *The Approximating Domain*

Figure 4.2

For $\delta > 0$ small we define a "δ-enlargement" of $\varepsilon\Sigma$:

$$\varepsilon\Sigma_\delta := \left\{ x' \in \mathbb{R}^{N-1} \setminus \varepsilon\Sigma \colon \operatorname{dist}(x', \Gamma_\varepsilon) < \delta \right\}.$$

Let $h \in C^\infty((0,+\infty)) \cap C([0,+\infty))$ such that $h(0) = \delta$, $h(s) = 0$ for $s \in [\delta, +\infty)$, $h'(s) < 0$ for all $s \in (0,\delta)$ and $h^{-1}(s) = h(s)$ for $s \in (0,\delta)$. We define $G \colon \varepsilon\Sigma_\delta \subset \mathbb{R}^{N-1} \to \mathbb{R}$ as

$$G(x') := -h(d(x')),$$

where $d(x') := \operatorname{dist}(x', \Gamma_\varepsilon)$. Now, let

$$Q_\delta = \left\{ x = (x', x_N) \in \mathbb{R}^N \colon (x', 0) \in \varepsilon\Sigma_\delta \text{ and } G(x') < x_N \leq 0 \right\},$$

see Figure 4.2.

For what concerns the regularity we observe that, since Γ_ε is of class $C^{1,1}$, then also d and the graph of G are of class $C^{1,1}$ (see [KP81]). Moreover it is easy to verify that the approximating domain $\Omega^{\varepsilon}_{r,\delta}$ satisfies the uniform exterior ball condition.

Remark 4.3.9. We point out that $\Omega^{\varepsilon}_{r,\delta}$ is starshaped. Indeed, first, if $x \in B'_r \setminus (\varepsilon\Sigma_\delta \cup \varepsilon\Sigma)$ (green part in Figure 4.2), trivially $x \cdot \boldsymbol{\nu}(x) = 0$ while if $x \in S_r^+$ (red part) $x \cdot \boldsymbol{\nu}(x) = r > 0$. If $x \in \{x_N = -1\} \cap \partial\Omega^{\varepsilon}_{r,\delta}$ (in orange), then $x \cdot \boldsymbol{\nu}(x) = -x_N = 1 > 0$. Third, if $x \in \Gamma_\varepsilon \times (-1, -\delta)$ (in purple), then $\boldsymbol{\nu}(x) = \boldsymbol{\nu}_{\varepsilon\Sigma}(x')$, where $\boldsymbol{\nu}_{\varepsilon\Sigma}(x')$ is the exterior unit normal of $\Gamma_\varepsilon = \partial(\varepsilon\Sigma)$; thus $x \cdot \boldsymbol{\nu}(x) = x' \cdot \boldsymbol{\nu}_{\varepsilon\Sigma}(x') \geq 0$ by (4.2.2). Finally, let $x \in \{(x', G(x') \colon x' \in \varepsilon\Sigma_\delta\}$ (in blue): in this case we have that

$$\boldsymbol{\nu}(x) = \frac{(\nabla G(x'), -1)}{\|(\nabla G(x'), -1)\|} = \frac{(-h'(d(x'))\nabla d(x'), -1)}{\|(-h'(d(x'))\nabla d(x'), -1)\|},$$

and so

$$x \cdot \boldsymbol{\nu}(x) = \frac{-h'(d(x'))\nabla d(x') \cdot x' + h(d(x'))}{\|(-h'(d(x'))\nabla d(x'), -1)\|}.$$

In fact, this quantity is nonnegative, since $h(s) > 0$, $h'(s) < 0$ for all $s \in (0,1)$ and

$$\nabla d(x') \cdot x' = \boldsymbol{\nu}_{\varepsilon\Sigma}(P(x')) \cdot (P(x') + d(x')\boldsymbol{\nu}_{\varepsilon\Sigma}(P(x')) = \boldsymbol{\nu}_{\varepsilon\Sigma}(x') \cdot x' + d(x') > 0,$$

where $P(x')$ is the unique point of Γ_ε that attains $d(x')$.

Approximating problems For $\alpha \in (0,1)$, let us fix $\tilde{r}, \tilde{\varepsilon} > 0$, with $0 < \tilde{\varepsilon} < \tilde{r} \leq R_{\max}$, such that

$$\left|\Omega^\varepsilon_{r,\delta}\right| \leq \left(\frac{1-\alpha}{C_N \lambda_0 \|p\|_\infty}\right)^{\frac{N}{2}} \quad \text{for all } \varepsilon \in (0, \tilde{\varepsilon}),\ r \in (\varepsilon, \tilde{r}),\ \delta \in (0, r - \varepsilon), \tag{4.3.12}$$

where C_N has been defined in (4.3.2).

For fixed $i \in \{1, \ldots, n_0\}$, $\varepsilon \in (0, \tilde{\varepsilon}]$, $r \in (\varepsilon, \tilde{r}]$ and for all $\delta \in (0, r - \varepsilon)$, let us consider the problem

$$\begin{cases} -\Delta u = \lambda_i^\varepsilon p u, & \text{in } D_\delta, \\ \quad u = 0, & \text{on } \partial D_\delta \setminus S_r^+, \\ \quad u = \varphi_i^\varepsilon, & \text{on } S_r^+, \end{cases} \tag{4.3.13}$$

where, for simplicity of notation, in this section we call $D_\delta := \Omega^\varepsilon_{r,\delta}$; we also denote $\delta_0 := r - \varepsilon$.

Theorem 4.3.10. *There exists a unique $u_\delta \in H^1(D_\delta)$ solution to problem (4.3.13). Moreover the family $\{u_\delta\}_{\delta \in (0,\delta_0)}$ is bounded in $H^1(D_{\delta_0})$ with respect to δ.*

Proof. We observe that, if we extend φ_i^ε to zero in $D_{\delta_0} \setminus \Omega_r^\varepsilon$ and we let $v = u - \varphi_i^\varepsilon$, then problem (4.3.13) is equivalent to

$$\begin{cases} -\Delta v = \lambda_i^\varepsilon p v + F, & \text{in } D_\delta, \\ \quad v = 0, & \text{on } \partial D_\delta, \end{cases} \tag{4.3.14}$$

where $F := \lambda_i^\varepsilon p \varphi_i^\varepsilon + \Delta\varphi_i^\varepsilon \in H^{-1}(D_\delta)$. Existence and uniqueness of a solution $v_\delta \in H_0^1(D_\delta)$ to (4.3.14) easily comes from Lax-Milgram Theorem. Indeed, the bilinear form

$$a(v,w) := \int_{D_\delta} (\nabla v \cdot \nabla w - \lambda_i^\varepsilon p v w)\, \mathrm{d}x, \quad v, w \in H_0^1(D_\delta)$$

is coercive, since, by (4.3.3) and (4.3.12), we have

$$\begin{aligned} a(v,v) &= \int_{D_\delta} \left(|\nabla v|^2 - \lambda_i^\varepsilon p\,|v|^2 \right) \mathrm{d}x \\ &\geq \left(1 - C_N \lambda_0 \, \|p\|_\infty \, |D_\delta|^{\frac{2}{N}}\right) \int_{D_\delta} |\nabla v|^2 \, \mathrm{d}x \geq \alpha \int_{D_\delta} |\nabla v|^2 \, \mathrm{d}x. \end{aligned} \tag{4.3.15}$$

From Lax-Milgram Theorem we also know that

$$\|\nabla v_\delta\|_{L^2(D_{\delta_0})} = \|\nabla v_\delta\|_{L^2(D_\delta)} \leq \frac{\|F\|_{H^{-1}(D_\delta)}}{\alpha},$$

where v_δ has been trivially extended in $D_{\delta_0} \setminus D_\delta$. One can easily prove that $\|F\|_{H^{-1}(D_\delta)} = O(1)$ as $\delta \to 0$. Indeed, for $w \in H^1_{D_\delta}$, we have, by Cauchy-Schwartz inequality

$$\begin{aligned} \left| {}_{H^{-1}(D_\delta)}\langle F, w\rangle_{H^1_0(D_\delta)} \right| &= \left| \int_{D_\delta} (\lambda_i^\varepsilon p \varphi_i^\varepsilon w - \nabla \varphi_i^\varepsilon \cdot \nabla w)\, \mathrm{d}x \right| \\ &\leq \lambda_i^\varepsilon \, \|p\|_\infty \, \|\varphi_i^\varepsilon\|_{L^2(D_\delta)} \, \|w\|_{L^2(D_\delta)} + \|\varphi\|_{H^1_0(D_\delta)} \, \|w\|_{H^1_0(D_\delta)} \,. \end{aligned}$$

Now, trivially extending w in $D_{\delta_0} \setminus D_\delta$, by (4.3.3) we have that the last term can by estimated by

$$(\lambda_i^\varepsilon \, \|p\|_\infty \sqrt{C_N}\, |D_{\delta_0}|^{1/N} \, \|\varphi_i^\varepsilon\|_{L^2(\Omega_r^\varepsilon)} + \|\varphi_i^\varepsilon\|_{H^1_0(\Omega_r^\varepsilon)}) \, \|w\|_{H^1_0(D_{\delta_0})} \,.$$

Hence

$$\|F\|_{H^{-1}(D_\delta)} \leq \lambda_i^\varepsilon \, \|p\|_\infty \sqrt{C_N}\, |D_{\delta_0}|^{1/N} \, \|\varphi_i^\varepsilon\|_{L^2(\Omega_r^\varepsilon)} + \|\varphi_i^\varepsilon\|_{H^1_0(\Omega_r^\varepsilon)} \,.$$

Then $u_\delta := v_\delta + \varphi_i^\varepsilon$ is the unique solution to (4.3.13) and $\{u_\delta\}_{\delta \in (0,\delta_0)}$ is bounded in $H^1(D_{\delta_0})$. ☐

Theorem 4.3.11. *If $u_\delta \in H^1(D_\delta)$ is the unique solution to (4.3.13), then*

$$u_\delta \longrightarrow \varphi_i^\varepsilon \quad \text{in } H^1(D_{\delta_0}), \quad \text{as } \delta \to 0.$$

Proof. Since $\{u_\delta\}_{\delta \in (0,\delta_0)}$ is bounded in $H^1(D_{\delta_0})$, then there exists $U \in H^1(D_{\delta_0})$ such that, up to a subsequence,

$$u_\delta \rightharpoonup U \quad \text{weakly in } H^1(D_{\delta_0}), \quad \text{as } \delta \to 0.$$

Actually $U \in H^1(\Omega_r^\varepsilon)$ and moreover it weakly solves

$$\begin{cases} -\Delta U = \lambda_i^\varepsilon p U, & \text{in } \Omega_r^\varepsilon, \\ \quad U = 0, & \text{on } \partial\Omega_r^\varepsilon \setminus S_r^+, \\ \quad U = \varphi_i^\varepsilon, & \text{on } S_r^+. \end{cases}$$

From the coercivity obtained in (4.3.15) we deduce that $U = \varphi_i^\varepsilon$.

In order to prove strong convergence in $H^1(D_{\delta_0})$, we notice that

$$\int_{D_{\delta_0}} |\nabla u_\delta|^2 \,\mathrm{d}x = \lambda_i^\varepsilon \int_{D_{\delta_0}} p u_\delta^2 \,\mathrm{d}x + \int_{S_r^+} \varphi_i^\varepsilon \frac{\partial u_\delta}{\partial \boldsymbol{\nu}} \,\mathrm{d}S. \tag{4.3.16}$$

From the compactness of the embedding $H^1(D_{\delta_0}) \hookrightarrow L^2(D_{\delta_0})$ we have that $u_\delta \to \varphi_i^\varepsilon$ in $L^2(D_{\delta_0})$ and so $\int_{D_{\delta_0}} p\,|u_\delta|^2 \,\mathrm{d}x \to \int_{D_{\delta_0}} p\,|\varphi_i^\varepsilon|^2 \,\mathrm{d}x$. Moreover, from the equation (4.3.13), we have that $\nabla u_\delta \rightharpoonup \nabla \varphi_i^\varepsilon$ weakly in $H(\mathrm{div}, D_{\delta_0})$ as $\delta \to 0$. Hence classical trace theorems for vector functions yield

$$\int_{S_r^+} \frac{\partial u_\delta}{\partial \boldsymbol{\nu}} \varphi_i^\varepsilon \,\mathrm{d}S \to \int_{S_r^+} \frac{\partial \varphi_i^\varepsilon}{\partial \boldsymbol{\nu}} \varphi_i^\varepsilon \,\mathrm{d}S, \quad \text{as } \delta \to 0.$$

Therefore, from (4.3.16) and from the equation satisfied by φ_i^ε, we conclude that, along a subsequence,

$$\begin{aligned}\lim_{\delta\to 0} \int_{D_{\delta_0}} |\nabla u_\delta|^2 \,\mathrm{d}x = \lambda_i^\varepsilon \int_{D_{\delta_0}} p\,|\varphi_i^\varepsilon|^2 \,\mathrm{d}x + \int_{S_r^+} \frac{\partial \varphi_i^\varepsilon}{\partial \boldsymbol{\nu}} \varphi_i^\varepsilon \,\mathrm{d}S \\ = \int_{\Omega_r^\varepsilon} |\nabla \varphi_i^\varepsilon|^2 \,\mathrm{d}x = \int_{D_{\delta_0}} |\nabla \varphi_i^\varepsilon|^2 \,\mathrm{d}x.\end{aligned}$$

Thanks to *Urysohn's Subsequence Principle* the proof is thereby complete. □

Theorem 4.3.12. *Let $u_\delta \in H^1(D_\delta)$ be the unique solution to (4.3.13). Then*

$$\nabla u_\delta \longrightarrow \nabla \varphi_i^\varepsilon \qquad \text{in } L^2(S_r^+), \quad \text{as } \delta \to 0.$$

Proof. Without loss of generality we assume that $\varepsilon + \delta < r/2$. Let us consider a cut off function $\xi \in C_c^\infty(\overline{B_r} \setminus B_{\frac{r}{2}})$ such that $\xi = 1$ in $B_r \setminus B_{\frac{3}{4}r}$. If we oddly extend u_δ and φ_i^ε in $B_r \setminus B_{\frac{r}{2}}$ with respect to $\{x_N = 0\}$ and we let $w_\delta := \xi(u_\delta - \varphi_i^\varepsilon)$, then we have

$$\begin{cases} -\Delta w_\delta - \lambda_i^\varepsilon p w_\delta = -(u_\delta - \varphi_i^\varepsilon)\Delta\xi - 2\nabla(u_\delta - \varphi_i^\varepsilon)\cdot\nabla\xi, & \text{in } B_r \setminus B_{\frac{r}{2}}, \\ w_\delta = 0, & \text{on } S_r \cup S_{\frac{r}{2}}, \end{cases}$$

in a weak sense, with B_R denoting the ball centered at the origin and with radius $R > 0$ and $S_R := \partial B_R$. From Theorem 4.3.11 we deduce that $w_\delta \to 0$ in $H^1(B_r \setminus B_{\frac{r}{2}})$ as $\delta \to 0$, indeed

$$\int_{B_r\setminus B_{\frac{r}{2}}} (|\nabla w_\delta|^2 + w_\delta^2) \,\mathrm{d}x \le C_\xi \,\|u_\delta - \varphi_i^\varepsilon\|^2_{H^1(B_r\setminus B_{\frac{r}{2}})} \le C_\xi \,\|u_\delta - \varphi_i^\varepsilon\|^2_{H^1(D_{\delta_0})} \to 0,$$

as $\delta \to 0$. Since we have that $f := -(u_\delta - \varphi_i^\varepsilon)\Delta\xi - 2\nabla(u_\delta - \varphi_i^\varepsilon)\cdot\nabla\xi \in L^2(B_r \setminus B_{\frac{r}{2}})$, by classical elliptic regularity theory we know that $w_\delta \in H^2(B_r \setminus B_{\frac{r}{2}})$ and that

$$\|w_\delta\|_{H^2(B_r\setminus B_{\frac{r}{2}})} \le C(\|w_\delta\|_{H^1(B_r\setminus B_{\frac{r}{2}})} + \|f\|_{L^2(B_r\setminus B_{\frac{r}{2}})}).$$

One can easily see that

$$\|f\|_{L^2(B_r\setminus B_{\frac{r}{2}})} \le \tilde{C}_\xi \|w_\delta\|_{H^1(B_r\setminus B_{\frac{r}{2}})},$$

hence $w_\delta \to 0$ in $H^2(B_r \setminus B_{\frac{r}{2}})$ as $\delta \to 0$. From the definition of w_δ, we deduce that $u_\delta \to \varphi_i^\varepsilon$ in $H^2(B_r \setminus B_{\frac{3}{4}r})$ as $\delta \to 0$, that is $\nabla u_\delta \to \nabla \varphi_i^\varepsilon$ in $H^1(B_r \setminus B_{\frac{3}{4}r})$ which, by trace embedding, implies the thesis.

□

Theorem 4.3.13. *Let u_δ be the unique solution to (4.3.13). Then the following identity holds*

$$\int_{S_r^+} |\nabla u_\delta|^2 \, \mathrm{d}S - \frac{N-2}{r} \int_{D_\delta} |\nabla u_\delta|^2 \, \mathrm{d}x$$
$$= \frac{1}{r} \int_{\partial D_\delta \setminus S_r^+} \left(\frac{\partial u_\delta}{\partial \boldsymbol{\nu}}\right)^2 x \cdot \boldsymbol{\nu} \, \mathrm{d}S + 2 \int_{S_r^+} \left(\frac{\partial u_\delta}{\partial \boldsymbol{\nu}}\right)^2 \mathrm{d}S + \frac{2\lambda_i^\varepsilon}{r} \int_{D_\delta} p \, u_\delta \, \nabla u_\delta \cdot x \, \mathrm{d}x.$$

Proof. We first observe that, by classical regularity theory, $u_\delta \in H^2(D_\delta)$ since D_δ verifies an exterior ball condition. Let us now test equation (4.3.13) with the function $\nabla u_\delta \cdot x \in H^1(D_\delta)$. Integrating by parts and using the following identity

$$\nabla u_\delta \cdot \nabla(\nabla u_\delta \cdot x) = \frac{1}{2} \operatorname{div}\left(|\nabla u_\delta|^2 x\right) - \frac{N-2}{2} |\nabla u_\delta|^2$$

we obtain the conclusion. □

Proof of Proposition 4.3.8. Let u_δ be the unique solution to (4.3.13). From Theorem 4.3.13 and Remark 4.3.9 we know that

$$\int_{S_r^+} |\nabla u_\delta|^2 \, \mathrm{d}S - \frac{N-2}{r} \int_{D_\delta} |\nabla u_\delta|^2 \, \mathrm{d}x \ge 2 \int_{S_r^+} \left(\frac{\partial u_\delta}{\partial \boldsymbol{\nu}}\right)^2 \mathrm{d}S + \frac{2\lambda_i^\varepsilon}{r} \int_{D_\delta} p \, u_\delta \, \nabla u_\delta \cdot x \, \mathrm{d}x.$$

Now, thanks to Theorems 4.3.11 and 4.3.12, we can pass to the limit as $\delta \to 0$ in the above inequality, thus obtaining (4.3.11). □

Remark 4.3.14. We observe that the assumption of $C^{1,1}$-regularity for $\partial\Sigma$ can be relaxed; indeed, if Σ is less regular (e.g. if $\partial\Sigma$ is Lipschitz continuous), we can approximate $\varepsilon\Sigma$ with a class of $C^{1,1}$- regular domains $(\varepsilon\Sigma)_\beta$ and start the procedure of Section 4.3.2 from the domain $(\varepsilon\Sigma)_\beta$.

4.3.3 Poincaré-type inequalities

In this section we consider the following spaces for $\varepsilon \in (0, 1]$ and $r > \varepsilon$:

$$V_\varepsilon(B_r^+) := H^1_{0, B_r' \setminus \varepsilon\Sigma} \left\{ u \in H^1(B_r^+) \colon u = 0 \text{ on } B_r' \setminus \varepsilon\Sigma \right\},$$
$$V_0(B_r^+) := H^1_{0, B_r'} \{ u \in H^1(B_r^+) \colon u = 0 \text{ on } B_r' \}.$$

We point out that, for $0 \le \varepsilon_1 \le \varepsilon_2 < r$,

$$H^1_0(B_r^+) \subseteq V_0(B_r^+) \subseteq V_{\varepsilon_1}(B_r^+) \subseteq V_{\varepsilon_2}(B_r^+) \subseteq H^1(B_r^+). \tag{4.3.17}$$

Lemma 4.3.15 (Poincaré-Type Inequality)**.** *Let $r > 0$. Then, for every $u \in H^1(B_r^+)$, the following inequality holds*

$$\frac{N-1}{r^2}\int_{B_r^+} |u|^2 \, \mathrm{d}x \le \int_{B_r^+} |\nabla u|^2 \, \mathrm{d}x + \frac{1}{r}\int_{S_r^+} |u|^2 \, \mathrm{d}S. \tag{4.3.18}$$

Proof. Integrating the equality $\operatorname{div}(u^2 x) = 2u\nabla u \cdot x + Nu^2$ over B_r^+ and recalling the elementary inequality $0 \le (u + \nabla u \cdot x)^2 = |u|^2 + |\nabla u \cdot x|^2 + 2u\nabla u \cdot x$, we obtain that

$$\begin{aligned}\int_{\partial B_r^+} |u|^2 \, x \cdot \boldsymbol{\nu} \, \mathrm{d}S &= \int_{B_r^+} \left(2u\nabla u \cdot x + N|u|^2\right) \mathrm{d}x \\ &\ge -\int_{B_r^+} \left(|u|^2 + |\nabla u \cdot x|^2\right) \mathrm{d}x + N\int_{B_r^+} |u|^2 \, \mathrm{d}x.\end{aligned}$$

Since $x \cdot \boldsymbol{\nu} = 0$ on B_r' and $|x| \le r$, then

$$r\int_{S_r^+} |u|^2 \, \mathrm{d}S \ge -r^2 \int_{B_r^+} |\nabla u|^2 \, \mathrm{d}x + (N-1)\int_{B_r^+} |u|^2 \, \mathrm{d}x.$$

Reorganizing the terms and dividing by r^2 yields the thesis. □

Lemma 4.3.16. *For $0 \le \sigma < 1$ the infimum*

$$m_\sigma = \inf_{\substack{u \in V_\sigma(B_1^+) \\ u \ne 0}} \frac{\int_{B_1^+} |\nabla u|^2 \, \mathrm{d}x}{\int_{S_1^+} |u|^2 \, \mathrm{d}S}$$

is achieved. Moreover $m_\sigma > 0$, the map $\sigma \mapsto m_\sigma$ is non-increasing in $[0,1)$ and continuous in 0 and $m_0 = 1$.

Proof. For $u \in V_\sigma(B_1^+)$, let us denote

$$F(u) := \frac{\int_{B_1^+} |\nabla u|^2 \, \mathrm{d}x}{\int_{S_1^+} |u|^2 \, \mathrm{d}S}.$$

Let $\{u_n\}_n \subseteq V_\sigma(B_1^+)$ be a minimizing sequence such that $\int_{S_1^+} |u_n|^2 \, \mathrm{d}S = 1$. From (4.3.18) it follows that $\{u_n\}_n$ is bounded in $H^1(B_1^+)$ and so there exists $\tilde{u} \in H^1(B_1^+)$ such that, up to a subsequence, $u_n \rightharpoonup \tilde{u}$ in $H^1(B_1^+)$. Taking into account the compact embedding $H^1(B_1^+) \hookrightarrow L^2(\partial B_1^+)$, we have that $\int_{S_1^+} |\tilde{u}|^2 \, \mathrm{d}S = 1$ and then $\tilde{u} \ne 0$. Moreover $\tilde{u} = 0$ on $B_r' \setminus \sigma\Sigma$, since $\{u_n\}$ do; in particular $\tilde{u} \in V_\sigma(B_1^+)$. By weak lower semicontinuity, we have that

$$\int_{B_1^+} |\nabla \tilde{u}|^2 \, \mathrm{d}x \le \liminf_{n \to +\infty} \int_{B_1^+} |\nabla u_n|^2 \, \mathrm{d}x = m_\sigma.$$

Then $m_\sigma = F(\tilde{u})$, i.e. $\tilde{u}$ attains the infimum m_σ. Trivially $m_\sigma > 0$, due to the null boundary conditions on $B_r' \setminus \sigma\Sigma$. The monotonicity of the map $\sigma \mapsto m_\sigma$ follows from (4.3.17).

Now we have to prove continuity. Let $\sigma_n \to 0^+$. For every n there exists $\tilde{u}_n \in V_{\sigma_n}(B_1^+)$ such that

$$\int_{S_1^+} |\tilde{u}_n|^2 \, \mathrm{d}S = 1, \qquad \int_{B_1^+} |\nabla \tilde{u}_n|^2 \, \mathrm{d}x = m_{\sigma_n}.$$

Then, since $m_{\sigma_n} \leq m_0$ for all n, we have that $\{\tilde{u}_n\}_n$ is bounded in $H^1(B_1^+)$. Thus there exists $u_0 \in H^1(B_1^+)$ sucht that, up to a subsequence, $\tilde{u}_n \rightharpoonup u_0$ weakly in $H^1(B_1^+)$. So, first

$$\int_{S_1^+} |u_0|^2 \, \mathrm{d}S = 1 \qquad \text{and} \qquad u_0 = 0 \quad \text{on} \quad B_r'.$$

Furthermore

$$m_0 \leq \int_{B_1^+} |\nabla u_0|^2 \, \mathrm{d}x \leq \liminf m_{\sigma_n} \leq \limsup m_{\sigma_n} \leq m_0.$$

Then, along the subsequence, $m_0 = \lim_{n\to+\infty} m_{\sigma_n}$. Thanks to *Urysohn's Subsequence Principle* we may conclude that $m_0 = \lim_{\sigma\to 0^+} m_\sigma$.

Finally we prove that $m_0 = 1$. Since the function u_0 achieving m_0 is harmonic in B_1^+, thanks to classical monotonicity arguments (we refer to [FFT11] for further details) we can say that the function

$$r \longmapsto M(r) := \frac{r \int_{B_r^+} |\nabla u_0|^2 \, \mathrm{d}x}{\int_{S_r^+} |u_0|^2 \, \mathrm{d}S}$$

is non-decreasing and that there exists $k \in \mathbb{N}$, $k \geq 1$, such that $\lim_{r\to 0^+} M(r) = k$. Hence $M(r) \geq 1$ for every $r \geq 0$. Then

$$m_0 = \frac{\int_{B_1^+} |\nabla u_0|^2 \, \mathrm{d}x}{\int_{S_1^+} |u_0|^2 \, \mathrm{d}S} = M(1) \geq 1.$$

Furthermore the function $v(x) = x_N$ belongs to $V_0(B_1^+)$ and $F(v) = 1$; hence $m_0 = 1$. $\square$

Corollary 4.3.17. *Let $\varepsilon \in (0,1]$ and $r > \varepsilon$. Then*

$$\frac{m_{\varepsilon/r}}{r} \int_{S_r^+} |u|^2 \, \mathrm{d}S \leq \int_{B_r^+} |\nabla u|^2 \, \mathrm{d}x \qquad \textit{for all } u \in V_\varepsilon(B_r^+).$$

Moreover, for every $\rho \in (0,1)$ there exists $\mu_\rho > 1$ such that, if $\varepsilon < \frac{r}{\mu_\rho}$, then

$$\frac{1-\rho}{r} \int_{S_r^+} |u|^2 \, \mathrm{d}S \leq \int_{B_r^+} |\nabla u|^2 \, \mathrm{d}x \qquad \textit{for all } u \in V_\varepsilon(B_r^+). \tag{4.3.19}$$

Proof. If we let $\sigma = \varepsilon/r$ in the previous Lemma, we have that

$$m_{\varepsilon/r} \int_{S_1^+} |u|^2 \, \mathrm{d}S \leq \int_{B_1^+} |\nabla u|^2 \, \mathrm{d}x \qquad \text{for all } u \in V_{\varepsilon/r}(B_1^+).$$

The first inequality follows by the change of variables $y = rx$, while the second one trivially comes from the continuity of m_σ in 0. $\square$

From (4.3.18) and Corollary 4.3.17 one can easily prove the following corollary.

Corollary 4.3.18. *For every $\rho \in (0,1)$ there exists $\mu_\rho > 1$ such that, for every $r > 0$ and $\varepsilon < \frac{r}{\mu_\rho}$,*

$$\frac{N-1}{r^2}\int_{B_r^+} |v|^2 \,\mathrm{d}x \leq \left(1 + \frac{1}{1-\rho}\right)\int_{B_r^+} |\nabla v|^2 \,\mathrm{d}x \quad \textit{for all } v \in V_\varepsilon(B_r^+). \tag{4.3.20}$$

4.3.4 Monotonicity formula

For any $0 < \varepsilon < r \leq R_{\max}$, $\lambda > 0$, $\varphi \in H^1(\Omega_r^\varepsilon)$, let us introduce the functions

$$E(\varphi, r, \lambda, \varepsilon) := \frac{1}{r^{N-2}}\int_{\Omega_r^\varepsilon} \left(|\nabla\varphi|^2 - \lambda p\,|\varphi|^2\right) \mathrm{d}x$$

and

$$H(\varphi, r) := \frac{1}{r^{N-1}}\int_{S_r^+} |\varphi|^2 \,\mathrm{d}S.$$

Moreover we define the Almgren type frequency function

$$\mathcal{N}(\varphi, r, \lambda, \varepsilon) := \frac{E(\varphi, r, \lambda, \varepsilon)}{H(\varphi, r)}. \tag{4.3.21}$$

Lemma 4.3.19 (Integration on the Tube)**.** *There exists a constant $\kappa = \kappa(N, \Sigma) > 0$, depending only on N and $|\Sigma|$, such that, for every $\varepsilon \in (0,1]$ and for every $u \in H^1(T_\varepsilon)$ such that $u = 0$ on $\partial T_\varepsilon \setminus \varepsilon\Sigma$,*

$$\int_{T_\varepsilon} |u|^2 \,\mathrm{d}x \leq \kappa\varepsilon^{2(N-1)/N}\int_{T_\varepsilon} |\nabla u|^2 \,\mathrm{d}x.$$

Proof. Let $\tilde{T}_\varepsilon = T_\varepsilon \cup \varsigma(T_\varepsilon)$, where ς is the reflection through the hyperplane $\{x_N = 0\}$, and let $\tilde{u}$ be the even extension of u on $\tilde{T}_\varepsilon$. Since $\tilde{u} \in H_0^1(\tilde{T}_\varepsilon)$, thanks to (4.3.3) we have that

$$\begin{aligned}\int_{T_\varepsilon} |u|^2 \,\mathrm{d}x = \frac{1}{2}\int_{\tilde{T}_\varepsilon} |\tilde{u}|^2 \,\mathrm{d}x \leq \frac{C_N}{2}|\tilde{T}_\varepsilon|^{2/N}\int_{\tilde{T}_\varepsilon} |\nabla\tilde{u}|^2 \,\mathrm{d}x \\ = C_N 2^{2/N}\,|\Sigma|^{2/N}\,\varepsilon^{2(N-1)/N}\int_{T_\varepsilon} |\nabla u|^2 \,\mathrm{d}x.\end{aligned}$$

Hence we can conclude the proof letting $\kappa = C_N 2^{2/N}\,|\Sigma|^{2/N}$. □

Lemma 4.3.20. *There exists $\varepsilon_1 \in (0,1]$, $R_1 > 0$, with $0 < \varepsilon_1 < R_1 \leq R_{\max}$, such that*

$$H(\varphi_i^\varepsilon, r) > 0 \qquad \textit{for all } \varepsilon \in (0, \varepsilon_1], \quad \textit{for all } r \in (\varepsilon, R_1], \quad \textit{for all } i = 1, \dots, n_0.$$

Proof. Suppose by contradiction that for every n there exists $\varepsilon_n \in (0,1]$, $r_n \in (\varepsilon_n, R_{\max}]$ and $i_n \in \{1,\dots,n_0\}$ such that $r_n \to 0$ and $H(\varphi_{i_n}^{\varepsilon_n}, r_n) = 0$. Let us denote $\nu_n := \lambda_{i_n}^{\varepsilon_n}$, $\xi_n := \varphi_{i_n}^{\varepsilon_n}$ and $\Omega_n := \Omega_{r_n}^{\varepsilon_n}$. From this it follows that $\xi_n = 0$ on $S_{r_n}^+$ and that

$$\int_{\Omega_n} |\nabla \xi_n|^2 \, dx = \nu_n \int_{\Omega_n} p\, |\xi_n|^2 \, dx \le \lambda_0 \|p\|_\infty \int_{\Omega_n} |\xi_n|^2 \, dx.$$

Using Lemma 4.3.19 when integrating on the tube we obtain

$$\int_{T_{\varepsilon_n}} |\xi_n|^2 \, dx \le \kappa \varepsilon_n^{2(N-1)/N} \int_{\Omega_n} |\nabla \xi_n|^2 \, dx.$$

Moreover (4.3.18) says that

$$\int_{B_{r_n}^+} |\xi_n|^2 \, dx \le \frac{r_n^2}{N-1} \int_{\Omega_n} |\nabla \xi_n|^2 \, dx.$$

Then we have that

$$\int_{\Omega_n} |\nabla \xi_n|^2 \, dx \le \lambda_0 \|p\|_\infty \left(\kappa \varepsilon_n^{2(N-1)/N} + \frac{r_n^2}{N-1} \right) \int_{\Omega_n} |\nabla \xi_n|^2 \, dx.$$

Thus $\xi_n \equiv 0$ in Ω_n, provided n is sufficiently large. Thanks to classical unique continuation properties for elliptic equations it follows that $\xi_n = 0$ in Ω^{ε_n}, which is a contradiction. $\square$

Lemma 4.3.21. *Let*

$$R_2 = \min \left\{ \left(\frac{N-1}{\lambda_0 \|p\|_\infty} \right)^{1/2}, R_{\max} \right\}.$$

For every $r \in (0, R_2]$ there exist $c_r > 0$ and $\varepsilon_r \in (0,1]$, with $\varepsilon_r < r$, such that

$$H(\varphi_i^\varepsilon, r) \ge c_r \qquad \textit{for all } \varepsilon \in (0, \varepsilon_r), \quad \textit{for all } i = 1, \dots, n_0.$$

Proof. We will prove the lemma for a fixed $i \in \{1,\dots,n_0\}$ and take as c_r the minimum among the constants found for each i. Suppose by contradiction that for a certain $r \in (0, R_2]$ and for every n (large enough) there exists $\varepsilon_n \in (0, 1/n)$ such that

$$H(\varphi_i^{\varepsilon_n}, r) < \frac{1}{n}. \tag{4.3.22}$$

We first note that, since $\varepsilon_n \to 0$, then $\lambda_i^{\varepsilon_n} \to \lambda_i$ (see [Dan03]). Moreover

$$\int_{\Omega^1} |\nabla \varphi_i^{\varepsilon_n}| \, dx = \int_{\Omega^{\varepsilon_n}} |\nabla \varphi_i^{\varepsilon_n}| \, dx = \lambda_i^{\varepsilon_n} \int_{\Omega^{\varepsilon_n}} p\, |\varphi_i^{\varepsilon_n}|^2 = \lambda_i^{\varepsilon_n} \le \lambda_0.$$

Hence there exists $v \in H_0^1(\Omega^1)$ such that $\varphi_i^{\varepsilon_n} \rightharpoonup v$ weakly in $H_0^1(\Omega^1)$ along a subsequence. Note that actually $v \in H_0^1(\Omega)$ and $v \not\equiv 0$ in Ω, since $\int_\Omega p\, |v|^2 \, dx = 1$. Moreover $\varphi_i^{\varepsilon_n} \to v$

strongly in $L^2(\Omega^1)$ and in $L^2(S_r^+)$, so that (4.3.22) implies that $v = 0$ on S_r^+. This tells us that v weakly solves

$$\begin{cases} -\Delta v = \lambda_i p v, & \text{in } B_r^+, \\ v = 0, & \text{on } \partial B_r^+. \end{cases}$$

Testing the above equation with v we obtain that $\int_{B_r^+} |\nabla v|^2 \,\mathrm{d}x = \lambda_i \int_{B_r^+} p\,|v|^2 \,\mathrm{d}x$ and then, thanks to (4.3.18) and the fact that $\lambda_i \leq \lambda_0$ and $r \leq R_2$,

$$\begin{aligned} 0 = \int_{B_r^+} \left(|\nabla v|^2 - \lambda_i p\,|v|^2 \right) \mathrm{d}x &\geq \left(\frac{N-1}{r^2} - \lambda_0 \|p\|_\infty \right) \int_{B_r^+} |v|^2 \,\mathrm{d}x \\ &\geq \left(\frac{N-1}{R_2^2} - \lambda_0 \|p\|_\infty \right) \int_{B_r^+} |v|^2 \,\mathrm{d}x. \end{aligned}$$

Due to the initial choice of R_2 we have that this last factor is positive; then $v = 0$ in B_r^+. This, together with classical unique continuation principles, implies that $v = 0$ in Ω, which is a contradiction. ☐

Let $\tilde\varepsilon$ and $\tilde r$ be the constants found in Proposition 4.3.8.

Proposition 4.3.22. *Let $i \in \{1, \dots, n_0\}$, $\varepsilon \in (0, \tilde\varepsilon]$ and $r \in (\varepsilon, \tilde r]$. Then*

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}r}(\varphi_i^\varepsilon, r, \lambda_i^\varepsilon, \varepsilon) \geq \frac{1}{r^{N-2}} \Bigg[2 \int_{S_r^+} \left(\frac{\partial \varphi_i^\varepsilon}{\partial \boldsymbol{\nu}} \right)^2 \mathrm{d}S + \\ + \frac{2\lambda_i^\varepsilon}{r} \int_{\Omega_r^\varepsilon} p\, \varphi_i^\varepsilon\, \nabla \varphi_i^\varepsilon \cdot x \,\mathrm{d}x - \lambda_i^\varepsilon \int_{S_r^+} p\,|\varphi_i^\varepsilon|^2 \,\mathrm{d}S + \frac{N-2}{r} \lambda_i^\varepsilon \int_{\Omega_r^\varepsilon} p\,|\varphi_i^\varepsilon|^2 \,\mathrm{d}x \Bigg] \end{aligned} \tag{4.3.23}$$

and

$$\frac{\mathrm{d}H}{\mathrm{d}r}(\varphi_i^\varepsilon, r) = \frac{2}{r^{N-1}} \int_{S_r^+} \varphi_i^\varepsilon \frac{\partial \varphi_i^\varepsilon}{\partial \boldsymbol{\nu}} \,\mathrm{d}S = \frac{2}{r} E(\varphi_i^\varepsilon, r, \lambda_i^\varepsilon, \varepsilon). \tag{4.3.24}$$

Proof. We compute the derivative

$$\frac{\mathrm{d}E}{\mathrm{d}r} = \frac{2-N}{r^{N-1}} \int_{\Omega_r^\varepsilon} \left(|\nabla \varphi_i^\varepsilon|^2 - \lambda_i^\varepsilon p\,|\varphi_i^\varepsilon|^2 \right) \mathrm{d}x + \frac{1}{r^{N-2}} \int_{S_r^+} \left(|\nabla \varphi_i^\varepsilon|^2 - \lambda_i^\varepsilon p\,|\varphi_i^\varepsilon|^2 \right) \mathrm{d}S.$$

Then, thanks to (4.3.11), we obtain (4.3.23). The proof of (4.3.24) follows from direct computations, the equation satisfied by φ_i^ε and integration by parts. ☐

Lemma 4.3.23. *Let $\rho \in (0, 1/2]$, μ_ρ be as in Corollary 4.3.17, $\varepsilon \in (0, 1]$ and $r \in (\varepsilon, R_{\max}]$. If $\varepsilon\mu_\rho < r$, then*

$$\int_{\Omega_r^\varepsilon} |u|^2 \,\mathrm{d}x \leq K_{\varepsilon,r}^1 \int_{\Omega_r^\varepsilon} |\nabla u|^2 \,\mathrm{d}x$$

for any $u \in H^1(\Omega_r^\varepsilon)$ such that $u = 0$ on $\partial\Omega_r^\varepsilon \setminus S_r^+$, where

$$K_{\varepsilon,r}^1 = \kappa \varepsilon^{2(N-1)/N} + \frac{3r^2}{N-1}$$

and κ is as in Lemma 4.3.19.

Proof. Thanks to Lemma 4.3.19 we have an estimate about the integral on the tube, i.e.

$$\int_{T_\varepsilon} |u|^2 \,\mathrm{d}x \le \kappa \varepsilon^{2(N-1)/N} \int_{T_\varepsilon} |\nabla u|^2 \,\mathrm{d}x \le \kappa \varepsilon^{2(N-1)/N} \int_{\Omega_r^\varepsilon} |\nabla u|^2 \,\mathrm{d}x.$$

On the other hand, by (4.3.20) we have that

$$\int_{B_r^+} |u|^2 \,\mathrm{d}x \le \frac{r^2}{N-1}\left(1+\frac{1}{1-\rho}\right) \int_{B_r^+} |\nabla u|^2 \,\mathrm{d}x \le \frac{3r^2}{N-1} \int_{\Omega_r^\varepsilon} |\nabla u|^2 \,\mathrm{d}x.$$

The conclusion follows by adding the two parts. □

Lemma 4.3.24. *Let $\rho \in (0,1/2)$, μ_ρ be as in Corollary 4.3.17, $\varepsilon \in (0,1]$ and $r \in (\varepsilon, R_{\max}]$. If $\varepsilon\mu_\rho < r$, then*

$$\int_{\Omega_r^\varepsilon} |u \nabla u \cdot x| \,\mathrm{d}x \le K_{\varepsilon,r}^2 \int_{\Omega_r^\varepsilon} |\nabla u|^2 \,\mathrm{d}x$$

for any $u \in H^1(\Omega_r^\varepsilon)$ such that $u = 0$ on $\partial\Omega_r^\varepsilon \setminus S_r^+$, where

$$K_{\varepsilon,r}^2 = \sqrt{2\kappa}\varepsilon^{(N-1)/N} + \sqrt{\frac{3}{N-1}}r^2$$

and κ is as in Lemma 4.3.19.

Proof. First we consider the integral over T_ε: thanks to Cauchy-Schwarz Inequality and Lemma 4.3.19 we know that

$$\int_{T_\varepsilon} |u\nabla u \cdot x| \,\mathrm{d}x \,\mathrm{d}x \le \sqrt{2\kappa}\varepsilon^{(N-1)/N} \int_{\Omega_r^\varepsilon} |\nabla u|^2 \,\mathrm{d}x.$$

From the Cauchy-Schwarz Inequality and (4.3.20) it follows that

$$\int_{B_r^+} |u\nabla u \cdot x| \,\mathrm{d}x \le \sqrt{\frac{3}{N-1}}r^2 \int_{\Omega_r^\varepsilon} |\nabla u|^2 \,\mathrm{d}x.$$

Adding the two parts we conclude the proof. □

Corollary 4.3.25. *Let $\rho \in (0,1/2)$, μ_ρ be as in Corollary 4.3.17, $\varepsilon \in (0,1]$, $r \in (\varepsilon, R_{\max}]$. If $\varepsilon\mu_\rho < r$ then*

$$\int_{\Omega_r^\varepsilon} \left(|\nabla u|^2 - \lambda_i^\varepsilon p\,|u|^2\right) \mathrm{d}x \ge \left(1 - \lambda_i^\varepsilon \|p\|_\infty K_{\varepsilon,r}^1\right) \int_{\Omega_r^\varepsilon} |\nabla u|^2 \,\mathrm{d}x \tag{4.3.25}$$

for any $u \in H^1(\Omega_r^\varepsilon)$ such that $u = 0$ on $\partial\Omega_r^\varepsilon \setminus S_r^+$ and for all $i \in \{1, \ldots, n_0\}$. Furthermore there exists $r_0 \le R_{\max}$ such that, for every r, ε satisfying $\varepsilon\mu_\rho < r \le r_0$, we have

$$\int_{\Omega_r^\varepsilon} |\nabla u|^2 \,\mathrm{d}x \le 2 \int_{\Omega_r^\varepsilon} \left(|\nabla u|^2 - \lambda_i^\varepsilon p\,|u|^2\right) \mathrm{d}x$$

for any $u \in H^1(\Omega_r^\varepsilon)$ such that $u = 0$ on $\partial\Omega_r^\varepsilon \setminus S_r^+$ and for all $i \in \{1, \ldots, n_0\}$.

Proof. The first statement (4.3.25) easily comes from Lemma 4.3.23. Besides, if we choose $r_0 \le R_{\max}$ such that

$$K^1_{\varepsilon,r} \le K^1_{r_0,r_0} \le \frac{1}{2\lambda_0 \|p\|_\infty},$$

from (4.3.25), we can conclude the proof. $\square$

Lemma 4.3.26. *Let $\rho \in (0,1/2)$ and μ_ρ be as in Corollary 4.3.17. Let R_1 and ε_1 be as in Lemma 4.3.20. Then there exists $\tau > 0$ depending only on N, λ_0, $\|p\|_\infty$ and $|\Sigma|$ such that, for every $\varepsilon \in (0,\varepsilon_1]$, r_1, r_2, with $0 < \mu_\rho\varepsilon < r_1 \le r_2 \le \min\{1, R_1\}$, we have that*

$$\frac{H(\varphi_i^\varepsilon, r_2)}{H(\varphi_i^\varepsilon, r_1)} \ge \exp\left(-\tau R_1^{2(N-1)/N}\right)\left(\frac{r_2}{r_1}\right)^{2(1-\rho)} \qquad \textit{for all } i \in \{1,\dots,n_0\}.$$

Proof. With the notation $E(r) = E(\varphi_i^\varepsilon, r, \lambda_i^\varepsilon, \varepsilon)$, $H(r) = H(\varphi_i^\varepsilon, r)$ and $\frac{\mathrm{d}H}{\mathrm{d}r}(\varphi_i^\varepsilon, r) = H'(r)$, we have that, from Proposition 4.3.22 and Corollary 4.3.25

$$\begin{aligned} H'(r) = \frac{2}{r}E(r) = \frac{2}{r^{N-1}}\int_{\Omega_r^\varepsilon}\left(|\nabla\varphi_i^\varepsilon|^2 - \lambda_i^\varepsilon p\,|\varphi_i^\varepsilon|^2\right)\mathrm{d}x \\ \ge \frac{2}{r^{N-1}}(1 - \lambda_i^\varepsilon \|p\|_\infty K^1_{\varepsilon,r})\int_{\Omega_r^\varepsilon}|\nabla\varphi_i^\varepsilon|^2\,\mathrm{d}x \end{aligned}$$

for all $\varepsilon\mu_\rho < r \le \min\{1, R_1\}$. Hence, since $\lambda_i^\varepsilon \le \lambda_0$, thanks to (4.3.19)

$$H'(r) \ge \frac{2}{r^{N-1}}(1 - \lambda_0\|p\|_\infty K^1_{\varepsilon,r})\frac{1-\rho}{r}\int_{S_r^+}|\varphi_i^\varepsilon|^2\,\mathrm{d}S = \frac{2(1-\rho)}{r}(1 - \lambda_0\|p\|_\infty K^1_{\varepsilon,r})H(r).$$

So we have

$$\frac{H'(r)}{H(r)} \ge \frac{2(1-\rho)}{r}\left[1 - \tau_1\varepsilon^{2(N-1)/N} - \tau_2 r^2\right]$$

where $\tau_1 = \lambda_0\|p\|_\infty \kappa$ and $\tau_2 = \lambda_0\|p\|_\infty \frac{3}{N-1}$. Since $\varepsilon < r$ and $r \le 1$, if $\tau_0 = \tau_1 + \tau_2$, then

$$(\log H(r))' = \frac{H'(r)}{H(r)} \ge \frac{2(1-\rho)}{r}\left[1 - \tau_0 r^{2(N-1)/N}\right] \ge \frac{2(1-\rho)}{r} - 2\tau_0 r^{1-\frac{2}{N}}.$$

Integrating from r_1 to r_2 and letting $\tau := \tau_0 N/(N-1)$, we obtain

$$\log\frac{H(\varphi_i^\varepsilon, r_2)}{H(\varphi_i^\varepsilon, r_1)} \ge 2(1-\rho)\log\frac{r_2}{r_1} - \tau(r_2^{2(N-1)/N} - r_1^{2(N-2)/N}) \ge 2(1-\rho)\log\frac{r_2}{r_1} - \tau R_1^{2(N-1)/N}.$$

Taking the exponentials yields the thesis. $\square$

Hereafter let $R_0 := \min\{1, R_1, R_2, r_0\}$ where R_1, R_2, r_0 are defined in Lemma 4.3.20, Lemma 4.3.21 and Corollary 4.3.25 respectively. Moreover let $\varepsilon_0 = \min\{1, \tilde{\varepsilon}, \varepsilon_1\}$ where $\tilde{\varepsilon}, \varepsilon_1$ are defined in Proposition 4.3.8 and Lemma 4.3.20 respectively.

Proposition 4.3.27. *Let $\rho \in (0, 1/2)$ and μ_ρ be as in Corollary 4.3.17. Then, for every $r \in (0, R_0]$, $\varepsilon \in (0, \varepsilon_0]$ such that $0 < \varepsilon\mu_\rho < r \le R_0$*

$$\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}r}(\varphi_i^\varepsilon, r, \lambda_i^\varepsilon, \varepsilon) \ge -f(r)\mathcal{N}(\varphi_i^\varepsilon, r, \lambda_i^\varepsilon, \varepsilon) \quad \text{for all } i \in \{1, \dots, n_0\},$$

where

$$f(r) = c_1 r + c_2 r^{(N-2)/N} + c_3 r^{-1/N}$$

and c_n's are positive constants depending only on ρ, $\|p\|_\infty$, λ_0, the dimension N and the geometry of the problem (in particular on Ω and on $|\Sigma|_{N-1}$).

Proof. With the usual notation

$$\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}r}(\varphi_i^\varepsilon, r, \lambda_i^\varepsilon, \varepsilon) =: \mathcal{N}'(r), \quad \frac{\mathrm{d}E}{\mathrm{d}r}(\varphi_i^\varepsilon, r, \lambda_i^\varepsilon, \varepsilon) := E'(r), \quad \frac{\mathrm{d}H}{\mathrm{d}r}(\varphi_i^\varepsilon, r) := H'(r),$$

from Proposition 4.3.22 we have that

$$\mathcal{N}'(r) \ge \frac{1}{H^2}\frac{2}{r^{2N-3}}\left\{\left[\left(\int_{S_r^+}\left(\frac{\partial\varphi_i^\varepsilon}{\partial\boldsymbol{\nu}}\right)^2 \mathrm{d}S\right)\left(\int_{S_r^+}|\varphi_i^\varepsilon|^2\,\mathrm{d}S\right) - \left(\int_{S_r^+}\varphi_i^\varepsilon\frac{\partial\varphi_i^\varepsilon}{\partial\boldsymbol{\nu}}\,\mathrm{d}S\right)^2\right]\right.$$
$$\left.+\left[\frac{\lambda_i^\varepsilon}{r}\int_{\Omega_r^\varepsilon} p\,\varphi_i^\varepsilon\,\nabla\varphi_i^\varepsilon\cdot x\,\mathrm{d}x - \frac{\lambda_i^\varepsilon}{2}\int_{S_r^+} p\,|\varphi_i^\varepsilon|^2\,\mathrm{d}S + \frac{N-2}{2r}\lambda_i^\varepsilon\int_{\Omega_r^\varepsilon} p\,|\varphi_i^\varepsilon|^2\,\mathrm{d}x\right]\int_{S_r^+}|\varphi_i^\varepsilon|^2\,\mathrm{d}S\right\}.$$

By Cauchy-Schwartz Inequality we have that

$$\mathcal{N}'(r) \ge \frac{2\lambda_i^\varepsilon}{\int_{S_r^+}|\varphi_i^\varepsilon|^2}\left[\int_{\Omega_r^\varepsilon} p\,\varphi_i^\varepsilon\,\nabla\varphi_i^\varepsilon\cdot x\,\mathrm{d}x + \frac{N-2}{2}\int_{\Omega_r^\varepsilon} p\,|\varphi_i^\varepsilon|^2\,\mathrm{d}x - \frac{r}{2}\int_{S_r^+} p\,|\varphi_i^\varepsilon|^2\,\mathrm{d}S\right].$$

Thanks to Lemmas 4.3.23, 4.3.24, Corollary 4.3.17 and Corollary 4.3.25 we can say that

$$\begin{aligned}\mathcal{N}'(r) &\ge -\frac{2\lambda_i^\varepsilon\,\|p\|_\infty}{\int_{S_r^+}|\varphi_i^\varepsilon|^2}\left[K_{\varepsilon,r}^2 + \frac{(N-2)}{2}K_{\varepsilon,r}^1 + \frac{r^2}{2(1-\rho)}\right]\int_{\Omega_r^\varepsilon}|\nabla\varphi_i^\varepsilon|^2\,\mathrm{d}x \\ &\ge -\frac{4\lambda_i^\varepsilon\,\|p\|_\infty}{r^{N-1}H(r)}r^{N-2}E(r)\left[K_{\varepsilon,r}^2 + \frac{(N-2)}{2}K_{\varepsilon,r}^1 + r^2\right].\end{aligned}$$

Taking into account that $K_{\varepsilon,r}^n < K_{r,r}^n$, $n = 1, 2$, we have

$$\mathcal{N}'(r) \ge -\left(c_1 r + c_2 r^{(N-2)/N} + c_3 r^{-1/N}\right)\mathcal{N}(r) = -f(r)\mathcal{N}(r)$$

by some constants $c_1, c_2, c_3 > 0$ independent of r and ε. ☐

Corollary 4.3.28. *Let $\rho \in (0, \frac{1}{2})$ and μ_ρ be as in Corollary 4.3.17. Then, for every $\mu > \mu_\rho$, $r \in (0, R_0]$, and $\varepsilon \in (0, \varepsilon_0]$ such that $\varepsilon\mu \le r \le R_0$, we have that*

$$\mathcal{N}(\varphi_i^\varepsilon, r, \lambda_i^\varepsilon, \varepsilon) \le e^{\int_r^{R_0} f(t)\,dt}\mathcal{N}(\varphi_i^\varepsilon, R_0, \lambda_i^\varepsilon, \varepsilon).$$

Proof. Form Proposition 4.3.27 it follows that $(e^{-\int_r^{R_0} f(t)\,dt}\mathcal{N}(r))' \ge 0$, which, by integration over (r, R_0), yields the conclusion. ☐

Energy Estimates Thanks to the almost monotonicity of the perturbed Almgren-type frequency functional (4.3.21), computed at φ_i^ε, we obtain the following energy estimates.

Proposition 4.3.29. *Let $\rho \in (0, 1/2)$. Then there exists $K_\rho > 0$ such that, for every $R \geq K_\rho$ and for every $i \in \{1, \dots, j\}$, we have*

$$\int_{\Omega^\varepsilon_{R\varepsilon}} |\nabla\varphi_i^\varepsilon|^2 \, \mathrm{d}x = O(\varepsilon^{N-2} H(\varphi_i^\varepsilon, K_\rho\varepsilon)) \quad \text{as } \varepsilon \to 0^+, \tag{4.3.26}$$

$$\int_{\Omega^\varepsilon_{R\varepsilon}} |\varphi_i^\varepsilon|^2 \, \mathrm{d}x = O(\varepsilon^{N-\frac{2}{N}} H(\varphi_i^\varepsilon, K_\rho\varepsilon)) \quad \text{as } \varepsilon \to 0^+, \tag{4.3.27}$$

$$\int_{S^+_{R\varepsilon}} |\varphi_i^\varepsilon|^2 \, \mathrm{d}S = O(\varepsilon^{N-1} H(\varphi_i^\varepsilon, K_\rho\varepsilon)) \quad \text{as } \varepsilon \to 0^+. \tag{4.3.28}$$

Proof. For $\rho \in (0, 1/2)$ let us consider μ_ρ as in Corollary 4.3.17, $\varepsilon_0 = \min\{1, \tilde{\varepsilon}, \varepsilon_1\}$, ε_{R_0} as in Lemma 4.3.21 and let $K_\rho > \max\{\mu_\rho, R_0/\varepsilon_0, R_0/\varepsilon_{R_0}\}$. From Corollary 4.3.28 we deduce that, if $R \geq K_\rho$ and $R\varepsilon < R_0$

$$\mathcal{N}(\varphi_i^\varepsilon, R\varepsilon, \lambda_i^\varepsilon, \varepsilon) \leq e^{\int_{R\varepsilon}^{R_0} f(t)\, dt} \mathcal{N}(\varphi_i^\varepsilon, R_0, \lambda_i^\varepsilon, \varepsilon). \tag{4.3.29}$$

Now let us analyze the frequency function $\mathcal{N}$ at radius R_0:

$$E(\varphi_i^\varepsilon, R_0, \lambda_i^\varepsilon, \varepsilon) = \frac{1}{R_0^{N-2}} \int_{\Omega^\varepsilon_{R_0}} \left(|\nabla\varphi_i^\varepsilon|^2 - \lambda_i^\varepsilon\, |\varphi_i^\varepsilon|^2\right) \mathrm{d}x \leq \frac{1}{R_0^{N-2}} \int_{\Omega^\varepsilon} |\nabla\varphi_i^\varepsilon|^2 \, \mathrm{d}x \leq \frac{\lambda_0}{R_0^{N-2}}.$$

Moreover, thanks to Lemma 4.3.21

$$H(\varphi_i^\varepsilon, R_0) \geq c_{R_0}.$$

Thus we have that

$$\mathcal{N}(\varphi_i^\varepsilon, R_0, \lambda_i^\varepsilon, \varepsilon) \leq \frac{\lambda_0}{c_{R_0} R_0^{N-2}}. \tag{4.3.30}$$

Then, from (4.3.29)

$$\int_{\Omega^\varepsilon_{R\varepsilon}} \left(|\nabla\varphi_i^\varepsilon|^2 - \lambda_i^\varepsilon\, |\varphi_i^\varepsilon|^2\right) \mathrm{d}x \leq \text{const}\, H(\varphi_i^\varepsilon, R\varepsilon)(R\varepsilon)^{N-2}. \tag{4.3.31}$$

From the second statement of Corollary 4.3.25 we have that

$$\int_{\Omega^\varepsilon_{R\varepsilon}} |\nabla\varphi_i^\varepsilon|^2 \, \mathrm{d}x \leq 2\, \text{const}\, H(\varphi_i^\varepsilon, R\varepsilon)(R\varepsilon)^{N-2}. \tag{4.3.32}$$

Now let $K_\rho\varepsilon \leq r \leq R_0$. Then, from Proposition 4.3.22

$$\frac{H'(\varphi_i^\varepsilon, r)}{H(\varphi_i^\varepsilon, r)} = \frac{2}{r} \mathcal{N}(\varphi_i^\varepsilon, r, \lambda_i^\varepsilon, \varepsilon)$$

and from Corollary 4.3.28 and (4.3.30)

$$\frac{H'(\varphi_i^\varepsilon, r)}{H(\varphi_i^\varepsilon, r)} \le \frac{C}{r}. \tag{4.3.33}$$

Now, integrating the previous inequality from $K_\rho\varepsilon$ to $R\varepsilon$, we obtain

$$\log \frac{H(\varphi_i^\varepsilon, R\varepsilon)}{H(\varphi_i^\varepsilon, K_\rho\varepsilon)} \le C \log \frac{R\varepsilon}{K_\rho\varepsilon},$$

hence $H(\varphi_i^\varepsilon, R\varepsilon) \le \mathrm{const}_{\rho,R}\, H(\varphi_i^\varepsilon, K_\rho\varepsilon)$, i.e. $H(\varphi_i^\varepsilon, R\varepsilon) = O(H(\varphi_i^\varepsilon, K_\rho\varepsilon))$ as $\varepsilon \to 0$. Then (4.3.26) follows from (4.3.32), whereas (4.3.28) is a direct consequence of the previous estimate and definition of H. Finally, thanks to Lemma 4.3.23 and (4.3.26), we have

$$\int_{\Omega^\varepsilon_{R\varepsilon}} |\varphi_i^\varepsilon|^2 \,\mathrm{d}x \le (c_1\varepsilon^{2(N-1)/N} + c_2(R\varepsilon)^2) \int_{\Omega^\varepsilon_{R\varepsilon}} |\nabla\varphi_i^\varepsilon| \,\mathrm{d}x = O(\varepsilon^{N-\frac{2}{N}} H(\varphi_i^\varepsilon, K_\rho\varepsilon)),$$

as $\varepsilon \to 0$, thus proving (4.3.27). $\square$

Proposition 4.3.30. *Let $\rho \in (0, 1/2)$ and K_ρ be as in Proposition 4.3.29. Then there exists $C_\rho > 0$ such that, for every $R \ge K_\rho$, for every $\varepsilon \in (0, \varepsilon_0]$ such that $R\varepsilon \le R_0$, and for every $i \in \{1, \dots, n_0\}$ we have*

$$\int_{\Omega^\varepsilon_{R\varepsilon}} |\nabla\varphi_i^\varepsilon|^2 \,\mathrm{d}x \le C_\rho (R\varepsilon)^{N-2\rho},$$
$$\int_{\Omega^\varepsilon_{R\varepsilon}} |\varphi_i^\varepsilon|^2 \,\mathrm{d}x \le C_\rho (R\varepsilon)^{N+2-2\rho-2/N},$$
$$\int_{S^+_{R\varepsilon}} |\varphi_i^\varepsilon|^2 \,\mathrm{d}x \le C_\rho (R\varepsilon)^{N+1-2\rho}.$$

Proof. From Lemma 4.3.26 we know that

$$H(\varphi_i^\varepsilon, R\varepsilon) \le \exp\left(\tau R_1^{2(N-1)/N}\right) \left(\frac{R\varepsilon}{R_0}\right)^{2(1-\rho)} H(\varphi_i^\varepsilon, R_0) \tag{4.3.34}$$

and, from (4.3.19), we have

$$H(\varphi_i^\varepsilon, R_0) = \frac{1}{R_0^{N-1}} \int_{S^+_{R_0}} |\varphi_i^\varepsilon|^2 \,\mathrm{d}S \le \frac{1}{R_0^{N-2}(1-\rho)} \int_{\Omega^\varepsilon_{R_0}} |\nabla\varphi_i^\varepsilon|^2 \,\mathrm{d}x \le \frac{\lambda_0}{R_0^{N-2}(1-\rho)}. \tag{4.3.35}$$

Combining (4.3.34) and (4.3.35) with Proposition 4.3.29 (in particular estimate (4.3.32) in the proof) we can deduce all the claims. $\square$

As a consequence of Proposition 4.3.30 we can say that

$$H(\varphi_i^\varepsilon, K_\rho\varepsilon) = O(\varepsilon^{2-2\rho}) \quad \text{as } \varepsilon \to 0. \tag{4.3.36}$$

As a byproduct of the proof of Proposition 4.3.29 we obtain the following result.

Corollary 4.3.31. *Let $\rho \in (0, 1/2)$ and K_ρ be as in Proposition 4.3.29. Then there exist $\bar{C}, q > 0$ such that, if $\varepsilon \in (0, \varepsilon_0]$ and $K_\rho \varepsilon < R_0$,*

$$H(\varphi_\varepsilon, K_\rho \varepsilon) \geq \bar{C} \varepsilon^q. \tag{4.3.37}$$

Proof. If we integrate (4.3.33) over $(K_\rho \varepsilon, R_0)$ and take the exponentials, we obtain

$$\frac{H(\varphi_\varepsilon, R_0)}{H(\varphi_\varepsilon, K_\rho \varepsilon)} \leq \left(\frac{R_0}{K_\rho \varepsilon} \right)^q,$$

denoting by q the constant C in (4.3.33). Then Lemma 4.3.21 implies that

$$H(\varphi_\varepsilon, K_\rho \varepsilon) \geq c_{R_0} \left(\frac{K_\rho \varepsilon}{R_0} \right)^q.$$

Hence the claim is proved with $\bar{C} := c_{R_0} (K_\rho / R_0)^q$. □

4.3.5 Estimates on the difference $\lambda_0 - \lambda_\varepsilon$

We star this section by recalling from [AF15] the following lemma about maxima of quadratic forms depending on a parameter, which plays a central role in estimating the difference $\lambda_0 - \lambda_\varepsilon$.

Lemma 4.3.32. *For every $\varepsilon > 0$ let us consider a quadratic form*

$$Q_\varepsilon : \mathbb{R}^{n_0} \longrightarrow \mathbb{R},$$
$$Q_\varepsilon(\xi_1, \dots, \xi_{n_0}) = \sum_{i,n=1}^{n_0} M_{i,n}(\varepsilon) \xi_i \xi_n,$$

with real coefficients $M_{i,n}(\varepsilon)$ such that $M_{i,n}(\varepsilon) = M_{n,i}(\varepsilon)$. Let us assume that there exist $\alpha > 0$, $\varepsilon \mapsto \sigma(\varepsilon) \in \mathbb{R}$ with $\sigma(\varepsilon) \geq 0$ and $\sigma(\varepsilon) = O(\varepsilon^{2\alpha})$ as $\varepsilon \to 0$, and $\varepsilon \mapsto \mu(\varepsilon) \in \mathbb{R}$ with $\mu(\varepsilon) = O(1)$ as $\varepsilon \to 0$, such that the coefficients $M_{i,n}(\varepsilon)$ satisfy the following conditions:

$M_{n_0,n_0}(\varepsilon) = \sigma(\varepsilon)\mu(\varepsilon)$,
for all $i < n_0$ $M_{i,i}(\varepsilon) \to M_i < 0$, *as* $\varepsilon \to 0$,
for all $i < n_0$ $M_{i,n_0}(\varepsilon) = O(\varepsilon^\alpha \sqrt{\sigma(\varepsilon)})$ *as* $\varepsilon \to 0$,
for all $i, n < n_0$ *with* $i \neq n$ $M_{i,n} = O(\varepsilon^{2\alpha})$ *as* $\varepsilon \to 0$,
there exists $M \in \mathbb{N}$ *such that* $\varepsilon^{(2+M)\alpha} = o(\sigma(\varepsilon))$ *as* $\varepsilon \to 0$.

Then

$$\max_{\substack{\xi \in \mathbb{R}^{n_0} \\ \|\xi\|=1}} Q_\varepsilon(\xi) = \sigma(\varepsilon)(\mu(\varepsilon) + o(1)) \quad \text{as } \varepsilon \to 0,$$

where $\|\xi\| = \|(\xi_1, \dots, \xi_{n_0})\| = \left(\sum_{i=1}^{n_0} \xi_i^2 \right)^{1/2}$.

Upper Bound For any $i \in \{0, \dots, n_0\}$, $R > 1$ and $\varepsilon \in (0,1]$, with $R\varepsilon \leq R_{\max}$, let us consider the following minimization problem

$$\min\left\{\int_{B^+_{R\varepsilon}} |\nabla u|^2 \, \mathrm{d}x \colon u \in H^1(B^+_{R\varepsilon}),\ u = 0 \text{ on } B'_{R\varepsilon},\ u = \varphi_i^\varepsilon \text{ on } S^+_{R\varepsilon}\right\}.$$

One can prove that this problem has a unique solution $v^{\mathrm{int}}_{i,R,\varepsilon}$, which weakly solves

$$\begin{cases} -\Delta v^{\mathrm{int}}_{i,R,\varepsilon} = 0, & \text{in } B^+_{R\varepsilon}, \\ v^{\mathrm{int}}_{i,R,\varepsilon} = 0, & \text{on } B'_{R\varepsilon}, \\ v^{\mathrm{int}}_{i,R,\varepsilon} = \varphi_i^\varepsilon, & \text{on } S^+_{R\varepsilon}. \end{cases}$$

Proposition 4.3.33. *Let $\rho \in (0, 1/2)$ and K_ρ be as in Proposition 4.3.29. Then*

$$\int_{B^+_{R\varepsilon}} \left|\nabla v^{\mathrm{int}}_{i,R,\varepsilon}\right|^2 \mathrm{d}x = O(\varepsilon^{N-2} H(\varphi_i^\varepsilon, K_\rho \varepsilon)) \quad \text{as } \varepsilon \to 0^+, \tag{4.3.38}$$

$$\int_{B^+_{R\varepsilon}} \left|v^{\mathrm{int}}_{i,R,\varepsilon}\right|^2 \mathrm{d}x = O(\varepsilon^{N} H(\varphi_i^\varepsilon, K_\rho \varepsilon)) \quad \text{as } \varepsilon \to 0^+, \tag{4.3.39}$$

$$\int_{S^+_{R\varepsilon}} \left|v^{\mathrm{int}}_{i,R,\varepsilon}\right|^2 \mathrm{d}x = O(\varepsilon^{N-1} H(\varphi_i^\varepsilon, K_\rho \varepsilon)) \quad \text{as } \varepsilon \to 0^+. \tag{4.3.40}$$

for all $R \geq 2$ and for any $i = 1, \dots, n_0$. Moreover there exists $\hat{C}_\rho$ such that, if $R \geq \max\{2, K_\rho\}$ and $\varepsilon < R_0/R$,

$$\int_{B^+_{R\varepsilon}} \left|\nabla v^{\mathrm{int}}_{i,R,\varepsilon}\right|^2 \mathrm{d}x \leq \hat{C}_\rho (R\varepsilon)^{N-2\rho}, \tag{4.3.41}$$

$$\int_{B^+_{R\varepsilon}} \left|v^{\mathrm{int}}_{i,R,\varepsilon}\right|^2 \mathrm{d}x \leq \hat{C}_\rho (R\varepsilon)^{N+2-2\rho}, \tag{4.3.42}$$

$$\int_{S^+_{R\varepsilon}} \left|v^{\mathrm{int}}_{i,R,\varepsilon}\right|^2 \mathrm{d}x \leq \hat{C}_\rho (R\varepsilon)^{N+1-2\rho}. \tag{4.3.43}$$

Proof. Proving (4.3.40) is trivial due to (4.3.28), since $v^{\mathrm{int}}_{i,R,\varepsilon} = \varphi_i^\varepsilon$ on $S^+_{R\varepsilon}$. Let $\eta = \eta_R(\frac{\cdot}{\varepsilon})$, with η_R defined in (4.3.1); then

$$\begin{aligned} \int_{B^+_{R\varepsilon}} \left|\nabla v^{\mathrm{int}}_{i,R,\varepsilon}\right|^2 \mathrm{d}x &\leq \int_{B^+_{R\varepsilon}} |\nabla(\eta \varphi_i^\varepsilon)|^2 \, \mathrm{d}x \leq \\ &\leq 2\left(\int_{B^+_{R\varepsilon}} |\nabla \varphi_i^\varepsilon|^2 + \frac{16}{(R\varepsilon)^2} \int_{B^+_{R\varepsilon}} |\varphi_i^\varepsilon|^2 \, \mathrm{d}x\right) \leq \mathrm{const}_\rho \int_{\Omega^\varepsilon_{R\varepsilon}} |\nabla \varphi_i^\varepsilon|^2 \, \mathrm{d}x, \end{aligned}$$

where the last step comes from (4.3.20). Combining this inequality with (4.3.26) we obtain (4.3.38). Moreover (4.3.20) and (4.3.38) yield (4.3.39). Finally estimates (4.3.41)–(4.3.43) follow from the above argument and Proposition 4.3.30. □

Now let us define, for all $i \in \{1, \dots, n_0\}$, for all $R > 1$ and $\varepsilon \in (0, 1]$ such that $R\varepsilon \le R_{\max}$,

$$v_{i,R,\varepsilon} := \begin{cases} v^{\text{int}}_{i,R,\varepsilon}, & \text{in } B^+_{R\varepsilon}, \\ \varphi^\varepsilon_i, & \text{in } \Omega \setminus B^+_{R\varepsilon}, \end{cases} \tag{4.3.44}$$

and

$$Z^\varepsilon_R(x) := \frac{v^{\text{int}}_{n_0,R,\varepsilon}(\varepsilon x)}{\sqrt{H(\varphi_\varepsilon, K_\rho\varepsilon)}}, \qquad \tilde{\varphi}^\varepsilon(x) := \frac{\varphi_\varepsilon(\varepsilon x)}{\sqrt{H(\varphi_\varepsilon, K_\rho\varepsilon)}}. \tag{4.3.45}$$

It is easy to prove that the family of functions $\{v_{1,R,\varepsilon}, \dots, v_{n_0,R,\varepsilon}\}$ is linearly independent in $H^1_0(\Omega)$.

Lemma 4.3.34. *For all $R \ge \max\{2, K_\rho\}$, we have that, as $\varepsilon \to 0^+$,*

$$\int_\Omega |\nabla v_{n_0,R,\varepsilon}|^2 \, \mathrm{d}x = \lambda_\varepsilon + \varepsilon^{N-2} H(\varphi_\varepsilon, K_\rho\varepsilon) \left(\int_{B^+_R} |\nabla Z^\varepsilon_R|^2 \, \mathrm{d}x - \int_{\Pi_R} |\nabla \tilde{\varphi}^\varepsilon|^2 \, \mathrm{d}x \right), \tag{4.3.46}$$

$$\int_\Omega |\nabla v_{i,R,\varepsilon}|^2 \, \mathrm{d}x = \lambda^\varepsilon_i + O(\varepsilon^{N-2\rho}) \quad \text{for all } i \in \{1, \dots, n_0\}, \tag{4.3.47}$$

$$\int_\Omega \nabla v_{i,R,\varepsilon} \cdot \nabla v_{n_0,R,\varepsilon} \, \mathrm{d}x = O\left(\varepsilon^{N-1-\rho} \sqrt{H(\varphi_\varepsilon, K_\rho\varepsilon)} \right) \quad \text{for all } i \in \{1, \dots, n_0 - 1\}, \tag{4.3.48}$$

$$\int_\Omega \nabla v_{i,R,\varepsilon} \cdot \nabla v_{n,R,\varepsilon} \, \mathrm{d}x = O(\varepsilon^{N-2\rho}) \quad \text{for all } i, n \in \{1, \dots, n_0\}, \ i \ne n, \tag{4.3.49}$$

$$\int_\Omega p \, |v_{n_0,R,\varepsilon}|^2 \, \mathrm{d}x = 1 + O(\varepsilon^{N-2/N} H(\varphi_\varepsilon, K_\rho\varepsilon)), \tag{4.3.50}$$

$$\int_\Omega p \, |v_{i,R,\varepsilon}|^2 \, \mathrm{d}x = 1 + O(\varepsilon^{N+2-2\rho-2/N}) \quad \text{for all } i \in \{1, \dots, n_0\}, \tag{4.3.51}$$

$$\int_\Omega p v_{i,R,\varepsilon} v_{n_0,R,\varepsilon} \, \mathrm{d}x = O\left(\varepsilon^{N+1-\rho-2/N} \sqrt{H(\varphi_\varepsilon, K_\rho\varepsilon)} \right) \quad \text{for all } i \in \{1, \dots, n_0 - 1\}, \tag{4.3.52}$$

$$\int_\Omega p v_{i,R,\varepsilon} v_{n,R,\varepsilon} \, \mathrm{d}x = O(\varepsilon^{N+2-2\rho-2/N}) \quad \text{for all } i, n \in \{1, \dots, n_0\}, \ i \ne n, \tag{4.3.53}$$

where, in (4.3.46), $\tilde{\varphi}^\varepsilon$ has been trivially extended in Π_R outside its domain.

Proof. We will only prove the first part of the estimates, i.e. (4.3.46), (4.3.47), (4.3.48), (4.3.49), since the second part is completely analogous. To prove (4.3.46) we observe that, by scaling,

$$\begin{aligned} \int_\Omega |\nabla v_{n_0,R,\varepsilon}|^2 \, \mathrm{d}x &= \int_{B^+_{R\varepsilon}} \left| \nabla v^{\text{int}}_{n_0,R,\varepsilon} \right|^2 \mathrm{d}x + \int_{\Omega^\varepsilon} |\nabla \varphi_\varepsilon|^2 \, \mathrm{d}x - \int_{\Omega^\varepsilon_{R\varepsilon}} |\nabla \varphi_\varepsilon|^2 \, \mathrm{d}x \\ &= \lambda_\varepsilon + \varepsilon^{N-2} H(\varphi_\varepsilon, K_\rho\varepsilon) \left(\int_{B^+_R} |\nabla Z^\varepsilon_R|^2 \, \mathrm{d}x - \int_{\Pi_R} |\nabla \tilde{\varphi}^\varepsilon|^2 \, \mathrm{d}x \right). \end{aligned}$$

Thanks to Propositions 4.3.30 and 4.3.33 we have that

$$\begin{aligned}\int_\Omega |\nabla v_{i,R,\varepsilon}|^2 \,\mathrm{d}x &= \int_{B^+_{R\varepsilon}} \left|\nabla v^{\mathrm{int}}_{i,R,\varepsilon}\right|^2 \mathrm{d}x + \int_{\Omega^\varepsilon} |\nabla \varphi_i^\varepsilon|^2 \,\mathrm{d}x - \int_{\Omega^\varepsilon_{R\varepsilon}} |\nabla \varphi_i^\varepsilon|^2 \,\mathrm{d}x \\ &= \lambda_i^\varepsilon + O(\varepsilon^{N-2\rho}),\end{aligned}$$

as $\varepsilon \to 0^+$, thus proving (4.3.47) and, by Cauchy-Schwarz Inequality, for $i < n_0$

$$\begin{aligned}\int_\Omega \nabla v_{i,R,\varepsilon} \cdot \nabla v_{n_0,R,\varepsilon} &= \int_{B^+_{R\varepsilon}} \nabla v^{\mathrm{int}}_{i,R,\varepsilon} \nabla v^{\mathrm{int}}_{n_0,R,\varepsilon} \,\mathrm{d}x + \int_{\Omega^\varepsilon} \nabla \varphi_i^\varepsilon \cdot \nabla \varphi_\varepsilon \,\mathrm{d}x - \int_{\Omega^\varepsilon_{R\varepsilon}} \nabla \varphi_i^\varepsilon \cdot \nabla \varphi_\varepsilon \,\mathrm{d}x \\ &= O(\varepsilon^{\frac{N-2\rho}{2}}) O\left(\varepsilon^{\frac{N-2}{2}} \sqrt{H(\varphi_\varepsilon, K_\rho \varepsilon)}\right) = O\left(\varepsilon^{N-1-\rho} \sqrt{H(\varphi_\varepsilon, K_\rho \varepsilon)}\right),\end{aligned}$$

as $\varepsilon \to 0^+$, thus proving (4.3.48). Similarly, for $i \neq n$

$$\begin{aligned}\int_\Omega \nabla v_{i,R,\varepsilon} \cdot \nabla v_{n,R,\varepsilon} &= \int_{B^+_{R\varepsilon}} \nabla v^{\mathrm{int}}_{i,R,\varepsilon} \nabla v^{\mathrm{int}}_{n,R,\varepsilon} \,\mathrm{d}x + \int_{\Omega^\varepsilon} \nabla \varphi_i^\varepsilon \cdot \nabla \varphi_n^\varepsilon \,\mathrm{d}x - \int_{\Omega^\varepsilon_{R\varepsilon}} \nabla \varphi_i^\varepsilon \cdot \nabla \varphi_n^\varepsilon \,\mathrm{d}x \\ &= O(\varepsilon^{N-2\rho}),\end{aligned}$$

as $\varepsilon \to 0^+$, which provides (4.3.49). □

We construct a basis $\{\hat{v}_{1,R,\varepsilon}, \dots, \hat{v}_{n_0,R,\varepsilon}\}$ of the space $\operatorname{span}\{v_{1,R,\varepsilon}, \dots, v_{n_0,R,\varepsilon}\}$ such that

$$\int_\Omega p\, \hat{v}_{n,R,\varepsilon} \hat{v}_{m,R,\varepsilon} \,dx = 0 \quad \text{for } n \neq m,$$

by defining

$$\hat{v}_{n_0,R,\varepsilon} = v_{n_0,R,\varepsilon}, \quad \hat{v}_{i,R,\varepsilon} = v_{i,R,\varepsilon} - \sum_{n=i+1}^{n_0} d^\varepsilon_{i,n} \hat{v}_{n,R,\varepsilon}, \quad \text{for all } i = 1, \dots, n_0 - 1\ ,$$

where

$$d^\varepsilon_{i,n} = \frac{\int_\Omega p\, v_{i,R,\varepsilon}\, \hat{v}_{n,R,\varepsilon} \,\mathrm{d}x}{\int_\Omega p\, |\hat{v}_{n,R,\varepsilon}|^2 \,\mathrm{d}x}.$$

Using the estimates established in Lemma 4.3.34, one can prove the following

$$\int_\Omega |\nabla \hat{v}_{n_0,R,\varepsilon}|^2 \,\mathrm{d}x = \lambda_\varepsilon + \varepsilon^{N-2} H(\varphi_\varepsilon, K_\rho \varepsilon) \left(\int_{B^+_R} |\nabla Z^\varepsilon_R|^2 \,\mathrm{d}x - \int_{\Pi_R} |\nabla \tilde{\varphi}^\varepsilon|^2 \,\mathrm{d}x\right), \tag{4.3.54}$$

$$\int_\Omega |\nabla \hat{v}_{i,R,\varepsilon}|^2 \,\mathrm{d}x = \lambda_i^\varepsilon + O(\varepsilon^{N-2\rho}) \quad \text{for all } i \in \{1, \dots, n_0\}, \tag{4.3.55}$$

$$\int_\Omega \nabla \hat{v}_{n_0,R,\varepsilon} \cdot \nabla \hat{v}_{i,R,\varepsilon} \,\mathrm{d}x = O(\varepsilon^{N-1-\rho} \sqrt{H(\varphi_\varepsilon, K_\rho \varepsilon)}) \quad \text{for all } i \in \{1, \dots, n_0 - 1\}, \tag{4.3.56}$$

$$\int_\Omega \nabla \hat{v}_{i,R,\varepsilon} \cdot \nabla \hat{v}_{m,R,\varepsilon} \,\mathrm{d}x = O(\varepsilon^{N-2\rho}) \quad \text{for all } i, m \in \{1, \dots, n_0\}, \quad i \neq m, \tag{4.3.57}$$

$$\int_\Omega p\, |\hat{v}_{n_0,R,\varepsilon}|^2 \,\mathrm{d}x = 1 + O(\varepsilon^{N-2/N} H(\varphi_\varepsilon, K_\rho \varepsilon)), \tag{4.3.58}$$

$$\int_\Omega p\, |\hat{v}_{i,R,\varepsilon}|^2 \,\mathrm{d}x = 1 + O(\varepsilon^{N+2-2\rho-2/N}) \quad \text{for all } i \in \{1, \dots, n_0\}, \tag{4.3.59}$$

as $\varepsilon \to 0$.

Proposition 4.3.35. *Let $\rho \in (0, 1/2)$, K_ρ as defined in Proposition 4.3.29 and $R \geq K_\rho$. For $\varepsilon < R_0/R$ there exists $f_R(\varepsilon)$ such that*

$$\lambda_0 - \lambda_\varepsilon \leq \varepsilon^{N-2} H(\varphi_\varepsilon, K_\rho \varepsilon)(f_R(\varepsilon) + o(1)) \quad \text{as } \varepsilon \to 0$$

and

$$f_R(\varepsilon) = \int_{B_R^+} |\nabla Z_R^\varepsilon|^2 \, \mathrm{d}x - \int_{\Pi_R} |\nabla \tilde{\varphi}^\varepsilon|^2 \, \mathrm{d}x,$$

where $\tilde{\varphi}^\varepsilon$ has been trivially extended in Π_R outside its domain.

Proof. By the *Courant-Fischer Min-Max* characterization of eigenvalues (see Proposition 3.0.5)

$$\lambda_0 = \min \left\{ \max_{\substack{\alpha_1, \dots, \alpha_{n_0} \in \mathbb{R} \\ \sum_{i=1}^{n_0} |\alpha_i|^2 = 1}} \frac{\int_\Omega |\nabla (\sum_{i=1}^{n_0} \alpha_i u_i)|^2 \, dx}{\int_\Omega p \, |\sum_{i=1}^{n_0} \alpha_i u_i|^2 \, dx} : \begin{array}{l} \{u_1 \dots, u_{n_0}\} \subseteq H_0^1(\Omega) \\ \text{linearly independent} \end{array} \right\}.$$

Testing the Rayleigh quotient with the family of functions

$$\tilde{v}_{i,R,\varepsilon} = \frac{\hat{v}_{i,R,\varepsilon}}{\sqrt{\int_\Omega p \, |\hat{v}_{i,R,\varepsilon}|^2 \, \mathrm{d}x}}$$

we obtain that

$$\lambda_0 - \lambda_\varepsilon \leq \max_{\substack{\alpha_1, \dots, \alpha_{n_0} \in \mathbb{R} \\ \sum_{i=1}^{n_0} |\alpha_i|^2 = 1}} \int_\Omega \left| \nabla \left(\sum_{i=1}^{n_0} \alpha_i \tilde{v}_{i,R,\varepsilon} \right) \right|^2 \mathrm{d}x - \lambda_\varepsilon = \max_{\substack{\alpha_1, \dots, \alpha_{n_0} \in \mathbb{R} \\ \sum_{i=1}^{n_0} |\alpha_i|^2 = 1}} \sum_{i,n=1}^{n_0} M_{i,n}^\varepsilon \alpha_i \alpha_n$$

where

$$M_{i,n}^\varepsilon = \frac{\int_\Omega \nabla \hat{v}_{i,R,\varepsilon} \cdot \nabla \hat{v}_{n,R,\varepsilon} \, \mathrm{d}x}{\left(\int_\Omega p \, |\hat{v}_{i,R,\varepsilon}|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega p \, |\hat{v}_{n,R,\varepsilon}|^2 \, \mathrm{d}x \right)^{1/2}} - \lambda_\varepsilon \delta_i^n,$$

with δ_i^n denoting the usual *Kronecker delta*, i.e. $\delta_i^n = 0$ for $i \neq n$ and $\delta_i^n = 1$ for $i = n$. From estimates (4.3.54)–(4.3.59) one can derive the following estimates

$$\begin{aligned}
&M_{n_0,n_0}(\varepsilon) = \varepsilon^{N-2} H(\varphi_\varepsilon, K_\rho \varepsilon)(f_R(\varepsilon) + O(\varepsilon^{2-2/N})), \\
&M_{i,n_0}(\varepsilon) = O\left(\varepsilon^{N-1-\rho} \sqrt{H(\varphi_\varepsilon, K_\rho \varepsilon)}\right) \quad \text{and} \quad M_{i,i}(\varepsilon) = \lambda_i^\varepsilon - \lambda_\varepsilon + o(1) \quad \text{for all } i < n_0, \\
&M_{i,n}(\varepsilon) = O(\varepsilon^{N-2\rho}) \quad \text{for all } i, n < n_0, \quad i \neq n,
\end{aligned}$$

as $\varepsilon \to 0$. Moreover, from Corollary 4.3.31, we know that $H(\varphi_\varepsilon, K_\rho \varepsilon) \geq \bar{C} \varepsilon^q$ for some $\bar{C}, q > 0$. Therefore, taking also into account (4.3.36) and the fact that $f_R(\varepsilon) = O(1)$ as $\varepsilon \to 0$ in view of (4.3.26) and (4.3.38), the hypotheses of Lemma 4.3.32 are satisfied with

$$\sigma(\varepsilon) = \varepsilon^{N-2} H(\varphi_\varepsilon, K_\rho \varepsilon), \quad \mu(\varepsilon) = f_R(\varepsilon) + o(1), \quad \alpha = \frac{N}{2} - \rho, \quad M > (2\rho - 2 + q) \frac{2}{N - 2\rho}.$$

The proof is thereby complete. $\square$

Lower Bound For any $R > 1$ and $\varepsilon \in (0,1]$, with $R\varepsilon \leq R_{\max}$, let us consider the following minimization problem

$$\min\left\{\int_{\Omega^\varepsilon_{R\varepsilon}} |\nabla u|^2 \,\mathrm{d}x \colon u \in H^1(\Omega^\varepsilon_{R\varepsilon}),\ u = 0 \text{ on } \partial\Omega^\varepsilon_{R\varepsilon} \setminus S^+_{R\varepsilon},\ u = \varphi_0 \text{ on } S^+_{R\varepsilon}\right\}. \tag{4.3.60}$$

One can prove that this problem has a unique solution $w^{\mathrm{int}}_{j,R,\varepsilon}$, which weakly verifies

$$\begin{cases} -\Delta w^{\mathrm{int}}_{n_0,R,\varepsilon} = 0, & \text{in } \Omega^\varepsilon_{R\varepsilon}, \\ \quad w^{\mathrm{int}}_{n_0,R,\varepsilon} = 0, & \text{on } \partial\Omega^\varepsilon_{R\varepsilon} \setminus S^+_{R\varepsilon}, \\ \quad w^{\mathrm{int}}_{n_0,R,\varepsilon} = \varphi_0, & \text{on } S^+_{R\varepsilon}. \end{cases}$$

Let us define

$$w_{n_0,R,\varepsilon} := \begin{cases} w^{\mathrm{int}}_{n_0,R,\varepsilon}, & \text{in } \Omega^\varepsilon_{R\varepsilon}, \\ \varphi_0, & \text{in } \Omega \setminus B^+_{R\varepsilon}. \end{cases}$$

Lemma 4.3.36. *There exists $\tilde{C} > 0$ such that, for all $i \in \{1,\dots,n_0-1\}$, for all $R > 1$ and $\varepsilon \in (0,1]$, with $R\varepsilon \leq R_{\max}$,*

$$\int_{B^+_{R\varepsilon}} |\nabla\varphi_i|^2 \,\mathrm{d}x \leq \tilde{C}(R\varepsilon)^N, \quad \int_{B^+_{R\varepsilon}} |\varphi_i|^2 \,\mathrm{d}x \leq \tilde{C}(R\varepsilon)^{N+2}, \quad \int_{S^+_{R\varepsilon}} |\varphi_i|^2 \,\mathrm{d}x \leq \tilde{C}(R\varepsilon)^{N+1},$$

and

$$\int_{B^+_{R\varepsilon}} |\nabla\varphi_0|^2 \,\mathrm{d}x \leq \tilde{C}(R\varepsilon)^{N+2k-2}, \quad \int_{B^+_{R\varepsilon}} |\varphi_0|^2 \,\mathrm{d}x \leq \tilde{C}(R\varepsilon)^{N+2k},$$
$$\int_{S^+_{R\varepsilon}} |\varphi_0|^2 \,\mathrm{d}x \leq \tilde{C}(R\varepsilon)^{N+2k-1}.$$

Proof. It follows from classical asymptotic estimates at the boundary, see e.g. [FFT11, Th. 1.3] and (4.2.8),(4.2.9). □

Lemma 4.3.37. *There exists $\hat{C} > 0$ such that, for all $R > 1$ and $\varepsilon \in (0,1]$, with $R\varepsilon \leq R_{\max}$,*

$$\int_{\Omega^\varepsilon_{R\varepsilon}} \left|\nabla w^{\mathrm{int}}_{n_0,R,\varepsilon}\right|^2 \,\mathrm{d}x \leq \hat{C}(R\varepsilon)^{N+2k-2}, \tag{4.3.61}$$
$$\int_{S^+_{R\varepsilon}} \left|w^{\mathrm{int}}_{n_0,R,\varepsilon}\right|^2 \,\mathrm{d}x \leq \hat{C}(R\varepsilon)^{N+2k-1}. \tag{4.3.62}$$

Furthermore, for all $R > \mu_{1/2}$ and $\varepsilon \in (0,1]$, with $R\varepsilon \leq R_{\max}$,

$$\int_{\Omega^\varepsilon_{R\varepsilon}} \left|w^{\mathrm{int}}_{n_0,R,\varepsilon}\right|^2 \,\mathrm{d}x \leq \hat{C}(R\varepsilon)^{N+2k-2/N}. \tag{4.3.63}$$

Proof. (4.3.62) is trivial since $w^{\mathrm{int}}_{n_0,R,\varepsilon} = \varphi_0$ on $S^+_{R\varepsilon}$. Also (4.3.61) is simple since φ_0 is an admissible test function for (4.3.60). Finally (4.3.63) comes from (4.3.61), (4.3.62) and Lemma 4.3.23. □

Now let us define, for all $R > 1$ and $\varepsilon \in (0, 1]$ such that $R\varepsilon \le R_{\max}$,

$$U_R^\varepsilon(x) = \frac{w_{n_0,R,\varepsilon}^{\mathrm{int}}(\varepsilon x)}{\varepsilon^k}, \qquad W^\varepsilon(x) = \frac{\varphi_0(\varepsilon x)}{\varepsilon^k}. \tag{4.3.64}$$

From (4.2.8), we easily deduce that

$$W^\varepsilon \longrightarrow \psi_k \quad \text{ in } H^1(B_R^+) \quad \text{as } \varepsilon \to 0, \quad \text{for all } R > 0,$$

where ψ_k has been defined in (4.2.10).

Lemma 4.3.38. *We have that*

$$U_R^\varepsilon \longrightarrow U_R \quad \textit{ in } \mathcal{H}_R \quad \textit{as } \varepsilon \to 0, \quad \textit{for all } R > 1,$$

where U_R is defined in Lemma 4.3.5.

Proof. From Lemma 4.3.37 and from the definition of U_R^ε we know that

$$\int_{\Pi_R} |\nabla U_R^\varepsilon|^2 \,\mathrm{d}x = O(1) \qquad \text{as } \varepsilon \to 0.$$

where U_R^ε has been trivially extended in Π_R outside its domain. So there exists $V = V_R \in \mathcal{H}_R$ such that, along a sequence $\varepsilon = \varepsilon_n \to 0$,

$$U_R^\varepsilon \rightharpoonup V \qquad \text{weakly in } \mathcal{H}_R \quad \text{as } \varepsilon = \varepsilon_n \to 0.$$

This means that

$$\nabla U_R^\varepsilon \rightharpoonup \nabla V \qquad \text{weakly in } L^2(\Pi_R) \quad \text{as } \varepsilon = \varepsilon_n \to 0.$$

Since $U_R^\varepsilon = W^\varepsilon$ on S_R^+ and $W^\varepsilon \to \psi_k$ in $L^2(S_R^+)$, then V satisfies (in a weak sense) the same equation as U_R, defined in Lemma 4.3.5. So, by uniqueness, $V = U_R$. Since the limit $V = U_R$ is the same for every subsequence, *Urysohn's Subsequence Principle* implies that the convergence $U_R^\varepsilon \rightharpoonup U_R$ holds as $\varepsilon \to 0$ (not only along subsequences).

To prove strong convergence it is enough to show that $\|U_R^\varepsilon\|_{\mathcal{H}_R} \to \|U_R\|_{\mathcal{H}_R}$ as $\varepsilon \to 0$. First we notice that, trivially, $-\Delta U_R^\varepsilon \rightharpoonup -\Delta U_R$ weakly in $L^2(\Pi_R)$: so, we have that $\nabla U_R^\varepsilon \rightharpoonup \nabla U_R$ in $H(\mathrm{div}, \Pi_R)$, thus

$$\frac{\partial U_R^\varepsilon}{\partial \boldsymbol{\nu}} \rightharpoonup \frac{\partial U_R}{\partial \boldsymbol{\nu}} \qquad \text{in } \left(H_{00}^{1/2}(S_R^+)\right)^* \quad \text{as } \varepsilon \to 0,$$

where $(H_{00}^{1/2}(S_R^+))^*$ is the dual of the Lions-Magenes space $H_{00}^{1/2}(S_R^+)$. Then, since $W^\varepsilon \to \psi_k$ in $H_{00}^{1/2}(S_R^+)$ as $\varepsilon \to 0$, we obtain that

$$\int_{\Pi_R} |\nabla U_R^\varepsilon|^2 \,\mathrm{d}x = \int_{S_R^+} \frac{\partial U_R^\varepsilon}{\partial \boldsymbol{\nu}} W^\varepsilon \,\mathrm{d}S \to \int_{S_R^+} \frac{\partial U_R}{\partial \boldsymbol{\nu}} \psi_k \,\mathrm{d}S = \int_{\Pi_R} |\nabla U_R|^2 \quad \text{as } \varepsilon \to 0,$$

thus completing the proof. □

It is easy to prove that the family of functions $\{\varphi_1, \varphi_2, \ldots, \varphi_{n_0-1}, w_{n_0,R,\varepsilon}\}$ is linearly independent in $H^1_0(\Omega^\varepsilon)$. As in the previous section, we construct a new basis of the space

$$\mathrm{span}\{\varphi_1, \varphi_2, \ldots, \varphi_{n_0-1}, w_{n_0,R,\varepsilon}\} \subseteq H^1_0(\Omega^\varepsilon)$$

by defining, for all $i = 1, \ldots, n_0 - 1$

$$\hat{w}_{i,R,\varepsilon} = \varphi_i$$

and

$$\hat{w}_{n_0,R,\varepsilon} = w_{n_0,R,\varepsilon} - \sum_{i=1}^{j-1} c_i^\varepsilon \varphi_i,$$

where

$$c_i^\varepsilon = \int_{\Omega^\varepsilon} p w_{n_0,R,\varepsilon} \varphi_i \, \mathrm{d}x.$$

In this way we have that $\int_{\Omega^\varepsilon} p\, \hat{w}_{n,R,\varepsilon} \hat{w}_{m,R,\varepsilon} \, dx = 0$ if $n \neq m$.

Using the estimates established in Lemmas 4.3.36 and 4.3.37, one can prove the following

$$\int_{\Omega^\varepsilon} |\nabla \hat{w}_{n_0,R,\varepsilon}|^2 \, \mathrm{d}x = \lambda_0 + \varepsilon^{N+2k-2} \left(\int_{\Pi_R} |\nabla U_R^\varepsilon|^2 \, \mathrm{d}x - \int_{B_R^+} |\nabla W^\varepsilon|^2 \, \mathrm{d}x + o(1) \right), \tag{4.3.65}$$

$$\int_{\Omega^\varepsilon} \nabla \hat{w}_{n_0,R,\varepsilon} \cdot \nabla \hat{w}_{i,R,\varepsilon} \, \mathrm{d}x = O(\varepsilon^{N+k-1}) \quad \text{for all } i \in \{1, \ldots, n_0 - 1\}, \tag{4.3.66}$$

$$\int_{\Omega^\varepsilon} p\, |\hat{w}_{n_0,R,\varepsilon}|^2 \, \mathrm{d}x = 1 + O(\varepsilon^{N+2k-2/N}), \tag{4.3.67}$$

as $\varepsilon \to 0$.

Proposition 4.3.39. *Let $\rho \in (0, 1/2)$, K_ρ as defined in Proposition 4.3.29 and $R \geq K_\rho$. For $\varepsilon < R_0/R$ there exists $g_R(\varepsilon)$ such that*

$$\lambda_\varepsilon - \lambda_0 \leq \varepsilon^{N+2k-2}(g_R(\varepsilon) + o(1)) \quad \text{as } \varepsilon \to 0$$

and

$$g_R(\varepsilon) = \int_{\Pi_R} |\nabla U_R^\varepsilon|^2 \, \mathrm{d}x - \int_{B_R^+} |\nabla W^\varepsilon|^2 \, \mathrm{d}x,$$

where U_R^ε has been trivially extended in Π_R outside its domain.

Proof. By the *Courant-Fischer Min-Max* characterization of eigenvalues (see Proposition 3.0.5) we have that

$$\lambda_\varepsilon = \min \left\{ \max_{\substack{\alpha_1, \ldots, \alpha_{n_0} \in \mathbb{R} \\ \sum_{i=1}^{n_0} |\alpha_i|^2 = 1}} \frac{\int_{\Omega^\varepsilon} \left|\nabla \left(\sum_{i=1}^{n_0} \alpha_i u_i \right)\right|^2 \mathrm{d}x}{\int_{\Omega^\varepsilon} p \left| \sum_{i=1}^{n_0} \alpha_i u_i \right|^2 \mathrm{d}x} : \begin{array}{l} \{u_1 \ldots, u_{n_0}\} \subseteq H^1_0(\Omega^\varepsilon) \\ \text{linearly independent} \end{array} \right\}.$$

Testing the Rayleigh quotient with the family of functions

$$\tilde{w}_{i,R,\varepsilon} = \frac{\hat{w}_{i,R,\varepsilon}}{\sqrt{\int_{\Omega^\varepsilon} p \left|\hat{w}_{i,R,\varepsilon}\right|^2 \, \mathrm{d}x}}$$

we obtain that

$$\lambda_\varepsilon - \lambda_0 \leq \max_{\substack{\alpha_1,\ldots,\alpha_{n_0}\in\mathbb{R} \\ \sum_{i=1}^{n_0} |\alpha_i|^2=1}} \int_{\Omega^\varepsilon} \left|\nabla \left(\sum \alpha_i \tilde{w}_{i,R,\varepsilon}\right)\right|^2 \mathrm{d}x - \lambda_0 = \max_{\substack{\alpha_1,\ldots,\alpha_{n_0}\in\mathbb{R} \\ \sum_{i=1}^{n_0} |\alpha_i|^2=1}} \sum_{i,n=1}^{n_0} L^\varepsilon_{i,n} \alpha_i \alpha_n,$$

where

$$L^\varepsilon_{i,n} = \frac{\int_{\Omega^\varepsilon} \nabla \hat{w}_{i,R,\varepsilon} \cdot \nabla \hat{w}_{n,R,\varepsilon} \, \mathrm{d}x}{\left(\int_{\Omega^\varepsilon} p \left|\hat{w}_{i,R,\varepsilon}\right|^2 \, \mathrm{d}x\right)^{1/2} \left(\int_{\Omega^\varepsilon} p \left|\hat{w}_{n,R,\varepsilon}\right|^2 \, \mathrm{d}x\right)^{1/2}} - \lambda_0 \delta_i^n,$$

with δ_i^n denoting the usual *Kronecker delta*. From estimates (4.3.65)–(4.3.67) it follows that

$$L^\varepsilon_{n_0,n_0} = \varepsilon^{N+2k-2}(g_R(\varepsilon) + o(1)), \quad L^\varepsilon_{i,n_0} = O(\varepsilon^{N+k-1}) \quad \text{for all } i < n_0,$$
$$L^\varepsilon_{i,i} = \lambda_i - \lambda_0 \quad \text{for all } i < n_0, \quad L^\varepsilon_{i,n} = 0 \quad \text{for all } i, n < n_0, \quad i \neq n,$$

as $\varepsilon \to 0$. Therefore, taking into account that $g_R(\varepsilon) = O(1)$ as $\varepsilon \to 0$ in view of Lemma 4.3.36 and (4.3.61), the hypotheses of Lemma 4.3.32 are satisfied with

$$\sigma(\varepsilon) = \varepsilon^{N+2k-2}, \quad \mu(\varepsilon) = g_R(\varepsilon) + o(1), \quad \alpha = \frac{N}{2}, \quad M > \frac{4(k-1)}{N}.$$

The proof is thereby complete. □

From the fact that $W^\varepsilon \to \psi_k$ in $H^1(B_R^+)$, as $\varepsilon \to 0$, for all $R > 0$, and from Lemma 4.3.38 we can deduce the following result.

Lemma 4.3.40. *For all $R > 1$ we have that*

$$g_R(\varepsilon) \longrightarrow g_R \qquad \text{as } \varepsilon \to 0,$$

where

$$g_R := \int_{\Pi_R} |\nabla U_R|^2 \, \mathrm{d}x - \int_{B_R^+} |\nabla \psi_k|^2 \, \mathrm{d}x. \tag{4.3.68}$$

In order to compute the limit $\lim_{R\to\infty} g_R$, we introduce the functions

$$\zeta(r) := \int_{S_1^+} \Phi(r\theta)\Psi_k(\theta) \, \mathrm{d}S(\theta) \qquad \text{for } r \geq 1, \tag{4.3.69}$$

$$\chi_R(r) := \int_{S_1^+} U_R(r\theta)\Psi_k(\theta) \, \mathrm{d}S(\theta) \qquad \text{for } 1 \leq r \leq R. \tag{4.3.70}$$

Moreover, we denote

$$\gamma_N := \int_{S_1^+} |\Psi_k(\theta)|^2 \,\mathrm{d}S(\theta). \tag{4.3.71}$$

We immediatly notice, thanks to Lemma 4.3.6 and to the embedding $H^1(B_1^+) \hookrightarrow L^2(S_1^+)$, that

$$\zeta(1) = \lim_{R\to+\infty} \chi_R(1). \tag{4.3.72}$$

Lemma 4.3.41. *Let ζ be the function defined in (4.3.69), γ_N the constant defined in (4.3.71) and $m_k(\Sigma)$ the one defined in (4.2.11). Then*

$$\zeta(1) = \gamma_N - \frac{2m_k(\Sigma)}{N+2k-2}. \tag{4.3.73}$$

Proof. From the definition of Φ, given in (4.3.9), one can easily prove that ζ satisfies the following ODE

$$\left(r^{N+2k-1}(r^{-k}\zeta(r))'\right)' = 0 \qquad \text{in } (1,+\infty).$$

This yields

$$r^{-k}\zeta(r) = \zeta(1) + C\frac{1-r^{-N-2k+2}}{N+2k-2} \tag{4.3.74}$$

for some constant $C \in \mathbb{R}$. Now we note that $r^{-k}\zeta(r) \to \gamma_N$ as $r \to +\infty$. Indeed, since $\Phi = w_k + \psi_k$, we can rewrite

$$\zeta(r) = \int_{S_1^+} w_k(r\theta)\Psi_k(\theta)\,\mathrm{d}S(\theta) + \gamma_N r^k.$$

By evaluating the vanishing order at 0 of the Kelvin transform of the restriction of the function w_k on $\Pi \setminus \Pi_1$, we can prove that

$$|w_k(x)| \le \text{const}\,|x|^{1-N} \qquad \text{for } |x| > 1.$$

Hence, when $r \to +\infty$

$$\left|r^{-k}\zeta(r) - \gamma_N\right| \le \int_{S_1^+} \frac{|w_k(r\theta)|}{r^k}\,|\Psi_k(\theta)|\,\mathrm{d}S(\theta) \le \text{const}\,r^{1-N-k} \to 0.$$

Then we can find the constant C in (4.3.74), letting $r \to +\infty$; so we can rewrite ζ as

$$\zeta(r) = \gamma_N r^k + (\zeta(1)-\gamma_N)r^{-N-k+2} \qquad \text{in } (1,+\infty). \tag{4.3.75}$$

Taking the derivative leads to

$$\begin{aligned} \zeta'(r) &= k\gamma_N r^{k-1} + (N+k-2)(\gamma_N - \zeta(1))r^{-N-k+1} \\ &= (N+2k-2)\gamma_N r^{k-1} - \frac{(N+k-2)\zeta(r)}{r}. \end{aligned} \tag{4.3.76}$$

Hence, taking into account the definition of ζ and evaluating its derivative at $r = 1$, we obtain

$$\int_{S_1^+} \frac{\partial \Phi}{\partial \boldsymbol{\nu}}(\theta)\Psi_k(\theta)\,\mathrm{d}S(\theta) = (N+2k-2)\gamma_N - (N+k-2)\zeta(1). \tag{4.3.77}$$

Since $-\Delta\Phi = 0$ in B_1^+, multiplying this equation by ψ_k and integrating by parts we obtain that

$$\int_{B_1^+} \nabla\Phi\cdot\nabla\psi_k\,\mathrm{d}x = \int_{S_1^+} \frac{\partial \Phi}{\partial \boldsymbol{\nu}}\psi_k\,\mathrm{d}S = \int_{S_1^+} \frac{\partial \Phi}{\partial \boldsymbol{\nu}}\Psi_k\,\mathrm{d}S. \tag{4.3.78}$$

Then, let us test the equation $-\Delta\psi_k = 0$ with Φ. From (4.2.12) and (4.3.10) it follows that

$$\begin{aligned}\int_{B_1^+} \nabla\psi_k\cdot\nabla\Phi\,\mathrm{d}x = \int_{S_1^+} \frac{\partial \psi_k}{\partial \boldsymbol{\nu}}\Phi\,\mathrm{d}S - \int_{\Sigma} \frac{\partial \psi_k}{\partial x_N}\Phi\,\mathrm{d}x' &= \int_{S_1^+} \frac{\partial \psi_k}{\partial \boldsymbol{\nu}}\Phi\,\mathrm{d}S - \int_{\Sigma} \frac{\partial \psi_k}{\partial x_N}w_k\,\mathrm{d}x' \\ &= \int_{S_1^+} \frac{\partial \psi_k}{\partial \boldsymbol{\nu}}\Phi\,\mathrm{d}S + 2m_k(\Sigma).\end{aligned} \tag{4.3.79}$$

Moreover we note that

$$\frac{\partial \psi_k}{\partial \boldsymbol{\nu}}(\theta) = k\Psi_k(\theta) \qquad \text{on } S_1^+. \tag{4.3.80}$$

Then, from (4.3.79) and (4.3.80) we obtain

$$\int_{B_1^+} \nabla\psi_k\cdot\nabla\Phi\,\mathrm{d}x = k\int_{S_1^+} \Phi\Psi_k\,\mathrm{d}S + 2m_k(\Sigma) = k\zeta(1) + 2m_k(\Sigma). \tag{4.3.81}$$

Finally, combining (4.3.77), (4.3.78) and (4.3.81) leads to the thesis. □

Lemma 4.3.42. *Let g_R be as defined in (4.3.68) and $m_k(\Sigma)$ as in (4.2.11). Then $\lim_{R\to+\infty} g_R = 2m_k(\Sigma)$.*

Proof. Integrating by parts we have that

$$g_R = \int_{S_R^+} \left(\frac{\partial U_R}{\partial \boldsymbol{\nu}} - \frac{\partial \psi_k}{\partial \boldsymbol{\nu}}\right)\psi_k\,\mathrm{d}S.$$

If χ_R is the function defined in (4.3.70), then

$$\chi_R'(r) = \int_{S_1^+} \frac{\partial U_R}{\partial \boldsymbol{\nu}}(r\theta)\Psi_k(\theta)\,\mathrm{d}S(\theta)$$

and, by a change of variable,

$$\chi_R'(r) = r^{1-N-k}\int_{S_r^+} \frac{\partial U_R}{\partial \boldsymbol{\nu}}\psi_k\,\mathrm{d}S. \tag{4.3.82}$$

By simple computations one can prove that χ_R solves

$$\left(r^{N+2k-1}(r^{-k}\chi_R(r))'\right)' = 0 \qquad \text{in } (1, R).$$

By integration, we arrive at

$$r^{-k}\chi_R(r) = \chi_R(1) + C\frac{1 - r^{-N-2k+2}}{N + 2k - 2}. \tag{4.3.83}$$

From the fact that $U_R = R^K\Psi_k$ on S_R^+, we have that $\chi_R(R) = R^k\gamma_N$, and this allows us to know the constant C. After some computations, the expression (4.3.83) then becomes as follows

$$r^{-k}\chi_R(r) = \chi_R(1) + (\gamma_N - \chi_R(1))\frac{1 - r^{-N-2k+2}}{1 - R^{-N-2k+2}}, \qquad r \in (1, R).$$

From (4.3.82), we get

$$\begin{aligned}\int_{S_R^+} \frac{\partial U_R}{\partial \boldsymbol{\nu}}\psi_k \, \mathrm{d}S &= \chi_R'(R)R^{N+k-1} \\ &= \frac{[\gamma_N(N+k-2) - \chi_R(1)(N+2k-2)]R^{-N-k+1} + k\gamma_N R^{k-1}}{1 - R^{-N-2k+2}}R^{N+k-1} \\ &= \frac{\gamma_N(N+k-2) - \chi_R(1)(N+2k-2) + k\gamma_N R^{N+2k-2}}{1 - R^{-N-2k+2}}.\end{aligned} \tag{4.3.84}$$

For what concerns the second part of g_R, it is easy to see that

$$\frac{\partial \psi_k}{\partial \boldsymbol{\nu}}(r\theta) = kr^{k-1}\Psi_k(\theta).$$

Therefore

$$\int_{S_R^+} \frac{\partial \psi_k}{\partial \boldsymbol{\nu}}\psi_k \, \mathrm{d}S = \int_{S_1^+} kR^{N+2k-2}\left|\Psi_k\right|^2 \mathrm{d}S(\theta) = kR^{N+2k-2}\gamma_N. \tag{4.3.85}$$

Finally, combining (4.3.72), (4.3.84), (4.3.85) and Lemma 4.3.41 and taking the limit when $R \to +\infty$ we reach the conclusion. $\square$

Combining Propositions 4.3.35 and 4.3.39 with Lemmas 4.3.40 and 4.3.42 we obtain the following upper-lower estimate for the eigenvalue variation.

Proposition 4.3.43. *Let $\rho \in (0, 1/2)$, K_ρ as defined in Proposition 4.3.29 and $m_k(\Sigma)$ as in (4.2.11). Then, for all $R \geq K_\rho$, we have that, as $\varepsilon \to 0$,*

$$-2m_k(\Sigma) + o(1) \leq \frac{\lambda_0 - \lambda_\varepsilon}{\varepsilon^{N+2k-2}} \leq \frac{H(\varphi_\varepsilon, K_\rho\varepsilon)}{\varepsilon^{2k}}(f_R(\varepsilon) + o(1)).$$

Since $-2m_k(\Sigma) > 0$, as a direct consequence of Proposition 4.3.43 we obtain the following estimate from below for $H(\varphi_\varepsilon, K_\rho\varepsilon)$.

Corollary 4.3.44. *We have that*

$$\frac{\varepsilon^{2k}}{H(\varphi_\varepsilon, K_\rho\varepsilon)} = O(1) \qquad \text{as } \varepsilon \to 0.$$

4.3.6 Blow-up analysis

Let us introduce the functional

$$\begin{aligned} F\colon \mathbb{R}\times H_0^1(\Omega) &\longrightarrow \mathbb{R}\times H^{-1}(\Omega) \\ (\lambda,\varphi) &\longmapsto (\|\varphi\|^2_{H_0^1(\Omega)}-\lambda_0, -\Delta\varphi-\lambda p\varphi) \end{aligned}$$

where $\|\varphi\|^2_{H_0^1(\Omega)}=\int_\Omega |\nabla\varphi|^2\,dx$ and

$$_{H^{-1}(\Omega)}\langle -\Delta\varphi-\lambda p\varphi, v\rangle_{H_0^1(\Omega)}=\int_\Omega (\nabla\varphi\cdot\nabla v-\lambda p\varphi v)\,\mathrm{d}x.$$

From the assumptions we know that $F(\lambda_0,\varphi_0)=(0,0)$. Moreover, from the simplicity assumption (4.2.4) and Fredholm Alternative, one can easily prove the following result (see e.g. [AF15] for details for a similar operator).

Lemma 4.3.45. *The functional F is differentiable at (λ_0,φ_0) and its differential*

$$\begin{aligned} &\mathrm{d}F(\lambda_0,\varphi_0)\colon \mathbb{R}\times H_0^1(\Omega)\longrightarrow \mathbb{R}\times H^{-1}(\Omega) \\ &\mathrm{d}F(\lambda_0,\varphi_0)(\lambda,\varphi)=\left(2\int_\Omega \nabla\varphi_0\cdot\nabla\varphi\,\mathrm{d}x, -\Delta\varphi-\lambda p\varphi_0-\lambda_0 p\varphi\right) \end{aligned}$$

is invertible.

Lemma 4.3.46. *Let $\rho\in(0,1/2)$, K_ρ as defined in Proposition 4.3.29 and $R\geq K_\rho$. Then, when $\varepsilon\to 0$,*

$$v_{n_0,R,\varepsilon}\longrightarrow\varphi_0 \qquad \text{in} \quad H_0^1(\Omega),$$

where $v_{n_0,R,\varepsilon}$ is defined in (4.3.44).

Proof. First note that

$$\begin{aligned} \int_\Omega |\nabla(v_{n_0,R,\varepsilon}-\varphi_0)|^2\,\mathrm{d}x &= \int_{\Omega^\varepsilon} |\nabla(\varphi_\varepsilon-\varphi_0)|^2\,\mathrm{d}x \\ &\quad -\int_{\Omega^\varepsilon_{R\varepsilon}} |\nabla(\varphi_\varepsilon-\varphi_0)|^2\,\mathrm{d}x+\int_{B^+_{R\varepsilon}}\left|\nabla(v^{\mathrm{int}}_{n_0,R,\varepsilon}-\varphi_0)\right|^2\,\mathrm{d}x. \end{aligned}$$

The first term tends to zero because of (4.2.6). For the second and the third term we can exploit the energy estimates in Proposition 4.3.30, Lemma 4.3.36 and Proposition 4.3.33 to conclude. ☐

Lemma 4.3.47. *Let $\rho\in(0,1/2)$, K_ρ as defined in Proposition 4.3.29 and $R\geq K_\rho$. Then*

$$\|v_{n_0,R,\varepsilon}-\varphi_0\|_{H_0^1(\Omega)}=O\left(\varepsilon^{N/2-1}\sqrt{H(\varphi_\varepsilon,K_\rho\varepsilon)}\right) \qquad \text{as} \quad \varepsilon\to 0$$

and, in particular,

$$\int_{\Omega\setminus B^+_{R\varepsilon}} |\nabla(\varphi_\varepsilon-\varphi_0)|^2\,\mathrm{d}x=O\left(\varepsilon^{N-2}H(\varphi_\varepsilon,K_\rho\varepsilon)\right) \qquad \text{as} \quad \varepsilon\to 0. \tag{4.3.86}$$

Proof. Taking into account Lemma 4.3.46 and (4.2.5), from the differentiability of the functional F it follows that

$$F(\lambda_\varepsilon, v_{n_0,R,\varepsilon}) = \mathrm{d}F(\lambda_0,\varphi_0)(\lambda_\varepsilon-\lambda_0, v_{n_0,R,\varepsilon}-\varphi_0) + o\left(|\lambda_\varepsilon-\lambda_0| + \|v_{n_0,R,\varepsilon}-\varphi_0\|_{H^1_0(\Omega)}\right)$$

as $\varepsilon \to 0$. Now let us apply $\mathrm{d}F(\lambda_0,\varphi_0)^{-1}$ to both members and obtain

$$\begin{aligned}|\lambda_\varepsilon-\lambda_0| &+ \|v_{n_0,R,\varepsilon}-\varphi_0\|_{H^1_0(\Omega)} \\ &\leq \left\|\mathrm{d}F(\lambda_0,\varphi_0)^{-1}\right\|_{\mathcal{L}(\mathbb{R}\times H^{-1}(\Omega),\mathbb{R}\times H^1_0(\Omega))} \|F(\lambda_\varepsilon, v_{n_0,R,\varepsilon})\|_{\mathbb{R}\times H^{-1}(\Omega)}(1+o(1))\end{aligned}$$

and so

$$\|v_{n_0,R,\varepsilon}-\varphi_0\|_{H^1_0(\Omega)} \leq C\left(\left|\|v_{n_0,R,\varepsilon}\|^2_{H^1_0(\Omega)}-\lambda_0\right| + \|-\Delta v_{n_0,R,\varepsilon} - \lambda_\varepsilon p v_{n_0,R,\varepsilon}\|_{H^{-1}(\Omega)}\right). \tag{4.3.87}$$

Thanks to (4.3.46), Proposition 4.3.43, and the fact that $f_R(\varepsilon) = O(1)$ as $\varepsilon \to 0$ in view of (4.3.26) and (4.3.38),

$$\left|\|v_{n_0,R,\varepsilon}\|^2_{H^1_0(\Omega)}-\lambda_0\right| \leq \left|\|v_{n_0,R,\varepsilon}\|_{H^1_0(\Omega)}-\lambda_\varepsilon\right| + |\lambda_\varepsilon-\lambda_0| = O(\varepsilon^{N-2}H(\varphi_\varepsilon, K_\rho\varepsilon)). \tag{4.3.88}$$

Let $u \in H^1_0(\Omega)$ be such that $\|u\|_{H^1_0(\Omega)} \leq 1$. Note that

$$\begin{aligned}\int_\Omega \nabla v_{n_0,R,\varepsilon}\cdot\nabla u\,\mathrm{d}x &= \int_{B^+_{R\varepsilon}} \nabla v^{\mathrm{int}}_{n_0,R,\varepsilon}\cdot\nabla u\,\mathrm{d}x + \int_{\Omega^\varepsilon}\nabla\varphi_\varepsilon\cdot\nabla u\,\mathrm{d}x - \int_{\Omega^\varepsilon_{R\varepsilon}}\nabla\varphi_\varepsilon\cdot\nabla u\,\mathrm{d}x \leq \\ &\leq \sqrt{\int_{B^+_{R\varepsilon}}\left|\nabla v^{\mathrm{int}}_{n_0,R,\varepsilon}\right|^2\mathrm{d}x} + \lambda_\varepsilon\int_{\Omega^\varepsilon} p\varphi_\varepsilon u\,\mathrm{d}x + \sqrt{\int_{\Omega^\varepsilon_{R\varepsilon}}|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x}.\end{aligned}$$

So we have that

$$\begin{aligned}&\int_\Omega \nabla v_{n_0,R,\varepsilon}\cdot\nabla u\,\mathrm{d}x - \lambda_\varepsilon\int_\Omega p v_{n_0,R,\varepsilon}u\,\mathrm{d}x \leq \\ &\leq \sqrt{\int_{B^+_{R\varepsilon}}\left|\nabla v^{\mathrm{int}}_{n_0,R,\varepsilon}\right|^2\mathrm{d}x} + \lambda_\varepsilon\left(\int_{\Omega^\varepsilon} p\varphi_\varepsilon u\,\mathrm{d}x - \int_\Omega p v_{n_0,R,\varepsilon}u\,\mathrm{d}x\right) + \sqrt{\int_{\Omega^\varepsilon_{R\varepsilon}}|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x}.\end{aligned} \tag{4.3.89}$$

Now let us analyze the middle term

$$\begin{aligned}\int_{\Omega^\varepsilon} p\varphi_\varepsilon u\,\mathrm{d}x - \int_\Omega p v_{n_0,R,\varepsilon}u\,\mathrm{d}x &= \int_{B^+_{R\varepsilon}} p\varphi_\varepsilon u\,\mathrm{d}x - \int_{B^+_{R\varepsilon}} p v^{\mathrm{int}}_{n_0,R,\varepsilon}u\,\mathrm{d}x \leq \\ &\leq \mathrm{const}\left(\sqrt{\int_{B^+_{R\varepsilon}}|\varphi_\varepsilon|^2\,\mathrm{d}x} + \sqrt{\int_{B^+_{R\varepsilon}}|v^{\mathrm{int}}_{n_0,R,\varepsilon}|^2\,\mathrm{d}x}\right)\end{aligned}$$

where we implicitly used the Poincaré Inequality. Thanks to inequality (4.3.18) and to the energy estimates made in Proposition 4.3.29

$$\int_{B^+_{R\varepsilon}}|\varphi_\varepsilon|^2\,\mathrm{d}x \leq \frac{(R\varepsilon)^2}{N-1}\int_{B^+_{R\varepsilon}}|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x + \frac{R\varepsilon}{N-1}\int_{S^+_{R\varepsilon}}|\varphi_\varepsilon|^2\,\mathrm{d}x = O(\varepsilon^N H(\varphi_\varepsilon, K_\rho\varepsilon)),$$

as $\varepsilon \to 0$. Then, from (4.3.89), Proposition 4.3.30 and Proposition 4.3.33 we obtain that

$$\int_\Omega \nabla v_{n_0,R,\varepsilon} \cdot \nabla u \,\mathrm{d}x - \lambda_\varepsilon \int_\Omega p v_{n_0,R,\varepsilon} u \,\mathrm{d}x = O\left(\varepsilon^{N/2-1}\sqrt{H(\varphi_\varepsilon, K_\rho\varepsilon)}\right) \quad \text{as } \varepsilon \to 0$$

uniformly with respect to $u \in H^1_0(\Omega)$ with $\|u\|_{H^1_0(\Omega)} \le 1$ and hence

$$\|-\Delta v_{n_0,R,\varepsilon} - \lambda_\varepsilon p v_{n_0,R,\varepsilon}\|_{H^{-1}(\Omega)} = O\left(\varepsilon^{N/2-1}\sqrt{H(\varphi_\varepsilon, K_\rho\varepsilon)}\right) \quad \text{as } \varepsilon \to 0. \tag{4.3.90}$$

The conclusion follows by combining (4.3.87), (4.3.88), and (4.3.90). □

Corollary 4.3.48. *Let $\rho \in (0, 1/2)$, K_ρ as defined in Proposition 4.3.29 and $R \ge K_\rho$. Then*

$$\int_{\frac{1}{\varepsilon}\Omega\setminus B_R^+} \left|\nabla\tilde{\varphi}^\varepsilon - \varepsilon^k H(\varphi_\varepsilon, K_\rho\varepsilon)^{-1/2}\nabla W^\varepsilon\right|^2 \mathrm{d}x = O(1), \qquad \textit{as } \varepsilon \to 0,$$

where $\tilde{\varphi}^\varepsilon$ is defined in (4.3.45) and W^ε in (4.3.64), while $\frac{1}{\varepsilon}\Omega = \{\frac{1}{\varepsilon}x \colon x \in \Omega\}$.

Proof. It directly follows from a change of variables in (4.3.86). □

The following Theorem provides a blow-up analysis for scaled eigenfunctions, which contains Theorem 4.2.2.

Theorem 4.3.49. *Let $\rho \in (0, 1/2)$ and K_ρ as defined in Proposition 4.3.29. Then*

$$\tilde{\varphi}^\varepsilon \longrightarrow \frac{1}{\sqrt{\Lambda_\rho}}\Phi \qquad \textit{in } \mathcal{H}_R \quad \textit{for all } R > 2, \tag{4.3.91}$$

$$\frac{H(\varphi_\varepsilon, K_\rho\varepsilon)}{\varepsilon^{2k}} \longrightarrow \Lambda_\rho, \tag{4.3.92}$$

$$\frac{\varphi_\varepsilon(\varepsilon x)}{\varepsilon^k} \longrightarrow \Phi(x) \qquad \textit{in } \mathcal{H}_R \quad \textit{for all } R > 2, \tag{4.3.93}$$

as $\varepsilon \to 0$, where

$$\Lambda_\rho := \frac{1}{K_\rho^{N-1}} \int_{S^+_{K_\rho}} |\Phi|^2 \,\mathrm{d}S.$$

Proof. Let $\varepsilon_n \to 0$. From Corollary 4.3.44 we deduce that, up to a subsequence,

$$\frac{(\varepsilon_n)^k}{\sqrt{H(\varphi_{\varepsilon_n}, K_\rho\varepsilon_n)}} \longrightarrow c \ge 0.$$

Since, in view of Proposition 4.3.29, $\{\tilde{\varphi}^{\varepsilon_n}\}$ is bounded in $\mathcal{H}_R$, by a diagonal process there exists $\tilde{\Phi}$, with $\tilde{\Phi} \in \mathcal{H}_R$ for all $R > 2$, and a subsequence (still denoted by ε_n) such that

$$\tilde{\varphi}^{\varepsilon_n} \rightharpoonup \tilde{\Phi} \qquad \text{weakly in } \mathcal{H}_R \quad \text{for all } R > 2. \tag{4.3.94}$$

Moreover $\int_{S^+_{K_\rho}} |\tilde\varphi^{\varepsilon_n}|^2 \, \mathrm{d}S = K_\rho^{N-1}$, hence, by compactness of trace embeddings,

$$\int_{S^+_{K_\rho}} |\tilde\Phi|^2 \, \mathrm{d}S = K_\rho^{N-1}, \tag{4.3.95}$$

thus implying that $\tilde\Phi \not\equiv 0$.

Actually we can prove that the convergence in (4.3.94) is strong. Indeed, consider the equation solved by $\tilde\varphi^{\varepsilon_n}$:

$$\begin{cases} -\Delta\tilde\varphi^{\varepsilon_n} = (\varepsilon_n)^2\lambda_{\varepsilon_n} p\,\tilde\varphi^{\varepsilon_n}, & \text{in } \Big((-\frac{1}{\varepsilon_n},0]\times\Sigma\Big)\cup B_R^+, \\ \quad\tilde\varphi^{\varepsilon_n} = 0, & \text{on } \partial\Big(\Big((-\frac{1}{\varepsilon_n},0]\times\Sigma\Big)\cup B_R^+\Big)\setminus S_R^+, \\ \tilde\varphi^{\varepsilon_n}(x) = \frac{\varphi_{\varepsilon_n}(\varepsilon_n x)}{\sqrt{H(\varphi_{\varepsilon_n},K_\rho\varepsilon_n)}}, & \text{on } S_R^+. \end{cases}$$

If we consider the restriction to $B_R^+\setminus B_{R/2}^+$ and the odd reflection through the hyperplane $x_N = 0$, we have that $\{\tilde\varphi^{\varepsilon_n}\}$ is bounded in $H^2(B_R\backslash B_{R/2})$, where $B_R = \{x\in\mathbb{R}^N\colon\ |x|<R\}$. Hence, up to a subsequence, $\frac{\partial\tilde\varphi^{\varepsilon_n}}{\partial\boldsymbol{\nu}} \to \frac{\partial\tilde\Phi}{\partial\boldsymbol{\nu}}$ in $L^2(S_R^+)$ and therefore

$$\int_{\Pi_R} |\nabla\tilde\varphi^{\varepsilon_n}|^2 \, \mathrm{d}x = (\varepsilon_n)^2\lambda_{\varepsilon_n}\int_{\Pi_R} p\,|\tilde\varphi^{\varepsilon_n}|^2 \, \mathrm{d}x \\ + \int_{S_R^+}\frac{\partial\tilde\varphi^{\varepsilon_n}}{\partial\boldsymbol{\nu}}\tilde\varphi^{\varepsilon_n}\,\mathrm{d}S \to \int_{S_R^+}\frac{\partial\tilde\Phi}{\partial\boldsymbol{\nu}}\tilde\Phi\,\mathrm{d}S = \int_{\Pi_R}|\nabla\tilde\Phi|^2\,\mathrm{d}x.$$

Then we conclude that $\tilde\varphi^{\varepsilon_n}\to\tilde\Phi$ strongly in $\mathcal{H}_R$ for all $R>2$.

From Corollary 4.3.48 it follows that there exist $c'>0$ and $n_0\in\mathbb{N}$ such that, for all $n\geq n_0$ and $\tilde R > R$,

$$\int_{B^+_{\tilde R}\backslash B^+_R}\Big|\nabla\tilde\varphi^{\varepsilon_n} - (\varepsilon_n)^k H(\varphi_{\varepsilon_n},K_\rho\varepsilon_n)^{-1/2}\nabla W^{\varepsilon_n}\Big|^2\,\mathrm{d}x \le c'.$$

Let us recall that $W^{\varepsilon_n}\to\psi_k$ in $H^1(B^+_{\tilde R})$ and (since the norms are equivalent) also in $\mathcal{H}_{\tilde R}$: so, passing to the limit as $n\to\infty$ in the above estimate, we obtain that

$$\int_{B^+_{\tilde R}\backslash B^+_R}|\nabla\tilde\Phi - c\nabla\psi_k|^2\,\mathrm{d}x\le c'.$$

Since the constant c' is independent on $\tilde R$, we deduce that

$$\int_\Pi|\nabla\tilde\Phi - c\nabla\psi_k|^2\,\mathrm{d}x < +\infty. \tag{4.3.96}$$

Moreover, the function $\tilde\Phi$ satisfies the following equation

$$\begin{cases} -\Delta\tilde\Phi = 0, & \text{in } \Pi, \\ \quad\tilde\Phi = 0, & \text{on } \partial\Pi. \end{cases} \tag{4.3.97}$$

We claim that $c > 0$. Otherwise, if $c = 0$ then, by (4.3.96) and (4.3.97), we could say that $\tilde{\Phi} = 0$, which would contradict (4.3.95).

From Proposition 4.3.4 we conclude that $\tilde{\Phi} = c\,\Phi$ and hence, in view of (4.3.95), $c = \Lambda_\rho^{-1/2}$. Since the limit of the sequence $\{\tilde{\varphi}^{\varepsilon_n}\}$ is the same for any choice of the subsequence, we conclude the proof by invoking the *Urysohn's Subsequence Principle.* $\square$

Corollary 4.3.50. *For all $R > 2$ we have that*

$$Z_R^\varepsilon \longrightarrow \frac{1}{\sqrt{\Lambda_\rho}} Z_R \qquad \text{in } H^1(B_R^+) \quad \text{as } \varepsilon \to 0.$$

Proof. From the definitions of the functions Z_R^ε and Z_R (in (4.3.45) and Lemma 4.3.7 respectively)

$$\begin{cases} -\Delta(\sqrt{\Lambda_\rho} Z_R^\varepsilon - Z_R) = 0, & \text{in } B_R^+, \\ \sqrt{\Lambda_\rho} Z_R^\varepsilon - Z_R = 0, & \text{on } B_R', \\ \sqrt{\Lambda_\rho} Z_R^\varepsilon - Z_R = \sqrt{\Lambda_\rho}\tilde{\varphi}^\varepsilon - \Phi, & \text{on } S_R^+. \end{cases}$$

So $Z_R^\varepsilon - Z_R$ is the unique, least energy solution with these prescribed boundary conditions. Now, let $\eta = \eta_R$ be as defined in (4.3.1). We have that

$$\begin{aligned} &\int_{B_R^+} \left|\nabla(\sqrt{\Lambda_\rho} Z_R^\varepsilon - Z_R)\right|^2 \mathrm{d}x \le \int_{B_R^+} \left|\nabla(\eta(\sqrt{\Lambda_\rho}\tilde{\varphi}^\varepsilon - \Phi))\right|^2 \mathrm{d}x \le \\ &\le 2\int_{B_R^+} |\nabla\eta|^2 \left|\sqrt{\Lambda_\rho}\tilde{\varphi}^\varepsilon - \Phi\right|^2 \mathrm{d}x + 2\int_{B_R^+} \eta^2 \left|\nabla(\sqrt{\Lambda_\rho}\tilde{\varphi}^\varepsilon - \Phi)\right|^2 \mathrm{d}x \le \\ &\le \frac{32}{R^2}\int_{B_R^+} \left|\sqrt{\Lambda_\rho}\tilde{\varphi}^\varepsilon - \Phi\right|^2 \mathrm{d}x + 2\int_{B_R^+} \left|\nabla(\sqrt{\Lambda_\rho}\tilde{\varphi}^\varepsilon - \Phi)\right|^2 \mathrm{d}x \to 0 \end{aligned}$$

as $\varepsilon \to 0$, thanks to (4.3.91) and to the embedding $\mathcal{H}_R \subset L^2(\Pi_R)$. The conclusion follows taking into account Poincaré Inequality for functions vanishing on a portion of the boundary. $\square$

4.3.7 Proof of Theorem 4.2.1

Thanks to Theorem 4.3.49 and Corollary 4.3.50, we know that

$$f_R := \lim_{\varepsilon \to 0} f_R(\varepsilon) = \frac{1}{\Lambda_\rho}\int_{B_R^+} |\nabla Z_R|^2 \,\mathrm{d}x - \frac{1}{\Lambda_\rho}\int_{\Pi_R} |\nabla\Phi|^2 \,\mathrm{d}x.$$

Moreover, in view of Proposition 4.3.43 and (4.3.92), we have that, for any $R > \max\{2, K_\rho\}$

$$C_k(\Sigma) \le \liminf_{\varepsilon \to 0} \frac{\lambda_0 - \lambda_\varepsilon}{\varepsilon^{N+2k-2}} \le \limsup_{\varepsilon \to 0} \frac{\lambda_0 - \lambda_\varepsilon}{\varepsilon^{N+2k-2}} \le \Lambda_\rho f_R, \tag{4.3.98}$$

where $C_k(\Sigma) = -2m_k(\Sigma) > 0$. To complete the proof of our main result it is then enough to show that

$$\lim_{R \to +\infty} \Lambda_\rho f_R = C_k(\Sigma).$$

For every $R > 2$ let us define

$$\xi_R(r) := \int_{S_1^+} Z_R(r\theta)\Psi_k(\theta)\,\mathrm{d}S(\theta) \qquad \text{for } 0 \le r \le R. \tag{4.3.99}$$

Lemma 4.3.51. *There holds*

$$\int_{S_R^+} \frac{\partial(Z_R - \psi_k)}{\partial\boldsymbol{\nu}}(\Phi - \psi_k)\,\mathrm{d}S \longrightarrow 0 \qquad \text{as } R \to +\infty, \tag{4.3.100}$$

$$\int_{S_R^+} \frac{\partial(\psi_k - \Phi)}{\partial\boldsymbol{\nu}}(\Phi - \psi_k)\,\mathrm{d}S \longrightarrow 0 \qquad \text{as } R \to +\infty. \tag{4.3.101}$$

Proof. In order to prove (4.3.100), we first take into account the equation solved by $Z_R - \psi_k$, i.e.

$$\begin{cases} -\Delta(Z_R - \psi_k) = 0, & \text{in } B_R^+, \\ Z_R - \psi_k = 0, & \text{on } B_R', \\ Z_R - \psi_k = \Phi - \psi_k, & \text{on } S_R^+. \end{cases} \tag{4.3.102}$$

Let $\eta = \eta_R$ as defined in (4.3.1). Testing (4.3.102) with $\eta(\Phi - \psi_k)$, we obtain that

$$\int_{B_R^+} \nabla(Z_R - \psi_k) \cdot \nabla(\eta(\Phi - \psi_k))\,\mathrm{d}x = \int_{S_R^+} \frac{\partial(Z_R - \psi_k)}{\partial\boldsymbol{\nu}}(\Phi - \psi_k)\,\mathrm{d}S.$$

Then, by the Dirichlet principle,

$$\begin{aligned} \int_{S_R^+} \frac{\partial(Z_R - \psi_k)}{\partial\boldsymbol{\nu}}(\Phi - \psi_k)\,\mathrm{d}S &\le \sqrt{\int_{B_R^+} |\nabla(Z_R - \psi_k)|^2\,\mathrm{d}x}\sqrt{\int_{B_R^+} |\nabla(\eta(\Phi - \psi_k))|^2\,\mathrm{d}x} \\ &\le \int_{B_R^+} |\nabla(\eta(\Phi - \psi_k))|^2\,\mathrm{d}x \\ &\le \frac{32}{R^2}\int_{B_R^+\setminus B_{R/2}^+} |\Phi - \psi_k|^2\,\mathrm{d}x + 2\int_{B_R^+\setminus B_{R/2}^+} |\nabla(\Phi - \psi_k)|^2\,\mathrm{d}x. \end{aligned}$$

The last terms vanish as $R \to +\infty$, thanks to the fact that $\Phi - \psi_k \in \mathcal{D}^{1,2}(\Pi)$ and to Hardy's inequality (reasoning as in Lemma 4.3.6).

For the second part, since $-\Delta(\Phi - \psi_k) = 0$ in $\Pi \setminus \Pi_R$ and $\Phi - \psi_k = 0$ on $\{x_N = 0\} \setminus \Sigma$, then

$$\int_{S_R^+} \frac{\partial(\psi_k - \Phi)}{\partial\boldsymbol{\nu}}(\Phi - \psi_k)\,\mathrm{d}S = \int_{\Pi\setminus\Pi_R} |\nabla(\Phi - \psi_k)|^2\,\mathrm{d}x \to 0$$

as $R \to +\infty$. □

Lemma 4.3.52. *We have that* $\lim_{R\to+\infty} \Lambda_\rho f_R = -2m_k(\Sigma)$.

Proof. Thanks to Lemma 4.3.51 we know that

$$\lim_{R\to+\infty} \Lambda_\rho f_R = \lim_{R\to+\infty} \int_{S_R^+} \left(\frac{\partial Z_R}{\partial\boldsymbol{\nu}} - \frac{\partial\Phi}{\partial\boldsymbol{\nu}}\right)\psi_k\,\mathrm{d}S. \tag{4.3.103}$$

From the definition of ζ (4.3.69) and from (4.3.76) we deduce that

$$\int_{S_R^+} \frac{\partial \Phi}{\partial \boldsymbol{\nu}} \psi_k \, \mathrm{d}S = R^{N+k-1} \zeta'(R) = k\gamma_N R^{N+2k-2} + (N+k-2)(\gamma_N - \zeta(1)). \qquad (4.3.104)$$

It's easy to verify that the function ξ_R defined in (4.3.99) satisfies the following ODE

$$\left(r^{N+2k-1}(r^{-k}\xi_R(r))'\right)' = 0 \qquad \text{in } (0, R).$$

By integration, we obtain

$$r^{N+k-2}\xi_R(r) = r^{N+2k-2}R^{-k}\xi_R(R) - \frac{C}{N+2k-2} + \frac{C}{N+2k-2} r^{N+2k-2} R^{-N-2k+2}.$$

Since Z_R is regular at 0, we have necessarily that $C = 0$; hence

$$\xi_R(r) = \left(\frac{r}{R}\right)^k \xi_R(R).$$

From the definition of ξ_R (4.3.99) we have

$$\int_{S_R^+} \frac{\partial Z_R}{\partial \boldsymbol{\nu}} \psi_k \, \mathrm{d}S = R^{N+k-1} \xi_R'(R) = kR^{N+k-2}\xi_R(R) = kR^{N+k-2}\zeta(R). \qquad (4.3.105)$$

Then, from (4.3.103), (4.3.104), (4.3.105), (4.3.75) and (4.3.73)

$$\begin{aligned}
\lim_{R\to+\infty} \Lambda_\rho f_R &= \lim_{R\to+\infty} \left(kR^{N+k-2}\zeta(R) - k\gamma_N R^{N+2k-2} - (N+k-2)(\gamma_N - \zeta(1))\right) \\
&= \lim_{R\to+\infty} R^{N+k-2}(N+2k-2)(\zeta(R) - \gamma_N R^k) \\
&= (N+2k-2)(\zeta(1) - \gamma_N) = -2m_k(\Sigma),
\end{aligned}$$

thus concluding the proof. □

We are now able to prove our main result.

Proof of Theorem 4.2.1. From (4.3.98) and Lemma 4.3.52 we conclude that

$$\liminf_{\varepsilon\to 0} \frac{\lambda_0 - \lambda_\varepsilon}{\varepsilon^{N+2k-2}} = \limsup_{\varepsilon\to 0} \frac{\lambda_0 - \lambda_\varepsilon}{\varepsilon^{N+2k-2}} = C_k(\Sigma),$$

thus completing the proof. □

CHAPTER 5

PERTURBATION OF THE BOUNDARY CONDITIONS

5.1 Spectral stability under mixed varying boundary conditions

This chapter is devoted to the study of another kind of perturbation which consist of a drastic change in the boundary conditions. In few words, we consider an eigenvalue problem with homogeneous (Dirichlet or Neumann) boundary conditions and we modify it by imposing another kind of boundary condition in a small portion of the boundary of the domain. More precisely, we deal with mixed Dirichlet-Neumann problems in which one of the two regions is disappearing and the problem is converging to a homogeneous one, with Dirichlet or Neumann boundary conditions. We now introduce the problem in broad terms and we refer to Section 5.2 for all the details. In fact, in this thesis, we focus on the case in which the portion with Dirichlet boundary conditions is vanishing and in the limit the pure Neumann problem is recovered. Nevertheless, the complementary situation is now under investigation in a forthcoming paper [FNO] and is not included here.

Let $N \geq 2$ and let Ω be a bounded open and connected set with Lipschitz boundary. Let us consider two subsets $\Gamma_N, \Gamma_D \subseteq \partial\Omega$ and assume, for sake of simplicity, that they are both regular submanifold of $\partial\Omega$. Moreover, let $\overline{\Gamma_D} \cap \Gamma_N = \emptyset$ and $\overline{\Gamma_D} \cup \Gamma_N = \partial\Omega$, with respect to the topology induced by $\partial\Omega$. Let us now consider the mixed Dirichlet-Neumann eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{\nu}} = 0, & \text{on } \Gamma_N, \\ u = 0, & \text{on } \overline{\Gamma_D}, \end{cases}$$

to be thought in a weak sense that will be specified later, see Section 5.2. By classical spectral theory the problem above admits nontrivial solutions just for a diverging sequence of nonnegative values of λ, which we denote by $(\lambda_n^{\mathrm{DN}})_n$, see Chapter 3. Clearly, the sequence $(\lambda_n^{\mathrm{DN}})_n$ has a strong connection with the geometry (in a wide sense) of the two regions Γ_N and Γ_D: our purpose is to investigate this dependence when one of these two portions disappears. In particular we have that, in both cases, the values $(\lambda_n^{\mathrm{DN}})_n$ converge towards the eigenvalues of a certain limit problem, as one of the regions Γ_D or Γ_N vanishes (see Proposition 5.3.13 and [FNO] respectively). As one may expect, in the limit problem the same homogeneous boundary condition is imposed in the whole $\partial\Omega$. In this sense, we may say that the eigenvalues of the limit problems are stable with respect to the singular perturbation consisting of the prescription of the other boundary condition in a small part of $\partial\Omega$. Furthermore, we are able to detect the sharp asymptotic behavior of the perturbed eigenvalues in the case in which they are converging, in a suitable sense, to a simple eigenvalue of the limit problem.

In this framework, the first attempts of such an investigation we are aware of date back to the late '80s and are due to Gadyl'shin. More precisely, in [Gad86] (see also [Gad96]) the author started the analysis by taking into account the case in which the Neumann portion is disappearing on a flat part of the boundary of a three dimensional domain, and proved an estimate on the rate of convergence of the first eigenvalue and of the corresponding eigenfunction. Here the perturbing set Γ_N is obtained by rescaling a fixed shape of a parameter $\varepsilon \to 0^+$. After some years, in [Gad92] the author improved those results in two dimensional domains: in particular, by means of the concordance method of asymptotic expansion, he proved the existence of a power expansion of any perturbed eigenvalue in terms of ε, see formula (2.1) in [Gad92]. Due to the strong localized nature of the singular perturbation, as one may expect, the coefficients and the exponents in the expansion depend on the local behavior of a suitably chosen basis of unperturbed eigenfunctions (corresponding to the limit eigenvalue), near the point where the perturbing set is concentrating. In addition, as a byproduct, the author obtain the ramification of limit multiple eigenvalues, in the sense that there are m simple perturbed eigenvalues converging to a limit eigenvalue of multiplicity m. Also, recently in [GS13] the authors investigated the sharp asymptotic behavior of the first eigenvalue on a disk, again when the Neumann part is shrinking, and applied the results in the context of Friedrichs-Poincaré inequalities. In [AFL18] the authors provided a different proof of the main result of [Gad92] in the case of simple limit eigenvalues and derived a more explicit expression for the coefficient of the leading term appearing in the expansion obtained by Gadyl'shin. Besides, [AFL18] contains a blow-up convergence result for the scaled, perturbed eigenfunction and some applications to eigenvalue problems for operators with Aharonov-Bohm potentials. Finally, in [Pla04, Pla05], in a three dimensional domain, the case in which the Neumann part Γ_N is a small strip is investigated and eigenvalue asymptotics are provided in terms of the width of the strip.

Moving to the complementary problem, i.e. when the Dirichlet portion Γ_D is disappearing and the limit problem has homogeneous Neumann boundary conditions, the first remarkable result to our knowledge is given in [Gad93]. In that work, the author considered a planar domain Ω and took into account the case in which Γ_D is obtained by rescaling of a parameter $\varepsilon \to 0^+$ of a fixed relatively open set $\omega \subseteq \partial\Omega$, with ω contained in a flat part of $\partial\Omega$. In this framework, the author proved the complete power asymptotic expansion of any perturbed eigenvalue in terms of the parameter ε, together with a ramification result for multiple limit eigenvalues, see Theorem 1 and 2 in [Gad93]. Again, the coefficients and the exponents in the asymptotic expansion depend on the local behavior of the limit eigenfunctions. For what concerns perturbations in a non flat part of the boundary, in the same framework, in [GRS16] and [GRS16] the authors investigated the case in which the domain Ω is the unit disk and the perturbing set with length of order ε is, respectively, one arc and the union of finitely many disjoint arcs.

We also mention [CP03], which concerns the spectral stability of the first eigenvalue, in both cases of shrinking Neumann and Dirichlet part, and [LMP+18], where again the first eigenvalue of the mixed problem is considered, but in a nonlocal framework. Furthermore, it is worth citing [ABIN20], in which the authors investigate a perturbed, mixed eigenvalue problem in two dimension, and apply the results in optimization of wave motion in bodies with small cavities (see also [ABN15, AB17]).

Finally, we give a brief overview of other significant possible kinds of perturbation, which are somehow related to the problems investigated in this section. In particular, a relevant problem concerns the case in which a small hole is excised from the interior of a domain. For instance, if Γ_D is disappearing and it is contained in a flat part of $\partial\Omega$, then, by even reflection through this flat part (if Ω does not intersect its reflection), one can see that the perturbation is equivalent to the removal of a small $(N-1)$-dimensional manifold from the interior of a domain, with zero Dirichlet boundary conditions on the removed manifold and Neumann on the rest of the boundary. In this context, the first notable contribution has been [RT75], when the authors first discovered the role played by the capacity of a set when dealing with spectral stability. Since then, a great amount of works has been written concerning domain perforated with finitely many holes. We mention, among many others, [Cou95], where a differentiability result for perturbed eigenvalue is proved, [Flu95], in which the author provides some more explicit formulas for quantitatively estimating the eigenvalue variation and [AFHL19], where the sharp asymptotics of perturbed eigenvalues are proved, with application to Aharonov-Bohm operators with coalescing poles. See also [AFN20] for analogous result in a nonlocal framework, which is also linked to Steklov-type eigenvalues. For what concerns perturbation of Steklov problems by imposing Dirichlet boundary conditions on a small part of the boundary, we mention [AC14] and references therein. Finally, we cite [GLdC14] and [Naz14], where the authors take into consideration the stability of Steklov eigenvalues on perforated domains.

5.2 The case of disappearing Dirichlet region

In this section we present the eigenvalue problem for the Laplacian with mixed Dirichlet-Neumann homogeneous boundary conditions, with focus on the case in which the region where Dirichlet boundary conditions are prescribed is disappearing, in a suitable sense that will be specified later. Actually, the methods we developed to derive eigenvalue asymptotics under this kind of perturbation turn out to be quite flexible and capable of treating also more general kinds of perturbation, e.g. the eigenvalue problem for the Neumann-Laplacian with a shrinking hole in the interior of the domain where homogeneous Dirichlet boundary conditions are assigned.

Let us introduce some basic assumptions and the functional setting. Let $\Omega \subseteq \mathbb{R}^N$ (with $N \geq 2$) be an open, bounded, Lipschitz and connected set and let $K \subseteq \overline{\Omega}$ be compact. Let $c \in L^\infty(\mathbb{R}^N)$ be such that

$$c(x) \geq c_0 > 0 \quad \text{a.e. in } \mathbb{R}^N, \quad \text{for some } c_0 \in \mathbb{R}. \tag{5.2.1}$$

We define the bilinear form $q = q_c : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ as

$$q(u,v) := \int_\Omega (\nabla u \cdot \nabla v + cuv)\,\mathrm{d}x \quad \text{for any } u, v \in H^1(\Omega). \tag{5.2.2}$$

For simplicity of notation we denote by $q(\cdot)$ also the quadratic form corresponding to (5.2.2), i.e. $q(u) = q(u,u)$. Thanks to assumption (5.2.1), the square root of the quadratic form $q(\cdot)$ is a norm on $H^1(\Omega)$, equivalent to the standard one

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega (|\nabla u|^2 + u^2)\,\mathrm{d}x\right)^{1/2}.$$

We also introduce the Sobolev space $H^1_{0,K}(\Omega)$ defined as the closure in $H^1(\Omega)$ of $C_c^\infty(\overline{\Omega}\setminus K)$. We observe that, if $\partial\Omega$ is smooth and K is a regular submanifold of $\partial\Omega$, the space $H^1_{0,K}(\Omega)$ can be characterized as

$$H^1_{0,K}(\Omega) = \{u \in H^1(\Omega) \colon u = 0 \text{ on } K\},$$

where $u = 0$ on K, for functions in $H^1(\Omega)$, is meant in the trace sense, see [Ber11]. Defining q_K as the restriction of the form q to $H^1_{0,K}(\Omega)$, we say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of q_K if there exists $u \in H^1_{0,K}(\Omega)$, $u \not\equiv 0$, called *eigenfunction*, such that

$$q_K(u,v) = \lambda (u,v)_{L^2(\Omega)} \quad \text{for all } v \in H^1_{0,K}(\Omega), \tag{5.2.3}$$

where $(\cdot,\cdot)_{L^2(\Omega)}$ is the usual scalar product in $L^2(\Omega)$. From classical spectral theory (see Chapter 3) we have that problem (5.2.3) admits a diverging sequence of positive eigenvalues

$$0 < \lambda_1(\Omega; K) < \lambda_2(\Omega; K) \leq \cdots \leq \lambda_n(\Omega; K) \leq \cdots,$$

where each one is repeated as many times as its multiplicity. Moreover, we denote by $(\varphi_n(\Omega; K))_n$ a sequence of eigenfunctions, which we choose so that it forms an orthonormal family in $L^2(\Omega)$. Hereafter we denote, for any integer $n \in \mathbb{N}_*$,

$$\lambda_n := \lambda_n(\Omega; \emptyset), \quad \varphi_n := \varphi_n(\Omega; \emptyset), \tag{5.2.4}$$

where $\mathbb{N}_* := \mathbb{N} \setminus \{0\}$. We notice that the connectedness of the domain Ω is not a restrictive assumption, since the spectrum of q_K in a non connected domain is the union of the spectra on the single connected components. We also point out that the assumption (5.2.1) is not substantial and it can be dropped, since, up to a translation of the spectrum, we can recover a coercive form as in (5.2.2). Besides, we notice that, in the particular case when K is the empty set and $c(x) \equiv c > 0$, $(\lambda_n(\Omega;\emptyset) - c)_n$ coincides with the sequence of eigenvalues of the standard Laplacian with homogeneous Neumann boundary condition. If K is smooth (e.g. if K is the closure of a smooth open set of $\mathbb{R}^N$ or a regular submanifold of $\partial\Omega$), problem (5.2.3) admits the following classical formulation

$$\begin{cases} -\Delta u + cu = \lambda u, & \text{in } \Omega \setminus K, \\ \qquad\quad u = 0, & \text{on } K, \\ \qquad\quad \dfrac{\partial u}{\partial \boldsymbol{\nu}} = 0, & \text{on } \partial\Omega \setminus K. \end{cases} \tag{5.2.5}$$

When $K \subseteq \partial\Omega$, (5.2.5) is an elliptic problem with mixed Dirichlet-Neumann homogeneous boundary conditions and one can interpret the spectrum $(\lambda_n(\Omega;K))_n$ as the square roots of the frequencies of oscillation of an elastic, vibrating membrane, whose boundary is clamped on K and free in the rest of $\partial\Omega$.

In this section we start from the unperturbed situation corresponding to the Neumann eigenvalue problem, i.e. the case $K = \emptyset$, and then we introduce a singular perturbation of it, which consist in considering a "small", nonempty $K \subseteq \overline{\Omega}$ and zero Dirichlet boundary conditions on it. Our aim is to study the eigenvalue variation due to this perturbation and to find the sharp asymptotics of the perturbed eigenvalue, in the limit when K is "disappearing" as a function of a certain parameter.

In the following we give the basic definitions and we present our main results. Let us recall that, throughout this section, Ω denotes an open, bounded, Lipschitz and connected subset of $\mathbb{R}^N$, where $N \geq 2$.

As in [AFHL19, AFN20, Cou95], the quantity that measures the "smallness" of the perturbation set $K \subseteq \overline{\Omega}$, which is suitable for the development of an eigenvalue stability theory for our problem, is a notion of capacity, as defined below.

Definition 5.2.1. Let $K \subseteq \overline{\Omega}$ be compact. We define the *relative Sobolev capacity* of K in $\overline{\Omega}$ as follows

$$\mathrm{Cap}_{\bar{\Omega}}(K) := \inf\left\{ \int_\Omega (|\nabla u|^2 + u^2)\,dx \colon u \in H^1(\Omega),\ u - 1 \in H^1_{0,K}(\Omega) \right\}.$$

We refer to Section 5.3 (see also [AW03]) for the mathematical description of this set function. A first taste of the fact that the relative Sobolev capacity defined above is a good perturbation parameter for our purposes is given by Proposition 5.3.8, which states that the space $H^1_{0,K}(\Omega)$ coincides with $H^1(\Omega)$ if and only if $\mathrm{Cap}_{\bar{\Omega}}(K) = 0$. This means that zero capacity sets are negligible for H^1 functions. Furthermore, the following theorem yields continuity of perturbed eigenvalues when $\mathrm{Cap}_{\bar{\Omega}}(K) \to 0$.

Theorem 5.2.2. *Let $K \subseteq \overline{\Omega}$ be compact and let $\lambda_n(\Omega;K)$ be an eigenvalue of problem (5.2.3) for some $n \in \mathbb{N}_*$. Let also λ_n be as in (5.2.4). Then there exist $C > 0$ and $\delta > 0$ (independent of K) such that, if $\mathrm{Cap}_{\bar{\Omega}}(K) < \delta$, then*

$$0 \leq \lambda_n(\Omega;K) - \lambda_n \leq C\,(\mathrm{Cap}_{\bar{\Omega}}(K))^{1/2}.$$

We observe that the left inequality is an easy consequence of the Min-Max variational characterization of the eigenvalues, namely

$$\lambda_n(\Omega;K) = \min\left\{ \max_{u\in V_n} \frac{q(u)}{\|u\|^2_{L^2(\Omega)}} : V_n \subseteq H^1_{0,K}(\Omega)\ n\text{-dimensional subspace} \right\}. \tag{5.2.6}$$

In order to state the first main result of this part, we introduce the following notion of convergence of sets.

Definition 5.2.3. Let $K \subseteq \overline{\Omega}$ be compact and let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact subsets of $\overline{\Omega}$. We say that K_ε is *concentrating* at K, as $\varepsilon \to 0$, if for any open set $U \subseteq \mathbb{R}^N$ such that $K \subseteq U$ there exists $\varepsilon_U > 0$ such that $K_\varepsilon \subseteq U$ for all $\varepsilon \in (0, \varepsilon_U)$.

We observe that the "limit" set of a concentrating family is not unique. Indeed, if K_ε is concentrating at K, then it is also concentrating at any compact set $\tilde{K}$ such that $K \subseteq \tilde{K} \subseteq \overline{\Omega}$. Nevertheless, in the cases considered in the present manuscript (e.g. when the limit set has zero capacity) this notion of convergence is the one that ensures the continuity of the capacity (see Proposition 5.3.13); furthermore, it is related to the convergence of sets in the sense of Mosco, see [AFHL19]. An example of family of concentrating sets is given by a decreasing family of compact sets, see Example 5.3.12.

In order to sharply describe the eigenvalue variation, the following definition of capacity associated to a H^1-function plays a fundamental role.

Definition 5.2.4. For any $f \in H^1(\Omega)$ and $K \subseteq \overline{\Omega}$ compact, we define the *relative Sobolev f-capacity* of K in $\overline{\Omega}$ as follows

$$\mathrm{Cap}_{\bar{\Omega},c}(K,f) := \inf\left\{ q(u) \colon u \in H^1(\Omega),\ u - f \in H^1_{0,K}(\Omega) \right\}. \tag{5.2.7}$$

We remark that, if $K \subseteq \partial\Omega$, the above definition actually only depends on the trace of f on $\partial\Omega$, which belongs to $H^{1/2}(\partial\Omega)$, and in particular on its values on K, if K is regular. Moreover one can prove that, if a family of compact sets $K_\varepsilon \subseteq \overline{\Omega}$ is concentrating to a compact $K \subseteq \overline{\Omega}$ such that $\mathrm{Cap}_{\bar{\Omega}}(K) = 0$, then $\mathrm{Cap}_{\bar{\Omega},c}(K_\varepsilon, f) \to 0$ as $\varepsilon \to 0$ for all $f \in H^1(\Omega)$, see Proposition 5.3.13 and Remark 5.3.9.

Hereafter we assume $n_0 \in \mathbb{N}_*$ to be such that

$$\lambda_0 := \lambda_{n_0} \text{ is simple} \tag{5.2.8}$$

and we denote as

$$\varphi_0 := \varphi_{n_0}, \tag{5.2.9}$$

a corresponding $L^2(\Omega)$-normalized eigenfunction. Our first main result is the following sharp asymptotic expansion of the eigenvalue variation.

Theorem 5.2.5. *Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact subsets of $\overline{\Omega}$ concentrating to $K \subseteq \overline{\Omega}$ compact such that $\mathrm{Cap}_{\bar{\Omega}}(K) = 0$. Let λ_0, φ_0 be as in (5.2.8), (5.2.9) respectively and let $\lambda_\varepsilon := \lambda_{n_0}(\Omega; K_\varepsilon)$. Then*

$$\lambda_\varepsilon - \lambda_0 = \mathrm{Cap}_{\bar{\Omega},c}(K_\varepsilon, \varphi_0) + o(\mathrm{Cap}_{\bar{\Omega},c}(K_\varepsilon, \varphi_0))$$

as $\varepsilon \to 0$.

In order to give some relevant examples of explicit expansions, we finally provide the sharp asymptotic behavior of the function $\varepsilon \mapsto \mathrm{Cap}_{\overline{\Omega},c}(K_\varepsilon, \varphi_0)$ appearing above, in a particular case. More precisely, we consider a family $\{K_\varepsilon\}_{\varepsilon>0} \subseteq \overline{\Omega}$ which is concentrating at a point $\bar{x} \in \overline{\Omega}$ in an appropriate way, that resembles the situation where a fixed set is being scaled and it is therefore maintaining the same shape while shrinking to the point. Hereafter we illustrate these results by distinguishing the cases $\bar{x} \in \partial\Omega$ and $\bar{x} \in \Omega$. Without losing generality we can assume that $\bar{x} = 0$. We perform this analysis under the assumption $N \geq 3$, since a detailed study of the case $N = 2$, with $K_\varepsilon, K \subseteq \partial\Omega$, has been already pursued in [Gad93]; nevertheless, our method, which is based on a blow-up analysis for the capacitary potentials, could be adapted to the 2-dimensional case by using a logarithmic Hardy inequality to derive energy estimates, instead of the Hardy-type inequality of Lemma 5.4.8, which does not hold in dimension 2.

Sets scaling to a boundary point We first focus on the case in which the perturbing compact sets $K_\varepsilon \subseteq \overline{\Omega}$ are concentrating to a point of the boundary of Ω, which, up to a translation, can be assumed to be the origin. In this situation, we assume that the boundary $\partial\Omega$ is of class $C^{1,1}$ in a neighborhood of $0 \in \partial\Omega$, namely

$$\begin{gathered} \text{there exists } r_0 > 0 \text{ and } g \in C^{1,1}(B'_{r_0}) \text{ such that} \\ B_{r_0} \cap \Omega = \{x \in B_{r_0} \colon x_N > g(x')\}, \\ B_{r_0} \cap \partial\Omega = \{x \in B_{r_0} \colon x_N = g(x')\}, \end{gathered} \tag{5.2.10}$$

where $B_{r_0} = \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : |x| < r_0\}$ is the ball in $\mathbb{R}^N$ centered at the origin with radius r_0, $x' = (x_1, \dots, x_{N-1})$ and $B'_{r_0} = \{(x', x_N) \in B_{r_0} : x_N = 0\}$. It is not restrictive to assume that $\nabla g(0) = 0$, i.e. that $\partial\Omega$ is tangent to the coordinate hyperplane $\{x_N = 0\}$ in the origin. Let us introduce the following class of diffeomorphisms that "straighten" the boundary near 0:

$$\begin{aligned} \mathcal{C} := \{\Phi : \mathcal{U} \to B_R \colon & \mathcal{U} \text{ is an open neighborhood of } 0,\ R > 0, \\ & \Phi \text{ is a diffeomorphism of class } C^{1,1}(\mathcal{U}; B_R),\ \Phi(0) = 0, \\ & J_\Phi(0) = I_N,\ \Phi(\mathcal{U} \cap \Omega) = \mathbb{R}^N_+ \cap B_R \text{ and } \Phi(\mathcal{U} \cap \partial\Omega) = B'_R\}, \end{aligned} \tag{5.2.11}$$

where $\mathbb{R}^N_+ := \{(x_1, \dots, x_N) \in \mathbb{R}^N \colon x_N > 0\}$ and I_N is the identity $N \times N$ matrix. Let us assume that, for any $\varepsilon \in (0,1)$, $K_\varepsilon \subseteq \overline{\Omega}$ is a compact set and the family $\{K_\varepsilon\}_\varepsilon$ satisfies the following properties:

$$\text{there exists } M \subseteq \overline{\mathbb{R}^N_+} \text{ compact such that } \Phi(K_\varepsilon)/\varepsilon \subseteq M \quad \text{for all } \varepsilon \in (0,1), \tag{5.2.12}$$

$$\begin{gathered} \text{there exists } K \subseteq \overline{\mathbb{R}^N_+} \text{ compact such that} \\ \mathbb{R}^N \setminus (\Phi(K_\varepsilon)/\varepsilon) \to \mathbb{R}^N \setminus K \quad \text{in the sense of Mosco, as } \varepsilon \to 0, \end{gathered} \tag{5.2.13}$$

for some $\Phi \in \mathcal{C}$, where $\Phi(K_\varepsilon)/\varepsilon := \{x/\varepsilon \colon x \in \Phi(K_\varepsilon)\}$. With reference to [Dan03, Mos69], we recall below the definition of convergence of sets in the sense of Mosco.

Definition 5.2.6. Let $\varepsilon \in (0,1)$ and let $U_\varepsilon, U \subseteq \mathbb{R}^N$ be open sets. We say that U_ε *is converging to* U *in the sense of Mosco* as $\varepsilon \to 0$ if the following two properties hold:

(i) the weak limit points (as $\varepsilon \to 0$) in $H^1(\mathbb{R}^N)$ of every family of functions $\{u_\varepsilon\}_\varepsilon \subseteq H^1(\mathbb{R}^N)$, such that $u_\varepsilon \in H^1_0(U_\varepsilon)$ for every $\varepsilon > 0$, belong to $H^1_0(U)$;

(ii) for every $u \in H^1_0(U)$ there exists a family $\{u_\varepsilon\}_\varepsilon \subseteq H^1(\mathbb{R}^N)$ such that $u_\varepsilon \in H^1_0(U_\varepsilon)$ for every $\varepsilon > 0$ and $u_\varepsilon \to u$ in $H^1(\mathbb{R}^N)$, as $\varepsilon \to 0$.

We may also say that $H^1_0(U_\varepsilon)$ is converging to $H^1_0(U)$ in the sense of Mosco.

In order to clarify hypotheses (5.2.12) and (5.2.13) we adduce below a bunch of examples in which they hold for subsets K_ε of $\partial\Omega$.

Examples 5.2.7.

(i) The easiest case is when $\partial\Omega$ is flat in a neighborhood of the origin and

$$K_\varepsilon := \varepsilon K = \{\varepsilon x \colon x \in K\},$$

for a certain fixed $K \subseteq \mathbb{R}^{N-1}$ compact. Here we can choose as Φ the identity so that $\Phi(K_\varepsilon)/\varepsilon \equiv K$, which clearly satisfies both hypotheses (5.2.12) and (5.2.13).

(ii) Another interesting example (always in the case of flat boundary) is when Φ is the identity and $K_\varepsilon/\varepsilon$ is a perturbation of a compact set. More precisely, let $K_1, K_2 \subseteq \mathbb{R}^{N-1}$ be two compact sets containing the origin and let $f \colon (0,1) \to (0,+\infty)$ be such that $f(s)/s \to 0$ as $s \to 0$. If we consider

$$K_\varepsilon := \varepsilon K_1 + f(\varepsilon) K_2 = \{\varepsilon x + f(\varepsilon) y \colon x \in K_1,\ y \in K_2\}$$

then $\Phi(K_\varepsilon)/\varepsilon$ fulfills (5.2.12) and (5.2.13) with $K = K_1$. We remark that it is possible to generalize this idea and produce other examples.

(iii) In the case of non-flat boundary, we have that conditions (5.2.12) and (5.2.13) hold e.g. when K_ε is the image through Φ^{-1}, for some $\Phi \in \mathcal{C}$, of sets like the ones in (i)–(ii). A remarkable case is when $\Phi(x', x_N) = (x', x_N - g(x'))$ in a neighborhood of the origin, so that the restriction of Φ to $\partial\Omega$ is the orthogonal projection of $\partial\Omega$ onto its tangent hyperplane at 0. Hence assumption (5.2.13) is satisfied, for example, if the each set K_ε is a compact subset of $\partial\Omega$ whose orthogonal projection on the hyperplane tangent to $\partial\Omega$ at 0 is of the form εK, for K being a compact subset of $\mathbb{R}^{N-1}$.

(iv) Finally, it is easy to prove that assumptions (5.2.12) and (5.2.13) hold for

$$K_\varepsilon := B_\varepsilon \cap \partial\Omega, \quad \text{with } K = \mathbb{R}^{N-1} \cap \overline{B_1}.$$

The last ingredient needed to detect the sharp asymptotics of the Sobolev capacity of shrinking sets is the notion of vanishing order for the limit eigenfunction. Let

$$L^2(\mathbb{S}^{N-1}_+) := \left\{ \psi \colon \mathbb{S}^{N-1}_+ \to \mathbb{R} : \ \psi \text{ is measurable and } \int_{\mathbb{S}^{N-1}_+} |\psi|^2 \, \mathrm{d}S < \infty \right\},$$

where $\mathbb{S}^{N-1}_+ := \{(x_1, \dots, x_N) \in \mathbb{R}^N \colon \ |x| = 1, \ x_N > 0\}$. Moreover let

$$H^1(\mathbb{S}^{N-1}_+) := \{\psi \in L^2(\mathbb{S}^{N-1}_+) \colon \nabla_{\mathbb{S}^{N-1}} \psi \in L^2(\mathbb{S}^{N-1}_+)\}.$$

The following proposition asserts that the limit eigenfunction φ_0 behaves like a harmonic polynomial near the origin.

Proposition 5.2.8. *Let Ω satisfy assumption (5.2.10) and let φ_0 be as in (5.2.9). Then there exists $\gamma \in \mathbb{N}$ (possibly 0) and $\Psi \in C^\infty(\overline{\mathbb{S}^{N-1}_+})$, $\Psi \neq 0$ such that, for all $\Phi \in \mathcal{C}$, there holds*

$$\frac{\varphi_0(\Phi^{-1}(\varepsilon x))}{\varepsilon^\gamma} \to |x|^\gamma \, \Psi\left(\frac{x}{|x|}\right) \quad \textit{in } H^1(B_R^+) \textit{ as } \varepsilon \to 0, \tag{5.2.14}$$

for all $R > 0$, where $B_R^+ := B_R \cap \mathbb{R}^N_+$.

Furthermore, for every $R > 0$,

$$\varepsilon^{-N-2\gamma} \int_{\Omega \cap B_{R\varepsilon}} \varphi_0^2(x) \, \mathrm{d}x \to \int_{B_R^+} \psi_\gamma^2(x) \, \mathrm{d}x \quad \textit{as } \varepsilon \to 0 \tag{5.2.15}$$

and

$$\varepsilon^{-N-2\gamma+2} \int_{\Omega \cap B_{R\varepsilon}} |\nabla \varphi_0(x)|^2 \, \mathrm{d}x \to \int_{B_R^+} |\nabla \psi_\gamma(x)|^2 \, \mathrm{d}x \quad \textit{as } \varepsilon \to 0 \tag{5.2.16}$$

where

$$\psi_\gamma(x) := |x|^\gamma \, \Psi\left(\frac{x}{|x|}\right). \tag{5.2.17}$$

We observe that the function Ψ appearing in (5.2.14) and (5.2.17) is necessarily a spherical harmonic of degree γ which is symmetric with respect to the equator $x_N = 0$, hence satisfying homogeneous Neumann boundary conditions on $\{x_N = 0\}$. More precisely, Ψ solves

$$\begin{cases} -\Delta_{\mathbb{S}^{N-1}} \Psi = \gamma(N + \gamma - 2)\Psi, & \text{in } \mathbb{S}^{N-1}_+, \\ \nabla_{\mathbb{S}^{N-1}} \Psi \cdot \boldsymbol{e}_N = 0, & \text{on } \partial\mathbb{S}^{N-1}_+, \end{cases}$$

where $\boldsymbol{e}_N = (0, \dots, 0, 1)$. The integer number γ is called the *vanishing order* of φ_0 in the origin. We also mention [Ber55, FT13, Nir59, Rob88] for asymptotic behavior of solutions to elliptic PDEs.

Remark 5.2.9. We observe that the restriction of Ψ to the $N-2$ dimensional unit sphere $\partial\mathbb{S}^{N-1}_+$ cannot vanish everywhere. Indeed this would mean that the nonzero harmonic function ψ_γ defined in (5.2.17) vanishes on $\partial\mathbb{R}^N_+$ together with is normal derivative; but then the trivial extension of ψ_γ to the whole $\mathbb{R}^N$ would violate the classical unique continuation principle (see [Wol92]), thus giving rise to a contradiction.

Now let us introduce the Beppo Levi space $\mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+})$ defined as the completion of $C_c^\infty(\overline{\mathbb{R}^N_+})$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+})} := \left(\int_{\mathbb{R}^N_+} |\nabla u|^2 \, \mathrm{d}x\right)^{1/2}.$$

Furthermore, for any compact $K \subseteq \overline{\mathbb{R}^N_+}$, we define the space $\mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+} \setminus K)$ as the closure of $C_c^\infty(\overline{\mathbb{R}^N_+} \setminus K)$ in $\mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+})$. Thereafter we introduce a notion of capacity that will appear in the asymptotic expansion of $\mathrm{Cap}_{\bar{\Omega},c}(K_\varepsilon, \varphi_0)$, when $\varepsilon \to 0$.

Definition 5.2.10. For any compact $K \subseteq \overline{\mathbb{R}^N_+}$ and for any $f \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+})$ we define the *relative f-capacity* of K in $\overline{\mathbb{R}^N_+}$ as

$$\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K, f) := \inf \left\{ \int_{\mathbb{R}^N_+} |\nabla u|^2 \, \mathrm{d}x \colon u \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+}),\ u - f \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+} \setminus K) \right\}.$$

If $f \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+})$ is equal to 1 in a neighborhood of K, we denote by

$$\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K) := \mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K, f)$$

the relative capacity of K in $\overline{\mathbb{R}^N_+}$. The definition can be extended to functions $f \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^N_+})$ by letting

$$\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K, f) := \inf \left\{ \int_{\mathbb{R}^N_+} |\nabla u|^2 \, \mathrm{d}x \colon u \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+}),\ u - \eta_K f \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+} \setminus K) \right\},$$

where $\eta_K \in C_c^\infty(\overline{\mathbb{R}^N_+})$ is such that $\eta_K = 1$ in a neighborhood of K.

Remark 5.2.11. We remark that the relative Sobolev capacity in $\overline{\mathbb{R}^N_+}$ of a compact set $K \subseteq \partial\mathbb{R}^N_+$, here denoted by $\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K)$, actually coincides with half of the capacity of K, in the classical sense (see [MZ97, Chapter 2.1]), defined as

$$\mathrm{cap}_{\mathbb{R}^N}(K) = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x \colon u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ u - \eta_K \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus K) \right\}.$$

Moreover we notice that $\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K)$ coincides, up to a constant, with the Gagliardo $\frac{1}{2}$-fractional capacity of K in $\mathbb{R}^{N-1}$, see e.g. [AFN20] for the definition.

In this framework we are able to state the second main result of this section, which concerns the sharp behavior of the function $\varepsilon \mapsto \mathrm{Cap}_{\bar{\Omega},c}(K_\varepsilon, \varphi_0)$ as $\varepsilon \to 0^+$.

Theorem 5.2.12. *Let $N \geq 3$. Assume (5.2.10) holds true. Let $\{K_\varepsilon\}_{\varepsilon>0} \subseteq \overline{\Omega}$ be a family of compact sets concentrating at $\{0\} \subseteq \partial\Omega$ as $\varepsilon \to 0$ and let (5.2.12)-(5.2.13) hold for* *some* $\Phi \in \mathcal{C}$ *and for some compact set* $K \subseteq \overline{\mathbb{R}^N_+}$ *satisfying* $\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K) > 0$*. Let* φ_0 *be as in* (5.2.9) *and let* γ*,* ψ_γ *be as in* (5.2.14)-(5.2.17)*. Then*

$$\mathrm{Cap}_{\bar{\Omega},c}(K_\varepsilon, \varphi_0) = \varepsilon^{N+2\gamma-2}\big(\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K, \psi_\gamma) + o(1)\big), \quad \textit{as } \varepsilon \to 0,$$

with $\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K, \psi_\gamma)$ *being as in Definition 5.2.10.*

Combining Theorems 5.2.12 and 5.2.5 we directly obtain the following corollary.

Corollary 5.2.13. *Under the same assumptions and with the same notations of both Theorems 5.2.12 and 5.2.5, we have that*

$$\lambda_\varepsilon - \lambda_0 = \varepsilon^{N+2\gamma-2}\big(\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K, \psi_\gamma) + o(1)\big), \quad \textit{as } \varepsilon \to 0.$$

The expansion stated in Corollary 5.2.13 provides the sharp asymptotics of the eigenvalue variation if $\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K, \psi_\gamma) > 0$. This happens e.g. whenever $K \subseteq \partial\mathbb{R}^N_+$ is a compact set such that $\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K) > 0$, as proved in Proposition 5.3.17; we observe that the validity of such result strongly relies on the position of the nodal set of ψ_γ with respect to the set K.

On the other hand, if $K \subseteq \partial\mathbb{R}^N_+$ is compact, we have that $\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K) > 0$ if its $N-1$ dimensional Lebesgue measure is nonzero, see Proposition 5.3.18.

Sets scaling to an interior point Although the present study was mainly motivated by our interest in the eigenvalue asymptotics for moving mixed Dirichlet-Neumann boundary conditions, our techniques also apply to another class of perturbations, without any substantial difference, in view of the various possibilities embraced by Theorem 5.2.5. In particular, it is possible to state a result analogous to Theorem 5.2.12 in the case in which the perturbing sets K_ε are concentrating at a point that lies in the interior of Ω. In this case the limit problem is the one with homogeneous Neumann boundary conditions on $\partial\Omega$ and the perturbed problem can be thought of as Ω without a "small" hole, on which zero Dirichlet boundary conditions are prescribed. We assume that $0 \in \Omega$ is the "limit" of the concentrating subsets K_ε and we ask assumptions similar to (5.2.12)-(5.2.13) to be satisfied, that is

$$\text{there exists } M \subseteq \mathbb{R}^N \text{ compact such that } K_\varepsilon/\varepsilon \subseteq M \quad \text{for all } \varepsilon \in (0,1), \tag{5.2.18}$$

$$\begin{gathered}\text{there exists } K \subseteq \mathbb{R}^N \text{ compact such that} \\ \mathbb{R}^N \setminus (K_\varepsilon/\varepsilon) \to \mathbb{R}^N \setminus K \quad \text{in the sense of Mosco, as } \varepsilon \to 0.\end{gathered} \tag{5.2.19}$$

As before, these assumptions are fulfilled, for instance, in the case $K_\varepsilon := \varepsilon K$, for a certain compact $K \subseteq \mathbb{R}^N$ such that $K_\varepsilon \subseteq \Omega$ for every $\varepsilon \in (0,1)$. Since $0 \in \Omega$ is an interior point, from classical regularity results for elliptic equations (see e.g. [Rob88]), there exist $\kappa \in \mathbb{N}$ and a spherical harmonic Z of degree κ such that

$$-\Delta_{\mathbb{S}^{N-1}} Z = \kappa(N+\kappa-2)Z \quad \text{in } \mathbb{S}^{N-1}$$

and
$$\frac{\varphi_0(\varepsilon x)}{\varepsilon^\kappa} \to \zeta_\kappa(x) := |x|^\kappa Z\left(\frac{x}{|x|}\right) \quad \text{in } H^1(B_R) \text{ as } \varepsilon \to 0, \tag{5.2.20}$$
for all $R > 0$. We can now state the last main result of this part, which is analogous to Theorem 5.2.12.

Theorem 5.2.14. *Let $N \geq 3$ and $\{K_\varepsilon\}_{\varepsilon>0} \subseteq \Omega$ be a family of compact sets concentrating at $\{0\} \subseteq \Omega$ as $\varepsilon \to 0$. Let (5.2.18)-(5.2.19) hold for some compact set $K \subseteq \mathbb{R}^N$ satisfying $\mathrm{cap}_{\mathbb{R}^N}(K) > 0$. Let φ_0 be as in (5.2.9) and κ, ζ_κ be as in (5.2.20). Then*
$$\mathrm{Cap}_{\bar{\Omega},c}(K_\varepsilon, \varphi_0) = \varepsilon^{N+2\kappa-2}(\mathrm{cap}_{\mathbb{R}^N}(K, \zeta_\kappa) + o(1)), \quad \text{as } \varepsilon \to 0,$$
where and $\mathrm{cap}_{\mathbb{R}^N}(K, \zeta_\kappa)$ is the standard Newtonian ζ_κ-capacity of K, see Definition 5.3.15.

We point out that, in general, $\mathrm{cap}_{\mathbb{R}^N}(K, \zeta_\kappa)$ may not be strictly positive, since K, still having positive capacity, may happen to be a subset of the zero level set of ζ_κ. In Lemma 5.3.16 we provide sufficient conditions for $\mathrm{cap}_{\mathbb{R}^N}(K, \zeta_\kappa)$ to be strictly positive: e.g. this happens when K has nonzero capacity whereas the intersection of K with the nodal set of ζ_κ has zero capacity. We refer to [EG15, Theorem 4.15] for a sufficient condition for $\mathrm{cap}_{\mathbb{R}^N}(K) > 0$: more precisely, we have that $\mathrm{cap}_{\mathbb{R}^N}(K) > 0$ if its N-dimensional Lebesgue measure is nonzero.

5.3 Preliminaries on Sobolev capacity and concentration of sets

In this section we focus on the notions of capacity and concentration of sets (see definitions 5.2.1, 5.2.10 and 5.2.3): we prove some basic properties (e.g. propositions 5.3.17 and 5.3.18) and we investigate their mutual relations. Since this is an introductory passage, we may temporarily relax the hypothesis of K being compact and put ourselves in a slightly more general framework. In particular we are going to define the present notion of capacity for any subset $A \subseteq \overline{\Omega}$ and then we are going to point out its particular features when restricted to a compact set $K \subseteq \overline{\Omega}$, thus justifying Definition 5.2.1. One of the main references in this passage is [AW03] (see also [War15]), in which the authors employed the notion of capacity we are presenting in order to define a realization of the Laplacian with generalized Robin boundary conditions, on arbitrary sets. We also refer to [MZ97, Section 2.1] for a comprehensive, classical exposition.

For any $A \subseteq \overline{\Omega}$ we call the *Sobolev capacity of A relative to $\overline{\Omega}$* the following quantity
$$\begin{aligned}\mathrm{Cap}_{\bar{\Omega}}(A) :&= \inf\left\{\|u\|^2_{H^1(\Omega)} : u \in H^1(\Omega),\ u \geq 1 \text{ a.e. in an open neighborhood of } A\right\}, \\ &= \inf\left\{\|u\|^2_{H^1(\Omega)} : u \in \mathcal{U}_{\bar{\Omega}}(A)\right\},\end{aligned}$$
where
$$\mathcal{U}_{\bar{\Omega}}(A) := \{u \in H^1(\Omega) \colon u \geq 1 \text{ in } O \supseteq A \text{ relatively open in } \overline{\Omega}\}.$$

One can immediately see that, by definition

$$\mathrm{Cap}_{\overline{\Omega}}(A) = \inf\{\mathrm{Cap}_{\overline{\Omega}}(O) \colon O \subseteq \overline{\Omega} \text{ relatively open and such that } A \subseteq O\}. \tag{5.3.1}$$

Let us now briefly explain the nomenclature we adopt in the present chapter, concerning the capacity. The word *Sobolev* in the definition stands for the fact that we are minimizing the square of a Sobolev norm; indeed, on the contrary, in the classical definition of capacity one seeks for the infimum of the energy $\int_\Omega |\nabla u|^2 \,\mathrm{d}x$, with u being in a Beppo-Levi type space. In the literature classical capacities are usually denoted with "cap", while Sobolev capacities with "Cap". Secondly, the word *relative* appears because $\overline{\Omega}$ can be seen as a reference domain with respect to which one computes the capacity of A, that does depend on $\overline{\Omega}$. In fact, we call the *Sobolev capacity of* A the following quantity

$$\mathrm{Cap}(A) := \inf\left\{\|u\|^2_{H^1(\mathbb{R}^N)} : u \in H^1(\mathbb{R}^N),\ u \geq 1 \text{ a.e. in an open neighborhood of } A\right\},$$

where there is no reference domain.

The theory of capacities was basically born with the seminal work of Choquet [Cho54], where the author laid the first theoretical foundation of this kind of set functions. Since then, the following definition has been generally accepted.

Definition 5.3.1. Let $\mathcal{T}$ be a topological space and let $\mathcal{P}(\mathcal{T})$ be the power set of $\mathcal{T}$. A function $\mathfrak{C} \colon \mathcal{P}(\mathcal{T}) \to [0, +\infty]$ is called a *Choquet capacity* of $\mathcal{T}$ if the following axioms are satisfied:

($\mathfrak{C}$1) $\mathfrak{C}(\emptyset) = 0$;

($\mathfrak{C}$2) *(monotonicity)* if $A \subseteq B$ then $\mathfrak{C}(A) \leq \mathfrak{C}(B)$;

($\mathfrak{C}$3) *(lower continuity)* for any sequence $\{A_n\}_n \subseteq \mathcal{P}(\mathcal{T})$ such that $A_n \subseteq A_{n+1}$, there holds $\mathfrak{C}(\cup_{n=1}^\infty A_n) = \lim_{n\to\infty} \mathfrak{C}(A_n)$;

($\mathfrak{C}$4) *(upper continuity for compact sets)* if $\{K_n\}_n \subseteq \mathcal{P}(\mathcal{T})$ is a sequence of compact sets such that $K_{n+1} \subseteq K_n$, then $\mathfrak{C}(\cap_{n=1}^\infty K_n) = \lim_{n\to\infty} \mathfrak{C}(K_n)$;

We have that the notion of capacity introduced above is Choquet. The result is essentially classical, but for sake of completeness we report a proof here.

Lemma 5.3.2. *The set function*

$$\begin{aligned} \mathrm{Cap}_{\overline{\Omega}} \colon \mathcal{P}(\overline{\Omega}) &\to [0, +\infty] \\ A &\mapsto \mathrm{Cap}_{\overline{\Omega}}(A) \end{aligned}$$

is a Choquet capacity, as in Definition 5.3.1.

Proof. Axiom ($\mathfrak{C}$1) is trivial, since $\emptyset$ is open and $\mathcal{U}_{\overline{\Omega}}(\emptyset)$ contains the null function, while ($\mathfrak{C}$2) follows from the fact that $A \subseteq B$ implies $\mathcal{U}_{\overline{\Omega}}(B) \subseteq \mathcal{U}_{\overline{\Omega}}(A)$.

In order to prove ($\mathfrak{C}$4), let us consider a decreasing family of compact sets $\{K_n\}_n \subseteq \mathcal{P}(\overline{\Omega})$ and let $K := \cap_{n=1}^\infty K_n$. We first observe that $\lim_{n\to\infty} \mathrm{Cap}_{\overline{\Omega}}(K_n)$ does exists in view of ($\mathfrak{C}2$). If $O \subseteq \overline{\Omega}$ is any relatively open set such that $K \subseteq O$, then there exists $\bar{n}$ such that $K_n \subseteq O$ for all $n \geq \bar{n}$. Therefore

$$\mathrm{Cap}_{\bar{\Omega}}(K) \leq \lim_{n\to\infty} \mathrm{Cap}_{\bar{\Omega}}(K_n) \leq \mathrm{Cap}_{\overline{\Omega}}(O).$$

($\mathfrak{C}4$) is thus proved thanks to characterization (5.3.1).

Let us finally prove ($\mathfrak{C}3$). Let $\{A_n\}_n \subseteq \mathcal{P}(\overline{\Omega})$ be an increasing sequence and let $A := \cup_{n=1}^{\infty} A_n$. We observe that $l_A := \lim_{n\to\infty} \mathrm{Cap}_{\bar{\Omega}}(A_n)$ exists and it might be $+\infty$. In addition

$$l_A \leq \mathrm{Cap}_{\bar{\Omega}}(A) \in [0, +\infty].$$

In particular, if $l_A = +\infty$ the thesis is trivial, so let us assume $l_A < +\infty$. In order to prove the converse inequality, let $u_n \in \mathcal{U}_{\bar{\Omega}}(A_n)$ be such that

$$\|u_n\|^2_{H^1(\Omega)} \leq \mathrm{Cap}_{\bar{\Omega}}(A_n) + 2^{-n}.$$

Hence $\{u_n\}_n$ is bounded in $H^1(\Omega)$ and so there exists $u \in H^1(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$ and $u_n \to u$ a.e. in Ω, as $n \to \infty$. In addition there exist a sequence of open sets $O_n \supseteq A_n$ such that $u_n \geq 1$ in O_n and it is not restrictive to assume that $O_n \subseteq O_{n+1}$. Therefore we have that, for any fixed $\bar{n} \in \mathbb{N}$, $u_n \geq 1$ a.e. in $O_{\bar{n}}$ for all $n \geq \bar{n}$; passing to the limit as $n \to \infty$, this implies that $u \geq 1$ a.e. in $O_{\bar{n}}$ for any $\bar{n} \in \mathbb{N}$ and so $u \geq 1$ a.e. in $O := \cup_{n=1}^{\infty} O_n$. By construction $A \subseteq O$, hence u is an admissible test function for $\mathrm{Cap}_{\bar{\Omega}}(A)$. Thus

$$\mathrm{Cap}_{\bar{\Omega}}(A) \leq \|u\|^2_{H^1(\Omega)} \leq \liminf_{n\to\infty} \|u_n\|_{H^1(\Omega)} = l_A$$

and the proof is thereby complete.

□

We can also observe that $\mathrm{Cap}_{\bar{\Omega}}$ possesses a monotonicity property with respect to the reference domain $\overline{\Omega}$: indeed, $\mathrm{Cap}_{\bar{\Omega}}(A)$ is nondecreasing with respect to $\overline{\Omega}$ and then behaves conversely to the classical relative capacity, which is nonincreasing.

Lemma 5.3.3. *Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^N$ be two open, bounded sets with Lipschitz boundary, such that $\Omega_1 \subseteq \Omega_2$ and let $A \subseteq \Omega_1$. Then*

$$\mathrm{Cap}_{\bar{\Omega}_1}(A) \leq \mathrm{Cap}_{\bar{\Omega}_2}(A).$$

Proof. Let $u \in \mathcal{U}_{\bar{\Omega}_2}(A)$ and let $v := u|_{\Omega_1}$. Then $v \in \mathcal{U}_{\bar{\Omega}_1}(A)$ and so

$$\mathrm{Cap}_{\bar{\Omega}_1}(A) \leq \|v\|^2_{H^1(\Omega_1)} \leq \|u\|^2_{H^1(\Omega_2)}.$$

The thesis easily comes by taking the infimum for $u \in \mathcal{U}_{\bar{\Omega}_2}(A)$. □

Another remarkable property of the notion of capacity we are studying is strong subadditivity.

Lemma 5.3.4. *Let $A_1, A_2 \subseteq \overline{\Omega}$. Then*

$$\mathrm{Cap}_{\bar\Omega}(A_1 \cup A_2) + \mathrm{Cap}_{\bar\Omega}(A_1 \cap A_2) \le \mathrm{Cap}_{\bar\Omega}(A_1) + \mathrm{Cap}_{\bar\Omega}(A_2).$$

Proof. Let $u_i \in \mathcal{U}_{\bar\Omega}(A_i)$ for $i = 1, 2$ and let $\bar u := \max\{u_1, u_2\}$ and $\underline{u} := \min\{u_1, u_2\}$. We have that $\bar u, \underline{u} \in H^1(\Omega)$ and

$$\|\bar u\|^2_{H^1(\Omega)} + \|\underline{u}\|^2_{H^1(\Omega)} = \|u_1\|^2_{H^1(\Omega)} + \|u_2\|^2_{H^1(\Omega)}.$$

In addition $\bar u \in \mathcal{U}_{\bar\Omega}(A_1 \cup A_2)$ and $\underline{u} \in \mathcal{U}_{\bar\Omega}(A_1 \cap A_2)$. Therefore

$$\mathrm{Cap}_{\bar\Omega}(A_1 \cup A_2) + \mathrm{Cap}_{\bar\Omega}(A_1 \cap A_2) \le \|u_1\|^2_{H^1(\Omega)} + \|u_2\|^2_{H^1(\Omega)}.$$

By taking the infimum for u_1, u_2 we may conclude the proof. □

We now start to focus on the particular case of capacity of a compact set. The following lemma justifies the definition of relative Sobolev capacity given in this chapter (see Definition 5.2.1). Here the Lipschitz regularity of $\partial\Omega$ does enter: indeed it ensures $C^\infty(\overline{\Omega})$ to be dense in $H^1(\Omega)$.

Lemma 5.3.5. *Let $K \subseteq \overline{\Omega}$ be a compact set. Then*

$$\inf\left\{\|u\|^2_{H^1(\Omega)} : u \in \mathcal{U}_{\bar\Omega}(K)\right\} = \inf\left\{\|u\|^2_{H^1(\Omega)} : u \in C^\infty(\overline{\Omega}),\ u - 1 \in C^\infty_c(\overline{\Omega} \setminus K)\right\}.$$

Proof. Let us consider $u \in \mathcal{U}_{\overline{\Omega}}(K)$, that is $u \in H^1(\Omega)$ and $u \ge 1$ on $O \cap \overline{\Omega}$, for some open $O \subseteq \mathbb{R}^N$ such that $K \subseteq O \cap \overline{\Omega}$. Moreover, let $O' \subseteq O$ be open and such that $K \subseteq O' \cap \overline{\Omega}$ and $\mathrm{dist}(\partial O, \partial O') > 0$. Let us consider a cut-off function $\eta \in C^\infty_c(O)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on O'. Also, let $v = \min\{u, 1\}$ and $(v_n)_n \subseteq C^\infty(\overline{\Omega})$ be such that $v_n \to v$ in $H^1(\Omega)$, as $n \to \infty$. Let us now define

$$u_n := \eta v + (1 - \eta) v_n.$$

We can observe that $u_n - 1 \in C^\infty_c(\overline{\Omega} \setminus K)$ and that $u_n \to v$ in $H^1(\Omega)$, as $n \to \infty$. Therefore

$$\begin{aligned}\inf\left\{\|u\|^2_{H^1(\Omega)} : u \in C^\infty(\overline{\Omega}),\ u - 1 \in C^\infty_c(\overline{\Omega} \setminus K)\right\} &\le \|u_n\|^2_{H^1(\Omega)} \\ &\le \|v\|^2_{H^1(\Omega)} + o(1) \\ &\le \|u\|^2_{H^1(\Omega)} + o(1),\end{aligned}$$

as $n \to \infty$. By letting $n \to \infty$ and taking the infimum among all functions $u \in \mathcal{U}_{\bar\Omega}(K)$ we prove that

$$\inf\left\{\|u\|^2_{H^1(\Omega)} : u \in C^\infty(\overline{\Omega}),\ u - 1 \in C^\infty_c(\overline{\Omega} \setminus K)\right\} \le \inf\left\{\|u\|^2_{H^1(\Omega)} : u \in \mathcal{U}_{\bar\Omega}(K)\right\}.$$

The converse is trivial, since

$$\left\{u \in C^\infty(\overline{\Omega}) \colon u - 1 \in C^\infty_c(\overline{\Omega} \setminus K)\right\} \subseteq \mathcal{U}_{\bar\Omega}(K).$$

□

In view of the previous result, we may now justify the definition of relative Sobolev capacity, that we gave in Definition 5.2.1. Indeed, by density of $C_c^\infty(\overline{\Omega}\setminus K)$ in $H^1_{0,K}(\Omega)$, we have that

$$\inf\left\{\|u\|^2_{H^1(\Omega)} : u\in C^\infty(\overline{\Omega}),\ u-1\in C_c^\infty(\overline{\Omega}\setminus K)\right\}$$
$$= \inf\left\{\|u\|^2_{H^1(\Omega)} : u\in H^1(\Omega),\ u-1\in H^1_{0,K}(\Omega)\right\}.$$

Given this brief and rather general introduction, we now focus on the capacity (or f-capacity) of compact sets. We start by mentioning the existence of a capacitary potential, that is to say a function that achieves the infimum in the definition of capacity. We omit the proof since it follows the classical one.

Proposition 5.3.6 (Capacity is Achieved)**.** *Let $K\subseteq\overline{\Omega}$ be compact, $f\in H^1(\Omega)$ and $c\in L^\infty(\Omega)$ satisfying (5.2.1). The f-capacity of K, as introduced in Definition 5.2.4, is uniquely achieved, i.e. there exists a unique $V_{K,f,c}\in H^1(\Omega)$ which satisfies*

$$V_{K,f,c}-f\in H^1_{0,K}(\Omega) \qquad \text{and} \qquad \mathrm{Cap}_{\bar\Omega,c}(K,f)=q(V_{K,f,c}).$$

Since in the following the function c is fixed, for the sake of brevity we will write

$$V_{K,f}:=V_{K,f,c}$$

omitting the dependence on c in the notation. We observe that $V_{K,f}$ satisfies

$$\begin{cases} -\Delta V_{K,f}+cV_{K,f}=0, & \text{in } \Omega\setminus K,\\ \dfrac{\partial V_{K,f}}{\partial \boldsymbol{\nu}}=0, & \text{on } \partial\Omega\setminus K,\\ V_{K,f}=f, & \text{on } K,\end{cases}$$

in a weak sense, that is $V_{K,f}-f\in H^1_{0,K}(\Omega)$ and

$$q(V_{K,f},\varphi)=\int_\Omega(\nabla V_{K,f}\cdot\nabla\varphi+cV_{K,f}\varphi)\,\mathrm{d}x=0 \quad \text{for all } \varphi\in H^1_{0,K}(\Omega). \tag{5.3.2}$$

Remark 5.3.7. In the particular case $c,f\equiv 1$ (as in Definition 5.2.1), we have that the potential $V_K:=V_{K,1}\in H^1(\Omega)$ satisfies

$$V_K-1\in H^1_{0,K}(\Omega) \qquad \text{and} \qquad \int_\Omega(\nabla V_K\cdot\nabla\varphi+V_K\varphi)\,\mathrm{d}x=0 \quad \text{for all } \varphi\in H^1_{0,K}(\Omega). \tag{5.3.3}$$

It is easy to verify that $V_K^-,(V_K-1)^+\in H^1_{0,K}(\Omega)$, so that we can choose $\varphi=V_K^-$ and $\varphi=(V_K-1)^+$ as test functions in the above equation, thus obtaining that $V_K^-\equiv 0$ and $(V_K-1)^+\equiv 0$, i.e.

$$0\le V_K(x)\le 1 \quad \text{for a.e. } x\in\Omega. \tag{5.3.4}$$

The following proposition asserts that the Sobolev spaces $H^1(\Omega)$ and $H^1_{0,K}(\Omega)$ coincide if and only if the set K has zero capacity, and draws conclusions on the eigenvalues of (5.2.3).

Proposition 5.3.8. *Let $K \subseteq \overline{\Omega}$ be compact. The following three assertions are equivalent:*

(i) $\mathrm{Cap}_{\bar{\Omega}}(K) = 0$*;*

(ii) $H^1(\Omega) = H^1_{0,K}(\Omega)$*;*

(iii) $\lambda_n(\Omega; K) = \lambda_n$ *for every* $n \in \mathbb{N}_*$*.*

Proof. In order to prove that (i) implies (ii) it is sufficient to prove that $H^1(\Omega) \subseteq H^1_{0,K}(\Omega)$ since the converse is trivial. We actually prove that $C^\infty(\overline{\Omega}) \subseteq H^1_{0,K}(\Omega)$ and the claim follows by density. By assumption (i), there exists $\{u_n\}_{n\geq 1} \subset H^1(\Omega)$ such that $u_n - 1 \in H^1_{0,K}(\Omega)$ for every $n \in \mathbb{N}_*$ and $\|u_n\|_{H^1(\Omega)} \to 0$ as $n \to \infty$. Let $u \in C^\infty(\overline{\Omega})$ and let us consider the sequence $\{u(1-u_n)\}_{n\geq 1} \subset H^1_{0,K}(\Omega)$. We claim that $u(1-u_n) \to u$ in $H^1(\Omega)$. Indeed

$$\begin{aligned}\|u - u(1-u_n)\|^2_{H^1(\Omega)} &= \|uu_n\|^2_{H^1(\Omega)} \\ &\leq 2\int_\Omega \left(u^2\,|\nabla u_n|^2 + u_n^2\,|\nabla u|^2\right) \mathrm{d}x + \int_\Omega u^2 u_n^2 \,\mathrm{d}x \\ &\leq 4\max\{\|u\|^2_{L^\infty(\Omega)}, \|\nabla u\|^2_{L^\infty(\Omega)}\}\, \|u_n\|^2_{H^1(\Omega)} \to 0,\end{aligned}$$

as $n \to \infty$.

We now prove that (ii) implies (i). Let us consider the equation (5.3.3) solved by V_K. Since $H^1_{0,K}(\Omega) = H^1(\Omega)$, we can choose $\varphi = V_K$ in (5.3.3) and then reach the conclusion.

Finally, let us show that (ii) is equivalent to (iii). The fact that (ii) implies (iii) follows from the min-max characterization (5.2.6). Conversely, suppose that (iii) holds, i.e. $\lambda_n(\Omega; K) = \lambda_n$ for every $n \in \mathbb{N}_*$. Then for every $n \in \mathbb{N}_*$ there exists an eigenfunction belonging to $H^1_{0,K}(\Omega)$ associated to λ_n. By the Spectral Theorem, there exists an orthonormal basis of $H^1(\Omega)$ made of $H^1_{0,K}$-functions, which implies that property (ii) holds. □

Remark 5.3.9. An inspection of the proof of Proposition 5.3.8 shows that (ii) actually implies that $\mathrm{Cap}_{\bar{\Omega},c}(K, f) = 0$ for all $f \in H^1(\Omega)$, and so

$$\mathrm{Cap}_{\bar{\Omega}}(K) = 0 \quad \text{if and only if} \quad \mathrm{Cap}_{\bar{\Omega},c}(K, f) = 0 \quad \text{for all } f \in H^1(\Omega).$$

Moreover, for any $f \in H^1(\Omega)$, we trivially have that $\mathrm{Cap}_{\bar{\Omega},c}(K, f) = 0$ if and only if $V_{K,f} = 0$.

Example 5.3.10 (Capacity of a Point)**.** Let $x_0 \in \overline{\Omega}$, then $\mathrm{Cap}_{\bar{\Omega}}(\{x_0\}) = 0$.

Proof. If $N \geq 3$, let $v_n \in C^\infty(\overline{\Omega})$ be such that $v_n(x) = 1$ if $x \in B(x_0, \frac{1}{n}) \cap \overline{\Omega}$, $v_n(x) = 0$ if $x \in \overline{\Omega} \setminus B(x_0, \frac{2}{n})$, $0 \leq v_n(x) \leq 1$ and $|\nabla v_n(x)| \leq 2n$ for all $x \in \overline{\Omega}$. It is easy to prove that $q(v_n) \to 0$, as $n \to \infty$ thus concluding the proof for $N \geq 3$. If $N = 2$, we can instead consider $v_n \in H^1(\Omega)$ defined as $v_n(x) = 1$ if $x \in B(x_0, \frac{1}{n}) \cap \overline{\Omega}$, $v_n(x) = 0$ if $x \in \overline{\Omega} \setminus B(x_0, \frac{1}{\sqrt{n}})$, $v_n(x) = (\log n)^{-1}(-\log n - 2\log|x - x_0|)$ if $x \in \overline{\Omega} \cap (B(x_0, \frac{1}{\sqrt{n}}) \setminus B(x_0, \frac{1}{n}))$. It is easy to prove that $q(v_n) \to 0$, as $n \to \infty$ thus concluding the proof for $N = 2$. □

Remark 5.3.11. Let $\{K_\varepsilon\}_{\varepsilon>0}, K \subseteq \overline{\Omega}$ be compact sets such that K_ε is concentrating at K as $\varepsilon \to 0$. Then, for any $\varphi \in C_c^\infty(\overline{\Omega} \setminus K)$, there exists $\varepsilon_\varphi > 0$ such that $\varphi \in C_c^\infty(\overline{\Omega} \setminus K_\varepsilon)$ for all $\varepsilon < \varepsilon_\varphi$. Moreover $\bigcap\limits_{\varepsilon>0} K_\varepsilon \subseteq K$.

Example 5.3.12. An example of concentrating sets is a family of compact sets decreasing as $\varepsilon \to 0$. Precisely, let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact subsets of $\overline{\Omega}$ such that $K_{\varepsilon_2} \subseteq K_{\varepsilon_1}$ for any $\varepsilon_2 \leq \varepsilon_1$ and let $K \subseteq \overline{\Omega}$ be a compact set such that $K = \cap_{\varepsilon>0} K_\varepsilon$. Then, arguing by contradiction, thanks to Bolzano-Weierstrass Theorem in $\mathbb{R}^N$, it is easy to prove that K_ε is concentrating at K.

With the next proposition we emphasize what is the relation between the notion of concentration of sets and that of convergence of capacities: it turns out that convergence holds if the limit set has zero capacity.

Proposition 5.3.13. *Let $K \subseteq \overline{\Omega}$ be a compact set and let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact subsets of $\overline{\Omega}$ concentrating at K. If $\mathrm{Cap}_{\overline{\Omega}}(K) = 0$ then*

$$V_{K_\varepsilon,f} \to V_{K,f} \quad \textit{in } H^1(\Omega) \quad \textit{and} \quad \mathrm{Cap}_{\overline{\Omega},c}(K_\varepsilon, f) \to \mathrm{Cap}_{\overline{\Omega},c}(K, f) \quad \textit{as } \varepsilon \to 0$$

for all $f \in H^1(\Omega)$ and all $c \in L^\infty(\Omega)$ satisfying (5.2.1). This result holds true, in particular, for the Sobolev capacity (see Definition 5.2.1) and its potentials, corresponding to the case in which $c \equiv 1$.

Proof. Since $q(V_{K_\varepsilon,f}) \leq q(f)$ for all $\varepsilon > 0$, then $\{V_{K_\varepsilon,f}\}_\varepsilon$ is bounded in $H^1(\Omega)$ and so there exists $W \in H^1(\Omega)$ such that, along a sequence $\varepsilon_n \to 0$,

$$V_{K_{\varepsilon_n},f} \rightharpoonup W \quad \text{weakly in } H^1(\Omega) \text{ as } n \to \infty,$$

that is

$$\int_\Omega (\nabla V_{K_{\varepsilon_n},f} \cdot \nabla\varphi + cV_{K_{\varepsilon_n},f}\varphi)\,\mathrm{d}x \to \int_\Omega (\nabla W \cdot \nabla\varphi + cW\varphi)\,\mathrm{d}x \quad \text{for all } \varphi \in H^1(\Omega). \tag{5.3.5}$$

Therefore, taking into account Remark 5.3.11 and the equation solved by $V_{K_\varepsilon,f}$ (5.3.2), we have that

$$\int_\Omega (\nabla W \cdot \nabla\varphi + cW\varphi)\,\mathrm{d}x = 0 \tag{5.3.6}$$

for all $\varphi \in C_c^\infty(\overline{\Omega} \setminus K)$ and then, by density, for all $\varphi \in H^1_{0,K}(\Omega)$. Moreover, taking $\varphi = V_{K,f} - f$ (respectively $\varphi = V_{K_\varepsilon,f} - f$) in the equation (5.3.2) for $V_{K,f}$ (respectively $V_{K_\varepsilon,f}$), we obtain

$$\mathrm{Cap}_{\overline{\Omega},c}(K, f) = \int_\Omega (\nabla V_{K,f} \cdot \nabla f + cV_{K,f}f)\,\mathrm{d}x, \tag{5.3.7}$$

respectively

$$\mathrm{Cap}_{\bar{\Omega},c}(K_\varepsilon, f) = \int_\Omega (\nabla V_{K_\varepsilon,f} \cdot \nabla f + cV_{K_\varepsilon,f} f)\,\mathrm{d}x. \tag{5.3.8}$$

From Proposition 5.3.8, we have that $H^1_{0,K}(\Omega) = H^1(\Omega)$ and then (5.3.6) yields $W = V_{K,f} = 0$; on the other hand, from (5.3.8) and (5.3.7) it follows that

$$\mathrm{Cap}_{\bar{\Omega}}(K_{\varepsilon_n}, f) \to \mathrm{Cap}_{\bar{\Omega},c}(K, f) = 0 \quad \text{as } n \to \infty.$$

Urysohn's Subsequence Principle concludes the proof. □

The following lemma is a fundamental step in the proof of our main results. It states that, when a sequence of sets is concentrating, as $\varepsilon \to 0$, at a zero capacity set, the squared $L^2(\Omega)$-norm of the associated capacitary potentials is negligible, as $\varepsilon \to 0$, with respect to the capacity.

Lemma 5.3.14. *Let $K \subseteq \overline{\Omega}$ be compact and let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact subsets of $\overline{\Omega}$ concentrating at K. If $\mathrm{Cap}_{\bar{\Omega}}(K) = 0$, then*

$$\int_\Omega |V_{K_\varepsilon,f}|^2 \,\mathrm{d}x = o(\mathrm{Cap}_{\bar{\Omega},c}(K_\varepsilon, f)) \quad \text{as } \varepsilon \to 0$$

for all $f \in H^1(\Omega)$ and all $c \in L^\infty(\Omega)$ satisfying (5.2.1).

Proof. Assume by contradiction that, for a certain $f \in H^1(\Omega)$, there exists $\varepsilon_n \to 0$ and $C > 0$ such that

$$\int_\Omega |V_{K_{\varepsilon_n},f}|^2 \,\mathrm{d}x \geq \frac{1}{C}\,\mathrm{Cap}_{\bar{\Omega},c}(K_{\varepsilon_n}, f).$$

Let

$$W_n := \frac{V_{K_{\varepsilon_n},f}}{\|V_{K_{\varepsilon_n},f}\|_{L^2(\Omega)}}.$$

Then $\|W_n\|_{L^2(\Omega)} = 1$ and

$$\|\nabla W_n\|^2_{L^2(\Omega)} + \int_\Omega c W_n^2 \,\mathrm{d}x = \frac{\mathrm{Cap}_{\bar{\Omega},c}(K_{\varepsilon_n}, f)}{\|V_{K_{\varepsilon_n},f}\|^2_{L^2(\Omega)}} \leq C.$$

Hence $\{W_n\}_n$ is bounded in $H^1(\Omega)$ and so there exists $W \in H^1(\Omega)$ such that $W_n \rightharpoonup W$ weakly in $H^1(\Omega)$, up to a subsequence, as $n \to \infty$. By compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ we have that $\|W\|_{L^2(\Omega)} = 1$. Using Remark 5.3.11, we can pass to the limit in the equation satisfied by W_n and then obtain

$$\int_\Omega (\nabla W \cdot \nabla \varphi + cW\varphi)\,\mathrm{d}x = 0 \quad \text{for all } \varphi \in C_c^\infty(\overline{\Omega} \setminus K).$$

On the other hand, since $\mathrm{Cap}_{\bar{\Omega}}(K) = 0$, in view of Proposition 5.3.8 $C_c^\infty(\overline{\Omega} \setminus K)$ is dense in $H^1(\Omega)$ and so $W = 0$, thus a contradiction arises. □

We are now going to prove that the term $\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K,\psi_\gamma)$, appearing in the expansion stated in Corollary 5.2.13, is nonzero whenever $K \subseteq \partial\mathbb{R}^N_+$ is a compact set such that $\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K) > 0$; to this aim we prove a more general lemma concerning the standard (Newtonian) capacity of a set, whose definition we recall below. For any open set $U \subseteq \mathbb{R}^N$, we denote by $\mathcal{D}^{1,2}(U)$ the completion of $C_c^\infty(U)$ with respect to the $L^2(U)$-norm of the gradient.

Definition 5.3.15. Let $K \subseteq \mathbb{R}^N$ be a compact set and let $\eta_K \in C_c^\infty(\mathbb{R}^N)$ be such that $\eta_K = 1$ in a neighborhood of K. If $f \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$, the following quantity

$$\mathrm{cap}_{\mathbb{R}^N}(K,f) = \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x : u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ u - f\eta_K \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus K)\right\}$$

is called the f-*capacity* of K. For $f = 1$, $\mathrm{cap}_{\mathbb{R}^N}(K) := \mathrm{cap}_{\mathbb{R}^N}(K,1)$ is called the *capacity* of K (as already introduced in Remark 5.2.11).

Lemma 5.3.16. *Let $K \subseteq \mathbb{R}^N$ be a compact set such that $\mathrm{cap}_{\mathbb{R}^N}(K) > 0$. Let $f \in C^\infty(\mathbb{R}^N)$ and let $Z_f := \{x \in \mathbb{R}^N : f(x) = 0\}$. If $\mathrm{cap}_{\mathbb{R}^N}(Z_f \cap K) < \mathrm{cap}_{\mathbb{R}^N}(K)$, then*

$$\mathrm{cap}_{\mathbb{R}^N}(K,f) > 0.$$

Proof. In this proof we make use of some properties of the classical Newtonian capacity of a set and we refer to [MZ97, Chapter 2] for the details. Let us consider a sequence of bounded open sets $\mathcal{U}_n \subseteq \mathbb{R}^N$ such that

$$Z_f \cap K \subseteq \mathcal{U}_{n+1} \subseteq \mathcal{U}_n, \quad \text{for all } n \text{ and} \quad \bigcap_{n\ge1} \overline{\mathcal{U}_n} = Z_f \cap K.$$

Let $K_n := K \setminus \mathcal{U}_n$. Since $K \subseteq K_n \cup \overline{\mathcal{U}_n}$, by subadditivity and monotonicity of the capacity

$$\mathrm{cap}_{\mathbb{R}^N}(K_n) \ge \mathrm{cap}_{\mathbb{R}^N}(K) - \mathrm{cap}_{\mathbb{R}^N}(\overline{\mathcal{U}_n}).$$

Moreover, since $\cap_{n\ge1}\overline{\mathcal{U}_n} = Z_f \cap K$, then $\mathrm{cap}_{\mathbb{R}^N}(Z_f \cap K) = \lim_{n\to\infty} \mathrm{cap}_{\mathbb{R}^N}(\overline{\mathcal{U}_n})$, and so

$$\mathrm{cap}_{\mathbb{R}^N}(K_n) > 0 \tag{5.3.9}$$

for large n, by the assumption $\mathrm{cap}_{\mathbb{R}^N}(K) - \mathrm{cap}_{\mathbb{R}^N}(Z_f \cap K) > 0$. Now we claim that

$$\mathrm{cap}_{\mathbb{R}^N}(K,|f|) > 0. \tag{5.3.10}$$

Let us fix a sufficiently large n in order for (5.3.9) to hold and let us set

$$c_n := \frac{1}{2}\inf_{K_n}|f| = \frac{1}{2}\min_{K_n}|f| > 0.$$

By definition of K_n we have $|f| \ge 2c_n > c_n$ on K_n and therefore, by continuity, $|f| > c_n$ in an open neighborhood of K_n. Let $\eta_K \in C_c^\infty(\mathbb{R}^N)$ be such that $\eta_K = 1$ in a neighborhood of K and let $u_n \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ be an arbitrary function such that $u_n - \eta_K|f| \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus K_n)$. We define

$$v_n := \min\{1, u_n/c_n\} \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

We have that $v_n - \eta_{K_n} \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus K_n)$, where $\eta_{K_n} \in C_c^\infty(\mathbb{R}^N)$ is equal to 1 in a neighborhood of K_n. Therefore v_n is an admissible competitor for $\mathrm{cap}_{\mathbb{R}^N}(K_n)$ and also, by truncation, the energy of v_n is lower than the energy of u_n/c_n. Hence

$$\mathrm{cap}_{\mathbb{R}^N}(K_n) \le \int_{\mathbb{R}^N} |\nabla v_n|^2 \,\mathrm{d}x \le \int_{\mathbb{R}^N} \frac{|\nabla u_n|}{c_n^2} \,\mathrm{d}x.$$

By arbitrariness of u_n, we have that $\mathrm{cap}_{\mathbb{R}^N}(K_n, |f|) \ge c_n^2 \,\mathrm{cap}_{\mathbb{R}^N}(K_n) > 0$. Moreover, by monotonicity, $\mathrm{cap}_{\mathbb{R}^N}(K, |f|) \ge \mathrm{cap}_{\mathbb{R}^N}(K_n, |f|) > 0$ and so (5.3.10) is proved. Finally, we claim that

$$\mathrm{cap}_{\mathbb{R}^N}(K, f) \ge \mathrm{cap}_{\mathbb{R}^N}(K, |f|). \tag{5.3.11}$$

Indeed, if $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ is such that $\xi - \eta_K f \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus K)$, then $|\xi| - \eta_K|f| \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus K)$. Hence

$$\mathrm{cap}_{\mathbb{R}^N}(K, |f|) \le \int_{\mathbb{R}^N} |\nabla|\xi||^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla \xi|^2 \,\mathrm{d}x$$

for all $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $\xi - \eta_K f \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus K)$, which implies (5.3.11). Combining (5.3.10) and (5.3.11) we can conclude the proof. □

As an application of the previous lemma, we obtain the following result.

Proposition 5.3.17. *Let $K \subseteq \partial\mathbb{R}^N_+$ be a compact set such that $\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K) > 0$ and let ψ_γ be as in (5.2.17). Then*

$$\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K, \psi_\gamma) > 0.$$

Proof. Let $Z_{\psi_\gamma} = \{x \in \mathbb{R}^N : \psi_\gamma(x) = 0\}$ as in the statement of Lemma 5.3.16. We notice that $\mathrm{cap}_{\mathbb{R}^N}(K) = 2\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K) > 0$ (see Remark 5.2.11), so that the first assumption of Lemma 5.3.16 holds. Concerning the second assumption, we have that $\mathrm{cap}_{\mathbb{R}^N}(Z_{\psi_\gamma} \cap K) = 2\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(Z_{\psi_\gamma} \cap K) = 0$, since the set $Z_{\psi_\gamma} \cap K$ is $(N-2)$-dimensional, in view of Remark 5.2.9, and $(N-2)$-dimensional sets have zero capacity in $\mathbb{R}^N$, see e.g. [MZ97, Theorem 2.52]. Then Lemma 5.3.16 provides $\mathrm{cap}_{\mathbb{R}^N}(K, \psi_\gamma) > 0$ and the proof follows by applying again Remark 5.2.11. □

We conclude this section with the following lower bound of $\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K)$ in terms of its $N-1$ dimensional Lebesgue measure, in case $N \ge 3$.

Proposition 5.3.18. *Let $N \ge 3$ and $K \subseteq \partial\mathbb{R}^N_+$ be compact. Then there exists a constant $C > 0$ (only depending on N) such that*

$$(|K|_{N-1})^{\frac{N-2}{N-1}} \le C \,\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K),$$

where $|\cdot|_{N-1}$ denotes the $N-1$ dimensional Lebesgue measure.

Proof. By definition of $\mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+})$ and $\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K)$, for every $\varepsilon > 0$ there exists $u \in C^\infty_c(\overline{\mathbb{R}^N_+})$ such that $u = 1$ in an open neighborhood U of K and

$$\int_{\mathbb{R}^N_+} |\nabla u|^2 \,\mathrm{d}x \leq \mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K) + \varepsilon.$$

On the other hand

$$|K|_{N-1} \leq |U|_{N-1} = \int_U |u|^{\frac{2(N-1)}{N-2}} \,\mathrm{d}S \leq \int_{\mathbb{R}^{N-1}} |u|^{\frac{2(N-1)}{N-2}} \,\mathrm{d}S.$$

By combining the two previous inequalities with the embedding

$$\mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+}) \hookrightarrow L^{\frac{2(N-1)}{N-2}}(\mathbb{R}^{N-1})$$

we can conclude the proof. □

5.4 Proof of the asymptotic expansion

The section is organized as follows: in Subsection 5.4.1 we prove the continuity of eigenvalues $\lambda_n(\Omega; K)$ with respect to $\mathrm{Cap}_{\bar\Omega}(K)$, i.e. Theorem 5.2.2. In Subsection 5.4.2 we prove our first main result Theorem 5.2.5. In Subsection 5.4.3 we prove our second main result Theorem 5.2.12 and finally, in Subsection 5.4.4, we prove Theorem 5.2.14.

5.4.1 Continuity of the eigenvalues with respect to the capacity

The aim of this section is to prove continuity of the eigenvalues $\lambda_n(\Omega; K)$, in the limit as $\mathrm{Cap}_{\bar\Omega}(K) \to 0$.

Proof of Theorem 5.2.2. If $\mathrm{Cap}_{\bar\Omega}(K) = 0$ the conclusion follows obviously from Proposition 5.3.8. Let us assume that $\mathrm{Cap}_{\bar\Omega}(K) > 0$. Then, by definition of $\mathrm{Cap}_{\bar\Omega}(K)$, there exists $v \in C^\infty(\overline{\Omega})$ such that $v - 1 \in C^\infty_{\mathrm{c}}(\overline{\Omega} \setminus K)$ and $\|v\|^2_{H^1(\Omega)} \leq 2\,\mathrm{Cap}_{\bar\Omega}(K)$. Letting $w = (1 - (1 - v)^+)^+$, we have that $w \in W^{1,\infty}(\Omega)$, $0 \leq w \leq 1$ a.e. in Ω, $w - 1 \in H^1_{0,K}(\Omega)$, and $\|w\|^2_{H^1(\Omega)} \leq \|v\|^2_{H^1(\Omega)} \leq 2\,\mathrm{Cap}_{\bar\Omega}(K)$.

Let $\varphi_1, \dots, \varphi_n$ be the eigenfunctions corresponding to $\lambda_1, \dots, \lambda_n$ and let $\Phi_i := \varphi_i(1 - w)$, $i = 1, \dots, n$. It's easy to prove that $\Phi_i \in H^1_{0,K}(\Omega)$ for all $i = 1, \dots, n$. Let us consider the linear subspace of $H^1_{0,K}(\Omega)$

$$E_n := \mathrm{span}\{\Phi_1, \dots, \Phi_n\}.$$

We claim that, if $\mathrm{Cap}_{\bar\Omega}(K)$ is sufficiently small, $\{\Phi_i\}_{i=1}^n$ is linearly independent, thus implying that $\dim E_n = n$. In order to compute $q(\Phi_i, \Phi_j)$, we test the equation satisfied by φ_i with $\varphi_j(1 - w)^2$. It follows that

$$\int_\Omega [(1-w)^2 \nabla\varphi_i \cdot \nabla\varphi_j + c(1-w)^2 \varphi_i\varphi_j] \,\mathrm{d}x = \int_\Omega [\lambda_i (1-w)^2 \varphi_i\varphi_j + 2(1-w)\varphi_j \nabla\varphi_i \cdot \nabla w] \,\mathrm{d}x.$$

Thanks to the previous identity, we are able to compute

$$q(\Phi_i,\Phi_j)=\int_\Omega[\varphi_j(1-w)\nabla\varphi_i\cdot\nabla w-\varphi_i(1-w)\nabla\varphi_j\cdot\nabla w+\varphi_i\varphi_j\,|\nabla w|^2+\lambda_i\varphi_i\varphi_j(1-w)^2]\,\mathrm{d}x.$$

From classical elliptic regularity theory (see e.g. [Sta58, Proposition 5.3]) it is well-known that $\varphi_i\in L^\infty(\Omega)$. Then, thanks also to Hölder inequality and (5.2.1), we have that

$$|q(\Phi_i,\Phi_j)-\delta_{ij}\lambda_i|\le C_1[(\mathrm{Cap}_{\bar\Omega}(K))^{1/2}+\mathrm{Cap}_{\bar\Omega}(K)],$$

for a certain $C_1>0$ (depending only on $\|\varphi_i\|_{L^\infty(\Omega)}$ and λ_i, $i=1,\dots,n$), where δ_{ij} is the *Kronecker's Delta*. The above inequality implies that

$$q(\Phi_i,\Phi_j)=\delta_{ij}\lambda_i+O((\mathrm{Cap}_{\bar\Omega}(K))^{1/2})\quad\text{as }\mathrm{Cap}_{\bar\Omega}(K)\to 0,$$

hence there exists $\delta>0$ such that, if $\mathrm{Cap}_{\bar\Omega}(K)<\delta$, then $\Phi_1,\dots,\Phi_n$ are linearly independent. Let us now compute the L^2 scalar products

$$\int_\Omega\Phi_i\Phi_j\,\mathrm{d}x=\int_\Omega\varphi_i\varphi_j(1-w)^2\,\mathrm{d}x=\delta_{ij}-2\int_\Omega\varphi_i\varphi_j w\,\mathrm{d}x+\int_\Omega\varphi_i\varphi_j w^2\,\mathrm{d}x.$$

Arguing as before, by Hölder inequality we obtain that

$$\begin{aligned}\left|\int_\Omega\Phi_i\Phi_j\,\mathrm{d}x-\delta_{ij}\right|&\le 2\sqrt{2}\,\|\varphi_i\varphi_j\|_{L^2(\Omega)}\,(\mathrm{Cap}_{\bar\Omega}(K))^{1/2}+2\,\|\varphi_i\varphi_j\|_{L^\infty(\Omega)}\,\mathrm{Cap}_{\bar\Omega}(K)\\&\le C_2[(\mathrm{Cap}_{\bar\Omega}(K))^{1/2}+\mathrm{Cap}_{\bar\Omega}(K)],\end{aligned}$$

for a certain $C_2>0$ (depending only on $\|\varphi_i\|_{L^\infty(\Omega)}$, $i=1,\dots,n$), i.e.

$$\int_\Omega\Phi_i\Phi_j\,\mathrm{d}x=\delta_{ij}+O((\mathrm{Cap}_{\bar\Omega}(K))^{1/2})\quad\text{as }\mathrm{Cap}_{\bar\Omega}(K)\to 0.$$

Now, from the min-max characterization (5.2.6), we have that

$$\begin{aligned}\lambda_n(\Omega;K)&\le\max_{\substack{\alpha_1,\dots,\alpha_n\in\mathbb{R}\\\sum_{i=1}^n\alpha_i^2=1}}\frac{q\left(\sum_{i=1}^n\alpha_i\Phi_i\right)}{\sum_{i,j=1}^n\alpha_i\alpha_j\int_\Omega\Phi_i\Phi_j\,\mathrm{d}x}\\&=\max_{\substack{\alpha_1,\dots,\alpha_n\in\mathbb{R}\\\sum_{i=1}^n\alpha_i^2=1}}\frac{\sum_{i,j=1}^n\alpha_i\alpha_j q(\Phi_i,\Phi_j)}{\sum_{i,j=1}^n\alpha_i\alpha_j(\delta_{ij}+O((\mathrm{Cap}_{\bar\Omega}(K))^{1/2}))}\\&=\max_{\substack{\alpha_1,\dots,\alpha_n\in\mathbb{R}\\\sum_{i=1}^n\alpha_i^2=1}}\frac{\sum_{i=1}^n\alpha_i^2\lambda_i+O((\mathrm{Cap}_{\bar\Omega}(K))^{1/2})}{1+O((\mathrm{Cap}_{\bar\Omega}(K))^{1/2})}\\&\le\frac{\lambda_n+O((\mathrm{Cap}_{\bar\Omega}(K))^{1/2})}{1+O((\mathrm{Cap}_{\bar\Omega}(K))^{1/2})}=\lambda_n+O((\mathrm{Cap}_{\bar\Omega}(K))^{1/2})\end{aligned}$$

as $\mathrm{Cap}_{\bar\Omega}(K)\to 0$. $\square$

5.4.2 Sharp asymptotics of perturbed eigenvalues

This section is devoted to the proof of Theorem 5.2.5. To this aim, let us give a preliminary lemma concerning the inverse of the operator $-\Delta+c$, when it acts on functions that vanish on a compact set.

Lemma 5.4.1. *For $K \subseteq \overline{\Omega}$ compact, let $A_K : H^1_{0,K}(\Omega) \to H^1_{0,K}(\Omega)$ be the linear bounded operator defined by*

$$q(A_K(u), v) = (u, v)_{L^2(\Omega)} \quad \textit{for every } u, v \in H^1_{0,K}(\Omega). \tag{5.4.1}$$

Then

(i) A_K is symmetric, non-negative and compact; in particular, 0 belongs to its spectrum $\sigma(A_K)$.

(ii) $\sigma(A_K) \setminus \{0\} = \{\mu_n(\Omega; K)\}_{n\in\mathbb{N}_*}$ and $\mu_n(\Omega; K) = 1/\lambda_n(\Omega; K)$ for every $n \in \mathbb{N}_*$.

(iii) For every $\mu \in \mathbb{R}$ and $u \in H^1_{0,K}(\Omega) \setminus \{0\}$ it holds

$$(\mathrm{dist}(\mu, \sigma(A_K)))^2 \leq \frac{q(A_K(u) - \mu u)}{q(u)}. \tag{5.4.2}$$

Proof. (i) A_K is clearly symmetric and non-negative; let us show that it is compact. We write $A_K = \mathcal{R} \circ \mathcal{I}$, where $\mathcal{I} : H^1_{0,K}(\Omega) \to (H^1_{0,K}(\Omega))^*$ is the compact immersion

$${}_{(H^1_{0,K}(\Omega))^*}\langle \mathcal{I}(u), v\rangle_{H^1_{0,K}(\Omega)} = \int_\Omega uv \,\mathrm{d}x \quad \text{for every } u, v \in H^1_{0,K}(\Omega),$$

and $\mathcal{R} : (H^1_{0,K}(\Omega))^* \to H^1_{0,K}(\Omega)$ is the Riesz isomorphism (on $H^1_{0,K}(\Omega)$ endowed with the scalar product q) given by

$$q(\mathcal{R}(F), v) = {}_{(H^1_{0,K}(\Omega))^*}\langle F, v\rangle_{H^1_{0,K}(\Omega)}$$

for every $v \in H^1_{0,K}(\Omega)$ and $F \in (H^1_{0,K}(\Omega))^*$. Then A_K is compact and $0 \in \sigma(A_K)$ (see for example [Hel13, Theorem 6.16]).
(ii) Again by [Hel13, Theorems 6.16], $\sigma(A_K) \setminus \{0\}$ consists of isolated eigenvalues having finite multiplicity. Being $q(\cdot)$ a norm over $H^1_{0,K}(\Omega)$, we have that $\mu \neq 0$ is an eigenvalue of A_K if and only if there exists $u \in H^1_{0,K}(\Omega)$, $u \not\equiv 0$, such that

$$q(A_K(u), v) = \mu q(u, v) \quad \text{for every } v \in H^1_{0,K}(\Omega),$$

so that $1/\mu = \lambda_n(\Omega; K)$ for some $n \in \mathbb{N}_*$.
(ii) This is a consequence of the Spectral Theorem (see for example [Hel13, Theorem 6.21 and Proposition 8.20]). □

We have now all the ingredients to give the proof of Theorem 5.2.5. It is inspired by [AFHL19, Theorem 1.4] (see also [AFN20, Theorem 1.5]).

Proof of Theorem 5.2.5. Let us recall that, by assumption, $\lambda_0 = \lambda_{n_0} = \lambda_{n_0}(\Omega;\emptyset)$ is simple and that φ_0 is an associated $L^2(\Omega)$-normalized eigenfunction. Recall also that $\lambda_\varepsilon = \lambda_{n_0}(\Omega;K_\varepsilon)$. For simplicity of notation we write $V_\varepsilon := V_{K_\varepsilon,\varphi_0}$ and $C_\varepsilon := \mathrm{Cap}_{\bar{\Omega},c}(K_\varepsilon,\varphi_0) = q(V_\varepsilon)$. Moreover we let $\psi_\varepsilon := \varphi_0 - V_\varepsilon$, that is ψ_ε is the orthogonal projection of φ_0 on $H^1_{0,K_\varepsilon}(\Omega)$ with respect to q. Indeed there holds

$$q(\psi_\varepsilon - \varphi_0, \varphi) = 0 \quad \text{for all } \varphi \in H^1_{0,K_\varepsilon}(\Omega).$$

We split the proof into three steps.

Step 1. We claim that

$$|\lambda_\varepsilon - \lambda_0| = o(C_\varepsilon^{1/2}) \quad \text{as } \varepsilon \to 0. \tag{5.4.3}$$

For any $\varphi \in H^1_{0,K_\varepsilon}(\Omega)$, being λ_0 an eigenvalue of q, we have

$$q(\psi_\varepsilon,\varphi) - \lambda_0(\psi_\varepsilon,\varphi)_{L^2(\Omega)} = q(\varphi_0,\varphi) - \lambda_0(\psi_\varepsilon,\varphi)_{L^2(\Omega)} = \lambda_0(V_\varepsilon,\varphi)_{L^2(\Omega)}. \tag{5.4.4}$$

According to the notation introduced in Lemma 5.4.1, we can rewrite (5.4.4) as

$$(\psi_\varepsilon,\varphi)_{L^2(\Omega)} = \mu_0 q(\psi_\varepsilon,\varphi) - (V_\varepsilon,\varphi)_{L^2(\Omega)}, \tag{5.4.5}$$

where $\mu_0 = \mu_{n_0}(\Omega;\emptyset) = \mu_{n_0}(\Omega;K) = 1/\lambda_0$. By (5.4.2) we have

$$\big(\mathrm{dist}(\mu_0,\sigma(A_{K_\varepsilon}))\big)^2 \le \frac{q(A_{K_\varepsilon}(\psi_\varepsilon) - \mu_0\psi_\varepsilon)}{q(\psi_\varepsilon)}. \tag{5.4.6}$$

From Proposition 5.3.13 it follows that

$$|q(\varphi_0,V_\varepsilon)| \le \sqrt{q(\varphi_0)}\sqrt{q(V_\varepsilon)} = \sqrt{\lambda_0}\sqrt{C_\varepsilon} = o(1)$$

as $\varepsilon \to 0$, so that, using the definition of ψ_ε, the denominator in the right hand side of (5.4.6) can be estimated as follows

$$q(\psi_\varepsilon) = q(\varphi_0) + C_\varepsilon - 2q(\varphi_0,V_\varepsilon) = \lambda_0 + o(1) \tag{5.4.7}$$

as $\varepsilon \to 0$. Concerning the numerator in the right hand side of (5.4.6), the definition of A_{K_ε} and relation (5.4.5) provide

$$q(A_{K_\varepsilon}(\psi_\varepsilon),\varphi) = (\psi_\varepsilon,\varphi)_{L^2(\Omega)} = \mu_0 q(\psi_\varepsilon,\varphi) - (V_\varepsilon,\varphi)_{L^2(\Omega)},$$

for every $\varphi \in H^1_{0,K_\varepsilon}$, so that, choosing $\varphi = A_{K_\varepsilon}(\psi_\varepsilon) - \mu_0\psi_\varepsilon$ in the previous identity, we arrive at

$$q(A_{K_\varepsilon}(\psi_\varepsilon) - \mu_0\psi_\varepsilon) = -(V_\varepsilon, A_{K_\varepsilon}(\psi_\varepsilon) - \mu_0\psi_\varepsilon)_{L^2(\Omega)}.$$

The Cauchy-Schwartz inequality, assumption (5.2.1) and Lemma 5.3.14, together with the previous equality, provide

$$\big(q(A_{K_\varepsilon}(\psi_\varepsilon) - \mu_0\psi_\varepsilon)\big)^{1/2} \le \frac{1}{\sqrt{c_0}}\|V_\varepsilon\|_{L^2(\Omega)} = o(C_\varepsilon^{1/2})$$

as $\varepsilon \to 0$. By combining the last inequality with (5.4.6) and (5.4.7) we see that

$$\operatorname{dist}(\mu_0, \sigma(A_{K_\varepsilon})) = o(C_\varepsilon^{1/2}) \quad \text{as } \varepsilon \to 0. \tag{5.4.8}$$

We know from Theorem 5.2.2 that $\lambda_n(\Omega; K_\varepsilon) \to \lambda_n$ as $\varepsilon \to 0$ and so, since λ_0 is assumed to be simple, also λ_ε is simple for $\varepsilon > 0$ sufficiently small. Hence, denoting $\mu_\varepsilon = \mu_{n_0}(\Omega; K_\varepsilon) = 1/\lambda_\varepsilon$, we have

$$\operatorname{dist}(\mu_0, \sigma(A_{K_\varepsilon})) = |\mu_0 - \mu_\varepsilon|$$

for $\varepsilon > 0$ small enough. Then, using (5.4.8),

$$|\lambda_0 - \lambda_\varepsilon| = \lambda_0 \lambda_\varepsilon |\mu_0 - \mu_\varepsilon| = o(C_\varepsilon^{1/2})$$

as $\varepsilon \to 0$, so that claim (5.4.3) is proved.

Let now $\Pi_\varepsilon \colon L^2(\Omega) \to L^2(\Omega)$ be the orthogonal projection onto the one-dimensional eigenspace corresponding to λ_ε, that is to say

$$\Pi_\varepsilon \psi = (\psi, \varphi_\varepsilon)_{L^2(\Omega)} \varphi_\varepsilon \quad \text{for every } \psi \in L^2(\Omega),$$

where we denoted by φ_ε a $L^2(\Omega)$-normalized eigenfunction associated to λ_ε.

Step 2. We claim that

$$q(\psi_\varepsilon - \Pi_\varepsilon \psi_\varepsilon) = o(C_\varepsilon) \quad \text{as } \varepsilon \to 0. \tag{5.4.9}$$

Let

$$\Phi_\varepsilon := \psi_\varepsilon - \Pi_\varepsilon \psi_\varepsilon \quad \text{and} \quad \xi_\varepsilon := A_{K_\varepsilon}(\Phi_\varepsilon) - \mu_\varepsilon \Phi_\varepsilon.$$

Using the fact that $\Pi_\varepsilon \psi_\varepsilon$ belongs to the eigenspace associated to λ_ε and relation (5.4.4), we have, for every $\varphi \in H^1_{0,K_\varepsilon}$,

$$\begin{aligned} q(\Phi_\varepsilon, \varphi) - \lambda_\varepsilon (\Phi_\varepsilon, \varphi)_{L^2(\Omega)} &= q(\psi_\varepsilon, \varphi) - \lambda_0 (\psi_\varepsilon, \varphi)_{L^2(\Omega)} + (\lambda_0 - \lambda_\varepsilon)(\psi_\varepsilon, \varphi)_{L^2(\Omega)} \\ &= \lambda_0 (V_\varepsilon, \varphi)_{L^2(\Omega)} + (\lambda_0 - \lambda_\varepsilon)(\psi_\varepsilon, \varphi)_{L^2(\Omega)}. \end{aligned}$$

Thanks to the previous relation, with $\varphi = \xi_\varepsilon$, and the definition of A_{K_ε}, we obtain

$$\begin{aligned} q(\xi_\varepsilon) = q(A_{K_\varepsilon}(\Phi_\varepsilon), \xi_\varepsilon) - \mu_\varepsilon q(\Phi_\varepsilon, \xi_\varepsilon) &= -\mu_\varepsilon \left[q(\Phi_\varepsilon, \xi_\varepsilon) - \lambda_\varepsilon (\Phi_\varepsilon, \xi_\varepsilon)_{L^2(\Omega)} \right] \\ &= -\frac{\lambda_0}{\lambda_\varepsilon} (V_\varepsilon, \xi_\varepsilon)_{L^2(\Omega)} - \frac{\lambda_0 - \lambda_\varepsilon}{\lambda_\varepsilon} (\psi_\varepsilon, \xi_\varepsilon)_{L^2(\Omega)}. \end{aligned} \tag{5.4.10}$$

Combining (5.4.5) and (5.4.7) we obtain that $\|\psi_\varepsilon\|_{L^2(\Omega)} = 1 + o(1)$ as $\varepsilon \to 0$. Therefore, from (5.4.10), taking into account (5.4.7), we deduce the existence of a constant C independent from ε such that

$$\sqrt{q(\xi_\varepsilon)} \leq C \left(\|V_\varepsilon\|_{L^2(\Omega)} + |\lambda_0 - \lambda_\varepsilon| \right).$$

Thus, using the definition of ξ_ε, Lemma 5.3.14 and (5.4.3), we obtain that

$$q(A_{K_\varepsilon}(\Phi_\varepsilon) - \mu_\varepsilon \Phi_\varepsilon) = o(C_\varepsilon) \quad \text{as } \varepsilon \to 0. \tag{5.4.11}$$

Let

$$N_\varepsilon = \left\{ w \in H^1_{0,K_\varepsilon} : (w, \varphi_\varepsilon)_{L^2(\Omega)} = 0 \right\}.$$

Note that, by definition, $\Phi_\varepsilon \in N_\varepsilon$. Moreover, being φ_ε an eigenfunction associated to λ_ε, from the definition of A_{K_ε} in (5.4.1) it follows that

$$A_{K_\varepsilon}(w) \in N_\varepsilon \quad \text{for every} \quad w \in N_\varepsilon.$$

In particular, the following operator

$$\tilde{A}_\varepsilon = A_{K_\varepsilon}\big|_{N_\varepsilon} : N_\varepsilon \to N_\varepsilon$$

is well defined. One can easily check that $\tilde{A}_\varepsilon$ satisfies properties (i)-(iii) in Lemma 5.4.1; moreover $\sigma(\tilde{A}_\varepsilon) = \sigma(A_{K_\varepsilon}) \setminus \{\mu_\varepsilon\}$. In particular, letting $\delta > 0$ be such that $(\mathrm{dist}(\mu_\varepsilon, \sigma(\tilde{A}_\varepsilon)))^2 \geq \delta$ for every ε small enough, estimate (5.4.11), combined with (5.4.2), provides

$$q(\psi_\varepsilon - \Pi_\varepsilon\psi_\varepsilon) = q(\Phi_\varepsilon) \leq \frac{1}{\delta}(\mathrm{dist}(\mu_\varepsilon, \sigma(\tilde{A}_\varepsilon)))^2\, q(\Phi_\varepsilon) \leq \frac{1}{\delta} q(\tilde{A}_\varepsilon(\Phi_\varepsilon) - \mu_\varepsilon\Phi_\varepsilon) = o(C_\varepsilon)$$

as $\varepsilon \to 0$, thus proving claim (5.4.9).

Step 3. We claim that

$$\lambda_\varepsilon - \lambda_0 = C_\varepsilon + o(C_\varepsilon) \quad \text{as } \varepsilon \to 0. \tag{5.4.12}$$

From the definition of ψ_ε, Lemma 5.3.14, and the previous step we deduce that

$$\|\varphi_0 - \Pi_\varepsilon\psi_\varepsilon\|_{L^2(\Omega)} \leq \|V_\varepsilon\|_{L^2(\Omega)} + \|\psi_\varepsilon - \Pi_\varepsilon\psi_\varepsilon\|_{L^2(\Omega)} = o(C_\varepsilon^{1/2}) \quad \text{as } \varepsilon \to 0,$$

which yields both

$$\|\Pi_\varepsilon\psi_\varepsilon\|_{L^2(\Omega)} = 1 + o(C_\varepsilon^{1/2}) \quad \text{as } \varepsilon \to 0 \tag{5.4.13}$$

and, consequently,

$$\|\varphi_0 - \hat{\varphi}_\varepsilon\|_{L^2(\Omega)} = o(C_\varepsilon^{1/2}) \quad \text{as } \varepsilon \to 0, \tag{5.4.14}$$

where $\hat{\varphi}_\varepsilon = \Pi_\varepsilon\psi_\varepsilon / \|\Pi_\varepsilon\psi_\varepsilon\|_{L^2(\Omega)}$. Using the fact that $\hat{\varphi}_\varepsilon$ is an eigenfunction associated to λ_ε and (5.4.5) with $\varphi = \hat{\varphi}_\varepsilon$, we obtain

$$(\lambda_\varepsilon - \lambda_0)(\psi_\varepsilon, \hat{\varphi}_\varepsilon)_{L^2(\Omega)} = \lambda_0(V_\varepsilon, \hat{\varphi}_\varepsilon)_{L^2(\Omega)}. \tag{5.4.15}$$

But actually

$$\lambda_0(V_\varepsilon, \hat{\varphi}_\varepsilon) = C_\varepsilon + o(C_\varepsilon) \quad \text{as } \varepsilon \to 0. \tag{5.4.16}$$

Indeed, since ψ_ε and V_ε are orthogonal with respect to q and φ_0 is an eigenfunction corresponding to λ_0, we have that

$$\begin{aligned} C_\varepsilon = q(V_\varepsilon) = q(V_\varepsilon, \varphi_0 - \psi_\varepsilon) = q(V_\varepsilon, \varphi_0) = \lambda_0(V_\varepsilon, \varphi_0)_{L^2(\Omega)} \\ = \lambda_0(V_\varepsilon, \hat{\varphi}_\varepsilon)_{L^2(\Omega)} + \lambda_0(V_\varepsilon, \varphi_0 - \hat{\varphi}_\varepsilon)_{L^2(\Omega)}, \end{aligned}$$

so that Lemma 5.3.14 and relation (5.4.14) allow us to prove (5.4.16). Concerning the left hand side of (5.4.15) we have, exploiting (5.4.13) and (5.4.9),

$$(\psi_\varepsilon, \hat{\varphi}_\varepsilon)_{L^2(\Omega)} = \frac{(\psi_\varepsilon - \Pi_\varepsilon\psi_\varepsilon, \Pi_\varepsilon\psi_\varepsilon)_{L^2(\Omega)} + \|\Pi_\varepsilon\psi_\varepsilon\|^2_{L^2(\Omega)}}{\|\Pi_\varepsilon\psi_\varepsilon\|_{L^2(\Omega)}} = 1 + o(C_\varepsilon^{1/2}) \quad \text{as } \varepsilon \to 0.$$

By combining the last estimate with (5.4.15) and (5.4.16), we complete the proof. □

5.4.3 Set scaling to a boundary point

Hereafter we assume $N \geq 3$. The purpose of this section is to find, in some particular cases, the explicit behavior of the function $\varepsilon \mapsto \mathrm{Cap}_{\bar{\Omega},c}(K_\varepsilon, \varphi_0)$ and therefore to give a more concrete connotation to the asymptotic expansion proved in Theorem 5.2.5. We consider a particular class of families of concentrating sets, that includes the case in which K_ε is obtained by rescaling a fixed compact set K by a factor $\varepsilon > 0$.

First, we prove Proposition 5.2.8.

Proof of Proposition 5.2.8. The proof is organized as follows: we first derive (5.2.14) for a certain diffeomorsphism in the class $\mathcal{C}$ and then we prove that it holds true for any diffeomorphism in $\mathcal{C}$. Hence we start by taking into consideration a particular diffeomorphism $\Phi_{\mathrm{AE}} \in \mathcal{C}$, first introduced in [AE97]. Let $g \in C^{1,1}(B'_{r_0})$ be, as in (5.2.10), the function that describes $\partial\Omega$ locally near the origin and let $\rho \in C_c^\infty(\mathbb{R}^{N-1})$ be such that $\operatorname{supp}\rho \subseteq B'_1$, $\rho \geq 0$ in $\mathbb{R}^{N-1}$, $\rho \not\equiv 0$ and $-\nabla\rho(y') \cdot y' \geq 0$ in $\mathbb{R}^{N-1}$. Then, for any $\delta > 0$, let

$$\rho_\delta(y') = c_\rho^{-1}\delta^{-N+1}\rho\left(\frac{y'}{\delta}\right) \quad \text{with} \quad c_\rho = \int_{\mathbb{R}^{N-1}} \rho(y')\,\mathrm{d}y',$$

be a family of mollifiers. Now, for $j = 1, \dots, N-1$ and $y_N > 0$, we let

$$u_j(y', y_N) := y_j - y_N\left(\rho_{y_N} \star \frac{\partial g}{\partial y_j}\right)(y'),$$

where $\star$ denotes the convolution product. Moreover, we define

$$\psi_j(y', y_N) := \begin{cases} u_j(y', y_N), & \text{for } y_N > 0, \\ 4u_j(y', -\frac{y_N}{2}) - 3u_j(y', -y_N), & \text{for } y_N < 0. \end{cases}$$

One can prove that $\psi_j \in C^{1,1}(B_{r_0/2})$. Finally, we let $F\colon B_{r_0/2} \to \mathbb{R}^N$ be defined as follows

$$F(y', y_N) := (\psi_1(y', y_N), \dots, \psi_{N-1}(y', y_N), y_N + g(y')).$$

Computations show that the Jacobian matrix of F on the hyperplane $\{y_N = 0\}$ is as follows

$$J_F(y', 0) = \begin{pmatrix} 1 & 0 & \cdots & 0 & -\frac{\partial g}{\partial y_1}(y') \\ 0 & 1 & \cdots & 0 & -\frac{\partial g}{\partial y_2}(y') \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\frac{\partial g}{\partial y_{N-1}}(y') \\ \frac{\partial g}{\partial y_1}(y') & \frac{\partial g}{\partial y_2}(y') & \cdots & \frac{\partial g}{\partial y_{N-1}}(y') & 1 \end{pmatrix},$$

and so $|\det J_F(0)| = 1 + |\nabla g(0)|^2 = 1$. Hence, by the inverse function theorem, F is invertible in a neighborhood of the origin: namely there exists $r_1 \in (0, r_0/2)$ such that F

is a diffeomorphism of class $C^{1,1}$ from B_{r_1} to $\mathcal{U} = F(B_{r_1})$ for some $\mathcal{U}$ open neighborhood of 0. Moreover, it is possible to choose r_1 sufficiently small so that

$$F^{-1}(\mathcal{U} \cap \Omega) = \mathbb{R}^N_+ \cap B_{r_1} = B^+_{r_1},$$
$$F^{-1}(\mathcal{U} \cap \partial\Omega) = \partial\mathbb{R}^N_+ \cap B_{r_1} = B'_{r_1},$$

which means that, near the origin, the image of Ω through F^{-1} has flat boundary (coinciding with $\partial\mathbb{R}^N_+$). In particular we have that the diffeomorphism

$$\Phi_{\mathrm{AE}} : \mathcal{U} \to B_{r_1}, \quad \Phi_{\mathrm{AE}} := F^{-1}$$

belongs to the class $\mathcal{C}$ defined in (5.2.11).

For $y \in \Phi_{\mathrm{AE}}(\mathcal{U} \cap \Omega) = B^+_{r_1}$, we let $\hat{\varphi}_0(y) := \varphi_0(\Phi^{-1}_{\mathrm{AE}}(y))$; from the equation satisfied by φ_0 in Ω, we deduce that

$$\int_{B^+_{r_1}} (A(y)\nabla\hat{\varphi}_0(y) \cdot \nabla\varphi(y) + \hat{c}(y)\hat{\varphi}_0(y)\varphi(y))\,\mathrm{d}y = \lambda_0 \int_{B^+_{r_1}} p(y)\hat{\varphi}_0(y)\varphi(y)\,\mathrm{d}y \tag{5.4.17}$$

for all $\varphi \in H^1_{0,S^+_{r_1}}(B^+_{r_1})$, where $S^+_{r_1} := \partial B_{r_1} \cap \overline{\mathbb{R}^N_+}$ and

$$\begin{aligned} A(y) &= J_{\Phi_{\mathrm{AE}}}(\Phi^{-1}_{\mathrm{AE}}(y))J_{\Phi_{\mathrm{AE}}}(\Phi^{-1}_{\mathrm{AE}}(y))^T \left|\det J_{\Phi_{\mathrm{AE}}}(\Phi^{-1}_{\mathrm{AE}}(y))\right|^{-1}, \\ \hat{c}(y) &= c(\Phi^{-1}_{\mathrm{AE}}(y)) \left|\det J_{\Phi_{\mathrm{AE}}}(\Phi^{-1}_{\mathrm{AE}}(y))\right|^{-1}, \\ p(y) &= \left|\det J_{\Phi_{\mathrm{AE}}}(\Phi^{-1}_{\mathrm{AE}}(y))\right|^{-1}. \end{aligned} \tag{5.4.18}$$

We point out that equation (5.4.17) is the weak formulation of the problem

$$\begin{cases} -\operatorname{div}(A(y)\nabla\hat{\varphi}_0(y)) + \hat{c}(y)\hat{\varphi}_0(y) = \lambda_0\, p(y)\hat{\varphi}_0(y), & \text{in } B^+_{r_1}, \\ \nabla\hat{\varphi}_0(y)A(y) \cdot \boldsymbol{\nu}(y) = 0, & \text{on } B'_{r_1}. \end{cases}$$

One can prove that A is symmetric and uniformly elliptic in $B^+_{r_1}$ (if r_1 is choosen sufficiently small); moreover, if we denote $A(y) = (a_{i,j}(y))_{i,j=1,\dots,N}$, then $a_{i,j} \in C^{0,1}(B^+_{r_1} \cup B'_{r_1})$ and

$$\begin{aligned} &a_{i,i}(y',0) = 1 + |\nabla g(y')|^2 - \left(\frac{\partial g}{\partial y_i}(y')\right)^2, && \text{for all } i = 1,\dots,N-1, \\ &a_{i,j}(y',0) = -\frac{\partial g}{\partial y_i}(y')\frac{\partial g}{\partial y_j}(y'), && \text{for all } i,j = 1,\dots,N-1,\ i \neq j, \\ &a_{i,N}(y',0) = 0, && \text{for all } i = 1,\dots,N-1, \\ &a_{N,N} = 1. \end{aligned} \tag{5.4.19}$$

Therefore, if we consider an even reflection of $\hat{\varphi}_0$ (which we still denote as $\hat{\varphi}_0$) through the hyperplane $\{x_N = 0\}$ in B_{r_1}, then it satisfies, in this ball, an elliptic equation in divergence form with Lipschitz continuous second order coefficients. More in particular $\hat{\varphi}_0$ weakly satisfies

$$-\operatorname{div}(\bar{A}(y)\nabla\hat{\varphi}_0(y)) = h(y)\hat{\varphi}_0(y) \quad \text{in } B_{r_1} \tag{5.4.20}$$

where

$$\bar{A}(y) := \begin{cases} A(y_1, \dots, y_{N-1}, y_N), & \text{if } y_N > 0, \\ QA(y_1, \dots, y_{N-1}, -y_N)Q, & \text{if } y_N < 0, \end{cases}$$

with

$$Q := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}$$

and $h \in L^\infty(B_{r_1})$. We point out that Lipschitz continuity of the coefficients of the matrix $\bar{A}$ comes from the fact that $a_{i,j} \in C^{0,1}(B^+_{r_1} \cup B'_{r_1})$ and that $a_{i,N}(y', 0) = 0$ for all $i < N$. Hence we deduce, from [Rob88], that there exists a homogeneous harmonic polynomial ψ_γ of degree $\gamma \in \mathbb{N}$ such that

$$\hat{\varphi}_0(y) = \psi_\gamma(y) + R_\gamma(y)$$

where $\|R_\gamma\|_{H^1(B_r)} = O(r^{\gamma + \frac{N}{2} + 1 - \delta})$ for some $\delta \in (0,1)$ as $r \to 0$. In particular (5.2.14) holds.

Now let $\Phi \in \mathcal{C}$. We can rewrite

$$\varphi_0(\Phi^{-1}(\varepsilon x)) = \varphi_0\left(\Phi^{-1}_{\mathrm{AE}}\left(\varepsilon G_\varepsilon(x)\right)\right), \quad \text{with } G_\varepsilon(x) := \frac{(\Phi_{\mathrm{AE}} \circ \Phi^{-1})(\varepsilon x)}{\varepsilon}.$$

Thanks to regularity properties of Φ_{AE} and Φ, we have that

$$G_\varepsilon(x) = x + |x|^2 O(\varepsilon), \quad \text{as } \varepsilon \to 0.$$

As a consequence, one can prove that

$$\varepsilon^{-\gamma}\varphi_0\left(\Phi^{-1}_{\mathrm{AE}}\left(\varepsilon G_\varepsilon(x)\right)\right) \to \psi_\gamma(x), \quad \text{in } H^1(B^+_R) \text{ as } \varepsilon \to 0,$$

for all $R > 0$, thus concluding the proof of (5.2.14).

The proofs of (5.2.15) and (5.2.16) follow from (5.2.14) by making a change of variable in the integral and taking into account that, for all $R > 0$,

$$\chi_{\varepsilon^{-1}\Phi(\Omega \cap B_{R\varepsilon})} \to \chi_{B^+_R} \quad \text{a.e. in } \mathbb{R}^N,$$

with χ_A denoting as usual the characteristic function of a set $A \subset \mathbb{R}^N$, as one can easily deduce from the fact that $\Phi^{-1}(y) = y + O(|y|^2)$ as $y \to 0$. □

In this section we consider a particular class of families of compact sets concentrating to the origin (which is assumed to belong to $\partial\Omega$), as described in Paragraph 5.2. Let us fix $\Phi \in \mathcal{C}$ with

$$\Phi : \mathcal{U}_0 \to B_{R_0} \tag{5.4.21}$$

being $\mathcal{U}_0$ an open neighborhood of 0 and $R_0 > 0$ such that $\Phi(\mathcal{U}_0 \cap \Omega) = B^+_{R_0}$ and $\Phi(\mathcal{U}_0 \cap \partial\Omega) = B'_{R_0}$. In the rest of this section, we will use the same notation as in the proof of Proposition 5.2.8 defining A and $\hat{c}$ as in (5.4.18) (with Φ instead of Φ_{AE}).

Since $A \in C^{0,1}(B^+_{R_0}, \mathcal{M}_{N\times N})$ (with $\mathcal{M}_{N\times N}$ denoting the space of $N \times N$ real matrices) and $A(0) = I_N$, it is possibile to choose $R_0 > 0$ small enough in order to have

$$\|A(x) - I_N\|_{\mathcal{M}_{N\times N}} \le \frac{1}{2} \quad \text{and} \quad \hat{c}(x) \ge \frac{c_0}{2} \quad \text{for a.e. } x \in B^+_{R_0}$$

(with $\|\cdot\|_{\mathcal{M}_{N\times N}}$ denoting the operator norm on $\mathcal{M}_{N\times N}$). With this choice of R_0, we have that

$$\begin{aligned} \int_{B^+_{R_0/\varepsilon}} A(\varepsilon x)\nabla u(x)\cdot\nabla u(x)\,\mathrm{d}x &= \int_{B^+_{R_0/\varepsilon}} (A(\varepsilon x) - I_N)\nabla u(x)\cdot\nabla u(x)\,\mathrm{d}x + \int_{B^+_{R_0/\varepsilon}} |\nabla u(x)|^2\,\mathrm{d}x \\ &\ge \frac{1}{2}\int_{B^+_{R_0/\varepsilon}} |\nabla u(x)|^2\,\mathrm{d}x \end{aligned} \tag{5.4.22}$$

and

$$\int_{B^+_{R_0/\varepsilon}} \hat{c}(\varepsilon x) u^2(x)\,\mathrm{d}x \ge \frac{c_0}{2}\int_{B^+_{R_0/\varepsilon}} u^2(x)\,\mathrm{d}x \tag{5.4.23}$$

for all $u \in H^1(B^+_{R_0/\varepsilon})$.

Let $K_\varepsilon \subseteq \overline{\Omega} \cap \mathcal{U}_0$ be a compact set for any $\varepsilon \in (0,1)$ such that (5.2.12) and (5.2.13) hold. In the following we denote

$$\tilde{K}_\varepsilon := \Phi(K_\varepsilon)/\varepsilon.$$

For any compact set $H \subseteq \mathbb{R}^N$ we define the *radius* of H as follows

$$r(H) := \max_{x\in H} |x|\,. \tag{5.4.24}$$

Remark 5.4.2. Concerning hypothesis (5.2.13), one can prove that the convergence of $\mathbb{R}^N \setminus \tilde{K}_\varepsilon$ to $\mathbb{R}^N \setminus K$ in the sense of Mosco, as $\varepsilon \to 0$, introduced in Definition 5.2.6, is equivalent to the convergence of the space $H^1_{0,\tilde{K}_\varepsilon}(B^+_R)$ to the space $H^1_{0,K}(B^+_R)$ in the sense of Mosco for all $R > r(M)$. We recall that $H^1_{0,\tilde{K}_\varepsilon}(B^+_R)$ is said to *converge to* $H^1_{0,K}(B^+_R)$ *in the sense of Mosco* if the following holds:

(1) the weak limit points (as $\varepsilon \to 0$) in $H^1(B^+_R)$ of every family of functions $\{u_\varepsilon\}_\varepsilon \subseteq H^1(B^+_R)$, such that $u_\varepsilon \in H^1_{0,\tilde{K}_\varepsilon}(B^+_R)$ for every ε, belong to $H^1_{0,K}(B^+_R)$;

(2) for every $u \in H^1_{0,K}(B^+_R)$ there exists a family $\{u_\varepsilon\}_\varepsilon \subseteq H^1(B^+_R)$ such that $u_\varepsilon \in H^1_{0,\tilde{K}_\varepsilon}(B^+_R)$ for every ε and $u_\varepsilon \to u$ in $H^1(B^+_R)$, as $\varepsilon \to 0$.

The proof of this equivalence is essentially based on the continuity of the extension operator for functions in $H^1(B_R^+)$ and of the restriction operator on B_R^+ for functions in $H^1(\mathbb{R}^N)$. Analogously, one can also prove that they are both also equivalent to the convergence of $H^1_{0,\tilde{K}_\varepsilon\cup S_R^+}(B_R^+)$ to $H^1_{0,K\cup S_R^+}(B_R^+)$ in the sense of Mosco, as $\varepsilon \to 0$. These equivalent hypotheses turn out to be more adequate for our needs in this final part of the section.

In view of Proposition 5.2.8 it is natural to consider the following rescalings of the limit eigenfunction φ_0 and of the φ_0-capacitary potential of K_ε

$$\tilde{\varphi}_\varepsilon(y) := \frac{\varphi_0(\Phi^{-1}(\varepsilon y))}{\varepsilon^\gamma} = \frac{\hat{\varphi}_0(\varepsilon y)}{\varepsilon^\gamma}, \quad \tilde{V}_\varepsilon(y) := \frac{V_{K_\varepsilon,\varphi_0}(\Phi^{-1}(\varepsilon y))}{\varepsilon^\gamma}, \quad y \in B^+_{R_0/\varepsilon}, \tag{5.4.25}$$

where $\hat{\varphi}_0(z) = \varphi_0(\Phi^{-1}(z)) \in H^1(B^+_{R_0})$. We have that $\tilde{\varphi}_\varepsilon \in H^1(B^+_{R_0/\varepsilon})$, $\tilde{V}_\varepsilon - \tilde{\varphi}_\varepsilon \in H^1_{0,\tilde{K}_\varepsilon}(B^+_{R_0/\varepsilon})$, and they satisfy

$$\int_{B^+_{R_0/\varepsilon}} (A(\varepsilon y)\nabla\tilde{\varphi}_\varepsilon(y)\cdot\nabla\varphi(y) + \varepsilon^2\hat{c}(\varepsilon y)\tilde{\varphi}_\varepsilon(y)\varphi(y))\,\mathrm{d}y = \lambda_0\varepsilon^2\int_{B^+_{R_0/\varepsilon}} p(\varepsilon y)\tilde{\varphi}_\varepsilon(y)\varphi(y)\,\mathrm{d}y, \tag{5.4.26}$$

for all $\varphi \in H^1_{0,S^+_{R_0/\varepsilon}}(B^+_{R_0/\varepsilon})$ and

$$\int_{B^+_{R_0/\varepsilon}} (A(\varepsilon y)\nabla\tilde{V}_\varepsilon(y)\cdot\nabla\varphi(y) + \varepsilon^2\hat{c}(\varepsilon y)\tilde{V}_\varepsilon(y)\varphi(y))\,\mathrm{d}y = 0, \tag{5.4.27}$$

for all $\varphi \in H^1_{0,\tilde{K}_\varepsilon\cup S^+_{R_0/\varepsilon}}(B^+_{R_0/\varepsilon})$.

The following Lemma provides a first, rough estimate of the boundary Sobolev capacity appearing in the asymptotic expansion in Theorem 5.2.5.

Lemma 5.4.3. *We have that*

$$\mathrm{Cap}_{\bar{\Omega},c}(K_\varepsilon,\varphi_0) = O(\varepsilon^{N+2\gamma-2}), \quad \text{as } \varepsilon \to 0.$$

Proof. Recall the definition of M in (5.2.12) and that of $r(M)$ in (5.4.24). Since $K_\varepsilon \subseteq \Phi^{-1}(\varepsilon\overline{B_{r(M)}})$ and $\Phi^{-1}(y) = y + O(|y|^2)$ as $|y| \to 0$, there exists $R > 0$ such that $K_\varepsilon \subset B_{R\varepsilon}$ for ε sufficiently small. Now let $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^N)$ be such that

$$0 \le \eta_\varepsilon(y) \le 1, \qquad \eta_\varepsilon(y) = \begin{cases} 0, & \text{for } y \in \mathbb{R}^N \setminus B_{2\varepsilon R},\\ 1, & \text{for } y \in B_{\varepsilon R}.\end{cases}$$
$$|\nabla\eta_\varepsilon| \le \frac{2}{\varepsilon R},$$

Since $\eta_\varepsilon\varphi_0 \in H^1_{0,K_\varepsilon}(\Omega)$, from (5.2.7) we have that

$$\begin{aligned}\mathrm{Cap}_{\bar{\Omega},c}(K_\varepsilon,\varphi_0) &\le q(\eta_\varepsilon\varphi_0)\\ &\le 2\int_{\Omega\cap B_{2R\varepsilon}} |\nabla\eta_\varepsilon(x)|^2\varphi_0(x)\,\mathrm{d}x\\ &\quad + 2\int_{\Omega\cap B_{2R\varepsilon}} |\eta_\varepsilon(x)|^2|\nabla\varphi_0(x)|^2\,\mathrm{d}x + \|c\|_{L^\infty(\Omega)}\int_{\Omega\cap B_{2R\varepsilon}} \varphi_0^2(x)\,\mathrm{d}x\\ &\le \left(\frac{8}{\varepsilon^2R^2} + \|c\|_{L^\infty(\Omega)}\right)\int_{\Omega\cap B_{2R\varepsilon}} \varphi_0^2(x)\,\mathrm{d}x + 2\int_{\Omega\cap B_{2R\varepsilon}} |\nabla\varphi_0(x)|^2\,\mathrm{d}x\end{aligned}$$

The conclusion then follows from (5.2.15) and (5.2.16). □

The following lemma, whose proof is classical, is useful in order to pass from a global scale (functions in $\mathcal{D}^{1,2}(\mathbb{R}^N_+)$) to a local one (meaning functions in $H^1(B_R^+)$) and vice versa.

Lemma 5.4.4. *Let $K \subseteq \overline{\mathbb{R}^N_+}$ be compact. If $f \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+})$ is such that $f|_{B_R^+} \in H^1_{0,K}(B_R^+)$ for some $R > r(K)$, then $f \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+} \setminus K)$. Conversely, if $f \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+} \setminus K)$, then $f|_{B_R^+} \in H^1_{0,K}(B_R^+)$ for all $R > r(K)$.*

In the lemma below we compare the two notions of capacity arising in our work, namely Definition 5.2.1 and Definition 5.2.10.

Lemma 5.4.5 (Equivalence of capacities)**.** *Let $K \subseteq \overline{\mathbb{R}^N_+}$ be a compact set and let $R > r(K)$. Then there exists a constant $\alpha = \alpha(R) > 0$ such that*

$$\alpha^{-1} \operatorname{cap}_{\overline{\mathbb{R}^N_+}}(K) \leq \operatorname{Cap}_{\overline{B_R^+}}(K) \leq \alpha \operatorname{cap}_{\overline{\mathbb{R}^N_+}}(K).$$

Proof. Let $\eta_K \in C_c^\infty(\mathbb{R}^N)$ be such that $\eta_K = 1$ in a neighborhood of K.

In order to prove the left hand inequality, let $W_R \in H^1(B_R^+)$ be the potential achieving $\operatorname{Cap}_{\overline{B_R^+}}(K)$ and let $\hat{W}_R \in H^1(\mathbb{R}^N_+)$ be its extension to $\mathbb{R}^N_+$. Obviously $\hat{W}_R \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+})$ and, by Lemma 5.4.4, we have that $\hat{W}_R - \eta_K \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+} \setminus K)$. Therefore $\hat{W}_R$ is admissible for $\operatorname{cap}_{\overline{\mathbb{R}^N_+}}(K)$ and hence

$$\begin{aligned} \operatorname{cap}_{\overline{\mathbb{R}^N_+}}(K) &\leq \int_{\mathbb{R}^N_+} |\nabla \hat{W}_R|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^N_+} (|\nabla \hat{W}_R|^2 + \hat{W}_R^2) \,\mathrm{d}x \\ &\leq C_1(R) \int_{B_R^+} (|\nabla W_R|^2 + W_R^2) \,\mathrm{d}x = C_1(R) \operatorname{Cap}_{\overline{B_R^+}}(K), \end{aligned}$$

where $C_1(R)$ is the constant related to the extension operator for Sobolev functions. In order to prove the other inequality, let $W_K \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+})$ be the potential achieving $\operatorname{cap}_{\overline{\mathbb{R}^N_+}}(K)$. Since $W_K - \eta_K \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+} \setminus K)$, then $W_K \in H^1(B_R^+)$ and, in view of Lemma 5.4.4, $W_K - 1 \in H^1_{0,K}(B_R^+)$. Hence W_K is admissible for $\operatorname{Cap}_{\overline{B_R^+}}(K)$. Moreover, by Hölder and Sobolev inequalities, we have that

$$\|W_K\|^2_{L^2(B_R^+)} \leq \|W_K\|^2_{L^{2^*}(B_R^+)} |B_R^+|^{2/N} \leq C_2(R) \, \|\nabla W_K\|^2_{L^2(\mathbb{R}^N_+)},$$

for some $C_2(R) > 0$ depending only on N and R. Then the following estimates hold

$$\operatorname{Cap}_{\overline{B_R^+}}(K) \leq \int_{B_R^+} (|\nabla W_K|^2 + W_K^2) \,\mathrm{d}x \leq (1 + C_2(R)) \operatorname{cap}_{\overline{\mathbb{R}^N_+}}(K),$$

thus concluding the proof. □

In order to prove Theorem 5.2.12 the following Poincaré type inequality is needed.

Lemma 5.4.6 (Poincaré Inequality). *Let $M, K \subseteq \overline{\mathbb{R}^N_+}$ and $\{K_\varepsilon\}_{\varepsilon\in(0,1)}$ satisfy (5.2.12)–(5.2.13) for some $\Phi \in \mathcal{C}$ and let $\tilde{K}_\varepsilon := \Phi(K_\varepsilon)/\varepsilon$. Let us assume that $\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K) > 0$. For any $R > r(M)$ there exist $\varepsilon_0 \in (0,1)$ and $C > 0$ (both depending on R and K) such that*

$$\int_{B_R^+} u^2 \,\mathrm{d}x \le C \int_{B_R^+} |\nabla u|^2 \,\mathrm{d}x$$

for all $u \in H^1_{0,\tilde{K}_\varepsilon}(B_R^+)$ and for all $\varepsilon < \varepsilon_0$.

Proof. By way of contradiction, suppose that, for a certain $R > r(M)$, there exist a sequence of real numbers $\varepsilon_n \to 0^+$ and a sequence of functions $u_n \in H^1_{0,\tilde{K}_{\varepsilon_n}}(B_R^+)$ such that

$$\int_{B_R^+} u_n^2 \,\mathrm{d}x > n \int_{B_R^+} |\nabla u_n|^2 \,\mathrm{d}x.$$

Now let us consider the sequence

$$v_n := \frac{u_n}{\|u_n\|_{L^2(B_R^+)}},$$

so that

$$\|v_n\|_{L^2(B_R^+)} = 1 \quad \text{and} \quad \int_{B_R^+} |\nabla v_n|^2 \,\mathrm{d}x < \frac{1}{n}.$$

Therefore, since $\|v_n\|_{H^1(B_R^+)}$ is uniformly bounded with respect to n, there exists $v \in H^1(B_R^+)$ such that, up to a subsequence, $v_n \rightharpoonup v$ weakly in $H^1(B_R^+)$. By compactness $v_n \to v$ strongly in $L^2(B_R^+)$; this implies that $\|v\|_{L^2(B_R^+)} = 1$ and hence that $v \not\equiv 0$. On the other hand, by weak lower semicontinuity,

$$\int_{B_R^+} |\nabla v|^2 \,\mathrm{d}x \le \liminf_{n\to\infty} \int_{B_R^+} |\nabla v_n|^2 \,\mathrm{d}x = 0,$$

hence there exists a constant $\kappa \neq 0$ such that $v = \kappa$ a.e. in B_R^+. Finally, since $\mathbb{R}^N \setminus \tilde{K}_{\varepsilon_n}$ is converging to $\mathbb{R}^N \setminus K$ in the sense of Mosco, $v \in H^1_{0,K}(B_R^+)$ (see Remark 5.4.2) and this implies the existence of a sequence $\{w_n\}_n \subset C_c^\infty(\overline{B_R^+} \setminus K)$ such that $\|w_n - \kappa\|_{H^1(B_R^+)} \to 0$ as $n \to +\infty$. Letting $z_n = (\kappa - w_n)/\kappa$, we have that

$$z_n - 1 \in H^1_{0,K}(B_R^+) \quad \text{and} \quad \|z_n\|_{H^1(B_R^+)} \to 0$$

as $n \to +\infty$, thus implying $\mathrm{Cap}_{\overline{B_R^+}}(K) = 0$ and hence contradicting, in view of Lemma 5.4.5, the fact that $\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K) > 0$. □

In the same spirit of Proposition 5.3.6, we have that the relative ψ_γ-capacity of the set K, denoted by $\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K, \psi_\gamma)$ (see Definition 5.2.10), is uniquely achieved, as asserted in the following lemma.

Lemma 5.4.7. *Let $\eta \in C_c^\infty(\overline{\mathbb{R}^N_+})$ be a cut-off function such that $\eta = 1$ in a neighborhood of K. There exists a unique $\tilde{V} \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+})$ such that $\tilde{V} - \eta\psi_\gamma \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+} \setminus K)$ and*

$$\int_{\mathbb{R}^N_+} \nabla\tilde{V} \cdot \nabla\varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+} \setminus K),$$

i.e. weakly solving

$$\begin{cases} -\Delta\tilde{V} = 0, & \text{in } \mathbb{R}^N_+ \setminus K, \\ \tilde{V} = \psi_\gamma, & \text{on } K, \\ \dfrac{\partial\tilde{V}}{\partial\boldsymbol{\nu}} = 0, & \text{on } \partial\mathbb{R}^N_+ \setminus K. \end{cases}$$

Moreover

$$\operatorname{cap}_{\overline{\mathbb{R}^N_+}}(K,\psi_\gamma) = \int_{\mathbb{R}^N_+} |\nabla\tilde{V}|^2 \,\mathrm{d}x$$

and $\tilde{V}$ does not depend on the choice of the cut-off function η.

Since we are in the case $N \geq 3$, the following Hardy-type inequality on half balls holds.

Lemma 5.4.8 (Hardy-type inequality)**.** *For all $R > 0$ and $u \in H^1(B_R^+)$*

$$\frac{N-2}{2}\int_{B_R^+} \frac{u^2}{|x|^2}\,\mathrm{d}x \leq \frac{N+1}{R^2}\int_{B_R^+} u^2\,\mathrm{d}x + \frac{N}{N-2}\int_{B_R^+} |\nabla u|^2\,\mathrm{d}x.$$

Proof. By integrating over B_R^+ the identity

$$\operatorname{div}\left(u^2\frac{x}{|x|^2}\right) = (N-2)\frac{u^2}{|x|^2} + 2u\nabla u \cdot \frac{x}{|x|^2},$$

we obtain that, for any $u \in C^\infty(\overline{B_R^+})$,

$$\begin{aligned}(N-2)\int_{B_R^+} \frac{u^2}{|x|^2}\,\mathrm{d}x &= \int_{\partial B_R^+} u^2\frac{x\cdot\boldsymbol{\nu}}{|x|^2}\,\mathrm{d}S - 2\int_{B_R^+} u\nabla u \cdot \frac{x}{|x|^2}\,\mathrm{d}x \\ &\leq \frac{1}{R}\int_{S_R^+} u^2\,\mathrm{d}S + \frac{N-2}{2}\int_{B_R^+}\frac{u^2}{|x|^2}\,\mathrm{d}x + \frac{2}{N-2}\int_{B_R^+}|\nabla u|^2\,\mathrm{d}x,\end{aligned}$$

thanks to the fact that $x \cdot \boldsymbol{\nu} = 0$ on $\{x_1 = 0\} \cap \partial B_R^+$ and $x = R\boldsymbol{\nu}$ on S_R^+.

On the other hand, integrating over B_R^+ the identity

$$\operatorname{div}(u^2 x) = 2u\nabla u \cdot x + Nu^2$$

and arguing in a similar way, we deduce that

$$\int_{S_R^+} u^2\,\mathrm{d}S \leq \frac{N+1}{R}\int_{B_R^+} u^2\,\mathrm{d}x + R\int_{B_R^+}|\nabla u|^2\,\mathrm{d}x.$$

Combining those two relations the lemma is proved. □

The following proposition provides a blow-up analysis for scaled capacitary potentials, which will be the core of the proof of Theorem 5.2.12.

Proposition 5.4.9. *Let $\{K_\varepsilon\}_{\varepsilon>0} \subseteq \overline{\Omega}$ be a family of compact sets concentrating at $\{0\} \subseteq \partial\Omega$ as $\varepsilon \to 0$ and satisfying (5.2.12)-(5.2.13) for some $\Phi \in \mathcal{C}$ and for some compact sets $M, K \subseteq \overline{\mathbb{R}^N_+}$ with $\operatorname{cap}_{\overline{\mathbb{R}^N_+}}(K) > 0$. Let φ_0 be as in (5.2.9) and let γ, ψ_γ be as in (5.2.14)-(5.2.17). Let $\tilde{V}_\varepsilon$ be as in (5.4.25) and $\tilde{V}$ as in Lemma 5.4.7. Then*

$$\begin{aligned} \tilde{V}_\varepsilon &\rightharpoonup \tilde{V} \quad \text{weakly in } H^1(B_R^+), \\ A(\varepsilon x)\nabla\tilde{V}_\varepsilon(x) &\rightharpoonup \nabla\tilde{V}(x) \quad \text{weakly in } L^2(B_R^+), \\ \varepsilon^2\hat{c}(\varepsilon x)\tilde{V}_\varepsilon(x) &\rightharpoonup 0 \quad \text{weakly in } L^2(B_R^+), \end{aligned}$$

as $\varepsilon \to 0$ for all $R > r(M)$, where A and $\hat{c}$ are as in (5.4.18) (with Φ instead of Φ_{AE}).

Proof. From Lemma 5.3.14 and Lemma 5.4.3 we have that

$$\int_\Omega |\nabla V_{K_\varepsilon,\varphi_0}|^2 \, \mathrm{d}x = \operatorname{Cap}_{\bar{\Omega},c}(K_\varepsilon,\varphi_0)(1+o(1)) = O(\varepsilon^{N+2\gamma-2}),$$

as $\varepsilon \to 0$. On the other hand, letting $\mathcal{U}_0$ and R_0 be as in (5.4.21), by a change of variables we have that

$$\int_\Omega |\nabla V_{K_\varepsilon,\varphi_0}|^2 \, \mathrm{d}x \geq \int_{\Omega\cap\mathcal{U}_0} |\nabla V_{K_\varepsilon,\varphi_0}|^2 \, \mathrm{d}x = \varepsilon^{N+2\gamma-2} \int_{B^+_{R_0/\varepsilon}} A(\varepsilon x)\nabla\tilde{V}_\varepsilon(x)\cdot\nabla\tilde{V}_\varepsilon(x)\,\mathrm{d}x$$

and so

$$\int_{B^+_{R_0/\varepsilon}} A(\varepsilon x)\nabla\tilde{V}_\varepsilon(x)\cdot\nabla\tilde{V}_\varepsilon(x)\,\mathrm{d}x = O(1) \tag{5.4.28}$$

as $\varepsilon \to 0$. From (5.4.28) and (5.4.22) it follows that

$$\int_{B^+_{R_0/\varepsilon}} |\nabla\tilde{V}_\varepsilon(x)|^2\,\mathrm{d}x \leq 2\int_{B^+_{R_0/\varepsilon}} A(\varepsilon x)\nabla\tilde{V}_\varepsilon(x)\cdot\nabla\tilde{V}_\varepsilon(x)\,\mathrm{d}x = O(1) \quad \text{as } \varepsilon \to 0. \tag{5.4.29}$$

On the other hand, in view of (5.4.23),

$$\begin{aligned} \operatorname{Cap}_{\bar{\Omega},c}(K_\varepsilon,\varphi_0) &\geq \int_{\Omega\cap\mathcal{U}_0} c(x)|V_{K_\varepsilon,\varphi_0}|^2\,\mathrm{d}x \\ &= \varepsilon^{N+2\gamma}\int_{B^+_{R_0/\varepsilon}} \hat{c}(\varepsilon x)|\tilde{V}_\varepsilon(x)|^2\,\mathrm{d}x \geq \frac{c_0}{2}\,\varepsilon^{N+2\gamma}\int_{B^+_{R_0/\varepsilon}} |\tilde{V}_\varepsilon(x)|^2\,\mathrm{d}x \end{aligned}$$

so that from Lemma 5.4.3 we deduce that

$$\varepsilon^2\int_{B^+_{R_0/\varepsilon}} |\tilde{V}_\varepsilon(x)|^2\,\mathrm{d}x = O(1) \quad \text{as } \varepsilon \to 0. \tag{5.4.30}$$

From (5.4.29) we deduce that there exists $C_1 > 0$ (not depending on R) such that, for every $R > 0$,

$$\int_{B_R^+} |\nabla\tilde{V}_\varepsilon|^2\,\mathrm{d}x \leq C_1 \quad \text{for all } \varepsilon \in (0, R_0/R). \tag{5.4.31}$$

From the Poincaré inequality proved in Lemma 5.4.6, we deduce that, for every $R > r(M)$, there exist $C_2 = C_2(R) > 0$ and $\varepsilon_{1,R} > 0$ (both depending on R) such that, if $\varepsilon \in (0, \varepsilon_{1,R})$,

$$\begin{aligned}\int_{B_R^+} |\tilde{V}_\varepsilon|^2 \,\mathrm{d}x &\le 2\int_{B_R^+} |\tilde{V}_\varepsilon - \tilde{\varphi}_\varepsilon|^2 \,\mathrm{d}x + 2\int_{B_R^+} |\tilde{\varphi}_\varepsilon|^2 \,\mathrm{d}x \\ &\le 2C_2 \int_{B_R^+} |\nabla(\tilde{V}_\varepsilon - \tilde{\varphi}_\varepsilon)|^2 \,\mathrm{d}x + 2\int_{B_R^+} |\tilde{\varphi}_\varepsilon|^2 \,\mathrm{d}x \\ &\le 4C_2 \int_{B_R^+} (|\nabla\tilde{V}_\varepsilon|^2 + |\nabla\tilde{\varphi}_\varepsilon|^2) \,\mathrm{d}x + 2\int_{B_R^+} |\tilde{\varphi}_\varepsilon|^2 \,\mathrm{d}x.\end{aligned}$$

Hence, from (5.2.14) and (5.4.31) we have that, for every $R > r(M)$, there exist $C_3 = C_3(R) > 0$ and $\varepsilon_{2,R} > 0$ (both depending on R) such that, if $\varepsilon \in (0, \varepsilon_{2,R})$,

$$\int_{B_R^+} |\tilde{V}_\varepsilon|^2 \,\mathrm{d}x \le C_3. \tag{5.4.32}$$

Combining (5.4.31) and (5.4.32) with a diagonal process, we deduce that there exists $W \in H^1_{\mathrm{loc}}(\mathbb{R}^N_+)$ (not depending on R) such that, along a subsequence $\varepsilon = \varepsilon_n \to 0^+$,

$$\begin{aligned} &\tilde{V}_\varepsilon \rightharpoonup W \quad \text{weakly in } H^1(B_R^+), \\ &\tilde{V}_\varepsilon \to W \quad \text{strongly in } L^2(B_R^+), \\ &\tilde{V}_\varepsilon \to W \quad \text{a.e. in } \mathbb{R}^N_+, \end{aligned} \tag{5.4.33}$$

for all $R > r(M)$. Since c is bounded and $\|A(\varepsilon x) - I_N\|_{\mathcal{M}_{N\times N}} \le C\,R\,\varepsilon$ for all $x \in B_R^+$, from (5.4.33) it follows easily that

$$A(\varepsilon x)\nabla\tilde{V}_\varepsilon(x) \rightharpoonup \nabla W(x) \quad \text{weakly in } L^2(B_R^+), \tag{5.4.34}$$

$$\varepsilon^2 \hat{c}(\varepsilon x)\tilde{V}_\varepsilon(x) \rightharpoonup 0 \quad \text{weakly in } L^2(B_R^+), \tag{5.4.35}$$

as $\varepsilon = \varepsilon_n \to 0$, for all $R > r(M)$.

Now let $\varphi \in C_c^\infty(\overline{\mathbb{R}^N_+} \setminus K)$. Then there exists $R > 0$ such that

$$\varphi \in C_c^\infty(B_R^+ \cup (B_R' \setminus K)) \subseteq H^1_{0,K\cup S_R^+}(B_R^+).$$

Since $H^1_{0,\tilde{K}_\varepsilon\cup S_R^+}(B_R^+)$ is converging to $H^1_{0,K\cup S_R^+}(B_R^+)$ in the sense of Mosco (see hypothesis (5.2.13) and Remark 5.4.2), there exists a sequence $\psi_\varepsilon \in H^1_{0,\tilde{K}_\varepsilon\cup S_R^+}(B_R^+)$ such that

$$\psi_\varepsilon \to \varphi \quad \text{strongly in } H^1(B_R^+), \text{ as } \varepsilon \to 0. \tag{5.4.36}$$

From the equation (5.4.27) satisfied by $\tilde{V}_\varepsilon$ we know that, for all $\varepsilon \in (0, R_0/R)$ (so that $B_R^+ \subset B_{R_0/\varepsilon}^+$)

$$\int_{B_R^+} (A(\varepsilon x)\nabla\tilde{V}_\varepsilon(x) \cdot \nabla\psi_\varepsilon(x) + \varepsilon^2\hat{c}(\varepsilon x)\tilde{V}_\varepsilon(x)\psi_\varepsilon(x)) \,\mathrm{d}x = 0.$$

Therefore, combining (5.4.34),(5.4.35) and (5.4.36) we obtain that

$$\int_{B_R^+} \nabla W \cdot \nabla \varphi \, \mathrm{d}x = 0.$$

Summing up we have that

$$\int_{\mathbb{R}^N_+} \nabla W \cdot \nabla \varphi \, \mathrm{d}x = 0 \quad \text{for all } \varphi \in C_c^\infty(\overline{\mathbb{R}^N_+} \setminus K). \tag{5.4.37}$$

By weak lower semicontinuity and (5.4.31) we have that

$$\int_{\mathbb{R}^N_+} |\nabla W|^2 \, \mathrm{d}x < \infty. \tag{5.4.38}$$

Now let $R > r(M)$ and $\varepsilon < R_0/R$. Thanks to Lemma 5.4.8 and estimates (5.4.29) and (5.4.30) we have that

$$\begin{aligned}\int_{B_R^+} \frac{|\tilde{V}_\varepsilon|^2}{|x|^2} \, \mathrm{d}x &\leq \int_{B^+_{R_0/\varepsilon}} \frac{|\tilde{V}_\varepsilon|^2}{|x|^2} \, \mathrm{d}x \\ &\leq \frac{2(N+1)}{(N-2)R_0^2} \varepsilon^2 \int_{B^+_{R_0/\varepsilon}} \tilde{V}_\varepsilon^2 \, \mathrm{d}x + \frac{2N}{(N-2)^2} \int_{B^+_{R_0/\varepsilon}} |\nabla \tilde{V}_\varepsilon|^2 \, \mathrm{d}x \leq C_4,\end{aligned}$$

for a certain $C_4 > 0$ not depending on ε and R. By weak lower semicontinuity we deduce that $\int_{B_R^+} \frac{|W|^2}{|x|^2} \, \mathrm{d}x \leq C_4$ for all $R > r(M)$, hence

$$\int_{\mathbb{R}^N_+} \frac{|W|^2}{|x|^2} \, \mathrm{d}x < \infty.$$

Thanks to this and to (5.4.38), we have that $W \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+})$; moreover, by density of $C_c^\infty(\overline{\mathbb{R}^N_+} \setminus K)$ in $\mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+} \setminus K)$ we have that (5.4.37) holds for any $\varphi \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+} \setminus K)$.

Let $\eta \in C_c^\infty(\overline{\mathbb{R}^N_+})$ be such that $\eta = 1$ in a neighborhood of M: since, for every $R > r(M)$, $\tilde{V}_\varepsilon - \eta \tilde{\varphi}_\varepsilon \in H^1_{0,\tilde{K}_\varepsilon}(B_R^+)$ and since $H^1_{0,\tilde{K}_\varepsilon}(B_R^+)$ is converging to $H^1_{0,K}(B_R^+)$ in the sense of Mosco, then, passing to the weak limit, there holds $W - \eta\psi_\gamma \in H^1_{0,K}(B_R^+)$ (see (5.2.14)). Hence, in view of Lemma 5.4.4, we have that $W - \eta\psi_\gamma \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+} \setminus K)$. Combining this fact with (5.4.37), by uniqueness (stated in Lemma 5.4.7) we have that $W = \tilde{V}$ and, by Urysohn's Subsequence Principle, we conclude that the convergences (5.4.33),(5.4.34), and(5.4.35) hold as $\varepsilon \to 0$ (not only along a sequence $\varepsilon_n \to 0^+$). □

Now we are ready for the proof of our second main result.

Proof of Theorem 5.2.12. Let $R > r(M)$ and let $\eta \in C_c^\infty(\mathbb{R}^N)$ be such that $\eta = 1$ in B_R. Also let $\tilde{R} > 0$ be such that $\operatorname{supp} \eta \subseteq B_{\tilde{R}}$. For $\varepsilon > 0$ small we define

$$\eta_\varepsilon(x) = \begin{cases} \eta\left(\frac{1}{\varepsilon}\Phi(x)\right), & \text{if } x \in \mathcal{U}_0, \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \mathcal{U}_0, \end{cases}$$

and observe that, if ε is sufficiently small, $\eta_\varepsilon \in C^1_c(\mathbb{R}^N)$ and $\eta_\varepsilon \equiv 1$ in a neighborhood of K_ε. Testing equation (5.3.2) with $V_{K_\varepsilon,\varphi_0} - \varphi_0\eta_\varepsilon$ leads to

$$\begin{aligned}\mathrm{Cap}_{\bar\Omega,c}(K_\varepsilon,\varphi_0) &= \int_\Omega \left(|\nabla V_{K_\varepsilon,\varphi_0}|^2 + c|V_{K_\varepsilon,\varphi_0}|^2\right)\mathrm{d}x \\ &= \int_\Omega (\nabla V_{K_\varepsilon,\varphi_0}\cdot\nabla(\varphi_0\eta_\varepsilon) + cV_{K_\varepsilon,\varphi_0}\varphi_0\eta_\varepsilon)\,\mathrm{d}x \\ &= \int_{\Omega\cap\mathcal{U}_0} (\nabla V_{K_\varepsilon,\varphi_0}\cdot\nabla(\varphi_0\eta_\varepsilon) + cV_{K_\varepsilon,\varphi_0}\varphi_0\eta_\varepsilon)\,\mathrm{d}x.\end{aligned}$$

Then, by the change of variable $x = \Phi^{-1}(\varepsilon y)$, we obtain

$$\mathrm{Cap}_{\bar\Omega,c}(K_\varepsilon,\varphi_0) = \varepsilon^{2\gamma+N-2}\int_{B^+_{\tilde R}} (A(\varepsilon y)\nabla\tilde V_\varepsilon(y)\cdot\nabla(\eta\tilde\varphi_\varepsilon)(y) + \varepsilon^2\hat c(\varepsilon y)\tilde V_\varepsilon(y)\eta(y)\tilde\varphi_\varepsilon(y))\,\mathrm{d}y \tag{5.4.39}$$

where $\tilde V_\varepsilon$ and $\tilde\varphi_\varepsilon$ are defined in (5.4.25). From Proposition 5.4.9 and Proposition 5.2.8 it follows that

$$\lim_{\varepsilon\to 0}\int_{B^+_{\tilde R}} (A(\varepsilon y)\nabla\tilde V_\varepsilon(y)\cdot\nabla(\eta\tilde\varphi_\varepsilon)(y) + \varepsilon^2\hat c(\varepsilon y)\tilde V_\varepsilon(y)\eta(y)\tilde\varphi_\varepsilon(y))\,\mathrm{d}y = \int_{\mathbb{R}^N_+}\nabla\tilde V\cdot\nabla(\eta\psi_\gamma)\,\mathrm{d}y. \tag{5.4.40}$$

Finally, testing the equation satisfied by $\tilde V$ (see Lemma 5.4.7) with $\tilde V - \eta\psi_\gamma \in \mathcal{D}^{1,2}(\overline{\mathbb{R}^N_+}\setminus K)$ we have that

$$\mathrm{cap}_{\overline{\mathbb{R}^N_+}}(K,\psi_\gamma) = \int_{\mathbb{R}^N_+}|\nabla\tilde V|^2\,\mathrm{d}x = \int_{\mathbb{R}^N_+}\nabla\tilde V\cdot\nabla(\eta\psi_\gamma)\,\mathrm{d}x.$$

This, combined with (5.4.39) and (5.4.40), concludes the proof. □

5.4.4 Set scaling to an interior point

In this last section we consider the case in which the perturbing sets K_ε are concentrating to an interior point in a way that resembles (and comprehends) the scaling of a fixed compact set and we sketch the steps that lead to the proof of Theorem 5.2.14 (counterpart of Theorem 5.2.12). Always in the case $N \geq 3$, we assume that $0 \in \Omega$ and that the family of compact sets $K_\varepsilon \subseteq \Omega$ satisfy (5.2.18) and (5.2.19). Heuristically speaking, in the previous section the rescaled domain Ω/ε was "approaching" the half space $\mathbb{R}^N_+$, due to the fact that $0 \in \partial\Omega$ and that $\partial\Omega$ was smooth in a neighborhood of the origin. In this section, since $0 \in \Omega$ the "limit" domain of Ω/ε turns out to be the whole space $\mathbb{R}^N$. For the same reason the role of half balls B^+_R is played, in this section, by balls B_R.

Let κ and ζ_κ be as in (5.2.20). As in Lemma 5.4.3, by testing $\mathrm{Cap}_{\bar\Omega,c}(K_\varepsilon,\varphi_0)$ with φ_0 suitably cutted off, it is possible to prove that

$$\mathrm{Cap}_{\bar\Omega,c}(K_\varepsilon,\varphi_0) = O(\varepsilon^{N+2\kappa-2}), \quad \text{as } \varepsilon\to 0.$$

Also in this framework, a Poincaré type inequality holds and the proof follows the same steps as Lemma 5.4.6.

Lemma 5.4.10 (Poincaré Inequality). *Let $M, K \subseteq \mathbb{R}^N$ and $\{K_\varepsilon\}_{\varepsilon\in(0,1)}$ satisfy* (5.2.18) *and* (5.2.19) *and let $\tilde{K}_\varepsilon := K_\varepsilon/\varepsilon$. Let us assume that $\operatorname{cap}_{\mathbb{R}^N}(K) > 0$. For any $R > r(M)$ there exist $\varepsilon_0 \in (0,1)$ and $C > 0$ (both depending on R and K) such that*

$$\int_{B_R} u^2 \, \mathrm{d}x \leq C \int_{B_R} |\nabla u|^2 \, \mathrm{d}x$$

for all $u \in H^1_{0,\tilde{K}_\varepsilon}(B_R)$ and for all $\varepsilon < \varepsilon_0$.

Furthermore, the capacity $\operatorname{cap}_{\mathbb{R}^N}(K, \zeta_\kappa)$, whose definition is recalled in Definition 5.3.15, is attained by a potential $\widehat{V} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, analogously to what is stated in Lemma 5.4.7. Also in this context it is possible to prove an Hardy type inequality, which reads as follows.

Lemma 5.4.11 (Hardy-type inequality). *We have that*

$$\frac{N-2}{2} \int_{B_R} \frac{u^2}{|x|^2} \, \mathrm{d}x \leq \frac{N+1}{R^2} \int_{B_R} u^2 \, \mathrm{d}x + \frac{N}{N-2} \int_{B_R} |\nabla u|^2 \, \mathrm{d}x$$

for all $u \in H^1(B_R)$ and for all $R > 0$.

Following the same steps as in the proof of Proposition 5.4.9 and Theorem 5.2.12 and adapting the ideas and the computations to the current framework, it is possible to prove Theorem 5.2.14.

Part II

Positivity principles and free boundary problems for fractional operators

CHAPTER 6

AN INTRODUCTION TO THE FRACTIONAL LAPLACIAN

The second part of this thesis is devoted to the analysis of two differential problems that are governed by the same operator, namely the fractional Laplacian. The purpose of this introduction is to rigorously define this pseudodifferential operator and to get acquainted with the basic notions and with the most relevant properties of concern for our work. In addition, we are going to illustrate an interesting relation of fractional powers of the (negative) Laplace operator with a Dirichlet-to-Neumann type map: this relation is based on the so called Caffarelli-Silvestre extension, which has been established in [CS07].

Fractional powers of differential operators, and nonlocal operators in general, attracted a lot of attentions among mathematicians in the recent years, and the literature is still blossoming. One of the reasons of their appeal is the abundance and great variety of possible applications. Without aiming at giving a complete picture, we enumerate some of them. Nonlocal operators appear in physics, for instance concerning crystal dislocation (see [GM12, Tol97, CDdP20]) or in the theory of water waves (see e.g. [dlLV09, GG03, MV20] or the books [Whi74, NS94]) or are of concern for semipermeable membranes (see [DL76, DJ21]), for flame propagation (see [CMS12]) and for mathematical finance (see [CT04a]). Moreover, fractional operators manifest strong relevance in numerous mathematical problems, for instance with reference to thin obstacle problems (see [Sil07, CSS08, GP09, BFRO18, GRO19, CROS17], and see also Chapter 8), nonlocal minimal surfaces (see [CRS10, BDLV20, DV19]) and mean field games (see [CCD$^+$19, EJ20]).

Among a number of surveys concerning the topic, we refer to [DNPV12], [Gar19] and [BV16] for self-contained expositions (see also [Dip20]).

Let $s \in (0,1)$ and $N > 2s$. If $\mathcal{S}(\mathbb{R}^N)$ denotes the Schwartz class of smooth, rapidly decreasing functions, then we define the fractional Laplacian of order s, for functions

$u \in \mathcal{S}(\mathbb{R}^N)$, as follows

$$(-\Delta)^s u(x) := C(N,s)\,\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y$$
$$= C(N,s) \lim_{\rho\to 0^+} \int_{|x-y|>\rho} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y,$$

where P.V. means that the integral has to be seen in the principal value sense and

$$C(N,s) = \pi^{-\frac{N}{2}} 2^{2s} \frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma(2-s)} s(1-s),$$

with Γ denoting the usual Euler's Gamma function. The reason of the presence of this constant lies in an alternative, equivalent definition, based on the Fourier transform, that is

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}\,\mathcal{F}(u)).$$

This equivalence also highlights the nature of fractional powers of the Laplacian as pseudodifferential operators with symbol $|\xi|^{2s}$. One can observe that, in the range $s \in (0,1/2)$, the integral

$$\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y$$

is in fact finite for any $x \in \mathbb{R}^N$ and any $u \in \mathcal{S}(\mathbb{R}^N)$. Moreover, one can prove that it is possible to rewrite the singular integral defining $(-\Delta)^s u$ as a weighted second order differential quotient, see [DNPV12, Lemma 3.2]. Namely

$$(-\Delta)^s u(x) = -\frac{C(N,s)}{2} \int_{\mathbb{R}^N} \frac{u(x-y)+u(x+y)-2u(x)}{|y|^{N+2s}}\,\mathrm{d}y.$$

We now seek to define the fractional Laplacian for a wider class of functions. There are several choices for this aim. One of the most general domain of definition is a family of tempered distributions satisfying certain decay properties; we are not going to pursue this path, but we refer to [Sil07, Section 2] for the specifics.

However, a very general interpretation of the fractional Laplacian is in the distributional sense. Indeed, if u belongs to the following space

$$\mathcal{L}^1_s := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2s}}\,\mathrm{d}x < \infty \right\}$$

then it is possible to define the fractional Laplacian of u to be a distribution acting as

$$\langle (-\Delta)^s u, \varphi \rangle := \int_{\mathbb{R}^N} u (-\Delta)^s \varphi \,\mathrm{d}x, \quad \text{for any } \varphi \in C^\infty_c(\mathbb{R}^N).$$

The proof of well posedness of this definition for functions in $\mathcal{L}^1_s$ can be seen, for instance, in [Sil07, Proposition 2.4].

On the other hand, it is possible to provide a definition of fractional Laplacian, in a "weak sense", which is the one we are going to employ in the present manuscript. The advantage of this approach is that it founded on an Hilbertian structure, which is obviously a truly powerful boost in variational frameworks. By direct computations one can see that

$$\int_{\mathbb{R}^N} u(-\Delta)^s v \,\mathrm{d}x = \int_{\mathbb{R}^N} v(-\Delta)^s u \,\mathrm{d}x = \frac{C(N,s)}{2} \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\mathrm{d}y$$

for all $u, v \in C_c^\infty(\mathbb{R}^N)$. Therefore, we naturally define the following scalar product on $C_c^\infty(\mathbb{R}^N)$

$$(u,v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} := \frac{C(N,s)}{2} \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\mathrm{d}y, \quad u,v \in C_c^\infty(\mathbb{R}^N)$$

and we define the space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm induced by $(\cdot,\cdot)_{\mathcal{D}^{s,2}(\mathbb{R}^N)}$. In this sense we can see the fractional Laplacian as an operator

$$(-\Delta)^s \colon \mathcal{D}^{s,2}(\mathbb{R}^N) \to (\mathcal{D}^{s,2}(\mathbb{R}^N))^*,$$

which acts as follows

$${}_{(\mathcal{D}^{s,2}(\mathbb{R}^N))^*}\langle (-\Delta)^s u, v\rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = (u,v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)}, \quad \text{for all } u,v \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

The space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is known in the literature as a fractional Beppo Levi space or fractional homogeneous Sobolev space, and it may be denoted also by $\dot{H}^s(\mathbb{R}^N)$ (or $\dot{W}^{s,2}(\mathbb{R}^N)$).

Fractional Beppo Levi spaces enjoy a variety of important functional inequalities. It is a truly hard job to exhaust a list of them, and therefore we limit ourselves to state a couple of results that turn out to be crucial for our work. The first inequality we recall is the fractional Hardy inequality, which has been proved by Herbst in [Her77] (see also [FLS08, Proposition 4.1] and [FS08] for a version with a remainder term).

Theorem 6.0.1 (Fractional Hardy inequality)**.** *There holds*

$$\gamma_H \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} \,\mathrm{d}x \le \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}, \quad \textit{for all } u \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

where

$$\gamma_H = \gamma_H(N,s) := 2^{2s} \frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}$$

is the best (largest) possible constant, which is never attained.

The other result which is worth to be stated here is the continuous embedding of the space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ into $L^{2_s^*}(\mathbb{R}^N)$, being

$$2_s^* := \frac{2N}{N-2s}$$

the critical Sobolev exponent in the fractional framework. For the proof we refer to [CT04b, Theorem 1.1], which actually establishes it for any $s \in \mathbb{R}$ (see also [SV11, Theorem 7] in the Appendix).

Theorem 6.0.2 (Fractional Sobolev inequality)**.** *There holds*

$$S\,\|u\|^2_{L^{2^*_s}(\mathbb{R}^N)} \leq \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}\,, \quad \textit{for all } u \in \mathcal{D}^{s,2}(\mathbb{R}^N), \tag{6.0.1}$$

where

$$S = S(N,s) := \frac{2^{2s}\pi^s\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma\left(\frac{N-2s}{2}\right)}\left(\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)}\right)^{\frac{2s}{N}}$$

is the best (largest) possible constant. Moreover the equality holds if and only if $u(x) = \alpha U((x-x_0)/\beta)$ *for some* $x_0 \in \mathbb{R}^N, \alpha \in \mathbb{R}$*,* $\beta > 0$*, where*

$$U(x) := \frac{1}{(1+|x|^2)^{\frac{N-2s}{2}}}$$

is the fractional Talenti function.

6.1 The Caffarelli-Silvestre extension

It is easy to observe that, by its definition, the fractional Laplacian is an integro-differential operator, thus displaying a nonlocal nature. Indeed, in order to compute $(-\Delta)^s$ of a function in a certain point, one must take into account the values of this function in the whole space. The nonlocal behavior brings in a number of difficulties. However, it is possible to recover from $(-\Delta)^s$ a differential, local operator by means of a boundary value problem in a space with a dimension more $\mathbb{R}^{N+1}_+ := \mathbb{R}^N \times (0,+\infty)$. This procedure has been established by Caffarelli and Silvestre in their seminal paper [CS07].

Let us now describe the approach. To this end, we fist consider the weighted Beppo Levi space $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$, defined as the completion of $C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm induced by the scalar product

$$(u,v)_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})} := \int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla u\cdot\nabla v\,\mathrm{d}x\mathrm{d}t, \quad u,v \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+}).$$

First of all, we observe that the weight

$$w(x,t) := t^{1-2s}$$

can be either singular or degenerate on the characteristic manifold $\mathbb{R}^N \cong \{(x,t)\colon t=0\}$, depending on the value of $s \in (0,1)$. Nevertheless, it still enjoys notable properties, in particular it belongs to the second Muckenhoupt class A_2, in the sense that is satisfies

$$\sup_{B \text{ ball in } \mathbb{R}^{N+1}_+}\left(\frac{1}{|B|_{N+1}}\int_B w\,\mathrm{d}x\mathrm{d}t\right)\left(\frac{1}{|B|_{N+1}}\int_B w^{-1}\,\mathrm{d}x\mathrm{d}t\right) < \infty,$$

with $|\cdot|_{N+1}$ denoting the $N+1$ dimensional Lebesgue measure. This property allows building a theory of weighted Sobolev spaces. In particular, there exists a well-defined and continuous trace map

$$\mathrm{Tr}\colon \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s}) \to \mathcal{D}^{s,2}(\mathbb{R}^N) \tag{6.1.1}$$

which is onto, see, for instance, [BCdPS13]. Let us now consider, for $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, the following minimization problem

$$\min\left\{\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\left|\nabla\Phi\right|^2\,\mathrm{d}x\mathrm{d}t\colon\, \Phi\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s}),\ \ \operatorname{Tr}\Phi=u\right\}. \tag{6.1.2}$$

By standard, direct methods of the calculus of variations, one can prove that there exists a unique function $\mathcal{H}(u)=U\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$ (which we call the *extension* of u) attaining (6.1.2), i.e.

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\left|\nabla U\right|^2\,\mathrm{d}x\mathrm{d}t\le\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\left|\nabla\Phi\right|^2\,\mathrm{d}x\mathrm{d}t \tag{6.1.3}$$

for all $\Phi\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$ such that $\operatorname{Tr}\Phi=u$. Furthermore, in [CS07] it has been proven that

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla U\cdot\nabla\Phi\,\mathrm{d}x\mathrm{d}t=\kappa_s(u,\operatorname{Tr}\Phi)_{\mathcal{D}^{s,2}(\mathbb{R}^N)}\quad\text{for all }\Phi\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s}), \tag{6.1.4}$$

where

$$\kappa_s:=\frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}. \tag{6.1.5}$$

We observe that (6.1.4) is the variational formulation of the following problem

$$\begin{cases}-\operatorname{div}(t^{1-2s}\nabla U)=0, & \text{in }\mathbb{R}^{N+1}_+,\\ -\lim\limits_{t\to 0}t^{1-2s}\dfrac{\partial U}{\partial t}=\kappa_s(-\Delta)^s u, & \text{on }\mathbb{R}^N.\end{cases} \tag{6.1.6}$$

Therefore, one can interpret the fractional Laplacian as a Dirichlet-to-Neumann operator. Namely, given the extension operator

$$\begin{aligned}\mathcal{H}\colon\mathcal{D}^{s,2}(\mathbb{R}^N)&\to\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})\\ u&\mapsto\mathcal{H}(u),\end{aligned}$$

and the "Neumann" operator

$$\begin{aligned}\boldsymbol{n}\colon\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})&\to(\mathcal{D}^{s,2}(\mathbb{R}^N))^*\\ U&\mapsto\boldsymbol{n}(U):=-\lim_{t\to0}t^{1-2s}\frac{\partial U}{\partial t},\end{aligned}$$

then, in view of (6.1.4), we can rewrite

$$(-\Delta)^s=\kappa_s^{-1}\boldsymbol{n}\circ\mathcal{H}\colon\mathcal{D}^{s,2}(\mathbb{R}^N)\to(\mathcal{D}^{s,2}(\mathbb{R}^N))^*.$$

Thanks to the extension tool, we are able to obtain the Hardy and Sobolev functional inequalities also in the space $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$. Namely, combining (6.1.3) and (6.1.4) with theorems 6.0.1 and 6.0.2, we obtain, respectively

$$\kappa_s\gamma_H\int_{\mathbb{R}^N}\frac{\left|\operatorname{Tr}U\right|^2}{\left|x\right|^{2s}}\,\mathrm{d}x\le\int_{\mathbb{R}^{N+1}_+}t^{1-2s}\left|\nabla U\right|^2\,\mathrm{d}x\mathrm{d}t\quad\text{for all }U\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s}) \tag{6.1.7}$$

and

$$\kappa_s S \, \|\operatorname{Tr} U\|^2_{L^{2^*_s}(\mathbb{R}^N)} \leq \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \, |\nabla U|^2 \, \mathrm{d}x\mathrm{d}t \quad \text{for all } U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}). \tag{6.1.8}$$

Moreover, just as a consequence of (6.1.3) and (6.1.4), we have that

$$\kappa_s \, \|\operatorname{Tr} U\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \leq \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \, |\nabla U|^2 \, \mathrm{d}x\mathrm{d}t \quad \text{for all } U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}). \tag{6.1.9}$$

We now state a result providing a compact trace embedding in weighted Lebesgue spaces, which will be useful in Chapter 7.

Lemma 6.1.1. *Let $p \in L^{N/2s}(\mathbb{R}^N)$. If*

$$(U_n)_n \subseteq \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \quad \textit{and} \quad U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$$

are such that

$$U_n \rightharpoonup U \quad \textit{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \textit{ as } n \to \infty,$$

then $\int_{\mathbb{R}^N} p |\operatorname{Tr} U_n|^2 \, \mathrm{d}x \to \int_{\mathbb{R}^N} p |\operatorname{Tr} U|^2 \, \mathrm{d}x$ as $n \to \infty$. In particular, if $p > 0$ a.e. in $\mathbb{R}^N$, the trace operator

$$\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \hookrightarrow L^2(\mathbb{R}^N; p \, \mathrm{d}x)$$

is compact, where $L^2(\mathbb{R}^N; p \, \mathrm{d}x) := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} p \, |u|^2 \, \mathrm{d}x < \infty \right\}$.

Proof. Let $(U_n)_n \subseteq \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ be such $U_n \rightharpoonup U$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ as $n \to \infty$. Hence, in view of continuity of the trace operator (6.1.1) and classical compactness results for fractional Sobolev spaces (see e.g. [DNPV12, Theorem 7.1]), we have that $\operatorname{Tr} U_n \to \operatorname{Tr} U$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Furthermore, by continuity of the trace operator (6.1.1) and (6.1.8), we have that, for every $\omega \subset \mathbb{R}^N$ measurable,

$$\int_\omega |p| |\operatorname{Tr}(U_n - U)|^2 \, \mathrm{d}x \leq C \|p\|_{L^{N/(2s)}(\omega)} \|U_n - U\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})}$$

for some positive constant $C > 0$ independent of ω and n. Therefore, by Vitali's Convergence Theorem we can conclude that $\lim_{n\to\infty} \int_{\mathbb{R}^N} |p| |\operatorname{Tr}(U_n - U)|^2 \, \mathrm{d}x = 0$, from which the conclusion follows. □

We finally introduce a class of weighted Lebesgue and Sobolev spaces, on bounded open Lipschitz sets $\omega \subseteq \mathbb{R}^{N+1}_+$ in the upper half-space. Namely, we define

$$L^2(\omega; t^{1-2s}) := \left\{ U \colon \omega \to \mathbb{R} \text{ measurable} \colon \int_\omega t^{1-2s} \, |U|^2 \, \mathrm{d}x\mathrm{d}t < \infty \right\}$$

and the weighted Sobolev space

$$H^1(\omega; t^{1-2s}) := \left\{ U \in L^2(\omega; t^{1-2s}) \colon \nabla U \in L^2(\omega; t^{1-2s}) \right\}.$$

From the fact that the weight t^{1-2s} belongs to the second Muckenhoupt class A_2 and thanks to well known weighted functional inequalities, one can prove that the embedding

$$H^1(\omega; t^{1-2s}) \hookrightarrow L^2(\omega; t^{1-2s})$$

is compact, see for details [OK90, Theorem 19.11] and [FF18, Proposition 7.1]. In addition, one can prove that the trace operator

$$H^1(\omega; t^{1-2s}) \hookrightarrow L^2(\partial\omega^+; t^{1-2s}),$$

is well defined and compact, where $\partial\omega^+ := \partial\omega \cap \mathbb{R}^{N+1}_+$ and

$$L^2(\partial\omega^+; t^{1-2s}) := \left\{ \psi\colon \partial\omega^+ \to \mathbb{R} \text{ measurable} : \int_{\partial\omega^+} t^{1-2s} |\psi|^2 \, \mathrm{d}S < \infty \right\}.$$

Finally, in the particular case when $\omega' := \partial\omega \cap \partial\mathbb{R}^{N+1}_+$ is the closure (in the $\mathbb{R}^N$ topology) of an open set, the same holds for the trace operator

$$H^1(\omega; t^{1-2s}) \hookrightarrow L^2(\omega').$$

For the proof of these facts concerning traces of A_2 weighted Sobolev functions we refer to [Nek93] and [STV19, Section 3.1].

Besides the connection with the fractional Laplacian, a theory of weighted Sobolev spaces has been largely developed, also in very general frameworks. In particular, we acknowledge their central role in the context of elliptic PDEs, driven by second order, weighted differential operators. Among a rich literature, we refer to the seminal paper [FKS82], where the authors investigated for the first time qualitative properties of solutions, such as maximum principles, Harnack inequalities, Hölder regularity etc. We also cite [STV19, STV20], in which the authors deeply examined regularity of solutions, in terms of Schauder estimates.

Notation We gather here some notation used throughout this part:

- $\mathbb{R}^{N+1}_+ := \{(x,t)\colon x \in \mathbb{R}^N,\ t > 0\} = \mathbb{R}^N \times \mathbb{R}_+$ denotes the positive half space;
- $B_r^+(x_0) := \{z \in \mathbb{R}^{N+1}_+\colon |z - x_0| < r\}$ and $S_r^+(x_0) := \{z \in \mathbb{R}^{N+1}_+\colon |z - x_0| = r\}$ denote, respectively, the positive half balls and half spheres, for any $x_0 \in \mathbb{R}^N$ and $r > 0$;
- $B_r'(x_0) := \partial B_r^+(x_0) \setminus \overline{S_r^+(x_0)}$ for balls on the thin space;
- in case of balls and spheres centered at the origin, we drop the center in the notation, i.e. $B_r^+ := B_r^+(0)$, $S_r^+ := S_r^+(x_0)$ and $B_r' := B_r'(x_0)$;
- $\mathbb{S}^N := \{z \in \mathbb{R}^{N+1}\colon |z| = 1\}$ is the unit N-dimensional sphere;
- $\mathbb{S}^N_+ := \mathbb{S}^N \cap \mathbb{R}^{N+1}_+$ and $\mathbb{S}^{N-1} := \partial\mathbb{S}^N_+$;
- for $y \in \mathbb{R}$, $y^+ := \max\{0, y\}$ and $y^- := \max\{0, -y\}$;
- $\mathrm{d}S$ and $\mathrm{d}\sigma$ denote the volume element in N and $N-1$ dimensional manifolds, respectively.

CHAPTER 7

FRACTIONAL MULTIPOLAR SCHRÖDINGER OPERATORS

The present chapter is committed to the study of positivity properties of a particular class of fractional Schrödinger operators defined in the whole space $\mathbb{R}^N$. However, before stepping in the fractional framework and going into the details of the problem under consideration, we think it is better to start with a more basic, classical model. This allows us to better understand the aims and to start familiarizing with the essential techniques, which are stand-alone subjects of interest. In particular, the original results contained in the present chapter crucially rely on two ingredients: Agmon-Allegretto-Piepenbrink's positivity principle and localization of binding, suitably adapted to the our framework. We are going to introduce them in the following section.

7.1 Agmon-Allegretto-Piepenbrink's principle and localization of binding: a brief overview

Let $N \geq 3$ and let $\Omega \subseteq \mathbb{R}^N$ be an open and connected set. For $V \in L^1_{\mathrm{loc}}(\Omega)$ let us consider the Schrödinger operator $-\Delta - V$, together with its corresponding quadratic form

$$Q_V(\varphi) := \int_\Omega (|\nabla\varphi|^2 - V\varphi^2)\,\mathrm{d}x, \tag{7.1.1}$$

defined for $\varphi \in C_c^\infty(\Omega)$. In addition, it is natural to consider the associated partial differential equation

$$-\Delta u - Vu = f, \quad \text{in } \Omega. \tag{7.1.2}$$

with $f \in L^1_{\mathrm{loc}}(\Omega)$. All the results we are going to cite in this introduction are well known and they hold with pretty light hypotheses on the potential V, which may change from time to time. However, for sake of simplicity in the exposition, we prefer to emphasize the ideas behind and not to focus on the sharpness of the assumptions. For this reason we restrict to a small class of potentials and we assume $V \in C_c^\infty(\Omega)$.

In 1974, W. Allegretto and J. Piepenbrink (see [All74] and [Pie74]) found out an intriguing relation between the positivity of the quadratic form (7.1.1) and the existence of a positive weak supersolution to (7.1.2). More precisely they proved that

$$Q_V(\varphi) \geq 0, \quad \text{for all } \varphi \in C_c^\infty(\Omega)$$

if and only if there exists $u \in H^1_{\mathrm{loc}}(\Omega)$ such that $u \geq 0$ a.e. in Ω and

$$\int_\Omega (\nabla u \cdot \nabla \varphi - V u \varphi) \, \mathrm{d}x \geq 0, \text{ for all } \varphi \in H^1_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega), \ \varphi \geq 0 \tag{7.1.3}$$

that are compactly supported in Ω. We say that such u is a positive *supersolution* of $-\Delta - V$ on Ω. These ideas have been deepened in the subsequent years by S. Agmon, who laid the foundations of the so called "Criticality Theory". This connection between existence of positive supersolutions and nonnegativity of the corresponding quadratic form is then known as the Agmon-Allegretto-Piepenbrink's principle (AAP principle in the following). Since then, this concept has been investigated in various flavours and a vast literature flourished, see among many others [Agm83, Pin89, Pin90, PV20] and the comprehensive work by Murata [Mur86]. We also refer to [Mor20] for an introduction to the AAP principle and to some of its possible applications.

The AAP principle is based, in turn, on the well known Picone's identity (frequently used as inequality), which we here recall.

Lemma 7.1.1 (Picone's identity)**.** *Let $u, v \in H^1_{\mathrm{loc}}(\Omega)$ be such that $v \geq 0$, $u > 0$ a.e. in Ω. Then*

$$|\nabla v|^2 - \nabla \left(\frac{v^2}{u} \right) \cdot \nabla u = \left| \nabla \left(\frac{v}{u} \right) \right|^2 u^2 \geq 0 \quad \text{a.e. in } \Omega.$$

We now state the AAP principle. For the first part of the statement, the minimal assumptions on the potential V are $V \in L^1_{\mathrm{loc}}(\Omega)$ and $V^- \in L^\infty_{\mathrm{loc}}(\Omega)$ (with V^- denoting the negative part of V), while for the second part $V \in L^\infty_{\mathrm{loc}}(\Omega)$.

Theorem 7.1.2 (AAP Positivity Principle)**.** *Let $f \in L^1_{\mathrm{loc}}(\Omega)$ be such that $f \geq 0$ a.e. If there exists a positive supersolution to (7.1.2), that is a function $u \in H^1_{\mathrm{loc}}(\Omega)$ such that $u \geq 0$ a.e. in Ω and*

$$\int_\Omega (\nabla u \cdot \nabla \varphi - V u \varphi) \, \mathrm{d}x \geq \int_\Omega f \varphi, \quad \text{for all } \varphi \in C_c^\infty(\Omega)$$

then we have that

$$Q_V(\varphi) = \int_\Omega \left(|\nabla \varphi|^2 - V \varphi^2 \right) \mathrm{d}x \geq \int_\Omega f \frac{\varphi^2}{u} \, \mathrm{d}x \geq 0$$

for all $\varphi \in C_c^\infty(\Omega)$. Conversely, if $Q_V(\varphi) \geq 0$ for all $\varphi \in C_c^\infty(\Omega)$, then there exists $u \in H^1_{\mathrm{loc}}(\Omega)$ positive supersolution of $-\Delta - V$.

Proof. We prove only the first part, since the second one requires more involved arguments, which go beyond the scope of this introduction. We refer to [Agm83, Theorem 3.1] for the proof of the second part.

Since $f \geq 0$ then u is a supersolution of $-\Delta - V$ and then

$$\operatorname*{ess\,sup}_K u > 0 \quad \text{for any compact } K \subseteq \Omega,$$

thus implying that $u^{-1} \in L^\infty_{\mathrm{loc}}(\Omega)$ This is a consequence of the weak Harnack inequality, see e.g. [GT83, Theorem 8.19]. Hence, for any $\varphi \in C_c^\infty(\Omega)$ we have that φ^2/u is an admissible test function for (7.1.3) and so

$$\int_\Omega \left(\nabla u \cdot \nabla \left(\frac{\varphi^2}{u} \right) - V\varphi^2 \right) \mathrm{d}x \geq \int_\Omega f \frac{\varphi^2}{u} \,\mathrm{d}x.$$

By Picone's identity Lemma 7.1.1 we have that

$$\nabla u \cdot \nabla \left(\frac{\varphi^2}{u} \right) = |\nabla \varphi|^2 - \left| \nabla \left(\frac{\varphi}{u} \right) \right|^2 u^2,$$

therefore we obtain that

$$\begin{aligned} Q_V(\varphi) &= \int_\Omega \left(\nabla u \cdot \nabla \left(\frac{\varphi^2}{u} \right) + \left| \nabla \left(\frac{\varphi}{u} \right) \right|^2 u^2 - V\varphi^2 \right) \mathrm{d}x \\ &\geq \int_\Omega \left(f \frac{\varphi^2}{u} + \left| \nabla \left(\frac{\varphi}{u} \right) \right|^2 u^2 \right) \mathrm{d}x \geq \int_\Omega f \frac{\varphi^2}{u} \,\mathrm{d}x. \end{aligned}$$

□

Remark 7.1.3. The functional

$$Q_u(\varphi) := \int_\Omega \left| \nabla \left(\frac{\varphi}{u} \right) \right|^2 u^2 \,\mathrm{d}x$$

is known in the literature as the "u-ground state transform" quadratic form and it is generated by the nonnegative self-adjoint operator

$$-\Delta - 2\nabla \log(u) \cdot \nabla$$

in $L^2(\Omega, u^2 \,\mathrm{d}x)$.

We now state a straightforward corollary to Theorem 7.1.2 useful in order to prove nonexistence of positive supersolutions.

Corollary 7.1.4. *Let $f \in L^1_{\mathrm{loc}}(\Omega)$ be such that $f \geq 0$ a.e. in Ω. If there exists $\varphi_0 \in C_c^\infty(\Omega)$ such that*

$$Q_V(\varphi_0) < 0,$$

then there are no positive supersolutions to (7.1.2).

Now, for any nonnegative $p \in L^1_{\text{loc}}(\Omega)$ such that $p^{-1} \in L^\infty_{\text{loc}}(\Omega)$, we define

$$m_p(V) := \inf_{\varphi \in C_c^\infty(\Omega)} \frac{Q_V(\varphi)}{\displaystyle\int_\Omega p\varphi^2 \,\mathrm{d}x}. \tag{7.1.4}$$

Definition 7.1.5. We say that the quadratic form Q_V *satisfies the p-property* if there exists a nonnegative $p \in L^1_{\text{loc}}(\Omega)$ such that $p^{-1} \in L^\infty_{\text{loc}}(\Omega)$ for which $m_p(V) > 0$.

The aforementioned criticality theory is based on the following classification.

Definition 7.1.6. We say that

1. $-\Delta - V$ is *subcritical* if Q_V satisfies the p-property;

2. $-\Delta - V$ is *critical* if Q_V is nonnegative, but does not satisfies the p-property;

3. $-\Delta - V$ is *supercritical* if Q_V is not nonnegative.

It is possible to prove that this classifications is meaningful in terms of existence and uniqueness of positive supersolutions. Indeed the following result holds true. We refer to [Agm83] for the proofs.

Proposition 7.1.7. *The following holds:*

1. *If $-\Delta - V$ is subcritical, then for any $f \in L^2(\Omega, p^{-1}\,\mathrm{d}x)$, with $f \geq 0$, there exists a positive supersolution to (7.1.2);*

2. *if $-\Delta - V$ is critical, then there exists a unique (up to a scalar factor) positive supersolution to $-\Delta - V$, which is actually a solution;*

3. *if $-\Delta - V$ is supercritical, then $-\Delta - V$ has no positive supersolutions, see Corollary 7.1.4.*

We also observe that the Schrödinger operator is subcritical if and only if the inequality

$$\int_\Omega V\varphi^2 \,\mathrm{d}x \leq \int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x \quad \text{for all } \varphi \in C_c^\infty(\Omega)$$

can be improved with a positive term $\int_\Omega p\varphi^2 \,\mathrm{d}x$ on the left hand side.

The notion of criticality of a Schrödinger operator is closely related to the notion of coercivity of the operator in the Beppo-Levi space $\mathcal{D}^{1,2}(\Omega)$, which basically is, in some sense, the object of our investigation in the next section. Here $\mathcal{D}^{1,2}(\Omega)$ denotes the completion of $C_c^\infty(\Omega)$ with respect to the L^2-norm of the gradient. Let us now define

$$\sigma(V) := \inf_{\varphi \in C_c^\infty(\Omega)} \frac{Q_V(\varphi)}{\displaystyle\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x}.$$

The quantities $m_p(V)$ and $\sigma(V)$ are related in the following way when one considers more regular weights p (see [FMT07, Remark 7.4] for the justification).

Lemma 7.1.8. *If $p \in L^{N/2}(\Omega) \cap C(\Omega)$ is such that $p > 0$ in Ω, then $m_p(V)$ and $\sigma(V)$ have the same sign.*

This result essentially relies on the following positivity principle, whose proof is inspired by AAP principle's one and exploits ground state transform, Picone's inequality and continuity of $\sigma(V)$ with respect to the potential V. We refer to [FMT07, Lemma 2.1] for the proof in a more general setting.

Lemma 7.1.9. *Let $V \in C_c^\infty(\Omega)$. The following are equivalent:*

(i) $\sigma(V) > 0$;

(ii) there exists $\varepsilon > 0$ and $u \in \mathcal{D}^{1,2}(\Omega) \cap C^1(\Omega)$ such that u is a positive supersolution to $-\Delta - (V + \varepsilon V^+)$.

The other ingredient we employ for our aims is the so called "localization of binding". Let us first introduce the setting. For $\Omega = \mathbb{R}^N$, let us consider two potentials $V_i \in C_c^\infty(\mathbb{R}^N)$, $i = 1, 2$ (again, the results we present actually hold for a more general class of potentials) and let us assume that the quadratic forms Q_{V_i}'s are nonnegative. The question we address is the following:

is it true that the quadratic form $Q_{V_1 - V_2(\cdot - y)}$, associated to the potential

$$x \mapsto V_1(x) + V_2(x - y)$$

is nonnegative, for $y \in \mathbb{R}^N$ and $|y|$ sufficiently large? (7.1.5)

The motivation for this type of questions comes from physics and, in particular, from a notable phenomenon discovered in 1970, called the Efimov's effect, named after the scientist who first noticed and studied it, see [Efi70, Efi73]. This effect concerns a three-body Schrödinger operator

$$-\Delta - \sum_{\substack{i,j=1 \\ i<j}}^{3} V_{i,j}$$

with $V_{i,j} : \mathbb{R}^3 \to \mathbb{R}$, and reads as follows: if the three operators $-\Delta - V_{i,j}$, corresponding to the two-body subsystems, are subcritical, then the three-particle operator $-\Delta - \sum_{i<j} V_{i,j}$, by the Efimov's effect, has an infinite number of negative L^2-eigenvalues accumulating to zero. This phenomenon, from the mathematical point of view, turns out to be closely related to the problem of localization of binding, stated in (7.1.5). This was first pointed out in [OS79], where the authors gave a variational proof of the Efimov's effect. We also refer to [Tam91, Tam93] for further generalizations. On the other hand, the problem of localization of binding became of independent interest in the mathematical literature. Its study started in [KS79], where the authors examined the behavior of the ground state energy $E(y)$ of the operator $-\Delta - (V_1 + V_2(\cdot - y))$, as a function of the translation vector $y \in \mathbb{R}^N$ (see also [Sim80]). The most complete positive answer to (7.1.5), to the best of our knowledge, has been given by Pinchover in [Pin95]. We report here one of the main

theorems, assuming the potentials in the so called Kato class at infinity. We first recall the definition and then state the result.

Definition 7.1.10 ([Pin95], Definition 2.3)**.** We define the *Kato class at infinity* as follows

$$K_N^\infty := \left\{ V \in C^{0,\alpha}(\mathbb{R}^N) \colon \lim_{M\to\infty} \sup_{x\in\mathbb{R}^N} \int_{|y|>M} \frac{|V(x)|}{|x-y|^{N-2}} \, \mathrm{d}x = 0 \right\}.$$

Theorem 7.1.11 ([Pin95], Theorem 3.1)**.** *Let V_i be in the Kato's class at infinity, for $i = 1, 2$. If $-\Delta - V_i$ are subcritical for $i = 1, 2$, then*

$$-\Delta - (V_1 + V_2(\cdot - y))$$

is subcritical for $y \in \mathbb{R}^N$ and $|y|$ sufficiently large.

In view of the connection asserted in Lemma 7.1.8, we finally cite [FMT07, Theorem 1.5]. This is an analogous result which allows the potentials to be outside the Kato's class and to takes into consideration also combinations of Hardy-type homogeneous functions.

7.2 The fractional multipolar problem

Let $s \in (0, 1)$ and $N > 2s$. Let us consider $m \geq 1$ real numbers $\mu_1, \dots, \mu_m$ (sometimes called *masses*) and m *poles* $a_1, \dots, a_m \in \mathbb{R}^N$ such that $a_i \neq a_j$ for all $i, j = 1, \dots, m,\ i \neq j$. The main object of our investigation is the operator

$$\mathcal{L}_{\mu_1,\dots,\mu_m,a_1,\dots,a_m} := (-\Delta)^s - \sum_{i=1}^m \frac{\mu_i}{|x - a_i|^{2s}} \qquad \text{in } \mathbb{R}^N. \tag{7.2.1}$$

Here $(-\Delta)^s$ denotes the fractional Laplace operator, which acts on functions $\varphi \in C_c^\infty(\mathbb{R}^N)$ as

$$(-\Delta)^s\varphi(x) := C(N,s)\,\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} \, \mathrm{d}y = C(N,s) \lim_{\rho\to 0^+} \int_{|x-y|>\rho} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} \, \mathrm{d}y,$$

where P.V. means that the integral has to be seen in the principal value sense and

$$C(N,s) = \pi^{-\frac{N}{2}} 2^{2s} \frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma(2-s)} s(1-s),$$

with Γ denoting the Euler's Gamma function. Hereafter, we refer to an operator of the type $(-\Delta)^s - V$ as a *fractional Schrödinger operator* with potential V.

One of the reasons of mathematical interest in operators of type (7.2.1) lies in the criticality of potentials of order $-2s$, which have the same scaling rate as the s-fractional Laplacian.

We introduce, on $C_c^\infty(\mathbb{R}^N)$, the following positive definite bilinear form, associated to $(-\Delta)^s$

$$(u,v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} := \frac{1}{2}C(N,s)\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \tag{7.2.2}$$

and we define the space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}$ induced by the scalar product (7.2.2). Moreover, the following quadratic form is naturally associated to the operator $\mathcal{L}_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}$

$$\begin{aligned} Q_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}(u) :&= \frac{1}{2}C(N,s)\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y - \sum_{i=1}^m \mu_i \int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x-a_i|^{2s}}\,\mathrm{d}x \\ &= \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^m \mu_i \int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x-a_i|^{2s}}\,\mathrm{d}x. \end{aligned} \tag{7.2.3}$$

We observe that $Q_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}$ is well-defined on $\mathcal{D}^{s,2}(\mathbb{R}^N)$ thanks to the validity of the fractional Hardy inequality:

$$\gamma_H \int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^{2s}}\,\mathrm{d}x \le \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \quad \text{for all } u\in\mathcal{D}^{s,2}(\mathbb{R}^N), \tag{7.2.4}$$

where the constant

$$\gamma_H = \gamma_H(N,s) := 2^{2s}\frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}$$

is optimal and not attained.

One goal of the present chapter is to find necessary and sufficient conditions (on the masses $\mu_1,\dots,\mu_m$) for the existence of a configuration of poles $(a_1,\dots,a_m)$ that guarantees the positivity of the quadratic form (7.2.3), extending to the fractional case some results obtained in [FMT07] for the classical Laplacian. The quadratic form $Q_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}$ is said to be *positive definite* if

$$\inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{Q_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}(u)}{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}} > 0.$$

In the case of a single pole (i.e. $m=1$), the fractional Hardy inequality (7.2.4) immediately answers the question of positivity: the quadratic form $Q_{\mu,a}$ is positive definite if and only if $\mu < \gamma_H$. Hence our interest in multipolar potentials is justified by the fact that the location of the poles (in particular the shape of the configuration) could play some role in the positivity of (7.2.3). Furthermore, one could expect that some other conditions on the masses may arise when $m>1$. We mention that several authors have approached the problem of multipolar singular potentials, both for the classical Laplacian, see e.g. [BGG20, BDE08, CZ13, dAFP17, FFK18, FM13] and for the fractional case, see [FCnC17].

In the classical (local) case, the problem of positivity of Schrödinger operators with multi-singular Hardy-type potentials was addressed in [FMT07]. In that article, the authors tackled the problem making use of a *localization of binding* result that provides, under certain assumptions, the positivity of the sum of two positive operators, by translating one of them through a sufficiently long vector. This argument is based, in turn, on a criterion which relates the positivity of an operator to the existence of a positive supersolution, in the spirit of Allegretto-Piepenbrink Theory (see [All74, Pie74] and Section 7.1). As one can observe in [FMT07], the strong suit of the local case is that the study of the action of the operator can be substantially reduced to neighborhoods of the singularities. However, this is not possible in the fractional context due to nonlocal effects: we overcome this issue by taking into consideration the Caffarelli-Silvestre extension (6.1.6) described in Section 6.1, which yields a local formulation of the problem.

The equivalence between the fractional problem in $\mathbb{R}^N$ and the Caffarelli-Silvestre extension problem in $\mathbb{R}^{N+1}_+$ allows us to characterize the coercivity properties of quadratic forms on $\mathcal{D}^{s,2}(\mathbb{R}^N)$ in terms of quadratic forms on $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$. We say that a function $V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies the *form-bounded condition* if

$$\sup_{\substack{u\in\mathcal{D}^{s,2}(\mathbb{R}^N)\\ u\not\equiv 0}} \frac{\int_{\mathbb{R}^N}|V(x)|u^2(x)\,\mathrm{d}x}{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}} < +\infty. \tag{FB}$$

Let $\mathcal{H}$ be the class of potentials satisfying the form-bounded condition, i.e.

$$\mathcal{H} = \{V \in L^1_{\mathrm{loc}}(\mathbb{R}^N) : V \text{ satisfies } (FB)\}.$$

It is easy to understand that, if $V \in \mathcal{H}$, then $Vu \in (\mathcal{D}^{s,2}(\mathbb{R}^N))^\star$ for all $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ and the quadratic form $u \mapsto \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} Vu^2$ is well defined in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. For all $V \in \mathcal{H}$ we define

$$\sigma(V) = \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} Vu^2\,\mathrm{d}x}{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}} \tag{7.2.5}$$

and observe that $\sigma(V) > -\infty$. The first taste of the relation between the fractional and the extended problem is given by the following result.

Lemma 7.2.1. *Let $V \in \mathcal{H}$. Then*

$$\sigma(V) = \inf_{\substack{U\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})\\ U\not\equiv 0}} \frac{\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2\,\mathrm{d}x\mathrm{d}t - \kappa_s\int_{\mathbb{R}^N} V|\operatorname{Tr} U|^2\,\mathrm{d}x}{\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2\,\mathrm{d}x\mathrm{d}t}. \tag{7.2.6}$$

In the present chapter we focus our attention on the following class of potentials

$$\Theta := \Bigg\{V(x) = \sum_{i=1}^m \frac{\mu_i\chi_{B'(a_i,r_i)}(x)}{|x-a_i|^{2s}} + \frac{\mu_\infty\chi_{\mathbb{R}^N\setminus B'_R}(x)}{|x|^{2s}} + W(x)\colon r_i, R>0,\ m\in\mathbb{N},$$
$$a_i\in\mathbb{R}^N,\ a_i\neq a_j \text{ for } i\neq j,\ \mu_i,\mu_\infty<\gamma_H,\ W\in L^{N/2s}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\Bigg\},$$

where, for any $r>0$ and $x\in\mathbb{R}^N$, we denote

$$B'(x,r) := \{y\in\mathbb{R}^N:\ |y-x|<r\} \quad \text{and} \quad B'_r := B'(0,r).$$

We observe that, when considering a potential $V\in\Theta$, it is not restrictive to assume that the sets $B'(a_i,r_i)$ and $\mathbb{R}^N\setminus B'_R$ appearing in its representation are mutually disjoint, up to redefining the remainder W.

It is easy to see that, for instance,

$$\sum_{i=1}^m \frac{\mu_i}{|x-a_i|^{2s}} \in \Theta, \quad \text{when } \mu_i<\gamma_H \text{ for all } i=1,\dots,m \text{ and } \sum_{i=1}^m \mu_i<\gamma_H.$$

We observe that any $V\in\Theta$ satisfies the form-bounded condition, i.e. $\Theta\subset\mathcal{H}$, thanks to the fractional Hardy and Sobolev inequalities stated in theorems 6.0.1 and 6.0.2 respectively.

Our first main result is a criterion that provides the equivalence between the positivity of $\sigma(V)$ for potentials $V\in\Theta$ and the existence of a positive supersolution to a certain (possibly perturbed) problem. This criterion is reminiscent of the Agmon-Allegretto-Piepenbrink Theory, developed in 1974 in [All74, Pie74] (see also [Agm83, Agm85, MP78, PP16]), that we briefly recalled in Section 7.1 . As far as we know, the result contained in the following lemma is new in the nonlocal framework; nevertheless, some tools from the Agmon-Allegretto-Piepenbrink Theory have been used in [FLS08, MVS12] to prove some Hardy-type fractional inequalities.

Lemma 7.2.2 (Positivity Criterion)**.** *Let* $V=\sum_{i=1}^m \frac{\mu_i\chi_{B'(a_i,r_i)}(x)}{|x-a_i|^{2s}}+\frac{\mu_\infty\chi_{\mathbb{R}^N\setminus B'_R}(x)}{|x|^{2s}}+W(x)\in\Theta$ *and let* $\tilde V\in L^\infty_{\mathrm{loc}}(\mathbb{R}^N\setminus\{a_1,\dots,a_m\})$ *be such that* $V\le\tilde V\le|V|$ *a.e. in* $\mathbb{R}^N$*. The following two assertions hold true.*

(I) Assume that there exist some $\varepsilon>0$ *and a function* $\Phi\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$ *such that* $\Phi>0$ *in* $\overline{\mathbb{R}^{N+1}_+}\setminus\{(0,a_1),\dots,(0,a_m)\}$*,* $\Phi\in C^0\left(\overline{\mathbb{R}^{N+1}_+}\setminus\{(0,a_1),\dots,(0,a_m)\}\right)$*, and*

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla\Phi\cdot\nabla U\,\mathrm{d}x\mathrm{d}t \ge \kappa_s\int_{\mathbb{R}^N}(V+\varepsilon\tilde V)\operatorname{Tr}\Phi\operatorname{Tr}U\,\mathrm{d}x, \tag{7.2.7}$$

for all $U\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$*,* $U\ge0$ *a.e. in* $\mathbb{R}^{N+1}_+$*. Then*

$$\sigma(V)\ge\varepsilon/(\varepsilon+1). \tag{7.2.8}$$

(II) Conversely, assume that $\sigma(V)>0$*. Then there exist* $\varepsilon>0$ *(not depending on* $\tilde V$*) and* $\Phi\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$ *such that* $\Phi\in C^0\left(\overline{\mathbb{R}^{N+1}_+}\setminus\{(0,a_1),\dots,(0,a_m)\}\right)$*,* $\Phi>0$ *in* $\mathbb{R}^{N+1}_+$*,* $\Phi\ge0$ *in* $\overline{\mathbb{R}^{N+1}_+}\setminus\{(0,a_1),\dots,(0,a_m)\}$*, and (7.2.7) holds for every* $U\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$ *satisfying* $U\ge0$ *a.e. in* $\mathbb{R}^{N+1}_+$*. If, in addition, we assume that* V *and* $\tilde V$ *are locally Hölder continuous in* $\mathbb{R}^N\setminus\{a_1,\dots,a_m\}$*, then* $\Phi>0$ *in* $\overline{\mathbb{R}^{N+1}_+}\setminus\{(0,a_1),\dots,(0,a_m)\}$*.*

In order to use statement (I) to obtain positivity of a given Schrödinger operator with potential in Θ, it is crucial to exhibit a weak supersolution to the corresponding Schrödinger equation, i.e. a function satisfying (7.2.7), which is *strictly positive* outside the poles. Nevertheless, the application of maximum principles to prove positivity of solutions to singular/degenerate extension problems is more delicate than in the classic case, due to regularity issues (see the Hopf type principle proved in [CS14, Proposition 4.11] and recalled in Proposition 7.3.2). For this reason, in order to apply the above criterion in Sections 7.7 and 7.8, we will develop an approximation argument introducing a class of more regular potentials (see (7.7.2)).

The following theorem, whose proof heavily relies on Lemma 7.2.2, fits in the theory of *localization of binding*, whose aim is study the bottom of the spectrum of Schrödinger operators of the type

$$-\Delta + V_1 + V_2(\cdot - y), \quad y \in \mathbb{R}^N,$$

in relation to the potentials V_1 and V_2 and to the translation vector $y \in \mathbb{R}^N$. The case in which V_1 and V_2 belong to the Kato class has been studied in [Pin95] (see Theorem 7.1.11), while Simon in [Sim80] analyzed the case of compactly supported potentials; singular inverse square potentials were instead considered in [FMT07] (we refer to Section 7.1 for a more detailed overview). Our result concerns the fractional case and provides sufficient conditions on the potentials and on the length of the translation for the positivity of the corresponding fractional Schrödinger operator.

Theorem 7.2.3 (Localization of Binding)**.** *Let*

$$V_1(x) = \sum_{i=1}^{m_1} \frac{\mu_i^1 \chi_{B'(a_i^1, r_i^1)}(x)}{|x - a_i^1|^{2s}} + \frac{\mu_\infty^1 \chi_{\mathbb{R}^N \setminus B'_{R_1}}(x)}{|x|^{2s}} + W_1(x) \in \Theta,$$
$$V_2(x) = \sum_{i=1}^{m_2} \frac{\mu_i^2 \chi_{B'(a_i^2, r_i^2)}(x)}{|x - a_i^2|^{2s}} + \frac{\mu_\infty^2 \chi_{\mathbb{R}^N \setminus B'_{R_2}}(x)}{|x|^{2s}} + W_2(x) \in \Theta,$$

and assume $\sigma(V_1), \sigma(V_2) > 0$ and $\mu_\infty^1 + \mu_\infty^2 < \gamma_H$. Then there exists $R > 0$ such that, for every $y \in \mathbb{R}^N \setminus \overline{B'_R}$,

$$\sigma(V_1(\cdot) + V_2(\cdot - y)) > 0.$$

Combining the previous theorem with an inductive procedure on the number of poles m, we obtain a necessary and sufficient condition for positivity of the operator (7.2.1).

Theorem 7.2.4. *Let $(\mu_1, \dots \mu_m) \in \mathbb{R}^m$. Then*

$$\mu_i < \gamma_H \quad \text{for all } i = 1, \dots, m, \quad \text{and} \quad \sum_{i=1}^m \mu_i < \gamma_H \tag{7.2.9}$$

is a necessary and sufficient condition for the existence of at least a configuration of poles $(a_1, \dots, a_m)$ such that the quadratic form $Q_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}$ associated to the operator $\mathcal{L}_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}$ is positive definite.

Besides the interest in the existence of a configuration of poles making $Q_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}$ positive definite, one can search for a condition on the masses $\mu_1,\dots,\mu_m$ that guarantees the positivity of this quadratic form for every configuration of poles; in this direction, an answer is given by the following theorem (we refer to [FT06, Proposition 1.2] for an analogous result in the classical case of the Laplacian with multipolar inverse square potentials).

Theorem 7.2.5. *Let* $t^+ := \max\{0,t\}$. *If*

$$\sum_{i=1}^{m} \mu_i^+ < \gamma_H, \tag{7.2.10}$$

then the quadratic form $Q_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}$ *is positive definite for all* $a_1,\dots,a_m \in \mathbb{R}^N$. *Conversely, if*

$$\sum_{i=1}^{m} \mu_i^+ > \gamma_H$$

then there exists a configuration of poles $(a_1,\dots,a_m)$ *such that* $Q_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}$ *is not positive definite.*

Finally, it is natural to ask whether $\sigma(V)$, defined as an infimum in (7.2.6), is attained or not. In the case of a single pole, it is known that the infimum is not achieved, see e.g. [FLS08]; however, when dealing with multiple singularities, the outcome can be different. Indeed, for V in the class Θ, we have that $\sigma(V) \leq 1 - \frac{1}{\gamma_H} \max_{i=1,\dots,m,\infty} \mu_i$, see Lemma 7.6.1, and the infimum is attained in the case of strict inequality, as established in the following proposition.

Proposition 7.2.6. *If* $V \in \Theta$ *is such that*

$$\sigma(V) < 1 - \frac{1}{\gamma_H} \max\{0, \mu_1, \dots, \mu_m, \mu_\infty\}, \tag{7.2.11}$$

then $\sigma(V)$ *is attained.*

The rest of the chapter is organized as follows. In Section 7.3 we prove some estimates for solutions of certain differential equations in subsets of the upper half space. In Section 7.4 we prove Theorem 7.2.5. In Section 7.5 we prove the positivity criterion, i.e. Lemma 7.2.2, while in Section 7.6 we look for upper and lower bounds of the quantity $\sigma(V)$. In Section 7.7 we investigate the persistence of the positivity of $\sigma(V)$, when the potential V is subject to a perturbation far from the origin or close to a pole. Section 7.8 is devoted to the proof of Theorem 7.2.3, that is the primary tool used in the proof of Theorem 7.2.4, pursued in Section 7.9. Finally, in Section 7.10 we prove Proposition 7.2.6.

7.3 Preliminaries

In this section we recall a couple of essential regularity results for solutions of A_2-weighted elliptic boundary value problems and we prove asymptotic estimates for these solutions; this passes through the introduction of an auxiliary eigenvalue problem on the upper half sphere.

7.3.1 Regularity of solutions

The first regularity result is due to Fall and Felli [FF15b] and Jin, Li and Xiong [JLX14] and reads as follows.

Proposition 7.3.1 ([FF15b] Proposition 3, [JLX14] Proposition 2.6)**.** *Let $a, b \in L^p(B_1')$, for some $p > \frac{N}{2s}$ and $c, d \in L^q(B_1^+; t^{1-2s})$, for some $q > \frac{N+2-2s}{2}$. Let $w \in H^1(B_1^+; t^{1-2s})$ be a weak solution of*

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla w) + t^{1-2s}c(z)w = t^{1-2s}d(z), & \text{in } B_1^+, \\ -\lim_{t\to 0^+} t^{1-2s}\frac{\partial w}{\partial t} = a(x)w + b(x), & \text{on } B_1'. \end{cases}$$

Then $w \in C^{0,\beta}(\overline{B_{1/2}^+})$ and in addition

$$\|w\|_{C^{0,\beta}(\overline{B_{1/2}^+})} \leq C\left(\|w\|_{L^2(B_1^+)} + \|b\|_{L^p(B_1')} + \|d\|_{L^q(B_1^+;t^{1-2s})}\right),$$

with $C, \beta > 0$ depending only on $N, s, \|a\|_{L^p(B_1')}, \|c\|_{L^q(B_1^+;t^{1-2s})}$.

Now we recall, from [CS14], an Hopf-type Lemma.

Proposition 7.3.2 ([CS14] Proposition 4.11)**.** *Let $\Phi \in C^0(B_R^+ \cup B_R') \cap H^1(B_R^+; t^{1-2s})$ satisfy*

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla\Phi) \geq 0, & \text{in } B_R^+, \\ \Phi > 0, & \text{in } B_R^+, \\ \Phi(0,0) = 0. \end{cases}$$

Then

$$-\limsup_{t\to 0^+} t^{1-2s}\frac{\Phi(t,0)}{t} < 0.$$

In addition, if

$$t^{1-2s}\frac{\partial\Phi}{\partial t} \in C^0(B_R^+ \cup B_R'), \tag{7.3.1}$$

then

$$-\left(t^{1-2s}\frac{\partial\Phi}{\partial t}\right)(0,0) < 0.$$

In several points of the present chapter we use the following result from [CS14] to verify the validity of assumption (7.3.1) needed to apply Proposition 7.3.2.

Lemma 7.3.3 ([CS14] Lemma 4.5)**.** *Let $s \in (0,1)$ and $R > 0$. Let $\varphi \in C^{0,\alpha}(B_{2R}')$ for some $\alpha \in (0,1)$ and $\Phi \in L^\infty(B_{2R}^+) \cap H^1(B_{2R}^+; t^{1-2s})$ be a weak solution to*

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla\Phi) = 0, & \text{in } B_{2R}^+, \\ -\lim_{t\to 0^+} t^{1-2s}\frac{\partial\Phi}{\partial t} = \varphi(x), & \text{on } B_{2R}'. \end{cases}$$

Then there exists $\beta \in (0,1)$ depending only on N, s, α such that

$$\Phi \in C^{0,\beta}(\overline{B_R^+}) \quad \text{and} \quad t^{1-2s}\frac{\partial\Phi}{\partial t} \in C^{0,\beta}(\overline{B_R^+}).$$

7.3.2 The Angular Eigenvalue Problem

Let us consider, for any $\mu \in \mathbb{R}$, the problem

$$\begin{cases} -\operatorname{div}_{\mathbb{S}^N}(\theta_1^{1-2s}\nabla_{\mathbb{S}^N}\psi) = \sigma\theta_1^{1-2s}\psi, & \text{in } \mathbb{S}^N_+, \\ -\lim\limits_{\theta_1\to 0^+} \theta_1^{1-2s}\nabla_{\mathbb{S}^N}\psi\cdot \boldsymbol{e}_1 = \kappa_s\mu\psi, & \text{on } \mathbb{S}^{N-1}, \end{cases} \tag{7.3.2}$$

where $\boldsymbol{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^{N+1}_+$ and $\nabla_{\mathbb{S}^N}$ denotes the gradient on the unit N-dimensional sphere $\mathbb{S}^N$. In order to give a variational formulation of (7.3.2) we introduce the following Sobolev space

$$H^1(\mathbb{S}^N_+;\theta_1^{1-2s}) := \left\{\psi \in L^2(\mathbb{S}^N_+;\theta_1^{1-2s})\colon \int_{\mathbb{S}_N^+} \theta_1^{1-2s}|\nabla_{\mathbb{S}^N}\psi|^2\, dS < +\infty\right\}.$$

We say that $\psi \in H^1(\mathbb{S}^N_+;\theta_1^{1-2s})$ and $\sigma\in\mathbb{R}$ weakly solve (7.3.2) if

$$\int_{\mathbb{S}^N_+} \theta_1^{1-2s}\nabla_{\mathbb{S}^N}\psi(\theta)\cdot\nabla_{\mathbb{S}^N}\varphi(\theta)\,\mathrm{d}S = \sigma\int_{\mathbb{S}^N_+}\theta_1^{1-2s}\psi(\theta)\varphi(\theta)\,\mathrm{d}S + \kappa_s\mu\int_{\mathbb{S}^{N-1}}\psi(0,\theta')\varphi(0,\theta')\,\mathrm{d}\sigma$$

for all $\varphi \in H^1(\mathbb{S}^N_+;\theta_1^{1-2s})$. By standard spectral arguments, if $\mu < \gamma_H$, there exists a diverging sequence of real eigenvalues of problem (7.3.2)

$$\sigma_1(\mu) \le \sigma_2(\mu) \le \cdots \le \sigma_n(\mu) \le \cdots$$

Moreover, each eigenvalue has finite multiplicity (which is counted in the enumeration above) and $\sigma_1(\mu) > -\left(\frac{N-2s}{2}\right)^2$ (see [FF14, Lemma 2.2]). For every $n \ge 1$ we choose an eigenfunction $\psi_n \in H^1(\mathbb{S}^N_+;\theta_1^{1-2s})\setminus\{0\}$, corresponding to $\sigma_n(\mu)$, such that

$$\int_{\mathbb{S}^N_+}\theta_1^{1-2s}\,|\psi_n|^2\,\mathrm{d}S = 1.$$

In addition, we choose the family $\{\psi_n\}_n$ in such a way that it is an orthonormal basis of $L^2(\mathbb{S}^N_+;\theta_1^{1-2s})$. We refer to [FF14] for further details.

In [FF14] the following implicit characterization of $\sigma_1(\mu)$ is given. For any $\alpha \in \left(0, \frac{N-2s}{2}\right)$ we define

$$\Lambda(\alpha) := 2^{2s}\frac{\Gamma\left(\frac{N+2s+2\alpha}{4}\right)\Gamma\left(\frac{N+2s-2\alpha}{4}\right)}{\Gamma\left(\frac{N-2s+2\alpha}{4}\right)\Gamma\left(\frac{N-2s-2\alpha}{4}\right)}. \tag{7.3.3}$$

It is known (see e.g. [FLS08] and [FF14, Proposition 2.3]) that the map $\alpha\mapsto\Lambda(\alpha)$ is continuous and monotone decreasing. Moreover

$$\lim_{\alpha\to 0^+}\Lambda(\alpha) = \gamma_H, \qquad \lim_{\alpha\to\frac{N-2s}{2}}\Lambda(\alpha) = 0. \tag{7.3.4}$$

Furthermore, in [FF14, Proposition 2.3] it is proved that, for every $\alpha \in \left(0, \frac{N-2s}{2}\right)$,

$$\sigma_1(\Lambda(\alpha)) = \alpha^2 - \left(\frac{N-2s}{2}\right)^2. \tag{7.3.5}$$

In particular, for every $\mu \in (0, \gamma_H)$ there exists one and only one $\alpha \in (0, \frac{N-2s}{2})$ such that $\Lambda(\alpha) = \mu$ and hence $\sigma_1(\mu) = \alpha^2 - (\frac{N-2s}{2})^2 < 0$.

We recall the following result from [Fal20].

Lemma 7.3.4 ([Fal20, Lemma 4.1]). *For every $\alpha \in \left(0, \frac{N-2s}{2}\right)$ there exists $\Upsilon_\alpha : \overline{\mathbb{R}^{N+1}_+} \setminus \{0\} \to \mathbb{R}$ such that Υ_α is locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{0\}$, $\Upsilon_\alpha > 0$ in $\overline{\mathbb{R}^{N+1}_+} \setminus \{0\}$, and*

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Upsilon_\alpha) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ \Upsilon_\alpha(0,x) = |x|^{-\frac{N-2s}{2}+\alpha}, & \text{on } \mathbb{R}^N, \\ -\lim\limits_{t\to 0} t^{1-2s}\dfrac{\partial \Upsilon_\alpha}{\partial t} = \kappa_s \Lambda(\alpha)\, |x|^{-2s}\, \Upsilon_\alpha, & \text{on } \mathbb{R}^N, \end{cases} \tag{7.3.6}$$

in a weak sense. Moreover, $\Upsilon_\alpha \in H^1(B_R^+; t^{1-2s})$ for every $R > 0$.

The first eigenvalue $\sigma_1(\mu)$ satisfies the properties described in the following lemma.

Lemma 7.3.5. *Let $\mu < \gamma_H$. Then the first eigenvalue of problem (7.3.2) can be characterized as*

$$\sigma_1(\mu) = \inf_{\substack{\psi \in H^1(\mathbb{S}^N_+;\theta_1^{1-2s}) \\ \psi \not\equiv 0}} \frac{\displaystyle\int_{\mathbb{S}^N_+} \theta_1^{1-2s}\, |\nabla_{\mathbb{S}^N}\psi|^2 \, dS - \kappa_s\mu \int_{\mathbb{S}^{N-1}} |\psi|^2 \, d\sigma}{\displaystyle\int_{\mathbb{S}^N_+} \theta_1^{1-2s}\, |\psi|^2 \, dS}$$

and the above infimum is attained by $\psi_1 \in H^1(\mathbb{S}^N_+; \theta_1^{1-2s})$, which weakly solves (7.3.2) for $\sigma = \sigma_1(\mu)$. Moreover

1. *$\sigma_1(\mu)$ is simple, i.e. if ψ attains $\sigma_1(\mu)$ then $\psi = \delta\psi_1$ for some $\delta \in \mathbb{R}$;*
2. *either $\psi_1 > 0$ or $\psi_1 < 0$ in $\mathbb{S}^N_+$;*
3. *if $\mu > 0$ and $\psi_1 > 0$, then the trace of ψ_1 on $\mathbb{S}^{N-1}$ is positive and constant;*
4. *if $\mu = 0$ then ψ_1 is constant in $\mathbb{S}^N_+$.*

Proof. The proof of the fact that $\sigma_1(\mu)$ is reached is classical, as well as the proofs of points (1) and (2), see for instance [Sal16, Section 8.3.3].

In order to prove (3), let us first observe that, if $\mu \in (0, \gamma_H)$, there exists one and only one $\alpha \in (0, \frac{N-2s}{2})$ such that $\Lambda(\alpha) = \mu$. For this α let $\Upsilon_\alpha > 0$ be the solution of (7.3.6). Thanks to [FF14, Theorem 4.1], it is possible to describe the behavior of Υ_α near the origin: in particular, since $\Upsilon_\alpha > 0$, we have that there exists $C > 0$ such that

$$\tau^{-a_{\Lambda(\alpha)}}\Upsilon_\alpha(0, \tau\theta') \to C\psi_1(0, \theta') \qquad \text{in } C^{1,\beta}(\mathbb{S}^{N-1}) \quad \text{as } \tau \to 0^+,$$

where

$$a_{\Lambda(\alpha)} = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \sigma_1(\Lambda(\alpha))}$$

Thanks to (7.3.5) we have that $a_{\Lambda(\alpha)} = -\frac{N-2s}{2} + \alpha$; then $\tau^{-a_{\Lambda(\alpha)}}\Upsilon_\alpha(0,\tau\theta') \equiv 1$ and so $\psi_1(0,\theta')$ is positive and constant in $\mathbb{S}^{N-1}$.

Finally, if $\mu = 0$ then $\sigma_1(0) = 0$ is clearly attained by every constant function. □

We note that, in view of well known regularity results (see Proposition 7.3.1 in the Appendix), $\psi_1 \in C^{0,\beta}(\overline{\mathbb{S}^N_+})$ for some $\beta \in (0,1)$. Hereafter, we choose the first eigenfunction ψ_1 of problem (7.3.2) to be strictly positive in $\mathbb{S}^N_+$. With this choice of ψ_1, we also have that, in view of the Hopf type principle proved in [CS14, Proposition 4.11] (see Proposition 7.3.2),

$$\min_{\overline{\mathbb{S}^N_+}} \psi_1 > 0. \tag{7.3.7}$$

7.3.3 Asymptotic Estimates of Solutions

In this section, we describe the asymptotic behavior of solutions to equations of the type $-\operatorname{div}(t^{1-2s}\nabla\Phi) = 0$, with singular potentials appearing in the Neumann-type boundary conditions, either on positive half-balls B_r^+ or on their complement in $\mathbb{R}^{N+1}_+$.

Lemma 7.3.6. *Let $R_0 > 0$, $\mu < \gamma_H$ and let $\Phi \in H^1(B^+_{R_0}; t^{1-2s})$, $\Phi \geq 0$ a.e. in $B^+_{R_0}$, $\Phi \not\equiv 0$, be a weak solution of the following problem*

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Phi) = 0, & \text{in } B^+_{R_0}, \\ -\lim\limits_{t\to 0} t^{1-2s}\dfrac{\partial\Phi}{\partial t} = \kappa_s(\mu\,|x|^{-2s} + q)\Phi, & \text{on } B'_{R_0}, \end{cases}$$

where $q \in C^1(B'_{R_0} \setminus \{0\})$ is such that

$$|q(x)| + |x\cdot\nabla q(x)| = O(|x|^{-2s+\varepsilon}) \qquad \text{as } |x| \to 0,$$

for some $\varepsilon > 0$. Then there exist $C_1 > 0$ and $R \leq R_0$ such that

$$\frac{1}{C_1}|z|^{a_\mu} \leq \Phi(z) \leq C_1 |z|^{a_\mu} \qquad \text{for all } z \in B^+_R, \tag{7.3.8}$$

where

$$a_\mu = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \sigma_1(\mu)}. \tag{7.3.9}$$

Furthermore, if $0 \leq \mu < \gamma_H$, then there exists $C_2 > 0$ such that

$$\lim_{|x|\to 0} |x|^{-a_\mu}\Phi(0,x) = C_2. \tag{7.3.10}$$

Proof. Since $\Phi \geq 0$ a.e., $\Phi \not\equiv 0$, from [FF14, Theorem 4.1] we know that there exists $C > 0$ such that

$$\tau^{-a_\mu}\Phi(\tau\theta) \to C\psi_1(\theta) \qquad \text{in } C^{0,\beta}(\overline{\mathbb{S}^N_+}) \quad \text{as } \tau \to 0. \tag{7.3.11}$$

Estimate (7.3.8) follows from the above convergence and (7.3.7). Convergence (7.3.10) follows from (7.3.11) and statements (3–4) of Lemma 7.3.5. □

Lemma 7.3.7. *Let $R_0 > 0$, $\mu < \gamma_H$ and let $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, $\Phi \geq 0$ a.e. in $\mathbb{R}^{N+1}_+$, $\Phi \not\equiv 0$, be a weak solution of the following problem*

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Phi) = 0, & \text{in } \mathbb{R}^{N+1}_+ \setminus B^+_{R_0}, \\ -\lim\limits_{t\to 0} t^{1-2s}\dfrac{\partial \Phi}{\partial t} = \kappa_s(\mu\,|x|^{-2s} + q)\Phi, & \text{on } \mathbb{R}^N \setminus B'_{R_0}, \end{cases}$$

where $q \in C^1(\mathbb{R}^N \setminus B'_{R_0})$ is such that

$$|q(x)| + |x \cdot \nabla q(x)| = O(|x|^{-2s-\varepsilon}) \qquad \text{as } |x| \to +\infty,$$

for some $\varepsilon > 0$. Then there exist $C_3 > 0$ and $R \geq R_0$ such that

$$\frac{1}{C_3}\,|z|^{-(N-2s)-a_\mu} \leq \Phi(z) \leq C_3\,|z|^{-(N-2s)-a_\mu} \qquad \text{for all } z \in \mathbb{R}^{N+1}_+ \setminus B^+_R.$$

Furthermore, if $0 \leq \mu < \gamma_H$, then there exists $C_4 > 0$ such that

$$\lim_{|x|\to\infty} |x|^{N-2s+a_\mu}\,\Phi(0,x) = C_4.$$

Proof. The proof follows by considering the equation solved by the Kelvin transform of Φ

$$\tilde{\Phi}(z) := |z|^{-(N-2s)}\,\Phi\!\left(\frac{z}{|z|^2}\right) \tag{7.3.12}$$

(see [FW12, Proposition 2.6]) and applying Lemma 7.3.6. □

Lemma 7.3.8. *Let $R_0 > 0$ and let $\Phi \in H^1(B^+_{R_0}; t^{1-2s})$, $\Phi \geq 0$ a.e. in $\mathbb{R}^{N+1}_+$, $\Phi \not\equiv 0$, be a weak solution of the following problem*

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Phi) = 0, & \text{in } B^+_{R_0}, \\ -\lim\limits_{t\to 0} t^{1-2s}\dfrac{\partial \Phi}{\partial t} = \kappa_s q\Phi, & \text{on } B'_{R_0}, \end{cases}$$

where $q \in C^1(B'_{R_0} \setminus \{0\})$ is such that

$$|q(x)| + |x \cdot \nabla q(x)| = O(|x|^{-2s+\varepsilon}) \qquad \text{as } |x| \to 0,$$

for some $\varepsilon > 0$. Then there exists $C_5 > 0$ such that

$$\lim_{|z|\to 0} \Phi(z) = \lim_{|x|\to 0} \Phi(0,x) = C_5.$$

Proof. The thesis is a direct consequence of the regularity result of [JLX14, Proposition 2.4] (see Proposition 7.3.1 in the Appendix) combined with the Hopf type principle in [CS14, Proposition 4.11] (see Proposition 7.3.2). It can be also derived as a particular case of [FF14, Theorem 4.1] with $\mu = 0$, taking into account that, for $\mu = 0$, ψ_1 is a positive constant on $\mathbb{S}^N_+$, as observed in Lemma 7.3.5. $\square$

Lemma 7.3.9. *Let $R_0 > 0$ and let $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, $\Phi \geq 0$ a.e. in $\mathbb{R}^{N+1}_+$, $\Phi \not\equiv 0$, be a weak solution of the following problem*

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Phi) = 0, & \text{in } \mathbb{R}^{N+1}_+ \setminus B^+_{R_0}, \\ -\lim\limits_{t\to 0} t^{1-2s}\dfrac{\partial\Phi}{\partial t} = \kappa_s q\Phi, & \text{on } \mathbb{R}^N \setminus B'_{R_0}, \end{cases}$$

where $q \in C^1(\mathbb{R}^N \setminus B'_{R_0})$ is such that

$$|q(x)| + |x \cdot \nabla q(x)| = O(|x|^{-2s-\varepsilon}) \qquad \text{as } |x| \to +\infty,$$

for some $\varepsilon > 0$. Then there exists $C_6 > 0$ such that

$$\lim_{|z|\to\infty} |z|^{N-2s}\,\Phi(z) = \lim_{|x|\to\infty} |x|^{N-2s}\,\Phi(0,x) = C_6$$

Proof. The proof follows by considering the equation solved by the Kelvin transform of Φ given in (7.3.12) and applying Lemma 7.3.8. $\square$

7.3.4 A density result

Finally, we prove a density result: the idea behind is that removing a point does not impair the definition of $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and $\mathcal{D}^{s,2}(\mathbb{R}^N)$; in other words, a point in $\mathbb{R}^N$ has null fractional s-capacity if $N > 2s$, see also [AFN20, Example 2.5].

Lemma 7.3.10. *Let $z_0 \in \overline{\mathbb{R}^{N+1}_+}$, $N > 2s$. Then $C^\infty_c(\overline{\mathbb{R}^{N+1}_+} \setminus \{z_0\})$ is dense in the space $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. As a consequence, if $x_0 \in \mathbb{R}^N$, then $C^\infty_c(\mathbb{R}^N \setminus \{x_0\})$ is dense in $\mathcal{D}^{s,2}(\mathbb{R}^N)$.*

Proof. Assume $z_0 \in \partial\overline{\mathbb{R}^{N+1}_+} = \mathbb{R}^N$ (the proof is completely analogous if $z_0 \in \mathbb{R}^{N+1}_+$). Moreover, without loss of generality, we can assume $z_0 = 0$. Let $U \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ and let $\xi_n \in C^\infty(\overline{\mathbb{R}^{N+1}_+})$ be a cut-off function such that

$$\xi_n(z) = \begin{cases} 1, & \text{if } z \in \overline{\mathbb{R}^{N+1}_+ \setminus B^+_{2/n}}, \\ 0, & \text{if } z \in \overline{B^+_{1/n}}, \end{cases}$$
$$\xi_n \quad \text{is radial, i.e. } \xi_n(z) = \xi_n(|z|), \qquad |\xi_n| \leq 1, \quad |\nabla\xi_n| \leq 2n.$$

Trivially $\xi_n U \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+} \setminus \{0\})$. We claim that $\xi_n U \to U$ in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Indeed, thanks to Dominated Convergence Theorem,

$$\begin{aligned}\int_{\mathbb{R}^{N+1}_+} & t^{1-2s} \left|\nabla((\xi_n - 1)U)\right|^2 \mathrm{d}x\mathrm{d}t \\ &\leq 2\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \left|\xi_n - 1\right|^2 \left|\nabla U\right|^2 \mathrm{d}x\mathrm{d}t + 2\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \left|U\right|^2 \left|\nabla \xi_n\right|^2 \mathrm{d}x\mathrm{d}t \\ &\leq o(1) + Cn^2 \int_{B^+_{2/n}\setminus B^+_{1/n}} t^{1-2s}\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Moreover

$$n^2 \int_{B^+_{2/n}\setminus B^+_{1/n}} t^{1-2s}\,\mathrm{d}x\mathrm{d}t = O(n^{2s-N}),$$

which concludes the proof of the claim, in view of the assumption $N > 2s$ and the density of $C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$.

For what concerns the second statement, as before, without loss of generality, we can assume $x_0 = 0$. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ and let $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ be its extension. By the density of $C_c^\infty(\overline{\mathbb{R}^{N+1}_+} \setminus \{0\})$ in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ just proved, there exists a sequence $\{U_n\} \subset C_c^\infty(\overline{\mathbb{R}^{N+1}_+} \setminus \{0\})$ such that $U_n \to U$ in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Then $\mathrm{Tr}(U_n) \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ and $\mathrm{Tr}(U_n) \to \mathrm{Tr}\, U = u$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, thanks to the continuity of the trace map $\mathrm{Tr}\colon \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \to \mathcal{D}^{s,2}(\mathbb{R}^N)$. □

7.4 Proof of Theorem 7.2.5

Proof of Theorem 7.2.5. First, assume $\sum_{i=1}^m \mu_i^+ < \gamma_H$. By Hardy inequality (7.2.4) we deduce that

$$Q_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}(u) \geq \left(1 - \frac{\sum_{i=1}^m \mu_i^+}{\gamma_H}\right) \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \quad \text{for all } u \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

thus implying that $Q_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}$ is positive definite.

Now we assume that $\sum_{i=1}^m \mu_i^+ > \gamma_H$. By optimality of the constant γ_H in Hardy inequality, it follows that there exists $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that

$$\|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^m \mu_i^+ \int_{\mathbb{R}^N} \frac{|\varphi|^2}{|x|^{2s}}\,\mathrm{d}x < 0. \tag{7.4.1}$$

Let $\varphi_\rho(x) := \rho^{-\frac{N-2s}{2}}\varphi(x/\rho)$. Then, taking into account Lemma 7.9.1, we have that

$$\begin{aligned}\|\varphi_\rho\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^m \mu_i^+ \int_{\mathbb{R}^N} \frac{|\varphi_\rho|^2}{|x-a_i|^{2s}}\,\mathrm{d}x &= \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^m \mu_i^+ \int_{\mathbb{R}^N} \frac{|\varphi|^2}{|x-a_i/\rho|^{2s}}\,\mathrm{d}x \\ &\to \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^m \mu_i^+ \int_{\mathbb{R}^N} \frac{|\varphi|^2}{|x|^{2s}}\,\mathrm{d}x < 0,\end{aligned}$$

as $\rho \to +\infty$. Therefore, there exists $\tilde{\rho}$ such that

$$\|\psi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^{m} \mu_i^+ \int_{\mathbb{R}^N} \frac{|\psi|^2}{|x-a_i|^{2s}} \,\mathrm{d}x < 0, \tag{7.4.2}$$

where $\psi := \varphi_{\tilde{\rho}}$. Let $R > 0$ be such that $\operatorname{supp}\psi \subset B'_R$. Then

$$\int_{\mathbb{R}^N} \frac{|\psi|^2}{|x-a|^{2s}} \leq \frac{1}{(|a|-R)^{2s}} \int_{B'_R} |\psi|^2 \,\mathrm{d}x \tag{7.4.3}$$

for $|a|$ sufficiently large. Hence, from (7.4.2) and (7.4.3) it follows that

$$\begin{aligned} Q_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}(\psi) &= \|\psi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^{m} \mu_i^+ \int_{\mathbb{R}^N} \frac{|\psi|^2}{|x-a_i|^{2s}} \,\mathrm{d}x + \sum_{i=1}^{m} \mu_i^- \int_{\mathbb{R}^N} \frac{|\psi|^2}{|x-a_i|^{2s}} \,\mathrm{d}x \\ &\leq \|\psi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^{m} \mu_i^+ \int_{\mathbb{R}^N} \frac{|\psi|^2}{|x-a_i|^{2s}} \,\mathrm{d}x + \sum_{i=1}^{m} \mu_i^- \frac{1}{(|a_i|-R)^{2s}} \int_{B'_R} |\psi|^2 \,\mathrm{d}x < 0 \end{aligned}$$

if the poles a_i's, corresponding to negative μ_i's, are sufficiently far from the origin. The proof is thereby complete. □

Remark 7.4.1. We observe that, in the case of two poles (i.e. $m = 2$), Theorem 7.2.5 implies the sufficiency of condition (7.2.9) for the existence of a configuration of poles that makes the quadratic form $Q_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}$ positive definite. Indeed, if $m = 2$ condition (7.2.9) directly implies (7.2.10).

7.5 A positivity criterion in the class Θ

In this section, we provide the proof of Lemma 7.2.2, that is a criterion for establishing positivity of Schrödinger operators with potentials in Θ, in relation with existence of positive supersolutions, in the spirit of Allegretto-Piepenbrink theory.

We first prove the equivalent formulation of the infimum in (7.2.5) stated in Lemma 7.2.1.

Proof of Lemma 7.2.1. Let's fix $\tilde{u} \in \mathcal{D}^{s,2}(\mathbb{R}^N) \setminus \{0\}$ and let's call $\tilde{U} \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ its extension. Since

$$\kappa_s \|\tilde{u}\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla \tilde{U}|^2 \,\mathrm{d}x\mathrm{d}t,$$

where κ_s is defined in (6.1.5), then

$$\frac{\|\tilde{u}\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} V \tilde{u}^2 \,\mathrm{d}x}{\|\tilde{u}\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}} = \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla \tilde{U}|^2 \,\mathrm{d}x\mathrm{d}t - \kappa_s \int_{\mathbb{R}^N} V |\operatorname{Tr} \tilde{U}|^2 \,\mathrm{d}x}{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla \tilde{U}|^2 \,\mathrm{d}x\mathrm{d}t}.$$

Therefore

$$\frac{\|\tilde{u}\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} V\,|\tilde{u}|^2\,\mathrm{d}x}{\|\tilde{u}\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}} \geq \inf_{\substack{U\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})\\ U\not\equiv 0}} \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2\,\mathrm{d}x\mathrm{d}t - \kappa_s\int_{\mathbb{R}^N} V|\operatorname{Tr}U|^2\,\mathrm{d}x}{\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2\,\mathrm{d}x\mathrm{d}t}$$

and then we can pass to the inf also on the left-hand quotient.

On the other hand, from (6.1.9), we have that, for any $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \setminus \{0\}$

$$\frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2\,\mathrm{d}x\mathrm{d}t - \kappa_s\int_{\mathbb{R}^N} V|\operatorname{Tr}U|^2\,\mathrm{d}x}{\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2\,\mathrm{d}x\mathrm{d}t} \geq 1 - \frac{\int_{\mathbb{R}^N} V\,|u|^2\,\mathrm{d}x}{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}}$$

where $u = \operatorname{Tr}U$. Taking the infimum to both sides concludes the proof. □

Now we are able to provide the proof of the positivity criterion.

Proof of Lemma 7.2.2. Let us first prove (I). Let $U \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+} \setminus \{(0,a_1),\dots,(0,a_m)\}) \setminus \{0\}$, $U \not\equiv 0$ on $\mathbb{R}^N$. Note that $U^2/\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and $U^2/\Phi \geq 0$, hence we can choose U^2/Φ in (7.2.7) as a test function. Easy computations yield

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\,|\nabla U|^2\,\mathrm{d}x\mathrm{d}t - \kappa_s\int_{\mathbb{R}^N} V\,|\operatorname{Tr}U|^2\,\mathrm{d}x \geq \varepsilon\kappa_s\int_{\mathbb{R}^N} \tilde{V}\,|\operatorname{Tr}U|^2\,\mathrm{d}x$$

which, taking into account the hypothesis on $\tilde{V}$, implies that

$$\kappa_s\int_{\mathbb{R}^N} V\,|\operatorname{Tr}U|^2\,\mathrm{d}x \leq \frac{1}{1+\varepsilon}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\,|\nabla U|^2\,\mathrm{d}x\mathrm{d}t.$$

Hence

$$\frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\,|\nabla U|^2\,\mathrm{d}x\mathrm{d}t - \kappa_s\int_{\mathbb{R}^N} V\,|\operatorname{Tr}U|^2\,\mathrm{d}x}{\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\,|\nabla U|^2\,\mathrm{d}x\mathrm{d}t} \geq \frac{\varepsilon}{1+\varepsilon}. \tag{7.5.1}$$

Therefore (I) follows by density of $C^\infty_c(\overline{\mathbb{R}^{N+1}_+} \setminus \{(0,a_1),\dots,(0,a_m)\})$ in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ (see Lemma 7.3.10).

Now we prove (II). First of all we notice that, thanks to Hölder's inequality, (6.1.7) and (6.1.8)

$$\begin{aligned}\kappa_s\int_{\mathbb{R}^N} \tilde{V}\,|\operatorname{Tr}U|^2\,\mathrm{d}x &\leq \kappa_s\int_{\mathbb{R}^N} |V|\,|\operatorname{Tr}U|^2\,\mathrm{d}x\\ &\leq \left[\frac{1}{\gamma_H}\left(\sum_{i=1}^m|\mu_i| + |\mu_\infty|\right) + S^{-1}\,\|W\|_{L^{N/2s}(\mathbb{R}^N)}\right]\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\,|\nabla U|^2\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. If

$$0 < \varepsilon < \frac{\sigma(V)}{2}\left[\frac{1}{\gamma_H}\left(\sum_{i=1}^m|\mu_i| + |\mu_\infty|\right) + S^{-1}\,\|W\|_{L^{N/2s}(\mathbb{R}^N)}\right]^{-1}, \tag{7.5.2}$$

then

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}x\mathrm{d}t - \kappa_s \int_{\mathbb{R}^N} (V + \varepsilon \tilde{V}) |\mathrm{Tr}\, U|^2 \, \mathrm{d}x \geq \frac{\sigma(V)}{2} \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}x\mathrm{d}t \tag{7.5.3}$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Hence, for any fixed $p \in L^{N/2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, Hölder continuous and positive, the infimum

$$m_p = \inf_{\substack{U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \\ \mathrm{Tr}\, U \not\equiv 0}} \frac{\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}x\mathrm{d}t - \kappa_s \int_{\mathbb{R}^N} (V + \varepsilon \tilde{V}) |\mathrm{Tr}\, U|^2 \, \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} p \, |\mathrm{Tr}\, U|^2 \, \mathrm{d}x}$$

is nonnegative. Also m_p is achieved by some function $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \setminus \{0\}$, that (by evenness) can be chosen to be nonnegative: indeed, thanks to Hardy inequality (6.1.7) and (7.5.3) it's easy to prove that the map

$$\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \ni U \mapsto \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}x\mathrm{d}t - \kappa_s \int_{\mathbb{R}^N} (V + \varepsilon \tilde{V}) |\mathrm{Tr}\, U|^2 \, \mathrm{d}x$$

is weakly lower semicontinuous (since its square root is an equivalent norm in the space $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$), while Lemma 6.1.1 yields the compactness of the trace map from $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ into $L^2(\mathbb{R}^N, p\, \mathrm{d}x)$. Moreover Φ satisfies in a weak sense

$$\begin{cases} -\operatorname{div}(t^{1-2s} \nabla \Phi) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t \to 0^+} t^{1-2s} \dfrac{\partial \Phi}{\partial t} = \kappa_s (V + \varepsilon \tilde{V}) \, \mathrm{Tr}\, \Phi + m_p p \, \mathrm{Tr}\, \Phi, & \text{in } \mathbb{R}^N, \end{cases} \tag{7.5.4}$$

i.e.

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla \Phi \cdot \nabla W \, \mathrm{d}x\mathrm{d}t = \kappa_s \int_{\mathbb{R}^N} (V + \varepsilon \tilde{V}) \, \mathrm{Tr}\, \Phi \, \mathrm{Tr}\, W \, \mathrm{d}x + m_p \int_{\mathbb{R}^N} p \, \mathrm{Tr}\, \Phi \, \mathrm{Tr}\, W \, \mathrm{d}x$$

for all $W \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. From [JLX14, Proposition 2.6] (see also Proposition 7.3.1 in the Appendix) we have that Φ is locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1), \dots, (0, a_m)\}$; in particular $\Phi \in C^0\left(\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1), \dots, (0, a_m)\}\right)$. Moreover, the classical Strong Maximum Principle implies that $\Phi > 0$ in $\mathbb{R}^{N+1}_+$; then, in the case when $V, \tilde{V}$ are locally Hölder continuous in $\mathbb{R}^N \setminus \{a_1, \dots, a_m\}$, the Hopf type principle proved in [CS14, Proposition 4.11] (which is recalled in the Proposition 7.3.2 of the Appendix) ensures that $\Phi(0, x) > 0$ for all $x \in \mathbb{R}^N \setminus \{a_1, \dots, a_m, \}$; we observe that assumption (7.3.1) of Proposition 7.3.2 is satisfied thanks to [CS14, Lemma 4.5], see Lemma 7.3.3. □

7.6 Upper and lower bounds for $\sigma(V)$

In this section we prove bounds from above and from below (in Lemma 7.6.1 and 7.6.2, respectively) for the quantity $\sigma(V)$.

Lemma 7.6.1. *For any* $V(x) = \sum_{i=1}^m \frac{\mu_i \chi_{B'(a_i,r_i)}(x)}{|x-a_i|^{2s}} + \frac{\mu_\infty \chi_{\mathbb{R}^N \setminus B'_R}(x)}{|x|^{2s}} + W(x) \in \Theta$, *there holds:*

(i) $\sigma(V) \le 1$;

(ii) if $\max_{i=1,\dots,m,\infty} \mu_i > 0$, *then* $\sigma(V) \le 1 - \frac{1}{\gamma_H} \max_{i=1,\dots,m,\infty} \mu_i$.

Proof. Let us fix $u \in C_c^\infty(\mathbb{R}^N)$, $u \not\equiv 0$ and $P \in \mathbb{R}^N \setminus \{a_1, \dots, a_m\}$. For every $\rho > 0$ we define $u_\rho(x) := \rho^{-\frac{(N-2s)}{2}} u(\frac{x-P}{\rho})$ and we notice that, by scaling properties,

$$\|u_\rho\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \quad \text{and} \quad \|u_\rho\|_{L^{2^*_s}(\mathbb{R}^N)} = \|u\|_{L^{2^*_s}(\mathbb{R}^N)}. \tag{7.6.1}$$

Moreover, since $\operatorname{supp}(u_\rho) = P + \rho \operatorname{supp}(u)$, we have that $a_1, \dots, a_m \notin \operatorname{supp}(u_\rho)$ for $\rho >$ sufficiently small, hence

$$V \in L^{N/2s}(\operatorname{supp}(u_\rho)). \tag{7.6.2}$$

Therefore, from the definition of $\sigma(V)$, thanks also to (7.6.1), (7.6.2), Hölder inequality, and (6.0.1), we deduce that

$$\begin{aligned}
\sigma(V) &\le 1 - \frac{\int_{\mathbb{R}^N} V |u_\rho|^2 \,\mathrm{d}x}{\|u_\rho\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}} \le 1 + \frac{\int_{\operatorname{supp}(u_\rho)} |V| |u_\rho|^2 \,\mathrm{d}x}{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}} \\
&\le 1 + \frac{\|V\|_{L^{N/2s}(\operatorname{supp}(u_\rho))} \|u\|^2_{L^{2^*_s}(\mathbb{R}^N)}}{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}} \le 1 + S^{-1} \|V\|_{L^{N/2s}(\operatorname{supp}(u_\rho))} = 1 + o(1),
\end{aligned}$$

as $\rho \to 0^+$. By density we may conclude the first part of the proof.

Now let us assume $\max_{i=1,\dots,m,\infty} \mu_i > 0$ and let us first consider the case

$$\max_{i=1,\dots,m,\infty} \mu_i = \mu_j \text{ for a certain } j = 1, \dots, m.$$

From optimality of the best constant in Hardy inequality (7.2.4) and from the density of $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_m, 0\})$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ (see Lemma 7.3.10), we have that, for any $\varepsilon > 0$, there exists $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_m, 0\})$ such that

$$\|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} < (\gamma_H + \varepsilon) \int_{\mathbb{R}^N} \frac{|\varphi|^2}{|x|^{2s}} \,\mathrm{d}x. \tag{7.6.3}$$

Now, for any $\rho > 0$ we define $\varphi_\rho(x) := \rho^{-\frac{(N-2s)}{2}} \varphi(\frac{x-a_j}{\rho})$. From the definition of $\sigma(V)$ and from (7.6.1) we deduce that

$$\sigma(V) \le 1 - \frac{\int_{\mathbb{R}^N} V |\varphi_\rho|^2 \,\mathrm{d}x}{\|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}}. \tag{7.6.4}$$

On the other hand, we can split the numerator as

$$\int_{\mathbb{R}^N} V\,|\varphi_\rho|^2\,\mathrm{d}x = \mu_j \int_{B'(a_j,r_j)} |x-a_j|^{-2s}\,|\varphi_\rho|^2\,\mathrm{d}x + \sum_{i\neq j}\mu_i \int_{B'(a_i,r_i)} |x-a_i|^{-2s}\,|\varphi_\rho|^2\,\mathrm{d}x$$
$$+\,\mu_\infty \int_{\mathbb{R}^N\setminus B'_R} |x|^{-2s}\,|\varphi_\rho|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} W\,|\varphi_\rho|^2\,\mathrm{d}x.$$

From Hölder inequality and (7.6.1) we have that

$$\left|\int_{\mathbb{R}^N} W\varphi_\rho^2\,\mathrm{d}x\right| \le \|W\|_{L^{N/2s}(\mathrm{supp}(\varphi_\rho))}\,\|\varphi\|^2_{L^{2^*_s}} \to 0, \quad \text{as } \rho\to 0^+, \tag{7.6.5}$$

while, just by a change of variable

$$\int_{B'(a_j,r_j)} |x-a_j|^{-2s}\,|\varphi_\rho|^2\,\mathrm{d}x = \int_{B'_{r_j/\rho}} |x|^{-2s}\,|\varphi|^2\,\mathrm{d}x \to \int_{\mathbb{R}^N} |x|^{-2s}\,|\varphi|^2\,\mathrm{d}x, \tag{7.6.6}$$

as $\rho\to 0^+$. Moreover $\mathrm{supp}(\varphi_\rho) = a_j + \rho\,\mathrm{supp}(\varphi)$, and therefore, thanks to (7.6.5) and (7.6.6), we have that, as $\rho\to 0^+$,

$$\int_{\mathbb{R}^N} V\,|\varphi_\rho|^2\,\mathrm{d}x = \mu_j \int_{\mathbb{R}^N} |x|^{-2s}\,|\varphi|^2\,\mathrm{d}x + o(1). \tag{7.6.7}$$

Hence, combining (7.6.4) with (7.6.7) and (7.6.3), we obtain that

$$\sigma(V) \le 1 - \mu_j(\gamma_H+\varepsilon)^{-1},$$

for all $\varepsilon>0$, which implies that $\sigma(V)\le 1-\mu_j/\gamma_H$. Finally, let us assume $\max_{i=1,\dots,m,\infty}\mu_i = \mu_\infty$. Letting $\varphi_\rho(x) := \rho^{-\frac{(N-2s)}{2}}\varphi(x/\rho)$, we observe that $\varphi_\rho\to 0$ uniformly, as $\rho\to+\infty$. So, arguing as before, one can similarly obtain that $\sigma(V)\le 1-\frac{\mu_\infty}{\gamma_H}$. The proof is thereby complete. □

The following result provides the positivity in the case of potentials with subcritical masses supported in sufficiently small neighborhoods of the poles. In the following we fix two cut-off functions $\zeta,\tilde\zeta:\mathbb{R}^N\to\mathbb{R}$ such that $\zeta,\tilde\zeta\in C^\infty(\mathbb{R}^N)$, $0\le\zeta(x)\le 1$, $0\le\tilde\zeta(x)\le 1$, and

$$\zeta(x)=1 \quad \text{for } |x|\le\frac{1}{2}, \qquad \zeta(x)=0 \quad \text{for } |x|\ge 1,$$
$$\tilde\zeta(x)=0 \quad \text{for } |x|\le 1, \qquad \tilde\zeta(x)=1 \quad \text{for } |x|\ge 2.$$

Lemma 7.6.2. *Let $\{a_1,a_2,\dots,a_m\}\subset B'_R$, $a_i\neq a_j$ for $i\neq j$, and $\mu_1,\mu_2,\dots,\mu_m,\mu_\infty\in\mathbb{R}$ be such that $M:=\max_{i=1,\dots,m,\infty}\mu_i<\gamma_H$. For any $0<h<1-\frac{M}{\gamma_H}$, there exists $\delta=\delta(h)>0$ such that*

$$\sigma\left(\sum_{i=1}^m \frac{\mu_i\zeta(\frac{x-a_i}{\delta})}{|x-a_i|^{2s}} + \frac{\mu_\infty\tilde\zeta(\frac{x}{R})}{|x|^{2s}}\right) \ge \begin{cases} 1-\frac{M}{\gamma_H}-h, & \text{if } M>0\\ 1, & \text{if } M\le 0.\end{cases}$$

Proof. Let us assume that $M > 0$, otherwise the statement is trivial. First, let us fix $0 < \varepsilon < \frac{\gamma_H}{M} - 1$, so that

$$\tilde{\mu}_i := \mu_i + \varepsilon\mu_i^+ < \gamma_H \quad \text{for all } i = 1, \dots, m, \infty.$$

In order to prove the statement, it is sufficient to find $\delta = \delta(\varepsilon) > 0$ and $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that $\Phi \in C^0(\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1/\delta), \dots, (0, a_m/\delta)\})$,

$$\Phi > 0 \text{ in } \overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1/\delta), \dots, (0, a_m/\delta)\}$$

, and there holds

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla\Phi\cdot\nabla U \,\mathrm{d}x\mathrm{d}t - \sum_{i=1}^m \kappa_s \int_{\mathbb{R}^N} V_i \operatorname{Tr}\Phi \operatorname{Tr} U \,\mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} V_\infty \operatorname{Tr}\Phi \operatorname{Tr} U \,\mathrm{d}x \geq 0 \quad (7.6.8)$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, $U \geq 0$ a.e., where

$$V_i(x) = \frac{\tilde{\mu}_i \zeta(x - \frac{a_i}{\delta})}{|x - a_i/\delta|^{2s}}, \qquad V_\infty(x) = \frac{\tilde{\mu}_\infty \tilde{\zeta}(\frac{\delta}{R}x)}{|x|^{2s}}.$$

Indeed, thanks to scaling properties in (7.6.8) and to Lemma 7.2.2, (7.6.8) implies that

$$\sigma\left(\sum_{i=1}^m \frac{\mu_i \zeta(\frac{x-a_i}{\delta})}{|x - a_i|^{2s}} + \frac{\mu_\infty \tilde{\zeta}(\frac{x}{R})}{|x|^{2s}}\right) \geq \frac{\varepsilon}{1+\varepsilon},$$

so that, letting $h := 1 - \frac{M}{\gamma_H} - \frac{\varepsilon}{\varepsilon+1}$, we obtain the result. Hence, we seek for some Φ positive and continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1/\delta), \dots, (0, a_m/\delta)\}$ satisfying (7.6.8). Let us set, for some $0 < \tau < 1$,

$$p_i(x) := p\left(x - \frac{a_i}{\delta}\right) \quad \text{for } i = 1, \dots, m, \quad p_\infty(x) = \left(\frac{\delta}{R}\right)^{2s} p\left(\frac{\delta x}{R}\right),$$

where $p(x) = \frac{1}{|x|^{2s-\tau}(1+|x|^2)^\tau}$. We observe that $p_i, p_\infty \in L^{N/2s}(\mathbb{R}^N)$. Therefore, thanks to Lemma 6.1.1, the weighted eigenvalue

$$\sigma_i = \inf_{\substack{\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \\ \operatorname{Tr}\Phi \not\equiv 0}} \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla\Phi|^2 \,\mathrm{d}x\mathrm{d}t - \kappa_s \int_{\mathbb{R}^N} V_i |\operatorname{Tr}\Phi|^2 \,\mathrm{d}x}{\int_{\mathbb{R}^N} p_i |\operatorname{Tr}\Phi|^2 \,\mathrm{d}x}$$

is positive and attained by some nontrivial, nonnegative function $\Phi_i \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ that weakly solves

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Phi_i) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to 0} t^{1-2s}\dfrac{\partial\Phi_i}{\partial t} = (\kappa_s V_i + \sigma_i p_i)\operatorname{Tr}\Phi_i, & \text{on } \mathbb{R}^N. \end{cases}$$

From the classical Strong Maximum Principle we deduce that $\Phi_i > 0$ in $\mathbb{R}^{N+1}_+$, while Proposition 7.3.1 yields that Φ_i is locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_i/\delta)\}$. Moreover, from the Hopf type lemma in Proposition 7.3.2 (whose assumption (7.3.1) is satisfied thanks to Lemma 7.3.3 outside $\{a_i/\delta\}$) we deduce that $\operatorname{Tr}\Phi_i > 0$ in $\mathbb{R}^N \setminus \{a_i/\delta\}$. Similarly

$$\sigma_\infty = \inf_{\substack{\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \\ \operatorname{Tr}\Phi \not\equiv 0}} \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \left|\nabla \Phi\right|^2 \mathrm{d}x\mathrm{d}t - \kappa_s \int_{\mathbb{R}^N} V_\infty \left|\operatorname{Tr}\Phi\right|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} p_\infty \left|\operatorname{Tr}\Phi\right|^2 \mathrm{d}x}$$

is positive and reached by some nontrivial, nonnegative function $\Phi_\infty \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that Φ_∞ is locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+}$ and $\Phi_\infty > 0$ in $\overline{\mathbb{R}^{N+1}_+}$. Moreover, Φ_∞ weakly solves

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Phi_\infty) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to 0} t^{1-2s}\dfrac{\partial \Phi_\infty}{\partial t} = (\kappa_s V_\infty + \sigma_\infty p_\infty)\operatorname{Tr}\Phi_\infty, & \text{on } \mathbb{R}^N. \end{cases}$$

Lemmas 7.3.6–7.3.9 (and continuity of the Φ_i's outside the poles) imply that there exists $C_0 > 0$ (independent of δ) such that

$$\frac{1}{C_0}\left|x - \frac{a_i}{\delta}\right|^{a_{\tilde{\mu}_i}} \le \operatorname{Tr}\Phi_i \le C_0 \left|x - \frac{a_i}{\delta}\right|^{a_{\tilde{\mu}_i}}, \qquad \text{in } B'(a_i/\delta, 1), \tag{7.6.9}$$

$$\frac{1}{C_0}\left|x - \frac{a_i}{\delta}\right|^{-(N-2s)} \le \operatorname{Tr}\Phi_i \le C_0 \left|x - \frac{a_i}{\delta}\right|^{-(N-2s)}, \qquad \text{in } \mathbb{R}^N \setminus B'(a_i/\delta, 1), \tag{7.6.10}$$

$$\frac{1}{C_0}\left|\frac{\delta x}{R}\right|^{-(N-2s)-a_{\tilde{\mu}_\infty}} \le \operatorname{Tr}\Phi_\infty \le C_0 \left|\frac{\delta x}{R}\right|^{-(N-2s)-a_{\tilde{\mu}_\infty}}, \qquad \text{in } \mathbb{R}^N \setminus B'_{R/\delta}, \tag{7.6.11}$$

$$\frac{1}{C_0} \le \operatorname{Tr}\Phi_\infty \le C_0, \qquad \text{in } B'_{R/\delta}. \tag{7.6.12}$$

Let $\Phi := \sum_{i=1}^m \Phi_i + \eta\Phi_\infty$, with $0 < \eta < \inf\{\frac{\sigma_i}{4C_0^2\tilde{\mu}_i} : i = 1, \ldots, m,\ \tilde{\mu}_i > 0\}$. Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla\Phi \cdot \nabla U \,\mathrm{d}x\mathrm{d}t - \sum_{i=1}^m \int_{\mathbb{R}^N} V_i \operatorname{Tr}\Phi \operatorname{Tr}U \,\mathrm{d}x - \int_{\mathbb{R}^N} V_\infty \operatorname{Tr}\Phi \operatorname{Tr}U \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \left[\sum_{i=1}^m \Big(\sigma_i p_i - V_\infty - \sum_{j\neq i} V_j\Big)\operatorname{Tr}\Phi_i + \eta\Big(\sigma_\infty p_\infty - \sum_{i=1}^m V_i\Big)\operatorname{Tr}\Phi_\infty\right]\operatorname{Tr}U \,\mathrm{d}x \\ &=: \int_{\mathbb{R}^N} g(x)\operatorname{Tr}U(x)\,\mathrm{d}x \end{aligned} \tag{7.6.13}$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Hereafter, let us assume $U \ge 0$ a.e. in $\mathbb{R}^{N+1}_+$. We will split the integral into three parts and prove that each of these is nonnegative. First, let us consider $x \in B'_{R/\delta} \setminus (\cup_{i=1}^m B'(a_i/\delta, 1))$: here $V_i = V_\infty = 0$ and we have that

$$g(x) = \sum_{i=1}^m \sigma_i p_i \operatorname{Tr}\Phi_i + \eta\sigma_\infty p_\infty \operatorname{Tr}\Phi_\infty \ge 0.$$

Now let us take $n \in \{1, \dots, m\}$ and $x \in B'(a_n/\delta, 1)$, where $V_\infty = V_i = 0$ for $i \neq n$. Then

$$\begin{aligned} g(x) &= \sum_{i=1}^{m} \sigma_i p_i \operatorname{Tr} \Phi_i - V_n \sum_{i \neq n} \operatorname{Tr} \Phi_i + \eta(\sigma_\infty p_\infty - V_n) \operatorname{Tr} \Phi_\infty \\ &\geq \sigma_n p_n \operatorname{Tr} \Phi_n - V_n \bigg(\sum_{\substack{i=1 \\ i \neq n}}^{m} \operatorname{Tr} \Phi_i + \eta \operatorname{Tr} \Phi_\infty \bigg). \end{aligned}$$

If $\tilde{\mu}_n \leq 0$ this is clearly nonnegative; so let us assume $\tilde{\mu}_n > 0$. Thanks to (7.6.9), (7.6.10) and (7.6.12) we can estimate this quantity from below by

$$\left| x - \frac{a_n}{\delta} \right|^{-2s} \left[\frac{\sigma_n}{2^\tau C_0} \left| x - \frac{a_n}{\delta} \right|^{\tau + a_{\tilde{\mu}_n}} - \tilde{\mu}_n C_0 \bigg(\sum_{i \neq n} \left| x - \frac{a_i}{\delta} \right|^{-(N-2s)} + \eta \bigg) \right]. \tag{7.6.14}$$

We observe that $|x - a_n/\delta|^{\tau + a_{\tilde{\mu}_n}} \geq 1$, since $\tilde{\mu}_n > 0$ implies that $a_{\tilde{\mu}_n} < 0$ and we can choose $\tau < -a_{\tilde{\mu}_n}$. Moreover it's not hard to prove that, for $i \neq n$,

$$\left| x - \frac{a_i}{\delta} \right|^{-(N-2s)} \leq \left(\frac{2}{|a_n - a_i|} \right)^{N-2s} \delta^{N-2s} < \frac{\eta}{m-1},$$

for $\delta > 0$ sufficiently small. Thanks to this and to the choice of η we have that the expression in (7.6.14) and then $g(x)$ is nonnegative in $B'(a_n/\delta, 1)$. Finally, if $x \in \mathbb{R}^N \setminus B'_{R/\delta}$, then the function g in (7.6.13) becomes

$$\sum_{i=1}^{m} (\sigma_i p_i - V_\infty) \operatorname{Tr} \Phi_i + \eta \sigma_\infty p_\infty \operatorname{Tr} \Phi_\infty. \tag{7.6.15}$$

Again, if $\tilde{\mu}_\infty \leq 0$ this quantity is nonnegative. If $\tilde{\mu}_\infty > 0$, thanks to (7.6.10) and (7.6.11), we have that the function in (7.6.15) is greater than or equal to

$$|x|^{-2s} \left[-C_0 \tilde{\mu}_\infty \sum_{i=1}^{m} \left| x - \frac{a_i}{\delta} \right|^{-(N-2s)} + \frac{\eta \sigma_\infty}{2^\tau C_0} \left| \frac{\delta x}{R} \right|^{-(N-2s) - a_{\tilde{\mu}_\infty} + \tau} \right]. \tag{7.6.16}$$

Now, one can easily see that

$$\left| x - \frac{a_i}{\delta} \right| \geq \left(1 - \frac{a}{R} \right) |x| \qquad \text{for all } x \in \mathbb{R}^N \setminus B'_{R/\delta}, \quad \text{where } a = \max_{j=1,\dots,m} |a_j| ,$$

so that we can estimate (7.6.16) from below obtaining that, for all $x \in \mathbb{R}^N \setminus B'_{R/\delta}$,

$$\begin{aligned} g(x) &\geq |x|^{-N} \left[-C_0 \tilde{\mu}_\infty m \Big(1 - \frac{a}{R} \Big)^{-(N-2s)} + \frac{\eta \sigma_\infty}{2^\tau C_0} \left| \frac{\delta}{R} \right|^{-(N-2s) - a_{\tilde{\mu}_\infty} + \tau} |x|^{-a_{\tilde{\mu}_\infty} + \tau} \right] \\ &\geq |x|^{-N} \left[-C_0 \tilde{\mu}_\infty m \Big(1 - \frac{a}{R} \Big)^{-(N-2s)} + \frac{\eta \sigma_\infty}{2^\tau C_0} \left| \frac{\delta}{R} \right|^{-(N-2s)} \right] \geq 0 \end{aligned}$$

for $\delta > 0$ sufficiently small, since $a_{\tilde{\mu}_\infty} < 0$ if $\tilde{\mu}_\infty > 0$. The proof is thereby complete. □

7.7 Perturbation at infinity and at poles

In this section, we investigate the persistence of the positivity when the mass is increased at infinity (Theorem 7.7.3) and at poles (Theorem 7.7.4).

In order to make use of Lemmas 7.3.6–7.3.9, we may need to restrict the class Θ to some more regular potentials and to have a control on their growth at infinity.

For any $\delta > 0$, we define

$$\begin{aligned}\mathcal{P}^{\delta}_{\infty} := \Big\{ f \colon \mathbb{R}^N \to \mathbb{R} \colon\, & f \in C^1(\mathbb{R}^N \setminus B'_{R_\infty}) \text{ for some } R_\infty > 0 \\ & \text{and } |f(x)| + |x \cdot \nabla f(x)| = O(|x|^{-2s-\delta}) \text{ as } |x| \to +\infty \Big\}.\end{aligned} \tag{7.7.1}$$

Moreover, in order to prove some intermediary, technical lemmas based on the positivity criterion Lemma 7.2.2, the need for even more regular potentials occasionally arises. So, let us introduce the class

$$\Theta^* := \Big\{ V \in \Theta \colon V \in C^1(\mathbb{R}^N \setminus \{a_1, \dots, a_m\}) \Big\}. \tag{7.7.2}$$

Then, we will recover the full generality of the class Θ, thanks to an approximation procedure, which is based on the following lemma.

Lemma 7.7.1. *Let $V_1, V_2 \in \Theta$ be such that $V_1 - V_2 \in L^{N/2s}(\mathbb{R}^N)$. Then*

$$|\sigma(V_2) - \sigma(V_1)| \le S^{-1} \|V_2 - V_1\|_{L^{N/2s}(\mathbb{R}^N)},$$

where $S > 0$ is the best constant in the Sobolev embedding (6.0.1).

Proof. From the definition of $\sigma(V_2)$, Hölder inequality and (6.1.8), we have that, for any choice of $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$,

$$\begin{aligned}\int_{\mathbb{R}^{N+1}_+} & t^{1-2s} |\nabla U|^2 \,\mathrm{d}x\mathrm{d}t - \kappa_s \int_{\mathbb{R}^N} V_1 |\mathrm{Tr}\, U|^2 \,\mathrm{d}x \\ & \ge \Big(\sigma(V_2) - S^{-1} \|V_2 - V_1\|_{L^{N/2s}(\mathbb{R}^N)}\Big) \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \,\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{7.7.3}$$

which implies that

$$\sigma(V_1) \ge \sigma(V_2) - S^{-1} \|V_2 - V_1\|_{L^{N/2s}(\mathbb{R}^N)}.$$

Analogously one can prove that $\sigma(V_2) \ge \sigma(V_1) - S^{-1} \|V_2 - V_1\|_{L^{N/2s}(\mathbb{R}^N)}$, thus concluding the proof. □

Lemma 7.7.2. *Let $V \in \mathcal{H}$, $a_1, \dots, a_m \in \mathbb{R}^N$, and $R > 0$ be such that*

$$V \in C^1(\mathbb{R}^N \setminus \{a_1, \dots, a_m\}) \quad \textit{and} \quad V(x) = \frac{\mu_\infty}{|x|^{2s}} + W(x) \quad \textit{in } \mathbb{R}^N \setminus B'_R,$$

where $\mu_\infty < \gamma_H$ and $W \in \mathcal{P}^{\delta}_{\infty} \cap L^\infty(\mathbb{R}^N)$ for some $\delta > 0$. Assume that $\sigma(V) > 0$ and let $\nu_\infty \in \mathbb{R}$ be such that $\mu_\infty + \nu_\infty < \gamma_H$. Then there exist $\tilde{R} > R$ and $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$

such that Φ *is locally Hölder continuous in* $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0,a_1),\dots,(0,a_m)\}$, $\Phi > 0$ *in* $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0,a_1),\dots,(0,a_m)\}$, *and*

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla\Phi \cdot \nabla U \, dxdt - \kappa_s \int_{\mathbb{R}^N} \left[V + \frac{\nu_\infty}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}} \right] \operatorname{Tr}\Phi \operatorname{Tr} U \, dx \geq 0,$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ *with* $U \geq 0$ *a.e.*

Proof. By (7.3.4) we can fix $\varepsilon \in (0, \frac{N-2s}{2})$ such that

$$\Lambda(\varepsilon) - \mu_\infty > 0 \quad \text{and} \quad \Lambda(\varepsilon) - \mu_\infty - \nu_\infty > 0. \tag{7.7.4}$$

Since $W \in \mathcal{P}^\delta_\infty \cap L^\infty(\mathbb{R}^N)$, there exists $C_0 > 0$ such that

$$W(x) \leq \frac{C_0}{|x|^{2s+\delta}} \qquad \text{in } \mathbb{R}^N. \tag{7.7.5}$$

Let $R_0 \geq \max\left\{ R, \frac{1}{2}\left[\frac{C_0}{\Lambda(\varepsilon)-\mu_\infty}\right]^{1/\delta} \right\}$, so that

$$\Lambda(\varepsilon) - \mu_\infty - C_0(2R_0)^{-\delta} \geq 0. \tag{7.7.6}$$

From Lemma 7.3.4 there exists a positive, locally Hölder continuous function $\Upsilon_\varepsilon : \overline{\mathbb{R}^{N+1}_+} \setminus \{0\} \to \mathbb{R}$ such that $\Upsilon_\varepsilon \in \bigcap_{r>0} H^1(B_r^+; t^{1-2s})$ and

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Upsilon_\varepsilon), = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \Upsilon_\varepsilon(0,x) = |x|^{-\frac{N-2s}{2}+\varepsilon}, & \text{on } \mathbb{R}^N, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\dfrac{\partial \Upsilon_\varepsilon}{\partial t}, = \kappa_s \Lambda(\varepsilon)|x|^{-2s} \operatorname{Tr}\Upsilon_\varepsilon & \text{on } \mathbb{R}^N, \end{cases} \tag{7.7.7}$$

in a weak sense. Direct calculations (see e.g. [FW12, Proposition 2.6]) yield that the Kelvin transform

$$\tilde{\Upsilon}_\varepsilon(z) = |z|^{-(N-2s)}\, \Upsilon_\varepsilon(z/|z|^2)$$

of Υ_ε weakly satisfies

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\tilde{\Upsilon}_\varepsilon) = 0, & \text{in } \mathbb{R}^{N+1}_+ \setminus \{0\}, \\ \tilde{\Upsilon}_\varepsilon(0,x) = |x|^{\frac{2s-N}{2}-\varepsilon}, & \text{on } \mathbb{R}^N \setminus \{0\}, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\dfrac{\partial \tilde{\Upsilon}_\varepsilon}{\partial t} = \kappa_s \Lambda(\varepsilon)|x|^{-2s} \operatorname{Tr}\tilde{\Upsilon}_\varepsilon, & \text{on } \mathbb{R}^N \setminus \{0\}, \end{cases} \tag{7.7.8}$$

$\tilde{\Upsilon}_\varepsilon > 0$ in $\overline{\mathbb{R}^{N+1}_+} \setminus \{0\}$ and $\tilde{\Upsilon}_\varepsilon$ is locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{0\}$. Moreover we have that

$$\int_{\mathbb{R}^{N+1}_+ \setminus B_r^+} t^{1-2s}|\nabla\tilde{\Upsilon}_\varepsilon|^2 \, dxdt + \int_{\mathbb{R}^{N+1}_+ \setminus B_r^+} t^{1-2s}\frac{|\tilde{\Upsilon}_\varepsilon|^2}{|x|^2+t^2} \, dxdt < +\infty \quad \text{for all } r > 0. \tag{7.7.9}$$

Let $\eta \in C^\infty(\overline{\mathbb{R}^{N+1}_+})$ be a cut-off function such that η is radial, i.e. $\eta(z) = \eta(|z|)$, $|\nabla \eta| \leq \frac{2}{R_0}$ in $\overline{\mathbb{R}^{N+1}_+}$,

$$\eta(z) := \begin{cases} 0, & \text{in } B^+_{R_0} \cup B'_{R_0} \\ 1, & \text{in } \left(\mathbb{R}^{N+1}_+ \setminus B^+_{2R_0}\right) \cup \left(\mathbb{R}^N \setminus B'_{2R_0}\right), \end{cases}$$

and $\eta > 0$ in $\overline{\mathbb{R}^{N+1}_+} \setminus \overline{B^+_{R_0}}$. We point out that

$$\frac{\partial \eta}{\partial t}(0,x) = 0 \quad \text{and} \quad \frac{1}{t}\left|\frac{\partial \eta}{\partial t}(x,t)\right| = O(1) \quad \text{as } t \to 0 \text{ (uniformly in } x).$$

We let $\Phi_1 := \eta \tilde{\Upsilon}_\varepsilon$. By its construction, Φ_1 is continuous on the whole $\overline{\mathbb{R}^{N+1}_+}$ and $\Phi_1 > 0$ in $\overline{\mathbb{R}^{N+1}_+} \setminus \overline{B^+_{R_0}}$, whereas (7.7.9) implies that $\Phi_1 \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Moreover direct computations yield that Φ_1 weakly solves

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla \Phi_1) = t^{1-2s} F_1, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t \to 0^+} t^{1-2s}\dfrac{\partial \Phi_1}{\partial t} = \kappa_s \Lambda(\varepsilon)|x|^{-2s} \operatorname{Tr}\Phi_1, & \text{on } \mathbb{R}^N, \end{cases} \tag{7.7.10}$$

where

$$F_1 := (2s-1)\frac{1}{t}\frac{\partial \eta}{\partial t}\tilde{\Upsilon}_\varepsilon - 2\nabla\tilde{\Upsilon}_\varepsilon \cdot \nabla \eta - \tilde{\Upsilon}_\varepsilon \Delta \eta.$$

We observe that $F_1 \in C^\infty(\mathbb{R}^{N+1}_+)$ and $\mathrm{supp}(F_1) \subset \overline{B^+_{2R_0} \setminus B^+_{R_0}}$. Given

$$f_1(x) := \kappa_s \Lambda(\varepsilon)\, |x|^{-2s} \, \chi_{B'_{2R_0} \setminus B'_{R_0}} \Phi_1(0,x),$$

we can choose a smooth, compactly supported function $f_2 \colon \mathbb{R}^N \to \mathbb{R}$ such that

$$f_1 + f_2 \geq 0 \quad \text{in } \mathbb{R}^N, \qquad H := f_2 + \left[W + \mu_\infty \, |x|^{-2s}\right] \chi_{B'_{2R_0} \setminus B'_{R_0}} \operatorname{Tr}\Phi_1 \geq 0 \quad \text{in } \mathbb{R}^N. \tag{7.7.11}$$

We also choose another smooth, positive, compactly supported function $F_2 \colon \overline{\mathbb{R}^{N+1}_+} \to \mathbb{R}$ such that $F_1 + F_2 \geq 0$ in $\overline{\mathbb{R}^{N+1}_+}$. Since $\sigma(V) > 0$ and $H \in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$, by Lax-Milgram Lemma there exists $\Phi_2 \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla \Phi_2) = t^{1-2s} F_2, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t \to 0^+} t^{1-2s}\dfrac{\partial \Phi_2}{\partial t} = \kappa_s \left[V \operatorname{Tr}\Phi_2 + H\right], & \text{on } \mathbb{R}^N, \end{cases} \tag{7.7.12}$$

holds in a weak sense. From Proposition 7.3.1 we know that Φ_2 is locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0,a_1), \dots, (0,a_m)\}$.

In order to prove that Φ_2 is strictly positive in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0,a_1), \dots, (0,a_m)\}$, we compare it with the unique weak solution $\Phi_3 \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ to the problem

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla \Phi_3) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t \to 0^+} t^{1-2s}\dfrac{\partial \Phi_3}{\partial t} = \kappa_s \left[V \operatorname{Tr}\Phi_3 + H\right], & \text{on } \mathbb{R}^N, \end{cases} \tag{7.7.13}$$

whose existence is again ensured by the Lax-Milgram Lemma. The difference $\widetilde{\Phi} = \Phi_2 - \Phi_3$ belongs to $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and weakly solves

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla\widetilde{\Phi}) = t^{1-2s}F_2, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\dfrac{\partial\widetilde{\Phi}}{\partial t} = \kappa_s V \operatorname{Tr}\widetilde{\Phi}, & \text{on } \mathbb{R}^N. \end{cases}$$

By directly testing the above equation with $-\widetilde{\Phi}^-$, since $\sigma(V) > 0$ we obtain that $\widetilde{\Phi} \geq 0$ in $\mathbb{R}^{N+1}_+$, i.e. $\Phi_2 \geq \Phi_3$. Furthermore, testing the equation for Φ_3 with $-\Phi_3^-$, we also obtain that $\Phi_3 \geq 0$ in $\mathbb{R}^{N+1}_+$. The classical Strong Maximum Principle, combined with Proposition 7.3.2 (whose assumption (7.3.1) for (7.7.13) is satisfied thanks to the assumption $V \in C^1(\mathbb{R}^N \setminus \{a_1, \dots, a_m\})$ and Lemma 7.3.3), yields $\Phi_3 > 0$ in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0,a_1), \dots, (0,a_m)\}$ and hence

$$\Phi_2 > 0 \quad \text{in} \quad \overline{\mathbb{R}^{N+1}_+} \setminus \{(0,a_1), \dots, (0,a_m)\}.$$

Finally, from Lemma 7.3.7 and from the continuity of Φ_2, there exists $C_1 > 0$ such that

$$\frac{1}{C_1}|x|^{-(N-2s)-a_{\mu_\infty}} \leq \Phi_2(0,x) \leq C_1|x|^{-(N-2s)-a_{\mu_\infty}} \quad \text{in } \mathbb{R}^N \setminus B'_{2R_0}. \tag{7.7.14}$$

Now we set $\Phi = \Phi_1 + \Phi_2$. We immediately observe that $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ is locally Hölder continuous and strictly positive in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0,a_1), \dots, (0,a_m)\}$. We claim that, for $\tilde{R} > 0$ sufficiently large,

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla\Phi\cdot\nabla U \,\mathrm{d}x\mathrm{d}t - \kappa_s\int_{\mathbb{R}^N}\left[V + \frac{\nu_\infty}{|x|^{2s}}\chi_{\mathbb{R}^N\setminus B'_{\tilde{R}}}\right]\operatorname{Tr}\Phi\operatorname{Tr}U\,\mathrm{d}x \geq 0, \tag{7.7.15}$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ with $U \geq 0$ a.e.

The function Φ weakly satisfies

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla\Phi) = t^{1-2s}(F_1 + F_2), & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\dfrac{\partial\Phi}{\partial t} = \kappa_s\left[\Lambda(\varepsilon)\,|x|^{-2s}\operatorname{Tr}\Phi_1 + V\operatorname{Tr}\Phi_2 + H\right], & \text{on } \mathbb{R}^N. \end{cases} \tag{7.7.16}$$

Hence, if $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, $U \geq 0$ a.e.,

$$\begin{aligned} &\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla\Phi\cdot\nabla U\,\mathrm{d}x\mathrm{d}t - \kappa_s\int_{\mathbb{R}^N}\left[V + \frac{\nu_\infty}{|x|^{2s}}\chi_{\mathbb{R}^N\setminus B'_{\tilde{R}}}\right]\operatorname{Tr}\Phi\operatorname{Tr}U\,\mathrm{d}x \\ \geq &\int_{\mathbb{R}^N}\left[\kappa_s\frac{\Lambda(\varepsilon)-\mu_\infty-\nu_\infty}{|x|^{2s}}\chi_{\mathbb{R}^N\setminus B'_{\tilde{R}}}\operatorname{Tr}\Phi_1 + \kappa_s\frac{\Lambda(\varepsilon)-\mu_\infty-|x|^{2s}W}{|x|^{2s}}\chi_{B'_{\tilde{R}}\setminus B'_{2R_0}}\operatorname{Tr}\Phi_1\right. \\ &\left.-\kappa_s\left(W\operatorname{Tr}\Phi_1 + \frac{\nu_\infty}{|x|^{2s}}\operatorname{Tr}\Phi_2\right)\chi_{\mathbb{R}^N\setminus B'_{\tilde{R}}} + f_1 + f_2\right]\operatorname{Tr}U\,\mathrm{d}x =: \int_{\mathbb{R}^N}F(x)\operatorname{Tr}U(x)\,\mathrm{d}x. \end{aligned}$$

If $x \in B'_{2R_0}$, then $F(x) = f_1(x) + f_2(x) \geq 0$. If $x \in B'_{\tilde{R}} \setminus B'_{2R_0}$, then from (7.7.11), (7.7.5) and (7.7.6)

$$F(x) \geq \kappa_s(\Lambda(\varepsilon)-\mu_\infty-|x|^{2s}W)\,|x|^{-2s}\operatorname{Tr}\Phi_1 \geq \kappa_s(\Lambda(\varepsilon)-\mu_\infty-C_0(2R_0)^{-\delta})\,|x|^{-2s}\operatorname{Tr}\Phi_1 \geq 0.$$

Finally, if $x \in \mathbb{R}^N \setminus B'_{\tilde{R}}$, then from the definition of Φ_1, (7.7.11), (7.7.14) and (7.7.5) we have that

$$F(x) \geq \kappa_s(\Lambda(\varepsilon) - \mu_\infty - \nu_\infty)\, |x|^{-\frac{N+2s}{2}-\varepsilon} - \kappa_s C_0\, |x|^{-\frac{N+2s}{2}-\varepsilon-\delta} - \kappa_s C_1 \nu_\infty\, |x|^{-N-a_{\mu_\infty}}.$$

Since the function $\mu \mapsto \sigma_1(\mu)$ is strictly decreasing and $\mu_\infty < \Lambda(\varepsilon)$, from (7.3.5) it follows that $\sigma_1(\mu_\infty) > \varepsilon^2 - \left(\frac{N-2s}{2}\right)^2$ which yields $-N - a_{\mu_\infty} < -\frac{N+2s}{2} - \varepsilon$. Hence, if $\tilde{R}$ is sufficiently large, $F(x) \geq 0$ for all $x \in \mathbb{R}^N \setminus B'_{\tilde{R}}$. This concludes the proof. □

Combining Lemma 7.7.2 with the positivity criterion Lemma 7.2.2 and an approximation procedure based on Lemma 7.7.1, we prove the persistence of the positivity under perturbations at infinity for potentials in the class Θ.

Theorem 7.7.3. *Let*

$$V(x) = \sum_{i=1}^m \frac{\mu_i \chi_{B'(a_i,r_i)}(x)}{|x-a_i|^{2s}} + \frac{\mu_\infty \chi_{\mathbb{R}^N \setminus B'_R}(x)}{|x|^{2s}} + W(x) \in \Theta.$$

Assume $\sigma(V) > 0$ and let $\nu_\infty \in \mathbb{R}$ be such that $\mu_\infty + \nu_\infty < \gamma_H$. Then there exists $\tilde{R} > R$ such that

$$\sigma\left(V + \frac{\nu_\infty}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}}\right) > 0.$$

Proof. Since $V \in \Theta$ and $\sigma(V) > 0$, arguing as in (7.5.3) we have that, for ε chosen sufficiently small as in (7.5.2), $\sigma(V + \varepsilon V) > \frac{\sigma(V)}{2} > 0$. Moreover we can choose ε such that $\mu_\infty + \nu_\infty + \varepsilon(\mu_\infty + \nu_\infty) < \gamma_H$ and $\mu_i + \varepsilon\mu_i < \gamma_H$ for all $i = 1, \dots, m, \infty$. Let $\omega = \omega(\varepsilon)$ be such that

$$0 < \omega < \min\{S\varepsilon, S\sigma(V)/2\}. \tag{7.7.17}$$

By density of $C_c^\infty(\mathbb{R}^N)$ in $L^{\frac{N}{2s}}(\mathbb{R}^N)$ there exists

$$\hat{V}(x) = \sum_{i=1}^m \frac{\mu_i \chi_{B'(a_i,r_i)}(x)}{|x-a_i|^{2s}} + \frac{\mu_\infty \chi_{\mathbb{R}^N \setminus B'_R}(x)}{|x|^{2s}} + \hat{W}(x) \in \Theta^*$$

such that $\hat{W} \in \mathcal{P}_\infty^\delta$ for some $\delta > 0$ and

$$\|\hat{V} - V\|_{L^{N/2s}(\mathbb{R}^N)} < \frac{\omega}{1+\varepsilon}. \tag{7.7.18}$$

Then from Lemma 7.7.1, taking into account (7.7.17) and (7.7.18), we have that

$$\sigma(\hat{V} + \varepsilon\hat{V}) \geq \sigma(V + \varepsilon V) - (1+\varepsilon) S^{-1} \|\hat{V} - V\|_{L^{N/2s}(\mathbb{R}^N)} > 0.$$

Now, thanks to Lemma 7.7.2, there exists $\tilde{R} > R$ and a function $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that Φ is strictly positive and locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0,a_1), \dots, (0,a_m)\}$ and

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla\Phi \cdot \nabla U \,\mathrm{d}x\mathrm{d}t - \kappa_s \int_{\mathbb{R}^N} \left[\hat{V} + \varepsilon\hat{V} + \frac{\nu_\infty + \varepsilon\nu_\infty}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}}\right] \operatorname{Tr}\Phi \operatorname{Tr} U \,\mathrm{d}x \geq 0,$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ with $U \geq 0$ a.e. Therefore Lemma 7.2.2 yields

$$\sigma\left(\hat{V} + \frac{\nu_\infty}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}}\right) \geq \frac{\varepsilon}{1+\varepsilon}.$$

Finally, thanks to Lemma 7.7.1, (7.7.17) and (7.7.18), we have the estimate

$$\sigma\left(V + \frac{\nu_\infty}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}}\right) \geq \sigma\left(\hat{V} + \frac{\nu_\infty}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}}\right) - S^{-1} \|\hat{V} - V\|_{L^{N/2s}(\mathbb{R}^N)} > 0$$

which yields the conclusion. □

Swapping the singularity at a pole for a singularity at infinity through the Kelvin transform, we obtain the analog of Theorem 7.7.3 when perturbing the mass of a pole.

Theorem 7.7.4. *Let*

$$V(x) = \sum_{i=1}^m \frac{\mu_i \chi_{B'(a_i, r_i)}(x)}{|x - a_i|^{2s}} + \frac{\mu_\infty \chi_{\mathbb{R}^N \setminus B'_R}(x)}{|x|^{2s}} + W(x) \in \Theta.$$

Assume $\sigma(V) > 0$ and let $i_0 \in \{1, \dots, m\}$ and $\nu \in \mathbb{R}$ be such that $\mu_{i_0} + \nu < \gamma_H$. Then there exists $\delta \in (0, r_{i_0})$ such that

$$\sigma\left(V + \frac{\nu}{|x - a_{i_0}|^{2s}} \chi_{B'(a_{i_0}, \delta)}\right) > 0.$$

Before proving Theorem 7.7.4, it is convenient to make the following remark.

Remark 7.7.5. (i) By the invariance by translation of the norm $\|\cdot\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}$, we have that, if $V \in \mathcal{H}$, then, for any $a \in \mathbb{R}^N$, the translated potential $V_a := V(\cdot + a)$ belongs to $\mathcal{H}$ and $\sigma(V_a) = \sigma(V)$.

(ii) If $V \in \mathcal{H}$ and $V_K(x) := |x|^{-4s} V(\frac{x}{|x|^2})$, then $V_K \in \mathcal{H}$ and $\sigma(V_K) = \sigma(V)$. To prove this statement, we observe that, by the change of variables $y = \frac{x}{|x|^2}$,

$$\int_{\mathbb{R}^N} |V_K(x)|^2 u^2(x)\, dx = \int_{\mathbb{R}^N} |V(y)|^2 (\mathcal{K}u)^2(y)\, dy \quad \text{for any } u \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

where $(\mathcal{K}u)(x) := |x|^{2s-N} u(\frac{x}{|x|^2})$ is the Kelvin transform of u. The claim then follows from the fact that $\mathcal{K}$ is an isometry on $\mathcal{D}^{s,2}(\mathbb{R}^N)$ (see [FW12, Lemma 2.2]).

Proof of Theorem 7.7.4. Let $V_1(x) := V(x + a_{i_0})$. We have that

$$V_1(x) = \frac{\mu_{i_0} \chi_{B'_{r_{i_0}}}(x)}{|x|^{2s}} + \sum_{i \neq i_0} \frac{\mu_i \chi_{B'(a_i - a_{i_0}, r_i)}(x)}{|x - (a_i - a_{i_0})|^{2s}} + \frac{\mu_\infty \chi_{\mathbb{R}^N \setminus B'_R}(x)}{|x|^{2s}} + W_1(x) \in \Theta$$

and, in view of Remark 7.7.5 (i), $\sigma(V_1) = \sigma(V) > 0$. Then we can choose some ε sufficiently small so that $\sigma(V_1 + \varepsilon V_1) > \frac{\sigma(V)}{2} > 0$ (see (7.5.3)) and $\mu_{i_0} + \nu + \varepsilon(\mu_{i_0} + \nu) < \gamma_H$,

$\mu_i + \varepsilon\mu_i < \gamma_H$ for all $i = 1, \dots, m, \infty$. Let $\omega = \omega(\varepsilon) \in (0, \min\{S\varepsilon, S\sigma(V)/2\})$. By density of $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_m\})$ in $L^{\frac{N}{2s}}(\mathbb{R}^N)$ there exists

$$V_2(x) = \frac{\mu_{i_0}\chi_{B'_{r_{i_0}}}(x)}{|x|^{2s}} + \sum_{i \neq i_0} \frac{\mu_i \chi_{B'(a_i - a_{i_0}, r_i)}(x)}{|x-(a_i - a_{i_0})|^{2s}} + \frac{\mu_\infty \chi_{\mathbb{R}^N \setminus B'_R}(x)}{|x|^{2s}} + W_2(x) \in \Theta^*$$

such that $W_2 \in L^{N/2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ vanishes in a neighborhood of any pole and in a neighborhood of ∞ and

$$\|V_2 - V_1\|_{L^{N/2s}(\mathbb{R}^N)} < \frac{\omega}{1+\varepsilon}.$$

Let $V_3(x) := |x|^{-4s} V_2(\frac{x}{|x|^2})$. Then

$$V_3 \in C^1\Big(\mathbb{R}^N \setminus \Big\{0, \tfrac{a_i - a_{i_0}}{|a_i - a_{i_0}|^2}\Big\}_{i \neq i_0}\Big)$$

and there exists $r > 0$ such that

$$V_3(x) = \frac{\mu_{i_0}}{|x|^{2s}} \quad \text{in } \mathbb{R}^N \setminus B'_r.$$

Moreover, from Remark 7.7.5 (ii) and Lemma 7.7.1 it follows that $V_3 \in \mathcal{H}$ and

$$\begin{aligned}\sigma(V_3 + \varepsilon V_3) &= \sigma(V_2 + \varepsilon V_2) \\ &\geq \sigma(V_1 + \varepsilon V_1) - S^{-1}(1+\varepsilon)\|V_1 - V_2\|_{L^{N/2s}(\mathbb{R}^N)} > \frac{\sigma(V)}{2} - S^{-1}\omega > 0.\end{aligned}$$

From Lemma 7.7.2 we deduce that there exists $\tilde{R} > r$ and a function $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that Φ is strictly positive and locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \Big\{0, \frac{a_i - a_{i_0}}{|a_i - a_{i_0}|^2}\Big\}_{i \neq i_0}$ and

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla\Phi \cdot \nabla U \, dx dt - \kappa_s \int_{\mathbb{R}^N} \left[V_3 + \varepsilon V_3 + \frac{\nu + \varepsilon\nu}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}}\right] \operatorname{Tr}\Phi \operatorname{Tr} U \, dx \geq 0,$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ with $U \geq 0$ a.e. Therefore Lemma 7.2.2 yields

$$\sigma\left(V_3 + \frac{\nu}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}}\right) \geq \frac{\varepsilon}{1+\varepsilon}.$$

From Remark 7.7.5 (ii) we have that $\sigma\left(V_3 + \frac{\nu}{|x|^{2s}}\chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}}\right) = \sigma\left(V_2 + \frac{\nu}{|x|^{2s}}\chi_{B'_{1/\tilde{R}}}\right) \geq \frac{\varepsilon}{1+\varepsilon}$. Hence, letting $\delta = 1/\tilde{R}$, from Remark 7.7.5 (i) and Lemma 7.7.1 we deduce that

$$\begin{aligned}\sigma\left(V + \frac{\nu}{|x - a_{i_0}|^{2s}} \chi_{B'(a_{i_0}, \delta)}\right) &= \sigma\left(V_1 + \frac{\nu}{|x|^{2s}} \chi_{B'_\delta}\right) \\ &\geq \sigma\left(V_2 + \frac{\nu}{|x|^{2s}} \chi_{B'_\delta}\right) - S^{-1}\|V_1 - V_2\|_{L^{N/2s}(\mathbb{R}^N)} \geq \frac{\varepsilon}{1+\varepsilon} - \frac{S^{-1}\omega}{1+\varepsilon} > 0\end{aligned}$$

which yields the conclusion. □

Corollary 7.7.6. *Let*

$$V(x) = \sum_{i=1}^{m} \frac{\mu_i \chi_{B'(a_i,r_i)}(x)}{|x-a_i|^{2s}} + \frac{\mu_\infty \chi_{\mathbb{R}^N \setminus B'_R}(x)}{|x|^{2s}} + W(x) \in \Theta$$

be such that $\sigma(V) > 0$*. Then there exists*

$$\widetilde{V}(x) = \sum_{i=1}^{m} \frac{\tilde{\mu}_i \chi_{B'(a_i,\tilde{r}_i)}(x)}{|x-a_i|^{2s}} + \frac{\mu_\infty \chi_{\mathbb{R}^N \setminus B'_R}(x)}{|x|^{2s}} + \widetilde{W}(x) \in \Theta \tag{7.7.19}$$

such that

$\widetilde{V} - V \in C^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_m\})$, $\quad \widetilde{V} \geq V$, $\quad \sigma(\widetilde{V}) > 0$, $\quad$ *and* $\quad \tilde{\mu}_i > 0$ *for all* $i = 1, \dots, m$.

Proof. For every $i = 1, \dots, m$, let ν_i be such that $\nu_i > 0$ and $\mu_i + \nu_i \in (0, \gamma_H)$. From Theorem 7.7.4 we have that, for every $i = 1, \dots, m$, there exists δ_i such that, letting

$$\widehat{V} = V + \sum_{i=1}^{m} \frac{\nu_i}{|x-a_i|^{2s}} \chi_{B'(a_i,\delta_i)},$$

$\sigma(\widehat{V}) > 0$. Let us consider a cut-off function $\zeta : \mathbb{R}^N \to \mathbb{R}$ such that $\zeta \in C^\infty(\mathbb{R}^N)$, $0 \leq \zeta(x) \leq 1$, $\zeta(x) = 1$ for $|x| \leq \frac{1}{2}$, and $\zeta(x) = 0$ for $|x| \geq 1$. Let

$$\widetilde{V}(x) = V + \sum_{i=1}^{m} \frac{\nu_i}{|x-a_i|^{2s}} \zeta\left(\frac{x-a_i}{\delta_i}\right).$$

Then $\widetilde{V} - V \in C^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_m\})$ and $\widetilde{V} \geq V$. Moreover $\widetilde{V}$ is of the form (7.7.19) with $\tilde{\mu}_i = \mu_i + \nu_i > 0$ and, in view of (7.2.5) and the fact that $\widetilde{V} \leq \widehat{V}$, $\sigma(\widetilde{V}) \geq \sigma(\widehat{V}) > 0$. The proof is thereby complete. □

7.8 Localization of Binding

This section is devoted to the proof of Theorem 7.2.3, which is the main tool needed in order to prove our main result. Indeed this tool ensures, inside the class Θ, that the sum of two positive operators is positive, provided one of them is translated sufficiently far.

For any $\delta > 0$ and $a_1, \dots, a_m \in \mathbb{R}^N$, we define

$$\mathcal{P}^\delta_{a_1,\dots,a_m} := \mathcal{P}^\delta_\infty \cap \left(\bigcap_{j=1}^{m} \mathcal{P}^\delta_{a_j} \right) \tag{7.8.1}$$

where $\mathcal{P}^\delta_\infty$ is defined in (7.7.1) and, for all $j = 1, \dots, m$,

$$\mathcal{P}^\delta_{a_j} = \Big\{ f\colon \mathbb{R}^N \to \mathbb{R}\colon f \in C^1(B'(a_j, R_j) \setminus \{a_j\}) \text{ for some } R_j > 0$$
$$\text{and } |f(x)| + |(x-a_j) \cdot \nabla f(x)| = O(|x-a_j|^{-2s+\delta}) \text{ as } x \to a_j \Big\}.$$

Lemma 7.8.1. *Let*

$$V_1(x) = \sum_{i=1}^{m_1} \frac{\mu_i^1 \chi_{B'(a_i^1, r_i^1)}(x)}{|x - a_i^1|^{2s}} + \frac{\mu_\infty^1 \chi_{\mathbb{R}^N \setminus B'_{R_1}}(x)}{|x|^{2s}} + W_1(x) \in \Theta^*,$$

$$V_2(x) = \sum_{i=1}^{m_2} \frac{\mu_i^2 \chi_{B'(a_i^2, r_i^2)}(x)}{|x - a_i^2|^{2s}} + \frac{\mu_\infty^2 \chi_{\mathbb{R}^N \setminus B'_{R_2}}(x)}{|x|^{2s}} + W_2(x) \in \Theta^*,$$

with $W_1 \in \mathcal{P}^\delta_{a_1^1, \dots, a_{m_1}^1}$, $W_2 \in \mathcal{P}^\delta_{a_1^2, \dots, a_{m_2}^2}$ *for some* $\delta > 0$. *If* $\sigma(V_1), \sigma(V_2) > 0$ *and* $\mu_\infty^1 + \mu_\infty^2 < \gamma_H$, *then there exists* $R > 0$ *such that for every* $y \in \mathbb{R}^N \setminus \overline{B'_R}$ *there exists* $\Phi_y \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ *such that* Φ_y *is strictly positive and locally Hölder continuous in* $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_i^1), (0, a_i^2 + y)\}_{i=1,\dots,m_j, j=1,2}$ *and*

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla \Phi_y \cdot \nabla U \, \mathrm{d}x\mathrm{d}t \geq \kappa_s \int_{\mathbb{R}^N} (V_1(x) + V_2(x-y)) \operatorname{Tr} \Phi_y \operatorname{Tr} U \, \mathrm{d}x \tag{7.8.2}$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, *with* $U \geq 0$ *a.e.*

Proof. First of all we observe that it is not restrictive to assume that $\mu_i^j > 0$ for all $i = 1, \dots, m_j$, $j = 1, 2$. Indeed, letting V_1, V_2 as in the assumptions, from Corollary 7.7.6 there exist $\widetilde{V}_1, \widetilde{V}_2 \in \Theta^*$ with positive masses at poles such that $\widetilde{V}_j \geq V_j$ and $\sigma(\widetilde{V}_j) > 0$ for $j = 1, 2$. If the theorem is true under the further assumption of positivity of masses at poles, we conclude that, for every y with $|y|$ sufficiently large, there exists $\Phi_y \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ strictly positive and locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_i^1), (0, a_i^2 + y)\}_{i=1,\dots,m_j, j=1,2}$ such that (7.8.2) holds with $\widetilde{V}_1(x) + \widetilde{V}_2(x-y)$ in the right hand side integral instead of $V_1(x) + V_2(x-y)$. Since $\widetilde{V}_1(x) + \widetilde{V}_2(x-y) \geq V_1(x) + V_2(x-y)$ we obtain (7.8.2). Then we can assume that $\mu_i^j > 0$ for all $i = 1, \dots, m_j$, $j = 1, 2$, without loss of generality.

Let $\varepsilon \in (0, \gamma_H)$ be such that $\mu_\infty^1 + \mu_\infty^2 < \gamma_H - \varepsilon$, $\mu_\infty^1 < \gamma_H - \varepsilon$, and $\mu_\infty^2 < \gamma_H - \varepsilon$ and let $\Lambda := \gamma_H - \varepsilon$. Let us set

$$\nu_\infty^1 := \Lambda - \mu_\infty^1, \quad \nu_\infty^2 := \Lambda - \mu_\infty^2,$$

so that $\nu_\infty^1, \nu_\infty^2 > 0$. Let $0 < \eta < 1$ be such that

$$\mu_\infty^2 < \nu_\infty^1 (1 - 2\eta) \quad \text{and} \quad \mu_\infty^1 < \nu_\infty^2 (1 - 2\eta). \tag{7.8.3}$$

Let us choose $\bar{R} > 0$ large enough so that

$$\cup_{i=1}^{m_j} B'(a_i^j, r_i^j) \subset B'_{\bar{R}} \quad \text{for } j = 1, 2.$$

We observe that, by Theorem 7.7.3, there exists $\tilde{R}_j > 0$ such that

$$\sigma \left(V_j + \frac{\nu_\infty^j}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}_j}} \right) > 0.$$

Since $\mu_i^j > 0$ implies that $a_{\mu_i^j} < 0$, we can fix some $\omega > 0$ such that $\omega < 2s$ and $\omega < -a_{\mu_i^j}$ for all $i = 1, \dots, m_j$, $j = 1, 2$. Let us consider, for $j = 1, 2$, $p_j \in C^\infty(\mathbb{R}^N \setminus \{a_1^j, \dots, a_{m_j}^j\}) \cap \mathcal{P}^\omega_{a_1^j, \dots, a_{m_j}^j}$ such that $p_j(x) > 0$ for all $x \in \mathbb{R}^N$ and

$$p_j(x) \geq \frac{1}{|x - a_i^j|^{2s-\omega}} \text{ if } x \in B'(a_i^j, r_i^j), \quad p_j(x) \geq 1 \text{ if } x \in B'_{\tilde{R}} \setminus \cup_{i=1}^{m_j} B(a_i^j, r_i^j). \tag{7.8.4}$$

Since $p_j \in L^{\frac{N}{2s}}(\mathbb{R}^N)$ satisfies the hypotheses of Lemma 6.1.1 the infimum

$$\sigma_j = \inf_{\substack{U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \\ \operatorname{Tr} U \not\equiv 0}} \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \,\mathrm{d}x\mathrm{d}t - \kappa_s \int_{\mathbb{R}^N} \left[V_j + \frac{\nu_\infty^j}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}_j}} \right] |\operatorname{Tr} U|^2 \,\mathrm{d}x}{\int_{\mathbb{R}^N} p_j \, |\operatorname{Tr} U|^2 \,\mathrm{d}x} > 0$$

is achieved by some nonnegative $\Psi_j \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, for $j = 1, 2$. In addition, Ψ_j weakly solves

$$\begin{cases} -\operatorname{div}(t^{1-2s} \nabla \Psi_j) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t \to 0^+} t^{1-2s} \dfrac{\partial \Psi_j}{\partial t} = \kappa_s \left[V_j + \dfrac{\nu_\infty^j}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}_j}} + \sigma_j p_j \right] \operatorname{Tr} \Psi_j, & \text{on } \mathbb{R}^N. \end{cases} \tag{7.8.5}$$

From Proposition 7.3.1 we know that Ψ_j is locally Hölder continuous in

$$\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1^j), \dots, (0, a_{m_j}^j)\}.$$

In order to prove that Ψ_j is strictly positive in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1^j), \dots, (0, a_{m_j}^j)\}$, we compare it with the unique weak solution $\widetilde{\Psi}_j \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ to the problem

$$\begin{cases} -\operatorname{div}(t^{1-2s} \nabla \widetilde{\Psi}_j) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t \to 0^+} t^{1-2s} \dfrac{\partial \widetilde{\Psi}_j}{\partial t} = \kappa_s V_j \operatorname{Tr} \widetilde{\Psi}_j + \kappa_s \sigma_j p_j \operatorname{Tr} \Psi_j, & \text{on } \mathbb{R}^N, \end{cases} \tag{7.8.6}$$

whose existence directly follows from the Lax-Milgram Lemma. The difference $\widetilde{\Phi}_j = \Psi_j - \widetilde{\Psi}_j$ belongs to $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and weakly solves

$$\begin{cases} -\operatorname{div}(t^{1-2s} \nabla \widetilde{\Phi}_j) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t \to 0^+} t^{1-2s} \dfrac{\partial \widetilde{\Phi}_j}{\partial t} = \kappa_s V_j \operatorname{Tr} \widetilde{\Phi}_j + \kappa_s \dfrac{\nu_\infty^j}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}_j}} \operatorname{Tr} \Psi_j, & \text{on } \mathbb{R}^N. \end{cases}$$

By testing the above equation with $-\widetilde{\Phi}_j^-$ and recalling that $\sigma(V_j) > 0$, we obtain that $\widetilde{\Phi}_j \geq 0$ in $\mathbb{R}^{N+1}_+$ and hence $\Psi_j \geq \widetilde{\Psi}_j$. Moreover, testing (7.8.6) with $-\widetilde{\Psi}_j^-$, we also obtain that $\widetilde{\Psi}_j \geq 0$ in $\mathbb{R}^{N+1}_+$. From the classical Strong Maximum Principle and Proposition

7.3.2 (whose assumption (7.3.1) for (7.8.6) is satisfied thanks to Lemma 7.3.3 and the assumption $V_j \in \Theta^*$) it follows that $\widetilde{\Psi}_j > 0$ in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a^j_1), \dots, (0, a^j_{m_j})\}$ and hence

$$\Psi_j > 0 \quad \text{in} \quad \overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a^j_1), \dots, (0, a^j_{m_j})\}.$$

Lemma 7.3.7 yields

$$\lim_{|x|\to\infty} \Psi_j(0, x)|x|^{N-2s+a_\Lambda} = \ell_j > 0,$$

for some $\ell_j > 0$ (see (7.3.9) for the notation a_Λ). Hence, the function $\Phi_j(x, t) := \frac{\Psi_j(x,t)}{\ell_j}$ satisfies (7.8.5) and $\Phi_j(0, x) \sim |x|^{-(N-2s+a_\Lambda)}$ for $|x| \to \infty$. Therefore, there exists $\rho > \max\{\tilde{R}_1, \tilde{R}_2, \bar{R}\}$ such that

$$(1 - \eta^2)|x|^{-(N-2s+a_\Lambda)} \leq \Phi_j(0, x) \leq (1 + \eta)|x|^{-(N-2s+a_\Lambda)} \tag{7.8.7}$$

and

$$|W_1(x)| \leq \frac{\eta\nu^2_\infty}{|x|^{2s}}, \qquad |W_2(x)| \leq \frac{\eta\nu^1_\infty}{|x|^{2s}} \tag{7.8.8}$$

for all $x \in \mathbb{R}^N \setminus B'_\rho$. Also, form Lemma 7.3.6 we know that there exists $C > 0$ such that

$$\frac{1}{C}|x - a^j_i|^{a_{\mu^j_i}} \leq \Phi_j(0, x) \leq C|x - a^j_i|^{a_{\mu^j_i}} \quad \text{in } B'(a^j_i, r^j_i), \tag{7.8.9}$$

for $i = 1, \dots m_j$, $j = 1, 2$. For any $y \in \mathbb{R}^N$, we define

$$\Phi_y(x, t) := \nu^2_\infty\Phi_1(x, t) + \nu^1_\infty\Phi_2(t, x - y) \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}).$$

Then

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla\Phi_y \cdot \nabla U \,\mathrm{d}x\mathrm{d}t - \kappa_s \int_{\mathbb{R}^N} (V_1(x) + V_2(x - y)) \operatorname{Tr}\Phi_y \operatorname{Tr} U \,\mathrm{d}x = \int_{\mathbb{R}^N} g_y(x) \operatorname{Tr} U \,\mathrm{d}x$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, where

$$\begin{aligned} g_y(x) := \kappa_s \Big[&\sigma_1\nu^2_\infty p_1(x)\Phi_1(0, x) + \frac{\nu^1_\infty\nu^2_\infty}{|x|^{2s}}\chi_{\mathbb{R}^N\setminus B'_{\tilde{R}_1}}(x)\Phi_1(0, x) + \sigma_2\nu^1_\infty p_2(x - y)\Phi_2(0, x - y) \\ &+\frac{\nu^1_\infty\nu^2_\infty}{|x - y|^{2s}}\chi_{\mathbb{R}^N\setminus B'(y,\tilde{R}_2)}(x)\Phi_2(0, x - y) - \nu^1_\infty V_1(x)\Phi_2(0, x - y) - \nu^2_\infty V_2(x - y)\Phi_1(0, x)\Big]. \end{aligned}$$

Therefore, to conclude the proof it is enough to show that $g_y \geq 0$ a.e. in $\mathbb{R}^N$.

From (7.8.3), (7.8.7) and (7.8.8), it follows that in $\mathbb{R}^N \setminus (B'_\rho \cup B'(y,\rho))$

$$\begin{aligned}
g_y(x) \geq \kappa_s \Bigg[& \frac{\nu_\infty^1 \nu_\infty^2}{|x|^{2s}} \Phi_1(0,x) + \frac{\nu_\infty^1 \nu_\infty^2}{|x-y|^{2s}} \Phi_2(0,x-y) \\
& - \nu_\infty^1 \left(\frac{\mu_\infty^1}{|x|^{2s}} + W_1(x) \right) \Phi_2(0,x-y) - \nu_\infty^2 \left(\frac{\mu_\infty^2}{|x-y|^{2s}} + W_2(x-y) \right) \Phi_1(0,x) \Bigg] \\
> \kappa_s \nu_\infty^1 \nu_\infty^2 (1-\eta^2) \Big[& |x|^{-(N+a_\Lambda)} + |x-y|^{-(N+a_\Lambda)} \\
& - |x|^{-(N-2s+a_\Lambda)} |x-y|^{-2s} - |x|^{-2s} |x-y|^{-(N-2s+a_\Lambda)} \Big] \\
= \kappa_s \nu_\infty^1 \nu_\infty^2 (1-\eta^2) & \left(\frac{1}{|x|^{N-2s+a_\Lambda}} - \frac{1}{|x-y|^{N-2s+a_\Lambda}} \right) \left(\frac{1}{|x|^{2s}} - \frac{1}{|x-y|^{2s}} \right) \geq 0.
\end{aligned}$$

For $|y| > R > 2\rho$, we have $B'_\rho \cap B'(y,\rho) = \emptyset$. From (7.8.4), (7.8.7), (7.8.8), (7.8.9) and the choice of ω we have that, in $B(a_i^1, r_i^1)$,

$$\begin{aligned}
g_y(x) \geq \kappa_s \Bigg[& \sigma_1 \nu_\infty^2 p_1(x) \Phi_1(0,x) + \frac{\nu_\infty^1 \nu_\infty^2}{|x-y|^{2s}} \Phi_2(0,x-y) \\
& - \nu_\infty^1 V_1(x) \Phi_2(0,x-y) - \nu_\infty^2 V_2(x-y) \Phi_1(0,x) \Bigg] \\
\geq \kappa_s |x-a_i^1|^{a_{\mu_i^1} - 2s + \omega} \Bigg[& \frac{\sigma_1 \nu_\infty^2}{C} \\
& - \nu_\infty^1 (1+\eta) |x-y|^{-(N-2s+a_\Lambda)} |x-a_i^1|^{-a_{\mu_i^1} - \omega} (\mu_i^1 + \|W_1\|_{L^\infty(\mathbb{R}^N)} |x-a_i^1|^{2s}) \\
& - \nu_\infty^2 \nu_\infty^1 (1-\eta) C |x-y|^{-2s} |x-a_i^1|^{2s-\omega} \Bigg] \\
\geq \kappa_s |x-a_i^1|^{a_{\mu_i^1} - 2s + \omega} & \left[\frac{\sigma_1 \nu_\infty^2}{C} + o(1) \right],
\end{aligned}$$

as $|y| \to \infty$. Now let $x \in B'_\rho \setminus (\cup_{i=1}^{m_1} B'(a_i^1, r_i^1))$: since Φ_1 is positive and continuous we have $\tilde{C}^{-1} > \Phi_1(0,x) > \tilde{C}$, for some $\tilde{C} > 0$, and so, thanks to (7.8.4), (7.8.7) and (7.8.8), there holds

$$g_y(x) \geq \kappa_s \sigma_1 \nu_\infty^2 \tilde{C} + o(1),$$

as $|y| \to \infty$. One can similarly prove that, for $|y|$ sufficiently large, $g_y(x) \geq 0$ in $B'(y,\rho)$ as well. The proof is thereby complete. $\square$

Proof of Theorem 7.2.3. First, let

$$0 < \varepsilon < \min \left\{ 2S\sigma(V_j), \frac{\sigma(V_j)}{2} \left[\frac{1}{\gamma_H} \left(\sum_{i=1}^{m} |\mu_i^j| + |\mu_\infty^j| \right) + S^{-1} \|W\|_{L^{N/2s}(\mathbb{R}^N)} \right]^{-1} \right\} \tag{7.8.10}$$

for $j = 1,2$, such that, in addition, $\mu_\infty^1 + \mu_\infty^2 + \varepsilon(\mu_\infty^1 + \mu_\infty^2) < \gamma_H$ and $\mu_i^j + \varepsilon \mu_i^j < \gamma_H$ for all $i = 1, \dots, m_j, \infty$. Similarly to (7.5.3), one can prove that $\sigma(V_j + \varepsilon V_j) > \frac{\sigma(V_j)}{2} > 0$ for

$j = 1, 2$. Moreover, let $\omega = \omega(\varepsilon)$ be such that

$$0 < \omega < \min\left\{\frac{S\sigma(V_1)}{2}, \frac{S\sigma(V_2)}{2}, \frac{S\varepsilon}{2}\right\}. \tag{7.8.11}$$

Let, for $j = 1, 2$,

$$\hat{V}_j(x) = \sum_{i=1}^{m_j} \frac{\mu_i^j \chi_{B'(a_i^j, r_i^j)}(x)}{|x - a_i^j|^{2s}} + \frac{\mu_\infty^j \chi_{\mathbb{R}^N \setminus B'_{R_j}}(x)}{|x|^{2s}} + \hat{W}_j(x) \in \Theta^*,$$

be such that $\hat{W}_j \in \mathcal{P}^\delta_{a_1^j, \dots, a_{m_j}^j}$ for some $\delta > 0$ and

$$\|\hat{V}_j - V_j\|_{L^{N/2s}(\mathbb{R}^N)} < \frac{\omega}{1 + \varepsilon}. \tag{7.8.12}$$

From Lemma 7.7.1, (7.8.11) and (7.8.12) we deduce that

$$\sigma(\hat{V}_j + \varepsilon \hat{V}_j) \geq \sigma(V_j + \varepsilon V_j) - (1 + \varepsilon) S^{-1} \|\hat{V}_j - V_j\|_{L^{N/2s}(\mathbb{R}^N)} > 0.$$

Hence we infer from Lemma 7.8.1 that there exists $R > 0$ such that, for all $y \in \mathbb{R}^N \setminus \overline{B'_R}$, there exists $\Phi_y \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that Φ_y is strictly positive and locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_i^1), (0, a_i^2 + y)\}_{i=1,\dots,m_j, j=1,2}$ and

$$\begin{aligned} &\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla \Phi_y \cdot \nabla U \, \mathrm{d}x\mathrm{d}t \\ &\qquad - \kappa_s \int_{\mathbb{R}^N} \left[\hat{V}_1(x) + \varepsilon \hat{V}_1(x) + \hat{V}_2(x - y) + \varepsilon \hat{V}_2(x - y)\right] \operatorname{Tr} \Phi_y \operatorname{Tr} U \, \mathrm{d}x \geq 0 \end{aligned}$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, with $U \geq 0$ a.e. Therefore, thanks to the positivity criterion (Lemma 7.2.2), we know that

$$\sigma(\hat{V}_1(\cdot) + \hat{V}_2(\cdot - y)) \geq \frac{\varepsilon}{\varepsilon + 1}.$$

Combining Lemma 7.7.1 with (7.8.11) and (7.8.12), we finally deduce that

$$\begin{aligned} \sigma(V_1(\cdot) + V_2(\cdot - y)) &\geq \sigma(\hat{V}_1(\cdot) + \hat{V}_2(\cdot - y)) \\ &\quad - S^{-1}\|V_1 - \hat{V}_1\|_{L^{N/2s}(\mathbb{R}^N)} - S^{-1}\|V_2(\cdot - y) - \hat{V}_2(\cdot - y)\|_{L^{N/2s}(\mathbb{R}^N)} > 0, \end{aligned}$$

thus completing the proof. □

7.9 Proof of Theorem 7.2.4

In order to prove Theorem 7.2.4, we first need the following lemma, concerning the left-hand side in Hardy inequality (7.2.4).

Lemma 7.9.1. *We have that*

$$\lim_{|\xi|\to 0}\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x+\xi|^{2s}}\,\mathrm{d}x=\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^{2s}}\,\mathrm{d}x \quad \text{and} \quad \lim_{|\xi|\to +\infty}\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x+\xi|^{2s}}\,\mathrm{d}x=0$$

for any $u\in\mathcal{D}^{s,2}(\mathbb{R}^N)$.

Proof. The proof easily follows from density of $C_c^\infty(\mathbb{R}^N\setminus\{0\})$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ (see Lemma 7.3.10), the Dominated Convergence Theorem and the fractional Hardy inequality (7.2.4). □

We are now able to prove Theorem 7.2.4.

Proof of Theorem 7.2.4. First we prove that condition (7.2.9) is sufficient for the existence of at least one configuration of poles $a_1,\dots,a_m$ such that the quadratic form associated to $\mathcal{L}_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}$ is positive definite. In order to do this, we argue by induction on the number of poles m. For any m we assume the masses to be sorted in increasing order $\mu_1\le\dots\le\mu_m$. If $m=2$ the claim is proved in Remark 7.4.1. Suppose now the claim is proved for $m-1$. If $\mu_m\le 0$ the proof is trivial, so let us assume $\mu_m>0$: since (7.2.9) holds, it is true also for $\mu_1,\dots,\mu_{m-1}$, hence there exists a configuration of poles $a_1,\dots,a_{m-1}$ such that $Q_{\mu_1,\dots,\mu_{m-1},a_1,\dots,a_{m-1}}$ is positive definite. If we let

$$V_1(x)=\sum_{i=1}^{m-1}\frac{\mu_i}{|x-a_i|^{2s}} \quad \text{and} \quad V_2(x)=\frac{\mu_m}{|x|^{2s}},$$

we have that $V_1,V_2\in\Theta$ satisfy the assumptions of Theorem 7.2.3. Therefore there exists $a_m\in\mathbb{R}^N$ such that

$$\mathcal{L}_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}=(-\Delta)^s-(V_1+V_2(\cdot-a_m))$$

is positive definite. This concludes the first part.

We now prove the necessity of condition (7.2.9). Let $\varepsilon>0$ be such that

$$\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}-\sum_{i=1}^m\mu_i\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x-a_i|^{2s}}\,\mathrm{d}x\ge\varepsilon\,\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \tag{7.9.1}$$

for all $u\in\mathcal{D}^{s,2}(\mathbb{R}^N)$ and let $\delta\in(0,\varepsilon\gamma_H)$. Assume by contradiction that $\mu_j\ge\gamma_H$ for some $j\in\{1,\dots,m\}$. By optimality of γ_H in Hardy inequality (7.2.4) and by density of $C_c^\infty(\mathbb{R}^N)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, we have that there exists $\varphi\in C_c^\infty(\mathbb{R}^N)$ such that

$$\|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}-\mu_j\int_{\mathbb{R}^N}\frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x<\delta\int_{\mathbb{R}^N}\frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x. \tag{7.9.2}$$

If we let $\varphi_\rho := \rho^{-\frac{N-2s}{2}}\varphi(x/\rho)$, we have that

$$\begin{aligned} Q_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}(\varphi_\rho(\cdot - a_j)) &= \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \mu_j \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x - \sum_{i\neq j}\mu_i \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{\left|x - \frac{a_i-a_j}{\rho}\right|^{2s}}\,\mathrm{d}x \\ &\to \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \mu_j \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x \qquad \text{as } \rho\to 0^+, \end{aligned} \tag{7.9.3}$$

in view of Lemma 7.9.1. Combining (7.9.1), (7.9.2), (7.9.3) and Hardy inequality (7.2.4) we obtain

$$\varepsilon \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \le \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \mu_j \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x < \delta \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x \le \frac{\delta}{\gamma_H}\|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)},$$

which is a contradiction, because of the choice of δ.

Now suppose that $K := \sum_{i=1}^m \mu_i \ge \gamma_H$. Arguing analogously, there exists $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that

$$\|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - K \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x < \delta \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x.$$

The function $\varphi_\rho(x) := \rho^{-\frac{N-2s}{2}}\varphi(x/\rho)$ satisfies

$$\begin{aligned} Q_{\mu_1,\dots,\mu_m,a_1,\dots,a_m}(\varphi_\rho) &= \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^m \mu_i \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x - a_i/\rho|^{2s}}\,\mathrm{d}x \\ &\to \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - K \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x \qquad \text{as } \rho\to +\infty, \end{aligned}$$

thanks to Lemma 7.9.1. With the same argument as above, we again reach a contradiction. □

7.10 Proof of Proposition 7.2.6

Finally, in this section we present the proof of Proposition 7.2.6, that is independent of the previous results from the point of view of the technical approach.

Proof of Proposition 7.2.6. First, let us denote $\bar\mu = \max\{0,\mu_1,\dots,\mu_m,\mu_\infty\}$. By hypothesis there exists $\alpha \in \left(0, 1 - \frac{\bar\mu}{\gamma_H}\right)$ such that $\sigma(V) \le 1 - \frac{\bar\mu}{\gamma_H} - \alpha$. From Lemma 7.6.2 we know that there exists $\delta > 0$ such that, denoting by

$$\bar V = \sum_{i=1}^m \frac{\mu_i \zeta(\frac{x-a_i}{\delta})}{|x-a_i|^{2s}} + \frac{\mu_\infty \tilde\zeta(\frac{x}{R})}{|x|^{2s}},$$

with $\zeta, \tilde\zeta$ being as in Lemma 7.6.2, we have that

$$\sigma(\bar V) \ge 1 - \frac{\bar\mu}{\gamma_H} - \frac{\alpha}{2}. \tag{7.10.1}$$

if $\bar{\mu} > 0$ and $\sigma(\bar{V}) \geq 1$ if $\bar{\mu} = 0$. We can write $V = \bar{V} + \bar{W}$ for some $\bar{W} \in L^{N/2s}(\mathbb{R}^N)$. Now let $\{U_n\}_n \subseteq \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ be a minimizing sequence for $\sigma(V)$, i.e.

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U_n|^2 \,\mathrm{d}x\mathrm{d}t - \kappa_s \int_{\mathbb{R}^N} V \,|\mathrm{Tr}\, U_n|^2 \,\mathrm{d}x = \sigma(V) + o(1), \quad \text{as } n \to \infty \tag{7.10.2}$$

and $\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U_n|^2 \,\mathrm{d}x\mathrm{d}t = 1$. Since $\{U_n\}_n$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, there exists $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that, up to a subsequence (still denoted by $\{U_n\}_n$),

$$U_n \rightharpoonup U \quad \text{weamly in } \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \quad \text{and} \quad U_n \to U \quad \text{a.e. in } \mathbb{R}^{N+1}_+, \tag{7.10.3}$$

as $n \to \infty$. There holds

$$\begin{aligned}\sigma(\bar{V}) &\leq \int_{\mathbb{R}^N} t^{1-2s}|\nabla U_n|^2 \,\mathrm{d}x\mathrm{d}t - \kappa_s \int_{\mathbb{R}^N} V \,|\mathrm{Tr}\, U_n|^2 \,\mathrm{d}x + \kappa_s \int_{\mathbb{R}^N} \bar{W} \,|\mathrm{Tr}\, U_n|^2 \,\mathrm{d}x \\ &= \sigma(V) + \kappa_s \int_{\mathbb{R}^N} \bar{W} \,|\mathrm{Tr}\, U_n|^2 \,\mathrm{d}x + o(1), \quad \text{as } n \to \infty.\end{aligned}$$

Hence, from (7.10.3), (7.10.1), the choice of α, and Lemma 6.1.1 we deduce that (if $\bar{\mu} > 0$)

$$1 - \frac{1}{\gamma_H}\bar{\mu} - \frac{\alpha}{2} \leq \sigma(V) + \kappa_s \int_{\mathbb{R}^N} \bar{W} \,|\mathrm{Tr}\, U|^2 \,\mathrm{d}x \leq 1 - \frac{\bar{\mu}}{\gamma_H} - \alpha + \kappa_s \int_{\mathbb{R}^N} \bar{W} \,|\mathrm{Tr}\, U|^2 \,\mathrm{d}x,$$

and so $\kappa_s \int_{\mathbb{R}^N} \bar{W} \,|\mathrm{Tr}\, U|^2 \,\mathrm{d}x \geq \frac{\alpha}{2} > 0$, which implies that $U \not\equiv 0$. The same conclusion easily follows in the case $\bar{\mu} = 0$. From the weak convergence $U_n \rightharpoonup U$ in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, the continuity of the trace map $\mathrm{Tr} : \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \to L^{N/2s}(\mathbb{R}^N)$ and the definition of $\sigma(V)$, we have that

$$\begin{aligned}\sigma(V) &\leq \frac{\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2 \,\mathrm{d}x\mathrm{d}t - \kappa_s \int_{\mathbb{R}^N} V \,|\mathrm{Tr}\, U|^2 \,\mathrm{d}x}{\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2 \,\mathrm{d}x\mathrm{d}t} \\ &= \frac{\sigma(V) - \left[\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla (U_n - U)|^2 \,\mathrm{d}x\mathrm{d}t - \kappa_s \int_{\mathbb{R}^N} V \,|\mathrm{Tr}\, U_n - \mathrm{Tr}\, U|^2 \,\mathrm{d}x\right] + o(1)}{\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U_n|^2 \,\mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla (U_n - U)|^2 \,\mathrm{d}x\mathrm{d}t + o(1)} \\ &\leq \sigma(V)\, \frac{1 - \displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla (U_n - U)|^2 \,\mathrm{d}x\mathrm{d}t + o(1)}{1 - \displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla (U_n - U)|^2 \,\mathrm{d}x\mathrm{d}t + o(1)} \\ &= \sigma(V) + \frac{o(1)}{\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2 \,\mathrm{d}x\mathrm{d}t + o(1)}.\end{aligned}$$

Letting $n \to \infty$ yields the fact that $\sigma(V)$ is attained by U and this concludes the proof. $\square$

CHAPTER 8

AN OBSTACLE PROBLEM FOR THE FRACTIONAL LAPLACIAN

8.1 Introduction

In the present chapter we investigate local properties of solutions of a stationary two-phase fractional obstacle-type problem, with particular emphasis on regularity aspects and on the structure of the nodal set. Our investigations focus on the study of weak solutions of the following boundary value problem

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla u) = 0, & \text{in } B_1^+, \\ -\lim_{t\to 0^+} t^{1-2s}\dfrac{\partial u}{\partial t} = \lambda_-((u-h)^-)^{p-1} - \lambda_+((u-h)^+)^{p-1}, & \text{on } B_1'. \end{cases} \tag{8.1.1}$$

Here $y^+ := \max\{y, 0\}$ and $y^- := \max\{-y, 0\}$ denote the positive and negative parts of $y \in \mathbb{R}$, $\lambda_-, \lambda_+ \geq 0$ are nonnegative constants, $p \geq 2$ and, for $r > 0$,

$$B_r^+ := \{z = (x,t) \in \mathbb{R}^{N+1}_+ \colon |z| < r\}, \quad B_r' := \{x \in \mathbb{R}^N \colon |x| < r\},$$

with

$$\mathbb{R}^{N+1}_+ := \{(x,t) \colon x \in \mathbb{R}^N,\ t > 0\} = \mathbb{R}^N \times (0, +\infty),$$

for $N \geq 2$. We refer to the function $h \colon B_1' \to \mathbb{R}$ as the *(thin) obstacle*. Thanks to the extension procedure established in [CS07], one can see that solutions to (8.1.1) are closely related to solutions of an equation in N dimensions, driven by the fractional Laplacian

$$(-\Delta)^s u(x_0) = C(N,s)\lim_{\varepsilon\to 0}\int_{|x-x_0|>\varepsilon}\frac{u(x_0)-u(x)}{|x_0-x|^{N+2s}}\,\mathrm{d}x,$$

where $C(N, s)$ is a positive constant. Loosely speaking, it is shown in [CS07] that, in a suitable sense,

$$(-\Delta)^s u(x_0) = -\kappa_s^{-1} \lim_{t \to 0^+} t^{1-2s} \frac{\partial u}{\partial t}(x_0, t), \quad \text{for } x_0 \in B_1'$$

for a certain (explicit) positive constant κ_s. Therefore, one can think of the trace on $\{t = 0\}$ of u in (8.1.1) as a solution to

$$(-\Delta)^s u = \lambda_-((u-h)^-)^{p-1} - \lambda_+((u-h)^+)^{p-1} \quad \text{in } B_1'. \tag{8.1.2}$$

We refer to Chapter 6 for a more accurate description of the operator $(-\Delta)^s$, the fractional setting, and the extension procedure. Thanks to this connection, the problem we study can be read in two different ways. On one hand, it can be seen as a two-phase penalized obstacle problem for the fractional Laplacian on $\mathbb{R}^N$, as in (8.1.2). On the other hand, in its extended formulation (8.1.1), one can understand it as a (weighted) two-phase boundary penalized obstacle problem associated to the operator $-\operatorname{div}(t^{1-2s}\nabla(\cdot))$. Although we keep in mind the parallel just outlined, in this work we focus on the study of the extended problem (8.1.1). More precisely, our main objectives are to investigate the regularity of the solution and the structure of the *free boundary*

$$\partial\{u(\cdot, 0) \neq 0\} \cap B_1'.$$

We now observe a couple of limit problems that may emerge from (8.1.1). If we consider both $\lambda_- = \lambda_+ = 0$ then the boundary condition in (8.1.1) becomes of homogeneous Neumann type, while if we let $\lambda_+ \to +\infty$, then the conditions boils down to

$$(u-h)^+ t^{1-2s} \frac{\partial u}{\partial t} = 0 \quad \text{on } B_1'.$$

In the case $s = 1/2$, this is known as the *thin obstacle problem* (or even *Signorini problem*, see [Sig59]).

Free boundary problems in relation with the fractional Laplace operator are pervasive objects of investigation in the recent years and we refer the interested reader to [DS18, DPP19] for thorough monographs on the subject. For instance, in [AP12, All12] the authors studied the so called lower dimensional two-phase membrane problem, while in [ALP15, All19] and [DJ21] the fractional version of a penalized obstacle problem is under consideration. On the other hand, the works [GP09, GRO19] together with [AC04, ACS08, CSS08] provide a fairly complete picture for what pertains to the regularity of the free boundary for the "classical" lower dimensional obstacle problem. In [STT20] the authors investigated geometric theoretical features of the nodal set of s-harmonic functions. We finally quote [ST18, ST19], where the authors developed new techniques in order to study regularity properties of zero level sets of solutions to local equations with sublinear (and even singular) powers; [Tor20] extends some of the results to the nonlocal framework.

In order to state our main results, we first introduce the appropriate functional setting. For any positive half ball B_r^+, we consider the weighted Sobolev space $H^1(B_r^+; t^{1-2s})$, defined in Chapter 6, that coincides with the completion of $C^\infty(\overline{B_r^+})$ with respect to the norm

$$\|u\|_{H^1(B_r^+;t^{1-2s})} := \left(\int_{B_r^+} t^{1-2s}(|\nabla u|^2 + u^2)\,\mathrm{d}x\mathrm{d}t\right)^{1/2}.$$

Moreover, we denote by $H^1_{0,S_r^+}(B_r^+;t^{1-2s})$ the closure of $C_c^\infty(\overline{B_r^+}\setminus\overline{S_r^+})$ in $H^1(B_r^+;t^{1-2s})$, where $S_r^+ := \partial B_r^+ \cap \mathbb{R}^{N+1}_+$. For a fixed function $g \in H^1(B_1^+;t^{1-2s}) \cap C(\overline{B_1^+})$, we consider the problem

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla u) = 0, & \text{in } B_1^+, \\ u = g, & \text{on } S_1^+, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\dfrac{\partial u}{\partial t} = \lambda_-(u^-)^{p-1} - \lambda_+(u^+)^{p-1}, & \text{on } B_1'. \end{cases} \tag{8.1.3}$$

In particular, we say that $u \in H^1(B_1^+;t^{1-2s}) \cap L^p(B_1')$ is a *solution* of the problem above if $u - g \in H^1_{0,S_1^+}(B_1^+;t^{1-2s})$ and

$$\int_{B_1^+} t^{1-2s}\nabla u\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t = \int_{B_1'}(\lambda_-(u^-)^{p-1} - \lambda_+(u^+)^{p-1})\varphi\,\mathrm{d}x, \tag{8.1.4}$$

for all $\varphi \in H^1_{0,S_1^+}(B_1^+;t^{1-2s})$. The Dirichlet boundary datum on S_1^+ in (8.1.3) appears (in contrast to (8.1.1)) in order to establish existence and uniqueness of a weak solution, see Proposition 8.2.2. Also, we explicitly note that, for the sake of simplicity, we have taken the thin obstacle $h \equiv 0$.

We carry out our analysis in the following steps. First of all, we establish the optimal regularity of the solution u in Hölder spaces, see Lemma 8.2.7. We point out that there is a substantial difference in this respect between the cases $s = 1/2$ and $s \neq 1/2$. While in the former the optimal Hölder space depends on the exponent p, this is no longer true in the latter case. This is a clear indication of the strong influence of the weight t^{1-2s}, which can be degenerate or singular when $t \to 0^+$, and affects the behavior of the solution near the thin space $\partial\mathbb{R}^{N+1}_+$. Nevertheless, this weight belongs to the second Muckenhoupt class A_2 and therefore it enjoys nice properties, including e.g. Sobolev and trace embeddings, see for instance [FKS82, MS68]. At this point, we investigate the local behavior of $u(x,t)$ when $t \to 0^+$. In order to do so, we establish Almgren and Monneau type monotonicity formulas that, together with a blow up analysis for a proper rescaling of u, provide the asymptotic rate and shape of the solution near a free boundary point, in terms of t^{1-2s}-harmonic polynomials (see Definition 8.2.3), which are even in the variable t and homogeneous of some degree. More precisely, for any integer $k \geq 0$, let $\mathbb{P}_k^s$ denote the space of polynomials $p\colon \mathbb{R}^{N+1} \to \mathbb{R}$ such that

$$p(x,-t) = p(x,t) \quad \text{for all } (x,t) \in \mathbb{R}^{N+1}, \tag{8.1.5}$$

$$p(\mu z) = \mu^k p(z) \quad \text{for all } z \in \mathbb{R}^{N+1} \text{ and all } \mu \geq 0, \tag{8.1.6}$$

$$-\operatorname{div}(|t|^{1-2s}\,\nabla p) = 0 \quad \text{in } \mathbb{R}^{N+1}. \tag{8.1.7}$$

One can immediately notice that, given the other conditions, the first one is equivalent to

$$|t|^{1-2s}\frac{\partial p}{\partial t} = 0 \quad \text{on } \mathbb{R}^N \times \{0\}.$$

Also, thanks to [STV19, Theorem 1.1], we know that, if the conditions above are satisfied, no other values of k apart from integer ones are allowed.

We are now able to state our first main result.

Theorem 8.1.1. *Let $u \in H^1(B_1^+; t^{1-2s})$ be the unique solution to (8.1.3) and let $x_0 \in B_1'$. If*

$$u_r^{x_0}(z) := u(rz + x_0),$$

then there exists an integer $k \geq 0$ and $p_k^{x_0} \in \mathbb{P}_k^s$ such that

$$\begin{aligned} r^{-k}u_r^{x_0} \to p_k^{x_0} \quad &\text{in } H^1(B_1^+; t^{1-2s}) \text{ and } C^{0,\alpha}(\overline{B_1^+}), \\ r^{-k}u_r^{x_0} \to p_k^{x_0} \quad &\text{in } C^{1,\alpha}(B_1'), \end{aligned}$$

as $r \to 0$, for some $\alpha \in (0,1)$.

Since the polynomial $p_k^{x_0}$ (at which we refer as *blow-up limit* of u at x_0) cannot vanish everywhere on the thin space $\mathbb{R}^N \times \{0\}$ (see Remark 8.4.3), then the previous theorem readily implies the boundary strong unique continuation principle.

Corollary 8.1.2 (Strong unique continuation)**.** *Let $u \in H^1(B_1^+; t^{1-2s})$ be the unique solution of (8.1.3) and assume that $u(z) = o(|z|^n)$ as $|z| \to 0$, $z \in \overline{B_1^+}$ for all $n \in \mathbb{N}$. Then $u \equiv 0$ on B_1^+.*

This result tells us that the nodal set of u on the thin space

$$\mathcal{Z}(u) := \partial\{x \in B_1' \colon u(x,0) = 0\}$$

has empty interior in the $\mathbb{R}^N$ topology. Therefore, thanks to the continuity of u, it coincides with the free boundary that separates the two phases of the solution

$$\begin{aligned} \mathcal{Z}(u) &= \partial\{u(\cdot,0) \neq 0\} \\ &= \partial\{u(\cdot,0) > 0\} \cup \partial\{u(\cdot,0) < 0\}. \end{aligned}$$

The last part of the present work is devoted to the analysis of the regularity and structural properties of the nodal set $\mathcal{Z}(u)$. A fundamental tool is the notion of frequency. More precisely, we introduce, for any $v \in H^1(B_1^+; t^{1-2s})$, $x_0 \in B_1'$ and $r > 0$

$$\mathcal{N}_{x_0}(v,r) := \frac{r\displaystyle\int_{B_r^+(x_0)} t^{1-2s}\,|\nabla v|^2 \,\mathrm{d}x\mathrm{d}t}{\displaystyle\int_{S_r^+(x_0)} t^{1-2s} v^2 \,\mathrm{d}S},$$

where

$$B_r^+(x_0) := \{z \in \mathbb{R}^{N+1}_+ \colon |z-(x_0,0)| < r\} \quad \text{and} \quad S_r^+(x_0) := \partial B_r^+(x_0) \cap \mathbb{R}^{N+1}_+.$$

We call $\mathcal{N}_{x_0}(v,r)$, as a function of r, the *frequency function* of v at x_0. In Proposition 8.3.9 we prove the existence of the following limit

$$\mathcal{N}_{x_0}(u,0^+) := \lim_{r\to 0^+} \mathcal{N}_{x_0}(u,r),$$

with u denoting the unique solution of (8.1.3). In addition, we show in Theorem 8.4.1 that $\mathcal{N}_{x_0}(u,0^+)$ is a nonnegative integer, and it is the degree of the blow-up limit of u at x_0. We refer to $\mathcal{N}_{x_0}(u,0^+)$ as the *frequency* of u at x_0. To proceed, we split the nodal set $\mathcal{Z}(u)$ into two disjoint parts

$$\begin{aligned} \mathcal{R}(u) &:= \mathcal{Z}_1(u), \\ \mathcal{S}(u) &:= \bigcup_{k\geq 2} \mathcal{Z}_k(u), \end{aligned}$$

where, for any integer $k \geq 1$,

$$\mathcal{Z}_k(u) := \{x_0 \in \mathcal{Z}(u) \colon \mathcal{N}_{x_0}(u,0^+) = k\}.$$

We call $\mathcal{R}(u)$ and $\mathcal{S}(u)$ the *regular* and *singular* part of the free boundary, respectively. At this point of our analysis, we are confronted again with the strong influence of the weight t^{1-2s} on the behavior of the solution u near the thin space. Indeed, we perform another classification of the zero points of u, based on whether the blow-up limit of the solution at $x_0 \in \mathcal{Z}(u)$ depends on the variable t or not. Namely, let us consider the following two spaces of polynomials

$$\mathbb{P}_k^* := \{p \in \mathbb{P}_k^s \colon p(x,t) = p(x) \text{ for all } (x,t) \in \mathbb{R}^{N+1}\} \quad \text{and} \quad \mathbb{P}_k^t := \mathbb{P}_k \setminus \mathbb{P}_k^*.$$

It is readily seen that the former can be characterized as

$$\mathbb{P}_k^* = \{p \in \mathbb{P}_k^s \colon \Delta_x p = 0 \text{ in } \mathbb{R}^{N+1}\}.$$

Moreover it is easy to observe that $\mathbb{P}_1^t = \emptyset$: this leads to our first result about the free boundary, concerning its regular part.

Proposition 8.1.3. *The regular part of the free boundary $\mathcal{R}(u)$ is a $C^{1,\alpha}$-hypersurface of B_1', for some $\alpha \in (0,1)$, and*

$$\mathcal{R}(u) = \{x_0 \in \mathcal{Z}(u) \colon |\nabla_x u(x_0,0)| \neq 0\}.$$

On the other hand, since $\mathbb{P}_k^t \neq \emptyset$ for $k \geq 2$, we cannot expect the same regularity for the singular set. However, we are able to prove a stratification result which describes the structure of $\mathcal{S}(u)$. Once more, we must distinguish between points for which the blow-up limit depends on the variable t, and points for which this does not happen. If we define, for $p_k^{x_0}$ as in Theorem 8.1.1,

$$\begin{aligned}\mathcal{Z}_k^*(u) &:= \{x_0 \in \mathcal{Z}_k(u) \colon p_k^{x_0} \in \mathbb{P}_k^*\},\\ \mathcal{Z}_k^t(u) &:= \{x_0 \in \mathcal{Z}_k(u) \colon p_k^{x_0} \in \mathbb{P}_k^t\} = \mathcal{Z}_k(u) \setminus \mathcal{Z}_k^*(u),\end{aligned}$$

then the singular set naturally splits into the following two disjoint parts

$$\mathcal{S}^t(u) := \bigcup_{k\geq 2} \mathcal{Z}_k^t(u) \quad \text{and} \quad \mathcal{S}^*(u) := \bigcup_{k\geq 2} \mathcal{Z}_k^*(u).$$

In order to prove their stratified structure, we need the notion of dimension of the singular set at one of its points.

Definition 8.1.4. For any $x_0 \in \mathcal{Z}_k(u)$, we define the *dimension* of $\mathcal{Z}_k(u)$ at x_0 as

$$d_k^{x_0} := \dim \left\{ \xi \in \mathbb{R}^N \colon \nabla_x p_k^{x_0}(x,0) \cdot \xi = 0 \text{ for all } x \in \mathbb{R}^N \right\}.$$

We observe that, since $p_k^{x_0} \not\equiv 0$ on $\mathbb{R}^N \times \{0\}$, we have that $0 \leq d_k^{x_0} \leq N-1$. Finally, let us define

$$\mathcal{Z}_k^n(u) := \left\{ x_0 \in \mathcal{Z}_k(u) \colon d_k^{x_0} = n \right\}. \tag{8.1.8}$$

We are now able to describe the structure of the singular set. Roughly speaking, we show that $\mathcal{Z}_k(u)$ is contained in a $d_k^{x_0}$-dimensional manifold near x_0.

Theorem 8.1.5. *The set $\mathcal{S}^t(u)$ is contained in a countable union of $(N-1)$-dimensional C^1-manifolds, while $\mathcal{S}^*(u)$ is contained in a countable union of $(N-2)$-dimensional C^1-manifolds. Furthermore*

$$\mathcal{S}^t(u) = \bigcup_{n=0}^{N-1} \mathcal{S}_n^t(u) \quad \text{and} \quad \mathcal{S}^*(u) = \bigcup_{n=0}^{N-2} \mathcal{S}_n^*(u),$$

where

$$\mathcal{S}_n^t(u) := \bigcup_{k\geq 2} \mathcal{Z}_k^t(u) \cap \mathcal{Z}_k^n(u) \quad \text{and} \quad \mathcal{S}_n^*(u) := \bigcup_{k\geq 2} \mathcal{Z}_k^*(u) \cap \mathcal{Z}_k^n(u)$$

and they are contained in a countable union of n-dimensional C^1 manifolds.

The last result of our paper yields estimates on the Hausdorff dimension of the nodal set and of its regular and singular part.

Theorem 8.1.6. *Let $u \in H^1(B_1^+; t^{1-2s})$ be the unique solution of (8.1.3) and assume $u \not\equiv 0$ on B_1'. Then*

$$\text{either} \quad \mathcal{Z}(u) = \emptyset \quad \text{or} \quad \dim_H(\mathcal{Z}(u)) = N-1. \tag{8.1.9}$$

Furthermore, in the latter case

$$\text{either} \quad \mathcal{R}(u) = \emptyset \quad \text{or} \quad \dim_H(\mathcal{R}(u)) = N-1 \tag{8.1.10}$$

and

$$\text{either} \quad \mathcal{S}(u) = \emptyset \quad \text{or} \quad \dim_H(\mathcal{S}(u)) \leq N-1. \tag{8.1.11}$$

The proof relies on Theorem 8.1.1 and Federer's Reduction Principle. We observe that the previous result immediately implies the boundary unique continuation from sets of positive N-dimensional measure.

Corollary 8.1.7. *Let $u \in H^1(B_1^+; t^{1-2s})$ be the unique solution to (8.1.3). If $|\mathcal{Z}(u)|_N > 0$, then $u \equiv 0$ in B_1^+.*

Outline of the chapter The rest of the chapter is organized as follows. In Section 8.2 we provide some basic properties of the solution, including the optimal regularity in Hölder spaces. Section 8.3 is devoted to the proof of monotonicity of a perturbed Almgren type frequency function and of a Monneau functional. These results are then applied in Section 8.4 to perform a blow-up analysis for a suitable scaling of the solution, which leads to the proof of Theorem 8.1.1. Finally, in Section 8.5 we prove the structure of the free boundary and Hausdorff dimension estimates.

8.2 Optimal regularity and main properties of the solution

In this section we higlight some important features of the solution and we establish its optimal regularity.

In order to prove existence and uniqueness of a solution to problem (8.1.3), it is useful to set up a variational framework. Hence, we introduce, for any $v \in H^1(B_1^+; t^{1-2s}) \cap L^p(B_1')$, the following energy functional

$$J(v) := \frac{1}{2}\int_{B_1^+} t^{1-2s} |\nabla v|^2 \, \mathrm{d}x\mathrm{d}t + \frac{1}{p}\int_{B_1'} (\lambda_-(v^-)^p + \lambda_+(v^+)^p) \, \mathrm{d}x \tag{8.2.1}$$

and the constraint

$$\Theta = \Theta_g := \left\{ v \in H^1(B_1^+; t^{1-2s}) \cap L^p(B_1') \colon v - g \in H^1_{0,S_1^+}(B_1^+; t^{1-2s}) \right\}. \tag{8.2.2}$$

We need the following Poincaré inequality.

Lemma 8.2.1. *For all $r > 0$ and for all $v \in H^1(B_r^+; t^{1-2s})$, there holds*

$$\frac{N+1-2s}{r^2}\int_{B_r^+} t^{1-2s} v^2 \, \mathrm{d}x\mathrm{d}t \le \int_{B_r^+} t^{1-2s} |\nabla v|^2 \, \mathrm{d}x\mathrm{d}t + \frac{1}{r}\int_{S_r^+} t^{1-2s} v^2 \, \mathrm{d}S.$$

Proof. Just by integration of the identity

$$\operatorname{div}(t^{1-2s} v^2 z) = 2t^{1-2s} v \nabla v \cdot z + (N + 2(1-s)) t^{1-2s} v^2$$

over B_r^+ and Young's inequality (with $z = (x,t)$). ☐

This variational setting allows us to prove the following.

Proposition 8.2.2. *There exists a unique solution to problem (8.1.3), i.e. a function $u \in \Theta$ such that*

$$\int_{B_1^+} t^{1-2s} \nabla u \cdot \nabla \varphi \,\mathrm{d}x\mathrm{d}t = \int_{B_1'} (\lambda_-(u^-)^{p-1} - \lambda_+(u^+)^{p-1})\varphi \,\mathrm{d}x,$$

for all $\varphi \in H^1_{0,S_1^+}(B_1^+; t^{1-2s})$. In particular, the solution u is the unique minimum of the functional J under the constraint Θ. Here Θ and J are as in (8.2.1) and (8.2.2).

Proof. Thanks to the form of the functional J, the proof of existence is standard and is based on Lemma 8.2.1, weak lower-semicontinuity of the weighted H^1-norm, compactness of the trace embedding $H^1(B_1^+; t^{1-2s}) \hookrightarrow L^2(B_1')$, and Fatou's Lemma. Uniqueness follows from strict convexity of the functions $t \mapsto (t^{\pm})^p$. $\square$

The first notable property of the solution is that its positive and negative part are subharmonic functions, with respect to the operator $-\operatorname{div}(t^{1-2s}\nabla(\cdot))$, in $B_1^+ \cup B_1'$. We first give the precise definition.

Definition 8.2.3. We say that a function $v \in H^1(B_1^+; t^{1-2s})$ is t^{1-2s}*-subharmonic* in $B_1^+ \cup B_1'$ if

$$\int_{B_1^+} t^{1-2s} \nabla v \cdot \nabla \varphi \,\mathrm{d}x\mathrm{d}t \leq 0,$$

for all $\varphi \in H^1_{0,S_1^+}(B_1^+; t^{1-2s})$ such that $\varphi \geq 0$ a.e. Moreover, we say that $v \in H^1(B_1^+; t^{1-2s})$ is t^{1-2s}*-harmonic* in $B_1^+ \cup B_1'$ if

$$\int_{B_1^+} t^{1-2s} \nabla v \cdot \nabla \varphi \,\mathrm{d}x\mathrm{d}t = 0,$$

for all $\varphi \in H^1_{0,S_1^+}(B_1^+; t^{1-2s})$.

Finally, we say that $v \in H^1_{\mathrm{loc}}(\Omega; t^{1-2s})$ is t^{1-2s}*-harmonic* in an open $\Omega \subseteq \mathbb{R}^{N+1}$ if

$$\int_{\Omega} t^{1-2s} \nabla v \cdot \nabla \varphi \,\mathrm{d}x\mathrm{d}t = 0,$$

for all $\varphi \in C_c^\infty(\Omega)$.

We then prove t^{1-2s}-subharmonicity of the positive and negative part of solutions to (8.1.3).

Lemma 8.2.4. *Let $u \in H^1(B_1^+; t^{1-2s})$ be the unique solution to (8.1.3). Then u^+ and u^- are t^{1-2s}-subharmonic in $B_1^+ \cup B_1'$.*

Proof. Let $\varphi \in H^1_{0,S_1^+}(B_1^+; t^{1-2s})$ such that $\varphi \geq 0$ a.e. and, for $\varepsilon > 0$ small, let $u_\varepsilon := (u - \varepsilon\varphi)^+ - u^-$. One can immediately notice that

$$u_\varepsilon^+ = (u - \varepsilon\varphi)^+ \leq u^+ \quad \text{and that} \quad u_\varepsilon^- = u^-. \tag{8.2.3}$$

Since $u_\varepsilon = u$ on S_1^+, by minimality and (8.2.3) we have

$$\int_{B_1^+} t^{1-2s} \left|\nabla u^+\right|^2 \, \mathrm{d}x\mathrm{d}t \leq \int_{B_1^+} t^{1-2s} \left|\nabla (u - \varepsilon\varphi)^+\right|^2 \, \mathrm{d}x\mathrm{d}t.$$

This yields

$$\begin{aligned}\int_{B_1^+} t^{1-2s} \left|\nabla u\right|^2 \chi_{\{u \geq 0\}} \, \mathrm{d}x\mathrm{d}t &\leq \int_{B_1^+} t^{1-2s} \left|\nabla (u - \varepsilon\varphi)\right|^2 \chi_{\{u \geq \varepsilon\varphi\}} \, \mathrm{d}x\mathrm{d}t \\ &\leq \int_{B_1^+} t^{1-2s} \left|\nabla (u - \varepsilon\varphi)\right|^2 \chi_{\{u \geq 0\}} \, \mathrm{d}x\mathrm{d}t.\end{aligned}$$

By expanding the last term we obtain

$$\int_{B_1^+} t^{1-2s} \nabla u^+ \cdot \nabla\varphi \, \mathrm{d}x\mathrm{d}t \leq \frac{\varepsilon}{2} \int_{B_1^+} t^{1-2s} \left|\nabla\varphi\right|^2 \chi_{\{u \geq 0\}} \, \mathrm{d}x\mathrm{d}t$$

for all $\varepsilon > 0$, which implies t^{1-2s}-subharmonicity of u^+. Similarly, by letting $u_\varepsilon := u^+ - (u + \varepsilon\varphi)^-$ we can prove t^{1-2s}-subharmonicity of u^-. □

Since u^+ and u^- are t^{1-2s}-subharmonic, the weak maximum principle (together with boundedness of g on S_1^+ and a reflection argument) immediately implies the following, see [FKS82, Theorem 2.2.2] and [MS68, Theorem 6.7].

Lemma 8.2.5 (Weak maximum principle)**.** *Let $u \in H^1(B_1^+; t^{1-2s})$ be the unique solution of problem (8.1.3). Then*

$$\min\{0, \inf_{S_1^+} g\} \leq u \leq \max\{0, \sup_{S_1^+} g\}$$

a.e. in B_1^+. In particular $u \in L^\infty(B_1^+)$ and

$$\|u\|_{L^\infty(B_1^+)} \leq \sup_{S_1^+} |g| \, .$$

We also recall the validity of mean value inequalities for for t^{1-2s}-subharmonic functions. The following result immediately comes from [WW16, Lemma A.1] and an even-in-t reflection.

Lemma 8.2.6. *Let $v \in H^1(B_1^+; t^{1-2s}) \cap C(B_1^+ \cup B_1')$ be t^{1-2s}-subharmonic in $B_1^+ \cup B_1'$. Then*

$$v(x_0, 0) \leq \frac{\alpha_{N,s}}{r^{N+2(1-s)}} \int_{B_r^+(x_0)} t^{1-2s} v \, \mathrm{d}x\mathrm{d}t \quad \text{and} \quad v(x_0, 0) \leq \frac{\beta_{N,s}}{r^{N+1-2s}} \int_{S_r^+(x_0)} t^{1-2s} v \, \mathrm{d}S$$

for all $x_0 \in B_1'$ and $r > 0$ such that $B_r^+(x_0) \subset\subset B_1^+$, where

$$\alpha_{N,s} := \int_{B_1^+} t^{1-2s} \, \mathrm{d}x\mathrm{d}t \quad \text{and} \quad \beta_{N,s} := \int_{S_1^+} t^{1-2s} \, \mathrm{d}S.$$

Boundedness of the solution, stated in Lemma 8.2.5, allows us to deduce its regularity. In particular, from [ALP15, Theorem 6.4] (see also [STV19, Theorem 1.5 and 1.6]) and [DJ21, Theorem 1.1] we deduce the following.

Lemma 8.2.7. *Let $u \in H^1(B_1^+; t^{1-2s})$ be the unique solution to problem (8.1.3). There holds*

- *if $s < 1/2$ then $u \in C^{0,2s}(B_{1/4}^+)$ and*
$$\|u\|_{C^{0,2s}(B_{1/4}^+)} \leq C(N, s, p, \|u\|_{L^\infty(B_1^+)});$$
- *if $s > 1/2$ then $u \in C^{1,2s-1}(B_{1/4}^+ \cup B'_{1/4})$ and*
$$\|u\|_{C^{1,2s-1}(B_{1/4}^+ \cup B'_{1/4})} \leq C(N, s, p, \|u\|_{L^\infty(B_1^+)});$$
- *if $s = 1/2$ then $u \in C^{1,\alpha}(B_{1/4}^+ \cup B'_{1/4})$ for some $\alpha \in (0,1)$ (depending on p) and*
$$\|u\|_{C^{1,\alpha}(B_{1/4}^+ \cup B'_{1/4})} \leq C(N, \|u\|_{L^\infty(B_1^+)}).$$

Remark 8.2.8. We point out that the result is optimal when $s \neq 1/2$. Indeed, let $\Psi(x)$ be harmonic in $\mathbb{R}^N$. Then, the function $u(x,t) = (t^{2s} + 2s)\Psi(x)$ is a solution of (8.1.3) with $\lambda_+ = \lambda_- = 1$ and $p = 2$, and so Lemma 8.2.7 is sharp. On the contrary, when $s = 1/2$ the regularity is actually higher, depending on the exponent p (we refer to [DJ21] for a detailed exposition).

Even though the global regularity cannot exceed the one just stated, if we look separately at the derivatives in x and the weighted derivative in t, we are able to prove something more. Let us first recall a regularity result, which can be easily derived from [TX11, Proposition 3.1].

Lemma 8.2.9. *Let $U \in H^1(B_1^+; t^{1-2s})$ be a weak solution of*
$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla U) = 0, & \text{in } B_1^+, \\ -\lim\limits_{t\to 0} t^{1-2s}\dfrac{\partial U}{\partial t} = b\,U + d, & \text{on } B'_1, \end{cases}$$
with $b, d \in L^\infty(B'_1)$. Then $U \in L^\infty(B_{1/2}^+)$ and
$$\|U\|_{L^\infty(B_{1/2}^+)} \leq C(\|U\|_{L^2(B_1^+; t^{1-2s})} + \|d\|_{L^\infty(B'_1)}),$$
for a certain $C > 0$ depending on N, s and $\|b\|_{L^\infty(B'_1)}$.

Remark 8.2.10. By inspecting the proof of the previous lemma, which is based on an iterative Moser-type argument, one can find a more explicit expression for the constant C and emphasize the dependence on $\|b\|_{L^\infty(B'_1)}$. More precisely
$$C = c_1(\max\{c_2, \|b\|_{L^\infty(B'_1)} + 1\})^{c_3},$$
where the c_i's are positive constant depending on N and s.

Lemma 8.2.11. *Let $u \in H^1(B_1^+; t^{1-2s})$ be the unique solution to problem (8.1.3). Then there exists $\alpha \in (0,1) \cap (0,2s)$, $R_0 \in (0,1/4)$ and $C > 0$, possibly depending on $N, s, \|g\|_{L^\infty(S_1^+)}$, such that*

$$\|\nabla_x u\|_{C^{0,\alpha}(B_{R_0}^+)} \leq C \|\nabla u\|_{L^2(B_1^+;t^{1-2s})}, \tag{8.2.4}$$

$$\left\| t^{1-2s} \frac{\partial u}{\partial t} \right\|_{C^{0,\alpha}(B_{R_0}^+)} \leq C. \tag{8.2.5}$$

Proof. Estimate (8.2.5) comes from Lemma 8.2.7 and [CS14, Lemma 4.5] (recalled in Lemma 7.3.3).

On the other hand, for $i = 1, \dots, N$, let $\boldsymbol{e}_i$ be the i-th vector of the standard basis of $\mathbb{R}^{N+1}$ and, for $h \in \mathbb{R}$ sufficiently small and $z = (x,t) \in B_{1/4}^+$, let

$$u^h(x,t) = u_i^h(x,t) := \frac{u(x + h\boldsymbol{e}_i, t) - u(x,t)}{|h|}.$$

This function weakly satisfies

$$\begin{cases} -\operatorname{div}(t^{1-2s} \nabla u^h) = 0, & \text{in } B_{1/4}^+, \\ -\lim\limits_{t \to 0^+} t^{1-2s} \dfrac{\partial u^h}{\partial t} = V_h u^h, & \text{on } B_{1/4}', \end{cases}$$

where

$$V_h(x) = \left[\lambda_- \frac{(u^-(x + h\boldsymbol{e}_i, 0))^{p-1} - (u^-(x,0))^{p-1}}{u(x + h\boldsymbol{e}_i, 0) - u(x,0)} \right. \\ \left. - \lambda_+ \frac{(u^+(x + h\boldsymbol{e}_i, 0))^{p-1} - (u^+(x,0))^{p-1}}{u(x + h\boldsymbol{e}_i, 0) - u(x,0)} \right] \chi_{\{u^h(x) \neq 0\}}(x).$$

From Lemma 8.2.9 we know that $u^h \in L^\infty(B_{r_1}^+)$, for some $r_1 \in (0,1/4)$ and that

$$\|u_h\|_{L^\infty(B_{r_1}^+)} \leq C_1 \left\| u^h \right\|_{L^2(B_{1/4}^+;t^{1-2s})}, \tag{8.2.6}$$

with $C_1 > 0$ depending on N, s and $\left\| V^h \right\|_{L^\infty(B_{1/4}')}$. Thanks to the Lipschitz continuity of the function $t \mapsto (t^\pm)^{p-1}$ on real compact intervals and to Lemma 8.2.5, we have that $V_h \in L^\infty(B_{1/4}')$ and its L^∞-norm is bounded by a constant that depends on p, λ_-, λ_+ and $\|u\|_{L^\infty(B_1^+)}$, uniformly with respect to h. More precisely, a direct study of the function

$$(y_1, y_2) \mapsto \frac{\left| (y_1^\pm)^{p-1} - (y_2^\pm)^{p-1} \right|}{|y_1 - y_2|}$$

tells us that

$$\|V_h\|_{L^\infty(B_{1/4}')} \leq (\lambda_- + \lambda_+)(p-1) \|u\|_{L^\infty(B_1^+)}^{p-2} \quad \text{uniformly in } h. \tag{8.2.7}$$

Combining this fact with (8.2.6), in view of Remark 8.2.10 and Lemma 8.2.5, we have that

$$\left\|u^h\right\|_{L^\infty(B_{r_1}^+)} \le \tilde{C}_1 \left\|u^h\right\|_{L^2(B_{1/4}^+;t^{1-2s})}, \tag{8.2.8}$$

with $\tilde{C}_1 > 0$ depending on $N, s, \lambda_+, \lambda_-, p$ and $\|g\|_{L^\infty(S_1^+)}$. From [STV19, Theorem 1.5] we know that, for $\alpha \in (0,1) \cap (0,2s)$ for which (8.2.5) holds, there exists $r_2 \in (0, r_1)$ and $C_2 > 0$, depending on N, s, α such that

$$\left\|u^h\right\|_{C^{0,\alpha}(B_{r_2}^+)} \le C_2 \left[\left\|u^h\right\|_{L^2(B_{r_1}^+;t^{1-2s})} + \left\|V_h\, u^h\right\|_{L^\infty(B_{r_1}^+)}\right]. \tag{8.2.9}$$

Actually, [STV19, Theorem 1.5] (see also [STV19, Theorem 8.2]) states this result for $V_h u^h \in L^q$ for some q sufficiently large; nevertheless, by scanning the proof, one can check that it still holds true for $q = \infty$ without any change. On the other hand, by (8.2.7) and (8.2.8) we have that

$$\left\|V_h\, u^h\right\|_{L^\infty(B_{r_1}^+)} \le \bar{C}_2 \left\|u^h\right\|_{L^2(B_{1/4}^+;t^{1-2s})},$$

with $\bar{C}_2 > 0$ depending on $N, s, \lambda_+, \lambda_-, p$. Therefore, combining this with (8.2.9), we deduce that

$$\left\|u^h\right\|_{C^{0,\alpha}(B_{r_2}^+)} \le \tilde{C}_3 \left\|u^h\right\|_{L^2(B_{1/4}^+;t^{1-2s})} \tag{8.2.10}$$

with $\tilde{C}_3 > 0$ depending on $N, s, \alpha, \lambda_+, \lambda_-, p$. Since $u \in H^1(B_1^+; t^{1-2s})$, we have that

$$\left\|u^h\right\|_{L^2(B_{1/4}^+;t^{1-2s})} \to \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(B_{1/4}^+;t^{1-2s})} \quad \text{as } |h| \to 0.$$

Therefore, by the Ascoli-Arzelà Theorem, up to a subsequence,

$$u^h \to \frac{\partial u}{\partial x_i} \quad \text{uniformly in } \overline{B_{r_2}^+} \text{ as } |h| \to 0$$

and the, passing to the limit in (8.2.10) yields

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{C^{0,\alpha}(B_{r_2}^+)} \le \tilde{C}_3 \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(B_{1/4}^+;t^{1-2s})}.$$

By iterating the argument for any $i = 1, \dots, N$ we have that (8.2.4) holds. □

Remark 8.2.12. By standard elliptic regularity theory, since the weight t^{1-2s} is uniformly bounded in every compact subset of B_1^+, we have that $u \in C^\infty(B_1^+)$. Moreover, by a covering argument in $B_1^+ \cup B_1'$ we can say that u, $\nabla_x u$ and $t^{1-2s}\partial_t u$ are in $C^{0,\alpha}_{\mathrm{loc}}(B_1^+ \cup B_1')$.

In view of the regularity of the solution, we have that also a strong maximum principle holds.

Lemma 8.2.13. *Let $u \in H^1(B_1^+; t^{1-2s})$ be the unique solution to (8.1.3) and assume $g \not\equiv 0$ on S_1^+. Then*

$$\begin{aligned} u > 0 \quad &\text{in } B_1^+ \cup B_1' \quad \text{if} \quad g \geq 0 \quad \text{on } S_1^+, \\ u < 0 \quad &\text{in } B_1^+ \cup B_1' \quad \text{if} \quad g \leq 0 \quad \text{on } S_1^+. \end{aligned}$$

Proof. In view of Remark 8.2.12, the proof follows from Lemma 8.2.5, Hopf Lemma and boundary Hopf Lemma for A_2-weighted elliptic equations, which can be found respectively in [FKS82, Corollary 2.3.10] and [CS14, Proposition 4.11] (also recalled in Proposition 7.3.2). □

8.3 Monotonicity Formulas

In this section we introduce a *perturbed* Almgren type frequency function, suitable for our problem, and we prove its monotonicity. As a consequence we deduce the existence of the limit of the *standard* Almgren frequency function of the solution, as the radius of the underlying ball vanishes. Finally, we introduce a Monneau type functional and we show it is nondecreasing with respect to the radius.

For any $v \in H^1(B_1^+; t^{1-2s}) \cap L^p(B_1')$, $x_0 \in B_1'$ and $r \in (0, R_0)$ (with R_0 as in Lemma 8.2.11), such that $B_r^+(x_0) \subseteq B_{R_0}^+$, we define the functions

$$E_{x_0}(v,r) := \frac{1}{r^{N-2s}} \int_{B_r^+(x_0)} t^{1-2s} |\nabla v|^2 \, \mathrm{d}x\mathrm{d}t, \quad H_{x_0}(v,r) := \frac{1}{r^{N+1-2s}} \int_{S_r^+(x_0)} t^{1-2s} v^2 \, \mathrm{d}S \tag{8.3.1}$$

and the classical *frequency* function

$$\mathcal{N}_{x_0}(v,r) := \frac{E_{x_0}(v,r)}{H_{x_0}(v,r)}. \tag{8.3.2}$$

Moreover we let

$$F(v) := \lambda_-(v^-)^p + \lambda_+(v^+)^p, \quad G_{x_0}(v,r) := \frac{1}{r^{N-2s}} \int_{B_r'(x_0)} F(v) \, \mathrm{d}x. \tag{8.3.3}$$

Then we introduce the *perturbed frequency* function

$$\tilde{\mathcal{N}}_{x_0}(v,r) := \frac{\tilde{E}_{x_0}(v,r)}{H_{x_0}(v,r)}, \quad \text{where} \quad \tilde{E}_{x_0}(v,r) := E_{x_0}(v,r) + \frac{2}{p} G_{x_0}(v,r).$$

The first goal of the section is to prove monotonicity of $\tilde{\mathcal{N}}_{x_0}(u,r)$ with respect to $r \in (0, R_0)$, where u is the unique solution to problem (8.1.3). Hereafter, when we compute the functionals above at the origin $x_0 = 0$ and for $v = u$, we may drop this dependence in our notation and simply write, e.g. $\tilde{\mathcal{N}}(r)$. Without loss of generality, in this section we can assume $x_0 = 0$.

Lemma 8.3.1. *Let $v \in C^\infty(\overline{B_1^+})$. There holds the following formula, for $z \in B_1^+$,*

$$\mathrm{div}(t^{1-2s} |\nabla v|^2 z - 2t^{1-2s}(\nabla v \cdot z)\nabla v) = (N-2s)t^{1-2s} |\nabla v|^2 - 2(\nabla v \cdot z) \,\mathrm{div}(t^{1-2s}\nabla v). \tag{8.3.4}$$

Proof. Since the proof is essentially just computations, we only sketch it. We first have that

$$\begin{aligned} &\operatorname{div}(t^{1-2s}|\nabla v|^2 z - 2t^{1-2s}(\nabla v\cdot z)\nabla v) \\ &= (N+2(1-s))t^{1-2s}|\nabla u|^2 + t^{1-2s} z\cdot\nabla(|\nabla u|^2) - 2(\nabla v\cdot z)\operatorname{div}(t^{1-2s}\nabla v) - 2t^{1-2s}\nabla v\cdot\nabla(\nabla v\cdot z). \end{aligned} \tag{8.3.5}$$

Moreover it's easy to check that the following identity holds

$$z\cdot\nabla(|\nabla v|^2) = 2\nabla v\cdot\nabla(\nabla v\cdot z) - 2|\nabla v|^2.$$

By plugging this into (8.3.5) we obtain the thesis. □

Lemma 8.3.2. *For a.e. $r\in(0,R_0)$ we have that*

$$\begin{aligned} \frac{1}{r^{N-2s}}\int_{S_r^+} t^{1-2s}|\nabla u|^2\,\mathrm{d}x\mathrm{d}t = \frac{N-2s}{r}\tilde{E}(r) + \frac{2}{r^{N-2s}}\int_{S_r^+} t^{1-2s}\left(\frac{\partial u}{\partial\boldsymbol{\nu}}\right)^2\mathrm{d}S \\ - \frac{2}{pr^{N-2s}}\int_{\partial B_r'} F(u)\,\mathrm{d}\sigma + \frac{4s}{pr}G(r). \end{aligned} \tag{8.3.6}$$

and

$$\int_{S_r^+} t^{1-2s} u\frac{\partial u}{\partial\boldsymbol{\nu}}\,\mathrm{d}S = \int_{B_r^+} t^{1-2s}|\nabla u|^2\,\mathrm{d}x\mathrm{d}t + \int_{B_r'}(\lambda_-(u^-)^p + \lambda_+(u^+)^p)\,\mathrm{d}x. \tag{8.3.7}$$

Proof. Since similar computations can be found in literature, we give a sketch of the proof and, among others, we refer to [FF14, Theorem 3.7] for a precise justification. In order to prove (8.3.6) we first integrate (8.3.4) over B_r^+, which, taking into account the equation (8.1.3) satisfied by u, gives

$$\begin{aligned} &(N-2s)\int_{B_r^+} t^{1-2s}|\nabla u|^2\,\mathrm{d}x\mathrm{d}t \\ &= r\int_{S_r^+} t^{1-2s}|\nabla u|^2\,\mathrm{d}S - 2r\int_{S_r^+} t^{1-2s}\left(\frac{\partial u}{\partial\boldsymbol{\nu}}\right)^2\mathrm{d}S - 2\int_{B_r'}\left(\lambda_-(u^-)^{p-1} - \lambda_+(u^+)^{p-1}\right)(\nabla_x u\cdot x)\,\mathrm{d}x. \end{aligned} \tag{8.3.8}$$

Now, integrating by parts in B_r' and taking into account the identity

$$(u^\pm)^{p-1}\nabla_x u^\pm\cdot x = \frac{1}{p}\nabla_x(u^\pm)^p\cdot x$$

we obtain the following expression for the last term in (8.3.8)

$$\int_{B_r'}\left(\lambda_-(u^-)^{p-1} - \lambda_+(u^+)^{p-1}\right)(\nabla_x u\cdot x)\,\mathrm{d}x = -\frac{r}{p}\int_{\partial B_r'} F(u)\,\mathrm{d}\sigma + \frac{N}{p}\int_{B_r'} F(u)\,\mathrm{d}x,$$

where F is as in (8.3.3). Combining the previous identity with (8.3.8) and reorganizing the terms yields (8.3.6). Finally, (8.3.7) comes from multiplying by u the equation satisfied by u itself and integrating over B_r^+. □

Lemma 8.3.3. *We have that $H(r) > 0$ for all $r \in (0, R_0)$.*

Proof. If we assume $H(\bar{r}) = 0$ for a certain $\bar{r} \in (0, R_0)$, then we reach a contradiction thanks to (8.3.7) and unique continuation principle (see e.g. [SS80, Theorem 7.1]). □

In the following lemma we state the regularity of the function H and we compute its derivative.

Lemma 8.3.4. *We have that $H \in C^1(0, R_0)$ and*

$$H'(r) = \frac{2}{r^{N+1-2s}} \int_{S_r^+} t^{1-2s} u \frac{\partial u}{\partial \boldsymbol{\nu}} \, \mathrm{d}S \tag{8.3.9}$$

$$= \frac{2}{r} \left[\tilde{E}(r) + \left(1 - \frac{2}{p}\right) G(r) \right], \tag{8.3.10}$$

for all $r \in (0, R_0)$.

Proof. Let us fix $r_0 \in (0, R_0)$ and let us examine the limit

$$\lim_{r \to r_0} \frac{H(r) - H(r_0)}{r - r_0} = \lim_{r \to r_0} \int_{S_1^+} t^{1-2s} \frac{u^2(rz) - u^2(r_0 z)}{r - r_0} \, \mathrm{d}S(z),$$

where $z = (x, t) \in S_1^+$. Since $u \in C^\infty(B_1^+)$ (see Remark 8.2.12) we know that

$$\lim_{r \to r_0} \frac{u^2(rz) - u^2(r_0 z)}{r - r_0} = 2u(r_0 z) \nabla u(r_0 z) \cdot z.$$

In order to pass to the limit under the integral sign, we make use of Lebesgue Dominate Convergence Theorem. In fact, from Lagrange Theorem and regularity of the solution (see Lemma 8.2.11) we derive that, for all $z \in S_1^+$ and $r \in (r_0/2, R_0)$,

$$\left| \frac{u^2(rz) - u^2(r_0 z)}{r - r_0} \right| \leq C \sup_{B_{R_0}^+ \setminus B_{r_0/2}^+} |u| \left[\sup_{B_{R_0}^+ \setminus B_{r_0/2}^+} |\nabla_x u| + \sup_{B_{R_0}^+ \setminus B_{r_0/2}^+} \left| t^{1-2s} \frac{\partial u}{\partial t} \right| \right],$$

with $C > 0$ depending on R_0 and s. We also recall that the weight t^{1-2s} is integrable; this, together with Lemma 8.2.11, allows us to conclude the proof of (8.3.9). Finally, (8.3.10) is a consequence of (8.3.9), (8.3.7) and the definition of $\tilde{E}$ (8.3.1). The proof is thereby complete.

□

Remark 8.3.5. In view of the assumption $p \geq 2$, we infer from Lemma 8.3.4 that $H'(r) \geq 0$ for all $r \in (0, R_0)$.

Lemma 8.3.6. *We have that $\tilde{E} \in W^{1,1}_{\mathrm{loc}}(0, R_0)$ and*

$$\tilde{E}'(r) = \frac{2}{r^{N-2s}} \int_{S_r^+} t^{1-2s} \left(\frac{\partial u}{\partial \boldsymbol{\nu}} \right)^2 \, \mathrm{d}S + \frac{4s}{pr} G(r)$$

in a distributional sense and for a.e. $r \in (0, R_0)$.

Proof. For any $r \in (0, R_0)$ we let

$$E(r) := \int_{B_r^+} t^{1-2s} |\nabla u|^2 \, \mathrm{d}x\mathrm{d}t + \frac{2}{p} \int_{B_r'} F(u) \, \mathrm{d}x,$$

so that

$$\tilde{E}(r) = \frac{1}{r^{N-2s}} E(r).$$

First of all it's easy to verify that $E \in L^1(0, R_0)$. Secondly, we compute its distributional derivative, that is, by coarea formula

$$E'(r) = \int_{S_r^+} t^{1-2s} |\nabla u|^2 \, \mathrm{d}S + \frac{2}{p} \int_{\partial B_r'} F(u) \, \mathrm{d}\sigma, \tag{8.3.11}$$

where $\mathrm{d}\sigma$ denotes the surface $N-1$ dimensional measure of $\mathbb{R}^N$. Again it's easy to see that $E' \in L^1(0, R_0)$, thus implying that $E \in W^{1,1}(0, R_0)$. Now, by definition we have that $\tilde{E} \in W^{1,1}_{\mathrm{loc}}(0, R_0)$ and

$$\tilde{E}'(r) = r^{-(N+1-2s)}[rE'(r) - (N-2s)E(r)].$$

This, together with identity (8.3.6), yields the thesis. □

Theorem 8.3.7. *The perturbed Almgren frequency function $\tilde{\mathcal{N}}$ belongs to $W^{1,1}_{\mathrm{loc}}(0, R_0)$ and it is nondecreasing in $(0, R_0)$, i.e. $\tilde{\mathcal{N}}'(r) \geq 0$ for all $r \in (0, R_0)$ in a distributional sense. In particular there exists*

$$k := \lim_{r \to 0} \tilde{\mathcal{N}}(r) \in [0, \infty).$$

Proof. Combining Lemma 8.3.4 and Lemma 8.3.6 we have that

$$\begin{aligned}
H^2(r)\tilde{\mathcal{N}}'(r) &= \tilde{E}'(r)H(r) - \tilde{E}(r)H'(r) \\
&= \frac{2}{r^{2N+1-4s}} \left(\int_{S_r^+} t^{1-2s} \left(\frac{\partial u}{\partial \boldsymbol{\nu}} \right)^2 \mathrm{d}S \right) \left(\int_{S_r^+} t^{1-2s} u^2 \, \mathrm{d}S \right) + \frac{4s}{pr} G(r)H(r) \\
&\quad - \frac{r}{2}(H'(r))^2 + \frac{r}{2}\left(1 - \frac{2}{p}\right) G(r)H'(r) \\
&= \frac{2}{r^{2N+1-4s}} \left[\left(\int_{S_r^+} t^{1-2s} \left(\frac{\partial u}{\partial \boldsymbol{\nu}} \right)^2 \mathrm{d}S \right) \left(\int_{S_r^+} t^{1-2s} u^2 \, \mathrm{d}S \right) - \left(\int_{S_r^+} t^{1-2s} u \frac{\partial u}{\partial \boldsymbol{\nu}} \, \mathrm{d}S \right)^2 \right] \\
&\quad + \frac{4s}{pr} G(r)H(r) + \left(1 - \frac{2}{p}\right) G(r)H'(r) \geq 0,
\end{aligned}$$

where the last inequality follows from Cauchy-Schwartz inequality, Remark 8.3.5 and the assumption $s \in (0, 1)$. □

From [FF14, Lemma 2.5] we immediately deduce the following trace inequality.

Lemma 8.3.8. *There exists a constant $C_1 = C_1(N,s) > 0$ such that*

$$\int_{B_r'} v^2 \,\mathrm{d}x \le C_1 \left(r^{2s} \int_{B_r^+} t^{1-2s} |\nabla v|^2 \,\mathrm{d}x\mathrm{d}t + r^{2s-1} \int_{S_r^+} t^{1-2s} v^2 \,\mathrm{d}S \right)$$

for all $v \in H^1(B_r^+; t^{1-2s})$ and for all $r > 0$.

We obtain as a consequence that the "unperturbed" frequency function $\mathcal{N}$ has a limit at 0^+.

Proposition 8.3.9. *Let $k \in [0,\infty)$ be as in Theorem 8.3.7. Then there exists the limit of $\mathcal{N}(r)$ as $r \to 0^+$ and $\lim_{r\to 0^+} \mathcal{N}(r) = k$.*

Proof. Let $r > 0$ be sufficiently small. Trivially $\mathcal{N}(r) \le \tilde{\mathcal{N}}(r)$. On the other hand, thanks to the boundedness of u, we have that

$$\int_{B_r'} F(u) \,\mathrm{d}x \le 2 \max\{\lambda_-, \lambda_+\} \|u\|_{L^\infty(B_1)}^{p-2} \int_{B_r'} u^2 \,\mathrm{d}x.$$

Therefore, thanks to Lemma 8.3.8, we have that

$$G(r) = \frac{1}{r^{N-2s}} \int_{B_r'} F(u) \,\mathrm{d}x \le C r^{2s} (E(r) + H(r)), \tag{8.3.12}$$

(with $C > 0$ depending on $N, s, \lambda_\pm, p, \|u\|_{L^\infty(B_1)}, p$) which in turn implies

$$\tilde{\mathcal{N}}(r) \le (1 + Cr^{2s}) \mathcal{N}(r) + Cr^{2s}.$$

Hence

$$\frac{\tilde{\mathcal{N}}(r) - Cr^{2s}}{1 + Cr^{2s}} \le \mathcal{N}(r) \le \tilde{\mathcal{N}}(r)$$

and the proof is thereby complete. □

Corollary 8.3.10. *Let $k = \lim_{r\to 0} \tilde{\mathcal{N}}(r)$. Then the following hold:*

(i) there exists a constant $K_1 > 0$, depending on $N, s, \|u\|_{L^\infty(B_1^+)}, k$ such that

$$H(r) \le K_1 r^{2k} \quad \text{for all } r \in (0, R_0);$$

(ii) for any $\delta > 0$ there exists a constant $K_2 = K_2(\delta) > 0$, depending also on N, s, u such that

$$H(r) \ge K_2 r^{2k+\delta} \quad \text{for all } r \in (0, R_0).$$

Proof. From Lemma 8.3.4 and Theorem 8.3.7 we know that

$$\frac{H'(r)}{H(r)} = \frac{2}{r} \left[\tilde{\mathcal{N}}(r) + \left(1 - \frac{2}{p}\right) \frac{G(r)}{H(r)} \right] \ge \frac{2}{r} k.$$

By integrating the inequality above in (r, R_0), thanks to the boundedness of u, we conclude the proof of point (i). On the other hand, we recall that

$$\frac{H'(r)}{H(r)} = \frac{2}{r}\left[\mathcal{N}(r) + \frac{G(r)}{H(r)}\right].$$

Recalling (8.3.12), we obtain

$$\frac{H'(r)}{H(r)} \le \frac{2}{r}\left[\mathcal{N}(r) + Cr^{2s}(\mathcal{N}(r)+1)\right].$$

Hence, since $\lim_{r\to 0}\mathcal{N}(r) = k$, for any $\delta > 0$ there exists $r_0 = r_0(\delta) > 0$ such that, for all $r \in (0, r_0)$

$$\frac{H'(r)}{H(r)} \le \frac{2k+\delta}{r}$$

Integrating the inequality above and taking into account the boundedness of u, we obtain that

$$H(r) \ge C_2 r^{2k+\delta} \quad \text{for all } r \in (0, r_0).$$

Thanks to continuity and positivity of H we may conclude the proof of (ii). □

The previous result readily implies an estimate on the growth rate of the solution u near the origin.

Corollary 8.3.11. *Let $x_0 \in B'_{R_0/2}$ and let $k = \lim_{r\to 0}\tilde{\mathcal{N}_{\S'}}(u, r)$. Then there exists $\tilde{C} > 0$ depending on N, s, $\|g\|_{L^\infty(S_1^+)}$ and k (independent of x_0) such that*

$$|u(x + x_0, 0)| \le \tilde{C}\,|x|^k \quad \textit{for all } x \in B'_{R_0/2}.$$

Proof. Let $x \in B'_{R_0/2}$ and let $r := |x|$. Thanks to Lemma 8.2.4 and Lemma 8.2.6 we have that

$$|u(x + x_0, 0)| \le \frac{\alpha_{N,s}}{r^{N+2(1-s)}}\int_{B_r^+(x+x_0)} t^{1-2s}\,|u|\,\mathrm{d}x\mathrm{d}t.$$

Moreover

$$\int_{B_r^+(x+x_0)} t^{1-2s}\,|u|\,\mathrm{d}x\mathrm{d}t \le \int_{B_{2r}^+(x_0)} t^{1-2s}\,|u|\,\mathrm{d}x.$$

From Cauchy-Schwartz inequality we deduce that

$$\int_{B_{2r}^+(x_0)} t^{1-2s}\,|u|\,\mathrm{d}x\mathrm{d}t \le C_1 r^{\frac{1}{2}(N+2(1-s))}\left(\int_{B_{2r}^+(x_0)} t^{1-2s}u^2\,\mathrm{d}x\mathrm{d}t\right)^{1/2},$$

for some $C_1 > 0$ depending on N, s and independent of x_0. Gathering in chain all the previous inequalities we obtain that

$$|u(x + x_0, 0)| \le \frac{C_2}{r^{\frac{1}{2}(N+2(1-s))}}\left(\int_{B_{2r}^+(x_0)} t^{1-2s}u^2\,\mathrm{d}x\mathrm{d}t\right)^{1/2}, \tag{8.3.13}$$

with $C_2 > 0$ again depending only on N, s. Now, Corollary 8.3.10 point (i) implies that

$$\int_{B_{2r}^+(x_0)} t^{1-2s} u^2 \,\mathrm{d}x\mathrm{d}t = \int_0^{2r} \left(\int_{S_\rho^+(x_0)} t^{1-2s} u^2 \,\mathrm{d}S \right) \mathrm{d}\,\rho \leq C_3 r^{N+2(1-s)+2k},$$

where $C_3 > 0$ depends on $N, s, \|g\|_{L^\infty(S_1^+)}, k$. Indeed the proof of Corollary 8.3.10 point (i) can be trivially adapted to the function $H_{x_0}(u,r)$ with the same constant K_1. The proof may now be concluded by combining the inequality above with (8.3.13). □

We now introduce the following functional of Monneau type

$$M_\gamma^{x_0}(v,w,r) := \frac{1}{r^{2\gamma}} H_{x_0}(v-w,r),$$

defined for $v, w \in H^1(B_1^+; t^{1-2s})$, $x_0 \in B_1'$, $r \in (0,1)$ and $\gamma \geq 0$, with H_{x_0} as in (8.3.1). As indicated before, we drop the index x_0 in the notation for $M_\gamma^{x_0}$ when $x_0 = 0$. In the following proposition we prove the monotonicity of this functional with respect to $r \in (0, R_0/2)$ when the solution u and a polynomial in $\mathbb{P}_k^s$ are "compared".

Proposition 8.3.12. *Let $k = \lim_{r\to 0} \tilde{\mathcal{N}}(r)$ and let $p_k \in \mathbb{P}_k^s$ be an t^{1-2s}-harmonic homogeneous polynomial of degree k, even in t, i.e. respectively satisfying* (8.1.7), (8.1.6), (8.1.5). *Then $M_k(u, p_k, \cdot) \in C^1(0, R_0/2)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}r} M_k(u, p_k, r) \geq -\hat{C} r^{k(p-2)+2s-1},$$

for some $\hat{C} > 0$ depending on N, s, $\|g\|_{L^\infty(S_1^+)}$ and k.

Proof. Let $r \in (0, R_0/2)$. Since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r} M_k(u,p_k,r) &= \frac{\mathrm{d}}{\mathrm{d}r}(r^{-2k} H(u-p_k,r)) \\ &= -2kr^{-2k-1} H(u-p_k,r) + r^{-2k} \frac{\mathrm{d}}{\mathrm{d}r} H(u-p_k,r), \end{aligned} \tag{8.3.14}$$

we thus need to compute the latter. By integrating the identity

$$\operatorname{div}(t^{1-2s} u \nabla p_k) = t^{1-2s} \nabla u \cdot \nabla p_k + u \operatorname{div}(t^{1-2s} \nabla p_k)$$

in B_r^+ we obtain

$$\int_{B_r^+} t^{1-2s} \nabla u \cdot \nabla p_k \,\mathrm{d}x\mathrm{d}t = \int_{S_r^+} t^{1-2s} u \frac{\partial p_k}{\partial \boldsymbol{\nu}} \,\mathrm{d}S, \tag{8.3.15}$$

since p_k is t^{1-2s}-harmonic and $t^{1-2s} \frac{\partial p_k}{\partial t} = 0$ on B_r'. Moreover, on S_r^+, we have

$$\frac{\partial p_k}{\partial \boldsymbol{\nu}} = \frac{1}{r} \nabla p_k \cdot z = \frac{k}{r} p_k$$

by the homogeneity of p_k. Combining this fact with (8.3.15) we have that

$$r\int_{B_r^+} t^{1-2s}\nabla u\cdot\nabla p_k\,\mathrm{d}x\mathrm{d}t = k\int_{S_r^+} t^{1-2s}up_k\,\mathrm{d}S. \tag{8.3.16}$$

On the other hand, we have that the function $u-p_k$ weakly satisfies

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla(u-p_k))=0, & \text{in } B_r^+,\\ -\lim\limits_{t\to 0^+} t^{1-2s}\dfrac{\partial}{\partial t}(u-p_k)=\lambda_-(u^-)^{p-1}-\lambda_+(u^+)^{p-1}, & \text{on } B_r', \end{cases}$$

that, multiplied by $u-p_k$ and integrated gives that

$$\begin{aligned}\int_{S_r^+} t^{1-2s}(u-p_k)\frac{\partial}{\partial\boldsymbol{\nu}}(u-p_k)\,\mathrm{d}S &= \int_{B_r^+} t^{1-2s}\left|\nabla(u-p_k)\right|^2\,\mathrm{d}x\mathrm{d}t\\ &+\int_{B_r'}(\lambda_-(u^+)^p+\lambda_+(u^+)^p)\,\mathrm{d}x+\int_{B_r'}(\lambda_-(u^-)^{p-1}-\lambda_+(u^+)^{p-1})p_k\,\mathrm{d}x'.\end{aligned} \tag{8.3.17}$$

Let us denote

$$I(u,p_k,r):=\frac{1}{r^{N-2s}}\int_{B_r'}(\lambda_-(u^-)^{p-1}-\lambda_+(u^+)^{p-1})p_k\,\mathrm{d}x'.$$

Reasoning as in the proof of Lemma 8.3.4, we have that

$$\frac{\mathrm{d}}{\mathrm{d}r}H(u-p_k,r)=\frac{2}{r^{N+1-2s}}\int_{S_r^+} t^{1-2s}(u-p_k)\frac{\partial}{\partial\boldsymbol{\nu}}(u-p_k)\,\mathrm{d}S.$$

Therefore, combining it with (8.3.17) and (8.3.16) we deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}H(u-p_k,r) &= \frac{2}{r}[E(u-p_k,r)+G(u,r)+I(u,p_k,r)]\\ &= \frac{2}{r}\left[E(u,r)+E(p_k,r)+G(u,r)-\frac{2}{r^{N-2s}}\int_{B_r^+}t^{1-2s}\nabla u\cdot\nabla p_k\,\mathrm{d}x\mathrm{d}t+I(u,p_k,r)\right]\\ &= \frac{2}{r}\left[E(u,r)+E(p_k,r)+G(u,r)-\frac{2k}{r^{N+1-2s}}\int_{S_r^+}t^{1-2s}up_k\,\mathrm{d}S+I(u,p_k,r)\right].\end{aligned}$$

In addition, since p_k is homogeneous, then $E(p_k,r)=kH(p_k,r)$; hence

$$\frac{\mathrm{d}}{\mathrm{d}r}H(u-p_k,r)=\frac{2}{r}\left[E(u,r)+kH(p_k,r)+G(u,r)-\frac{2k}{r^{N+1-2s}}\int_{S_r^+}t^{1-2s}up_k\,\mathrm{d}S+I(u,p_k,r)\right]. \tag{8.3.18}$$

Finally, let us compute the other term in (8.3.14)

$$H(u-p_k,r)=H(u,r)+H(p_k,r)-\frac{2}{r^{N+1-2s-1}}\int_{S_r^+}t^{1-2s}up_k\,\mathrm{d}S. \tag{8.3.19}$$

Putting together (8.3.18) and (8.3.19) with (8.3.14) we obtain that $M_k(u,p_k,\cdot)\in C^1(0,R_0/2)$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}M_k(u,p_k,r)=&\frac{2}{r^{2k+1}}\left[\tilde{E}(u,r)-kH(u,r)+\left(1-\frac{2}{p}\right)G(u,r)+I(u,p_k,r)\right]\\ =&\frac{2}{r^{2k+1}}\left[\tilde{E}(u,r)-kH(u,r)+\left(1-\frac{2}{p}\right)G(u,r)\right]\\ &+\frac{2}{r^{N+1-2s+2k}}\int_{B_r'}(\lambda_-(u^-)^{p-1}-\lambda_+(u^+)^{p-1})p_k\,\mathrm{d}x'.\end{aligned}\tag{8.3.20}$$

From Corollary 8.3.11 we have that

$$\|u\|_{L^\infty(B_r')}\le\tilde{C}r^k,$$

while, being p_k an homogeneous polynomial of degree k,

$$\|p_k\|_{L^\infty(B_r')}\le\tilde{C}_1r^k$$

for some $\tilde{C}_1>0$. Therefore

$$\left|\frac{2}{r^{N+1-2s+2k}}\int_{B_r'}(\lambda_-(u^-)^{p-1}-\lambda_+(u^+)^{p-1})p_k\,\mathrm{d}x'\right|\le\hat{C}r^{k(p-2)+2s-1}$$

for some $\hat{C}>0$ depending on N, s, $\|g\|_{L^\infty(S_1^+)}$ and k. In view of the inequality above and (8.3.20), together with the fact that $\tilde{\mathcal{N}}(r)\ge k$ and that $p\ge2$, the proof is complete. $\square$

Corollary 8.3.13. *There exists* $\lim_{r\to0}M_k(u,p_k,r)=:M_k(u,p_k,0^+)\in[0,+\infty)$. *Moreover*

$$M_k(u,p_k,0^+)\le M_k(u,p_k,r)+\hat{C}_1r^{k(p-2)+2s},\quad \textit{for all } r\in(0,R_0/2),\tag{8.3.21}$$

where $\hat{C}_1:=\hat{C}(k(p-2)+2s)^{-1}$ *and* $\hat{C}>0$ *is as in Proposition 8.3.12.*

Proof. From Proposition 8.3.12 we can derive that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[M_k(u,p_k,r)+\frac{\hat{C}r^{k(p-2)+2s}}{k(p-2)+2s}\right]\ge0.\tag{8.3.22}$$

Hence the following limit exists and it is finite

$$\lim_{r\to0}\left[M_k(u,p_k,r)+\frac{\hat{C}r^{k(p-2)+2s}}{k(p-2)+2s}\right].$$

Since $s\in(0,1)$ and $p\ge2$, we know that $r^{k(p-2)+2s}$ vanishes as $r\to0$, therefore

$$\lim_{r\to0}\left[M_k(u,p_k,r)+\frac{\hat{C}r^{k(p-2)+2s}}{k(p-2)+2s}\right]=\lim_{r\to0}M_k(u,p_k,r)=:M_k(u,p_k,0^+).$$

In addition $M_k(u,p_k,r)\ge0$ for all $r\in(0,R_0/2)$ and so $M_k(u,p_k,0^+)\ge0$ as well. Finally, by integration in $(0,r)$ of (8.3.22) we obtain (8.3.21) and we conclude the proof. $\square$

8.4 Blow-up analysis

This section is devoted the blow-up analysis of a particular normalization of the solution, which then leads to the proof of Theorem 8.1.1.

Hereafter, we fix $\tilde{R} \in (0, R_0/2)$. In this section we denote, for $z \in B_1^+$ and $r \in (0, R)$,

$$u_r(z) := \frac{u(rz)}{\sqrt{H(r)}}. \tag{8.4.1}$$

The first step is the blow-up analysis for the function u_r.

Theorem 8.4.1. *Let $k = \lim_{r\to 0} \mathcal{N}(r)$ and u_r as in (8.4.1). Then $k \in \mathbb{N} \cup \{0\}$ and for any sequence $r_n \to 0^+$ there exists a subsequence $r_{n_i} \to 0^+$ and a function $\tilde{u} \in \mathbb{P}_k^s$, $\tilde{u} \not\equiv 0$, such that*

$$u_{r_{n_i}} \to \tilde{u} \quad \text{in } H^1(B_1^+; t^{1-2s}), \text{ and } C^{0,\alpha}(B_1^+) \tag{8.4.2}$$

$$u_{r_{n_i}} \to \tilde{u} \quad \text{in } C^{1,\alpha}(B_1') \tag{8.4.3}$$

as $i \to \infty$.

Proof. Let $r_n \to 0^+$. First, we have that $u_{r_n} \in H^1(B_1^+; t^{1-2s})$ and

$$\int_{S_1^+} t^{1-2s} u_{r_n}^2 \, \mathrm{d}S = H(u_{r_n}, 1) = 1.$$

Moreover

$$\int_{B_1^+} t^{1-2s} \left|\nabla u_{r_n}\right|^2 \mathrm{d}x\mathrm{d}t = \mathcal{N}(u_{r_n}, 1) = \mathcal{N}(u, r_n) \le \tilde{\mathcal{N}}(u, R_0).$$

Therefore, thanks to the Poincaré type inequality Lemma 8.2.1, we know that $\{u_{r_n}\}_n$ is bounded in $H^1(B_1^+; t^{1-2s})$. Thus there exists a subsequence $r_{n_i} \to 0$ and a function $\tilde{u} \in H^1(B_1^+; t^{1-2s})$ such that

$$u_{r_{n_i}} \rightharpoonup \tilde{u} \quad \text{weakly in } H^1(B_1^+; t^{1-2s})$$

as $i \to \infty$ and, due to compact embedding, $u_{r_{n_i}} \to \tilde{u}$ strongly in $L^2(B_1^+; t^{1-2s})$ which implies that $\|\tilde{u}\|_{L^2(B_1^+; t^{1-2s})} = 1$ and then $\tilde{u} \not\equiv 0$. We have that $u_r \in H^1(B_1^+; t^{1-2s})$ satisfies

$$\int_{B_1^+} t^{1-2s} \nabla u_r \cdot \nabla \varphi \, \mathrm{d}x\mathrm{d}t = r^{2s} (H(r))^{\frac{p-2}{2}} \int_{B_1'} (\lambda_-(u_r^-)^{p-1} - \lambda_+(u_r^+)^{p-1}) \varphi \, \mathrm{d}x$$

for all $\varphi \in C_c^\infty(B_1^+ \cup B_1')$. Hence, for $\varphi \in C_c^\infty(B_1^+ \cup B_1')$ we have

$$\left| \int_{B_1^+} t^{1-2s} \nabla u_{r_{n_i}} \cdot \nabla \varphi \, \mathrm{d}x\mathrm{d}t \right| \le C r_{n_i}^{2s} (H(r_{n_i}))^{\frac{p-2}{2}} \int_{B_1'} \left| u_{r_{n_i}} \right|^{p-1} \mathrm{d}x, \tag{8.4.4}$$

where $C = 2\max\{\lambda_-, \lambda_+\} \|\varphi\|_{L^\infty(B_1^+)}$. We now observe that, thanks to Corollary 8.3.11,

$$\left|u_{r_{n_i}}(x,0)\right|^{p-1} \leq \frac{C r_{n_i}^{k(p-1)}}{(H(r_{n_i}))^{\frac{p-1}{2}}},$$

for another constant $C > 0$. Combining this with (8.4.4) we obtain that

$$\left|\int_{B_1^+} t^{1-2s} \nabla u_{r_{n_i}} \cdot \nabla\varphi \,\mathrm{d}x\mathrm{d}t\right| \leq \frac{C r_{n_i}^{k(p-1)+2s}}{(H(r_{n_i}))^{1/2}}.$$

up to renaming the constant C. Now, by choosing $\delta = 2s$ in Corollary 8.3.10 point (ii), and plugging the estimate on H in the previous equation, we derive that

$$\left|\int_{B_1^+} t^{1-2s} \nabla u_{r_{n_i}} \cdot \nabla\varphi \,\mathrm{d}x\mathrm{d}t\right| \leq C r_{n_i}^{k(p-2)+s},$$

possibly renaming again the constant C. Thus, passing to the limit for $i \to \infty$ yields that $\tilde{u}$ is t^{1-2s}-harmonic in $B_1^+ \cup B_1'$. Thanks to the regularity of u_r in Hölder spaces (see Lemma 8.2.7 and Lemma 8.2.11) we can easily deduce that

$$u_{r_{n_i}} \to \tilde{u}, \quad \text{in } C^{0,\alpha}(\overline{B_1^+}), \quad \text{as } i \to \infty,$$

as well as

$$\nabla_x u_{r_{n_i}} \to \nabla_x \tilde{u} \quad \text{and} \quad t^{1-2s}\frac{\partial u_{r_{n_i}}}{\partial_t} \to t^{1-2s}\frac{\partial u_{r_{n_i}}}{\partial_t}, \quad \text{in } C^{0,\alpha}(\overline{B_1^+}), \quad \text{as } i \to \infty,$$

which, after integration by parts, imply that $\|u_{r_{n_i}}\|_{H^1(B_1^+;t^{1-2s})} \to \|\tilde{u}\|_{H^1(B_1^+;t^{1-2s})}$ and then $u_{r_{n_i}} \to \tilde{u}$ strongly in $H^1(B_1^+;t^{1-2s})$ as $i \to \infty$. Hence (8.4.2) and (8.4.3) are proved.

Now we can pass to the limit in the Almgren frequency function and obtain that, for any $R \in (0,1)$

$$k = \lim_{i\to\infty} \mathcal{N}(u, R r_{n_i}) = \lim_{i\to\infty} \mathcal{N}(u_{r_{n_i}}, R) = \mathcal{N}(\tilde{u}, R),$$

which implies that $\tilde{u}$ is homogeneous of degree k. Therefore, the even reflection of $\tilde{u}$ through $\{t = 0\}$ is an t^{1-2s}-harmonic homogeneous polynomial of degree k. But such a polynomial must be smooth (see e.g. [STV19, Theorem 1.1]) and so we know that k must be a nonnegative integer and that $\tilde{u} \in \mathbb{P}_k^s$. The proof is thereby complete. □

Next, we prove a nondegeneracy result of the solution at any boundary point.

Proposition 8.4.2. *There exists two constants $C_1, C_2 > 0$ such that the following estimates from above and below hold:*

$$|u(z)| \leq C_1 |z|^k \quad \textit{for all } z \in B_r^+ \tag{8.4.5}$$

and

$$\sup_{S_r^+} |u| \geq C_2 r^k \tag{8.4.6}$$

for all $r \in (0, \tilde{R})$. Moreover, there exists $\lim_{r\to 0} r^{-2k} H(u,r) \in (0,+\infty)$.

Proof. In order to prove (8.4.5), let us assume by contradiction that there exists a sequence $r_n \to 0$ such that

$$\sup_{z \in S^+_{r_n}} |u(z)| \geq n r_n^k.$$

Together with Corollary 8.3.10 and a change of variable this means that

$$\sup_{S_1^+} |u_{r_n}|^2 \geq K_1 n \int_{S_1^+} t^{1-2s} u_{r_n}^2 \, \mathrm{d}S.$$

But in view of Theorem 8.4.1 this gives rise to a contradiction for n sufficiently large.

Now let us prove (8.4.6). We observe that (8.4.6) is equivalent to

$$r^{-2k} \sup_{S_r^+} |u|^2 \geq C_2^2 > 0. \tag{8.4.7}$$

It is also possible to prove that

$$r^{-2k} H(u,r) \leq \bar{C} r^{-2k} \sup_{S_r^+} |u|^2 .$$

for some $\bar{C} > 0$. So let us contradict (8.4.7) and assume that, up to a subsequence

$$0 \leq r^{-2k} H(u,r) \leq \bar{C} r^{-2k} \sup_{S_r^+} |u|^2 \to 0,$$

and so that

$$H(u,r) = o(r^{2k}), \quad \text{as } r \to 0. \tag{8.4.8}$$

Let $\tilde{u} \in H^1(B_1^+; t^{1-2s})$ be, as in Theorem 8.4.1, the blow-up limit of $u_r(z) = [H(u,r)]^{-1/2} u(rz)$, possibly passing to another subsequence as $r \to 0$. By definition of the Monneau-type functional, we have

$$M_k(u,\tilde{u},r) = r^{-2k} \left(H(u,r) + H(\tilde{u},r) - \frac{2}{r^{N+1-2s}} \int_{S_r^+} t^{1-2s} u \tilde{u} \, \mathrm{d}S \right). \tag{8.4.9}$$

Thanks to homogeneity of $\tilde{u}$, we notice that

$$H(\tilde{u},r) = r^{2k} H(\tilde{u},1). \tag{8.4.10}$$

On the other hand, by Hölder's inequality, (8.4.8) and (8.4.10)

$$\left| r^{-N+2s-2k} \int_{S_r^+} t^{1-2s} u \tilde{u} \, \mathrm{d}x \mathrm{d}t \right| \leq \sqrt{r^{-2k} H(u,r)} \sqrt{r^{-2k} H(\tilde{u},r)} = o(1)$$

as $r \to 0$. Combining this with (8.4.9) and (8.4.10) we obtain that

$$M_k(u,\tilde{u},r) \to H(\tilde{u},1),$$

as $r \to 0$. Then, by monotonicity of $M_k(u, \tilde{u}, r)$, proved in Proposition 8.3.12, we have that

$$r^{-2k} H(\tilde{u}, r) = H(\tilde{u}, 1) \leq r^{-2k} H(u - \tilde{u}, r) + \hat{C}_1 r^{k(p-2)+2s},$$

which, after expansion, says that

$$\frac{1}{r^{N+1-2s}} \int_{S_r^+} t^{1-2s} (u^2 - 2u\tilde{u}) \, \mathrm{d}S \geq -\hat{C}_1 r^{kp+2s} \tag{8.4.11}$$

for the chosen sequence $r \to 0$. But, after rescaling and dividing by $\sqrt{r^{2k} H(u, r)}$, thanks again to the homogeneity of $\tilde{u}$, (8.4.11) becomes

$$\int_{S_1^+} t^{1-2s} (\sqrt{r^{-2k} H(u, r)} u_r^2 - 2u_r \tilde{u}) \, \mathrm{d}S \geq -\frac{\hat{C}_1 r^{k(p-1)+2s}}{\sqrt{H(u, r)}}.$$

Now, let us apply Corollary 8.3.10 point (ii) with $\delta = 2s$. We have that there exists $K_2 > 0$ such that $H(u, r) \geq K_2 r^{2k+2s}$; going on with the inequality above we get

$$\int_{S_1^+} t^{1-2s} (\sqrt{r^{-2k} H(u, r)} u_r^2 - 2u_r \tilde{u}) \, \mathrm{d}S \geq -\hat{C}_1 r^{k(p-2)+s}$$

and passing to the limit as $r \to 0$ we obtain that

$$-2H(\tilde{u}, 1) = -2 \int_{S_1^+} t^{1-2s} \tilde{u}^2 \, \mathrm{d}S \geq 0$$

thus giving rise to a contradiction. Indeed $H(\tilde{u}, 1) > 0$ because, if not the unique continuation principle would be violated, since $H(\tilde{u}, 1) = r^{-2k} H(\tilde{u}, r)$ for all $r \in (0, 1)$. Then there exists a constant $C > 0$ such that

$$C \leq r^{-2k} H(u, r) \leq \bar{C} r^{-2k} \sup_{S_r^+} |u|^2$$

for all $r \in (0, 1)$ and this proves (8.4.6). The second part of the thesis comes from the inequality above and Corollary 8.3.10 point (i).

□

Here the monotonicity of the Monneau functional (defined in (8.3)) enters and it allows to uniquely identify the blow-up limit and to prove Theorem 8.1.1.

Proof of Theorem 8.1.1. Let $r_n \to 0$. By Theorem 8.4.1 there exists a subsequence $r_{n_i} \to 0$ and an t^{1-2s}-harmonic homogeneous polynomial $\tilde{u}$, even in t, such that

$$u_{r_{n_i}} \to \tilde{u} \quad \text{in } H^1(B_1^+; t^{1-2s}),\ C^{0,\alpha}(\overline{B_1^+}) \text{ and } C^{1,\alpha}(B_1')$$

as $i \to \infty$. Therefore, if we let $\gamma := \lim_{r \to 0} r^{-2k} H(u, r) \in (0, +\infty)$, whose existence in ensured by Proposition 8.4.2, we have that

$$u_{r_{n_i}}^k \to u_0 := \sqrt{\gamma}\, \tilde{u} \quad \text{in } H^1(B_1^+; t^{1-2s}),\ C^{0,\alpha}(\overline{B_1^+}) \text{ and } C^{1,\alpha}(B_1')$$

as $i \to \infty$, where

$$u_r^k(z) := \frac{u(rz)}{r^k}.$$

By Corollary 8.3.13 the functional $M_k(u, u_0, r)$ admits a limit as $r \to 0$, therefore

$$\lim_{r\to 0} M_k(u, u_0, r) = \lim_{i\to\infty} M_k(u, u_0, r_{n_i}) = \lim_{i\to\infty} M_k(u^k_{r_{n_i}}, u_0, 1) = 0.$$

Now we prove that the limit does not depend on the choice of the subsequence. Let $r_{n_i} \to 0$ another subsequence and $\tilde{u}_0$ another possbile limit: again $\tilde{u}_0$ is an t^{1-2s}-harmonic homogeneous polynomial of degree k and even in t. Hence, analogously

$$\lim_{r\to 0} M_k(u, \tilde{u}_0, r) = \lim_{i\to\infty} M_k(u, \tilde{u}_0, r_{n_i}) = \lim_{i\to\infty} M_k(u^k_{r_{n_i}}, \tilde{u}_0, 1) = 0.$$

Furthermore

$$\int_{S_1^+} t^{1-2s}(u_0 - \tilde{u}_0)^2 \,\mathrm{d}x\mathrm{d}t = M_k(u_0, \tilde{u}_0, r) \leq 2M_k(u, u_0, r) + 2M_k(u, \tilde{u}_0, r) \to 0$$

as $r \to 0$ and so $u_0 = \tilde{u}_0$ and the proof is concluded.

□

Remark 8.4.3. First of all we point out that the functions $\tilde{u}$ and u_0, defined respectively in Theorem 8.4.1 and Theorem 8.1.1 are such that $u_0 = \sqrt{\gamma}\,\tilde{u}$, where $\gamma = \lim_{r\to 0} r^{-2k} H(u, r) > 0$.

Moreover, since u_0 is an t^{1-2s}-harmonic polynomial in $\mathbb{R}^{N+1}_+$ and since $\lim_{t\to 0^+} t^{1-2s}\frac{\partial u_0}{\partial t} = 0$, then u_0 cannot vanish everywhere on the thin space $\{t = 0\}$, because otherwise its trivial extension to the whole $\mathbb{R}^{N+1}$ would violate the unique continuation principle for A_2-weighted elliptic equations, see [TZ08, Corollary 1.4] (see also [GRO19, Lemma 5.2]).

Remark 8.4.4. We recall that

$$\mathcal{Z}(u) = \{x_0 \in B_1' \colon u(x_0, 0) = 0\},$$
$$\mathcal{Z}_k(u) = \{x_0 \in \mathcal{Z}(u) \colon \mathcal{N}_{x_0}(u, 0^+) = k\}.$$

Thanks to Theorem 8.1.1, for any $x_0 \in \mathcal{Z}(u)$ there exists a unique integer $k \geq 1$ and a unique t^{1-2s}-harmonic polynomial $p_k^{x_0} \in \mathbb{P}_k^s$, homogeneous of degree k such that

$$u_r^{k,x_0}(z) := \frac{u(rz + x_0)}{r^k} \to p_k^{x_0}(z) \quad \text{in } H^1(B_1^+; t^{1-2s}).$$

We refer to this polynomial as the *blow-up limit* of u at the point x_0.

We conclude the section by proving the continuous dependence of the blow-up limits on the nodal points. This result plays a pivotal role in the analysis of the structure of the free boundary.

Theorem 8.4.5. *Let $k \in \mathbb{N}$, $k \geq 1$. Then the map*

$$\begin{aligned} \mathcal{Z}_k(u) &\longrightarrow \mathbb{P}_k^s \\ x_0 &\longmapsto p_k^{x_0} \end{aligned}$$

is continuous, where $\mathcal{Z}_1(u) = \mathcal{R}(u)$ and $p_k^{x_0}$ as in Remark 8.4.4. Moreover for any compact $K \subseteq \mathcal{Z}_k(u)$

$$\left\| u_r^{k,x_0} - p_k^{x_0} \right\|_{L^\infty(B_{1/2}^+)} \to 0 \quad \text{as } r \to 0, \text{ uniformly for } x_0 \in K.$$

In particular there exists a modulus of continuity $\sigma_K(t) \geq 0$, $\sigma_K(0^+) = 0$ such that

$$\left| u(z) - p_k^{x_0}(z - x_0) \right| \leq \sigma_K(|z - x_0|) \, |z - x_0|^k ,$$

for any $x_0 \in K$.

Proof. First we observe that $\mathbb{P}_k^s$ is a subset of finite dimensional space, namely the space of homogeneous polynomial of degree k, and so all the norms are equivalent: we choose to endow this space with the $L^2(S_1^+; t^{1-2s})$-norm. Thanks to Theorem 8.1.1, for any $x_0 \in \mathcal{Z}_k(u)$ and any $\varepsilon > 0$ there exists $r_\varepsilon = r_\varepsilon(x_0) \in (0, \varepsilon)$ such that

$$M_k^{x_0}(u, p_k^{x_0}, r) = r^{-N+2s-2k} \int_{S_r^+} t^{1-2s} (u(z + x_0) - p_k^{x_0}(z))^2 \, \mathrm{d}S < \varepsilon$$

for all $r < r_\varepsilon$. Moreover, due to α-Hölder continuity of u and the previous inequality, there exists $\delta_\varepsilon = \delta_\varepsilon(x_0, k, \alpha) > 0$ such that

$$M_k^{\bar{x}}(u, p_k^{x_0}, r) = r^{-N+2s-2k} \int_{S_r^+} t^{1-2s} (u(z + \bar{x}) - p_k^{x_0}(z))^2 \, \mathrm{d}S < 2\varepsilon, \tag{8.4.12}$$

for any $\bar{x} \in \mathcal{Z}_k(u)$ satisfying $|\bar{x} - x_0| < \delta_\varepsilon$ and for any $r < r_\varepsilon$. Then, the almost monotonicity of $M_k^{\bar{x}}(u, p_k^{x_0}, r)$ (see Corollary 8.3.13) yields

$$\begin{aligned} \int_{S_1^+} t^{1-2s} (p_k^{\bar{x}} - p_k^{x_0})^2 \, \mathrm{d}S &= \lim_{r \to 0} M_k^{\bar{x}}(u, p_k^{x_0}, r) \\ &\leq M_k^{\bar{x}}(u, p_k^{x_0}, r) + \hat{C}_1 r^{k(p-2)+2s} < 3\varepsilon \end{aligned} \tag{8.4.13}$$

and for any $\bar{x} \in \mathcal{Z}_k(u)$ such that $|\bar{x} - x_0| < \delta_\varepsilon$ and for any $r < r_\varepsilon$ (up to taking a smaller r_ε, chosen independently of $\bar{x}$). This proves the first part of the statement.

Now, let us fix a compact $K \subseteq \mathcal{Z}_k(u)$ and let $x_0 \in K$. From now until the end of the proof, we denote by C a positive constant, whose value may change from line to line, that is independent of ε and of the choice of $\bar{x}$. From (8.4.12) we deduce that

$$\int_{S_r^+} t^{1-2s} (u(z + \bar{x}) - p_k^{x_0}(z))^2 \, \mathrm{d}S \leq 2\varepsilon r^{N-2s+2k}$$

which, integrated in r and up to a change of variable, yields,

$$\left\| u_r^{k,\bar{x}} - p_k^{x_0} \right\|^2_{L^2(B_1^+;t^{1-2s})} \leq C\varepsilon, \tag{8.4.14}$$

for any $\bar{x} \in \mathcal{Z}_k(u)$ such that $|\bar{x} - x_0| < \delta_\varepsilon$ and for any $r < r_\varepsilon$. By a change of variable in (8.4.13) we obtain that

$$\int_{S_r^+} t^{1-2s}(p_k^{\bar{x}} - p_k^{x_0})^2 \,\mathrm{d}S \leq C\varepsilon r^{N-2s+2k}.$$

Again, integrating in r and with a backwards change of variable, we can infer

$$\left\| p_k^{\bar{x}} - p_k^{x_0} \right\|^2_{L^2(B_1^+;t^{1-2s})} \leq C\varepsilon$$

for any $\bar{x} \in \mathcal{Z}_k(u)$ such that $|\bar{x} - x_0| < \delta_\varepsilon$ and for any $r < r_\varepsilon$. Combining the previous estimate with (8.4.14) we have that

$$\left\| u_r^{k,\bar{x}} - p_k^{\bar{x}} \right\|_{L^2(B_1^+;t^{1-2s})} \leq C\sqrt{\varepsilon} \tag{8.4.15}$$

for any $\bar{x} \in \mathcal{Z}_k(u)$ such that $|\bar{x} - x_0| < \delta_\varepsilon$ and for any $r < r_\varepsilon$. We now observe that the function $w_r^{k,\bar{x}} := u_r^{k,\bar{x}} - p_k^{\bar{x}} \in H^1(B_1^+; t^{1-2s})$ weakly solves

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla w_r^{k,\bar{x}}) = 0, & \text{in } B_1^+, \\ -\displaystyle\lim_{t\to 0} t^{1-2s}\frac{\partial w_r^{k,\bar{x}}}{\partial t} = r^{k(p-2)+2s}(\lambda_-((u_r^{k,\bar{x}})^-)^{p-1} - \lambda_+((u_r^{k,\bar{x}})^+)^{p-1}), & \text{on } B_1'. \end{cases}$$

If we let

$$d := r^{k(p-2)+2s}(\lambda_-((u_r^{k,\bar{x}})^-)^{p-1} - \lambda_+((u_r^{k,\bar{x}})^+)^{p-1}),$$

in view of Corollary 8.3.11 we have that

$$\|d\|_{L^\infty(B_1')} \leq Cr^{k(p-2)+2s}.$$

Therefore, from Lemma 8.2.9, by virtue of the previous inequality and (8.4.15), we can say that

$$\begin{aligned} \left\| w_r^{k,\bar{x}} \right\|_{L^\infty(B_{1/2}^+)} &\leq C\left(\left\| w_r^{k,\bar{x}} \right\|_{L^2(B_1^+;t^{1-2s})} + \|d\|_{L^\infty(B_1')} \right), \\ &\leq C(\sqrt{\varepsilon} + r^{k(p-2)+2s}) \end{aligned}$$

for any $\bar{x} \in \mathcal{Z}_k(u)$ such that $|\bar{x} - x_0| < \delta_\varepsilon$ and for any $r < r_\varepsilon$. In particular we can choose r_ε sufficiently small (independently of $\bar{x}$) in such a way that

$$\left\| u_r^{k,\bar{x}} - p_k^{\bar{x}} \right\|_{L^\infty(B_{1/2}^+)} \leq C\sqrt{\varepsilon}.$$

The conclusion of the proof follows by a covering argument of the compact set K.

□

8.5 Regularity of the free boundary

The goal of this last section is to prove the results regarding the structure of the free boundary, more precisely its rectifiability and estimates on the Hausdorff dimension of its parts. A first step is the proof of the regularity of the regular part $\mathcal{R}(u)$.

Proposition 8.5.1. *The regular part of the free boundary $\mathcal{R}(u)$ is relatively open in $\mathcal{Z}(u)$ while, for $k \geq 2$, the k-singular part $\mathcal{Z}_k(u)$ is of type F_σ, i.e. the union of countably many closed sets.*

Proof. The proof of the first claim immediately follows from the upper-semicontinuity of the map $x_0 \mapsto \mathcal{N}(x_0, u, 0^+)$, combined with the fact that the frequency function takes no values in the range $(1, 2)$. In view of Proposition 8.4.2, the proof of the second claim is the same as [GP09, Lemma 1.5.3], so we omit it. □

Proof of Proposition 8.1.3. First of all, we recall that any t^{1-2s}-harmonic, 1-homogeneous polynomial $p_1(x, t)$ must be smooth (see e.g. [STV19]) and so it must depend only on the variable x and be harmonic in x. In particular $p_1(x, t) = x \cdot \boldsymbol{\nu}$ for some $\boldsymbol{\nu} \in \mathbb{S}^{N-1} = \mathbb{S}^N \cap \{t = 0\}$. Then, from Theorem 8.1.1 we know that, for any $x_0 \in \mathcal{R}(u)$ there exists $\boldsymbol{\nu}_{x_0} \in \mathbb{S}^{N-1}$ such that

$$u(x, 0) = (x - x_0) \cdot \boldsymbol{\nu}_{x_0} + o(|x - x_0|)$$

as $x \to x_0$. Moreover, from Theorem 8.4.5 we know that the map $x_0 \mapsto \boldsymbol{\nu}_{x_0}$ is continuous on $\mathcal{R}(u)$. Since $u(\cdot, 0) \in C^{1,\alpha}(B_1')$ we can compute the directional derivative and observe that $\nabla_x u(x_0, 0) = \boldsymbol{\nu}_{x_0}$ thus proving that $\mathcal{R}(u) \subseteq \{x_0 \in \mathcal{Z}(u) \colon |\nabla_x u(x_0, 0)| \neq 0\}$. The opposite inclusion trivially comes from Theorem 8.1.1. Finally, the implicit function theorem allows us to conclude the proof. □

The following result is a key step in the proof of Theorem 8.1.5, and we refer to [STT20, Theorem 7.7] for the proof, which basically relies on Theorem 8.4.5, Proposition 8.5.1, Whitney's extension theorem (as stated in [Whi34]) and the implicit function theorem.

Lemma 8.5.2. *For any $k \geq 2$ and for any $n = 0, \ldots, N-1$, the sets $\mathcal{Z}_k^n(u)$ (defined in (8.1.8)) are contained in a countable union of n-dimensional C^1 manifolds.*

As a consequence, we are now able to prove Theorem 8.1.5.

Proof of Theorem 8.1.5. For any fixed $n \geq 0$, the sets $\mathcal{Z}_{k_1}^t(u) \cap \mathcal{Z}_{k_1}^n(u)$ and $\mathcal{Z}_{k_1}^t(u) \cap \mathcal{Z}_{k_1}^n(u)$ are disjoint for $k_1 \neq k_2$; therefore

$$\mathcal{S}_n^t(u) = S^t(u) \cap \Big(\bigcup_{k \geq 2} \mathcal{Z}_k^n(u) \Big)$$

and so it is contained in a countable union of n-dimensional C^1 manifold, thanks to Lemma 8.5.2. Then, taking the union for $n = 0, \ldots, N-1$, we obtain the desired result for $\mathcal{S}^t(u)$. The same applies for the stratum $\mathcal{S}^*(u)$, but in this case the dimension at any point cannot exceed the threshold $N - 2$, since the polynomials in $\mathbb{P}_k^*$ are harmonic in $\mathbb{R}^N$. □

The last part of this section is dedicated to the proof of Theorem 8.1.6, regarding the Hausdorff dimension of the nodal set $\mathcal{Z}(u)$ and of its regular and singular part. The main tool we exploit is a generalization of the Federer's Reduction Principle. The basic idea is the following. Consider a family of functions $\mathcal{F}$ of $\mathbb{R}^N$ invariant under rescaling and translation and a map Σ which associates to every function in $\mathcal{F}$ a subset of $\mathbb{R}^N$. This principles establishes that, under suitable conditions on $\mathcal{F}$ and Σ, in order to control the Hausdorff dimension of $\Sigma(f)$ for every $f \in \mathcal{F}$, you just need to control the Hausdorff dimension of $\Sigma(f)$ for the functions f that are homogeneous of some degree. Up to our knowledge, this principle (originally proved by Federer) appears for the first time in the form we need in [Sim83, Appendix A]. We make use of the following version of the principle, which is a particular case of the generalization made by Chen, see [Che98b, Theorem 8.5] and [Che98a, Proposition 4.5].

Theorem 8.5.3 (Federer's reduction principle). *Let $\mathcal{F} \subseteq C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ and let, for any $u \in \mathcal{F}$, $x_0 \in \mathbb{R}^N$ and $r > 0$*

$$u_r^{x_0}(x) := u(x_0 + rx).$$

We say that $u_n \to u$ in $\mathcal{F}$ as $n \to \infty$ if $u_n \to u$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. Let us assume that the family $\mathcal{F}$ satisfies the following conditions:

(F1) (closure under appropriate translations and rescalings) *for any $r, \rho \in (0,1)$, $x_0 \in B'_{1-r}$ and $u \in \mathcal{F}$ we have that $\rho u_r^{x_0} \in \mathcal{F}$;*

(F2) (existence of a homogeneous blow-up limit) *for any $x_0 \in B'_1$, $r_n \to 0$ and $u \in \mathcal{F}$, there exists a sequence $\rho_n > 0$, a real number $\mu \geq 0$ and an μ-homogeneous function $\hat{u} \in \mathcal{F}$ such that, up to a subsequence $\rho_n u_{r_n}^{x_0} \to \hat{u}$ in $\mathcal{F}$;*

(F3) (singular set hypotheses) *there exists a map $\Sigma \colon \mathcal{F} \to \mathcal{C}$, where*

$$\mathcal{C} := \{A \subseteq \mathbb{R}^N \colon A \cap B'_1 \text{ is relatively closed in } B'_1\},$$

such that

(i) for any $r, \rho \in (0,1)$, $x_0 \in B'_{1-r}$ and $u \in \mathcal{F}$ we have that

$$\Sigma(\rho u_r^{x_0}) = \frac{\Sigma(u) - x_0}{r};$$

(ii) for any $x_0 \in B'_1$, $r_n \to 0$, $u, \hat{u} \in \mathcal{F}$ such that there exists $\rho_n > 0$ for which $\rho_n u_{r_n}^{x_0} \to \hat{u}$ in $\mathcal{F}$, the following holds true:

$$\text{for any } \varepsilon > 0 \text{ there exists } n_\varepsilon > 0 \text{ such that}$$
$$\Sigma(\rho_n u_{r_n}^{x_0}) \subseteq \{x \in B'_1 \colon \mathrm{dist}(x, \Sigma(\hat{u})) \leq \varepsilon\} \quad \text{for all } n \geq n_\varepsilon.$$

Then either $\Sigma(u) = \emptyset$ for every $u \in \mathcal{F}$ or $\dim_H(\Sigma(u)) \leq d$ for every $u \in \mathcal{F}$, where

$$d := \max\{\dim V \colon V \text{ is a subspace of } \mathbb{R}^N \text{ and there exists } \mu \geq 0 \text{ and } u \in \mathcal{F} \text{ such that } \Sigma(u) \neq \emptyset \text{ and } u_r^{x_0} = r^\mu u \text{ for all } x_0 \in V,\ r > 0\},$$

Furthermore, in the latter case there exists a function $\psi \in \mathcal{F}$, a d-dimensional subspace $V \subseteq \mathbb{R}^N$ and a real number $\mu \geq 0$ such that

$$\psi_r^b = r^\mu \psi \text{ for all } b \in V,\ r > 0 \text{ and } \Sigma(\psi) = V \cap B_1.$$

Finally, if $d = 0$ the set $\Sigma(u) \cap B_r'$ is discrete for every $u \in \mathcal{F}$ and $r \in (0,1)$.

Let us consider the following class of problems, to be intended in a weak sense

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla u) = 0, & \text{in } B_R^+, \\ -\lim\limits_{t\to 0} t^{1-2s}\dfrac{\partial u}{\partial t} = \lambda\left(\lambda_-(u^-)^{p-1} - \lambda_+(u^+)^{p-1}\right), & \text{on } B_R', \end{cases} \tag{8.5.1}$$

for $R \geq 1$, $\lambda \geq 0$ and let us introduce the following family of functions:

$$\mathcal{F} := \Big\{ u(\cdot,0)\colon u \in C^{0,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+}),\ \nabla_x u \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N),\ u(\cdot,0) \not\equiv 0 \text{ in } \mathbb{R}^N \\ \text{and solves (8.5.1) for some } R \geq 1 \text{ and } \lambda \geq 0 \Big\}. \tag{8.5.2}$$

Remark 8.5.4. We observe that if $u \in \mathcal{F}$ is homogeneous of degree $\mu > 0$, then it must satisfy

$$-\lim_{t\to 0} t^{1-2s}\frac{\partial u}{\partial t} = 0, \quad \text{on } B_R'$$

because the function $y \mapsto \lambda_-(y^-)^{p-1} - \lambda_+(y^+)^{p-1}$ is homogeneous of degree $p-1$ with $p \geq 2$. This means that the even reflection of such u is harmonic in B_1.

Now we are ready to prove the last main result of our paper.

Proof of Theorem 8.1.6. Let $\mathcal{F}$ be as in (8.5.2). It is straightforward to check that assumptions (F1) and (F2) in Theorem 8.5.3 are satisfied by $\mathcal{F}$, in view also of Theorem 8.1.1. Then, we choose the map Σ depending on the claim.
Hausdorff dimension of $\mathcal{Z}(u)$. In order to prove (8.1.9), we let $\Sigma(u) := \mathcal{Z}(u)$. It's easy to prove that the singular set hypotheses in (F3) are fulfilled by this choice of the map Σ. Therefore Theorem 8.5.3 applies and we have that either $\mathcal{Z}(u) = \emptyset$ or $\dim_H(\mathcal{Z}(u)) \leq d$. Assume by contradiction that $d = N$. Then there exists a function $\hat{u} \in \mathcal{F}$, homogeneous of some degree $\mu > 0$, that, in view of Remark 8.5.4, weakly solves

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla \hat{u}) = 0, & \text{in } B_1^+, \\ \hat{u} = 0, & \text{on } B_1', \\ -\lim\limits_{t\to 0} t^{1-2s}\dfrac{\partial \hat{u}}{\partial t} = 0, & \text{on } B_1'. \end{cases}$$

We notice that it must be $\hat{u} \equiv 0$, because otherwise its trivial extension to the whole $B_1 \subseteq \mathbb{R}^{N+1}$ would violate the unique continuation principle. But this is a contradiction, since $0 \notin \mathcal{F}$.

Hausdorff dimension of $\mathcal{R}(u)$. (8.1.10) is an immediate consequence of Proposition 8.1.3.

Hausdorff dimension of $\mathcal{S}(u)$. Finally, we let $\Sigma(u) := \mathcal{S}(u)$. Again, it's not hard to verify the hypotheses in (F3), hence we can apply the theorem in this case as well. Moreover, since $\mathcal{S}(u) \subseteq \mathcal{Z}(u)$, we have that $\dim_H(\mathcal{S}(u)) \leq N-1$. In fact, this bound is optimal. In order to see this, we consider [STT20, Proposition 4.13]. There, the authors found explicit expressions for 2-dimensional t^{1-2s}-harmonic polynomials, homogeneous of any possible integer degree. In particular, for $(y,t) \in \mathbb{R}^2$ and $k \in 2\mathbb{N}$, the following is a k-homogeneous, t^{1-2s}-harmonic polynomial of degree k, even in the variable t

$$p(y,t) = \frac{(-1)^{\frac{k}{2}}\Gamma\left(\frac{1}{2}+\frac{1-2s}{2}\right)}{2^k\Gamma\left(1+\frac{k}{2}\right)\Gamma\left(\frac{1}{2}+\frac{1-2s}{2}+\frac{k}{2}\right)}\,{}_2F_1\left(-\frac{k}{2}, -\frac{k}{2}-\frac{1-2s}{2}+\frac{1}{2}, \frac{1}{2}, -\frac{y^2}{t^2}\right)t^k,$$

where Γ is the usual Gamma function and ${}_2F_1$ is the hypergeometric function. To conclude the proof of (8.1.11), it is sufficient to consider $V = \{x \in \mathbb{R}^N : x_N = 0\}$ and $\hat{u}(x,t) = p(x_N,t)$.

$\square$

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