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A Novel Perturbative Approach to Stochastic Partial Differential Equations

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Al mio Jeff

Nel tempo fatto di attimi
e settimane enigmistiche. . .
(Paolo Conte)

Contents

Introduction

The topic of this thesis is the application of techniques proper of *algebraic quantum field theory* (AQFT) to the analysis of *stochastic partial differential equations* (SPDEs), in particular to non-linear ones. Despite being apparently so far apart, these two frameworks have a lot in common and, probably, the most unexpected shared feature is the need of invoking *renormalization*.

For the sake of brevity, we shall not enter into the detail about AQFT and SPDEs in this introduction. Instead, we shall refer to the literature [13, 14, 15, 16, 26, 48] for what concerns AQFT, which consists of a mathematically rigorous approach to quantum field theory, and to Chapter 1 for a brief survey of SPDE theory. The aim of this introduction is solely to guide the reader through this manuscript in order to smoothen the reading.

Chapter 1 is devoted to recollecting some basic material about stochastic partial differential equations, starting from some motivating examples, presenting a brief survey of the theory of regularity structures and highlighting some notable technical results.

More into the detail, in Section 1.1 we discuss some examples motivating the mathematical interest in the field of SPDEs. Next we introduce a few technical tools, coming both from probability theory and from analysis aiming at a rigorous definition of *white noise*. This is a Gaussian random distributions which, in the cases we are going to consider, is the source of the randomness of the SPDEs.

Moving on, in Section 1.3 we discuss quite in some detail the technical difficulties one has to face when dealing with non-linear SPDEs run by a white noise. These hurdles are due to the very singular nature of this random noise and they call for *renormalization* in order for non-linear SPDEs to be meaningful, as we discuss in Section 1.3.

In Section 1.4 we briefly outline the theory of *regularity structure*, which contains a large part of the state of the art of the field and which provides a framework where to study a large class of SPDEs.

Finally, we underline that in Chapter 1 some additional contributions of the author are briefly discussed:

- in Section 1.2.2, in particular in the paragraph "**Excursus: A Microlocal Version of Young's Theorem**", the results of [28] are outlined: They consist of a microlocal version of the Young's product theorem for Hölder functions and distributions;

- in Section 1.4, in particular in the paragraph "**Excursus: The Reconstruction Theorem on Smooth Manifolds**", the results of [85] are presented: They concern the extension to a smooth manifold of the reconstruction theorem – one of the cornerstones of the theory of regularity structures, formulated in the framework of *coherent germs of distributions* recently introduced in [17].

The remaining Chapters, 2, 3 and 4, are devoted to the main contribution of this Ph.D. thesis, namely the *microlocal approach to SPDEs* introduced in [27]. This provides a novel framework for the perturbative analysis of a vast class of non-linear SPDEs. In particular, adapting techniques proper of AQFT, such as microlocal analysis and the theory of the scaling degree, it allows to deal with renormalization avoiding any ε-regularization procedures and subtraction of infinities. On the contrary, it allows the explicit construction of finite renormalization constants and the classification of the ambiguities arising as a consequence of the renormalization procedure.

In Chapter 2 one of the basic tools of the microlocal approach to SPDEs, namely the notion of *functional-valued distribution*, is introduced. In particular, we shall be interested in a specific class of these objects, whose microlocal behaviour, codified by the wave-front set of their functional derivatives, is determined by the distributional properties of the white noise. More into the detail, inspired by the algebraic approach to quantum field theory, in Section 2.1 we shall endow the space of functional-valued distributions over a smooth manifold M, denoted by $\mathcal{D}'_{\mathrm{C}}(M;\mathrm{Fun})$, with an algebra structure whose product is the pointwise one, yielding an algebra that we call $\mathcal{A}$.

At this level, this construction is purely deterministic and unaware of the white noise behaviour. Again inspired by the algebraic approach to quantum field theory, the stochastic nature of the functional-valued distributions in $\mathcal{A}$ is introduced by means of a *deformation* of the algebra product. This is constructed out of the linear part of the underlying SPDE and of the random noise playing the rôle of source of the equation, namely the white noise in the case in hand. This new product, which is not unique due to renormalization ambiguities, is dubbed $\cdot_Q$ and it yields the algebra $\mathcal{A}_{\cdot_Q}$.

As a matter of fact, in order to recover the white noise scenario, this deformation strategy requires a renormalization procedure since, *a priori*, this would give rise to ill-defined objects. In order to disentangle at first sight the deformation argument and the renormalization procedure, in Section 2.2 we discuss the deformation argument starting from a *smooth regularization* of the white noise scenario. This allows us to consider the deformation argument avoiding any renormalization issue.

The white noise scenario is postponed to Chapter 3, where the renormalization side is discussed quite in detail. As usual in the algebraic approach to quantum field theory, renormalization consists of an *extension* to the whole space of distributions which are everywhere defined but on a submanifold. We refer to Chapter 3 and to Appendix B for further details. In particular, in Section 3.1 we discuss the construction of the algebra $\mathcal{A}_{\cdot_Q}$. Within this deformed and renormalized algebra it is possible to compute, at any order in perturbation theory, the expectation values of the solution of a vast class of SPDEs.

The algebra $\mathcal{A}_{\cdot_Q}$ is not the only one constructed in Chapter 3: Indeed, in Section 3.2 we introduce an additional algebra, dubbed $\mathcal{A}_{\bullet_Q}$ endowed with a product $\bullet_Q$ which allows us to compute the multi-local n-point correlation functions of the perturbative solution at any order in perturbation theory.

Again in Chapter 3 we discuss the uniqueness properties of the aforementioned algebras. The results are that these algebras are not unique due to the presence of *renormalization ambiguities*, which are studied and classified in Section 3.3.

Chapter 4 is devoted to the application of the general machinery discussed in the previous two chapters to a specific example, namely the *stochastic quantization equation* represented by the so-called Φ^4_d model, where d is the dimension of the underlying manifold. In this chapter we make some explicit computations at first order in perturbation theory both for the expectation value of the solution and for the two-point correlation function.

In addition, in Section 4.3 we discuss an argument allowing to classify sub-critical cases for a generic Φ^k_d model, with $k \in \mathbb{N}$, from the point of view of renormalization. This is based on a graph argument as well as on a scaling degree one.

Finally, in Appendix A we recollect some basic notions and results of microlocal analysis and wave front set theory. They mainly serve as a useful reference throughout this whole manuscript.

In appendix B we recall the notion of scaling degree of a distribution and its main properties. In addition, in this appendix, in particular in Section B.4, we state and prove some technical results which are exploited in the main body of this thesis, *e.g.*, in Chapter 3.

Notation In the following, given a smooth manifold M, $\mathcal{D}(M)$, $\mathcal{S}(M)$ and $\mathcal{E}(M)$ shall denote the space of smooth compactly supported functions over M, the space of Schwartz functions over M (also called rapidly decreasing functions) and the space of smooth functions over M, respectively [61].

Moreover, we shall denote with a prime their topological dual spaces, namely the space of distributions $\mathcal{D}'(M)$, the space of tempered distributions $\mathcal{S}'(M)$ and the space of compactly supported distributions $\mathcal{E}'(M)$ respectively [61].

The symbol $\lesssim$ denotes an inequality up to a multiplicative constants.

Preliminaries

Chapter 1

Stochastic Partial Differential Equations

The aim of this chapter is twofold: On the one hand we collect some examples of stochastic partial differential equations (henceforth SPDEs), both linear and non-linear, arising from various fields of research, ranging from quantum field theory to interface dynamics and many others. On the other hand, we shall also discuss quite in some detail the problem associated with non-linear stochastic partial differential equations and the need for *renormalization* in order to give a meaning to this class of equations. Eventually, we shall also briefly discuss the state of the art of this field, in particular the *theory of regularity structures* [50]. We underline that this is not the only approach to this kind of equation but we limit ourselves to a brief outline of this theory for the sake of brevity. Other important approaches to these equations are the approach based on *paracontrolled distributions* [43, 47] and one based on renormalization group techniques [74].

Roughly speaking, a stochastic partial differential equation is an equation of the form

$$\mathcal{L}u = F(u, \xi)\,, \tag{1.0.1}$$

where $\mathcal{L}$ is a linear partial differential operator (tipically parabolic, elliptic or hyperbolic), ξ is a random input which is responsible for the stochastic nature of the equation and where F is tipically a non-linear operator in the variable u. We underline that the non-linearity F might also be function of derivatives of u, with the assumptions that these are of lower order of those of $\mathcal{L}$. For what concerns the random input ξ, we shall consider the case of the *white noise*, which is a centred Gaussian random distribution whose covariance is a Dirac delta distribution (white noise will be rigorously defined in the following), formally

$$\mathbb{E}[\xi(x)] = 0\,, \qquad \mathbb{E}[\xi(x)\xi(y)] = \delta(x, y)\,, \tag{1.0.2}$$

where $\mathbb{E}$ denotes the expectation value and where x, y may be space-time or only space points depending on the underlying setting. Nonetheless, also other kinds of random inputs might be considered. In this manuscript we shall be interested in scenarios where the dependence on the random noise ξ of the function F is affine. As above, also in this case one might consider more general cases with some minor differences.

1.1 Some Motivating Examples

In this section we shall discuss some examples arising from applications ranging from applied mathematics to quantum field theory and interface dynamics.

Generally speaking, SPDEs arise as a combination of PDEs and of randomness which are both ubiquitous tools used to model and describe mathematical and physical phenomena. On the one hand, PDEs have always been one of the main building blocks in the formulation of physical laws, as one can see thinking of heat diffusion, fluid and interface dynamics, just to cite some of them. Furthermore, the description of any physical phenomenon having *locality* among its features involves a PDE. On the other hand, randomness is an important tool when dealing with systems with underlying uncertainties due both to complicated or chaotic microscopic interactions or to an effective modeling of a phenomenon.

1.1.1 A First Example

To start with, we discuss the case of the stochastic heat equation, which is a linear SPDE which represents also the linear part of several non-linear SPDEs which are of interest for applications. The stochastic heat equation is given by

$$\partial_t u = \Delta u + \xi \,, \qquad u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R} \,, \tag{1.1.1}$$

where ∂_t denotes the partial derivative with respect to the time variable in $\mathbb{R}_+$, Δ denotes the Laplace operator on $\mathbb{R}^d$ and ξ the white noise, whose rigorous definition is postponed to the next section, see Definition 1.2.7. Following the notation of Equation 1.0.1, in this case $\mathcal{L} = \partial_t - \Delta$ and $F(u, \xi) = \xi$.

Remark 1.1.1: *The stochastic heat equation can be derived as the scaling limit of a microscopical model describing a polymer in a liquid, where randomness is due to the interaction between the bead of the polymer and liquid molecules [24, 39].*

The solution theory for Equation (1.1.1) can be discussed starting from the one of the heat equation, namely

$$\partial_t u = \Delta u \,, \qquad u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R} \,. \tag{1.1.2}$$

We recall that, given any bounded continuous initial condition $u_0 : \mathbb{R}^n \to \mathbb{R}$, there exists a unique solution $u(t, x)$ of Equation (1.1.2) such that $u(0, x) = u_0(x)$ for any $x \in \mathbb{R}^d$. In particular, this solution is

$$u(t, x) = e^{t\Delta} u_0(x) := \frac{\theta(t)}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{2t}} u_0(y)\, dy \,,$$

having introduced the notation $e^{t\Delta}$ for the solution map of the heat operator. We underline that $e^{t\Delta} u_0(x) = \int_{\mathbb{R}^n} P(t, x - y) u_0(y)\, dy$ where P is the fundamental solution of the heat operator, whose integral kernel on $\mathbb{R} \times \mathbb{R}^n$ is

$$P(t, x) = \frac{\theta(t)}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{2t}} \,, \tag{1.1.3}$$

with $\theta(t)$ denoting the Heaviside function. On account of this, we can move to the case of Equation (1.1.1) with a suitable deterministic initial condition u_0. From an analytic viewpoint and neglecting for the moment the stochastic behaviour of ξ, the only difference with respect to Equation (1.1.2) is represented by ξ, which can be seen as a source term, *a priori* of distributional nature. As usual in PDEs theory, the solution to this equation can be obtained again by means of the fundamental solution P of the heat operator, *i.e.*,

$$u = P * \xi + e^{t\Delta} u_0 \,, \tag{1.1.4}$$

where with $P * \xi$ we denote the convolution between the fundamental solution P and the white noise, which at the formal level of integral kernel is

$$(P * \xi)(t, x) := \int_{\mathbb{R}_+ \times \mathbb{R}^d} P(t - s, x - y)\xi(ds, dy) \,, \tag{1.1.5}$$

where we used the notation $\xi(ds, dy)$ to underline the *a priori* distributional nature of ξ.

Remark 1.1.2: *We observe that the convolution in Equation* (1.1.5) *is locally well defined on account of the smoothing and microlocal properties of P – we shall come back to these properties later on in this manuscript. We actually observe that the integral in Equation* (1.1.5) *might present some infrared divergences due the fact that we are considering an integral over $\mathbb{R}^d$ for what concerns the spatial variable. This divergent behaviour can be avoided by replacing $\mathbb{R}^d$ with, e.g., the d-dimensional torus $\mathbb{T}^d$. A different approach, which we shall implicitly assume throughout this section is that of considering a cut-off, namely instead of P we shall consider $P \cdot \chi$ with $\chi \in \mathcal{D}(\mathbb{R}^d)$ smooth and compactly supported function. This will guarantee the convergence of integrals such as the one of Equation* (1.1.5) *without spoiling the local behaviour we are interested in. We will come back to the cut-off function in Chapter 3.*

To conclude this heuristic survey of the stochastic heat equation we observe that the solution, in the sense of Equation (1.1.4), is unique. Indeed, the difference of any two solutions would be solution of Equation (1.1.2) with a vanishing initial condition. We observe in addition that the convolution $P * \xi$ is still a Gaussian random distribution although with a different covariance with respect to ξ. Roughly speaking, this is a consequence of the Gaussianity of ξ together with the fact that any *linear* combination of Gaussian random variables is still Gaussian.

Remark 1.1.3: *The previous example shows a more general fact, namely that solution theory for linear SPDEs is quite well-understood in the sense that it shares the same challenges of the classical counterpart, such as existence of fundamental solutions, locality or globality of the solutions, regularity properties and so on and so forth. In other words, at the level of* linear *SPDEs, the stochastic nature of the noise and, as a consequence, its highly irregular nature, do not come as a further conceptual layer of difficulty. This is no longer true when considering* non-linear *SPDEs, for the reason we are going to discuss in Sections 1.2 and 1.3.*

1.1.2 Non-Linear Examples

Before discussing the reasons for the greater difficulty in the analysis of non-linear SPDEs with respect to the linear scenario, we shall present some examples arising from applications in various fields of mathematics and physics showing how linear SPDEs are quite often insufficient to effectively model some interesting phenomena.

We shall only cite these examples and discuss them very briefly, giving references for further details.

- **KPZ Equation:** This equation, which was first proposed by Kardar, Parisi and Zhang [68], is given by the following parabolic equation on $\mathbb{R}_+ \times \mathbb{R}$,

$$\partial_t H = \partial_x^2 H + (\partial_x H)^2 + \xi\,, \tag{1.1.6}$$

denoting with $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$ the time and space variables, respectively. This equation arises as the scaling limit of models of interface growth, where each point of the interface randomly grows or drops over time. In particular, the KPZ equation arises from an asymmetric model where a growth is more likely than a fall [23, 31, 50]. As an example of such a model, one can think of an interface grown in the ocean out of a volcanic eruption [24].

- **Stochastic Quantization:** Stochastic quantization actually refers to a large class of SPDEs having their roots in Euclidean quantum field theory which was first introduced in [83]. Roughly speaking, if we consider a functional $\mathcal{H}(\varphi)$ of a configuration φ and if we interpret it as an Hamiltonian, the *stochastic quantization equation* associated with $\mathcal{H}$ is the following gradient flow perturbed by space-time white noise ξ, *i.e.*

$$\partial_t \varphi = -\frac{\delta \mathcal{H}(\varphi)}{\delta \varphi} + \xi\,, \tag{1.1.7}$$

where $\frac{\delta \mathcal{H}(\varphi)}{\delta \varphi}$ denotes the functional derivatives of $\mathcal{H}$ with respect to the configuration φ. We observe that if one considers the free Hamiltonian, *i.e.*, the Dirichlet form $\mathcal{H}(\varphi) = \frac{1}{2}\int (\nabla\varphi)^2\, dx$, since $\frac{\delta \mathcal{H}(\varphi)}{\delta \varphi} = -\Delta\varphi$ where Δ is the Laplace operator, it descends that Equation (1.1.7) reduces to the stochastic heat equation (1.1.1).

A particularly relevant case is the φ^4_d-equation, where d denotes the number of spatial dimensions. This is obtained out of the Hamiltonian

$$\mathcal{H}(\varphi) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla\varphi(x)|^2 + \frac{1}{4}\varphi^4(x) \right) dx\,,$$

yielding the following stochastic quantization equation

$$\partial_t \varphi = \Delta\varphi - \varphi^3 + \xi\,, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\,, \tag{1.1.8}$$

with tipically $d = 1, 2, 3$. In general, the interest on the stochastic quantization equation is justified by the notion of *invariant measure*. Indeed, given a Hamiltonian $\mathcal{H}(\varphi)$, the formal Gibbs measure

$$\frac{1}{Z} e^{-\mathcal{H}(\varphi)} D\varphi\,, \tag{1.1.9}$$

where Z is a "normalization constant" and where $D\varphi$ is the formal "Lebesgue measure" on the space of configurations, formally plays the rôle of the *invariant measure*[1] of Equation (1.1.7). We underline that the above constructions are purely formal since, among other issues, there exists no Lebesgue measure $D\varphi$ on an infinite dimensional space. In spite

[1]Roughly speaking, we say that a measure μ is *invariant* with respect to a flow such as the one of Equation (1.1.7) if, given a random initial condition whose probability law is μ, then the solution at any time t is distributed according to the same probability law.

of these difficulties, these measures are of relevance in *Euclidean Quantum Field Theory* since, in its *path integral* formulation, observables quantities are defined as expectation values with respect to such a measure. In particular, the aim of *constructive quantum field theory* is that of giving a precise meaning to these measures [63].

Coming back to the stochastic quantization equation (1.1.7), this provides an approach for the construction of such a measure. In particular, the basic idea is that of constructing the measure (1.1.9) by constructing the long-time solution of Equation (1.1.7) and by averaging its distribution over time. This approach proved to be successful in the cases of Equation (1.1.8) for $d \leq 3$ [6, 41, 42, 65, 77]. In recent paper [88], the authors analyse from a perturbative viewpoint this equation.

- **Non-Linear Parabolic Anderson Model:** This is the class of parabolic equations in $\mathbb{R}_+ \times \mathbb{R}^d$, with spatial dimension $d = 2, 3$, given by

$$\partial_t u = \Delta u + f(u)\zeta \,, \tag{1.1.10}$$

where f is a continuous function and where ζ is a space white noise, namely it is a white noise in its spatial entry and it is constant in time. This equation describes the motion of a massive particle through a random media. We observe that the property of being constant in time of the noise here is due to the assumption that the particle motion is much faster than the time scale of changing of the medium. For further details on the case $f(u) = u$ see [19]. A nice review of where the paracontrolled distributions approach is applied to the parabolic Anderson model is [47].

If one consider Equation (1.1.10) with a space-time white noise, one gets the so called *multiplicative stochastic heat equation*, which has a strong connection with the KPZ equation by means of the Hopf-Cole transformation [23].

- **Stochastic Complex Ginzburg-Landau Equation:** This is the equation,

$$\partial_t u = (i + \mu)\Delta u + \nu(1 - |u|^2)u + \xi \,, \qquad \mu > 0 \,, \quad \nu \in \mathbb{C} \,, \tag{1.1.11}$$

where the spatial dimension is $d \leq 3$, u is complex valued and ξ is the complex-valued space-time white noise [62]. This equation is of relevance in the description of several physical phenomena such as non-linear waves, phase transitions, superconductivity and superfluidity [3].

- **Dynamical sine-Gordon Equation:** This is the equation

$$\partial_t u = \Delta u + \sin(\beta u) + \xi \,, \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{T}^2 \,, \tag{1.1.12}$$

where $\mathbb{T}^2$ denotes the two-dimensional torus. This SPDE describes the dynamics of some two-dimensional systems exhibiting the *Berezinskii-Kosterlitz-Thouless* phase transition [8, 66, 71] such as a two-dimensional Coulomb gas and some condensed matter materials. In particular, here β^2 represents the inverse of the temperature.

- **Non-Linear Stochastic Wave Equation:** This is the equation

$$\partial_t^2 u - \Delta u + F(u) = \xi \,, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^d \,, \tag{1.1.13}$$

where F is a non-linear operator. This class of equations, for different non-linearities F and dimension d of the spatial variable has been studied in [2, 44, 45, 46, 82, 91].

Other interesting examples of non-linear SPDEs are the stochastic Navier-Stokes equation [37, 73], the stochastic Yang-Mills flow [20, 87] and the equation describing the random motion of a curve in a smooth manifold [11, 55], just to cite some of them.

Remark 1.1.4: *In the previous examples, except for the non-linear wave equations, we only discussed parabolic SPDEs. This is due to historical reasons since this class of SPDEs is the first one which has been considered. Nonetheless, as we have seen in the last example, also hyperbolic SPDEs are studied and the same can be said for the elliptic case, in particular from the point of view of stochastic quantization [1, 7, 41].*

1.2 Some Preliminaries

The aim of this section is to give a rigorous definition of white noise as a random distribution, codifying its properties from both the probabilistic and the analytical viewpoint. In particular, we shall introduce the notion of (generalized) Hölder functions together with some results coming from harmonic analysis. We shall use these results so to clarify why non-linear SPDEs are much more difficult than their linear counterpart and at the same time, why *renormalization* is needed in order to deal with such equations.

Remark 1.2.1: *Throughout this section and in the discussion of the analytical tools we shall refer to the case of a non-linear* parabolic *SPDEs, namely a non-linear SPDEs whose linear part $\mathcal{L}$, in the sense of Equation* (1.0.1), *is a parabolic operator.*
This will reverberate in the definition of the spaces of Hölder distributions $\mathcal{C}^\alpha$ since these are characterized by means of a scaling condition which in the parabolic scenario is defined starting from the parabolic scaling. More precisely, if u is a solution of the heat equation (1.1.2), *then also $\widetilde{u}(t,x) := u(\lambda^{-2}t, \lambda^{-1}x)$, with $\lambda > 0$, is also a solution to the same equation.*

This leads to the following scaling transformation: let $\varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^d)$ be a smooth and compactly supported function, $\lambda \in (0,1)$ and let moreover $z \equiv (t,x) \in \mathbb{R}\times\mathbb{R}^d$ be generic point. We will henceforth write

$$\varphi_z^\lambda(s,y) := \lambda^{-2-d}\varphi(\lambda^{-2}(s-t), \lambda^{-1}(y-x))\,, \qquad (s,y) \in \mathbb{R}\times\mathbb{R}^d\,. \tag{1.2.1}$$

Furthermore, considering the parabolic scaling on $\mathbb{R}\times\mathbb{R}^d$, we introduce the parabolic distance *between points $z = (t,x)$ and $z' = (t',x')$ in $\mathbb{R}\times\mathbb{R}^d$ as*

$$\|z - z'\|_{\mathfrak{p}} := |t - t'|^{\frac{1}{2}} + \sum_{j=1}^{d} |x_j - x'_j|\,.$$

This also yields to the notion of parabolic degree of a polynomial: given a multi-index $k = (k_0, \ldots, k_d)$, we define the monomial z^k, with $z = (t,x)$, in the usual way and we define its parabolic degree as $|k|_{\mathfrak{p}} := 2k_0 + \sum_{j=1}^d k_j$.

We underline that the same arguments we are going to use, mutatis mutandis, *hold true for other scenarios such as the elliptic one.*

1.2.1 Probabilistic Tools: Random Distributions

Before giving the rigorous definition of the white noise, we need some preliminary notions from probability theory. As a starting point, we introduce the notion of *random distribution*.

Definition 1.2.2: *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, namely Ω is a set, $\mathcal{F}$ is a σ-algebra built out of subsets of Ω while $\mathbb{P}$ is a probability measure over $\mathcal{F}$. Let $L^2(\Omega, \mathcal{F}, \mathbb{P})$ be the space of square integrable random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.* A random distribution *η is a linear and continuous map $\eta : \mathcal{D}(\mathbb{R} \times \mathbb{R}^d) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, we say that a random distribution η satisfies the* equivalence of moments *if for any $p \geq 1$ there exists a constant K_p such that*

$$\mathbb{E}|\eta(f)|^{2p} \leq K_p(\mathbb{E}|\eta(f)|^2)^p\,, \qquad \forall\, f \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)\,.$$

Remark 1.2.3: *We observe that the notion of random distribution is a superposition of the notions of distribution and of random variable. In particular, it can be seen as a distribution taking values in the space of square summable random variables, namely $\eta \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d; L^2(\Omega, \mathcal{F}, \mathbb{P}))$.*

Analogously one can view a random distribution η as a random variable taking values in the space of distributions $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$, namely one could view η as a $\mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$-indexed stochastic process.

Remark 1.2.4: *In particular, we observe that the condition of equivalence of moments is satisfied if η is a Gaussian random distribution, namely if for any $f \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$, the random variable $\eta(f)$ is Gaussian.*

Definition 1.2.5: *Let $\eta \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d; L^2(\Omega, \mathcal{F}, \mathbb{P}))$ be a random distribution as per Definition 1.2.2 and let $C \in \mathcal{D}'((\mathbb{R} \times \mathbb{R}^d) \times (\mathbb{R} \times \mathbb{R}^d))$ be a (bi-)distribution. We say that η has covariance C if for any $f, g \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$ it holds*

$$\mathbb{E}[\eta(f)\eta(g)] = \langle C * f, g \rangle_{L^2(\mathbb{R} \times \mathbb{R}^d)}\,,$$

where $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R} \times \mathbb{R}^d)}$ denotes the $L^2(\mathbb{R} \times \mathbb{R}^d, d^{d+1}x)$ inner product while $C * f$ denotes the convolution between the distribution C and the test-function f.

Finally, we recall the notion of *version* of a random distribution, which is borrowed from the framework of stochastic processes.

Definition 1.2.6: *With the above notation, let $\eta \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d; L^2(\Omega, \mathcal{F}, \mathbb{P}))$ be a random distribution, we say that $\widetilde{\eta} \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d; L^2(\Omega, \mathcal{F}, \mathbb{P}))$ is a* version *of η if for any test-function $f \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$ it holds $\widetilde{\eta}(f) = \eta(f)$, $\mathbb{P}$-almost surely.*

We have now all the ingredients for defining the white noise, making rigorous the heuristic definition we gave in Equation (1.0.2) and for establishing its analytical properties.

Definition 1.2.7: *We define the (space-time)* ***white noise*** *ξ as the real-valued centred Gaussian random distribution with covariance given by the Dirac delta distribution δ.*

Remark 1.2.8: *In other words, ξ is the Gaussian random distribution such that*

$$\mathbb{E}[\xi(f)] = 0\,, \qquad \mathbb{E}[\xi(f)\xi(g)] = \langle f, g \rangle_{L^2}\,, \tag{1.2.2}$$

for any test-functions $f, g \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$.

1.2.2 Analytical Tools: Hölder Spaces

As a starting point we introduce the class of Hölder spaces, both of positive and of negative regularity. We recall that the former are spaces of continuous functions having suitable regularity properties while the latter are spaces of distributions.

Definition 1.2.9: *Let $\alpha \geq 0$. We say that a function $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ belongs to $\mathcal{C}^\alpha$ if for any $z \in \mathbb{R} \times \mathbb{R}^d$ there exists a polynomial P_z of (parabolic) degree at most α such that for any compact set $K \subset \mathbb{R} \times \mathbb{R}^d$*

$$\sup_{\substack{z,z' \in K \\ z \neq z'}} \frac{|f(z) - P_z(z')|}{\|z - z'\|_{\mathfrak{p}}} < \infty\,. \tag{1.2.3}$$

Moreover, we define

$$\|f\|_{\mathcal{C}^\alpha(K)} := \max\left\{ \|f\|_{C^{\lfloor \alpha \rfloor}(K)}\,,\, \sup_{x,y \in K} \frac{|f(y) - P_x(y)|}{|x-y|^\alpha} \right\}.$$

Remark 1.2.10: *We observe that P_z appearing in the above definition is actually the Taylor polynomial of degree $\lfloor \alpha \rfloor$ of f centred at z, where $\lfloor \alpha \rfloor$ denotes the biggest integer number smaller than α. We observe in addition that in the case $\alpha \in (0,1)$, the above definition restricts to the usual notion of Hölder functions. Indeed in such a scenario Equation* (1.2.3) *boils down to*

$$|f(z) - f(z')| \lesssim \|z - z'\|_{\mathfrak{p}}\,,$$

locally uniformly with respect to z and z'.

Remark 1.2.11: *We observe that there are some important results on $\mathcal{C}^\alpha$ spaces, such as Schauder estimates, which fail to hold when $\alpha \in \mathbb{N}$. As a consequence, since this will not let us lose in generality, we shall assume $\alpha \notin \mathbb{N}$.*

In order to introduce the notion of Hölder spaces of negative regularity, we need to consider the framework of distributions. To this end, for $\alpha < 0$ a natural choice is the definition of the spaces $\mathcal{C}^\alpha$ as the class of (parabolically scaled) *Besov spaces* $B^\alpha_{\infty,\infty}$ [4]. The naturality of this choice relies on the fact that for $\alpha \in (0,1)$ the Holder spaces $\mathcal{C}^\alpha$ and $B^\alpha_{\infty,\infty}$ actually coincide[2].

There are several characterization of this class of Besov spaces, *e.g.*, the one relying on the notion of Littlewood-Paley partition of unity [4].

In this manuscript, we shall use a simple definition which is convenient for our purposes, since it is more in the spirit of distribution theory and it presents some analogies with the notion of scaling degree, see Appendix B.

Remark 1.2.12: *As a premise to the following definition, we need to fix some notations. Let $r \in \mathbb{N}$, we call $\mathcal{B}_r$ the space of smooth and compactly supported functions f whose support is contained in the ball $B(0,1) \subset \mathbb{R} \times \mathbb{R}^d$ of center the origin and (parabolic) radius 1 such that $\|f\|_{C^r} \leq 1$, where $\|\cdot\|_{C^r}$ denotes the norm on the space C^r, namely*

$$\|f\|_{C^r} := \sup_{\substack{\alpha \\ |\alpha| \leq r}} \sup_{x \in \mathbb{R}^{d+1}} |\partial^\alpha f(x)|\,,$$

where α here is a multi-index.

Definition 1.2.13: *Let $\alpha < 0$. We define $\mathcal{C}^\alpha$ the space of distributions $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$ such that for any compact set $K \subset \mathbb{R} \times \mathbb{R}^d$ it holds*

$$\|u\|_{\mathcal{C}^\alpha(K)} := \sup_{z \in K} \sup_{\substack{f \in \mathcal{B}_r \\ \lambda \in (0,1]}} \frac{|u(f^\lambda_z)|}{\lambda^\alpha} < \infty\,, \tag{1.2.4}$$

[2]Some slight modifications are required if one want to consider the case $\alpha \geq 1$, yielding the notion of *Zygmund spaces*, but we shall not enter into these details [86].

where $r = \lceil -\alpha \rceil$ is the smallest integer greater than $-\alpha$ and where we adopted the notation of Remark 1.2.1 for what concerns the scaling transformation of f.

Having introduced Hölder spaces, we can now recall some important results concerning the product of Hölder functions and the convolutions between the fundamental solutions of the heat operator and a Hölder distribution. On account of the discussion of Section 1.1 and of the characterization of the white noise as per Definition 1.2.7, these two operations are the main ingredients for the discussion of the solution theory for non-linear stochastic SPDEs.

We start by considering the product of Hölder functions/distributions. We observe that in general one may expect this operation to be ill-defined since it might involves the product of distributions, which it is known not to be an *a priori* well-defined operation [61, 80, 81].

The multiplication results for Hölder spaces is also known as *Young product theorem* [92]. The following result, whose proof is based on techniques borrowed from paradifferential calculus and Bony paraproducts [9], is a particular case of the general theorem [4, Thm. 2.52].

Theorem 1.2.14: *Let $\mathcal{E}(\mathbb{R} \times \mathbb{R}^d) \times \mathcal{E}(\mathbb{R} \times \mathbb{R}^d) \ni (f, g) \mapsto f \cdot g$ be the pointwise product of smooth functions. This map extends to a continuous and bilinear map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to the space of distributions $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$ if and only if $\alpha + \beta > 0$. Moreover, in such scenario, the image of the product lies in $\mathcal{C}^{\alpha \wedge \beta}$, where $\alpha \wedge \beta$ denotes the minimum between α and β.*

We now move to *Schauder estimates*, which provide a regularity result for the convolution between suitable kernels with Hölder functions and distributions. In particular, at this level we shall only consider Schauder estimates for the heat kernel as per Equation (1.1.3), recalling also the argument of Remark 1.1.2 for what concerns infra-red divergences. For further references on this topic we refer to [72, 90]. For analogous results in the elliptic scenario we refer to [40].

In particular, for the heat kernel the result is the following.

Theorem 1.2.15: *Let P be the fundamental solution of the heat operator on $\mathbb{R}_+ \times \mathbb{R}^d$ with integral kernel as per Equation* (1.1.3) *and let $f \in \mathcal{C}^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)$. We define*

$$u(t, x) := \int_{\mathbb{R}_+ \times \mathbb{R}^d} P(t - s, x - y) f(s, y) \, ds \, dy \,,$$

with a distributional interpretation if $\alpha < 0$. Then

$$\|u\|_{\mathcal{C}^{\alpha+2}} \lesssim \|f\|_{\mathcal{C}^\alpha} \,.$$

Remark 1.2.16: *We recall that on account of Remark 1.2.11 we are considering Hölder spaces with $\alpha \notin \mathbb{N}$, otherwise results such as Theorem 1.2.15 do not hold true.*

Remark 1.2.17: *Roughly speaking, Schauder estimates entails that the heat kernel P is a regularizing kernel. In particular, it improves Hölder regularity by two derivatives. A similar improvement is obtained in the elliptic framework through the elliptic regularity theorem [40].*

Excursus: A Microlocal Version of Young's Theorem In a recent paper [28], jointly written with C. Dappiaggi and F. Sclavi, we analysed the product between Hölder distributions of suitable regularity, namely the scenario described by Young's product Theorem 1.2.14, from the viewpoint of *microlocal analysis*. In this paragraph we briefly review these results without getting into the details, since this goes beyond the scope of this manuscript. Moreover, from the point of view of the discussion about the problem with non-linearities in SPDEs theory, one may skip this paragraph. We refer to [28] for all the details.

Generally speaking, the product of two distributions is ill-defined and this issue has been thoroughly studied in the literature, mainly through the notion of wave front set [61], see also Appendix A for a brief survey. In particular, this provides some sufficient condition on the singular structure of the underlying distributions in order for their product to be well-defined.

In spite of this, in order to fit with the scenario of multiplying two Hölder distributions, we actually use the notion of *Sobolev wave-front set*, which is a refined version of the usual smooth wave-front set which was introduced in [60] and which has the merit of catching a more specific singular behaviour with respect to the standard smooth wave-front set (see also [67] for a review). In particular, Sobolev wave-front set appears to be more fine-tuned in order to codify the Hölder regularity in a microlocal sense.

This intermingle between these two approaches, namely microlocal analysis and Hölder classes, results in a *case by case* improvement of Young's product Theorem 1.2.14, giving sufficient conditions allowing the product of Hölder distributions in scenarios where the classical Young's product Theorem 1.2.14 fails.

One of the main result is the following theorem, where $\mathrm{singsupp}(f)$ denotes the singular support of a distribution $f \in \mathcal{D}'$ (see Remark A.0.3).
We recall that given a compact set $K \subset \mathbb{R}^d$ and $R > 0$, the R-enlargement of K is defined as

$$\overline{K}_R := \{x \in \mathbb{R}^d \mid |x - y| \leq R\,,\ \text{for some } y \in K\}\,.$$

In addition, let $g \in \mathcal{C}^\beta(\mathbb{R}^d)$ with $\beta < 0$, we define

$$\beta_g^* := \sup\{\gamma \leq 0 : g \in B^\gamma_{p,\infty}(\mathbb{R}^d) \text{ for } p \in [2, \infty]\}\,, \tag{1.2.5}$$

where $B^\gamma_{p,\infty}$ denotes the class of *Besov spaces* of suitable parameters [4, 28]. We shall not enter into detail about these spaces referring to the aforementioned literature for their definition. We only observe that, being $g \in \mathcal{C}^\beta(\mathbb{R}^d) \subset B^\beta_{\infty,\infty}$, β^* is always finite.

Theorem 1.2.18: *[28, Thm. 33] Let $f \in \mathcal{C}^\alpha_{\mathrm{loc}}(\mathbb{R}^d)$ and $g \in \mathcal{C}^\beta_{\mathrm{loc}}(\mathbb{R}^d)$ with $\alpha > 0$ and $\beta < 0$. If singsupp$(f) \cap$ singsupp$(g) \neq \emptyset$ and*

$$\alpha + \beta_g^* > 0\,, \tag{1.2.6}$$

there exists the product $f \cdot g \in \mathcal{C}^\beta_{\mathrm{loc}}(\mathbb{R}^d)$. Moreover for any compact set $K \subset \mathbb{R}^d$, one has

$$\|f \cdot g\|_{\mathcal{C}^\beta(K)} \lesssim \|f\|_{\mathcal{C}^\alpha(\overline{K}_1)} \|g\|_{\mathcal{C}^\beta(\overline{K}_1)}\,, \tag{1.2.7}$$

where $\overline{K}_1$ denotes the 1*-enlargement of K.*

Remark 1.2.19: *In Theorem 1.2.18 with canonical product we mean a product defined in the sense of [61, Thm. 8.2.10], namely defined as the pull-back through the diagonal map of the tensor product of the underlying distributions (see [28] for a more in detail discussion on this point).*

To better grasp why Theorem 1.2.18 allows the product of distributions in cases where Theorem 1.2.14 fails, we consider the following example. Further examples of this kind can be found in [28].

Example 1.2.20: *Let $f \in \mathcal{C}^d(\mathbb{R}^d)$ be such that* $\mathrm{singsupp}(f) = \{0\}$ *and let $\delta_0 \in \mathcal{C}^{-d}(\mathbb{R}^d)$ be the Dirac delta distribution centred at the origin. On the one hand, we observe that Theorem*

1.2.14 does not apply since here $\alpha + \beta = 0$. On the other hand, the hypothesis of Theorem 1.2.18 are met since Equation (1.2.6) boils down to

$$d - \frac{d}{2} = \frac{d}{2} > 0\,,$$

where we exploited $\mathrm{ord}(\delta_0) = 0$. By Theorem 1.2.18, the product $f \cdot \delta_0 \in \mathcal{D}'(\mathbb{R}^d)$ exists. Moreover, for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, it is defined by $(f\delta_0)(\varphi) = \delta_0(f\varphi)$, yielding $f\delta_0 = f(0)\delta_0$. Notice that we could have also picked $f \in \mathcal{C}^{\frac{d}{2}+\varepsilon}(\mathbb{R}^d)$ for any $\varepsilon > 0$ without hindering the existence of the product $f \cdot \delta_0$. We further observe that is this case, with the notations of Theorem 1.2.14, $\alpha + \beta < 0$.

We observe that Theorem 1.2.18 requires the Hölder regularity α of one of the two factors to be positive. In [28] we also discuss some scenarios where the product can be canonically constructed also if both α and β are negative. This is achieved through an extension procedure making use of the notion of *scaling degree*, which we briefly outline in Appendix B, in particular see Theorem B.2.1.

Theorem 1.2.21: *[28, Thm. 38] Let $f \in \mathcal{C}^{\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$ and $g \in \mathcal{C}^{\beta}_{\mathrm{loc}}(\mathbb{R}^d)$, with $\alpha, \beta \in \mathbb{R}$. Assume that $\mathrm{singsupp}(f) \cap \mathrm{singsupp}(g) = \{x\}$ with $x \in \mathbb{R}^d$. Then*

(i) If $\alpha > 0$ and $\beta \in (-d, 0)$ are such that

$$\alpha + \beta^*_g \leq 0\,,$$

there exists a unique extension $\widetilde{f \cdot g} \in \mathcal{C}^{\beta}_{\mathrm{loc}}(\mathbb{R}^d)$ of $f \cdot g \in \mathcal{C}^{\beta}_{\mathrm{loc}}(\mathbb{R}^d \setminus \{x\})$ which preserves the scaling degree;

(ii) If $\alpha < 0$ and $\beta < 0$, there exists an extension $\widetilde{f \cdot g} \in \mathcal{C}^{\alpha+\beta}_{\mathrm{loc}}(\mathbb{R}^d)$ of $f \cdot g \in \mathcal{C}^{\alpha+\beta}_{\mathrm{loc}}(\mathbb{R}^d \setminus \{x\})$ which preserves the scaling degree. Moreover, if $\alpha + \beta > -d$, such extension is unique.

For an example of application of Theorem 1.2.21 we refer to [28, Example 39].

Remark 1.2.22: *In the previous theorem we assumed the intersection of the singular supports of f and g to be a single point x. Nonetheless, the above result can be generalized straightforwardly to more complicated scenarios [28].*

1.3 The Problem with Non-Linearities

In order to understand the hurdle one has to face in order to discuss non-linear SPDEs, we first need to clarify the connection between white noise and Hölder spaces of functions and distributions. This is codified by a "Kolmogorov like" theorem [50, 78].

Theorem 1.3.1: *Let $\alpha < 0$ and let η be a random distribution satisfying the equivalence of moments condition as per Definition 1.2.2 and such that for any $z \in \mathbb{R} \times \mathbb{R}^d$*

$$\mathbb{E}[|\eta(f^{\lambda}_{z})|^2] \lesssim \lambda^{2\alpha}\,, \tag{1.3.1}$$

uniformly for $\lambda \in (0, 1]$ and $f \in \mathcal{B}_r$, with $r = \lceil -\alpha \rceil$. Then for any $\varepsilon > 0$ there exists a version $\widetilde{\eta}$ of η which is a $\mathcal{C}^{\alpha-\varepsilon}$-valued random variable.

Remark 1.3.2: *This theorem entails that whenever a random distribution satisfies suitable scaling properties, then from the distributional viewpoint it can be characterized as a Hölder distribution with negative regularity. In the following we shall omit the terminology* version *and we shall not distinguish between versions of a random distribution.*

Remark 1.3.3: *For what concerns the space-time white noise ξ, one can apply Theorem 1.3.1 by exploiting the scaling behaviour of the L^2-norm. In particular, recalling Equation* (1.2.2) *as well as Equation* (1.2.1)*, it holds, for any $z \in \mathbb{R} \times \mathbb{R}^d$ and $f \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$,*

$$\mathbb{E}[|\xi(f_z^\lambda)|^2] = \|f_z^\lambda\|_{L^2} = \lambda^{-d-2}\|f\|_{L^2}\,,$$

uniformly for $\lambda \in (0,1]$. By comparison with Equation (1.3.1)*, Theorem 1.3.1 entails that for any $\varepsilon > 0$, $\xi \in \mathcal{C}^{-\frac{d}{2}-1-\varepsilon}(\mathbb{R} \times \mathbb{R}^d)$.*

This concludes the discussion regarding the definition and the properties of the white noise as a random distribution. We are now in position of discussing quite in some detail the solution theory for non-linear SPDEs.

Solution Theory for Non-Linear SPDEs When dealing with a non-linear scenario, the Duhamel approach adopted for the linear case in Section 1.1.1 and culminating into Equation (1.1.4) becomes a *fixed point* argument.

To illustrate it, we consider for simplicity the Φ^4_d-equation as per Equation (1.1.8) with one spatial dimension, *i.e.*, $d = 1$. Furthermore, again for simplicity, we shall consider the case of zero initial condition and we shall consider as a domain $\mathbb{R}_+ \times \mathbb{S}^1$, with $\mathbb{S}^1$ a one-dimensional circle. Hence, by writing Equation (1.1.8) in its mild form, *i.e.*, by means of the fundamental solution $P = (\partial_t - \Delta)^{-1}$ of the heat operator as in Equation (1.3.2), we have at the level of integral kernel

$$\begin{aligned}\varphi(t,x) = &\int_{\mathbb{R}_+}\int_{\mathbb{S}^1} P(t-s,y-x)\xi(s,y)\,ds\,dy \\ &- \int_{\mathbb{R}_+}\int_{\mathbb{S}^1} P(t-s,y-x)\varphi^3(s,y)\,ds\,dy\,.\end{aligned} \tag{1.3.2}$$

We set

$$\overset{\bullet}{|} := \int_{\mathbb{R}_+}\int_{\mathbb{S}^1} P(t-s,y-x)\xi(s,y)\,ds\,dy\,, \tag{1.3.3}$$

where we adopted a convenient graphical notation, which we shall use extensively in the following, where a line represents a *propagator* P and the bullet on top of it the white noise ξ. On account of Remark 1.3.3 and of Schauder estimates as per Theorem 1.2.15, we see that $\overset{\bullet}{|} \in \mathcal{C}^{\frac{1}{2}-\varepsilon}(\mathbb{R}_+ \times \mathbb{S}^1)$ for any $\varepsilon > 0$. $\overset{\bullet}{|}$ is also called *stochastic convolution*.

It follows that, to consider Equation (1.3.2) as a fixed point problem, the natural space to consider is $\mathcal{C}^{\frac{1}{2}-\varepsilon}(\mathbb{R}_+ \times \mathbb{S}^1)$ for any $\varepsilon > 0$. In particular, we observe that since we are considering the scenario with spatial dimension $d = 1$, the non-linearity φ^3 appearing in Equation (1.3.2) is harmless on account of Young's product Theorem 1.2.14.

As a result, for any but fixed realization ξ of the white noise, there exists a "time" $T(\xi) > 0$

such that the operator

$$\Psi : \varphi \mapsto \int_{\mathbb{R}_+} \int_{\mathbb{S}^1} P(t-s, y-x)\xi(s,y)\, ds\, dy \\ - \int_{\mathbb{R}_+} \int_{\mathbb{S}^1} P(t-s, y-x)\varphi^3(s,y)\, ds\, dy\,, \tag{1.3.4}$$

is a contraction on a bounded balls of the space $\mathcal{C}^{\frac{1}{2}-\varepsilon}([0, T(\xi)] \times \mathbb{S}^1)$ for any $\varepsilon > 0$ [22].

Remark 1.3.4: *We observe that the second term on the right hand side of Equation* (1.3.2), *namely*

$$v(t,x) := \varphi(t,x) - \overset{\bullet}{|}(t,x) = -\int_{\mathbb{R}_+} \int_{\mathbb{S}^1} P(t-s, y-x)\varphi^3(s,y)\, ds\, dy\,,$$

is more regular than φ itself, since $v \in \mathcal{C}^{\frac{5}{2}-\varepsilon}([0, T(\xi)] \times \mathbb{S}^1)$, due to Schauder estimates.

Remark 1.3.5: *We further observe that the above results are independent of the relative minus sign of the non-linearity φ^3 in Equation* (1.1.8). *This minus sign is relevant when studying properties such as the absence of divergences at finite time of the solution or when replacing the circle $\mathbb{S}^1$ with $\mathbb{R}$.*

The above approach, which works for the Φ^4_1 model, fails when the number of spatial dimensions is $d \geq 2$. The main issue is the following: on account of Remark 1.3.3 and of Schauder estimates as per Theorem 1.2.15, it holds that $\overset{\bullet}{|} \in \mathcal{C}^{\frac{d-2}{2}-\varepsilon}$ for any $\varepsilon > 0$. In particular, whenever $d \geq 2$, the Hölder regularity of the stochastic convolution is strictly negative. Hence, on account of Theorem 1.2.14 there is no canonical way of constructing the map $\varphi \mapsto \varphi^3$ for $\varphi = \overset{\bullet}{|}$.

This hurdle appears, *e.g.*, when running a *Picard iteration*, namely when studying the behaviour of the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ iteratively defined as

$$\varphi_{n+1} := \Psi(\varphi_n)\,, \qquad \forall\, n \in \mathbb{N}\,,$$

setting for simplicity $\varphi_0 = 0$ and where Ψ is the fixed point operator introduced in Equation (1.3.4). The first step of the Picard iteration yields $\varphi_1 = \overset{\bullet}{|} \in \mathcal{C}^{\frac{d-2}{2}-\varepsilon}$ for any $\varepsilon > 0$. The problem arises already at the second iteration: indeed, in order to evaluate φ_2, one needs to compute $\Psi(\overset{\bullet}{|})$, which involves $\overset{\bullet}{|}^3$ which we have seen being ill-defined by Theorem 1.2.14. We observe that the higher the spatial dimension d is, the higher the singularity of the equation, since the white noise ξ becomes more and more singular.

Remark 1.3.6: *The take home message of this argument is that when increasing the spatial dimension d not only existence and uniqueness of the solution are difficult to discuss, but also the very notion of solution is somewhat problematic. As we outlined, this is mainly due to the problem of considering the product of distributions which cannot a priori be multiplied. As we will see in the following,* renormalization *will be the tool to overcome this hurdle.*

Wick Powers In this paragraph we shall briefly recall the construction of *Wick powers* without giving all the details, for which we refer to [76, 79, 51].
In a sense, the above discussion was deterministic since we considered any but fixed realization of the white noise ξ and this way we met the analytical difficulties discussed above.

Some of these difficulties can be dealt with by exploiting some probabilistic features of the random distributions under investigations, yielding the construction of the so-called *Wick powers*.

We recall that denoting with ξ the space-time white noise on $\mathbb{R}^{d+1}$ and with $L^2(\xi)$ the space of square summable random variables which are measurable with respect to the σ-algebra generated by ξ, one can introduce a decomposition of $L^2(\xi)$ as

$$L^2(\xi) = \bigoplus_{\ell \geq 0} \mathcal{H}^\ell(\xi)\,,$$

where $\mathcal{H}^0$ contains the constants, $\mathcal{H}^1$ random variables of the form $\xi(\psi)$, with $\psi \in L^2(\mathbb{R} \times \mathbb{R}^d)$ and where $\mathcal{H}^m$, for $m \in \mathbb{N}$, contains generalised Hermite polynomials of the elements of $\mathcal{H}^1$ [76, 79].

The elements of $\mathcal{H}^m$ can be represented by square-summable kernels of m variables. This representation is unique if one imposes that such kernels are symmetric with respect to permutations of their arguments, namely there is a surjective map $I^{(m)} : L^2(\mathbb{R} \times \mathbb{R}^d)^{\otimes m} \to \mathcal{H}^m$ such that $I^{(m)}(A) = I^{(m)}(B)$ if the symmetrisations of A and B coincide, with $A, B \in L^2(\mathbb{R} \times \mathbb{R}^d)^{\otimes m}$.

As we have seen before, we are particularly interested in computing the powers of $\dot{|}$ or more generally of a random distribution $\eta = K * \xi$, with ξ the space-time white noise and K a suitable kernel, such as the heat one.

To start with, we consider the simpler case where K is a smooth kernel: in such a scenario, η is a random functions and its n-th power can be represented as

$$\eta^n(\varphi) = \sum_{\substack{j \in \mathbb{N} \\ 2j < n}} P_{j,n} C^j I^{(n-2j)}(K_\varphi^{(n-2j)})\,, \tag{1.3.5}$$

where $P_{j,n}$ are some combinatorial coefficients, where we introduced[3]

$$K_\varphi^{(\ell)}(z_1, \ldots, z_\ell) := \int_{\mathbb{R} \times \mathbb{R}^d} K(z - z_1) \ldots K(z - z_\ell)\varphi(z)\, dz\,,$$

and where $C := \int_{\mathbb{R} \times \mathbb{R}^d} K^2(z)\, dz$.

We are now in position to introduce Wick powers in the particular case where K is the heat kernel P introduced above, namely when $\eta = \dot{|}$.

In particular, we define the n-th *Wick power* $:\eta^n:$ as the random distribution one gets from Equation (1.3.5) by considering the higher order term, namely

$$:\eta^n:(\varphi) = I^{(n)}(K_\varphi^{(n)})\,, \qquad \varphi \in L^2(\mathbb{R} \times \mathbb{R}^d)\,.$$

Wick powers have some nice regularity properties such as the next proposition, which is a particular case of a more general result [32, 51].

Proposition 1.3.7: *Let P be the heat kernel on $\mathbb{R} \times \mathbb{R}^d$ and let $\eta = P * \xi$ be the random distribution introduced above. The Wick power $:\eta^n:$ is well-defined as a random distribution and it belongs almost surely to $\mathcal{C}^\alpha$ for any $\alpha < (2-d)\frac{n}{2}$.*

[3]Here we are considering a cut-off version of K which is compactly supported. Such hypothesis is non-restrictive, *e.g.*, in the case of the heat kernel P since one can always decompose P as $P = K + R$ with K a compactly supported term and R a smooth remainder.

Remark 1.3.8: *We observe that the above result is stable with respect to regularizations [51]. In particular, if one considers a regularized version P_ε of the kernel P and the associated family of Wick powers $:\eta^n_\varepsilon:$, then for $\varepsilon \to 0^+$ one can prove convergence in probability of $:\eta^n_\varepsilon:$ to $:\eta^n:$ in $\mathcal{C}^\alpha$ with the superscripts α as in Proposition 1.3.7. Moreover, for any $\varepsilon > 0$ one can give an explicit formula relating the regularized Wick power $:\eta^n_\varepsilon:$ with η^n_ε, which we underline is well-defined due to the regularization. This formula is, with $z \in \mathbb{R}^{d+1}$,*

$$:\eta^n_\varepsilon:(z) = H_n(\eta^n_\varepsilon, C_\varepsilon)\,,$$

where $H_n(\cdot, C)$ are the rescaled Hermite polynomials, *which are related to the standard Hermite polynomials $H_n(C)$ through the relation $H_n(t,C) = C^{\frac{n}{2}} H_n(C^{-\frac{1}{2}}t)$ and where $C_\varepsilon := \int_{\mathbb{R}\times\mathbb{R}^d} P^2_\varepsilon(z)\,dz$.*

1.3.1 The Da Prato-Debussche Argument

In the particular case where the spatial dimension is $d = 2$, the renormalization as well as the fixed point argument for the stochastic quantization equation (1.1.8) has been discussed in [29] with an argument known as the *Da Prato-Debussche argument*. In this section we shall discuss such an argument for the 2-dimensional scenario and we shall also discuss why it fails when the spatial dimension increases to $d \geq 3$.

The starting point consists of considering an ε-regularized version of the white noise ξ_ε. This is achieved by introducing a *mollifier* $\rho \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^d)$, namely a smooth and compactly supported function whose integral is normalized, *i.e.*, $\int_{\mathbb{R}\times\mathbb{R}^d} \rho = 1$. Then we set

$$\xi_\varepsilon(t,x) := (\rho^\varepsilon * \xi)(t,x)\,, \tag{1.3.6}$$

where $*$ here denotes the convolution between ρ^ε, which is a (parabolically) scaling of ρ according to Equation (1.2.1), and the random distribution ξ. We underline that on account of standard results in distribution theory ξ_ε is a smooth function. Moreover, we observe that in a distributional sense $\xi_\varepsilon \to \xi$ for $\varepsilon \to 0^+$.

We shall now consider the stochastic quantization equation (1.1.8) having the regularized noise ξ_ε as a source, namely

$$\partial_t\varphi_\varepsilon = \Delta\varphi_\varepsilon - \varphi^3_\varepsilon + \xi_\varepsilon\,.$$

As long as $\varepsilon > 0$, the above equation is non-singular being ξ_ε a smooth function. Clearly, we are interested in the limit for $\varepsilon \to 0^+$ of the above equation. This cannot be taken at this level and thus we need to slightly modify the equation – introducing the so-called *renormalized equation*, so that its solution admits a non-trivial limit for $\varepsilon \to 0^+$. On account of the previous discussion on Wick powers and in particular of Proposition 1.3.7 and Remark 1.3.8, the renormalized equation reads

$$\partial_t\varphi_\varepsilon = \Delta\varphi_\varepsilon - (\varphi^3_\varepsilon - 3C_\varepsilon\varphi_\varepsilon) + \xi_\varepsilon\,, \tag{1.3.7}$$

where we observe that $\varphi^3_\varepsilon - 3C_\varepsilon\varphi_\varepsilon = H_3(\varphi_\varepsilon, C_\varepsilon)$ is the Hermite polynomials with C_ε as in Remark 1.3.8.

The idea at the heart of the Da Prato-Debussche argument is the fact that, as discussed in Remark 1.3.4, if we write the solution of the singular Equation (1.1.8) as $\varphi = \eta + v$, with

$\eta = P * \xi = \dot{|}$, then we expect the remainder term v to be of better regularity than η. In other words, the idea is that of introducing a sort of *expansion* in "singular objects" of the solution φ where we isolate η, which is expected to be the most singular term.

To clarify this aspect, let η_ε be the solution of the regularized stochastic heat equation $\partial_t \eta_\varepsilon = \Delta \eta_\varepsilon + \xi_\varepsilon$, namely $\eta_\varepsilon = \dot{|}_\varepsilon = P * \xi_\varepsilon$, and let $\varphi_\varepsilon = \eta_\varepsilon + v_\varepsilon$. As a consequence, exploiting Equation (1.3.7) and the defining property of η_ε,

$$\begin{aligned}\partial_t v_\varepsilon &= \Delta v_\varepsilon + \big((v_\varepsilon + \eta_\varepsilon)^3 - 3C_\varepsilon(v_\varepsilon + \eta_\varepsilon)\big)\\ &= \Delta v_\varepsilon - v_\varepsilon^3 - 3\eta_\varepsilon v_\varepsilon^2 - 3(\eta_\varepsilon^2 - C_\varepsilon)v_\varepsilon - (\eta_\varepsilon^3 - 3C_\varepsilon \eta_\varepsilon)\,. \end{aligned} \tag{1.3.8}$$

First of all we observe that Equation (1.3.8) is more tantalizing than the original one, since the noise ξ_ε is no longer present due to the decomposition of the solution and since the terms appearing in Equation (1.3.8) are more regular.

Moreover, on account of Proposition (1.3.7) and of Remark 1.3.8 the polynomials of η_ε appearing in Equation (1.3.8) converge in probability to the associated Wick powers as $\varepsilon \to 0^+$. As a consequence, we can consider the equation in the limit for $\varepsilon \to 0^+$, yielding, in its mild form,

$$v = P * (-v^3 - 3 :\eta: v^2 - 3 :\eta^2: v + :\eta^3:)\,. \tag{1.3.9}$$

Equation (1.3.9) is now suitable to be studied as a fixed point equation. In particular, on account of Proposition 1.3.7 and of the assumption on the spatial dimension, *i.e.*, $d = 2$, we have that $:\eta:, :\eta^2:, :\eta^3: \in \mathcal{C}^{-\kappa}$ for any $\kappa > 0$, with a particular interest for small κ. As a consequence of Schauder estimates as per Theorem 1.2.15 and of Young's product theorem as per Theorem 1.2.14, Equation (1.3.9) can be formulated in $\mathcal{C}^{2-\kappa}$ for any $\kappa > 0$. We underline in particular that all the products involved in the right hand side of Equation (1.3.9) are well-defined in $\mathcal{C}^{2-\kappa}$.

Remark 1.3.9: *We observe that the above fixed point argument is local in time. Moreover, this argument is independent of the minus sign in front of the non-linearity. This minus sign is fundamental when proving global existence in time for v [78].*

Failure of the Argument in $d \geq 3$ As we have seen above, the Da Prato-Desbussche argument provides an elegant way of dealing with renormalization and Picard iteration for the Φ^4_2 model. As a matter of fact, this argument falls short of working already for the Φ^4_3 model, namely for the stochastic quantization when the spatial dimension is $d = 3$.

The main issue behind this failing is the lack of regularity of the white noise: indeed, if $d = 3$ we have $\xi \in \mathcal{C}^{-\frac{5}{2}-\kappa}$ for any $\kappa > 0$, yielding by Schauder estimates $\eta = \dot{|} \in \mathcal{C}^{-\frac{1}{2}-\kappa}$. The problem arises at the level of the fixed point argument. Indeed Equation (1.3.9) still holds true but in the $d = 3$ scenario we have $:\eta^3: \in \mathcal{C}^{-\frac{3}{2}-\kappa}$ for any $\kappa > 0$ as a consequence of Proposition 1.3.7.

Exploiting Equation (1.3.9) as well as Theorem 1.2.15 we cannot hope for a better regularity for v than $\mathcal{C}^{\frac{1}{2}-\kappa}$. Since by Proposition 1.3.7 we have $:\eta^2: \in \mathcal{C}^{-1-\kappa}$, the Young's product Theorem 1.2.14 does not apply for the term $v :\eta^2:$ since there is a lack of regularity.

Remark 1.3.10: *A possible strategy to bypass this hurdle could consist of taking a further step in the strategy initially adopted in the Da Prato-Desbussche argument. In particular, by considering the second Picard iteration, one could expand the solution a bit further by writing*

$$\varphi =: \eta : - P * (:\eta^3:) + v\,,$$

yielding the equation

$$
\begin{aligned}
v = P * (-v^3 - 3 :\eta: v^2 - 3 :\eta^2: v - 3P * (:\eta^3:)^2 v - 3P * (:\eta^3:) v^2 \\
- 6 :\eta: P * (:\eta^3:) v^2 - 3 :\eta^2: P * (:\eta^3:) - 3 :\eta: P * (:\eta^3:)^2) \,.
\end{aligned}
$$

As before we have a problem with ill-defined product of distributions. Indeed, the hurdle here arises from the contribution $:\eta^2: P * (:\eta^3:)$ *since* $P * (:\eta^3:) \in \mathcal{C}^{\frac{1}{2}-\kappa}$ *and* $:\eta^2: \in \mathcal{C}^{-1-\kappa}$ *and thus Theorem 1.2.14 does not apply.*

Actually, this ill-defined product can be tamed by introducing a second renormalization constant similarly to what we did in Equation (1.3.7)*. We shall comment thoroughly about this divergent term in Chapter 3. The real problem in the* $d = 3$ *scenario is another one: Even introducing this second renormalization constant, we cannot complete the Picard iteration. This is due to the term* $:\eta^2: v$ *which cannot be avoided even by further expanding the solution. The hurdle with such a term is that* $:\eta^2: \in \mathcal{C}^{-1-\kappa}$ *and then, if such a product exists it is of regularity at most* $\mathcal{C}^{-1-\kappa}$*. This entails that in the best case scenario we would have* $v \in \mathcal{C}^{1-\kappa}$ *by Schauder estimates as per Theorem 1.2.15, which is however not regular enough in order to define the product* $:\eta^2: v$*.*

1.4 Theory of Regularity Structures

As we have seen in the previous section, the Da Prato-Debussche argument allows the analysis of the stochastic quantization equation (1.1.8) in spatial dimension $d = 2$ but not in higher dimension. In particular, we have seen that this approach consists of isolating the most singular contribution to the solution and then, by using tools borrowed from stochastic analysis, of solving a fixed point argument for the better behaving remainder.

Furthermore, we have seen that the fixed point argument does not work in spatial dimension $d = 3$, due to a lack of regularity. This issue has been solved by Martin Hairer in the *theory of regularity structures* [52]. Within this approach, which expands the Da Prato-Debussche one, it is again postulated an expansion for the solution of the form

$$\Phi(z) = \Phi_1(z) :\eta: + \Phi_2(z) P * (:\eta^3:) + \ldots + \Phi_n(z)\mathbf{1} \,,$$

where now, as one may see, the coefficients of the expansion are no longer constants but they are functions of suitable regularity. In this sense, with respect to the Da Prato-Debussche argument where a *global* expansion is considered, in the theory of regularity structures one deals with a *local* expansion. As a consequence, in this theory instead of solving a fixed point problem for a single function v one has to solve a fixed point problem for a family of functions $(\Phi_1, \ldots, \Phi_n)$.

To this end, one has to introduce a whole new mathematical framework, namely that of *regularity structures*, which allows to formulate and to solve this problem for a large class of SPDEs. In this section we shall briefly survey the theory of regularity structures, without any attempt of completeness, for which we refer to the literature, mainly to [52] – see also [10, 11, 21, 22, 51, 53, 54].

As we shall see, both the renormalization and the fixed point argument are discussed at an abstract level. Then the analytical content of the solution is recovered by means of the *reconstruction theorem*, which is a *pivôtal* result of the theory of regularity structures. To begin with, we introduce the abstract setting of the theory, starting from the very notion of *regularity structure*.

Definition 1.4.1: *We define* a regularity structure *as a triple* $\mathcal{T} = (A, T, G)$ *such that:*

- $A \subset \mathbb{R}$ *is a discrete set bounded from below which plays the rôle of an indexing set. Moreover we assume* $0 \in A$*;*

- $T = \oplus_{\alpha \in A} T_\alpha$ *is a graded space where each* T_α *is a Banach space whose elements are said to have* homogeneity $\alpha \in A$*. We further assume that* T_0 *is a one-dimensional space with a distinguished basis vector dubbed* $\mathbf{1}$*. For* $\tau \in T$*,* $\|\tau\|_\alpha$ *will denote the norm of its* T_α*-component;*

- G *is group, dubbed the* structure group*, of continuous linear maps acting on* T *such that for any* $\Gamma \in G$*,* $\alpha \in A$ *and* $\tau_\alpha \in T_\alpha$ *it holds that*

$$\Gamma \tau_\alpha - \tau_\alpha \in \bigoplus_{\beta < \alpha} T_\alpha =: T_{<\alpha} \, ,$$

  as well as $\Gamma \mathbf{1} = \mathbf{1}$ *for any* $\Gamma \in G$.

Remark 1.4.2: *In the triple* (A, T, G)*, the set* T *is the set containing the abstract objects with respect to which we will expand, whereas the indexing set* A *lists the* orders *– or also the* homogeneities*, of the abstract objects in* T*. Finally, the structure group* G*, which plays a fundamental rôle in the technical part of the theory, codifies the information on the dependence on points* $z \in \mathbb{R}^{d+1}$ *of the expansions, encoding the behaviour with respect to changes of such point. This is in agreement with the more local nature of the expansion we where discussing before.*

Having introduced the main abstract construction of the theory, we can introduce the first bridge between the abstract setting of the regularity structure and more concrete setting of space-time distributions in $\mathcal{D}'(\mathbb{R}^{d+1})$, namely the notion of *model*.

Definition 1.4.3: *Let* $\mathcal{T} = (A, T, G)$ *be a regularity structure. A* model *for* $\mathcal{T}$ *over* $\mathbb{R}^{d+1}$ *is defined as a pair* (Π, Γ) *satisfying the following conditions:*

- $\Gamma : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to G$ *is a map* $(x, y) \mapsto \Gamma_{xy} \in G$ *such that*

$$\Gamma_{xx} = \mathrm{Id} \, , \qquad \Gamma_{xy}\Gamma_{yz} = \Gamma_{xz} \, , \qquad \forall \, x, y, z \in \mathbb{R}^{d+1} \, ;$$

- $\Pi = \{\Pi_x\}_{x \in \mathbb{R}^{d+1}}$ *is a family of linear maps* $\Pi_x : T \to \mathcal{D}'(\mathbb{R}^{d+1})$*;*

- *it holds that*

$$\Pi_y = \Pi_x \Gamma_{xy} \, , \qquad \forall \, x, y \in \mathbb{R}^{d+1} \, .$$

Moreover, we also assume that for any $\alpha \in A$ *and for any compact set* $K \subset \mathbb{R}^{d+1}$ *the bounds*

$$|(\Pi_x \tau)(f_x^\lambda)| \lesssim \|\tau\|_\alpha \lambda^\alpha \, , \qquad \sup_{\beta < \alpha} \frac{\|\Gamma_{xy}\tau\|_\beta}{|x - y|_{\mathfrak{p}}^{\alpha - \beta}} \lesssim \|\tau\|_\alpha \, ,$$

hold uniformly for $\tau \in T_\alpha$*,* $x, y \in K$*,* $\lambda \in (0, 1]$ *and* $f \in \mathcal{B}_r$*, with* $r = \lceil -\min A \rceil$*.*

Remark 1.4.4: *We observe that through the notion of model one can see that the* homogeneity $\alpha \in A$ *of an abstract object* $\tau \in T_\alpha$ *is related to the Hölder regularity of* $\Pi_x \tau$*. Moreover, the notion of model also clarifies the rôle of structure group* G *with respect to changes of the base point.*

We are now in position of introducing the space of *modelled distribution*, which is where both the renormalization and the fixed point argument will take place in the theory of regularity structures.

Definition 1.4.5: *Let $\mathcal{T} = (A, T, G)$ be a regularity structure and let (Π, Γ) be a model for $\mathcal{T}$ over $\mathbb{R}^{d+1}$. For any $\gamma \in \mathbb{R}$, we define the space of* modelled distributions *$\mathcal{D}^\gamma$, where γ plays the rôle of a regularity, as the space of functions $F : \mathbb{R}^{d+1} \to T_{<\gamma}$ such that for any compact set $K \subset \mathbb{R}^{d+1}$ it holds that*

$$\|F\|_{\gamma,K} := \sup_{x\in K}\sup_{\beta<\gamma}\|F(x)\|_\beta + \sup_{\substack{x,y\in K\\ |x-y|_{\mathfrak{p}}\le 1}}\sup_{\beta<\gamma}\frac{\|F(x)-\Gamma_{xy}F(y)\|_\beta}{|x-y|_{\mathfrak{p}}^{\gamma-\beta}} < \infty\,. \tag{1.4.1}$$

Remark 1.4.6: *We underline that the spirit of regularity structures is that of operating with a fixed regularity structure $\mathcal{T} = (A, T, G)$ and of varying the underlying models (Π, Γ) in a suitable way, in particular when dealing with renormalization of SPDEs. As a consequence it is important to keep in mind that the definition of $\mathcal{D}^\gamma$ strongly depends on the chosen model.*

We further observe that the $\mathcal{D}^\gamma$-regularity is a natural lift of the Hölder regularity at the abstract level of modelled distributions. Example 1.4.11 will clarify this point.

Remark 1.4.7: *As we have anticipated, the space of modelled distributions $\mathcal{D}^\gamma$ is the arena where an SPDE will be formulated in the framework of regularity structures, in a way we shall sketch in Paragraph 1.4.*

As a consequence, in order to formulate non-linear *SPDEs in such a space, we need both a notion of product between modelled distributions and a notion of integration of modelled distributions against an integral kernel which is a lift to the space of modelled distributions of the heat kernel. We shall not discuss these two operations, referring to [52, Sec. 4] and [52, Sec. 5] respectively for details.*

In order to have a proper solution theory for SPDEs, once we have obtained an *abstract solution* to the equation in the sense of modelled distributions, we need a bridge between this abstract setting and the concrete setting of space-time distributions. This fundamental link, which is at the very heart of the theory of regularity structures, is the *reconstruction theorem*, which we report here for the case of modelled distributions of strictly positive regularity (see also [56]).

Theorem 1.4.8: *Let $\mathcal{T} = (A, T, G)$ be a regularity structure and let (Π, Γ) be a model for $\mathcal{T}$ over $\mathbb{R}^{d+1}$. Let $\alpha := \min A$ and let $\mathcal{D}^\gamma$ be the space of modelled distributions associated with (Π, Γ) for any $\gamma > 0$. There exists a continuous linear map $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{C}^\alpha$ such that, for any compact set $K \subset \mathbb{R}^{d+1}$ and $F \in \mathcal{D}^\gamma$, $\mathcal{R}F$ is the unique distribution satisfying*

$$|(\mathcal{R}F - \Pi_z F(z))(f_z^\lambda)| \lesssim \lambda^\gamma\,, \tag{1.4.2}$$

uniformly for $z \in K$, $\lambda \in (0, 1]$ and $f \in \mathcal{B}_r$ with $r = \lceil -\alpha \rceil$.

Remark 1.4.9: *This fundamental result states that, whenever a modelled distribution $F \in \mathcal{D}^\gamma$ is of positive regularity $\gamma > 0$, then there exists a unique space-time distributions $\mathcal{R}F \in \mathcal{C}^\alpha$ which is* locally *a good approximation, in the sense of Equation (1.4.2), of $\Pi_z F(z)$. As a consequence, the aim of the abstract setting of the theory is that of solving both the renormalization part and the fixed point argument in a space of modelled distribution of positive regularity. On account of this, the reconstruction theorem will provide both existence and uniqueness of a space-time distribution which locally looks like the obtained solution. This approach provides itself a proper notion of solution of the underlying SPDE.*

Remark 1.4.10: *We observe that in scenarios where the regularity γ of the modelled distributions is non-positive, the existence side of the reconstruction theorem still holds true whereas the uniqueness side fails [52, 17].*

Example 1.4.11: *As an example of regularity structure, we report the one associated with Taylor polynomials on a space-time $\mathbb{R}^{d+1}$. In such a case, the abstract symbols living in the vector space T are all the polynomials in $d+1$ indeterminates $X_0, \dots, X_d$, with X_0 playing the rôle of a time coordinate. A canonical basis of such a space is given by $X^k = X_0^{k_1} \dots X_d^{k_d}$ for any multi-index $k \in \mathbb{N}^{d+1}$, with the basis vector $\mathbf{1}$ corresponding to the zero multi-index.*

Moreover, a natural (parabolic) grading is achieved by setting $|X^k| = 2k_0 + \sum_{j=1}^d k_j$ for the homogeneity of the generator X^k yielding $\mathbb{N}$ as the indexing set A, whereas a norm on each T_k is obtained by setting $\|X^k\|$ for any multi-index k. Finally, the structure group is $\mathbb{R}^{d+1}$ with respect to the sum.

For what concerns a model (Π, Γ) for this regularity structure, a natural one is given by the realization of the abstract symbols centred at any point z of the space-time $\mathbb{R}^{d+1}$, namely

$$(\Pi_z X^k)(z') = (z - z')^k = (z_0 - z'_0)^{k_0} \dots (z_d - z'_d)^{k_d} \,,$$

and for any $h \in \mathbb{R}^{d+1}$,

$$\Gamma_h X^k = (X - h)^k = (X_0 - h'_0)^{k_0} \dots (X_d - h'_d)^{k_d} \,.$$

By direct inspection one can see that all conditions required by the definition both of regularity structure and of model are fulfilled by this construction.

To conclude we observe that given $\alpha > 0$ and a function $f \in \mathcal{C}^\alpha(\mathbb{R}^{d+1})$ as per Definition 1.2.9, then we can associate with it the function $F : \mathbb{R}^{d+1} \to T$ given by

$$F(z) = \sum_{|k|_{\mathfrak{p}} \leq \alpha} \frac{\partial^k f(z)}{k!} X^k \,. \tag{1.4.3}$$

By Equation (1.4.1)*, it follows that F is a modelled distribution in $\mathcal{D}^\alpha$. We observe that also the inverse holds true, namely if a function is such that F as per Equation* (1.4.3) *is a modelled distribution in $\mathcal{D}^\alpha$, then $f \in \mathcal{C}^\alpha$ [22].*

Regularity Structures and SPDEs In this paragraph we shall briefly sketch the explicit construction of a regularity structure associated with an SPDE, omitting the construction of the structure group, referring again to the aforementioned literature for details. Moreover, we shall discuss here only the example of the stochastic quantization equation.

The Model Space First of all, we observe that, given an SPDE, the vector space T is fine tuned so that it only contains the elements that are strictly necessary in order to make sense of the objects involved into the fixed point argument. In particular we shall require that T contains the polynomial regularity structures as per Example 1.4.11 which is needed in order to discuss smooth contributions. In addition, since we want to deal with an SPDE, we need an abstract symbol Ξ associated with the space-time white noise ξ. Moreover, recalling the Hölder regularity of ξ, *cf.* Remark 1.3.3, we set $|\Xi| = \alpha = -\frac{d}{2} - 1 - \kappa$ for an arbitrary (small) $\kappa > 0$.

A further symbol we are going to need is $\mathcal{I}(\Xi)$, which represents the solution of the underlying linear SPDE and where $\mathcal{I}$ stands for *integration* against the kernel of the linear operator. More in general, given $\tau \in T$, we shall denote with $\mathcal{I}(\tau)$ the symbols representing the convolution "$K * \tau$". For what concerns the parabolic scenario we are considering, on account of Schauder estimates as per Theorem 1.2.15, we set $|\mathcal{I}(\tau)| = 2 + |\tau|$.

Finally, given two abstract symbols $\tau_1, \tau_2 \in T$, we shall denote with $\tau_1\tau_2$ their product, setting $|\tau_1\tau_2| = |\tau_1| + |\tau_2|$.

We observe that the outcome of such a construction is not a regularity structure. Indeed if we consider all these abstract symbols then the index set A fails to be bounded from below since we are allowing objects such as Ξ^n for $n \in \mathbb{N}$. Nonetheless we underline that such symbols are unnecessary for our task and thus we shall consider a smaller family of symbols. More precisely, we consider a set $\mathcal{U}$ which is the smallest collection of symbols containing $\mathbf{1}, X, \mathcal{I}(\Xi)$ such that

$$\tau_1, \tau_2, \tau_3 \in \mathcal{U} \qquad \Rightarrow \qquad \mathcal{I}(\tau_1\tau_2\tau_3) \in \mathcal{U}\,.$$

Then we set $\mathcal{W} := \{\Xi\} \cup \{\tau_1\tau_2\tau_3 \,|\, \tau_1, \tau_2, \tau_3 \in \mathcal{U}\}$. Finally we define T as the span of $\mathcal{W}$.

Remark 1.4.12: *It can be seen [22, 52] that the above regularity structure is rich enough in order to formulate the stochastic quantization equation in a space of modelled distribution $\mathcal{D}^\gamma$ for some γ. In particular, assuming for simplicity a vanishing initial condition, the equation is*

$$\Psi = \mathcal{K}(\Xi - \Psi^3)\,,$$

where $\mathcal{K}$ is the lift to the space of modelled distributions of the integration against the heat kernel discussed in Remark 1.4.7.

Remark 1.4.13: *A very important observation concerns the notion of* subcriticality *of an SPDEs [22, 52]. This is a scaling behaviour characterizing a class of SPDEs which is fundamental in order for the theory of regularity structures to apply. Indeed, subcriticality guarantees that, at the level of modelled distributions, the SPDE can be formulated as a fixed point problem in $\mathcal{D}^\gamma$ for a big enough γ. We underline that this is fundamental in order to get a proper solution theory on account of the reconstruction Theorem 1.4.8. Dwelling more into the details, we say that an SPDE*

$$\partial_t u = \Delta u + F(u, \nabla u) + \xi\,, \qquad \text{in} \quad \mathbb{R}^{d+1}\,,$$

is subcritical *if under the scaling*

$$\widehat{u}(t,x) := \lambda^{\frac{d}{2}-1} u(\lambda^2 t, \lambda x)\,, \qquad \widehat{\xi}(t,x) := \lambda^{\frac{d}{2}+1} \xi(\lambda^2 t, \lambda x)\,, \tag{1.4.4}$$

the non-linear term $F(u, \nabla u)$ is transformed in $F_\lambda(u, \nabla u)$ which formally vanishes for $\lambda \to 0^+$.

To better grasp this notion, we shall discuss the example of the stochastic quantization Φ^4_d model. The starting equation is

$$\partial_t u = \Delta u - u^3 + \xi\,,$$

which under the scaling transformation of Equation (1.4.4) *becomes*

$$\partial_t \widehat{u} = \Delta \widehat{u} - \lambda^{4-d} \widehat{u}^3 + \widehat{\xi}\,.$$

As a consequence, the stochastic quantization equation is subcritical in spatial dimension $d \leq 3$. Roughly speaking, subcriticality entails that at "small scales" the equation behaves like its linear part since the non-linearity vanishes.

The Canonical Model The next step consists of introducing a particular model for the above constructed regularity structure, the so-called *canonical model* $(\Pi^\varepsilon, \Gamma^\varepsilon)$. In particular, on polynomials it acts exactly as discussed in Example 1.4.11 whereas on the symbol Ξ it acts as $\Pi^\varepsilon(\Xi) := \xi_\varepsilon$, with $\xi_\varepsilon := \rho^\varepsilon * \xi$ and $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R}^{1+3})$ a mollifier. We recall that ρ^ε is defined as per Equation (1.2.1).
In other words, ξ_ε is a smooth regularization of the white noise such that $\xi_\varepsilon \to \xi$ for $\varepsilon \to 0^+$ in a distributional sense. The canonical model is then defined in a suitable recursive way on symbols of the form $\mathcal{I}(\tau)$ [22, 52], whereas on products $\tau_1\tau_2$ as $\Pi^\varepsilon(\tau_1\tau_2) = \Pi^\varepsilon(\tau_1)\Pi^\varepsilon(\tau_2)$.

Remark 1.4.14: *We underline that the product property of the canonical model is legitimate since, on account of the regularization of the white noise, Π^ε maps abstract symbols onto smooth functions, whose pointwise product is always well defined.*

We also underline that the canonical model $(\Pi^\varepsilon, \Gamma^\varepsilon)$ does not converge to a model for the underlying regularity structure for $\varepsilon \to 0^+$ due to the distributional behaviour of ξ_ε in such limit.

As we have seen in Section 1.3.1, Da Prato-Debussche solved a similar issue for the Φ^4_2 model by implementing a renormalization procedure, consisting of a subtraction of the divergent part, yielding to the notion of Wick powers.

From this viewpoint, the approach of the theory of regularity structures is rather similar [12, 52]. Without entering into the details, it consists of a deformation of the multiplicative structure of the canonical model, implementing directly the renormalization subtraction proper of the Da Prato-Debussche argument.

In this way one gets a *modified* equation, dubbed the *renormalized equation* similar to Equation (1.3.7) in the space of modelled distribution. Provided that the original SPDE is subcritical, as it is the case for the Φ^4_3 model, then this renormalized equation can be solved through a fixed point argument in $\mathcal{D}^\gamma$ for a γ big enough. As we discussed in Remark 1.4.9, one can then get the proper solution of the renormalized equation through the reconstruction operator introduced in Theorem 1.4.8.

Finally we underline that, thanks to the renormalization procedure, taking the limit for $\varepsilon \to 0^+$ one obtains a non-trivial limit, which is the proper solution of the SPDE. This last discussion can be summarized in the following theorem.

Theorem 1.4.15: *Consider a the family of equations in $\mathbb{R}^{1+3}$*

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + C_\varepsilon u_\varepsilon - u_\varepsilon^3 + \xi_\varepsilon \, ,$$

where $\xi_\varepsilon := \rho^\varepsilon * \xi$, with $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R}^{1+3})$ a mollifier and with ρ^ε as per Equation (1.2.1) with $d = 3$. There exists a choice of constants C_ε such that the family u_ε converges in probability to a limit distributional process u which is independent of the mollifier ρ.

Excursus: The Reconstruction Theorem on Smooth Manifolds As we have seen in the previous section, the *reconstruction theorem* is one of the cornerstones of the theory of regularity structures since it provides a bridge between the abstract framework of modelled distributions and the concrete framework of space-time distributions.

Nonetheless, the reconstruction theorem can be formulated also in a purely distributional language without any reference to the setting of regularity structures, as it has been done in [17], where the authors introduced the notion of *coherent germs of distributions* which is a family of distributions satisfying some generalized Hölder condition inspired by the notions both of *model* and of *modelled distributions.*

The problem they address is the following: If for any $x \in \mathbb{R}^d$ we are given a distribution $F_x \in \mathcal{D}'(\mathbb{R}^d)$, one may wonder whether there exists a suitable distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ which is *locally*, namely in a neighbourhood of $x \in \mathbb{R}^d$, approximated by F_x. This for any $x \in \mathbb{R}^d$. In [17], the authors prove that this is actually the case if one requires a further assumption, dubbed *coherence*, for the family $\{F_x\}_{x\in\mathbb{R}^d}$ of distributions.

Dwelling into the details, the notion of coherent germs of distribution is the following.

Definition 1.4.16: *[17, Def. 4.3] We call germ of distributions $\{F_x\}_{x\in\mathbb{R}^d} \subset \mathcal{D}'(\mathbb{R}^d)$ a family of distributions such that for any $\psi \in \mathcal{D}(\mathbb{R}^d)$, the map $\mathbb{R}^d \ni x \mapsto F_x(\psi) \in \mathbb{R}$ is measurable. Moreover, given $\gamma \in \mathbb{R}$, we say that a germ of distributions $\{F_x\}_{x\in\mathbb{R}^d} \subset \mathcal{D}'(\mathbb{R}^d)$ is γ-coherent if there exists $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that for any compact set $K \subset \mathbb{R}^d$ there exists $\alpha_K \leq \min\{0,\gamma\}$ such that*

$$|(F_z - F_x)(\varphi_x^\lambda)| \lesssim \lambda^{\alpha_K}(|z-x|+\lambda)^{\gamma-\alpha_K}\,,$$

uniformly for $x, z \in K$ and $\lambda \in (0,1]$.

In a recent paper [85], jointly written with F. Sclavi, we extended the results on the reconstruction theorem obtained in [17] to the setting of a smooth d-dimensional manifold replacing the Euclidean space $\mathbb{R}^d$. In particular, we introduced a notion of *coherence* for germs of distribution in the manifold setting, yielding a more local definition with respect to that of [17].

Definition 1.4.17: *[85, Def. 4] Let M be a smooth d-dimensional manifold and let $\mathcal{A} = \{(U_\alpha,\phi_\alpha)\}_\alpha$ be a smooth atlas on M. Let $F = (F_p)_{p\in M}$ be a germ of distributions on M and $\gamma \in \mathbb{R}$. We say that F is γ-coherent on $(M,\mathcal{A})$ if for any $(U,\phi) \in \mathcal{A}$ there exists $f \in \mathcal{D}(\phi(U))$, with $\int_{\mathbb{R}^d} dx\, f(x) \neq 0$, such that for any compact set $K \subset U$ there exists $\alpha_K^U \leq \min\{0,\gamma\}$ such that*

$$|(\phi_*(F_p) - \phi_*(F_q))(f_{\phi(q)}^\lambda)| \lesssim \lambda^{\alpha_K^U}(|\phi(p)-\phi(q)|+\lambda)^{\gamma-\alpha_K^U}\,,$$

uniformly for $p, q \in K$, $\lambda \in (0,1]$.

Eventually we proved the reconstruction theorem, both for positive and negative coherence parameter γ, also in this setting, whose proof [85] is based on the very definition of distributions on a smooth manifold together with a localized version of the results of [17, Thm. 4.4].

Theorem 1.4.18: *[85, Thm. 18] Let M be a d-dimensional smooth manifold and let $\mathcal{A} = \{(U_j,\phi_j)\}_j$ be an atlas over M. Let $\gamma > 0$ and let $F = (F_p)_{p\in M}$ be a γ-coherent germ of distributions on $(M,\mathcal{A})$. There exists a unique distribution $\mathcal{R}F \in \mathcal{D}'(M)$ such that, for any $(U,\phi) \in \mathcal{A}$, $\phi_*(\mathcal{R}F) \in \mathcal{D}'(\phi(U))$ and it satisfies, for any compact set $K \subset U$ and for any $h \in \mathcal{D}(\phi(U))$,*

$$|(\phi_*(\mathcal{R}F) - \phi_*(F_p))(h_{\phi(p)}^\lambda)| \lesssim \lambda^\gamma\,,$$

uniformly for $p \in K$ and $\lambda \in (0,1]$.

Theorem 1.4.19: *[85, Thm. 21] Let M be a d-dimensional smooth manifold and let $\mathcal{A} = \{(U_j,\phi_j)\}_{j\in J}$ be an atlas over M, with J index set. Let $\gamma \leq 0$ and let $F = (F_p)_{p\in M}$ be a γ-coherent germ of distributions on $(M,\mathcal{A})$. There exists a distribution $\mathcal{R}F \in \mathcal{D}'(M)$ such that, for any $(U,\phi) \in \mathcal{A}$, $\phi_*(\mathcal{R}F) \in \mathcal{D}'(\phi(U))$ and it satisfies, for any compact set $K \subset U$ and for any $h \in \mathcal{D}(\phi(U))$,*

$$|(\phi_*(\mathcal{R}F) - \phi_*(F_p))(h_{\phi(p)}^\lambda)| \lesssim \begin{cases} \lambda^\gamma & \text{if } \gamma < 0\,, \\ 1+|\log\lambda| & \text{if } \gamma = 0\,, \end{cases}$$

uniformly for $p \in K$ *and* $\lambda \in (0, 1]$. *This distribution* $\mathcal{R}F \in \mathcal{D}'(M)$ *is non-unique.*

Moreover, in the paper we also prove that both the notion of coherence of a germ and the reconstruction operator are atlas-independent, yielding a geometrical construction.

Microlocal Approach to SPDEs

In the remaining part of this manuscript we shall discuss the *microlocal approach to SPDEs*, which is a novel framework introduced in [27] in a joint work with C. Dappiaggi, N. Drago and L. Zambotti. Inspired by the *algebraic approach to quantum field theory*, this allows the study of a large class of non-linear SPDEs.

In particular, in Chapter 2 we shall introduce some useful technical tools which will be the main ingredients of this approach, in particular the notion of *functional-valued distribution.* As one may imagine from the name, a functional-valued distribution is a "superposition" of a functional and of a distribution. As a random distribution, see Definition 1.2.2, encodes both the information of a distribution and of a random variable, in our approach to the analysis of SPDEs, functional-valued distributions will encode the information regarding a distribution and the expectation values of the associated random distribution.

This is achieved by giving the space of functional-valued distributions an algebra structure and then by deforming the algebra product in a suitable way, determined by the properties of the underlying noise. We shall comment on this interpretation in Chapter 2. We observe in addition that this deformation procedure can be used also in scenarios where the random noise is not a white noise, *e.g.*, in cases where the two point function of ξ is not a Dirac delta but a different distribution.

This deformation procedure, allowing to encode both the information about expectation values and multi-local correlation functions, needs renormalization in order to be meaningful due to the aforementioned singular behaviour of the stochastic noise and of the fundamental solution of the linear operator. This topic is thoroughly discussed in Chapter 3.

Finally, in Chapter 4 we discuss in detail the application of the whole machinery of the microlocal approach to a concrete SPDE, in particular the Ψ^4_d model we introduced in Chapter 1.

Chapter 2

Functional-Valued Distributions

The aim of this chapter is twofold: On the one hand we set the general framework we are going to work in, fixing at the same time some notations. On the other hand, we introduce the key notion of *functional-valued distribution*, which is the basic ingredient of our approach to SPDEs, in particular to renormalization.

Moreover, we shall endow the space of functional valued distributions with an algebra structure and we shall also discuss how through a *deformation argument* applied to the algebra product, inspired by the algebraic approach to quantum field theory [26, 33, 34, 84], we can codify the stochastic properties of the white noise.

2.1 Main Definitions

In the following both M and N are smooth, connected and finite-dimensional manifolds such that $\partial M = \partial N = \emptyset$. Moreover, TM and T^*M will denote the tangent and cotangent bundle of M, respectively. Finally, $\mathbb{D}(M)$ and $\mathbb{D}(N)$ denote the density bundle on the associated manifold, namely $\mathbb{D}(M)$ is the line bundle over M whose fiber at each point $p \in M$ is $\mathbb{D}_p(M) \equiv \mathbb{D}(T_pM)$, *i.e.*, the set of densities over the vector space T_pM [75]. For definiteness, we assume that a reference top-density, say μ_M and μ_N respectively, has been fixed with the underlying assumption that whenever a manifold has a Riemannian structure, the chosen top-density is the metric induced volume form. Moreover, we shall also omit the subscript in μ_M when there is no chance of confusion.

On top of M we consider a linear partial differential operator E, which in the analysis of SPDEs plays the rôle the linear part of the equation. In particular we are interested in two classes of linear partial differential equations, namely

1. an *elliptic* second order partial differential operator E;
2. an operator of the form $E = \partial_t - \widetilde{E}$ if the manifold M is homeomorphic to $\mathbb{R} \times \Sigma$, with Σ a smooth manifold, where t is the standard Euclidean coordinate over $\mathbb{R}$ while $\widetilde{E}$ is an elliptic, t-independent partial differential operator over Σ.

Example 2.1.1: *To realize examples of the above scenarios, one can consider in the first case M to be endowed with a Riemannian metric g and $E = -\Delta_g$ as the Laplace-Beltrami operator over M associated to g. Similarly, in the second scenario one may take Σ to be endowed with a Riemannian metric h and consider $\widetilde{E} = -\Delta_h$ the Laplace-Beltrami operator over Σ associated to h. This is tantamount to considering the heat operator on the manifold M.*

Remark 2.1.2: *We observe that the two cases listed above are particular examples of a larger class of partial differential operators, namely the* microhypoelliptic *operators [59, Chapt. XXII]. We recall that an operator E falls into this class if for any distribution $u \in \mathcal{D}'(M)$,*

$$\mathrm{WF}(u) = \mathrm{WF}(Eu)\,, \tag{2.1.1}$$

where WF *denotes the wave-front set [61] – see Appendix A for a brief survey of this topic.*

We underline that the following discussion holds true for a generic microhypoelliptic operator, provided some minor changes in the proofs are implemented. In order to avoid burdening the reader with many technical side remarks, we only consider here the two scenarios listed above since they are moreover the most relevant in view of the applications to SPDEs.

Furthermore, given E, we denote with P a *parametrix* associated with E [59] and with E^* we denote the formal adjoint of E, whose parametrix is P^*. For the sake of readability, we recall the definition of parametrix [61, 89].

Definition 2.1.3: *Let M be a smooth manifold and let E be a scalar, pseudodifferential [59, Chapt. XVIII] operator over M.* A parametrix *of E is a bi-distribution $P \in \mathcal{D}'(M \times M)$ such that*

$$PE - \mathrm{id}|_{\mathcal{D}(M\times M)} \in \mathcal{E}(M \times M), \quad \text{and} \quad EP - \mathrm{id}|_{\mathcal{D}(M\times M)} \in \mathcal{E}(M \times M)\,.$$

Remark 2.1.4: *A very important property of microhypoelliptic operators concerns the microlocal behaviour of the parametrices. Indeed, if we consider P to be a parametrix of a microhypoelliptic operator E, then $EP = \delta_{\mathrm{Diag}_2} + w$, with $w \in C^\infty(M \times M)$ and $\delta_{\mathrm{Diag}_2} \in \mathcal{D}'(M \times M)$ the Dirac delta distribution supported on the total diagonal* Diag_2 *of $M \times M$. Exploiting Equation* (2.1.1)*, at the level of wave-front set it holds*

$$\mathrm{WF}(P) = \mathrm{WF}(EP) = \mathrm{WF}(\delta) = \{(x, k_x; y, k_y) \in T^*M^2 \setminus \{0\} \,|\, x = y\,, k_x = -k_y\}\,. \tag{2.1.2}$$

In other words, a parametrix or, if existent, the fundamental solution, behaves microlocally as a Dirac delta distribution – cf. Example A.0.5.

Remark 2.1.5: *In the following, given a parametrix P of an operator E as above, we shall be interested in considering expressions such as $P(f \otimes \varphi)$, with $f \in \mathcal{D}(M)$ and $\varphi \in \mathcal{E}(M)$. We observe that* a priori *these expressions may be meaningless due to infra-red divergences, the problem lying in the possible non-compactness of the support of φ.*

This issue can be avoided in two ways, either by assuming the underlying manifold M to be compact or by considering a cut-off function $\chi \in \mathcal{D}(M)$ and replacing $P \in \mathcal{D}'(M \times M)$ with $P\cdot(1\otimes\chi) \in \mathcal{D}'(M\times M)$. Here $\cdot$ denotes the pointwise multiplication between the bi-distribution $P \in \mathcal{D}'(M \times M)$ and the smooth function $1 \otimes \chi \in \mathcal{E}(M \times M)$.

This two approaches are equivalent in the elliptic scenario, whereas the second one is preferable in the parabolic one, due to the $\mathbb{R}$ factor in the product $M = \mathbb{R} \times \Sigma$. Since in the main body of this manuscript we shall consider the elliptic scenario, commenting step by step on the differences and the analogies with the parabolic case, we assume the manifold M to be compact.

In spite of this assumption, throughout this manuscript we shall keep the notations $\mathcal{D}(M)$ and $\mathcal{E}(M)$ even though they are equivalent on a compact manifold in order to highlight which is the correct space to use whenever the manifold M is non-compact.

We are now in position of introducing the key notion of *functional-valued distribution*. As one may imagine, this is a combination of two objects: a functional and a distribution.

In particular, *a priori*, we shall have a distribution on the manifold N and a functional on the manifold M.

Definition 2.1.6: *We define a* **functional-valued distribution** $\tau \in \mathcal{D}'(N;\mathrm{Fun})$ *to be a map*

$$\tau\colon \mathcal{D}(N)\times\mathcal{E}(M)\ni(f,\varphi)\mapsto\tau(f;\varphi)\in\mathbb{C}\,,$$

which is linear in its first component and continuous with respect to the locally convex topology of $\mathcal{D}(N)\times\mathcal{E}(M)$. *Moreover, we indicate with* $\tau^{(k)}\in\mathcal{D}'(N\times M^k;\mathrm{Fun})$ *the k-th order functional derivative of* $\tau\in\mathcal{D}'(N;\mathrm{Fun})$ *such that, for any* $f\in\mathcal{D}(N)$ *and* $\psi_1,\dots,\psi_k,\varphi\in\mathcal{E}(M)$,

$$\tau^{(k)}(f\otimes\psi_1\otimes\ldots\otimes\psi_k;\varphi)\doteq\frac{\partial^k}{\partial s_1\cdots\partial s_k}\tau(f;s_1\psi_1+\ldots+s_k\psi_k+\varphi)\bigg|_{s_1=\ldots s_k=0}. \tag{2.1.3}$$

Finally, we say that $\tau\in\mathcal{D}'(M;\mathrm{Fun})$ *is φ-polynomial if there exists* $\bar{k}\in\mathbb{N}$ *such that* $\tau^{(k)}=0$ *for any* $k\geq\bar{k}$. *We denote by* $\mathcal{D}'(N;\mathrm{Pol})$ *the vector space of functional-valued φ-polynomial distributions.*

Remark 2.1.7: *We observe that for what concerns the functional derivatives* $\tau^{(k)}\in\mathcal{D}'(N\times M^k;\mathrm{Fun})$ *of* τ *as per Equation* (2.1.3), *in its last k entries this is a symmetric and compactly supported distribution. We also observe that the existence of the functional derivatives* $\tau^{(k)}\in\mathcal{D}'(N\times M^k;\mathrm{Fun})$ *of any order* $k\in\mathbb{N}$ *is a request on the chosen class of functionals. In this sense, these functionals are smooth [26].*

In the following it will be useful to have a counterpart of the notion of directional derivative at the level of functionals, namely, for a given $\psi\in\mathcal{E}(M)$, *we define*

$$\delta_\psi\colon\mathcal{D}'(N;\mathrm{Fun})\to\mathcal{D}'(N;\mathrm{Fun})\,,\qquad[\delta_\psi\tau](f;\varphi)\doteq\tau^{(1)}(f\otimes\psi;\varphi)\,. \tag{2.1.4}$$

Example 2.1.8: *In order to better grasp the nature of functional-valued distributions, we consider some clarifying examples which will be relevant in the following discussion. As starting point we fix* $M=N$ *for the manifolds and we introduce the following functional-valued distributions*

$$\Phi(f;\varphi):=\int_M\varphi(x)f(x)\mathrm{d}\mu(x)\,,\qquad\mathbf{1}(f;\varphi):=\int_M f(x)\mathrm{d}\mu(x)\,, \tag{2.1.5}$$

where $\varphi\in\mathcal{E}(M)$, $f\in\mathcal{D}(M)$ *and where we recall that μ is a reference top-density*[1] *on the manifold M. It holds* $\Phi,\mathbf{1}\in\mathcal{D}'(M;\mathrm{Pol})$ *and we observe that both these functional valued distributions are also linear in their first argument. With a similar notation, a non-linear example is* $\Phi^2\in\mathcal{D}'(M;\mathrm{Pol})$, *given by*

$$\Phi^2(f;\varphi):=\int_M\varphi^2(x)f(x)\mathrm{d}\mu(x)\,,\qquad\varphi\in\mathcal{E}(M)\,,\quad f\in\mathcal{D}(M)\,. \tag{2.1.6}$$

Finally, another notable class of functional-valued distributions we shall be interested in involves the parametrix P of the underlying differential operator E. As an example of these functionals, consider $\Phi P\circledast\Phi\in\mathcal{D}'(M;\mathrm{Pol})$,

$$[\Phi\,P\circledast\Phi]\,(f;\varphi)=\int\limits_M\varphi(x)\,(P\circledast\varphi)(x)\,f(x)\mathrm{d}\mu(x)\,,\qquad\varphi\in\mathcal{E}(M)\,,\quad f\in\mathcal{D}(M)\,,$$

[1]We omit the subscript M in μ_M for simplicity of notation at this level.

where $(P \circledast \varphi)(f) := P(f \otimes \varphi)$ is well-defined on account of the discussion of Remark 2.1.5.

To conclude this example, we also compute the functional derivatives of the functional-valued distributions introduced above. First of all we observe that the functional $\mathbf{1}$ *is constant with respect to φ and thus all its functional derivatives are vanishing.*
Focusing on the functional Φ, it holds, at the level of integral kernel,

$$\Phi^{(1)}(x, z; \varphi) = \delta_{\mathrm{Diag}_2}(x, z)\,,$$

while all higher order derivatives are vanishing. Here $\delta_{\mathrm{Diag}_n}(x, z_1, \ldots, z_{n-1})$, $n \geq 2$ is the integral kernel of the Dirac-delta distribution on M^n.
Eventually, for the last two examples of functional-valued distributions we have

$$[\Phi^2]^{(1)}(x, z; \varphi) = 2\varphi(x)\delta_{\mathrm{Diag}_2}(x, z)\,, \qquad [\Phi^2]^{(2)}(x, z_1, z_2; \varphi) = 2\delta_{\mathrm{Diag}_3}(x, z_1, z_2)\,, \tag{2.1.7}$$

and

$$[\Phi P \circledast \Phi]^{(1)}(x, z; \varphi) = (P \circledast \varphi)(x)\delta_{\mathrm{Diag}_2}(x, z) + \varphi(x)P(x, z)\,, \tag{2.1.8}$$

$$[\Phi P \circledast \Phi]^{(2)}(x, z_1, z_2; \varphi) = 2\delta_{\mathrm{Diag}_2}(x, z_1)P(x, z_2)\,. \tag{2.1.9}$$

Remark 2.1.9: *On account of Definition 2.1.6, let $\tau \in \mathcal{D}'(N; \mathrm{Pol})$, $f \in \mathcal{D}(N)$ and $\varphi \in \mathcal{E}(M)$, then it holds, for $\lambda \in \mathbb{C}$,*

$$\tau(f; \lambda\varphi) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}\mu^k} \tau(f; \mu\varphi)\bigg|_{\mu=0} \lambda^k = \sum_{k=0}^{\infty} \frac{1}{k!} \tau^{(k)}(f \otimes \varphi^{\otimes k}; 0)\lambda^k\,, \tag{2.1.10}$$

where $\varphi^{\otimes k} = \underbrace{\varphi \otimes \cdots \otimes \varphi}_{k}$, while

$$\begin{aligned}
\frac{\mathrm{d}^k}{\mathrm{d}\mu^k} \tau(f; \mu\varphi)\bigg|_{\mu=0} &= \frac{\mathrm{d}^k}{\mathrm{d}\mu^k} \tau(f; (\mu + \mu_1 + \ldots + \mu_k)\varphi)\bigg|_{\mu=\mu_1=\ldots=\mu_k=0} \\
&= \frac{\partial^k}{\partial\mu_1 \cdots \partial\mu_k} \tau(f; (\mu_1 + \ldots + \mu_k)\varphi)\bigg|_{\mu_1=\ldots=\mu_k=0} = \tau^{(k)}(f \otimes \varphi^{\otimes k}; 0)\,.
\end{aligned}$$

We underline that the right hand side of Equation (2.1.10) *is well-defined since only a finite number of derivatives are non-vanishing due to the polynomial behaviour of τ in φ. Moreover, if $\tau^{(k)}(x, z_1, \ldots, z_k)$ denotes the integral kernel of the distribution $\tau^{(k)}(\cdot\,; 0)$, it descends*

$$\tau(f; \varphi) = \sum_{k=0}^{\infty} \frac{1}{k!} \int\limits_{N \times M^k} \tau^{(k)}(x, z_1, \ldots, z_k) f(x)\varphi(z_1) \cdots \varphi(z_k) \mathrm{d}\mu_N(x)\mathrm{d}\mu_M(z_1) \cdots \mathrm{d}\mu_M(z_k)\,,$$

yielding an isomorphism of topological vector spaces

$$\mathcal{D}'(N; \mathrm{Pol}) \simeq \bigoplus_{k \geq 0} \mathcal{D}'(N \times M^k)\,.$$

Remark 2.1.10: *The space of functional-valued distributions $\mathcal{D}'(M; \mathrm{Pol})$ contains a large class of elements. In particular we shall be interested in some distinguished subspaces of $\mathcal{D}'(M; \mathrm{Pol})$. As it is customary in the functional approach to algebraic quantum field theory, see, e.g., [26],* *polynomial functionals can be further classified on account of the microlocal behaviour of their functional derivatives.*

Important classes of polynomial functionals are those of regular *functionals and of* local *functionals, the former being those whose functional derivatives are distributions generated by smooth functions whereas the latter having derivatives behaving microlocally as Dirac delta distributions. We refer to [26, Def. 7] for a more in detail characterization of such classes.*

In the following definition we introduce a class of subspaces of $\mathcal{D}'(M; \mathrm{Pol})$ whose properties are exploited several times in the forthcoming analysis of elliptic and parabolic SPDEs.

To this end, we first need to fix some notation. For any but fixed $k \in \mathbb{N}$, $I_1 \uplus \ldots \uplus I_\ell$ denotes an arbitrary partition of $\{1, \ldots, k\}$ into ℓ disjoint non-empty subsets and we set $|I_i|$, $i = 1, \ldots, \ell$, the cardinality of the set I_i. In addition, $\delta_{\mathrm{Diag}_{|I_i|}}$ shall indicate the Dirac delta distribution supported on the submanifold $\mathrm{Diag}_{|I_i|} = \{(x, \ldots, x) \in M^{|I_i|} \mid x \in M\} \in M^{|I_i|}$ and we adopt the convention that, if the cardinality of $|I_i| = 1$, then $\mathrm{Diag}_{|I_i|} = \emptyset$.

Definition 2.1.11: *Let $k \in \mathbb{N}$ and, adopting the notation $\widehat{x}_k = (x_1, \ldots, x_k) \in M^k$, let*

$$\begin{aligned} \mathrm{C}_k := \{(\widehat{x}_k, \widehat{\xi}_k) \in T^*M^k \setminus \{0\} \mid & \\ & \exists \ell \in \{1, \ldots, k-1\}\,,\ \{1 \ldots, k\} = I_1 \uplus \ldots \uplus I_\ell\,,\ \textit{such that} \\ & \qquad \forall i \neq j\,,\ \forall (a, b) \in I_i \times I_j\,,\ \textit{then } x_a \neq x_b\,, \\ & \qquad\qquad \textit{and } \forall j \in \{1, \ldots, \ell\}\,,\ (\widehat{x}_{I_j}, \widehat{\xi}_{I_j}) \in \mathrm{WF}(\delta_{\mathrm{Diag}_{|I_j|}})\}\,, \end{aligned} \tag{2.1.11}$$

where $(\widehat{x}_{I_j}, \widehat{\xi}_{I_j}) = (x_i, \xi_i)_{i \in I_j} \in T^*M^{|I_j|}$ for all $j \in \{1, \ldots, \ell\}$ and where WF *stands for the wavefront set – cf. Appendix A. We define the subspaces $\mathcal{D}'_{\mathrm{C}}(M^k; \mathrm{Pol}) \subset \mathcal{D}'(M^k; \mathrm{Pol})$ as*

$$\mathcal{D}'_{\mathrm{C}}(M^k; \mathrm{Pol}) := \{\tau \in \mathcal{D}'(M^k; \mathrm{Pol}) \mid \mathrm{WF}(\tau^{(n)}) \subseteq \mathrm{C}_{k+n}\ \forall n \geq 0\}\,. \tag{2.1.12}$$

Remark 2.1.12: *We notice that* $\mathrm{WF}(\tau^{(n)})$ *as it appears in Equation* (2.1.12) *might depend on the configuration $\varphi \in \mathcal{E}(M)$ as $\tau^{(n)}$ is a functional-valued distribution. In this sense, Equation* (2.1.12) *in Definition 2.1.11 is required to hold true for any $\varphi \in \mathcal{E}(M)$.*

As we shall see in Chapter 3, this space of functional-valued distributions contains the objects which are needed in order to deal with SPDEs driven by a white noise as those discussed in Chapter 1.

Remark 2.1.13: *As already anticipated, functional-valued distributions are a combination of the functional and of the distributional structures. Definition 2.1.11 is adding some information on the distributional behaviour – or, more properly, the microlocal behaviour, of the functional-valued distribution we are interested in.*

From this viewpoint, the additional functional structure reverberates on the distributional one by forcing the distributions $\tau(\cdot; \varphi)$ to be generated by a smooth function, for any $\varphi \in \mathcal{E}(M)$ and $\tau \in \mathcal{D}'_{\mathrm{C}}(M; \mathrm{Pol})$. This can be seen directly from Equation (2.1.11) *in the case $k = 1$ since, $C_1 = \emptyset$ entails* $\mathrm{WF}(\tau) = \emptyset$ *for any $\tau \in \mathcal{D}'_{\mathrm{C}}(M; \mathrm{Pol})$.*

Remark 2.1.14: *In order to better grasp the microlocal properties of functional-valued distributions satisfying the condition of Equation* (2.1.12)*, we observe that the sets C_k as per Equation* (2.1.11) *are sets of points in the cotangent bundle $T^*M^k \setminus \{0\}$ clustering in subfamilies behaving like singular directions of Dirac delta distributions.*

In this sense Equation (2.1.12) *tells us that functional-valued distributions in $\mathcal{D}'_{\mathrm{C}}(M; \mathrm{Pol})$ have k-th functional derivatives that, from a microlocal viewpoint, behaves like Dirac delta distributions on sub-diagonal of M^k. We shall come back in Example 2.1.17 on this point.*

Remark 2.1.15: *We observe that Equation* (2.1.11) *entails that* $\mathrm{C}_{k_1} \otimes \mathrm{C}_{k_2} \subseteq \mathrm{C}_{k_1+k_2}$. *In addition, the following property also holds true:*

$$(x, \widehat{y}_{k_1}; \xi_1, \widehat{\nu}_{k_1}) \in \mathrm{C}_{k_1+1}\,,\ (x, \widehat{y}_{k_2}; \xi_2, \widehat{\nu}_{k_2}) \in \mathrm{C}_{k_2+1} \Longrightarrow (x, \widehat{y}_{k_1}, \widehat{y}_{k_2}; \xi_1 + \xi_2, \widehat{\nu}_{k_1}, \widehat{\nu}_{k_2}) \in \mathrm{C}_{k_1+k_2+1}\,. \tag{2.1.13}$$

Remark 2.1.16: *We underline that* $\mathcal{D}'_{\mathrm{C}}(M; \mathrm{Pol})$ *is stable under* δ_ψ *for any* $\psi \in \mathcal{E}(M)$ *– cf. Equation* (2.1.4). *Moreover, for* $n_1, \ldots, n_m \in \mathbb{N}$ *and* $\tau_j \in \mathcal{D}'(M^{n_j}; \mathrm{Pol})$ *for any* $j \in \{1, \ldots, m\}$*, we define the functional-valued tensor product distribution* $\tau_1 \otimes \ldots \otimes \tau_m \in \mathcal{D}'(M^n; \mathrm{Pol})$*, where* $n = n_1 + \ldots + n_m$*, as*

$$(\tau_1 \otimes \ldots \otimes \tau_m)(f_1 \otimes \ldots \otimes f_m; \varphi) = \tau_1(f_1; \varphi) \cdots \tau_m(f_m; \varphi)\,, \tag{2.1.14}$$

for any $f_j \in \mathcal{D}(M^{n_j})$*,* $j \in \{1, \ldots, m\}$*, and* $\varphi \in \mathcal{E}(M)$*. By direct inspection, for all* $k \geq 0$*,*

$$(\tau_1 \otimes \ldots \otimes \tau_m)^{(k)} = \sum_{\substack{k_1, \ldots, k_m \\ k_1 + \ldots + k_m = k}} \tau_1^{(k_1)} \otimes \ldots \otimes \tau_m^{(k_n)}\,.$$

Hence, if $\tau_j \in \mathcal{D}'_{\mathrm{C}}(M; \mathrm{Pol})$ *for any* $j \in \{1, \ldots, m\}$ *then* $\tau_1 \otimes \ldots \otimes \tau_m \in \mathcal{D}'_{\mathrm{C}}(M^n; \mathrm{Pol})$.

Example 2.1.17: *We notice that the functional-valued distributions* Φ^2 *and* $\Phi P \circledast \Phi$ *introduced in Example 2.1.8 are elements of* $\mathcal{D}'_{\mathrm{C}}(M; \mathrm{Pol})$*, as one can see from Equations* (2.1.7) *and* (2.1.8) *where the functional derivatives of these functional-valued distributions are explicitly computed.. The same holds true trivially for the functionals* Φ *and* $\mathbf{1}$.

We observe that Equations (2.1.7) *and* (2.1.8) *together with the micro-hypoellipticity of the operator* E*, implying* $\mathrm{WF}(P) = \mathrm{WF}(\delta_{\mathrm{Diag}_2})$*, provide an intuition of why we are interested in the sets of Equation* (2.1.11). *Indeed, functional derivatives (except for first order ones) of polynomial functional-valued distributions results in Dirac delta distributions on sub-diagonals as one can see from Equations* (2.1.7) *and* (2.1.8). *A priori, as it happens in the case of* $\Phi P \circledast \Phi$*, these sub-diagonals might be related through the integral kernel of the parametrix* P*, which again behaves like a Dirac delta distributions from a microlocal point of view.*

We observe that as a consequence the explicit form of the sets C_k *strongly depends on the micro-hypoellitpic nature of the partial differential operator. Indeed, a different class of operators, e.g., hyperbolic ones, would yield a different structure for these sets.*

A further important property of the space $\mathcal{D}'_{\mathrm{C}}(M; \mathrm{Pol})$ is its stability with respect to the "convolution" against the parametrix P. Furthermore, in the following Lemma we also prove an important result concerning the scaling degree – see Appendix B for a short survey of this topic – of this class of functional-valued distributions which will be of relevance in the forthcoming analysis of SPDEs.

Lemma 2.1.18: *Let* $\tau \in \mathcal{D}'_{\mathrm{C}}(M; \mathrm{Pol})$*. Then, setting*

$$[P \circledast \tau](f; \varphi) := \tau(P \circledast f; \varphi)\,, \qquad \forall\, f \in \mathcal{D}(M)\,, \forall\, \varphi \in \mathcal{E}(M)\,,$$

where $P \circledast f \in \mathcal{D}'(M)$ *is defined by* $(P \circledast f)(h) := P(h \otimes f)$*, it holds* $P \circledast \tau \in \mathcal{D}'_{\mathrm{C}}(M; \mathrm{Pol})$*. Furthermore, if* $\mathrm{sd}_{\mathrm{Diag}_{p+1}}(\tau)^{(p)} < \infty$ *for* $p \in \mathbb{N}$*, it holds*

$$\mathrm{sd}_{\mathrm{Diag}_{p+1}}(P \circledast \tau)^{(p)} < \infty\,, \tag{2.1.15}$$

where $\mathrm{sd}_{\mathrm{Diag}_{p+1}}$ *stands for the scaling degree with respect to the submanifold* $\mathrm{Diag}_{p+1} = \{(x, \ldots, x) \in M^{p+1} \mid x \in M\}$*, cf. Appendix B.*

Proof. To start with, we prove that $P \circledast \tau \in \mathcal{D}'(M;\mathrm{Fun})$ is well-defined and it is actually a functional-valued distribution. To this end, we observe that, on account of Equation (2.1.2), which in turns implies $\mathrm{WF}_2'(P) = \emptyset$ with the notation of Theorem A.0.6, the condition in Equation (A.0.8) is satisfied and thus $P \circledast \tau$ is well-defined as a distribution. Moreover, since τ is generated by a smooth function, it descends by Equation (A.0.10) that $\mathrm{WF}(P \circledast \tau) = \emptyset$ and thus $P \circledast \tau \in \mathcal{D}'(M;\mathrm{Fun})$.

In addition, assuming without loss of generality $\psi_1 = \ldots = \psi_k = \psi \in \mathcal{E}(M)$ in Equation (2.1.3), it descends

$$(P \circledast \tau)^{(k)}(f \otimes \psi^{\otimes k};\varphi) = \tau^{(k)}((P \circledast f) \otimes \psi^{\otimes k};\varphi) = [(P \otimes \delta_{\mathrm{Diag}_2}^{\otimes k}) \circledast \tau^{(k)}](f \otimes \psi^{\otimes k};\varphi)\,,$$

guaranteeing that $P \circledast \tau \in \mathcal{D}'(M;\mathrm{Pol})$ since $(P \otimes \delta_{\mathrm{Diag}_2}^{\otimes k})$ is a compactly supported distribution.

In order to conclude the proof we need an estimate for $\mathrm{WF}((P \circledast \tau)^{(k)})$ so to prove that it is actually contained in C_{k+1} as per Equation (2.1.11). To this end, Equations (A.0.2) and (A.0.17) together with item *2.* of Theorem A.0.6 entail

$$\begin{aligned}\mathrm{WF}(P \otimes \delta_{\mathrm{Diag}_2}^{\otimes k}) = \{&(x_1, x_2, \widehat{z}_k, \widehat{y}_k, \xi_1, \xi_2, \widehat{\zeta}_k, \widehat{\eta}_k) \in T^*M^{2+2k} \setminus \{0\} \mid \\ &(x_1, x_2, \xi_1, \xi_2) \in \mathrm{WF}(\delta_{\mathrm{Diag}_2})\,,\, (\widehat{z}_k, \widehat{y}_k, \widehat{\zeta}_k, \widehat{\eta}_k) \in \mathrm{WF}(\delta_{\mathrm{Diag}_2}^{\otimes k})\} \\ = \{&(x, x, \widehat{z}_k, \widehat{z}_k, \xi, -\xi, \widehat{\zeta}_k, -\widehat{\zeta}_k) \in T^*M^{2+2k} \setminus \{0\}\}\,.\end{aligned}$$

Equation (A.0.7) yields

$$\begin{aligned}\mathrm{WF}_2(P \otimes \delta_{\mathrm{Diag}_2}^{\otimes k}) = \{&(x_2, \widehat{y}_k, \xi_2, \widehat{\eta}_k) \in T^*M^{1+k} \setminus \{0\} \mid \\ &\exists x_1 \in M\,,\ \widehat{z}_k \in M^k\,, (x_1, x_2, \widehat{z}_k, \widehat{y}_k, 0, \xi_2, 0, \widehat{\eta}_k) \in \mathrm{WF}(P \otimes \delta_{\mathrm{Diag}_2}^{\otimes k})\} = \emptyset\,,\end{aligned}$$

and similarly $\mathrm{WF}_1(P \otimes \delta_{\mathrm{Diag}_2}^{\otimes k}) = \emptyset$. This entails that the condition expressed by Equation (A.0.8) is satisfied, so that $(P \otimes \delta_{\mathrm{Diag}_2}^{\otimes k}) \circledast \tau^{(k)}$ is a well-defined functional-valued distribution. Moreover, Theorem A.0.6 and Equation (A.0.10) imply

$$\begin{aligned}\mathrm{WF}((P \otimes \delta_{\mathrm{Diag}_2}^{\otimes k}) \circledast \tau^{(k)}) &\subseteq \mathrm{WF}(P \otimes \delta_{\mathrm{Diag}_2}^{\otimes k}) \circ \mathrm{WF}(\tau^{(k)}) \\ &= \{(x_1, \widehat{z}_k, \xi_1, \widehat{\zeta}_k) \in T^*M^{1+k} \setminus \{0\} \mid \\ &\exists (x_2, \widehat{y}_k, \xi_2, \widehat{\eta}_k) \in \mathrm{WF}(\tau^{(k)})\,,\, (x_1, x_2, \widehat{z}_k, \widehat{y}_k, \xi_1, \xi_2, \widehat{\zeta}_k, \widehat{\eta}_k) \in \mathrm{WF}(P \otimes \delta_{\mathrm{Diag}_2}^{\otimes k})\} \\ &\subseteq \mathrm{C}_{k+1}\,,\end{aligned}$$

where we used the bound $\mathrm{WF}(\tau^{(k)}) \subseteq \mathrm{C}_{k+1}$ as well as the explicit form of $\mathrm{WF}(P \circledast \delta_{\mathrm{Diag}_2}^{\otimes k})$.

To conclude, in order to prove Equation (2.1.15) it suffices to show that, when dealing with $P \circledast \tau^{(k)}$, Lemma B.4.1 applies. Indeed, it implies

$$\mathrm{sd}_{\mathrm{Diag}_{k+1}}((P \circledast \tau)^{(k)}) < \infty\,.$$

To this end, we have to prove that $P \in \mathcal{D}'(M \times M)$ while $\tau^{(k)} \in \mathcal{D}'(M^{k+1})$ satisfies the hypotheses of Lemma B.4.1. First of all, we observe that, since $\mathrm{WF}_2'(P) = \emptyset$, item *1.* of Lemma B.4.1 is satisfied. The same holds true for items *2.* and *3.* on account of Remark 2.1.5 and of Example B.1.5, respectively. It remains to check item *4.* of Lemma B.4.1. In particular we have to prove that, setting $T := (P \otimes \tau^{(k)}) \cdot (1 \otimes \delta_{\mathrm{Diag}_2} \otimes 1_k) \in \mathcal{D}'(M^{k+3})$, Equation (B.4.2) holds true, namely

$$\mathrm{WF}(T) \cap \{(x, y, z, \hat{x}; \xi, 0, 0, \hat{\eta}_k) \in T^*M^{k+3} \setminus \{0\} \,|\, x \neq y \,,\, x \neq z\} = \emptyset \,,$$

On account of Theorem A.0.6, in particular of item *2.* and *3.*, respectively,

$$\begin{aligned}\mathrm{WF}(P \otimes \tau^{(k)}) \subseteq \underbrace{\big(\mathrm{WF}(P) \times \mathrm{WF}(\tau^{(k)})\big)}_{A} &\cup \underbrace{\big(\mathrm{WF}(P) \times (\mathrm{spt}(\tau^{(k)}) \times \{0\})\big)}_{B} \\ &\cup \underbrace{\big((\mathrm{spt}(P) \times \{0\}) \times \mathrm{WF}(\tau^{(k)})\big)}_{C} \,, \end{aligned} \tag{2.1.16}$$

and

$$\begin{aligned}\mathrm{WF}(T) \subseteq \{(x_1, x_2, x_3, \widehat{y}_k; &\xi_1 + \eta_1, \xi_2 + \eta_2, \xi_3 + \eta_3, \widehat{\zeta}_k + \widehat{\sigma}_k) \in T^*M^{k+3} \setminus \{0\} \,| \\ &(x_1, x_2, x_3, \widehat{y}_k; \xi_1, \xi_2, \xi_3, \widehat{\zeta}_k) \in \mathrm{WF}(P \otimes \tau^{(k)}) \text{ or } \widehat{\zeta}_k = 0 \,, \widehat{\xi}_3 = 0 \,, \\ &(x_1, x_2, x_3, \widehat{y}_k; \eta_1, \eta_2, \eta_3, \widehat{\sigma}_k) \in \mathrm{WF}(1 \otimes \delta_{\mathrm{Diag}_2} \otimes 1_k) \text{ or } \widehat{\sigma}_k = 0 \,, \widehat{\eta}_3 = 0 \,\} \,. \end{aligned} \tag{2.1.17}$$

As a matter of fact, from

$$\mathrm{WF}(1 \otimes \delta_{\mathrm{Diag}_2} \otimes 1_k) = \{(x_1, x_2, x_2, \widehat{y}_k; 0, \eta, -\eta, \widehat{0}) \in T^*M^{k+3} \setminus \{0\}\} \,,$$

we can estimate separately the contributions to $\mathrm{WF}(T)$ coming from A, B and C. Referring to them for convenience as $\mathrm{WF}^{\#}(T)$ with $\# = A, B, C$, we start from the middle term. It descends

$$\mathrm{WF}^{B}(T) \subseteq \{(x, x, x, \widehat{y}_k; \xi, -\xi + \eta, -\eta, \widehat{0}) \in T^*M^{k+3} \setminus \{0\}\} \,. \tag{2.1.18}$$

This set cannot contain elements of the form $\{(x, y, z, \hat{x}; \xi, 0, 0, \hat{\eta}_k) \in T^*M^{k+3} \setminus \{0\} \,|\, x \neq y \,,\, x \neq z\}$ due to the explicit form of the singular support as per Equation (2.1.18).
An analogous argument holds true for the contribution $\mathrm{WF}^{A}(T)$ and therefore we omit it.
Focusing on C, it holds

$$\mathrm{WF}^{C}(T) \subseteq \{(x_1, x_2, x_2, \widehat{y}_k; 0, \eta, \xi - \eta, \widehat{\zeta}_k) \in T^*M^{k+3} \setminus \{0\} \,|\, (x_2, \widehat{y}_k; \xi, \widehat{\zeta}_k) \in C_{k+1}\} \,.$$

with C_{k+1} as per Equation (2.1.11). This set cannot contain elements of the form

$$\{(x, y, z, \hat{x}; \xi, 0, 0, \hat{\eta}_k) \in T^*M^{k+3} \setminus \{0\} \,|\, x \neq y \,,\, x \neq z\} \,,$$

since this would imply $(x_2, \widehat{x}; 0, \widehat{\zeta}_k) \in C_{k+1}$ with $x_2 \neq x$. This is incompatible with the definition of C_{k+1} as per Definition 2.1.11.
This concludes the proof. □

Finally, before moving to the *deformation* argument applied to the algebra product, which will be one of the key ingredients in the microlocal approach to SPDEs, we introduce a suitable subclass of elements of $\mathcal{D}'_C(M; \mathrm{Pol})$. These have an algebra structure and they contain all the functional-valued distributions needed for describing the kinematics of an SPDE. In particular, this algebra $\mathcal{A}$ does not contain any information about expectation values and white noise. Yet it will play a key rôle in the construction of a counterpart, which we denote $\mathcal{A}_{\cdot_Q}$ encoding the stochastic properties. From the point of view of algebraic quantum field theory, this is tantamount to introducing the algebra of the observables of the underlying model.

Remark 2.1.19: *In the following, for any subset $\mathcal{O} \subseteq \mathcal{D}'(M; \mathrm{Fun})$, we shall denote by $\mathcal{E}[\mathcal{O}]$ the smallest $\mathcal{E}(M)$-ring containing $\mathcal{O}$, where the action of the ring $\mathcal{E}(M)$ on functionals in $\mathcal{O}$ is defined, for any $\psi \in \mathcal{E}(M)$ and $F \in \mathcal{O}$, as $\psi F(f; \varphi) = F(f; \psi\varphi)$ for any $f \in \mathcal{D}(M)$ and $\varphi \in \mathcal{E}(M)$. We refer in addition to $\mathcal{E}[\mathcal{O}]$ as the polynomial ring on $\mathcal{E}(M)$ generated by elements in $\mathcal{O}$ – though $\mathcal{O}$ is not required to be a countable set.*

Definition 2.1.20: *Let $\mathbf{1}, \Phi \in \mathcal{D}'(M; \mathrm{Pol})$ be the functional-valued distributions*

$$\Phi(f; \varphi) := \int_M f\varphi\mu\,, \qquad \mathbf{1}(f; \varphi) = \int_M f\mu\,. \tag{2.1.19}$$

We define a unital, commutative $\mathbb{C}$-algebra $\mathcal{A}$ as follows. We set recursively

$$\mathcal{A}_0 := \mathcal{E}[\mathbf{1}, \Phi]\,, \qquad \mathcal{A}_j := \mathcal{E}[\mathcal{A}_{j-1} \cup P \circledast \mathcal{A}_{j-1}]\,, \qquad \forall j \in \mathbb{N}\,, \tag{2.1.20}$$

denoting $P \circledast \mathcal{A}_{j-1} := \{P \circledast \tau \,|\, \tau \in \mathcal{A}_{j-1}\}$. These algebras are ordered by inclusion, i.e., $\mathcal{A}_{j_1} \subseteq \mathcal{A}_{j_2}$ whenever $j_1 \leq j_2$, and, as a consequence, we can introduce the direct limit

$$\mathcal{A} = \varinjlim \mathcal{A}_j\,, \tag{2.1.21}$$

which is thus a commutative and associative $\mathbb{C}$-algebra, as well as an $\mathcal{E}(M)$-module. The $\mathbb{C}$-algebra structure is codified by the pointwise product

$$[\tau_1\tau_2](f; \varphi) := (\tau_1 \otimes \tau_2)(f\delta_{\mathrm{Diag}_2}; \varphi)\,, \qquad \forall\, \tau_1, \tau_2 \in \mathcal{A}\,. \tag{2.1.22}$$

Example 2.1.21: *To see how the product of Equation* (2.1.22) *works explicitly we consider the following example recalling the functional-valued distributions discussed in Example 2.1.8. In particular we observe that $\Phi\Phi = \Phi^2$. Indeed, for any $\varphi \in \mathcal{E}(M)$ and $f \in \mathcal{D}(M)$,*

$$\begin{aligned}[\Phi\Phi](f; \varphi) = [\Phi \otimes \Phi](f\delta_{\mathrm{Diag}_2}; \varphi) &= \int_{M\times M} \varphi(x)\varphi(y)f(x)\delta_{\mathrm{Diag}_2}(x, y)\, d\mu(x)d\mu(y) \\ &= \int_M \varphi^2(x)f(x)\, d\mu(x) = \Phi^2(f; \varphi)\,.\end{aligned}$$

Remark 2.1.22: *More in general, we observe that the pointwise product on $\mathcal{A}$ is well-defined because any $\tau \in \mathcal{A}$ is a functional-valued distribution generated by a smooth function, as we discussed in Remark 2.1.13. Furthermore, we notice that the algebra $\mathcal{A}$ is stable under the action of δ_ψ for any $\psi \in \mathcal{E}(M)$ – cf. Equation* (2.1.4).

Remark 2.1.23: *An important feature of the algebra $\mathcal{A}$ (*resp.* of each $\mathcal{A}_j$, $j \geq 0$), which we are going to use extensively in the results of Chapter 3, is that it is a positively bigraded algebra over the ring $\mathcal{E}(M)$, i.e.,*

$$\mathcal{A} = \bigoplus_{l,k\in\mathbb{N}_0} \mathcal{M}_{l,k}\,, \qquad \mathcal{A}_j = \bigoplus_{l,k\in\mathbb{N}_0} \mathcal{M}^j_{l,k}\,,$$

where $\mathcal{M}_{l,k}$ is the $\mathcal{E}(M)$-module generated by the elements of $\mathcal{A}$ in which the parametrix P acts l-times while Φ appears only in degree k, e.g., $P \circledast (\Phi^2 P \circledast \Phi^3) \in \mathcal{M}_{2,5}$. At the same time $\mathcal{M}^j_{l,k} \doteq \mathcal{M}_{l,k} \cap \mathcal{A}_j$. For later convenience, we also introduce $\mathcal{M}_k := \bigoplus_{\substack{l\in\mathbb{N}_0\\ p\leq k}} \mathcal{M}_{l,p}$ as well as $\mathcal{M}^j_k := \bigoplus_{\substack{l\in\mathbb{N}_0\\ p\leq k}} \mathcal{M}^j_{l,p}$.
In addition it holds that

$$\mathcal{A} = \varinjlim \mathcal{M}_k\,, \qquad \mathcal{M}_k\mathcal{M}_{k'} = \mathcal{M}_{k+k'},\ \forall k, k' \in \mathbb{N}_0\,. \tag{2.1.23}$$

Remark 2.1.24: *We focus now on the functional derivatives of the elements in* $\mathcal{A}$*. These are distributions whose wavefront set is controlled as per Definition 2.1.11. In particular, for any* $\tau \in \mathcal{A}$ *and* $p \in \mathbb{N}$, $\tau^{(p)}$ *may be singular only on the total diagonal* $\mathrm{Diag}_{p+1} \subset M^{p+1}$*. In the next lemma we prove a bound for the scaling degree of* $\tau^{(p)}$ *with respect to* Diag_{p+1} *– cf. Definition B.1.1.*

Lemma 2.1.25: *Let* $\mathcal{A}$ *be the algebra as per Definition 2.1.20. For any* $\tau \in \mathcal{A}$ *and* $p \in \mathbb{N}$*, let* $\sigma_p(\tau) := \mathrm{sd}_{\mathrm{Diag}_{p+1}}(\tau^{(p)})$ *– cf. Definition B.1.1. It follows,*

$$\sigma_p(\tau) < \infty\,, \qquad \forall\, \tau \in \mathcal{A}\,, \quad \forall\, p \in \mathbb{N}\,. \tag{2.1.24}$$

Proof. To start with, we observe that, if $\tau = \mathbf{1}$, Equation (2.1.24) holds true by direct inspection, while, for $\tau = \Phi$, *cf.* Equation (2.1.19), the thesis follows from the identities

$$\Phi^{(1)} = \delta_{\mathrm{Diag}_2}\,, \qquad \Phi^{(p)} = 0, \;\; \forall p > 1\,,$$

together with Example B.1.4. As second step we show that, if Equation (2.1.24) holds true for $\tau_1, \tau_2 \in \mathcal{A}$, then the same applies to the product $\tau_1\tau_2$. As a matter of fact, by Leibniz rule

$$(\tau_1\tau_2)^{(p)} = \sum_{\substack{p_1,p_2\\ p_1+p_2=p}} \tau_1^{(p_1)} \otimes \tau_2^{(p_2)}\,,$$

which, in turn, implies the inequality

$$\sigma_p(\tau_1\tau_2) \leq \max_{p_1+p_2=p} \sigma_p(\tau_1^{(p_1)} \otimes \tau_2^{(p_2)}) \leq \max_{p_1+p_2=p} \Big(\sigma_{p_1}(\tau_1^{(p_1)}) + \sigma_{p_2}(\tau_2^{(p_2)})\Big) < \infty\,. \tag{2.1.25}$$

To conclude it suffices to observe that, if Equation (2.1.24) holds true for $\tau \in \mathcal{A}$, then the same applies to $\psi\tau$ and to $P \circledast \tau$ with $\psi \in \mathcal{E}(M)$. While the first statement is straightforward, the second is a consequence of Lemma B.4.1. Indeed $(P \circledast \tau)^{(k)} = P \circledast \tau^{(k)}$ and we notice that the hypotheses of Lemma B.4.1 are met on account of the microlocal behaviour of $\tau^{(k)}$ which is codified by the space C_{k+1}, *cf.* Equation (2.1.11), together with the properties of the parametrix[2] P. Summarizing, one obtains

$$\sigma_p(P \circledast \tau) < \infty\,.$$

On account of Definition 2.1.20 we can cover inductively all cases, hence proving the sought statement. □

2.2 Smooth Deformation of the Algebra Product

In this section we introduce a smooth deformation of the product of algebra $\mathcal{A}$ which encodes the stochastic nature induced by the white noise. As we shall see in the remaining part of this manuscript, this deformation argument together with the renormalization procedure is at the heart of the microlocal approach to SPDEs developed in [27].

[2]This argument is analogous to the one used in the proof of Equation (2.1.15).

Remark 2.2.1: *The starting point is the underlying linear SPDEs, which serves as motivation for the chosen deformation. As we anticipated at the beginning of Section 2.1, this is represented by the equation $E\widehat{\psi} = \widehat{\xi}$, where we are using that hat-notation in order to underline that these objects are the stochastic ones and in order to distinguish them from the smooth configurations of the functional-valued distributions.*
As we saw in Chapter 1, the solution of the equation $E\widehat{\psi} = \widehat{\xi}$ is the so-called stochastic convolution $\widehat{\varphi} = P * \widehat{\xi}$ with P the fundamental solution, or also a parametrix, of E. We recall furthermore that, in general, $\widehat{\varphi}$ is a Gaussian random distribution such that, for any test-functions $f, g \in \mathcal{D}(M)$,

$$\mathbb{E}[\widehat{\varphi}(f)] = 0\,, \qquad \mathbb{E}[\widehat{\varphi}(f)\widehat{\varphi}(g)] = (P \circ P^*)(f \otimes g) =: Q(f \otimes g)\,, \tag{2.2.1}$$

where we introduced the bi-distribution[3] $Q := P \circ P^*$, where $\circ$ denotes the composition of distributions as per item 5. of Theorem A.0.6. Furthermore we observe that $Q \in \mathcal{D}'(M \times M)$ is a well-defined distribution on account of the microlocal properties of P as well as of the discussion in Remark 2.1.5.

Finally, we notice that had we considered a regularized version of the equation $E\widehat{\psi} = \widehat{\xi}$, namely replacing $\widehat{\xi}$ with $\widehat{\xi}_\varepsilon := \widehat{\xi} * \rho^\varepsilon$ for a mollifier $\rho \in \mathcal{D}(M)$ as in Equation (1.3.6), the solution would be $\widehat{\varphi}_\varepsilon = P * \widehat{\xi}_\varepsilon$. Then Equation (2.2.1) would become

$$\mathbb{E}[\widehat{\varphi}_\varepsilon(f)] = 0\,, \qquad \mathbb{E}[\widehat{\varphi}_\varepsilon(f)\widehat{\varphi}_\varepsilon(g)] = (P_\varepsilon \circ P_\varepsilon^*)(f \otimes g) =: Q_\varepsilon(f \otimes g)\,,$$

with $P_\varepsilon \in \mathcal{E}(M \times M)$ an ε-regularization of P, i.e., a smooth function satisfying $\lim_{\varepsilon\to 0^+} P_\varepsilon = P$ in $\mathcal{D}'(M \times M)$.

In the next chapter we shall comment quite in detail the problems arising when taking the limit for $\varepsilon \to 0^+$ of the expectation values of arbitrary functions of φ_ε.

In the previous section we constructed the algebra $\mathcal{A}$, whose product is the pointwise one. We observe that it is possible to construct algebras which are isomorphic to $\mathcal{A}$ by deforming in a suitable sense the pointwise product. On account of the argument of Remark 2.2.1, consider an ε-regularization $Q_\varepsilon \in \mathcal{E}(M \times M)$ of $Q \in \mathcal{D}'(M \times M)$, *i.e.*, given a family of smooth maps $P_\varepsilon \subset \mathcal{E}(M \times M)$ such that $\lim_{\varepsilon\to 0^+} P_\varepsilon = P$ in the sense of $\mathcal{D}'(M \times M)$, let $Q_\varepsilon := P_\varepsilon \circ P_\varepsilon$. Then, for any $\tau_1, \tau_2 \in \mathcal{A}$, let

$$(\tau_1 \cdot_{Q_\varepsilon} \tau_2)(f;\varphi) := \sum_{n=0}^{\infty} \frac{1}{n!}\big[(\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes n}) \cdot (\tau_1^{(n)} \widetilde{\otimes} \tau_2^{(n)})\big](f \otimes 1_{1+2n};\varphi)\,, \tag{2.2.2}$$

for all $f \in \mathcal{D}(M)$ and $\varphi \in \mathcal{E}(M)$. Here the symbol $\widetilde{\otimes}$ indicates that the integral kernel of $(\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes n}) \cdot (\tau_1^{(n)} \widetilde{\otimes} \tau_2^{(n)})$ reads

$$\delta(x_1, x_2) \prod_{i=1}^{n} Q_\varepsilon(z_i, y_i)\, \tau_1^{(n)}(x_1, z_1, \ldots, z_n)\, \tau_2^{(n)}(x_2, y_1, \ldots, y_n)\,. \tag{2.2.3}$$

More in general, Equation (2.2.2) defines an algebra product on the whole space $\mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$, as per the following proposition.

[3] We observe that, when considering the elliptic scenario, then P is formally self-adjoint and thus $P = P^*$. This is not true if one considers the parabolic case.

Proposition 2.2.2: *Let M be a compact manifold and let $P_\varepsilon \in \mathcal{E}(M\times M)$ be a family of smooth maps such that $\lim_{\varepsilon\to 0^+} P_\varepsilon = P$ in $\mathcal{D}'(M\times M)$ and let $Q_\varepsilon := P_\varepsilon \circ P_\varepsilon^* \in \mathcal{E}(M^2)$. For any $\tau_1,\tau_2 \in \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$ let $\tau_1 \cdot_{Q_\varepsilon} \tau_2 \in \mathcal{D}'(M;\mathrm{Pol})$ be*

$$(\tau_1 \cdot_{Q_\varepsilon} \tau_2)(f;\varphi) = \sum_{k\geq 0} \frac{1}{k!} [(\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes k}) \cdot (\tau_1^{(k)} \widetilde{\otimes} \tau_2^{(k)})](f\otimes 1_{1+2k};\varphi)\,, \tag{2.2.4}$$

for any $f \in \mathcal{D}(M)$ and $\varphi \in \mathcal{E}(M)$. Here, for all $\ell \in \mathbb{N}$, $1_\ell \in \mathcal{E}(M^\ell)$ denotes the function $1_\ell(\widehat{x}_\ell) = 1$. As above, $\widetilde{\otimes}$ is defined as per Equation (2.2.3). Then $\tau_1 \cdot_{Q_\varepsilon} \tau_2 \in \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$ and $(\mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol}), \cdot_{Q_\varepsilon})$ is a unital commutative and associative algebra.

Proof. The proof is divided in three steps. First we show that Equation (2.2.4) actually defines a functional-valued distribution $\tau_1 \cdot_{Q_\varepsilon} \tau_2 \in \mathcal{D}'(M;\mathrm{Pol})$. In the second part we check the microlocal properties, showing that $\tau_1 \cdot_{Q_\varepsilon} \tau_2 \in \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$. Finally we discuss the ensuing algebraic structure.

Proof that $\tau_1 \cdot_{Q_\varepsilon} \tau_2 \in \mathcal{D}'(M;\mathrm{Pol})$. On account of the polynomial nature of $\tau_1,\tau_2 \in \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$, the right hand side of Equation (2.2.4) contains only a finite number of non-vanishing terms. As a consequence, it suffices to show that each of these terms is well-defined. Let $k\geq 0$ and consider

$$[(\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes k}) \cdot (\tau_1^{(k)} \widetilde{\otimes} \tau_2^{(k)})]\,.$$

On account of Equation (A.0.2) and since $Q_\varepsilon \in \mathcal{E}(M\times M)$, it holds

$$\mathrm{WF}(\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes k}) = \{(x,x,\widehat{z}_k,\widehat{y}_k;\xi,-\xi,0,0) \in T^*M^{2+2k}\setminus\{0\}\}\,, \tag{2.2.5}$$

where $\widehat{z}_k,\widehat{y}_k \in M^k$ while $(x,\xi)\in T^*M$.
In addition, Remark 2.1.16 entails that $\tau_1^{(k)} \widetilde{\otimes} \tau_2^{(k)} \in \mathcal{D}'_{\mathrm{C}}(M^{2+2k};\mathrm{Pol})$ and

$$\begin{aligned}\mathrm{WF}(\tau_1^{(k)} \widetilde{\otimes} \tau_2^{(k)}) \subseteq \{&(x_1,x_2,\widehat{z}_k,\widehat{y}_k,\xi_1,\xi_2,\widehat{\zeta}_k,\widehat{\eta}_k) \in T^*M^{2+2k}\setminus\{0\} \mid \\ &(x_1,\widehat{z}_k,\xi_1,\widehat{\zeta}_k),(x_2,\widehat{y}_k,\xi_2,\widehat{\eta}_k) \in \mathrm{C}_k\}\,.\end{aligned}$$

As a consequence Equation (A.0.5) holds true since

$$\begin{aligned}(x_1,x_2,\widehat{z}_k,\widehat{y}_k,0,0,0,0) \notin \{&(x_1,x_2,\widehat{z}_k,\widehat{y}_k,\xi_1+\xi_1',\xi_2+\xi_2',\widehat{\zeta}_k+\widehat{\zeta}_k',\widehat{\eta}_k+\widehat{\eta}_k') \in T^*M^{2+2k} \mid \\ &(x_1,x_2,\widehat{z}_k,\widehat{\eta}_k,\xi_1,\xi_2,\widehat{\zeta}_k,\widehat{\eta}_k) \in \mathrm{WF}(\delta_{\mathrm{Diag}_2}\otimes Q_\varepsilon^{\otimes k})\,,\\ &(x_1,\widehat{z}_k,\xi_1',\widehat{\zeta}_k') \in \mathrm{WF}(\tau_1^{(k)})\,,\,(x_2,\widehat{y}_k,\xi_2',\widehat{\eta}_k') \in \mathrm{WF}(\tau_2^{(k)})\}\,.\end{aligned}$$

Hence, by item *3* of Theorem A.0.6 it holds

$$[(\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes k}) \cdot (\tau_1^{(k)} \widetilde{\otimes} \tau_2^{(k)})] \in \mathcal{D}'(M^{2+2k};\mathrm{Fun})\,.$$

Moreover, since for any $\ell\in\mathbb{N}$ it holds

$$[(\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes k}) \cdot (\tau_1^{(k)} \widetilde{\otimes} \tau_2^{(k)})]^{(\ell)} = \sum_{\substack{\ell_1,\ell_2\\ \ell_1+\ell_2=\ell}} [(\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes k}\otimes 1_\ell) \cdot (\tau_1^{(k+\ell_1)} \widetilde{\otimes} \tau_2^{(k+\ell_2)})]\,,$$

and taking into account that both τ_1 and τ_2 are polynomial functionals, it descends that $[(\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes k}\otimes 1_\ell) \cdot (\tau_1^{(k+\ell_1)} \widetilde{\otimes} \tau_2^{(k+\ell_2)})] \in \mathcal{D}'(M^{2+2k+\ell};\mathrm{Pol})$.

Proof that $\tau_1 \cdot_{Q_\varepsilon} \tau_2 \in \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$. We need to prove that, for any $\ell \geq 0$ it holds

$$\mathrm{WF}([\tau_1 \cdot_{Q_\varepsilon} \tau_2]^{(\ell)}) \subseteq \mathrm{C}_{\ell+1} .$$

By direct computation,

$$\begin{aligned}
[\tau_1 \cdot_{Q_\varepsilon} \tau_2]^{(\ell)}(f \otimes \psi_\ell;\varphi) &= \sum_{k\geq 0} \frac{1}{k!} \sum_{\ell_1+\ell_2=\ell} [(\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes k} \otimes 1_\ell) \cdot (\tau_1^{(k+\ell_1)} \widetilde{\otimes} \tau_2^{(k+\ell_2)})](f \otimes 1_{1+2k} \otimes \psi_\ell;\varphi) \\
&=: \sum_{k\geq 0} \frac{1}{k!} \sum_{\ell_1+\ell_2=\ell} T_{\ell_1,\ell_2}(f \otimes \psi_\ell;\varphi) .
\end{aligned}$$

where $f \in \mathcal{D}(M)$, $\psi_\ell \in \mathcal{E}(M^\ell)$ and $\varphi \in \mathcal{E}(M)$. We shall estimate the wave front set separately for each term T_{ℓ_1,ℓ_2} in the above sum. Let $\ell_1, \ell_2 \in \mathbb{N} \cup \{0\}$ be such that $\ell_1 + \ell_2 = \ell$ and consider

$$[(\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes k} \otimes 1_\ell) \cdot (\tau_1^{(k+\ell_1)} \widetilde{\otimes} \tau_2^{(k+\ell_2)})] ,$$

whose wavefront set reads, in view of Equation (2.2.5),

$$\begin{aligned}
&\mathrm{WF}([(\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes k} \otimes 1_\ell) \cdot (\tau_1^{(k+\ell_1)} \widetilde{\otimes} \tau_2^{(k+\ell_2)})]) \subseteq \\
&\qquad \{(x, x, \widehat{z}_k, \widehat{y}_k, \widehat{u}_\ell, \xi_1 + \xi_1', -\xi_1 + \xi_2', \widehat{\zeta}_k, \widehat{\eta}_k, \widehat{\nu}_\ell) \in T^*M^{2+2k+\ell} \setminus \{0\} \mid \\
&\qquad\qquad (x, \widehat{z}_k, \widehat{u}_{\ell_1}, \xi_1', \widehat{\zeta}_k, \widehat{\nu}_{\ell_1}) \in \mathrm{C}_{k+\ell_1} ,\ (x, \widehat{y}_k, \widehat{u}_{\ell_2}, \xi_2', \widehat{\eta}_k, \widehat{\nu}_{\ell_2}) \in \mathrm{C}_{k+\ell_2}\} .
\end{aligned} \tag{2.2.6}$$

For our purposes, we only need to consider the functional-valued distribution obtained from $[(\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes k} \otimes 1_\ell) \cdot (\tau_1^{(k+\ell_1)} \widetilde{\otimes} \tau_2^{(k+\ell_2)})]$ after its partial evaluation against the constant function 1_{1+2k},

$$T_{\ell_1,\ell_2}(f \otimes \psi_\ell;\varphi) := [(\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes k} \otimes 1_\ell) \cdot (\tau_1^{(k+\ell_1)} \otimes \tau_2^{(k+\ell_2)})](f \otimes 1_{1+2k} \otimes \psi_\ell;\varphi) .$$

On account of the argument in Remark A.0.8, we have

$$\begin{aligned}
\mathrm{WF}(T_{\ell_1,\ell_2}) \subseteq \{&(x_1, \widehat{u}_\ell, \xi_1, \widehat{\nu}_\ell) \in T^*M^{1+\ell} \setminus \{0\} \mid \exists x_2 \in M \text{ and } \exists \widehat{z}_k, \widehat{y}_k \in M^k , \\
&(x_1, x_2, \widehat{z}_k, \widehat{y}_k, \widehat{u}_\ell, \xi_1, 0, 0, 0, \widehat{\nu}_\ell) \in \mathrm{WF}((\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes k}) \cdot (\tau_1^{(k+\ell_1)} \widetilde{\otimes} \tau_2^{(k+\ell_2)}))\} \subseteq \mathrm{C}_{1+\ell} ,
\end{aligned}$$

where the last inclusion follows from the bound on the wave front set of $(\delta_{\mathrm{Diag}_2} \otimes Q_\varepsilon^{\otimes k} \otimes 1_\ell) \cdot (\tau_1^{(k+\ell_1)} \widetilde{\otimes} \tau_2^{(k+\ell_2)})$ – *cf.* Equation (2.2.6), as well as from the definition of the sets C_ℓ as per Equation (2.1.11) and Remark 2.1.15. This entails $\tau_1 \cdot_{Q_\varepsilon} \tau_2 \in \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$.

Algebraic properties of $\mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$. Eventually, in order to conclude the proof, we need to show that $\cdot_{Q_\varepsilon}$ endows the space $\mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$ with a unital and commutative algebra structure. In the preceding steps we have shown that

$$\cdot_{Q_\varepsilon} \colon \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol}) \times \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol}) \to \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol}) ,$$

entailing that $(\mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol}), \cdot_{Q_\varepsilon})$ is an algebra. It is unital, since, being $\mathbf{1}(f;\varphi) := 1(f)$ as per Equation (2.1.19), then,

$$\tau \cdot_{Q_\varepsilon} \mathbf{1} = \mathbf{1} \cdot_{Q_\varepsilon} \tau = \tau , \qquad \forall \tau \in \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol}) .$$

In addition, Equation (2.2.4) is symmetric in τ_1 and τ_2, since $Q_\varepsilon = P_\varepsilon \circ P_\varepsilon^*$ is symmetric by itself. It only remains to be shown that $\cdot_{Q_\varepsilon}$ is associative. To this end, we introduce a useful notation:

(i) for $\tau_1,\tau_2\in\mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$ we denote by $\tau_1\widehat{\otimes}\tau_2\in\mathcal{D}'(M^2;\mathrm{Pol}^{\otimes 2})$ the functional-valued distribution

$$[\tau_1\widehat{\otimes}\tau_2](f_1\otimes f_2;\varphi_1\otimes\varphi_2):=\tau_1(f_1;\varphi_1)\tau_2(f_2;\varphi_2)\,,$$

for any $f_1,f_2\in\mathcal{D}(M)$ and $\varphi_1,\varphi_2\in\mathcal{E}(M)$. Notice that $\tau_1\widehat{\otimes}\tau_2$ is a functional defined on pairs of configurations φ_1,φ_2.

(ii) We call $\mathsf{m}\colon\mathcal{D}'(M^2;\mathrm{Pol}^{\otimes 2})\to\mathcal{D}'(M^2;\mathrm{Pol})$

$$\mathsf{m}(\tau_1\widehat{\otimes}\tau_2):=\tau_1\otimes\tau_2\,,$$

where $\tau_1\otimes\tau_2\in\mathcal{D}'(M^2;\mathrm{Pol})$ is defined as in Remark (2.1.16).

(iii) We call $\Upsilon_{Q_\varepsilon}\colon\mathcal{D}'_{\mathrm{C}}(M^2;\mathrm{Pol}^{\otimes 2})\to\mathcal{D}'_{\mathrm{C}}(M^2;\mathrm{Pol}^{\otimes 2})$ the linear map

$$[\Upsilon_{Q_\varepsilon}(\tau_1\widehat{\otimes}\tau_2)](f_1\otimes f_2;\varphi_1\otimes\varphi_2):=[(1_2\otimes Q_\varepsilon)\cdot(\tau_1^{(1)}\widehat{\otimes}\tau_2^{(1)})](f_1\otimes f_2\otimes 1_2;\varphi_1\otimes\varphi_2)\,,$$

for all $f_1,f_2\in\mathcal{D}(M)$ and $\varphi_1,\varphi_2\in\mathcal{E}(M)$.

With these notations, Equation (2.2.4) boils down to

$$[\tau_1\cdot_{Q_\varepsilon}\tau_2](f;\varphi)=[\delta_{\mathrm{Diag}_2}\cdot\mathsf{m}\circ\exp[\Upsilon_{Q_\varepsilon}](\tau_1\widehat{\otimes}\tau_2)](f\otimes 1;\varphi)\,,$$

where $\circ$ stands here for the composition between maps while $\exp[\Upsilon_{Q_\varepsilon}]=\sum_{k\geq 0}\frac{1}{k!}\Upsilon^k_{Q_\varepsilon}$. We observe that only a finite number of terms in the above sum is non-vanishing due to the polynomial behaviour of the underlying functionals. If $\tau_1,\tau_2,\tau_3\in\mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$ it follows, for any $f\in\mathcal{D}(M)$,

$$\begin{aligned}[(\tau_1\cdot_{Q_\varepsilon}\tau_2)\cdot_{Q_\varepsilon}\tau_3](f;\varphi)&=[\delta_{\mathrm{Diag}_2}\cdot\mathsf{m}\exp[\Upsilon_{Q_\varepsilon}]([\tau_1\cdot_{Q_\varepsilon}\tau_2]\widehat{\otimes}\tau_3)](f\otimes 1;\varphi)\\&=[\delta_{\mathrm{Diag}_2}\cdot\mathsf{m}\exp[\Upsilon_{Q_\varepsilon}][(\delta_{\mathrm{Diag}_2}\cdot\mathsf{m}\exp[\Upsilon_{Q_\varepsilon}])\otimes\mathrm{Id}](\tau_1\widehat{\otimes}\tau_2\widehat{\otimes}\tau_3)](f\otimes 1_2;\varphi)\,,\end{aligned}$$

where Id denotes the identity operator on $\mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$. Denoting with Υ_{12} the linear map acting as

$$\Upsilon_{12}(\tau_1\widehat{\otimes}\tau_2\widehat{\otimes}\tau_3):=\Upsilon_{Q_\varepsilon}(\tau_1\widehat{\otimes}\tau_2)\widehat{\otimes}\tau_3\,,$$

and similarly $\Upsilon_{13},\Upsilon_{23}$, for any $f\in\mathcal{D}(M)$ it holds

$$\begin{aligned}[(\tau_1\cdot_{Q_\varepsilon}\tau_2)\cdot_{Q_\varepsilon}\tau_3](f;\varphi)&=\delta_{\mathrm{Diag}_3}\cdot\mathsf{m}(\mathsf{m}\otimes\mathrm{Id})\exp[\Upsilon_{13}+\Upsilon_{23}]\exp[\Upsilon_{12}](\tau_1\widehat{\otimes}\tau_2\widehat{\otimes}\tau_3)(f\otimes 1_2;\varphi)\\&=\delta_{\mathrm{Diag}_3}\cdot\mathsf{m}(\mathrm{Id}\otimes\mathsf{m})\exp[\Upsilon_{12}+\Upsilon_{13}]\exp[\Upsilon_{23}](\tau_1\widehat{\otimes}\tau_2\widehat{\otimes}\tau_3)(f\otimes 1_2;\varphi)\\&=[\tau_1\cdot_{Q_\varepsilon}(\tau_2\cdot_{Q_\varepsilon}\tau_3)](f;\varphi)\,.\end{aligned}$$

This concludes the proof. □

To better grasp the stochastic interpretation of the $\cdot_{Q_\varepsilon}$ product, we consider the following example.

Remark 2.2.3: *Proposition 2.2.2 as well as Equation* (2.2.4) *in particular codify the expectation values of polynomial expressions of the shifted, regularized random field* $\widehat{\varphi}_\varepsilon(x) = P \circledast \widehat{\xi}_\varepsilon$ *discussed in Chapter 1. For concreteness, let* $\Phi \in \mathcal{D}'_{\mathrm{C}}(M; \mathrm{Pol})$ *be the functional-valued distribution*

$$\Phi(f;\varphi) = \int_M f\varphi\mu\,, \qquad f \in \mathcal{D}(M)\,, \quad \varphi \in \mathcal{E}(M)\,.$$

A direct application of Equation (2.2.4) *yields*

$$[\Phi \cdot_{Q_\varepsilon} \Phi](f;\varphi) = \int_M f(x)[\varphi^2(x) + Q_\varepsilon(x,x)]\mathrm{d}\mu(x) = \Phi^2(f;\varphi) + \int_M f(x)Q_\varepsilon(x,x)\mathrm{d}\mu(x)\,, \tag{2.2.7}$$

where Φ^2 *is the functional-valued distribution we defined in Example 2.1.8. Thus, the last expression coincides with the expectation value* $\mathbb{E}(\widehat{\varphi}^2_\varepsilon(f))$ *of the regularized random field* $\widehat{\varphi}_\varepsilon(x)^2 = (P \circledast \widehat{\xi}_\varepsilon + \varphi)^2(x)$ *smeared against a test function* $f \in \mathcal{D}(M)$*. It follows that, in order to recover the expectation value of the proper regularized stochastic convolution* $(P \circledast \widehat{\xi}_\varepsilon)(f)$*, smeared against a test function* $f \in \mathcal{D}(M)$*, it suffices to evaluate the functional of Equation* (2.2.7) *at the configuration* $\varphi = 0$*.*

Remark 2.2.4: *We observe that we can also endow the algebra* $\mathcal{A}$ *with the product* $\cdot_{Q_\varepsilon}$*, cf. Proposition 2.2.2 turning it into a subalgebra of* $\mathcal{D}'_{\mathrm{C}}(M; \mathrm{Pol})$ *with respect to the* $\cdot_{Q_\varepsilon}$ *product.. Furthermore, notice that, since the integral kernel of* Q_ε *is smooth, one can prove by means of a standard argument, see e.g. [15, Ch. 5], that the algebra* $(\mathcal{A}, \cdot_{Q_\varepsilon})$ *is isomorphic to* $\mathcal{A}$ *endowed with the pointwise product. The isomorphism is*

$$\tau\tau' = \alpha_{Q_\varepsilon}\left(\alpha^{-1}_{Q_\varepsilon}(\tau) \cdot_{Q_\varepsilon} \alpha^{-1}_{Q_\varepsilon}(\tau')\right),$$

where

$$\alpha_{Q_\varepsilon} : \mathcal{A} \to \mathcal{A}, \qquad \alpha_{Q_\varepsilon}(\tau)(f;\varphi) := \sum_{n=0}^{\infty} \frac{1}{n!}[(1 \otimes Q_\varepsilon^{\otimes k}) \cdot \tau^{(2k)}](f \otimes 1_{2k};\varphi)\,, \tag{2.2.8}$$

We recall that the sum is finite because τ *is a polynomial functional-valued distribution. Notice that* $\alpha^{-1}_{Q_\varepsilon} = \alpha_{-Q_\varepsilon}$*.*

As we have seen, the product $\cdot_{Q_\varepsilon}$ captures the information on the expectation value of polynomial expressions in the shifted regularized random field $\widehat{\varphi}_\varepsilon$. However, this falls short of our target of describing the stochastic behaviour of the stochastic convolution since it does not contain any information about the correlations. These can be encoded via a different product structure, called $\bullet_{Q_\varepsilon}$, on a suitable tensor algebra built out of $\mathcal{D}'_{\mathrm{C}}(M; \mathrm{Pol})$. The following proposition makes this statement precise. Since the proof is similar to that of Proposition 2.2.2, we omit it.

Proposition 2.2.5: *We define* $\mathcal{T}'_{\mathrm{C}}(M; \mathrm{Pol})$ *the vector space*

$$\mathcal{T}'_{\mathrm{C}}(M; \mathrm{Pol}) := \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathcal{D}'_{\mathrm{C}}(M^n; \mathrm{Pol})\,. \tag{2.2.9}$$

This space is a commutative and associative algebra if endowed with the product $\bullet_{Q_\varepsilon}$ which is defined as follows: For any $\tau_1 \in \mathcal{D}'_{\mathrm{C}}(M^{n_1};\mathrm{Pol})$ and $\tau_2 \in \mathcal{D}'_{\mathrm{C}}(M^{n_2};\mathrm{Pol})$, where $n_1, n_2 \in \mathbb{N} \cup \{0\}$

$$(\tau_1 \bullet_{Q_\varepsilon} \tau_2)(f_1 \otimes f_2; \varphi) = \sum_{k \geq 0} \frac{1}{k!}[(1_{n_1+n_2} \otimes Q_\varepsilon^{\otimes k}) \cdot (\tau_1^{(k)} \widetilde{\otimes} \tau_2^{(k)})](f_1 \otimes f_2 \otimes 1_{2k}; \varphi)\,, \tag{2.2.10}$$

for any $f_1 \in \mathcal{D}(M^{n_1})$, $f_2 \in \mathcal{D}(M^{n_2})$ and $\varphi \in \mathcal{E}(M)$, where $\cdot$ denotes again the pointwise product of distributions.

Remark 2.2.6: *We underline that in Equation (2.2.10) we are using, with a slight abuse of notation, the symbol $\widetilde{\otimes}$ which first appeared in Equation (2.2.3). Here it still indicates a re-ordering of the underlying variables, namely the integral kernel of $(1_{n_1+n_2} \otimes Q_\varepsilon^{\otimes k}) \cdot (\tau_1^{(k)} \widetilde{\otimes} \tau_2^{(k)})$ reads*

$$\prod_{i=1}^{k} Q_\varepsilon(z_i, y_i)\tau_1^{(k)}(x_1, \ldots, x_{n_1}, z_1, \ldots, z_k)\tau_2^{(k)}(x_{n_1+1}, \ldots, x_{n_2}, y_1, \ldots, y_k).$$

Remark 2.2.7: *The product $\bullet_{Q_\varepsilon}$ codifies the correlation between polynomial expressions in the shifted regularized random field $\widehat{\varphi}_\varepsilon$. Similarly to the case of $\cdot_{Q_\varepsilon}$, to better grasp the meaning of this product we consider as an example, for Φ defined as per Example 2.1.8, Equation (2.2.10) yields*

$$[\Phi \bullet_{Q_\varepsilon} \Phi](f_1 \otimes f_2; \varphi) = \int_{M \times M} f_1(x_1)f_2(x_2)[\varphi(x_1)\varphi(x_2) + Q_\varepsilon(x_1, x_2)]\mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2)\,, \tag{2.2.11}$$

which coincides with $\mathbb{E}(\widehat{\varphi}_\varepsilon(f_1)\widehat{\varphi}_\varepsilon(f_2))$ where $\widehat{\varphi}_\varepsilon(x) = (P \circledast \widehat{\xi}_\varepsilon + \varphi)(x)$. As before, we observe that, in order to recover the expressions of Remark 2.2.1, we have to evaluate the functional of Equation (2.2.11) at the configuration $\varphi = 0$.

Remark 2.2.8: *Propositions 2.2.2 and 2.2.5 clarify how $\mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$ encodes the information on the moments of any polynomial whose variable is a regularized random field $\widehat{\varphi}_\varepsilon$.*

Yet, one should pay attention to the fact that, for $\dim M \geq 4$ or $\dim(\Sigma) \geq 2$ in the parabolic case with $M = \mathbb{R} \times \Sigma$, neither $\cdot_{Q_\varepsilon}$ nor $\bullet_{Q_\varepsilon}$ do converge when taking the weak limit as $\varepsilon \to 0^+$. As a matter of fact, if one starts from Equations (2.2.4) and (2.2.10) and if one replaces formally Q_ε with $Q := P \circ P^*$, the ensuing expressions are ill-defined e.g., the one in Equation (2.2.7). This should not come as a surprise since we have seen that this feature occurs often in the analysis of stochastic PDEs as in Chapter 1.

These divergences, which in a large class of scenarios can be tamed by means of a renormalization procedure*, will be the main topic of Chapter 3.*

Chapter 3

Microlocal Renormalization of SPDEs

3.1 Construction of the Algebra $\mathcal{A}_{\cdot_Q}$

In the last part of Chapter 2 we endowed the pointwise algebra $\mathcal{A}$ with a product, dubbed $\cdot_{Q_\varepsilon}$, allowing to recover the information on the stochastic behaviour of the expectation values of arbitrary polynomial functions of the regularized stochastic convolution $P*\xi_\varepsilon$. Similarly, we also introduced, again by means of a deformation procedure, the product $\bullet_{Q\varepsilon}$ on a suitable tensor algebra constructed out of $\mathcal{A}$ allowing to recover also the information concerning multi-local correlation functions.

As we discussed in the previous chapter and in particular in Remark 2.2.8, the previous construction strongly relies on an ε-regularization of the white noise. In particular, at the level of expectation values and correlation functions, the ε-regularization reverberates in a smooth regularization P_ε of the parametrix P of the underlying linear differential operator E, yielding a smooth regularization $Q_\varepsilon := P_\varepsilon \circ P_\varepsilon^*$ of the kernel $Q := P \circ P^*$. As we have seen in Remark 2.2.1, this is associated to the two-point correlation function of the stochastic convolution $\widehat{\varphi} = P \circledast \widehat{\xi}$.

As per Remark 2.2.8, depending on the dimension $d = \dim(M)$ of the underlying manifold M, the limit for $\varepsilon \to 0^+$ of the above construction may be ill-defined, mainly due to the distributional nature of the kernel Q, which in turn is a consequence of the singular behaviour of the white noise as well as of the parametrix P.

To better grasp the hurdle behind these difficulties, we consider the following examples, having their roots in Remark 2.2.3.

Example 3.1.1: *As we have seen in Equation* (2.2.7), *it holds that*

$$[\Phi \cdot_{Q_\varepsilon} \Phi](f;\varphi) = \int_M f(x)[\varphi^2(x) + Q_\varepsilon(x,x)]\mathrm{d}\mu(x) = \Phi^2(f;\varphi) + \int_M f(x)Q_\varepsilon(x,x)\mathrm{d}\mu(x)\,, \tag{3.1.1}$$

We would like to take the limit of Equation (3.1.1) *for $\varepsilon \to 0^+$. We observe that the possible problems may arise from the last term on the right hand side of Equation* (3.1.1)*. This is due to the fact that, formally, taking such limit would yield the kernel*

$$Q(x,x) := \lim_{\varepsilon\to 0^+} Q_\varepsilon(x,x) = \lim_{\varepsilon\to 0^+}\int_M P_\varepsilon^2(x,y)\, d\mu(y)\,, \qquad x \in M\ .$$

At this point, we would like to exchange the limit with the integral but, in general, this is not possible. Another way to realize this issue is the following: If one formally take this limit under *the integral, this yields*

$$Q(x,x)\text{``}=\text{''}\int_M P^2(x,y)\,d\mu(y)\,,$$

where $P^2(x,y)$ should be interpreted as the kernel of P^2, i.e., the square of the parametrix P. In general, it is ill-defined since it involves the product between distributions which cannot be multiplied. This is again the signature of the issues we have discussed in Chapter 1.

Goal of this section – and also of the whole microlocal approach to SPDEs, is to prove that the deformation strategy discussed in Section 2.2 can be extended also starting from a formula such as Equation (2.2.2) for the product though with $Q_\varepsilon \in \mathcal{E}(M\times M)$ replaced by $Q \in \mathcal{D}'(M\times M)$. This scenario has greater relevance for the applications to the study of non-linear SPDEs since it encodes the information on the expectation values of polynomial expressions of the shifted random field $\widehat{\varphi}$.

On account of the previous example, this step is not for free: Indeed, if $d=\dim(M)\geq 4$, replacing the smooth function Q_ε with the distribution Q gives rise to so-called *ultra-violet* divergences. This hurdle can be overcome by means of *renormalization* which is at the heart of the next theorem. In particular, inspired by the algebraic approach to quantum field theory, we shall view the renormalization procedure as an *extension* to the whole space, of distributions which are defined everywhere but on a sub-manifold. This strongly relies on the notion of scaling degree, which we survey in Appendix B, and on Theorem B.2.1.

Remark 3.1.2: *This approach to renormalization has several advantages: First of all, it avoids any ε-regularization scheme as well as any subtraction of infinities. In addition, since this procedure takes place in* position space *[35, 36] it allows to handle renormalization in a mathematically rigorous way also on curved backgrounds, without resorting to any Fourier transform. For further details concerning this approach to renormalization in the framework of quantum field theory, we refer to [5, 13, 18, 25, 26, 38, 57, 58, 70].*

Remark 3.1.3: *In the following theorem we shall discuss in detail the elliptic scenario, briefly commenting at the end of the proof on the minor differences with respect to the parabolic scenario.*

We tackle the problem in two steps.

1. In the first one we define a suitable deformation of $\mathcal{A}$ similar in spirit to the map α_{Q_ε} introduced in Remark 2.2.4;

2. In the second one, we endow $\mathcal{A}$ with a new algebra structure by means of an expression similar to that of Equation (2.2.8).

Summarizing, the spirit of the next theorem is that of introducing a deformation similar to that of Remark 2.2.4, which does not involve any regularization of the kernel Q. As a consequence of our premise, this will need renormalization in order to be meaningful. We underline that the renormalization procedure yields non-uniqueness in the construction of these maps. We shall discuss in detail this feature in Section 3.3.

Theorem 3.1.4: *Let $\mathcal{A}$ be the the algebra as per Definition 2.1.20. There exists a linear map $\Gamma_{\cdot_Q}:\mathcal{A}\to\mathcal{D}'_C(M;\mathrm{Pol})$ satisfying the following properties:*

1. *for any $\tau\in\mathcal{M}_1$ – with the notation of Remark 2.1.23, it holds that*

$$\Gamma_{\cdot_Q}(\tau)=\tau\,; \tag{3.1.2}$$

2. *for any $\tau \in \mathcal{A}$, it holds that*

$$\Gamma_{\cdot_Q}(P \circledast \tau) = P \circledast \Gamma_{\cdot_Q}(\tau)\,; \tag{3.1.3}$$

3. *for any $\psi \in \mathcal{E}(M)$, it holds that*

$$\Gamma_{\cdot_Q} \circ \delta_\psi = \delta_\psi \circ \Gamma_{\cdot_Q}\,, \qquad \Gamma_{\cdot_Q}(\psi\tau) = \psi\Gamma_{\cdot_Q}(\tau)\,; \tag{3.1.4}$$

4. *for any $\tau \in \mathcal{A}$ and $p \geq 1$,*

$$\sigma_p(\Gamma_{\cdot_Q}(\tau)) < \infty\,, \tag{3.1.5}$$

where we recall that $\sigma_p(\tau) := \mathrm{sd}_{\mathrm{Diag}_{p+1}}(\tau^{(p)})$ – cf. Lemma 2.1.25, and where $\mathrm{Diag}_{p+1} \subset M^{p+1}$ denotes the total diagonal of M^{p+1}, namely $\mathrm{Diag}_{p+1} = \{(x_1, \ldots, x_{p+1}) \in M^{p+1} \mid x_1 = \ldots = x_{p+1}\}$.

Proof. Due to its length and following [27, Thm. 3.1], we divide the proof in steps.

The Cases $\mathbf{d} \in \{\mathbf{2}, \mathbf{3}\}$. First we observe that, when $d \in \{2,3\}$, we set $\Gamma_{\cdot_Q}(\tau) = \tau$ for all $\tau \in \mathcal{M}_1$ and

$$\Gamma_{\cdot_Q}(\tau_1 \cdots \tau_\ell) := \tau_1 \cdot_Q \ldots \cdot_Q \tau_\ell\,,$$

for all $\tau_1, \ldots, \tau_\ell \in \mathcal{M}_1$, where $\cdot_Q$ is defined by giving meaning to an expression analogous to Equation (2.2.4) with Q_ε replaced by Q.

Notice that, under the assumption that $d \in \{2,3\}$, $\cdot_Q$ is actually well-defined on account of the scaling degree $\mathrm{sd}_{\mathrm{Diag}_2}(P) = d-2$ of P with respect to the total diagonal of M^2. Indeed, in these cases the kernel $Q(x,x)$ generates an element in $\mathcal{D}'(M)$ due to item *1* of Theorem B.2.1. All other properties required by $\Gamma_{\cdot_Q}$ are verified by direct inspection.

We further observe that this fact makes the following construction unnecessary if $d \in \{2,3\}$. Nonetheless, the following proof holds true also in these cases the only difference being in that if $d \in \{2,3\}$ then the extension arising from renormalization is unique.

Strategy of the proof The idea at the heart of the proof consists of constructing $\Gamma_{\cdot_Q}$ inductively exploiting Equation (2.1.23) and in particular the decomposition $\mathcal{A} = \bigoplus\limits_{k\in\mathbb{N}_0} \mathcal{M}_k$. Starting from Equation (3.1.2), for $\tau = \tau_1 \cdots \tau_n$, $\tau_i \in \mathcal{A}$, $i = 1, \ldots, n$, one would set

$$\Gamma_{\cdot_Q}(\tau) := \Gamma_{\cdot_Q}(\tau_1) \cdot_Q \ldots \cdot_Q \Gamma_{\cdot_Q}(\tau_n)\,.$$

However, contrary to the cases $d \in \{2,3\}$, the product $\cdot_Q$ is in general ill-defined due to the singular behaviour of Q on the total diagonal. To cope with this hurdle we shall proceed by renormalizing the ill-defined expressions arising in the product $\cdot_Q$ in a way consistent with the grading of $\mathcal{A}$. In particular, we shall prove that, whenever $\Gamma_{\cdot_Q}$ has been defined on the sub-module $\mathcal{M}_k$, then it can be extended, possibly non-uniquely, to $\mathcal{M}_{k+1}$. The extension procedure requires not only an induction over k, but also over the index j controlling the direct limit $\mathcal{A} = \varinjlim \mathcal{A}_j$ where $\mathcal{A}_j = \bigoplus_{k\in\mathbb{N}_0} \mathcal{M}_k^j$.

Step 1. To begin with, we prove that by assuming $\Gamma_{\cdot_Q}$ has been properly defined for $\tau \in \mathcal{A}$ in such a way that properties *3.* and *4.* are satisfied, then Equations (3.1.4) and (3.1.5) hold true also for $P \circledast \tau$. To this end, we observe that for any $\tau \in \mathcal{A}$, $\Gamma_{\cdot_Q}(P \circledast \tau)$ is completely defined via Equation (3.1.3) as

$$\Gamma_{\cdot_Q}(P \circledast \tau)(f;\varphi) := P \circledast \Gamma_{\cdot_Q}(\tau)(f;\varphi) = \Gamma_{\cdot_Q}(\tau)(P \circledast f;\varphi)\,.$$

Since $\Gamma_{\cdot_Q}(\tau) \in \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$, Lemma 2.1.18 entails that $P \circledast \Gamma_{\cdot_Q}(\tau) \in \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$. In addition, for any $k \geq 0$ and $\psi \in \mathcal{E}(M)$ it holds that

$$\Gamma_{\cdot_Q}(P \circledast \tau)^{(k)}(f \otimes \psi^{\otimes k};\varphi) = [(P \otimes \delta_{\mathrm{Diag}_2}^{\otimes k}) \circledast \Gamma_{\cdot_Q}(\tau)^{(k)}](f \otimes \psi^{\otimes k};\varphi)\,.$$

Equation (3.1.4) follows directly from

$$\begin{aligned}
[P \circledast \Gamma_{\cdot_Q}(\tau)]^{(1)}(f \otimes \psi;\varphi) &= \Gamma_{\cdot_Q}(\tau)^{(1)}(P \circledast f \otimes \psi;\varphi)\\
&= \Gamma_{\cdot_Q}(\delta_\psi \tau)(P \circledast f;\varphi) = \Gamma_{\cdot_Q}(P \circledast \delta_\psi \tau)(f;\varphi) = \Gamma_{\cdot_Q}(\delta_\psi P \circledast \tau)(f;\varphi)\,.
\end{aligned}$$

At the same time, Equation (3.1.5) comes as a consequence of Lemma 2.1.18 – *cf.* Equation (2.1.15)– together with the assumption that Equation (3.1.5) holds true for $\Gamma_{\cdot_Q}(\tau)$, yielding

$$\sigma_p(\Gamma_{\cdot_Q}(P \circledast \tau)) < \infty\,.$$

Step 2 – First induction procedure: $k = 1, 2$. We focus on defining inductively $\Gamma_{\cdot_Q}$ on $\mathcal{M}_k$ –see Remark (2.1.23). The case $k = 1$ is trivial since it is ruled by Equation (3.1.2) and it represents the first step in the induction procedure. We can focus directly on the case $k = 2$. We shall discuss it thoroughly and eventually we shall generalize the same procedure to arbitrary[1] k.

In order to extend $\Gamma_{\cdot_Q}$ from $\mathcal{M}_1$ to $\mathcal{M}_2$, we shall exploit $\mathcal{M}_2 = \bigcup_{j\in\mathbb{N}_0} \mathcal{M}_2^j$ – *cf.* Remark 2.1.23, proceeding inductively over j. For $j = 0$, we recall that $\mathcal{M}_2^0$ is the $\mathcal{E}(M)$-module

$$\mathcal{M}_2^0 = \mathrm{span}_{\mathcal{E}(M)}(\mathbf{1}, \Phi, \Phi^2)\,.$$

On account of Equation (3.1.2), the only unknown is $\Gamma_{\cdot_Q}(\Phi^2)$. As a consequence, it suffices to properly define $\Gamma_{\cdot_Q}(\Phi^2)$ and to extend $\Gamma_{\cdot_Q}$ per linearity to the whole $\mathcal{M}_2^0$. Hence, for any $f \in \mathcal{D}(M)$ and $\varphi \in \mathcal{E}(M)$, recalling that Φ^2 has been defined in Example 2.1.8, we set formally

$$\Gamma_{\cdot_Q}(\Phi^2)(f;\varphi) := \Gamma_{\cdot_Q}(\Phi) \cdot_Q \Gamma_{\cdot_Q}(\Phi)(f;\varphi) = \Phi^2(f;\varphi) + P^2(f \otimes 1)\,. \tag{3.1.6}$$

The second equality is nothing but Equation (2.2.4) with Q_ε replaced by $Q = P \circ P^*$, where we have also used both that $\Gamma_{\cdot_Q}^{(1)}(\Phi) = \delta_{\mathrm{Diag}_2}$ and that

$$\begin{aligned}
(\delta_{\mathrm{Diag}_2} \otimes Q) \cdot (\Gamma_{\cdot_Q}(\Phi)^{(1)} \otimes \Gamma_{\cdot_Q}(\Phi)^{(1)})(f \otimes 1_3;\varphi) &= \int_M Q(x,x) f(x) \mathrm{d}\mu(x)\\
&= \int_M P^2(x,y) f(x) 1(y) \mathrm{d}\mu(x) \mathrm{d}\mu(y) = P^2(f \otimes 1)\,.
\end{aligned}$$

[1]From the viewpoint of the induction procedure, we could hop directly to the case of arbitrary k since the first step is represented by $k = 1$ and thus the scenario discussing explicitly $k = 2$ is unnecessary. Nonetheless, we analyse it quite in some details since we think it can be useful in order to clarify the argument in the general case.

Observe that, since we are working with parametrices of elliptic operators $P^*(x,y) = P(y,x)$ and this justifies why in the last formula $P^2(x,y)$ is present, as we computed in Example 3.1.1. We underline that both this last expression and Equation (3.1.6) are purely formal since P^2, the square of the parametrix P, is ill-defined as a bi-distribution over $M \times M$.

To face this hurdle, we start by observing that $P^2 \in \mathcal{D}'(M^2 \setminus \mathrm{Diag}_2)$ is a well-defined distribution because $\mathrm{WF}(P) = \mathrm{WF}(\delta_{\mathrm{Diag}_2})$ – *cf.* Example A.0.7, entailing that P is a smooth function over $M^2 \setminus \mathrm{Diag}_2$. In addition, using Equation (B.1.4) and Example B.1.5, $\mathrm{sd}_{\mathrm{Diag}_2}(P^2) \leq 2(d-2) < +\infty$ and, therefore, on account of Theorem B.2.1 there exists an extension $\widehat{P}_2 \in \mathcal{D}'(M \times M)$ of P^2 such that $\mathrm{sd}_{\mathrm{Diag}_2}(\widehat{P}_2) = \mathrm{sd}_{\mathrm{Diag}_2}(P^2)$. Furthermore, $\mathrm{WF}(\widehat{P}_2) = \mathrm{WF}(\delta_{\mathrm{Diag}_2})$.

We assume that one such extension, dubbed $\widehat{P}_2$, has been chosen once and for all. Notice that this extension is unique when $\dim M = d \in \{2,3\}$, consistently with the definition of $\cdot_Q$ and consistently with the comments on the cases $\dim M = d \in \{2,3\}$ at the beginning of the proof.
Accordingly and in view of Equation (3.1.6), we set

$$\Gamma_{\cdot_Q}(\Phi^2)(f;\varphi) := \Phi^2(f;\varphi) + \widehat{P}_2(f \otimes 1)\,. \tag{3.1.7}$$

In addition, we can infer that $\Gamma_{\cdot_Q}(\Phi^2) \in \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$. In particular Remark A.0.8 yields that $\widehat{P}_2 \circledast 1 \in \mathcal{E}(M)$ since $\mathrm{WF}_1(\widehat{P}_2) = \emptyset$, while the bound $\mathrm{WF}([\Gamma_{\cdot_Q}(\Phi)^2]^{(\ell)}) \subseteq \mathrm{C}_{\ell+1}$ holds true combining Example 2.1.8 with the wavefront set of $\widehat{P}_2$. Moreover it descends that

$$\begin{aligned}
&\Gamma_{\cdot_Q}(\Phi^2)^{(1)}(f \otimes \psi;\varphi) = [\Phi^2]^{(1)}(f \otimes \psi;\varphi)\,,\\
&\Gamma_{\cdot_Q}(\Phi^2)^{(2)}(f \otimes \psi_1 \otimes \psi_2;\varphi) = [\Phi^2]^{(2)}(f \otimes \psi_1 \otimes \psi_2;\varphi)\,,
\end{aligned}$$

from which both Equations (3.1.4) and (3.1.5) hold true. This completes the definition of $\Gamma_{\cdot_Q}$ on $\mathcal{M}_2^0$.

Proceeding inductively with respect to the index j, we assume that $\Gamma_{\cdot_Q}$ has been defined on $\mathcal{M}_2^j$ iterating the procedure used in Equation (3.1.6) and we extend it to $\mathcal{M}_2^{j+1}$. We observe that, for any $\tau \in \mathcal{M}_2^{j+1}$, it suffices to prove the induction step for those elements either of the form $P \circledast \tau'$, $\tau' \in \mathcal{M}_2^j$ or of the form $\tau = \tau_1\tau_2$ with $\tau_1, \tau_2 \in \mathcal{M}_1^j \cup P \circledast \mathcal{M}_1^j$ – see Definition 2.1.20. In the first scenario, it suffices to invoke both the induction hypothesis for $\tau' \in \mathcal{M}_2^j$ and the first step of the proof to conclude for $\tau = P \circledast \tau'$.

In the second scenario, we consider the formal expression

$$\Gamma_{\cdot_Q}(\tau_1) \cdot_Q \Gamma_{\cdot_Q}(\tau_2)(f;\varphi) = \big(\Gamma_{\cdot_Q}(\tau_1)\Gamma_{\cdot_Q}(\tau_2)\big)(f;\varphi) + \Big[(\delta_{\mathrm{Diag}_2} \otimes Q) \cdot (t_1^{(1)} \widetilde{\otimes} t_2^{(1)})\Big](f \otimes 1_3;\varphi)\,,$$

where we set $t_i^{(1)} := \Gamma_{\cdot_Q}(\tau_i)^{(1)}$, for $i = 1,2$. *A priori* the above formula is not well-defined on account of the presence of the product

$$T := (\delta_{\mathrm{Diag}_2} \otimes Q) \cdot (t_1^{(1)} \widetilde{\otimes} t_2^{(1)})\,.$$

To bypass this hurdle, we observe that $\Gamma_{\cdot_Q}(\tau_1), \Gamma_{\cdot_Q}(\tau_2) \in \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$. Hence it descends

$$\mathrm{WF}(t_1^{(1)}) \cup \mathrm{WF}(t_2^{(1)}) \subseteq \mathrm{C}_2 = \mathrm{WF}(\delta_{\mathrm{Diag}_2}) = \mathrm{WF}(Q)\,.$$

It follows that T identifies an element of $\mathcal{D}'(M^4 \setminus \mathrm{Diag}_4^{\mathrm{big}})$ where

$$\mathrm{Diag}_4^{\mathrm{big}} := \{(x_1,\ldots,x_4) \in M^4 \,|\, \exists\, a,b \in \{1,2,3,4\}\,,\ x_a = x_b\,,\ a \neq b\}\,. \tag{3.1.8}$$

Moreover observe that, whenever $x \in \mathrm{Diag}_4^{\mathrm{big}} \setminus \mathrm{Diag}_4$, namely outside the total diagonal, one of the factors between $\delta_{\mathrm{Diag}_2} \otimes Q$, $t_1^{(1)}$, $t_2^{(1)}$ is smooth while, recalling in particular Equation (2.2.3), the product of the other two is well-defined.

This implies $T \in \mathcal{D}'(M^4 \setminus \mathrm{Diag}_4)$. Furthermore, on account of the inductive hypothesis over j it holds that

$$\mathrm{sd}_{\mathrm{Diag}_4}(T) \leq \mathrm{sd}_{\mathrm{Diag}_4}(\delta_{\mathrm{Diag}_2} \otimes Q) + \mathrm{sd}_{\mathrm{Diag}_2}(t_1^{(1)}) + \mathrm{sd}_{\mathrm{Diag}_2}(t_2^{(1)}) < \infty\,,$$

where in the last inequality we also used Corollary B.4.2 to prove finiteness of $\mathrm{sd}_{\mathrm{Diag}_2}(Q)$. Thanks to Theorem B.2.1 we can conclude that there exists a possibly non-unique extension $\widehat{T} \in \mathcal{D}'(M^4)$ of T such that $\mathrm{sd}_{\mathrm{Diag}_4}(\widehat{T}) = \mathrm{sd}_{\mathrm{Diag}_4}(T)$. In addition it holds that $\mathrm{WF}(\widehat{T}) = \mathrm{WF}(T)$. Choosing an extension $\widehat{T}$, we set for any $f \in \mathcal{D}(M)$ and $\varphi \in \mathcal{E}(M)$,

$$\Gamma_{\cdot_Q}(\tau)(f;\varphi) := \left(\Gamma_{\cdot_Q}(\tau_1)\Gamma_{\cdot_Q}(\tau_2)\right)(f;\varphi) + \widehat{T}(f \otimes 1_3;\varphi)\,.$$

By direct inspection we see that $\Gamma_{\cdot_Q}(\tau) \in \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$. The condition codified by Equation (3.1.4) is satisfied by construction, while, in order to check the inequality in Equation (3.1.5) we observe that

$$\begin{aligned}
\Gamma_{\cdot_Q}(\tau)^{(1)}(f \otimes \psi;\varphi) &= \left(\Gamma_{\cdot_Q}(\tau_1)\Gamma_{\cdot_Q}(\tau_2)^{(1)} + \Gamma_{\cdot_Q}(\tau_1)^{(1)}\Gamma_{\cdot_Q}(\tau_2)\right)(f \otimes \psi;\varphi)\,,\\
\Gamma_{\cdot_Q}(\tau)^{(2)}(f \otimes \psi_1 \otimes \psi_2;\varphi) &= \left(\Gamma_{\cdot_Q}^{(1)}(\tau_1)\Gamma_{\cdot_Q}(\tau_2)^{(1)}\right)(f \otimes \psi_1 \otimes \psi_2;\varphi)\,.
\end{aligned}$$

It descends that,

$$\begin{aligned}
&\sigma_1(\Gamma_{\cdot_Q}(\tau)) \leq \max_{i=1,2} \sigma_1(\Gamma_{\cdot_Q}(\tau_i)) < \infty\,,\\
&\sigma_2(\Gamma_{\cdot_Q}(\tau)) \leq \sigma_1(\Gamma_{\cdot_Q}(\tau_1)) + \sigma_1(\Gamma_{\cdot_Q}(\tau_2)) < \infty\,,
\end{aligned}$$

where we used Equation (3.1.5) applied to $\Gamma_{\cdot_Q}(\tau_i)$, $i = 1, 2$. Its validity is guaranteed by the induction step.

Step 3 – Second induction procedure. In the preceding step we have proven the sought after statement for $\mathcal{M}_k$ with $k = 1, 2$. We proceed now by induction over k. We assume that $\Gamma_{\cdot_Q}$ has been defined on $\mathcal{M}_k$, for an arbitrary but fixed k, and we prove that $\Gamma_{\cdot_Q}$ can be consistently defined on $\mathcal{M}_{k+1} = \bigcup_{j \in \mathbb{N}_0} \mathcal{M}_{k+1}^j$, *cf.* Remark 2.1.23, in agreement with the statement of the theorem.

Step 3a – The special case $\mathcal{M}^{\mathbf{0}}_{\mathbf{k+1}}$. If $j = 0$, we are considering the $\mathcal{E}(M)$-module

$$\mathcal{M}_{k+1}^0 = \mathrm{span}_{\mathcal{E}(M)}(\mathbf{1}, \Phi, \ldots, \Phi^{k+1})\,.$$

By the inductive hypothesis over k and on account of Equation (3.1.3), we are left with defining $\Gamma_{\cdot_Q}(\Phi^{k+1})$. Following the same strategy as in Equation (3.1.6) and recalling the identities

$$\Gamma_{\cdot_Q}(\Phi)^{(1)} = \delta_{\mathrm{Diag}_2}\,, \qquad \Gamma_{\cdot_Q}(\Phi)^{(k)} = 0, \quad \forall\, k \geq 2\,,$$

we consider the formal expression

$$\Gamma_{\cdot_Q}(\Phi^{k+1}) = \underbrace{\Gamma_{\cdot_Q}(\Phi)\cdot_Q \ldots \cdot_Q \Gamma_{\cdot_Q}(\Phi)}_{k+1}(f;\varphi) = \sum_{\ell=0}^{\lfloor\frac{k+1}{2}\rfloor} \frac{(2\ell)!}{(2\ell)!!}\binom{k+1}{2\ell}(Q_{2\ell}\cdot\Gamma_{\cdot_Q}(\Phi)^{k+1-2\ell})(f;\varphi)\,, \tag{3.1.9}$$

where $\lfloor\frac{k+1}{2}\rfloor \leq \frac{k+1}{2}$ denotes the integer part of $\frac{k+1}{2}$ while $n!! := n(n-2)(n-4)\cdots = 2^n n!$ denotes the double factorial. Here $Q_{2\ell}$ is given by

$$Q_{2\ell}(f) = (P^2)^{\otimes\ell}\cdot(\delta_{\mathrm{Diag}_\ell}\widetilde{\otimes}1_\ell)(f\otimes 1_{2\ell-1})\,, \tag{3.1.10}$$

where $\widetilde{\otimes}$ means that, at the level of integral kernels, the product on the right hand side of Equation (3.1.10) reads

$$\prod_{j=1}^{\ell} P^2(x_j,y_j)\delta_{\mathrm{Diag}_\ell}(x_1,\ldots,x_\ell)1_\ell(y_1,\ldots,y_\ell)\,.$$

We observe that the expression in Equation (3.1.9) is only formal due to the presence of $Q_{2\ell}$, which is built out of P^2, the square of the parametrix P. Nevertheless, we have already shown that P^2 admits a possibly non-unique extension $\widehat{P}_2 \in \mathcal{D}'(M^2)$ such that $\mathrm{sd}_{\mathrm{Diag}_2}(\widehat{P}_2) = \mathrm{sd}_{\mathrm{Diag}_2}(P^2)$. Given any but fixed choice for $\widehat{P}_2$, we denote by

$$\widehat{Q}_{2\ell}(f) := \widehat{P}_2^{\otimes\ell}\cdot(\delta_{\mathrm{Diag}_\ell}\widetilde{\otimes}1_\ell)(f\otimes 1_{2\ell-1})\,,$$

the associated extension of $Q_{2\ell}$. Notice that the product $\widehat{P}_2^{\otimes\ell}\cdot(\delta_{\mathrm{Diag}_\ell}\otimes 1_\ell)(\cdot\otimes 1_{2\ell-1}) \in \mathcal{D}'(M)$ is well-defined on account of Theorem A.0.6 – *cf.* Equation (A.0.5). We consider the extension $\widehat{Q}_{2\ell}$ associated to the choice of $\widehat{P}_2$ which we outlined in Step 2 of the proof. Hence we can set[2]

$$\Gamma_{\cdot_Q}(\Phi^{k+1})(f;\varphi) := \sum_{\ell=0}^{\lfloor\frac{k+1}{2}\rfloor} \frac{(2\ell)!}{(2\ell)!!}\binom{k+1}{2\ell}[\widehat{Q}_{2\ell}\cdot\Gamma_{\cdot_Q}(\Phi)^{k+1-2\ell}](f;\varphi)\,. \tag{3.1.11}$$

Since $\Gamma_{\cdot_Q}(\Phi)$ is a functional-valued distribution generated by a smooth function, it descends that the product $\widehat{Q}_{2\ell}\cdot\Gamma_{\cdot_Q}(\Phi)^{k+1-2\ell}$ is well-defined. As a matter of fact, a direct application of Equation (A.0.6) and of Remark A.0.8 shows that $[\widehat{Q}_{2\ell}\cdot\Gamma_{\cdot_Q}(\Phi)^{k+1-2\ell}]$ is a functional-valued distribution generated by a smooth function.

We observe that Equation (3.1.11), defining $\Gamma_{\cdot_Q}(\Phi^{k+1})$, is coherent with Equation (3.1.4) since, for any $f\in\mathcal{D}(M)$ and $\varphi,\psi\in\mathcal{E}(M)$,

$$\begin{aligned}\Gamma_{\cdot_Q}(\Phi^{k+1})^{(j)}(f\otimes\psi^{\otimes j};\varphi) &= \sum_{\ell=0}^{\lfloor\frac{k+1-j}{2}\rfloor}\frac{(k+1)!}{(2\ell)!!}(k+1-2\ell-j)![\widehat{Q}_{2\ell}\cdot\Gamma_{\cdot_Q}(\Phi)^{k+1-2\ell-j}](f\psi^j;\varphi)\\ &= \Gamma_{\cdot_Q}(\delta^j_{\psi^{\otimes j}}\Phi^k)(f;\varphi)\,,\end{aligned}$$

[2]This construction shows that in order to build the renormalized expectation value of any power Φ^k of Φ it suffices to renormalize Φ^2, as a consequence of the Gaussian behaviour of Φ.

where we used the identity $[\Gamma_{\cdot_Q}(\Phi)^a]^{(b)} = \frac{a!}{(a-b)!}\Gamma_{\cdot_Q}(\Phi)^{a-b}\cdot\delta_{\mathrm{Diag}_{b+1}}$, while $\delta^j_{\psi^{\otimes j}} := \delta_\psi\circ\cdots\circ\delta_\psi$. By Equation (2.1.4), it holds that

$$\begin{aligned}\Gamma_{\cdot_Q}(\Phi^{k+1})^{(j)}(f\otimes\psi^{\otimes j};\varphi) &= \frac{(k+1)!}{(k+1-j)!}\Gamma_{\cdot_Q}(\underbrace{\psi\cdots\psi}_{j}\,\Phi^{k+1-j})(f;\varphi)\\ &= \frac{(k+1)!}{(k+1-j)!}\Gamma_{\cdot_Q}(\Phi^{k+1-j})(\underbrace{\psi\cdots\psi}_{j}\,f;\varphi)\,.\end{aligned}$$

Thanks to the arbitrariness of ψ, f and φ, this yields

$$\Gamma_{\cdot_Q}(\Phi^{k+1})^{(j)} = \frac{(k+1)!}{(k+1-j)!}\Gamma_{\cdot_Q}(\Phi^{k+1-j})\cdot\delta_{\mathrm{Diag}_{j+1}}\,,$$

implying in turn $\mathrm{WF}(\Gamma_{\cdot_Q}(\Phi^{k+1})^{(j)})\subseteq \mathrm{C}_{j+1}$. Therefore $\Gamma_{\cdot_Q}(\Phi^{k+1})\in\mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$.
Finally Equation (3.1.5) is a direct consequence of the bound

$$\sigma_p(\Gamma_{\cdot_Q}(\Phi^{k+1}))\leq \mathrm{sd}_{\mathrm{Diag}_{p+1}}(\Gamma_{\cdot_Q}(\Phi^{k+1-p})\cdot\delta_{\mathrm{Diag}_{p+1}})\leq \mathrm{sd}_{\mathrm{Diag}_{p+1}}(\delta_{\mathrm{Diag}_{p+1}}) = pd\,.$$

Step 3b: The general case $\mathcal{M}^{\mathbf{j}}_{\mathbf{k+1}}$. The preceding step allows us to proceed in the inductive construction of $\Gamma_{\cdot_Q}$. In particular we shall assume that the sought after result to be valid for $\mathcal{M}^j_{k+1}$ and we show that then it holds true also for $\mathcal{M}^{j+1}_{k+1}$.

Let us consider a generic $\tau\in\mathcal{M}^{j+1}_{k+1}$. In view of Definition 2.1.20, of Remark 2.1.23 and of the linearity of $\Gamma_{\cdot_Q}$, it suffices to focus on those elements which are either of the form $P\circledast\tau'$, with $\tau'\in\mathcal{M}^j_{k+1}$, or

$$\tau=\tau_{k_1}\cdots\tau_{k_\ell}\,,\qquad \tau_{k_n}\in\mathcal{M}^j_{k_n}\cup P\circledast\mathcal{M}^j_{k_n}\,,$$

where $\ell\in\mathbb{N}\cup\{0\}$, $k_n\in\mathbb{N}$ for all $n\in\{1,\dots,\ell\}$, $\sum\limits_{n=1}^{\ell}k_n = k+1$. While in the first scenario we can resort to Step 1 of the proof, in the second one we can start by recalling that the inductive hypothesis entails that $\Gamma_{\cdot_Q}(\tau_{k_n})$ is known for all n.
Following the same strategy of Steps *2* and *3a*, we set formally

$$\Gamma_{\cdot_Q}(\tau) := \Gamma_{\cdot_Q}(\tau_{k_1})\cdot_Q\cdots_Q\Gamma_{\cdot_Q}(\tau_{k_\ell})\,.$$

Denoting $t_{k_i} := \Gamma_{\cdot_Q}(\tau_{k_i})$ and exploiting Equation (2.2.4) with Q_ε replaced by Q, we get

$$\begin{aligned}&[t_{k_1}\cdot_Q\cdots_Q t_{k_\ell}](f;\varphi) =\\ &\quad= \sum_{\substack{N\geq 0\\ N_1+\ldots+N_\ell=2N}}\frac{1}{(2N)!!}\frac{(2N)!}{N_1!\cdots N_\ell!}[(\delta_{\mathrm{Diag}_\ell}\otimes Q^{\otimes N})\cdot(t^{(N_1)}_{k_1}\widetilde{\otimes}\ldots\widetilde{\otimes}t^{(N_\ell)}_{k_\ell})](f\otimes 1_{\ell-1+2N};\varphi)\,,\end{aligned}$$

where $\cdot$ indicates once more the pointwise product between distributions. As in the preceding cases, this is only a formal expression and, in order to make it well-defined, we start by noticing that, being all functionals polynomial, then $N_i\leq k_i$ for all $i\in\{1,\dots,\ell\}$ and thus $2N\leq k+1$. In addition, for later convenience we set

$$T_N := [(\delta_{\mathrm{Diag}_\ell}\otimes Q^{\otimes N})\cdot(t^{(N_1)}_{k_1}\widetilde{\otimes}\ldots\widetilde{\otimes}t^{(N_\ell)}_{k_\ell})]\,. \tag{3.1.12}$$

Furthermore, it holds that – *cf.* item *2* of Theorem A.0.6,

$$\mathrm{WF}(\delta_{\mathrm{Diag}_\ell}\otimes Q^{\otimes N})\subseteq\{(\widehat{x}_\ell,\widehat{z}_{2N},\widehat{\xi}_\ell,\widehat{\zeta}_{2N})\in T^*M^{\ell+2N}\setminus\{0\}\,|\\ (\widehat{x}_\ell,\widehat{\xi}_\ell)\in\mathrm{WF}(\delta_{\mathrm{Diag}_\ell})\,,\,(\widehat{z}_{2N},\widehat{\zeta}_{2N})\in\mathrm{WF}(Q^{\otimes N})\}\,.$$

By the inductive hypothesis it holds $\mathrm{WF}(t_{k_i}^{(N_i)})\subseteq\mathrm{C}_{N_i+1}$ for any $i\in\{1,\ldots,\ell\}$. Thus, by applying Theorem A.0.6 – *cf.* Equation (A.0.5), $T_N\in\mathcal{D}'(M^{\ell+2N}\setminus\mathrm{Diag}^{\mathrm{big}}_{\ell+2N})$, where

$$\mathrm{Diag}^{\mathrm{big}}_{\ell+2N}:=\{x\in M^{\ell+2N}\,|\,\exists i,j\in\{1,\ldots,\ell+2N\}\,,\,x_i=x_j\,,\,i\neq j\}\,. \tag{3.1.13}$$

These data in combination with Equation (A.0.6) yield

$$\mathrm{WF}(T_N)\subseteq\{(\widehat{x}_\ell,\widehat{z}_{2N},\widehat{\xi}_\ell+\widehat{\xi}'_\ell,\widehat{\zeta}_{N_1}+\widehat{\zeta}'_{N_1},\ldots,\widehat{\zeta}_{N_\ell}+\widehat{\zeta}'_{N_\ell})\in T^*M^{\ell+2N}\setminus\{0\}\,|\\ (\widehat{x}_\ell,\widehat{\xi}_\ell)\in\mathrm{WF}(\delta_{\mathrm{Diag}_\ell})\,,\,(\widehat{z}_{2N},\widehat{\zeta}_{2N})\in\mathrm{WF}(Q^{\otimes N})\,,\\ \forall p\in\{1,\ldots,\ell\}\,(x_p,\widehat{z}_{N_p},\xi'_p,\widehat{\zeta}'_{N_p})\in\mathrm{C}_{N_p}\}\,, \tag{3.1.14}$$

where $\widehat{z}_{2N}=(\widehat{z}_{N_1},\ldots,\widehat{z}_{N_\ell})$. Let $\{A,B\}$ be a disjoint partition of $\{1,\ldots,\ell+2N\}$ – that is, $\{1,\ldots,\ell+2N\}=A\cup B$, $A\cap B=\emptyset$ – such that, if $(x_1,\ldots,x_{\ell+2N})=(\widehat{x}_A,\widehat{x}_B)$, then $x_a\neq x_b$ for all $x_a\in\widehat{x}_A$ and for all $x_b\in\widehat{x}_B$. As a consequence, the integral kernel of T_N decomposes as

$$T_N(\widehat{x}_A,\widehat{x}_B)=K_{N,1}(\widehat{x}_A)S_N(\widehat{x}_A,\widehat{x}_B)K_{J,N}(\widehat{x}_B)\,,$$

where S_N is a smooth kernel while $K_{N,1}$, $K_{N,2}$ are the integral kernels of the distributions appearing in the definition of $\Gamma_{\cdot_Q}$ on $\mathcal{M}^b_a$ for $b<j+1$ and $a\leq k+1$.
By the inductive hypothesis $K_{N,1}$, $K_{N,2}$ are well-defined and the same holds true for $(K_{J,1}\otimes K_{J,2})\cdot S_J$ on account of Theorem A.0.6 – in particular of Equation (A.0.5). Accordingly we can conclude that[3]

$$T_N\in\mathcal{D}'(M^{\ell+2N}\setminus\mathrm{Diag}_{\ell+2N})\,.$$

In addition, because of Lemma B.4.1, of Corollary B.4.2 and of the inductive hypothesis on $\Gamma_{\cdot_Q}(\tau_{k_1}),\ldots,\Gamma_{\cdot_Q}(\tau_{k_\ell})$, it holds that

$$\begin{aligned}\mathrm{sd}_{\mathrm{Diag}_{\ell+2N}}(T_N)&=\mathrm{sd}_{\mathrm{Diag}_{\ell+2N}}((\delta_{\mathrm{Diag}_\ell}\otimes Q^{\otimes N})\cdot(t_{k_1}^{(N_1)}\otimes\ldots\otimes t_{k_\ell}^{(N_\ell)}))\\&\leq\mathrm{sd}_{\mathrm{Diag}_{\ell+2N}}((\delta_{\mathrm{Diag}_\ell}\otimes Q^{\otimes N})+\sum_{i=1}^{\ell}\mathrm{sd}_{\mathrm{Diag}_{N_i+1}}(t_{k_i}^{(N_i)})<\infty\,,\end{aligned}$$

As a consequence, Theorem B.2.1 applies and there exists an extension $\widehat{T}_N\in\mathcal{D}'(M^{\ell+2N})$ of T_N with $\mathrm{sd}_{\mathrm{Diag}_{\ell+2N}}(\widehat{T}_N)=\mathrm{sd}_{\mathrm{Diag}_{\ell+2N}}(T_N)$ and with $\mathrm{WF}(\widehat{T}_N)=\mathrm{WF}(T_N)$. We recall that such extension may be non-unique. Consequently, for any $f\in\mathcal{D}(M)$ and $\varphi\in\mathcal{E}(M)$, we can set

$$\Gamma_{\cdot_Q}(\tau)(f;\varphi):=\sum_{\substack{N\geq 0\\ N_1+\ldots+N_\ell=2J}}\frac{1}{(2N)!!}\frac{(2N)!}{N_1!\cdots N_\ell!}\widehat{T}_N(f\otimes 1_{\ell-1+2N};\varphi)\,. \tag{3.1.15}$$

[3] This argument is the generalization of the one of Equation (3.1.8).

As in the preceding steps, once an extension $\widehat{T}_N$ is chosen, this last formula is well-defined and Remark A.0.8 yields $\mathcal{D}(M) \ni f \mapsto \widehat{T}_N(f \otimes 1_{\ell+2N}; \varphi) \in \mathcal{E}(M)$ – *cf.* Equations (3.1.14) and (A.0.10).

For all $p \in \mathbb{N} \cup \{0\}$ and $\psi \in \mathcal{E}(M)$, it holds

$$\Gamma_{\cdot_Q}(\tau)^{(p)}(f \otimes \psi^{\otimes p}; \varphi) := \sum_{\substack{N \geq 0 \\ N_1+\ldots+N_\ell=2N}} \frac{1}{(2N)!!} \frac{(2N)!}{N_1! \cdots N_\ell!} \widehat{T}_N^{(p)}(f \otimes 1_{\ell-1+2N} \otimes \psi^{\otimes p}; \varphi) \,,$$

where $0 \leq 2N \leq k+1-p$. In order to ensure that Equation (3.1.4) holds true, consider the formal expression

$$\begin{aligned}
&[\Gamma_{\cdot_Q}(\tau_{k_1}) \cdot_Q \cdots_Q \Gamma_{\cdot_Q}(\tau_{k_\ell})]^{(p)}(f \otimes \psi^{\otimes p}; \varphi) \\
&\quad := \sum_{\substack{N \geq 0 \\ N_1+\ldots+N_\ell=2N \\ p_1+\ldots+p_\ell=p}} \frac{1}{(2N)!!} \frac{(2N)!p!}{N_1!p_1! \cdots N_\ell!p_\ell!} [(\delta_{\mathrm{Diag}_\ell} \otimes Q^{\otimes N} \otimes 1_p) \cdot \\
&\qquad\qquad \cdot (t_{k_1}^{(N_1+p_1)} \otimes \ldots \otimes t_{k_\ell}^{(N_\ell+p_\ell)})](f \otimes 1_{\ell-1+2N} \otimes \psi^{\otimes p}; \varphi) = \\
&\quad = \sum_{\substack{N \geq 0 \\ N_1+\ldots+N_\ell=2N \\ p_1+\ldots+p_\ell=p}} \frac{1}{(2N)!!} \frac{(2N)!p!}{N_1!p_1! \cdots N_\ell!p_\ell!} T_N^{[\widehat{p}_\ell]}(f \otimes 1_{\ell-1+2N} \otimes \psi^{\otimes p}; \varphi) \,,
\end{aligned}$$

where $T_N^{[\widehat{p}_\ell]}$ is a functional-valued distribution on $M^{\ell+2N+p} \setminus \mathrm{Diag}_{\ell+2N+p}$ with finite scaling degree at $\mathrm{Diag}_{\ell+2N+p}$. Then Equation (3.1.4) is satisfied provided we choose the extension $\widehat{T}_N$ in such a way that

$$\widehat{T}_N^{(p)} = \sum_{p_1+\ldots+p_\ell=p} \frac{p!}{p_1! \cdots p_\ell!} \widehat{T_N^{[\widehat{p}_\ell]}} \,, \tag{3.1.16}$$

where $\widehat{T_N^{[\widehat{p}_\ell]}} \in \mathcal{D}'(M^{\ell+2N+p})$ is a scaling degree preserving extension of $T_N^{[\widehat{p}_\ell]}$, whose existence is guaranteed by the finiteness of $\mathrm{sd}_{\mathrm{Diag}_{\ell+2N+p}}(T_N^{[\widehat{p}_\ell]})$ together with Theorem B.2.1. Notice that we can impose Equation (3.1.16) on account of the fairly explicit construction of $\widehat{T}_N$ – *cf.* Theorem B.2.1. The proof that $\Gamma_{\cdot_Q}(\tau) \in \mathcal{D}'_{\mathrm{C}}(M; \mathrm{Pol})$ follows by estimating the wave front set of the distribution

$$\mathcal{D}(M) \otimes \mathcal{E}(M)^{\otimes p} \ni f \otimes \psi^{\otimes p} \to \widehat{T}_N^{(p)}(f \otimes 1_{\ell-1+2N} \otimes \psi^{\otimes p}; \varphi) \,.$$

This is obtained using Theorem A.0.6 – *cf.* Remark (A.0.8) – together with the estimate

$$\begin{aligned}
\mathrm{WF}(\widehat{T}_N^{(p)}) = \{&(\widehat{x}_\ell, \widehat{z}_{2N}, \widehat{y}_p, \widehat{\xi}_\ell + \widehat{\xi}'_\ell, \widehat{\zeta}_{2N} + \widehat{\zeta}'_{2N}, \widehat{\eta}_p + \widehat{\eta}'_p) \in T^*M^{\ell+2N+p} \setminus \{0\} \mid \\
&(\widehat{x}_\ell, \widehat{\xi}_\ell) \in \mathrm{WF}(\delta_{\mathrm{Diag}_\ell}) \,, \, (\widehat{z}_{2N}, \widehat{\zeta}_{2N}) \in \mathrm{WF}(Q^{\otimes N}) \,, \\
&\forall h \in \{1, \ldots, \ell\} \,, \, (x_h, \widehat{z}_{N_h}, \widehat{y}_{p_h}, \xi_h, \widehat{\zeta}'_{n_h}, \widehat{\eta}'_{p_h}) \in \mathrm{C}_{N_h+p_h+1}\} \,,
\end{aligned}$$

where we wrote $\widehat{\zeta}_{2N} = (\widehat{\zeta}_{N_1}, \ldots, \widehat{\zeta}_{N_\ell})$ and similarly for $\widehat{y}_p$, $\widehat{\zeta}_{2N}$ and $\widehat{\eta}_p$.

Finally the bound (3.1.5) is satisfied by direct inspection, since

$$\begin{aligned}\sigma_p(\Gamma_{\cdot_Q}(\tau)) &\leq \max_{0\leq 2N\leq k+1-p} \mathrm{sd}_{\mathrm{Diag}_{\ell+2N+p}}(T_N^{(p)})(\cdot\otimes 1_{\ell-1+2N}\otimes\cdot;\varphi)\\ &\leq \mathrm{sd}_{\mathrm{Diag}_\ell}(\delta_{\mathrm{Diag}_\ell}) + \max_{0\leq 2N\leq k+1-p}\left[N\,\mathrm{sd}_{\mathrm{Diag}_2}(Q) + \sum_{i=1}^{q}\sigma_{1+N_i+p_i}(\Gamma_{\cdot_Q}(\tau_{k_i}))\right] < \infty\,,\end{aligned}$$

where we exploited the inductive hypothesis on $\mathcal{M}_k$, Examples B.1.4 and B.1.5 as well as a minor generalization of Lemma B.4.1.

This concludes the induction procedure and the proof. □

Remark 3.1.5 (Parabolic Case)**:** *As anticipated in Remark 3.1.3, we comment shortly on Theorem 3.1.4 in the parabolic scenario. In particular, this result holds true,* mutatis mutandis, *also in the parabolic case with $M=\mathbb{R}\times\Sigma$ and $E=\partial_t-\widetilde{E}$ as discussed in Chapter 2.*

We observe that in this case renormalization enters the game if $\dim(\Sigma)=d\geq 2$. *The proof goes along the same lines, the main point being the renormalization procedure, namely the extension of the singular distributions involved in the definition of $\cdot_Q$. This is based on the finiteness of the weighted scaling degree – cf. Section B.3 and Example B.3.3.*

Finally, in agreement with the discussion preceding Theorem 3.1.4, by means of the map $\Gamma_{\cdot_Q}$ we induce the sought after deformation of the algebra structure of $\mathcal{A}$.

Theorem 3.1.6: *Let $\Gamma_{\cdot_Q}:\mathcal{A}\to\mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$ be a map defined as per Theorem 3.1.4. The vector space*

$$\mathcal{A}_{\cdot_Q} := \Gamma_{\cdot_Q}(\mathcal{A}) \subseteq \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})\,,$$

is a unital, commutative and associative $\mathbb{C}$-algebra if endowed with the product

$$\tau_1\cdot_{\Gamma_{\cdot_Q}}\tau_2 := \Gamma_{\cdot_Q}[\Gamma_{\cdot_Q}^{-1}(\tau_1)\Gamma_{\cdot_Q}^{-1}(\tau_2)]\,,\qquad \forall\,\tau_1,\tau_2\in\mathcal{A}_{\cdot_Q}\,. \tag{3.1.17}$$

Proof. First of all we observe that any map $\Gamma_{\cdot_Q}:\mathcal{A}\to\mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$ built as per Theorem 3.1.4 is such that $\ker\Gamma_{\cdot_Q}=\{0\}$. To this end, let $\tau\in\ker\Gamma_{\cdot_Q}\setminus\{0\}$ be of polynomial degree k in Φ, *i.e.*, $\delta^k_\psi\tau\neq 0$ while $\delta^{k+1}_\psi\tau=0$ – *cf.* Remark 2.1.23. Equation (3.1.2) implies that $\mathbf{1}\notin\ker\Gamma_{\cdot_Q}$, so that $k>0$. In turn this entails that for all $\psi\in\mathcal{E}(M)$ there exists $0\neq f_\psi\in\mathcal{E}(M^k)$ such that

$$\delta^k_\psi\tau = f_\psi\mathbf{1}\,.$$

Moreover Equation (3.1.4) implies that, if $\tau\in\ker\Gamma_{\cdot_Q}$, then also $\delta_\psi\tau\in\ker\Gamma_{\cdot_Q}$. This yields $f_\psi\mathbf{1}\in\ker\Gamma_{\cdot_Q}$, implying in turn $f_\psi=0$, which is a contradiction. Thus $\Gamma_{\cdot_Q}$ is injective and therefore $\cdot_{\Gamma_{\cdot_Q}}$ is well-defined being $\Gamma_{\cdot_Q}$ an algebra isomorphism. All remaining properties are straightforwardly verified. In particular $\mathbf{1}\cdot_{\Gamma_{\cdot_Q}}\tau=\tau\cdot_{\Gamma_{\cdot_Q}}\mathbf{1}=\tau$ for any $\tau\in\mathcal{A}_{\cdot_{\Gamma_{\cdot_Q}}}$ while associativity and commutativity of $\mathcal{A}_{\cdot_{\Gamma_{\cdot_Q}}}$ are inherited from the corresponding properties of $\mathcal{A}$. □

Remark 3.1.7: *By means of Theorems 3.1.4 and 3.1.6 we identify a suitable algebra $\mathcal{A}_{\cdot_Q}$ whose interpretation in terms of the stochastic underlying process has already been explained. A few comments about the* covariance *of such construction are in due order since one may wonder which is the interplay between this construction and diffeomorphism invariance, namely to* *which extent the algebra $\mathcal{A}_{\cdot_Q} = \mathcal{A}_{\cdot_Q}(M, E)$ depends on the geometrical data M and E, i.e., the underlying manifold and the linear partial differential operator.*

Dwelling more into the details, one may want the assignment

$$(M, E) \mapsto \mathcal{A}_{\cdot_Q}(M, E)\,,$$

to be functorial, namely to satisfy the following property: Whenever $\iota\colon M_1 \to M_2$, with M_1 and M_2 suitable smooth manifolds, is a smooth map such that $E_1 \circ \iota^* = \iota^* \circ E_2$ for suitable microhypoelliptic differential operators E_1 and E_2 and where ι^* denotes the pull-back through ι, then there exists a corresponding injective algebras isomorphism $\mathcal{A}_{\cdot_Q}(\iota)\colon \mathcal{A}_{\cdot_Q}(M_1, E_1) \to \mathcal{A}_{\cdot_Q}(M_2, E_2)$.

This statement can be read as the translation to this setting of the principle of general covariance *[16] which is adopted in algebraic quantum field theory, cf. [13, Ch. 4] and which is often stated adopting the language of category theory. Here, we avoid entering into the details, and we just observe that, similarly to the analysis presented in [26], such kind of properties are hard to implement as our construction depends on the choice of a parametrix P for the differential operator E.*

It is a well-known issue that the choice of P is not covariant and this has a repercussion in the failure of $\mathcal{A}_{\cdot_Q}$ being invariant under the induced action of the diffeomorphism group of the underlying manifold M. Following the same rationale of [26] a possible way out from this quandary consists of working with all possible parametrices P at once, introducing a further structure given by a bundle of algebras whose base space is the space of parametrices of E. This functorial formulation, even though in a different framework with respect to stochastic PDEs, has been introduced by the author together with C. Dappiaggi and N. Drago in [26] in the general context of Euclidean quantum field theory and with M. Carfora, C. Dappiaggi and N. Drago in [18] in the context of non-linear Sigma models.

This discussion is beyond the scope of this manuscript and therefore we omit it.

3.2 Correlations and the $\bullet_Q$ Product

As we have seen in the last part of Chapter 2, the product $\cdot_{\Gamma_{\cdot_Q}}$, or analogously its regularized counterpart $\cdot_{Q_\varepsilon}$, captures the information on the expectation values of polynomial expressions of the shifted random field $\widehat{\varphi}$.

Subsequently, in order to compute multi-local correlation functions, we have also introduced a second product, dubbed $\bullet_{Q_\varepsilon}$ on a suitable tensor algebra constructed out of $\mathcal{A}$.

Similarly to the discussion in Section 3.1, the aim of this section is to construct a renormalized version of the product $\bullet_Q$, namely to give a meaning to Equation (2.2.10) with Q_ε replaced by Q.

As one may imagine, also in this scenario ultra-violet divergences arise due to the singular behaviour of the bi-distribution $Q \in \mathcal{D}'(M \times M)$ on the total diagonal of $M \times M$. To see why this is the case, we consider the following concrete example.

Example 3.2.1: *Consider $\Phi^2 \in \mathcal{D}'_{\mathrm{C}}(M; \mathrm{Pol})$ as per Example 2.1.8. Formally replacing Q_ε with*

Q in Equation (2.2.10) *yields*

$$[\Phi^2 \bullet_Q \Phi^2](f_1 \otimes f_2; \varphi) = \\ = \int\limits_{M\times M} f_1(x_1) f_2(x_2) \big[\varphi^2(x_1)\varphi^2(x_2) + 4\varphi(x_1) Q(x_1,x_2)\varphi(x_2) + 2Q^2(x_1,x_2)\big] \mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2)\,,$$

where $f_1, f_2 \in \mathcal{D}(M)$ *and* $\varphi \in \mathcal{E}(M)$.

A priori, due to the distributional nature of Q*, the last term on the right hand side of the previous identity is a priori well-defined only outside the total diagonal of* $M \times M$*. This is due to the fact that, by Equation* (A.0.16)*, since* $\mathrm{WF}_1(P) = \mathrm{WF}_2(P) = \emptyset$*, it holds* $\mathrm{WF}(Q) \subseteq \mathrm{WF}(\delta_{\mathrm{Diag}_2})$*, cf. Equation* (A.0.2)*. This entails that* Q *is a smooth function over* $(M\times M)\setminus \mathrm{Diag}_2$*, yielding* $Q^2 \in \mathcal{D}'(M \times M \setminus \mathrm{Diag}_2)$*.*
Yet, we observe that, on account of Remark B.1.6, Lemma B.4.1 and Corollary B.4.2, it holds

$$\mathrm{sd}_{\mathrm{Diag}_2}(Q^2) \leq 2\,\mathrm{sd}_{\mathrm{Diag}_2}(Q) < \infty\,.$$

As a consequence, in view of Theorem B.2.1, there exists $\widehat{Q}_2 \in \mathcal{D}'(M\times M)$ *which is an extension of* Q^2 *satisfying* $\mathrm{sd}_{\mathrm{Diag}_2}(\widehat{Q}_2) = \mathrm{sd}_{\mathrm{Diag}_2}(Q^2)$*. As a matter of fact, exploiting the local behaviour of the parametrix* P *as well as Example B.1.5, it descends* $\mathrm{sd}_{\mathrm{Diag}_2}(Q) \leq d-4$*. Hence, if* $\dim M = d < 8$ *the above extension is unique, cf. Theorem B.2.1. Following the same strategy as in the preceding section, we can conceive to use such extension to give meaning to* $\Phi^2 \bullet_Q \Phi^2$*. The remainder of the section is devoted to making this idea precise and general in order to be able to multiply any element of the algebra* $\mathcal{A}_{\cdot_Q}$ *with respect to the product* $\bullet_Q$*.*

Remark 3.2.2: *The same strategy of Example 3.2.1 applies to the parabolic case. For the sake of clarity, we underline that the only difference is that* $\mathrm{wsd}_{\mathrm{Diag}_2}(Q^2) = 2(d-3)$*, cf. Section B.3, where* $d = \dim M$*. As a consequence, for the same reasons as in the above example, the extension* $\widehat{Q}_2 \in \mathcal{D}'(M \times M)$ *of* $Q^2 \in \mathcal{D}'(M \times M \setminus \mathrm{Diag}_2)$ *preserving the weighted scaling degree is unique if* $\dim(\Sigma) < 6$ *where* $M = \mathbb{R} \times \Sigma$*.*

Following the same approach of Section 3.1, throughout this section we shall discuss quite in some detail the elliptic scenario, commenting here and thereon the parabolic case, which is analogous, mutatis mutandis.

As for Theorems 3.1.4 and 3.1.6, we divide the result in two pieces: In the first one we introduce a map $\Gamma_{\bullet_Q}$ which serves as main ingredient in the deformation of the algebra structure, which we shall discuss in the second theorem.

Remark 3.2.3: *As a premise, we observe that the decomposition* $\mathcal{A} = \bigoplus\limits_{k\geq 0} \mathcal{M}_k = \varinjlim\limits_{j} \bigoplus\limits_{k\geq 0} \mathcal{M}_k^j$*, as per Remark 2.1.23, induces a counterpart at the level of the universal tensor module* $\mathcal{T}(\mathcal{A}_{\cdot_Q}) = \mathcal{E}(M) \oplus \bigoplus_{\ell>0} \mathcal{A}_{\cdot_Q}^{\otimes \ell}$*, i.e.*

$$\begin{aligned}\mathcal{T}(\mathcal{A}_{\cdot_Q}) &= \mathcal{E}(M) \oplus \bigoplus_{\ell>0} \bigoplus_{k=0}^{\infty} \bigoplus_{\substack{k_1,\dots,k_\ell \\ k_1+\dots+k_\ell = k}} \Gamma_{\cdot_Q}(\mathcal{M}_{k_1}) \otimes \ldots \otimes \Gamma_{\cdot_Q}(\mathcal{M}_{k_\ell}) \\ &= \mathcal{E}(M) \oplus \bigoplus_{\ell>0} \bigoplus_{k=0}^{\infty} \bigoplus_{\substack{k_1,\dots,k_\ell \\ k_1+\dots+k_\ell = k}} \varinjlim_{j_1,\dots,j_\ell} \Gamma_{\cdot_Q}(\mathcal{M}_{k_1}^{j_1}) \otimes \ldots \otimes \Gamma_{\cdot_Q}(\mathcal{M}_{k_\ell}^{j_\ell})\,.\end{aligned}$$

Theorem 3.2.4: *Let $\mathcal{A}_{\cdot_Q}$ be the algebra defined in Theorem 3.1.6 with a map $\Gamma_{\cdot_Q}$ built as per Theorem 3.1.4. Let moreover $\mathcal{T}'_{\mathrm{C}}(M;\mathrm{Pol})$ be defined as per Equation* (2.2.9) *and let $\mathcal{T}(\mathcal{A}_{\cdot_Q})$ be the universal tensor module constructed out of $\mathcal{A}_{\cdot_Q}$.*

There exists a linear map $\Gamma_{\bullet_Q}\colon \mathcal{T}(\mathcal{A}_{\cdot_Q}) \to \mathcal{T}'_{\mathrm{C}}(M;\mathrm{Pol})$ satisfying the following properties:

(i) for all $\tau_1,\dots,\tau_\ell \in \mathcal{A}_{\cdot_Q}$ with $\tau_1 \in \Gamma_{\cdot_Q}(\mathcal{M}_1)$ it holds

$$\Gamma_{\bullet_Q}(\tau_1\otimes\ldots\otimes\tau_\ell) := \tau_1 \bullet_Q \Gamma_{\bullet_Q}(\tau_2\otimes\ldots\otimes\tau_\ell)\,, \tag{3.2.1}$$

where $\bullet_Q$ is defined as in Equation (2.2.10) *with Q_ε replaced by Q.*

(ii) Let $\tau_1,\dots,\tau_\ell \in \mathcal{A}_{\cdot_Q}$ and $f_1,\dots,f_\ell \in \mathcal{D}(M)$. If there exists $I \subsetneq \{1,\dots,\ell\}$ for which

$$\bigcup_{i\in I}\mathrm{supp}(f_i)\cap\bigcup_{j\notin I}\mathrm{supp}(f_j)=\emptyset\,,$$

then

$$\Gamma_{\bullet_Q}(\tau_1\otimes\ldots\otimes\tau_\ell)(f_1\otimes\ldots\otimes f_\ell) = \left[\Gamma_{\bullet_Q}\left(\bigotimes_{i\in I}\tau_i\right)\bullet_Q\Gamma_{\bullet_Q}\left(\bigotimes_{j\notin I}\tau_j\right)\right](f_1\otimes\ldots\otimes f_\ell)\,. \tag{3.2.2}$$

(iii) for all $\ell\geq 0$, $\Gamma_{\bullet_Q}\colon \mathcal{A}_{\cdot_Q}^{\otimes\ell} \to \mathcal{T}'_{\mathrm{C}}(M;\mathrm{Pol})$ is a symmetric map,

(iv) $\Gamma_{\bullet_Q}$ satisfies the following identities:

$$\Gamma_{\bullet_Q}(\tau)=\tau\,,\qquad \forall\tau\in\mathcal{A}_{\cdot_Q}\,, \tag{3.2.3a}$$

$$\Gamma_{\bullet_Q}\circ\delta_\psi=\delta_\psi\circ\Gamma_{\bullet_Q}\,,\qquad \forall\psi\in\mathcal{E}(M)\,, \tag{3.2.3b}$$

$$\Gamma_{\bullet_Q}(\tau_1\otimes\ldots\otimes P\circledast\tau_j\otimes\ldots\otimes\tau_\ell)=(\delta_{\mathrm{Diag}_2}^{\otimes j-1}\otimes P\otimes\delta_{\mathrm{Diag}_2}^{\otimes\ell-j})\circledast\Gamma_{\bullet_Q}(\tau_1\otimes\ldots\otimes\tau_j\otimes\ldots\otimes\tau_\ell)\,, \tag{3.2.3c}$$

for all $\tau_1,\dots,\tau_\ell\in\mathcal{A}_{\cdot_Q}$ and for all $\ell\in\mathbb{Z}_+$.

Proof. The strategy of the proof is very similar in spirit to the one of Theorem 3.1.4, since also here we proceed by induction. As in the previous case, we divide what follows in separate steps.

As a preliminary observation we stress that Equation (3.2.1) and the map $\bullet_Q$ are well-defined when applied to elements lying in $\Gamma_{\cdot_Q}(\mathcal{M}_1)$, since no divergences occur in such scenario. In particular all equations in item (iv) are automatically satisfied. In addition we observe that, under the assumption made on the test functions $f_1,\dots,f_\ell$, the product $\bullet_Q$ appearing in Equation (3.2.2) is well-defined.

Step 1 – Induction over ℓ. The first step consists of controlling $\Gamma_{\bullet_Q}$ as the number of arguments in the tensor product increases, recalling that $\mathcal{T}(\mathcal{A}_{\cdot_Q})=\mathcal{E}(M)\bigoplus_{\ell=1}^\infty\mathcal{A}_{\cdot_Q}^{\otimes\ell}$.

Stated differently, we are proceeding inductively over ℓ and we observe that Equation (3.2.3a) defines completely $\Gamma_{\bullet_Q}$ when $\ell=1$. By the inductive hypothesis, we assume that the $\Gamma_{\bullet_Q}$ has been defined on $\mathcal{A}_{\cdot_Q}^{\otimes p} \subseteq \mathcal{T}'_{\mathrm{C}}(M;\mathrm{Pol})$ for any $p<\ell$ and we prove the existence of $\Gamma_{\bullet_Q}$ on $\mathcal{A}_{\cdot_Q}^{\otimes \ell}$. Since, by Remark 3.2.3,

$$\mathcal{A}_{\cdot_Q}^{\otimes \ell} = \bigoplus_{k\geq 0} \bigoplus_{\substack{k_1,\dots k_l \\ k_1+\dots+k_\ell=k}} \Gamma_{\cdot_Q}(\mathcal{M}_{k_1}) \otimes \dots \otimes \Gamma_{\cdot_Q}(\mathcal{M}_{k_\ell}) \,,$$

the problem reduces to constructing, for any $k\in\mathbb{N}\cup\{0\}$, $\Gamma_{\bullet_Q}(\tau_{k_1}\otimes\dots\otimes\tau_{k_\ell})$ for all $\tau_{k_1},\dots,\tau_{k_\ell}$ with $\tau_{k_p}\in\Gamma_{\cdot_Q}(\mathcal{M}_{k_p})$, while $k_1,\dots,k_p\in\mathbb{N}\cup\{0\}$ are such that $k_1+\dots+k_\ell=k$.

Step 2 – Induction over k. We thus proceed inductively over $k\in\mathbb{N}\cup\{0\}$. The cases $k=0,1$ are readily verified since no singularity can occur in

$$\Gamma_{\bullet_Q}(\tau_{k_1}\otimes\dots\otimes\tau_{k_\ell}) = \tau_{k_1}\bullet_Q\dots\bullet_Q\tau_{k_\ell} \,.$$

Furthermore both items *(ii)* and *(iii)* are satisfied per construction. Hence we can make the inductive hypothesis assuming that $\Gamma_{\bullet_Q}(\tau_{k_1}\otimes\dots\otimes\tau_{k_\ell})$ has been defined for all $k_1,\dots,k_\ell$ such that $k_1+\dots+k_\ell=k-1$. In order to extend the definition of $\Gamma_{\bullet_Q}$ to the case when $k_1+\dots k_\ell=k$ we can invoke Remark 3.2.3 to write

$$\Gamma_{\cdot_Q}(\mathcal{M}_{k_1})\otimes\dots\otimes\Gamma_{\cdot_Q}(\mathcal{M}_{k_\ell}) = \varinjlim_{j_1,\dots,j_\ell} \Gamma_{\cdot_Q}(\mathcal{M}_{j_1}^{k_1})\otimes\dots\otimes\Gamma_{\cdot_Q}(\mathcal{M}_{j_\ell}^{k_\ell}) \,,$$

proceeding inductively over $j_1,\dots,j_\ell$.

Step 2a – Induction over $\mathbf{j_1,\dots,j_\ell}$: starting case. If $j_1=\dots=j_\ell=0$, this amounts to considering only the case $\tau_{k_p}=\Gamma_{\cdot_Q}(\Phi^{k_p})$ for all $p\in\{1,\dots,\ell\}$ – *cf.* Definition 2.1.20. For any $f_i\in\mathcal{D}(M)$, $i=1,\dots,\ell$ with disjoint supports and $\varphi\in\mathcal{E}(M)$, Equation (3.2.2) implies, together with Equation (2.2.10) with Q_ε replaced by Q,

$$\begin{aligned}
\Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi^{k_1})\otimes\dots\otimes\Gamma_{\cdot_Q}(\Phi^{k_\ell}))(f_1\otimes\dots\otimes f_\ell;\varphi) &= \Gamma_{\cdot_Q}(\Phi^{k_1})\bullet_Q\dots\bullet_Q\Gamma_{\cdot_Q}(\Phi^{k_\ell})(f_1\otimes\dots\otimes f_\ell;\varphi) \\
= \sum_{N=0}^{\infty}\frac{(2N)!}{(2N)!!} \sum_{\substack{N_1,\dots N_\ell \\ N_1+\dots+N_\ell=2N}} & \binom{\widehat{k}_\ell}{\widehat{N}_\ell}(1_\ell\otimes Q^{\otimes N})\cdot \\
&\cdot[\delta_{\mathrm{Diag}_{N_1+1}}\Gamma_{\cdot_Q}(\Phi^{k_1-N_1})\otimes\dots\otimes\delta_{\mathrm{Diag}_{N_\ell+1}}\Gamma_{\cdot_Q}(\Phi^{k_\ell-N_\ell})](f_1\otimes\dots\otimes f_\ell\otimes 1_{2N};\varphi) \,,
\end{aligned}$$

where $\cdot$ denotes the product of distributions, while $\binom{\widehat{k}_\ell}{\widehat{N}_\ell} := \prod_{p=1}^{\ell}\binom{k_p}{N_p}$ and

$$\Gamma_{\cdot_Q}(\Phi^k)^{(j)} = \frac{k!}{(k-j)!}\Gamma_{\cdot_Q}(\Phi^{k-j})\delta_{\mathrm{Diag}_{j+1}} \,.$$

The above expression defines $\Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi^{k_1})\otimes\dots\otimes\Gamma_{\cdot_Q}(\Phi^{k_\ell}))$ as a functional-valued distribution on $M^\ell\setminus\mathrm{Diag}_\ell$. This is codified by the fact that

$$S_{\ell,N} := (1_\ell\otimes Q^{\otimes N})\cdot[\delta_{\mathrm{Diag}_{N_1+1}}\otimes\dots\otimes\delta_{\mathrm{Diag}_{N_\ell+1}}] \,,$$

is a well-defined functional-valued distribution only on $M^{\ell+2N}\setminus\mathrm{Diag}_{\ell+2N}$. To complete the definition of $\Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi^{k_1})\otimes\dots\otimes\Gamma_{\cdot_Q}(\Phi^{k_\ell}))$ we notice that $\mathrm{sd}_{\mathrm{Diag}_{\ell+2N}}(S_{\ell,N})$ is finite and, as a consequence, Theorem B.2.1 guarantees the existence of an extension $\widehat{S}_{\ell,N}$ of $S_{\ell,N}$ to the whole space $M^{\ell+2N}$ which is scaling degree preserving, namely $\mathrm{sd}_{\mathrm{Diag}_{\ell+2N}}(\widehat{S}_{\ell,N}) = \mathrm{sd}_{\mathrm{Diag}_{\ell+2N}}(S_{\ell,N})$.

Having chosen one such extension we set for all $f_i \in \mathcal{D}(M)$, $i = 1, \dots, \ell$ and for all $\varphi \in \mathcal{E}(M)$

$$\begin{aligned}
&\Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi^{k_1}) \otimes \dots \otimes \Gamma_{\cdot_Q}(\Phi^{k_\ell}))(f_1 \otimes \dots \otimes f_\ell; \varphi) \\
&= \sum_{\substack{N \geq 0 \\ N_1+\dots+N_\ell=2N}} \frac{(2N)!}{(2N)!!} \binom{\widehat{k}_\ell}{\widehat{N}_\ell} \widehat{S}_{\ell,N} \cdot [\Gamma_{\cdot_Q}(\Phi^{k_1-N_1}) \otimes \dots \otimes \Gamma_{\cdot_Q}(\Phi^{k_\ell-N_\ell})](f_1 \otimes \dots \otimes f_\ell \otimes 1_{2N}; \varphi) \,,
\end{aligned}$$

which identifies an element in $\mathcal{D}'(M^\ell; \mathrm{Pol})$. In addition, for any $p \in \mathbb{N} \cup \{0\}$ and $\psi \in \mathcal{E}(M)$, it holds

$$\begin{aligned}
&\Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi^{k_1}) \otimes \dots \otimes \Gamma_{\cdot_Q}(\Phi^{k_\ell}))^{(p)}(f_1 \otimes \dots \otimes f_\ell \otimes \psi^{\otimes p}; \varphi) \\
&\quad = \sum_{\substack{N \geq 0 \\ N_1+\dots+N_\ell=2N \\ p_1+\dots+p_\ell=p}} \frac{(2N)!p!}{(2N)!!} \prod_{h=1}^{q} C(k_h, N_h, p_h) \cdot \\
&\qquad\qquad \cdot \widehat{S}^{[p]}_{\ell,N} \cdot [\Gamma_{\cdot_Q}(\Phi^{k_1-N_1-p_1}) \otimes \dots \otimes \Gamma_{\cdot_Q}(\Phi^{k_\ell-N_\ell-p_\ell})](f_1 \otimes \dots \otimes f_\ell \otimes 1_{2N} \otimes \psi^{\otimes p}; \varphi) \,,
\end{aligned}$$

where $C(k_h, N_h, p_h) = \frac{k_h!}{p_h! N_h! (k_h - N_h - p_h)!}$ while we denote with $\widehat{S}^{[p]}_{\ell,N}$ the distribution

$$\begin{aligned}
\widehat{S}^{[p]}_{\ell,N} &:= (\widehat{S}_{\ell,N} \otimes 1_p) \cdot [1_N \otimes \delta_{\mathrm{Diag}_{p_1+1}} \otimes \dots \otimes \delta_{\mathrm{Diag}_{p_\ell+1}}] \supseteq \\
&\qquad \supseteq (1_\ell \otimes Q^{\otimes N} \otimes 1_p) \cdot [\delta_{\mathrm{Diag}_{p_1+N_1+1}} \otimes \dots \otimes \delta_{\mathrm{Diag}_{p_\ell+N_\ell+1}}] \,,
\end{aligned}$$

where $\supseteq$ means that the two distributions coincide on $\mathcal{D}(M^{\ell+2N+p} \setminus \mathrm{Diag}_{\ell+2N+p})$. In addition it holds that $\Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi^{k_1}) \otimes \dots \otimes \Gamma_{\cdot_Q}(\Phi^{k_\ell})) \in \mathcal{D}'_{\mathrm{C}}(M^\ell; \mathrm{Pol})$. This follows by estimating the wave front set of the distribution

$$\mathcal{D}(M)^{\otimes \ell} \otimes \mathcal{E}(M)^{\otimes p} \ni f_1 \otimes \dots \otimes f_\ell \otimes \psi^{\otimes p} \mapsto \widehat{S}^{[p]}_{\ell,N}(f_1 \otimes \dots \otimes f_\ell \otimes 1_{2N} \otimes \psi^{\otimes p}) \,,$$

by means of Theorem A.0.6 and Remark A.0.8. Furthermore a direct application of Equation (3.1.4) yields, for any $f_i \in \mathcal{D}(M)$, $i = 1, \dots, \ell$ and $\varphi \in \mathcal{E}(M)$,

$$\begin{aligned}
&\Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi^{k_1}) \otimes \dots \otimes \Gamma_{\cdot_Q}(\Phi^{k_\ell}))^{(p)}(f_1 \otimes \dots \otimes f_\ell \otimes \psi^{\otimes p}; \varphi) = \\
&= \sum_{\substack{N \geq 0 \\ N_1+\dots+N_\ell=2N \\ p_1+\dots+p_\ell=p}} \frac{p!}{p_1! \cdots p_\ell!} \binom{\widehat{k}_\ell}{\widehat{N}_\ell} \widehat{S}_{\ell,N} \cdot [\Gamma_{\cdot_Q}(\delta^{p_1}_\psi \Phi^{k_1-N_1}) \otimes \dots \otimes \Gamma_{\cdot_Q}(\delta^{p_\ell}_\psi \Phi^{k_\ell-N_\ell})](f_1 \otimes \dots \otimes f_\ell \otimes 1_{2N}; \varphi) \,,
\end{aligned}$$

which entails Equation (3.2.3b). This completes the inductive proof for the initial case $n_1 = \dots = n_\ell = 0$.

Step 2b – Induction over $j_1, \dots, j_\ell$: general case. In view of the preceding step we can assume that $\Gamma_{\bullet_Q}(\tau_{k_1} \otimes \dots \otimes \tau_{k_\ell})$ has been defined for all $\tau_{k_p} \in \Gamma_{\cdot_Q}(\mathcal{M}^{m_p}_{k_p})$ for $m_p < j_p$. Our final goal is to show how to extend $\Gamma_{\bullet_Q}$ to the case $m_p = j_p$ for all $p \in \{1, \dots, \ell\}$.

Let $\tau_{k_1}, \dots, \tau_{k_\ell}$ be such that $\tau_{k_p} \in \Gamma_{\cdot_Q}(\mathcal{M}^{j_p}_{k_p})$ for all $p \in \{1, \dots, \ell\}$ and $k_1 + \dots + k_\ell = k$. If $\tau_{k_p} = P \circledast \sigma_{k_p}$ for $\sigma_{k_p} \in \mathcal{M}^{j_p-1}_{k_p}$, then $\Gamma_{\bullet_Q}(\tau_{k_1} \otimes \dots \otimes \tau_{k_\ell})$ is defined via Equation (3.2.3c). In the general case, for any $f_1, \ldots, f_\ell \in \mathcal{D}(M)$ with disjoint supports and $\varphi \in \mathcal{E}(M)$, Equation (3.2.2) yields

$$\begin{aligned}
&\Gamma_{\bullet_Q}(\tau_{k_1} \otimes \ldots \otimes \tau_{k_\ell})(f_1 \otimes \ldots \otimes f_\ell; \varphi) = (\tau_{k_1} \bullet_Q \ldots \bullet_Q \tau_{k_\ell})(f_1 \otimes \ldots \otimes f_\ell; \varphi) \\
&= \sum_{N=0}^{\infty} \frac{1}{(2N)!!} \sum_{\substack{N_1, \ldots N_\ell \\ N_1 + \ldots + N_\ell = 2N}} \frac{(2N)!}{N_1! \cdots N_\ell!} \big[(1_\ell \otimes Q^{\otimes J}) \cdot (\tau_{k_1}^{(N_1)} \widetilde{\otimes} \ldots \widetilde{\otimes} \tau_{k_\ell}^{(N_\ell)})\big](f_1 \otimes \ldots \otimes f_\ell \otimes 1_{2N}; \varphi) \\
&= \sum_{N=0}^{\infty} \frac{1}{(2N)!!} \sum_{\substack{N_1, \ldots N_\ell \\ N_1 + \ldots + N_\ell = 2N}} \frac{(2N)!}{N_1! \cdots N_\ell!} S_{\ell,N}(f_1 \otimes \ldots \otimes f_\ell \otimes 1_{2N}; \varphi) \,,
\end{aligned}$$

where as usual $\cdot$ stands for the product of distributions. As before, this defines $\Gamma_{\bullet_Q}(\tau_{k_1} \otimes \ldots \otimes \tau_{k_\ell})$ as a functional-valued distribution on $M^\ell \setminus \mathrm{Diag}_\ell$. Notice that with the same argument used in the proof of Theorem 3.1.4 we can infer that $S_{\ell,N}$ is a well-defined functional-valued distribution on $M^{\ell+2N} \setminus \mathrm{Diag}_{\ell+2N}$.

To complete the definition of $\Gamma_{\bullet_Q}(\tau_{k_1} \otimes \ldots \otimes \tau_{k_\ell})$ we ought to identify a suitable extension of $S_{\ell,N} \in \mathcal{D}'(M^{\ell+2N} \setminus \mathrm{Diag}_{\ell+2N})$. Since $\mathrm{sd}_{\mathrm{Diag}_{\ell+2N}}(S_{\ell,N}) < +\infty$, we can apply Theorem B.2.1 to ensure the existence of an extension $\widehat{S}_{\ell,N}$ of $S_{\ell,N}$ to the whole $M^{\ell+2J}$ such that $\mathrm{sd}_{\mathrm{Diag}_{\ell+2N}}(\widehat{S}_{\ell,N}) = \mathrm{sd}_{\mathrm{Diag}_{\ell+2N}}(S_{\ell,N})$. Again, having chosen one such extension, we set for any $f_1, \ldots, f_\ell \in \mathcal{D}(M)$ and for all $\varphi \in \mathcal{E}(M)$

$$\Gamma_{\bullet_Q}(\tau_{k_1} \otimes \ldots \otimes \tau_{k_\ell})(f_1 \otimes \ldots \otimes f_\ell; \varphi) := \sum_{\substack{N \geq 0 \\ N_1 + \ldots + N_\ell = 2N}} \frac{1}{(2N)!!} \frac{(2N)!}{N_1! \cdots N_\ell!} \widehat{S}_{\ell,N}(f_1 \otimes \ldots \otimes f_\ell \otimes 1_{2N}; \varphi) \,.$$

This formula entails that $\Gamma_{\bullet_Q}(\tau_{k_1} \otimes \ldots \otimes \tau_{k_\ell}) \in \mathcal{D}'(M^\ell; \mathrm{Pol})$ which is per construction symmetric in $\tau_{k_1}, \ldots, \tau_{k_\ell}$. Furthermore, for any $p \in \mathbb{N} \cup \{0\}$ it holds

$$\begin{aligned}
&\Gamma_{\bullet_Q}(\tau_{k_1} \otimes \ldots \otimes \tau_{k_\ell})^{(p)}(f_1 \otimes \ldots \otimes f_\ell \otimes \psi^{\otimes p}; \varphi) \\
&\qquad = \sum_{N=0}^{\infty} \frac{1}{(2N)!!} \sum_{\substack{N_1, \ldots N_\ell \\ N_1 + \ldots + N_\ell = 2N}} \frac{(2N)!}{N_1! \cdots N_\ell!} \widehat{S}_{\ell,N}^{(p)}(f_1 \otimes \ldots \otimes f_\ell \otimes 1_{2N} \otimes \psi^{\otimes p}; \varphi) \,.
\end{aligned}$$

With reference to Equation (3.2.3b) consider the formal expression

$$\begin{aligned}
&(\tau_{k_1} \bullet_Q \ldots \bullet_Q \tau_{k_\ell})^{(p)}(f_1 \otimes \ldots \otimes f_\ell \otimes \psi^{\otimes p}; \varphi) = \\
&\sum_{\substack{N \geq 0 \\ N_1 + \ldots + N_\ell = 2N \\ p_1 + \ldots + p_\ell = p}} \frac{(2N)! p!}{(2N)!! \prod_{i=1}^{\ell} N_i! p_i!} \big[(1_\ell \otimes Q^{\otimes N} \otimes 1_p) \cdot (\tau_{k_1}^{(N_1+p_1)} \widetilde{\otimes} \ldots \widetilde{\otimes} \tau_{k_\ell}^{(N_\ell+p_\ell)})\big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (f_1 \otimes \ldots \otimes f_\ell \otimes 1_{2N} \otimes \psi^{\otimes p}; \varphi) \\
&= \sum_{\substack{N \geq 0 \\ N_1 + \ldots + N_\ell = 2N \\ p_1 + \ldots + p_\ell = p}} \frac{(2N)! p!}{(2N)!! \prod_{i=1}^{\ell} N_i! p_i!} S_{\ell,N}^{[\widehat{p}_\ell]}(f_1 \otimes \ldots \otimes f_\ell \otimes 1_{2N} \otimes \psi^{\otimes p}; \varphi) \,,
\end{aligned}$$

where $S^{[\widehat{p}_\ell]}_{\ell,N}$ is a functional-valued distribution defined on $M^{\ell+2N+p}\setminus \mathrm{Diag}_{\ell+2N+p}$ with finite scaling degree at $\mathrm{Diag}_{\ell+2N+p}$. Here we set $\widehat{p}_\ell=(p_1,\ldots,p_\ell)$. On account of the explicit form of such $\widehat{S}_{\ell,M}$ – *cf.* Theorem B.2.1 – we can choose $\widehat{S}_{\ell,M}$ so that

$$\widehat{S}^{(p)}_{\ell,M}=\sum_{p_1+\ldots+p_\ell=p}\frac{p!}{p_1!\cdots p_\ell!}\widehat{S^{[\widehat{p}_\ell]}_{\ell,N}}\,,$$

where $\widehat{S^{[\widehat{p}_\ell]}_{\ell,N}}$ denotes a scaling degree preserving extension of $S^{[\widehat{p}_\ell]}_{\ell,N}$ on $M^{\ell+2N+p}$. With this choice it follows that $\Gamma_{\bullet_Q}(\tau_{k_1}\otimes\ldots\otimes\tau_{k_\ell})$ satisfies Equation (3.2.3b). In addition, $\Gamma_{\bullet_Q}(\tau_{k_1}\otimes\ldots\otimes\tau_{k_\ell})\in\mathcal{D}'_{\mathrm{C}}(M^\ell;\mathrm{Pol})$ as it can be shown by evaluating the wave front set of

$$\mathcal{D}(M)^{\otimes\ell}\otimes\mathcal{E}(M)^{\otimes p}\ni f_1\otimes\ldots\otimes f_\ell\otimes\psi^{\otimes p}\mapsto\widehat{S}^{(p)}_{\ell,N}(f_1\otimes\ldots\otimes f_\ell\otimes 1_{2N}\otimes\psi^{\otimes p};\varphi)\,.$$

This concludes the proof by induction over $j_1,\ldots,j_\ell$ and consequently also that over k and over ℓ. Finally, this also concludes the whole proof. □

Remark 3.2.5: *As for Theorem 3.1.4, also Theorem 3.2.4 holds true* mutatis mutandis *also in the parabolic case with $M=\mathbb{R}\times\Sigma$ and $E=\partial_t-\widetilde{E}$. As we discussed in Remark 3.1.5, this is a consequence of the finiteness of the weighted scaling degree, cf. Section B.3.*

Theorem 3.2.6: *Let $\Gamma_{\bullet_Q}:\mathcal{T}(\mathcal{A}_{\cdot_Q})\to\mathcal{T}'_{\mathrm{C}}(M;\mathrm{Pol})$ be a map as per Theorem 3.2.4. Let*

$$\mathcal{A}_{\bullet_Q}:=\Gamma_{\bullet_Q}(\mathcal{A}_{\cdot_Q})\subseteq\mathcal{T}'_{\mathrm{C}}(M;\mathrm{Pol})\,. \tag{3.2.4}$$

Then the bilinear map $\bullet_Q\colon\mathcal{A}_{\bullet_Q}\times\mathcal{A}_{\bullet_Q}\to\mathcal{A}_{\bullet_Q}$ defined by

$$\tau\bullet_Q\bar{\tau}:=\Gamma_{\bullet_Q}(\Gamma^{-1}_{\bullet_Q}(\tau)\otimes\Gamma^{-1}_{\bullet_Q}(\bar{\tau}))\,,\qquad\forall\tau,\bar{\tau}\in\mathcal{A}_{\bullet_Q}\,, \tag{3.2.5}$$

defines a unital, commutative and associative product on $\mathcal{A}_{\bullet_Q}$.

Proof. The proof of this theorem is similar to that of Theorem 3.1.6. Once $\Gamma_{\bullet_Q}$ has been defined, the algebraic properties of $\bullet_Q$ follows by direct inspection. In particular commutativity of the product is due to the symmetry of $\Gamma_{\bullet_Q}$. Associativity follows instead by a direct computation. For all $\tau_1,\tau_2,\tau_3\in\mathcal{A}_{\bullet_Q}$ it holds

$$\begin{aligned}(\tau_1\bullet_Q\tau_2)\bullet_{\Gamma_{\bullet_Q}}\tau_3=\Gamma_{\bullet_Q}(\Gamma^{-1}_{\bullet_Q}(\tau_1\bullet_Q\tau_2)\otimes\Gamma^{-1}_{\bullet_Q}(\tau_3))&=\Gamma_{\bullet_Q}(\Gamma^{-1}_{\bullet_Q}(\tau_1)\otimes\Gamma^{-1}_{\bullet_Q}(\tau_2)\otimes\Gamma^{-1}_{\bullet_Q}(\tau_3))\\&=\tau_1\bullet_Q(\tau_2\bullet_Q\tau_3)\,.\end{aligned}$$

□

3.3 Uniqueness Results

In the previous sections we discussed the existence of the algebras $\mathcal{A}_{\cdot_Q}$ and $\mathcal{A}_{\bullet_Q}$ – *cf.* Theorems 3.1.6 and 3.2.6, commenting also about their interpretation from the stochastic viewpoint.

In particular, we have seen that these constructions strongly rely on the existence of suitable maps $\Gamma_{\cdot_Q}$ and $\Gamma_{\bullet_Q}$ as per Theorems 3.1.4 and 3.2.4 respectively. We recall also that, due to the ultra-violet divergences arising because of the singular behaviour of the white noise, the construction of the aforementioned maps involves a renormalization procedure.

As we discussed, such procedure is an extension of *a priori* ill-defined distributions based on the notion of scaling degree – see Appendix B. In particular, as one can see from Theorem B.2.1, this procedure may be non-unique depending on the dimension d of the underlying manifold M and on the singular behaviour of the specific distribution, as we have seen in the proofs of Theorems 3.1.4 and 3.2.4.

As a consequence, in general one cannot hope for uniqueness of the maps $\Gamma_{\cdot_Q}$ and $\Gamma_{\bullet_Q}$, prompting the question whether it may be possible to classify the freedom in their definition. A further related question arising from this observation is how the algebras $\mathcal{A}_{\cdot_Q}$ and $\mathcal{A}_{\bullet_Q}$, constructed out of the maps $\Gamma_{\cdot_Q}$ and $\Gamma_{\bullet_Q}$, actually depend on these freedoms.

This is the issue we address in this section, inspired by the results on the classification of renormalization ambiguities both in relativistic [57, 58, 69] and in Euclidean [26] quantum field theory.

Example 3.3.1: *Aiming to a better understanding of the rationale behind the characterization of the freedom in defining the maps $\Gamma_{\cdot_Q}$ and $\Gamma_{\bullet_Q}$, we feel worth focusing first on a concrete example, which is traded from Equation* (3.1.7)*. The notation we are going to use will be consistent with the one of Theorem 3.3.2.*

Suppose that $\Gamma_{\cdot_Q}$ and $\widetilde{\Gamma}_{\cdot_Q}$ are two different maps as per Theorem 3.1.4. In addition, consider the functional-valued distribution $\tau = \Phi^2 \in \mathcal{A}$, cf. Example 2.1.8, and let $\mathcal{C}_0 \in \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$ be defined as

$$\mathcal{C}_0(f;\varphi) := \Gamma_{\cdot_Q}(\Phi^2)(f;\varphi) - \widetilde{\Gamma}_{\cdot_Q}(\Phi^2)(f;\varphi) \qquad \forall (f;\varphi) \in \mathcal{D}(M) \times \mathcal{E}(M)\,.$$

Resorting to Equation (3.1.7)*, it turns out that for any $\psi \in \mathcal{E}(M)$, it holds*

$$\begin{aligned}\mathcal{C}_0^{(1)}(f\otimes\psi;\varphi) &= \Gamma_{\cdot_Q}(\Phi^2)^{(1)}(f\otimes\psi;\varphi) - \widetilde{\Gamma}_{\cdot_Q}(\Phi^2)^{(1)}(f\otimes\psi;\varphi)\\ &= 2\Gamma_{\cdot_Q}(\psi\Phi)(f;\varphi) - 2\widetilde{\Gamma}_{\cdot_Q}(\psi\Phi)(f;\varphi) = 0\,,\end{aligned}$$

where we exploited Equation (3.1.4) *together with Equation* (3.1.2)*, recalling that $\Phi \in \mathcal{M}_1$ – cf. Remark 2.1.23. It descends that $\mathcal{C}_0$ does not depend on the configuration $\varphi \in \mathcal{E}(M)$.*

Combining this statement with Equation (2.1.11)*, it turns out that $\mathcal{C}_0 \in \mathcal{D}'(M)$ and $\mathrm{WF}(\mathcal{C}_0) = \emptyset$. Hence there must exist $c_0 \in \mathcal{E}(M)$ such that $\mathcal{C}_0 = c_0\mathbf{1}$. It follows that*

$$\Gamma_{\cdot_Q}(\Phi^2) = \widetilde{\Gamma}_{\cdot_Q}(\Phi^2) + c_0\mathbf{1}\,,$$

that is, on the functional-valued distribution Φ^2, $\widetilde{\Gamma}_{\cdot_Q}$ and $\Gamma_{\cdot_Q}$ differ by a distribution generated by a smooth function. This argument shows that a classification of the renormalization ambiguities is actually possible. In the following, we shall make this reasoning more systematic.

3.3.1 The Case of $\cdot_Q$

To start with, we focus our attention on the case of the map $\Gamma_{\cdot_Q}$.

Theorem 3.3.2: *Let $\widetilde{\Gamma}_{\cdot_Q} : \mathcal{A} \to \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$ and $\Gamma_{\cdot_Q} : \mathcal{A} \to \mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$ be two linear maps compatible with the constraints listed in Theorem 3.1.4. Recalling that $\mathcal{A} = \bigoplus_{k\in\mathbb{N}_0}\mathcal{M}_k$, cf.*

Remark 2.1.23, there exists a family $\{\mathcal{C}_\ell\}_{\ell\in\mathbb{N}_0}$ of linear maps $\mathcal{C}_\ell\colon \mathcal{A}\to\mathcal{M}_\ell$, such that:

$$\mathcal{C}_\ell[\mathcal{M}_k]=0\,,\qquad \forall k\leq \ell+1\,, \tag{3.3.1a}$$

$$\mathcal{C}_k[P\circledast\tau]=P\circledast\mathcal{C}_k[\tau]\,,\qquad \forall k\in\mathbb{N}_0\,,\forall\tau\in\mathcal{A} \tag{3.3.1b}$$

$$\delta_\psi\circ\mathcal{C}_k=\mathcal{C}_{k-1}\circ\delta_\psi\,,\qquad \forall k\in\mathbb{N}\,,\forall\psi\in\mathcal{E}(M) \tag{3.3.1c}$$

$$\widetilde{\Gamma}_{\cdot_Q}(\tau)=\Gamma_{\cdot_Q}\left(\tau+\mathcal{C}_{k-2}[\tau]\right)\,,\qquad \forall\tau\in\mathcal{M}_k\,. \tag{3.3.1d}$$

Proof. The proof proceeds by induction over k, namely we show that, whenever the map $\mathcal{C}_\ell$ has been consistently defined on all $\widetilde{\tau}\in\mathcal{M}_{k-1}$ for any $\ell\in\mathbb{N}_0$, then $\mathcal{C}_\ell[\tau]$ is known also for $\tau\in\mathcal{M}_k$.

If $k=1$, then both Equation (3.1.2) and Equation (3.3.1a) entail that $\mathcal{C}_\ell[\mathcal{M}_1]=0$ for all $\ell\in\mathbb{N}_0$. As induction hypothesis we assume that, for all $\ell\in\mathbb{N}_0$, $\mathcal{C}_\ell[\tau]$ has been defined for all $\tau\in\mathcal{M}_{k-1}$.

In order to define $\mathcal{C}_\ell[\tau]$ for all $\ell\in\mathbb{N}\cup\{0\}$ and for all $\tau\in\mathcal{M}_k$, we observe that Equation (3.3.1a) yields that $\mathcal{C}_\ell[\mathcal{M}_k]=0$ whenever $\ell\geq k-1$. To construct $\mathcal{C}_\ell$, for $0\leq\ell\leq k-2$, we consider $\tau\in\mathcal{M}_k$. If there exists $\bar{\tau}\in\mathcal{M}_k$ such that $\tau=P\circledast\bar{\tau}$, in agreement with Equation (3.3.1b), we set $\mathcal{C}_\ell[P\circledast\bar{\tau}]=P\circledast\mathcal{C}_\ell[\bar{\tau}]$. If this is not case, for any $\ell\in\{1,\dots,k-2\}$, we define – *cf.* Remark 2.1.9,

$$\mathcal{C}_\ell[\tau](f;\varphi)=\mathcal{C}_\ell[\tau](f;0)+\int_0^1\mathcal{C}_{\ell-1}[\delta_\varphi\tau](f;s\varphi)\mathrm{d}s\,.\qquad \forall\, f\in\mathcal{D}(M)\,,\forall\,\varphi\in\mathcal{E}(M)\,.$$

Notice that $\delta_\varphi\tau\in\mathcal{M}_{k-1}$ so that $\mathcal{C}_{\ell-1}[\delta_\varphi\tau]\in\mathcal{M}_{\ell-1}$ is known by the inductive hypothesis, whereas $\mathcal{C}_\ell[\tau](f;0)$ can be seen as an arbitrary additive constant. In addition, $\mathcal{C}_\ell[\tau]$ satisfies Equation (3.3.1c), since for all $\psi\in\mathcal{E}(M)$,

$$\begin{aligned}(\delta_\psi\mathcal{C}_\ell[\tau])(\varphi)&=\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{C}_\ell[\tau](\varphi+\lambda\psi)\Big|_{\lambda=0}\\&=\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_0^1\mathcal{C}_{\ell-1}[\delta_{\varphi+\lambda\psi}\tau](s\varphi+s\lambda\psi)\mathrm{d}s\Big|_{\lambda=0}\\&=\int_0^1\mathcal{C}_{\ell-1}[\delta_\psi\tau](s\varphi)\mathrm{d}s+\int_0^1 s\delta_\psi\mathcal{C}_{\ell-1}[\delta_\varphi\tau](s\varphi)\mathrm{d}s\,.\end{aligned}$$

We observe that, on account of Equation (3.3.1c),

$$\delta_\psi\mathcal{C}_{\ell-1}[\delta_\varphi\tau](s\varphi)=\delta_\varphi\mathcal{C}_{\ell-1}[\delta_\psi\tau](s\varphi)=\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{C}_{\ell-1}[\delta_\psi\tau](s\varphi)\,.$$

Integration by parts yields

$$\begin{aligned}(\delta_\psi\mathcal{C}_\ell[\tau])(\varphi)&=\int_0^1\mathcal{C}_{\ell-1}[\delta_\psi\tau](s\varphi)\mathrm{d}s+\int_0^1 s\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{C}_{\ell-1}[\delta_\psi\tau](s\varphi)\mathrm{d}s\\&=s\mathcal{C}_{\ell-1}[\delta_\psi\tau](s\varphi)\Big|_{s=0}^{s=1}=\mathcal{C}_{\ell-1}[\delta_\psi\tau](\varphi)\,.\end{aligned}$$

We have defined $\mathcal{C}_\ell[\tau]$ for all $\tau\in\mathcal{M}_k$ and $\ell\geq1$ compatibly with the constraints in Equations (3.3.1a)-(3.3.1b). We are left with working with $\mathcal{C}_0[\tau]$. In addition, we have to prove that Equation (3.3.1d) is valid. For $\tau\in\mathcal{M}_k$ we set

$$\mathcal{C}_0[\tau]:=\widetilde{\Gamma}_{\cdot_Q}(\tau)-\Gamma_{\cdot_Q}(\tau)-\Gamma_{\cdot_Q}(\mathcal{C}_{k-2}[\tau])\,.$$

The assignment $\tau \to \mathcal{C}_0[\tau]$ is linear and, moreover, $\mathcal{C}_0[\tau] \in \mathcal{M}_0$. This follows by direct inspection of $\delta_\psi \mathcal{C}_0[\tau]$ since

$$\delta_\psi \mathcal{C}_0[\tau] = \widetilde{\Gamma}_{\cdot_Q}(\delta_\psi \tau) - \Gamma_{\cdot_Q}(\delta_\psi \tau) - \Gamma_{\cdot_Q}(\mathcal{C}_{k-3}[\delta_\psi \tau]) = 0\,,$$

where we exploited Equations (3.2.3b) and (3.3.1c) together with the inductive hypothesis for $\delta_\psi \tau \in \mathcal{M}_{k-1}$. Equation (3.3.1d) is satisfied since $\Gamma_{\cdot_Q}(\mathcal{C}_0[\tau]) = \mathcal{C}_0[\tau]$. This concludes the proof. □

Theorem 3.3.2 and, in particular, Equation (3.3.1d) state that given two prescriptions $\widetilde{\Gamma}_{\cdot_Q}$ and $\Gamma_{\cdot_Q}$ as per Theorem 3.1.4 and an element $\tau \in \mathcal{A}$, their action on τ is the same, up to a suitable affine modification of τ represented by the maps $\mathcal{C}_k$. Roughly speaking Theorem 3.3.2 ensures that, for $\tau \in \mathcal{M}_k$, $\widetilde{\Gamma}_{\cdot_Q}(\tau)$ and $\Gamma_{\cdot_Q}(\tau)$ differ by $\Gamma_{\cdot_Q}(\mathcal{C}_{k-2}[\tau])$, being $\mathcal{C}_{k-2}[\tau] \in \mathcal{M}_{k-2}$. This yields the following corollary, encoding the relation between the algebras constructed out of different maps $\widetilde{\Gamma}_{\cdot_Q}$ and $\Gamma_{\cdot_Q}$ as per Theorem 3.1.4.

Corollary 3.3.3: *Let $\widetilde{\Gamma}_{\cdot_Q} : \mathcal{A} \to \mathcal{D}'_C(M;\mathrm{Pol})$ and $\Gamma_{\cdot_Q} : \mathcal{A} \to \mathcal{D}'_C(M;\mathrm{Pol})$ be two linear maps compatible with the constraints listed in Theorem 3.1.4. The algebras $\mathcal{A}_{\cdot_Q} = \Gamma_{\cdot_Q}(\mathcal{A})$ and $\widetilde{\mathcal{A}}_{\cdot_Q} = \widetilde{\Gamma}_{\cdot_Q}(\mathcal{A})$, defined as per Theorem 3.1.6, coincide.*

Remark 3.3.4: *We observe that up to Theorem 3.3.2 the ambiguity in the form of $\mathcal{C}_{k-2}[\tau]$ in Equation* (3.3.1d) *is quite large since it is only known that $\mathcal{C}_{k-2}[\tau] \in \mathcal{M}_{k-2}$. Nonetheless, this ambiguity can be further restricted by means of Equation* (3.3.1c)*. This can be seen explicitly by considering specific algebra elements, namely Φ^k as per Example 2.1.8.*

Proposition 3.3.5: *Let $\widetilde{\Gamma}_{\cdot_Q}, \Gamma_{\cdot_Q} : \mathcal{A} \to \mathcal{D}'(M;\mathrm{Pol})$ be two linear maps compatible with the constraints listed in Theorem 3.1.4. Then there exists $\{c_\ell\}_{\ell\in\mathbb{N}_0} \subset \mathcal{E}(M)$, a family of smooth functions, such that for all $k \in \mathbb{N}$*

$$\widetilde{\Gamma}_{\cdot_Q}(\Phi^k) = \Gamma_{\cdot_Q}\left(\Phi^k + \sum_{\ell=0}^{k-2} \binom{k}{\ell} c_{k-\ell}\Phi^\ell\right). \tag{3.3.2}$$

Proof. For $k = 1$ Equation (3.3.2) holds true by construction as a consequence of Equation (3.1.2), while for $k = 2$ it reduces to Example 3.3.1. The proof can be concluded by induction over k exploiting the argument used in Example 3.3.1 and using Equation (3.3.1d). □

Remark 3.3.6: *Proposition 3.3.5 prompts the question whether one can write an expression of the form of Equation* (3.3.2) *for a generic $\tau \in \mathcal{A}$ and not only for objects such as Φ^k. A possible strategy could be that of considering, for $\tau \in \mathcal{M}_k$,*

$$\widetilde{\Gamma}_{\cdot_Q}(\tau)(f;\varphi) = \Gamma_{\cdot_Q}(\tau)(f;\varphi) + \sum_{\ell=0}^{k-2} \frac{1}{(k-\ell)!}\Gamma_{\cdot_Q}(\tau)^{(k-\ell)}(f\otimes \tilde{c}_{k-\ell};\varphi)\,, \tag{3.3.3}$$

where $\tilde{c}_{k-\ell} \in \mathcal{E}(M^{k-\ell})$. If $\tau = \Phi^k$, this last formula boils down to Equation (3.3.2) *by setting*

$$\tilde{c}_{k-\ell}(\widehat{x}_{k-\ell}) := \frac{1}{k-\ell}\sum_{j=1}^{k-\ell} c_{k-\ell}(x_j)\,,$$

and by observing that $(\Phi^k)^{(k-\ell)} = \frac{k!}{\ell!}\Phi^\ell \delta_{\mathrm{Diag}_{\ell+1}}$. We recall that here $\widehat{x}_{k-\ell} = (x_1, \dots, x_{k-\ell})$.

Yet, in general Equation (3.3.3) *does not hold true, as one can infer considering as counterexample the functional-valued distribution* $\tau = \Phi P \circledast \Phi$. *On account of Equations* (3.2.1) and (3.2.3b), *it holds that there must exist* $c \in \mathcal{E}(M)$ *such that*

$$\widetilde{\Gamma}_{\cdot_Q}(\tau)(f;\varphi) - \Gamma_{\cdot_Q}(\tau)(f;\varphi) = \int\limits_M c\, f\mu .$$

At the same time Equation (3.3.3) *entails that there must exist* $\tilde{c}_2 \in \mathcal{E}(M^2)$ *such that*

$$\int_M c\, f\, \mu = \Gamma_{\cdot_Q}(\tau)^{(2)}(f \otimes \tilde{c}_2;\varphi) = \int\limits_{M\times M} f(x)P(x,y)[\tilde{c}_2(x,y) + \tilde{c}_2(y,x)]\mathrm{d}\mu(x)\mathrm{d}\mu(y)\,.$$

In general there exists no such $\tilde{c}_2$.

3.3.2 The Case of $\bullet_Q$

To conclude this section we state a counterpart of Theorem 3.3.2 aimed at characterizing the non uniqueness in constructing $\Gamma_{\bullet_Q}$.

Theorem 3.3.7: *Let* $\Gamma_{\bullet_Q}\colon \mathcal{T}(\mathcal{A}_{\cdot_Q}) \to \mathcal{T}'_{\mathrm{C}}(M;\mathrm{Pol})$ *and* $\widetilde{\Gamma}_{\bullet_Q}\colon \mathcal{T}(\mathcal{A}_{\cdot_Q}) \to \mathcal{T}'_{\mathrm{C}}(M;\mathrm{Pol})$ *be two linear maps compatible with the constraints listed in Theorem 3.2.4. Then there exists a family* $\{\mathcal{C}_{\underline{k}}\}_{\underline{k}\in(\mathbb{N}_0)^{\mathbb{N}_0}}$ *of linear maps* $\mathcal{C}_{\underline{k}}\colon \mathcal{T}(\mathcal{A}) \to \mathcal{T}(\mathcal{A})$ *such that:*

1. *for all* $\ell \in \mathbb{N}\cup\{0\}$, $\mathcal{C}_{\underline{k}}[\mathcal{A}^{\otimes\ell}] \subseteq \mathcal{M}_{k_1}\otimes\ldots\otimes\mathcal{M}_{k_\ell}$ *while* $\mathcal{C}_{\underline{j}}[\mathcal{M}_{k_1}\otimes\ldots\otimes\mathcal{M}_{k_\ell}] = 0$ *whenever* $k_i \leq j_i - 1$ *for some* $i \in \{1,\ldots,\ell\}$

2. *for all* $\ell \in \mathbb{N}\cup\{0\}$ *and* $\tau_1,\ldots,\tau_\ell \in \mathcal{A}$, *it holds*

$$\mathcal{C}_{\underline{k}}(\tau_1\otimes\ldots\otimes P\circledast\tau_k\otimes\ldots\otimes\tau_\ell) = (\delta_{\mathrm{Diag}_2}^{\otimes(k-1)}\otimes P\otimes\delta_{\mathrm{Diag}_2}^{\otimes\ell-k})\circledast\mathcal{C}_{\underline{k}}(\tau_1\otimes\ldots\otimes\tau_\ell) \tag{3.3.4}$$

$$\delta_\psi\mathcal{C}_{\underline{k}}(\tau_1\otimes\ldots\otimes\tau_\ell) = \sum_{a=1}^{\ell}\mathcal{C}_{\underline{k}(a)}(\tau_1\otimes\ldots\otimes\delta_\psi\tau_a\otimes\ldots\otimes\tau_\ell)\,, \tag{3.3.5}$$

   where $\underline{k}(a)_i = k_i$ *if* $i\neq a$ *and* $\underline{k}(a)_a = k_a - 1$.

3. *for all* $\tau_{k_1},\ldots,\tau_{k_\ell} \in \mathcal{A}$, *with* $\tau_{k_j} \in \mathcal{M}_{k_j}$ *for all* $j \in \{1,\ldots,\ell\}$, *and* $f_1,\ldots,f_\ell \in \mathcal{D}(M)$ *it holds*

$$\begin{aligned}\widetilde{\Gamma}_{\bullet_Q}(\widetilde{\Gamma}_{\cdot_Q}^{\otimes\ell}(\tau_{k_1}\otimes\ldots\otimes\tau_{k_\ell}))(f_1\otimes\ldots\otimes f_\ell) &= \Gamma_{\bullet_Q}\left[\Gamma_{\cdot_Q}^{\otimes\ell}(\tau_{k_1}\otimes\ldots\otimes\tau_{k_\ell})\right](f_1\otimes\ldots\otimes f_\ell)\\ &+ \sum_{\wp\in\mathrm{P}(1,\ldots,\ell)}\Gamma_{\bullet_Q}\left[\Gamma_{\cdot_Q}^{\otimes|\wp|}\mathcal{C}_{\underline{k}_\wp}\Big(\bigotimes_{I\in\wp}\prod_{i\in I}\tau_{k_i}\Big)\right]\Big(\bigotimes_{I\in\wp}\prod_{i\in I}f_i\Big)\,.\end{aligned} \tag{3.3.6}$$

   Here $\mathrm{P}(1,\ldots,\ell)$ *denotes the set of all partitions of* $\{1,\ldots,\ell\}$ *into non-empty disjoint sets while* $\underline{k}_\wp = (k_I)_{I\in\wp}$ *where* $k_I := \sum_{i\in I}k_i$ *whereas* $\underline{j}\leq\underline{k}_\wp$ *means that* $j_I \leq k_I$ *for all* $I\in\wp$.

Proof. The proof follows the same steps as those in the proof of Theorem 3.3.7. Notice in particular that for $\ell = 1$ Equation (3.3.6) reduces to Equation (3.3.1d). At this stage we can proceed by induction exactly as in Theorem 3.3.2 just taking into account more indices. For this reason we omit giving all details. $\square$

Example 3.3.8: *For concreteness we now specialize Equation* (3.3.6) *to the case of two elements $\tau_{k_1} = \tau_{k_2} = \Phi^2$. The admissible partitions are $\wp = \{1,2\}$ and $\wp = \{\{1\},\{2\}\}$ which lead to*

$$\begin{aligned}\widetilde{\Gamma}_{\bullet_Q}(\widetilde{\Gamma}_{\cdot_Q}(\Phi^2)\otimes\widetilde{\Gamma}_{\cdot_Q}(\Phi^2))(f_1\otimes f_2) &= \Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi^2)\otimes\Gamma_{\cdot_Q}(\Phi^2))(f_1\otimes f_2)\\ &+\Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}^{\otimes 2}(\mathcal{C}_{2,2}(\Phi^2\otimes\Phi^2)))(f_1\otimes f_2)+\Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\mathcal{C}_4(\Phi^4)))(f_1f_2)\\ &= \Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi^2)\otimes\Gamma_{\cdot_Q}(\Phi^2))(f_1\otimes f_2)\\ &+\Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}^{\otimes 2}(\mathcal{C}_{2,2}(\Phi^2\otimes\Phi^2)))(f_1\otimes f_2)+\Gamma_{\cdot_Q}(\mathcal{C}_4(\Phi^4))(f_1f_2)\,,\end{aligned}$$

where we used Equation (3.2.1)*. In the previous identity $\mathcal{C}_4(\Phi^4)\in\mathcal{M}_4$ while $\mathcal{C}_{2,2}(\Phi^2\otimes\Phi^2)\in\mathcal{M}_2\otimes\mathcal{M}_2$.*

We shall now compute $\mathcal{C}_4(\Phi^4)$ and $\mathcal{C}_{2,2}(\Phi^2\otimes\Phi^2)$ explicitly. On account of Proposition (3.3.5) *we know that $\widetilde{\Gamma}_{\cdot_Q}(\Phi^2)=\Gamma_{\cdot_Q}(\Phi^2)+c_0\mathbf{1}$, where $c_0\in\mathcal{E}(M)$. This entails that*

$$\begin{aligned}\widetilde{\Gamma}_{\bullet_Q}(\widetilde{\Gamma}_{\cdot_Q}(\Phi^2)\otimes\widetilde{\Gamma}_{\cdot_Q}(\Phi^2)) &= \widetilde{\Gamma}_{\bullet_Q}((\Gamma_{\cdot_Q}(\Phi^2)+c_0\mathbf{1})\otimes(\Gamma_{\cdot_Q}(\Phi^2)+c_0\mathbf{1}))\\ &=\widetilde{\Gamma}_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi^2)\otimes\Gamma_{\cdot_Q}(\Phi^2))+2\Gamma_{\cdot_Q}(\Phi^2)\vee c_0\mathbf{1}+c_0\mathbf{1}\vee c_0\mathbf{1}\,,\end{aligned}$$

where $\vee$ denotes the symmetrized tensor product. It remains to be evaluated

$$R:=\widetilde{\Gamma}_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi^2)\otimes\Gamma_{\cdot_Q}(\Phi^2))-\Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi^2)\otimes\Gamma_{\cdot_Q}(\Phi^2))\,.$$

A direct inspection using Equation (3.2.3b) *shows that $\delta_\psi R=0$. Moreover, Equation* (3.2.2) *entails that $R(f_1\otimes f_2)=0$ whenever $f_1,f_2\in\mathcal{D}(M)$ are such that $\mathrm{supp}(f_1)\cap\mathrm{supp}(f_2)=\emptyset$. It descends that there exists $c_1\in\mathcal{E}(M)$ such that*

$$\begin{aligned}\widetilde{\Gamma}_{\bullet_Q}(\widetilde{\Gamma}_{\cdot_Q}(\Phi^2)\otimes\widetilde{\Gamma}_{\cdot_Q}(\Phi^2))(f_1\otimes f_2) &= \Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi^2)\otimes\Gamma_{\cdot_Q}(\Phi^2))(f_1\otimes f_2)\\ &+2\Gamma_{\cdot_Q}(\Phi^2)\vee c_0\mathbf{1}(f_1\otimes f_2)+c_0\mathbf{1}\vee c_0\mathbf{1}(f_1\otimes f_2)+c_1\mathbf{1}(f_1f_2)\,.\end{aligned}$$

It follows that

$$\mathcal{C}_{2,2}(\Phi^2\otimes\Phi^2)=2\Gamma_{\cdot_Q}(\Phi^2)\vee c_0\mathbf{1}+c_0\mathbf{1}\vee c_0\mathbf{1}\,,\qquad \mathcal{C}_4(\Phi^4)=c_1\mathbf{1}\,.$$

Similarly to what happens for Theorem 3.3.2, Theorem 3.3.7 leads to the following important result as a corollary.

Corollary 3.3.9: *Let $\widetilde{\Gamma}_{\cdot_Q}:\mathcal{A}\to\mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$ and $\Gamma_{\cdot_Q}:\mathcal{A}\to\mathcal{D}'_{\mathrm{C}}(M;\mathrm{Pol})$ be two linear maps compatible with the constraints listed in Theorem 3.1.4. Moreover, let $\Gamma_{\bullet_Q}:\mathcal{T}(\mathcal{A}_{\cdot_Q})\to\mathcal{T}'_{\mathrm{C}}(M;\mathrm{Pol})$ and $\widetilde{\Gamma}_{\bullet_Q}:\mathcal{T}(\mathcal{A}_{\cdot_Q})\to\mathcal{T}'_{\mathrm{C}}(M;\mathrm{Pol})$ be linear maps as per Theorem 3.2.4 – here $\mathcal{A}_{\cdot_Q}=\Gamma_{\cdot_Q}(\mathcal{A})=\widetilde{\mathcal{A}}_{\cdot_Q}=\widetilde{\Gamma}_{\cdot_Q}(\mathcal{A})$ because of Corollary 3.3.3. Then the algebras $\mathcal{A}_{\bullet_Q}=\Gamma_{\bullet_Q}(\mathcal{A}_{\cdot_Q})$ and $\widetilde{\mathcal{A}}_{\bullet_Q}=\widetilde{\Gamma}_{\bullet_Q}(\widetilde{\mathcal{A}}_{\cdot_Q})$, defined as per Equation* (3.2.4)*, coincide.*

Chapter 4

An Application: The Stochastic Quantisation Model

In the previous chapters, in particular in Chapters 2 and 3 we have introduced the abstract setting of the microlocal approach to the renormalization of SPDEs.

At this level, we have seen how to encode, from an algebraic and analytical viewpoint, the information on the expectation-values of polynomial functionals constructed out of φ. This is interpreted as a smooth counterpart of the expectation-values of $P * \xi$, with ξ a white noise and P a parametrix of a microhypoelliptic operator E. In addition, we have encoded the information on multi-local correlation functions of these objects.

Adopting a physical language, the machinery outlined so far allows the analysis of the *kinematics* associated to an SPDE whose linear part is codified by the operator E while the source term is a white noise ξ. At this level, what is still missing is an interacting *dynamic* which, in the cases we are interested in, is represented by a non-linearity in the SPDE, as those seen in Chapter 1, in particular in Section 1.1.2.

As a consequence, the aim of this chapter is to apply the machinery of the microlocal approach to the *perturbative analysis* of a specific SPDE, in particular the so-called stochastic quantisation equation as per Equation (1.1.8).

To start with, we introduce the perturbative setting in which we analyse this equation. For the sake of clarity, we also discuss quite in some detail the solution at first order in perturbation theory. In addition, since in the previous chapter we presented renormalization as an extension procedure for distributions, in Section 4.1.2 we shall discuss an explicit construction of such extensions in a specific framework.

Finally, in Section 4.3, by means of a graphical argument as well as of Theorem B.2.1, we shall discuss sub-critical scenarios.

4.1 Perturbative Setting

Remark 4.1.1: *As a premise, we underline that, for simplicity and in order to make some computation more explicit, we shall avoid in this section the generic smooth manifold setting. We consider a simpler setting represented by the parabolic scenario, in the sense of Section 2.1, where $M = \mathbb{R} \times \mathbb{R}^d$ and where $E = \partial_t - \Delta$, denoting as usual with t the time coordinate along $\mathbb{R}$ and with Δ the Laplace operator on $\mathbb{R}^d$, built out the flat Euclidean metric.*

In spite of this, we underline that except for some explicit computation of Section 4.1.1, all the following arguments remain valid also in the general scenario as per Section 2.1.

Remark 4.1.2 (Notation)**:** *It is important to notice that here $M = \mathbb{R} \times \mathbb{R}^d$ has dimension $d + 1$, differently from the generic parabolic scenario as per Section 2.1 where we assumed $d = \dim(M)$. This is to make contact with the existing literature on this equation, see Section 1.1.2 for references.*

Dwelling more into the detail, we focus on the perturbative analysis of the following stochastic PDE on $\mathbb{R} \times \mathbb{R}^d$

$$\partial_t \widehat{\psi} = \Delta \widehat{\psi} - \lambda \widehat{\psi}^3 + \widehat{\xi}\,, \tag{4.1.1}$$

where $\lambda \in \mathbb{R}$ is a coupling constant which will serve as perturbative expansion parameter and where $\widehat{\xi}$ denotes the space-time white noise.

Remark 4.1.3: *Recalling the discussion of Remark 2.1.5, since $\mathbb{R} \times \mathbb{R}^d$ in not a compact manifold, in order give meaning to expressions such as $P \circledast \eta$ – cf. Equation* (A.0.9) *– with $\eta \in \mathcal{E}(\mathbb{R} \times \mathbb{R}^d)$, we need to introduce a cut-off function $\chi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$ and to replace $P \in \mathcal{D}'((\mathbb{R} \times \mathbb{R}^d)^2)$ with*

$$P_\chi := P \cdot (1 \otimes \chi) \in \mathcal{D}'((\mathbb{R} \times \mathbb{R}^d)^2)\,,$$

where $\cdot$ denotes the pointwise product between the distribution $P \in \mathcal{D}'((\mathbb{R} \times \mathbb{R}^d)^2)$ and the smooth function $1 \otimes \chi \in \mathcal{E}'((\mathbb{R} \times \mathbb{R}^d)^2)$. This allows to avoid infrared divergences when dealing with convolutions such as $P \circledast \eta$ with $\eta \in \mathcal{E}(\mathbb{R} \times \mathbb{R}^d)$. In order to make this assumption clear, throughout this section we shall employ the notation P_χ. We underline that the bi-distribution P_χ obtained as above is an actual parametrix, in the sense of Definition 2.1.3 of the heat operator $\partial_t - \Delta$ over $\mathbb{R} \times \mathbb{R}^d$.

We observe that in order to get the perturbative solution of the SPDE one should at the end consider the so-called adiabatic limit where the infra-red cut-off χ is removed, namely considering the limit where $\chi \to 1$. We shall not address this issue in this manuscript.

Following the rationale outlined in the previous chapters, the first step consists of formulating Equation (4.1.1), in its mild form, *i.e.*, as in Equation (1.3.2), in the framework of functional-valued distributions, namely within the algebra $\mathcal{A}$, *i.e.*,

$$\Psi = \Phi - \lambda P_\chi \circledast (\Psi^3)\,, \tag{4.1.2}$$

where $P_\chi := P \cdot (1 \otimes \chi)$ while $\Phi \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d; \mathrm{Pol})$ is the functional defined in Equation (2.1.19).

In this sense, Equation (4.1.2) is to be seen as the formulation of Equation (1.3.2) in the algebra $\mathcal{A}$, where the functional-valued distribution $\Phi \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d; \mathrm{Pol})$ is the one associated to the stochastic convolution as it appears in Equation (1.3.3).

We observe that the formulation of Equation (4.1.2) in the algebra $\mathcal{A}$ is purely formal since we cannot claim that the solution Ψ to this equation is an element of $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^d; \mathrm{Pol})$. Indeed, generally speaking, the solution is not expected to be polynomial in the configuration $\varphi \in \mathcal{E}(\mathbb{R} \times \mathbb{R}^d)$.

In order to bypass this hurdle, the idea is to use a perturbative scheme to construct Ψ. In particular we read $\Psi = \Psi[\![\lambda]\!]$, the perturbative solution of Equation (4.1.2), as a *formal power series* in the parameter $\lambda \in \mathbb{R}$ having coefficients in $\mathcal{A}$. Dwelling more into the detail, we have the following definition.

Definition 4.1.4: *Let $\mathcal{A}$ be the algebra as per Definition 2.1.20 and let $\lambda \in \mathbb{R}$. We define $\mathcal{A}[\![\lambda]\!]$ the space of formal power series in λ having coefficients in $\mathcal{A}$, whose elements are*

$$\Omega[\![\lambda]\!] = \sum_{j\geq 0} \lambda^j \Omega_j \,, \qquad \{\Omega_j\}_{j\geq 0} \subset \mathcal{A} \,.$$

Coming back to Equation (4.1.2), we look for a perturbative solution $\Psi[\![\lambda]\!] \in \mathcal{A}[\![\lambda]\!]$, in other words we look for a suitable formal power series

$$\Psi[\![\lambda]\!] = \sum_{j\geq 0} \lambda^j F_j \,, \qquad F_j \in \mathcal{A} \,. \tag{4.1.3}$$

The rationale of the perturbative approach is to exploit the dynamics, given by Equation (4.1.2), in order to determine order by order in λ the coefficients $F_j \in \mathcal{A}$ of the solution.
As a matter of fact, observing that from Equation (4.1.2) it descends that $F_0 = \Phi$, inserting Equation (4.1.3) into Equation (4.1.2) one obtains by direct computation

$$F_j = -\sum_{j_1+j_2+j_3=j-1} P_\chi \circledast (F_{j_1} F_{j_2} F_{j_3}) \,, \qquad j \in \mathbb{N} \,. \tag{4.1.4}$$

Example 4.1.5: *As example of application of Equation* (4.1.4), *we observe that the first orders in λ of Equation* (4.1.3) *are*

$$F_0 = \Phi \,, \quad F_1 = -P_\chi \circledast \Phi^3 \,, \quad F_2 = 3P_\chi \circledast (\Phi^2 P_\chi \circledast \Phi^3) \,. \tag{4.1.5}$$

Remark 4.1.6: *We observe that the perturbative expansion as per Equations* (4.1.4) *and* (4.1.5) *is not the unique iterative approximation scheme one can construct. Indeed, as common in the analysis both of PDE and of SPDEs – see, e.g., [42, 52] – one may think of the* Picard iteration *looking to establish a fixed point argument.*

Without entering into the details, we observe that the perturbative series is a re-ordering *of the series one obtains through the Picard iteration. In terms of the latter, Equation* (4.1.2) *can be written as*

$$\Psi = \mathcal{F}(\Psi) \,, \qquad \mathcal{F}(\Psi) := \Phi - \lambda P_\chi \circledast \Psi^3 \,.$$

The idea behind the Picard iteration is to start from a suitable $\widetilde{\Psi}_0$, say $\widetilde{\Psi}_0 = 0$ and to iteratively compute

$$\widetilde{\Psi}_n = \mathcal{F}(\widetilde{\Psi}_{n-1}) \,.$$

As an example, consider

$$\begin{aligned}
&\widetilde{\Psi}_1 = \Phi \,, \\
&\widetilde{\Psi}_2 = \Phi - \lambda P_\chi \circledast \Phi^3 \,, \\
&\widetilde{\Psi}_3 = \Phi - \lambda P_\chi \circledast \Phi^3 + 3\lambda^2 P_\chi \circledast (\Phi^2 P_\chi \circledast \Phi^3) - 3\lambda^3 P_\chi \circledast (\Phi (P_\chi \circledast \Phi^3)^2) + \lambda^4 P_\chi \circledast (P_\chi \circledast \Phi^3)^3 \,.
\end{aligned}$$

This shows that Picard iterations yield a mixture of different perturbative orders, resulting thus in a different re-ordering of the series. In the following we shall discuss renormalization of the perturbative solution, nonetheless the same arguments are valid for the Picard iteration. This might be a starting point in the direction of pushing to a non-perturbative level the microlocal approach to SPDEs.

So far we have only worked at the level of the algebra $\mathcal{A}$ and thus, on account of the discussion of Chapters 2 and 3, the argument is purely deterministic.

The strategy in order to recover the stochastic nature of Equation (4.1.1) consists of deforming the algebra $\mathcal{A}$ into its *renormalized* counterpart $\mathcal{A}_{\cdot_Q}$ as per Theorem 3.1.6. We recall that this is achieved by means of the map $\Gamma_{\cdot_Q}$ constructed in Theorem 3.1.4 and by Equation (3.1.17) in particular. We observe that, in order to take into account the cut-off χ, we consider $Q = Q_\chi = P_\chi \circ P_\chi$.

In this spirit, the idea is to represent, through the map $\Gamma_{\cdot_Q}$, the perturbative solution $\Psi[\![\lambda]\!] \in \mathcal{A}[\![\lambda]\!]$ in the algebra $\mathcal{A}_{\cdot_Q}$, namely

$$\Psi[\![\lambda]\!] \mapsto \Psi_{\cdot_Q}[\![\lambda]\!] := \Gamma_{\cdot_Q}(\Psi[\![\lambda]\!]) .$$

As outlined in Section 3.1, $\Psi_{\cdot_Q}[\![\lambda]\!] \in \mathcal{A}_{\cdot_Q}[\![\lambda]\!]$ and, for any $\varphi \in \mathcal{E}(\mathbb{R} \times \mathbb{R}^d)$ and for any $f \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$, $\Psi_{\cdot_Q}[\![\lambda]\!](f;\varphi)$ is the expectation value of the φ-shifted, f-localized (*cf.* Remark 2.2.3) perturbative solution $\widehat{\psi}_\varphi[\![\lambda]\!](f)$ of Equation (4.1.1), *i.e.*,

$$\mathbb{E}(\widehat{\psi}_\varphi[\![\lambda]\!](f)) = \Gamma_{\cdot_Q}(\Psi[\![\lambda]\!])(f;\varphi) = \sum_{j \geq 0} \lambda^j \Gamma_{\cdot_Q}(F_j)(f;\varphi) , \tag{4.1.6}$$

where $\widehat{\psi}$ is the formal perturbative solution of Equation (4.1.1), while $\mathbb{E}$ stands for the expectation value. The subscript φ is to remind that we are free to consider a shifted white noise, namely $\mathbb{E}(P_\chi \circledast \xi) = \varphi$. As remarked in Section 2.2, the usual scenario of a Gaussian white noise centered at 0 can be recovered by evaluating the right hand side of Equation (4.1.6) at the configuration $\varphi = 0$.

Remark 4.1.7: *We recall that, as we discussed in Section 3.3, the map $\Gamma_{\cdot_Q}$ is not unique. As a consequence, it is important to bear in mind that, in the above procedure, there is the arbitrariness in picking $\Gamma_{\cdot_Q}$ as discussed in Theorem 3.3.2.*

So far, in this chapter, we have implicitly picked one of the possible choices and different ones lead to far reaching consequences in the construction of the perturbative solutions of Equation (4.1.2). *In the following we shall discuss in detail these consequences.*

Before dwelling into specific computations, we stress that a noteworthy advantage of our approach is the predictability at a perturbative level also of the correlations between the solutions of Equation (4.1.2). As we have thoroughly discussed in Section 3.2, this can be obtained by means of the product $\bullet_Q$ and working therefore with the algebra $\mathcal{A}_{\bullet_Q}$ or, rather, with $\mathcal{A}_{\bullet_Q}[\![\lambda]\!]$, the algebra of formal power series in λ with coefficients in $\mathcal{A}_{\bullet_Q}$.
We recollect all interesting expressions involving Ψ in two single algebraic objects.

Definition 4.1.8: *Let $\Gamma_{\cdot_Q}$, $\Gamma_{\bullet_Q}$ be two maps as per Theorems 3.1.4-3.2.4. We define $\mathcal{A}^{\Psi}_{\cdot_Q} \subset \mathcal{A}_{\cdot_Q}[\![\lambda]\!]$ as the subalgebra of $\mathcal{A}_{\cdot_Q}[\![\lambda]\!]$ generated by $\Psi_{\cdot_Q} := \Gamma_{\cdot_Q}(\Psi)$. Similarly we denote with $\mathcal{A}^{\Psi}_{\bullet_Q} \subseteq \mathcal{A}_{\bullet_Q}[\![\lambda]\!]$ the smallest subalgebra of $\mathcal{A}_{\bullet_Q}[\![\lambda]\!]$ containing $\mathcal{A}^{\Psi}_{\cdot_Q}$.*

As an example, in the following we shall consider the two-point correlation function, but the reader should keep in mind that, barring the sheer computational difficulties, we could analyse with the same methods all higher order multi-local correlation functions.

In view of this comment, on account of Theorem 3.2.4 and in particular of Equation (3.2.3a), it holds that, for any $f_1, f_2 \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$ and for ant $\varphi \in \mathcal{E}(\mathbb{R} \times \mathbb{R}^d)$, the linearity of the map

$\Gamma_{\bullet_Q}$ entails that, denoting with ω_2 the two-point function of the perturbative solution,

$$\begin{aligned}\omega_2(f_1\otimes f_2;\varphi) :&= \big(\Gamma_{\cdot_Q}(\Psi[\![\lambda]\!]) \bullet_{\Gamma_{\bullet_Q}} \Gamma_{\cdot_Q}(\Psi[\![\lambda]\!])\big)(f_1\otimes f_2;\varphi)\\ &= \Gamma_{\bullet_Q}\big(\Gamma_{\cdot_Q}(\Psi[\![\lambda]\!])\otimes\Gamma_{\cdot_Q}(\Psi[\![\lambda]\!])\big)(f_1\otimes f_2;\varphi)\\ &= \sum_{k\geq 0}\lambda^k\sum_{j=0}^{k}\Gamma_{\bullet_Q}\big(\Gamma_{\cdot_Q}(F_j)\otimes\Gamma_{\cdot_Q}(F_{k-j})\big)(f_1\otimes f_2;\varphi)\,. \end{aligned} \tag{4.1.7}$$

4.1.1 First Order in Perturbation Theory

With the goal clarifying the procedure outlined above, we compute in this section both the expectation value and the two-point correlation function of the perturbative solution of Equation (4.1.1), limiting ourselves to the first order in λ for simplicity. Nonetheless, we observe that higher order computations involve only higher difficulties from a computational viewpoint and not at a conceptual level.

Expectation value of the solution

Using Equation (4.1.3) together with Equation (4.1.4), the solution of Equation (4.1.2) to order $O(\lambda^2)$ reads, omitting for simplicity the dependence on $(f;\varphi)$,

$$\Psi[\![\lambda]\!] = \Phi - \lambda P_\chi \circledast \Phi^3 + O(\lambda^2)\,, \tag{4.1.8}$$

and thus, exploiting the linearity of $\Gamma_{\cdot_Q}$,

$$\Psi_{\cdot_Q}[\![\lambda]\!] := \Gamma_{\cdot_Q}(\Psi[\![\lambda]\!]) = \Gamma_{\cdot_Q}(\Phi) - \lambda\Gamma_{\cdot_Q}(P_\chi\circledast\Phi^3) + O(\lambda^2)\,.$$

On the one hand, on account of Equation (3.1.2), it holds $\Gamma_{\cdot_Q}(\Phi)=\Phi$. On the other hand, from Equations (3.1.3) and (3.1.7) together with the general construction of the map $\Gamma_{\cdot_Q}$ in the proof of Theorem 3.1.4 – *cf.* Equation (3.1.7),

$$\Gamma_{\cdot_Q}(P_\chi\circledast\Phi^3) = P_\chi\circledast\Gamma_{\cdot_Q}(\Phi^3) = P_\chi\circledast(\Phi^3+3C\Phi)\,.$$

We recall that $C\in\mathcal{E}(\mathbb{R}\times\mathbb{R}^d)$ is a smooth function as a consequence of Theorem 3.1.4 and of Remark A.0.8 together with $\mathrm{WF}_1(P)=\emptyset$, *cf.* Equation (A.0.11). More precisely $C(z)=\chi(z)\widehat{P}_2(\delta_z\otimes\chi)$, with $\widehat{P}_2$ the chosen extension to the whole $(\mathbb{R}\times\mathbb{R}^d)^2$ of the bi-distribution $P^2\in\mathcal{D}'((\mathbb{R}\times\mathbb{R}^d)^2\setminus\mathrm{Diag}_2)$.

At this stage we do not enter into the details of the explicit form of $\widehat{P}_2$ postponing its construction to Section 4.1.2.

Remark 4.1.9: *It is worth emphasizing that, choosing a specific product $\Gamma_{\cdot_Q}$ entails in turn that we have also made a specific choice of $\widehat{P}_2$, namely an explicit extension of P^2. From this point of view, the function C is fixed. Yet, as we have thoroughly discussed in the previous sections, there is a freedom in selecting $\Gamma_{\cdot_Q}$ codified by Theorem 3.3.2.*

In the case in hand, see also Remark 3.3.1, this translates in different choices for the function C, which is thus referred to as a renormalization ambiguity*.*

Putting together all this information, we end up with

$$\Psi_{\cdot_Q}[\![\lambda]\!] = \Gamma_{\cdot_Q}(\Psi[\![\lambda]\!])(\varphi) = \Phi(\varphi) - \lambda P_\chi\circledast(\Phi^3+3C\Phi)(\varphi) + O(\lambda^2)\,. \tag{4.1.9}$$

Up to order $O(\lambda^2)$, this is the expectation value of the φ-shifted solution $\widehat{\psi}$ of Equation (4.1.1). We recall that, in order to reproduce the standard white noise behaviour, one has to evaluate this functional at $\varphi = 0$. Using the notation of Equation (4.1.6), this yields

$$\mathbb{E}(\widehat{\psi}_0[\![\lambda]\!]) = O(\lambda^2)\,.$$

This entails that, up to the second order in perturbation theory, the expectation value of the perturbative solution is vanishing. Actually, this is true at all orders in perturbation theory, as shown by the following lemma.

Lemma 4.1.10: *Let $\Psi[\![\lambda]\!] \in \mathcal{A}[\![\lambda]\!]$ be a perturbative solution of Equation* (4.1.2) *defined as per Equation* (4.1.4). *Then $\Gamma_{\cdot_Q}(\Psi)(f;0) = 0$ for all $f \in \mathcal{D}(M)$.*

Proof. Let $\mathcal{O} \subset \mathcal{A}$ be the vector space of $\mathcal{A}$ made by those elements $\tau \in \mathcal{A}$ such that $\tau^{(2n)}(\cdot;0) = 0$ for any $n \in \mathbb{N}_0$. Notice that the superscript indicates the $2n$-th functional derivative and, with a slight abuse of notation, we avoid indicating the directions of derivation as well as the test function, *cf.* Definition 2.1.6. We observe that, for any $\tau \in \mathcal{O}$, it holds $\Gamma_{\cdot_Q}(\tau) \in \mathcal{O}$ and, moreover, $\tau_1\tau_2\tau_3 \in \mathcal{O}$ for any $\tau_1, \tau_2, \tau_3 \in \mathcal{O}$. This is a consequence of the fact that all coefficients F_j as per Equation (4.1.4) contain an odd number of factors Φ.

We prove the thesis by showing that $\Psi[\![\lambda]\!] \in \mathcal{O}[\![\lambda]\!]$, namely that $F_j \in \mathcal{O}$ for any $j \in \mathbb{N}_0$, being $F_j \in \mathcal{A}$ defined as per Equation (4.1.4). For $j = 0$ we have $F_0 = \Phi \in \mathcal{O}$. The proof goes by induction. We assume the thesis to hold true for $j \leq \ell$ and we prove that the same is valid for $j = \ell + 1$. Since, by Equation (4.1.4),

$$F_{\ell+1} = -\sum_{j_1+j_2+j_3=\ell} P_\chi \circledast (F_{j_1}F_{j_2}F_{j_3})\,,$$

and $j_1, j_2, j_3 \leq \ell$, it follows that $F_{j_1}, F_{j_2}, F_{j_3} \in \mathcal{O}$ and, as we have seen before, $F_{j_1}F_{j_2}F_{j_3} \in \mathcal{O}$. This implies that $F_{\ell+1} \in \mathcal{O}$. □

Two-point correlation function

We compute, up to order $O(\lambda^2)$, the two-point correlation function of the solution of Equation (4.1.1). The starting point is Equation (4.1.7) limiting ourselves up to $k = 1$, namely, for $f_1, f_2 \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$ and for $\varphi \in \mathcal{E}(\mathbb{R} \times \mathbb{R}^d)$,

$$\begin{aligned}\omega_2(f_1 \otimes f_2;\varphi) = \Gamma_{\bullet_Q}\big(\Gamma_{\cdot_Q}(F_0)\otimes\Gamma_{\cdot_Q}(F_0)\big)(f_1 \otimes f_2;\varphi)&\\ + \Gamma_{\bullet_Q}\big(\Gamma_{\cdot_Q}(F_0) \otimes \Gamma_{\cdot_Q}(F_1)\big)(f_1 \otimes f_2 + f_2 \otimes f_1;\varphi)\,.&\end{aligned}$$

As a consequence, at zeroth-order in λ, we need to evaluate for $f_1, f_2 \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$ and for $\varphi \in \mathcal{E}(\mathbb{R} \times \mathbb{R}^d)$

$$\Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi) \otimes \Gamma_{\cdot_Q}(\Phi))(f_1 \otimes f_2;\varphi) = (\Phi \otimes \Phi)(f_1 \otimes f_2;\varphi) + Q(f_1 \otimes f_2)\,,$$

whereas at first order

$$\begin{aligned}&\Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi) \otimes \Gamma_{\cdot_Q}(P_\chi \circledast \Phi^3))(f_1 \otimes f_2 + f_2 \otimes f_1;\varphi)\\ &\quad= (\Phi \otimes (P_\chi \circledast (\Phi^3 + 3C\Phi))(f_1 \otimes f_2 + f_2 \otimes f_1;\varphi)\\ &\qquad+ Q \cdot (1 \otimes 3P_\chi \circledast (\Phi^2 + C\mathbf{1})(f_1 \otimes f_2 + f_2 \otimes f_1;\varphi)\,,\end{aligned}$$

where C is the smooth function introduced above, $\mathbf{1}$ denotes the identity functional while $Q\cdot(1\otimes 3P_\chi\circledast(\Phi^2+C\mathbf{1})(\varphi)$ is the bi-distribution whose integral kernel is $3Q(x,y)(P_\chi\circledast(\varphi^2+C))(y)$ – we recall that $Q = P_\chi \circ P_\chi$. As a consequence, we get

$$\omega_2(f_1\otimes f_2;\varphi) = [\Phi\otimes\Phi + Q](f_1\otimes f_2;\varphi) - \lambda\big[\big(\Phi\otimes(P_\chi\circledast(\Phi^3+3C\Phi)) \\ +3Q\cdot(1\otimes(P_\chi\circledast(\Phi^2+C\mathbf{1}))\big)\big](f_1\otimes f_2 + f_2\otimes f_1;\varphi) + O(\lambda^2)\,. \tag{4.1.10}$$

As in the previous case, the scenario with the standard white noise as a source can be recovered evaluating Equation (4.1.10) at $\varphi = 0$. This yields

$$\mathbb{E}(\widehat{\psi}[\![\lambda]\!]\otimes\widehat{\psi}[\![\lambda]\!])(f_1\otimes f_2) = \omega_2(f_1\otimes f_2;0) = \\ = Q(f_1\otimes f_2) - 3\lambda Q\cdot(1\otimes(P_\chi\circledast C))(f_1\otimes f_2 + f_2\otimes f_1) + O(\lambda^2).$$

Remark 4.1.11: *Notice that in the computation the only freedom appearing is still codified by the lone function C and thus the same comments as per Remark 4.1.9 apply also to this scenario.*

4.1.2 Explicit Construction of P^n

As we have seen several times in Chapter 3, the microlocal approach to renormalization consists of extending to the whole space distributions which are everywhere defined but on a submanifold, tipically the total diagonal. The aim of this section is to show that in this scenario, such extensions can be written explicitly.

Dwelling more into the details, in the previous formulae, particularly Equation (4.1.9) and (4.1.10), the fundamental solution of the heat operator enters the game together with the arbitrarily chosen extension $\widehat{P}_2 \in \mathcal{D}'((\mathbb{R}\times\mathbb{R}^d)^2)$ of $P^2 \in \mathcal{D}'((\mathbb{R}\times\mathbb{R}^d)^2\setminus \mathrm{Diag}_2)$, which contributes through the function C – *cf.* Equation (4.1.10).

For concreteness, in this paragraph we wish to discuss an explicit extension procedure for powers of the fundamental solution of the heat operator, to prove that our method can actually yield explicit expressions.

To this avail we observe that the fundamental solution $P \in \mathcal{D}'((\mathbb{R}\times\mathbb{R}^d)^2)$ of $\partial_t - \Delta$ is translation invariant – that is, $P(t,x;s,y) = \mathsf{p}(t-s, x-y)$ where $\mathsf{p} \in \mathcal{D}'(\mathbb{R}\times\mathbb{R}^d)$ is the fundamental solution of $\partial_t - \Delta$ on the Euclidean space $\mathbb{R}^{d+1}$ – *cf.* Equation (4.1.11).

In the following we discuss the extensions of $\mathsf{p}^n \in \mathcal{D}'(\mathbb{R}^{d+1}\setminus\{0\})$ and, to this end, it is convenient to start from the fundamental solution of $\partial_t - \kappa\Delta$ on $\mathbb{R}^{d+1}$, with $\kappa \in \mathbb{R}_+$,

$$\mathsf{p}_\kappa(t,x) := \frac{1}{(4\pi\kappa t)^{\frac{d}{2}}}\vartheta(t)e^{-\frac{|x|^2}{4\kappa t}}\,, \tag{4.1.11}$$

where ϑ is the Heaviside step function. Per construction $\mathsf{p}_\kappa \in \mathcal{D}'(\mathbb{R}^{d+1})$ satisfies

$$(\partial_t - \kappa\Delta)\mathsf{p}_\kappa = \delta\,, \tag{4.1.12}$$

where $\delta \in \mathbb{D}'(\mathbb{R}^{d+1})$ is the Dirac delta distribution centred at the origin. For simplicity, in what follows we shall denote $\mathsf{p}_1 = \mathsf{p}$.

For any $n \in \mathbb{N}$ let us consider p^{n+1}. Since $\partial_t - \kappa\Delta$ is a microhypoelliptic operator – see Equation (2.1.1) – it holds $\mathrm{WF}(\mathsf{p}) = \mathrm{WF}(\delta)$ and thus $\mathsf{p}^{n+1} \in \mathcal{D}'(\mathbb{R}^{d+1} \setminus \{0\})$. If $d = 1$ and $n = 1$, then $\mathsf{p}^2 \in \mathcal{D}'(\mathbb{R}^2)$ since

$$\mathsf{p}^2(f) = \int_0^{+\infty} \frac{\mathrm{d}t}{\sqrt{t}} \int_{\mathbb{R}} \mathrm{d}x \frac{1}{4\pi} e^{-\frac{|x|^2}{2}} f(t, \sqrt{t}x) < +\infty \,,$$

On the contrary, for $d \geq 2$ and $n \geq 2$, the singularity at the origin calls for an extension procedure. To this end we use the identity

$$\frac{1}{t^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \mathrm{d}z z^{\alpha-1} e^{-tz} \,.$$

Replacing it in the integral kernel of $\mathsf{p}^{n+1} \in \mathcal{D}'(\mathbb{R}^{d+1} \setminus \{0\})$, we obtain the following Källén-Lehmann type formula

$$\begin{aligned} \mathsf{p}(t,x)^{n+1} &= \frac{1}{(4\pi t)^{\frac{(n+1)d}{2}}} \Theta(t) e^{-\frac{(n+1)|x|^2}{4t}} \,. \\ &= \frac{1}{(4\pi)^{\frac{nd}{2}}} \frac{1}{(n+1)^{\frac{d}{2}}} \frac{1}{\Gamma(\frac{nd}{2})} \int_0^{+\infty} \mathrm{d}z z^{\frac{nd}{2}-1} \mathsf{p}_{\frac{1}{n+1},z}(t,x) \,. \end{aligned} \tag{4.1.13}$$

where $\mathsf{p}_{\frac{1}{n+1},z}$ is the fundamental solution of the parabolic equation

$$\left[\partial_t - \frac{1}{n+1}\Delta + z\right] \mathsf{p}_{\frac{1}{n+1},z} = \delta \,, \qquad \mathsf{p}_{\frac{1}{n+1},z}(t,x) := \frac{(n+1)^{\frac{d}{2}}}{(4\pi t)^{\frac{d}{2}}} \Theta(t) e^{-\frac{(n+1)|x|^2}{4t}} e^{-zt} \,. \tag{4.1.14}$$

Equation (4.1.13) represents the singularity of p^{n+1} at the origin in terms of the divergent integral in z. This suggests a specific extension $\widehat{\mathsf{p}}^{n+1}$ of p^{n+1}. To wit, for $a > 0$ we define ${}_a\mathsf{p}^{n+1} \in \mathcal{D}'(\mathbb{R}^{d+1})$ via the integral kernel

$${}_a\mathsf{p}^{n+1}(t,x) := \frac{1}{(4\pi)^{\frac{nd}{2}}} \frac{1}{(n+1)^{\frac{d}{2}}} \frac{1}{\Gamma(\frac{nd}{2})} \left[-\partial_t + \frac{1}{n+1}\Delta + a\right]^\ell \int_0^{+\infty} \mathrm{d}z \frac{z^{\frac{nd}{2}-1}}{(z+a)^\ell} \mathsf{p}_{\frac{1}{n+1},z}(t,x) \,, \tag{4.1.15}$$

where $\ell = \lfloor \frac{nd}{2} \rfloor \leq \frac{nd}{2}$. We observe that this choice makes the z-integral convergent after smearing it against a compactly supported function $f \in \mathcal{D}(\mathbb{R}^{d+1})$.

By direct inspection, once restricted to $(t,x) \neq (0,0)$, ${}_a\mathsf{p}^{n+1}(t,x) = \mathsf{p}(t,x)^{n+1}$. Furthermore, the weighted scaling degree of ${}_a\mathsf{p}^{n+1}$ at the origin coincides with the one of p^{n+1}, *cf.* Section B.3, though one should keep in mind that here the dimension of M is $d+1$.

Remark 4.1.12: *An important observation is that different choices for $a > 0$ lead to results consistent with Theorem B.2.1 and Section B.3.*

Getting more into the detail, for any $a, b > 0$, the difference ${}_a\mathsf{p}^{n+1} - {}_b\mathsf{p}^{n+1}$ results in a linear combination of derivatives of Dirac delta distributions with weighted scaling degree at most $(n+1)d$. To see this, we observe that, for all $m \in \mathbb{N}$ and $a \in \mathbb{R}$,

$$(H_{\frac{1}{n+1}} + a)^m \mathsf{p}_{\frac{1}{n+1},z} = (z+a)^m \mathsf{p}_{\frac{1}{n+1},z} - \sum_{j=0}^{m-1} (z+a)^{m-1-j} (H_{\frac{1}{n+1}} + a)^j \delta \,,$$

where $H_{\frac{1}{n+1}} := -\partial_t + \frac{1}{n+1}\Delta$ *is a short and useful notation. It follows that, setting*

$$c_{n,d} := \frac{1}{(4\pi)^{\frac{nd}{2}}} \frac{1}{(n+1)^{\frac{d}{2}}} \frac{1}{\Gamma(\frac{nd}{2})}\,,$$

it holds

$$\begin{aligned}
&{}_{a+b}\mathsf{p}^{n+1}(t,x) - {}_{a}\mathsf{p}^{n+1}(t,x) \\
&\qquad = c_{n,d}\Big[(H_{\frac{1}{n+1}}+a+b)^\ell - (H_{\frac{1}{n+1}}+a)^\ell\Big]\int_0^{+\infty} \mathrm{d}z \frac{z^{\frac{nd}{2}-1}}{(z+a+b)^\ell}\mathsf{p}_{\frac{1}{n+1},z}(t,x) \\
&\qquad + c_{n,d}(H_{\frac{1}{n+1}}+a)^\ell \int_0^{+\infty} \mathrm{d}z z^{\frac{nd}{2}-1}\left[\frac{1}{(z+a+b)^\ell} - \frac{1}{(z+a)^\ell}\right]\mathsf{p}_{\frac{1}{n+1},z}(t,x)\,.
\end{aligned}$$

With standard algebraic manipulations we find

$$\begin{aligned}
&{}_{a+b}\mathsf{p}^{n+1} - {}_{a}\mathsf{p}^{n+1} \\
&= c_{n,d}\sum_{j=0}^{\ell-1}\binom{\ell}{j} b^{\ell-j}\int_0^{+\infty}\mathrm{d}z \frac{z^{\frac{nd}{2}-1}}{(z+a+b)^\ell (z+a)^\ell}\cdot \\
&\qquad\qquad \Big[(H_{\frac{1}{n+1}}+a)^j (z+a)^\ell - (z+a)^j (H_{\frac{1}{n+1}}+a)^\ell\Big]\mathsf{p}_{\frac{1}{n+1},z} \\
&= c_{n,d}\sum_{j=0}^{\ell-1}\binom{\ell}{j} b^{\ell-j}\int_0^{+\infty}\mathrm{d}z \frac{z^{\frac{nd}{2}-1}}{(z+a+b)^\ell (z+a)^\ell}\sum_{q=j}^{\ell-1}(z+a)^{\ell+j-q-1}(H_{\frac{1}{n+1}}+a)^q\delta \\
&= \sum_{q=0}^{\ell-1}\zeta_q (H_{\frac{1}{n+1}}+a)^q\delta\,,
\end{aligned}$$

where $\zeta_j \in \mathbb{C}$. *Notice that the weighted scaling degree at the origin of* $(H_{\frac{1}{n+1}}+a)^q\delta$ *is at most* $2\ell - 2 + d - 2 = (n+1)d$ *as required.*

Remark 4.1.13: *It is worth mentioning that our analysis can be repeated almost varbatim also in the case where the underlying space is* $\mathbb{R}\times\mathbb{T}^d$, $\mathbb{T}^d$ *being the flat d-torus. In this case the counterpart of* p *is played by the distribution* p *obtained via a Poisson formula as*

$$p(t,x) := \sum_{n\in\mathbb{Z}} \mathsf{p}(t,x+n)\,, \qquad t\in\mathbb{R}\,,\ x\in(0,1]\,. \tag{4.1.16}$$

Here we realized $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d \simeq (0,1]^d$. *Equation (4.1.16) is particularly relevant because it shows that the singular structure of* p *is the same as that of* p *and in particular* $p^n - \mathsf{p}^n$ *is smooth for all* $n\in\mathbb{N}$.

4.2 Renormalized Equation

As we have seen in Section 4.1.1, the deformation procedure as per Section 3.1, which allows to recover the stochastic content of the underlying SPDE, Equation (4.1.1), affects the structure of Equation (4.1.1) itself. This is a consequence primarily of the deformation procedure and secondarily of the renormalization one.

In this section we construct the equation of which $\Psi_{\cdot_Q}$ is a formal solution and we shall refer to it as the *renormalized equation*.

To this end we recall that $\Psi[\![\lambda]\!] \in \mathcal{A}[\![\lambda]\!]$ is the perturbative solution of Equation (4.1.2), constructed in Equation (4.1.3) and (4.1.4). If one applies $\Gamma_{\cdot_Q}$ to Equation (4.1.2), Theorem 3.1.6 – *cf.* in particular Equation (3.1.17) – entails that $\Psi_{\cdot_Q}$ obeys to the following equation

$$\Psi_{\cdot_Q} = \Phi - \lambda P_\chi \circledast (\Psi_{\cdot_Q} \cdot_Q \Psi_{\cdot_Q} \cdot_Q \Psi_{\cdot_Q}) \,. \tag{4.2.1}$$

We observe that the presence of the product $\cdot_Q$ in the non-linear term $\Psi_{\cdot_Q} \cdot_Q \Psi_{\cdot_Q} \cdot_Q \Psi_{\cdot_Q}$ is a signature of the fact that we are representing the full stochastic content of Equation (4.1.2), according to the interpretation we discussed in the previous chapters.

However, for practical purposes, it may be desirable to get rid of the $\cdot_Q$-product, recasting Equation 3.1.17 in terms of the pointwise product.

The price to pay for such change is a modification of Equation (4.1.1), yielding what we called the *renormalized equation*.

Theorem 4.2.1: *Let $\Psi[\![\lambda]\!] \in \mathcal{A}[\![\lambda]\!]$ be the perturbative solution of Equation* (4.1.2)*, as per Equation* (4.1.4)*, and let $\Psi_{\cdot_Q} := \Gamma_{\cdot_Q}(\Psi)$. There exists a sequence of functional-valued linear operators $\{M_n\}_{n\in\mathbb{N}}$ such that*

(i) for any $n \in \mathbb{N}$ and $\varphi \in \mathcal{E}(\mathbb{R}^{d+1})$, $M_n(\varphi)\colon \mathcal{E}(\mathbb{R}^{d+1}) \to \mathcal{E}(\mathbb{R}^{d+1})$;

(ii) for any $n \in \mathbb{N}$, $M_n(\varphi)$ has a polynomial dependence on φ and, in addition, for all $j \in \mathbb{N}$, $M_n^{(2j+1)}(0) = 0$, where the superscript indicates the $(2j+1)$-th functional derivative;

(iii) $\Psi_{\cdot_Q}$ satisfies the following equation

$$\Psi_{\cdot_Q} = \Phi - \lambda P_\chi \circledast \Psi^3_{\cdot_Q} - P_\chi \circledast M\Psi_{\cdot_Q} \,, \tag{4.2.2}$$

where $M := \sum_{n\geq 1} \lambda^n M_n$, while χ is the cut-off function introduced in Remark 4.1.3 and in Equation (4.1.2)*.*

Proof. The proof is constructive and it proceeds by induction over the perturbative order n. In particular, for what concerns the case $n = 0$, Equation (4.2.2) is satisfied since $\Psi_{\cdot_Q} = \Phi + O(\lambda)$. For $n = 1$, focusing first on the left hand side (LHS) of Equation (4.2.2), it holds

$$\text{LHS} = \Phi - \lambda P_\chi \circledast \Gamma_{\cdot_Q}(\Phi^3) + O(\lambda^2) = \Phi - \lambda P_\chi \circledast \Phi^3 - 3\lambda P_\chi \circledast C_1\Phi + O(\lambda^2) \,,$$

whereas for the right hand side (RHS) it holds,

$$\text{RHS} = \Phi - \lambda P_\chi \circledast \Phi^3 - \lambda P_\chi \circledast M_1\Phi + O(\lambda^2) \,,$$

where $C_1(x) := \chi(x)(\widehat{P}_2 \circledast \chi)(x)$, $\widehat{P}_2 \in \mathcal{D}'((\mathbb{R}^{d+1})^2)$ is an extension of $P^2 \in \mathcal{D}'((\mathbb{R}^{d+1})^2 \setminus \mathrm{Diag}_2)$, *cf.* Section 4.1.1.

As a consequence, we can set $M_1 := 3C_1$ and we observe that this choice satisfies items *(i-ii)* of the theorem as well as Equation (4.2.2) up to order $O(\lambda^2)$.

By induction, we assume that M_k has been defined for all $k \leq n$ coherently with all requirements *(i-ii)* and in such a way that Equation (4.2.2) is satisfied up to order $O(\lambda^{n+1})$. We then show that we can construct M_{n+1} so that that Equation (4.2.2) holds true up to order $O(\lambda^{n+2})$.

To this end, we expand Equation (4.2.2) up to order $O(\lambda^{n+2})$ and we find

$$\begin{aligned}
\mathrm{LHS} &= R_n - \lambda^{n+1} \sum_{j_1+j_2+j_3=n} P_\chi \circledast \Gamma_{\cdot_Q}(\Psi_{j_1}\Psi_{j_2}\Psi_{j_3}) + O(\lambda^{n+2})\,,\\
\mathrm{RHS} &= R_n - \lambda^{n+1} \sum_{j_1+j_2+j_3=n} P_\chi \circledast \Gamma_{\cdot_Q}(\Psi_{j_1})\Gamma_{\cdot_Q}(\Psi_{j_2})\Gamma_{\cdot_Q}(\Psi_{j_3})\\
&\quad - \lambda^{n+1} \sum_{\substack{k_1+k_2=n+1\\ k_2\geq 1}} P_\chi \circledast M_{k_1}\Gamma_{\cdot_Q}(\Psi_{k_2}) - \lambda^{n+1}P_\chi \circledast M_{n+1}\Phi + O(\lambda^{n+2})\,,
\end{aligned}$$

where we isolated the summand containing M_{n+1}. In this last formulae, R_n represents the lowest order contributions for which Equation (4.2.2) holds true by the inductive hypothesis. We scrutinize thoroughly the remaining terms. To this avail, we recall that the vector space $\mathcal{O} \subset \mathcal{A}$, introduced in the proof of Lemma 4.1.10, is built out of elements $\tau \in \mathcal{A}$ such that $\tau^{(2n)}(\cdot;0) = 0$ for all $n \in \mathbb{N}_0$. We also recall that the proof of Lemma 4.1.10 entails that $\Psi \in \mathcal{O}$ and we observe that all contributions under analysis are of the form $P_\chi \circledast F$ where $F \in \mathcal{O}$.

This is a by product of the left hand side of Equation (4.2.2) since $\Psi \in \mathcal{O}$, *cf.* Lemma 4.1.10. This holds true also for $M_{k_1}\Gamma_{\cdot_Q}(\Psi_{k_2})$ on account of the inductive hypothesis for M_k.

In addition, we observe that any element $\tau \in \mathcal{O}$ can be written as $\tau = L\Phi$ where, for all $\varphi \in \mathcal{E}(\mathbb{R}^{d+1})$, $L(\varphi)\colon \mathcal{E}(\mathbb{R}^{d+1}) \to \mathcal{E}(\mathbb{R}^{d+1})$ and, moreover, $L(\varphi)$ has a polynomial dependence on φ with $L^{(2j+1)}(0) = 0$ for all $j \in \mathbb{N}$. The superscript still indicates the order of the functional derivative. Overall we obtained

$$\Psi_{\cdot_Q} - \Phi + \lambda P_\chi \circledast \Psi^3_{\cdot_Q} + P_\chi \circledast M\Psi_{\cdot_Q} = \lambda^{n+1}[P_\chi \circledast L\Phi - P_\chi \circledast M_{n+1}\Phi] + O(\lambda^{n+2})\,.$$

Finally, we set $M_{n+1} := L$, which satisfies *(i-ii)*. Hence the induction step is complete and we have proven the sought after result. □

Example 4.2.2: *In order to pinpoint the non-local behaviour encoded in the operator M_2, it is worth computing M at second order in perturbation theory as per Theorem 4.2.1. To this end, we observe that, up to $O(\lambda^3)$, Equation* (4.2.2) *leads to*

$$\begin{aligned}
&LHS = R_1 + 3\lambda^2 P_\chi \circledast \Gamma_{\cdot_Q}(\Phi^2 P_\chi \circledast \Phi^3) + O(\lambda^3)\,,\\
&RHS = R_1 + 3\lambda^2 P_\chi \circledast (\Phi^2\Gamma_{\cdot_Q}(P_\chi \circledast \Phi^3)) + \lambda^2 P_\chi \circledast (M_1\Gamma_{\cdot_Q}(P_\chi \circledast \Phi^3)) - \lambda^2 P_\chi \circledast M_2\Phi + O(\lambda^3)\,,
\end{aligned}$$

where, as above, $M_1 = 3C_1$, with $C_1(x) = \chi(x)(\widehat{P}_2 \circledast \chi)(x)$ while R_1 encompasses lower order contributions. Fulfilling Equation (4.2.2) *modulo $O(\lambda^3)$ entails that*

$$P_\chi \circledast M_2\Phi = -18P_\chi \circledast \left[[(P_\chi \circ P_\chi)\cdot P_\chi \circledast (\Phi^2 + C_1) + C_2]\Phi\right],$$

where $C_2(x,y) = P_\chi \cdot \widehat{(P_\chi \circ P_\chi)^2}(x,y) \in \mathcal{D}'((\mathbb{R}^{d+1})^2)$ denotes the chosen extension of the bi-distribution $P_\chi \cdot (P_\chi \circ P_\chi)^2 \in \mathcal{D}'((\mathbb{R}^{d+1})^2 \setminus \mathrm{Diag}_2)$ on the whole space. This leads to setting M_2 as

$$M_2 := -18[(P_\chi \circ P_\chi)P_\chi \circledast (\Phi^2 + C_1) + C_2]\,,$$

which is a non local operator.

Remark 4.2.3: *A reader might wonder whether the renormalized equation that we construct is connected with the one derived, e.g., in [54, Prop 4.9]. This is not a straightforward comparison since on the one hand, in [54], the rôle of* $\widehat{\xi}$ *in Equation* (4.1.1) *is played by a smooth function, which could be chosen for example as an* ε*-regularized smooth version of a white noise, obtained by means of a mollifier. On the other hand, our analysis is devised intrinsically to encode the singularities of the white noise in the product of the algebra of functionals.*

In order to connect the two approaches, a possible path could be that of starting from [54] and to discuss the limit as $\varepsilon \to 0$ *of the renormalized equation.*

4.3 Classification of Sub-Critical Cases

In the perturbative analysis of the Ψ^4_d model discussed in this chapter, we have seen that, up to order $\mathcal{O}(\lambda^2)$, the only ultra-violet divergence one has to deal with is the one associated to P^2.

In particular, we observe that all the above expressions, concerning both the expectation values and the two-point correlation functions, are completely determined by the function C obtained through renormalization of P^2, *cf.* Equation (4.1.9).

At this point, a natural question one may ask is whether, carrying the perturbative analysis of the model to all orders in λ, the number of contributions needing to be renormalized in order for the formal power series in Equation (4.1.6) to be meaningful, at least order by order, is finite or infinite.

In order to answer this question, we shall consider a slightly more general scenario. In particular we focus on the following SPDE

$$\partial_t \widehat{\psi} = \Delta \widehat{\psi} - \lambda \widehat{\psi}^k + \widehat{\xi}\,, \tag{4.3.1}$$

where we introduced a generic $k \in \mathbb{N}$ as polynomial degree of the non-linearity.

This behaviour of the model is clearly dependent on the value of two parameters, k and d, codifying respectively the non-linearity of the model and the dimension of the underlying space-time. As a consequence, depending on (k, d) there can be a finite or infinite number of contributions needing renormalization. In the literature, the former cases are called *sub-critical.* We shall discuss this nomenclature in the following. To recover the specific results about the case of Equation (4.1.1), it suffices to set $k = 3$ in the following constructions and results.

In this section, exploiting an argument based on a graph description of the models of Equation (4.3.1), we provide a characterization of sub-critical models.

First of all, we observe that, in order to describe perturbatively the dynamics of Equation (4.1.2), it is not necessary to construct the map $\Gamma_{\cdot_Q}$ as per Theorem 3.1.4 on the whole algebra $\mathcal{A}$.

As a matter of fact, the perturbative expansion as per Equations (4.1.3) and (4.1.4) involves only a suitable restricted number of elements lying in $\mathcal{A}$.

In addition we observe that in the case with a generic $k \in \mathbb{N}$, the coefficients appearing in Equation (4.1.3) are of the form

$$F_0 = \Phi\,, \qquad F_j = - \sum_{j_1 + \ldots + j_k = j-1} P \circledast (F_{j_1} \ldots F_{j_k})\,, \quad j \in \mathbb{N}\,.$$

Example 4.3.1: *For example, on account of the argument of Lemma 4.1.10, one can see that, with the notation of Remark 2.1.23, in the case $k = 3$ and for any $\ell \in \mathbb{N}$, no elements lying in $M_{2\ell}$ can appear in the perturbative solution of Equations* (4.1.3) *and* (4.1.4).

On account of this, we introduce the following sets of functional-valued distributions which are fine-tuned in order to describe the chosen dynamics.

Definition 4.3.2: *We define $\mathcal{U} \subset \mathcal{A}$ the collection of formal expressions which is the smallest family of elements of $\mathcal{A}$ containing both $\mathbf{1}$ and Φ and such that the following implication holds true*

$$\tau_1, \dots, \tau_k \in \mathcal{U} \qquad \Rightarrow \qquad P \circledast (\tau_1 \dots \tau_k) \in \mathcal{U} \,.$$

Then we set

$$\mathcal{W} := \big\{ \tau_1 \dots \tau_k \,|\, \tau_i \in \mathcal{U} \,, i \in \{1, \dots, k\} \big\} \,, \qquad \text{and} \qquad T := \mathrm{Span}_{\mathcal{E}(\mathbb{R}^d)}\{\mathcal{W}\} \subseteq \mathcal{A} \,.$$

By direct inspection, one can see that the vector space T contains all elements needed in the description of the right hand side of Equation (4.1.2).

Remark 4.3.3: *To scrutinize all possible divergences generated by applying the map $\Gamma_{\cdot_Q}$ to elements of T, it is convenient, as customary in this kind of problems, to introduce a graph representation for the functional-valued distributions. This also allows to keep a light notation. In particular*

- *we associate with Φ the symbol* ,
- *we join at their roots any two graphs to denote their pointwise product as functionals,*
- *we denote with an edge* *the composition with the parametrix P.*

As concrete examples, consider

$\Phi^2 =$ , $\qquad P \circledast \Phi^2 =$ .

We observe that the notation $\Phi =$ *is motivated by the fact that the functional Φ encodes the expectation value of $P \circledast \xi$. In this sense, the symbol $\bullet$ on top of* *can be seen as denoting the noise ξ. This graphical notation is inspired by the one of [52].*

In the following, we shall be interested in the description of the image through $\Gamma_{\cdot_Q}$ of the elements of T and, as a consequence, we need to codify at the level of graphs the action of the map $\Gamma_{\cdot_Q}$. In view of the discussion of Sections 2.2 and 3.1, this action can be represented by a collapse of two leaves into an integration vertex.

As an example, cf. Equation (3.1.6), *consider*

$\Gamma_{\cdot_Q}(\Phi^2) =$  $+$  .

Remark 4.3.4: *In addition, wee observe that, when considering graphs, divergences only occur from closed subgraphs appearing as a consequence of the action $\Gamma_{\cdot_Q}$ collapsing two leaves into a vertex. To better grasp this observation, consider the example*

$\Gamma_{\cdot_Q}(\Phi^2 P \circledast \Phi^2) =$  $+$  $+$  $+$  $+2$  $+$  .

Once we have discussed the closed diagrams  *and* *, all terms contributing to* $\Gamma_{\cdot_Q}(\Phi^2 P \circledast \Phi^2)$ *are known. To this end, recall that by construction any branch* *contributes to a diagram with a smooth factor* φ *and thus no divergence occurs.*

The general strategy we adopt is based on the following observation: When acting with $\Gamma_{\cdot_Q}$ on a generic element of the vector space T, we obtain a functional-valued distribution $t \in \mathcal{D}'(U)$, with $U \subseteq (\mathbb{R}^d)^N$ for a suitable $N \in \mathbb{N}$, depending on the specific element of T.

As we discussed in Theorem 3.1.4, typically one has $U = (\mathbb{R}^{d+1})^N \setminus \mathrm{Diag}_N$, with Diag_N being the total diagonal of $(\mathbb{R}^{d+1})^N$. As we have outlined in Section 3.1, the idea is to employ the results of Appendix B to discuss the existence of an extension $\widehat{t}$ of t to the whole space $\mathbb{R}^{(d+1)N}$.

Within the pictorial representation introduced above, we associate to each distribution $t \equiv t_{\mathcal{G}}$ a graph $\mathcal{G}$ with the following features:

- $\mathcal{G}$ has N vertices of valency at most $k+1$;
- each edge e of $\mathcal{G}$ corresponds to a parametrix $P(x_{s(e)}, x_{t(e)})$, where $s(e)$ (*resp.* $t(e)$) denotes the source (*resp.* the target) of the edge e;
- for a given $\mathcal{G}$, the integral kernel of the distribution $t_{\mathcal{G}}$ reads

$$t_{\mathcal{G}}(x_1, \ldots, x_N) := \prod_{e \in \mathbf{E}} P(x_{s(e)}, x_{t(e)}) \,,$$

  where $\mathbf{E}$ denotes the set of edges of $\mathcal{G}$.

In the following, for a given graph $\mathcal{G}$ we shall denote by L (*resp.* N) the number of edges (*resp.* vertices) of $\mathcal{G}$. Recalling that the weighted scaling degree of P_χ on the total diagonal of $\mathbb{R}^{2(d+1)}$ is d – *cf.* Example B.3.3, it follows that the degree of divergence of the distribution $t_{\mathcal{G}}$ – *cf.* Theorem B.2.1 – is

$$\rho(t_{\mathcal{G}}) = Ld - (N-1)(d+2) =: \rho(\mathcal{G}) \,, \tag{4.3.2}$$

since $(N-1)(d+2)$ is the effective codimension of the total diagonal of $\mathbb{R}^{(d+1)N}$ – *cf.* Remark B.2.2.

Remark 4.3.5: *Exploiting Theorem B.2.1, we underline that, on the one hand, if* $\rho(\mathcal{G}) < 0$ *then the associated distribution* $t_{\mathcal{G}}$ *admits a unique extension* $\widehat{t}$ *to the whole space which preserves the scaling degree. On the other hand, if* $\rho(\mathcal{G}) \geq 0$*, then some renormalization ambiguities shall occur.*

Remark 4.3.6: *We observe that had we considered the elliptic version of Equation* (4.3.1)*, namely the same equation with the Laplace operator* Δ *replacing the heat operator and with* $\mathbb{R}^d$ *as underlying manifold, then Equation* (4.3.2) *would read*

$$\rho(t_{\mathcal{G}}) = L(d-2) - (N-1)d =: \rho(\mathcal{G}) \,,$$

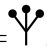
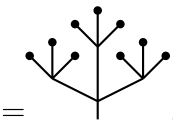
exploiting the fact that in the elliptic scenario the scaling degree of P_χ *on the total diagonal of* $\mathbb{R}^d \times \mathbb{R}^d$ *is* $d-2$ *– cf. Example B.1.5.*

Remark 4.3.7: *We observe that in Equation* (4.3.2)*, the number L of edges contributes to $\rho(\mathcal{G})$ by increasing the degree of divergence whereas the number N of vertices reduces it. In order to get a bound from above for the degree of divergence, we shall exploit the properties of the graph contained in the set T to bound the possible number L of edges in terms of the number N of vertices. This is the spirit of the next lemma.*

Lemma 4.3.8: *Let $\tau \in \mathcal{W}$ as per Definition 4.3.2 and let $\mathcal{G}$ be the graph associated with $\Gamma_{\cdot_Q}(\tau)$ as discussed above. Let L and N denote respectively the number of edges and vertices of $\mathcal{G}$. Then*

$$L \leq \frac{2k}{k+1} N\,.$$

Proof. To prove this result we introduce a family of trees lying in the space $\mathcal{W}$ which is constructed by an iteration procedure and which maximizes the number of edges with respect to the number of vertices. It is constructed as follows: $\mathcal{T}_0 = P \circledast \Phi^k$ and $\mathcal{T}_{n+1} = P \circledast (\mathcal{T}_n)^k$ for all $n \in \mathbb{N}$. For example the first two elements of this family in the case of a cubic interaction ($k = 3$) are

$\mathcal{T}_0 =$ , $\mathcal{T}_1 =$ .

This is the family with the highest possible ratio $\frac{L}{N}$ since all the intermediate vertices have the maximum valency, which is $k + 1$.

At the n-th iteration, the number L_n of edges of $\mathcal{T}_n$

$$L_n = \sum_{j=0}^{n+1} k^j = \frac{k^{n+2} - 1}{k - 1}\,.$$

Focusing on the number of vertices, we recall that we are interested in the graph describing $\Gamma_{\cdot_Q}(\mathcal{T}_n)$. As we discussed above, the map $\Gamma_{\cdot_Q}$ acting on the tree $\mathcal{T}_n$ collapses in pairs the leaves of the graph. The number of edges remains untouched by this operation. As a consequence, the total number of vertices is

$$N_n = \sum_{j=0}^{n+1} k^j + 1 - \frac{1}{2}k^{n+1} = \frac{k^{n+2} + k^{n+1} + 2k - 4}{2(k-1)}\,.$$

Hence,

$$r_n := \frac{L_n}{N_n} = \frac{2k^{n+2} - 2}{k^{n+2} + k^{n+1} + 2k - 4}\,.$$

This is an increasing and bounded function of n and its limit is $r = \lim_{n\to\infty} r_n = \frac{2k}{k+1}$. We conclude that $\frac{2k}{k+1}$ is an upper bound for the fraction L/N for a generic element $\tau \in \mathcal{W}$. $\square$

Theorem 4.3.9: *With the above notation, if*

$$d < \frac{2(k+1)}{k-1}\,, \tag{4.3.3}$$

the model of Equation (4.3.1) *is sub-critical, namely only a finite number of contributions needs to be renormalized.*

Proof. First of all we notice that the previous equation is meaningful since $k \geq 2$. The thesis is a direct consequence of Equation (4.3.2) and of Lemma 4.3.8.

Given a graph $\mathcal{G} = (L, N)$ associated with a distribution $\Gamma_{\cdot_Q}(\tau)$ with $\tau \in \mathcal{W}$, it holds

$$\rho(\mathcal{G}) = Ld - (N-1)(d+2) \leq \frac{2k}{k+1}Nd - (N-1)(d+2) = N\left(\frac{kd - d - 2k - 2}{k+1}\right) + d + 2\,. \tag{4.3.4}$$

It turns out that if $kd - d - 2k - 2 < 0$, which is equivalent to Equation (4.3.3), then $\rho(\mathcal{G})$ becomes negative increasing sufficiently N. This concludes the proof. □

Remark 4.3.10: *The above result proves that when Equation* (4.3.3) *is satisfied, then only a finite number of renormalization ambiguities do occur to all orders in the perturbative series of Equations* (4.1.3) *and* (4.1.4)

Example 4.3.11: *First of all, we observe that in the case of Equation* (4.1.1)*, i.e., if* $k = 3$*, Theorem 4.3.9 and, in particular, Equation* (4.3.4) *entail that Equation* (4.1.1) *is* sub-critical *for* $d < 4$*, as expected.*

Remark 4.3.12: *From Equation* (4.3.4) *one can see that, if* $d > \frac{2(k+1)}{k-1}$*, then the number of divergences is expected to grow linearly with the number of vertices and, thus, with the perturbative order. This is the so-called* super-critical *scenario.*

Finally, if $d = \frac{2(k+1)}{k-1}$*, which is the so-called* critical *case, the number of divergences is independent from the number of vertices and one has to scrutinize case by case.*

Remark 4.3.13: *We observe that Equation* (4.3.3) *can also be read the other way round, i.e., noticing that the model of Equation* (4.3.1) *is sub-critical for* $k < \frac{d+2}{d-2}$ *if* $d > 2$ *and for* $k > \frac{d+2}{d-2}$ *if* $d < 2$*. In addition, we observe that if* $d = 2$ *the condition* $kd - d - 2k - 2 < 0$ *which comes from Equation* (4.3.4)*, is identically satisfied for any* k*. These inequalities imply that for spatial dimension* $d \leq 2$*, every model with a polynomial non-linearity is sub-critical.*

Remark 4.3.14: *Finally we comment on the elliptic scenario discussed in Remark 4.3.6. In such a case, Equation* (4.3.3) *is replaced by*

$$d < \frac{4k}{k-1}\,, \tag{4.3.5}$$

as one can deduce following the same steps as in the proof of Theorem 4.3.9 tough starting from the formula in Remark 4.3.6, i.e.,

$$\rho(t_{\mathcal{G}}) = L(d-2) - (N-1)d\,.$$

On account of Lemma 4.3.8, it yields

$$\rho(t_{\mathcal{G}}) = L(d-2) - (N-1)d \leq \frac{2k}{k+1}N(d-2) - (N-1)d = N\frac{kd - 4k - d}{k-1}\,,$$

from which Equation (4.3.5) *descends.*

This inequality implies that, e.g., in the case $k = 3$*, the elliptic version of Equation* (4.3.1) *on* $\mathbb{R}^d$ *is* sub-critical *for* $d < 6$*. In addition, following the same lines of Remark 4.3.13, we see that, in the elliptic scenario, if* $d \leq 4$ *then every model with a polynomial non-linearity is sub-critical.*

Conclusions and Perspecitves

In this thesis we have presented a novel approach to the analysis of non-linear stochastic partial differential equations on a generic smooth manifold. This framework is based on techniques proper of the algebraic approach to quantum field theory, in particular for what concerns the renormalization procedure.

Starting from a generic SPDE, this novel approach, based on [27], starts from the construction of a deterministic abstract algebra $\mathcal{A}$, made of functional-valued distributions having suitable microlocal properties. This algebra is fine-tuned so to contain all the elements needed to formulate the original SPDE in a perturbative framework. This has been shown in Chapter 4 for the particular case of the Φ^4_d model.

Subsequently, inspired by the algebraic approach to quantum field theory, we recover the stochastic behaviour of the perturbative solution, due to the presence of the random noise playing the rôle of a source in the equation, through a *deformation* of the algebra product, yielding the algebra $\mathcal{A}_{\cdot_Q}$. As we have seen in Chapter 2, this deformation depends uniquely on the parametrix of the linear part of the SPDE and on the properties of the random noise.

As we have thoroughly discussed in Chapter 3, this construction needs renormalization in order to be meaningful.

As a result, this machinery allows the computation, at any order in perturbation theory, of the expectation values and of the multi-local n-point correlation functions of the solutions of a vast class of SPDEs.

Among the main advantages of this approach to non-linear SPDEs there is the absence of ε-regularizations and *subtraction of infinities* when dealing with renormalization. This is a consequence of the viewpoint on renormalization proper of algebraic quantum field theory, where renormalization consists of an extension procedure of *a priori* ill-defined distributions rather than a subtraction of singular contributions. In addition, this extension procedure is *explicit*, as we have seen, *e.g.*, in Section 4.1.2, and algorithmic.

Another important feature of this approach is that it works for a relatively large class of stochastic partial differential equations, also for those formulated on a generic smooth manifold.

In particular, in this thesis we considered SPDEs whose linear part is given by a micro-hypoelliptic operator – a class of partial differential operators containing both elliptic and parabolic ones – and where the additive noise is the white one.

We postpone to future works the extension and application of these methods to the analysis of *hyperbolic* SPDEs – such as the non-linear stochastic wave equation – as well as to the analysis of complex valued fields, *e.g.*, the non-linear stochastic Schrödinger equation.

At this level, the main limit of this approach to non-linear SPDEs lies in its perturbative nature, contrary to other approaches such as the theory of regularity structures [52] or paracontrolled calculus [43].

From this viewpoint, a future quest might be the introduction of a suitable topology on the space of functional-valued distributions in order to be able to use a fixed point argument allowing a non-perturbative analysis, akin to what happens in [43, 52].

Appendices

Appendix A

Microlocal Analysis

The aim of this appendix is to introduce the main concepts and tools proper of *microlocal analysis*, in particular the notion of *wave front set* as well as to recollect the main associated results since they play a key rôle in the main body of this manuscript. Throughout this section, our main reference will be [61, Chapt. 8], to which we also refer for the proof of the various statements.

Remark A.0.1 (Notation)**:** *In the following we shall denote with $\mathcal{D}(U)$ the space of smooth and compactly supported functions on the open set $U \subseteq \mathbb{R}^d$ and with $\mathcal{D}'(U)$ its topological dual, namely the space of distributions over U. Moreover, we denote with T^*U the cotangent bundle of U.*

Definition A.0.2: *Let $U \subseteq \mathbb{R}^n$, $x_0 \in U$ and let $t \in \mathcal{D}'(U)$. We say that $\xi_0 \in T^*_{x_0}U$ is not a singular direction for t at x_0 if there exists:*

(i) an open neighbourhood $V_{x_0} \subseteq U$ containing x_0;

(ii) an open conic neighbourhood $V_{\xi_0} \subseteq \mathbb{R}^n$ of $\xi_0 \in \mathbb{R}^n$;

(iii) a test function $f \in \mathcal{D}(V_{x_0})$ with $f(x_0) \neq 0$,

such that

$$\sup_{\xi \in V_{\xi_0}} |\xi|^k \, |\mathcal{F}(ft)(\xi)| < +\infty \,, \qquad \forall\, k \in \mathbb{N} \,,$$

where $\mathcal{F}$ denotes the Fourier transform, that is if $\mathcal{F}(ft)$ is rapidly decreasing in V_{ξ_0}. The collection of singular directions of t at x_0 is denoted by $\Sigma_x(t)$. Finally we define the ***wave front set*** *of t as the closed subset*

$$\mathrm{WF}(t) := \{(x;\xi) \in T^*U \setminus \{0\} \,|\, \xi \in \Sigma_x(t)\} \,. \tag{A.0.1}$$

Remark A.0.3: *In other words, the wave front set of a distribution t provides an estimate on the singular behaviour of u at any of its singular points x. This is achieved by considering the decay properties of the Fourier transform of a localization of the distribution t on an arbitrarily small neighbourhood of x. From this viewpoint, the notion of wave front set is a refinement of that of singular support, which is the set of point in U having no neighbourhood such that the restriction thereon of t is generated by a smooth function. As a matter of fact, the singular support of t is the projection on U of its wave front set.*

Remark A.0.4: *We observe that, on account of Definition A.0.2,* $\mathrm{WF}(t)$, *with* $t \in \mathcal{D}'(U)$, *is a closed subset of* $T^*U \setminus \{0\}$ *which is invariant with respect to diffeomorphisms. As a consequence, Definition A.0.2 can be naturally extended to distributions in* $\mathcal{D}'(M)$, *where* M *is a smooth manifold.*

Example A.0.5: *As an example, we compute the wave front set of the Dirac delta distribution centred at the diagonal of* $\mathbb{R}^n$. *Let* $\mathrm{Diag}_n \subseteq \mathbb{R}^n$ *be the total diagonal of* $\mathbb{R}^n$ *– i.e.* $(x_1, \ldots, x_n) \in \mathbb{R}^n$ *such that* $x_1 = \ldots = x_n$. *By* $\delta_{\mathrm{Diag}_n} \in \mathcal{D}'(\mathbb{R}^n)$ *we denote the Dirac delta distribution centred at* Diag_n, *that is,*

$$\delta_{\mathrm{Diag}_n}(f) := \int_{\mathbb{R}} f(x, \ldots, x) \mathrm{d}x \qquad \forall\, f \in \mathcal{D}(\mathbb{R}^n)\,.$$

A direct inspection shows that

$$\mathrm{WF}(\delta_{\mathrm{Diag}_n}) = \Big\{(x_1, \ldots, x_n; \xi_1, \ldots, \xi_n) \in T^*\mathbb{R}^n \setminus \{0\} \,|\, x_1 = \ldots = x_n\,,\ \sum_{\ell=1}^{n} \xi_\ell = 0\Big\}\,. \tag{A.0.2}$$

In the following theorem we collect some of the main properties of the wave front set.

Theorem A.0.6: *Let* $U \subseteq \mathbb{R}^n$ *be an open subset. The following statements hold true:*

1. *If* E *is a differential operator and* $t \in \mathcal{D}'(U)$,

$$\mathrm{WF}(Et) \subseteq \mathrm{WF}(t) \subseteq \mathrm{WF}(Et) \cup \mathrm{Char}(E)\,, \tag{A.0.3}$$

   where $\mathrm{Char}(E)$ *denotes the characteristic set of* E, *namely*

$$\mathrm{Char}(E) := \{(x;\xi) \in T^*U \setminus \{0\} \,|\, \sigma_E(x,\xi) = 0\}\,,$$

   σ_E *being the principal symbol*[1] *of* E.

2. *- [61, Thm. 8.2.9]: Let* $V \subseteq \mathbb{R}^k$ *be an open subset and let* $t_1 \in \mathcal{D}'(U)$ *and* $t_2 \in \mathcal{D}'(V)$, *then*

$$\begin{aligned}\mathrm{WF}(t_1 \otimes t_2) \subset \big(\mathrm{WF}(t_1) \times \mathrm{WF}(t_2)\big) &\cup \big((\mathrm{supp}(t_1) \times \{0\}) \times \mathrm{WF}(t_2)\big)\\ &\cup \big(\mathrm{WF}(t_1) \times (\mathrm{supp}(t_2) \times \{0\})\big)\end{aligned} \tag{A.0.4}$$

3. *- [61, Thm.8.2.10]: Let* $t_1, t_2 \in \mathcal{D}'(U)$. *If*

$$(x, 0) \notin \{(x; \xi_1 + \xi_2) \in T^*U \mid (x;\xi_1) \in \mathrm{WF}(t_1)\,,\, (x;\xi_2) \in \mathrm{WF}(t_2)\}\,, \tag{A.0.5}$$

   then the product $t_1 \cdot t_2 := (t_1 \otimes t_2) \cdot \delta_{\mathrm{Diag}_2}$, *acting on a test function* $f \in\in \mathcal{D}(U)$ *as* $(t_1 \cdot t_2)(f) = (t_1 \otimes t_2)(f\delta_{\mathrm{Diag}_2})$ *is a well-defined distribution in* $\mathcal{D}'(U)$ *and*

$$\mathrm{WF}(t_1 \cdot t_2) \subseteq \{(x; \xi_1 + \xi_2) \in T^*U \mid (x;\xi_1) \in \mathrm{WF}(t_1)\,,\, (x;\xi_2) \in \mathrm{WF}(t_2)\}\,. \tag{A.0.6}$$

[1] We recall that, given a differential operator with smooth coefficients of order $m \in \mathbb{N}$ on an open set $U \subset \mathbb{R}^{d+1}$, which is of the form $P = P(x, D) = \sum_{|j| \leq m} a_j(x) D^j$, where $D_j = -i\partial_j$ for a multi-index j, its principal symbol is $\sigma_E(x,\xi) = \sum_{|j|=m} a_j(x)\xi^j$ [61, Sect. 8.3].

4. - *[61, Thm. 8.2.12-13]. Let* $V \subseteq \mathbb{R}^k$ *be an open subset, let* $K \in \mathcal{D}'(U \times V)$, $t \in \mathcal{E}'(V)$ *and set*

$$\mathrm{WF}'_2(K) := \{(x_2; \xi_2) \in T^*V \,|\, \exists x_1 \in U\,,\ (x_1, x_2; 0, -\xi_2) \in \mathrm{WF}(K)\}\,. \tag{A.0.7}$$

It holds that, if

$$\mathrm{WF}'_2(K) \cap \mathrm{WF}(t) = \emptyset\,, \tag{A.0.8}$$

then $K \circledast t \in D'(U)$ *where*

$$[K \circledast t](f) := K(f \otimes t) := [K \cdot (1_n \otimes t)](f \otimes 1_k)\,, \qquad \forall\, f \in \mathcal{D}(U)\,. \tag{A.0.9}$$

Furthermore

$$\mathrm{WF}(K \circledast t) \subseteq \mathrm{WF}_1(K) \cup \mathrm{WF}'(K) \circ \mathrm{WF}(t)\,, \tag{A.0.10}$$

where

$$\mathrm{WF}_1(K) := \{(x_1; \xi_1) \in T^*U \setminus \{0\} \mid \exists x_2 \in V\,,\ (x_1, x_2; \xi_1, 0) \in \mathrm{WF}(K)\}\,, \tag{A.0.11}$$

$$\mathrm{WF}'(K) := \{(x_1, x_2; \xi_1, \xi_2) \in T^*(U \times U) \setminus \{0\} \mid (x_1, x_2; \xi_1, -\xi_2) \in \mathrm{WF}(K)\} \tag{A.0.12}$$

$$\begin{aligned}\mathrm{WF}'(K) \circ \mathrm{WF}(t) := \{(x_1; \xi_1) \in T^*U \setminus \{0\} \mid \\ \exists (x_2; \xi_2) \in \mathrm{WF}(t)\,,\ (x_1, x_2; \xi_1, -\xi_2) \in \mathrm{WF}(K)\}\,.\end{aligned} \tag{A.0.13}$$

5. - *[61, Thm. 8.2.14]: Let* $V \subseteq \mathbb{R}^k$ *be an open subset and let* $K_1 \in \mathcal{D}'(U \times V)$ *and* $K_2 \in \mathcal{D}'(V \times U)$ *be such that*

$$\mathrm{WF}'_2(K_1) \cap \mathrm{WF}_1(K_2) = \emptyset\,. \tag{A.0.14}$$

In addition assume that $\mathrm{pr}_1(\mathrm{supp}(K_2)) \cap \mathrm{pr}_2(\mathrm{supp}(K_1))$ *is compact, where* $\mathrm{pr}_1 \colon V \times U \to V$, $\mathrm{pr}_2 \colon U \times V \to V$ *are the canonical projections. Then* $K_1 \circ K_2 \in \mathcal{D}'(U \times U)$ *is completely defined by*

$$(K_1 \circ K_2)(f_1 \otimes f_2) := [(K_1 \otimes K_2) \cdot (1_n \otimes \delta_{\mathrm{Diag}_2} \otimes 1_n)](f_1 \otimes 1_{2k} \otimes f_2) \quad \forall\, f_1, f_2 \in \mathcal{D}(U)\,, \tag{A.0.15}$$

where $\delta_{\mathrm{Diag}_2} \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ *is the Dirac delta distribution centred on the diagonal* $\mathrm{Diag}_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ *– i.e.* $\mathrm{Diag}_2 := \{(x, x) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, x \in \mathbb{R}^n\}$, *cf. Example A.0.5. In addition, it holds that*

$$\mathrm{WF}'(K_1 \circ K_2) \subseteq \mathrm{WF}'(K_1) \circ \mathrm{WF}'(K_2) \cup (\mathrm{WF}_1(K_1) \times U \times \{0\}) \cup (U \times \{0\} \times \mathrm{WF}_2(K_2))\,, \tag{A.0.16}$$

where

$$\begin{aligned}\mathrm{WF}'(K) &:= \{(x_1, x_2; \xi_1, \xi_2) \in T^*(U \times U) \setminus \{0\} \mid (x_1, x_2; \xi_1, -\xi_2) \in \mathrm{WF}(K)\}\,,\\ \mathrm{WF}'(K_1) \circ \mathrm{WF}'(K_2) &:= \{(x_1, x_2; \xi_1, \xi_2) \in T^*(U \times U) \setminus \{0\} \mid \exists (z_3; \zeta_3) \in T^*V,\\ &\qquad (x_1, z_3; \xi_1, \zeta_3) \in \mathrm{WF}'(K_1)\,, (z_3, x_2; \zeta_3, \xi_2) \in \mathrm{WF}'(K_2)\}\,.\end{aligned}$$

Example A.0.7: *As application of Equation* (A.0.3) *we compute the wave front set of a parametrix P of an elliptic operator D,namely we consider $P \in \mathcal{D}'(U \times U)$ satisfying*

$$P \circledast D = DP\circledast = \delta_{\mathrm{Diag}_2} \qquad \mathrm{mod}\ \mathcal{E}(U)\,,$$

i.e., $P \circledast Df - f, DP \circledast f - f \in \mathcal{E}(U)$. Since D is elliptic [61] by hypothesis, its principal symbol σ_D is nowhere vanishing. As a consequence $\mathrm{Char}(D) = \emptyset$. *Thus, Equation* (A.0.3) *implies*

$$\mathrm{WF}(P) = \mathrm{WF}(DP) = \mathrm{WF}(\delta_{\mathrm{Diag}_2})\,. \tag{A.0.17}$$

Remark A.0.8: *A particularly relevant case of Equation* (A.0.10) *is the one where $K \in \mathcal{D}'(U \times V)$ and $t \in \mathcal{D}(V)$. In such a case* $\mathrm{WF}(t) = \emptyset$ *and therefore the condition stated in Equation* (A.0.8) *is fulfilled and $K \circledast t \in \mathcal{D}'(U)$ is well-defined. Moreover equation* (A.0.10) *entails that*

$$\mathrm{WF}(K \circledast t) \subseteq \mathrm{WF}_1(K)\,. \tag{A.0.18}$$

Appendix B

Scaling Degree

In this appendix we shall introduce the key notions and results of the theory of the **scaling degree** of a distribution. These are at the heart of the microlocal approach to renormalization developed in the framework of algebraic quantum field theory [14]. We refer to [14] for more detailed discussion and for the the proofs of the various statements.

B.1 Definition and Properties

Definition B.1.1: *Let $U \subseteq \mathbb{R}^d$ be a conic open set and, for any $x_0 \in \mathbb{R}^d$, let $U_{x_0} := U + x_0$. Let $f \in \mathcal{D}(U)$ and $\lambda > 0$, we define $f^\lambda_{x_0} := \lambda^{-d} f(\lambda^{-1}(x - x_0)) \in \mathcal{D}(U_{x_0})$. By duality, for $t \in \mathcal{D}'(U_{x_0})$ we define $t^\lambda_{x_0} \in \mathcal{D}'(U)$ via $t^\lambda_{x_0}(f) := t(f^\lambda_{x_0})$ for all $f \in \mathcal{D}(U)$. Finally, we define the scaling degree of t at x_0 as*

$$\mathrm{sd}_{x_0}(t) := \inf \left\{ \omega \in \mathbb{R} \mid \lim_{\lambda \to 0^+} \lambda^\omega t^\lambda_{x_0} = 0 \,, \text{ in } \mathcal{D}' \right\} . \tag{B.1.1}$$

Example B.1.2: *As an example, we consider $\delta_x \in \mathcal{D}'(\mathbb{R}^d)$. By direct inspection, $\delta^\lambda_x = \lambda^{-d}\delta_x$ and thus $\mathrm{sd}_x(\delta_x) = d$. More generally, if a distribution t is homogeneous of degree β at x_0 [61], then $\mathrm{sd}_{x_0}(t) = -\beta$. From this point of view, the scaling degree is a generalization of the notion of homogeneity of distributions.*

Remark B.1.3: *We observe that since the scaling degree is a local notion, it can be easily extended to the manifold setting [14], namely replacing $\mathbb{R}^d$ with a smooth manifold M. We also underline that Definition B.1.1 can be generalized so to encompass the notion of scaling degree with respect to an embedded submanifold $N \subseteq M$ – cf. [14]. For our purposes it suffices to consider only the case $N = \mathrm{Diag}_n \subseteq M^n$. To this end, let g be an arbitrary (pseudo-)Riemannian metric on M and let U be a star-shaped neighbourhood of the zero section of $T_N M := TM|_N$ and let $\alpha \colon U \to N \times M$ be the smooth map*

$$\alpha(x, \xi) := (x, \exp_x(\xi)) \qquad \forall (x, \xi) \in U \,,$$

where $\exp_x$ denotes the exponential map of M centred at x. We observe that $\alpha(x, 0) = (x, x)$ so that $\mathrm{pr}_2(\alpha(U))$ is an open neighbourhood of N in M, where $\mathrm{pr}_2 \colon N \times M \to M$. Let $t \in \mathcal{D}'(\mathrm{pr}_2(\alpha(U)))$ and let $t^\alpha := (1 \otimes t) \circ \alpha_* \in \mathcal{D}'(U)$, where $\alpha_* \colon \mathcal{D}(U) \to \mathcal{D}(\alpha(U))$ is defined by $(\alpha_* f)(z) := f(\alpha^{-1}(z))$. Let μ_U be a reference top density on TU, if t is generated by a smooth *function we have*

$$t^\alpha(f) = \int_U t(\exp_x \xi) f_{\mu_U}(x,\xi)\,,$$

where $f_{\mu_U} := f\mu_U$. *Similarly, for all* $0 < \lambda \leq 1$, *we define* $t^\alpha_\lambda \in \mathcal{D}'(U)$ *via* $t^\alpha_\lambda(f) := t^\alpha(f^\lambda)$ *where* $f^\lambda(x,\xi) := \lambda^{-d} f(x, \lambda^{-1}\xi)$ *for all* $f \in \mathcal{D}(U)$. *Once more, if* t *is generated by a smooth function, it holds that*

$$t^\alpha_\lambda(f) = \int_U t(\exp_x \lambda\xi) f_{\mu_U}\,.$$

The scaling degree of t *with respect to* N *is*

$$\mathrm{sd}_N(t) := \inf\left\{\omega \in \mathbb{R} \,|\, \lim_{\lambda\to 0^+} \lambda^\omega t^\alpha_\lambda = 0\right\}. \tag{B.1.2}$$

Example B.1.4: *Similarly to Example B.1.2, it holds that* $\mathrm{sd}_{\mathrm{Diag}_n}(\delta_{\mathrm{Diag}_n}) = (n-1)d$.

Example B.1.5: *For any elliptic or hyperbolic operator* E *on a smooth manifold* M *of dimension* $\dim M = d$ *it holds* $\mathrm{sd}_{\mathrm{Diag}_2} P = d-2$, P *being any parametrix of* E.

Remark B.1.6: *A useful property of the scaling degree is its additivity with respect to tensor product of distributions, namely [14, Lem. 5.1]*

$$\mathrm{sd}_{(x_1,x_2)}(T_1 \otimes T_2) = \mathrm{sd}_{x_1}(T_1) + \mathrm{sd}_{x_2}(T_2)\,. \tag{B.1.3}$$

Moreover, as a consequence of [14, Lem. 6.6] if $T_1, T_2 \in \mathcal{D}'(\mathbb{R}^d)$ *satisfy condition* (A.0.5), *then* $T_1T_2 \in \mathcal{D}'(\mathbb{R}^d)$ *satisfies*

$$\mathrm{sd}_x(T_1T_2) \leq \mathrm{sd}_x(T_1) + \mathrm{sd}_x(T_2)\,. \tag{B.1.4}$$

B.2 Extension of Distributions

As already anticipated, the scaling degree is the key tool for the microlocal approach to renormalization. This is due to the following theorem (see [14, Thm. 5.2-5.3] – see also [14, Thm 6.9] for the corresponding result for submanifolds).

Theorem B.2.1: *Let* $x \in \mathbb{R}^d$ *and* $t \in \mathcal{D}'(\mathbb{R}^d_x)$ *where* $\mathbb{R}^d_x := \mathbb{R}^d \setminus \{x\}$ *and set* $\rho := \mathrm{sd}_x(t) - d$. *Then*

1. *if* $\rho < 0$, *there exists a unique* $\widehat{t} \in \mathcal{D}'(\mathbb{R}^d)$ *such that*[1] $t \subseteq \widehat{t}$ *as well as* $\mathrm{sd}_x(\widehat{t}) = \mathrm{sd}_x(t)$.
2. *if* $\rho \geq 0$, *all distributions* $\widehat{t} \in \mathcal{D}'(\mathbb{R}^d)$ *such that* $t \subseteq \widehat{t}$ *as well as* $\mathrm{sd}_x(\widehat{t}) = \mathrm{sd}_x(t)$ *are of the form*

$$\widehat{t} = t \circ W_\rho + \sum_{|\alpha|\leq\rho} a_\alpha \partial^\alpha \delta_x\,,$$

where $\{a_\alpha\}_\alpha \subset \mathbb{C}$ *while* $W_\rho \colon \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}(\mathbb{R}^d)$ *is defined by*

$$W_\rho f := f - \sum_{|\alpha|\leq\rho} \frac{1}{\alpha!} (\partial^\alpha f)(x) \psi_\alpha\,,$$

being $\{\psi_\alpha\}_\alpha \subseteq \mathcal{D}(\mathbb{R}^d)$ *any family of smooth compactly supported functions such that* $\partial^\beta \psi_\alpha(x) = \delta^\beta_\alpha$.

[1] The notation $t \subseteq \widehat{t}$ means that $\widehat{t}$ is an extension of t, namely for any $f \in \mathcal{D}(\mathbb{R}^d_x)$ it holds $\widehat{t}(f) = t(f)$.

3. *if $\rho = +\infty$, there are no extensions $\widehat{t} \in \mathcal{D}'(\mathbb{R}^d)$ of t.*

Remark B.2.2: *The above result can be generalized to the problem of extending a given distribution $t \in \mathcal{D}'(M_N)$ – with N a smooth submanifold of M and where $M_N := M \setminus N$ – to a distribution $\widehat{t} \in \mathcal{D}'(M)$ on the whole space such that $\mathrm{sd}_N(\widehat{t}) = \mathrm{sd}_N(t)$ – cf. remark B.1.3. In this latter scenario the parameter ρ appearing in Theorem B.2.1 has to be replaced by*

$$\rho := \mathrm{sd}_N(t) - \mathrm{codim}(N)\,.$$

In the particular case of $M = Z^n$, $N = \mathrm{Diag}_n$, this reduces to $\mathrm{sd}_{\mathrm{Diag_n}}(t) - (n-1)\dim Z$. For further details we refer to [14, Thm. 6.9].

B.3 Weighted Scaling Degree

In this section we slightly modify the definition of scaling degree so to allow also the use of a *weighted scaling*, yielding the notion of *weighted scaling degree* sd_ω. This is necessary in order to discuss the microlocal renormalization in the framework of parabolic stochastic PDEs, where the natural scaling transformation is the parabolic one. Hence the time coordinate is scaled with a dimension which is twice that of any of the spatial variables (*cf.* Example B.3.3 and also Remark 1.2.1).

Remark B.3.1 (Notation)**:** *As anticipated above, we are interested in a weighted scaling transformation where different sets of coordinates of $\mathbb{R}^d$ are rescaled differently. Let $n \in \mathbb{N}$ and $\{d_j\}_{j=1}^n \subset \mathbb{N}_0 := \mathbb{N} \setminus \{0\}$ be such that $d = d_1 + \ldots + d_n$. Finally, let $\omega := (\omega_1, \ldots, \omega_n) \in [0, +\infty)^n$ such that $0 \leq \omega_1 \leq \ldots \leq \omega_n$. In this setting ω represents the weights of the different coordinates. We shall call $d_\omega := \sum_{j=1}^n \omega_j d_j$ the effective dimension of $\mathbb{R}^d$.*
If $x = (x_1, \ldots, x_n) \in \mathbb{R}^d$, with $x_j \in \mathbb{R}^{d_j}$ for all $j \in \{1, \ldots, n\}$, and $\lambda > 0$ we shall denote with $\frac{x}{\lambda^\omega} \in \mathbb{R}^d$

$$\frac{x}{\lambda^\omega} = \left(\frac{x_1}{\lambda^{\omega_1}}, \ldots, \frac{x_n}{\lambda^{\omega_n}}\right)\,.$$

We are now in position to introduce the notion of weighted scaling degree.

Definition B.3.2: *We define the ω-weighted scaling transformation on test-functions $\Lambda^\omega \colon \mathbb{R}_+ \times \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}(\mathbb{R}^d)$ by*

$$(\Lambda(\lambda, f))(x) = (\Lambda^\omega_\lambda f)(x) := f^\omega_\lambda(x) := \frac{1}{\lambda^{d_\omega}} f\left(\frac{x}{\lambda^\omega}\right)\,. \tag{B.3.1}$$

By duality we define $\Lambda^\omega \colon \mathbb{R}_+ \times \mathcal{D}'(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ by setting

$$(\Lambda^\omega_\lambda T)(f) := T^\omega_\lambda(f) := T(f^\omega_\lambda)\,, \tag{B.3.2}$$

for all $\lambda > 0$, $T \in \mathcal{D}'(\mathbb{R}^d)$ and $f \in \mathcal{D}(\mathbb{R}^d)$. The ω-weighted scaling degree of $T \in \mathcal{D}'(\mathbb{R}^d)$ at the origin is defined by

$$\mathrm{sd}_\omega(T) := \inf\{\sigma > 0 \,|\, \lim_{\lambda \to 0^+} \lambda^\sigma T^\omega_\lambda = 0\}\,.$$

We observe that if $\omega_1 = \ldots = \omega_n = 1$, then $\mathrm{sd}_\omega = \mathrm{sd}$ and we recover the usual notion of scaling degree as per Definition B.1.1.

Example B.3.3: *We consider*

$$d = k + 1\,, \qquad d_1 = k\,, \qquad d_2 = 1\,, \qquad \omega_1 = 1\,, \qquad \omega_2 = 2\,. \tag{B.3.3}$$

This corresponds to the parabolic scaling. Let $G \in \mathcal{D}'(\mathbb{R}^{k+1})$ be the distribution whose integral kernel reads

$$G(x,t) := \frac{\vartheta(t)}{(4\pi t)^{k/2}} e^{-\frac{|x|^2}{4t}}\,, \tag{B.3.4}$$

where ϑ denotes the Heaviside step-function. This is the fundamental solution associated with the heat operator on $\mathbb{R}^d$, namely $(\partial_t - \Delta)G = \delta$. A direct inspection shows that G is homogeneous with respect to parabolic scaling, i.e.,

$$G^\omega_\lambda = \lambda^{-k} G\,,$$

yielding $\mathrm{sd}_\omega(G) = k$. It is interesting to compute $\mathrm{sd}(G)$, namely the scaling degree of G with respect to standard scaling – that is, $\omega = \mathbf{1}$. For that, we shall employ that $G(t,x) \sim \vartheta(t)\delta(x)$ as $t, x \to 0$. Actually, let $T \in \mathcal{D}'(\mathbb{R}^{k+1})$ be the distribution whose integral kernel reads $S(x,t) := \vartheta(t)\delta(x)$. Since $S^{\mathbf{1}}_\lambda = \lambda^{-k} S$ it follows that $\mathrm{sd}(S) = k$. Then, for all $f \in \mathcal{D}(\mathbb{R}^{k+1})$ we find

$$\begin{aligned}
G^{\mathbf{1}}_\lambda(f) - S^{\mathbf{1}}_\lambda(f) &= \int_0^{+\infty} \mathrm{d}t \int_{\mathbb{R}^k} \frac{\mathrm{d}^k x}{(4\pi\lambda t)^{k/2}} f(x,t) e^{-\frac{\lambda |x|^2}{4t}} - \lambda^{-k} \int_0^\infty \mathrm{d}t f(0,t) \\
&= \lambda^{-k} \left[\int_0^{+\infty} \mathrm{d}t \int_{\mathbb{R}^k} \frac{\mathrm{d}^k z}{(4\pi t)^{k/2}} f\left(\frac{z}{\sqrt{\lambda}}, t\right) e^{-\frac{|z|^2}{4t}} - \int_0^\infty \mathrm{d}t f(0,t) \int_{\mathbb{R}^k} \frac{\mathrm{d}^k z}{(2\pi t)^{k/2}} e^{-\frac{|z|^2}{4t}} \right] \\
&= \lambda^{-k} \int_0^{+\infty} \mathrm{d}t \int_{\mathbb{R}^k} \frac{\mathrm{d}^k z}{(4\pi t)^{k/2}} e^{-\frac{|z|^2}{4t}} \left[f\left(\frac{z}{\sqrt{\lambda}}, t\right) - f(0,t) \right].
\end{aligned}$$

Neglecting the prefactor λ^{-k}, by the uniform convergence theorem the limit for $\lambda \to 0^+$ of the above integral converges to

$$C := -\int_0^{+\infty} \mathrm{d}t \int_{\mathbb{R}^k} \frac{\mathrm{d}^k z}{(4\pi t)^{k/2}} e^{-\frac{|z|^2}{4t}} f(0,t)\,.$$

It follows that

$$\lim_{\lambda \to 0^+} \lambda^k G^{\mathbf{1}}_\lambda = S + C\,,$$

which entails that $\mathrm{sd}(G) = k$.

In the above example we discussed a case where the parabolic scaling degree and the standard one actually coincide. This is not a general feature, as one can conclude from the following example. As a consequence, the scaling degree of distribution may or may not depend on the underlying weight.

Example B.3.4: *Let $T \in \mathcal{D}'(\mathbb{R}^{k+1})$ be the distribution whose integral kernel is $T(x,t) := t \log|x|$. With reference with the parabolic and standard scaling we have*

$$\mathrm{sd}_\omega(T) = -2\,, \qquad \mathrm{sd}(T) = -1\,.$$

Remark B.3.5: *We observe that, mutatis mutandis, Theorem B.2.1 remains valid with respect to any scaling ω. The only difference lies in the fact that the scaling degree sd is replaced by the weighted one ω-sd and the dimension d is replaced by the effective dimension d_ω.*

B.4 Some Technical Results

In this section we state and prove a pair of technical results playing a pivotal rôle in the main body of this manuscript. Basically, the following lemma states that if we consider suitable distributions satisfying some nice microlocal properties as well as having finite scaling degree, then their convolution, in the sense of item *4* of Theorem A.0.6, has finite scaling degree.

Lemma B.4.1: *[27, Lemma B.11] Let* $K \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ *and* $t_\ell \in \mathcal{D}'(\underbrace{\mathbb{R}^d \times \ldots \times \mathbb{R}^d}_{\ell-\text{times}}) = \mathcal{D}'((\mathbb{R}^d)^\ell)$ *be such that, with the notation of Theorem A.0.6,*

1. $\mathrm{WF}_2'(K) \cap \mathrm{WF}_1(t_\ell) = \emptyset$*;*

2. *the set* $\mathcal{K} := \mathrm{pr}_2(\mathrm{supp}(K)) \cap \mathrm{pr}_1(\mathrm{supp}(t_\ell))$ *is compact in* $\mathbb{R}^d$*;*

3. $\mathrm{sd}_{\mathrm{Diag}_2}(K) < \infty$ *and* $\mathrm{sd}_{\mathrm{Diag}_\ell}(t_\ell) < \infty$*;*

4. *setting*

$$T := (K \otimes t_\ell) \cdot (1 \otimes \delta_{\mathrm{Diag}_2} \otimes 1_{\ell-1}) \in \mathcal{D}'((\mathbb{R}^d)^{\ell+2}) \,, \tag{B.4.1}$$

it holds

$$\mathrm{WF}(T) \cap \{(x, y, z, \hat{x}; \xi, 0, 0, \hat{\eta}_\ell) \in T^*(\mathbb{R}^d)^{\ell+2} \setminus \{0\} \,|\, x \neq y \,,\, x \neq z\} = \emptyset \,, \tag{B.4.2}$$

where $\hat{x} = \underbrace{(x, \ldots, x)}_{\ell-\text{times}}$ *and similarly* $\hat{\eta}_\ell = (\eta_1, \ldots, \eta_\ell)$*.*

Then

$$\mathrm{sd}_{\mathrm{Diag}_k}(K \circledast t_\ell) < +\infty \,. \tag{B.4.3}$$

Proof. On account of Theorem A.0.6, we observe that T as per Equation (B.4.1) identifies an element of $\mathcal{D}'((\mathbb{R}^d)^{\ell+2})$. Moreover, for any $f, h \in \mathcal{D}(\mathbb{R}^d)$,

$$(K \circledast t_\ell)(f \otimes h^{\otimes \ell}) = T(f \otimes 1_2 \otimes h^{\otimes \ell}) \,.$$

Let $\chi \in \mathcal{D}(\mathbb{R}^d)$ and $\lambda \in (0, 1)$, it follows

$$\begin{aligned}
\lambda^\omega (K \circledast t_\ell)^\lambda_\alpha (\chi \otimes f \otimes h^{\otimes \ell}) &= \lambda^\omega \int_{\mathbb{R}^d} T[f^\lambda_x \otimes 1_2 \otimes (h^{\otimes \ell})^\lambda_{\hat{x}}] \chi(x) \, dx \\
&= \lambda^{\omega + 2d} \int_{\mathbb{R}^d} T[f^\lambda_x \otimes (g^{\otimes 2})^\lambda_{(x,x)} \otimes (h^{\otimes \ell})^\lambda_{\hat{x}}] \chi(x) \, dx \\
&+ \lambda^\omega \int_{\mathbb{R}^d} T[f^\lambda_x \otimes (1_2 - \lambda^{2d} (g^{\otimes 2})^\lambda_{(x,x)}) \otimes (h^{\otimes \ell})^\lambda_{\hat{x}}] \chi(x) \, dx \,,
\end{aligned}$$

with $g \in \mathcal{D}(\mathbb{R}^d)$ such that $\mathrm{supp}(g) \subset B(0,1)$ and $g|_{B(0,1/2)} = 1$. After a few algebraic manipulations, the above expression can be written as

$$\begin{aligned}\lambda^\omega (K \circledast t_\ell)^\lambda_\alpha(\chi \otimes f \otimes h^{\otimes \ell}) &= \underbrace{\lambda^{\omega+2d} T^\lambda_\alpha(\chi \otimes f \otimes g^{\otimes 2} \otimes h^{\otimes \ell})}_{A} \\ &+ \underbrace{\lambda^\omega \int_{\mathbb{R}^d} T[f^\lambda_x \otimes (1_2 - \lambda_0^{2d}(g^{\otimes 2})^{\lambda_0}_{(x,x)}) \otimes (h^{\otimes \ell})^\lambda_{\hat{x}}]\chi(x)\,dx}_{B} \\ &+ \underbrace{\lambda^\omega \int_{\mathbb{R}^d} \chi(x)\,dx \int_\lambda^{\lambda_0} T[f^\lambda_x \otimes (Dg^{\otimes 2})^\mu_{(x,x)}) \otimes (h^{\otimes \ell})^\lambda_{\hat{x}}]\mu^{2d-1}\,d\mu}_{C}\,,\end{aligned} \tag{B.4.4}$$

where $\lambda_0 \in (\lambda, 1)$ will be fixed later in the proof, while $(Dg)(z) := (z \cdot \nabla g)(z)$.
Focusing on A, we observe that $A \to 0$ for $\lambda \to 0^+$ for any $\omega > \mathrm{sd}_{\mathrm{Diag}_{\ell+2}}(T) - 2d < \infty$ exploiting, in the last inequality, the bound $\mathrm{sd}_{\mathrm{Diag}_{\ell+2}}(T) \leq \mathrm{sd}_{\mathrm{Diag}_2}(K) + \mathrm{sd}_{\mathrm{Diag}_\ell}(t_\ell) + d$ (see Remark B.1.6 and Equation (B.4.1)).
For what concerns B, we have

$$\lim_{\lambda \to 0^+} \int_{\mathbb{R}^d} T[f^\lambda_x \otimes (1_2 - \lambda_0^{2d}(g^{\otimes 2})^\lambda_{(x,x)}) \otimes (h^{\otimes \ell})^\lambda_{\hat{x}}]\chi(x)\,dx = \int_{\mathbb{R}^d} T_{(x,\hat{x})}[1_2 - \lambda_0^{2d}(g^{\otimes 2})^{\lambda_0}_{(x,x)}]\chi(x)\,dx\,,$$

where $T_{(x,\hat{x})} \in \mathcal{E}'(\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,\hat{x})\})$ is such that, for any $w \in \mathcal{E}(\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,\hat{x})\})$, $T_{(x,\hat{x})}(w) := T(\delta_x \otimes w \otimes \delta_x^{\otimes \ell})$. Well-definiteness of $T_{(x,\hat{x})}$ as a distribution is granted by the hypothesis of Equation (B.4.2) jointly with Theorem A.0.6. As a consequence, we see that, for any value of λ_0, $B \to 0$ for $\lambda \to 0^+$ whenever $\omega > 0$.
Eventually, we focus on C. As a starting point, we recall that, whenever $\omega > \mathrm{sd}_{\mathrm{Diag}_{\ell+2}}(T)$, for any compact set $\mathfrak{K} \subset (\mathbb{R}^d)^{\ell+3}$ there exists a polynomial function P of degree $p \in \mathbb{N}$ satisfying

$$|\lambda^\omega T^\lambda_\alpha(\chi \otimes f)| \leq \sup_{\mathfrak{K}} |P(\partial)(\chi \otimes f)|\,, \tag{B.4.5}$$

for any $\chi \otimes f \in \mathcal{D}((\mathbb{R}^d)^{\ell+3})$ such that $\mathrm{supp}(\chi \otimes f) \subset \mathfrak{K}$. For what concerns C, it holds that

$$C = \lambda^\omega \int_\lambda^{\lambda_0} T^\mu_\alpha(\chi \otimes f^{\frac{\lambda}{\mu}} \otimes (Dg)^{\otimes 2} \otimes (h^{\otimes \ell})^{\frac{\lambda}{\mu}})\mu^{2d-1} d\mu\,.$$

Since $0 < \lambda/\mu < 1$, we have

$$\mathrm{supp}(\chi \otimes f^{\frac{\lambda}{\mu}} \otimes (Dg)^{\otimes 2} \otimes (h^{\otimes \ell})^{\frac{\lambda}{\mu}}) \subset \mathbf{K} \subset (\mathbb{R}^d)^{\ell+3}\,, \tag{B.4.6}$$

with $\mathbf{K}$ a compact set. To conclude it suffices to prove that the compact set $\mathbf{K}$ is independent of χ. Indeed in such scenario we can apply uniformly the bound of Equation (B.4.5). This yields that, for any $\varepsilon > 0$ and uniformly in λ,

$$|C| \leq \lambda^\omega \int_\lambda^{\lambda_0} \mu^{-\mathrm{sd}_{\mathrm{Diag}_{\ell+2}}(T) - \varepsilon} \mu^{2d+p} \lambda^{-2d-p} \mu^{2d-1}\,d\mu = \mathcal{O}(\lambda^{\omega - p - 2d}) + \mathcal{O}(\lambda^{\omega - \mathrm{sd}_{\mathrm{Diag}_{\ell+2}}(T) - \varepsilon + 2d})\,.$$

It follows that $C \to 0$ for $\lambda \to 0^+$ and for $\omega > \max\{2d+p, \mathrm{sd}_{\mathrm{Diag}_{\ell+2}}(T)-2d\}$. As a consequence, recollecting the estimates for A, B and C, we conclude that

$$\lim_{\lambda\to 0^+} \lambda^\omega (K \circledast t_\ell)^\lambda_\alpha(\chi\otimes f\otimes h^{\otimes\ell}) = 0\,, \qquad \forall\,\omega > \max\{2d+p, \mathrm{sd}_{\mathrm{Diag}_{\ell+2}}(T)-2d\} < \infty\,,$$

i.e., $\mathrm{sd}(K \circledast t_\ell) < \infty$.

To conclude, we focus on the compact set $\mathbf{K}$ introduced in Equation (B.4.6) and we prove its independence from the choice of χ. This descends from the hypothesis that $\mathcal{K} := \mathrm{pr}_2(\mathrm{supp}(K)) \cap \mathrm{pr}_1(\mathrm{supp}(t_\ell))$ is compact, since the only non vanishing contribution to C comes from the portion of $\mathrm{supp}(\chi)$ contained in $\mathcal{K}$. More in detail, consider C at the level of integral kernels, *i.e.*,

$$\lambda^\omega \int_\lambda^{\lambda_0} \int_{(\mathbb{R}^d)^{\ell+3}} K(x+\lambda z, y) t_\ell(y, \hat{x}+\lambda\hat{z}_\ell) f(z) h^{\otimes\ell}(\hat{z}_\ell)(Dg)^{\otimes 2}\big(\frac{y-x}{\mu}\big)\chi(x)\mu^{2d-1}\, dx\, dy\, dz\, d\hat{z}_\ell\, d\mu\,.$$

We recall that Dg is supported in an annulus whose external radius is μ. If we consider the case $\mathrm{supp}(\chi)\cap\mathcal{K} = \emptyset$, then $d(\mathrm{supp}(\chi),\mathcal{K}) := R > 0$, where $d(\mathrm{supp}(\chi),\mathcal{K})$ denotes the Euclidean distance between the compact sets. As a consequence, it suffices to choose λ_0 small enough in Equation (B.4.4) so that the above contribution vanishes for any $\mu \in (\lambda, \lambda_0)$. This concludes the proof. □

Notice that the above result is formulated for the $\circledast$ operation between distributions. Nonetheless, it can be proven also for the composition $\circ$ of distributions introduced in item *5* of Theorem A.0.6.

Corollary B.4.2: *[27, Cor. B12] Let $K_1, K_2 \in \mathcal{D}'(\mathbb{R}^d\times\mathbb{R}^d)$ be such that:*

(i) $\mathrm{sd}_{\mathrm{Diag}_2}(K_1) < \infty$, $\mathrm{sd}_{\mathrm{Diag}_2}(K_2) < \infty$*;*

(ii) $\mathrm{WF}'_2(K_1)\cap\mathrm{WF}_1(K_2) = \emptyset$*;*

(iii) $\mathrm{pr}_2(\mathrm{supp}(K_1))\cap\mathrm{pr}_1(\mathrm{supp}(K_2))$ *is compact;*

(iv) $\mathrm{WF}(T)\cap\{(x,y,z,x;\xi,0,0,\eta)\in T^*((\mathbb{R}^d\times\mathbb{R}^d)^2)\setminus\{0\}\,|\,x\neq y\,,\,x\neq z\} = \emptyset$, *with* $T := (K_1\otimes K_2)\cdot(1\otimes\delta_{\mathrm{Diag}_2}\otimes 1)$

Then $K_1\circ K_2 \in \mathcal{D}'(\mathbb{R}^d\times\mathbb{R}^d)$ *and*

$$\mathrm{sd}_{\mathrm{Diag}_2}(K_1\circ K_2) < \infty\,. \tag{B.4.7}$$

Proof. Well-definiteness of $K_1\circ K_2$ is a consequence of item *5.* of Theorem A.0.6. To prove the estimate (B.4.7) it suffices to observe that, for any $f_1, f_2 \in \mathcal{D}(\mathbb{R}^d)$,

$$(K_1\circ K_2)(f_1\otimes f_2) := [(K_1\otimes K_2)\cdot(1_d\otimes\delta_{\mathrm{Diag}_2}\otimes 1_d)](f_1\otimes 1_{2d}\otimes f_2)\,.$$

The statement follows applying Lemma B.4.1 together with the identifications

$$\ell = 2\,, \qquad T = (K_1\otimes K_2)\cdot(1_d\otimes\delta_{\mathrm{Diag}_2}\otimes 1_d)\,.$$

□

Remark B.4.3: *The statements and the proofs of Lemma B.4.1 and of Corollary B.4.2 remain valid, mutatis mutandis, if one considers a generic weighted scaling ω.*

List of Publications

1. M. Carfora, C. Dappiaggi, N. Drago, P. Rinaldi
   Ricci Flow from the Renormalization of Nonlinear Sigma Models in the Framework of Euclidean Algebraic Quantum Field Theory,
   Commun. Math. Phys. 374, 241-276 (2020).

2. C. Dappiaggi, N. Drago, P. Rinaldi
   The algebra of Wick polynomials of a scalar field on a Riemannian manifold,
   Reviews in Mathematical Physics, Vol. 32, No. 08 2050023 (2020).

3. P. Rinaldi, F. Sclavi,
   Reconstruction Theorem for Germs of Distributions on Smooth Manifolds,
   Journal of Mathematical Analysis and Applications, Vol 501, Issue 2 (2021)

4. C. Dappiaggi, N. Drago, P. Rinaldi, L. Zambotti,
   A Microlocal Approach to Renormalization in Stochastic PDEs,
   Communication in Contemporary Mathematics, 2150075 (2021)

5. C. Dappiaggi, P. Rinaldi, F. Sclavi,
   On a Microlocal Version of Young Product Theorem,
   Preprint, ArXiv:2104.12423 (2021)

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