

Thirring quantum cellular automaton

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We analytically diagonalize a discrete-time on-site interacting fermionic cellular automaton in the two-particle sector. Important features of the solutions sensibly differ from those of analogous Hamiltonian models. In particular, we find a wider variety of scattering processes, we have bound states for every value of the total momentum, and there exist bound states also in the free case, where the coupling constant is null.

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Quantum cellular automata and quantum walks constitute an increasingly attractive arena for research in many-body systems [1–3], quantum computation [4–7], and foundations of quantum field theory [8–12]. The notion of the quantum cellular automaton introduced by Feynman [13] as a universal quantum simulator was mathematically formalized in Refs. [14,15]. In the case of noninteracting theories the evolution of field operators is linear and its simulation through quantum cellular automata reduces to simulation of a single particle through a quantum walk [16–18]. The interacting case is largely unexplored and was mainly approached by extending the quantum walk formalism, introducing decoherence [19,20], or a classical external field [10,21–23]. Notable exceptions are Ref. [24], where bound states in interacting Hadamard quantum walks are studied, and Refs. [25,26], where quantum walks coupled by nonlinear terms are considered.

In the present paper we study a one-dimensional massive fermionic cellular automaton with a four-fermion on-site interaction. The main result consists in the complete analytical solution in the two-particle sector. The linear part of the evolution corresponds to a one-dimensional Dirac walk [10], with an interaction having the most general on-site number-preserving form. The same kind of interaction characterizes the most studied integrable quantum systems [27–30] such as Hubbard's [31] and Thirring's [32] models. For this reason we call the present model Thirring quantum cellular automaton.

Despite the similarities, the present quantum cellular automaton differs from the above models mainly in the discreteness of time evolution. This feature produces nontrivial differences in the dynamical solutions of the model, in particular a wider spectrum of scattering states and the existence of bound states for every value of the total momentum. As a consequence of the departure of the present discrete-time evolution from the usual Hamiltonian paradigm, we are not allowed to borrow the common Bethe ansatz technique straightforwardly.

We start by defining a quantum cellular automaton for interacting particles on the lattice \mathbb{Z} , assuming the particle statistics to be fermionic. First we introduce the walk W for a free two-component fermionic field ψ defined at any lattice point $x \in \mathbb{Z}$ and at any discrete time $t \in \mathbb{Z}$,

$$\psi(x, t+1) = W\psi(x, t), \quad \psi(x, t) = \begin{pmatrix} \psi_{\uparrow}(x, t) \\ \psi_{\downarrow}(x, t) \end{pmatrix}$$

$$W = \begin{pmatrix} vT_x^{\dagger} & -i\mu \\ -i\mu & vT_x \end{pmatrix}, \quad v, \mu > 0, \quad v^2 + \mu^2 = 1, \quad (1)$$

where T_x is the translation operator $T_x\phi(x) = \phi(x+1)$ and ψ_{\uparrow} and ψ_{\downarrow} denote the two components of the field. Since the evolution of Eq. (1) describes noninteracting particles, the one-particle sector completely specifies the dynamics. The one-particle walk is a unitary operator W_1 over the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \ell^2(\mathbb{Z})$ for which we will use the factorized orthonormal basis $|a\rangle|x\rangle$, with $a \in \{\uparrow, \downarrow\}$. Accordingly, the evolution of N free fermions is given by $W_N = W_1^{\otimes N}$. It is convenient to express the one-particle walk in the momentum basis $(|p\rangle := (2\pi)^{-1/2} \sum_x e^{-ipx}|x\rangle, p \in (-\pi, \pi])$ as

$$W_1 = \int dp W_1(p) \otimes |p\rangle\langle p|, \quad W_1(p) = \begin{pmatrix} ve^{ip} & -i\mu \\ -i\mu & ve^{-ip} \end{pmatrix}$$

$$W_1(p)\mathbf{v}_p^s = e^{-is\omega(p)}\mathbf{v}_p^s, \quad \mathbf{v}_p^s := \frac{1}{|N_s|} \begin{pmatrix} -i\mu \\ g_s(p) \end{pmatrix},$$

$$\omega(p) := \arccos(v \cos p), \quad s \in \{+, -\}, \quad (2)$$

with $g_s(p) = -i[s \sin \omega(p) + v \sin p]$ and $|N_s|^2 = \mu^2 + |g_s|^2$. Notice that the walk evolution is local, with the field at time t and at site x depending only on the field at sites $x \pm 1$ at time $t-1$, and it recovers the dynamics of a free Dirac field of mass μ in the limit of small p [8,9,33].

The Thirring quantum cellular automaton is now defined as

$$U = W V(\chi), \quad V(\chi) = e^{i\chi n_{\uparrow}(x)n_{\downarrow}(x)}, \quad (3)$$

where $n_a(x) = \psi_a^{\dagger}(x)\psi_a(x)$ is the number operator at site x and with internal state $a \in \{\uparrow, \downarrow\}$ and $\chi \in [-\pi, \pi]$ is the automaton coupling constant. The interacting term $V(\chi)$ corresponds to an on-site coupling, namely, the action of $V(\chi)$ is nontrivial if and only if two fermions lie at the same site

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of the lattice. Moreover, $V(\chi)$ is the most general on-site coupling of fermions that preserves the number of particles [34] (one can easily verify that $V(\chi)$ commutes with the total number operator $n = \sum_x [\psi_\uparrow^\dagger(x)\psi_\uparrow(x) + \psi_\downarrow^\dagger(x)\psi_\downarrow(x)]$). Since the Thirring quantum cellular automaton is number preserving, the evolution of N fermions can be written as $U_N := W_N V_N = W_1^{\otimes N} V_N(\chi)$ where the Hilbert space of the system is $\mathcal{H}_N = \mathcal{H}^{\otimes N}$.

Typically, an interacting dynamics cannot be solve analytically and one resorts to analytical approximations or numerics (or both). However, there exist dynamical models in which the complete analytical solution can be derived. These are called quantum integrable systems, and paradigmatic examples are the Thirring model and the Hubbard model in one dimension. A rigorous characterization of quantum integrable systems is missing; however, roughly speaking, we may say that a quantum field theory is integrable if the many-body dynamics can be reduced to a two-body dynamics (free theories can be thought of as the case in which the dynamics is one-body reducible). Quantum integrable models are rather peculiar and occur in one spatial dimension. Nevertheless, they provide paradigmatic frameworks for many phenomena, which can be studied in full detail, and provide benchmarks for approximate and numerical methods.

Since the quantum cellular automaton (QCA) theory that we introduced in Eq. (3) has the same interaction term as the Hubbard and Thirring models, one could wonder whether this is a quantum integrable model as well. All the quantum integrable systems that are known today are solved via the so-called Bethe ansatz. The technique works as follows: (i) Solve the two-particle dynamics, (ii) create an ansatz for the solution of the N -particle case from the two-particle solution, and (iii) verify that the ansatz gives all the solutions. Despite some additional technical difficulties, the first step of the procedure can be accomplished also for the two-particle sector of the Thirring quantum cellular automaton. Surprisingly, we will show that the solution of the two-particle case substantially differs from the analogous solution of the Hubbard or Thirring model. These unexpected features of the two-particle dynamics give rise to a rich phenomenology (most notably, a richer family of scattering processes) that presents interesting scenarios for research. Unfortunately, because of these differences from the known integrable models, the usual N -particle ansatz cannot be applied to the Thirring quantum cellular automaton. Whether the dynamical model (3) could be a different quantum integrable system remains an open question.

We now focus our attention on the analytical solution of the two-particle sector of our QCA model. For $N = 2$ the Thirring quantum cellular automaton becomes

$$U_2 := W_2 V_2(\chi), \quad V_2(\chi) := e^{i\chi\delta_{y,0}(1-\delta_{a_1,a_2})}, \quad (4)$$

where we introduced the center-of-mass basis $|a_1, a_2\rangle|y\rangle|w\rangle$ for the two-particle Hilbert space $\mathcal{H}_2 = \mathbb{C}^4 \otimes \ell^2(\mathbb{Z})$, with $a_1, a_2 \in \{\uparrow, \downarrow\}$ and $y = x_1 - x_2$ and $w = x_1 + x_2$ the relative and the center-of-mass coordinate, respectively. Defining the (half) relative momentum as $k = \frac{1}{2}(p_1 - p_2)$ and the (half) total momentum as $p = \frac{1}{2}(p_1 + p_2)$, the free eigenstates W_2

in the momentum representation is written as

$$W_2 = \int dk dp W_2(p, k) \otimes |k\rangle\langle k| \otimes |p\rangle\langle p|, \\ W_2(p, k) \mathbf{v}_{p,k}^{sr} = e^{-i\omega_{sr}(p,k)} \mathbf{v}_{p,k}^{sr}, \quad \mathbf{v}_{p,k}^{sr} := \mathbf{v}_{p+k}^s \otimes \mathbf{v}_{p-k}^r, \\ \omega_{sr}(p, k) := s\omega(p+k) + r\omega(p-k), \quad s, r \in \{+, -\}, \quad (5)$$

where the eigenvectors of $W_2(p, k) := W_1(p+k) \otimes W_1(p-k)$ are easily computed as the tensor product of the single-particle eigenstates in Eq. (2).

Since the interacting dynamics U_2 commutes with translation T_w in the center-of-mass coordinate w , it is convenient to write the walk in the hybrid basis $|a_1, a_2\rangle|y\rangle|p\rangle$, in the block-diagonal form

$$U_2 = \int dp U_2(\chi, p) \otimes |p\rangle\langle p|, \quad U_2(\chi, p) := W_2(p) \tilde{V}_2(\chi), \\ W_2(p) := \mu\nu \begin{pmatrix} \frac{\nu}{\mu} e^{i2p} & -i e^{ip} T_y & -i e^{ip} T_y^\dagger & -\frac{\mu}{\nu} \\ -i e^{ip} T_y & \frac{\nu}{\mu} T_y^2 & -\frac{\mu}{\nu} & -i e^{-ip} T_y \\ -i e^{ip} T_y^\dagger & -\frac{\mu}{\nu} & \frac{\nu}{\mu} T_y^{\dagger 2} & -i e^{-ip} T_y^\dagger \\ -\frac{\mu}{\nu} & -i e^{-ip} T_y & -i e^{-ip} T_y^\dagger & \frac{\nu}{\mu} e^{-i2p} \end{pmatrix}, \\ \tilde{V}_2(\chi) := \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & e^{i\chi\delta_{y,0}} I & 0 & 0 \\ 0 & 0 & e^{i\chi\delta_{y,0}} I & 0 \\ 0 & 0 & 0 & I \end{pmatrix},$$

with T_y the translation in the relative coordinate y . Then, solving the two-particle dynamics means diagonalizing the infinite-dimensional operator $U_2(\chi, p)$. Typically, an infinite-dimensional eigenvalue problem cannot be analytically solved. However, we will show that the eigenvalue problem $U_2(\chi, p)|\psi\rangle = e^{i\omega}|\psi\rangle$ can be suitably reduced to a finite set of algebraic equations that can be analytically solved.

First, let us consider the linear difference equation

$$U_2(\chi, p) \mathbf{f}_{p,\omega,\chi} = e^{i\omega} \mathbf{f}_{p,\omega,\chi}, \\ \mathbf{f}_{p,\omega,\chi} : \mathbb{Z} \rightarrow \mathbb{C}^4, \quad \omega \in \mathbb{C} \quad (6)$$

for any possible value of χ and p . Among all the possible solutions of Eq. (6) we will then choose those which are eigenvectors (or generalized eigenvectors) of $U_2(\chi, p)$ considered as an operator on the Hilbert space $\mathbb{C}^4 \otimes \ell^2(\mathbb{Z})$.

Since the interacting particles are fermions, we are only interested in the solutions that are antisymmetric under the exchange of the two particles, i.e.,

$$\mathbf{f}_{p,\omega,\chi}(y) = -E \mathbf{f}_{p,\omega,\chi}(-y),$$

where E is represented as $E = \frac{1}{2} \sum_{i=0}^3 \sigma_i \otimes \sigma_i$ (with $\sigma_0 = I$ and $\sigma_i, i = 1, 2, 3$, the Pauli matrices). In the following, in order to simplify the notation, we will omit the explicit dependence of the solutions from p, ω, χ and we will write $\mathbf{f}(y)$ for $\mathbf{f}_{p,\omega,\chi}(y)$.

Since the interacting term acts only at the origin, for $y > 0$ Eq. (6) becomes a linear recurrence relation with constant coefficients whose most general solutions [35] are of two forms, $\mathbf{f}_\infty(y)$ or $\mathbf{f}(y)$, given by

$$\mathbf{f}_\infty(y) = (\zeta_\infty, 0, 0, \zeta'_\infty)^T \delta_{y,1}, \quad \zeta_\infty, \zeta'_\infty \in \mathbb{C}, \quad y > 0, \quad (7)$$

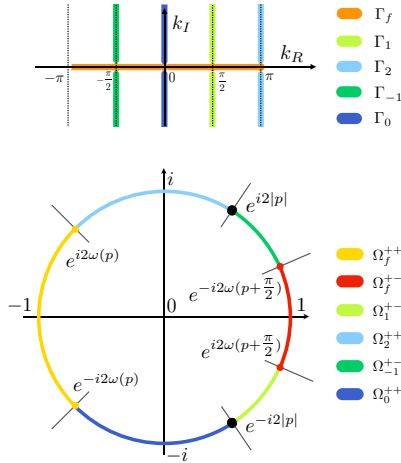


FIG. 1. The top plot shows k_R and k_I representing the real and the imaginary part of the relative momentum k in the two-fermion state. The highlighted regions collect the values of $k \in \mathbb{C}$ providing a real value of the quasienergy ω . On the bottom of the unit circle are the images under $k \mapsto e^{i\omega_{sr}(p,k)}$ of the disjoint regions in the top figure, for fixed values of the total momentum $p = 0.55$ and mass $\mu = 0.8$. Here Ω_f coincides with the continuous spectrum of $U_2(\chi, p)$ [see Eq. (4)]. The discrete spectrum lies in the other regions and for a fixed value of the coupling constant χ it consists of a single point. Varying the value of χ , the unit circle is covered and the boundary points of the arcs depend on p .

$$\mathbf{f}(y) = \sum_{s,r=\pm} \int_{\mathcal{S}} dk e^{-iky} g_{\omega}(s,r,k) \mathbf{v}_k^{sr}, \quad y > 0,$$

$$k = k_R + ik_I, \quad \mathcal{S} := \{k \in \mathbb{C} \mid k_R \in (-\pi, \pi]\},$$

$$g_{\omega}(s,r,k) \in \mathbb{C} : e^{-i\omega} \neq e^{-i\omega_{sr}(p,k)} \Rightarrow g_{\omega}(s,r,k) = 0, \quad (8)$$

where the function $\omega_{rs}(p,k)$ and the vectors $\mathbf{v}_k^{sr} := \mathbf{v}_{p,k}^{sr}$ have been defined in Eq. (5), and for a complex argument \arccos is the principal value of the arccosine function.

Let us first consider the functions given by Eq. (8). A necessary condition for a function obeying Eq. (8) to be a (proper or improper) eigenvector of $U_2(\chi, p)$ is that $\omega_{sr}(p,k) \in \mathbb{R}$. In order to analyze this condition, it is

By Lemma 1, Eq. (8) yields two classes of solutions:

$$\mathbf{f}_k^+(y) \quad \text{with } k \in \Gamma_f \cup \Gamma_0 \cup \Gamma_2,$$

$$\mathbf{f}_k^-(y) \quad \text{with } k \in \Gamma_f \cup \Gamma_{-1} \cup \Gamma_1,$$

$$\mathbf{f}_k^{\pm}(y) = \begin{cases} [\alpha_{\pm} \mathbf{v}_k^{\pm\pm} - (-1)^y \delta_{\pm} \mathbf{v}_{k-\pi}^{\mp\mp}] e^{-iyk} - [\beta_{\pm} \mathbf{v}_{-k}^{\pm\pm} - (-1)^y \gamma_{\pm} \mathbf{v}_{\pi-k}^{\mp\mp}] e^{iyk}, & y > 0 \\ (0, \eta_{\pm}, -\eta_{\pm}, 0)^T, & y = 0 \\ \text{antisymmetrized,} & y < 0, \end{cases} \quad (13)$$

where $\alpha_{\pm}, \beta_{\pm}, \dots$ are complex coefficients which depend on p, k, m, χ . We now determine these coefficients by requiring that Eq. (6) is satisfied. Because of the locality of the evolution, this constraint needs to be verified only for $y = 0, 1, 2$. A tedious albeit straightforward calculation allows one to put Eq. (13) into the following form after suitable reparametrization:

$$\mathbf{f}_k^{\pm}(y) = c_1 \mathbf{f}_k^{\pm, f}(y) + c_2 \mathbf{f}_k^{\pm, i}(y),$$

$$\mathbf{f}_k^{\pm, f}(y) = [\mathbf{v}_k^{\pm\pm} + (-1)^y \mathbf{v}_{k-\pi}^{\mp\mp}] e^{-iyk} - [\mathbf{v}_{-k}^{\pm\pm} + (-1)^y \mathbf{v}_{\pi-k}^{\mp\mp}] e^{iyk},$$

useful to introduce the following sets (see Fig. 1):

$$\Gamma_f := \{k \in \mathcal{S} \mid k_I = 0\},$$

$$\Gamma_z := \{k \in \mathcal{S} \mid k_R = z\frac{\pi}{2}\}, \quad z = 0, \pm 1, 2, \quad (9)$$

$$\Omega_f^{sr} := \{\exp[-i\omega_{sr}(p,k)] \mid k \in \Gamma_f\},$$

$$\Omega_z^{sr} := \{\exp[-i\omega_{sr}(p,k)] \mid k \in \Gamma_z, (-1)^z = sr\}. \quad (10)$$

Recalling that $\omega(x + \pi) = \pi - \omega(x)$, one can verify that

$$\Omega_f^{++} = \Omega_f^{--}, \quad \Omega_f^{+-} = \Omega_f^{-+}, \quad \Omega_0^{++} = \Omega_2^{--},$$

$$\Omega_2^{++} = \Omega_0^{--}, \quad \Omega_{-1}^{+-} = \Omega_1^{-+}, \quad \Omega_1^{+-} = \Omega_{-1}^{-+}. \quad (11)$$

The following technical result, proved in Appendix A, marks the first important difference from the Hamiltonian integrable models, relying on the degeneracy of two-particle levels. The degeneracy in the Hamiltonian case is 2, corresponding to the intuitive one-dimensional picture where either a classical elastic bounce or a quantum tunneling where the particles cross each other occurs. On the other hand, in the discrete case, due to periodicity of the quasienergy spectrum, the degeneration is 4. As we will see, this feature allows for scattering events where hopping to a distant region in the Brillouin zone can occur and for a wider family of bound states. From a technical point of view, the larger degeneracy of the two-particle levels is also responsible for the failure of the N -particle Bethe ansatz.

Lemma 1. Let $\omega_{sr}(p,k)$ be defined as in Eq. (5) and let us assume $p \neq z\frac{\pi}{2}$ ($z \in \mathbb{Z}$).

(a) Then we have

$$\omega_{\pm\pm}(p,k) \in \mathbb{R} \Rightarrow k \in \Gamma_f \cup \Gamma_0 \cup \Gamma_2,$$

$$\omega_{\pm\mp}(p,k) \in \mathbb{R} \Rightarrow k \in \Gamma_f \cup \Gamma_{-1} \cup \Gamma_1. \quad (12)$$

(b) The six sets $\Omega_f^{++}, \Omega_f^{+-}, \Omega_0^{++}, \Omega_2^{++}, \Omega_{-1}^{+-}, \Omega_1^{+-}$ are disjoint and their union is the whole unit circle except for the points $e^{\pm i2p}$.

(c) For any $\omega \in \mathbb{R}$ such that $e^{-i\omega} \neq e^{\pm i2p}$ the equation $e^{-i\omega} = e^{-i\omega_{sr}(p,k)}$ has four distinct solutions. If the triple $(+, +, k)$ is a solution then also $(+, +, -k)$, $(-, -, \pi - k)$, and $(-, -, k - \pi)$ are solutions. If the triple $(+, -, k)$ is a solution then $(-, +, -k)$, $(+, -, \pi - k)$, and $(-, +, k - \pi)$ are solutions.

$$\mathbf{f}_k^{\pm,i}(y) = \begin{cases} e^{-i\delta_y,0\chi} \{[\mathbf{v}_k^{\pm\pm} - (-1)^y \mathbf{v}_{k-\pi}^{\mp\mp}] e^{-iyk} - T_{\pm}[\mathbf{v}_{-k}^{\pm\pm} - (-1)^y \mathbf{v}_{\pi-k}^{\mp\mp}] e^{iyk}\}, & y \geq 0 \\ \text{antisymmetrized}, & y < 0, \end{cases}$$

$$T_{\pm} := \frac{g_+(p+k) + e^{-i\chi} g_{\pm}(p-k)}{g_{\pm}(p-k) + e^{-i\chi} g_+(p+k)}, \quad c_1, c_2 \in \mathbb{C}. \quad (14)$$

The next step of the analysis is to identify, among the set of functions of Eq. (14), those which correspond to eigenvectors or generalized eigenvectors of $U_2(\chi, p)$.

For $k \in \Gamma_f$, Eq. (14) gives the generalized eigenvector of $U_2(\chi, p)$ corresponding to the continuous spectrum $\sigma_c = \Omega_f^{++} \cup \Omega_f^{+-}$. Since $U_2 - W_2$ is a finite rank operator, the continuous spectrum of the two-particle interacting case is the same as that of the free walk (see Theorem IV 5.35 of Ref. [36]). From Eq. (14) we have that the solutions of the kind $\mathbf{f}_k^{\pm,f}$ are generalized eigenvectors of the free theory which are also generalized eigenvectors of the interacting theory. This is easily understood since $\mathbf{f}_k^{\pm,f}(0) = \mathbf{0}$ and therefore those eigensolutions are not affected by the presence of the interaction, which is localized at $y = 0$. On the other hand, we can interpret the solution of the kind $\mathbf{f}_k^{\pm,i}$ as a scattering of plane waves with the T_{\pm} playing the role of transmission coefficients. We notice that a scattering process involves four different values of the relative momentum between the two particles, namely, k , $-k$, $\pi - k$, and $k - \pi$. This is in sharp contrast to the scattering in the usual Thirring model in continuous space-time, where relative momentum can be only k or $-k$.

For $k \notin \Gamma_f$, necessary conditions for \mathbf{f}_k^{\pm} to be a (proper or generalized) eigenvector of $U_2(\chi, p)$ are that $k_I = \text{Im}(k) < 0$, $c_1 = 0$, $c_2 \neq 0$, and $T_{\pm} = 0$ (otherwise \mathbf{f}_k^{\pm} is exponentially divergent). In Appendix B we prove the following result.

Lemma 2. Let T_{\pm} be defined as in Eq. (14) and let us assume $p \neq z\frac{\pi}{2}$. If $e^{i\chi} \notin \{e^{\pm i2p}, 1, -1\}$, then there exists a unique $k \in \Gamma_0 \cup \Gamma_{-1} \cup \Gamma_1 \cup \Gamma_2$ with $k_I < 0$ such that either $T_+ = 0$ or $T_- = 0$. On the other hand, if $e^{i\chi} \in \{e^{\pm i2p}, 1, -1\}$ then $T_+ \neq 0$ and $T_- \neq 0$ for all $k \in \Gamma_0 \cup \Gamma_{-1} \cup \Gamma_1 \cup \Gamma_2$ with $k_I < 0$.

The above result tells us that, for $e^{i\chi} \notin \{e^{\pm i2p}, 1, -1\}$, the two-particle interacting evolution $U_2(\chi, p)$ has one proper eigenvector whose corresponding eigenvalue constitutes the discrete spectrum of $U_2(\chi, p)$. This eigenstate is easily interpreted as a bound state of two particles.

We now consider the functions given by Eq. (7) which lead to the antisymmetric functions

$$\mathbf{f}_{\infty}(y) = \begin{cases} (\zeta_{\infty}, 0, 0, \zeta'_{\infty})^T \delta_{y,1}, & y > 0 \\ (0, \eta_{\infty}, -\eta_{\infty}, 0)^T, & y = 0 \\ (-\zeta_{\infty}, 0, 0, -\zeta'_{\infty})^T \delta_{y,-1}, & y < 0. \end{cases} \quad (15)$$

Imposing the condition (6), we obtain the following solutions:

$$\mathbf{f}_{\pm\infty}(y) = \begin{cases} i e^{\pm ip} \left(-\frac{1\pm 1}{2}, 0, 0, -\frac{1\pm 1}{2}\right)^T \delta_{y,1}, & y > 0 \\ (0, \frac{\mu}{v}, -\frac{\mu}{v}, 0)^T, & y = 0 \\ i e^{\pm ip} \left(\frac{1\pm 1}{2}, 0, 0, -\frac{1\pm 1}{2}\right)^T \delta_{y,-1}, & y < 0, \end{cases}$$

$$U_2(\chi, p) \mathbf{f}_{\pm\infty} = e^{\pm i2p} \mathbf{f}_{\pm\infty} \text{ for } e^{i\chi} = e^{\pm i2p}. \quad (16)$$

Equation (16) provides the proper eigenstate of $U_2(\chi, p)$ for the cases $e^{i\chi} = e^{\pm i2p}$, which were missing in Lemma 2.

We can then write, for $p \neq z\frac{\pi}{2}$, the spectral resolution of $U_2(\chi, p)$, i.e.,

$$U_2(\chi, p) = \sum_{\substack{s=\pm, \\ j=f,i}} \int_{-\pi}^{\pi} dk e^{-i\omega_{\pm}(p,k)} |\phi_{p,\chi}^{s,j}(k)\rangle \langle \phi_{p,\chi}^{s,j}(k)| \\ + e^{-i\tilde{\omega}} |\varphi_{p,\chi}\rangle \langle \varphi_{p,\chi}|,$$

where we defined

$$\langle y | \phi_{p,\chi}^{s,j}(k) \rangle := N_{p,\chi,s,j,k} \mathbf{f}_k^{s,j}(y),$$

$$\langle y | \varphi_{p,\chi} \rangle := \begin{cases} M_{p,\chi,\tilde{k}} \mathbf{f}_{\tilde{k}}^{\pm,i}(y), & e^{i\chi} \neq e^{\pm i2p}, T_+(\tilde{k}) = 0 \\ M_{p,\chi,\tilde{k}} \mathbf{f}_{\tilde{k}}^{\mp,i}(y), & e^{i\chi} \neq e^{\pm i2p}, T_-(\tilde{k}) = 0 \\ M_p^{\pm} \mathbf{f}_{\pm\infty}(y), & e^{i\chi} = e^{\pm i2p}, \end{cases}$$

$$\tilde{\omega} := \begin{cases} \omega_{++}(p,\tilde{k}), & e^{i\chi} \neq e^{\pm i2p}, T_+(\tilde{k}) = 0 \\ \omega_{+-}(p,\tilde{k}), & e^{i\chi} \neq e^{\pm i2p}, T_-(\tilde{k}) = 0 \\ \pm 2p, & e^{i\chi} = e^{\pm i2p}, \end{cases}$$

and N and M are normalization factors such that

$$\langle \phi_{p,\chi}^{s,j}(k) | \phi_{p,\chi}^{s',j'}(k') \rangle = \delta_{s,s'} \delta_{j,j'} \delta(k - k'),$$

$$\langle \varphi_{p,\chi} | \varphi_{p,\chi} \rangle = 1.$$

We conclude our analysis with the discussion of the cases $p = z\frac{\pi}{2}$ starting from $p = 0$. We have $\omega_{\pm\pm}(0,k) = \pm 2\omega(k)$, with $\omega(k) \in (-\pi, \pi]$ and $\omega(k) \neq 0$, if and only if $k \in \Gamma_f \cup \Gamma_0 \cup \Gamma_2$. On the other hand, $\omega_{\pm\mp}(0,k) = 0$ for all $k \in \mathbb{C}$ and thus $\omega_{\pm\mp}(0,k) \neq \omega_{\pm\pm}(0,k')$ for all values of k and k' . Therefore, the previous analysis still holds for $e^{-i\omega} \neq 1$ and, by setting $p = 0$, the solutions \mathbf{f}_k^{\pm} of Eq. (14) are (proper and improper) eigenvectors of $U_2(\chi, 0)$. Thus, the spectrum of $U_2(\chi, 0)$ decomposes into a continuous spectrum, which is the arc of the unit circle which contains -1 and has $e^{\pm 2i\omega(0)}$ as extremes, and a point spectrum made of two distinct points $e^{-i2\omega(\tilde{k})}$ (where \tilde{k} is the solution of $T_+ = 0$ when $p = 0$) and 1 . Since $U_2(\chi, 0)$ is unitary, if $e^{-i\omega}$ belongs to the point spectrum then it is a proper eigenvalue of $U_2(\chi, 0)$. Let us denote by P_0^- the projection on the eigenspace of the eigenvalue 1 and by P_p^- the projection

$$P_p^s := \sum_{j=f,i} \int_{-\pi}^{\pi} dk |\phi_{p,\chi}^{s,j}(k)\rangle \langle \phi_{p,\chi}^{s,j}(k)|.$$

Now, since $\lim_{p \rightarrow 0} \|U_2(\chi, p) - U_2(\chi, 0)\| = 0$ and 1 is a separate part of the spectrum of $U_2(\chi, 0)$, then $\lim_{p \rightarrow 0} \|P_p^- - P_0^-\| = 0$ (see Theorem IV 3.16 of Ref. [36]). We have then

that

$$P_0^- = \sum_{\substack{n \in \mathbb{Z} \\ j=f,i}} |\psi_0^-(n,j)\rangle \langle \psi_0^-(n,j)|,$$

$$|\psi_0^-(n,j)\rangle := \int_{-\pi}^{\pi} g_n(k) |\phi_{0,\chi}^-(k)\rangle,$$

where $g_n(k)$ is an orthonormal basis for $L^2(-\pi, \pi]$. The cases $p = \pi, \pm \frac{\pi}{2}$ can be analyzed in the same way. The eigenspace corresponding to the eigenvalue 1 is thus a separable Hilbert space of stationary bound states. This result marks an important departure from the behavior of analogous Hamiltonian models. Remarkably, it occurs even in the noninteracting case $\chi = 0$.

The diagonalization of $U_2(\chi, p)$ is summarized by the following proposition.

Proposition 1. Let $U_2(\chi, p)$ be defined as in Eq. (5). Then its spectral resolution is

$$U_2(\chi, p) = \begin{cases} \sum_{\substack{s=\pm, \\ j=f,i}} U_{p,\chi}^{s,j} + e^{-i\tilde{\omega}} P_{p,\chi}, & p \neq z \frac{\pi}{2} \\ \sum_{j=f,i} U_{z\pi,\chi}^{+,j} + P_{z\pi}^- + e^{-i\tilde{\omega}} P_{z\pi,\chi}, & p = 0, \pi, \\ \sum_{j=f,i} U_{\pm\pi/2,\chi}^{-,j} - P_{\pm\pi/2}^+ + e^{-i\tilde{\omega}} P_{\pm\pi/2,\chi}, & p = \pm \frac{\pi}{2}, \end{cases}$$

$$U_{p,\chi}^{s,j} := \int_{-\pi}^{\pi} dk e^{-i\omega_{\pm}(p,k)} |\phi_{p,\chi}^{s,j}(k)\rangle \langle \phi_{p,\chi}^{s,j}(k)|,$$

$$P_{p,\chi} = |\varphi_{p,\chi}\rangle \langle \varphi_{p,\chi}|.$$

In Fig. 2 we plot the spectrum of $U_2(\chi, p)$ as a function of p for different values of χ .

We introduced a fermionic quantum cellular automaton with a number-preserving on-site interaction. The model, which we named Thirring quantum cellular automaton, is given by the product of a free evolution, which in a suitable limit recovers the Dirac equation, and an interacting term. We analytically diagonalized the two-particle sector. Differently from analogous Hamiltonian models, (i) bound states exist for

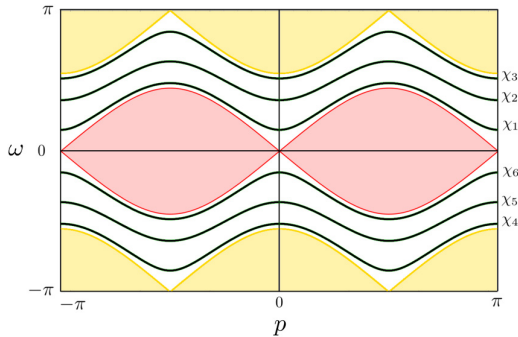


FIG. 2. Spectrum of the two-particle automaton of Eq. (4): The continuous spectrum bands are depicted in red (inner band) and yellow (outer band); depicted in black is the discrete band for different values of the coupling: $\chi_1 = -\frac{\pi}{5}$, $\chi_2 = -\frac{\pi}{2}$, $\chi_3 = -\frac{4\pi}{5}$, $\chi_4 = \frac{4\pi}{5}$, $\chi_5 = \frac{\pi}{2}$, and $\chi_6 = \frac{\pi}{5}$.

every value of the total momentum, (ii) there are four classes of scattering solutions instead of two, and (iii) the bound states exist also in the free case.

Another striking consequence of our result is that the Thirring model in the continuum cannot be straightforwardly recovered from the Thirring quantum cellular automaton. This is in sharp contrast to the free case, in which the low-momentum limit of the quantum automaton recovers the Dirac equation in the continuum. When interactions are turned on, the discreteness of the time evolution introduces physical phenomena that are prevented in the continuous-time counterpart. For example, a scattering process in which the relative momentum changes from k to $\pi - k$ is possible within the Thirring quantum cellular automaton evolution but cannot be recovered within the framework of the usual Thirring model in continuous space-time.

Finally, because of the distinctive features of the two particle dynamics, the usual Bethe ansatz solution does not apply and the problem of finding a general solution for the N -particle case is an open line of research.

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APPENDIX A: PROOF OF LEMMA 1

1. Proof of item (a)

Let us define $\hat{\omega}_{\pm} + i\tilde{\omega}_{\pm} := \omega(p \pm k)$. Since $\omega(z^*) = \omega^*(z)$, we have that both $\hat{\omega}_{\pm}$ and $\tilde{\omega}_{\pm}$ are real. Then we have

$$\text{Im}[\omega_{r,s}(p,k)] = 0 \iff r\tilde{\omega}_+ = -s\tilde{\omega}_-$$

$$\Rightarrow \cosh \tilde{\omega}_+ = \cosh \tilde{\omega}_- =: \cosh \tilde{\omega}. \quad (\text{A1})$$

Recalling that $\cos \omega(p \pm k) = v \cos(p \pm k)$, Eq. (A1) implies that

$$\cos^2 \hat{\omega}_{\pm} \cosh^2 \tilde{\omega} = v^2 \cos^2(p \pm k_R) \cosh^2 k_I,$$

$$\sin^2 \hat{\omega}_{\pm} \sinh^2 \tilde{\omega} = v^2 \sin^2(p \pm k_R) \sinh^2 k_I. \quad (\text{A2})$$

From the above relations we find that

$$\cos^2(p \pm k_R) \frac{\cosh^2 k_I}{\cosh^2 \tilde{\omega}} + \sin^2(p \pm k_R) \frac{\sinh^2 k_I}{\sinh^2 \tilde{\omega}} = \frac{1}{v^2}, \quad (\text{A3})$$

which gives

$$[\sin^2(p + k_R) - \sin^2(p - k_R)] \left(\frac{\sinh^2 k_I}{\sinh^2 \tilde{\omega}} - \frac{\cosh^2 k_I}{\cosh^2 \tilde{\omega}} \right) = 0.$$

Now, since $\frac{\sinh^2 k_I}{\sinh^2 \tilde{\omega}} - \frac{\cosh^2 k_I}{\cosh^2 \tilde{\omega}} = 0$ implies $\frac{\sinh^2 k_I}{\sinh^2 \tilde{\omega}} = \frac{\cosh^2 k_I}{\cosh^2 \tilde{\omega}} = 1$, which is not compatible with Eq. (A3), it must be $\sin^2(p + k_R) = \sin^2(p - k_R)$, which gives

$$k_R = \frac{z}{2}\pi \vee p = \frac{z}{2}\pi, \quad z \in \mathbb{Z}.$$

By explicit computation one obtains

$$\begin{aligned} \operatorname{Im}[\omega_{\pm\pm}(p,k)] = 0 \wedge k_I \neq 0 &\Rightarrow k_R = 0, \quad \pi \vee p = \pm \frac{\pi}{2} \\ \operatorname{Im}[\omega_{\pm\mp}(p,k)] = 0 \wedge k_I \neq 0 &\Rightarrow k_R = \pm \frac{\pi}{2}, \quad \vee p = 0, \pi, \end{aligned}$$

which proves the first item of Lemma 1.

2. Proof of item (b)

Let us consider the case in which $p \in (0, \frac{\pi}{2})$. The function $k \mapsto \omega_{++}(p,k)$ is smooth and periodic with period 2π and therefore it ranges between its maximum and minimal values. The maximum and minimum values are found by setting $\partial_k \omega_{++}(p,k) = 0$. By explicit computation one obtains

$$\frac{\sin(p+k)}{\sqrt{1-v^2 \cos^2(p+k)}} = \frac{\sin(p-k)}{\sqrt{1-v^2 \cos^2(p-k)}},$$

which implies, for $p \neq z\frac{\pi}{2}$, that $k = 0, \pi$. We have then that $\omega_{++}(p,k)$ ranges between $2\omega(p)$ and $2\pi - 2\omega(p)$. By noting that $\omega_{++}(p, \frac{\pi}{2}) = \pi$ we have that Ω_f^{++} is the arc which connects $e^{i2\omega(p)}$ and $e^{-i2\omega(p)}$ and which includes -1 (see Fig. 1). With the same procedure we find that Ω_f^{+-} is the arc connecting $e^{i[2\omega(p+\pi/2)-\pi]}$ and $e^{-i[2\omega(p+\pi/2)-\pi]}$ which includes 1 (see Fig. 1). We now verify that Ω_f^{++} and Ω_f^{+-} are disjoint. Since $\omega(p) < \frac{\pi}{2}$ and $\omega(p + \frac{\pi}{2}) > \frac{\pi}{2}$ we have $e^{-i\omega} \in \Omega_f^{++}$ if and only if $\omega \bmod 2\pi \in (-\pi, -2\omega(p)] \cup [2\omega(p), \pi]$ and $e^{-i\omega} \in \Omega_f^{+-}$ if and only if $\omega \bmod 2\pi \in [\pi - 2\omega(p + \frac{\pi}{2}), 2\omega(p + \frac{\pi}{2}) - \pi]$. Then, from the inequality $|\frac{d}{dx}\omega(x)| < 1 \forall x \in \mathbb{R}$ we have

$$\omega\left(p + \frac{\pi}{2}\right) - \omega(p) = \int_p^{p+\pi/2} dx \frac{d}{dx}\omega(x) < \int_p^{p+\pi/2} dx < \frac{\pi}{2},$$

which implies that the sets $(-\pi, -2\omega(p)] \cup [2\omega(p), \pi]$ and $[\pi - 2\omega(p + \frac{\pi}{2}), 2\omega(p + \frac{\pi}{2}) - \pi]$ are disjoint.

Let us now consider the set Ω_0^{++} . For $\pi \neq 0, \pi$, the function $\mathbb{R} \ni k_I \mapsto \omega_{++}(p, ik_I) = \omega(p + ik_I) + \omega(p - ik_I)$ is smooth. Therefore, the extremal points of its range occur either in its stationary points or at its limiting values for $k_I \rightarrow \pm\infty$. By setting $\partial_{k_I} \omega_{++}(p, ik_I) = 0$ we obtain

$$\begin{aligned} \frac{\sin(p+ik_I)}{\sqrt{1-v^2 \cos^2(p+ik_I)}} &= \frac{\sin(p-ik_I)}{\sqrt{1-v^2 \cos^2(p-ik_I)}} \\ \Rightarrow \sin^2(p+ik_I) &= \sin^2(p-ik_I) \\ \Rightarrow \sin(p+ik_I) &= \pm \sin(p-ik_I) \Rightarrow k_I = 0, \end{aligned}$$

where we used the hypothesis $p \neq z\frac{\pi}{2}$. When $k_I = 0$ we clearly have $\omega_{++}(p, 0) = 2\omega(p)$. Let us now compute $\lim_{k_I \rightarrow +\infty} \omega_{++}(p, ik_I)$. Since $\omega_{++}(p, ik_I)$ is an even function of k_I the limits $k_I \rightarrow +\infty$ and $k_I \rightarrow -\infty$ coincide. We have

$$\begin{aligned} \omega_{++}(p, ik_I) &= \omega(p + ik_I) + \omega(p - ik_I) \\ &= 2\operatorname{Re} \omega(p + ik_I) = 2\operatorname{Re} \arccos[v \cos(p + ik_I)] \\ &= 2\operatorname{Re} \arccos(v \cos p \cosh k_I - i \sin p \cosh k_I) \\ &= 2 \arccos 2^{-1} [\sqrt{(1 + \cos p \cosh k_I)^2 + \sin^2 p \cosh^2 k_I} \\ &\quad - \sqrt{(1 - \cos p \cosh k_I)^2 + \sin^2 p \cosh^2 k_I}] \\ &\xrightarrow{k_I \rightarrow +\infty} 2 \arccos \cos p = 2|p|. \end{aligned}$$

Since we are assuming $p \in (0, \frac{\pi}{2})$ we have that

$$\begin{aligned} \frac{d}{dp}[\omega(p) - p] &= \frac{d}{dp}\omega(p) - 1 < 0, \\ \omega(0) > 0, \quad \omega\left(\frac{\pi}{2}\right) &= \frac{\pi}{2}, \end{aligned}$$

which imply $\omega(p) - p > 0$ for $p \in (0, \frac{\pi}{2})$. Similarly, one can show $\omega(p + \frac{\pi}{2}) - \frac{\pi}{2} < p$ for $p \in (0, \frac{\pi}{2})$. From $\omega(p + \frac{\pi}{2}) - \frac{\pi}{2} < p < \omega(p)$ we have that $e^{-i\omega} \in \Omega_0^{++}$ if and only if $\omega \bmod 2\pi \in (-2\omega(p), -2p)$. Moreover, we have that $e^{-i\omega} \in \Omega_2^{++}$ if and only if $e^{i\omega} \in \Omega_0^{++}$ if and only if $\omega \bmod 2\pi \in (2p, 2\omega(p))$. This proves that, for $p \in (0, \frac{\pi}{2})$, Ω_0^{++} , Ω_f^{++} , Ω_f^{+-} , and Ω_2^{++} are disjoint sets (see Fig. 1). Following the same derivation it is easy to show that $e^{-i\omega} \in \Omega_1^{+-}$ if and only if $\omega \bmod 2\pi \in (-2p, \pi - 2\omega(p + \frac{\pi}{2}))$ and $e^{-i\omega} \in \Omega_{-1}^{+-}$ if and only if $\omega \bmod 2\pi \in (2\omega(p + \frac{\pi}{2}) - \pi, 2p)$, which proves item (b) of Lemma 1 for $p \in (0, \frac{\pi}{2})$ (see Fig. 1). The same line of derivation can be followed for the cases $p \in (-\frac{\pi}{2}, 0)$, $p \in (\frac{\pi}{2}, \pi)$, and $p \in (-\pi, -\frac{\pi}{2})$, thus completing the proof.

3. Proof of item (c)

Let us consider a value $e^{-i\omega} \neq e^{\pm i2p}$. From item (b) of Lemma 1 we have that the sets $\Omega_f^{++}, \Omega_f^{+-}, \Omega_0^{++}, \Omega_2^{++}, \Omega_{-1}^{+-}, \Omega_1^{+-}$ cover the whole unit circle except the points $e^{\pm i2p}$ and therefore $e^{-i\omega}$ must belong to one of those sets. We prove the thesis for the case $e^{-i\omega} \in \Omega_f^{++}$ and the remaining cases can be proved in the same way. If $e^{-i\omega} \in \Omega_f^{++}$, then there exists $k \in \Gamma_f$ such that $\omega_{++}(p, k) = \omega \bmod 2\pi$. By direct computation one can verify that also $\omega_{++}(p, -k) = \omega_{--}(p, k - \pi) = \omega_{--}(p, \pi - k) \bmod 2\pi = \omega$. In order to prove that these are the only admissible solutions we must check that $k' \neq \pm k$ implies $\omega_{++}(p, k') \neq \omega \bmod 2\pi$. By contradiction let us suppose that there exists $k' \neq \pm k$ such that $\omega_{++}(p, k') = \omega \bmod 2\pi$. This clearly implies $\omega_{++}(p, k') = \omega_{++}(p, k')$ since the range of ω_{++} is smaller than 2π . Let us consider the case $0 < k' < k$. Since ω_{++} is smooth, there must exist k'' such that $k' < k'' < k$ and $[\frac{d}{dk}\omega_{++}](p, k'') = 0$. By direct computation one proves that this is impossible. The generalization to the cases $-k < k' < 0$, $k < k' < \pi$, and $-\pi < k' < k$ is straightforward. The analysis of the cases $k' = 0, \pi$ is easily done by direct computation.

APPENDIX B: PROOF OF LEMMA 2

In order to prove the lemma it is convenient to introduce the following function from the negative half line $k_I \in (-\infty, 0]$ to the unit circle S^1 :

$$G_z : \mathbb{R}^- \rightarrow S^1, \quad G_z(k_I) = -\frac{A_z(p, k)}{A_z^*(p, k)}, \quad j = 0, 2, \pm 1$$

$$A_0(p, k) = \sin[\omega(p - ik_I)] + v \sin(p - ik_I),$$

$$A_2(p, k) = \sin[\omega(p - ik_I)] - v \sin(p - ik_I),$$

$$A_{\pm 1}(p, k) = \sin\left(\omega\left(p \mp \frac{\pi}{2} - ik_I\right)\right) + v \sin\left(p \mp \frac{\pi}{2} - ik_I\right).$$

The above function allows us to study the constraint $T_{\pm}(p, k, \chi) = 0$ through the following five properties: (i)

$$k \in \Gamma_0 : T_+(p, k, \chi) = 0 \iff G_0(k_I) = e^{i\chi},$$

$$k \in \Gamma_2 : T_+(p, k, \chi) = 0 \iff G_2(k_I) = e^{i\chi},$$

$$k \in \Gamma_{\pm 1} : T_-(p, k, \chi) = 0 \iff G_{\pm 1}(k_I) = e^{i\chi};$$

(ii)

$$G_0(0) = G_2(0) = -1, \quad G_{\pm 1}(0) = 1;$$

(iii)

$$G_0(k_I) \neq 1, \quad G_2(k_I) \neq 1, \quad G_{\pm 1}(k_I) \neq -1 \quad \forall k_I;$$

(iv)

$$\frac{d}{dk_I} G_z(k_I) \neq 0 \quad \forall k_I, \quad z = 0, 2, \pm 1;$$

and (v)

$$\lim_{k_I \rightarrow -\infty} G_0(k_I) = e^{i2|p|},$$

$$\lim_{k_I \rightarrow -\infty} G_2(k_I) = e^{-i2|p|},$$

$$\lim_{k_I \rightarrow -\infty} G_{\pm 1}(k_I) = \begin{cases} e^{\pm i2p}, & p \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ e^{\mp i2p}, & p \in \left(-\pi, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right). \end{cases}$$

The statement of the lemma is now proved, noticing that the functions $G_z(k_I)$, $z = 0, 2, \pm 1$, from the negative half line $k_I \in (-\infty, 0]$ to the unit circle S^1 , are injective and their ranges are the range of G_0 , the smallest arc having -1 and $e^{i2|p|}$ as its extremal points; the range of G_2 , the smallest arc having -1 and $e^{-i2|p|}$ as its extremal points; and the range of $G_{\pm 1}$, the smallest arc having 1 and (depending on the value of p) $e^{\pm i2|p|}$ as its extremal points.

Here we provide the proofs of items (i)–(v) for the case $k \in \Gamma_0$. The proof for the other three cases $k \in \Gamma_2$ and $k \in \Gamma_{\pm 1}$ is almost identical.

1. Proof of item (i) for $k \in \Gamma_0$

If $k \in \Gamma_0$, $k = ik_I$, and starting from the definition of Eq. (14) we can rewrite $T_+(p, k, \chi)$ as

$$T_+ = \frac{A_0^*(p, k) + e^{-i\chi} A_0(p, k)}{e^{-i\chi} A_0^*(p, k) + A_0(p, k)}.$$

Let us replace $A_0(p, k)$ with A and $T_+(p, k, \chi)$ with T_+ in order to simplify the notation. We have that $T_+ = 0 \iff A^* + e^{-i\chi} A = 0 \wedge e^{-i\chi} A^* + A \neq 0$. First we observe that $A \neq 0$ and indeed $A = 0 \iff \sin[\omega(p - ik)] = -n \sin(p - ik) \Rightarrow \sin^2[\omega(p - ik)] = n^2 \sin^2(p - ik) \Rightarrow v^2 = 1$, which is not an admissible value. Accordingly, a straightforward computation shows that $T_+ = 0 \iff A^* + e^{-i\chi} A = 0 \wedge e^{-i\chi} A^* + A = 0$ if and only if $\chi = m\pi$ ($m \in \mathbb{Z}$). However, $T_+(p, k, m\pi) = (-1)^m \neq 0$ and we conclude that $T_+ = 0$ if and only if $A^* + e^{-i\chi} A = 0$, which proves item (i).

2. Proof of item (ii) for $k \in \Gamma_0$

Notice that $G_0(k_I) = -1 \Rightarrow A = A^*$, which implies $\text{Im}[A] = 0 \Rightarrow \text{Im}[A^2] = 0$ [where we replaced $A_0(p, k)$ with A in order to simplify the notation]. Since we have $A^2 = 1 -$

$v^2 + 2v \sin(p - ik_I)A$ the condition $\text{Im}[A] = 0 \wedge \text{Im}[A^2] = 0$ implies $\text{Im} \sin(p - ik_I) = 0$, that is, $\cos(p) \sinh(k_I) = 0$, and then $k_I = 0 \vee p = \frac{\pi}{2} + m\pi$ ($m \in \mathbb{Z}$), which proves item (ii).

3. Proof of item (iii) for $k \in \Gamma_0$

We have $G_0(k_I) = 1 \Rightarrow A = -A^*$, which implies $\text{Re}[A] = 0 \Rightarrow \text{Im}[A^2] = 0$ [where we replaced $A_0(p, k)$ with A in order to simplify the notation]. Since $A^2 = 1 - v^2 + 2v \sin(p - ik_I)A$ the condition $\text{Re}[A] = 0 \wedge \text{Im}[A^2] = 0$ implies $\text{Re} \sin(p - ik_I) = 0$, that is, $\sin(p) \cosh(k_I) = 0$. Since the last equality is satisfied only for $p = m\pi$ ($m \in \mathbb{Z}$), which are not admissible values of p , item (iii) is proved.

4. Proof of item (iv) for $k \in \Gamma_0$

We prove that $\frac{d}{dk_I} G_0(k_I) = 0 \Rightarrow p = m\frac{\pi}{2}$ ($m \in \mathbb{Z}$), which are not admissible values of p . Again we replace $A_0(p, k)$ with A in order to simplify the notation. Consider

$$\frac{d}{dk_I} G_0(k_I) = \frac{A'A^* - AA'^*}{(A^*)^2},$$

where $A' = \partial_{k_I} A$ and $A'^* := \partial_{k_I} A^* = A'^*$. Then, recalling that $A \neq 0$ [see the proof of item (i)] and noting that

$$A'A^* - AA'^* = -i|1 + \omega'_-|^2 \sin(\omega_- + \omega_+),$$

$$\omega_{\pm} := \omega(p \pm ik), \quad \omega'(x) := \frac{d}{dx} \omega(x)$$

(this can be verified by rewriting A as $A = \sin[\omega(p - ik)][1 + \omega'(p - ik)]$), one has

$$\frac{d}{dk_I} G_0(k_I) = 0 \iff 1 + \omega'_- = 0 \vee \sin(\omega_- + \omega_+) = 0.$$

Let us investigate the two possible cases. In the first case $1 + \omega'_- = 0$ it must be

$$\omega'_-{}^2 = \frac{v^2 \sin^2(p - ik_I)}{\sin^2[\omega(p - ik_I)]} = 1 \Rightarrow v = 1,$$

which is not admissible. Let us now consider the case $\sin(\omega_- + \omega_+) = 0$, that is, $\omega_- + \omega_+ = m\pi$. We have for m even $\cos \omega_- = \cos \omega_+ \Rightarrow \sin p = 0 \vee k_I = 0$. On the other hand, if m is odd we have $\cos \omega_- = -\cos \omega_+ \Rightarrow \cos p = 0$. Item (iv) is thus proved.

5. Proof of item (v) for $k \in \Gamma_0$

For convenience in the following we replace $A_0(p, k)$ with A . First we rewrite the function G_0 as

$$G_0(k_I) = -\frac{Z}{Z^*},$$

$$Z := -iA = e^{-i\omega(p - ik_I)} - v e^{i(p - ik_I)}.$$

Recalling that in this case $k \in \Gamma_0$, $k = k_R + ik_I$ with $k_R = 0$, from Appendix C we have the expressions of $e^{-i\omega(p - ik_I)}$ for $k_I \rightarrow -\infty$,

$$p > 0 \Rightarrow e^{-i\omega(p - ik_I)} = \frac{1}{v} e^{-ip} e^{k_I},$$

$$p < 0 \Rightarrow e^{-i\omega(p - ik_I)} = v e^{ip} e^{-k_I} - \frac{\mu^2}{v} e^{-ip} e^{k_I},$$

from which item (v) follows.

APPENDIX C: ASYMPTOTIC BEHAVIOR OF THE WALK EIGENVALUES

The one-particle Dirac walk in momentum space is defined through the matrix valued function of Eq. (2). Since $U(p) \in \mathbb{S}\mathbb{U}(2)$, its eigenvalues are $e^{-i\omega(p)}$ and $e^{i\omega(p)}$, where $\omega(p)$ is the solution of the equation $\cos \omega = \nu \cos p$ with positive value. Then we write

$$\omega : (-\pi, \pi] \rightarrow [0, \pi], \quad p \mapsto \omega(p) = \arccos(\nu \cos p) \geq 0.$$

For our purposes it is convenient to consider the analytic continuation of $U(p)$ to the subset $\mathcal{S} := \{p \in \mathbb{C} \mid \operatorname{Re}(p) = p_R \in (-\pi, \pi], \operatorname{Im}(p) = p_I \leq 0\}$ of the complex plane. The eigenvalues of $U(p)$, with $p \in \mathcal{S}$, are $e^{-i\omega(p)}$ and $e^{i\omega(p)}$, where now $\omega(p) = \arccos(\nu \cos p)$, with \arccos denoting the principal value of the multivalued analytic function \arccos . We note that $\arg(e^{i\omega(p)}) = \operatorname{Re}[\omega(p)] = \operatorname{Re}[\arccos(\nu \cos p)] \in [0, \pi]$.

In the two-particles case we introduced the center-of-mass coordinates p and k , representing, respectively, the total and the relative momentum. While p is always real, k can have an imaginary part. Let us study the eigenvalues of $U(p-k)$ in

the limit $k_I \rightarrow -\infty$. We have

$$U(p-k) = \nu e^{i(p-k_R)} e^{-k_I} \begin{pmatrix} e^{2k_I} & -i \frac{\mu}{\nu} e^{-i(p-k_R)} e^{k_I} \\ -i \frac{\mu}{\nu} e^{-i(p-k_R)} e^{k_I} & e^{-i2(p-k_R)} \end{pmatrix},$$

and denoting by λ'_1 and λ'_2 the two eigenvalues of $\nu^{-1} e^{-i(p-k_R)} e^{k_I} D(p-k)$ we have, for $k_I \rightarrow -\infty$,

$$\lambda'_1 = 1 - \frac{\mu^2}{\nu^2} e^{-2i(p-k_R)} e^{2k_I}, \quad \lambda'_2 = \frac{1}{\nu^2} e^{-2i(p-k_R)} e^{2k_I}.$$

Accordingly, for $k_I \rightarrow -\infty$, the eigenvalues λ_1 and λ_2 of $D(p-k)$ are

$$\lambda_1 = \nu e^{i(p-k_R)} e^{-k_I} - \frac{\mu^2}{\nu} e^{-i(p-k_R)} e^{k_I},$$

$$\lambda_2 = \frac{1}{\nu} e^{-i(p-k_R)} e^{k_I},$$

and noting that $\lim_{k_I \rightarrow -\infty} \arg(\lambda_1) = p - k_R$ and $\lim_{k_I \rightarrow -\infty} \arg(\lambda_2) = -(p - k_R)$ we get

$$p - k_R > 0 \Rightarrow e^{-i\omega(p-k)} = \lambda_2, \quad e^{i\omega(p-k)} = \lambda_1$$

$$p - k_R < 0 \Rightarrow e^{-i\omega(p-k)} = \lambda_1, \quad e^{i\omega(p-k)} = \lambda_2.$$

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- [1] C. M. Dawson, J. Eisert, and T. J. Osborne, *Phys. Rev. Lett.* **100**, 130501 (2008).
- [2] B. M. Boghosian and W. Taylor, *Phys. Rev. E* **57**, 54 (1998).
- [3] J. I. Cirac, D. Perez-Garcia, N. Schuch, and F. Verstraete, *J. Stat. Mech.* (2017), 083105.
- [4] A. M. Childs, *Phys. Rev. Lett.* **102**, 180501 (2009).
- [5] N. B. Lovett, S. Cooper, M. Everitt, M. Trevers, and V. Kendon, *Phys. Rev. A* **81**, 042330 (2010).
- [6] A. Schreiber, K. N. Cassemiro, V. Potocek, A. Gabris, P. J. Mosley, E. Andersson, I. Jex, and C. Silberhorn, *Phys. Rev. Lett.* **104**, 050502 (2010).
- [7] L. Sansoni, F. Sciarrino, G. Vallone, P. Mataloni, A. Crespi, R. Ramponi, and R. Osellame, *Phys. Rev. Lett.* **108**, 010502 (2012).
- [8] I. Bialynicki-Birula, *Phys. Rev. D* **49**, 6920 (1994).
- [9] D. Meyer, *J. Stat. Phys.* **85**, 551 (1996).
- [10] A. Bisio, G. M. D'Ariano, and A. Tosini, *Phys. Rev. A* **88**, 032301 (2013).
- [11] P. Arrighi, S. Facchini, and M. Forets, *Quantum Inf. Process.* **15**, 3467 (2016).
- [12] G. M. D'Ariano and P. Perinotti, *Phys. Rev. A* **90**, 062106 (2014).
- [13] R. Feynman, *Int. J. Theor. Phys.* **21**, 467 (1982).
- [14] B. Schumacher and R. Werner, [arXiv:quant-ph/0405174](https://arxiv.org/abs/quant-ph/0405174).
- [15] D. Gross, V. Nesme, H. Vogts, and R. Werner, *Commun. Math. Phys.* **310**, 419 (2012).
- [16] A. Ambainis, E. Bach, A. Nayak, A. Vishwanath, and J. Watrous, in *Proceedings of the 33rd Annual ACM Symposium on Theory of Computing* (ACM, New York, 2001), pp. 37–49.
- [17] Y. Aharonov, L. Davidovich, and N. Zagury, *Phys. Rev. A* **48**, 1687 (1993).
- [18] G. Grössing and A. Zeilinger, *Complex Syst.* **2**, 197 (1988).
- [19] Y. Shikano, K. Chisaki, E. Segawa, and N. Konno, *Phys. Rev. A* **81**, 062129 (2010).
- [20] C. M. Chandrashekar, R. Srikanth, and S. Banerjee, *Phys. Rev. A* **76**, 022316 (2007).
- [21] P. Arnault and F. Debbasch, *Phys. Rev. A* **93**, 052301 (2016).
- [22] P. Arnault, G. Di Molfetta, M. Brachet, and F. Debbasch, *Phys. Rev. A* **94**, 012335 (2016).
- [23] D. A. Meyer, *Phys. Rev. E* **55**, 5261 (1997).
- [24] A. Ahlbrecht, A. Alberti, D. Meschede, V. B. Scholz, A. H. Werner, and R. F. Werner, *New J. Phys.* **14**, 073050 (2012).
- [25] C. Navarrete-Benlloch, A. Pérez, and E. Roldán, *Phys. Rev. A* **75**, 062333 (2007).
- [26] C.-W. Lee, P. Kurzyński, and H. Nha, *Phys. Rev. A* **92**, 052336 (2015).
- [27] E. H. Lieb and F. Y. Wu, *Phys. Rev. Lett.* **20**, 1445 (1968).
- [28] S. Coleman, *Phys. Rev. D* **11**, 2088 (1975).
- [29] V. E. Korepin, *Theor. Math. Phys.* **41**, 953 (1979).
- [30] F. H. Essler, H. Frahm, F. Göhmann, A. Klümper, and V. E. Korepin, *The One-Dimensional Hubbard Model* (Cambridge University Press, Cambridge, 2005).
- [31] J. Hubbard, *Proc. R. Soc. London Ser. A* **276**, 238 (1963).
- [32] W. E. Thirring, *Ann. Phys. (NY)* **3**, 91 (1958).
- [33] A. Bisio, G. M. D'Ariano, and A. Tosini, *Ann. Phys. (NY)* **354**, 244 (2015).
- [34] S. Östlund and E. Mele, *Phys. Rev. B* **44**, 12413 (1991).
- [35] P. Cull, M. Flahive, and R. Robson, *Difference Equations: From Rabbits to Chaos* (Springer Science + Business Media, Berlin, 2005).
- [36] T. Kato, *Perturbation Theory for Linear Operators* (Springer Science + Business Media, Berlin, 2013), Vol. 132.