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Theoretical framework for Higher-Order Quantum Theory

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Higher-order quantum theory is an extension of quantum theory where one introduces transformations whose input and output are transformations, thus generalizing the notion of channels and quantum operations. The generalization then goes recursively, with the construction of a full hierarchy of maps of increasingly higher order. The analysis of special cases already showed that higher-order quantum functions exhibit features that cannot be tracked down to the usual circuits, such as indefinite causal structures, providing provable advantages over circuital maps. The present treatment provides a general framework where this kind of analysis can be carried out in full generality. The hierarchy of higher-order quantum maps is introduced axiomatically with a formulation based on the language of types of transformations. Complete positivity of higher-order maps is derived from the general admissibility conditions instead of being postulated as in previous approaches. The recursive characterization of convex sets of maps of a given type is used to prove equivalence relations between different types. The axioms of the framework do not refer to the specific mathematical structure of quantum theory, and can therefore be exported in the context of any operational probabilistic theory.

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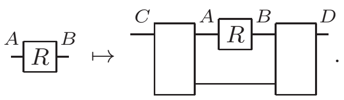
1. Introduction

The key idea behind the higher-order quantum theory is the promotion of quantum channels, which are normally considered as the logical gates in a quantum circuit, to the role of inputs, thus introducing "second-order" gates that transform channels into channels. In order to analyse this kind of question, the first step to take is to consider the channels as exquisitely mathematical objects, and then define the class of maps from the set of channels to itself. This class must satisfy some minimal *admissibility* requirement, that are the loosest constraints for the maps to respect the probabilistic structure of quantum theory, as in the axiomatisation of completely positive maps [1]. An admissible map from quantum operations to quantum operations must then i) respect convex combinations (i.e. it must be linear), ii) respect the set of channels also when applied locally to bipartite channels, and iii) preserve the normalization of channels. In Ref. [2] it was proved that this kind of transformations precisely corresponds to inserting the input channel into a fixed open circuit as in the following diagram



(1.1)

This idea is then immediately brought to its most general scenario: every kind of map can be raised to the level of the input of a computation at a further level in the hierarchy. Such a construction is not exclusive to quantum computation, and can be made also in the case of classical gates [3,4]. Actually, the first instance of higher-order computation can be tracked back to the invention of Lambda calculus. A quantum version constituting a model for higher-order quantum computation was elaborated in Ref. [5].

A relevant sub-hierarchy of maps is the one consisting of *quantum combs*, that can be thought of as the generalization of maps of eq. (1.1) with more than two "teeth", where one comb with n teeth maps a comb with $n-1$ teeth to a channel. This hierarchy was extensively studied in the last decade (for exhaustive reviews see [6–9]). The distinctive feature of maps in this sub-hierarchy is that they can be implemented by modular connection of networks of quantum gates.

As soon as one makes one step further, e.g. considering transformations from combs to combs, maps that cannot be implemented by a quantum circuit appear [7,10]. A paradigmatic example is the quantum SWITCH map [10] which takes as an input two quantum channels, say $\mathcal{A}$ and $\mathcal{B}$, and outputs the coherent superposition of the sequential applications of the two channels in two different order, i.e. $\mathcal{A}\circ\mathcal{B}$ and $\mathcal{B}\circ\mathcal{A}$. In some special case, these maps can be thought of as mixtures or "superpositions" of causally ordered circuits [10,11], as precognized in the pioneering proposals of Hardy [12]. Important results followed in the subsequent years, showing advantages over standard quantum computation in non local games [11], in gate discrimination [13], and oracle permut ation [14,15]. This opened the route to the study of operational tests for indefinite causal structures based on the idea of witnesses of a convex set [16], as well as to a notion of dynamics of causal structures [17]. The theoretical effort in this field inspired pioneering experiments [18,19].

The wealth of theoretical results about special cases of higher-order quantum maps calls for a thorough unified theoretical framework. This was initiated in Ref. [20] and formalized in Ref. [21] in the language of categorical quantum mechanics [22,23]. In the present paper, we complete the picture with a fully operational formulation. Every approach so far postulates complete positivity as a purely mathematical requirement on higher-order maps. Here we make the definition of admissibility fully operational, avoiding explicit reference to the mathematical properties of maps in the hierarchy—in particular complete positivity is not postulated but derived—and provide a characterization of admissible maps thus defined. Higher-order quantum theory must be thought of as an extension of quantum theory, which provides a natural unfolding of a part of the theory that is implicitly contained in any of its formulations. As such, it has a fundamental value, being a new standpoint for the analysis of the peculiarities of quantum theory. The formulation of the theory of higher-order maps in terms of operational axioms can indeed be applied to any operational probabilistic theory—taking in due care the fact that in general theories the notion of a transformation is more complex [24–26]—and allows for a comparison between the extended structures thus obtained.

The study of the hierarchy of higher-order maps requires a formal language that accounts for all the kinds of maps that can be defined. Following Ref. [20] we define a *type system* for higher-order maps. Every map comes then with a type, which summarizes basic information such as its domain and its range. For example, provided that elementary types such as A, B denote the sets of states of elementary systems, the type $(A \rightarrow B)$ denotes the set of quantum operations with input is A and output B.

Let us conclude this section with a short summary of the paper. After a review of preliminary linear algebra and the Choi isomorphism in Sec. 2, the type system of higher-order quantum maps is reviewed in Sec. 3, where the notion of extension by an elementary type is introduced, which plays a crucial role in the definition of admissibility. In Sec. 4 the operational axioms of higher-order quantum theory are presented. We show that the property of complete positivity follows from the operational definition of admissibility provided. Moreover, we prove a necessary and sufficient condition for admissible maps to be deterministic, that will be used in the subsequent analysis. In Sec. 5 we introduce the notion of a type structure, which summarizes the important features of a type. Then we prove a characterization theorem for deterministic admissible maps of an arbitrary type which makes explicit the results of the previous section. We then apply the result to some remarkable special cases, such as the proof of the uncurryng rule and the spelling out of the definition of tensor product of types. We also introduce the hierarchy of generalized combs, and show some structural identities for this family of maps. In Sec. 6 we pose the problem of inverting the characterization of deterministic types, namely, given a convex set of maps, finding, if any, the type to which it corresponds. Finally, Sec. 7 we close with some comments and remarks.

2. Linear maps and the Choi isomorphism

Let us start with some notational remarks. We denote quantum systems with capital letters $A, B \ldots, Z$ and the corresponding Hilbert spaces with $\mathcal{H}_A, \mathcal{H}_B, \ldots, \mathcal{H}_Z$. Throughout this paper we restrict ourselves to quantum systems with finitely many degrees of freedom, i.e. finite dimensional Hilbert spaces. The dimension of a Hilbert space $\mathcal{H}_A$ is denoted by d_A and since $d_A < \infty$ we have $\mathcal{H}_A \equiv \mathbb{C}^{d_A}$. The system with dimension 1, called the *trivial system*, is denoted by I. The parallel composition of systems A and B is denoted by AB and therefore we have $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. The parallel composition between a system A with the trivial system I gives back the same sytem A, i.e. $AI = A$. We denote with $\mathcal{L}((\mathcal{H}_A))$ the set of linear operators on $\mathcal{H}_A$ and with $\mathcal{L}(\mathcal{L}(\mathcal{H}_A), \mathcal{L}(\mathcal{H}_B))$ the set of linear maps from $\mathcal{L}(\mathcal{H}_A)$ to $\mathcal{L}(\mathcal{H}_B)$.

A *state* of a quantum system A is a positive operator $0 \leq \rho \in \mathcal{L}(\mathcal{H}_A)$ such that $\mathrm{Tr}[\rho] \leq 1$. States such that $\mathrm{Tr}[\rho] = 1$ are called normalized states or deterministic states. Physical transformations from system A to B are described by completely positive trace non increasing maps $\mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_A), \mathcal{L}(\mathcal{H}_B))$ also known as *quantum operations*. The requirements of complete positivity and trace non increasing guarantee that the transformation $\mathcal{M}$ is physically *admissible*, i.e. i) it is compatible with the probabilistic structure of quantum theory, and ii) it maps quantum states to quantum states even when locally applied to bipartite states. A quantum operation which is trace preserving is called *quantum channel*. A set $\{\mathcal{M}_i\}_{i \in \mathsf{S}}$ of quantum operations from system A to system B such that $\mathcal{M} := \sum_{i \in \mathsf{S}} \mathcal{M}_i$ is trace preserving, is called *quantum instrument*. A special instance of instrument is given by positive-operator-values measures POVMs, which maps states into probabilities, and are described by a collection of positive operators that sums to the identity. Moreover, states of a quantum system A can be considered as a special case of completely positive maps from the trivial system I to A.

The Choi isomorphism [27] between linear maps and linear operators will play a key role in the following.

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Theorem 1 (Choi isomorphism)**.** *Consider the map* $\mathsf{Ch} : \mathcal{L}(\mathcal{L}(\mathcal{H}_A), \mathcal{L}(\mathcal{H}_B)) \to \mathcal{L}(\mathcal{H}_B \otimes \mathcal{H}_A)$ *defined as*

$$\mathsf{Ch} : \mathcal{M} \mapsto M \qquad M := \mathcal{I}_A \otimes \mathcal{M}(|I\rangle\rangle\langle\langle I|) \tag{2.1}$$

where $\mathcal{I}_A$ *is the identity map on* $\mathcal{L}(\mathcal{H}_A)$ *and* $|I\rangle\rangle := \sum_{n=1}^{d_A} |n\rangle|n\rangle$, $\{|n\rangle\}_{n=1}^{d_A}$ *denoting an orthonormal basis of* $\mathcal{H}_A$. *Then* Ch *defines an isomorphism between* $\mathcal{L}(\mathcal{L}(\mathcal{H}_A), \mathcal{L}(\mathcal{H}_B))$ *and* $\mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$. *The operator* $M = \mathsf{Ch}(\mathcal{M})$ *is called the* Choi operator *of* $\mathcal{M}$. *Moreover one has* [1]*:*

$$\begin{aligned}
&\mathrm{Tr}[\mathcal{M}(X)] = \mathrm{Tr}[X] \quad \forall X \in \mathcal{L}(\mathcal{H}_A) \Leftrightarrow \mathrm{Tr}_B[M] = I_A,\\
&\mathcal{M}(X)^\dagger = \mathcal{M}(X^\dagger) \Leftrightarrow M^\dagger = M,\\
&\mathcal{M} \text{ is completely positive} \Leftrightarrow M \geq 0.
\end{aligned}$$

The inverse of the map Ch is given by the following expression:

$$\begin{aligned}
&[\mathsf{Ch}^{-1}(M)](O) = \mathrm{Tr}_A[(O^T \otimes I_B)M] \\
&O \in \mathcal{L}(\mathcal{H}_A) \quad M \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B),
\end{aligned} \tag{2.2}$$

where O^T denotes the transpose opearator with respect to the orthonormal basis we used to define $|I\rangle\rangle$ in Theorem 1.

3. Type system

In this section we lay the foundations of higher-order quantum theory. The notions of quantum operation and POVM allow for a complete and effective description of processing of quantum information encoded into quantum states. However, this set of tools is unsuitable for describing processes in which the input and output of the transformation are transformations themselves. Our goal is to introduce a formal language which enables us to overcome such a limitation. This language can be regarded as the *type system* for higher-order quantum maps. Starting with a set of *elementary types*, corresponding to finite dimensional quantum systems, by using appropriate *type constructors* one recursively builds new types from old ones. This procedure generates the whole hierarchy of types of admissible quantum maps, which maps from quantum trasformations to quantum transformations are a special case of.

Definition 1 (Types)**.** *Every finite dimensional quantum system corresponds to a* Type A. *The elementary type corresponding to the tensor product of quantum systems* A *and* B *is denoted with* AB. *The type of the* trivial system *is denoted by* I. *We denote with* $\mathsf{EleTypes}$ *the set of elementary types. Let* $\mathsf{A} := \mathsf{EleTypes} \cup \{(\} \cup \{)\} \cup \{\to\}$ *be an alphabet. We define the* set of types *as the smallest subset* $\mathsf{Types} \subset \mathsf{A}^*$ *such that*[2]

- $\mathsf{EleTypes} \subset \mathsf{Types}$,
- *if* $x, y \in \mathsf{Types}$ *then* $(x \to y) \in \mathsf{Types}$.

As one can easily verify, a type x is given a by a string like $x = (((A_1 \to A_2) \to (A_3 \to A_1)) \to (A_4 \to A_1))$ where A_i are elementary types. According to the above definition, for every pair of types x and y, one can form a new type $(x \to y)$, where x is the tail (input) and y the head (output) of an arrow. The new type $(x \to y)$ must be thought of as a new single entity that can be the head or the tail of a further arrow. In order to lighten the notation, the outermost parentheses are usually omitted. As it will be clear soon, if A, B, C and D are elementary types, then the type $(A \to B)$ is the type of maps from system A to system B and the type $(A \to B) \to (C \to D)$ is the

[1] Tr_B denotes the partial trace on system B and I_A is the identity operator on system A
[2] Please note that A^* stands for the set of words of the alphabet A

type of maps from "maps from A to B" to "maps from C to D". It is worth noticing that, for each type x there exist a positive integer n, n types x_i and an elementary type A such that

$$x = x_1 \rightarrow (x_2 \rightarrow (x_3 \rightarrow \cdots (x_n \rightarrow A)) \cdots) \tag{3.1}$$

The following definition will allow us to extend the notion of admissible map to the whole hierarchy.

Definition 2 (Extension with an elementary type)**.** *Let $x \in$ Types be a type and $E \in$ EleTypes be an elementary type. The* extension *$x \parallel E$ of x by the elementary type E is defined recursively as follows:*

- *for any $A, E \in$ EleTypes we have $A \parallel E := AE$;*
- *for any $x, y \in$ Types, $(x \rightarrow y) \parallel E := (x \rightarrow y \parallel E)$.*

From the first item of definition 2 we see that the parallel composition of elementary events is recovered. From the recursive definition, it is immediate to compute the parallel composition $x \parallel E$ when x is given explicitly. For example we have:

$$\begin{aligned}
&(((A_1 \rightarrow A_2) \rightarrow (A_3 \rightarrow A_1)) \rightarrow (A_4 \rightarrow A_1)) \parallel E = \\
&(((A_1 \rightarrow A_2) \rightarrow (A_3 \rightarrow A_1)) \rightarrow (A_4 \rightarrow A_1) \parallel E) = \\
&(((A_1 \rightarrow A_2) \rightarrow (A_3 \rightarrow A_1)) \rightarrow (A_4 \rightarrow A_1 \parallel E)) = \\
&(((A_1 \rightarrow A_2) \rightarrow (A_3 \rightarrow A_1)) \rightarrow (A_4 \rightarrow A_1 E))
\end{aligned}$$

From Equation (3.1) we clearly have $x \parallel E = x_1 \rightarrow (x_2 \rightarrow \cdots (x_n \rightarrow AE) \cdots)$. Clearly, the parallel composition with the trivial type I, leaves the type x unaffected, i.e. $x \parallel I = x$. Since many of the results of this paper are proved by induction, it is useful to introduce the following partial ordering between types.

Definition 3 (Partial ordering $\preceq$)**.** *We say that type x* is a parent *of type y and we write $x \preceq_p y$ if there exists a type z such that either $y = (x \rightarrow z)$ or $y = (z \rightarrow x)$. The relation $x \preceq y$ is defined as the transitive closure of the binary relation $\preceq_p$*

From the previous definition we have, for example,

$$x = (y \rightarrow w) \rightarrow z \implies y, w, z \preceq x.$$

The relation $\preceq$ is a well founded relation and Noetherian induction can be used. If we want to show that some proposition $\mathfrak{P}(x)$ holds for all types x of the set Types, we need to show that:

1. $\mathfrak{P}(y)$ is true for all elementary types (which are the minimal elements of the set Types).
2. If $\mathfrak{P}(y)$ is true for all y such that $y \preceq x$, then $\mathfrak{P}(x)$ is true for x.

In most of the cases, we will be required to prove that a statement holds for the type $x||E$ for any arbitrary elementary type E. Then item 2 becomes:

2′ If $\mathfrak{P}(y \parallel E)$ is true for all y such that $y \preceq x$ and for any E, then $\mathfrak{P}(x \parallel E')$ is true for x and any E'.

4. Axioms for higher-order quantum theory

It is worth stressing that the hierarchy of types has been defined as an abstract set of strings, with no relationship with the set of linear maps on Hilbert space. We now introduce such a connection through the notion of event.

Definition 4 (Generalized events). *If x is a type in* Types*, the set of* generalized events of type x*, denoted by $\mathsf{T}_{\mathbb{R}}(x)$, is defined by the following recursive definition.*

- *if A is an elementary type, then every $M \in \mathcal{L}(\mathcal{H}_A)$ is a generalized event of type A, i.e. $\mathsf{T}_{\mathbb{R}}(A) := \mathcal{L}(\mathcal{H}_A)$.*
- *if x, y are two types, then every Choi operator of linear maps $\mathcal{M} : \mathsf{T}_{\mathbb{R}}(x) \to \mathsf{T}_{\mathbb{R}}(y)$, is a generalized event M of type $(x \to y)$.*

The following lemma immediately follows from of Definition 4.

Lemma 1 (Characterization of events). *Let x be a type. Then $\mathsf{T}_{\mathbb{R}}(x) = \mathcal{L}(\mathcal{H}_x)$ where $\mathcal{H}_x := \bigotimes_i \mathcal{H}_i$ and $\mathcal{H}_i$ are the Hilbert spaces corresponding to the elementary types $\{A_i\}$ occuring in the expression of x.*

Proof. First we notice that the thesis holds for elementary types $x = A$. We then prove that if the thesis holds for arbitrary types x, y than it holds for $x \to y$. Let us then suppose that $\mathsf{T}_{\mathbb{R}}(x) = \mathcal{L}(\mathcal{H}_x)$ and $\mathsf{T}_{\mathbb{R}}(y) = \mathcal{L}(\mathcal{H}_y)$. An event of type $\mathsf{T}_{\mathbb{R}}(x \to y)$ is the Choi operator M of a map $\mathcal{M} : \mathcal{L}(\mathcal{H}_x) \to \mathcal{L}(\mathcal{H}_y)$ and therefore $M \in \mathcal{L}(\mathcal{H}_x \otimes \mathcal{H}_y)$. ■

An explicit example can be useful. Let A, B, C and D be elementary quantum systems with Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C, \mathcal{H}_D$ and let us consider the type $x := ((A \to B) \to C) \to D$. According to Definition 4 and Lemma 1, an event of type x is an operator $M \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D)$. Obviously, the type of an event cannot be inferred by the operator alone. Indeed, the same M can also define an event of a different type $y := (A \to B) \to (C \to D)$. Therefore, when we define an event, we need to explicitly declare its type.

Given two quantum systems A and B, not every operator $M \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ represents the Choi operator of a physical transformation from A to B. An operator M represents a physical transformation if and only of it is the Choi operator of a completely positive trace non increasing map, i.e. if and only if $0 \leq M \leq N$ with $\mathrm{Tr}_B[N] = I$. In an analogous way, we now want to characterise those events that correspond to physical maps. The key step toward achieving this goal is to formulate a notion of *admissible event* which generalises the requirement of complete positivity. In order to do that, we start with the following definition.

Definition 5 (Extended event). *Let x be a non-elementary type, E an elementary type and $M \in \mathsf{T}_{\mathbb{R}}(x)$. We denote with M_E the* extension of M by E *which is defined recursively as follows: If x, y are two types and $M \in \mathsf{T}_{\mathbb{R}}(x \to y)$ then $M_E \in \mathsf{T}_{\mathbb{R}}(x \parallel E \to y \parallel E)$ is the Choi operator of the map $\mathcal{M} \otimes \mathcal{I}_E : \mathsf{T}_{\mathbb{R}}(x \parallel E) \to \mathsf{T}_{\mathbb{R}}(y \parallel E)$, where $\mathcal{I}_E : \mathcal{L}(\mathcal{H}_E) \to \mathcal{L}(\mathcal{H}_E)$ is the identity map.*

If A and B are elementary types then $M \in \mathsf{T}_{\mathbb{R}}(A \to B)$ is the Choi of a map $\mathcal{M} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$. Therefore, M_E is the Choi operator of the map $\mathcal{M} \otimes \mathcal{I}_E : \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_E) \to \mathcal{L}(\mathcal{H}_B \otimes \mathcal{H}_E)$.

The notion of extended event allows us to give the definition of admissible event. We split the definition into two parts. The first part defines admissible elementary events and it is the usual definition of quantum states as positive operators.

Definition 6 (Admissible elementary event). *Let A be an elementary type and $M \in \mathsf{T}_{\mathbb{R}}(A)$. We say that:*

- *M is a* deterministic *event if $M \geq 0$ and $\mathrm{Tr}[M] = 1$. $\mathsf{T}_1(A)$ denotes the set of deterministic events of type A.*
- *M is* admissible *if $M \geq 0$ and there exists $N \in \mathsf{T}_1(A)$ such that $M \leq N$. $\mathsf{T}(A)$ is the set of admissible events of type A.*

Admissible elementary events which are not deterministic, i.e. the strict inequality $M < N$ holds, are called *probabilistic* elementary events. Up to this point, Definition 6 just introduced

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a new notation for well known objects. However, the use of this new language simplifies the statement of the second part of the definition of admissible events.

Definition 7 (Admissible event)**.** *Let $x, y \in \mathsf{Types}$ be two types, $M \in \mathsf{T}_{\mathbb{R}}(x \to y)$ be an event of type $x \to y$ and $M_E \in \mathsf{T}_{\mathbb{R}}(x \parallel E \to y \parallel E)$ be the extension of M by E. Let $\mathcal{M} : \mathsf{T}_{\mathbb{R}}(x) \to \mathsf{T}_{\mathbb{R}}(y)$ and $\mathcal{M} \otimes \mathcal{I}_E : \mathsf{T}_{\mathbb{R}}(x \parallel E) \to \mathsf{T}_{\mathbb{R}}(y \parallel E)$ be the linear maps whose Choi operator are M and M_E respectively.*

We say that M is admissible *if,*

(i) for all elementary types E, the map $\mathcal{M} \otimes \mathcal{I}_E$ sends admissible events of type $x \parallel E$ to admissible events of type $y \parallel E$.
(ii) there exist $\{N_i\}_{i=1}^n \subseteq \mathsf{T}_{\mathbb{R}}(x \to y)$, $0 \le n < \infty$ such that, for all elementary types E,
  - *$\forall 1 \le i \le n$ The map $\mathcal{N}_i$ satisfies item (i),*
  - *For all elementary types E, the map $(\mathcal{M} + \sum_{i=1}^n \mathcal{N}_i) \otimes \mathcal{I}_E$ maps deterministic events of type $x \parallel E$ to deterministic events of type $y \parallel E$*

The set of admissible events of type $x \to y$ is denoted with $\mathsf{T}(x \to y)$. An operator $D \in \mathsf{T}_{\mathbb{R}}(x \to y)$ is a deterministic *event of type $x \to y$, if $D \in \mathsf{T}(x \to y)$ and $(\mathcal{D} \otimes \mathcal{I}_E)$ maps deterministic admissible events of type $x \parallel E$ to deterministic admissible events of type $y \parallel E$.*

In lemma 6 we prove that if $M \in \mathsf{T}(x)$, and $\{N_i\}_{i=1}^n$ satisfy the requirements of Definition 7, then $N_i \in \mathsf{T}(x)$ and $M + \sum_{i=1}^n N_i \in \mathsf{T}_1(x)$. Clearly, we also have that if, for every $E \in \mathsf{EleTypes}$, $\mathcal{D} \otimes \mathcal{I}_E(\mathsf{T}(x \parallel E)) \subseteq \mathsf{T}(y \parallel E)$ and $\mathcal{D}(\mathsf{T}_1(x \parallel E)) \subseteq \mathsf{T}_1(y \parallel E)$, then $D \in \mathsf{T}_1(x \to y)$.

Definition 7 generalises Kraus' axiomatic definition of quantum operations [1] to higher-order maps. Indeed, one can easily verify that, for the simplest case $x = A \to B$, definition 7 reduces to the notion of completely positive trace non increasing map from $\mathcal{L}(\mathcal{H}_A)$ to $\mathcal{L}(\mathcal{H}_B)$. Let M be an admissible event of type $A \to B$. Since the set of admissible event of type A and B are the set of density matrices, $\mathcal{M}$ must be completely positive. Moreover, there exists a set of operators N_i such that, for any i, $\mathcal{N}_i$ must be completely positive as well. The condition that $\mathcal{M} + \sum_i \mathcal{N}_i$ maps deterministic events of A to deterministic events of B, implies that $\mathcal{M} + \sum_i \mathcal{N}_i$ must be trace preserving and therefore $\mathcal{M}$ is trace non increasing.

The following theorems characterise the set of admissible events.

Theorem 2 (Characterization of admissible events)**.** *Let x be a type and $M \in \mathsf{T}_{\mathbb{R}}(x)$. Then we have*

$$M \in \mathsf{T}(x) \Leftrightarrow M \ge 0 \wedge \exists D \in \mathsf{T}_1(x) \text{ s.t. } M \le D, \tag{4.1}$$

Proof. See Appendix A. ∎

The result of theorem 2 tells us that the only relevant cone in higher-order quantum theory is the cone of positive operators. This is a relevant improvement e.g. with respect to the previous literature on the subject, where complete positivity was assumed from the very beginning. The present definition of admissibility, on the contrary, can be extended to the case of general operational probabilistic theories [24,26,28] where in general the Choi correspondence, defined through the notion of a *faithful state*, is not surjective on the cone of states.

Notice that condition (4.1) reduces the characterization of the set $\mathsf{T}(x)$ to that of the set of deterministic events $\mathsf{T}_1(x)$. The latter is achieved by the next result.

Theorem 3 (Characterization of deterministic events)**.** *Let x, y be two types, $M \in \mathsf{T}_{\mathbb{R}}(x \to y)$ be an event of type $x \to y$. Then we have:*

$$M \in \mathsf{T}_1(x \to y) \Longleftrightarrow \begin{cases} M \ge 0, \\ [\mathsf{Ch}^{-1}(M)](\mathsf{T}_1(x)) \subseteq \mathsf{T}_1(y) \end{cases} \tag{4.2}$$

Proof. See Appendix B. ∎

Definition 7, Theorem 2 and Theorem 3 complete the construction of the hierarchy of higher-order quantum maps. Every type x corresponds to a convex set of positive operators which is the set $\mathsf{T}_1(x)$ of deterministic events of type x. The set $\mathsf{T}_1(x)$ uniquely determines the convex set PType_x of probabilistic events of type x. According to our framework, the colloquial sentence "M is an higher-order quantum map" translates into "M is a deterministic or probabilistic event of some kind x"

The main question in the theory of higher-order quantum theory is to characterize $\mathsf{T}_1(x)$ for any type x. For example, one could ask whether two different different types x and y have the same set of deterministic events, i.e. $\mathsf{T}_1(x) = \mathsf{T}_1(y)$. Whenever this is the case, we say that the types x and y are *equivalent*. We emphasize this concept by giving the following definition.

Definition 8 (Equivalent types)**.** *Let x and y be two types. We say that x and y are* equivalent, *and denote it as $x \equiv y$, if $\mathsf{T}_1(x) = \mathsf{T}_1(y)$.*

5. Characterization of higher-order quantum maps

In this section we further develop the framework of higher-order quantum theory that has been introduced in the previous section.

(a) Type structure

Many results we are going to prove depend only on the structure of the type x we are considering rather than on the specific elementary systems A_i that compose it. For example, the types $A_0 \to B_0$ and $A_1 \to B_1$ will be treated on the same footing, even if $d_{A_0} \neq d_{A_1}$ or $d_{B_0} \neq d_{B_1}$. It is then convenient to give the following definition.

Definition 9 (Type structure)**.** *Let $\Omega := \{*, I, (,), \to\}$ be an alphabet. We define the* set of type structures *as the smallest subset $\mathsf{Str} \subset \Omega^*$ such that*

- $*, I \in \mathsf{Str}$,
- *if* $\mathrm{x}, \mathrm{y} \in \mathsf{Str}$ *then* $(\mathrm{x} \to \mathrm{y}) \in \mathsf{Str}$.

We say that a type x belongs to the type structure x, and we write $x \in \mathrm{x}$, if x can be obtained by substituting arbitrary elementary types $A_i \in \mathsf{EleTypes}$ (that can possibly be the trivial type I) in place of the symbols $*$ in the expression of the type structure x.

One could think of a structure as an expression of the kind

$$\mathrm{x} := ((* \to *) \to I) \to (* \to *), \tag{5.1}$$

and the types that belong to x are, for example,

$$((A \to B) \to I) \to (C \to D) \in \mathrm{x}$$
$$((A \to I) \to I) \to (I \to D) \in \mathrm{x}$$

The type structure E is the type structure of the elementary types, $A \in \mathrm{E}\ \forall A \in \mathsf{EleTypes}$. Given a type structure y one can obtain another type structure y' by substituting the trivial type I in place of some of the symbols $*$ in the expression of y. This feature introduces a partial ordering among the type structures:

Definition 10 (Substructures)**.** *We say that a type strucure x is* substructure *of a type strucure x' ad we write $\mathrm{x} \subset \mathrm{x}'$ if x can be obtained by substituting the trivial type I in place of some of the symbols $*$ in the expression of x'.*

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For example we have:

$$\begin{aligned}
&\mathrm{y} := (* \to I) \to (* \to *) \\
&\mathrm{y}' := (* \to *) \to (* \to *) \\
&\mathrm{y} \subset \mathrm{y}'.
\end{aligned}$$

We notice that the same type x may belong to different type strucures, for example

$$\begin{aligned}
&\mathrm{y} := (* \to *) \to (* \to *) \\
&\mathrm{y}' := (* \to I) \to (* \to *) \\
&y := (A \to I) \to (C \to D) \quad y \in \mathrm{y}, \text{ and } y \in \mathrm{y}'.
\end{aligned}$$

However, among the type structures which a type x belongs to, there exists a privileged one.

Definition 11 (Natural type structure). *The* natural type structure *of a type x, is the type structure $[x]$ such that:*

- $x \in [x]$
- $x \notin \mathrm{y}$ *for any* $\mathrm{y} \subset [x]$

The expression of the natural type structure of a type x is obtained by replacing the all the elementary types but the trivial ones in the expression of x, with the elementary type structure $*$. The following example clarifies the meaning of Definition 11:

$$\begin{aligned}
x &:= (A \to I) \to ((C \to D) \to (F \to I)) \\
[x] &:= (* \to I) \to ((* \to *) \to (* \to I)).
\end{aligned}$$

(b) $\mathsf{L}_{\mathbf{b}}$ spaces

There is family of linear spaces of operators that plays a central role in higher-order quantum theory. In this subsection, we will introduce a notation which will allow us to more efficiently manipulate those linear spaces.

For a given an Hilbert space $\mathcal{H}$, we denote with $\mathsf{Herm}(\mathcal{H})$ the the linear (real) subspace of the Hermitian operators on $\mathcal{H}$. It is useful to split $\mathsf{Herm}(\mathcal{H})$ as the direct sum of the subspace of traceless operators and the one dimensional subspace generated by the identity operator:

$$\begin{aligned}
&\mathsf{L}_1 := \mathrm{span}\{I\} \quad \mathsf{L}_0 := \{X \mid \mathrm{Tr}[X] = 0,\ X^\dagger = X\} \\
&\mathsf{Herm}(\mathcal{H}) = \mathsf{L}_0 \oplus \mathsf{L}_1
\end{aligned} \tag{5.2}$$

where I is the identity operator on $\mathcal{H}$. Therefore, if O is in $\mathsf{Herm}(\mathcal{H})$ we can write the decomposition $O = \lambda I + X$ where $\lambda \in \mathbb{R}$ and T is a traceless selfadjoint operator $X \in \mathsf{L}_0$, X. When we are dealing with a tensor product of l Hilbert spaces, $\mathcal{H} := \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_l}$, we define

$$\mathsf{L}_{\mathbf{b}} := \mathsf{L}_{b_1} \otimes \mathsf{L}_{b_2} \otimes \cdots \otimes \mathsf{L}_{b_l} \quad b_i = 0, 1. \tag{5.3}$$

For example, for $\mathcal{H} := \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$, we have

$$\begin{aligned}
&\mathsf{L}_{00} = \mathrm{span}\{X \otimes Y\} \quad \mathsf{L}_{01} = \mathrm{span}\{X \otimes I\} \\
&\mathsf{L}_{10} = \mathrm{span}\{I \otimes Y\} \quad \mathsf{L}_{11} = \mathrm{span}\{I \otimes I\},
\end{aligned}$$

where the symbols X and Y denote $X \in \mathsf{L}_0$ and $Y \in \mathsf{L}_0$, respectively.

It is rather easy to verify that the spaces $\mathsf{L}_{\mathbf{b}}$ enjoy the following properties:

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Lemma 2 (Properties of the $\mathsf{L}_{\mathbf{b}}$ spaces)**.** *Given a binary string* $\mathbf{b}$ *of lenght* l*, let* $\mathsf{L}_{\mathbf{b}}$ *be the corresponding subset of* $\mathcal{H} := \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_l$ *defined as in Equation (5.3). If* $\mathbf{b} \neq \mathbf{b}'$ *then* $\mathsf{L}_{\mathbf{b}}$ *and* $\mathsf{L}_{\mathbf{b}'}$ *are orthogonal subspaces with respect the Hilbert-Schmidt product*[3]*.*

Proof. Since $\mathbf{b} \neq \mathbf{b}'$ there exist an some i such that $b_i \neq b'_i$. Without loss of generality we may suppose that $b_1 = 1$ and $b'_1 = 0$. From Equation (5.3) we have $\mathsf{L}_{\mathbf{b}} \ni A = I \otimes \tilde{A}$ and $\mathsf{L}_{\mathbf{b}'} \ni B = X \otimes \tilde{B}$. Taking the Hilbert-Schmidt product of A and B gives

$$\begin{aligned}(A, B)_{HS} &= \mathrm{Tr}[A^\dagger B] = \mathrm{Tr}[(I \otimes \tilde{A}^\dagger)(X \otimes \tilde{B})] \\ &= \mathrm{Tr}[X]\,\mathrm{Tr}[\tilde{A}^\dagger \tilde{B}] = 0.\end{aligned}$$

This proves that $\mathsf{L}_{\mathbf{b}}$ and $\mathsf{L}_{\mathbf{b}'}$ are Hilbert-Schmidt orthogonal. ■

From Lemma 2 we have that the sum $\mathsf{L}_{\mathbf{b}} + \mathsf{L}_{\mathbf{b}'}$ is actually a direct sum $\mathsf{L}_{\mathbf{b}} \oplus \mathsf{L}_{\mathbf{b}'}$. It is useful to intruduce the following notation:

$$W^{(l)} := \{0, 1\}^l, \quad T^{(l)} := W^{(l)} \setminus \{\mathbf{e}\}, \ \mathbf{e} := 11\ldots1, \tag{5.4}$$

$$\mathsf{L}_J := \bigoplus_{\mathbf{b} \in J} \mathsf{L}_{\mathbf{b}}, \quad J \subseteq W^{(l)}, \quad \mathsf{L}_\emptyset = \{0\}, \ \mathsf{L}_{\boldsymbol{\varepsilon}} = \mathbb{R}, \tag{5.5}$$

where $\boldsymbol{\varepsilon}$ is the null string in $W^{(0)}$ such that

$$\boldsymbol{\varepsilon}\mathbf{b} = \mathbf{b}\boldsymbol{\varepsilon} = \mathbf{b} \quad \forall \mathbf{b} \in W^{(l)}. \tag{5.6}$$

It is worth stressing that the notation $\mathsf{L}_{\mathbf{b}}$ is not reminiscent of the dimensions of the Hilbert spaces $\mathcal{H}_{A_i}$ occurring in the decomposition $\mathcal{H} = \bigotimes_i \mathcal{H}_{A_i}$. Therefore, if two types have the same natural type structure, i.e. $[x] = [x']$, they share the same set of strings.

Given a subspace

$$\Delta = \bigoplus_{\mathbf{b} \in J \subseteq T^{(l)}} \mathsf{L}_{\mathbf{b}},$$

we define the following two spaces related to Δ

$$\overline{\Delta} := \bigoplus_{\mathbf{b} \in \overline{J}} \mathsf{L}_{\mathbf{b}}, \qquad \overline{J} := T^{(l)} \setminus J. \tag{5.7}$$

$$\Delta^\perp := \bigoplus_{\mathbf{b} \in J^\perp} \mathsf{L}_{\mathbf{b}}, \qquad J^\perp := W^{(l)} \setminus J. \tag{5.8}$$

Given $J \subseteq W$, $J' \subseteq W^{(l')}$ and $\mathbf{w}' \in W^{(l')}$, we can define the sets $JJ' \subseteq W^{(l+l')}$ and $J'\mathbf{w}' \subseteq W^{(l+l')}$ as follows:

$$\begin{aligned} JJ' &:= \{\mathbf{b} = \mathbf{w}\mathbf{w}' \mid \mathbf{w} \in J, \ \mathbf{w}' \in J'\}, \\ J\mathbf{w}' &:= \{\mathbf{w}\mathbf{w}' \mid \mathbf{w} \in J\}. \end{aligned} \tag{5.9}$$

If $J = J'$ we will write $J^2 = JJ$ and J^n for the set $JJ\ldots$ (n times). In the following we will omit the label l from the symbols $W^{(l)}$ and $T^{(l)}$, whenever l is clear from the context. Eq. (5.7) defines the complement of Δ in the space of traceless operators, i.e. $\Delta \oplus \overline{\Delta} = \mathsf{Traceless}(\mathcal{H})$. Notice that, according to the definitions above, we have

$$\overline{\mathsf{L}}_J = \mathsf{L}_{\overline{J}}, \quad J \subseteq T \tag{5.10}$$

$$\mathsf{Herm}(\mathcal{H}) = \mathsf{L}_W = \bigoplus_{\mathbf{b} \in W} \mathsf{L}_{\mathbf{b}}, \tag{5.11}$$

$$\mathsf{Traceless}(\mathcal{H}) = \mathsf{L}_T = \bigoplus_{\mathbf{b} \in T} \mathsf{L}_{\mathbf{b}}. \tag{5.12}$$

[3]we remind that the Hilbert-Schimt product of two operators A and B is $(A, B)_{HS} := \mathrm{Tr}[A^\dagger B]$

Notice that when $\mathcal{H}=\bigotimes_i \mathcal{H}_{A_i}$ contains some trivial system $A_k = I$, one has $\mathcal{H}=\bigotimes_{i\neq k}\mathcal{H}_{A_i}$. Correspondingly, the non trivial spaces $\mathsf{L}_{\mathbf{b}}$ are determined only by the bits b_j in positions $j\neq k$ corresponding to systems different from the trivial system A_k. Indeed, if $b_k=0$ the space $\mathsf{L}_{\mathbf{b}}=\{0\}$ is trivial, while for $b_k=1$ one has that $\mathsf{L}_{\mathbf{b}}=\mathsf{L}_{\mathbf{b}'_k}$, where $\mathbf{b}'_k$ is the string obtained from $\mathbf{b}$ dropping the k-th bit. In formula

$$\mathsf{L}_J=\mathsf{L}_{J'_k},\quad J'_k:=\{\mathbf{b}'_k\mid \mathbf{b}\in J,\ b_k=1\},\tag{5.13}$$

$$\mathbf{b}'_k:=b_1b_2\ldots b_{k-1}b_{k+1}\ldots b_l,\tag{5.14}$$

having denoted by l the length of the string $\mathbf{b}$ so that the strings $\mathbf{b}'_k$ have length $l-1$. Repeatedly reducing the expression of the space $\mathcal{H}=\bigotimes_i\mathcal{H}_{A_i}$ to $\mathcal{H}=\bigotimes_{i\notin N}\mathcal{H}_{A_i}$, where $N:=\{i\mid A_i=I\}$, one obtains

$$\begin{aligned}&\mathsf{L}_J=\mathsf{L}_{J'_N},\quad J'_N:=\{\mathbf{b}'_N\mid \mathbf{b}\in J,\ \forall k\in N\, b_k=1\},\\ &\mathbf{b}'_N:=((\mathbf{b}'_{k_1})'_{k_2}\ldots)'_{k_n},\quad k_i\in N,\ n=|N|.\end{aligned}\tag{5.15}$$

Once a set J is reduced as above, dropping all the bits in positions $i\in N$ corresponding to trivial systems $A_i=I$, we call the resulting set of strings J'_N to be reduced to its *normal form*. The strings in J'_N have length $l-n$. Notice that for the trivial system I with $\mathcal{H}=\mathbb{C}$, we have $W=W^{(0)}=\{\varepsilon\}$, and correspondingly

$$\mathsf{Herm}(\mathcal{H})=\mathsf{L}_{\varepsilon}=\mathbb{R},\tag{5.16}$$

$$\mathsf{Traceless}(\mathcal{H})=\mathsf{L}_{\emptyset}=\{0\}.\tag{5.17}$$

(c) Characterization of $\mathsf{T}_1(x)$

We are now ready to present the characterization of the set $\mathsf{T}_1(x)$ of deterministic events of type x. The first step is to prove the following lemma.

Lemma 3 (Transpose of deterministic event). *Let x be a type and let $R\in\mathsf{T}_1(x)$ be a deterministic event of type x. Then also R^T, which is the transpose of R with respect to the basis used in the definition of the Choi isomorphism, is a deterministic event of type x, i.e. $R\in\mathsf{T}_1(x)\iff R^T\in\mathsf{T}_1(x)$.*

Proof. The statement is true for elementary events. Let us suppose that the statement is true for arbitrary types x and y and let R be a deterministic event of type $x\to y$. Then we have $\mathrm{Tr}_x[(S_x^T\otimes I_y)R]\in\mathsf{T}_1(y)$ for any $S_x\in\mathsf{T}_1(x)$. By hypothesis we have that $S_x^T\in\mathsf{T}_1(x)$ for any $S_x\in\mathsf{T}_1(x)$, and therefore $\mathrm{Tr}_x[(S_x\otimes I_y)R]\in\mathsf{T}_1(y)$ for any $S_x\in\mathsf{T}_1(x)$. By hypothesis we also have that $S_y^T\in\mathsf{T}_1(y)$ for any $S_y\in\mathsf{T}_1(y)$, and consequently $\mathrm{Tr}_x[(S_x^T\otimes I_y)R^T]=(\mathrm{Tr}_x[(S_x\otimes I_y)R])^T\in\mathsf{T}_1(y)$ which by theorem 3 proves $R^T\in\mathsf{T}_1(x\to y)$. ■

We now prove the main result of this section.

Proposition 1 (Characterization of $\mathsf{T}_1(x)$). *Let x be a type and let $\mathcal{H}_x:=\bigotimes_i\mathcal{H}_i$ be the Hilbert space given by the tensor product of the Hilbert spaces corresponding to the elementary types $\{A_i\}$ occuring in the definition of x. Then we have:*

$$\begin{aligned}&R\in\mathsf{T}_1(x)\iff R=\lambda_xI_x+X_x\\ &X_x\in\Delta_x\subseteq\mathsf{Traceless}(\mathcal{H}_x),\quad R\geq 0,\end{aligned}\tag{5.18}$$

where the real positive coefficient λ_x and and the linear subspace Δ_x are defined recursively as follows:

$$\begin{aligned}&\Delta_A=\mathsf{Traceless}(\mathcal{H}_A),\ \textit{if}\ A\in\mathsf{EleTypes}\\ &\Delta_{x\to y}=[\mathsf{Herm}(\mathcal{H}_x)\otimes\Delta_y]\oplus[\overline{\Delta}_x\otimes\Delta_y^{\perp}],\end{aligned}\tag{5.19}$$

$$\lambda_E=\frac{1}{d_E}\ \textit{if}\ E\in\mathsf{EleTypes},\quad \lambda_{x\to y}=\frac{\lambda_y}{d_x\lambda_x}.\tag{5.20}$$

Proof. For any elementary type A the set $\mathsf{T}_1(A)$ is the set of normalized states, and then the thesis holds. Let us consider the case in which x is not elementary and let us suppose that the thesis hold for any type $y \preceq x$. For $x = y \to z$, any $R \in \mathsf{T}_1(y \to z)$ is a positive operator that can be decomposed as $R = \lambda_R I + O_R$ where $O_R \in \mathsf{Traceless}(\mathcal{H}_y \otimes \mathcal{H}_z)$. Since R maps deterministic events of type y to deterministic events of type z, we must have $\mathrm{Tr}_y[(S_y^T \otimes I_z)R] \in \mathsf{T}_1(z)$ for all $S_y \in \mathsf{T}_1(y)$. Thanks to Lemma 3, this can be restated as $\mathrm{Tr}_y[(S_y \otimes I_z)R] \in \mathsf{T}_1(z)$ for all $S_y \in \mathsf{T}_1(y)$. First, let us consider the case $S_y = \lambda_y I_y$, which is in $\mathsf{T}_1(y)$ thanks to the inductive hypothesis. From the inductive hypothesis, there exists $Z \in \Delta_z$ such that

$$\mathrm{Tr}_y[(\lambda_y I_y \otimes I_z)(\lambda_R I + O_R)] = \lambda_z I_z + Z \tag{5.21}$$

Equation (5.21) implies that

$$\lambda_R = \frac{\lambda_z}{d_y \lambda_y} =: \lambda_{y \to z} = \lambda_x \tag{5.22}$$

$$\mathrm{Tr}_y[O_R] \in \Delta_z. \tag{5.23}$$

Let now Y be an arbitrary operator in Δ_y. There exists $\mu \neq 0$ such that $\lambda_y I + \mu Y \geq 0$. From the induction hypothesis we have $\lambda_y I + \mu Y \in \mathsf{T}_1(y)$, which implies, together with Equation (5.23),

$$\mathrm{Tr}_y[((\lambda_y I_y + \mu Y) \otimes I_z)(\lambda_x I_x + O_R)] = \lambda_z I_z + Z, \tag{5.24}$$

for some $Z \in \Delta_z$. From Equations (5.22) (5.23) and (5.24) we obtain that

$$\mathrm{Tr}_y[(X_y \otimes I_z)O_R] \in \Delta_z, \quad \forall X_y \in \Delta_y. \tag{5.25}$$

Equations (5.23) and (5.25) are satisfied if and only if

$$\mathrm{Tr}[(X_y \otimes \overline{S}_z)O_R] = \mathrm{Tr}[(I_y \otimes \overline{S}_z)O_R] = 0, \tag{5.26}$$

for any $X_y \in \Delta_y$ and any $\overline{S}_z \in \Delta_z^{\perp}$. Equation (5.26) finally implies

$$O_R \in [\mathsf{L}_W \otimes \Delta_z] \oplus [\overline{\Delta}_y \otimes \Delta_z^{\perp}]. \tag{5.27}$$

On the other hand, let us consider and arbitrary operator $O'_R \in [\mathsf{L}_W \otimes \Delta_z] \oplus [\overline{\Delta}_y \otimes \Delta_z^{\perp}]$. Clearly, there exist a real number $\mu \in \mathbb{R}$ such that $R' := \lambda_x I_x + \mu O'_R$ is a positive operator. Let $S_y \in \mathsf{T}_1(y)$ be an arbitrary deterministic event of type y. By the induction hypothesis we have that $S_y = \lambda_y I_y + Y$, where $Y \in \Delta_y$. By positivity of R' one has $0 \leq \mathrm{Tr}_y[(S_y \otimes I_z)R']$. By direct computation, one can show that $\mathrm{Tr}_y[(S_y \otimes I_z)R'] = \lambda_z I_z + Z$ for some $Z \in \Delta_z$. By the induction hypothesis we have $\mathrm{Tr}_y[(S_y \otimes I_z)R'] \in \mathsf{T}_1(z)$. This proves the following inclusion: $\Delta_x \supseteq [\mathsf{Herm}(\mathcal{H}_y) \otimes \Delta_z] \oplus [\overline{\Delta}_y \otimes \Delta_z^{\perp}]$. Thus, $\Delta_x = [\mathsf{Herm}(\mathcal{H}_y) \otimes \Delta_z] \oplus [\overline{\Delta}_y \otimes \Delta_z^{\perp}]$. ■

Corollary 1. *Let x and y be two types. Then we have*

$$x \equiv y \iff \lambda_x = \lambda_y \wedge \Delta_x = \Delta_y \tag{5.28}$$

Corollary 2. *Let x be a type and let A_i denote the elementary types occurring in the definition of x. Let I_x be the identity operator in $\mathcal{L}(\mathcal{H}_x)$ and let λ_x be defined as in Equation (5.20). Then we have*

$$\lambda_x I_x \in \mathsf{T}_1(x), \qquad \lambda_x = \prod_{A_i \in x} d_{A_i}^{-K_x(A_i)} \tag{5.29}$$

$$K_x(A_i) := \#[\text{“}\to\text{”}] + \#[\text{“}(\text{”}] \pmod 2 \tag{5.30}$$

$\#[\text{“}\to\text{”}]$ and $\#[\text{“}(\text{”}]$ denotes the number of arrows $\to$ and open round brackets (to the right of A_i in the expression of x, respectively.

Proof. The only non-trivial claim is that $\lambda_x = \prod_i d_i^{-K(i)}$. Let us prove this statement by induction. The thesis is true for elementary types. We now suppose that the thesis holds for any $y \preceq x$, and we consider a non elementary type $x = y \to z$.

Let A be any elementary type occuring in the expression of y. We now show that $K_x(A)$ is $K_y(A)+1 \pmod 2$. First we observe that the expression of type y must occurs in $y \to z$ with the outermost parenthesis. By definition 1, we have that any expression of a type, with the outermost parenthesis, contains as many "$\to$" as "(". Then we consider an elementary type A which occurs in the definition of y. The same type will occur in the expression of $y \to z$. It is easy to realize that the number of "$\to$" and "(" that follow A in the expression of $y \to z$ is changed by an odd number. Indeed, we now have all the "$\to$" and "(" that appear in y plus one more $\to$ which is the $\to$ that stays between y and z.

Let us now consider an elementary type B which occurs in the expression of z. The same B appears in the expression of $y \to z$ and the number "$\to$" and "(" to its right is unchanged and therefore $K_x(B) = K_z(B)$. Then we have

$$\prod_{A_i \in x} d_{A_i}^{-K_x(A_i)} = \prod_{A_i \in y} d_{A_i}^{-[K_y(A_i)+1 \pmod 2]} \prod_{A_i \in z} d_{A_i}^{-K_z(A_i)} =$$
$$= \left(d_y \prod_{A_i \in y} d_{A_i}^{-K_y(A_i)} \right)^{-1} \prod_{A_i \in z} d_{A_i}^{-K_z(A_i)}$$

which proves that recurrence relation of Equation (5.20) is satisfied. ■

Corollary 3. *For any type x, we have*

$$\Delta_x = \bigoplus_{\mathbf{b} \in D_x} \mathsf{L}_{\mathbf{b}} \tag{5.31}$$

for some set D_x of string.

Proof. The thesis is true for elementary types ($D_E = 0$). Let us suppose that Δ_x and Δ_y are the direct sum of $\mathsf{L}_{\mathbf{b}}$ spaces for two types x and y. Then, by Equation (5.19), also $\Delta_{x \to y}$ is the direct sum of $\mathsf{L}_{\mathbf{b}}$ spaces. ■

Notice that the expression on the right hand side of Eq. (5.31) can involve different choices of D_x depending on the number of trivial systems I that are explicitly considered in the expansion of $\mathcal{H}_x$. However, the space Δ_x on the right hand side is uniquely defined, independently of the choice of D_x. In particular, there is one preferred choice for D_x which is the one obtained after reducing the strings D_x to their normal form $(D_x)'_N$ as in Eq. (5.15). It is easy to realise that the set $(D_x)'_N$ depends only on the natural type structure of the type x. If two types x and x' have the same natural type structure, then we have

$$\mathrm{x} := [x] = [x'] \implies D_{\mathrm{x}} := (D_x)'_N = (D_{x'})'_N. \tag{5.32}$$

Moreover, it is possible to generalise Proposition 1 to type structures.

Corollary 4. *Let x be a type structure and let D_{x} be the set of strings defined according to Equation (5.32). Then D_{x} is such that:*

$$\begin{aligned} &D_I = \emptyset \quad D_I^{\perp} = \{\varepsilon\}, \quad D_* = \{0\}, \\ &D_{\mathrm{x} \to \mathrm{y}} = W_{\mathrm{x}} D_{\mathrm{y}} \cup \overline{D}_{\mathrm{x}} D_{\mathrm{y}}^{\perp}, \end{aligned} \tag{5.33}$$

where the sets W, T have been defined in Equation (5.4), the sets D have been defined in Equation (5.32) and juxtaposition of sets and strings has been defined in Equation (5.9).

The results that we presented in this section are the basic technical tools in the study of higher-order quantum maps. In particular, Proposition 1 unfolds the characterization of admissible events given in Theorems 2 and 3, and provides an explicit constructive formula. In the next subsections we will apply this result to prove some equivalence between types (and type structures).

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(d) Functionals

In this section we study the types of the kind $x \to I$ (we remind that I denotes the type of the trivial elementary system). Events of type $x \to I$ are linear functionals on events of of type x. It is convenient to introduce the shorthand notation

$$\overline{x} := x \to I. \tag{5.34}$$

By virtue of Proposition 1 we have the following lemma.

Lemma 4. *Let x be a type and let Δ_x and λ_x be defined as in Proposition 1. Then $\Delta_{\overline{x}} = \overline{\Delta}_x$ and $\lambda_{\overline{x}} = \frac{1}{\lambda_x d_x}$.*

Proof. For the trivial system I, we have $\lambda_I = 1$ and $\Delta_I = 0$. Then, from Equation (5.19) we immediately have

$$\Delta_{x\to I} = [\overline{\Delta}_x \otimes \mathsf{Herm}(\mathbb{C})] = \overline{\Delta}_x$$

and $\lambda_{\overline{x}} = \frac{1}{\lambda_x d_x}$. ■

We can now easily prove the following identity.

Proposition 2. *Let x be a type. Then $\overline{\overline{x}} \equiv x$.*

Proof. By definition 8, $\overline{\overline{x}} \equiv x$ iff $\mathsf{T}_1(x) = \mathsf{T}_1(\overline{\overline{x}})$. Now,

$$R \in \mathsf{T}_1(x) \iff R \geq 0, \quad R = \lambda_x I + X, \quad X \in \Delta_x.$$

Using Equation (5.19) we have

$$\lambda_{\overline{\overline{x}}} = \lambda_{(x\to I)\to I} = \frac{1}{d_{x\to I}\lambda_{x\to I}} = \frac{1}{d_x \frac{1}{d_x \lambda_x}} = \lambda_x. \tag{5.35}$$

Then we have

$$\begin{aligned} R \in \mathsf{T}_1(\overline{\overline{x}}) &\iff R \geq 0, \quad R = \lambda_{\overline{\overline{x}}} I_{\overline{\overline{x}}} + X, \quad X \in \Delta_{\overline{\overline{x}}} \\ &\iff R \geq 0, \quad R = \lambda_x I_x + X, \quad X \in \overline{\overline{\Delta}}_x \\ &\iff R \geq 0, \quad R = \lambda_x I_x + X, \quad X \in \Delta_x \\ &\iff R \in \mathsf{T}_1(x), \end{aligned}$$

where we used Equation (5.35) and Lemma 4 in the second line. ■

Let us clarify the previous discussion with some examples. We know that for an elementary type A the set $\mathsf{T}_1(A)$ is the set of positive operators on $\mathcal{H}_A$ with unit trace, (i.e. $\Delta_A = \mathsf{Traceless}(\mathcal{H}_A)$ and $\lambda_A = d_A^{-1}$), while for the trivial elementary type I we have $\mathsf{T}_1(I) = 1$ (i.e. $\Delta_I = \mathsf{Traceless}(\mathbb{C}) = 0$ and $\lambda_I = d_I^{-1} = 1$). By applying Equation (5.19) we have $\lambda_{A\to I} = 1$ and $\Delta_{A\to I} = 0$. Indeed, $\Delta_{A\to I} = [\overline{\Delta}_A \otimes \mathsf{Herm}(\mathbb{C})] \oplus [(\mathsf{L}_1 \oplus \Delta_A) \otimes \Delta_I] = [\overline{\mathsf{Traceless}(\mathcal{H}_A)} \otimes \mathsf{Herm}(\mathbb{C})] \oplus [(\mathsf{L}_1 \oplus \mathsf{Traceless}(\mathcal{H}_A)) \otimes \Delta_I] = 0$, since $\Delta_I = \overline{\mathsf{Traceless}(\mathcal{H}_A)} = 0$. Then we have $\mathsf{T}_1(A \to I) = I_A$, i.e. the set of deterministic events of type $A \to I$ has only one element, the identity operator on $\mathcal{H}_A$. The set of probabilistic events of type $A \to I$ is the set of positive operators bounded by I. We recover then the usual notion of effect (element of a POVM). The equivalence $\overline{\overline{A}} \equiv A$ tells us that a quantum state can be equivalently interpreted as the Choi operator of a map that sends a deterministic measurement (which is uniquely represented by the identity operator) to the number 1. It seems we have gone quite a long and devious way to prove an obviuos fact. However, as we will see, when considering more complex types, the equivalence between types can be far from obvious.

(e) Tensor product of types

In this section we introduce the following composition law for types:

$$x \otimes y := \overline{x \rightarrow \overline{y}}. \tag{5.36}$$

This operation can be thought of as the generalization of parallel composition of elementary types to the whole hierarchy.

Lemma 5 (Characterization of tensor product of types)**.** *Let x and y be two types and let $\Delta_x, \lambda_x, \Delta_y, \lambda_y$ be defined as in Proposition 1. We have:*

$$\begin{aligned} \Delta_{x\otimes y} &= (\mathsf{L_e} \otimes \Delta_x) \oplus (\Delta_y \otimes \Delta_x) \oplus (\Delta_y \otimes \mathsf{L_e}) \\ \lambda_{x\otimes y} &= \lambda_x \lambda_y \end{aligned} \tag{5.37}$$

Proof. The thesis can be easily proved by recursively applying Equation (5.19). ■

Proposition 3 (Properties of the tensor product of types)**.** *The following equivalences hold:*

$$A \otimes B \equiv AB, \quad \forall\, A, B \in \mathsf{EleTypes} \tag{5.38}$$

$$x \otimes y \equiv y \otimes x, \quad \forall\, x, y \in \mathsf{Types} \tag{5.39}$$

$$(x \otimes y) \otimes z \equiv x \otimes (y \otimes z), \quad \forall\, x, y, z \in \mathsf{Types} \tag{5.40}$$

Proof. By recursively applying Equation (5.37) one has $\lambda_{A\otimes B} = \lambda_{AB}$, $\lambda_{x\otimes y} = \lambda_{y\otimes x}$, $\lambda_{(x\otimes y)\otimes z} = \lambda_{x\otimes(y\otimes z)}$, $\Delta_{A\otimes B} = \Delta_{AB}$, $\Delta_{x\otimes y} = \Delta_{y\otimes x}$ and $\Delta_{(x\otimes y)\otimes z} = \Delta_{x\otimes(y\otimes z)}$. Since the cone of positive operators depends only on the elementary systems occurring in the definition of a type, we have $\mathsf{P}_{A\otimes B} = \mathsf{P}_{AB}$, $\mathsf{P}_{x\otimes y} = \mathsf{P}_{y\otimes x}$ and $\mathsf{P}_{(x\otimes y)\otimes z} = \mathsf{P}_{x\otimes(y\otimes z)}$ and the thesis follows. ■

We have seen that the tensor product of elementary types recovers the familiar notion of tensor product of quantum systems. However, when non trivial types are involved, the interpretation of the tensor product between two types is more subtle. Let us clarify this feature with an example. Let us consider the types $A \rightarrow B$ and $C \rightarrow D$. The deterministic events of type $A \rightarrow B$ and $C \rightarrow D$ are quantum channels from system A to system B and quantum channels from system C to system D, respectively. Then we have

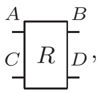
$$\begin{aligned} &R \in \mathsf{T}_1(A \rightarrow B) \iff R = \frac{1}{d_B} I + X \\ &R \in \mathsf{P}(\mathcal{H}_A \otimes \mathcal{H}_B), \\ &X \in \mathsf{L}_T \otimes \mathsf{L}_W, \end{aligned}$$

An analogous equation holds for $C \rightarrow D$. Let us now consider the type $(A \rightarrow B) \otimes (C \rightarrow D)$. From Equation (5.37) we have that $R \in \mathsf{T}_1((A \rightarrow B) \otimes (C \rightarrow D))$ iff

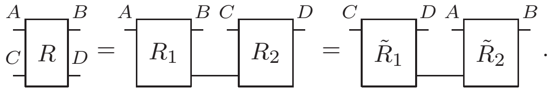
$$\begin{aligned} &R = \frac{1}{d_D d_B} I + X, \quad R \geq 0, \\ &X \in [\mathsf{L}_W \otimes \mathsf{L}_T \otimes \mathsf{L_e} \otimes \mathsf{L_e}] \oplus \\ &\qquad [\mathsf{L_e} \otimes \mathsf{L_e} \otimes \mathsf{L}_W \otimes \mathsf{L}_T] \oplus \\ &\qquad [\mathsf{L}_W \otimes \mathsf{L}_T \otimes \mathsf{L}_W \otimes \mathsf{L}_T]. \end{aligned} \tag{5.41}$$

Operators that obey Equation (5.41) are Choi operators of *non-signalling channels*,

$$\mathcal{R} : \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_C) \rightarrow \mathcal{L}(\mathcal{H}_B \otimes \mathcal{H}_D), \tag{5.42}$$



which send quantum states of the bipartite system AC to quantum states of the bipartite system BD, such that the output B does not depend on the input C and the output D does not depend on the input A. Non-signalling channels of this kind have two possible realisations as memory channels as follows [4]:



The previous equation means that for any non signalling channel $\mathcal{R}:\mathcal{L}(\mathcal{H}_A\otimes\mathcal{H}_C)\to\mathcal{L}(\mathcal{H}_B\otimes\mathcal{H}_D)$ there exist four channels $\mathcal{R}_1,\tilde{\mathcal{R}}_2:\mathcal{L}(\mathcal{H}_A)\to\mathcal{L}(\mathcal{H}_B)$ and $\mathcal{R}_2,\tilde{\mathcal{R}}_1:\mathcal{L}(\mathcal{H}_C)\to\mathcal{L}(\mathcal{H}_D)$ such that $\mathcal{R}$ can be realized as either the concatenation of $\mathcal{R}_1$ and $\mathcal{R}_2$ or the concatenation of $\tilde{\mathcal{R}}_1$ and $\tilde{\mathcal{R}}_2$. Given two channels $\mathcal{R}:\mathcal{L}(\mathcal{H}_A)\to\mathcal{L}(\mathcal{H}_B)$ and $\mathcal{S}:\mathcal{L}(\mathcal{H}_C)\to\mathcal{L}(\mathcal{H}_D)$, their tensor product $\mathcal{R}\otimes\mathcal{S}$ is a non-signalling channel. Also the convex combination $p\mathcal{R}\otimes\mathcal{S}+(1-p)\mathcal{R}'\otimes\mathcal{S}'$ of tensor product of channels is a non-signalling channel. However not every non-signalling channel is a convex combination of tensor products of channels. In the language of higher-order quantum theory, this means that the following strict inclusion holds:

$$\begin{aligned}\mathsf{T}_1(x\otimes y)=\mathsf{P}(\mathcal{H}_x\otimes\mathcal{H}_y)\cap\mathsf{Aff}\{\mathsf{T}_1(x)\otimes\mathsf{T}_1(y)\}\supset\\ \supset\mathsf{Conv}\{\mathsf{T}_1(x)\otimes\mathsf{T}_1(y)\}\end{aligned}$$

where $\mathsf{Aff}\{S\}$ denotes the affine hull of the set S and $\mathsf{Conv}\{S\}$ denotes the convex hull of the set S.

We conclude this subsection by proving the *uncurrying* identity for higher-order quantum maps

Proposition 4 (Quantum uncurrying)**.** *For any types x , y and z we have the equivalence*

$$x\to(y\to z)\equiv(x\otimes y)\to z \tag{5.43}$$

Proof. The equivalence (5.43) is consequence of the associativity of the tensor product of types Indeed, from Equation (5.40) and Proposition 2 we have

$$\begin{aligned}&\overline{(x\otimes y)\to\overline{z}}\equiv\overline{x\to\overline{(y\otimes z)}}\\ \iff&(x\otimes y)\to\overline{z}\equiv x\to(\overline{y\otimes z})\\ \iff&(x\otimes y)\to\overline{z}\equiv x\to(y\to\overline{z}).\end{aligned}$$

By substituting $\overline{z}$ with z we have he thesis. ∎

(f) Generalized comb

In this subsection, we study the following family of sub-hierarchies:

Definition 12 (n-comb with base x)**.** *Let* x *be a type structure. The type structure* $\mathbf{n}_{\mathrm{x}}$ *of* n-combs with base x *is defined recursively as follows:*

- $\mathbf{1}_{\mathrm{x}}=\mathrm{x}$,
- $\mathbf{n}_{\mathrm{x}}=(\mathbf{n}-\mathbf{1})_{\mathrm{x}}\to\mathrm{x}$.

The type structure x *is called the* base *of the type structure* $\mathbf{n}_{\mathrm{x}}$*. We denote with* $\mathbf{n}_x$ *a generic type such that* $\mathbf{n}_{\mathrm{x}}$ *is its natural type structure, i.e.* $[\mathbf{n}_x]=\mathbf{n}_{\mathrm{x}}$.

[4]further details about the realization of no-signalling bipartite channels can be found in Ref. [29].

According to Definition 12 a type $\mathbf{n}_x$ has the following expression:

$$\begin{aligned}\mathbf{n}_x &= ((\dots((x_1 \to x_2) \to x_3)\dots) \to x_{n-1}) \to x_n,\\ [x_i] &= \mathrm{x} \quad \forall i = 1,\dots,n.\end{aligned} \tag{5.44}$$

For example, $\mathbf{4}_\mathrm{x} = ((\mathrm{x} \to \mathrm{x}) \to \mathrm{x}) \to \mathrm{x}$ and $\mathbf{4}_x = ((x_1 \to x_2) \to x_3) \to x_4$. As it is known, the case in which $\mathrm{x} = * \to *$ gives rise to the *comb* hierarchy which is extensively studied in the literature [6–8,30,31].

As it will be soon clear, the language of type structures, which was unnecessary in subsections (d) and (e), simplifies the study of the quantum types introduced by Definition 12. Our first result is a characterisation theorem for n-combs of base x.

Proposition 5 (Characterisation of generalized n-combs)**.** *Let $\mathbf{n}_\mathrm{x}$ be a type structure defined as in Definition 12. Then we have*

$$D_n = \begin{cases} \bigcup_{l=1}^{\frac{n+1}{2}} W^{n-2l+1} D {D^\perp}^{2l-2} \\ \quad \cup \bigcup_{l=1}^{\frac{n-1}{2}} \mathbf{e}^{2l-1} \overline{D} {D^\perp}^{n-2l} & n \text{ odd} \\ \bigcup_{l=1}^{\frac{n}{2}} \Big(W^{n-2l-1} D {D^\perp}^{2l-2} \\ \quad \cup \mathbf{e}^{2l-2} \overline{D} {D^\perp}^{n-2l+1} \Big) & n \text{ even} \end{cases} \tag{5.45}$$

where the sets D_n are defined according to Equation (5.32) for the type structure $\mathbf{n}_\mathrm{x}$ with $D := D_1$, and $W := W_\mathrm{x}$, $\mathbf{e} := \mathbf{e}_\mathrm{x}$ are defined according to Equation (5.4). Moreover, for any type $\mathbf{n}_x$, we have

$$\lambda_n = \begin{cases} \lambda_{x_n} \prod_{i=1}^{\frac{n-1}{2}} \left[\lambda_{x_{2i-1}} \left(\lambda_{x_{2i}} d_{x_{2i}} \right)^{-1} \right] & n \text{ odd} \\ \prod_{i=1}^{\frac{n}{2}} \left[\lambda_{x_{2i}} \left(\lambda_{x_{2i-1}} d_{x_{2i-1}} \right)^{-1} \right] & n \text{ even}. \end{cases} \tag{5.46}$$

where λ_n is defined as in Proposition 1 and x_i are defined as in Equation (5.44).

Proof. Let us begin with the proof of Equation (5.45). The thesis hold for $\mathbf{1}_\mathrm{x}$. Let us then suppose that the thesis holds for any $m < n+2$ and m even. By applying corollary 4 twice, we have

$$\begin{aligned} D_{n+2} &= W_{n-1} D_1 \cup \overline{D}_{n-1} D_1^\perp = \\ &= W_{n-1} D_1 \cup \mathbf{e}_{n-2} D_1 D_1^\perp \cup D_{n-2} D_1^\perp D_1^\perp \end{aligned}$$

which, thanks to the induction hypothesis, proves the thesis for n even. The proof for n odd is analogous.

We now focus on Equation (5.46). Since $\mathbf{1}_x = x_1$ the thesis clearly holds. Let us fix an arbitrary odd n and let us suppose that the thesis hold for any $m < n$. Since $\mathbf{n}_x = (\mathbf{n}-\mathbf{1})_x \to x_n$, by

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combining Equation (5.46) and the induction hypothesis, we have

$$\lambda_n = \frac{\lambda_{x_n}}{d_{n-1}\lambda_{n-1}}$$
$$= \frac{\lambda_{x_n}}{\prod_{i=1}^{\frac{n-1}{2}} \left[d_{x_{2i}} d_{x_{2i-1}} \lambda_{x_{2i}} \left(\lambda_{x_{2i-1}} d_{x_{2i-1}}\right)^{-1}\right]} =$$
$$= \frac{\lambda_{x_n}}{\prod_{i=1}^{\frac{n-1}{2}} \left[d_{x_{2i}} \lambda_{x_{2i}} \left(\lambda_{x_{2i-1}}\right)^{-1}\right]} =$$
$$= \lambda_{x_n} \prod_{i=1}^{\frac{n-1}{2}} \left[\lambda_{x_{2i-1}} (\lambda_{x_{2i}} d_{x_{2i}})^{-1}\right],$$

which proves the thesis for odd n. The n even case can be proved by a similar calculation. ■

In order to clarify the discussion, it is convenient to analyze some examples in detail. Let us start with the case in which the base x is the elementary structure, i.e x $= *$, and

$$\mathbf{n}_E = (\dots((E_1 \to E_2) \to E_3)\dots) \to E_n, \tag{5.47}$$

Then we have

$$D = \{0\}, \quad \overline{D} = \emptyset, \quad D^\perp = \{1\}$$
$$\mathbf{b} \in D_n \iff \begin{array}{l}\mathbf{b} \text{ starts from the right} \\ \text{with an even number of 1s,}\end{array} \tag{5.48}$$
$$\lambda_n = \begin{cases} d_{E_n}^{-1} \prod_{i=1}^{\frac{n-1}{2}} d_{E_{2i-1}}^{-1} & n \text{ odd} \\ \prod_{i=1}^{\frac{n}{2}} d_{E_{2i}}^{-1} & n \text{ even.} \end{cases} \tag{5.49}$$

Then, let us analyze the comb hierarchy, i.e. the case x $= * \to *$,

$$\mathbf{n}_{A\to B} = (\dots(A_1 \to B_1) \to)\dots) \to (A_n \to B_n). \tag{5.50}$$

We have

$$W = \{00, 01, 10, 11\}$$
$$D = \{10, 00\}, \quad \overline{D} = \{01\}, \quad D^\perp = \{11, 01\}$$

From Proposition 5 we have that D_n has the following structure

$$D_n = W_{n-1}D \cup W_{n-3}D{D^\perp}^2 \cup \dots \cup D{D^\perp}^{n-1} \cup$$
$$\mathbf{e}\overline{D}{D^\perp}^{n-2} \cup \mathbf{e}^3\overline{D}{D^\perp}^{n-4} \dots \cup \mathbf{e}^{n-2}\overline{D}D^\perp \ (n \text{ odd}),$$
$$D_n = W_{n-1}D \cup W_{n-3}D{D^\perp}^2 \cup \dots \cup WD{D^\perp}^{n-2} \cup$$
$$\overline{D}{D^\perp}^{n-1} \cup \mathbf{e}^2\overline{D}{D^\perp}^{n-3} \cup \dots \cup \mathbf{e}^{n-2}\overline{D}D^\perp \ (n \text{ even}),$$

for example, for $\mathbf{3}_{A\to B}$ we have: $D_3 = WWD \cup WDD^\perp D^\perp \cup \mathbf{e}DD^\perp$. We see that type structure of $\mathbf{n}_{E\to E}$ induces a decomposition of the binary string $\mathbf{b}$ into n binary strings of two digit, i.e.

$$\mathbf{b} = \mathbf{w}_1\mathbf{w}_2\dots\mathbf{w}_n =$$
$$= w_1^A\, w_1^B \;\; w_2^A\, w_2^B \;\; \dots \;\; w_n^A\, w_n^B$$
$$w_i^E = 0, 1, \quad i = 1, \dots n, \quad E = A, B.$$

Let us then consider the following permuted string

$$\tilde{\mathbf{b}} := w_n^A \ldots w_2^A w_1^A w_1^B w_2^B \ldots w_n^B .$$

Thanks to Equation (5.48), we have

$$\begin{aligned} \mathbf{b} \in D_n^{A\to B} \iff & \tilde{\mathbf{b}} \text{ starts from the right} \\ & \text{with an even number of 1s,} \\ \iff & \tilde{\mathbf{b}} \in D_{2n}^E \end{aligned} \tag{5.51}$$

where the superscript to the sets D_n reminds us that we are considering two different comb hierarchies. We can therefore prove the following equivalence between types.

Proposition 6 (Equivalence between $\mathbf{n}_{A\to B}$ and $\mathbf{2n}_E$)**.** *Let A_i , B_i , $1 \leq i \leq n$ be elementary types. Then following equivalence holds:*

$$\begin{aligned} &(\cdots((A_1 \to B_1) \to (A_2 \to B_2))\cdots) \to (A_n \to B_n) \equiv \\ &\equiv (\cdots(A_n \to A_{n-1})\cdots \to A_1) \to B_1)\cdots) \to B_n \end{aligned} \tag{5.52}$$

Proof. The identity $\Delta_n^{A\to B}$ and Δ_{2n}^E follows from Equation (5.51), which holds unchanged also in the more general case in which the elementary types have different dimensions. Then, with the help of Equation (5.46) one can verify that $\lambda_n^{A\to B} = \lambda_{n2}^E$. ■

The proof of Equation (5.51), which leads to the non trivial type equivalence (5.52), is an example highlighting the relevance of the formalism introduced in Subsection (b).

We now further investigate the comb hierarchy of types $\mathbf{n}_{A\to B}$. From this point to the end of this subsection, the subscripts n or m or p will refer to the comb hierarchy, namely the types $\mathbf{n}_{A\to B}$.

From the type equilavence of Equation (5.52) and from Equation (5.48), we recover the usual normalization condition for comb:

$$R^{(n)} \in \mathsf{T}_1(\mathbf{n}_{A\to B}) \iff \begin{cases} R^{(n)} \geq 0 \\ \mathrm{Tr}_{2k}[R^{(k)}] = I_{2n-1} \otimes R^{(k-1)} \\ R^{(0)} = 1, \quad k = 1, \ldots, n, \end{cases}$$

$$E_i = \begin{cases} A_{n-i+1} & 1 \leq i \leq n \\ B_{i-n} & n+1 \leq i \leq 2n, \end{cases}$$

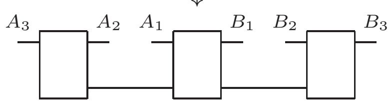
where Tr_i and I denote the partial trace and the identity operator on the Hilbert space of the system E_i. As it is well known, n-comb can be realized as causally order *quantum network* with n vertices (i.e. a sequence of channels with memory). For example we have

$$((((A_3 \to A_2) \to A_1) \to B_1) \to B_2) \to B_3$$

$$|||$$

$$((A_1 \to B_1) \to (A_2 \to B_2)) \to (A_3 \to B_3)$$

$$\updownarrow$$



Thanks to Equation (5.51) it easy to prove that, for $p = n + m$,

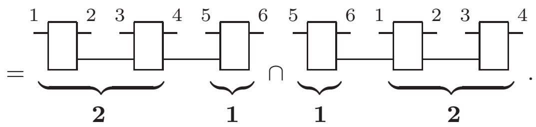
$$\begin{aligned} D_p = & \mathbf{e}_n D_m \cup D_n \mathbf{e}_m \cup \\ & \cup D_n D_m \cup \overline{D}_n \otimes D_m . \end{aligned} \tag{5.53}$$

By combining Equation (5.37) and Equation (5.53) we obtain the characterization of the type $\mathbf{n} \otimes \mathbf{m}$, i.e.

$$\begin{aligned}\Delta_{m\otimes n} =& \mathsf{L}_{\mathbf{e}\,n} \otimes \Delta_m \oplus \Delta_n \otimes \mathsf{L}_{\mathbf{e}\,m} \\ &\oplus \Delta_n \otimes \Delta_m = \\ =& \Delta_{m+n} \cap \Delta_{\sigma(m+n)}\end{aligned} \tag{5.54}$$

where $\sigma(m+n)$ is the permutation that exchanges the the m comb with the n comb, for example

$$\Delta_{2\otimes 1} = \Delta_{2+1} \cap \Delta_{\sigma(2+1)} =$$



Moreover, it is easy to verify that

$$\lambda_{m+n} = \lambda_{\sigma(m+n)}, \tag{5.55}$$

which, together with Equation (5.54) gives

$$\mathsf{T}_1(\mathbf{m} \otimes \mathbf{n}) = \mathsf{T}_1(\mathbf{m} + \mathbf{n}) \cap \mathsf{T}_1(\sigma(\mathbf{m} + \mathbf{n})). \tag{5.56}$$

Finally, let us consider the type $\mathbf{n} \to \mathbf{m}$. By definition we have $\mathbf{n} \to \mathbf{m} = \mathbf{n} \to (\mathbf{m} \to \mathbf{1})$ and from Proposition 4 we have $\mathbf{n} \to \mathbf{m} \equiv (\mathbf{n} \otimes \mathbf{m} - \mathbf{1}) \to \mathbf{1}$. Then, from Equation (5.19) together with Equation (5.54), we have

$$\begin{aligned}\Delta_{n\to m} =& \mathsf{Herm}(\mathcal{H}_{n\otimes(m-1)}) \otimes \Delta_1 \\ &\oplus \overline{\Delta}_{n\otimes(m-1)} \otimes (\mathsf{L}_{\mathbf{e}\,1} \oplus \overline{\Delta}_1) = \\ =& \operatorname{span}(\Delta_{(n+(m-1))\to 1} \cup \Delta_{\sigma(n+(m-1))\to 1}) \\ \Longrightarrow & \mathsf{T}_1(\mathbf{n} \to \mathbf{m}) = \\ =& \mathsf{Aff}\{\mathsf{T}_1((\mathbf{n} + (\mathbf{m} - \mathbf{1})) \to \mathbf{1}) \cup \\ & \mathsf{T}_1(\sigma(\mathbf{n} + (\mathbf{m} - \mathbf{1})) \to \mathbf{1})\}\end{aligned} \tag{5.57}$$

where in the last step we used Equation (5.55).

6. The inverse characterization problem

In the previous section we studied the following problem: given a type x characterize the convex set $\mathsf{T}_1(x)$ of deterministic events of type x. From Proposition 1 we have that the solution to this problem amounts to the evalution of the function

$$\begin{aligned}\Upsilon : \mathsf{Types} &\to \mathbb{R} \times \mathsf{S}(\mathsf{Traceless}(\mathcal{H}_x)) \\ x &\mapsto \begin{pmatrix}\Upsilon_1(x) = \lambda_x \\ \Upsilon_2(x) = \Delta_x\end{pmatrix}\end{aligned} \tag{6.1}$$

where $\mathsf{S}(\mathsf{Traceless}(\mathcal{H}_x))$ denotes the set of real subspaces of $\mathsf{Traceless}(\mathcal{H}_x)$. Both Υ_1 and Υ_2 can be evaluated by recursively applying Equations (5.20) and (5.19). All the relevant information about higher-order quantum theory is encoded in the map Υ and in the cone of positive operators. For example, let us consider the set $\Upsilon_2(\mathsf{Types})$, i.e. the range of the map Υ_2. This set contains a relevant information about the mathematical structure of quantum theory, namely what are the linear subspaces that are relevant in higher-order quantum theory. From this point of view, it is obvious that the set of quantum transformations and higher-order maps exhibits a much richer structure than the set of normalized quantum states, which are simply all the positive operators with unit trace.

For example, one could wonder what is the image under the map Υ_2 of the set of types which have the same Hilbert space (i.e. $\mathcal{H}_x = \mathcal{H}_y = \mathcal{H}$ for two types x and y). More generally, we can address the following question:

Inverse characterization problem: *Given an Hilbert space $\mathcal{H}$ and a linear subspace $\Delta \subseteq \mathsf{Traceless}(\mathcal{H})$, which are the a types x such that $\mathcal{H}_x = \mathcal{H}$ and $\Delta = \Delta_x$ (if any)?*

Roughly speaking, the inverse characterization problem amounts to computing the inverse map ${\Upsilon_2}^{-1}$. This is a much harder task than the direct one. We now address an instance of this problem, which we find particularly instructive.

Let $\mathcal{H} := \mathbb{C}^2 \otimes \mathbb{C}^2$ and $\Delta := \mathsf{Traceless}(\mathbb{C}^2) \otimes \mathsf{Traceless}(\mathbb{C}^2)$ and let us suppose that there exists a type z such that $\mathcal{H}_z = \mathcal{H}$ and $\Delta = \Delta_z$. First we notice that z cannot be an elementary type. If $z = A$ with $\mathcal{H}_A = \mathcal{H}$ it must be $\Delta_A = \mathsf{Traceless}(\mathcal{H})$ and $\dim(\mathsf{Traceless}(\mathcal{H})) = 15$ while $\dim(\Delta) = 9$. Let us then suppose that $z = x \to y$. Since $\dim(\mathcal{H}_z) = 4$ we must have $\dim(\mathcal{H}_x)\dim(\mathcal{H}_y) = 4$. Moreover, since $I \to y \equiv y$ we suppose that $\dim(\mathcal{H}_x) > 1$. We have therefore the following two possibilities:

$$\dim(\mathcal{H}_x) = 4 \text{ and } \dim(\mathcal{H}_y) = 1 \text{ or}$$
$$\dim(\mathcal{H}_x) = 2 \text{ and } \dim(\mathcal{H}_y) = 2.$$

From Equation (5.19) we have

$$\begin{aligned} \Delta_z &= [\mathsf{Herm}(\mathcal{H}_y) \otimes \overline{\Delta_x}] \oplus [\Delta_y \otimes (\mathsf{L_e} \oplus \Delta_x)] \implies \\ 9 = \dim(\Delta_z) &= d_y^2(d_x^2 - 1 - a_x) + a_y(1 + a_x) \end{aligned} \tag{6.2}$$

where $d_x = \dim(\mathcal{H}_x)$, $d_y = \dim(\mathcal{H}_y)$, $a_x = \dim(\Delta_x) < d_x^2$ and $a_y = \dim(\Delta_y) < d_y^2$. If we assume $d_y = 1$ and $d_x = 4$, i.e. $z \equiv x \to I$, then we must have $a_x = 6$. Since for any elementary type E we must have $a_E = d_E^2 - 1$, the type x cannot be elementary. Then there must exist $f \neq I$ and g such that $x = f \to g$. Since $\overline{\overline{x}} = x$ we must have that $g \neq I$, in order to avoid the tautology $z \equiv (z \to I) \to I$. Then, since $d_x = 4$, we must have $d_f = d_g = 2$ and then

$$6 = \dim(\Delta_{f\to g}) = 4(4 - 1 - a_f) + a_g(1 + a_f)$$

which cannot be satisfied for any couple a_x, a_y such that $0 \leq a_f, a_g \leq 3$. Therefore the case $d_x = 4$, $d_y = 1$ must be discarded. Let us then consider the case $d_x = d_y = 2$. Eq. (6.2) gives

$$9 = 4(4 - 1 - a_x) + a_y(1 + a_x)$$

which cannot be satisfied for any couple a_x, a_y such that $0 \leq a_x, a_y \leq 3$.

This result shows that, given $\mathcal{H}$ and $\Delta \subseteq \mathsf{Traceless}(\mathcal{H})$, it might be the case that there exists no type such that $\mathcal{H}_x = \mathcal{H}$ and $\Delta = \Delta_x$. Notice that this no-go result holds also in the simplified scenario where $\mathcal{H}$ is specified from the beginning as the tensor product of elementary type spaces. Therefore, for a given Hilbert space, the characterization of the set of subspaces $\Delta \subseteq \mathsf{Traceless}(\mathcal{H})$ which correspond to some type is far from trivial. In comparison to the first item in the notion of admissibility, that reduces to the positivity requirement, the second one, which involves the notion of deterministic, entails a much more complex mathematical structure.

7. Conclusions

We formulated a fully operational framework for higher-order quantum theory based on a set of axioms regarding the notion of admissible transformation. This definition is recursive, and requires a type system in the first place, allowing for the labelling of sets of transformations, basically through their common domain and range. This structure is shared with classical *typed* *lambda calculus* [32], where the typing rules are necessary to select well-formed expressions. We provide a recursive characterization of maps of an arbitrary type, which is then used to prove a set of basic type equivalences.

Although some similarities, it is worth stressing that our framework fundamentally differs from the works on denotational semantics for a quantum programming language and quantum lambda calculus [33–38]. In particular, one of the goals of our approach is to encompass quantum computation without a definite causal order. For example, the quantum SWITCH map, which we previously described, is a paradigmatic example of a higher order map which our formalism can describe, but that lies outside the framework of Ref [35], as first noticed in Ref. [10]. A categorical framework closely related to the one presented in this contriburion, has been presented by Kissinger and Uijlen in Ref. [21]. They introduce a categorical construction which sends certain compact closed categories $\mathcal{C}$ to a new category $\mathrm{Caus}[\mathcal{C}]$. This procedure can be applied to Selinger's CPM construction of Ref. [39], which does not take normalization, and hence causality, into account. On one hand, by combining this two results, one obtains the hierarchy of higher order quantum maps of our framework. On the other hand, from a foundational perspective, assuming CPM's construction amounts to assume complete positivity for all the maps in the hierarchy without any physical motivation. Moreover, several assumptions of the framework in Ref. [21], for example that second-order causal processes factorise, are also not operationally justified. The main goal of our work is to give a fully operational (i.e. avoiding explicit reference to the mathematical properties of maps in the hierarchy) formulation of higher order quantum theory which can encompass indefinite causal structures. In particular, we gave an operational definition of admissibility which does not assume complete positivity. In our setting, the proof that any positive operator (up to suitable and necessary rescaling) is an admissible higher order map is nontrivial. On the other hand, by assuming CPM and Ref. [21] construction, this same result becomes a rather straightforward observation.

Higher-order quantum theory must be thought of as an extension of quantum theory, which provides a natural unfolding of a part of the theory that is implicitly contained in any of its formulations. As such, higher-order quantum theory has a fundamental value, being a new standpoint for the analysis of the peculiarities of quantum theory. The axioms of our framework have a purely operational nature and do not rely on the specific mathematical structure of quantum theory. Therefore, with proper care, our framework can be applied to general probabilistic theories. In particular the most important ingredient we used is the Choi isomorphism, that can be always provided in theories where local discriminability holds. If the latter does not hold one must reformulate the recursive definition of admissible events avoiding the Choi correspondence. In this case, since parallel composition is not simply translated in the tensor product rule, a transformation is not simply a single matrix, but a possibly infinite family of matrices representing the action of the map on all possible extended systems.

The framework that we introduced leads to several open problems. An interesting question is to determine what types, if any, can be attributed to a given subspace of linear maps. An even harder problem is to determine all the possible types of a given linear map.

In this work, we proved a family of equivalences between types of higher order maps. Therefore, another question that naturally arises is whether there exists a complete set of type equivalences, i.e. a set of type equivalences such that their compositions provide an alternative characaterization of the hierarchy of higher order quantum maps. Moreover, following the case of causally ordered quantum networks, one would like to infer the causal structure of an higher order map from its type.

Finally, the present work only partially addresses the composition of types. It is implicit in our definition that, given a map of type x and a map of type $x \to y$, they can be composed and give a map of type y. However, our formalism does not provide any formal rules which would translate a partial application of a higher order map (apart from the easiest case of the extension with an elementary type). In order to have a theory of computation, a comprehensive set of rules that encompasses all the admissible composition of maps must be given.

A. Proof of Theorem 2

Lemma 6. *$X \in \mathsf{T}(x)$ if and only if it satisfies item (i) of definition 7, and there exist $\{X_i\}_{i=1}^n \subseteq \mathsf{T}(x)$ such that $X + \sum_{i=1}^n X_i \in \mathsf{T}_1(x)$.*

Proof. The proof proceeds by induction. The statement is straightforwardly true for $x \in \mathsf{EleTypes}$. Let it now be true for any y and $y \parallel E'$, for arbitrary $E' \in \mathsf{EleTypes}$ and $y \preceq x$. We need to prove that the statement holds for $x \parallel E$ for any arbitrary $E \in \mathsf{EleTypes}$. If x is not elementary we can write $x \parallel E = (y \to z) \parallel E = y \to z \parallel E$ for some $y, z \preceq x$.

Clearly, if $X \in \mathsf{T}(x \parallel E)$, by definition 7 there must exist $\{X_i\}_{i=1}^n \subseteq \mathsf{T}(x \parallel E)$ such that, upon defining $X_0 := X$ and $D := \sum_{i=0}^n X_i$, one has $[\mathsf{Ch}^{-1}(D) \otimes \mathcal{I}_{E'}](\mathsf{T}_1(y \parallel E')) \subseteq \mathsf{T}_1(z \parallel EE')$. Now, for $Y_0 \in \mathsf{T}(y \parallel E')$, there exist $\{N_j\}_{j=1}^k \subseteq \mathsf{T}(y \parallel E')$ such that, by the induction hypothesis, $G := Y_0 + \sum_{j=1}^k N_j \in \mathsf{T}_1(y \parallel E')$. Thus,

$$
\begin{aligned}
&[\mathsf{Ch}^{-1}(D) \otimes \mathcal{I}_{E'}](G)\\
&\qquad = \sum_{j=0}^k [\mathsf{Ch}^{-1}(D) \otimes \mathcal{I}_{E'}](Y_j) \in \mathsf{T}_1(z \parallel EE'),
\end{aligned}
$$

which means that $[\mathsf{Ch}^{-1}(D) \otimes \mathcal{I}_E](Y_0)$ is admissible, again using the induction hypothesis. Then D satisfies item (i) of definition 7.

The proof of the converse statement is trivial. ■

Lemma 7. *If $X, X' \in \mathsf{T}(x)$, then $X + X' \in \mathsf{T}(x)$ if and only if there exist $\{X_i\}_{i=1}^n \subseteq \mathsf{T}(x)$ such that $X + X' + \sum_{i=1}^n X_i \in \mathsf{T}_1(x)$.*

Proof. The direct statement can be proved by the same technique as for lemma 6. Now, for the converse, we proceed by induction. Suppose that the statement holds for y and $y \parallel E$, for every $y \prec x$ and $E \in \mathsf{EleTypes}$. Suppose now that $X, X' \in \mathsf{T}(x \parallel E)$, and that $\{X_i\}_{i=1}^n$ exists such that $D := X + X' + \sum_{i=1}^n X_i \in \mathsf{T}_1(x \parallel E)$. Since $\mathsf{Ch}^{-1}(X)$ and $\mathsf{Ch}^{-1}(X')$ satisfy item (i), then for every $Y \in \mathsf{T}(y \parallel E')$, both $[\mathsf{Ch}^{-1}(X) \otimes \mathcal{I}_{E'}](Y)$ and $[\mathsf{Ch}^{-1}(X) \otimes \mathcal{I}_{E'}](Y)$ are in $\mathsf{T}((z \parallel E) \parallel E') = \mathsf{T}(z \parallel EE')$. Moreover, there are $\{Y_i\}_{i=1}^n$ such that $Y_0 := Y + \sum_{j=1}^m Y_i \in \mathsf{T}_1(y \parallel E')$, and thus $[\mathsf{Ch}^{-1}(D) \otimes \mathcal{I}_E](Y_0) \in \mathsf{T}_1(z \parallel EE')$. On the other hand,

$$
\begin{aligned}
&[\mathsf{Ch}^{-1}(D) \otimes \mathcal{I}_{E'}](Y_0)\\
&= [\mathsf{Ch}^{-1}(X) \otimes \mathcal{I}_{E'}](Y) + [\mathsf{Ch}^{-1}(X') \otimes \mathcal{I}_{E'}](Y)\\
&+ \sum_{i=1}^n [\mathsf{Ch}^{-1}(X_i) \otimes \mathcal{I}_{E'}](Y) + \sum_{j=1}^m [\mathsf{Ch}^{-1}(D) \otimes \mathcal{I}_{E'}](Y_j).
\end{aligned}
$$

By the induction hypothesis, the above condition assures us that $[\mathsf{Ch}^{-1}(X) \otimes \mathcal{I}_{E'}](Y) + [\mathsf{Ch}^{-1}(X') \otimes \mathcal{I}_{E'}](Y) \in \mathsf{T}(z \parallel EE')$, and thus $[\mathsf{Ch}^{-1}(X + X') \otimes \mathcal{I}_{E'}](Y) \in \mathsf{T}(z \parallel EE')$. This implies that $\mathsf{Ch}^{-1}(X + X') \otimes \mathcal{I}_{E'}$ maps $\mathsf{T}(y \parallel E')$ into $\mathsf{T}(z \parallel EE')$. Thus, all the requirements of definition 7 are fulfilled by $X + X'$, and $X + X' \in \mathsf{T}(x \parallel E)$ for any E. ■

Lemma 8. *$X \in \mathsf{T}_{\mathbb{R}}(x)$ is admissible if and only if it satisfies item (i) of definition 7 and there exists X' satisfying item (i) such that $X + X' \in \mathsf{T}_1(x)$.*

Proof. Let X and X' satisfy item (i) of definition 7 and $X + X' \in \mathsf{T}_1(x)$. Then $X, X' \in \mathsf{T}(x)$. Viceversa, if X is admissible then it satisfies item (i) and there exist $\{X_i\}_{i=1}^n$ such that, for every $1 \leq i \leq n$, X_i satisfies item (i), and $X + \sum_{i=1}^n \in \mathsf{T}_1(x)$. Thus, for every $1 \leq i \leq n$ it is $X_i \in \mathsf{T}(x)$. By iterating lemma 7, we have $S := \sum_{i=1}^n X_i \in \mathsf{T}(x)$. Moreover, clearly $X + S \in \mathsf{T}_1(x)$. ■

Corollary 5. *$X\in\mathsf{T}_{\mathbb{R}}(x)$ is admissible if and only if it satisfies item (i) of definition 7 and there exists $X'\in\mathsf{T}(x)$ such that $X+X'\in\mathsf{T}_1(x)$.*

Lemma 9. *Let $X\in\mathsf{T}(x)$, and $\rho\in\mathsf{T}(E)$. Then $X\otimes\rho\in\mathsf{T}(x\parallel E)$. Moreover, if $X\in\mathsf{T}_1(x)$ and $\rho\in\mathsf{T}_1(E)$, then $X\otimes\rho\in\mathsf{T}_1(x\parallel E)$.*

Proof. Also in this case we proceed by induction. The statement is true for $x\in\mathsf{EleTypes}$. Let now the statement be true for $y\parallel F'$ for any $y\preceq x$ and arbitrary $F'\in\mathsf{EleTypes}$. Since x is not an elementary type, we have $x\parallel F=y\to z\parallel F$ for some $y,z\preceq x$. Let $X\in\mathsf{T}(x\parallel F)$. Let $R\in\mathsf{T}(y\parallel F')$. Then $[\mathsf{Ch}^{-1}(X\otimes\rho)\otimes\mathcal{I}_{F'}](R)=[\mathsf{Ch}^{-1}(X)\otimes\mathcal{I}_{F'}](R)\otimes\rho$, and since by definition $[\mathsf{Ch}^{-1}(X)\otimes\mathcal{I}_{F'}](R)\in\mathsf{T}(z\parallel FF')$, we have by the induction hypothesis that $[\mathsf{Ch}^{-1}(X\otimes\rho)\otimes\mathcal{I}_{F'}](R)=[\mathsf{Ch}^{-1}(X)\otimes\mathcal{I}_{F'}](R)\otimes\rho\in\mathsf{T}(z\parallel FF'E)$. Thus, $[\mathsf{Ch}^{-1}(X\otimes\rho)\otimes\mathcal{I}_{F'}][\mathsf{T}(y\parallel F')]\subseteq\mathsf{T}(z\parallel FF'E)$ for every $F'\in\mathsf{EleTypes}$. Now, if X is admissible, then by lemma 8, there is X' such that $\mathsf{Ch}^{-1}(D)$ is a deterministic map of type $x=y\to z\parallel F$, where $D:=X+X'$, and $[\mathsf{Ch}^{-1}(X')\otimes\mathcal{I}_{F'}][\mathsf{T}(y\parallel F')]\subseteq\mathsf{T}(z\parallel FF')$ for every $F'\in\mathsf{EleTypes}$. Similarly, there is $\sigma\in\mathsf{T}_1(E)$ such that $\sigma\geq\rho$. Thus, $[\mathsf{Ch}^{-1}(D\otimes\sigma)\otimes\mathcal{I}_{F'}](Y)=[\mathsf{Ch}^{-1}(D)\otimes\mathcal{I}_{F'}](Y)\otimes\sigma$ which is deterministic by the induction hypothesis, and $D\otimes\sigma=X\otimes\rho+X'\otimes\rho+X\otimes\tau+X'\otimes\tau$, where $\tau:=\sigma-\rho\geq 0$. Thus $X\otimes\rho$ is admissible. As to the second item in the thesis, if $X\in\mathsf{T}_1(x\parallel F)$, $\rho\in\mathsf{T}_1(E)$, and $Y\in\mathsf{T}_1(y\parallel F')$, then $[\mathsf{Ch}^{-1}(X)\otimes\mathcal{I}_{F'}](Y)\in\mathsf{T}_1(z\parallel FF')$ and thus, by the induction hypothesis, $[\mathsf{Ch}^{-1}(X\otimes\rho)\otimes\mathcal{I}_{F'}](Y)\in\mathsf{T}_1(z\parallel FF'E)$, and thus $[\mathsf{Ch}^{-1}(X\otimes\rho)\otimes\mathcal{I}_{F'}][\mathsf{T}_1(y\parallel F')]\subseteq\mathsf{T}_1(z\parallel FF'E)$. ■

We now prove the following crucial lemma. Let $\mathsf{T}_+(x)$ denote the set $\{P\in\mathsf{T}_{\mathbb{R}}(x)\mid\exists\lambda\geq 0,\ P'\in\mathsf{T}(x):\ P=\lambda P'\}$.

Lemma 10. *For every type $x\in\mathsf{Types}$, the set $\mathsf{T}_+(x)$ is the full positive cone in $\mathcal{L}(\mathcal{H}_x)$.*

Proof. Let us restate the hypothesis as follows. For every type $x\in\mathsf{Types}$, the sets $\mathsf{T}_+(x)$, $\mathsf{T}_+(x\to I)$, and $\mathsf{T}_+(x\parallel E)$ are the full positive cones in $\mathcal{L}(\mathcal{H}_x)$, $\mathcal{L}(\mathcal{H}_x)$, and $\mathcal{L}(\mathcal{H}_x\otimes\mathcal{H}_E)$, respectively. This new form of the thesis is amenable to a proof by induction as follows. The thesis holds for elementary types. Let now $x=y_1\to y_2$, and let us suppose that the thesis holds for y_1 and y_2. In the first place, this implies that a necessary condition for M to be admissible is that M is the Choi of a completely positive map, and thus it must be positive. We have then that the set $T_+(x)$ is contained in the cone of positive operators in $\mathcal{L}(H_x)$. Moreover, the induction hypothesis implies that there exist full-rank elements of type $\mathsf{T}_1(y_1\to I)$ and $\mathsf{T}_1(y_2)$. Let $\overline{Y}_1$ and Y_2 denote two such elements. We now claim that $X:=\overline{Y}_1\otimes Y_2$ is proportional to an admissible element of type x. Indeed, let Y_{1E} denote an arbitrary admissible element of type $y_1\parallel E$. One can easily check that $[\mathsf{Ch}^{-1}(X)](Y_{1E})=\rho_E\otimes Y_2$, where $\rho_E:=[\mathsf{Ch}^{-1}(\overline{Y}_1)](Y_{1E})$. Now, by lemma 9, $\rho_E\otimes Y_2$ is admissible. Thus, $\mathsf{Ch}^{-1}(X)$ maps admissible elements of $\mathsf{T}(y_1\parallel E)$ to admissible elements of $\mathsf{T}(y_2\parallel E)$. Moreover, $\mathsf{Ch}^{-1}(\overline{Y}_1)\otimes\mathcal{I}_E$ maps deterministic elements of $\mathsf{T}_1(y_1\parallel E)$ to elements of the form $\rho_E\otimes Y_2$, with $\rho_E\in\mathsf{T}_1(E)$ and $Y_2\in\mathsf{T}_1(y_2)$. By lemma 9 these elements are in $\mathsf{T}_1(y_2\parallel E)$. Finally, we proved that $X=Y_2\otimes\overline{Y}_1$ satisfies the conditions for an admissible element of $\mathsf{T}_1(y_1\to y_2)$, and it is full-rank. Exactly the same argument can be used to prove the statement for $x\parallel F=y_1\to y_2\parallel F$. Finally, for the case $x=(y_1\to y_2)\to I$ one can easily check that $\overline{Y}_2\otimes Y_1$ is in $\mathsf{T}_{\mathbb{R}}((y_1\to y_2)\to I)$. Moreover, it is admissible since $[\mathsf{Ch}^{-1}(\overline{Y}_2\otimes Y_1)\otimes\mathcal{I}_E](X)=X_E$ corresponds to the application of $\mathsf{Ch}^{-1}(\overline{Y}_2)$ to $[\mathsf{Ch}^{-1}(X)](Y_1)$. Now, since $\mathsf{Ch}^{-1}(X)$ is admissible by hypothesis as well as $\mathsf{Ch}^{-1}(Y_1)$ and $\mathsf{Ch}^{-1}(\overline{Y}_2)$, what we get in the end is an admissible element of $\mathsf{T}(E)$. Moreover, $\overline{Y}_2\otimes Y_1$ is deterministic, since $[\mathsf{Ch}^{-1}(X)](Y_1)$ is deterministic for deterministic X, and thus we have that $X_E=[\mathsf{Ch}^{-1}(\overline{Y}_2\otimes Y_1)\otimes\mathcal{I}_E](X)$ is deterministic for any deterministic X. Moreover, $\overline{Y}_2\otimes Y_1$ is full rank. Now, let $M\geq 0$ be a positive operator on $\mathcal{H}_x$. Then, since $\overline{Y}_1\otimes Y_2$ is positive and full-rank, there exist a positive coefficient λ such that

$\lambda M \leq \overline{Y}_1 \otimes Y_2$. One can then easily verify that λM and $\overline{Y}_1 \otimes Y_2 - \lambda M$ satisfy the hypotheses of Definition 7. Therefore the positive cone in $\mathcal{L}(\mathcal{H}_x)$ is contained in $T_+(x)$. The same result can be proved for $x \to I$ and $x \parallel E$. ■

We are now ready to prove theorem 2

Proof. By lemma 10, if $M \in \mathsf{T}(x)$ then $M \geq 0$. Moreover, by corollary 5 there exists $M' \geq 0$ such that $D := M + M' \in \mathsf{T}_1(x)$. Thus, $M' = D - M \geq 0 \implies M \leq D$.

To prove the converse, let us consider a non elementary type $x \parallel E$, i.e. $x = y \to z \parallel E$, for an arbitrary elementary type E (the thesis is trivially true if x is elementary). Let us suppose that the thesis holds for any $y \parallel E$, $y \preceq x$ and consider $0 \leq M \leq D \in \mathsf{T}_1(x \parallel E)$, $N := D - M$. From the induction hypothesis, one can verify that M and N satisfy the hypothesis of Definition 7 and therefore M is an admisible event of type $x \parallel E$ for arbitrary E. ■

B. Proof of Theorem 3

The proof relies on the following preliminary results.

Lemma 11. *For arbitrary x and y, consider the type $x \to y$. Let $R \in \mathsf{T}_{\mathbb{R}}(x \to y)$ such that $R \geq 0$ and, for all E, $[\mathsf{Ch}^{-1}(R) \otimes \mathcal{I}_E](\mathsf{T}_1(x \parallel E)) \subseteq \mathsf{T}_1(y \parallel E)$. Then $R \in \mathsf{T}_1(x \to y)$*

Proof. We first need to show that $[\mathsf{Ch}^{-1}(R) \otimes \mathcal{I}_E](\mathsf{T}(x \parallel E)) \subseteq \mathsf{T}(y \parallel E)$. Let us fix an arbitrary $O_{x\parallel E} \in \mathsf{T}(x \parallel E)$. From Theorem 2 we have that $0 \leq O_{x\parallel E} \leq D_{x\parallel E}$ for some $D_{x\parallel E} \in \mathsf{T}_1(x \parallel E)$. Since $R \geq 0$, the map $\mathsf{Ch}^{-1}(R)$ is completely positive. Therefore we have $[\mathsf{Ch}^{-1}(R) \otimes \mathcal{I}_E](D_{x\parallel E} - O_{x\parallel E}) \geq 0$, which implies

$$0 \leq [\mathsf{Ch}^{-1}(R) \otimes \mathcal{I}_E](O_{x\parallel E}) \\ \leq ([\mathsf{Ch}^{-1}(R) \otimes \mathcal{I}_E](D_{x\parallel E}) \in \mathsf{T}_1(y \parallel E).$$

We conclude that $[\mathsf{Ch}^{-1}(R) \otimes \mathcal{I}_E](O_{x\parallel E}) \in \mathsf{T}(y \parallel E)$. Finally, using the hypothesis that $[\mathsf{Ch}^{-1}(R) \otimes \mathcal{I}_E](\mathsf{T}_1(x \parallel E)) \subseteq \mathsf{T}_1(y \parallel E)$, by theorem 2 the thesis follows. ■

Lemma 12. *For every $E, E' \in \mathsf{EleTypes}$, and for every $R \in \mathsf{T}(x \parallel EE')$ one has that*

$$R \in \mathsf{T}_1(x \parallel EE') \iff \mathrm{Tr}_E[R] \in \mathsf{T}_1(x \parallel E') \tag{A 1}$$

Proof. The proof is by induction. For $x \in \mathsf{EleTypes}$ the thesis is easily verified. Let us suppose that the thesis holds for any $y \preceq x$, and let us write $x = y \to z$. Clearly, since $R \in \mathsf{T}(x \parallel EE')$, by lemma 10 we have that $R \geq 0$. By lemma 11 a necessary and sufficient condition for $R \in \mathsf{T}_1(y \to z \parallel EE')$ is then that $[\mathsf{Ch}^{-1}(R) \otimes \mathcal{I}_F](\mathsf{T}_1(y \parallel F)) \subseteq \mathsf{T}_1(z \parallel EE'F)$. Let us now fix an arbitrary F and an arbitrary $D \in \mathsf{T}_1(y \parallel F)$. By the induction hypothesis, a necessary and sufficient condition for $[\mathsf{Ch}^{-1}(R) \otimes \mathcal{I}_F](D)$ to be in $\mathsf{T}_1(z \parallel EE'F)$ is that

$$\mathrm{Tr}_F[(\mathsf{Ch}^{-1}(R) \otimes \mathcal{I}_F)(D)] \in \mathsf{T}_1(z \parallel EE').$$

However, we have

$$\mathrm{Tr}_F[\mathsf{Ch}^{-1}(R) \otimes \mathcal{I}_F](D)] = [\mathsf{Ch}^{-1}(R)](\mathrm{Tr}_F[D]).$$

We can now rewrite the necessary and sufficient condition for $R \in \mathsf{T}_1(y \to z \parallel EE')$ as

$$\mathrm{Tr}_{yF}[(D^T \otimes I_{z\parallel EE'})(R \otimes I_F)] \in \mathsf{T}_1(z \parallel EE').$$

By the induction hypothesis, again this is equivalent to

$$\mathrm{Tr}_{yEF} = [(D^T \otimes I_{z\parallel EE'})(R \otimes I_F)] \in \mathsf{T}_1(z \parallel E'),$$

namely

$$\begin{aligned}&\mathrm{Tr}_{yF}[(D^T\otimes I_{z\|E'})(\mathrm{Tr}_E[R]\otimes I_F)]\\&=\mathsf{Ch}^{-1}(\mathrm{Tr}_E[R])(\mathrm{Tr}_F[D])\\&=\mathrm{Tr}_F[(\mathsf{Ch}^{-1}(\mathrm{Tr}_E[R])\otimes\mathcal{I}_F)(D)]\in\mathsf{T}_1(z\parallel E'),\end{aligned}$$

which, again by the induction hypothesis, is equivalent to $(\mathsf{Ch}^{-1}(\mathrm{Tr}_E[R])\otimes\mathcal{I}_F)(D)\in\mathsf{T}_1(z\parallel E'F)$. If this holds for arbitrary F and arbitrary $D\in\mathsf{T}_1(y\parallel F)$, the above condition is finally equivalent to condition (A 1). ■

Let us now address the proof of Theorem 3.

Proof. By lemma 10 and lemma 11, a necessary and sufficient condition for $M\in\mathsf{T}_1(x\to y)$ is that $M\geq 0$ and $[\mathsf{Ch}^{-1}(M)\otimes\mathcal{I}_E](\mathsf{T}_1(x\parallel E))\subseteq\mathsf{T}_1(y\parallel E)$. Now, for $M\geq 0$, by lemma 12 the above necessary and sufficient condition is equivalent to the requirement that for every $D\in\mathsf{T}_1(x\parallel E)$ one has

$$\mathrm{Tr}_E[(\mathsf{Ch}^{-1}(M)\otimes\mathcal{I}_E)(D)]=\mathsf{Ch}^{-1}(M)(\mathrm{Tr}_E[D])\in\mathsf{T}_1(y).$$

However, for this condition to hold it is necessary and sufficient that $\mathsf{Ch}^{-1}(M)(\mathsf{T}_1(x))\subseteq\mathsf{T}_1(y)$. ■

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