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Besov wavefront set and germs of distributions on smooth manifolds

Federico Sclavi

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Federico Sclavi

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Chapter 1

Introduction

The topic of this thesis lies at the interface between microlocal analysis, quantum field theory and stochastic processes. Since the 1970s, it has emerged that a number of physical systems are modelled by what is known in the literature as a *stochastic partial differential equation* (SPDE). An SPDE arises as a combination of partial differential equations (PDEs) and of randomness which are both ubiquitous tools used to model several physical phenomena. On the one hand, PDEs have always been exploited to describe macroscopic phenomena such as heat diffusion, electro-magnetic dynamics, interface and fluid dynamics. On the other hand, randomness plays a prominent rôle in dealing with systems with uncertainty or with chaotic microscopic interactions and it is modelled by a stochastic process, which is tipically a Gaussian white noise. Therefore, SPDEs combine the best of both worlds and they arise in applications ranging from statistical mechanics to models for interface growth and many other areas of physics, such as quantum field theory - see [CS20, PW81, MW17]. As a basic example of an SPDE, we can mention the stochastic heat equation which is a linear parabolic PDE with an additive white noise arising, for instance, from a microscopic model of a polymer in a liquid - see [CS20]. At the same time, notable examples of nonlinear SPDEs include the stochastic Schrödinger equation, see [BDR21], arising from models of Bose-Einstein condensates or of superconductivity, the Kardar-Parisi-Zhang (KPZ) equation, see [KPZ86], which is used to model interface dynamics, and the nonlinear parabolic Anderson model, see [CM94, GIP15, HL18], which describes the motion of a massive particle through a random porous media.

In general, the solution theory for linear SPDEs with additive noise is well-understood and it shares the same mathematical challenges as that of their classical counterparts, such as proving existence, uniqueness as well as regularity of the solutions. However, this is not the case for nonlinear SPDEs because of the highly singular nature of the Gaussian white noise and of the underlying nonlinear terms, which are a priori meaningless from a mathematical viewpoint. In this regard, a breakthrough in the analysis of a large class of nonlinear SPDEs has been recently made thanks to the *theory of regularity structures* [Hai14, Hai15] and to *paracontrolled calculus* formulated in [GIP15]. The aim of these novel frameworks is to establish existence and uniqueness of the solutions to an underlying SPDE by means of a fixed point argument. Furthermore, in order to address the problem of singularities arising from nonlinear terms, these approaches adopt a regularization argument similar to that of quantum field theory. Despite the absolute relevance of these frameworks, they fall short in giving any information on the explicit form of the solutions and of their correlation functions, which would be important to make contact with physical experiments.

In order to strengthen the interplay between renormalization and SPDEs, it has been recently developed a novel framework to solve a large class of nonlinear SPDEs, which is based on *algebraic quantum field theory*, a mathematically rigorous approach to quantum field theory - see [DDRZ21, BDR21]. In contrast to the theory of regularity structures and paracontrolled calculus, this approach allows to construct both solutions and correlation functions by means of a perturbative series. Moreover, it has the advantage of providing a rigorous method to discuss renormalization by means of *microlocal analysis*, see [Hör03], a collection of techniques which gives a systematic and detailed description of the singularities of a given distribution. However, this framework suffers of a weak point shared by several approaches to quantum field theory, that is, the lack of any control on the convergence of the perturbative series. This hurdle can be mainly ascribed to the coarseness of microlocal techniques exploited to discuss quantum field singularities. As a matter of fact, in the context of quantum field theory, one is interested in establishing if a given physical quantity is either smooth or singular, while in the realm of SPDEs this is by far insufficient since one often considers a more refined class of distributions, elements of a suitable *Besov space*. These function spaces, which shall be discussed in Section 2.1, are endowed with a Banach structure whose norm estimates the scaling degree of a given distribution at any point of an Euclidean space. In addition, in the context of SPDEs, Besov spaces play a prominent rôle in formulating a fixed point argument to prove existence and uniqueness of the solutions to a given SPDE - see [Hai14, GIP15].

Therefore, as a consequence of these remarks, the main contribution of this thesis is to develop the mathematical tools which allow us to better study the behavior of the perturbative solutions to a large class of SPDEs.
To this end, in Chapter 3, we shall introduce a novel microlocal structure, called *Besov wavefront set*, which aims at describing the Besov singularities of an underlying distribution - see [DRS22].
Furthermore, in Chapter 4, we shall formulate the *reconstruction theorem*, one of the cornerstones of the theory of regularity structures, on smooth manifolds - see [RS21]. This formulation shall rely on the framework of *germs of distributions*, introduced in [CZ20], which allows to formulate and to prove the reconstruction theorem in the language of distribution theory without any reference to regularity structures. In addition, this generalization can be read as a first step to extend the Hairer's framework to smooth manifolds. As a matter of fact, on account of quantum field theory on curved spacetimes, it would be desiderable to formulate the theory of regularity structures on arbitrary smooth manifolds in order to strengthen the interplay between SPDEs and quantum field theory.

Synopsis: Chapter 2 is devoted to recalling the main notions and tools which shall be used in Chapters 3 and 4.

In Section 2.1, we shall give an overview of the theory of Besov spaces. In particular, we shall focus on recalling two equivalent characterizations. In Subsection 2.1.1, we shall discuss the Fourier-analytical formulation of Besov spaces, which is defined starting from a Littlewood-Paley partition of unity. At the same time, in Subsection 2.1.5, we shall discuss the characterization of Besov distributions by means of a scaling property. This characterization shall be the one mostly used in this thesis and it shall play a key rôle in Chapter 3. Furthermore, in Subsection 2.1.6, we shall recall the definition of Hölder and Sobolev spaces and their relations with Besov spaces. In particular, we shall emphasize the interplay bewteen Hölder spaces and the class of Besov spaces $B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$, where $\alpha \in \mathbb{R}$.

The Section 2.2 shall be devoted to recollecting the basic notions of the theory of pseudodifferential operators (ΨDOs), which are a generalization of differential operators. More into the detail, in Subsection 2.2.1, we shall give an overview of the theory of symbols. In Subsection 2.2.2, we shall introduce the space of ΨDOs and we shall recall its properties, such as its invariance with respect to the action of diffeomorphisms. In addition, we shall recall how pseudodifferential operators act on Besov spaces. At last, Subsection 2.2.3 shall be devoted to discussing the microlocal behavior of ΨDOs. More precisely, we shall recall the notion of operator wavefront set and that of characteristic set.

In Section 2.3, we shall recollect the basic notions and properties concerning the theory of the smooth wavefront set. More in detail, we shall see that the smooth wavefront set establishes sufficient conditions for the well-posedness of a priori ill-defined operations between distributions, such as the pullback and the product. At last, we shall recall the propagation of singularities theorem for a suitable class of hyperbolic partial differential equations. This result aims at characterizing the smooth wavefront set of a solution to a partial differential equation in terms of the principal symbol of the corresponding operator.

In Section 2.4, we shall present a succinct overview of the theory of germs of distributions [CZ20], which aims at formulating the reconstruction theorem without any reference to regularity structures. More precisely, in Subsection 2.4.1, we shall recall the notion of germ of distributions and that of coherence. In addition, we shall present some examples of coherent germs, such as the one given by the Taylor polynomial of an Hölder function. In Subsection 2.4.2, we shall recall the formulation of the reconstruction theorem in the context of germs of distributions. Lastly, in Subsection 2.4.3, we outline an application of the reconstruction theorem to prove Young's product theorem, which provides sufficient conditions to multiply two Besov distributions.

In Chapters 3, 4, we shall discuss the main contributions of this Ph.D. thesis, namely the *Besov wavefront set* introduced in [DRS22] and the *reconstruction theorem on smooth manifolds* as outlined in [RS21].

In Chapter 3, we shall introduce the notion of Besov wavefront set, which aims at characterizing the directions in Fourier space along which a given distribution lies or does not lie in a suitable Besov space $B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$, where $\alpha \in \mathbb{R}$. Throughout this chapter, we shall emphasize that this form of wavefront set is a refinement of its smooth counterpart.

In Section 3.1, we shall define the Besov wavefront set of an underlying distribution by means of its Fourier transform. Although this definition sets out correctly the notion of Besov wavefront set, it is rather involved to use concretely. For this reason, in Section 3.2, we shall establish two equivalent characterizations of the notion of Besov wavefront set, which shall be very useful from an operational viewpoint. The first one relies on pseudodifferential operators, while the second one characterizes the Besov wavefront set in terms of its smooth counterpart. These characterizations shall play a prominent rôle in the proofs of several structural properties of the Besov wavefront set.

In Section 3.3, we shall outline the structural properties of the Besov wavefront set. Analogously to the its smooth counterpart, this form of wavefront set shall allow us to formulate sufficient criteria for the well-posedness of a few operations between distributions. We shall emphasize that these sufficient conditions are weaker than those formulated by Hörmander in the smooth setting. Without entering into details, we summarize the main results of Section 3.3. In Subsection 3.3.2, given an embedding $f\colon \Omega_1 \to \Omega_2$ between two open sets $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$, we shall establish a criterion for the existence of the pullback of a given distribution along f, which generalizes the one formulated by Hörmander in the context of the smooth wavefront set. As a byproduct, we shall prove that the Besov wavefront set of a distribution is invariant under the action of diffeormophisms. This result shall allow us to define the Besov wavefront set of distributions on a smooth manifold. In Subsection 3.3.3, we shall establish a sufficient criterion for the existence of the product between two distributions with prescribed Besov wavefront sets. This result can be read as a microlocal formulation of Young's product theorem. In Subsection 3.3.4, we shall discuss the interplay between the Besov wavefront set and Schwartz kernels. As a first step, we shall prove an estimate for the Besov wavefront set of $\mathcal{K}u$, where $\mathcal{K}\colon \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ is a linear map with Schwartz kernel $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ while $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ are two open sets. Subsequently, we shall establish a sufficient criterion for the extension of $\mathcal{K}$ to $\mathcal{E}'(\Omega_2)$ by means of the Besov wavefront set of the Schwartz kernel K. This result shall entail a microlocal formulation of Schauder estimates, which are often used to estimate the regularity of a solution to a suitable (stochastic) partial differential equation.

At last, in Subsection 3.3.5, we shall discuss the propagation of singularities within the context of the Besov wavefront set. More precisely, we shall prove a propagation of singularities theorem for a certain class of hyperbolic partial differential equations.

In Section 3.4, we shall present an application of the results obtained in the previous sections in the context of coherent germs of distributions. More precisely, given a coherent germ defined as the tensor product between two distributions $u \in B^{\alpha_1}_{\infty,\infty}(\mathbb{R}^d)$ and $v \in B^{\alpha_2}_{\infty,\infty}(\mathbb{R}^d)$ with $\alpha_1 + \alpha_2 > 0$, we shall prove that its reconstruction coincides with the pullback of the germ along the diagonal, that is the product between u and v. This shall lead us to conjecture that, given an arbitrary coherent germ, its reconstruction coincides with the pullback of the germ along the diagonal.

In Chapter 4, we shall formulate Hairer's reconstruction theorem, one of the main results of the theory of regularity structures, on smooth manifolds. To this end, we shall rely on the framework of germs of distributions introduced in [CZ20]. This framework allows us to formulate and to prove the reconstruction theorem in the language of the theory of distributions, without any reference to regularity structures. For this reason, it shall turn out to be well-suited to our aims.

More precisely, in Section 4.1, we shall extend the notion of coherent germ of distributions to an arbitrary smooth manifold. In addition, we shall prove that the notion of coherence is independent of the choice of the atlas.

In Section 4.2, we shall prove the reconstruction theorem for coherent germs of distributions supported on a smooth manifold.

In Appendix A, we shall summarize the basic notions and results of the theory of distributions, which are used in this thesis.

In Appendix B, we shall discuss the coherence of germs of distributions on an open subset of $\mathbb{R}^d$. It plays a prominent rôle in the formulation of the theory of germs of distributions on smooth manifolds - see Chapter 4.

Chapter 2

Preliminaries

2.1 Besov Spaces

The aim of this section is to recall the main definitions and results concerning the theory of Besov spaces. These spaces have recently found several applications in many fields of mathematics, such as partial differential equations and Fourier analysis, e.g. [BCD11, Hai14, GIP15, BL22A].

In Subsections 2.1.1 and 2.1.5, we shall discuss two different albeit equivalent characterizations of Besov spaces, that is the *Fourier-analytical approach* and the *local means formulation*. At the same time, Sections 2.1.2 and 2.1.3 outline the most important embedding theorems and duality properties of Besov spaces. In Subsection 2.1.4, we shall give a succint overview of *paradifferential calculus*, which addresses the problem of multiplying two Besov distributions. To conclude, Subsection 2.1.6 shall be devoted to recalling Hölder and Sobolev spaces and to emphasizing their relations with Besov spaces. For further details concerning these topics, refer to [BCD11, Sect. 2.7], [Tri06, Chap. 1] and to [Saw18, Chap. 2].

In this section, we shall make use of the notions introduced in Appendix A, where we summarize the basic concepts of the theory of distributions as well as relevant results concerning the Fourier transform. In the following, we denote by $\mathcal{D}(\mathbb{R}^d)$ and $\mathcal{D}'(\mathbb{R}^d)$ the space of compactly supported smooth functions and its topological dual respectively. In addition, $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ stand for the space of rapidly decreasing functions and the space of tempered distributions respectively. At the same time, $\mathcal{E}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$ denotes the space of compactly supported distributions.

Given $u \in \mathcal{S}'(\mathbb{R}^d)$, we denote by $\widehat{u}$ its Fourier transform. At the same time, $\check{u}$ and $\mathcal{F}^{-1}u$ stand for the inverse Fourier transform of u.

2.1.1 The Fourier-analytical approach

In this subsection, we shall define Besov spaces according to the Fourier-analytical approach. For this purpose, the first step consists of introducing a *Littlewood-Paley partition of unity*, which is a suitable decomposition of unity in Fourier space. Nevertheless, it will first be convenient to give a definition of a Fourier multiplier operator. In what follows, we shall make use of the notions introduced in Appendix A, particularly Section A.11. Let

$$\mathcal{O}(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) : \forall \ell \in \mathbb{N}_0^d, \exists C, N > 0 \text{ s.t. } |\partial^\ell f(x)| \leq C\langle x\rangle^N\}, \tag{2.1.1}$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\langle \cdot \rangle$ denotes the Japanese bracket, defined by

$$\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}, \quad (x \in \mathbb{R}^d). \tag{2.1.2}$$

Definition 2.1.1: *Let $\psi \in \mathcal{O}(\mathbb{R}^d)$. The **Fourier multiplier operator** associated to ψ is the continuous map $\psi(D)\colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ defined as*

$$\psi(D)u = \mathcal{F}^{-1}\{\psi(\xi)\widehat{u}(\xi)\}, \quad \forall u \in \mathcal{S}'(\mathbb{R}^d).$$

Definition 2.1.2: *If N is a positive integer, a **Littlewood-Paley partition of unity** is a sequence $(\psi_j)_{j\in\mathbb{N}_0}$ of functions such that*

- *$\psi_j \in \mathcal{D}(\mathbb{R}^d)$ and $\psi_j \geq 0$ for all $j \geq 0$;*
- *ψ_j is supported in $\{2^{j-N} \leq |\xi| \leq 2^{j+N}\}$ for all $j \geq 1$ while ψ_0 is supported in $\{|\xi| \leq 2^N\}$;*
- *$\sum_{j=0}^{\infty} \psi_j(\xi) = 1$ for any $\xi \in \mathbb{R}^d$;*
- *for any multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, $\exists C_\alpha > 0$, such that*

$$|\partial^\alpha \psi_j(\xi)| \leq C_\alpha \langle \xi \rangle^{-|\alpha|}, \quad j \geq 1,$$

  where $|\alpha| := \sum_{i=1}^d \alpha_i$;
- *$\psi_j(\xi) = \psi_j(-\xi)$ for all $j \geq 0$.*

The existence of a Littlewood-Paley partition of unity as in Definition 2.1.2 is proven in [BCD11, Prop. 2.10]. Moreover, given a sequence $(\psi_j)_{j\in\mathbb{N}_0}$ as in Definition 2.1.2, the Fourier multiplier operator $\psi_j(D)$ is said to be a *Littlewood-Paley block*. It formally holds true the following *Littlewood-Paley decomposition*:

$$\mathrm{Id} = \sum_{j=0}^{\infty} \psi_j(D). \tag{2.1.3}$$

The previous identity makes sense in $\mathcal{S}'(\mathbb{R}^d)$.

Remark 2.1.3: *Here, we recall the usual construction of a Littlewood-Paley decomposition of unity. Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ be an even positive function supported in $\{2^{-N} \leq |\xi| \leq 2^N\}$ and let $\tilde{\psi} \in \mathcal{D}(\mathbb{R}^d)$ be such that*

$$\tilde{\psi}(\cdot) = \frac{\psi(\cdot)}{\sum_{k\in\mathbb{Z}} \psi(2^{-k}\cdot)}.$$

Then, a Littlewood-Paley partition of unity $(\chi_j)_{j\in\mathbb{N}_0}$ can be generated by setting

$$\chi_j(\cdot) := \tilde{\psi}(2^{-j}\cdot), \quad \chi_0(\cdot) := 1 - \sum_{j\geq 0} \tilde{\psi}(2^{-j}\cdot).$$

For further details, the interested reader may refer to [BCD11, Sect. 2.2].

Definition 2.1.4: *Let $\alpha \in \mathbb{R}$ and let $1 \leq p, q \leq \infty$. If $(\psi_j)_{j\in\mathbb{N}_0}$ is a Littlewood-Paley partition of unity, we call **Besov space** $B^\alpha_{p,q}(\mathbb{R}^d)$ the set of all $u \in \mathcal{S}'(\mathbb{R}^d)$ such that*

$$\|u\|_{B^\alpha_{p,q}(\mathbb{R}^d)} := \begin{cases} \left(\sum_{j\geq 0} 2^{jq\alpha} \|\psi_j(D)u\|^q_{L^p(\mathbb{R}^d)} \right)^{\frac{1}{q}} < \infty & \text{if } q < \infty, \\ \sup_{j\geq 0} 2^{j\alpha} \|\psi_j(D)u\|_{L^p(\mathbb{R}^d)} < \infty & \text{if } q = \infty. \end{cases} \tag{2.1.4}$$

In addition, we say that $u \in B^{\alpha,\mathrm{loc}}_{p,q}(\mathbb{R}^d)$ if $\phi u \in B^\alpha_{p,q}(\mathbb{R}^d)$ for any $\phi \in \mathcal{D}(\mathbb{R}^d)$.

Remark 2.1.5: *Observe that $u \in B^{\alpha}_{p,q}(\mathbb{R}^d)$ if and only if*

$$(\psi_j(D)u)_{j\in\mathbb{N}_0} \in \ell^{\alpha}_q(L^p(\mathbb{R}^d)).$$

Here, given a Banach space $(E, \|\cdot\|_E)$, $\ell^{\alpha}_q(E)$ denotes the space of all E-valued sequences $a = (a_j)_{j\in\mathbb{N}_0}$ such that it is finite

$$\|a\|_{\ell^{\alpha}_q(E)} := \begin{cases} \left(\sum_{j\geq 0} 2^{jq\alpha}\|a_j\|^q_E\right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \sup_{j\geq 0} 2^{j\alpha}\|a_j\|_E & \text{if } q = \infty. \end{cases} \tag{2.1.5}$$

Moreover, for the sake of simplicity, we set $\ell^q(E) := \ell^0_q(E)$ and $\ell^q := \ell^0_q(\mathbb{R})$.

Definition 2.1.4 does not depend on the choice of the Littlewood-Paley partition of unity. This property is stated in the following theorem - see [Saw18, Th. 2.1].

Theorem 2.1.6: *Let $(\psi_j)_{j\in\mathbb{N}_0}, (\tilde{\psi}_j)_{j\in\mathbb{N}_0}$ be two Littlewood-Paley partitions of unity as per Definition 2.1.2. Then $\|(\psi_j(D)u)_{j\in\mathbb{N}_0}\|_{\ell^{\alpha}_q(L^p(\mathbb{R}^d))}$ and $\|(\tilde{\psi}_j(D)u)_{j\in\mathbb{N}_0}\|_{\ell^{\alpha}_q(L^p(\mathbb{R}^d))}$ are equivalent, i.e. there exist $c, C > 0$ such that*

$$c\|(\tilde{\psi}_j(D)u)_{j\in\mathbb{N}_0}\|_{\ell^{\alpha}_q(L^p(\mathbb{R}^d))} \leq \|(\psi_j(D)u)_{j\in\mathbb{N}_0}\|_{\ell^{\alpha}_q(L^p(\mathbb{R}^d))} \leq C\|(\tilde{\psi}_j(D)u)_{j\in\mathbb{N}_0}\|_{\ell^{\alpha}_q(L^p(\mathbb{R}^d))}$$

for every $u \in \mathcal{S}'(\mathbb{R}^d)$.

A further property of Besov spaces is the completeness under the norm in Equation (2.1.4) - see [BCD11, Th. 2.72].

Theorem 2.1.7: *Let $1 \leq p, q \leq \infty$ and $\alpha \in \mathbb{R}$. $B^{\alpha}_{p,q}(\mathbb{R}^d)$ equipped with the norm as per Equation (2.1.4) is a Banach space.*

Convolutions inequalities for Besov spaces We outline the properties of the convolution between two Besov distributions. The following theorem asserts that the convolution is a continuous bilinear map between suitable Besov spaces as per Definition 2.1.4 - see [KS21, Th. 2.2].

Theorem 2.1.8: *Let $\alpha_1, \alpha_2 \in \mathbb{R}$. Let $1 \leq p, p_1, p_2 \leq \infty$ and let $1 \leq q, q_1, q_2 \leq \infty$ be such that*

$$\frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2}, \qquad 1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

If $u \in B^{\alpha_1}_{p_1,q_1}(\mathbb{R}^d)$ and $v \in B^{\alpha_2}_{p_2,q_2}(\mathbb{R}^d)$, then $u * v \in B^{\alpha_1+\alpha_2}_{p,q}(\mathbb{R}^d)$ and there exists a constant $C > 0$ such that

$$\|u * v\|_{B^{\alpha_1+\alpha_2}_{p,q}(\mathbb{R}^d)} \leq C\|u\|_{B^{\alpha_1}_{p_1,q_1}(\mathbb{R}^d)}\|v\|_{B^{\alpha_2}_{p_2,q_2}(\mathbb{R}^d)},$$

where $*$ stands for the convolution operator as per Definition A.7.1.

2.1.2 Embedding theorems

The purpose of this subsection is to sum up a few notable embedding theorems for Besov spaces.

Theorem 2.1.9: *Let $1 \leq p, q \leq \infty$ and $\alpha \in \mathbb{R}$. The following statements hold true:*

1. *$\mathcal{S}(\mathbb{R}^d) \hookrightarrow B^{\alpha}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$.*
2. *If $p, q < \infty$, then $\mathcal{D}(\mathbb{R}^d)$ is densely embedded in $B^{\alpha}_{p,q}(\mathbb{R}^d)$. In particular, $\mathcal{S}(\mathbb{R}^d)$ is dense in $B^{\alpha}_{p,q}(\mathbb{R}^d)$.*

Remark 2.1.10: *Being $\mathcal{S}(\mathbb{R}^d)$ continuously embedded in $B^\alpha_{p,q}(\mathbb{R}^d)$, the first statement of Theorem 2.1.9 entails that $B^\alpha_{p,q}(\mathbb{R}^d)$ is* nontrivial.

Remark 2.1.11: *If $q = \infty$, it is possible to prove that the closure of $\mathcal{D}(\mathbb{R}^d)$ with respect to $\|\cdot\|_{B^\alpha_{p,q}}$ is the space of all $u \in \mathcal{S}'(\mathbb{R}^d)$ such that*

$$\lim_{j\to\infty} 2^{j\alpha}\|\psi_j(D)u\|_{L^p(\mathbb{R}^d)} = 0.$$

The interested reader may refer to [BCD11, Remark 2.75].

As mentioned in Remark 2.1.5, $B^\alpha_{p,q}(\mathbb{R}^d)$ can be identified with the sequence space $\ell^\alpha_q(L^p(\mathbb{R}^d))$. As a matter of fact, the next propositions follow from the definition of ℓ^q and of $\ell^\alpha_q(E)$ - see [Saw18, Prop. 2.2, Prop. 2.3].

Proposition 2.1.12: *Let $\alpha \in \mathbb{R}$ and let $1 \le p, q_1, q_2 \le \infty$. If $q_1 \le q_2$, then*

$$B^\alpha_{p,q_1}(\mathbb{R}^d) \hookrightarrow B^\alpha_{p,q_2}(\mathbb{R}^d). \tag{2.1.6}$$

Proposition 2.1.13: *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and let $1 \le p, q \le \infty$. If $\alpha_1 \le \alpha_2$, then*

$$B^{\alpha_2}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{\alpha_1}_{p,q}(\mathbb{R}^d). \tag{2.1.7}$$

We conclude by stating a Sobolev-type embedding theorem for Besov spaces - see [BCD11, Th. 2.71].

Theorem 2.1.14: *Let $1 \le p_1 \le p_2 \le \infty$, $1 \le q_1 \le q_2 \le \infty$ and $\alpha \in \mathbb{R}$. Then*

$$B^\alpha_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow B^{\alpha-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,q_2}(\mathbb{R}^d). \tag{2.1.8}$$

2.1.3 Duality

In this subsection, we shall deal with the duality properties of Besov spaces. To this end, we first introduce some useful notations and definitions. Given $p \in [1,\infty]$, the *conjugate exponent* to p is the real number $p' \in [1,\infty]$ such that

$$\frac{1}{p} + \frac{1}{p'} = 1. \tag{2.1.9}$$

Furthermore, given a Banach space $(E, \|\cdot\|_E)$, its *dual space* E' is defined as the set of all continuous linear functionals $L\colon E \to \mathbb{R}$. E' is, in turn, a Banach space when endowed with the operator norm

$$\|L\|_{E'} := \sup_{\|v\|_E \le 1} |L(v)|, \quad (L \in E').$$

Roughly speaking, the duality of $B^\alpha_{p,q}(\mathbb{R}^d)$ can be deduced by the well-known dualities $(L^p(\mathbb{R}^d))' = L^{p'}(\mathbb{R}^d)$ and $(\ell^q)' = \ell^{q'}$ - see [Saw18, Th. 2.17], [BCD11, Prop. 2.76].

Theorem 2.1.15: *Let $1 \le p, q < \infty$ and $\alpha \in \mathbb{R}$. Then, for any $L \in (B^\alpha_{p,q}(\mathbb{R}^d))'$, there exists a unique $u \in B^{-\alpha}_{p',q'}(\mathbb{R}^d)$ such that*

$$L(\varphi) = \langle u, \varphi\rangle \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d),$$

where the pairing $\langle u, v\rangle$ is defined as

$$\langle u, \varphi\rangle := \sum_{j,j'} \int_{\mathbb{R}^d} (\psi_j(D)u)(x)(\psi_{j'}(D)\varphi)(x)dx.$$

Moreover, there exists a positive constant C such that

$$\|u\|_{B^{-\alpha}_{p',q'}(\mathbb{R}^d)} \leq C\|L\|_{(B^{\alpha}_{p,q}(\mathbb{R}^d))'}.$$

Conversely, if $u \in B^{-\alpha}_{p',q'}(\mathbb{R}^d)$, there is a constant $c > 0$ such that

$$|\langle u, \varphi\rangle| \leq c\|u\|_{B^{-\alpha}_{p',q'}(\mathbb{R}^d)}\|\varphi\|_{B^{\alpha}_{p,q}(\mathbb{R}^d)}, \quad \forall\varphi \in \mathcal{S}(\mathbb{R}^d).$$

Therefore, the functional $\mathcal{S}(\mathbb{R}^d) \ni \varphi \mapsto \langle u, \varphi\rangle \in \mathbb{C}$ continuously extends to $B^{\alpha}_{p,q}(\mathbb{R}^d)$.

Remark 2.1.16: *$B^{-\alpha}_{p',q'}(\mathbb{R}^d)$ can generally be identified with the dual space of the closure of $\mathcal{D}(\mathbb{R}^d)$ with respect to $\|\cdot\|_{B^{\alpha}_{p,q}}(\mathbb{R}^d)$ for $1 \leq p, q \leq \infty$. Therefore, when p and q are finite, it turns out that $B^{-\alpha}_{p',q'}(\mathbb{R}^d) = (B^{\alpha}_{p,q}(\mathbb{R}^d))'$ by using the second statement of Theorem 2.1.9.*

2.1.4 Paradifferential calculus

In this subsection, we shall focus on the analysis of the product between two Besov distributions. A renown approach to address this issue is *paradifferential calculus*. For this topic, we mainly refer to [BCD11, Sect. 2.8] and [GIP15].

Let $(\psi_j)_{j\in\mathbb{N}_0}$ be a Littlewood-Paley partition of unity as per Definition 2.1.2 and let $u, v \in \mathcal{S}'(\mathbb{R}^d)$. Taking into account the Littlewood-Paley decompositions, see Equation (2.1.3),

$$u = \sum_{j\geq 0} \psi_j(D)u, \quad v = \sum_{j\geq 0} \psi_j(D)v,$$

the product $u \cdot v$ can be writteny formall as

$$u \cdot v = \sum_{j\geq 0}\sum_{j'\geq 0} \psi_j(D)u \cdot \psi_{j'}(D)v. \tag{2.1.10}$$

The idea at the core of paradifferential calculus is to decompose the sum on the right-hand side of Equation (2.1.10) in three parts. The first one coincides with the product between the low frequencies of u with the high ones of v, while the second is the symmetric counterpart of the first. These two contributions are called *paraproducts*. The third term, dubbed *resonant term* or *remainder*, consists of those products where the indices j and j' are such that $|j - j'| \leq 1$. It encodes the potential ultraviolet divergences preventing the well-posedness of Equation (2.1.10). Therefore, we introduce the following definitions.

Definition 2.1.17: *Let $u, v \in \mathcal{S}'(\mathbb{R}^d)$ and let $(\psi_j)_{j\in\mathbb{N}_0}$ be a Littlewood-Paley partition of unity as per Definition 2.1.2. The* ***paraproduct*** *between v and u is defined as*

$$T_u v = T(u, v) := \sum_{j\geq 1} S_{j-1}u \cdot \psi_j(D)v, \tag{2.1.11}$$

where $S_{j-1}u := \sum_{i=0}^{j} \psi_i(D)u$. The ***resonant term*** *between u and v is defined by*

$$R(u, v) = \sum_{|j-j'|\leq 1} \psi_j(D)u \cdot \psi_{j'}(D)v. \tag{2.1.12}$$

On account of Equations (2.1.11) and (2.1.11), paraproducts and the resonant term are bilinear maps. In addition, at least formally, the operators T and R realize the so-called *Bony decomposition*, *cf.* [Bo81],

$$u \cdot v = T_u v + T_v u + R(u, v). \tag{2.1.13}$$

As anticipated, the only hurdle to give meaning to Equation (2.1.13) lies in the resonant term. At the same time, $T_u v$ and $T_v u$ are always well-defined distributions, see [BCD11, Th. 2.82]. In the following, we recall the continuity properties of the paraproduct and of the remainder. In particular, Theorem 2.1.19 shall clarify under which conditions the Bony decomposition is not formal, see [BCD11, Th. 2.85].

Theorem 2.1.18: *For any $(\alpha_1,\alpha_2)\in\mathbb{R}\times(-\infty,0)$ and any $(p,q_1,q_2)\in[1,\infty]^3$ there exists $C>0$ such that*

$$\|T(u,v)\|_{B^{\alpha_1}_{p,q}(\mathbb{R}^d)}\leq C^{|\alpha_1|+1}\|u\|_{L^\infty(\mathbb{R}^d)}\|v\|_{B^{\alpha_1}_{p,q}(\mathbb{R}^d)},$$

$$\|T(u,v)\|_{B^{\alpha_1+\alpha_2}_{p,q}(\mathbb{R}^d)}\leq \frac{C^{|\alpha_1+\alpha_2|+1}}{-t}\|u\|_{B^{\alpha_2}_{\infty,q_1}(\mathbb{R}^d)}\|v\|_{B^{\alpha_1}_{p,q_2}(\mathbb{R}^d)},\ \textit{with}\ \frac{1}{q}=\min\left\{1,\frac{1}{q_1}+\frac{1}{q_2}\right\}.$$

Theorem 2.1.19: *Let $(\alpha_1,\alpha_2)\in\mathbb{R}^2$ and let $(p_1,p_2,q_1,q_2)\in[1,\infty]^4$ be such that*

$$\frac{1}{p}:=\frac{1}{p_1}+\frac{1}{p_2}\leq 1,\qquad \frac{1}{q}:=\frac{1}{q_1}+\frac{1}{q_2}\leq 1.$$

If $\alpha_1+\alpha_2>0$, then there exists a constant $C>0$ such that

$$\|R(u,v)\|_{B^{\alpha_1+\alpha_2}_{p,q}(\mathbb{R}^d)}\leq\frac{C^{\alpha_1+\alpha_2+1}}{\alpha_1+\alpha_2}\|u\|_{B^{\alpha_1}_{p_1,q_1}(\mathbb{R}^d)}\|v\|_{B^{\alpha_2}_{p_2,q_2}(\mathbb{R}^d)},\quad \forall (u,v)\in B^{\alpha_1}_{p_1,q_1}(\mathbb{R}^d)\times B^{\alpha_2}_{p_2,q_2}(\mathbb{R}^d).$$

In addition, if $\alpha_1+\alpha_2=0$ and $q=1$, there exists $C>0$ such that

$$\|R(u,v)\|_{B^0_{p,\infty}(\mathbb{R}^d)}\leq C\|u\|_{B^{\alpha_1}_{p_1,q_1}(\mathbb{R}^d)}\|v\|_{B^{\alpha_2}_{p_2,q_2}(\mathbb{R}^d)},\quad \forall (u,v)\in B^{\alpha_1}_{p_1,q_1}(\mathbb{R}^d)\times B^{\alpha_2}_{p_2,q_2}(\mathbb{R}^d).$$

To conclude this subsection, we recall Young product theorem, which is a direct consequence of Theorems 2.1.18 and 2.1.19 applied to the class of Besov spaces with $p=q=\infty$, see [Hai14, Prop. 4.14].

Theorem 2.1.20: *Let $\alpha_1,\alpha_2\in\mathbb{R}$ be such that $\alpha_1+\alpha_2>0$. Then $(u,v)\mapsto u\cdot v$ extends to a bilinear continuous map from $B^{\alpha_1,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)\times B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ to $B^{\alpha_1\wedge\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$, where we set $\alpha_1\wedge\alpha_2:=\min\{\alpha_1,\alpha_2\}$.*

In Subsection 3.3.3, we shall discuss a reformulation of Theorem 2.1.20 from the viewpoint of microlocal analysis.

2.1.5 Local means

In the following, we recall an equivalent characterization of the Besov norm in Equation (2.1.4), called *local means formulation*. The content of this subsection is mainly inspired by [Tri06, Sect. 1.4]. Firstly, we introduce some helpful notations. If $\alpha\in\mathbb{R}$, we set

$$\lfloor\alpha\rfloor:=\max\{N\in\mathbb{Z}:N\leq\alpha\}.\tag{2.1.14}$$

Given $f\in C^\infty(\mathbb{R}^d)$, $x\in\mathbb{R}^d$ and $\lambda\in(0,1]$, we denote by $f^\lambda_x\colon\mathbb{R}^d\to\mathbb{R}$ the scaled version of f, defined as follows

$$f^\lambda_x(y):=\lambda^{-d}f(\lambda^{-1}(y-x)),\quad y\in\mathbb{R}^d.\tag{2.1.15}$$

Let $B(0,1):=\{y\in\mathbb{R}^d:|y|<1\}$. We shall denote by $\mathcal{D}(B(0,1))$ the space of smooth functions with compact support in $B(0,1)$.

Remark 2.1.21: *We say that a measurable function $\omega\colon(0,1]\to\mathbb{R}$ lies in $L^q((0,1),\lambda^{-1}d\lambda)$ if it is finite*

$$\|\omega\|_{L^q((0,1),\lambda^{-1}d\lambda)}:=\begin{cases}\left(\int_0^1|\omega(\lambda)|^q\frac{d\lambda}{\lambda}\right)^{\frac{1}{q}} & \text{if } q<\infty,\\ \sup_{\lambda\in(0,1]}|\omega(\lambda)| & \text{if } q=\infty.\end{cases}$$

In the following, we shall denote $L^q((0,1),\lambda^{-1}d\lambda)$ by L^q_λ to simplify the notation.

The next discussion explains the reason why the approach via local means has been introduced. Let $\psi_0 \in \mathcal{D}(\mathbb{R}^d)$ be as in Definition 2.1.2 and let $\psi \in \mathcal{D}(\mathbb{R}^d)$ be as in Remark 2.1.3. We set $\psi_j(\cdot) = \psi(2^{-j}\cdot)$ for any $j \geq 1$. Note that, on the basis of Definition 2.1.1, it descends

$$(\psi_j(D)u)(x) = \mathcal{F}^{-1}\{\psi_j(\xi)\widehat{u}(\xi)\}(x) = (\mathcal{F}^{-1}\{\psi_j\} * u)(x) = u(2^{jd}\check{\psi}(2^j(\cdot - x))) = u(\check{\psi}_x^{2^{-j}}), \tag{2.1.16}$$

where we used that $\mathcal{F}^{-1}\{uv\} = \check{u} * \check{v}$ and $\mathcal{F}^{-1}\{\psi_j\}(x) = 2^{jd}\check{\psi}(2^j x)$. For instance, if $u \in B^\alpha_{\infty,\infty}(\mathbb{R}^d)$,

$$|u(\check{\psi}_x^{2^{-j}})| \leq \|u\|_{B^\alpha_{\infty,\infty}} 2^{j\alpha}, \quad \forall x \in \mathbb{R}^d, \forall j \geq 1.$$

Since $\check{\psi} \in \mathcal{S}(\mathbb{R}^d)$, one needs to know u on the whole Euclidean space in order to estimate $u(\check{\psi}_x^{2^{-j}})$ at each point $x \in \mathbb{R}^d$. Therefore, it would be suitable to switch the compactness of the support from ψ to $\check{\psi}$. This is achievable by *local means*. Given $\kappa \in \mathcal{D}(B(0,1))$, local means are defined by

$$u(\kappa_x^\lambda) = \int_{\mathbb{R}^d} u(y)\kappa_x^\lambda(y)dy$$

with $x \in \mathbb{R}^d$ and $\lambda > 0$. In order to define Besov spaces via local means, we need to introduce a suitable subclass of test functions.

Definition 2.1.22: *Let $\alpha \in \mathbb{R}$. We call $\mathscr{B}_{\lfloor \alpha \rfloor}$ the subset of $\mathcal{D}(B(0,1))$ whose elements κ are such that there exists $\epsilon > 0$*

$$\check{\kappa}(\xi) \neq 0 \ \text{ if } \ \frac{\varepsilon}{2} < |\xi| \leq 2\varepsilon\,, \quad \text{and} \quad (\partial^\ell \check{\kappa})(0) = 0 \ \text{ if } |\ell| \leq \lfloor \alpha \rfloor. \tag{2.1.17}$$

Remark 2.1.23: *If $\alpha < 0$, then the second condition in Equation (2.1.17) is empty.*

Definition 2.1.24: *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathscr{B}_{\lfloor \alpha \rfloor}$ and let $\underline{\kappa} \in \mathcal{D}(B(0,1))$ be such that $\check{\underline{\kappa}}(0) \neq 0$. We call $B^\alpha_{p,q}(\mathbb{R}^d)$, $p, q \in [1,\infty]$, the space of distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ such that*

$$\|u\|^{\kappa,\underline{\kappa}}_{B^\alpha_{p,q}(\mathbb{R}^d)} := \|u(\underline{\kappa}_x)\|_{L^p_x(\mathbb{R}^d)} + \left\| \frac{\|u(\kappa_x^\lambda)\|_{L^p_x(\mathbb{R}^d)}}{\lambda^\alpha} \right\|_{L^q_\lambda} < \infty \tag{2.1.18}$$

where $L^p_x(\mathbb{R}^d)$ denotes $L^p(\mathbb{R}^d, dx)$. In addition, we say that $u \in \mathcal{S}'(\mathbb{R}^d)$ lies in $B^{\alpha,\mathrm{loc}}_{p,q}(\mathbb{R}^d)$ if, for any compact set $\mathfrak{K} \subset \mathbb{R}^d$,

$$\|u\|^{\kappa,\underline{\kappa}}_{B^\alpha_{p,q}(\mathfrak{K})} := \|u(\underline{\kappa}_x)\|_{L^p_x(\mathfrak{K})} + \left\| \frac{\|u(\kappa_x^\lambda)\|_{L^p_x(\mathfrak{K})}}{\lambda^\alpha} \right\|_{L^q_\lambda} < \infty\,, \tag{2.1.19}$$

where $L^p_x(\mathfrak{K}) := L^p(\mathfrak{K}, dx)$.

Remark 2.1.25: *On account of Equation (A.11.3), the second condition in Equation (2.1.17) and the condition $\check{\underline{\kappa}}(0) \neq 0$ in Definition 2.1.24 amount to $\int_{\mathbb{R}^d} x^\ell \kappa(x)dx = 0$ if $|\ell| \leq \lfloor \alpha \rfloor$ and $\int_{\mathbb{R}^d} \underline{\kappa}(x)dx \neq 0$ respectively.*

Remark 2.1.26: *As a result of Definition 2.1.22 and Definition 2.1.24, observe that the kernels κ and $\underline{\kappa}$ play the rôle of a Littlewood-Paley partition of unity. Moreover, different choices for κ and $\underline{\kappa}$ in Equation (2.1.18) yield equivalent norms. Therefore, we can avoid to write the superscripts κ and $\underline{\kappa}$.*

The following theorem states the equivalence between Equations (2.1.4) and (2.1.18) - see [Tri06, th. 1.10].

Theorem 2.1.27: *Let $\alpha \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Let $\kappa, \underline{\kappa}$ be as in Definition 2.1.24 and let $(\psi_j)_{j\in\mathbb{N}_0}$ be a Littlwood-Paley partition of unity as per Definition 2.1.2. Then, for any $u \in \mathcal{S}'(\mathbb{R}^d)$, there exist $C, c > 0$ such that*

$$c\|u\|^{\kappa,\underline{\kappa}}_{B^\alpha_{p,q}(\mathbb{R}^d)} \leq \|(\psi_j(D)u)_{j\in\mathbb{N}_0}\|_{\ell^\alpha_q(L^p(\mathbb{R}^d))} \leq C\|u\|^{\kappa,\underline{\kappa}}_{B^\alpha_{p,q}(\mathbb{R}^d)}.$$

In Chapter 3, we shall often refer to Definition 2.1.24 rather than to Definition 2.1.4.
In the following, we recall a few equivalent characterizations of the Besov norm defined in Equation 2.1.24 when $\alpha < 0$ - see [BL22B, Prop A.5], [Tri06, Cor. 1.12].

Proposition 2.1.28: *Let $\alpha < 0$, $1 \leq p, q \leq \infty$ and let $u \in \mathcal{S}'(\mathbb{R}^d)$. Then the following statements are equivalent:*

(i) u lies in $B^\alpha_{p,q}(\mathbb{R}^d)$.

(ii) For any $\underline{\kappa} \in \mathcal{D}(B(0,1))$ with $\check{\underline{\kappa}}(0) \neq 0$

$$\left\|\frac{\|u(\underline{\kappa}^\lambda_x)\|_{L^p_x(\mathbb{R}^d)}}{\lambda^\alpha}\right\|_{L^q_\lambda} < \infty. \tag{2.1.20}$$

(iii) There exists $\underline{\kappa} \in \mathcal{D}(B(0,1))$ with $\check{\underline{\kappa}}(0) \neq 0$

$$\left\|\frac{\|u(\underline{\kappa}^\lambda_x)\|_{L^p_x(\mathbb{R}^d)}}{\lambda^\alpha}\right\|_{L^q_\lambda} < \infty. \tag{2.1.21}$$

(iv) Let $r > -\alpha$. We set

$$\mathcal{B}_r := \{\phi \in \mathcal{D}(B(0,1)) : \|\phi\|_{C^r(\mathbb{R}^d)} \leq 1\},$$

where $\|\cdot\|_{C^r(\mathbb{R}^d)}$ has been introduced in Equation (A.1.3). Then it holds true that

$$\left\|\sup_{\phi\in\mathcal{B}_r}\left|\frac{\|u(\phi^\lambda_x)\|_{L^p_x(\mathbb{R}^d)}}{\lambda^\alpha}\right|\right\|_{L^q_\lambda} < \infty, \tag{2.1.22}$$

Remark 2.1.29: *Proposition 2.1.28 still holds true for Besov spaces $B^{\alpha,\mathrm{loc}}_{p,q}(\mathbb{R}^d)$ with $\alpha < 0$. In this case, in Equations (2.1.20), (2.1.21), (2.1.22) the norm $\|\cdot\|_{L^p_x(\mathbb{R}^d)}$ is replaced by $\|\cdot\|_{L^p_x(\mathfrak{K})}$ for any compact set $\mathfrak{K} \subset \mathbb{R}^d$.*

In the following, we give the definition of the Besov space $B^{\alpha,\mathrm{loc}}_{p,q}(\Omega)$, where Ω is an arbitrary domain of $\mathbb{R}^d$ - see [Tri06, Sect. 4.1.2].

Definition 2.1.30: *Let $\Omega \subset \mathbb{R}^d$, let $1 \leq p, q \leq \infty$ and let $\alpha \in \mathbb{R}$. The Besov space $B^{\alpha,\mathrm{loc}}_{p,q}(\Omega)$ is the set of all $u \in \mathcal{D}'(\Omega)$ such that ϕu lies in $B^\alpha_{p,q}(\mathbb{R}^d)$ as per Definition 2.1.24 for any $\phi \in \mathcal{D}(\Omega)$.*

The last part of this subsection shall be devoted to Besov spaces with $p = q = \infty$. The following proposition shows that the space of compactly supported distributions $\mathcal{E}'(\mathbb{R}^d)$ can be characterized in terms of Besov spaces $B^\alpha_{\infty,\infty}(\mathbb{R}^d)$.

Proposition 2.1.31: *Let $u \in \mathcal{E}'(\mathbb{R}^d)$ and let $\mathrm{ord}(u) \in \mathbb{N}_0$ be the order of u as per Definition A.2.5. Then $u \in B^{-d-\mathrm{ord}(u)}_{\infty,\infty}(\mathbb{R}^d)$. Furthermore*

$$\mathcal{E}'(\mathbb{R}^d) = \bigcup_{\alpha\in\mathbb{R}} B^\alpha_{\infty,\infty,c}(\mathbb{R}^d),$$

where we set

$$B^\alpha_{\infty,\infty,c}(\mathbb{R}^d) := B^\alpha_{\infty,\infty}(\mathbb{R}^d) \cap \mathcal{E}'(\mathbb{R}^d) \tag{2.1.23}$$

Proof. Fix $\underline{\kappa} \in \mathcal{D}(B(0,1))$ such that $\check{\underline{\kappa}}(0) \neq 0$. Given that $\|\partial^\ell \underline{\kappa}_x^\lambda\|_{L^\infty(\mathbb{R}^d)} = \lambda^{-|\ell|-d}\|\partial^\ell \underline{\kappa}_x^\lambda\|_{L^\infty(\mathbb{R}^d)}$, it turns out

$$\|\underline{\kappa}_x^\lambda\|_{C^m(\mathbb{R}^d)} \leq \|\underline{\kappa}\|_{C^m(\mathbb{R}^d)}\lambda^{-m-d}, \tag{2.1.24}$$

for some $m \in \mathbb{N}_0$. Therefore, from the definition of $\mathcal{E}'(\mathbb{R}^d)$ and from Equation (2.1.24), there exists a constant $C > 0$ such that

$$|u(\underline{\kappa}_x^\lambda)| \leq C\|\underline{\kappa}_x^\lambda\|_{C^{\mathrm{ord}(u)}(\mathbb{R}^d)} \leq C\|\underline{\kappa}\|_{C^{\mathrm{ord}(u)}(\mathbb{R}^d)}\lambda^{-\mathrm{ord}(u)-d}$$

uniformly for $\lambda \in (0,1]$ and $x \in \mathrm{supp}(u)$. We infer that $u \in B^{-d-\mathrm{ord}(u)}_{\infty,\infty}(\mathbb{R}^d)$ per Proposition 2.1.28. In addition, it follows immediately

$$\mathcal{E}'(\mathbb{R}^d) = \bigcup_{\alpha\in\mathbb{R}} B^\alpha_{\infty,\infty,c}(\mathbb{R}^d).$$

□

In the following, we prove an embedding result concerning Besov spaces $B^\alpha_{\infty,\infty,c}(\mathbb{R}^d)$ introduced in Equation (2.1.23). The following proposition shall play a key rôle as a technical tool in Subsection 3.3.5 when proving a regularity result for solutions to a specific class of first order hyperbolic partial differential equations.

Proposition 2.1.32: *Let $\alpha \in \mathbb{R}$. Then $B^\alpha_{\infty,\infty}(\mathbb{R}^d) \cap \mathcal{E}'(\mathbb{R}^d) \hookrightarrow B^\alpha_{2,\infty}(\mathbb{R}^d)$.*

Proof. Let $u \in B^\alpha_{\infty,\infty}(\mathbb{R}^d) \cap \mathcal{E}'(\mathbb{R}^d)$ and let $\kappa \in \mathscr{B}_{\lfloor\alpha\rfloor}$ as per Definition 2.1.22. Being u compactly supported and bearing in mind that $L^\infty(\mathfrak{K}) \hookrightarrow L^2(\mathfrak{K})$ for any compact set $\mathfrak{K} \subset \mathbb{R}^d$, it descends that there exists a constant $C > 0$ such that

$$\|u(\kappa_x^\lambda)\|_{L^2_x(\mathbb{R}^d)} \leq C\|u(\kappa_x^\lambda)\|_{L^\infty_x(\mathbb{R}^d)} \leq C\|u\|_{B^\alpha_{\infty,\infty}(\mathbb{R}^d)}\lambda^\alpha,$$

uniformly for $\lambda \in (0,1]$. Analogously, it turns out that there exists a constant $c > 0$ such that

$$\|u(\underline{\kappa}_x^\lambda)\|_{L^2_x(\mathbb{R}^d)} \leq c\|u\|_{B^\alpha_{\infty,\infty}(\mathbb{R}^d)}\lambda^\alpha,$$

uniformly for $\lambda \in (0,1]$. On account of Definition 2.1.24, we infer that $u \in B^\alpha_{2,\infty}(\mathbb{R}^d)$. □

Eventually, we prove an equivalent characterization of the elements lying in $B^\alpha_{\infty,\infty}(\mathbb{R}^d)$ in terms of their Fourier transform. The following result shall inspire a definition of Besov wavefront set in Section 3.1.

Proposition 2.1.33: *Let $u \in \mathcal{S}'(\mathbb{R}^d)$ and let $\alpha \in \mathbb{R}$. Then $u \in B^\alpha_{\infty,\infty}(\mathbb{R}^d)$ if and only if, for any $\kappa \in \mathscr{B}_{\lfloor\alpha\rfloor}$ and $\underline{\kappa} \in \mathcal{D}(B(0,1))$ such that $\check{\underline{\kappa}}(0) \neq 0$, it holds true that*

$$|\langle\widehat{u}(\xi), e^{ix\cdot\xi}\check{\underline{\kappa}}(\xi)\rangle| \lesssim 1\,, \qquad |\langle\widehat{u}(\xi), e^{ix\cdot\xi}\check{\kappa}(\lambda\xi)\rangle| \lesssim \lambda^\alpha\,, \tag{2.1.25}$$

uniformly for $\lambda \in (0,1]$ and $x \in \mathbb{R}^d$.

Proof. Bearing in mind that $u(\varphi) = \widehat{u}(\check{\varphi})$ and $\check{\varphi_x^\lambda}(\xi) = e^{ix\cdot\xi}\check{\varphi}(\lambda\xi)$, it descends

$$u(\underline{\kappa}_x) = \langle\widehat{u}(\xi), e^{ix\cdot\xi}\check{\underline{\kappa}}(\xi)\rangle, \qquad u(\kappa_x^\lambda) = \langle\hat{u}(\xi), e^{ix\cdot\xi}\check{\kappa}(\lambda\xi)\rangle. \tag{2.1.26}$$

The statement is an immediate consequence of the previous identities and of Definition 2.1.24. □

2.1.6 Relations with other function spaces

Hölder Spaces

In this subsection, we shall define Hölder spaces and we shall see under which conditions such class of functions fits into the framework of Besov spaces. For further details concerning this topic, the reader may refer to [Saw18, Sect. 2.2.2]

Definition 2.1.34: *Let $m \in \mathbb{N}_0$ and $\tau \in (0,1]$. The* ***Hölder space*** *$C^{m,\tau}(\mathbb{R}^d)$ is the set of all functions $f \in C^m(\mathbb{R}^d)$ such that*

$$\|f\|_{C^{m,\tau}(\mathbb{R}^d)} := \|f\|_{C^m(\mathbb{R}^d)} + \sum_{|\ell|=m} \sup_{\substack{x,y\in\mathbb{R}^d \\ x\neq y}} \frac{|\partial^\ell f(x) - \partial^\ell f(y)|}{|x-y|^\tau} < \infty. \tag{2.1.27}$$

In addition, we say that $f \in C^m(\mathbb{R}^d)$ lies in $C^{m,\tau}_{\text{loc}}(\mathbb{R}^d)$ if, for any compact set $\mathfrak{K} \subset \mathbb{R}^d$,

$$\|f\|_{C^{m,\tau}(\mathfrak{K})} := \|f\|_{C^m(\mathfrak{K})} + \sum_{|\ell|\leq m} \sup_{\substack{x,y\in\mathfrak{K} \\ x\neq y}} \frac{|\partial^\ell f(x) - \partial^\ell f(y)|}{|x-y|^\tau} < \infty. \tag{2.1.28}$$

The norm $\|f\|_{C^{m,\tau}(\mathbb{R}^d)}$ can be equivalently characterized in terms of the m-th order Taylor polynomial of f.

Proposition 2.1.35: *Let $f \in C^{m,\tau}_{\text{loc}}(\mathbb{R}^d)$ with $m \in \mathbb{N}_0$ and $\tau \in (0,1]$. Then, for any compact set $\mathfrak{K} \subset \mathbb{R}^d$, there exist $c, C > 0$ such that*

$$c\|f\|_{C^{m,\tau}(\mathfrak{K})} \leq \max\left\{ \|f\|_{C^m(\mathfrak{K})}, \sup_{\substack{x,y\in\mathfrak{K} \\ x\neq y}} \frac{|f(y) - P_x(y)|}{|x-y|^{m+\tau}} \right\} \leq C\|f\|_{C^{m,\gamma}(\mathfrak{K})},$$

where P_x is the the m-th order Taylor polynomial of f centered at x, that is,

$$P_x(y) = \sum_{|\ell|\leq m} \partial^\ell f(x) \frac{(y-x)^\ell}{\ell!} \quad \forall y \in \mathbb{R}^d. \tag{2.1.29}$$

Remark 2.1.36: *The space $C^{m,\tau}(\mathbb{R}^d)$ equipped with the norm in Equation (2.1.27) is a Banach space.*

The following theorem states that Hölder spaces are closely related to the class of Besov spaces with $p = q = \infty$ - see [Saw18, Th. 2.7, Th. 2.8].

Theorem 2.1.37: *Let $\alpha \in (0,\infty) \setminus \mathbb{N}$. Then $B^\alpha_{\infty,\infty}(\mathbb{R}^d) = C^{\lfloor\alpha\rfloor,\alpha-\lfloor\alpha\rfloor}(\mathbb{R}^d)$, that is, the norms in Equations (2.1.4) and (2.1.27) are equivalent, where $\lfloor\alpha\rfloor$ has been defined in Equation (2.1.14).*

If $\alpha \in \mathbb{N}$, the space $B^\alpha_{\infty,\infty}(\mathbb{R}^d)$ is *strictly larger* than the space $C^{\alpha-1,1}(\mathbb{R}^d)$. As a matter of fact, the following theorem holds true.

Theorem 2.1.38: *Let $\alpha \in \mathbb{N}$. Then $f \in B^\alpha_{\infty,\infty}(\mathbb{R}^d)$ if and only if $f \in C^{\alpha-1}(\mathbb{R}^d)$ is such that*

$$\|f\|_{C^{\alpha-1}(\mathbb{R}^d)} + \sum_{|\ell|=\alpha-1} \sup_{\substack{x,h\in\mathbb{R}^d \\ h\neq 0}} \frac{|\partial^\ell f(x+h) - 2\partial^\ell f(x) + \partial^\ell f(x-h)|}{|h|} < \infty. \tag{2.1.30}$$

Remark 2.1.39: *In general, if $\alpha > 0$, it shows that the Besov space $B^\alpha_{\infty,\infty}(\mathbb{R}^d)$ coincides with the* ***Hölder-Zygmund space*** *$C^\alpha_*(\mathbb{R}^d)$. Here, $C^\alpha_*(\mathbb{R}^d)$ is defined as the Hölder space $C^{\lfloor\alpha\rfloor,\alpha-\lfloor\alpha\rfloor}(\mathbb{R}^d)$ if $\alpha \in (0,\infty)\setminus\mathbb{N}$. Otherwise if $\alpha \in \mathbb{N}$, the space $C^\alpha_*(\mathbb{R}^d)$ is the set of all $f \in C^{\alpha-1}(\mathbb{R}^d)$ that satisfy Equation (2.1.30).*

Theorems 2.1.37 and 2.1.38 still hold true for the local analogues of function spaces at hand.

In view of Chapter 3, we recall a result regarding the behaviour of the Fourier transform of a Hölder function - see [SS07, Chap. 3, Chap. 5]. First we introduce the Landau notation. Given two real-valued functions f, g on $\mathbb{R}^d$, we write

$$f(x) = \mathcal{O}(g(x)) \text{ as } |x| \to +\infty,$$

if there exist $C, L > 0$ such that

$$|f(x)| \leq C|g(x)|, \quad \forall |x| \geq L.$$

Proposition 2.1.40: *Let $\tau \in (0,1]$ and let $f\colon \mathbb{R}^d \to \mathbb{C}$ be such that there exists $\varepsilon > 0$*

$$\sup_{x\in\mathbb{R}^d} (1+|x|)^{d+\varepsilon}|f(x)| < \infty. \tag{2.1.31}$$

If

$$\widehat{f}(\xi) = \mathcal{O}(|\xi|^{-d-\tau}) \text{ as } |\xi| \to +\infty, \tag{2.1.32}$$

then there exists $c > 0$ such that

$$|f(x+h) - f(x)| \leq c|h|^\tau, \quad \forall x \in \mathbb{R}^d, \forall h \in \overline{B(0,1)},$$

where $\overline{B(0,1)} = \{y \in \mathbb{R}^d : |y| \leq 1\}$.

Proof. By using the Fourier inversion formula, it turns out that

$$\begin{aligned}|f(x+h) - f(x)| &= \left|(2\pi)^{-d}\int_{\mathbb{R}^d}\widehat{f}(\xi)e^{i(x+h)\cdot\xi}d\xi - (2\pi)^{-d}\int_{\mathbb{R}^d}\widehat{f}(\xi)e^{ix\cdot\xi}d\xi\right| \\ &= \left|(2\pi)^{-d}\int_{\mathbb{R}^d}\widehat{f}(\xi)(e^{ih\cdot\xi}-1)e^{ix\cdot\xi}d\xi\right| \leq (2\pi)^{-d}\int_{\mathbb{R}^d}|\widehat{f}(\xi)||e^{ih\cdot\xi}-1|d\xi \\ &= \underbrace{(2\pi)^{-d}\int_{|\xi|\leq|h|^{-1}}|\widehat{f}(\xi)||e^{ih\cdot\xi}-1|d\xi}_{=:|A|} + \underbrace{(2\pi)^{-d}\int_{|\xi|>|h|^{-1}}|\widehat{f}(\xi)||e^{ih\cdot\xi}-1|d\xi}_{=:|B|}\end{aligned}$$

We start by analyzing $|B|$. It is useful to recall

$$|e^{ix}-1| \leq 2\min\{|x|,1\}. \tag{2.1.33}$$

On account of Equation (2.1.32), it descends that there exists $C > 0$ such that

$$|B| \leq 2^{-d+1}\pi^{-1}\int_{|\xi|>|h|^{-1}}|\widehat{f}(\xi)|d\xi \leq 2^{-d+1}\pi^{-1}C\int_{|\xi|>|h|^{-1}}\frac{1}{|\xi|^{d+\gamma}}d\xi = \frac{2^{-d+1}\pi^{-1}C}{1-\tau}|h|^\tau, \tag{2.1.34}$$

where we applied that $|e^{ih\cdot\xi}-1| \leq 2$ on $\{\xi : |\xi| \geq |h|^{-1}\}$ in the first inequality. Focusing on $|A|$, we split it as follows

$$|A| = \underbrace{(2\pi)^{-d}\int_{|\xi|\leq 1}|\widehat{f}(\xi)||e^{ih\cdot\xi}-1|d\xi}_{=:|A_1|} + \underbrace{(2\pi)^{-d}\int_{1<|\xi|\leq|h|^{-1}}|\widehat{f}(\xi)||e^{ih\cdot\xi}-1|d\xi}_{=:|A_2|}$$

Being $\hat{f}$ bounded on $\overline{B(0,1)}$, we get

$$|A_1| \leq (2\pi)^{-d}\|\widehat{f}\|_{L^\infty(\overline{B}(0,1))}\int_{|\xi|\leq 1}|\xi\cdot h|d\xi \leq (2\pi)^{-d}\mathcal{L}^d(\overline{B}(0,1))\|\widehat{f}\|_{L^\infty(\overline{B(0,1)})}|h|, \tag{2.1.35}$$

where $\mathcal{L}^d(\overline{B(0,1)})$ denotes the d-dimensional volume of $\overline{B(0,1)}$. At the same time, on account of Equations (2.1.32) and (2.1.33), it descends

$$|A_2| \le \pi^{-1} C \int_{1<|\xi|<|h|^{-1}} |\xi|^{-d-\tau} |\xi \cdot h| d\xi \le \pi^{-1} C|h| \int_{1<|\xi|<|h|^{-1}} |\xi|^{-\tau-d+1} d\xi$$
$$\le \pi^{-1} C \frac{|h|}{1-\tau}(|h|^{\tau-1}-1) = \frac{\pi^{-1}C}{1-\tau}|h|^{\tau}(1-|h|^{1-\tau}) \le \frac{\pi^{-1}C}{1-\tau}|h|^{\tau}. \tag{2.1.36}$$

As a result, combining Equations (2.1.34), (2.1.35) and (2.1.36), we conclude

$$|f(x+h)-f(x)| \le c(|h|+|h|^{\tau}) \le c|h|^{\tau}, \quad \forall x \in \mathbb{R}^d, \forall h \in \overline{B(0,1)},$$

where we set $c := \max\{\frac{2^{-d+1}\pi^{-1}C}{1-\gamma}, (2\pi)^{-1}\mathcal{L}^d(\overline{B(0,1)})\|\widehat{f}\|_{L^\infty(\overline{B(0,1)})}\}$. □

Sobolev Spaces

To conclude this chapter, we shall give a succint overview of the theory of Sobolev spaces. In particular, we shall emphasize their relations with Besov spaces. For further information on this topic, refer to [BCD11, Sect. 1.4].

Definition 2.1.41: *Let $s \in \mathbb{R}$. We call **fractional Sobolev space** $H^s(\mathbb{R}^d)$ the set of all $u \in \mathcal{S}'(\mathbb{R}^d)$ such that*

$$\langle D \rangle^s u := \mathcal{F}^{-1}\{\langle \xi \rangle^s \widehat{u}(\xi)\} \in L^2(\mathbb{R}^d). \tag{2.1.37}$$

The norm of $H^s(\mathbb{R}^d)$ is given by

$$\|u\|_{H^s(\mathbb{R}^d)} := \|\langle D \rangle^s u\|_{L^2(\mathbb{R}^d)}. \tag{2.1.38}$$

In addition, we say that $u \in \mathcal{S}'(\mathbb{R}^d)$ lies in $H^s_{\text{loc}}(\mathbb{R}^d)$ if $\varphi u \in H^s(\mathbb{R}^d)$ for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

The Fourier multiplier operator $\langle D \rangle^s$ defines a *Bessel potential* - see [BCCS22]. For this reason, $H^s(\mathbb{R}^d)$ is also called *Bessel potential space.*

Remark 2.1.42: *Observe that u lies in $H^s(\mathbb{R}^d)$ if and only if it holds true*

$$\int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 d\xi < \infty.$$

This fact is a direct consequence of Plancherel's theorem, see Theorem A.11.10.

Theorem 2.1.43: *Let $s \in \mathbb{R}$. Then $H^s(\mathbb{R}^d)$ equipped with the scalar product*

$$(u,v) := \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} \widehat{u}(\xi) \overline{\widehat{v}}(\xi) d\xi$$

is a Hilbert space.

The following result shows that the family of Sobolev spaces is a decreasing filtration with respect to s - see [FJ99, Cor. 9.3.1].

Theorem 2.1.44: *Let $s, s' \in \mathbb{R}$ be such that $s' \le s$. Then*

$$H^s(\mathbb{R}^d) \hookrightarrow H^{s'}(\mathbb{R}^d).$$

To conclude this compendium, we recall some notable duality properties of $H^s(\mathbb{R}^d)$ starting from the following density result - see [BCD11, Prop. 1.58].

Theorem 2.1.45: *Let $s \in \mathbb{R}$. Then $\mathcal{S}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$.*

Theorem 2.1.46: *Let $s \in \mathbb{R}$. Then, if $L \in (H^s(\mathbb{R}^d))'$, there exists a unique $u \in H^{-s}(\mathbb{R}^d)$ such that*

$$L(\psi) = \langle u, \psi \rangle, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d).$$

Conversely, if $u \in H^{-s}(\mathbb{R}^d)$, then

$$|\langle u, \psi \rangle| \leq \|u\|_{H^{-s}(\mathbb{R}^d)} \|\psi\|_{H^s(\mathbb{R}^d)}, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d),$$

and the functional $\mathcal{S}(\mathbb{R}^d) \ni \psi \mapsto \langle u, \psi \rangle \in \mathbb{C}$ continuously extends to $H^s(\mathbb{R}^d)$. Moreover, it holds true that $\|L\|_{(H^s(\mathbb{R}^d))'} = \|u\|_{H^{-s}(\mathbb{R}^d)}$.

We recall that the Sobolev norm in Equation (2.1.38) can be equivalently characterized in terms of a Littlewood-Paley partition of unity.

Theorem 2.1.47: *Let $s \in \mathbb{R}$ and let $(\psi_j)_{j \in \mathbb{N}_0}$ be a Littlewood-Paley partition of unity as per Definition 2.1.2. Then there exist two positive constants c, C such that*

$$c\|u\|_{H^s(\mathbb{R}^d)} \leq \|(\psi_j(D)u)_{j \in \mathbb{N}_0}\|_{\ell_2^s(L^2(\mathbb{R}^d))} \leq C\|u\|_{H^s(\mathbb{R}^d)},$$

for every $u \in \mathcal{S}'(\mathbb{R}^d)$. Hence $H^s(\mathbb{R}^d) = B^s_{2,2}(\mathbb{R}^d)$.

Resorting to Besov spaces, it is possible to show an improved version of Sobolev lemma, as stated by the proposition below - see [Hör97, Prop. 8.6.10].

Proposition 2.1.48: *Let $s \in \mathbb{R}$. Then $H^s(\mathbb{R}^d) \hookrightarrow B^{s-\frac{d}{2}}_{\infty,\infty}(\mathbb{R}^d)$.*

At last, we prove a few notable embeddings of Besov spaces into suitable Sobolev spaces - see [DRS21]. First we recall *Hölder inequality, c.f.* [Bre10, Th. 4.6].

Theorem 2.1.49: *Let (Ω, Σ, μ) be a measure space and let $1 \leq p, p' \leq \infty$ be such that $1/p + 1/p' = 1$. Then, if $f \in L^p(\Omega, \Sigma, \mu)$ and $g \in L^{p'}(\Omega, \Sigma, \mu)$,*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}. \tag{2.1.39}$$

Theorem 2.1.50: *Let $\alpha \in \mathbb{R}$. Then the following statements hold true:*

(i) If $p \geq 2$ and $q > 2$, $B^{\alpha,\mathrm{loc}}_{p,q}(\mathbb{R}^d) \hookrightarrow H^s_{\mathrm{loc}}(\mathbb{R}^d)$ for any $s < \alpha$.

(ii) If $p \geq 2$ and $q = 2$, $B^{\alpha,\mathrm{loc}}_{p,q}(\mathbb{R}^d) \hookrightarrow H^s_{\mathrm{loc}}(\mathbb{R}^d)$ for any $s \leq \alpha$.

Proof. We exploit the characterization of Besov spaces via local means as per Definition 2.1.24. We prove the two assertions separately.

(a) Fix an arbitrary compact set $\mathfrak{K} \subset \mathbb{R}^d$. Since $H^s(\mathfrak{K}) = B^s_{2,2}(\mathfrak{K})$ per Theorem 2.1.47, we show that, if $u \in B^{\alpha,\mathrm{loc}}_{p,q}(\mathbb{R}^d)$, then there exists $C_{\mathfrak{K}} > 0$ such that

$$\|u\|_{B^s_{2,2}(\mathfrak{K})} \leq C_{\mathfrak{K}} \|u\|_{B^s_{p,q}(\mathfrak{K})}$$

for any $s < \alpha$. To this end, let $\kappa \in \mathscr{B}_{\lfloor s \rfloor}$ and $\underline{\kappa} \in \mathcal{D}(B(0,1))$ such that $\check{\underline{\kappa}}(0) \neq 0$, where $\mathscr{B}_{\lfloor s \rfloor}$ has been defined as per Definition 2.1.22. Bearing in mind that $L^p(\mathfrak{K}) \hookrightarrow L^2(\mathfrak{K})$ for $p \geq 2$, it follows immediately that there exists $c'_K > 0$ such that

$$\|u(\underline{\kappa}_x)\|_{L^2_x(\mathfrak{K})} \leq c'_{\mathfrak{K}} \|u(\underline{\kappa}_x)\|_{L^p_x(\mathfrak{K})}.$$

Again by applying $L^p(\mathfrak{K}) \hookrightarrow L^2(\mathfrak{K})$ for $p \geq 2$, it descends

$$\|u\|^2_{B^s_{2,2}(\mathfrak{K})} = \int_0^1 \frac{\|u(\kappa^\lambda_x)\|^2_{L^2_x(\mathfrak{K})}}{\lambda^{2s}} \frac{d\lambda}{\lambda} \leq c''_{\mathfrak{K}} \int_0^1 \frac{\|u(\kappa^\lambda_x)\|^2_{L^p_x(\mathfrak{K})}}{\lambda^{2s}} \frac{d\lambda}{\lambda} = c''_{\mathfrak{K}} \int_0^1 \underbrace{\frac{\|u(\kappa^\lambda_x)\|^2_{L^p_x(\mathfrak{K})}}{\lambda^{2\alpha}}}_{L^{\frac{q}{2}}_\lambda} \underbrace{\frac{1}{\lambda^{2(s-\alpha)}}}_{L^{(\frac{q}{2})'}_\lambda} \frac{d\lambda}{\lambda}$$

$$\leq c''_{\mathfrak{K}} \|\lambda^{2(\alpha-s)}\|_{L^{(\frac{q}{2})'}_\lambda} \|\lambda^{-2\alpha}\|u(\kappa^\lambda_x)\|^2_{L^p_x(\mathfrak{K})}\|_{L^{\frac{q}{2}}_\lambda} = c''_{\mathfrak{K}} \|\lambda^{2(\alpha-s)}\|_{L^{(\frac{q}{2})'}_\lambda} \|u\|^2_{B^\alpha_{p,q}(\mathfrak{K})} < \infty$$

where we exploited Hölder inequality in the second bound, $(\frac{q}{2})'$ denotes the conjugate exponent of $\frac{q}{2}$ as per Equation (2.1.9) and $L^{\frac{q}{2}}_\lambda$ is the Lebesgue space as per Remark 2.1.21. Moreover we conclude $\|\lambda^{2(\alpha-s)}\|_{L^{(\frac{q}{2})'}_\lambda}$ is finite since $s < \alpha$.

(b) If $p \geq 2$ and $q = 2$, the proof of the second statement is analogous. The only difference is the presence of the L^∞_λ-norm of $\lambda^{2(\alpha-s)}$, which is finite for $s \leq \alpha$.

□

2.2 Pseudodifferential operators

In this section, we present a succinct overview of the theory of pseudodifferential operators, which are a generalization of differential operators. By means of a Fourier transform, any differential operator $P(x, D)$ with smooth coefficients acting on a tempered distribution can be written as

$$P(x, D)u = \mathcal{F}^{-1}\{P(x, \xi)\widehat{u}(\xi)\},$$

where $P(x, \xi)$ is a polynomial function of the Fourier variable ξ, see Example 2.2.16 below. In order to define pseudodifferential operators, we shall generalize this representation of differential operators by considering more general smooth functions, called *symbols*. In Chapter 3, pseudodifferential operators shall play a prominent rôle in providing an equivalent characterization of the Besov wavefront set - see Theorem 3.2.1.

This section is divided in three parts. In Subsection 2.2.1 we shall focus on the theory of symbols. Asymptotic sums of symbols and the notion of ellipticity are sketched. At the same time, in Subsection 2.2.2 we outline the leading concepts and properties concerning pseudodifferential operators. In particular, we shall see that the set of pseudodifferential operators is closed under composition and invariant under diffeomorphisms. At last, we concisely recall how pseudodifferential operators act on Besov spaces. In Subsection 2.2.3, we introduce the basic notions to analyze the microlocal behaviour of pseudodifferential operators. For further information on all these topics, see [Hin21], [GS94, Chap. 1, Chap. 3]. Moreover, the interested reader may refer to Appendix A where we summarize the basic elements of the theory of distributions, which shall be used in this section.

In the following, for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and a multi-index $\ell = (\ell_1, \ldots, \ell_d) \in \mathbb{N}^d_0$, we set

$$\partial^\ell_x := \partial^{\ell_1}_{x_1} \cdots \partial^{\ell_d}_{x_d}, \quad D^\ell_x := D^{\ell_1}_{x_1} \cdots D^{\ell_d}_{x_d}, \quad \ell! = \ell_1! \cdots \ell_d!,$$

where $D_{x_j} = -i\partial_{x_j}$.

2.2.1 Symbols

In order to define pseudodifferential operators, we need to introduce a class of symbols. To this end, this subsection shall be devoted to recalling the main definitions and properties concerning symbols. For this topic, we mainly refer to [Hin21, Chap. 3].

Definition 2.2.1: *Let $m \in \mathbb{R}$. The space of* ***symbols of order m****, $S^m(\mathbb{R}^d;\mathbb{R}^n)$, is the collection of all functions $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^n)$ such that, for every compact set $\mathfrak{K} \subset \mathbb{R}^d$ and for every $\gamma \in \mathbb{N}_0^d$, $\ell \in \mathbb{N}_0^n$,*

$$\|a\|_{\mathfrak{K},\gamma,\ell} := \sup_{(x,\xi)\in\mathfrak{K}\times\mathbb{R}^n} \frac{|\partial_x^\gamma \partial_\xi^\ell a(x,\xi)|}{\langle\xi\rangle^{m-|\ell|}} < \infty, \tag{2.2.1}$$

where $\langle\xi\rangle$ has been defined in Equation (2.1.2). In addition, we define the space of ***residual symbols*** *as*

$$S^{-\infty}(\mathbb{R}^d;\mathbb{R}^n) := \bigcap_{m\in\mathbb{R}} S^m(\mathbb{R}^d;\mathbb{R}^n). \tag{2.2.2}$$

$S^m(\mathbb{R}^d;\mathbb{R}^n)$ is a Fréchet space with the topology induced by the semi-norms in Equation (2.2.1).

Remark 2.2.2: *Let $m, m' \in \mathbb{R}$ be such that $m \le m'$. By Definition 2.2.1, $S^m(\mathbb{R}^d;\mathbb{R}^n) \hookrightarrow S^{m'}(\mathbb{R}^d;\mathbb{R}^n)$. Therefore, $S^m(\mathbb{R}^d;\mathbb{R}^n)$ identifies an increasing filtration with respect to m.*

Remark 2.2.3: *On account of Definition 2.2.1, the maps*

$$D_x^\gamma \colon S^m(\mathbb{R}^d;\mathbb{R}^n) \to S^m(\mathbb{R}^d;\mathbb{R}^n), \quad D_\xi^\ell \colon S^m(\mathbb{R}^d;\mathbb{R}^n) \to S^{m-|\ell|}(\mathbb{R}^d;\mathbb{R}^n),$$

are continuous.

Example 2.2.4: *Let $m \ge 0$ and let $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^n)$ be such that*

$$a(x,\xi) := \sum_{|\ell|\le m} a_\ell(x)\xi^\ell, \quad \forall (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^n,$$

where $a_\ell \in C^\infty(\mathbb{R}^d)$ for every $\ell \in \mathbb{N}_0^d$ such that $|\ell| \le m$. It descends that a lies in $S^m(\mathbb{R}^d;\mathbb{R}^n)$.

Example 2.2.5: *Let $m \in \mathbb{R}$. Then $\langle\xi\rangle^m$ lies in $S^m(\mathbb{R}^d;\mathbb{R}^n)$.*

We introduce the subclass of *homogeneous* symbols, which include those introduced in Example 2.2.4.

Definition 2.2.6: *Let $m \in \mathbb{R}$. A function $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^n)$ is said to be an* ***homogeneous symbol of order m*** *if*

$$a(x,\lambda\xi) = \lambda^m a(x,\xi), \quad \forall \lambda > 0, \quad \forall |\xi| \ge 1, \tag{2.2.3}$$

and, if for every compact set $\mathfrak{K} \subset \mathbb{R}^d$ and for every $\gamma \in \mathbb{N}_0^d$, $\ell \in \mathbb{N}_0^n$, there exists $C_{\mathfrak{K},\gamma,\ell} > 0$ such that

$$|\partial_x^\gamma \partial_\xi^\ell a(x,\xi)| \le C_{\mathfrak{K},\gamma,\ell} |\xi|^{m-|\ell|}, \quad \forall x \in \mathfrak{K}, \quad \forall |\xi| \ge 1. \tag{2.2.4}$$

We denote the space of all homogeneous symbols of order m by $S^m_{\text{hom}}(\mathbb{R}^d;\mathbb{R}^n)$.

By Definition 2.2.6, it holds true that $S^m_{\text{hom}}(\mathbb{R}^d;\mathbb{R}^n)$ is a subspace of $S^m(\mathbb{R}^d;\mathbb{R}^n)$.

Proposition 2.2.7: *Let $m, m' \in \mathbb{R}$. If $a \in S^m(\mathbb{R}^d;\mathbb{R}^n)$ and $b \in S^{m'}(\mathbb{R}^d;\mathbb{R}^n)$, then $ab \in S^{m+m'}(\mathbb{R}^d;\mathbb{R}^n)$. In particular, the pointwise multiplication of symbols*

$$S^m(\mathbb{R}^d;\mathbb{R}^n) \times S^{m'}(\mathbb{R}^d;\mathbb{R}^n) \ni (a,b) \mapsto ab \in S^{m+m'}(\mathbb{R}^d;\mathbb{R}^n)$$

is a continuous bilinear map.

Asymptotic sums of symbols In this paragraph, we see under which conditions an asymptotic sum of symbols converges to an element lying in $S^m(\mathbb{R}^d;\mathbb{R}^n)$ as per Definition 2.2.1 - see [Hör94, Prop. 18.1.3].

Proposition 2.2.8: *For $j \in \mathbb{N}_0$, let $a_j \in S^{m_j}(\mathbb{R}^d,\mathbb{R}^n)$, where $(m_j)_{j\in\mathbb{N}_0}$ is a sequence of real numbers with $\lim_{j\to\infty} m_j \to -\infty$. Define $m'_k := \sup_{j\geq k} m_j$. Then there exists $a \in S^{m'_0}(\mathbb{R}^d;\mathbb{R}^n)$ such that $\operatorname{supp}(a) \subset \bigcup_{j\in\mathbb{N}_0} \operatorname{supp}(a_j)$ and*

$$a - \sum_{j=0}^{k-1} a_j \in S^{m'_k}(\mathbb{R}^d;\mathbb{R}^n), \quad \forall k \in \mathbb{N}_0. \tag{2.2.5}$$

In addition, a is unique up to an element lying in $S^{-\infty}(\mathbb{R}^d;\mathbb{R}^n)$. We say that a is ***asymptotic*** *to $\sum_{j=0}^{\infty} a_j$ and we write*

$$a \sim \sum_{j=0}^{\infty} a_j.$$

Remark 2.2.9: *On account of Definition 2.2.1, Equation (2.2.5) can be read as follows: For every $k \in \mathbb{N}_0$, every compact set $\mathfrak{K} \subset \mathbb{R}^d$ and every $\gamma \in \mathbb{N}_0^d, \ell \in \mathbb{N}_0^n$, there exists $C_{k,\mathfrak{K},\gamma,\ell}$ such that*

$$\left|\partial_x^\gamma \partial_\xi^\ell \left(a(x,\xi) - \sum_{j=0}^{k-1} a_j(x,\xi)\right)\right| \leq C_{k,K,\gamma,\ell} \langle\xi\rangle^{m'_k - |\ell|}, \quad \forall (x,\xi) \in \mathfrak{K} \times \mathbb{R}^n. \tag{2.2.6}$$

If a is to be asymptotic to $\sum_{j=0}^{\infty} a_j$ as per Proposition 2.2.8, then Equation (2.2.6) can be simplified, as stated by the following proposition - see [Hör94, Prop. 18.1.4].

Proposition 2.2.10: *For $j \in \mathbb{N}_0$, let $a_j \in S^{m_j}(\mathbb{R}^d,\mathbb{R}^n)$, where $(m_j)_{j\in\mathbb{N}_0}$ is a sequence of real numbers such that $\lim_{j\to\infty} m_j \to -\infty$. Moreover, let $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^n)$ be such that*

(a) For every compact set $\mathfrak{K} \subset \mathbb{R}^d$ and for every $\gamma \in \mathbb{N}_0^d$, $\ell \in \mathbb{N}_0^n$, there exists $C, \mu > 0$ such that

$$|\partial_x^\gamma \partial_\xi^\ell a(x,\xi)| \leq C\langle\xi\rangle^\mu, \quad \forall (x,\xi) \in \mathfrak{K} \times \mathbb{R}^n; \tag{2.2.7}$$

(b) There exists a sequence $(\mu_k)_{k\in\mathbb{N}_0}$ with $\lim_{k\to\infty} \mu_k = -\infty$ such that for every compact set $\mathfrak{K} \subset \mathbb{R}^d$ and every $k \in \mathbb{N}$, there exists $C > 0$ such that

$$|a(x,\xi) - \sum_{j=0}^{k-1} a_j(x,\xi)| \leq C\langle\xi\rangle^{\mu_k} \quad \forall (x,\xi) \in \mathfrak{K} \times \mathbb{R}^n. \tag{2.2.8}$$

Then $a \in S^m(\mathbb{R}^d;\mathbb{R}^n)$, $m = \sup_{j\in\mathbb{N}_0} m_j$, and $a \sim \sum_{j=0}^{\infty} a_j$.

Ellipticity This paragraph is devoted to discussing *elliptic* symbols.

Definition 2.2.11: *Let $m \in \mathbb{R}$. A symbol $a \in S^m(\mathbb{R}^d;\mathbb{R}^n)$ is called* ***elliptic*** *if there exists a symbol $b \in S^{-m}(\mathbb{R}^d;\mathbb{R}^n)$ such that $ab - 1 \in S^{-1}(\mathbb{R}^d;\mathbb{R}^n)$.*

The following proposition collects a few notable conditions which are equivalent to ellipticity.

Proposition 2.2.12: *Let $m \in \mathbb{R}$ and let $a \in S^m(\mathbb{R}^d;\mathbb{R}^n)$. The following statements are equivalent:*

(1) a is elliptic.

(2) There exist $L, C > 0$ *such that, if* $|\xi| \geq L$,

$$|a(x,\xi)| \geq C|\xi|^m. \tag{2.2.9}$$

(3) There exist $C, C' > 0$ *such that*

$$|a(x,\xi)| \geq C|\xi|^m - C'|\xi|^{m-1}, \quad \forall |\xi| \geq 1. \tag{2.2.10}$$

Remark 2.2.13: *By statement (2) of Proposition 2.2.12, if* $a \in S^m(\mathbb{R}^d;\mathbb{R}^n)$ *is elliptic, it turns out that* $a+\tilde{a}$ *is also elliptic for any* $\tilde{a} \in S^{m-1}(\mathbb{R}^d;\mathbb{R}^n)$*. Therefore, ellipticity is a property of the equivalence class*

$$[a] \in S^m(\mathbb{R}^d;\mathbb{R}^n)/S^{m-1}(\mathbb{R}^d;\mathbb{R}^n).$$

Example 2.2.14: *Let* $m \in \mathbb{R}$ *and let* $a \in S^m(\mathbb{R}^d;\mathbb{R}^n)$ *be such that* $a(x,\xi) = \langle\xi\rangle^m$*. On account of Definition 2.2.11, then* a *is elliptic. As a matter of fact, there exists* $b(x,\xi) = \langle\xi\rangle^{-m}$*, lying in* $S^{-m}(\mathbb{R}^d;\mathbb{R}^n)$*, such that*

$$ab - 1 = 0 \in S^{-\infty}(\mathbb{R}^d;\mathbb{R}^n) \hookrightarrow S^{-1}(\mathbb{R}^d;\mathbb{R}^n).$$

2.2.2 Quantizations and Pseudodifferential Operators

After introducing a class of symbols in the previous subsection, we are in position to define the space of *pseudodifferential operators*. To this purpose, we start by giving the definition of *quantization* of a symbol. For this section, we mainly refer to [Hin21].

Definition 2.2.15: *Let* $m \in \mathbb{R}$ *and let* $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d;\mathbb{R}^d)$*. Its* ***quantization****,* $\mathrm{Op}(a)\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$*, is defined as*

$$(\mathrm{Op}(a)u)(x) := (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)dyd\xi, \quad u \in \mathcal{S}(\mathbb{R}^d). \tag{2.2.11}$$

$\mathrm{Op}(a)$ *is referred to as a* ***pseudodifferential operator (ΨDO) of order*** $\mathbf{m}$ *while* a *is dubbed* ***full symbol*** *of* $\mathrm{Op}(a)$*. We denote the space of all these operators by* $\Psi^m(\mathbb{R}^d)$*. In addition, we set*

$$\Psi^{-\infty}(\mathbb{R}^d) := \bigcap_{m\in\mathbb{R}} \Psi^m(\mathbb{R}^d). \tag{2.2.12}$$

Example 2.2.16: *Let* $m \in \mathbb{R}$ *and let* $a \in S^m(\mathbb{R}^d;\mathbb{R}^d)$ *be as in Example 2.2.4. Then* $\mathrm{Op}(a)$ *lies in* $\Psi^m(\mathbb{R}^d)$*. Moreover, on account of Equation (2.2.11), it descends that, for any* $u \in \mathcal{S}(\mathbb{R}^d)$*,*

$$\begin{aligned}(\mathrm{Op}(a)u)(x) &= (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi}a(x,\xi)u(y)dyd\xi = (2\pi)^{-d}\sum_{|\ell|\leq m}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi}a_\ell(x)\xi^\ell u(y)dyd\xi \\ &= \sum_{|\ell|\leq m} a_\ell(x)\int_{\mathbb{R}^d} e^{ix\cdot\xi}\xi^\ell\widehat{u}(\xi)\frac{d\xi}{(2\pi)^d} = \sum_{|\ell|\leq m} a_\ell(x)(D^\ell u)(x).\end{aligned}$$

Therefore,

$$\mathrm{Op}(a) = \sum_{|\ell|\leq m} a_\ell(x)D^\ell.$$

Example 2.2.17: *Let* $m \in \mathbb{R}$*. Let* $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d;\mathbb{R}^d)$ *be such that* $a(x,y,\xi) = \langle\xi\rangle^m$*, where* $\langle\xi\rangle$ *has been defined in Equation (2.1.2). Then* $\langle D\rangle^m := \mathrm{Op}(a)$ *lies in* $\Psi^m(\mathbb{R}^d)$*.*

On account of Equation (2.2.11), we note that the Schwartz kernel $K \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ of $\mathrm{Op}(a)$ is given by

$$K(x,y) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x,y,\xi) d\xi, \tag{2.2.13}$$

which can be understood as the inverse Fourier transform of $a(x,y,\xi)$ with respect to ξ. In particular, K is smooth away from the diagonal.

Proposition 2.2.18: *Let $K \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ be the Schwartz kernel of a pseudodifferential operator. Then*

$$\mathrm{singsupp}(K) \subset \{(x,x) : x \in \mathbb{R}^d\}.$$

Proposition 2.2.18 is a direct consequence of the theory of oscillatory integrals. For this topic, the interested reader can refer to [Hör03, Sect. 7.8].

By means of a duality argument, the action of a ΨDO of order m can be continuously extended to $\mathcal{S}'(\mathbb{R}^d)$. As a matter of fact, given $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$, it turns out that, for any $u, v \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}
\langle \mathrm{Op}(a)u, v\rangle &= (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x,y,\xi)u(y)v(x)e^{i(x-y)\cdot\xi} dy d\xi dx \\
&\overset{\substack{\xi\mapsto-\xi \\ x\mapsto y}}{=} (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(y,x,-\xi)v(y)u(x)e^{i(x-y)\cdot\xi} dy d\xi dx \\
&= \langle u, \mathrm{Op}(a^\dagger)v\rangle,
\end{aligned}$$

where we set

$$a^\dagger(x,y,\xi) = a(y,x,-\xi).$$

It descends that

$$\langle \mathrm{Op}(a)u, v\rangle = \langle u, \mathrm{Op}(a^\dagger)v\rangle, \quad \forall u, v \in \mathcal{S}(\mathbb{R}^d). \tag{2.2.14}$$

Being $\mathcal{S}(\mathbb{R}^d)$ dense in $\mathcal{S}'(\mathbb{R}^d)$, $\mathrm{Op}(a)$ continuously extends to $\mathcal{S}'(\mathbb{R}^d)$ via Equation (2.2.14).

In the following, we recall that Equation (2.2.11) can be equivalently replaced either by the *left quantization* of a *left symbol* $a_L \in S^m(\mathbb{R}^d; \mathbb{R}^d)$, independent from y, or by the *right quantization* of a *right symbol* $a_R \in S^m(\mathbb{R}^d; \mathbb{R}^d)$, independent from x:

$$(\mathrm{Op}_L(a_L)u)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a_L(x,\xi)u(y) dy d\xi, \quad u \in \mathcal{S}(\mathbb{R}^d), \tag{2.2.15}$$

$$(\mathrm{Op}_R(a_R)u)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a_R(y,\xi)u(y) dy d\xi, \quad u \in \mathcal{S}(\mathbb{R}^d). \tag{2.2.16}$$

This fact is the content of the following theorem - see [Hin21, Th. 4.8].

Theorem 2.2.19: *Let $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$. Then there exists a unique left symbol $a_L \in S^m(\mathbb{R}^d; \mathbb{R}^d)$ such that*

$$\mathrm{Op}(a) = \mathrm{Op}(a_L) = \mathrm{Op}_L(a_L),$$

and a unique right symbol $a_R \in S^m(\mathbb{R}^d; \mathbb{R}^d)$ such that

$$\mathrm{Op}(a) = \mathrm{Op}(a_R) = \mathrm{Op}_R(a_R).$$

In addition, a_L, a_R are given by the asymptotic sums

$$a_L(x,\xi) \sim \sum_{\ell\in\mathbb{N}_0^d} \frac{1}{\ell!} (\partial_\xi^\ell D_x^\ell a(x,y,\xi))|_{y=x},$$

$$a_R(y,\xi) \sim \sum_{\ell \in \mathbb{N}_0^d} \frac{(-1)^{|\ell|}}{\ell!} (\partial_\xi^\ell D_y^\ell a(x,y,\xi))|_{x=y}.$$

Definition 2.2.20: *We call a_L and a_R the left and right reductions of the full symbol a respectively. Setting $A = \mathrm{Op}(a)$, we write*

$$\sigma_L(A) := a_L, \quad \sigma_R(A) := a_R.$$

The following theorem states that $\Psi^{-\infty}(\mathbb{R}^d)$ coincides with the space of quantizations of residual symbols. In particular, it entails that the elements lying in $\Psi^{-\infty}(\mathbb{R}^d)$ are smoothing operators.

Proposition 2.2.21: *An operator $A\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ lies in $\Psi^{-\infty}(\mathbb{R}^d)$ if and only if its Schwartz kernel $K \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ is smooth and satisfies*

$$\sup_{x,y\in\mathbb{R}^d} \langle x-y\rangle^N |\partial_x^\ell \partial_y^k K(x,y)| < \infty,$$

for every $N \in \mathbb{N}_0$, $\ell, k \in \mathbb{N}_0^d$. In addition, there exist unique symbols $a_L, a_R \in S^{-\infty}(\mathbb{R}^d;\mathbb{R}^d)$ such that $A = \mathrm{Op}_L(a_L) = \mathrm{Op}_R(a_R)$.

Adjoints In this paragraph, we discuss *adjoints* of pseudodifferential operators - see [Hin21, Cor. 4.13].

Definition 2.2.22: *Let $A \in \Psi^m(\mathbb{R}^d)$. We define its adjoint $A^*: \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ by*

$$\int_{\mathbb{R}^d} (A^*u)(x)\overline{v(x)}dx = \int_{\mathbb{R}^d} u(x)\overline{(Av)(x)}dx, \quad \forall u, v \in \mathcal{S}(\mathbb{R}^d).$$

Proposition 2.2.23: *If $A \in \Psi^m(\mathbb{R}^d)$, then $A^* \in \Psi^m(\mathbb{R}^d)$. In particular, if $A = \mathrm{Op}(a)$, then $A^* = \mathrm{Op}(a^*)$, where $a^*(x,y,\xi) = \overline{a(x,y,\xi)}$.*

Composition In the following, we recall that the composition of ΨDOs yields another ΨDO - see [Hin21, Th. 4.16]. To this end, we recall the definition of proper map and we introduce an important subclass of pseudodifferential operators, called *properly supported*.

Definition 2.2.24: *A continuous map $f\colon \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is called* ***proper*** *if $f^{-1}(\mathfrak{K})$ is a compact subset in $\mathbb{R}^{d_1}$ for any compact set $\mathfrak{K} \subset \mathbb{R}^{d_2}$.*

Definition 2.2.25: *Let $\pi_1\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and let $\pi_2\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be the canonical projections defined by $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$ respectively. We say that $A \in \Psi^m(\mathbb{R}^d)$ with Schwartz kernel $K \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ is* ***properly supported*** *if π_1, π_2 are proper maps when restricted to $\mathrm{supp}(K)$.*

Proposition 2.2.26: *Let $A \in \Psi^m(\mathbb{R}^d)$, $B \in \Psi^{m'}(\mathbb{R}^d)$. Suppose that at least one among A, B is properly supported. Then $AB \in \Psi^{m+m'}(\mathbb{R}^d)$ and its left symbol is given by*

$$\sigma_L(AB) \sim \sum_{\ell\in\mathbb{N}_0^d} \frac{1}{\ell!} \partial_\xi^\ell \sigma_L(A) D_x^\ell \sigma_L(B).$$

On account of Proposition 2.2.26, it is possible to prove the *pseudolocality* of ΨDOs - see [Hin21, Prop. 4.17].

Proposition 2.2.27: *Let $A \in \Psi^m(\mathbb{R}^d)$. Then*

$$\mathrm{singsupp}(Au) \subset \mathrm{singsupp}(u), \quad \forall u \in \mathcal{S}'(\mathbb{R}^d). \tag{2.2.17}$$

A is said to be a ***pseudolocal operator****.*

Principal symbol In the following, we define the *principal symbol* of an operator $A \in \Psi^m(\mathbb{R}^d)$, which coincides with the leading contibution to a.

Definition 2.2.28: *Let $m \in \mathbb{R}$ and let $A \in \Psi^m(\mathbb{R}^d)$. Then the **principal symbol** of A is defined by*

$$\sigma_m(A) := [\sigma_L(A)] \in S^m(\mathbb{R}^d; \mathbb{R}^d)/S^{m-1}(\mathbb{R}^d; \mathbb{R}^d),$$

where $[\sigma_L(A)]$ denote the equivalence class of $\sigma_L(A)$. In order to lighten the notation, we shall omit the square brackets.

The following proposition collects a few notable properties of a principal symbol - [Hin21, Prop. 4.21].

Proposition 2.2.29: *The following statements hold true:*

- *If $A \in \Psi^m(\mathbb{R}^d)$, then $\sigma_m(A) = [\sigma_R(A)]$.*
- *Let $A \in \Psi^m(\mathbb{R}^d)$. Then $\sigma_m(A^*) = \overline{\sigma_m(A)}$, where A^* is the adjoint of A as per Definition 2.2.22.*
- *Let $A \in \Psi^m(\mathbb{R}^d)$, $B \in \Psi^{m'}(\mathbb{R}^d)$ with at least one among A, B properly supported. Then $\sigma_{m+m'}(AB) = \sigma_m(A)\sigma_{m'}(B)$.*

Elliptic ΨDOs By recalling Definition 2.2.11, we introduce the notion of *elliptic* operator.

Definition 2.2.30: *Let $m \in \mathbb{R}$. An operator $A \in \Psi^m(\mathbb{R}^d)$ is called **elliptic** if its principal symbol $\sigma_m(A)$ is elliptic.*

Example 2.2.31: *Let $m \in \mathbb{R}$. If $a(x, \xi) = \langle \xi \rangle^m$ with $\langle \xi \rangle$ defined as per Equation (2.1.2), then $\langle D \rangle^m := \mathrm{Op}_L(a)$ is an elliptic ΨDO of order m.*

We also recall that an elliptic ΨDO admits an inverse operator, called *parametrix* - see [Hin21, Th. 4.26], [GS94, Th. 4.1].

Theorem 2.2.32: *Let $m \in \mathbb{R}$ and let $A \in \Psi^m(\mathbb{R}^d)$ be elliptic. Then there exists $Q \in \Psi^{-m}(\mathbb{R}^d)$, properly supported, such that*

$$AQ - I \in \Psi^{-\infty}(\mathbb{R}^d), \quad QA - I \in \Psi^{-\infty}(\mathbb{R}^d).$$

*Q is said to be a **parametrix** of A.*

Invariance of ΨDOs under diffeomorpshims In this paragraph, we shall discuss the behaviour of ΨDOs under the action of a local diffeomorphism. We give a result in which the pullback of a diffeomorphism plays a key rôle. For details concerning the pullback of a distribution, the interested reader can refer to Appendix A.9.

We start by introducing the notion of *local* symbol.

Definition 2.2.33: *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $m \in \mathbb{R}$. A function $a \in C^\infty(\Omega \times \mathbb{R}^d)$ is a **local symbol** of order m if $\phi a \in S^m(\mathbb{R}^d; \mathbb{R}^d)$ for all $\phi \in \mathcal{D}(\Omega)$. We denote the space of local symbols of order m by $S^m(\Omega; \mathbb{R}^d)$.*

Given $a \in S^m(\Omega; \mathbb{R}^d)$, its left quantization identifies an operator

$$\mathrm{Op}_L(a) \colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{D}'(\Omega), \tag{2.2.18}$$

where $\mathrm{Op}_L(a)$ is defined as per Equation (2.2.15). By analogy with Definition 2.2.15, we denote the space of ΨDOs as in Equation 2.2.18 by $\Psi^m(\Omega)$.

Remark 2.2.34: *Since $\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}'(\Omega) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$, the domain of $\mathrm{Op}_L(a)$ in Equation (2.2.18) can be restricted to $\mathcal{D}(\Omega)$ or to $\mathcal{E}'(\Omega)$.*

Remark 2.2.35: *Let $\Omega \subset \mathbb{R}^d$ be an open set. If $A \in \Psi^m(\Omega)$ is properly supported as per Definition 2.2.25, then $A\colon \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$, $A\colon C^\infty(\Omega) \to C^\infty(\Omega)$, $A\colon \mathcal{E}'(\Omega) \to \mathcal{E}'(\Omega)$ and $A\colon \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ are continuous linear operators, see [GS94, Chap. 3].*

The following theorem states that the class of m-th order pseudodifferential operators is invariant under change of coordinates on $\mathbb{R}^d$ - see [Hin21, Th. 5.2].

Proposition 2.2.36: *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be two open sets, let $f\colon \Omega_1 \to \Omega_2$ be a diffeomorphism and let $f^*\colon \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$ be the pullback map along f. If $A \in \Psi^m(\Omega_2)$, then*

$$A_f\colon \mathcal{D}(\Omega_1) \to \mathcal{D}'(\Omega_1), \quad \mathcal{D}(\Omega_1) \ni u \mapsto f^* A((f^{-1})^* u)$$

lies in $\Psi^m(\Omega_1)$. In addition,

$$\sigma_m(A_f)(x,\xi) = \sigma_m(A)(f(x), ({}^t df(x))^{-1}\xi),$$

where $\sigma_m(A)$ and $\sigma_m(A_f)$ are the principal symbols of A and A_f respectively as per Definition 2.2.28 while df denotes the differential map of f.

Remark 2.2.37: *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be two open sets and let $f\colon \Omega_1 \to \Omega_2$ be a diffeomorphism. Suppose that A, A_f are defined as in Proposition 2.2.36. On account of Remark 2.2.35, if A is properly supported as per Definition 2.2.35, it descends that A_f is also properly supported.*

ΨDOs on Besov spaces To conclude this review, we state a result regarding the action of pseudodifferential operators on Besov spaces as per Definition 2.1.4 - see [Abe12, Sect. 6.6].

Theorem 2.2.38: *Let $m \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $1 \le p,q \le \infty$ and let $A \in \Psi^m(\mathbb{R}^d)$. Then $A\colon B^\alpha_{p,q}(\mathbb{R}^d) \to B^{\alpha-m}_{p,q}(\mathbb{R}^d)$ is a bounded linear operator. In particular, if A is properly supported, then the restriction of A to $B^{\alpha,\mathrm{loc}}_{p,q}(\mathbb{R}^d)$ is a continuous linear operator from $B^{\alpha,\mathrm{loc}}_{p,q}(\mathbb{R}^d)$ to $B^{\alpha-m,\mathrm{loc}}_{p,q}(\mathbb{R}^d)$.*

2.2.3 Microlocalization

This subsection is devoted to recalling the basic notions and results concerning the microlocal behaviour of a symbol or, equivalently, of its quantization. Here, we shall only consider properly supported ΨDOs, as per Definition 2.2.25. We start by giving the definition of *operator wavefront set*, which establishes the *microlocal non-triviality* of a ΨDO. The notion of triviality is closely related to the behaviour of symbols as $|\xi| \to \infty$. Roughly speaking, a pseudodifferential operator is microlocally trivial at (x_0, ξ_0) if its full symbol lies in $S^{-\infty}(\mathbb{R}^d; \mathbb{R}^n)$ in a *conic neighborhood* of (x_0, ξ_0). For the sake of completeness, we first recall the definition of a *conic* set.

Definition 2.2.39: *A subset $\Gamma \subset \mathbb{R}^n \setminus \{0\}$ is called **conic** if*

$$\xi \in \Gamma \Rightarrow \lambda\xi \in \Gamma, \quad \forall \lambda > 0.$$

*A **(proper) conic neighborhood** Γ of $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ is a conic set such that $\{\lambda\xi_0 : \lambda > 0\} \subsetneq \Gamma$.*

Example 2.2.40: *Given $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ and $\epsilon > 0$, an open conic neighborhood of ξ_0 can be defined by*

$$\Gamma := \left\{ \xi \in \mathbb{R}^n \setminus \{0\} : \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \epsilon \right\}.$$

Definition 2.2.41: *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\pi_\Omega\colon \Omega \times (\mathbb{R}^n \setminus \{0\}) \to \Omega$ be the canonical projection such that $\pi_\Omega(x,\xi) = x$ for any $(x,\xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$. A subset $V \subset \Omega \times (\mathbb{R}^n \setminus \{0\})$ is called **(fibrewise) conic** if*

$$(x,\xi) \in V \Rightarrow (x, \lambda\xi) \in V, \quad \forall \lambda > 0.$$

*A **conic neighborhood** of $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ is a (fibrewise) conic set $V \subset \Omega \times (\mathbb{R}^n \times \{0\})$ such that $\pi_\Omega(V)$ is a neighborhood of x_0 and $V \cap (\{x_0\} \times (\mathbb{R}^n \setminus \{0\}))$ is a conic neighborhood of ξ_0 as per Definition 2.2.39.*

Definition 2.2.42: *Let $a \in S^m(\mathbb{R}^d; \mathbb{R}^n)$. Then $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^n \setminus \{0\})$ does not lie in the **essential support** of a,*

$$\operatorname{ess\,supp}(a) \subset \mathbb{R}^d \times (\mathbb{R}^n \setminus \{0\}),$$

if there exist an open conic neighborhood V of (x_0, ξ_0) as per Definition 2.2.41 such that for all $\gamma \in \mathbb{N}_0^d$, $\ell \in \mathbb{N}_0^n$, $k \in \mathbb{R}$, there exists $C > 0$ such that

$$|\partial_x^\gamma \partial_\xi^\ell a(x, \xi)| \leq C \langle \xi \rangle^{-k}, \quad \forall (x, \xi) \in V, \quad |\xi| \geq 1, \tag{2.2.19}$$

where $\langle \xi \rangle$ has been defined as per Equation (2.1.2).

Remark 2.2.43: *By Definition 2.2.42, observe that $\operatorname{ess\,supp}(a)$ is a fiberwise closed conic set as per Definition 2.2.41.*

Definition 2.2.44: *Let $A = \mathrm{Op}_L(a) \in \Psi^m(\mathbb{R}^d)$. The **operator wavefront set** of A is*

$$WF'(A) := \operatorname{ess\,supp}(a). \tag{2.2.20}$$

The following proposition collects a few notable properties of the operator wavefront set - see [Hin21, Prop. 6.4].

Proposition 2.2.45: *Let $A, B \in \Psi^m(\mathbb{R}^d)$. Then the following statements hold true:*

- *If the Schwartz kernel of A lies in $\mathcal{E}'(\mathbb{R}^d \times \mathbb{R}^d)$, then $WF'(A) = \emptyset$ if and only if $A \in \Psi^{-\infty}(\mathbb{R}^d)$.*
- *$WF'(A + B) \subset WF'(A) \cup WF'(B)$.*
- *If at least one among A, B is properly supported, $WF'(AB) \subset WF'(A) \cap WF'(B)$.*
- *$WF'(A^*) = WF'(A)$, where A^* is the adjoint of A as per Definition 2.2.22.*

Example 2.2.46: *If $A = \mathrm{Op}_L(a) \in \Psi^m(\mathbb{R}^d)$ is elliptic as per Definition 2.2.30, then*

$$WF'(A) = \operatorname{supp}(a) \times (\mathbb{R}^d \setminus \{0\})$$

Example 2.2.47: *Let $A = \sum_{|\ell| \leq m} a_\ell(x) D^\ell$ where $a_\ell \in C^\infty(\mathbb{R}^d)$ for all $\ell \in \mathbb{N}_0^d$ with $|\ell| \leq m$. Then*

$$WF'(A) = \left(\bigcup_{|\ell| \leq m} \operatorname{supp}(a_\ell) \right) \times (\mathbb{R}^d \setminus \{0\})$$

A further important concept is that of *elliptic set*, which refines at a microlocal level the notion of ellipticity of ΨDOs, as per Definition 2.2.30.

Definition 2.2.48: *Let $A \in \Psi^m(\mathbb{R}^d)$. A point $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ lies in the **elliptic set** of A, denoted by $\mathrm{Ell}(A)$, if there exist an open conic neighborhood V of (x_0, ξ_0) as per Definition 2.2.41 and $C > 0$ such that*

$$|\sigma_m(A)(x, \xi)| \geq C|\xi|^m, \quad \forall (x, \xi) \in V, |\xi| \geq 1, \tag{2.2.21}$$

*where $\sigma_m(A)$ is the principal symbol as per Definition 2.2.28. If $(x_0, \xi_0) \in \mathrm{Ell}(A)$, we say that A is **elliptic** at (x_0, ξ_0). In addition, we define the **characteristic set** of A, denoted by $\mathrm{Char}(A)$, as the complement of $\mathrm{Ell}(A)$, that is,*

$$\mathrm{Char}(A) := \{(x, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) : \sigma_m(A)(x, \xi) = 0\}. \tag{2.2.22}$$

Remark 2.2.49: *On account of Definition 2.2.11, the notion of* $\mathrm{Ell}(A)$ *can be equivalently reformulated as follows:* $(x_0, \xi_0) \in \mathrm{Ell}(A)$ *if and only if there exists* $b \in S^{-m}(\mathbb{R}^d; \mathbb{R}^d)$ *such that* $\sigma_m(A)b - 1 \in S^{-1}(\mathbb{R}^d; \mathbb{R}^d)$ *in a conic neighborhood of* (x_0, ξ_0).

Remark 2.2.50: *Given* $A \in \Psi^m(\mathbb{R}^d)$*, its elliptic set is a fiberwise closed conic set as per Definition 2.2.41.*

At last, we point out that the construction of a parametrix, as per Theorem 2.2.32, can be microlocalized. As a matter of fact, the following proposition holds true - see [Hin21, Prop. 6.15].

Proposition 2.2.51: *Let* $A \in \Psi^m(\mathbb{R}^d)$ *and let* $\mathscr{C} \subset \mathbb{R}^d$ *be a closed subset. Then there exists* $Q \in \Psi^{-m}(\mathbb{R}^d)$*, properly supported, such that*

$$\mathscr{C} \cap WF'(AQ - I) = \emptyset, \quad \mathscr{C} \cap WF'(QA - I) = \emptyset. \tag{2.2.23}$$

Q *is a* ***microlocal parametrix*** *of* A *on* $\mathscr{C}$.

2.3 Smooth Wavefront Set

In this section, we shall concisely outline the definition and the main properties of the *smooth wavefront set* of a distribution. Its purpose is to establish the directions in Fourier space which cause the appearance of singularities of an underlying distribution.
As shown in Appendix A, some operations between distributions can be only defined in limited cases. For instance, the product of distributions is well-defined when the singular supports of the distributions are disjoint, see Theorem A.5.3. However, in Subsections 2.3.2 and 2.3.3, we shall see that the notion of smooth wavefront set establishes sufficient criteria to extend the definition of operations such as pullback and multiplication to the whole space of distributions. In Subsection 2.3.4, we shall see how the smooth wavefront set transforms under a push-forward along a projection map. In Subsection 2.3.5, given a linear map $\mathcal{K}\colon \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ with Schwartz kernel $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$, we shall discuss the smooth wavefront set of $\mathcal{K}u$ for any $u \in \mathcal{D}(\Omega_2)$. Moreover, we shall see under which conditions $\mathcal{K}$ can be extended to $\mathcal{E}'(\Omega_2)$. Lastly, in Subsection 2.3.6 we discuss the problem of the propagation of singularities, which aims at characterizing the smooth wavefront set of a solution to a partial differential equations. For further details concerning this topic, the reader may refer to [Hör03, Chap. VIII], [Hin21], [FJ99, Chap. 11]. In this section, we shall make use of the notions introduced in Appendix A.

2.3.1 Basic definitions

In this subsection, we shall define the notion of *smooth wavefront set* of a distribution. By means of the Paley-Wiener-Schwartz theorem, the singular behaviour of a distribution can be analyzed in terms of those directions along which its Fourier transform is not rapidly decreasing. Therefore, we recall a suitable version of the Paley-Wiener-Schwartz theorem - see [FJ99, Th. 10.2.2].

Proposition 2.3.1: *Let* $v \in \mathcal{E}'(\mathbb{R}^d)$*. Then* v *lies in* $\mathcal{D}(\mathbb{R}^d)$ *if and only if for all* $N \in \mathbb{N}_0$ *there exists* $C_N > 0$ *such that*

$$|\widehat{v}(\xi)| \leq C_N \langle \xi \rangle^{-N}, \quad \forall \xi \in \mathbb{R}^d, \tag{2.3.1}$$

where $\langle \xi \rangle$ *has been defined in Equation* (2.1.2).

Remark 2.3.2: *Proposition 2.3.1 yields an equivalent characterization of the singular support of a distribution. Let* $\Omega \subset \mathbb{R}^d$ *be an open set. If* $u \in \mathcal{D}'(\Omega)$*, then a point* $x \in \Omega$ *does not lie in* $\mathrm{singsupp}(u)$ *if and only if there exists* $\phi \in \mathcal{D}(\Omega)$*,* $\phi(x) = 1$*, such that* $\widehat{\phi u}$ *satisfies Equation* (2.3.1).

In the following, we give a precise notion of singular direction. In particular, we shall require that the complement of the set of all singularities, called *frequency set*, is an open conic neighbourhood as per Definition 2.2.39.

Definition 2.3.3: *Let $v \in \mathcal{E}'(\mathbb{R}^d)$. A direction $\xi_0 \in \mathbb{R}^d \setminus \{0\}$ does not lie in the* ***frequency set*** *of v, denoted by $\xi_0 \notin \Sigma(v)$, if there exists an open conic neighborhood Γ of ξ_0 such that Equation (2.3.1) is valid for any $\xi \in \Gamma$.*

On account of Definition 2.3.3, Proposition 2.3.1 can be equivalently stated as follows:

Lemma 2.3.4: *Let $v \in \mathcal{E}'(\mathbb{R}^d)$. Then v lies in $\mathcal{D}(\mathbb{R}^d)$ if and only if $\Sigma(v) = \emptyset$.*

The frequency set of v shrinks when v is localized by a compactly supported smooth function - see [Hör03, Lemma 8.1.1].

Lemma 2.3.5: *Let $v \in \mathcal{E}'(\mathbb{R}^d)$ and let $\phi \in \mathcal{D}(\mathbb{R}^d)$. Then*

$$\Sigma(\phi v) \subset \Sigma(v).$$

Let $\Omega \subset \mathbb{R}^d$ be an open set and let $u \in \mathcal{D}'(\Omega)$. Given $x \in \Omega$, we set

$$\Sigma_x(u) := \bigcap_{\substack{\phi \in \mathcal{D}(\Omega),\\ \phi(x) \neq 0}} \Sigma(\phi u)$$

In particular, notice that $\Sigma_x(u) = \emptyset$ if and only if there exists $\phi \in \mathcal{D}(\Omega)$ with $\phi(x) \neq 0$ such that $\phi u \in \mathcal{D}(\Omega)$, that is, $x \notin \operatorname{singsupp}(u)$.

Having introduced these basic notions, we observe that the singular behaviour of a distribution is characterized both by its singular support, which identifies the singular points, and by its frequency set, which expresses the singular directions. We give the definition of smooth wavefront set of a distribution as a combination of these two types of information.

Definition 2.3.6: *Let $\Omega \subset \mathbb{R}^d$ and let $u \in \mathcal{D}'(\Omega)$. The* ***smooth wavefront set*** *of u is*

$$WF(u) := \{(x, \xi) \in \Omega \times (\mathbb{R}^d \setminus \{0\}) : \xi \in \Sigma_x(u)\}. \tag{2.3.2}$$

Remark 2.3.7: *Denoting the cotangent bundle of Ω without the zero section by $T^*\Omega$ and taking into account the identification*

$$T^*\Omega \setminus \{0\} \simeq \Omega \times (\mathbb{R}^d \setminus \{0\}),$$

$WF(u)$ can be read as a subset of $T^*\Omega \setminus \{0\}$. This viewpoint is crucial if we wish to develop the theory in a more general geometrical setting, where Ω is replaced by a smooth manifold.

$WF(u)$ is a (fiberwise) closed conic set of $\Omega \times (\mathbb{R}^d \setminus \{0\})$ as per Definition 2.2.41. Moreover, as mentioned above, the notion of smooth wavefront set refines that of singular support. As matter of fact, the projection of $WF(u)$ on Ω coincides with $\operatorname{singsupp}(u)$ - see [FJ99, Prop 11.1.1].

Proposition 2.3.8: *Let $\Omega \subset \mathbb{R}^d$ and let $u \in \mathcal{D}'(\Omega)$. Then*

$$\operatorname{singsupp}(u) = \{x \in \Omega : \exists \xi \in \mathbb{R}^d \setminus \{0\}, (x, \xi) \in WF(u)\}.$$

Concerning the projection of $WF(u)$ on $\mathbb{R}^d \setminus \{0\}$, the following result holds true.

Proposition 2.3.9: *Let $\Omega \subset \mathbb{R}^d$ and let $\pi_\xi : \Omega \times (\mathbb{R}^d \setminus \{0\}) \to \mathbb{R}^d \setminus \{0\}$ be the canonical projection such that $\pi_\xi(x, \xi) = \xi$. If $u \in \mathcal{E}'(\Omega)$, then $\pi_\xi(WF(u)) = \Sigma(u)$.*

In the following example, we show how to compute the smooth wavefront set of the Dirac delta.

Example 2.3.10: *Let $\delta \in \mathcal{D}'(\mathbb{R}^d)$ be the Dirac delta centered at the origin. Since* $\mathrm{singsupp}(\delta) = \{0\}$*, it suffices to evaluate $\Sigma_0(u)$. Given $\phi \in \mathcal{D}(\Omega)$ with $\phi(0) \neq 0$, it turns out that*

$$\widehat{\phi\delta}(\xi) = \phi(0), \quad \forall \xi \in \mathbb{R}^d.$$

It descends that $\widehat{\phi\delta}$ is nowhere rapidly decreasing. Therefore, we infer that $\Sigma_0(\delta) = \mathbb{R}^d \setminus \{0\}$ and

$$WF(\delta) = \{(0, \xi) : \xi \in \mathbb{R}^d \setminus \{0\}\}.$$

Pseudodifferential characterization In this paragraph, we recall the characterization of the smooth wavefront set in terms of properly supported pseudodifferential operators as per Definition 2.2.25. This characterization is mainly outlined in [GS94, Chap. 7] and [Hin21, Sect. 6.3].

Proposition 2.3.11: *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $u \in \mathcal{D}'(\Omega)$. Then $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^d \setminus \{0\})$ does not lie in $WF(u)$ if and only if there exists $A \in \Psi^0(\Omega)$, elliptic at (x_0, ξ_0) as per Definition 2.2.48, such that $Au \in C^\infty(\Omega)$. Therefore,*

$$WF(u) = \bigcap_{\substack{A \in \Psi^0(\Omega) \\ Au \in C^\infty(\Omega)}} \mathrm{Char}(A),$$

where $\mathrm{Char}(A)$ *is the characteristic set of A introduced in Equation (2.2.22).*

Microlocality of Pseudodifferential Operators An important property of the smooth wavefront set of a distribution is that it is shrinked by the action of pseudodifferential operators as per Definition 2.2.15 - see [Hin21, Prop. 6.27], [GS94, Lemma 7.2]. On account of this property, we say that pseudodifferential operators are *microlocal*. In this paragraph, we only consider properly supported pseudodifferential operators as per Definition 2.2.25.

Proposition 2.3.12: *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $m \in \mathbb{R}$. If $A \in \Psi^m(\Omega)$, then*

$$WF(Au) \subset WF'(A) \cap WF(u), \quad u \in \mathcal{D}'(\Omega),$$

where $WF'(A)$ has been introduced in Definition 2.2.44.

We conclude by stating a *microlocal elliptic regularity* result - see [Hin21, Prop. 6.28].

Proposition 2.3.13: *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $m \in \mathbb{R}$. If $A \in \Psi^m(\Omega)$, then*

$$WF(u) \subset \mathrm{Char}(A) \cup WF(Au), \quad u \in \mathcal{D}'(\Omega).$$

In particular, if A is elliptic, then

$$WF(u) = WF(Au), \quad u \in \mathcal{D}'(\Omega).$$

2.3.2 Pullbacks and Smooth Wavefront Sets

In Appendix A.9, we proved that the pullback of a distribution along a submersion is always well-defined. However, the pullback of a distribution along an *embedding* is, in general, an ill-defined operation because of singularities. The aim of this subsection is to discuss under which conditions this operation can be extended to distributions. In particular, we shall see that the smooth wavefront set plays a key rôle in providing a sufficient criterion to pull back an underlying distribution along an embedding. To this end, we first introduce a topology on the space of distributions with a given bound for the wavefront set. The

content of this subsection is mainly inspired by [Hör03, Sect. 8.2].
Let $\Omega \subset \mathbb{R}^d$ be an open set and let V be a closed cone in $\Omega \times (\mathbb{R}^d \setminus \{0\})$ as per Definition 2.2.41. We define

$$\mathcal{D}'_V(\Omega) := \{u \in \mathcal{D}'(\Omega) : WF(u) \subset V\}.$$

Lemma 2.3.14: *Let $u \in \mathcal{D}'(\Omega)$. Then $u \in \mathcal{D}'_V(\Omega)$ if and only if for any $\phi \in \mathcal{D}(\Omega)$ and for any closed cone $\Gamma \subset \mathbb{R}^d$ as per Definition 2.2.39 such that*

$$V \cap (\mathrm{supp}(\phi) \times \Gamma) = \emptyset, \tag{2.3.3}$$

it follows that

$$\sup_{\xi \in \Gamma} |\xi|^N |\widehat{\phi u}(\xi)| < \infty, \quad \forall N \in \mathbb{N}_0. \tag{2.3.4}$$

In view of Lemma 2.3.14, we can endow $\mathcal{D}'_V(\Omega)$ with a notion of convergence.

Definition 2.3.15: *Let $(u_j)_{j\in\mathbb{N}_0}$ be a sequence in $\mathcal{D}'_V(\Omega)$ and let $u \in \mathcal{D}'_V(\Omega)$. We say that $(u_j)_{j\in\mathbb{N}_0}$ converges to u in $\mathcal{D}'_\Gamma(\Omega)$ if*

(i) $u_j \overset{\mathcal{D}'}{\to} u$ *as per Definition A.2.4,*

(ii) for any $\phi \in \mathcal{D}(\Omega)$ and for any closed cone $\Gamma \subset \mathbb{R}^d$ such that Equation (2.3.3) is satisfied, it holds true that

$$\lim_{j\to\infty} \sup_{\xi\in\Gamma} |\xi|^N |\widehat{\phi u_j}(\xi) - \widehat{\phi u}(\xi)| = 0 \quad \forall N \in \mathbb{N}_0.$$

We write $u_j \overset{\mathcal{D}'_V}{\to} u$.

We recall that $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}'_V(\Omega)$ - see [Hör03, Th. 8.2.3].

Theorem 2.3.16: *Let $\Omega \subset \mathbb{R}^d$. For any $u \in \mathcal{D}'_V(\Omega)$ there exists a sequence $(u_j)_{j\in\mathbb{N}_0} \subset \mathcal{D}(\Omega)$ such that* $u_j \overset{\mathcal{D}'_\Gamma}{\to} u$.

We are now in position to state the main result of the subsection, which gives a sufficient condition to pullback a distribution along an embedding. Moreover, this statement establishes how the smooth wavefront sets transform under pullbacks - see [Hör03, Th. 8.2.4].

Definition 2.3.17: *Let $\Omega_1 \subset \mathbb{R}^{d_1}$, $\Omega_2 \subset \mathbb{R}^{d_2}$ be open sets with $d_1 < d_2$ and let $f\colon \Omega_1 \to \Omega_2$ be a smooth map. We say that f is an* ***immersion*** *if its differential, $df(x)$, is injective for every $x \in \Omega_1$.*

Definition 2.3.18: *Let $\Omega_1 \subset \mathbb{R}^{d_1}$, $\Omega_2 \subset \mathbb{R}^{d_2}$ be open sets with $d_1 < d_2$ and let $f\colon \Omega_1 \to \Omega_2$ be an immersion. We say that $f\colon \Omega_1 \to \Omega_2$ is an* ***embedding*** *if f is a diffeomorphism between Ω_1 and $f[\Omega_1]$.*

Theorem 2.3.19: *Let $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ be open sets with $d_1 < d_2$ and let $f : \Omega_1 \to \Omega_2$ be an embedding. Moreover let*

$$N_f := \{(f(x), \eta) \in \Omega_2 \times \mathbb{R}^{d_2} : {}^t df(x)\eta = 0\} \tag{2.3.5}$$

be the set of normals of f. For any $u \in \mathcal{D}'(\Omega_2)$ such that

$$N_f \cap WF(u) = \emptyset, \tag{2.3.6}$$

there exists a unique $f^*u \in \mathcal{D}'(\Omega_1)$ so that $f^*u = u \circ f$ if $u \in C^\infty(\Omega_2)$. In addition, for any closed conic subset V of $\Omega_2 \times (\mathbb{R}^{d_2} \setminus \{0\})$ with $V \cap N_f = \emptyset$, $f^*\colon \mathcal{D}'_V(\Omega_2) \to \mathcal{D}'_{f^*V}(\Omega_1)$ is a continuous map, where

$$f^*V := \{(x, {}^t df(x)\eta) : (f(x), \eta) \in V\}.$$

In particular,

$$WF(f^*u) \subset f^*WF(u),$$

for every $u \in \mathcal{D}'(\Omega_2)$ abiding to Equation (2.3.6).

Remark 2.3.20: *Theorem 2.3.19 can be improved by assuming that $f\colon \Omega_1 \to \Omega_2$ is an immersion as per Definition 2.3.17. In the present subsection, we have focused on the particular case of pullbacks along embeddings. This is in light of the fact that the product between two distributions, when no issue arise in its definition, is expressed explicity as a suitable pullback along the diagonal embedding $\boldsymbol{\delta}\colon \Omega \to \Omega \times \Omega, \boldsymbol{\delta}(x) = (x, x)$, where $\Omega \subset \mathbb{R}^d$ is an open set - see Subsection 2.3.3.*

At last, we recall that the smooth wavefront set of a distribution is invariant under the action of diffeomorphisms - [FJ99, Prop. 11.1.2].

Proposition 2.3.21: *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be open sets. If $f\colon \Omega_1 \to \Omega_2$ is a diffeomorphism, then*

$$WF(f^*u) = f^*WF(u).$$

Remark 2.3.22: *Proposition 2.3.21 is especially important since it is the building block to extend the notion of smooth wavefront set to distributions supported on any arbitrary smooth manifold. Let M be a d-dimensional smooth manifold and let $\mathcal{A} = \{(U_i, h_i)\}_i$ be a smooth atlas thereon. On account of Remark 2.3.7, if $u \in \mathcal{D}'(M)$ as per Definition A.10.1, we define $WF(u)$ as the subset of $T^*M \setminus \{0\}$ such that its restriction to U_i is given by $(h_i)^*WF((h_i^{-1})^*u)$. On account of Proposition 2.3.21, this definition is well-posed and invariant under a change of local coordinates. For further details, the interested reader may refer to [Hör03, Chap. VIII].*

2.3.3 Product of distributions

In the framework of the theory of distrubutions, one of the questions is to establish under which conditions the product of two distributions is well-defined. We recall that this operation is always well-defined when the distributions have disjoint singular supports, see Theorem A.5.3. However, even if a point lies in both singular supports, we are able to define the product of two distributions after assuming a suitable condition on the smooth wavefront sets, called *Hörmander criterion*. To this purpose, we observe that, if u, v are two scalar functions on Ω, then the product $u(x)v(x)$ can be seen as the pullback of the tensor product $u(x)v(y)$ along the diagonal. Therefore, we first recall an estimate for the smooth wavefront set of the tensor product between two distributions - see [Hör03, Th. 8.2.9].

Theorem 2.3.23: *Let $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets. If $u \in \mathcal{D}'(\Omega_1)$ and $v \in \mathcal{D}'(\Omega_2)$, then*

$$WF(u \otimes v) \subset (WF(u) \times WF(v)) \cup ((\mathrm{supp}(u) \times \{0\}) \times WF(v)) \cup (WF(u) \times (\mathrm{supp}(v) \times \{0\})). \quad (2.3.7)$$

Theorem 2.3.24: *Let $u, v \in \mathcal{D}'(\Omega)$ be such that*

$$(x, \xi) \in WF(u) \Rightarrow (x, -\xi) \notin WF(v) \quad \text{(Hörmander criterion)}. \quad (2.3.8)$$

Then the product $uv \in \mathcal{D}'(\Omega)$ can be defined as

$$uv := \boldsymbol{\delta}^*(u \otimes v),$$

where $\boldsymbol{\delta}\colon \Omega \to \Omega \times \Omega$, $\boldsymbol{\delta}(x) := (x, x)$, is the diagonal map. In addition,

$$WF(uv) \subset \{(x, \xi_1 + \xi_2) : (x, \xi_1) \in WF(u) \text{ or } \xi_1 = 0, (x, \xi_2) \in WF(v) \text{ or } \xi_2 = 0\}.$$

Proof. Since

$$^t d\boldsymbol{\delta}(x)(\xi_1, \xi_2) = \xi_1 + \xi_2 \quad \forall \xi_1, \xi_2 \in \mathbb{R}^d,$$

then $N_{\boldsymbol{\delta}} = \{(\xi, -\xi) : \xi \in \mathbb{R}^d\}$, where $N_{\boldsymbol{\delta}}$ denotes the set of normals of the diagonal map, see Equation (2.3.5). On account of Equation (2.3.8) and Theorem 2.3.23, it descends that

$$N_{\boldsymbol{\delta}} \cap WF(u \otimes v) = \emptyset.$$

Therefore, by applying Theorem 2.3.19, we conclude that there exists $\delta^*(u \otimes v)$ and

$$WF(\boldsymbol{\delta}^*(u \otimes v)) \subset \boldsymbol{\delta}^* WF(u \otimes v) = \{(x, \xi_1 + \xi_2) : (x, \xi_1) \in WF(u) \text{ or } \xi_1 = 0, (x, \xi_2) \in WF(v) \text{ or } \xi_2 = 0\}.$$

□

2.3.4 Push-forwards and Smooth Wavefront Sets

In this subsection, we shall see how the smooth wavefront set transforms under *push-forwards* along projection maps - see [FJ99, Def. 11.2.1, Prop. 11.2.1]. To start with, we recall the definition of push-forward of a distribution along a projection map, which can be understood as a partial evaluation against the constant function 1.

Definition 2.3.25: *Let $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets, let $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ and let $\pi \colon \Omega_1 \times \Omega_2 \to \Omega_1$ be the canonical projection defined by $\pi(x_1, x_2) = x_1$. Suppose that π is a proper map when restricted to $\mathrm{supp}(K)$. The* ***push-forward*** *of K along π, $\pi_*(K)$, is the element in $\mathcal{D}'(\Omega_1)$ such that, for any $\phi \in \mathcal{D}(\Omega_1)$,*

$$\langle \pi_*(K), \phi \rangle := \langle K, \phi \otimes \psi \rangle, \tag{2.3.9}$$

where $\psi \in \mathcal{D}(\Omega_2)$ is identically one on a neighbourhood of

$$\{x_2 \in \Omega_2 : \exists x_1 \in \mathrm{supp}(\phi), (x_1, x_2) \in \mathrm{supp}(K)\}.$$

Remark 2.3.26: *Equation (2.3.9) is independent from the choice of ψ. Let $\phi \in \mathcal{D}(\Omega_1)$. If $\psi, \tilde{\psi} \in \mathcal{D}(\Omega_2)$ are chosen as in Definition 2.3.25, then*

$$\psi - \tilde{\psi} \equiv 0 \text{ on } \{x_2 \in \Omega_2 : \exists x_1 \in \mathrm{supp}(\phi), (x_1, x_2) \in \mathrm{supp}(K)\}.$$

Therefore, it descends that

$$\langle K, \phi \otimes \psi \rangle - \langle K, \phi \otimes \tilde{\psi} \rangle = \langle K, \phi \otimes (\psi - \tilde{\psi}) \rangle = 0.$$

Remark 2.3.27: *If $K \in L^1(\Omega_1 \times \Omega_2) \cap \mathcal{E}'(\Omega_1 \times \Omega_2)$, then*

$$\pi_*(K)(x_1) = \int_{\mathbb{R}^n} K(x_1, x_2) dx_2 \quad (x_1 \in \Omega_1).$$

Proposition 2.3.28: *Let $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ and let $\pi \colon \Omega_1 \times \Omega_2 \to \Omega_1$ be as in Definition 2.3.25. Then*

$$WF(\pi_*(K)) = \{(x_1, \xi_1) \in \Omega_1 \times (\mathbb{R}^{d_1} \setminus \{0\}) : \exists x_2 \in \Omega_2, (x_1, x_2, \xi_1, 0) \in WF(K)\}.$$

2.3.5 Schwartz Kernels and Smooth Wavefront Sets

This subsection is devoted to discussing the smooth wavefront set of $\mathcal{K}u$, where $\mathcal{K}\colon \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ is a linear map with Schwartz kernel $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ - see [Hör03, Th. 8.2.12]. For details concerning Schwartz kernels, the interested reader may refer to Appendix A.8.

Theorem 2.3.29: *Let $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets and let $\mathcal{K}\colon \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ be a linear map with Schwartz kernel $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$. Then, for any $u \in \mathcal{D}(\Omega_2)$,*

$$WF(\mathcal{K}u) \subset \{(x_1, \xi_1) \in \Omega_1 \times (\mathbb{R}^{d_1} \setminus \{0\}) : \exists x_2 \in \mathrm{supp}(u), (x_1, x_2, \xi_1, 0) \in WF(K)\}.$$

Proof. Let $\pi\colon \Omega_1 \times \Omega_2 \to \Omega_1$ be the projection on the first factor. If $u \in \mathcal{D}(\Omega_2)$, then $\mathcal{K}u$ can be defined as the distribution lying in $\mathcal{D}'(\Omega_1)$ such that

$$(\mathcal{K}u)(\phi) = \langle K(1 \otimes u), \phi \otimes 1\rangle \quad \forall \phi \in \mathcal{D}(\Omega_1).$$

On account of Definition 2.3.25, we infer that $\mathcal{K}u = \pi_*(K(1 \otimes u))$, where π_* is the push-forward along π as per Definition 2.3.25. Since $WF(1 \otimes u) = \emptyset$, then the product $K(1 \otimes u)$ is well-defined per Theorem 2.3.24. Here, $K(1 \otimes u)$ denotes the standard product between smooth functions and distributions as per Definition A.5.2. Furthermore, Theorem 2.3.24 implies that

$$WF(K(1 \otimes u)) \subset \{(x_1, x_2, \xi_1, \xi_2) \in WF(K) : x_2 \in \mathrm{supp}(u)\}.$$

To conclude, Proposition 2.3.28 entails that

$$WF(\pi_*(K(1 \otimes u))) \subset \{(x_1, \xi_1) \in \Omega_1 \times (\mathbb{R}^{d_1} \setminus \{0\}) : x_2 \in \mathrm{supp}(u), (x_1, x_2, \xi_1, 0) \in WF(K)\}.$$

□

The following theorem generalizes the previous one to the case in which u lies in $\mathcal{E}'(\Omega_2)$ - see [Hör03, Th. 8.2.13], [FJ99, Th. 11.4.1].

Theorem 2.3.30: *Let $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets, let $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ be the Schwartz kernel of $\mathcal{K}\colon \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ and let $u \in \mathcal{E}'(\Omega_2)$. Define*

$$-WF'_{\Omega_2}(K) = \{(x_2, \xi_2) \in \Omega_2 \times (\mathbb{R}^{d_2} \setminus \{0\}) : \exists x_1 \in \Omega_1, (x_1, x_2, 0, -\xi_2) \in WF(K)\}. \tag{2.3.10}$$

If

$$WF(u) \cap (-WF'_{\Omega_2}(K)) = \emptyset, \tag{2.3.11}$$

then there exists a unique $\mathcal{K}u \in \mathcal{D}'(\Omega_1)$. In addition,

$$WF(\mathcal{K}u) \subset WF_{\Omega_1}(K) \cup WF'(K) \circ WF(u),$$

where $WF'(K) \circ WF(u) := \{(x_1, \xi_1) : \exists (x_2, \xi_2) \in WF(u), (x_1, x_2, \xi_1, -\xi_2) \in WF(K)\}$ while $WF_{\Omega_1}(K) := \{(x_1, \xi_1) \in \Omega_1 \times (\mathbb{R}^{d_1} \setminus \{0\}) : \exists x_2 \in \Omega_2, (x_1, x_2, \xi_1, 0) \in WF(K)\}$.

Proof. Let $\pi\colon \Omega_1 \times \Omega_2 \to \Omega_1$ be the projection on the first factor. In analogy to the proof of Theorem 2.3.29, $\mathcal{K}u$, if it exists, is defined as the distribution in Ω_1 such that

$$\mathcal{K}u = \pi_*(K(1 \otimes u)),$$

where π_* is the push-forward by π as per Definition 2.3.25. We start to prove that the product between K and $1 \otimes u$ is well-defined. On account of Theorem 2.3.23, it descends that

$$WF(1 \otimes u) \subset (\Omega_1 \times \{0\}) \times WF(u) = \{(x_1, x_2, 0, \xi_2) : (x_2, \xi_2) \in WF(u)\}.$$

Combining Equation (2.3.11) and Theorem 2.3.24, it follows that there exists $K(1 \otimes u) \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ and

$$\begin{aligned} WF(K(1 \otimes u)) \subset & \{(x_1, x_2, \xi_1, \xi_2 + \xi_2') : (x_2, \xi_2) \in WF(u), (x_1, x_2, \xi_2, \xi_2') \in WF(K)\} \\ & \cup WF(K) \cup WF(1 \otimes u). \end{aligned}$$

At last, on account of Proposition 2.3.28, we deduce that

$$WF(\pi_*(K(1 \otimes u))) \subset \{(x_1, \xi_1) : \exists (x_2, \xi_2) \in WF(u), (x_1, x_2, \xi_1, -\xi_2) \in WF(K)\} \cup WF_{\Omega_1}(K).$$

□

2.3.6 The propagation of singularities

As explained in Subsection 2.3.1, the notion of smooth wavefront set characterizes all those directions in Fourier space along which an underlying distribution is singular - see Definition 2.3.6. The aim of this subsection is to recall the *propagation of singularities theorem*. According to this result, the smooth wavefront set of a solution to a suitable partial differential equation is characterized by means of the principal symbol of the corresponding differential operator. In what follows, we shall focus on the problem of the propagation of singularities related to a large class of hyperbolic partial differential equations. More precisely, it asserts that the singularities of the solution propagate along the flow induced by the principal symbol, which is reinterpreted as a *Hamiltonian function*. We shall mainly refer to [Hin21, Sect. 7.2, Chap. 8].

Let $A \in \Psi^1(\mathbb{R}^d)$ be properly supported as per Defintion 2.2.25. In addition, we assume that the principal symbol of A, denoted by $\sigma_1(A)$, lies in $S^1_{\text{hom}}(\mathbb{R}^d; \mathbb{R}^d)$ as per Definition 2.2.6 and it is real-valued. Given $u_0 \in \mathcal{D}'(\mathbb{R}^d)$, we want to analyze the microlocal behavior of the solution $u \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$ of an initial value problem of the form

$$\begin{cases} D_t u = Au, & (t, x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.3.12}$$

where $D_t := -i\partial_t$. Here, we denote by $u_0(x)$ and $u(t, x)$ the integral kernels of u_0 and u respectively as per Remark A.8.2. We denote the solution map associated to Equation (2.3.12) by

$$\mathscr{S}(t, 0) \colon u_0 \mapsto u(t), \quad t \in \mathbb{R}.$$

It shows that $\mathscr{S}(t, 0)$ is a continuous operator from the Sobolev space $H^s(\mathbb{R}^d)$ as per Definition 2.1.41 into itself - see [Hin21, Th. 7.1]. In addition, $\mathscr{S}(t, 0)$ is invertible and $\mathscr{S}(t, 0)^{-1} = \mathscr{S}(0, t)$. As already anticipated, the main intuition behind the propagation of singularities theorem is to read $\sigma_1(A)$ as a Hamiltonian function. Therefore, $\sigma_1(A)$ determines a unique Hamiltonian vector field $X_{\sigma_1(A)}$, defined by

$$X_{\sigma_1(A)}|_{(x,\xi)} = \sum_{j=1}^d \partial_{\xi_j} \sigma_1(A)(x, \xi) \partial_{x_j}|_{(x,\xi)} - \sum_{j=1}^d \partial_{x_j} \sigma_1(A)(x, \xi) \partial_{\xi_j}|_{(x,\xi)}, \quad \forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{2.3.13}$$

Let $(x_0, \xi_0) \in \mathbb{R}^d$ and let $\rho \in C^\infty(J_{(x_0,\xi_0)}, \mathbb{R}^d \times \mathbb{R}^d)$ be the local solution of the Cauchy problem

$$\begin{cases} \frac{d\rho(t)}{dt} = X_{\sigma_1(A)}|_{\rho(t)}, \\ \rho(0) = (x_0, \xi_0), \end{cases} \tag{2.3.14}$$

where $J_{(x_0,\xi_0)} \subset \mathbb{R}$ is an open interval containing 0. As a result, the *Hamiltonian flow* associated with $X_{\sigma_1(A)}$ is defined as the map

$$\Phi : J_{(x_0,\xi_0)} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d \quad \text{s.t.} \quad (t, x_0, \xi_0) \mapsto \Phi_t(x_0, \xi_0) := \rho(t). \tag{2.3.15}$$

In this setting, the proof of the propagation of singularities theorem is based on *Egorov's theorem* - see [Hin21, Th. 8.3].

Theorem 2.3.31: *Let $m \in \mathbb{R}$ and let $b \in S^m(\mathbb{R}^d; \mathbb{R}^d)$. Given $B_0 := \mathrm{Op}(b) \in \Psi^m(\mathbb{R}^d)$, we set*

$$B(t) := \mathscr{S}(t, 0) \circ B_0 \circ \mathscr{S}(0, t), \quad t \in \mathbb{R}.$$

Then for any $t \in \mathbb{R}$, $B(t) \in \Psi^m(\mathbb{R}^d)$ up to an element lying in $\Psi^{-\infty}(\mathbb{R}^d)$ as per Equation (2.2.12), i.e. *there exists $R \in C^\infty(\mathbb{R}, \Psi^{-\infty}(\mathbb{R}^d))$ such that $B(t) - R(t) \in \Psi^m(\mathbb{R}^d)$. In addition, the principal symbol of $B(t)$ is given by*

$$\sigma_m(B(t))(x, \xi) = b(\Phi_t(x, \xi)), \tag{2.3.16}$$

where Φ_t is the flow from t to 0 induced by the Hamiltonian vector field $X_{\sigma_1(A)}$ as per Equation (2.3.13). In other words $\Phi_t(x, \xi) = \rho(t)$ where ρ satisfies Equation 2.3.14.

We now are in position to prove the following propagation singularities theorem - see [Hin21, Th. 7.4].

Theorem 2.3.32: *Let $A \in \Psi^m(\mathbb{R}^d)$ be such that its principal symbol $\sigma_1(A)$ lies in $S^1_{\mathrm{hom}}(\mathbb{R}^d; \mathbb{R}^d)$ as per Definition 2.2.6 and it is real-valued. In addition, let $u_0 \in \mathcal{D}'(\mathbb{R}^d)$ and let $u \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$ be the solution of the initial value problem as per Equation (2.3.12). Then*

$$WF(u(t)) = \Phi_t WF(u_0), \tag{2.3.17}$$

where Φ_t is the flow from t to 0 induced by the Hamiltonian vector field $X_{\sigma_1(A)}$ as per Equation (2.3.15) while we set

$$\Phi_t WF(u_0) := \{\Phi_t(x, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) : (x, \xi) \in WF(u_0)\}.$$

Proof. We only prove the inclusion $\subset$, since the other inclusion follows immediately by inverting the time direction. Let $(x_0, \xi_0) \notin WF(u_0)$. On account of Proposition 2.3.11, there exists a properly supported pseudodifferential operator $B \in \Psi^0(\mathbb{R}^d)$, elliptic at (x_0, ξ_0) as per Definition 2.2.48, such that $Bu_0 \in C^\infty(\mathbb{R}^d)$. Therefore, we set $B(t) := \mathscr{S}(t, 0) \circ B \circ \mathscr{S}(0, t)$ so that $B(t)u(t) = \mathscr{S}(t, 0)Bu_0 \in C^\infty(\mathbb{R}^d)$. On account of Theorem 2.3.31, we infer that $B(t)$ lies in $\Psi^0(\mathbb{R}^d)$ and it is elliptic at $\Phi_t^{-1}(x_0, \xi_0)$. Therefore, this entail that $\Phi_t^{-1}(x_0, \xi_0) \notin WF(u(t))$. □

Theorem 2.3.32 asserts that the singularities of the initial data propagate along the Hamiltonian flow induced by $\sigma_1(A)$. In Subsection 3.3.5, we shall prove Theorem 2.3.32 for a more specific class of first order hyperbolic partial differential equations within the framework of the Besov wavefront set.

2.4 Germs of Distributions

In this section, we give a succinct overview of the theory of germs of distributions as outlined in [CZ20]. The aim of this theory is to formulate and to prove *Hairer's reconstruction theorem*, *c.f* [Hai14, Th. 3.10], in a purely distributional language without any reference to *regularity structures*. More precisely, in [CZ20], the authors deal with the following problem: if for any $x \in \mathbb{R}^d$ we are given a distribution $F_x \in \mathcal{D}'(\mathbb{R}^d)$, we wonder whether there exists a global distribution $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ which is well-approximated by F_x locally around each $x \in \mathbb{R}^d$. We shall see that, under a suitable assumption on the family of distributions $(F_x)_{x\in\mathbb{R}^d}$, called *coherence*, the existence of the desired distribution $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ is guaranteed. This is the content of the reconstruction theorem formulated in [CZ20]. We underline that the framework developed in [CZ20] is established in the $B^\alpha_{\infty,\infty}$ setting. In a recent paper [BL22B], the reconstruction theorem has been extended to Besov spaces $B^\alpha_{p,q}(\mathbb{R}^d)$ with $p, q \in [1,\infty]$. A few years earlier, this generalization was discussed in [HL17] within the framework of regularity structures. In this section, we shall point out that the framework of germs of distributions is well-suited to extend the reconstruction theorem to the case of distributions supported on any arbitrary smooth manifold as discussed in Chapter 4.

In Subsection 2.4.1, we recall the basic notions at the heart of the theory of germs of distributions. More precisely, we recall the definition of germ of distributions and the notion of coherence. In Subsection 2.4.2, we recall the formulation of the reconstruction theorem within the current framework. In Subsection 2.4.3, we discuss an application of the reconstruction theorem, which provides an alternative proof of Young's product theorem (Theorem 2.1.20).

Throughout this section, we denote by $\lesssim$ an inequality holding true up to a multiplicative finite constant. Given a compact set $\mathfrak{K} \subset \mathbb{R}^d$ and $R > 0$, we define the R-enlargement of $\mathfrak{K}$ as

$$\overline{\mathfrak{K}}_R := \{y \in \mathbb{R}^d : |y - x| \leq R,\, x \in \mathfrak{K}\}. \tag{2.4.1}$$

In addition, we denote by $B(0,1)$ the unit open ball in $\mathbb{R}^d$ centered at the origin. Given a function $f\colon \mathbb{R}^d \to \mathbb{R}$, $x \in \mathbb{R}^d$ and $\lambda > 0$, we recall that $f^\lambda_x\colon \mathbb{R}^d \to \mathbb{R}$ denotes the scaled version of f, defined as

$$f^\lambda_x(y) := \lambda^{-d} f(\lambda^{-1}(y - x)), \quad y \in \mathbb{R}^d.$$

2.4.1 Germs of Distributions and Coherence

In this subsection, we introduce the basic notions at the heart of the theory of germs of distributions, which aims at formulating Hairer's reconstruction theorem [Hai14, Th. 3.10] in a purely distributional language. We shall mainly refer to [CZ20].

We start by giving the definition of *germ of distributions*.

Definition 2.4.1: *A family $F = (F_x)_{x\in\mathbb{R}^d}$ of distributions, $F_x \in \mathcal{D}'(\mathbb{R}^d)$ for any $x \in \mathbb{R}^d$, is said to be a* ***germ*** *if, for any $\psi \in \mathcal{D}(\mathbb{R}^d)$, the map $x \mapsto F_x(\psi)$ is measurable.*

Remark 2.4.2: *A germ F can be read as an element of $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$, whose integral kernel $F(x, y) \equiv F_x(y)$ is such that the map $x \mapsto \langle F_x(y), \psi(y)\rangle$ is measurable for any $\psi \in \mathcal{D}(\mathbb{R}^d)$.*

On account of the previous definition, a germ $F = (F_x)_{x\in\mathbb{R}^d}$ can be read as a family of local approximations for a global distribution $\mathcal{R}F$. As a matter of fact, we are interested in finding a distribution $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ which is well-approximated by F_x around each point $x \in \mathbb{R}^d$. More precisely, we investigate the existence of $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ such that for any $\underline{\kappa} \in \mathcal{D}(\mathbb{R}^d)$ with $\check{\underline{\kappa}}(0) \neq 0$ and for any compact set $\mathfrak{K} \subset \mathbb{R}^d$

$$\lim_{\lambda\to 0^+} |(\mathcal{R}F - F_x)(\underline{\kappa}^\lambda_x)| = 0, \quad \forall x \in \mathfrak{K}. \tag{2.4.2}$$

In particular, Equation (2.4.2) entails the *uniqueness* of $\mathcal{R}F$ - see [CZ20, Lemma 4.2].

Lemma 2.4.3: *Let $F = (F_x)_{x\in\mathbb{R}^d}$ be a germ as per Definition 2.4.1, let $\underline{\kappa} \in \mathcal{D}(\mathbb{R}^d)$ with $\check{\underline{\kappa}}(0) \neq 0$ and let $\mathfrak{K} \subset \mathbb{R}^d$ be a compact set. If there exist two distributions $\mathcal{R}F_1, \mathcal{R}F_2 \in \mathcal{D}'(\mathbb{R}^d)$ satisfying Equation (2.4.2) uniformly for $x \in \mathfrak{K}$, then $\mathcal{R}F_1(\varphi) = \mathcal{R}F_2(\varphi)$ for any $\varphi \in \mathcal{D}(\mathfrak{K})$.*

In Subsection 2.4.2, we shall see that the existence of a distribution $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ satisfying Equation (2.4.2) is guaranteed as soon as one considers a *coherent* germ $F = (F_x)_{x\in\mathbb{R}^d}$, which we define in the following.

Definition 2.4.4: *Let $\gamma \in \mathbb{R}$ and let $F = (F_x)_{x\in\mathbb{R}^d}$ be a germ as per Definition 2.4.1. F is called γ-**coherent** if there exists $\underline{\kappa} \in \mathcal{D}(\mathbb{R}^d)$ with $\breve{\underline{\kappa}}(0) \neq 0$ such that for any compact set $\mathfrak{K} \subset \mathbb{R}^d$ there exists $\zeta_{\mathfrak{K}} \leq \min\{0, \gamma\}$ such that*

$$|(F_y - F_x)(\underline{\kappa}_x^\lambda)| \lesssim \lambda^{\zeta_{\mathfrak{K}}}(|x-y|+\lambda)^{\gamma-\zeta_{\mathfrak{K}}}, \tag{2.4.3}$$

*uniformly for $x, y \in \mathfrak{K}$ and for $\lambda \in (0,1]$. We say that F is $(\boldsymbol{\zeta}, \gamma)$-**coherent** where $\boldsymbol{\zeta} = (\zeta_{\mathfrak{K}})_{\mathfrak{K}}$ is the family of exponents in Equation (2.4.3). In particular, if $\zeta_{\mathfrak{K}} = \zeta$ for any compact set $\mathfrak{K}$, F is said to be (ζ, γ)-**coherent**.*

Remark 2.4.5: *In Definition 2.4.4, we can replace the constraint $\lambda \in (0,1]$ by $\lambda \in (0, \epsilon]$, for any fixed $\epsilon > 0$. As a matter of fact, if $\lambda \in (0, \epsilon]$, the bound in Equation (2.4.3) still holds true with a different multiplicative constant.*

Remark 2.4.6: *We point out that the coherence condition as per Equation (2.4.3) depends on the test function $\underline{\kappa} \in \mathcal{D}(\mathbb{R}^d)$ with $\breve{\underline{\kappa}}(0) \neq 0$. However, we shall recall in Proposition 2.4.11 that $\underline{\kappa}$ in Equation (2.4.3) can be replaced by any test function $\phi \in \mathcal{D}(B(0,1))$, provided that we adjust suitably the exponents $\zeta_{\mathfrak{K}}$. This entails that the set of γ-coherent germs is a vector space.*

Remark 2.4.7: *The bound in Equation (2.4.3) can be read as a generalized Hölder condition. Moreover, using the language of the theory of regularity structures, the notion of coherent germ is inspired by that of modelled distribution - see [Hai14, Def. 3.1].*

In light of Definition 2.4.4, we can introduce a family of semi-norms which establishes the coherence of a germ. Let $\gamma \in \mathbb{R}$ and let $F = (F_x)_{x\in\mathbb{R}^d}$ be a germ. Then F is $(\boldsymbol{\zeta}, \gamma)$-coherent as per Definition 2.4.4 if and only if there exists $\underline{\kappa} \in \mathcal{D}(\mathbb{R}^d)$ with $\breve{\underline{\kappa}}(0) \neq 0$ such that for every compact set $\mathfrak{K} \subset \mathbb{R}^d$

$$\|F\|^{\mathrm{coh}}_{\mathfrak{K},\underline{\kappa},\gamma,\zeta_{\mathfrak{K}}} := \sup_{\substack{x,y\in\mathfrak{K}\\ \lambda\in(0,1]}} \frac{|(F_y - F_x)(\underline{\kappa}_x^\lambda)|}{\lambda^{\zeta_{\mathfrak{K}}}(|x-y|+\lambda)^{\gamma-\zeta_{\mathfrak{K}}}} < \infty. \tag{2.4.4}$$

Example 2.4.8: *Let $F = (F_x)_{x\in\mathbb{R}^d}$ be a $(\boldsymbol{\zeta}, \gamma)$-coherent germ as per Definition 2.4.4 and let $u \in \mathcal{D}'(\mathbb{R}^d)$. Then the germ $G := (u - F_x)_{x\in\mathbb{R}^d}$ is still $(\boldsymbol{\zeta}, \gamma)$-coherent.*

Homogeneity In this paragraph, we recall that a coherent germ satisfies a *homogeneity* condition - see [CZ20, Lemma 4.12].

Lemma 2.4.9: *Let $F = (F_x)_{x\in\mathbb{R}^d}$ be a γ-coherent germ with $\gamma \in \mathbb{R}$ as per Definition 2.4.4. Then, for any compact set $\mathfrak{K} \subset \mathbb{R}^d$, there exists $\beta_{\mathfrak{K}} < \gamma$ such that*

$$|F_x(\underline{\kappa}_x^\lambda)| \lesssim \lambda^{\beta_{\mathfrak{K}}} \tag{2.4.5}$$

*uniformly for $x \in \mathfrak{K}$ and $\lambda \in (0,1]$, where $\underline{\kappa}$ is chosen as in Definition 2.4.4. We say that F is **locally homogeneous** with exponents $\boldsymbol{\beta} = (\beta_{\mathfrak{K}})_{\mathfrak{K}}$. If $\beta_{\mathfrak{K}} = \beta$ for any compact set $\mathfrak{K}$, F is said to be **globally homogeneous** of degree β.*

Remark 2.4.10: *It is worth pointing out that the case $\beta_{\mathfrak{K}} > 0$ is rather trivial. As a matter of fact, if $\beta_{\mathfrak{K}} > 0$ for a compact set $\mathfrak{K} \subset \mathbb{R}^d$, then $\mathcal{R}F = 0$ satisfies Equation (2.4.2) on $\mathfrak{K}$. In addition, on account of Lemma 2.4.3, $\mathcal{R}F = 0$ is the only solution to Equation (2.4.2).*

Analogously to the notion of coherence, we introduce a family of semi-norms which estimate the homogeneity of a coherent germ $F = (F_x)_{x\in\mathbb{R}^d}$. Let $\beta \in \mathbb{R}$. For any compact set $\mathfrak{K} \subset \mathbb{R}^d$, we define

$$\|F\|^{\mathrm{hom}}_{\mathfrak{K},\underline{\kappa},\beta} := \sup_{\substack{x\in\mathfrak{K},\\ \lambda\in(0,1]}} \frac{|F_x(\underline{\kappa}_x^\lambda)|}{\lambda^\beta}, \tag{2.4.6}$$

where $\underline{\kappa} \in \mathcal{D}(\mathbb{R}^d)$ is chosen as in Definition 2.4.4.

Enhanced coherence In this paragraph, we recall that the coherence condition as per Equation (2.4.3) can be *enhanced*. This allows us to introduce the notion of *enhanced coherence*. The idea at the heart of enhanced coherence is to replace $\underline{\kappa}$ in Equation (2.4.3) by an arbitrary test function, provided that the exponents $\zeta_{\mathfrak{K}}$ are suitably adjusted - see [CZ20, Prop. 3.1].

Proposition 2.4.11 (Enhanced coherence)**:** *Let $\gamma \in \mathbb{R}$ and let $F = (F_x)_{x\in\mathbb{R}^d}$ be a $(\boldsymbol{\zeta}, \gamma)$-coherent germ as per Definition 2.4.4. Then for any compact set $\mathfrak{K} \subset \mathbb{R}^d$ and for any integer $r > -\zeta_{\overline{\mathfrak{K}}_2}$,*

$$|(F_y - F_x)(\phi_x^\lambda)| \lesssim \|\phi\|_{C^r(\mathbb{R}^d)} \lambda^{\zeta_{\overline{\mathfrak{K}}_2}} (\lambda + |x-y|)^{\gamma - \zeta_{\overline{\mathfrak{K}}_2}} \tag{2.4.7}$$

uniformly for $x, y \in \mathfrak{K}$, $\lambda \in (0,1]$ and $\phi \in \mathcal{D}(B(0,1))$. In addition, the set of all γ-coherent germs is a vector space

Proposition 2.4.11 asserts that coherence implies its enhanced formulation. The converse implication holds true trivially. As a result, we give the following equivalent definition of coherence.

Definition 2.4.12: *Let $\gamma \in \mathbb{R}$. A germ $F = (F_x)_{x\in\mathbb{R}^d}$ is said to be γ-coherent if for any compact set $\mathfrak{K} \subset \mathbb{R}^d$ there exists $\zeta_{\mathfrak{K}} \leq \min\{\gamma, 0\}$ such that, for any integer $r > -\zeta_{\mathfrak{K}}$,*

$$|(F_y - F_x)(\phi_x^\lambda)| \lesssim \|\phi\|_{C^r(\mathbb{R}^d)} \lambda^{\zeta_{\mathfrak{K}}} (\lambda + |x-y|)^{\gamma - \zeta_{\mathfrak{K}}},$$

uniformly for $x, y \in \mathfrak{K}$, $\lambda \in (0,1]$ and $\phi \in \mathcal{D}(B(0,1))$.

Remark 2.4.13: *In Appendix B, we formulate coherence and enhanced coherence on open sets. This local formulation allows us to extend the same notions to distributions on smooth manifolds.*

At last, we give a few notable examples of coherent germs.

Example 2.4.14: *Let $u \in \mathcal{D}'(\mathbb{R}^d)$. We define $F_x := u$ for any $x \in \mathbb{R}^d$. Since $F_y - F_x = 0$ for any $x, y \in \mathbb{R}^d$, then $F = (F_x)_{x\in\mathbb{R}^d}$ if $(\boldsymbol{\zeta}, \gamma)$-coherent for any $\gamma \in \mathbb{R}$ and for any family of exponents $\boldsymbol{\zeta} = (\zeta_{\mathfrak{K}})_{\mathfrak{K}}$.*

A prototypical example of germ is given by the Taylor polynomial of a Hölder function - see Subsection 2.1.6. General germs can be read as *generalized* local Taylor expansions.

Example 2.4.15: *Let $f \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ with $\alpha \in (0,\infty) \setminus \mathbb{N}$. On account of Propositions 2.1.35 and 2.1.38, it descends that, for any compact set $\mathfrak{K} \subset \mathbb{R}^d$,*

$$|f(y) - P_x(y)| \lesssim |x-y|^\alpha \quad \forall x, y \in \mathfrak{K},$$

where P_x is the $\lfloor\alpha\rfloor$-th order Taylor polynomial of f centered at x as per Equation (2.1.29) while $\lfloor\alpha\rfloor$ has been defined in Equation (2.1.14). As shown in [CZ20, Example 4.11], the germ $(P_x)_{x\in\mathbb{R}^d}$ is $(0,\alpha)$-coherent.

2.4.2 Reconstruction Theorem

As mentioned in the previous subsection, given a germ $(F_x)_{x\in\mathbb{R}^d}$, we wish to find a global distribution $\mathcal{R}F$ which is approximated by F_x locally at any point $x \in \mathbb{R}^d$. The solution to this problem is known as *reconstruction theorem*, which is one of the cornerstones of the theory of regularity structures - see [Hai14, Th. 3.10]. In this Subsection, we recall the formulation of this theorem in the framework of the theory of germs of distributions, without any reference to regularity structures. As a matter of fact, under the assumption of coherence of the germ $(F_x)_{x\in\mathbb{R}^d}$ as per Definition 2.4.4, this result entails the existence of the desired distribution $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ - see [CZ20, Th. 5.1]. In addition, we point out that the reconstruction theorem formulated in [CZ20] is established in the $B^\alpha_{\infty,\infty}$ setting. In a recent paper [BL22B], it has been extended to Besov spaces $B^\alpha_{p,q}(\mathbb{R}^d)$ with $p, q \in [1,\infty]$ as per Definition 2.1.24. In

Section 4.2, we shall see that the framework introduced in [CZ20] is well-suited for the extension of the reconstruction theorem to the smooth manifold setting. In what follows, $\overline{\mathfrak{K}}_R$ denotes the R-enlargement of a compact set $\mathfrak{K}$ as per Equation (2.4.1).

Theorem 2.4.16: *Let $\gamma \in \mathbb{R}$ and let $F = (F_x)_{x\in\mathbb{R}^d}$ be a $(\boldsymbol{\zeta}, \gamma)$-coherent germ as per Definition 2.4.4. In addition, suppose that F is locally homogeneous with exponents $\boldsymbol{\beta} = (\beta_{\mathfrak{K}})_{\mathfrak{K}}$ as per Lemma 2.4.9. Then there exists $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$, called* ***reconstruction*** *of F, such that, for any compact set $\mathfrak{K} \subset \mathbb{R}^d$ and for any integer $r > \max\{-\alpha_{\overline{\mathfrak{K}}_2}, -\beta_{\overline{\mathfrak{K}}_2}\}$,*

$$|(\mathcal{R}F - F_x)(\phi_x^\lambda)| \lesssim \|\phi\|_{C^r(\mathbb{R}^d)} \|F\|^{\mathrm{coh}}_{\overline{\mathfrak{K}}_2, \underline{\kappa}, \alpha_{\overline{\mathfrak{K}}_2}, \gamma} \begin{cases} \lambda^\gamma & \text{if } \gamma \neq 0, \\ 1 + |\log \lambda| & \text{if } \gamma = 0, \end{cases} \tag{2.4.8}$$

uniformly for $\phi \in \mathcal{D}(B(0,1))$, $x \in \mathfrak{K}$, $\lambda \in (0,1]$, where $\|F\|^{\mathrm{coh}}_{K_2, \underline{\kappa}, \alpha_{\overline{\mathfrak{K}}_2}, \gamma}$ has been defined as per Equation (2.4.4). If $\lambda > 0$, then $\mathcal{R}F$ is unique. If $\lambda \leq 0$, the distribution $\mathcal{R}F$ is non-unique.

Remark 2.4.17: *As shown in [CZ20, Sect. 11], the reconstruction of a γ-coherent germ with $\gamma \leq 0$ is non-unique. As a matter of fact, its construction depends on the choice of the cover of $\mathbb{R}^d$ and of the partition of unity subordinated to such cover. This fact remains true also in the smooth manifold setting - see Theorem 4.2.4.*

Remark 2.4.18: *Let $\gamma < 0$. If $F = (F_x)_{x\in\mathbb{R}^d}$ is a γ-coherent germ as per Definition 2.4.4, then $\mathcal{R}F$ is defined up to an element lying in $B^{\gamma;\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ as per Definition 2.1.24. If $u \in B^{\gamma;\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$, we prove that $\mathcal{R}F + u$ satisfies the bound in Equation (2.4.8) for $\gamma < 0$. To this end, we recall that for any compact set $\mathfrak{K} \subset \mathbb{R}^d$ and for any integer $r > -\gamma$, it holds true that*

$$|u(\phi_x^\lambda)| \lesssim \|\phi\|_{C^r(\mathbb{R}^d)} \lambda^\gamma \tag{2.4.9}$$

uniformly for $x \in \mathfrak{K}$, $\lambda \in (0,1]$ and $\phi \in \mathcal{D}(B(0,1))$ - see Proposition 2.1.28. Therefore, given a compact set $\mathfrak{K} \subset \mathbb{R}^d$, it descends that

$$|(\mathcal{R}F + u - F_x)(\phi_x^\lambda)| \leq |(\mathcal{R}F - F_x)(\phi_x^\lambda)| + |u(\phi_x^\lambda)| \lesssim \|\phi\|_{C^r(\mathbb{R}^d)} \lambda^\gamma,$$

uniformly for $x \in \mathfrak{K}$, $\lambda \in (0,1]$ and $\phi \in \mathcal{D}(B(0,1))$, where the second inequality descends from Equations (2.4.8) and (2.4.9). As a result, we conclude that $\mathcal{R}F + u$ is a reconstruction of F. Conversely, let $\mathcal{R}F_1$ and $\mathcal{R}F_2$ be two distributions which satisfy the bound in Equation 2.4.8. Given a compact set $\mathfrak{K} \subset \mathbb{R}^d$, it descends that

$$|(\mathcal{R}F_1 - \mathcal{R}F_2)(\phi_x^\lambda)| \leq |(\mathcal{R}F_1 - F_x)(\phi_x^\lambda)| + |(\mathcal{R}F_2 - F_x)(\phi_x^\lambda)| \lesssim \lambda^\gamma$$

uniformly for $x \in \mathfrak{K}$, $\lambda \in (0,1]$ and $\phi \in \mathcal{D}(B(0,1))$. Therefore, Proposition 2.1.28 entails that $\mathcal{R}F_1 - \mathcal{R}F_2 \in B^{\gamma;\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. This proves that the reconstruction of a γ-coherent germ with $\gamma < 0$ is non-unique and it is defined up to an element lying in $B^{\gamma;\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$

In the following, we recall a regularity result concerning the reconstruction of a coherent germ - see [CZ20, Th. 12.7].

Theorem 2.4.19: *Let $F = (F_x)_{x\in\mathbb{R}^d}$ be a (ζ, γ)-coherent germ as per Definition 2.4.4. In addition, suppose that F is homogeneous of degree $\beta < \gamma$ as per Lemma 2.4.9. If $\beta > 0$, then $\mathcal{R}F = 0$. If $\beta \leq 0$, then $\mathcal{R}F \in B^{\beta,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ and, for any compact set $\mathfrak{K} \subset \mathbb{R}^d$, it holds true that*

$$\|\mathcal{R}F\|_{B^\beta_{\infty,\infty}(\mathfrak{K})} \lesssim (\|F\|^{\mathrm{coh}}_{\overline{\mathfrak{K}}_2, \underline{\kappa}, \zeta, \gamma} + \|F\|^{\mathrm{hom}}_{\overline{\mathfrak{K}}_2, \underline{\kappa}, \beta}),$$

where $\underline{\kappa} \in \mathcal{D}(\mathbb{R}^d)$ has been chosen as in Definition 2.4.4 while $\|F\|^{\mathrm{coh}}_{\overline{\mathfrak{K}}_2, \underline{\kappa}, \zeta, \gamma}$, $\|F\|^{\mathrm{hom}}_{\overline{\mathfrak{K}}_2, \underline{\kappa}, \beta}$ have been defined in Equations (2.4.4) and (2.4.6) respectively.

Example 2.4.20: *Let $u \in \mathcal{D}'(\mathbb{R}^d)$. We consider the constant germ $F_x = u$ introduced in Example 2.4.14. Being $F_x = u$ for any $x \in \mathbb{R}^d$, it holds true that $\mathcal{R}F = u$.*

Example 2.4.21: *Let $f \in B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$ with $\alpha \in (0,\infty) \setminus \mathbb{N}$ and let $P = (P_x)_{x\in\mathbb{R}^d}$ be the germ given by the $\lfloor\alpha\rfloor$-th order Taylor polynomial of f as per Example 2.4.15, where $\lfloor\alpha\rfloor$ is as per Equation (2.1.14). As already mentioned in Example 2.4.15, the germ $P = (P_x)_{x\in\mathbb{R}^d}$ is $(0,\alpha)$-coherent. Therefore, on account of Theorem 2.4.16, there exists a unique reconstruction $\mathcal{R}P$. In particular, we show that $\mathcal{R}P = f$. To this end, we recall that for any compact set $\mathfrak{K} \subset \mathbb{R}^d$*

$$|f(y) - P_x(y)| \lesssim |x-y|^{\alpha} \quad \forall x, y \in \mathfrak{K}, \tag{2.4.10}$$

see Proposition 2.1.35. Given a compact set $\mathfrak{K} \subset \mathbb{R}^d$, it descends that

$$\begin{aligned}\left|\int_{\mathbb{R}^d}(f(y)-P_x(y))\chi^{\lambda}_x(y)dy\right| &\leq \int_{\mathbb{R}^d}|f(y)-P_x(y)||\phi^{\lambda}_x(y)|dy\\ &\lesssim \int_{\mathbb{R}^d}|x-y|^{\alpha}|\phi^{\lambda}_x(y)|dy \lesssim \lambda^{\alpha}\int_{\mathbb{R}^d}|\phi^{\lambda}_x(y)|dy \lesssim \|\phi\|_{L^{\infty}(\mathbb{R}^d)}\lambda^{\alpha},\end{aligned}$$

uniformly for $x \in \mathfrak{K}$, $\lambda \in (0,1]$ and $\phi \in \mathcal{D}(B(0,1))$, where the second bound descends from Equation (2.4.10). As a result, we infer that f is the reconstruction of P.

To conclude, we recall a special case of Theorem 2.4.16. The following result shall play a leading rôle in the proof of the reconstruction theorem for germs of distributions on smooth manifolds in Section 4.2 - see [CZ20, Th. 4.4].

Theorem 2.4.22: *Let $\gamma \in \mathbb{R}$ and let $F = (F_x)_{x\in\mathbb{R}^d}$ be a $(\boldsymbol{\zeta},\gamma)$-coherent germ as per Definition 2.4.4. Then there exists $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ such that, for any compact set $\mathfrak{K} \subset \mathbb{R}^d$ and for any $\psi \in \mathcal{D}(\mathbb{R}^d)$,*

$$|(\mathcal{R}F - F_x)(\psi^{\lambda}_x)| \lesssim \begin{cases}\lambda^{\gamma} & \text{if } \gamma \neq 0,\\ 1 + |\log\lambda| & \text{if } \gamma = 0,\end{cases} \tag{2.4.11}$$

uniformly for $x \in \mathfrak{K}$ and for $\lambda \in (0,1]$. If $\lambda > 0$, $\mathcal{R}F$ is unique and it is called the reconstruction of F. If $\lambda \leq 0$, the distribution $\mathcal{R}F$ is non-unique.

Remark 2.4.23: *Since Theorems 2.4.22 and 2.4.16 are local statements, they still hold true for germs of distributions on open sets - see Appendix B.*

2.4.3 Young's Product Theorem

In Subsection 2.4.2, we recalled the formulation of the reconstruction theorem in the context of germs of distributions, without any reference to the theory of regularity structures. Given a coherence germ $(F_x)_{x\in\mathbb{R}^d}$ as per Definition 2.4.4, this result asserts the existence of a global distribution which is approximated by F_x locally at any point $x \in \mathbb{R}^d$.

As shown in [CZ20, Sect. 14], the reconstruction theorem allows to prove the existence of the product between two Besov distributions $u \in B^{\alpha_1,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ and $v \in B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ when $\alpha_1 + \alpha_2 > 0$. This result is known as Young's product theorem, see Theorem 2.1.20, which was proven by means either of Bony's paraproducts, see Subsection 2.1.4, or of wavelet analysis - see [Bo81], [BCD11, Th. 2.52] and [Hai14, Prop. 4.14]. Furthermore, if $\alpha_1 + \alpha_2 \leq 0$, the reconstruction theorem entails that there exists still a non-unique and non-canonical product.

Since $B^{\alpha_1}_{\infty,\infty}(\mathbb{R}^d) \hookrightarrow B^{\alpha_1-\varepsilon}_{\infty,\infty}(\mathbb{R}^d)$ for any $\varepsilon > 0$, see Theorem 2.1.13, we can assume without loss of generality that $\alpha_1 \notin \mathbb{N}$. Since $B^{\alpha_1}_{\infty,\infty}(\mathbb{R}^d)$ concides with the Hölder space $C^{\lfloor\alpha_1\rfloor,\alpha_1-\lfloor\alpha\rfloor}(\mathbb{R}^d)$ (see Theorem 2.1.37), this assumption is convenient for what follows. Otherwise, if $\alpha_1 \in \mathbb{N}$, it holds true that $C^{\alpha_1-1,1}(\mathbb{R}^d) \subset B^{\alpha_1}_{\infty,\infty}(\mathbb{R}^d)$ on account of Theorem 2.1.38.

Let $\alpha_1 \in (0,\infty)\setminus\mathbb{N}$ and let $\alpha_2 < 0$. In addition, let $u \in B^{\alpha_1,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ and $v \in B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. From the theory of distributions, the product between u and v, denoted by uv, is in general ill-defined. Nonetheless, on account of Theorem 2.1.37, we can approximate the product uv locally at a point $x \in \mathbb{R}^d$, replacing u by its $\lfloor \alpha_1 \rfloor$-order Taylor polynomial centered at x, that is,

$$P_x(y) = \sum_{|\ell| \le \lfloor \alpha_1 \rfloor} \partial^\ell u(x) \frac{(y-x)^\ell}{\ell!} \quad \forall y \in \mathbb{R}^d,$$

where $\lfloor \alpha \rfloor$ has been introduced in Equation (2.1.14). Therefore, we define the germ $F = (F_x)_{x\in\mathbb{R}^d}$, where, for any $x \in \mathbb{R}^d$,

$$F_x(\varphi) := (vP_x)(\varphi) = v(P_x\varphi), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d).$$

The germ F can be interpreted as a family of local approximations of the product uv. The following proposition asserts that F is a coherent germ - see [CZ20, Prop. 14.4].

Proposition 2.4.24: *Let $u \in B^{\alpha_1,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ and let $v \in B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ with $\alpha_1 \in (0,\infty)\setminus\mathbb{N}$ and $\alpha_2 < 0$. Then the germ $F = (F_x)_{x\in\mathbb{R}^d}$ is $(\alpha_2, \alpha_1+\alpha_2)$-coherent as per Definition 2.4.4. In addition, F is homogeneous of degree α_2 as per Lemma 2.4.9.*

On account of the previous proposition, the germ F fulfills the hypotheses of the reconstruction theorem. Therefore, the following result is a consequence of Theorem 2.4.16 by setting $\mathcal{M}(u,v) := \mathcal{R}F$ - see [CZ20, Th. 14.1].

Theorem 2.4.25: *Let $\alpha_1 \in (0,\infty)\setminus\mathbb{N}$ and let $\alpha_2 < 0$. If $\alpha_1+\alpha_2 > 0$, then there exists a bilinear continuous map $\mathcal{M}\colon B^{\alpha_1,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d) \times B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d) \to B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ such that it extends the usual product $\mathcal{M}(u,v) = uv$ when $u \in C^\infty(\mathbb{R}^d)$. In addition, for any compact set $\mathfrak{K} \subset \mathbb{R}^d$ and for any integer $r > -\alpha_2$, it holds true that*

$$|(\mathcal{M}(u,v) - vP_x)(\phi^\lambda_x)| \lesssim \|\phi\|_{C^r(\mathbb{R}^d)} \lambda^{\alpha_1+\alpha_2},$$

uniformly for $x \in \mathfrak{K}$, $\lambda \in (0,1]$ and $\phi \in \mathcal{D}(B(0,1))$.

If $\alpha_1+\alpha_2 \le 0$, there exists a bilinear continuous map $\mathcal{M}\colon B^{\alpha_1,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d) \times B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d) \to B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ such that, for any compact set $\mathfrak{K} \subset \mathbb{R}^d$ and for any integer $r > -\alpha_2$,

$$|(\mathcal{M}(u,v) - vP_x)(\phi^\lambda_x)| \lesssim \|\phi\|_{C^r(\mathbb{R}^d)} \begin{cases} \lambda^{\alpha_1+\alpha_2} & \text{if } \alpha_1+\alpha_2 < 0, \\ 1 + |\log\lambda| & \text{if } \alpha_1+\alpha_2 = 0 \end{cases} \tag{2.4.12}$$

uniformly for $x \in \mathfrak{K}$, $\lambda \in (0,1]$ and $\phi \in \mathcal{D}(B(0,1))$. In this case, the map $\mathcal{M}$ is neither unique nor canonical.

In addition, for any compact set $\mathfrak{K} \subset \mathbb{R}^d$, it holds true that

$$\|\mathcal{M}(u,v)\|_{B^{\alpha_2}_{\infty,\infty}(\mathfrak{K})} \lesssim \|u\|_{B^{\alpha_1}_{\infty,\infty}(\overline{\mathfrak{K}}_4)} \|v\|_{B^{\alpha_2}_{\infty,\infty}(\overline{\mathfrak{K}}_4)},$$

where $\overline{\mathfrak{K}}_4$ denotes the 4-enlargement of $\mathfrak{K}$ as per Equation (2.4.1).

Remark 2.4.26: *Let $\alpha_1 \in (0,\infty)\setminus\mathbb{N}$ and let $\alpha_2 < 0$ such that $\alpha_1+\alpha_2 > 0$. Since $C^\infty(\mathbb{R}^d)$ is not densely embedded into $B^{\alpha_1,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$, it descends that the map $\mathcal{M}\colon C^\infty(\mathbb{R}^d) \times B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d) \to B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$, $\mathcal{M}(u,v) = uv$, cannot be uniquely extended to $\mathcal{M}\colon B^{\alpha_1,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d) \times B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d) \to B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. For this reason, Theorem 2.4.25 does not entail the uniqueness of $\mathcal{M}\colon B^{\alpha_1,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d) \times B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d) \to B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. For further details, the interested reader may refer to [CZ20, Remark 4.12].*

Chapter 3

Besov Wavefront Set

Handling the singularities of a distribution represents one of the main issues of several physical theories, such as quantum field theory. Because of this, it became necessary to develop a theory which characterizes in detail the singularities of an underlying distribution. As a matter of fact, in the 1950s, Lars Hörmander moved the first steps in this direction by introducing a framework, known as *microlocal analysis* - see [Hör03],[Hör94]. It is a collection of mathematical techniques, which involves the study of singularities of a distribution resorting to Fourier theory. In particular, a key tool to analyze singularities is provided by the notion of smooth wavefront set, which is a refinement of that of singular support - see Section 2.3. The smooth wavefront set aims at characterizing the singularities of an underlying distribution, linking the singular points to their respective singular directions in Fourier space. Such information is provided by a suitable analysis of the behavior of the Fourier transform of the distribution under investigation. For these reasons, microlocal analysis and, in particular, the notion of wavefront set have increasingly become important in mathematical analysis and have found broad applications in theoretical and mathematical physics. As an example, microlocal techniques play a leading rôle in the construction of a quantum field theory on curved backgrounds, as well as in a rigorous mathematical formulation of renormalization using the language of distributions - see [FR16, BFDY15, BF09, BF00].

The smooth wavefront set, however, turns out to be a rough attempt at describing singularities, since it detects only the directions of rapid decrease in Fourier space of a given distribution. As a matter of fact, in many concrete situations, one might be interested in establishing other notions of regularity. As a consequence, this led to developing more refined forms of wavefront set such as the so-called Sobolev wavefront set (see [Hör97]), which aims at estimating the singular behavior of a distribution comparing it with that of an element lying in a suitable Sobolev space $H^s(\mathbb{R}^d)$ as per Definition 2.1.41. In [JS02], it has emerged that the Sobolev wavefront has relevant applications in quantum field theory. At the same time, in recent works (see [Vas08, Vas12]), the Sobolev wavefront set has been used to estimate the singular behavior of the solutions to wave equations on a large class of Lorentzian manifolds with boundary.

One field of mathematical analysis in which the singular behavior of distributions plays a key rôle is that of nonlinear stochastic partial differential equations (SPDEs). As a matter of fact, their analysis presents several mathematical challenges because of the singular behavior of the random source and of non-linearities. In the last few years, significant steps forward in the analysis of SPDEs have been made by the theory of regularity structures [Hai14] as well as by that of paracontrolled distributions [GIP15]. Although these frameworks apply suitable renormalization techniques to give meaning to ill-defined products between distributions, microlocal analysis never comes into play. As a matter of fact, since the solution to a SPDE is tipically an element lying in a suitable Besov space $B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$ with $\alpha \in \mathbb{R}$ [BL22A, GIP15, Hai15], one relies on Bony's paradifferential calculus [Bo81] (see Subsection 2.1.4), which appears to better capture the singular behavior of Besov distributions. For this reason, at first sight, the notion of smooth wavefront set seems to be far from the ideal tool to characterize the singular directions of a distribution lying in a Besov space $B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$. Nevertheless, in a few recent works, see [DDRZ21, BDR21], it has been developed a novel framework for the study of solutions to a large class of non-linear SPDEs resorting to microlocal techniques. In particular, microlocal analysis has been used to construct solutions by means of a recursive scheme as well as to discuss the renormalization and its associated freedom. However, this novel approach is not able to establish a convergence of the perturbative series with respect to the norm of a Banach space, such as a Besov one. This can be partly ascribed to the fact that the smooth wavefront set fails to characterize the Besov-type behavior of the underlying distributions. In the context of the theory of stochastic partial differential equations, Besov spaces, which are endowed with a Banach structure, play an important rôle in formulating a fixed point argument to prove the existence of solutions - see [Hai15, GIP15].

Therefore, having in mind these facts and inspired by the notion of Sobolev wavefront set, it seems natural to aim at formulating a notion of Besov spaces from a microlocal viewpoint. In a recent joint work with Claudio Dappiaggi and Paolo Rinaldi (see [DRS22]), we introduced a refinement of the smooth wavefront set, named *Besov wavefront set*, which characterizes the microlocal behavior of an underlying distribution comparing it with that of an element lying in a suitable Besov space $B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$. As a result, for instance, it is able to provide a refined estimate of singular directions of distributions such as $|x|^{\alpha}$ and the Dirac delta δ. We focused on the class of Besov spaces with $p = q = \infty$ since it is currently the most commonly used in the concrete applications. Following the same rationale of the smooth wavefront set, we shall first give the definition of Besov wavefront set of a distribution in terms of the behavior of its Fourier transform. Although this definition correctly characterizes the concept of Besov wavefront set, it is rather cumbersome to use in applications. For this reason, we shall prove an equivalent characterization in terms of a suitable class of pseudodifferential operators (ΨDOs) of order zero as per Definition 2.2.15. By employing this alternative formulation, we shall be able to prove several structural properties of the Besov wavefront set.

Contents. In Section 3.1, we give the definition of Besov wavefront set of a distribution resorting to Fourier space techniques - see Definition 3.1.1. This definition is based on Proposition 2.1.33 as well as on Definition 2.1.24. In addition, we shall prove a few basic properties of the Besov wavefront set which are an immediate consequence of its definition.

Since Definition 3.1.1 appears to be somewhat difficult to use from an operational viewpoint, we shall prove in Section 3.2 two alternative, albeit equivalent, characterizations of the Besov wavefront set, which shall be rather helpful in proving several structural properties. The first one is based on properly supported pseudodifferential operators as per Definition 2.2.25 (see Theorem 3.2.1), while the second one establishes that the Besov wavefront set can be characterized by means its smooth counterpart.

In Section 3.3, we prove a large set of structural properties of the Besov wavefront set, resorting to the characterizations discussed in Section 3.2. In Subsection 3.3.1, we prove the microlocal properties of ΨDOs and an elliptic regularity result in the framework of the Besov wavefront set, which are an adaptation of Proposition 2.3.12. In Subsection 3.3.2, we establish a sufficient criterion in terms of Besov wavefront set for the well-posedness of the pullback of an underlying distribution along an embedding - see Theorem 3.3.5. This result generalizes the one formulated by Hörmander within the framework of the smooth wavefront set, see Subsection 2.3.2. More precisely, being the Besov wavefront set a refinement of its smooth counterpart, it entails a weaker criterion than the one established by the smooth wavefront set. As a byproduct, we shall also prove that the Besov wavefront set is invariant under the action of diffeomorphisms. This result is noteworthy since it allows to extend the notion of Besov wavefront set to distributions supported on an arbitrary smooth manifold. In Subsection 2.3.3, similarly to the smooth setting, we address the issue of the multiplication between distributions in the context of the Besov wavefront set. In particular, we formulate a counterpart of the Hörmander criterion for the existence of the product of two distributions - see Theorem 3.3.10. If the product exists, we also establish an estimate of the associated Besov wavefront set. This result can be read as a microlocal version of the renowned Young's product theorem (Theorem 2.1.20). In Subsection 3.3.4, we prove an estimate for the Besov wavefront of $\mathcal{K}u$, where $\mathcal{K}\colon \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ is a linear map while $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ are two open sets. In addition, we establish a sufficient condition to extend the map $\mathcal{K}$ to $\mathcal{E}'(\Omega_2)$, which adapts the one formulated by Hörmander in the smooth setting as outlined in Subsection 3.3.4. This result entails a microlocal formulation of Schauder estimates [Sim97] - see Theorem 3.3.15 and Corollary 3.3.16. Lastly, in Subsection 3.3.5, we prove a propagation of singularities theorem for a certain class of hyperbolic partial differential equations - see Theorem 3.3.19. This result characterizes the singularities of a solution to a suitable hyperbolic partial differential equation in terms of the principal symbol of the corresponding differential operator.

In Section 3.4, we present an application of the results of the previous sections in the context of coherent germs of distributions as per Definition 2.4.4. More precisely, given a coherent germ defined as the tensor product of two Besov distributions, we prove that its reconstruction coincides with the pointwise product of the two distributions at hand.

Notations. Throughout of this chapter, we shall denote with $\lesssim$ an inequality holding true up to a multiplicative finite constant. In general, given a function $f\colon \mathbb{R}^d \to \mathbb{R}$ and a point $x \in \mathbb{R}^d$, we recall that $f_x^\lambda\colon \mathbb{R}^d \to \mathbb{R}^d$ denotes the rescaled version of f, defined as

$$f_x^\lambda(y) := \lambda^{-d} f(\lambda^{-1}(y-x)), \quad y \in \mathbb{R}^d,$$

for $\lambda \in (0,1]$. We denote by $B(0,1)$ the unit open ball in $\mathbb{R}^d$ centered at the origin. In addition, given $u \in \mathcal{S}'(\mathbb{R}^d)$, we denote by $\widehat{u}$ its Fourier transform. At the same time, $\check{u}$ denotes the inverse Fourier transform of u.

3.1 Basic Definitions and Properties

The smooth wavefront set, defined in Subsection 2.3.1, turns out to be a coarse first attempt at describing singularities, since its complement only captures those directions along which an underlying distribution, upon localization, is smooth. In many concrete situations, we might be interested in employing a more refined notion of regularity. For instance, since the random source of a stochastic partial differential equation is tipically a Besov distribution, it might be informative to estimate the singular behavior of a solution of such an equation by comparing it with that of an element lying in a suitable Besov space $B^\alpha_{\infty,\infty}(\mathbb{R}^d)$ as per Definition 2.1.24 - see [Hai14, Hai15, GIP15, BL22A]. With these scenarios in mind, we develop a more refined form of wavefront set, named *Besov wavefront set*, whose complement consists of all those directions in Fourier space along which an underlying distribution lies in a suitable $B^\alpha_{\infty,\infty}(\mathbb{R}^d)$. We focus on the Besov spaces $B^\alpha_{\infty,\infty}(\mathbb{R}^d)$, since they are the most commonly used in concrete applications. The following discussion is mainly based on [DRS22].

In order to introduce the Besov wavefront set, we proceed in two different, albeit equivalent ways. The first one we discuss in this Section is based on Fourier methods. The second one, outlined in Section 3.2, characterizes the Besov wavefront set by means of properly supported pseudodifferential operators as per Definition 2.2.25 - see Theorem 3.2.1. This alternative formulation is rather useful from an operational viewpoint since it proves instrumental in establishing several structural properties of the Besov wavefront set - see Section 3.3. In what follows, we rely on Proposition 2.1.33 as well as Definition 2.1.24.

Definition 3.1.1: *Let $\alpha \in \mathbb{R}$ and let $u \in \mathcal{D}'(\mathbb{R}^d)$. A point $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ does not lie in the $B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$-**wavefront set** of u, $(x_0, \xi_0) \notin WF^{\alpha}(u)$, if there exist $\phi \in \mathcal{D}(\mathbb{R}^d)$ with $\phi(x_0) \neq 0$ and an open conic neighborhood Γ of ξ_0 as per Definition 2.2.39 such that for any $\kappa \in \mathscr{B}_{\lfloor \alpha \rfloor}$ as per Definition 2.1.22, for any $\underline{\kappa} \in \mathcal{D}(B(0,1))$ with $\check{\underline{\kappa}}(0) \neq 0$ and for any compact set $\mathfrak{K} \subset \mathbb{R}^d$,*

$$\left| \int_{\Gamma} \widehat{\phi u}(\xi) \check{\underline{\kappa}}(\xi) e^{ix\cdot\xi} d\xi \right| \lesssim 1, \tag{3.1.1}$$

$$\left| \int_{\Gamma} \widehat{\phi u}(\xi) \check{\kappa}(\lambda\xi) e^{ix\cdot\xi} d\xi \right| \lesssim \lambda^{\alpha}, \tag{3.1.2}$$

uniformly for $\lambda \in (0,1]$ and for $x \in \mathfrak{K}$.

Remark 3.1.2: *Analogously to the smooth wavefront set, see Remark 2.3.7, from a geometrical viewpoint the Besov counterpart should be read as a subset of $T^*\mathbb{R}^d \setminus \{0\}$, which denotes the cotangent bundle of $\mathbb{R}^d$ without the zero section. In addition, on account of Definition 3.1.1, it descends that the Besov wavefront set is a closed conic set in $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ as per Definition 2.2.41.*

Remark 3.1.3: *On account of the localization via ϕ in Equations (3.1.1) and (3.1.2), without loss of generality, we can consider $u \in \mathcal{E}'(\mathbb{R}^d)$ in Definition 3.1.1.*

Remark 3.1.4: *On account of Propositions 2.1.28 and 2.1.33, if $\alpha < 0$ in Definition 3.1.1, it suffices to check that for any $\underline{\kappa} \in \mathcal{D}(B(0,1))$ with $\check{\underline{\kappa}}(0) \neq 0$*

$$\left| \int_{\Gamma} \widehat{\phi u}(\xi) \check{\underline{\kappa}}(\lambda\xi) e^{ix\cdot\xi} d\xi \right| \lesssim \lambda^{\alpha},$$

uniformly for $\lambda \in (0,1]$ and for x lying in compact sets.

In the following, we prove a few basic properties of the Besov wavefront set which follow directly from Definition 3.1.1 - see [DRS22, Prop. 25, Prop. 26, Cor. 27].

Proposition 3.1.5: *Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and let $\alpha \in \mathbb{R}$. Then $u \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ if and only if $WF^{\alpha}(u) = \emptyset$.*

Proof. Let $u \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ and let $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$. On account of Proposition 2.1.33, given $\kappa \in \mathscr{B}_{\lfloor \alpha \rfloor}$ as per Definition 2.1.22 and $\underline{\kappa} \in \mathcal{D}(B(0,1))$ with $\check{\underline{\kappa}}(0) \neq 0$, it holds true that for any $\phi \in \mathcal{D}(\mathbb{R}^d)$

$$\left| \int_{\mathbb{R}^d} \widehat{\phi u}(\xi) e^{ix\cdot\xi} \check{\kappa}(\lambda\xi) d\xi \right| \lesssim \lambda^{\alpha}, \qquad \left| \int_{\mathbb{R}^d} \widehat{\phi u}(\xi) e^{ix\cdot\xi} \check{\underline{\kappa}}(\xi) d\xi \right| \lesssim 1, \quad \forall x \in \mathbb{R}^d, \forall \lambda \in (0,1].$$

As a result, Equations (3.1.1) and (3.1.2) are satisfied by choosing $\Gamma = \mathbb{R}^d$ and $\phi \in \mathcal{D}(\mathbb{R}^d)$ with $\phi(x_0) \neq 0$. On account of Definition 3.1.1, it descends that $WF^{\alpha}(u) = \emptyset$.
Conversely, let $WF^{\alpha}(u) = \emptyset$. On account of Definition 3.1.1, Equations (3.1.1) and (3.1.2) hold true for any $\phi \in \mathcal{D}(\mathbb{R}^d)$ and for $\Gamma = \mathbb{R}^d$. Therefore, on account of Proposition 2.1.33, ϕu lies in $B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$ for any $\phi \in \mathcal{D}(\mathbb{R}^d)$, that is $u \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. ☐

Remark 3.1.6: *In view of the continuous embedding $C^{\infty}(\mathbb{R}^d) \hookrightarrow B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ for all $\alpha \in \mathbb{R}$ and on account of Proposition 3.1.5, it descends that, for any $f \in C^{\infty}(\mathbb{R}^d)$,*

$$WF^{\alpha}(f) = \emptyset, \quad \forall \alpha \in \mathbb{R}. \tag{3.1.3}$$

In particular, this result implies that, given $u \in \mathcal{D}'(\mathbb{R}^d)$, if $x_0 \notin \mathrm{singsupp}(u)$, then $(x_0, \xi_0) \notin WF^\alpha(u)$ for all $\alpha \in \mathbb{R}$ and for any $\xi_0 \in \mathbb{R}^d \setminus \{0\}$. Per hypotesis, Definition A.3.2 entails that there exists an open neighborhood U_{x_0} of x_0 such that $u|_{U_{x_0}}$ lies in $C^\infty(U_{x_0})$. Therefore, on account of Equation (3.1.3), it descends that $WF^\alpha(\phi u) = \emptyset$ for every $\phi \in \mathcal{D}(U_{x_0})$ with $\phi(x_0) \neq 0$ and for all $\alpha \in \mathbb{R}$. As a result, for any $\xi_0 \in \mathbb{R}^d \setminus \{0\}$, we infer that $(x_0, \xi_0) \notin WF^\alpha(u)$ for every $\alpha \in \mathbb{R}$.

Proposition 3.1.7: *If $u, v \in \mathcal{D}'(\mathbb{R}^d)$, then*

$$WF^\alpha(u+v) \subset WF^\alpha(u) \cup WF^\alpha(v).$$

Proof. Let $(x_0, \xi_0) \in WF^\alpha(u+v)$. On account of Definition 3.1.1, it holds true that, for any test function $\phi \in \mathcal{D}(\mathbb{R}^d)$ with $\phi(x_0) \neq 0$ and for any open conic neighborhood Γ of ξ_0, there exists a compact set $\mathfrak{K} \subset \mathbb{R}^d$ such that for all $N \in \mathbb{N}_0$

$$\left| \int_\Gamma \widehat{\phi(u+v)}(\xi) \check{\kappa}(\lambda' \xi) e^{ix' \cdot \xi} d\xi \right| > N \lambda'^\alpha, \qquad \left| \int_\Gamma \widehat{\phi(u+v)}(\xi) \check{\underline{\kappa}}(\xi) e^{ix' \cdot \xi} d\xi \right| > N$$

for some $x' \in \mathfrak{K}$ and $\lambda' \in (0,1]$. Therefore, it descends that

$$\begin{aligned} \left| \int_\Gamma \widehat{\phi v}(\xi) \check{\kappa}(\lambda' \xi) e^{ix' \cdot \xi} d\xi \right| + \left| \int_\Gamma \widehat{\phi u}(\xi) \check{\kappa}(\lambda' \xi) e^{ix' \cdot \xi} d\xi \right| > N \lambda'^\alpha, \\ \left| \int_\Gamma \widehat{\phi v}(\xi) \check{\underline{\kappa}}(\xi) e^{ix' \cdot \xi} d\xi \right| + \left| \int_\Gamma \widehat{\phi u}(\xi) \check{\underline{\kappa}}(\xi) e^{ix' \cdot \xi} d\xi \right| > N, \end{aligned}$$

where we applied the triangle inequality. As a result, we infer that $(x_0, \xi_0) \in WF^\alpha(u) \cup WF^\alpha(v)$. □

Corollary 3.1.8: *Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and let $\alpha_1, \alpha_2 \in \mathbb{R}$ be such that $\alpha_1 \leq \alpha_2$. Then*

$$WF^{\alpha_1}(u) \subset WF^{\alpha_2}(u). \tag{3.1.4}$$

Proof. Let $(x_0, \xi_0) \notin WF^{\alpha_2}(u)$. On account of Definition 3.1.1, particularly Equation (3.1.2), it descends immediately that $(x_0, \xi_0) \notin WF^{\alpha_1}(u)$. □

Remark 3.1.9: *Corollary 3.1.8 should be read as a microlocal reformulation of the inclusion $B^{\alpha_2}_{\infty,\infty}(\mathbb{R}^d) \hookrightarrow B^{\alpha_1}_{\infty,\infty}(\mathbb{R}^d)$ when $\alpha_1 \leq \alpha_2$, see Proposition 2.1.13.*

In the following, we provide some examples on how to compute the Besov wavefront set of specific distributions.

Example 3.1.10: *Let $u = \delta \in \mathcal{D}'(\mathbb{R}^d)$ be the Dirac delta centered at the origin and let $\xi_0 \in \mathbb{R}^d \setminus \{0\}$. Given $\phi \in \mathcal{D}(\mathbb{R}^d)$ with $\phi(0) \neq 0$, an open conic neighborhood Γ of ξ_0 and $\kappa \in \mathscr{B}_{-d}$, it descends that*

$$\left| \int_\Gamma \widehat{\phi u}(\xi) \check{\kappa}(\lambda \xi) e^{ix \cdot \xi} d\xi \right| = |\phi(0)| \left| \int_\Gamma \check{\kappa}(\lambda \xi) e^{ix \cdot \xi} d\xi \right| \lesssim \int_\Gamma |\check{\kappa}(\lambda \xi)| d\xi \lesssim \lambda^{-d} \int_\Gamma |\check{\kappa}(\xi')| d\xi', \quad \forall \lambda \in (0,1], \forall x \in \mathbb{R}^d,$$

where in the first equality we used that $\phi\delta = \phi(0)\delta$ while in the last inequality we applied the change of variable $\xi' = \lambda \xi$. Since $\check{\kappa} \in \mathcal{S}(\mathbb{R}^d)$, it holds true that

$$\left| \int_\Gamma \widehat{\phi u}(\xi) \check{\kappa}(\lambda \xi) e^{ix \cdot \xi} d\xi \right| \lesssim \lambda^{-d}, \quad \forall x \in \mathbb{R}^d, \forall \lambda \in (0,1].$$

We focus on Equation (3.1.1). Let $\underline{\kappa} \in \mathcal{D}(B(0,1))$ with $\check{\underline{\kappa}}(0) \neq 0$. Since $\check{\underline{\kappa}} \in \mathcal{S}(\mathbb{R}^d)$, it descends that

$$\left| \int_\Gamma \widehat{\phi u}(\xi) \check{\underline{\kappa}}(\xi) e^{ix \cdot \xi} d\xi \right| \lesssim |\phi(0)| \int_\Gamma |\check{\underline{\kappa}}(\xi)| d\xi \lesssim 1, \quad \forall x \in \mathbb{R}^d.$$

Definition 3.1.1 entails that $WF^\alpha(\delta)=\emptyset$ if $\alpha\le -d$. In order to obtain a sharp estimate, we put $x=0$ in Equation (3.1.2). Therefore, since $\check{\kappa}\in\mathcal{S}(\mathbb{R}^d)$ and $\phi(0)\neq 0$, it descends that

$$\left|\int_\Gamma \widehat{\phi u}(\xi)\check{\kappa}(\lambda\xi)d\xi\right| = \lambda^{-d}|\phi(0)|\left|\int_\Gamma \check{\kappa}(\xi')d\xi'\right| = c_{\kappa,\phi(0)}\lambda^{-d}$$

where the first equality descends from the change of variable $\xi'=\lambda\xi$. On account of Definition 3.1.1, we conclude that

$$\mathrm{WF}^\alpha(\delta)=\begin{cases}\emptyset & \alpha\le -d,\\ \{(0,\xi):\xi\in\mathbb{R}^d\setminus\{0\}\} & \alpha>-d.\end{cases}$$

Example 3.1.11: *Let $u=\partial_j\delta\in\mathcal{D}'(\mathbb{R}^d)$ be a derivative of the Dirac delta centered at the origin, where $\partial_j=\frac{\partial}{\partial x_j}$. In analogy with Example 3.1.10, given $\xi_0\in\mathbb{R}^d\setminus\{0\}$, we show that $(0,\xi_0)\notin WF^\alpha(u)$ for any $\alpha\le -d-1$. Let $\phi\in\mathcal{D}(\mathbb{R}^d)$ with $\phi(0)\neq 0$, let Γ be an open conic neighborhood of ξ_0 and $\kappa\in\mathscr{B}_{-d-1}$ as per Definition 2.1.22. For the sake of conciseness, we only focus on Equation (3.1.2). Bearing in mind that $\phi\partial_j\delta=\phi(0)\partial_j\delta-(\partial_j\phi)(0)\delta$ and $\check{\kappa}\in\mathcal{S}(\mathbb{R}^d)$, it descends that*

$$\begin{aligned}\left|\int_\Gamma \widehat{\phi u}(\xi)\check{\kappa}(\lambda\xi)e^{ix\cdot\xi}d\xi\right| &= \left|\phi(0)\int_\Gamma \xi_j\check{\kappa}(\lambda\xi)e^{ix\cdot\xi}d\xi-(\partial_j\phi)(0)\int_\Gamma\check{\kappa}(\lambda\xi)e^{ix\cdot\xi}d\xi\right|\\ &\lesssim \int_\Gamma|\xi_j||\check{\kappa}(\lambda\xi)|d\xi+\int_\Gamma|\check{\kappa}(\lambda\xi)|d\xi\lesssim\lambda^{-d-1}\int_\Gamma|\xi'_j||\check{\kappa}(\xi')|d\xi'+\lambda^{-d}\int_\Gamma|\check{\kappa}(\xi')|d\xi'\lesssim\lambda^{-d-1}\end{aligned}$$

uniformly for $x\in\mathbb{R}^d$ and for $\lambda\in(0,1]$, where in the second inequality we applied the change of variable $\xi'=\lambda\xi$. Similarly, we can prove Equation (3.1.1). On account of Definition 3.1.1, it descends that $WF^\alpha(u)=\emptyset$ if $\alpha\le -d-1$. In order to obtain a sharp estimate, we put $x=0$ in Equation (3.1.2). Moreover, we can focus our attention only on the contribution due to $\phi(0)\partial_j\delta$. Since $\check{\kappa}\in\mathcal{S}(\mathbb{R}^d)$, it descends that

$$\left|\phi(0)\int_\Gamma\xi_j\check{\kappa}(\lambda\xi)d\xi\right|=\lambda^{-d-1}|\phi(0)|\left|\int_\Gamma\xi_j\check{\kappa}(\xi')d\xi'\right|=c_{\kappa,\phi(0)}\lambda^{-d-1}\quad\forall\lambda\in(0,1],$$

where $\xi':=\lambda\xi$. As a result, we can conclude that

$$\mathrm{WF}^\alpha(\partial_j\delta)=\begin{cases}\emptyset & \alpha\le -d-1,\\ \{(0,\xi):\xi\in\mathbb{R}^d\setminus\{0\}\} & \alpha>-d-1.\end{cases}$$

Example 3.1.12: *Let $u\in\mathcal{E}'(\mathbb{R}^d)$. On account of Theorem A.11.11, there exists $C>0$ such that*

$$|\widehat{u}(\xi)|\le C\langle\xi\rangle^{\mathrm{ord}(u)}\quad\forall\xi\in\mathbb{R}^d,$$

where $\langle\xi\rangle$ has been defined in Equation (2.1.2) and $\mathrm{ord}(u)$ stands for the order of u as per Definition A.2.5. Let $\phi\in\mathcal{D}(\mathbb{R}^d)$ be such that $\phi=1$ on $\mathrm{supp}(u)$ and let Γ be an open conic neighborhood of $\xi_0\in\mathbb{R}^d\setminus\{0\}$. For the sake of coinciseness, we focus only on Equation (3.1.2). Given $\kappa\in\mathscr{B}_{-d-\mathrm{ord}(u)}$ as per Definition 2.1.22, it descends that

$$\begin{aligned}\left|\int_\Gamma\widehat{\phi u}(\xi)\check{\kappa}(\lambda\xi)e^{ix\cdot\xi}d\xi\right|&\le\int_\Gamma|\widehat{u}(\xi)||\check{\kappa}(\lambda\xi)|d\xi\le C\int_\Gamma\langle\xi\rangle^{\mathrm{ord}(u)}|\check{\kappa}(\lambda\xi)|d\xi\\ &\approx\int_\Gamma|\xi|^{\mathrm{ord}(u)}|\check{\kappa}(\lambda\xi)|d\xi=\lambda^{-d-\mathrm{ord}(u)}\int_\Gamma|\xi'||\check{\kappa}(\xi')|d\xi',\end{aligned}\tag{3.1.5}$$

uniformly for $x\in\mathbb{R}^d$ and for $\lambda\in(0,1]$, where the last equality descends from the change of variable $\xi'=\lambda\xi$. As a result, we infer that $WF^\alpha(u)=\emptyset$ if $\alpha\le -d-\mathrm{ord}(u)$, that is to say $u\in B^{-d-\mathrm{ord}(u)}_{\infty,\infty}(\mathbb{R}^d)$. This is nothing but Proposition 2.1.31.

Remark 3.1.13: *Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and let $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$. $\phi \in \mathcal{D}(\mathbb{R}^d)$ with $\phi(x_0) \neq 0$, Example 3.1.12 entails that Equation (3.1.2) is satisfied for any $\alpha \leq -d - \operatorname{ord}(\phi u)$. As a result, there exists $\tilde{\alpha} \in \mathbb{R}$ such that $(x_0, \xi_0) \notin WF^{\tilde{\alpha}}(u)$.*

Example 3.1.14: *Let $u\colon \mathbb{R}^2 \to \mathbb{R}$ be such that $u(x_1, x_2) = (x_1^2 + x_2^2)^{\frac{1}{4}}$. On account of Proposition A.11.14, it descends that $\widehat{u}(\xi_1, \xi_2) = (\xi_1^2 + \xi_2^2)^{-\frac{5}{4}}$, which is to be understood as an element lying in $\mathcal{S}'(\mathbb{R}^2)$. Since $\operatorname{singsupp}(u) = \{(0,0)\}$, it suffices to analyze the directions $(0, 0, \xi_1, \xi_2)$ with $(\xi_1, \xi_2) \neq (0, 0)$. Therefore, fix $\phi \in \mathcal{D}(\mathbb{R}^2)$ with $\phi(0, 0) = 1$ and an open conic neighborhood Γ of (ξ_1, ξ_2). We start by checking Equation (3.1.2). Given $\kappa \in \mathscr{B}_0$ as per Definition 2.1.22, $\lambda \in (0, 1]$ and $(x_1, x_2) \in \mathbb{R}^2$, it descends that*

$$\left| \int_\Gamma \widehat{\phi u}(\eta_1, \eta_2) \check{\kappa}(\lambda\eta_1, \lambda\eta_2) e^{ix_1\eta_1} e^{ix_2\eta_2} d\eta_1 d\eta_2 \right| \leq \int_\Gamma |\widehat{\phi u}(\eta_1, \eta_2)| |\check{\kappa}(\lambda\eta_1, \lambda\eta_2)| d\eta_1 d\eta_2$$

$$= \int_\Gamma (\eta_1^2 + \eta_2^2)^{-\frac{5}{4}} |\check{\kappa}(\lambda\eta_1, \lambda\eta_2)| d\eta_1 d\eta_2 \overset{\substack{\lambda\eta_1 \mapsto \eta_1 \\ \lambda\eta_2 \mapsto \eta_2}}{=} \lambda^{\frac{1}{2}} \int_\Gamma (\eta_1^2 + \eta_2^2)^{-\frac{5}{4}} |\check{\kappa}(\eta_1, \eta_2)| d\eta_1 d\eta_2 \lesssim \lambda^{\frac{1}{2}}.$$

Although $\widehat{u} \notin L^1(B(0, 1))$, we neglected the localization via ϕ since $\check{\kappa}$ is supported outside the origin. Focusing on Equation (3.1.1), given $\underline{\kappa} \in \mathcal{D}(B(0, 1))$ such that $\check{\underline{\kappa}}(0) \neq 0$, it descends that

$$\left| \int_\Gamma \widehat{\phi u}(\eta_1, \eta_2) \check{\underline{\kappa}}(\eta_1, \eta_2) e^{ix_1\eta_1} e^{ix_2\eta_2} d\eta_1 d\eta_2 \right| \leq \int_\Gamma |\widehat{\phi u}(\eta_1, \eta_2)| |\check{\underline{\kappa}}(\eta_1, \eta_2)| d\eta_1 d\eta_2 \lesssim 1,$$

uniformly for $(x_1, x_2) \in \mathbb{R}^2$. In order to obtain a sharp estimate, we set $(x_1, x_2) = (0, 0)$ in Equation (3.1.2). Hence it descends that

$$\left| \int_\Gamma (\eta_1^2 + \eta_2^2)^{-\frac{5}{4}} \check{\kappa}(\lambda\eta_1, \lambda\eta_2) d\eta_1 d\eta_2 \right| \overset{\substack{\lambda\eta_1 \mapsto \eta_1 \\ \lambda\eta_2 \mapsto \eta_2}}{=} \lambda^{\frac{1}{2}} \left| \int_\Gamma (\eta_1^2 + \eta_2^2)^{-\frac{5}{4}} \check{\kappa}(\eta_1, \eta_2) d\eta_1 d\eta_2 \right|. \tag{3.1.6}$$

On account Definition 3.1.1, we conclude that

$$\mathrm{WF}^\alpha(u) = \begin{cases} \emptyset & \alpha \leq \frac{1}{2}, \\ \{(0, 0, \xi_1, \xi_2) : (\xi_1, \xi_2) \neq (0, 0)\} & \alpha > \frac{1}{2}. \end{cases}$$

Example 3.1.15: *Let $u \in \mathcal{S}'(\mathbb{R})$ be such that*

$$u = \mathrm{p.v.}\left(\frac{1}{ix}\right) + \pi\delta,$$

where $\mathrm{p.v.}\left(\frac{1}{ix}\right)$ denotes the Cauchy principal value of $\frac{1}{ix}$ and $\delta \in \mathcal{D}'(\mathbb{R})$ stands for the Dirac delta centered at the origin. We shall denote by Θ the Heaviside function. Since $\operatorname{singsupp}(u) = \{0\}$ and $\widehat{u}(\cdot) = \Theta(\cdot)$, it suffices to analyze the directions $(0, \xi)$ such that $\xi > 0$. Therefore, fix $\phi \in \mathcal{D}(\mathbb{R})$ with $\phi(0) = 1$ and the open conic neighborhood $\Gamma = (0, +\infty)$. We start by checking Equation (3.1.2). Given $\kappa \in \mathscr{B}_0$ as per Definition 2.1.22, $\lambda \in (0, 1]$ and $x \in \mathbb{R}$, it descends that

$$\left| \int_\Gamma \widehat{\phi u}(\eta) \check{\kappa}(\lambda\eta) e^{ix\eta} d\eta \right| = \left| \int_\Gamma \Theta(\eta) \check{\kappa}(\lambda\eta) e^{ix\eta} d\eta \right| \leq \int_\Gamma |\check{\kappa}(\lambda\eta)| d\eta \overset{\lambda\eta \mapsto \eta}{=} \lambda^{-1} \int_\Gamma |\check{\kappa}(\eta)| d\eta \lesssim \lambda^{-1},$$

where we used that $\Theta = 1$ on Γ and we negleted the localization via ϕ since $\check{\kappa}$ is supported outside the origin. We now focus on Equation (3.1.1). Let $\underline{\kappa} \in \mathcal{D}(B(0, 1))$ with $\check{\underline{\kappa}}(0) = 0$. Since $\check{\kappa} \in \mathcal{S}(\mathbb{R})$, it descends that

$$\left| \int_\Gamma \widehat{\phi u}(\xi) \check{\underline{\kappa}}(\xi) e^{ix \cdot \xi} d\xi \right| \lesssim |\phi(0)| \int_\Gamma |\check{\underline{\kappa}}(\xi)| d\xi \lesssim 1, \quad \forall x \in \mathbb{R}.$$

In order to obtain a sharp estimate, we set $x = 0$ in Equation (3.1.2). Therefore, it descends that

$$\left|\int_\Gamma \Theta(\eta)\check{\kappa}(\lambda\eta)d\eta\right| = \left|\int_\Gamma \check{\kappa}(\lambda\eta)d\eta\right| \overset{\lambda\eta\mapsto\eta}{=} \lambda^{-1}\left|\int_\Gamma \check{\kappa}(\lambda\eta)d\eta\right|$$

On account of Definition 3.1.1, we infer that

$$\mathrm{WF}^\alpha(u) = \begin{cases} \emptyset & \alpha \leq 1, \\ \{(0,\xi) : \xi > 0\} & \alpha > 1. \end{cases}$$

3.2 Characterizations of the Besov Wavefront Set

In the previous section, we have introduced the concept of Besov wavefront, see Definition 3.1.1, which estimates the singular behavior of a given distribution in Fourier space comparing it with that of an element lying in a suitable Besov space $B^\alpha_{\infty,\infty}(\mathbb{R}^d)$ as per Definition 2.1.24. Although Definition 3.1.1 characterizes appropriately the concept of Besov wavefont set of an underlying distribution, it is rather difficult to use it concretely. For this reason, we shall give two equivalent characterizations of the Besov wavefront set, which shall be rather helpful in the proof of several results in Section 3.3. More precisely, the first one relies on properly supported pseudodifferential operators (ΨDOs) as per Definition 2.2.25, while the second one characterizes the Besov wavefront set in terms of the smooth counterpart as per Definition 2.3.6. As we shall see in Section 3.3, both characterizations turn out to be well-suited when trying to extend a few notable operations to distributions with a fixed Besov wavefront set. In particular, the formulation of the Besov wavefront set in terms of the smooth counterpart shall allow us to apply a few results stated in Section 2.3. In addition, in Subsection 3.3.5 we shall prove a theorem of propagation of singularities for a large class of hyperbolic partial differential equations by using the characterization in terms of pseudodifferential operators. We shall mainly refer to [DRS22, Sect. 3.1].

We start by proving the charaterization of the Besov wavefront set by means of properly supported ΨDOs, which adapts to the current scenario the content of Proposition 2.3.11 - see [DRS22, Prop. 33]. In the following, we shall mainly make use of the notions introduced in Sections 2.2 and 2.3.

Theorem 3.2.1: *Let $\alpha \in \mathbb{R}$. If $u \in \mathcal{D}'(\mathbb{R}^d)$, then*

$$WF^\alpha(u) = \bigcap_{\substack{A\in\Psi^0(\mathbb{R}^d),\\ Au\in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)}} \mathrm{Char}(A), \tag{3.2.1}$$

where the intersection is taken only over properly supported pseudodifferential operators as per Definition 2.2.25 and $\mathrm{Char}(A)$ stands for the characteristic set of A introduced in Equation (2.2.22).

Proof. Let $(x_0, \xi_0) \notin WF^\alpha(u)$. On account of Definition 3.1.1, there exist $\phi \in \mathcal{D}(\mathbb{R}^d)$ with $\phi(x_0) \neq 0$ and an open conic neighborhood Γ of ξ_0 such that for any $\kappa \in \mathscr{B}_{\lfloor\alpha\rfloor}$, for any $\underline{\kappa} \in \mathcal{D}(B(0,1))$ with $\check{\underline{\kappa}}(0) \neq 0$ and for any compact set $\mathfrak{K} \subset \mathbb{R}^d$

$$\left|\int_\Gamma \widehat{\phi u}(\xi)\check{\underline{\kappa}}(\xi)e^{ix\cdot\xi}d\xi\right| \lesssim 1, \quad \left|\int_\Gamma \widehat{\phi u}(\xi)\check{\kappa}(\lambda\xi)e^{ix\cdot\xi}d\xi\right| \lesssim \lambda^\alpha, \quad \forall\lambda \in (0,1], \forall x \in \mathfrak{K}, \tag{3.2.2}$$

On account of Theorem A.11.9, the estimates in Equation (3.2.2) can be equivalently written as

$$|\langle \mathcal{F}^{-1}[\mathbb{I}_\Gamma(\xi)\widehat{\phi u}(\xi)], \kappa^\lambda_x\rangle| \lesssim \lambda^\alpha, \quad |\langle \mathcal{F}^{-1}[\mathbb{I}_\Gamma(\xi)\widehat{\phi u}(\xi)], \underline{\kappa}_x\rangle| \lesssim 1 \quad \forall x \in \mathfrak{K}, \forall\lambda \in (0,1],$$

where $\mathbb{I}_\Gamma$ denotes the characteristic function on Γ, defined by

$$\mathbb{I}_\Gamma(\xi) := \begin{cases} 1 & \text{if } \xi \in \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

On account of Definition 2.1.24, it descends that

$$\mathcal{F}^{-1}[\mathbb{I}_\Gamma(\xi)\widehat{\phi u}(\xi)] \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d). \tag{3.2.3}$$

Denoting the $(d-1)$-dimensional sphere by $\mathbb{S}^{d-1}$ and given $\epsilon > 0$, we choose $\psi \in C^\infty(\mathbb{S}^{d-1})$ such that

$$\mathrm{supp}(\psi) \subset \left\{ \xi \in \mathbb{R}^d \setminus \{0\} : \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \epsilon \right\} \subset \Gamma,$$

and $\psi(\xi_0/|\xi_0|) \neq 0$. In addition, let $\chi \in C^\infty(\mathbb{R}^d)$ be such that $\chi(\xi) = 0$ if $|\xi| \leq c$ and $\chi(\xi) = 1$ if $|\xi| \geq 2c$, where c is a positive real constant chosen so that $\chi(\xi_0) \neq 0$. We set

$$a(x, y, \xi) := \phi(x)\psi\left(\frac{\xi}{|\xi|}\right)\chi(\xi)\phi(y) \in S^0(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d).$$

By Definition 2.2.15, $A := \mathrm{Op}(a)$ is a properly supported pseudodifferential operator lying in $\Psi^0(\mathbb{R}^d)$. In addition, on account of Definition 2.2.30, we infer that A is elliptic at (x_0, ξ_0). To conclude, combining Equation (3.2.3) and Theorem 2.2.38, it descends that $Au \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$.
Conversely, let $(x_0, \xi_0) \notin \bigcap_{\substack{A \in \Psi^0(\mathbb{R}^d) \\ Au \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)}} \mathrm{Char}(A)$. Therefore, there exists $A \in \Psi^0(\mathbb{R}^d)$, elliptic at (x_0, ξ_0) as per Definition 2.2.48, such that $Au \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. We can choose once more ϕ, ψ and χ as in the previous part of the proof so that

$$WF'(B) \subset \mathrm{Ell}(A),$$

where $B := \mathrm{Op}_R(\psi(\xi/|\xi|)\chi(\xi)\phi(y))$ and where $WF'(B)$ denotes the operator wavefront set of B as per Definition 2.2.44. We show that $Bu \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. On account of Proposition 2.2.51, there exists a properly supported microlocal parametrix $Q \in \Psi^0(\mathbb{R}^d)$ of A on $WF'(B)$ such that

$$WF'(I - QA) \cap WF'(B) = \emptyset.$$

Therefore, we can split Bu as follows:

$$Bu = (BQ)(Au) + B(I - QA)u.$$

Since $WF'(I - QA) \cap WF'(B) = \emptyset$, it descends that $B(I - QA)u \in C^\infty(\mathbb{R}^d)$. Chosen $\rho \in \mathcal{D}(\mathbb{R}^d)$ such that $\rho = 1$ on $\mathrm{supp}(\phi)$, it holds true that

$$(BQ)(Au) = (BQ)(\rho Au) + BQ((1 - \rho)Au).$$

On the one hand, being $\rho = 1$ on $\mathrm{supp}(\phi)$, then $(BQ)((1 - \rho)Au) = 0$. On the other hand, on account of Theorem 2.2.38, it descends that $(BQ)(\rho Au) \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. As a result, we conclude that $Bu \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. Therefore, on account of Proposition 2.1.33, it descends that for any $\kappa \in \mathscr{B}_{\lfloor \alpha \rfloor}$ as per Definition 2.1.22 and for any compact set $\mathfrak{K} \subset \mathbb{R}^d$,

$$\left| \int_{\mathrm{Ell}(\psi(D/|D|)\chi(D))} \psi\left(\frac{\xi}{|\xi|}\right)\chi(\xi)\widehat{\phi u}(\xi)\check{\kappa}(\lambda\xi)e^{ix\cdot\xi}d\xi \right| \lesssim \lambda^\alpha, \quad \forall \lambda \in (0, 1], \forall x \in \mathfrak{K}. \tag{3.2.4}$$

On account of Remark 2.2.49, there exists a symbol $s \in S^0(\mathbb{R}^d;\mathbb{R}^d)$ such that

$$R(\xi) := 1 - \psi\left(\frac{\xi}{|\xi|}\right)\chi(\xi)s(\xi) \in S^{-1}(\mathbb{R}^d;\mathbb{R}^d),$$

for any $\xi \in \mathrm{Ell}(\psi(D/|D|)\chi(D))$, where $\mathrm{Ell}(\psi(D/|D|)\chi(D))$ stands for the elliptic set of $\psi(D/|D|)\chi(D)$ as per Definition 2.2.48. Given $x \in \mathfrak{K}$ and $\lambda \in (0,1]$, it descends that

$$\begin{aligned}
&\left|\int_{\mathrm{Ell}(\psi(D/|D|)\chi(D))} \widehat{\phi u}(\xi)\check{\kappa}(\lambda\xi)e^{ix\cdot\xi}d\xi\right| = \left|\int_{\mathrm{Ell}(\psi(D/|D|)\chi(D))} \left(\psi\left(\frac{\xi}{|\xi|}\right)\chi(\xi)s(\xi) + R(\xi)\right)\widehat{\phi u}(\xi)\check{\kappa}(\lambda\xi)e^{ix\cdot\xi}d\xi\right| \\
&\leq \left|\int_{\mathrm{Ell}(\psi(D/|D|)} \psi\left(\frac{\xi}{|\xi|}\right)\chi(\xi)s(\xi)\widehat{\phi u}(\xi)\check{\kappa}(\lambda\xi)e^{ix\cdot\xi}d\xi\right| + \left|\int_{\mathrm{Ell}(\psi(D/|D|)} R(\xi)\widehat{\phi u}(\xi)\check{\kappa}(\lambda\xi)e^{ix\cdot\xi}d\xi\right| \\
&= \underbrace{\left|\left\langle s(D)\psi\left(\frac{D}{|D|}\right)\chi(D)(\phi u), \kappa_x^\lambda\right\rangle\right|}_{|I_1|} + \underbrace{\left|\int_{\mathrm{Ell}(\psi(D/|D|)} R(\xi)\widehat{\phi u}(\xi)\check{\kappa}(\lambda\xi)e^{ix\cdot\xi}d\xi\right|}_{|I_2|}. \qquad (3.2.5)
\end{aligned}$$

On the one hand, Theorem 2.2.38 entails that

$$|I_1| = \left|\left\langle s(D)(Au), \kappa_x^\lambda\right\rangle\right| \lesssim \lambda^\alpha, \quad \forall\lambda \in (0,1], \forall x \in \mathfrak{K}.$$

On the other hand, since $R(D) \in \Psi^{-1}(\mathbb{R}^d)$ and $s(D)\psi\left(\frac{D}{|D|}\right)\chi(D)(\phi u) \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$, it descends that

$$\begin{aligned}
|I_2| &\leq \left|\left\langle R(D)s(D)\psi\left(\frac{D}{|D|}\right)\chi(D)(\phi u), \kappa_x^\lambda\right\rangle\right| + \left|\int_{\mathrm{Ell}(\psi(D/|D|))} R^2(\xi)\widehat{\phi u}(\xi)\check{\kappa}(\lambda\xi)e^{ix\cdot\xi}d\xi\right| \\
&\lesssim \lambda^{\alpha+1} + \left|\int_{\mathrm{Ell}(\psi(D/|D|)\chi(D))} R^2(\xi)\widehat{\phi u}(\xi)\check{\kappa}(\lambda\xi)e^{ix\cdot\xi}d\xi\right|, \quad \forall\lambda \in (0,1], \forall x \in \mathfrak{K},
\end{aligned}$$

where we applied once more Theorem 2.2.38. Therefore, we conclude that $(x_0,\xi_0) \notin WF^\alpha(u)$. □

Next we prove the characterization of the Besov wavefront set of a distribution in terms of the smooth counterpart - see [DRS22, Prop. 35].

Proposition 3.2.2: *Let $\alpha \in \mathbb{R}$ and let $u \in \mathcal{D}'(\mathbb{R}^d)$. Then*

$$WF^\alpha(u) = \bigcap_{v \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)} WF(u-v), \qquad (3.2.6)$$

where WF denotes the smooth wavefront set introduced in Definition 2.3.6.

Proof. Let $(x_0,\xi_0) \in WF^\alpha(u)$ and let $v \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. On account of Proposition 2.3.11, it holds true that

$$WF(u-v) = \bigcap_{\substack{A \in \Psi^0(\mathbb{R}^d) \\ A(u-v) \in C^\infty(\mathbb{R}^d)}} \mathrm{Char}(A),$$

where $\mathrm{Char}(A)$ is the characteristic set of A defined in Equation (2.2.22). Fix $A \in \Psi^0(\mathbb{R}^d)$ such that $A(u-v) \in C^\infty(\mathbb{R}^d)$. Therefore, since $Av \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ per Theorem 2.2.38, it descends that

$$Au = A(u-v) + Av \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d).$$

Since $(x_0, \xi_0) \in WF^\alpha(u)$, Theorem 3.2.1 entails that $(x_0, \xi_0) \in \mathrm{Char}(A)$. Therefore,

$$WF^\alpha(u) \subset \bigcap_{v \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)} WF(u - v).$$

Conversely, let $(x_0, \xi_0) \notin WF^\alpha(u)$. On account of Definition 3.1.1, there exist $\phi \in \mathcal{D}(\mathbb{R}^d)$ with $\phi(x_0) = 1$ and an open conic neighborhood Γ of ξ_0 such that Equations (3.1.1) and (3.1.2) are fulfilled. Choose $v \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ such that

$$\widehat{v}(\xi) = \begin{cases} \widehat{\phi u}(\xi) & \text{if } \xi \in \Gamma, \\ 0 & \text{if } \xi \notin \Gamma. \end{cases} \tag{3.2.7}$$

Setting $\theta := \phi u - v$, on account of Equation (3.2.7), $\widehat{\theta}$ vanishes on Γ and therefore $(x_0, \xi_0) \notin WF(\theta)$. Let $\chi \in \mathcal{D}(\mathbb{R}^d)$ be such that $\chi\phi = 1$ in a neighborhood of x_0. Then, it holds true that $\chi v \in B^\alpha_{\infty,\infty}(\mathbb{R}^d)$ and $(x_0, \xi_0) \notin WF(\chi\theta)$. Therefore, observing that $u - \chi v = (1 - \chi\phi)u + \chi\theta$, it descends that $(x_0, \xi_0) \notin WF(u - \chi v)$ since $(1 - \chi\phi)u$ vanishes in a neighborhood of x_0. Since $\chi(x_0) = 1$, we can conclude that $(x_0, \xi_0) \notin WF(u - v)$. □

Remark 3.2.3: *Let $\alpha \in \mathbb{R}$ and let $u \in \mathcal{D}'(\mathbb{R}^d)$. Equation (3.2.6) can be equivalently formulated as follows: $(x_0, \xi_0) \notin WF^\alpha(u)$ if and only if there exists $v \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ such that $(x_0, \xi_0) \notin WF(u - v)$.*

At last, we prove that the smooth wavefront set of a distribution can be characterized in turn by means of the union of all Besov counterparts - see [DRS22, Cor. 36]. The following result is an adaptation to the case at hand of the one valid in the framework of the Sobolev wavefront set - see [Hin21, Prop. 6.32].

Corollary 3.2.4: *If $u \in \mathcal{D}'(\mathbb{R}^d)$, then*

$$WF(u) = \overline{\bigcup_{\alpha \in \mathbb{R}} WF^\alpha(u)}. \tag{3.2.8}$$

Proof. Let $(x_0, \xi_0) \in \bigcup_{\alpha \in \mathbb{R}} WF^\alpha(u)$. Hence there exists $\alpha \in \mathbb{R}$ such that $(x_0, \xi_0) \in WF^{\alpha'}(u)$ for every $\alpha' \geq \alpha$. On account of Proposition 3.2.2 and of Remark 3.1.6, we choose $v \in C^\infty(\mathbb{R}^d)$ such that $(x_0, \xi_0) \in WF(u - v)$. Since $WF(u - v) = WF(u)$, we infer that $\bigcup_{\alpha \in \mathbb{R}} WF^\alpha(u) \subset WF(u)$. Since $WF(u)$ is a closed conic set, it descends that $\overline{\bigcup_{\alpha \in \mathbb{R}} WF^\alpha(u)} \subset WF(u)$.
Conversely, let $(x_0, \xi_0) \notin WF^\alpha(u)$ for all $\alpha \in \mathbb{R}$. Therefore, there exists an open conic neighborhood $V \subset \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ of (x_0, ξ_0) as per Definition 2.2.41 such that $V \cap WF^\alpha(u) = \emptyset$ for every $\alpha \in \mathbb{R}$. On account of Theorem 3.2.1, there exists $A \in \Psi^0(\mathbb{R}^d)$, elliptic at (x_0, ξ_0) as per Definition 2.2.48, such that $WF'(A) \subset V$ and $Au \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ for every $\alpha \in \mathbb{R}$. Therefore, Remark 3.1.6 entails that $Au \in C^\infty(\mathbb{R}^d)$. On account of Proposition 2.3.11, it descends that $(x_0, \xi_0) \notin WF(u)$. □

Remark 3.2.5: *Definition 3.1.1 as well as the results proved so far can be straightforwardly adapted to distributions $u \in \mathcal{D}'(\Omega)$, where Ω is an arbitrary domain of $\mathbb{R}^d$.*

3.3 Structural Properties of the Besov Wavefront Set

As explained in Appendix A, a few notable operations such as pullback and multiplication are well-defined under restrictive assumptions on the distributions at hand. However, in Section 2.3 we have seen that the smooth wavefront set provides sufficient criteria to extend the definition of these operations to the whole space of distributions. In this section, we shall see that the notion of Besov wavefront set, introduced in

Definition 3.1.1, plays a similar rôle to its smooth counterpart by providing weaker conditions to extend the same operations. We shall also discuss the microlocal properties of pseudodifferential operators as per Definition 2.2.15 within the current scenario. From an operational viewpoint, the characterization of the Besov wavefront set in terms of pseudodifferential operators (Theorem 3.2.1) shall be rather useful in proving the results of this Section.

In Subsection 3.3.1, we discuss the microlocal properties of pseudodifferential operators within the framework of Besov wavefront set - see Proposition 3.3.1. Moreover, we prove a microlocal elliptic regularity result in terms of the Besov wavefront set - see Corollary 3.3.3. These results turn out to be relevant to analyze the Besov regularity of solutions to partial differential equations - see Example 3.3.4. In Subsection 3.3.2, we establish a sufficient condition in terms of the Besov wavefront set to pull back an underlying distribution along an embedding - see Theorem 3.3.5. In addition, as we shall see in Example 3.3.6, we emphasize that this condition is weaker than that of Theorem 2.3.19. We also prove that the Besov wavefront set is invariant under the action of a diffeomorphism. This result is the cornerstone to extend the notion of Besov wavefront set to distributions supported on an arbitrary smooth manifold. In Subsection 3.3.3, we deal with the question of the product between two distributions, again resorting to the concept of Besov wavefront set. Analogously to Theorem 2.3.24, we formulate a version of Hörmander's criterion adapted to the current framework and we provide an estimate for the Besov wavefront set of the product, see Theorem 3.3.10. This result can be read as a microlocal formulation of the Young's product theorem (Theorem 2.1.20), which is often used in the applications to *nonlinear* stochastic partial differential equations - see [Hai14, Hai15, GIP15, BL22A]. In Subsection 3.3.4, we discuss the Besov wavefront set of $\mathcal{K}u$, where $\mathcal{K}\colon \mathcal{D}(\mathbb{R}^{d_2}) \to \mathcal{D}'(\mathbb{R}^{d_1})$ is a linear map with Schwartz kernel $K \in \mathcal{D}'(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. Analogously to Theorem 2.3.29, if $u \in \mathcal{E}'(\mathbb{R}^{d_2})$, we establish a sufficient condition in terms of the Besov wavefront sets of K and u for the well-posedness of $\mathcal{K}u$ - see Theorem 3.3.15. In addition, we prove a bound on the Besov wavefront set of $\mathcal{K}u$. This result entails in turn a microlocal version of the renowned Schauder estimates ([Sim97]) which are often used to analyze the regularity of solutions to (stochastic) partial differential equations, *c.f.* [Hai14], [GIP15]. Lastly, in Subsection 3.3.5 we shall prove a theorem of propagation of singularities for a large class of hyperbolic partial differential equations resorting to the formulation of the Besov wavefront set in terms of pseudodifferential operators as per Theorem 3.2.1. More precisely, this result provides a characterization of singularities of a solution to a suitable hyperbolic partial differential equation in terms of the Besov wavefront set. In the following, we shall make use of the notions introduced in Sections 3.1, 3.2 and 2.2. The content of this section is mainly based on [DRS22, Sect. 4].

3.3.1 Microlocal Properties of Pseudodifferential Operators and Besov Wavefront Set

This subsection is devoted to discussing the interplay between pseudodifferential operators (ΨDOs) as per Definition 2.2.15 and distributions from a microlocal viewpoint. More precisely, we prove the microlocality of ΨDOs and an elliptic regularity result in the framework of the Besov wavefront set as per Definition 3.1.1 as well as Theorem 3.2.1. The following results are of considerable interest to analyze the Besov-type regularity of a solution to a partial differential equation. Throughout this subsection, we only consider properly supported ΨDOs as per Definition 2.2.25.

We start by proving the microlocality of pseudodifferential operators within the current scenario, which is an adaptation of Proposition 2.3.12 - see [DRS22, Prop. 41].

Proposition 3.3.1: *Let $\alpha, m \in \mathbb{R}$. If $A \in \Psi^m(\mathbb{R}^d)$ and $u \in \mathcal{D}'(\mathbb{R}^d)$, then*

$$WF^{\alpha-m}(Au) \subset WF'(A) \cap WF^{\alpha}(u), \tag{3.3.1}$$

where WF' stands for the operator wavefront set as per Definition 2.2.44.

Proof. Let $(x_0, \xi_0) \notin WF'(A)$. On account of Proposition 2.2.51, there exists $B \in \Psi^0(\mathbb{R}^d)$ such that $WF'(B) \cap WF'(A) = \emptyset$ and $(x_0, \xi_0) \in \mathrm{Ell}(B)$, where $\mathrm{Ell}(B)$ stands for the elliptic set of B as per Definition 2.2.48. As a result, Proposition 2.2.45 yields $WF'(BA) = \emptyset$, that is to say $BA \in \Psi^{-\infty}(\mathbb{R}^d)$. Therefore, $B(Au)$ lies in $C^\infty(\mathbb{R}^d) \hookrightarrow B^{\alpha-m,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. On account of Theorem 3.2.1, it descends that $(x_0, \xi_0) \notin WF^{\alpha-m}(Au)$.
Let $(x_0, \xi_0) \notin WF^\alpha(u)$. On account of Theorem 3.2.1, there exists $\tilde{A} \in \Psi^0(\mathbb{R}^d)$, elliptic at (x_0, ξ_0), such that $\tilde{A}u \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. Fix $B \in \Psi^0(\mathbb{R}^d)$, elliptic at (x_0, ξ_0), such that $WF'(B) \subset \mathrm{Ell}(\tilde{A})$. On account of Proposition 2.2.51, there exists a microlocal parametrix $Q \in \Psi^0(\mathbb{R}^d)$ of $\tilde{A}$ on $WF'(B)$ such that

$$WF'(B) \cap WF'(I - Q\tilde{A}) = \emptyset. \tag{3.3.2}$$

Hence,

$$B(Au) = BAQ(\tilde{A}u) + BA(I - Q\tilde{A})u.$$

Combining Equation (3.3.2) and Proposition 2.2.45, we infer that $BA(I - Q\tilde{A}) \in \Psi^{-\infty}(\mathbb{R}^d)$. This entails that $BA(I - Q\tilde{A})u \in C^\infty(\mathbb{R}^d)$. At the same time, on account of Proposition 2.2.26, $BAQ(\tilde{A}u)$ lies in $B^{\alpha-m,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ since $BAQ \in \Psi^m(\mathbb{R}^d)$ and $\tilde{A}u \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. To conclude, since $B(Au) \in B^{\alpha-m,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ and $(x_0, \xi_0) \in \mathrm{Ell}(B)$, the point (x_0, ξ_0) does not lie in $WF^{\alpha-m}(Au)$ per Theorem 3.2.1. □

Subsequently, we prove a kind of converse of Proposition 3.3.1 and a microlocal elliptic regularity result within the current framework - [DRS22, Prop. 42, Cor. 43]. The following results are an adaptation to the case at hand of Proposition 2.3.13.

Proposition 3.3.2: *Let $m, \alpha \in \mathbb{R}$ and let $A \in \Psi^m(\mathbb{R}^d)$. If $u \in \mathcal{D}'(\mathbb{R}^d)$, then*

$$WF^\alpha(u) \subset \mathrm{Char}(A) \cup WF^{\alpha-m}(Au). \tag{3.3.3}$$

Proof. Let $(x_0, \xi_0) \notin \mathrm{Char}(A) \cup WF^{\alpha-m}(Au)$. On account of Theorem 3.2.1, there exists $B \in \Psi^0(\mathbb{R}^d)$, elliptic at (x_0, ξ_0) as per Definition 2.2.48, such that $B(Au) \in B^{\alpha-m,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. Without loss of generality, we choose a properly supported ΨDO $P \in \Psi^{-m}(\mathbb{R}^d)$ such that $(x_0, \xi_0) \in \mathrm{Ell}(P)$. On account of Proposition 2.2.26, it descends that $PBA \in \Psi^0(\mathbb{R}^d)$. Therefore, Theorem 2.2.38 entails that $(PBA)u \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. Since $(x_0, \xi_0) \in \mathrm{Ell}(PBA)$, it descends that $(x_0, \xi_0) \notin WF^\alpha(u)$. □

Corollary 3.3.3: *Let $m, \alpha \in \mathbb{R}$ and let $A \in \Psi^m(\mathbb{R}^d)$ be elliptic as per Definition 2.2.30. If $u \in \mathcal{D}'(\mathbb{R}^d)$, then*

$$WF^\alpha(u) = WF^{\alpha-m}(Au). \tag{3.3.4}$$

Proof. Since $A \in \Psi^m(\mathbb{R}^d)$ is elliptic, it descends that $\mathrm{Char}(A) = \emptyset$. Therefore, combining Propositions 3.3.1 and 3.3.2, we conclude that $WF^\alpha(u) = WF^{\alpha-m}(Au)$. □

Proposition 3.3.2 and Corollary 3.3.3 play a key rôle to analyze the Besov-type regularity of solutions to partial differential equations.

Example 3.3.4: *Let $\Delta := \sum_{j=1}^d \partial_j^2$ be the Laplace operator on $\mathbb{R}^d$ and let $h \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. In addition, let $u \in \mathcal{D}'(\mathbb{R}^d)$ be such that*

$$-\Delta u = h.$$

Since A is an elliptic ΨDO of order 2, Corollary 3.3.3 yields that $u \in B^{\alpha+2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$.

3.3.2 Trasformations Properties under Pullback

As explained in Subsection 2.3.2, the pullback of a distribution along an embedding is in general an ill-defined operation. However, imposing a suitable condition which involves the smooth wavefront set of the distribution at hand, the feasibility of this operation is guaranteed - see Theorem 2.3.19. In this Subsection, we shall see that the notion of Besov wavefront set as per Definition 3.1.1 plays a similar rôle to that of its smooth counterpart but then again, as one should expect, it allows to establish a more accurate existence result for the pullback in comparison to the one formulated in the smooth setting - see Theorem 3.3.5. Since the product of two distributions can be defined via pullback as explained in Subsection 2.3.3, we shall see in Subsection 3.3.3 that this result entails a weaker sufficient condition than Hörmander's criterion for the existence of the product. Lastly, as a byproduct we prove that the Besov wavefront set is invariant under a change of coordinates. This result is noteworthy because it is instrumental in showing that the definition of Besov wavefront set can be extended to distributions supported on an arbitrary smooth manifold as per Definition A.10.1. In order to prove the following results, we shall exploit the characterizations of the Besov wavefront set outlined in Section 3.2.

We start by proving the main result of this subsection, which establishes a sufficient criterion for the well-posedness of the pullback of a distribution via an embedding within the framework of the Besov wavefront set - see [DRS22, Th. 38]. On account of Remark 3.2.5, we shall consider distributions lying in $\mathcal{D}'(\Omega)$, where Ω is an arbitrary domain of $\mathbb{R}^d$.

Theorem 3.3.5: *Let $\Omega_1 \subset \mathbb{R}^{d_1}$, $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets with $d_1 < d_2$ and let $f\colon \Omega_1 \to \Omega_2$ be an embedding as per Definition 2.3.18. Then there exists a unique $f^*u \in \mathcal{D}'(\Omega_1)$ for any $u \in \mathcal{D}'(\Omega_2)$ such that there exists $\alpha' > 0$ so that*

$$N_f \cap WF^{\alpha'}(u) = \emptyset, \tag{3.3.5}$$

where N_f stands for the set of normals of f, defined in Equation (2.3.5). In addition, for any $u \in \mathcal{D}'(\Omega_2)$ satisfying Equation (3.3.5) and for all $\alpha \in \mathbb{R}$, it holds true that

$$WF^\alpha(f^*u) \subset f^*WF^\alpha(u), \tag{3.3.6}$$

where

$$f^*WF^\alpha(u) := \{(x, {}^t df(x)\xi) : (f(x), \xi) \in WF^\alpha(u)\} \tag{3.3.7}$$

and df denotes the differential of f.

Proof. Combining Proposition 3.2.2 and Equation (3.3.5), it descends that there exists $v \in B^{\alpha',\mathrm{loc}}_{\infty,\infty}(\Omega_2)$ as per Definition 2.1.30 such that

$$N_f \cap WF(u - v) = \emptyset.$$

Therefore, on account of Theorem 2.3.19, there exists $f^*(u-v)$ lying $\mathcal{D}'(\Omega_1)$. Being $B^{\alpha',\mathrm{loc}}_{\infty,\infty}(\Omega_2) \hookrightarrow C^0(\Omega_2)$ per Theorems 2.1.37 and 2.1.38, f^*v is defined as the composition $v \circ f$. Therefore, since

$$f^*u = f^*(u - v) + f^*v,$$

we conclude that there exists $f^*u \in \mathcal{D}'(\Omega_1)$. Next we show Equation (3.3.6). Let $\alpha \in \mathbb{R}$ and let $(x_0, {}^t df(x)\xi_0) \notin f^*WF^\alpha(u)$. Hence, Equation (3.3.7) implies that $(f(x_0), \xi_0) \notin WF^\alpha(u)$. On account of Theorem 3.2.1, it descends that there exists $A \in \Psi^0(\Omega_2)$, properly supported as per Definition 2.2.25, such that $Au \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_2)$ and $(f(x_0), \xi_0) \in \mathrm{Ell}(A)$, where $\mathrm{Ell}(A)$ stands for the elliptic set of A as per Definition 2.2.48. Due to Proposition 2.2.36 and Remark 2.2.37, being f a diffeomorphism on its image, it descends that

$$A_f\colon \mathcal{D}'(\Omega_1) \to \mathcal{D}'(\Omega_1), \quad \mathcal{D}'(\Omega_1) \ni v \mapsto f^*A((f^{-1})^*v)$$

is a properly supported pseudodifferential operator lying in $\Psi^0(\Omega_1)$. Still on account of Proposition 2.2.36, it holds true that

$$\sigma_0(A_f)(x_0, {}^t df(x_0)\xi_0) = \sigma_0(A)(f(x_0), \xi_0) \neq 0.$$

To conclude that $(x_0, {}^t df(x_0)\xi_0) \notin WF^\alpha(f^*u)$, we show that $A_f(f^*u) \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_1)$ as per Definition 2.1.30. Given $\phi \in \mathcal{D}(\Omega_1)$ and $\kappa \in \mathscr{B}_{\lfloor\alpha\rfloor}$ as per Definition 2.1.22, it holds true that

$$\begin{aligned}|\langle \phi A_f(f^*u), \kappa^\lambda_x\rangle| &= |\langle f^*(Au), \phi\kappa^\lambda_x\rangle| = |\langle Au, (f^{-1})^*\phi((f^{-1})^*\kappa)^\lambda_{f(x)}|\det(df^{-1})|\rangle| \\ &\lesssim |\langle Au, (f^{-1})^*\phi((f^{-1})^*\kappa)^\lambda_{f(x)}\rangle| \lesssim \lambda^\alpha, \quad \forall\lambda\in(0,1], \quad \forall x\in\Omega_1,\end{aligned}$$

where in the last inequality we used that $Au \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_2)$. Analogous estimates yield

$$|\langle \phi A_f(f^*u), \underline{\kappa}_x\rangle| \lesssim 1,$$

for any $x \in \Omega_1$ and $\underline{\kappa} \in \mathcal{D}(B(0,1))$ with $\check{\underline{\kappa}}(0) \neq 0$. Since $(x_0, {}^t df(x)\xi_0) \notin \mathrm{Char}(A_f)$ and $A_f(f^*u) = f^*(Au) \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$, Theorem 3.2.1 entails that $(x_0, {}^t df(x)\xi_0)$ does not lie in $WF^\alpha(f^*u)$. □

Remark 3.3.6: *In the framework of the smooth wavefront set, a sufficient condition to pullback an $u \in \mathcal{D}'(\mathbb{R}^d)$ is*

$$N_f \cap WF(u) = \emptyset, \tag{3.3.8}$$

see Theorem 2.3.19. We show that Equation (3.3.5) codifies a weaker condition than that in Equation (3.3.8). On account of Corollary 3.2.4, it descends that Equation (3.3.8) implies Equation (3.3.5). However, the converse does not hold true in general. For instance, let $u \in \mathcal{D}'(\mathbb{R}^2)$ be such that its integral kernel is $\tilde{u}(x_1,x_2) = (x_1^2+x_2^2)^{\frac14}$ and let $\boldsymbol{\delta}\colon \mathbb{R}\to\mathbb{R}\times\mathbb{R}$ be the diagonal map, $\boldsymbol{\delta}(x) = (x,x)$. On account of Example 3.1.14 and being $N_{\boldsymbol{\delta}} = \{(x,x,\xi,-\xi)\}$, it descends that

$$N_{\boldsymbol{\delta}} \cap WF^{\frac12}(u) = \emptyset, \quad N_{\boldsymbol{\delta}} \cap WF^\alpha(u) \neq \emptyset,$$

for any $\alpha > 1/2$. This entails that $N_{\boldsymbol{\delta}} \cap WF(u) \neq \emptyset$. Moreover, on account of Theorem 3.3.5, there exists $\boldsymbol{\delta}^*u \in \mathcal{D}'(\mathbb{R})$, whose integral kernel is $(\boldsymbol{\delta}^*\tilde{u})(x) = 2^{\frac14}|x|^{\frac12}$.

We conclude this subsection by proving that the Besov wavefront set of a distribution is invariant under the action of diffeomorphisms - see [DRS22, Th. 39].

Theorem 3.3.7: *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$, let $f\colon \Omega_1 \to \Omega_2$ be a diffeormophism and let $\alpha \in \mathbb{R}$. If $u \in \mathcal{D}'(\Omega_2)$, then*

$$WF^\alpha(f^*u) = f^*WF^\alpha(u),$$

where $f^*WF^\alpha(u)$ has been defined in Equation (3.3.7).

Proof. To start with, we prove that $WF^\alpha(f^*u) \subset f^*WF^\alpha(u)$. Let $(x_0, {}^t df(x_0)\xi_0) \notin f^*WF^\alpha(u)$. Hence, Equation (3.3.7) entails that $(f(x_0), \xi_0) \notin WF^\alpha(u)$. On account of Theorem 3.2.1, there exists $A \in \Psi^0(\Omega_2)$, elliptic at $(f(x_0), \xi_0)$ as per Definition 2.2.48, such that $Au \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_2)$. Bearing in mind that $f\colon \Omega_1 \to \Omega_2$ is a diffeomorphism,

$$A_f\colon \mathcal{D}'(\Omega_1) \to \mathcal{D}'(\Omega_1), \quad \mathcal{D}'(\Omega_1) \ni v \mapsto f^*A((f^{-1})^*v)$$

is a properly supported pseudodifferential operator lying in $\Psi^0(\Omega_1)$ on account of Proposition 2.2.36 and of Remark 2.2.37. In addition, Proposition 2.2.36 entails that

$$\sigma_0(A_f)(x_0, {}^t df(x_0)\xi_0) = \sigma_0(A)(f(x_0), \xi_0) \neq 0,$$

that is to say A_f is elliptic at $(x_0, {}^t df(x_0)\xi_0)$, see Definition 2.2.48. In addition, reasoning as in the proof of Theorem 3.3.5, it turns out that $A_f(f^*u) \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_1)$ as per Definition 2.1.30. On account of Theorem 3.2.1, we deduce that $(x_0, {}^t df(x_0)\xi_0) \notin WF^\alpha(f^*u)$.
Conversely, let $(x_0, \xi_0) \notin WF^\alpha(f^*u)$. On account of Theorem 3.2.1, there exists $A_f \in \Psi^0(\Omega_1)$, properly supported, such that $A_f(f^*u) \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_1)$ and $(x_0, \xi_0) \in \mathrm{Ell}(A_f)$, where $\mathrm{Ell}(A_f)$ stands for the elliptic set of A_f introduced in Definition 2.2.48. For any $v \in \mathcal{D}'(\Omega_2)$, we set

$$Av := (f^{-1})^*(A_f(f^*v)).$$

On account of Proposition 2.2.36, it holds true that $A \in \Psi^0(\Omega_2)$ and

$$\sigma_0(A)(f(x_0), ({}^t df(x_0))^{-1}\xi_0) = \sigma_0(A_f)(x_0, \xi_0) \neq 0.$$

In view of Remark 2.2.37, A is also properly supported since f is a diffeomorphism. Reasoning as in the proof of Theorem 3.3.5, it turns out that $Au \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_2)$. On account of Theorem 3.2.1, it descends that $(x_0, \xi_0) \notin f^* WF^\alpha(u)$. □

Remark 3.3.8: *Analogously to Proposition 2.3.21, Theorem 3.3.7 is particularly noteworthy since it is the cornerstone to define the Besov wavefront set of a distribution supported on a smooth manifold, following the same rationale of Remark 3.1.2. Let M be a d-dimensional smooth manifold and let $\mathcal{A} = \{(U_i, h_i)\}_i$ be a smooth atlas thereon. On account of Remark 2.3.7, if $u \in \mathcal{D}'(M)$ as per Definition A.10.1, we define $WF^\alpha(u)$ as the subset of $T^*M \setminus \{0\}$ such that its restriction to U_i is given by $(h_i)^* WF^\alpha((h_i^{-1})^*u)$.*

3.3.3 Product of distributions and Besov Wavefront Set

A well-known result of the theory of distributions asserts that the product of two distributions is well-defined if their singular supports are disjoint - see Theorem A.5.3. However, even though the singular supports are not disjoint, in Subsection 2.3.3 we have shown that the product among two distributions can be defined imposing a suitable condition on the smooth wavefront sets, called Hörmander's criterion - see Theorem 2.3.24. In this Subsection, we address the same issue in the context of the Besov wavefront set as per Definition 3.1.1 as well as Theorem 3.2.1. More precisely, we formulate a version of Hörmander's criterion for the existence of the product of two distributions, adapted to the current framework. If the product exists, we also establish an estimate of the associated Besov wavefront set - see Theorem 3.3.10. This result should be read as a microlocal version of the Young's product theorem (Theorem 2.1.20), which is often applied to analyze the well-posedness of nonlinear stochastic partial differential equations - see [Hai14, Hai15, GIP15, BL22A]. Moreover, Theorem 3.3.10 shall play a prominent rôle in Subsection 3.3.4, which discusses the analysis of the Besov wavefront set of $\mathcal{K}u$, where $\mathcal{K}$ is a linear map from $\mathcal{D}(\Omega_2)$ to $\mathcal{D}'(\Omega_1)$ while $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ are two open sets. Lastly, we shall present in Section 3.4 an application of Theorem 3.3.10 in the context of coherent germs of distributions as per Definition 2.4.4.

As explained in Subsection 2.3.3, the product between two distributions can be thought as the pullback of their product tensor along the diagonal. For this reason, since the interplay between the Besov wavefront set and the pullback of a distribution has already been discussed in Subsection 3.3.2, we wish to establish an estimate on the singular behaviour of the tensor product of two distributions within the current framework - see [DRS22, Prop. 44]. We shall omit the proof of this result since it is an adaptation to the case in hand of the one valid in the context of the Sobolev wavefront set, see [JS02, Prop. B.5], which is based in turn on [Hör97, Lemma 11.6.3]. First of all we introduce some useful notations. Given $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\alpha \in \mathbb{R}$, we set

$$WF_0^\alpha(u) := WF^\alpha(u) \cup (\mathrm{supp}(u) \times \{0\}),\ WF_0(u) := WF(u) \cup (\mathrm{supp}(u) \times \{0\}). \quad (3.3.9)$$

In the following, on account of Remark 3.2.5, we shall consider distributions supported on an open set of $\mathbb{R}^d$.

Proposition 3.3.9: *Let $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets and let $\alpha_1, \alpha_2 \in \mathbb{R}$. If $u \in \mathcal{D}'(\Omega_1)$ and $v \in \mathcal{D}'(\Omega_2)$, then the following inclusions hold true:*

$$WF^{\alpha_1+\alpha_2}(u \otimes v) \subset (WF_0^{\alpha_1}(u) \times WF(v)) \cup (WF(u) \times WF_0^{\alpha_2}(v)), \tag{3.3.10}$$

and, setting $\alpha := \min\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$,

$$WF^{\alpha}(u \otimes v) \subset (WF^{\alpha_1}(u) \times WF_0(v)) \cup (WF_0(u) \times WF^{\alpha_2}(v)). \tag{3.3.11}$$

At last, we prove a counterpart of Hörmander's criterion within the context of the Besov wavefront set - see [DRS22, Th. 45]. In particular, the following result should be read as a microlocal formulation of the Young's product theorem, see Theorem 2.1.20.

Theorem 3.3.10: *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $u, v \in \mathcal{D}'(\Omega)$. Suppose that for any $(x, \xi) \in \Omega \times (\mathbb{R}^d \setminus \{0\})$ there exist $\alpha_1', \alpha_2' \in \mathbb{R}$ with $\alpha_1' + \alpha_2' > 0$ such that $(x, \xi) \notin WF^{\alpha_1'}(u)$ and $(x, -\xi) \notin WF^{\alpha_2'}(v)$. Then the product $uv \in \mathcal{D}'(\Omega)$ can be defined as*

$$uv := \boldsymbol{\delta}^*(u \otimes v),$$

where $\boldsymbol{\delta}^*$ denotes the pullback along the diagonal map $\boldsymbol{\delta}\colon \Omega \to \Omega \times \Omega$, $\boldsymbol{\delta}(x) := (x, x)$. In addition, for any $\alpha_1, \alpha_2 \in \mathbb{R}$, it holds true that

$$WF^{\alpha}(uv) \subset \{(x, \xi_1 + \xi_2) : (x, \xi_1) \in WF^{\alpha_1}(u), (x, \xi_2) \in WF_0(v) \,\text{or}\, (x, \xi_1) \in WF_0(u), (x, \xi_2) \in WF^{\alpha_2}(v)\}, \tag{3.3.12}$$

where we set $\alpha := \min\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$.

Proof. Per hypotesis, there exist $\alpha_1', \alpha_2' \in \mathbb{R}$ with $\alpha_1' + \alpha_2' > 0$ such that

$$N_{\boldsymbol{\delta}} \cap WF^{\alpha_1'+\alpha_2'}(u \otimes v) = \emptyset,$$

where $N_{\boldsymbol{\delta}} = \{(x, x, \xi, -\xi)\}$ is the set of normals of $\boldsymbol{\delta}$ defined in Equation (2.3.5). On account of Theorem 3.3.5, it descends that there exists $\boldsymbol{\delta}^*(u \otimes v) \in \mathcal{D}'(\Omega)$. Furthermore, on account of Theorem 3.3.5 combined with Equation (3.3.11), it descends that for any $\alpha_1, \alpha_2 \in \mathbb{R}$

$$\begin{gathered}\mathrm{WF}^{\alpha}(\boldsymbol{\delta}^*(u \otimes v)) \subset \boldsymbol{\delta}^* \mathrm{WF}^{\alpha}(u \otimes v) = \\ = \{(x, \xi_1 + \xi_2) : (x, \xi_1) \in \mathrm{WF}^{\alpha_1}(u), (x, \xi_2) \in WF_0(v) \text{ or } (x, \xi_1) \in WF_0(u), (x, \xi_2) \in \mathrm{WF}^{\alpha_2}(v)\},\end{gathered}$$

where $\alpha := \min\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. □

Remark 3.3.11: *Let $u \in B^{\alpha_1,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ and let $v \in B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ with $\alpha_1 + \alpha_2 > 0$. On account of Proposition 3.1.5, it descends that $WF^{\alpha_1}(u) = WF^{\alpha_2}(v) = \emptyset$. Therefore, on account of Theorem 3.3.10, there exists $uv \in \mathcal{D}'(\mathbb{R}^d)$. In addition, if we set $\alpha_1 \wedge \alpha_2 := \min\{\alpha_1, \alpha_2\}$, Equation 3.3.12 entails that $WF^{\alpha_1 \wedge \alpha_2}(uv) = \emptyset$, that is to say $uv \in B^{\alpha_1 \wedge \alpha_2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. As mentioned above, on account of Theorem 3.3.10, we recover Young's product theorem as per Theorem 2.1.20.*

3.3.4 Schwartz Kernels and Besov Wavefront Set

In Subsection 3.1, we introduced the notion of Besov wavefront set, which estimates the directions in Fourier space of an underlying distribution comparing them with those of an element lying in a suitable Besov space $B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$ as per Definition 2.1.24. The aim of this subsection is to discuss the Besov wavefront set of $\mathcal{K}u$, where $\mathcal{K}\colon \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ is a linear map with Schwartz kernel $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$. Similarly to a result shown in Subsection 2.3.5, given $u \in \mathcal{E}'(\Omega_2)$, we shall establish a criterion for the existence of $\mathcal{K}u$ that involves a condition on the Besov wavefront sets of K and u as well as an estimate on the singular behavior of $\mathcal{K}u$. This result also leads to a generalization of *Schauder estimates*, see [Sim97], which is often used to analyze the $B^{\alpha}_{\infty,\infty}$-type regularity of a solution to a partial differential equation, such as the heat equation. We shall mainly refer to [DRS22, Sect. 4]. For basic notions concerning Schwartz kernels, the reader may refer to Appendix A.8. On account of Remark 3.2.5, throughout this subsection we shall consider distributions lying in $\mathcal{D}'(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is an open set.

Let $\mathcal{K}\colon \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ be a linear map with Schwartz kernel $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$, where $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ are two open sets. As observed in the proof of Theorem 2.3.29, for any $u \in \mathcal{D}(\Omega_2)$ we can define $\mathcal{K}u$ as

$$\mathcal{K}u = \pi_*(K(1 \otimes u)),$$

where $\pi\colon \Omega_1 \times \Omega_2 \to \Omega_1$ is the canonical projection on the first factor while π_* denotes the push-forward map along π as per Definition 2.3.25. For this reason, we start by proving two ancillary results. The first one is a regularity result concerning the push-forward of a distribution lying in $B^{\alpha}_{\infty,\infty}(\Omega_1 \times \Omega_2)$ - see [DRS22, Cor. 47]. The second one asserts how the Besov wavefront set trasforms under push-forwards - see [DRS22, Prop. 48]. In what follows, on account of Remark 3.1.3, we can only consider compactly supported distributions.

Corollary 3.3.12: *Let $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets, let $K \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_1 \times \Omega_2)$ with $\alpha \in \mathbb{R}$ and let $\pi\colon \Omega_1 \times \Omega_2 \to \Omega_1$ be the canonical projection on the first factor defined by $\pi(x_1, x_2) = x_1$. In addition, suppose that π is a proper map when restricted to $\mathrm{supp}(K)$. Then $\pi_*(K)$ lies in $B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_1)$, where π_* is the push-forward map along π as per Definition 2.3.25.*

Proof. Without loss generality, we consider $K \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_1 \times \Omega_2) \cap \mathcal{E}'(\Omega_1 \times \Omega_2)$. By Definition 2.3.25, it holds true that

$$\langle \pi_*(K), \phi \rangle = K(\phi \otimes 1) \quad \forall \phi \in C^{\infty}(\Omega_1).$$

Let $\kappa \in \mathscr{B}_{\lfloor \alpha \rfloor}$ as per Definition 2.1.22. Since $\kappa \otimes 1 \in \mathscr{B}_{\lfloor \alpha \rfloor}$, it descends that

$$|\langle \pi_*(K), \kappa^{\lambda}_{x_1} \rangle| = |K((\kappa \otimes 1)^{\lambda}_{(x_1,x_2)})| \lesssim \lambda^{\alpha}, \quad \forall \lambda \in (0,1], \forall x_1 \in \Omega_1.$$

At the same time, given $\underline{\kappa} \in \mathcal{D}(B(0,1))$ with $\check{\underline{\kappa}}(0) \neq 0$, it descends that

$$|\langle \pi_*(K), \underline{\kappa}_{x_1} \rangle| = |K((\underline{\kappa} \otimes 1)_{(x_1,x_2)})| \lesssim 1, \quad \forall \lambda \in (0,1], \forall x_1 \in \Omega_1.$$

As a result, we conclude that $\pi_*(K) \in B^{\alpha}_{\infty,\infty}(\Omega_1)$ as per Definition 2.1.24. □

Proposition 3.3.13: *Let $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ where $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ are two open sets. In addition, suppose that the canonical projection on the first factor $\pi\colon \Omega_1 \times \Omega_2 \to \Omega_1$ is proper when restricted to $\mathrm{supp}(K)$. Then, for any $\alpha \in \mathbb{R}$, it holds true that*

$$WF^{\alpha}(\pi_*(K)) \subset \{(x_1, \xi_1) \in \Omega_1 \times (\mathbb{R}^{d_1} \setminus \{0\}) : \exists x_2 \in \mathrm{supp}(K), (x_1, x_2, \xi_1, 0) \in WF^{\alpha}(K)\}, \qquad (3.3.13)$$

where π_* is the push-forward map by π as per Definition 2.3.25.

Proof. Without loss of generality, we consider $K \in \mathcal{E}'(\Omega_1 \times \Omega_2)$. Since $K \in \mathcal{E}'(\Omega_1 \times \Omega_2)$, then $\pi_*(K) \in \mathcal{E}'(\Omega_1)$. As a matter of fact, by Definition 2.3.25, it holds true that

$$\langle \pi_*(K), \phi \rangle = K(\phi \otimes 1), \quad \forall \phi \in C^\infty(\Omega_1).$$

Therefore, in what follows we consider compactly supported distributions. Let $(x_1, \xi_1) \in WF^\alpha(\pi_*(K))$. On account of Proposition 3.2.2, it descends that $(x_1, \xi_1) \in WF^\alpha(\pi_*(K) - v)$ for any $v \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_1) \cap \mathcal{E}'(\Omega_1)$. In addition, Corollary 3.3.12 entails that for any $v \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_1) \cap \mathcal{E}'(\Omega_1)$ there exists $\tilde{v} \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_1 \times \Omega_2) \cap \mathcal{E}'(\Omega_1 \times \Omega_2)$ such that $v = \pi_*(\tilde{v})$. As a result, it holds true that $(x_1, \xi_1) \in WF(\pi_*(K - \tilde{v}))$ for any $\tilde{v} \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_1 \times \Omega_2) \cap \mathcal{E}'(\Omega_1 \times \Omega_2)$. Applying Proposition 2.3.28, it descends that

$$WF(\pi_*(K - \tilde{v})) \subset \{(x_1, \xi_1) : \exists x_2 \in \mathrm{supp}(K - \tilde{v}),\, (x_1, x_2, \xi_1, 0) \in WF(K - \tilde{v})\},$$

for any $\tilde{v} \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_1 \times \Omega_2) \cap \mathcal{E}'(\Omega_1 \times \Omega_2)$. The arbitrariness of $\tilde{v}$ and Proposition 3.2.2 entail that $x_2 \in \mathrm{supp}(K)$ and $(x_1, x_2, \xi_1, 0) \in WF^\alpha(K)$. This proves the statement. □

We now are in a position to prove the main results of this subsection. The following theorem establishes a bound on the Besov wavefront set of $\mathcal{K}u$ for any $u \in \mathcal{D}(\Omega_2)$ - see [DRS22, Th. 49].

Theorem 3.3.14: *Let $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets and let $\mathcal{K} \colon \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ be a linear map with Schwartz kernel $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$. Then, for all $\alpha \in \mathbb{R}$ and for any $u \in \mathcal{D}(\Omega_2)$,*

$$WF^\alpha(\mathcal{K}u) \subset \{(x_1, \xi_1) \in \Omega_1 \times (\mathbb{R}^{d_1} \setminus \{0\}) : \exists x_2 \in \mathrm{supp}(u), (x_1, x_2, \xi_1, 0) \in WF^\alpha(K)\}. \tag{3.3.14}$$

Proof. Let $\pi \colon \Omega_1 \times \Omega_2 \to \Omega_1$ be the projection map on the first factor. Moreover, assume that π is proper when restricted to $\mathrm{supp}(K)$. As observed in Theorem 2.3.29, if $u \in \mathcal{D}(\Omega_2)$, then $\mathcal{K}u$ can be defined as the element lying in $\mathcal{D}'(\Omega_1)$ such that

$$(\mathcal{K}u)(\phi) := \langle \pi_*(K(1 \otimes u)), \phi \rangle = \langle K(1 \otimes u), \phi \otimes 1 \rangle \quad \forall \phi \in \mathcal{D}(\Omega_1),$$

where π_* is the push-forward map along π as per Definition 2.3.25. Since $1 \otimes u \in C^\infty(\Omega_1 \times \Omega_2)$, then the product $K(1 \otimes u)$ is well-defined. Here, $K(1 \otimes u)$ denotes the standard product between smooth functions and distributions as per Definition A.5.2. Therefore, Theorem 3.3.10 entails that for any $\alpha \in \mathbb{R}$

$$WF^\alpha(K(1 \otimes u)) \subset \{(x_1, x_2, \xi_1, \xi_2) \in WF^\alpha(K) : x_2 \in \mathrm{supp}(K)\}.$$

To conclude, on account of Proposition 3.3.13, it descends that

$$WF^\alpha(\pi_*(K(1 \otimes u))) \subset \{(x_1, \xi_1) \in \Omega_1 \times (\mathbb{R}^{d_1} \setminus \{0\}) : \exists x_2 \in \mathrm{supp}(u), (x_1, x_2, \xi_1, 0) \in WF^\alpha(K)\}.$$

□

In the following, we generalize the previous theorem to the case when $u \in \mathcal{E}'(\Omega_2)$ and we establish a sufficient criterion for the well-posedness of $\mathcal{K}u$ - see [DRS22, Th. 50].

Theorem 3.3.15: *Let $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets, let $\mathcal{K} \colon \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ be a linear map with Schwartz kernel $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ and let $u \in \mathcal{E}'(\Omega_2)$. In addition, for any $\alpha \in \mathbb{R}$, we set*

$$-WF^\alpha_{\Omega_2}(K) := \{(x_2, \xi_2) \in \Omega_2 \times (\mathbb{R}^{d_2} \setminus \{0\}) : \exists x_1 \in \Omega_2, (x_1, x_2, 0, -\xi_2) \in WF^\alpha(K)\}. \tag{3.3.15}$$

If for any $(x_2, \xi_2) \in \Omega_2 \times (\mathbb{R}^d \setminus \{0\})$ there exist $\alpha'_1, \alpha'_2 \in \mathbb{R}$ with $\alpha'_1 + \alpha'_2 > 0$ such that

$$(x_2, \xi_2) \notin -WF^{\alpha'_1}_{\Omega_2}(K) \cup WF^{\alpha'_2}(u), \tag{3.3.16}$$

then there exists $\mathcal{K}u \in \mathcal{D}'(\Omega_1)$. In addition, for any $\alpha_1, \alpha_2 \in \mathbb{R}$, it holds true that

$$WF^\alpha(\mathcal{K}u) \subset \{(x_1,\xi_1) \in \Omega_1 \times (\mathbb{R}^{d_1} \setminus \{0\}) : \exists (x_2,\xi_2) \in \Omega_2 \times (\mathbb{R}^{d_2} \setminus \{0\}), (x_1,x_2,\xi_1,\xi_2) \in X \cup Y\}, \quad (3.3.17)$$

where we set $\alpha := \min\{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}, X := \{(x_1,x_2,\xi_1,\xi_2) \in WF^{\alpha_1}(K) : (x_2,-\xi_2) \in WF_0(u)\}, Y := \{(x_1,x_2,\xi_1,\xi_2) \in WF_0(K) : (x_2,-\xi_2) \in WF^{\alpha_2}(u)\}$ while WF_0 has been defined in Equation (3.3.9).

Proof. As in the proof of Theorem 3.3.14, we wish to define $\mathcal{K}u$ as

$$\mathcal{K}u := \pi_*(K(1 \otimes u)),$$

where π_* is the push-forward along the canonical projection on the first factor $\pi : \Omega_1 \times \Omega_2 \to \Omega_1$ as per Definition 2.3.25. In order to prove the first part of the statement, we show that the product $K(1 \otimes u) \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ is well-defined. Given $\alpha_2 \in \mathbb{R}$, on account of Equation (3.3.11), it descends that $WF^{\alpha_2}(1 \otimes u) \subset (\Omega_1 \times \{0\}) \times WF^{\alpha_2}(u)$. As a result, combining Theorem 3.3.10 and Equation (3.3.16), we infer that there exists $K(1 \otimes u) \in \mathcal{D}'(\Omega_1 \times \Omega_2)$. Yet, being u compactly supported, it descends that $\pi_*(K(1 \otimes u))$ is a well-defined element lying in $\mathcal{D}'(\Omega_1)$.

At this stage, we focus on Equation (3.3.17). On account of Proposition 3.3.13, it descends that, for any $\alpha \in \mathbb{R}$,

$$WF^\alpha(\pi_*(K(1 \otimes u))) \subset \{(x_1,\xi_1) \in \Omega_1 \times (\mathbb{R}^{d_1} \setminus \{0\}) : \exists x_2 \in \mathrm{supp}(u),\ (x_1,x_2,\xi_1,0) \in WF^\alpha(K(1 \otimes u))\}.$$

Given $\alpha_1, \alpha_2 \in \mathbb{R}$, we set $\alpha := \min\{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$. Theorem 3.3.10, in particular Equation (3.3.12), entails that a point $(x_1,x_2,\xi_1,0) \in WF^\alpha(K(1 \otimes u))$ if one of the following conditions is satisfied:

- $\exists \xi_2 \in \mathbb{R}^{d_2}$ such that $(x_1,x_2,\xi_1,\xi_2) \in WF^{\alpha_1}(K)$ and $(x_2,-\xi_2) \in WF_0(u)$,
- $\exists \xi_2 \in \mathbb{R}^{d_2}$ such that $(x_1,x_2,\xi_1,\xi_2) \in WF_0(K)$ and $(x_2,-\xi_2) \in WF^{\alpha_2}(u)$.

This concludes the proof. □

At last, we prove a result which should be read as a generalization of Schauder estimates - see [DRS22, Cor. 51]. In particular, the following result is an adaptation to the case in hand of an important result valid in the context of the Sobolev wavefront set , *c.f.* [JS02, Prop. B.9]. We first set

$$WF_{\Omega_2}(K) := \{(x_2,\xi_2) \in \Omega_2 \times \mathbb{R}^{d_2} : \exists x_1 \in \Omega_1,\ (x_1,x_2,0,\xi_2) \in WF(K)\}, \quad (3.3.18)$$

Corollary 3.3.16: *Let $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets, let $\mathcal{K} : \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ be a linear map with Schwartz kernel $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ and let $u \in \mathcal{E}'(\Omega_2)$. In addition, suppose that for any $(x_2,\xi_2) \in \Omega_2 \times (\mathbb{R}^{d_2} \setminus \{0\})$ there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 > 0$ such that*

$$(x_2,\xi_2) \notin -WF^{\alpha_1}_{\Omega_2}(K) \cup WF^{\alpha_2}(u), \quad (3.3.19)$$

where $-WF^{\alpha_1}_{\Omega_2}(K)$ has been defined in Equation (3.3.15). If $WF_{\Omega_2}(K) = \emptyset$ and if there exists $\varepsilon \in \mathbb{R}$ such that $\mathcal{K}(B^\alpha_{\infty,\infty}(\Omega_2) \cap \mathcal{E}'(\Omega_2)) \subset B^{\alpha+\varepsilon,\mathrm{loc}}_{\infty,\infty}(\Omega_1)$ for any $\alpha \in \mathbb{R}$, then it holds true that

$$WF^{\alpha-\varepsilon}(\mathcal{K}u) \subset WF'(K) \circ WF^\alpha(u) \cup WF_{\Omega_1}(K), \quad (3.3.20)$$

where $WF'(K) \circ WF^\alpha(u) := \{(x_1,\xi_1) : \exists (x_2,\xi_2) \in WF^\alpha(u),\ (x_1,x_2,\xi_1,-\xi_2) \in WF(K)\}, WF_{\Omega_1}(K) := \{(x_1,\xi_1) \in \Omega_1 \times \mathbb{R}^{d_1} : \exists x_2 \in \Omega_2, (x_1,x_2,\xi_1,0) \in WF(K)\}$.

Proof. On account of Theorem 3.3.15 combined with Equation (3.3.20), it descends that there exists $\mathcal{K}u \in \mathcal{D}'(\Omega_1)$. Let $V \subset \Omega_1 \times (\mathbb{R}^{d_2} \setminus \{0\})$ be an open conic neighborhood as per Definition 2.2.41. Due to Proposition 3.2.2, we infer that $WF(u - v) \subset V$ for any $v \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\Omega_2)$. Being $\mathcal{K}v \in B^{\alpha-\varepsilon,\mathrm{loc}}_{\infty,\infty}(\Omega_1)$ per assumption and on account of Theorem 2.3.30, it descends that

$$WF^{\alpha-\varepsilon}(\mathcal{K}u) \subset WF(\mathcal{K}(u - v)) \subset WF'(K) \circ V \cup WF_{\Omega_1}(K) \subset WF'(K) \circ V \cup WF_{\Omega_1}(K).$$

In view of the arbitrariness of V, we conclude that

$$WF^{\alpha-\varepsilon}(\mathcal{K}u) \subset WF'(K) \circ WF^{\alpha}(u) \cup WF_{\Omega_1}(K).$$

□

Example 3.3.17: *Let $K \in \mathcal{D}'(\mathbb{R}^{1+d} \times \mathbb{R}^{1+d})$ be the fundamental solution of the heat equation, whose integral kernel is*

$$K(t_1, x_1, t_2, x_2) = \frac{\Theta(t_1 - t_2)}{(4\pi(t_1 - t_2))^{d/2}} e^{-\frac{|x_1 - x_2|^2}{4(t_1 - t_2)}} \quad (t_1, x_1, t_2, x_2) \in \mathbb{R}^{1+d} \times \mathbb{R}^{1+d},$$

where Θ is the Heaviside function. On account of Schauder estimates, K is the kernel of a linear map $\mathcal{K}\colon B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^{1+d}) \to B^{\alpha+2,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^{1+d})$, defined by $\mathcal{K}u := K * u$, where $*$ denotes the convolution as per Definition A.7.1. In addition, since the heat operator is hypoelliptic, it holds true that

$$WF(K) = \{(t, x, t, x, \tau, \xi, -\tau, -\xi) : (t, x) \in \mathbb{R}^{1+d},\ (\tau, \xi) \in \mathbb{R}^{1+d} \setminus \{0\}\}, \tag{3.3.21}$$

see [Hör90, Sect. 11.1]. On account of Equation (3.3.21), it descends that $WF^{\alpha_1}_{\mathbb{R}^{1+d}}(K) = \emptyset$ for any $\alpha_1 \in \mathbb{R}$, where the subscript $\mathbb{R}^{d+1}$ should be read as in Equation (3.3.18). Moreover, given $u \in \mathcal{E}'(\mathbb{R}^{1+d})$, Example 3.1.12 entails that there exists $\alpha_2 < 0$ such that $WF^{\alpha_2}(u) = \emptyset$. Then, we are in position to apply Corollary 3.3.16. Therefore, on account of Equation 3.3.20, it descends that

$$WF^{\alpha+2}(\mathcal{K}u) \subset WF'(K) \circ WF^{\alpha}(u).$$

Observing that $WF'(K) \circ WF^{\alpha}(u) = WF^{\alpha}(u)$, we conclude that

$$WF^{\alpha+2}(\mathcal{K}u) \subset WF^{\alpha}(u).$$

3.3.5 Hyperbolic Partial Differential Equations

In Subsection 2.3.6, we discussed the propagation of singularities for a large class of first order hyperbolic partial differential equations in the context of the smooth wavefront set. The main theorem characterizes the smooth wavefront set of a solution to a partial differential equation in terms of the principal symbol of the corresponding differential operator. More precisely, it asserts that the singularities propagate along the flow induced by the principal symbol, which is read as a Hamiltonian function - see Theorem 2.3.32. The aim of this subsection is to discuss the same problem in the framework of the Besov wavefront set as per Definition 3.1.1 as well as Theorem 3.2.1. In particular, we shall prove a propagation of singularities theorem which adapts to the case in hand Theorem 2.3.32. Since the Besov wavefront set is a refinement of its smooth counterpart, our result entails a more refined characterization of the singularities of a solution to an hyperbolic partial differential equation. The formulation of the Besov wavefront set in terms of pseudodifferential operators as per Theorem 3.2.1 shall play a prominent rôle in the proof of our propagation of singularities result - see Theorem 3.3.19. In the following, we shall make use of the notions introduced in the previous parts, especially in Subsection 2.1.1 and in Section 2.2. We shall mainly refer to [DRS22, Subsect. 4.1].

In what follows, all pseudodifferential operators shall be assumed to be properly supported as per Definition 2.2.25. Let $A \in \Psi^1(\mathbb{R}^d)$ as per Definition 2.2.15 be such that its full symbol $a \in S^1(\mathbb{R}^d; \mathbb{R}^d)$ as per Definition 2.2.1 is independent of the spatial component, *i.e.* $a = a(\xi)$. In addition, we assume that the principal symbol of A, denoted by $\sigma_1(A)$, is real-valued and it lies in $S^1_{\text{hom}}(\mathbb{R}^d; \mathbb{R}^d)$ as per Definition 2.2.6. Resorting to the notion of Besov wavefront set, we wish to analyze the microlocal behavior of the distributional solution $u \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$ to an initial value problem of the form

$$\begin{cases} \partial_t u = iAu, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.3.22}$$

where $u_0 \in \mathcal{D}'(\mathbb{R}^d)$. Here, we denote by $u_0(x)$ and $u(t,x)$ the integral kernels of u_0 and u respectively as per Remark A.8.2. Let $K \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$ be the fundamental solution for the operator $\partial_t - iA$ such that its integral kernel $K(t,x)$ satisfies

$$\begin{cases} (\partial_t - iA)K(t,x) = \delta(t)\delta(x), & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ K(0,x) = \delta(x), & x \in \mathbb{R}^d. \end{cases} \tag{3.3.23}$$

Exploiting standard Fourier methods, it turns out that the integral kernel of $K \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$ is

$$K(t,x) = \Theta(t)[e^{itA}\delta](x) \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{3.3.24}$$

where Θ is the Heaviside function. We prove a regularity result concerning the fundamental solution K - [DRS22, Prop. 54].

Proposition 3.3.18: *Let $K \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$ be the fundamental solution of the operator $\partial_t - iA$ as per Equation (3.3.24). Then, $K(t,\cdot) \in B^{-\frac{d}{2}}_{2,\infty}(\mathbb{R}^d)$ for any $t \in \mathbb{R}$. In addition, for any $v \in B^{\alpha,\text{loc}}_{\infty,\infty}(\mathbb{R}^d)$ with $\alpha \in \mathbb{R}$,*

$$K(t,\cdot) * v \in B^{\alpha - \frac{d}{2},\text{loc}}_{\infty,\infty}(\mathbb{R}^d),$$

where $*$ stands for the convolution introduced in Definition A.7.1.

Proof. Let $\{\psi_j\}_{j \geq 0}$ be a Littlewood-Paley partition of unity as per Definition 2.1.2. On account of Plancherel's theorem (Theorem A.11.10), it descends that

$$\|\psi_j(D_x)e^{itA}\delta\|_{L^2(\mathbb{R}^d)} = \|\psi_j\|_{L^2(\mathbb{R}^d)} = 2^{j\frac{d}{2}}\|\psi\|_{L^2(\mathbb{R}^d)} \quad \forall j \geq 1.$$

Therefore, we infer that

$$\sup_{j \geq 0} 2^{-j\frac{d}{2}}\|\psi_j(D_x)e^{itA}\delta\|_{L^2(\mathbb{R}^d)} < \infty.$$

On account of Definition 2.1.4, it descends that $K(t,\cdot) \in B^{-\frac{d}{2}}_{2,\infty}(\mathbb{R}^d)$ for any $t \in \mathbb{R}$.

We focus on the second part of the statement. Let $v \in B^{\alpha,\text{loc}}_{\infty,\infty}(\mathbb{R}^d)$ with $\alpha \in \mathbb{R}$. On account of Definition 2.1.4, it holds true that $\phi v \in B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$ for every $\phi \in \mathcal{D}(\mathbb{R}^d)$. Therefore, on account of Theorem 2.1.8, we conclude that $K(t,\cdot) * (\phi v) \in B^{\alpha - \frac{d}{2}}_{\infty,\infty}(\mathbb{R}^d)$ for any $t \in \mathbb{R}$. □

Proposition 3.3.18 asserts that the solution map associated to Equation (3.3.22),

$$\mathscr{S}(t,0)\colon u_0 \mapsto u(t) \quad t \in \mathbb{R},$$

is continuous from $B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ to $B^{\alpha-\frac{d}{2},\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. In addition, $\mathscr{S}(t,0)$ is invertible and $\mathscr{S}^{-1}(0,t) = \mathscr{S}(0,t)$. As mentioned in Subsection 2.3.6, we recall that, when analyzing the problem of the propagation of singularities, $\sigma_1(A)$ is interpreted as a Hamiltonian function. Therefore, $\sigma_1(A)$ determines a unique Hamiltonian vector field given by

$$X_{\sigma_1(A)}|_{(x,\xi)} = \sum_{j=1}^{d} \partial_{\xi_j}\sigma_1(A)(\xi)\partial_{x_j}|_{(x,\xi)}, \quad \forall (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{3.3.25}$$

Let $\rho : \mathbb{R} \to \mathbb{R}^d \times \mathbb{R}^d$, $t \mapsto \rho(t) := (x(t),\xi(t))$, be the integral curve of $X_{\sigma_1(A)}$ such that $\rho(0) = (x_0,\xi_0)$. Then, the map

$$\Phi\colon \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d\times\mathbb{R}^d \quad \text{s.t.} \quad (t,x_0,\xi_0) \mapsto \Phi_t(x_0,\xi_0) := \rho(t),$$

is the Hamiltonian flow associated with $X_{\sigma_1(A)}$. After this premise, we are in position to prove a propagation of singularities theorem for an initial value problem as per Equation (3.3.22) in the framework of the Besov wavefront set - see [DRS22, Th. 55].

Theorem 3.3.19: *Let $A \in \Psi^1(\mathbb{R}^d)$ be such that its principal symbol $\sigma_1(A)$ lies in $S^1_{\mathrm{hom}}(\mathbb{R}^d;\mathbb{R}^d)$ as per Definition 2.2.6 and it is real-valued. In addition, let $u_0 \in \mathcal{D}'(\mathbb{R}^d)$ and let $u \in \mathcal{D}'(\mathbb{R}\times\mathbb{R}^d)$ be the solution of the initial value problem*

$$\begin{cases} \partial_t u = iAu, & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{3.3.26}$$

Then, for any $\alpha \in \mathbb{R}$,

$$WF^{\alpha-\frac{d}{2}}(u(t)) = \Phi_t WF^{\alpha}(u_0), \tag{3.3.27}$$

where Φ_t is the flow from t to 0 associated with $X_{\sigma_1(A)}$ while we set

$$\Phi_t WF^\alpha(u_0) := \{\Phi_t(x,\xi) \in \mathbb{R}^d \times (\mathbb{R}^d\setminus\{0\}) : (x,\xi) \in WF^\alpha(u_0)\}, \quad \forall t \in \mathbb{R}.$$

Proof. It suffices to prove the inclusion $\subset$. The other one follows inverting the time direction. Let $(x_0,\xi_0) \notin WF^\alpha(u_0)$. On account of Theorem 3.2.1, there exists $A \in \Psi^0(\mathbb{R}^d)$ such that $Au_0 \in B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. If we set $A(t) := \mathscr{S}(t,0)\circ A \circ \mathscr{S}(0,t)$, then Proposition 3.3.18 entails that $A(t)u(t) = \mathscr{S}(t,0)Au_0 \in B^{\alpha-\frac{d}{2},\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. On account of Egorov's theorem (Theorem 2.3.31), we infer that $A(t)$ lies in $\Psi^0(\mathbb{R}^d)$ and it is elliptic at $\Phi_t^{-1}(x_0,\xi_0)$. Therefore, Theorem 3.2.1 entails that $\Phi_t^{-1}(x_0,\xi_0) \notin WF^{\alpha-\frac{d}{2}}(u(t))$. □

Remark 3.3.20: *It is worth pointing out that the estimate on the Besov wavefront set in Equation (3.3.27) is not optimal. As a matter of fact, it might be improved if we established a more refined regularity of K in Proposition 3.3.18, which seems to be difficult to achieve at this stage.*

Example 3.3.21: *We consider the initial value problem in Equation 3.3.26 with $A = D_x \in \Psi^1(\mathbb{R})$, where $D_x := -i\partial_x$. Therefore, the principal symbol of A is $\sigma_1(A)(\xi) = \xi$. In this case, given $(x_0,\xi_0) \in \mathbb{R}\times\mathbb{R}$, the Hamilton equations read*

$$\begin{cases} \frac{dx(t)}{dt} = 1 & t \in \mathbb{R}, \\ \frac{d\xi(t)}{dt} = 0 & t \in \mathbb{R}, \\ (x(0),\xi(0)) = (x_0,\xi_0). \end{cases} \tag{3.3.28}$$

As a result, we infer that $(x(t), \xi(t)) = (x_0 + t, \xi_0)$*, that is to say* $\Phi_t(x_0, \xi_0) = (x_0 + t, \xi_0)$*. On account of Theorem 3.3.19, it descends that, for any* $\alpha \in \mathbb{R}$*,*

$$WF^{\alpha-\frac{d}{2}}(u(t)) = \{(x + t, \xi) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\}) : (x, \xi) \in WF^{\alpha}(u_0)\} \quad \forall t \in \mathbb{R}.$$

Example 3.3.22: *In Equation 3.3.26, we consider* $A \in \Psi^1(\mathbb{R}^d)$ *such that* $A := \langle D_x \rangle$ *as per Example 2.2.17. Observe that* $\partial_t - i\langle D_x \rangle$ *arises from the factorization*

$$\partial_t^2 - (\Delta + 1) = (\partial_t - i\langle D_x \rangle)(\partial_t + i\langle D_x \rangle),$$

where Δ *denotes the Laplace operator on* $\mathbb{R}^d$*. In this case, the principal symbol of* A *is* $\sigma_1(A)(\xi) = |\xi|$*. Taking into account that*

$$X_{\sigma_1(A)}|_{(x,\xi)} = |\xi|^{-1} \sum_{i=1}^{d} \xi_j \partial_{x_j}|_{(x,\xi)} \quad \forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d,$$

we deduce that $\Phi_t(x_0, \xi_0) := (|\xi_0|^{-1}\xi_0 t + x_0, \xi_0)$*, where* $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$*. On account of Theorem 3.3.19, it descends that, for any* $\alpha \in \mathbb{R}$*,*

$$WF^{\alpha-\frac{d}{2}}(u(t)) = \{(|\xi|^{-1}\xi t + x, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) : (x, \xi) \in WF^{\alpha}(u_0)\} \quad \forall t \in \mathbb{R}.$$

3.4 Application to Coherent Germs of Distributions

In Section 3.1, we introduced the notion of Besov wavefront set, which aims at characterizing all the directions in Fourier space along which an underlying distribution lies in a suitable Besov space $B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$ as per Definition 3.1.1. In Subsection 3.3.2, resorting to the notion of Besov wavefront set, we established a sufficient condition to extend the pullback map along an embedding to the whole space of distributions - see Theorem 3.3.5. As a result, since the product between two distributions can be defined as the pullback of their tensor product along the diagonal, this condition entails a criterion, analogous to the Hörmander one in the smooth setting, for the existence of the product - see Subsection 2.3.3. This is the content of Theorem 3.3.10, which can be read as a microlocal formulation of Young's product theorem (Theorem 2.1.20). In this section, we present an application of these results in the context of coherent germs of distributions as per Definitions 2.4.4 and 2.4.12. In particular, we shall consider coherent germs defined as the tensor product of two Besov distributions. By applying Theorems 3.3.5 and 3.3.10, we shall prove that the reconstruction of such a germ amounts to the product between the two distributions at hand, which coincides in turn with the pullback of the germ along the diagonal. In addition, Theorem 3.3.10, particularly Equation (3.3.12), entails an estimate of the Besov regularity of the reconstruction. As a result, since a coherent germ is an element lying in $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ (see Remark 2.4.2), we can conjecture that its reconstruction coincides with the pullback of the germ along the diagonal.

In the following, by applying Theorems 3.3.5 and 3.3.10, we provide an alternative proof of Theorem 2.4.25 when $\alpha_1 + \alpha_2 > 0$. The proof of the following result is inspired by those of [DRS21, Prop. 45, Prop. 30].

Proposition 3.4.1: *Let* $\Omega \subset \mathbb{R}^d$ *be an open set, let* $\alpha_1 \in (0, \infty) \setminus \mathbb{N}$ *and let* $\alpha_2 < 0$ *be such that* $\alpha_1 + \alpha_2 > 0$*. Let* $u \in B^{\alpha_1,\mathrm{loc}}_{\infty,\infty}(\Omega)$ *and let* $v \in B^{\alpha_2,\mathrm{loc}}_{\infty,\infty}(\Omega)$ *as per Definition 2.1.30. Then* $F = (F_x)_{x\in\Omega}$*, where*

$$F_x(\cdot) := (P_x v)(\cdot) = \sum_{|\ell| \leq \lfloor \alpha_1 \rfloor} \frac{\partial^\ell u(x)}{\ell!} (\cdot - x)^\ell v(\cdot) \quad \forall x \in \Omega, \tag{3.4.1}$$

is a $(\alpha_2, \alpha_1+\alpha_2)$-coherent germ, whose reconstruction as per Theorem 2.4.16 is $\mathcal{R}F = uv$. In addition, uv lies in $B^{\alpha_2;\mathrm{loc}}_{\infty,\infty}(\Omega)$ and, for any compact set $\mathfrak{K}\subset\Omega$, it holds true that

$$\|uv\|_{B^{\alpha_2}_{\infty,\infty}(\mathfrak{K})} \lesssim \|u\|_{B^{\alpha_1}_{\infty,\infty}(\overline{\mathfrak{K}}_1)}\|v\|_{B^{\alpha_2}_{\infty,\infty}(\overline{\mathfrak{K}}_1)}, \tag{3.4.2}$$

where $\overline{\mathfrak{K}}_1$ is the 1-enlargement of $\mathfrak{K}$ as per Equation 2.4.1.

Proof. On account of Proposition 2.4.24, the germ F is $(\alpha_2, \alpha_1+\alpha_2)$-coherent as per Definition 2.4.4. Being $\alpha_1+\alpha_2>0$, on account of the reconstruction theorem, *c.f* Theorem 2.4.16, it descends that there exists a unique $\mathcal{R}F\in\mathcal{D}'(\Omega)$ satisfying the bound in Equation (2.4.8) with $\gamma=\alpha_1+\alpha_2$.

Per hypotheses, Proposition 3.1.5 entails that $WF^{\alpha_1}(u) = WF^{\alpha_2}(v)=\emptyset$. Therefore, on account of Theorem 2.3.24, it descends that there exists a unique product $uv\in\mathcal{D}'(\Omega)$, defined as $uv := \boldsymbol{\delta}^*(u\otimes v)$ where $\boldsymbol{\delta}^*$ is the pullback along the diagonal map $\boldsymbol{\delta}\colon\Omega\to\Omega\times\Omega$, $\boldsymbol{\delta}(x)=(x,x)$. In addition, it holds true that $(uv)(\varphi)=v(u\varphi)$ for any $\varphi\in\mathcal{D}(\mathbb{R}^d)$. On account of Equation (3.3.12), we infer that $WF^{\alpha_2}(uv)=\emptyset$, that is $uv\in B^{\alpha_2;\mathrm{loc}}_{\infty,\infty}(\Omega)$. We show that uv is the reconstruction of F. Let $\mathfrak{K}\subset\Omega$ be a compact set. On account of Theorem 2.1.37 and of Definition 2.1.34, we recall that

$$u(y) = P_x(y) + R(x,y) \quad \forall x,y\in\mathfrak{K},$$

where the reminder $R(x,y)$ is such that

$$|R(x,y)| \le \|u\|_{B^{\alpha_1}_{\infty,\infty}(\mathfrak{K})}|x-y|^{\alpha_1},$$

uniformly for $x,y\in\mathfrak{K}$. In addition, R can be written as

$$R(x,y) = \sum_{|\ell|=\lfloor\alpha_1\rfloor} f_\ell(y)(y-x)^\ell, \tag{3.4.3}$$

where f_ℓ is such that

$$\lim_{y\to x}\frac{|f_\ell(y)|}{|y-x|^{\alpha_1-\lfloor\alpha_1\rfloor}} = C_u \le \|u\|_{B^{\alpha_1}_{\infty,\infty}(\mathfrak{K})}.$$

As a result, using Equation (3.4.3), it holds true that

$$|(uv-P_xv)(\phi^\lambda_x)| = |v((u-P_x)\phi^\lambda_x)| = |v(R(x,\cdot)\phi^\lambda_x)| \le \sum_{|\ell|=\lfloor\alpha_1\rfloor}|v(f_\ell(\cdot)(\cdot-x)^\ell\phi^\lambda_x)|. \tag{3.4.4}$$

Setting $\tilde\phi(y) = y^\ell\phi(y)$, it holds true that

$$(y-x)^\ell\phi^\lambda_x(y) = \lambda^{|\ell|}\tilde\phi^\lambda_x(y).$$

Therefore, we infer that

$$\begin{aligned}
|(uv-P_xv)(\phi^\lambda_x)| &\le \sum_{|\ell|=\lfloor\alpha_1\rfloor}|v(f_\ell(\cdot)(\cdot-x)^\ell\phi^\lambda_x)| = \sum_{|\ell|=\lfloor\alpha_1\rfloor}\lambda^{|\ell|}|v(f_\ell(\cdot)\tilde\phi^\lambda_x(y))|\\
&= \lambda^{\lfloor\alpha_1\rfloor}\sum_{|\ell|=\lfloor\alpha_1\rfloor}|v(f_\ell(\cdot)\tilde\phi^\lambda_x(y))| = \lambda^{\lfloor\alpha_1\rfloor}\sum_{|\ell|=\lfloor\alpha_1\rfloor}\sum_{|k|=\alpha_1-\lfloor\alpha_1\rfloor}|v(C_u(\cdot-x)^k\tilde\phi^\lambda_x)|\\
&\overset{\eta(y)=y^k\tilde\phi(y)}{\lesssim} \lambda^{\alpha_1}\|u\|_{B^{\alpha_1}_{\infty,\infty}(\overline{\mathfrak{K}}_1)}|v(\eta^\lambda_x)| \lesssim \|u\|_{B^{\alpha_1}_{\infty,\infty}(\overline{\mathfrak{K}}_1)}\|v\|_{B^{\alpha_2}_{\infty,\infty}(\overline{\mathfrak{K}}_1)}\lambda^{\alpha_1+\alpha_2},
\end{aligned} \tag{3.4.5}$$

uniformly for $x \in \mathfrak{K}$, $\lambda \in (0,1]$ and $\phi \in \mathcal{D}(B(0,1))$, where in the last inequality we exploited that $v \in B^{\alpha_2;\mathrm{loc}}_{\infty,\infty}(\Omega)$ as per Definition 2.1.30 as well as Proposition 2.1.28. This proves that $\mathcal{R}F = uv$.

It remains to be proven Equation (3.4.2). By the triangle inequality, for any $x \in \mathfrak{K}$, $\phi \in \mathcal{D}(B(0,1))$ and $\lambda \in (0,1]$, it holds true that

$$|(uv)(\phi^\lambda_x)| \leq |v(u\phi^\lambda_x)| \leq \underbrace{|v(P_x(\cdot)\phi^\lambda_x)|}_{|A|} + \underbrace{|v(R(x,\cdot)\phi^\lambda_x)|}_{|B|}.$$

Since $|B|$ has been already estimated as per Equation (3.4.5), we focus on $|A|$. On account of the triangle inequality and of Proposition 2.1.28, it descends that

$$\begin{aligned} |A| &\leq \sum_{|\ell|\leq\lfloor\alpha_1\rfloor} \frac{|\partial^\ell u(x)|}{\ell!} |v((\cdot - x)^\ell \phi^\lambda_x)| \overset{\eta(y):=y^\ell\phi(y)}{=} \sum_{|\ell|\leq\lfloor\alpha_1\rfloor} \lambda^{|\ell|} \frac{|\partial^\ell u(x)|}{\ell!} |v(\eta^\lambda_x)| \\ &\lesssim \|v\|_{B^{\alpha_2}_{\infty,\infty}(\overline{\mathfrak{K}}_1)} \sum_{|\ell|\leq\lfloor\alpha_1\rfloor} \frac{\|\partial^\ell u\|_{L^\infty(\mathfrak{K})}}{\ell!} \lambda^{\alpha_2+|\ell|} \lesssim \|u\|_{B^{\alpha_1}_{\infty,\infty}(\overline{\mathfrak{K}}_1)} \|v\|_{B^{\alpha_2}_{\infty,\infty}(\overline{\mathfrak{K}}_1)} \lambda^{\alpha_2}, \end{aligned} \tag{3.4.6}$$

uniformly for $x \in \mathfrak{K}$, $\lambda \in (0,1]$ and $\phi \in \mathcal{D}(B(0,1))$, where the last inequality descends from $\|\partial^\ell u\|_{L^\infty(\mathfrak{K})} \lesssim \|u\|_{B^{\alpha_1}_{\infty,\infty}(\overline{\mathfrak{K}}_1)}$. As a result, combining the estimates for $|A|$ and $|B|$, it descends that

$$|(uv)(\phi^\lambda_x)| \lesssim \|u\|_{B^{\alpha_1}_{\infty,\infty}(\overline{\mathfrak{K}}_1)} \|v\|_{B^{\alpha_2}_{\infty,\infty}(\overline{\mathfrak{K}}_1)} (\lambda^{\alpha_2} + \lambda^{\alpha_1+\alpha_2}) \lesssim \|u\|_{B^{\alpha_1}_{\infty,\infty}(\overline{\mathfrak{K}}_1)} \|v\|_{B^{\alpha_2}_{\infty,\infty}(\overline{\mathfrak{K}}_1)} \lambda^{\alpha_2},$$

uniformly for $x \in \mathfrak{K}$, $\phi \in \mathcal{D}(B(0,1))$ and $\lambda \in (0,1]$. This concludes the proof of the statement. □

Remark 3.4.2: *Let $\Omega \subset \mathbb{R}^d$ be an open set. Let u, v be such that they satisfy the hypotheses of Proposition 3.4.1. We denote by $u(x)$ and $v(x)$ the integral kernels of u and v respectively. Let $F \in \mathcal{D}'(\Omega \times \Omega)$ be the germ as per Proposition 3.4.1, whose integral kernel as per Remark A.8.2 is given by*

$$F(x,y) \equiv F_x(y) = \sum_{|\ell|\leq\lfloor\alpha_1\rfloor} \frac{\partial^\ell u(x)}{\ell!} (y-x)^\ell v(y), \quad \forall x, y \in \Omega.$$

Then, we observe that

$$\mathcal{R}F(x) = (\boldsymbol{\delta}^* F)(x) = (F \circ \boldsymbol{\delta})(x) = F(x,x) = u(x)v(x),$$

where $\mathcal{R}F(x)$ denotes the integral kernel of the reconstruction of F as per Remark A.8.2 while $\boldsymbol{\delta}^*$ is the pullback map along $\boldsymbol{\delta}\colon \Omega \to \Omega \times \Omega$, $\boldsymbol{\delta}(x) = (x,x)$. In addition, Theorem 3.3.5 provides an estimate of the Besov wavefront set of $\mathcal{R}F$.

On account of Proposition 3.4.1 and of Remark 3.4.2, we formulate the following conjecture.

Conjecture 3.4.3: *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $F \in \mathcal{D}'(\Omega \times \Omega)$ be a γ-coherent germ with $\gamma > 0$. Then, the unique reconstruction of F is $\mathcal{R}F = \boldsymbol{\delta}^* F$, where $\boldsymbol{\delta}^*$ is the pullback along $\boldsymbol{\delta}\colon \Omega \to \Omega \times \Omega$, $\boldsymbol{\delta}(x) = (x,x)$. In addition, for any $\alpha \in \mathbb{R}$, it holds true that*

$$WF^\alpha(\mathcal{R}F) \subset \boldsymbol{\delta}^* WF^\alpha(F),$$

where $\boldsymbol{\delta}^* WF^\alpha(F)$ is defined as per Equation (3.3.7).

Remark 3.4.4: *Let $\gamma \leq 0$ and let $F \in \mathcal{D}'(\Omega \times \Omega)$ be a γ-coherent germ such that satisfies the hypothesis of Theorem 3.3.5. On account of Theorem 2.4.16, the reconstruction of F is non-unique. Similarly to the case of positive coherence, we can conjecture that $\boldsymbol{\delta}^* F$ is a choice of reconstruction of F. Therefore, Theorem 3.3.5 provides a rationale to choose a distinguished reconstruction of F in cases where Theorem 2.4.16 fails to do so. For further details, the reader may refer to [DRS21].*

Chapter 4

Reconstruction theorem on smooth manifolds

Martin Hairer's reconstruction theorem is one of the main results of the theory of regularity structures (see [Hai14, Hai15]), a novel framework to investigate the well-posedness of a specific class of nonlinear stochastic partial differential equations (SPDEs) on the Euclidean setting.
Stochastic partial differential equations are closely related to quantum field theory (QFT), in particular to *stochastic quantization* [PW81]. As a matter of fact, the idea at the heart of stochastic quantization is to give meaning to the path integral formulation of an Euclidean quantum field theory by means of an invariant measure of a nonlinear SPDE. The interaction between SPDEs and quantum field theory has been also strenghtened by a few recent works [DDRZ21, BDR21], where techniques coming from the latter, such as renormalization, have been applied within the SPDEs solution theory. On account of the formulation of QFT on curved backgrounds, see e.g. [BFDY15, BF09, FR16, JS02, BF00], the extension of the theory of regularity structures on smooth manifolds would be crucial to further strengthen this interplay. To this end, an initial step should be the formulation of the reconstruction theorem on an arbitrary smooth manifold. In a recent joint work with Paolo Rinaldi [RS21], we extended this result to smooth manifolds relying on the theory of germs of distributions [CZ20]. As explained in Section 2.4, this framework allows to formulate and to prove the reconstruction theorem in the language of theory of distributions on the Euclidean space $\mathbb{R}^d$, without any reference to regularity structures. More precisely, the authors deal with the following problem: if for any $x \in \mathbb{R}^d$ we are given a distribution $F_x \in \mathcal{D}'(\mathbb{R}^d)$, we wonder whether there exists $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ which is locally approximated by F_x around each $x \in \mathbb{R}^d$. For instance, given a smooth function $f\colon \mathbb{R}^d \to \mathbb{R}$, if $F_x(\cdot)$ is its Taylor polynomial of order $m \in \mathbb{N}$ centered at $x \in \mathbb{R}^d$, then the sought global distribution is $\mathcal{R}F(x) = f(x)$. As a matter of fact, since $|f(y) - F_x(y)| \lesssim |y-x|^{m+1}$ for $y \in \mathbb{R}^d$ close to x, f is well-approximated by F_x in a neighborhood of x. However, if the local approximations F_x are singular objects, the solution to the problem is somewhat involved. In this situation, under a further assumption on the family $(F_x)_{x\in\mathbb{R}^d}$, dubbed *coherence*, the reconstruction theorem entails the existence of the desired global distribution.

Since the notion of distribution is local, it can be easily generalized to any smooth manifold - see Definition A.10.1. For this reason, since the framework introduced in [CZ20] is completely based on distribution theory, it shall turn out to be well-suited for the generalization of the reconstruction theorem to the case of distributions on smooth manifolds as shown in [RS21]. We stress that our results hold true at the level of smooth manifold without calling for further structures, such as a Riemannian one. Observe that, in [DDK19], the reconstruction theorem has been formulated within the Riemannian setting.

Contents. This chapter is divided in two sections.

In Section 4.1, we extend the notion of germ of distribution and the notion of coherence to the smooth manifold setting. In addition, we also formulate enhanced coherence on a smooth manifold. Lastly, we prove that the notion of coherence is independent of the choice of the atlas - see Proposition 4.1.14. This entails that coherence is a geometric notion.

In Section 4.2, we prove that the reconstruction theorem for coherent germs of distributions on a smooth manifold. If the coherence exponent is strictly positive, we show that the reconstruction is independent of the choice of the atlas - see Theorems 4.2.1 and 4.2.2. Otherwise if the coherence exponent is non-positive, similarly to the Euclidean case, see Remark 2.4.17, we prove that the reconstruction is non-unique and, in particular, it depends on the choice of the atlas and of the partition of unity used to construct it - see Theorem 4.2.4.

Notations. We shall denote by $\lesssim$ an inequality holding true up to a multiplicative constant. We denote by M a d-dimensional connected smooth manifold without boundary. In addition, we denote by (U, h) a local chart on U, where $U \subset M$ is an open set and $h\colon U \to h(U) \subset \mathbb{R}^d$ is a diffeomorphism. The pullback along h^{-1} is denoted by $(h^{-1})^*$. Given an open set $\Omega \subset \mathbb{R}^d$, a function $f\colon \Omega \to \mathbb{R}$ and a point $x \in \Omega$, we recall that $f_x^\lambda\colon \Omega \to \mathbb{R}^d$ denotes the rescaled version of f, defined as

$$f_x^\lambda(y) := \lambda^{-d} f(\lambda^{-1}(y - x)), \quad y \in \Omega,$$

for $\lambda \in (0, 1]$. Moreover, we denote by $B(0, 1)$ the unit open ball in $\mathbb{R}^d$ centered at the origin.

4.1 Germs of distributions on smooth manifolds

Stochastic partial differential equations (SPDEs) have become increasingly important in applications ranging from physics to finance. For instance, they play a prominent rôle in quantum field theory, in particular within stochastic quantization [PW81]. However, their analysis presents several mathematical challenges. In the last few years, a breakthrough in their study has been made thanks to Hairer's paper on the theory of regularity structures - see [Hai14, Hai15]. This novel framework allows to solve a certain class of nonlinear SPDEs on the Euclidean space $\mathbb{R}^d$ by means of a fixed point argument. Always having in mind applications of SPDEs in physics, on account of quantum field theory on curved backgrounds, see e.g. [BFDY15, BF09, FR16], it would be desiderable to formulate the theory of regularity structures on a more general geometrical ground. As a first step in this direction, we wish to extend Hairer's reconstruction theorem, one of the cornerstones of this theory, to an arbitrary smooth manifold - see Subsection 4.2. To this end, we shall rely on the theory of germs of distributions as outlined in Section 2.4 - see [CZ20]. As a matter of fact, this framework allows to formulate and to prove the reconstruction theorem in the language of the theory of distributions on the Euclidean space $\mathbb{R}^d$, without any reference to regularity structures. On account of the local nature of distributions, their definition can be easily extended to any smooth manifold - see Appendix A.10. As a result, the approach adopted in [CZ20] turns out to be well suited to our purposes. The following discussion is mainly inspired by [RS21].

In this subsection, we generalize the notion of germ of distributions and the notion of coherence to arbitrary smooth manifolds. Subsequently, in Subsection 4.2 we shall prove the reconstruction theorem on a smooth manifold. We point out that the following discussion holds true at the level of smooth manifolds, without resorting to any further structure, such as a Riemannian one. In what follows, we shall make use of the notions introduced in Section 2.4 and Appendix B.

We start by introducing the notion of germ of distributions on a smooth manifold.

Definition 4.1.1: *Let M be a d-dimensional smooth manifold. We say that a family $F = (F_{\mathsf{p}})_{\mathsf{p}\in M}$ of distributions, $F_{\mathsf{p}} \in \mathcal{D}'(M)$ as per Definition A.10.1 for any $\mathsf{p} \in M$, is a **germ** on M if, for any $\varphi \in \mathcal{D}(M)$ as per Definition A.1.7, the map $\mathsf{p} \mapsto F_{\mathsf{p}}(\varphi)$ is measurable with respect to the Borel σ-algebra of M.*

Remark 4.1.2: *Let M be a d-dimensional smooth manifold. A germ F on M can be read as an element lying in $\mathcal{D}'(M \times M)$, whose integral kernel $F(\mathsf{p},\mathsf{q}) \equiv F_{\mathsf{p}}(\mathsf{q})$ is such that the map $\mathsf{p} \mapsto \langle F_{\mathsf{p}}(\mathsf{q}), \varphi(\mathsf{q})\rangle$ is measurable with respect to the Borel σ-algebra of M for any $\varphi \in \mathcal{D}(M)$.*

Analogously to the Euclidean case, the idea at the heart of the notion of germ is that $F = (F_{\mathsf{p}})_{\mathsf{p}\in M}$ can be thought of as a collection of local approximations for a global distribution on M. As we shall see in Section 4.2, under the assumption of coherence of the germ, the existence of such global distribution is guaranteed. This is the content of the reconstruction theorem - see Theorem 4.2.1. For this reason, we extend the notion of coherent germ to the manifold setting. Since, as explained in Appendix B, the notion of coherence on an Euclidean space is local, relying on Definition B.0.2, we define its counterpart on a smooth manifold.

Definition 4.1.3: *Let M be a d-dimensional smooth manifold, let $\mathcal{A} = \{(U_i, h_i)\}_{i\in I}$ be a smooth atlas thereon and let $\gamma \in \mathbb{R}$. In addition, let $F = (F_{\mathsf{p}})_{\mathsf{p}\in M}$ be a germ of distributions on M as per Definition 4.1.1. The germ F is said to be γ-**coherent** on $(M,\mathcal{A})$ if for any $(U,h) \in \mathcal{A}$ there exists $\underline{\kappa} \in \mathcal{D}(h(U))$ with $\check{\underline{\kappa}}(0) \neq 0$ such that for any compact set $\mathfrak{K} \subset U$ there exists $\zeta^U_{\mathfrak{K}} \leq \min\{0,\gamma\}$ such that*

$$|((h^{-1})^*(F_{\mathsf{p}}) - (h^{-1})^*(F_{\mathsf{q}}))(\underline{\kappa}^{\lambda}_{h(\mathsf{q})})| \lesssim \lambda^{\zeta^U_{\mathfrak{K}}}(|h(\mathsf{p}) - h(\mathsf{q})| + \lambda)^{\gamma-\zeta^U_{\mathfrak{K}}}, \tag{4.1.1}$$

uniformly for $\mathsf{p},\mathsf{q} \in \mathfrak{K}$ and $\lambda \in (0,1]$, where $(h^{-1})^*$ is the pullback along h^{-1} as per Remark A.9.3.

Remark 4.1.4: *The previous definition seems to imply that the concept of coherent germ depends on the choice of the atlas. However, in Proposition 4.1.14 we shall prove that coherence is actually independent of such choice.*

Remark 4.1.5: *On account of Definition B.0.2, the supremum among all possible values of λ is $D_{\mathfrak{K}}/4$, where $D_{\mathfrak{K}} := \mathrm{dist}(\partial h(U), \mathfrak{K})$ while $\partial h(U)$ denotes the boundary of the open set $h(U)$. In Appendix B, we adopted this choice in order to prove enhanced coherence on an open set. However, since all bounds are established up to a multiplicative constant and the scaling operation is implicitly studied in the limit as $\lambda \to 0^+$, we can impose the constraint $\lambda \in (0,1]$ in Definition 4.1.3.*

Remark 4.1.6: *Using the language of regularity structures, a coherent germ as per Definition 4.1.3 can be read as a modelled distribution on a smooth manifold.*

In order to weaken the dependence of the notion of coherence on the atlas, we first prove that it is independent of the coordinates - see [RS21, Prop 7].

Proposition 4.1.7: *Let $F = (F_{\mathsf{p}})_{\mathsf{p}\in M}$ be a γ-coherent germ on $(M,\mathcal{A})$ as per Definition 4.1.3 and let $U \subset M$ be an open set. Then the γ-coherence condition in Equation (4.1.1) is independent of the choice of the local chart on U. In addition, the family of exponents $\boldsymbol{\zeta}^U = (\zeta^U_{\mathfrak{K}})$ is also independent of the choice of the coordinates.*

Proof. Let $(U,h), (U,\tilde{h})$ be two local charts on U. We suppose that the coherence bound in Equation (4.1.1) holds true with respect to (U,h). To prove the statement, we show that it holds true also with respect to $(U,\tilde{h})$. Per assumption, there exists $\underline{\kappa} \in \mathcal{D}(h(U))$ with $\check{\underline{\kappa}}(0) \neq 0$ such that for any compact set $\mathfrak{K} \subset \mathbb{R}^d$ there exists $\zeta^U_{\mathfrak{K}} \leq \min\{0,\gamma\}$ for which

$$|((h^{-1})^*(F_{\mathsf{p}}) - (h^{-1})^*(F_{\mathsf{q}}))(\underline{\kappa}^{\lambda}_{h(\mathsf{q})})| \lesssim \lambda^{\zeta^U_{\mathfrak{K}}}(|h(\mathsf{p}) - h(\mathsf{q})| + \lambda)^{\gamma-\zeta^U_{\mathfrak{K}}},$$

uniformly for $p,q \in \mathfrak{K}$ and for $\lambda \in (0,1]$, where $(h^{-1})^*$ denotes the pullback along h^{-1} as per Remark A.9.3. To conclude the proof, we seek a test function $\chi \in \mathcal{D}(\tilde{h}(U))$ with $\check{\chi}(0) \neq 0$ such that the coherence condition with respect to $(U,\tilde{h})$ is satisfied. For any $\tilde{\chi}\in\mathcal{D}(\tilde{h}(U))$, it holds true that

$$\begin{aligned}|((\tilde{h}^{-1})^*(F_{\mathsf{p}})-(\tilde{h}^{-1})^*(F_{\mathsf{q}}))(\tilde{\chi}^\lambda_{\tilde{h}(\mathsf{q})})| &= |\langle (h\circ\tilde{h}^{-1})^*(h^{-1})^*(F_{\mathsf{p}})-(h\circ\tilde{h}^{-1})^*(h^{-1})^*(F_{\mathsf{q}}),\tilde{\chi}^\lambda_{\tilde{h}(\mathsf{q})}\rangle| \\ &\lesssim |\langle (h^{-1})^*(F_{\mathsf{p}})-(h^{-1})^*(F_{\mathsf{p}}),((\tilde{h}\circ h^{-1})^*\tilde{\chi})^\lambda_{\tilde{h}(\mathsf{q})}\rangle|, \end{aligned} \tag{4.1.2}$$

where the last equality descends from Remark A.9.3 and from $\|\det d(h\circ\tilde{h}^{-1})\|_{L^\infty(\tilde{h}(U))}\lesssim 1$. Here, $d(h\circ\tilde{h}^{-1})$ denotes the differential of the coordinate change induced by $h\circ\tilde{h}^{-1}$. Therefore, we can choose $\chi\in\mathcal{D}(\tilde{h}(U))$ with $\check{\chi}(0)\neq 0$ such that $(h\circ\tilde{h}^{-1})^*\chi=\underline{k}$. As a result, it descends that

$$|((\tilde{h}^{-1})^*(F_{\mathsf{p}})-(\tilde{h}^{-1})^*(F_{\mathsf{q}}))(\chi^\lambda_{\tilde{h}(\mathsf{q})})|\lesssim \lambda^{\zeta^U_{\mathfrak{K}}}(|h(\mathsf{p})-h(\mathsf{q})|+\lambda)^{\gamma-\zeta^U_{\mathfrak{K}}}\lesssim \lambda^{\zeta^U_{\mathfrak{K}}}(|\tilde{h}(\mathsf{p})-\tilde{h}(\mathsf{q})|+\lambda)^{\gamma-\zeta^U_{\mathfrak{K}}},$$

uniformly for $\mathsf{p},\mathsf{q}\in\mathfrak{K}$ and for $\lambda\in(0,1]$, where the last inequality descends from the uniform bound

$$\sup_{\substack{\mathsf{p},\mathsf{q}\in\mathfrak{K},\\ \mathsf{p}\neq\mathsf{q}}}\frac{|h(\mathsf{p})-h(\mathsf{q})|}{|\tilde{h}(\mathsf{p})-\tilde{h}(\mathsf{q})|}\lesssim 1.$$

□

In the following, we prove that a coherent germ as per Definition 4.1.3 satisfies a homogeneity bound - [RS21, Lemma 16]. The following result is an adaptation of Lemma 2.4.9 to this setting.

Lemma 4.1.8: *Let $(M,\mathcal{A})$ be a d-dimensional smooth manifold and let $F=(F_{\mathsf{p}})_{\mathsf{p}\in M}$ be a γ-coherent germ of distributions as per Definition 4.1.3. In addition, let $(U,h)\in\mathcal{A}$ be a local chart. Then, for any compact set $\mathfrak{K}\subset U$, there exists $\beta^U_{\mathfrak{K}}<\gamma$ such that*

$$|\langle (h^{-1})^*(F_{\mathsf{p}}),\underline{\kappa}^\lambda_{h(\mathsf{p})}\rangle|\lesssim\lambda^{\beta^U_{\mathfrak{K}}},\quad \forall\mathsf{p}\in\mathfrak{K},\quad\forall\lambda\in(0,1],\tag{4.1.3}$$

*where $\underline{\kappa}\in\mathcal{D}(h(U))$ is chosen as in Definition 4.1.3 while $(h^{-1})^*$ denotes the pullback along h^{-1} as per Remark A.9.3. We say that F is **locally homogeneous** in U with exponents $\boldsymbol{\beta}^U=(\beta^U_{\mathfrak{K}})_{\mathfrak{K}}$. In addition, if $\beta^U_{\mathfrak{K}}=\beta^U$ for any compact set $\mathfrak{K}\subset U$, F is said to be **homogeneous** of degree β^U in U.*

Proof. The proof is similar to that of Lemma 2.4.9 - see [CZ20, Lemma 4.12]. We fix a compact set $\mathfrak{K}\subset\mathbb{R}^d$ and a point $\mathsf{q}\in\mathfrak{K}$. Since $(h^{-1})^*(F_{\mathsf{q}})\in\mathcal{D}'(h(U))$, Remark A.2.2 entails that there exists $r\in\mathbb{N}$ such that

$$|\langle (h^{-1})^*(F_{\mathsf{q}}),\underline{\kappa}^\lambda_{h(\mathsf{p})}\rangle|\lesssim\lambda^{-d-r},\quad\forall\mathsf{p}\in\mathfrak{K},\quad\forall\lambda\in(0,1].\tag{4.1.4}$$

In addition, being $\mathrm{Diam}(\phi(\mathfrak{K})):=\sup_{\mathsf{p},\mathsf{q}\in\mathfrak{K}}|h(\mathsf{p})-h(\mathsf{q})|<\infty$ and on account of Equation (4.1.1), it descends that

$$|((h^{-1})^*(F_{\mathsf{p}})-(h^{-1})^*(F_{\mathsf{q}}))(\underline{\kappa}^\lambda_{\varphi(\mathsf{q})})|\lesssim\lambda^{\zeta^U_{\mathfrak{K}}}(|h(\mathsf{p})-h(\mathsf{q})|+\lambda)^{\gamma-\zeta^U_{\mathfrak{K}}}\leq\lambda^{\zeta^U_{\mathfrak{K}}}(\mathrm{Diam}(h(\mathfrak{K}))+\lambda)^{\gamma-\zeta^U_{\mathfrak{K}}}\lesssim\lambda^{\zeta^U_{\mathfrak{K}}},\tag{4.1.5}$$

uniformly for $\mathsf{p},\mathsf{q}\in\mathfrak{K}$ and for $\lambda\in(0,1]$. As a result, combining Equations (4.1.4) and (4.1.5), it descends that

$$\begin{aligned}|\langle (h^{-1})^*(F_{\mathsf{p}}),\underline{\kappa}^\lambda_{\varphi(\mathsf{p})}\rangle| &\leq |\langle (h^{-1})^*(F_{\mathsf{q}}),\underline{\kappa}^\lambda_{h(\mathsf{p})}\rangle|+|((h^{-1})^*(F_{\mathsf{p}})-(h^{-1})^*(F_{\mathsf{q}}))(\underline{\kappa}^\lambda_{\varphi(\mathsf{q})})| \\ &\lesssim\lambda^{-d-r}+\lambda^{\zeta^U_{\mathfrak{K}}}\lesssim\lambda^{\min\{-d-r,\zeta^U_{\mathfrak{K}}\}},\quad\forall\mathsf{p}\in\mathfrak{K},\forall\lambda\in(0,1],\end{aligned}$$

where we applied the triangle inequality. As a result, we can choose $\beta^U_{\mathfrak{K}}<\min\{-d-r,\zeta^U_{\mathfrak{K}},\gamma\}$. This concludes the proof. □

In the following, we prove that the homogeneity exponents are independent of the coordinates - see [RS21, Prop. 17].

Proposition 4.1.9: *Let $(M, \mathcal{A})$ be a d-dimensional smooth manifold and let $F = (F_\mathsf{p})_{\mathsf{p}\in M}$ be a γ-coherent germ as per Definition 4.1.3. In addition, let $(U, h), (U, \tilde{h}) \in \mathcal{A}$ be two local charts on the open set $U \subset M$. Suppose that F satisfies the homogeneity condition in Equation (4.1.3) with respect to (U, h). Then F is also locally homogeneous with respect to $(U, \tilde{h})$.*

Proof. Let $\underline{\kappa} \in \mathcal{D}(h(U))$ be as per Definition 4.1.3. Given a compact set $\mathfrak{K} \subset U$, on account of Equation (4.1.3), it descends that

$$|\langle(\tilde{h}^{-1})^*(F_\mathsf{p}), ((h\circ\tilde{h}^{-1})^*\underline{\kappa})^\lambda_{\tilde{h}(\mathsf{p})}\rangle| \lesssim || \lesssim |\langle(h^{-1})^*(F_\mathsf{p}), \underline{\kappa}^\lambda_{h(\mathsf{p})}\rangle| \lesssim \lambda^{\beta^U_{\mathfrak{K}}}, \quad \forall \mathsf{p} \in \mathfrak{K}, \forall\lambda \in (0, 1].$$

where in the first inequality we exploited that $\|\det d(\tilde{h}\circ h^{-1})\|_{L^\infty(h(U))} \lesssim 1$ while in the last one we used the homogeneity bound with respect to (U, h) as per Equation (4.1.3). Therefore, we conclude that $(\tilde{h}^{-1})^*(F_\mathsf{p})$ is locally homogeneous in $\tilde{h}(U)$ with exponent $\beta^U_{\mathfrak{K}}$. □

Enhanched Coherence In this paragraph, similarly to the Euclidean case, we *enhance* the notion of coherence on a smooth manifold as per Definition 4.1.3. This leads us to defining the notion of *enhanced coherence* adapted to the smooth manifold setting. We recall that the idea behind it is to replace the test function $\underline{\kappa} \in \mathcal{D}(h(U))$ in Equation (4.1.1) by an arbitrary test function, provided that we modify suitably the family of exponents $\boldsymbol{\zeta} = (\zeta^U_{\mathfrak{K}})_{\mathfrak{K}}$. This can be achieved by resorting to the same arguments for the case of coherence on an open set of $\mathbb{R}^d$ - see Appendix B. As a matter of fact, given a γ-coherent germ $F = (F_\mathsf{p})_{\mathsf{p}\in M}$ of a smooth manifold $(M, \mathcal{A})$ and a local chart $(U, h) \in \mathcal{A}$, Definition 4.1.3 entails that the germ $\mathcal{F}_{h(\mathsf{p})} := (h^{-1})^*(F_\mathsf{p})$ is γ-coherent on the open set $h(U) \subset \mathbb{R}^d$ as per Definition B.0.2. As a result, we apply Proposition B.0.6 in order to formulate the following equivalent definition of coherence on a smooth manifold.

Definition 4.1.10: *Let M be a d-dimensional smooth manifold and let $\mathcal{A} = \{(U_i, h_i)\}_{i\in I}$ be a smooth atlas thereon. In addition, let $\gamma \in \mathbb{R}$ and let $F = (F_\mathsf{p})_{\mathsf{p}\in M}$ be a germ of distributions on M as per Definition 4.1.1. We say that F is γ-**coherent** on $(M, \mathcal{A})$ if for any $(U, h) \in \mathcal{A}$ and for any compact set $\mathfrak{K} \subset U$ there exists $\zeta^U_{\mathfrak{K}} \le \min\{0, \gamma\}$ such that, for any integer $r > -\zeta^U_{\mathfrak{K}}$,*

$$|((h^{-1})^*(F_\mathsf{p}) - (h^{-1})^*(F_\mathsf{q}))(\phi^\lambda_{h(\mathsf{q})})| \lesssim \|\phi\|_{C^r(\mathbb{R}^d)}\lambda^{\zeta^U_{\mathfrak{K}}}(|h(\mathsf{p}) - h(\mathsf{q})| + \lambda)^{\gamma-\zeta^U_{\mathfrak{K}}}, \tag{4.1.6}$$

uniformly for $\mathsf{p}, \mathsf{q} \in \mathfrak{K}$, $\lambda \in (0, 1]$ and $\phi \in \mathcal{D}(B(0, 1)) \subset \mathbb{R}^d$, where $B(0, 1)$ denotes the unit open ball centered at the origin.

Remark 4.1.11: *Definition 4.1.10 proves to be more advantageous than Definition 2.4.4 since the bound in Equation 4.1.6 is independent of the test function $\underline{\kappa}$. As a by product, it entails that the set of γ-coherent germs of distributions on a smooth manifold is a vector space. However, Definition 2.4.4 is rather useful from an operational viewpoint, since it allows to establish coherence by checking the bound in Equation 2.4.3 only for a test function.*

Remark 4.1.12: *In Proposition 4.1.7, we proved that the notion of coherence as per Definition 4.1.3 is independent of the underlying coordinates. As a result, on account of the equivalence between Definitions 4.1.3 and 4.1.10, it descends that also the notion of enhanced coherence is independent of the choice of the local chart. Equivalently, this independence descends directly from the same arguments used in the proof of Proposition 4.1.7.*

Remark 4.1.13: *On account of Proposition B.0.8, it descends that the notion of coherence on a smooth manifold is stable under restrictions. More precisely, if F is γ-coherent with respect to (U,h), then F is still γ-coherent with respect to (V,h) for any open subset $V\subset U$.*

In the following, we prove that coherence as per Definition 4.1.10 is independent of the atlas - see Proposition [RS21, Prop. 13].

Proposition 4.1.14: *Let M be a d-dimensional smooth manifold, let $\mathcal{A},\mathcal{A}'$ be two smooth atlases thereon and let $F=(F_{\mathsf{p}})_{p\in M}$ be a germ of distributions on M as per Definition 4.1.1. If $F=(F_{\mathsf{p}})_{\mathsf{p}\in M}$ is γ-coherent with respect to $(M,\mathcal{A})$, then it is also γ-coherent with respect to $(M,\mathcal{A}')$.*

Proof. On account of Definition 4.1.10, we shall prove that, for any $(U',h')\in\mathcal{A}'$, F satisfies the bound in Equation (4.1.6) on (U',h'). In addition, since the notion of coherence is independent of the underlying coordinates (Proposition 4.1.7), we can consider U' neglecting the local chart h'. In addition, there exists a family $\{U_i\}_{i\in J}\subset\mathcal{A}$ of open sets such that $U'=\bigcup_{i\in I}(U_i\cap U')=\bigcup_{i\in i}U'_i$, where we set $U'_j:=U\cap U_j$. On account of Proposition 4.1.7 and of Remark 4.1.13, it descends that F satisfies the bound of Equation (4.1.6) on U'_i for any $i\in i$. Moreover, being $U'_i\subset U$ for any $i\in I$, we can equip each of them with a local chart h on U. In order to prove the statement, we shall show that the coherence bound of Equation (4.1.6) holds true on the union of two open sets U'_j and U'_ℓ with $U'_j\cap U'_\ell\neq\emptyset$.
To this end, we fix a compact set $\mathfrak{K}\subset U'_j\cup U'_\ell$. We observe that, if the compact set $\mathfrak{K}$ is contained in one of the two open sets U'_J or U'_ℓ, then Equation (4.1.6) holds true on account of Remark 4.1.13. For this reason, we shall consider $\mathfrak{K}\subset U'_j\cup U'_\ell$ such that $\mathfrak{K}\cap U'_j\neq\emptyset$ and $\mathfrak{K}\cap U'_\ell\neq\emptyset$.
In this situation, we can split the compact set $\mathfrak{K}$ as $\mathfrak{K}=\mathfrak{K}_j\cup\mathfrak{K}_\ell$, where $\mathfrak{K}_j\subset U'_j$ and $\mathfrak{K}_\ell\subset U'_\ell$ are two compact sets such that $\mathfrak{K}_j\cap\mathfrak{K}_\ell\neq\emptyset$. In the following, we shall prove that F satisfies the coherence bound in Equation (4.1.6) uniformly for $\mathsf{p},\mathsf{q}\in\mathfrak{K}$. Being $\mathfrak{K}_j\subset U'_j$ and $\mathfrak{K}_\ell\subset U'_\ell$ and on account of Remark 4.1.13, if two points $\mathsf{p},\mathsf{q}\in\mathfrak{K}$ lie both in $\mathfrak{K}_j$ or in $\mathfrak{K}_\ell$, then F satisfies the coherence bound in Equation (4.1.6). Therefore, we only discuss the case with $\mathsf{p}\in\mathfrak{K}_j\setminus U'_\ell$ and $\mathsf{q}\in\mathfrak{K}_\ell\setminus U'_j$. It holds true that, for any $\mathsf{e}\in\mathfrak{K}_j\cap\mathfrak{K}_\ell$ and for any $\phi\in\mathcal{D}(B(0,1))$,

$$\begin{aligned}&|((h^{-1})^*(F_{\mathsf{p}})-(h^{-1})^*(F_{\mathsf{q}}))(\phi^\lambda_{h(\mathsf{q})})|\\&\qquad\le\underbrace{|((h^{-1})^*(F_{\mathsf{p}})-(h^{-1})^*(F_{\mathsf{e}}))(\phi^\lambda_{h(\mathsf{q})}))|}_{|A|}+\underbrace{|((h^{-1})^*(F_{\mathsf{e}})-(h^{-1})^*(F_{\mathsf{q}}))(\phi^\lambda_{h(\mathsf{q})})|}_{|B|},\end{aligned}\tag{4.1.7}$$

where we applied the triangle inequality. In addition, we fix $r\in\mathbb{N}_0$ such that $r>\max\{-\zeta^{U'_j}_{\mathfrak{K}_j},-\zeta^{U'_\ell}_{\mathfrak{K}_\ell}\}$. We start by estimating $|B|$. On account of Equation (4.1.6) and of the choice of r, it descends that

$$|B|\lesssim\|\phi\|_{C^r(\mathbb{R}^d)}\lambda^{\zeta^{U'_\ell}_{\mathfrak{K}_\ell}}(|h(\mathsf{e})-h(\mathsf{q})|+\lambda)^{\gamma-\zeta^{U'_\ell}_{\mathfrak{K}_\ell}},$$

uniformly for $\mathsf{e},\mathsf{q}\in\mathfrak{K}_\ell$ and $\lambda\in(0,1]$. In addition, noticing that

$$\sup_{\substack{\lambda\in(0,1],\\ e\in\mathfrak{K}_j\cap\mathfrak{K}_\ell}}\ \sup_{\substack{\mathsf{p}\in\mathfrak{K}_j\setminus U'_\ell,\\ \mathsf{q}\in\mathfrak{K}_\ell\setminus U'_j}}\frac{(|h(\mathsf{e})-h(\mathsf{q})|+\lambda)^{\gamma-\zeta^{U'_\ell}_{\mathfrak{K}_\ell}}}{(|h(\mathsf{p})-h(\mathsf{q})|+\lambda)^{\gamma-\zeta^{U'_\ell}_{\mathfrak{K}_\ell}}}\lesssim1,\tag{4.1.8}$$

it descends that

$$|B|\lesssim\|\phi\|_{C^r(\mathbb{R}^d)}\lambda^{\zeta^{U'_\ell}_{\mathfrak{K}_\ell}}(|h(\mathsf{p})-h(\mathsf{q})|+\lambda)^{\gamma-\zeta^{U'_\ell}_{\mathfrak{K}_\ell}},\tag{4.1.9}$$

uniformly for $\mathsf{p},\mathsf{q}\in\mathfrak{K}$. The estimate for $|A|$ is more involved. Although the test function in $|A|$ is centered at $h(\mathsf{q})$, it can be centered at $h(\mathsf{e})$ by exploiting the same argument used in the proof of [CZ20, Prop. 6.2]. As a matter of fact, one can observe that

$$\phi^{\lambda}_{h(\mathsf{q})}=\tilde{\phi}^{\lambda_1}_{h(\mathsf{e})},\quad\text{where}\quad \tilde{\phi}:=\phi^{\lambda}_{w},$$

with $\lambda_1,\lambda_2\in(0,1]$ and $w\in B(0,1)$ such that

$$\lambda_1=|h(\mathsf{q})-h(\mathsf{e})|+\lambda,\quad \lambda_2=\frac{\lambda}{\lambda_1},\quad w=\frac{h(\mathsf{q})-h(\mathsf{e})}{|h(\mathsf{q})-h(\mathsf{e})|+\lambda}.$$

As a result, the coherence bound on U'_j entails that

$$|A|=|((h^{-1})^*(F_{\mathsf{p}})-(h^{-1})^*(F_{\mathsf{e}}))(\tilde{\phi}^{\lambda_1}_{h(\mathsf{e})})|\lesssim\|\tilde{\phi}\|_{C^r(\mathbb{R}^d)}\lambda^{\zeta^{U'_j}_{\mathfrak{K}_j}}(|h(\mathsf{p})-h(\mathsf{e})|+\lambda)^{\gamma-\zeta^{U'_j}_{\mathfrak{K}_j}},$$

uniformly for $\mathsf{p}\in\mathfrak{K}_j\setminus U'_\ell$, $\mathsf{e}\in\mathfrak{K}_j\cap\mathfrak{K}_\ell$ and $\lambda\in(0,1]$. By definition of $\tilde{\phi}$ and on account of Remark A.2.2, we infer that

$$\|\tilde{\phi}\|_{C^r(\mathbb{R}^d)}\lesssim\lambda_2^{-r-d}\|\phi\|_{C^r(\mathbb{R}^d)}\lesssim\lambda^{-r-d}\|\phi\|_{C^r(\mathbb{R}^d)}.$$

Therefore, it descends that

$$|A|\lesssim\|\phi\|_{C^r(\mathbb{R}^d)}\lambda^{\zeta^{U'_j}_{\mathfrak{K}_j}-r-d}(|h(\mathsf{p})-h(\mathsf{e})|+\lambda)^{\gamma-\zeta^{U'_j}_{\mathfrak{K}_j}}\lesssim\|\phi\|_{C^r(\mathbb{R}^d)}\lambda^{\tilde{\zeta}^{U'_j}_{\mathfrak{K}_j}}(|h(\mathsf{p})-h(\mathsf{e})|+\lambda)^{\gamma-\tilde{\zeta}^{U'_j}_{\mathfrak{K}_j}},$$

where in the last inequality we set $\tilde{\zeta}^{U'_j}_{\mathfrak{K}_j}:=\zeta^{U'_j}_{\mathfrak{K}_j}-r-d$ and where we exploited

$$\sup_{\substack{\lambda\in(0,1],\mathsf{p}\in\mathfrak{K}_j\setminus U'_\ell,\\ e\in\mathfrak{K}_j\cap\mathfrak{K}_\ell}}(|h(\mathsf{p})-h(\mathsf{e})|+\lambda)^{-r-d}\lesssim 1.$$

Resorting to a bound similar to Equation (4.1.8), we infer that

$$|A|\lesssim\|\phi\|_{C^r(\mathbb{R}^d)}\lambda^{\tilde{\zeta}^{U'_j}_{\mathfrak{K}_j}}(|h(\mathsf{p})-h(\mathsf{q})|+\lambda)^{\gamma-\tilde{\zeta}^{U'_j}_{\mathfrak{K}_j}},\tag{4.1.10}$$

At last, combining Equations (4.1.7),(4.1.9) and (4.1.10) and setting $\zeta^{U'_j\cup U'_\ell}_{\mathfrak{K}}:=\min\{\zeta^{U'_\ell}_{\mathfrak{K}_\ell},\tilde{\zeta}^{U'_j}_{\mathfrak{K}_j}\}$, it descends that, for any integer $r>-\zeta^{U'_j\cup U'_\ell}_{\mathfrak{K}}$,

$$|((h^{-1})^*(F_{\mathsf{p}})-(h^{-1})^*(F_{\mathsf{q}}))(\phi^{\lambda}_{h(\mathsf{q})})|\lesssim\|\phi\|_{C^r(\mathbb{R}^d)}\lambda^{\zeta^{U'_j\cup U'_\ell}}(|h(\mathsf{p})-h(\mathsf{q})|+\lambda)^{\gamma-\zeta^{U'_j\cup U'_\ell}_{\mathfrak{K}}},$$

uniformly for $\mathsf{p},\mathsf{q}\in\mathfrak{K}$, $\lambda\in(0,1]$ and $\phi\in\mathcal{D}(B(0,1))$. This entails that F is γ-coherent on $U'_j\cup U'_\ell$. In order to conclude the proof, we analyze the following two scenarios. In the first one, we assume that the open set U' is bounded. In this case, since there exists a finite number of open sets $U_i\in\mathcal{A}$ such that $U'\subset\cup_{i\in I}U_i$, we can iterate the above procedure a finite number of times in order to conclude the proof. In the second scenario, we assume that the open set U' is unbounded. In this case, for any compact set $\mathfrak{K}\subset U'$, there exists a finite family of open sets $\{U_i\}_{i=1}^{N_{\mathfrak{K}}}\subset\mathcal{A}$ such that $\mathfrak{K}\subset\cup_{i=1}^{N_{\mathfrak{K}}}U_i$. As a result, the coherence bound in Equation (4.1.6) is satisfied by iterating the above procedure a finite number of times. □

Proposition 4.1.14 entails a geometric notion of coherence. To conclude this section, we give two examples of coherent germs on a smooth manifold.

Example 4.1.15: *Let M be a d-dimensional smooth manifold and let $u \in \mathcal{D}'(M)$ as per Definition A.10.1. We set $F_{\mathsf{p}} := u$ for any $\mathsf{p} \in M$. Since $F_{\mathsf{p}} - F_{\mathsf{q}} = 0$ for any $\mathsf{p}, \mathsf{q} \in M$, then we infer that $(F_{\mathsf{p}})_{\mathsf{p}\in M}$ is γ-coherent for any $\gamma \in \mathbb{R}$.*

Example 4.1.16: *We point out that our framework is a generalization of the one discussed in Section 2.4. As a matter of fact, we recover the Euclidean case if we choose $M = \mathbb{R}^d$ equipped with the trivial atlas $\mathcal{A} = \{(\mathbb{R}^d, \mathrm{Id})\}$, where $\mathrm{Id}\colon \mathbb{R}^d \to \mathbb{R}^d$ is the identity map. Therefore, for instance, the germ given by the Taylor polynomial of a Hölder function, see Example 2.4.15, is coherent with respect to this atlas.*

4.2 Reconstruction theorem

In the previous section, we introduced the notion of coherent germ of distributions on an arbitrary smooth manifold - see Definitions 4.1.3 and 4.1.10. Moreover, we proved that this notion is independent of the atlas, see Proposition 4.1.14. Since a germ can be interpreted as a family of local approximations for an unknown global distribution, the aim of this section is to investigate the existence of such a distribution. As explained in Section 2.4, the answer to this problem is given by the reconstruction theorem, which was originally formulated within the framework of regularity structures - see [Hai14]. With the objective of the extension to curved backgrounds, we shall rely on its formulation within the theory of germs of distributions [CZ20]. As a matter of fact, resorting to the notions introduced in Section 4.1, we shall prove the reconstruction theorem for coherent germs of distributions on a smooth manifold. This result can be read as a first step for the extension of the theory of regularity structures to smooth manifolds.

In this section, we shall prove the reconstruction theorem for γ-coherent germs of distributions on a smooth manifold with $\gamma > 0$ - see [RS21, Th. 18]. In this scenario, we shall prove that the reconstruction is independence of the choice of the atlas - see Theorem 4.2.2. In addition, in accordance to the Euclidean setting, if the coherence exponent is non-positive, we prove existence while showing the non-uniqueness of the reconstruction - see Theorem 4.2.4. In this case, we shall emphasize that a reconstruction depends on the atlas and on the partition of unity used to construct it. It is noteworthy to underline that the following results hold true in the smooth manifold setting, without, for instance, calling for a Riemannian structure. In addition, the following results are proven as a consequence of the local version of Theorem 2.4.22 - see Remark 2.4.23.

Theorem 4.2.1: *Let M be a d-dimensional smooth manifold and let $\mathcal{A} = \{(U_i, h_i)\}_{i\in I}$ be a smooth atlas thereon. Let $F = (F_{\mathsf{p}})_{\mathsf{p}\in M}$ be a γ-coherent germ on $(M, \mathcal{A})$ with $\gamma > 0$ as per Definition 4.1.3. Then there exists a unique distribution $\mathcal{R}F \in \mathcal{D}'(M)$ such that, for any $(U, h) \in \mathcal{A}$, for any compact set $\mathfrak{K} \subset U$ and for any $\psi \in \mathcal{D}(h(U))$,*

$$|((h^{-1})^*(\mathcal{R}F) - (h^{-1})^*(F_{\mathsf{p}}))(\psi^{\lambda}_{h(\mathsf{p})})| \lesssim \lambda^{\gamma}, \tag{4.2.1}$$

*uniformly for $\mathsf{p} \in \mathfrak{K}$ and $\lambda \in (0, 1]$, where $(h^{-1})^*$ is the pullback along h as per Remark A.9.3. We say that $\mathcal{R}F$ is the **reconstruction** of F.*

Proof. The proof of this statement relies on Theorems 2.4.22 and A.10.2. Definition 4.1.3 entails that, for any $(U, h) \in \mathcal{A}$, the germ $\mathcal{F}_{h(\mathsf{p})} := (h^{-1})^*(F_{\mathsf{p}})$ is γ-coherent on the open set $h(U) \subset \mathbb{R}^d$ with $\gamma > 0$ as per Definition B.0.2. As a result, on account of Theorem 2.4.22, it descends that there exists a unique distribution $(\mathcal{R}F)^{h(U)} \in \mathcal{D}'(h(U))$ such that, for any compact set $\mathfrak{K} \subset U$ and for any $\psi \in \mathcal{D}(h(U))$

$$|((\mathcal{R}F)^{h(U)} - (h^{-1})^*(F_{\mathsf{p}}))(\psi^{\lambda}_{h(\mathsf{p})})| \lesssim \lambda^{\gamma}, \quad \forall \mathsf{p} \in \mathfrak{K}, \forall \lambda \in (0, 1]. \tag{4.2.2}$$

Therefore, we infer that there exists a family of local distributions $\{(\mathcal{R}F)^{h(U)} \in \mathcal{D}'(h(U)) : (U,h) \in \mathcal{A}\}$. At this stage, if we prove that

$$(\mathcal{R}F)^{h(U)} = (\tilde{h} \circ h^{-1})^*(\mathcal{R}F)^{\tilde{h}(\tilde{U})} \quad \text{on } h(U \cap \tilde{U}) \tag{4.2.3}$$

for any pair of local charts $(U,h), (\tilde{U},\tilde{h}) \in \mathcal{A}$, Theorem A.10.2 entails that the family $\{(\mathcal{R}F)^{h(U)}\}_{(U,h)\in\mathcal{A}}$ identifies a unique distribution $\mathcal{R}F \in \mathcal{D}'(M)$ such that $(h^{-1})^*(\mathcal{R}F) = (\mathcal{R}F)^{h(U)}$ for any $(U,h) \in \mathcal{A}$. To this end, we fix a compact set $\mathfrak{K} \subset U \cap V$. By the triangle inequality, it holds true that

$$\begin{aligned} &|((\mathcal{R}F)^{h(U)} - (\tilde{h} \circ h^{-1})^*(\mathcal{R}F)^{\tilde{h}(\tilde{U})})(\psi^\lambda_{h(p)})| \\ &\quad \leq \underbrace{|((\mathcal{R}F)^{h(U)} - (h^{-1})^*(F_{\mathsf{p}}))(\psi^\lambda_{h(\mathsf{p})})|}_{|A|} + \underbrace{|((\tilde{h} \circ h^{-1})^*(\mathcal{R}F)^{\tilde{h}(\tilde{U})} - (h^{-1})^*(F_{\mathsf{p}}))(\psi^\lambda_{h(\mathsf{p})}))|}_{|B|}, \end{aligned} \tag{4.2.4}$$

for any $\psi \in \mathcal{D}(h(U \cap \tilde{U}))$. On the one hand, since $(\mathcal{R}F)^{h(U)}$ is the reconstruction of $(h^{-1})^*(F_{\mathsf{p}})$ as per Theorem 2.4.22, we infer that $|A| \lesssim \lambda^\gamma$ uniformly for $\mathsf{p} \in \mathfrak{K}$ and $\lambda \in (0,1]$. On the other hand, it descends that

$$\begin{aligned} |B| &= |((\tilde{h} \circ h^{-1})^*(\mathcal{R}F)^{\tilde{h}(\tilde{U})} - (h^{-1})^*(F_{\mathsf{p}}))(\psi^\lambda_{h(\mathsf{p})}))| \\ &\lesssim |((\mathcal{R}F)^{\tilde{h}(\tilde{U})} - (\tilde{h}^{-1})^*(F_{\mathsf{p}}))((h \circ \tilde{h}^{-1})^*\psi)^\lambda_{\tilde{h}(\mathsf{p})}| \lesssim \lambda^\gamma, \end{aligned} \tag{4.2.5}$$

uniformly for $\mathsf{p} \in \mathfrak{K}$ and $\lambda \in (0,1]$, where in the first inequality we used that $\|\det d(h \circ \tilde{h}^{-1})\|_{L^\infty(\tilde{h}(\tilde{U}))} \lesssim 1$ while in the last inequality we exploited that $(\mathcal{R}F)^{\tilde{h}(\tilde{U})}$ satisfies Equation (2.4.11). Therefore, for any $\psi \in \mathcal{D}(h(U \cap \tilde{U}))$, it descends that

$$|((\mathcal{R}F)^{h(U)} - (\tilde{h} \circ h^{-1})^*(\mathcal{R}F)^{\tilde{h}(\tilde{U})})(\psi^\lambda_{h(p)})| \lesssim \lambda^\gamma,$$

uniformly for $\mathsf{p} \in \mathfrak{K}$ and $\lambda \in (0,1]$. As a result, being $\gamma > 0$,

$$|((\mathcal{R}F)^{h(U)} - (\tilde{h} \circ h^{-1})^*(\mathcal{R}F)^{\tilde{h}(\tilde{U})})(\psi^\lambda_{h(p)})| \lesssim \lambda^\gamma \to 0 \quad \text{as} \quad \lambda \to 0^+.$$

Applying Lemma A.7.11 to the distribution $u := (\mathcal{R}F)^{h(U)} - (\tilde{h} \circ h^{-1})^*(\mathcal{R}F)^{\tilde{h}(\tilde{U})}$ on the open set $h(U \cap \tilde{U})$, it descends that

$$(\mathcal{R}F)^{h(U)} = (\tilde{h} \circ h^{-1})^*(\mathcal{R}F)^{\tilde{h}(\tilde{U})} \quad \text{on } h(U \cap \tilde{U}).$$

As a result, Theorem A.10.2 implies that there exists a unique distribution $\mathcal{R}F \in \mathcal{D}'(M)$ such that $(h^{-1})^*(\mathcal{R}F) = (\mathcal{R}F)^{h(U)}$ for any $(U,h) \in \mathcal{A}$. In addition, $(h^{-1})^*(\mathcal{R}F)$ satisfies the bound in Equation (4.2.2). □

In the following, given $\gamma > 0$, we prove that the reconstruction of a γ-coherent germ of distributions is independent of the choice of the atlas - see [RS21, Th. 20].

Theorem 4.2.2: *Let M be a d-dimensional smooth manifold and let $\mathcal{A}, \tilde{\mathcal{A}}$ be two atlases thereon. Let $\gamma > 0$ and let $F = (F_{\mathsf{p}})_{\mathsf{p}\in M}$ be a γ-coherent germ of distributions on M as per Definition 4.1.3. If $\mathcal{R}_{\mathcal{A}}F \in \mathcal{D}'(M)$ and $\mathcal{R}_{\tilde{\mathcal{A}}}F \in \mathcal{D}'(M)$ are the reconstructions of F with respect to $\mathcal{A}$ and $\tilde{\mathcal{A}}$ as per Theorem 4.2.1, then $\mathcal{R}_{\mathcal{A}}F = \mathcal{R}_{\tilde{\mathcal{A}}}F$, that is the reconstruction is independent of the atlas.*

Proof. Theorem 4.2.1 implies that there exists a unique reconstruction $\mathcal{R}_{\mathcal{A}}F \in \mathcal{D}'(M)$ of F with respect to the atlas $(M, \mathcal{A})$. On account of Theorem A.10.2, the global distribution $\mathcal{R}_{\mathcal{A}}F \in \mathcal{D}'(M)$ is identified by the family $\{(\mathcal{R}_{\mathcal{A}}F)^{h(U)}\}_{(U,h)\in\mathcal{A}}$, where $(\mathcal{R}_{\mathcal{A}}F)^{h(U)} \in \mathcal{D}'(h(U))$. In addition, we fix a local chart $(\tilde{U}, \tilde{h}) \in \tilde{\mathcal{A}}$. On the one hand, we can consider the distribution $(\tilde{h}^{-1})^*(\mathcal{R}_{\mathcal{A}}F) \in \mathcal{D}'(\tilde{h}(\tilde{U}))$. At the same time, on the other hand, Theorem 4.2.1 applied with reference to the atlas $\tilde{\mathcal{A}}$ entails that there exists a unique reconstruction of F, $\mathcal{R}_{\tilde{\mathcal{A}}}F$, such that $(\tilde{h}^{-1})^*(\mathcal{R}_{\tilde{\mathcal{A}}}F) = (\mathcal{R}_{\tilde{\mathcal{A}}}F)^{\tilde{h}(\tilde{U})}$. In order to conclude the proof of this theorem, on account of Definition A.10.1, it suffices to show that

$$(\tilde{h}^{-1})^*(\mathcal{R}_{\mathcal{A}}F) = (\mathcal{R}_{\tilde{\mathcal{A}}}F)^{\tilde{h}(\tilde{U})} \quad \text{in } \mathcal{D}'(\tilde{h}(\tilde{U})). \tag{4.2.6}$$

To this end, $\tilde{U}$ can be covered by a family of local charts $\{(U_i, h_i)\}_{i\in I} \subset \mathcal{A}$, for some index I, *i.e*, $\tilde{U} = \cup_{i\in I}\tilde{U}_i$ where we set $\tilde{U}_i := \tilde{U} \cap U_i$. Therefore, we shall prove Equation (4.2.6) restricted to a subset $\tilde{U}_i$. On account of Remark 4.1.13 and by uniqueness of the reconstruction, we infer that $(\mathcal{R}_{\tilde{\mathcal{A}}}F)^{\tilde{h}(\tilde{U})}|_{\tilde{h}(\tilde{U}_i)} = (\mathcal{R}_{\tilde{\mathcal{A}}}F)^{\tilde{h}(\tilde{U}_i)}$. Then, we shall prove that, for any $i \in I$,

$$(\tilde{h}^{-1})^*(\mathcal{R}_{\mathcal{A}}F|_{\tilde{U}_i}) = (\mathcal{R}_{\tilde{\mathcal{A}}}F)^{\tilde{h}(\tilde{U}_i)}. \tag{4.2.7}$$

Given a compact set $\mathfrak{K} \subset \tilde{U}_i$ and a test function $\psi \in \mathcal{D}(\tilde{h}(\tilde{U}_i))$, it holds true that

$$\begin{aligned} &|((\tilde{h}^{-1})^*(\mathcal{R}_{\mathcal{A}}F|_{\tilde{U}_i}) - (\tilde{h}^{-1})^*(F_{\mathsf{p}}))(\psi^{\lambda}_{\tilde{h}(\mathsf{p})}))| \\ &\qquad \lesssim |((h_i^{-1})^*(\mathcal{R}_{\mathcal{A}}F|_{\tilde{U}_i}) - (h_i^{-1})^*(F_{\mathsf{p}}))((\tilde{h}\circ h_i^{-1})^*\psi)^{\lambda}_{h_i(\mathsf{p})})| \lesssim \lambda^{\gamma}, \end{aligned} \tag{4.2.8}$$

uniformly for $\mathsf{p} \in \mathfrak{K}$ and $\lambda \in (0, 1]$, where in the first inequality we performed a change of coordinates while in the last one we exploited that $\mathcal{R}_{\mathcal{A}}F$ is the reconstruction of F with respect to the atlas $\mathcal{A}$. Being $\gamma > 0$ and on account of Theorem 4.2.1, we recall that $(\mathcal{R}_{\tilde{\mathcal{A}}}F)^{\tilde{h}(\tilde{U}_i)}$ is the unique distribution satisfying Equation (4.2.8). As a result, we infer that Equation (4.2.7) holds true. Eventually, Equation (4.2.6) descends from a partition of unity argument. This concludes the proof. □

Remark 4.2.3: *Let $(M, \mathcal{A})$ be a d-dimensional smooth manifold. On account of Remark 4.1.2, a γ-coherent germ F is an element of $\mathcal{D}'(M \times M)$ as per Definition A.10.1. Then, we can formulate a conjecture similar to that of the Euclidean setting - see Conjecture 3.4.3. If $\gamma > 0$, then it holds true that*

$$\mathcal{R}F = \boldsymbol{\delta}_M^* F,$$

where $\boldsymbol{\delta}_M \colon M \to M \times M$, $\boldsymbol{\delta}_M(\mathsf{p}) = (\mathsf{p}, \mathsf{p})$. In addition, Theorem 3.3.5 entails an estimate of the Besov wavefront set of $\mathcal{R}F$ starting from that of F.

In the following, we discuss the reconstruction theorem in the case of a non-positive coherence exponent - see [RS21, Th. 21]. In this scenario, similarly to the Euclidean space setting, we prove that the reconstruction is non-unique. As a matter of fact, we show that a reconstruction depends on the atlas and on the partition of unity used to define it.

Theorem 4.2.4: *Let M be a d-dimensional smooth manifold and let $\mathcal{A} = \{(U_i, h_i)\}_{i\in I}$ be a smooth atlas thereon, where I is an index set. Let $\gamma \le 0$ and let $F = (F_{\mathsf{p}})_{\mathsf{p}\in M}$ be a γ-coherent germ of distributions on $(M, \mathcal{A})$ as per Definition 4.1.3. Then there exists $\mathcal{R}F \in \mathcal{D}'(M)$ such that, for any $(U, h) \in \mathcal{A}$, $(h^{-1})^*(\mathcal{R}F) \in \mathcal{D}'(h(U))$ satisfies, for any compact set $\mathfrak{K} \subset U$ and for any $\psi \in \mathcal{D}(h(U))$,*

$$|((h^{-1})^*(\mathcal{R}F) - (h^{-1})^*(F_{\mathsf{p}}))(\psi^{\lambda}_{h(\mathsf{p})})| \lesssim \begin{cases} \lambda^{\gamma} & \text{if } \gamma < 0, \\ 1 + |\log \lambda| & \text{if } \gamma = 0, \end{cases} \tag{4.2.9}$$

uniformly for $\mathsf{p} \in \mathfrak{K}$ *and* $\lambda \in (0,1]$, *where* $(h^{-1})^*$ *is the pullback along* h^{-1} *as per Remark A.9.3. In addition, the distribution* $\mathcal{R}F$ *is non-unique.*

Proof. The proof of this result is similar to that of Theorem 4.2.1 and it is based on the local formulation of Theorem 2.4.22. For this reason, we only sketch the proof. In addition, we only focus on the case $\gamma < 0$. The proof of the case $\gamma = 0$ follows suit. On account of Theorem 2.4.22, for any $(U_i, h_i) \in \mathcal{A}$ there exists $(\mathcal{R}F)^{h_i(U_i)} \in \mathcal{D}'(h_i(U_i))$ such that, for any compact set $\mathfrak{K} \subset U$ and for any $\psi \in \mathcal{D}(h_i(U_i))$,

$$|((\mathcal{R}F)^{h_i(U_i)} - (h_i^{-1})^*(F_{\mathsf{p}}))(\psi^{\lambda}_{h_i(\mathsf{p})})| \lesssim \lambda^{\gamma} \quad \forall \mathsf{p} \in \mathfrak{K}, \forall \lambda \in (0,1]. \tag{4.2.10}$$

Being $\gamma < 0$, $(\mathcal{R}F)^{h_i(U_i)}$ is non-unique for any $(U_i, h_i) \in \mathcal{A}$. However, for any $(U_i, h_i) \in \mathcal{A}$, we can choose a reconstruction $(\mathcal{R}F)^{h_i(U_i)} \in \mathcal{D}'(h_i(U_i))$. We now introduce a partition of unity $(\rho_i)_{i\in I}$ subordinated to the open cover $(U_i)_{i\in I}$. As a result, analogously to [CZ20, Sect. 11], we can define a global reconstruction $\mathcal{R}F \in \mathcal{D}'(M)$ by

$$\mathcal{R}F := \sum_{i\in I} \rho_i (\mathcal{R}F)^{h_i(U_i)}.$$

We stress that $\mathcal{R}F$ is non-unique. As a matter of fact, it depends on the choice of local reconstructions $(\mathcal{R}F)^{h_i(U_i)}$, on the atlas $\mathcal{A}$ and on the partition of unity $(\rho_i)_{i\in I}$. The dependence on the partition of unity descends from the lack of the overlapping condition (Equation (A.10.1)). □

At last, we prove that a reconstruction $\mathcal{R}F$ locally lies in a suitable Besov space $B^{\alpha}_{\infty,\infty}$. The following result adapts Theorem 2.4.19 to the smooth manifold setting.

Theorem 4.2.5: *Let* $(M, \mathcal{A})$ *be a* d*-dimensional smooth manifold and let* $F = (F_{\mathsf{p}})_{\mathsf{p}\in M}$ *be a* γ*-coherent germ of distributions on* M *as per Definition 4.1.3, where* $\gamma \in \mathbb{R}$. *In addition, let* $\mathcal{R}F \in \mathcal{D}'(M)$ *be a reconstruction of* F *as per Theorems 4.2.4 and 4.2.1 and let* $(U,h) \in \mathcal{A}$ *be a local chart. Suppose that* F *is homogeneous of degree* $\beta^U < \gamma$ *in* U *as per Lemma 4.1.8. If* $\beta^U > 0$, *then* $(h^{-1})^*(\mathcal{R}F) = 0$ *in* $h(U)$. *If* $\beta^U \leq 0$, *then* $(h^{-1})^*(\mathcal{R}F)$ *lies in* $B^{\beta^U,\mathrm{loc}}_{\infty,\infty}(h(U))$ *as per Definition 2.1.30.*

Proof. If $\beta^U > 0$, on account of Remark 2.4.10, we infer that $(h^{-1})^*(\mathcal{R}F) = 0$ in $h(U)$. Therefore, we focus on the case $\beta^U \leq 0$. In addition, for the sake of simplicity, we consider the case $\gamma \neq 0$. The case $\gamma = 0$ follows suit. Let $\underline{\kappa} \in \mathcal{D}(h(U))$ be the test function as in Definition 4.1.3 and let $\mathfrak{K} \subset U$ be a compact set. Since $(h^{-1})^*(\mathcal{R}F)$ reconstructs locally the germ F, it holds true that

$$|((h^{-1})^*(\mathcal{R}F) - (h^{-1})^*(F_{\mathsf{p}}))(\underline{\kappa}^{\lambda}_{h(\mathsf{p})})| \lesssim \lambda^{\gamma}, \tag{4.2.11}$$

uniformly for $\mathsf{p} \in \mathfrak{K}$ and $\lambda \in (0,1]$. By the triangle inequality, it descends that

$$|\langle (h^{-1})^*(\mathcal{R}F), \underline{\kappa}^{\lambda}_{h(\mathsf{p})}\rangle| \leq |((h^{-1})^*(\mathcal{R}F) - (h^{-1})^*(F_{\mathsf{p}}))(\underline{\kappa}^{\lambda}_{h(\mathsf{p})})| + |\langle (h^{-1})^*(F_{\mathsf{p}}), \underline{\kappa}^{\lambda}_{h(\mathsf{p})}\rangle| \lesssim \lambda^{\beta^U} + \lambda^{\gamma} \lesssim \lambda^{\beta^U},$$

uniformly for $\mathsf{p} \in M$ and for $\lambda \in (0,1]$, where the last inequality descends from Equation (4.2.11) and from the homogeneity bound in Equation (4.1.3). As a consequence, Proposition 2.1.28 entails that $(h^{-1})^*(\mathcal{R}F)$ lies in $B^{\beta^U,\mathrm{loc}}_{\infty,\infty}(h(U))$. □

Chapter 5

Conclusions and Perspectives

In this thesis we developed two novel frameworks which we expect to play a notable rôle in the analysis of stochastic partial differential equations (SPDEs). In what follows, we present an overview of the main results obtained in Chapters 3 and 4 and we outline a few future perspectives.

Besov wavefront set. In Chapter 3, we introduced the novel notion of Besov wavefront set [DRS22], which is a refinement of its smooth counterpart. More precisely, it characterizes the directions in Fourier space along which an underlying distribution lies or not in a suitable Besov space $B^{\alpha,\mathrm{loc}}_{\infty,\infty}(\mathbb{R}^d)$. A key result of Chapter 3 is the characterization of the Besov wavefront set in terms of pseudodifferential operators - see Theorem 3.2.1. As a matter of fact, this characterization has been heavily exploited in Section 3.3 to prove several structural properties of the Besov wavefront set. In the following, we give a succinct overview of the main results of Section 3.3.

In Subsection 3.3.2, given an embedding $f\colon \Omega_1 \to \Omega_2$ between two open sets $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$, we established a sufficient condition for the well-posedness of the pullback along f of an underlying distribution with prescribed Besov wavefront set - see Theorem 3.3.5. As a byproduct, we proved that the Besov wavefront set is invariant under the action of diffeomorphisms - see Theorem 3.3.7. This result is crucial for the extension of the notion of Besov wavefront set to distributions supported on a smooth manifold.

In Subsection 3.3.3, we established a sufficient criterion for the well-posedness of the product between two distributions with prescribed Besov wavefront sets - see Theorem 3.3.10. This criterion is a generalization of the one formulated by Hörmander in the framework of the smooth wavefront set. In addition, if the product exists, we proved an estimate of the associated Besov wavefront set. This result can be read as a microlocal formulation of Young's product theorem, which is often applied to establishing the well-posedness of nonlinear SPDEs.

In Subsection 3.3.4, we established an estimate for the Besov wavefront set of $\mathcal{K}u$, where $\mathcal{K}\colon \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ is a linear map while $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ are two open sets - see Theorem 3.3.14. In addition, the notion of Besov wavefront set has allowed us to formulate a sufficient condition for the well-posedness of the extension of $\mathcal{K}$ to the whole space of compactly supported distributions - see Theorem 3.3.15. This result entails a microlocal version of the renown Schauder estimates, which are of relevance in the estimate of the regularity of a solution to a suitable class of SPDEs - see Corollary 3.3.16.

In Subsection 3.3.19, we proved a propagation of singularities theorem for a specific class of hyperbolic partial differential equations within the context of the Besov wavefront set - see Theorem 3.3.19. This result aims at characterizing the Besov wavefront set of a solution to a suitable hyperbolic partial differential equation in terms of the principal symbol of the corresponding differential operator.

Lastly, in Section 3.4, we have presented an application of the previous results in the context of coherent germs of distributions. More precisely, given a coherent germ defined as the tensor product of two distributions $u \in B^{\alpha_1}_{\infty,\infty}(\mathbb{R}^d)$ and $v \in B^{\alpha_2}_{\infty,\infty}(\mathbb{R}^d)$ with $\alpha_1 + \alpha_2 > 0$, we have proved that its reconstruction amounts to the product between u and v, which in turn coincides with the pullback of the germ along the diagonal. This led us to conjecture that, in general, the reconstruction of a γ-coherent germ with $\gamma > 0$ is the pullback of the germ along the diagonal - see Conjecture 3.4.3. We postpone to future works the investigation of this proposal.

Since the random forcing term of an SPDE is tipically a distribution lying in a suitable $B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$, we expect that the Besov wavefront set shall play a prominent rôle in estimating the singular behavior of a solution to such an equation. For instance, in [DDRZ21], a novel approach has been developed for the study of a large class of nonlinear SPDEs at a perturbative level resorting to microlocal techniques. More precisely, microlocal analysis, in particular the notion of smooth wavefront set, has been used to construct solutions of a suitable class of SPDEs by means of a recursive scheme. However, this approach is not able to establish a convergence of the perturbative series with respect to a suitable $B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$ norm. This lack of any control of convergence can be mainly ascribed to the fact that the smooth wavefront set is not well-suited to characterize the singular behavior of elements lying in a suitable Besov space $B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$. We expect that the notion of Besov wavefront set is the right tool to overcome this drawback. This shall be subject of investigation for future works.

Furthermore, a future goal shall be the formulation of a notion of wavefront set for any class of Besov spaces $B^{\alpha}_{p,q}(\mathbb{R}^d)$ with $1 \leq p, q \leq \infty$. This shall entail even more refined estimates of the singular behavior of an underlying distribution.

Reconstruction theorem on smooth manifolds. In Chapter 4, we extended the reconstruction theorem, one of the cornerstones of the theory of regularity structures, to the smooth manifold setting - see [RS21]. This generalization has been obtained by relying on the framework of germs of distributions [CZ20], which turned out to be well-suited for our purposes. In the following, we summarize the main results of this chapter.

In Section 4.1, we generalized the notion of coherent germ of distributions to the smooth manifold setting. In addition, we proved that this notion is independent of the choice of an atlas - see Proposition 4.1.14.

In Section 4.2, we proved the reconstruction theorem for γ-coherent germs of distributions on a smooth manifold. In particular, if $\gamma > 0$, we proved that the reconstruction is independent of the choice of atlas - see Theorem 4.2.2. Otherwise if $\gamma \leq 0$, the reconstruction is non-unique since it depends on the choice of an underlying atlas and of the partition of unity subordinated to it - see Theorem 4.2.4.

Our formulation of the reconstruction theorem can be read as a first step for the extension of the theory of regularity structures to smooth manifolds, which shall be a topic of investigation in future works.

Furthermore, we shall study the interplay between Besov wavefront sets and the reconstruction of a coherent germ of distributions on a smooth manifold. As a matter of fact, similarly to the Euclidean case, if the coherence exponent is strictly positive, we conjectured that the reconstruction coincides with the pullback of the germ along the diagonal - see Remark 4.2.3.

Appendix A

Theory of distributions

In this appendix, we outline the main concepts concerning the theory of distributions. We shall mainly refer to [FJ99] and to [Hör03].

A.1 Test functions

In this section, we introduce the space of *test functions*, which plays a leading rôle in the theory of distributions. First we recall the definition of *support* of a function. Let $\Omega \subset \mathbb{R}^d$ be an open set. Given a function $\varphi\colon \Omega \to \mathbb{C}$, the support of ϕ, denoted by $\operatorname{supp}(\phi)$, is defined as

$$\operatorname{supp}(\phi) := \overline{\{x \in \Omega : \phi(x) \neq 0\}}.$$

Note that $\operatorname{supp}(\phi)$ is a closed subset of Ω. Moreover, given $m \in \mathbb{N}_0$, we denote by $C^m(\Omega)$ the space of m-times continuously differentiable complex-valued functions in Ω, that is to say, $\phi\colon \Omega \to \mathbb{C}$ lies in $C^m(\Omega)$ if and only if all partial derivatives

$$\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial \phi}{\partial x_{i_j}}$$

of order $j \leq m$ exist and are continuous. In order to lighten the notation, we shall write them as

$$\partial_1^{\ell_1} \cdots \partial_d^{\ell_d} \phi = \partial^\ell \phi$$

where $\partial_j = \partial / \partial x_j$ and $\ell = (\ell_1, \dots, \ell_d)$ is a *multi-index*, that is, a d-tuple of non-negative integers. The order of differentiation is given by $|\ell| := \sum_{j=1}^d \ell_j$. Furthermore, we set

$$C^\infty(\Omega) = \bigcap_{k=0}^\infty C^m(\Omega).$$

The set $C^\infty(\Omega)$ identifies the space of smooth functions.

Definition A.1.1: *Let $\Omega \subset \mathbb{R}^d$ be an open set. The space $\mathcal{D}(\Omega)$ is the set of all $\phi \in C^\infty(\Omega)$ with compact support in Ω. The elements of $\mathcal{D}(\Omega)$ are called* ***test functions****. Furthermore, given $m \in \mathbb{N}_0$, we denote by $C_c^m(\Omega)$ the set of all $\phi \in C^m(\Omega)$ such that $\operatorname{supp}(\phi)$ is a compact subset of Ω.*

We shall sometimes denote the space of test functions by $C_c^\infty(\Omega)$. In the following, we endow $\mathcal{D}(\Omega)$ with a notion of convergence.

Definition A.1.2: *Let $\Omega \subset \mathbb{R}^d$ be an open set. We say that a sequence $(\phi_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ converges in $\mathcal{D}(\Omega)$ to $\phi \in \mathcal{D}(\Omega)$ if there exists a compact set $\mathfrak{K} \subset \Omega$ such that $\mathrm{supp}(\phi_j) \subset \mathfrak{K}$ for all $j \in \mathbb{N}$ and*

$$\lim_{j\to\infty} \sup_{x\in\mathfrak{K}} |\partial^\ell \phi_j(x) - \partial^\ell \phi(x)| = 0. \tag{A.1.1}$$

for any multi-index $\ell \in \mathbb{N}_0^d$. We denote it by $\phi_j \overset{\mathcal{D}}{\to} \phi$.

Remark A.1.3: *We can also give a notion of convergence in $C_c^m(\Omega)$. In this case, it suffices to require uniform convergence of all partial derivatives up to the m-th order.*

Fixed a compact set $\mathfrak{K} \subset \Omega$, we recall that $\mathcal{D}(\mathfrak{K})$ is a Fréchet space with the topology endowed by the family of semi-norms

$$\Big\{\phi \mapsto \|\phi\|_N := \sum_{|\ell|\le N} \sup_{x\in\mathfrak{K}} |\partial^\ell \phi(x)| : N \in \mathbb{N}_0\Big\}. \tag{A.1.2}$$

For further comments on the topology of $\mathcal{D}(\Omega)$, refer to [FJ99, Appendix].

We conclude by giving a notion of convergence in $C^\infty(\Omega)$.

Definition A.1.4: *A sequence $(\phi_j)_{j\in\mathbb{N}} \subset C^\infty(\Omega)$ is said to converge in $C^\infty(\Omega)$ to $\phi \in C^\infty(\Omega)$ if, for each multi-index $\ell \in \mathbb{N}_0^d$,*

$$\lim_{j\to\infty} \sup_{x\in\mathfrak{K}} |\partial^\ell \phi_j(x) - \partial^\ell \phi(x)| = 0$$

for every compact set $\mathfrak{K} \subset \Omega$. We write $\phi_j \overset{C^\infty}{\to} \phi$.

We recall that $C^\infty(\Omega)$ is a Fréchet space with respect to the semi-norms

$$\phi \mapsto \|\phi\|_{N,\mathfrak{K}} := \sum_{|\ell|\le N} \sup_{x\in\mathfrak{K}} |\partial^\ell \phi(x)|$$

where $N \in \mathbb{N}_0$ and $\mathfrak{K}$ ranges over the compact subsets of Ω - see [FJ99, Appendix].

For the sake of completeness, given $m \in \mathbb{N}_0$, we define the C^m-norm as follows

$$\phi \mapsto \|\phi\|_{C^m(\Omega)} := \sum_{|\ell|\le m} \sup_{x\in\Omega} |\partial^\ell \phi(x)|. \tag{A.1.3}$$

Smooth functions on a manifold We recall the notion of smooth function on a smooth manifold.

Definition A.1.5: *Let $(M, \mathcal{A})$ be a d-dimensional smooth manifold, where $\mathcal{A}$ is a smooth atlas thereon. We say that a function $\phi\colon M \to \mathbb{C}$ is **smooth** at $\mathsf{p} \in M$ if there exists a local chart $(U, h) \in \mathcal{A}$ such that $\mathsf{p} \in U$ and $\phi \circ h^{-1}\colon h(U) \to \mathbb{C}$ is smooth at $h(\mathsf{p})$. In addition, $\phi\colon M \to \mathbb{C}$ is said to be a **smooth function** on M if it is smooth at any point $\mathsf{p} \in M$. We denote the space of smooth functions on M by $C^\infty(M)$.*

Remark A.1.6: *The smoothness of a function supported on a smooth manifold is independent of the choice of local charts.*

In the following, we define the space of test functions on a smooth manifold. As usual, given a smooth manifold M, we denote the support of a function $\phi\colon M \to \mathbb{C}$ by

$$\mathrm{supp}(\phi) := \overline{\{\mathsf{p} \in M : \phi(\mathsf{p}) \neq 0\}}.$$

Definition A.1.7: *Let $(M, \mathcal{A})$ be a d-dimensional smooth manifold. The space $\mathcal{D}(M)$ is the set of all $\phi \in C^\infty(M)$ such that* supp(ϕ) *is compact in M. The elements of $\mathcal{D}(M)$ are called* ***test functions*** *on M.*

A.2 Distributions

This section shall be devoted to defining and to characterizing *distributions* - see [Hör03, Sect. 2.1], [FJ99, Sect. 1.3].

Definition A.2.1: *Let $\Omega \subset \mathbb{R}^d$ be an open set. A linear map $u : \mathcal{D}(\Omega) \to \mathbb{C}$ is a* ***distribution*** *in Ω if, for every compact set $\mathfrak{K} \subset \Omega$, there exists a constant $C_\mathfrak{K} > 0$ and a nonnegative integer $N_\mathfrak{K}$ such that*

$$|\langle u, \phi\rangle| = |u(\phi)| \leq C_\mathfrak{K} \|\phi\|_{C^{N_\mathfrak{K}}(\Omega)}, \quad \forall \phi \in \mathcal{D}(\mathfrak{K}), \tag{A.2.1}$$

where $\|\cdot\|_{C^{N_\mathfrak{K}}}$ has been defined in Equation (A.1.3). The set of all distributions in Ω is denoted by $\mathcal{D}'(\Omega)$.

Remark A.2.2: *Let $\phi \in C^m(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is an open set. If $\phi_x^\lambda \colon \Omega \to \mathbb{R}$ is the scaled version of ϕ as per Equation 2.1.15, then we can estimate its C^m-norm as follows:*

$$\|\phi_x^\lambda\|_{C^m(\Omega)} \leq \lambda^{-d-m} \|\phi\|_{C^m(\Omega)}. \tag{A.2.2}$$

Let $u \in \mathcal{D}'(\Omega)$. On account of Equation (A.2.2), given a compact set $\mathfrak{K} \subset \mathbb{R}^d$, it descends that there exist a constant $C_\mathfrak{K} > 0$ and a nonnegative integer $N_\mathfrak{K}$ such that

$$|u(\phi_x^\lambda)| \leq C_\mathfrak{K} \|\phi\|_{C^{N_\mathfrak{K}}(\Omega)} \lambda^{-N_\mathfrak{K}-d}, \quad \forall x \in \mathfrak{K}, \forall \lambda \in (0,1], \forall \phi \in \mathcal{D}(\mathfrak{K}).$$

The set $\mathcal{D}'(\Omega)$ is a vector space, whose structure is completely fixed by

$$(u+v)(\phi) = u(\phi) + v(\phi), \quad (au)(\phi) = au(\phi), \quad \forall u, v \in \mathcal{D}'(\Omega), \forall \phi \in \mathcal{D}(\Omega), \forall a \in \mathbb{R}.$$

Note that the bound in Equation (A.2.1) encodes a continuity property for the functional u with respect to the semi-norms on $\mathcal{D}(\mathfrak{K})$. Moreover, we recall that there is an equivalent characterization of Equation (A.2.1) in terms of *sequential continuity*. The following theorem codifies this fact.

Theorem A.2.3: *A linear map u on $\mathcal{D}(\Omega)$ is a distribution according to Definition A.2.1 if and only if $\lim_{j\to\infty} u(\phi_j) = 0$ for every sequence $(\phi_j)_{j\in\mathbb{N}_0} \subset \mathcal{D}(\Omega)$ such that $\phi_j \overset{\mathcal{D}}{\to} 0$.*

Next we introduce a notion of a convergent sequence of distributions.

Definition A.2.4: *Let $\Omega \subset \mathbb{R}^d$ be an open set. A sequence of distributions $(u_j)_{j\in\mathbb{N}_0} \subset \mathcal{D}'(\Omega)$ is said to converge in $\mathcal{D}'(\Omega)$ to $u \in \mathcal{D}'(\Omega)$ if*

$$\lim_{j\to\infty} u_j(\phi) = u(\phi), \quad \forall \phi \in \mathcal{D}(\Omega).$$

We write $u_j \overset{\mathcal{D}'}{\to} u$.

To conclude this section, we recall the definition of *distributions of finite order*.

Definition A.2.5: *Let $u \in \mathcal{D}'(\Omega)$. If there exists $N \in \mathbb{N}_0$ such that the bound in Equation (A.2.1) is valid for every compact set $K \subset \Omega$, then u is said to be of order $\leq N$. The vector space of such distributions is denoted by $\mathcal{D}'^N(\Omega)$. In addition, their union $\mathcal{D}'_F(\Omega) = \bigcup_N \mathcal{D}'^N(\Omega)$ is the space of distributions of finite order.*

A.3 Localization

Let $\tilde{\Omega} \subset \Omega \subset \mathbb{R}^d$ and let $u \in \mathcal{D}'(\Omega)$. Since $\mathcal{D}(\tilde{\Omega}) \subset \mathcal{D}(\Omega)$, one can restrict u to a distribution $u|_{\tilde{\Omega}} \in \mathcal{D}'(\tilde{\Omega})$ by setting

$$u|_{\tilde{\Omega}}(\phi) := u(\phi) \quad \forall \phi \in \mathcal{D}(\tilde{\Omega}). \tag{A.3.1}$$

The distribution $u|_{\tilde{\Omega}}$ is the *restriction* or *localization* of u to $\tilde{\Omega}$. Such a notion allows to define the support of a distribution.

Definition A.3.1: *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $u \in \mathcal{D}'(\Omega)$. The **support** of u is the complement of the set*

$$\{x \in \Omega : \exists U_x \subset \Omega \text{ a neighborhood of } x \text{ s.t. } u|_{U_x} = 0\}.$$

The support is denoted by $\operatorname{supp}(u)$.

Observe that $\operatorname{supp}(u)$ is a closed subset of Ω as the support of a function. In addition, we give the definition of *singular support* of a distribution.

Definition A.3.2: *Let $\Omega \subset \mathbb{R}^d$ be an open subset and let $u \in \mathcal{D}'(\Omega)$. The **singular support** of u, denoted by* $\operatorname{singsupp}(u)$, *is the complement of the set*

$$\{x \in \Omega : \exists U_x \subset \Omega \text{ a neighborhood of } x \text{ s.t. } u|_{U_x} \in C^\infty(U_x)\}.$$

Note that the singular support is a closed set in Ω. Generally speaking, a distribution in Ω can be defined starting from its localizations. As a matter of fact, the following theorem holds true - see [FJ99, Th. 1.4.3].

Theorem A.3.3: *Let $(\Omega_i)_{i\in I}$, where I is an index set, be a family of open subsets of $\mathbb{R}^d$ such that $\Omega = \bigcup_{i\in I} \Omega_i$. Moreover, for any $i \in I$, let $u_i \in \mathcal{D}'(\Omega_i)$ be such that the overlapping condition,*

$$u_i = u_j \text{ on } \Omega_i \cap \Omega_j,$$

holds true for all $i, j \in I$. Then there exists a unique $u \in \mathcal{D}'(\Omega)$ such that $u|_{\Omega_i} = u_i$ for every $i \in I$.

A.4 Distributions with compact support

The aim of this section is to discuss the space of compactly supported distributions.

Definition A.4.1: *Let $\Omega \subset \mathbb{R}^d$ be an open set. A linear map $u\colon C^\infty(\Omega) \to \mathbb{C}$ is called continuous if there exist a compact set $\mathfrak{K} \subset \Omega$, a constant $C > 0$ and $N \in \mathbb{N}_0$ such that*

$$|u(\phi)| \leq C \sum_{|\ell|\leq N} \sup_{x\in\mathfrak{K}} |\partial^\ell \phi(x)|, \quad \forall \phi \in C^\infty(\Omega). \tag{A.4.1}$$

The space of all continuous linear maps on $C^\infty(\Omega)$ is denoted by $\mathcal{E}'(\Omega)$.

Similarly to $\mathcal{D}'(\Omega)$, the continuity property in Equation (A.4.1) can be equivalently characterized in terms of sequential continuity.

Theorem A.4.2: *Let $\Omega \subset \mathbb{R}^d$ be an open set. Then u lies in $\mathcal{E}'(\Omega)$ if and only if $u(\phi_j) \to u(\phi)$ for any sequence $(\phi_j)_{j\in\mathbb{N}} \subset C^\infty(\Omega)$ such that $\phi_j \overset{C^\infty}{\to} \phi$.*

The following theorem states that $\mathcal{E}'(\Omega)$ can be identified with the space of distributions with compact support.

Theorem A.4.3: *Let $u \in \mathcal{D}'(\Omega)$ be with compact support. Then there exists a unique element $v \in \mathcal{E}'(\Omega)$ such that its restriction to $\mathcal{D}(\Omega)$ coincides with u.*

We conclude by stating a corollary which is a direct consequence of Definition A.4.1.

Corollary A.4.4: *Let $u \in \mathcal{E}'(\mathbb{R}^d)$. Then u is of finite order, as per Definition A.2.5.*

A.5 Differentiation and multiplication

We shall outline two basic operations on distributions: differentiation and multiplication by smooth functions. We start by introducing the concept of *distributional derivative*.

Definition A.5.1: *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $m \in \mathbb{N}_0$. For $u \in \mathcal{D}'(\Omega)$, its* ***distributional derivative*** *$\partial^\ell u$ of order m is defined by*

$$(\partial^\ell u)(\phi) := (-1)^{|\ell|} u(\partial^\ell \phi) \quad \forall \phi \in \mathcal{D}(\Omega), \tag{A.5.1}$$

where ℓ is a multi-index such that $|\ell| = m$.

As the space $\mathcal{D}(\Omega)$ is stable under differentiation and u lies in $\mathcal{D}'(\Omega)$, Equation (A.5.1) implies that $\partial^\ell u$ is, in turn, a distribution.

Next we extend to distributions the multiplication by smooth functions.

Definition A.5.2: *Let $\Omega \subset \mathbb{R}^d$ be an open set. If $f \in C^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$, then the product fu is the distribution defined by*

$$(fu)(\phi) := u(f\phi), \quad \forall \phi \in \mathcal{D}(\Omega). \tag{A.5.2}$$

Furthermore, we stress that the Leibniz rule holds true:

$$\partial_j(fu) = (\partial_j f)u + f(\partial_j u),$$

for any $f \in C^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$. At the same time, it is noteworthy to point out that the product among two distributions is, in general, ill-defined. The following theorem gives a sufficient condition on singular supports whereby the multiplication of distributions is well-defined - see [Hör03, Sect. 3.1].

Theorem A.5.3: *Let $\Omega \subset \mathbb{R}^d$ be an open set and $u, v \in \mathcal{D}'(\Omega)$. If*

$$\mathrm{singsupp}(u) \cap \mathrm{singsupp}(v) = \emptyset,$$

then the product uv is well-defined.

A.6 Tensor product

This section shall be devoted to defining the tensor product of distributions. For this part, we mainly refer to [Hör03, Sect. 5.1].

Definition A.6.1: *For $i = 1, 2$, let $\Omega_i \subset \mathbb{R}^{d_i}$ be an open set and let $f_i \in C^0(\Omega_i)$. We call* ***tensor product*** *of f_1 and f_2 the function $f_1 \otimes f_2$ in $\Omega_1 \times \Omega_2 \subset \mathbb{R}^{d_1+d_2}$ defined by*

$$(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1) f_2(x_2), \quad \forall (x_1, x_2) \in \Omega_1 \times \Omega_2.$$

The previous definition can be extended to distributions. As a matter of fact, since $f_i \in C^0(\Omega_i)$ for $i = 1, 2$, the tensor product $f_1 \otimes f_2$ lies in $C^0(\Omega_1 \times \Omega_2)$ and hence it extends to a distribution in $\Omega_1 \times \Omega_2$ such that

$$(f_1 \otimes f_2)(\phi) = \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} f_1(x_1) f_2(x_2) \phi(x_1, x_2) dx_1 dx_2, \quad \forall \phi \in \mathcal{D}(\Omega_1 \times \Omega_2).$$

In particular, if $\phi = \phi_1 \otimes \phi_2$ with $\phi_i \in \mathcal{D}(\Omega_i)$, then

$$(f_1 \otimes f_2)(\phi_1 \otimes \phi_2) = \int_{\mathbb{R}^{d_1}} f_1(x_1)\phi_1(x_1)dx_1 \int_{\mathbb{R}^{d_2}} f_2(x_2)\phi_2(x_2)dx_2 = f_1(\phi_1)f_2(\phi_2).$$

Theorem A.6.2: *For $i = 1, 2$, let $\Omega_i \subset \mathbb{R}^{d_i}$ be an open set and let $u_i \in \mathcal{D}'(\Omega_i)$. Then there exists a unique distribution $u_1 \otimes u_2 \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ such that*

$$(u_1 \otimes u_2)(\phi_1 \otimes \phi_2) = u_1(\phi_1)u_2(\phi_2), \quad \forall \phi_1 \in \mathcal{D}(\Omega_1), \quad \forall \phi_2 \in \mathcal{D}(\Omega_2).$$

The distribution $u_1 \otimes u_2$ is called the tensor product of u_1 and u_2.

To conclude this section, we list a few notable properties of the tensor product.

Theorem A.6.3: *The following statements hold true:*

(i) The tensor product $u_1 \otimes u_2$ can be continuously extended on $\mathcal{D}(\Omega_1 \times \Omega_2)$. Furthermore we have

$$(u_1 \otimes u_2)(\phi) = u_1(u_2(\phi(x_1, x_2))) = u_2(u_1(\phi(x_1, x_2))), \quad \forall \phi \in \mathcal{D}(\Omega_1 \times \Omega_2),$$

where u_i acts on ϕ as a function of x_i only, that is

$$u_1(\phi(\cdot, x_2)), \quad u_2(\phi(x_1, \cdot)).$$

(ii) The support of $u_1 \otimes u_2$ is $\mathrm{supp}(u_1) \times \mathrm{supp}(u_2)$.

(iii) Let ℓ_1, ℓ_2 be two multi-indexes. Then

$$\partial_{x_1}^{\ell_1} \partial_{x_2}^{\ell_2}(u_1 \otimes u_2) = (\partial_{x_1}^{\ell_1} u_1) \otimes (\partial_{x_2}^{\ell_2} u_2).$$

A.7 Convolution

In this section, we shall see how to define the convolution between two distributions, see [FJ99, Chap. 5]. To this end, we start by recalling the definition when considering two integrable functions. If $f, g \in L^1(\mathbb{R}^d)$, the convolution of f and g, $f * g$ on $\mathbb{R}^d$, is defined by

$$(f * g)(x) := \int_{\mathbb{R}^d} f(y)g(x - y)dy = \int_{\mathbb{R}^d} f(x - y)g(y)dy, \quad x \in \mathbb{R}^d.$$

Then $f * g$ is well defined and, in particular, it lies in $L^1(\mathbb{R}^d)$. Therefore, $f * g$ can be extended to a distribution in $\mathbb{R}^d$ such that

$$(f * g)(\phi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)g(y)\phi(x + y)dxdy, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d).$$

Bearing in mind this example, we define the convolution of two distributions.

Definition A.7.1: *Let $u \in \mathcal{E}'(\mathbb{R}^d)$ and $v \in \mathcal{D}'(\mathbb{R}^d)$. Then their convolution $u * v$ is the distribution lying in $\mathcal{D}'(\mathbb{R}^d)$ defined by*

$$(u * v)(\phi) = \langle u(x) \otimes v(y), \phi(x + y)\rangle, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d). \tag{A.7.1}$$

Remark A.7.2: *In general, if $u, v \in \mathcal{D}'(\mathbb{R}^d)$, the right hand side of Equation (A.7.1) may not exist because $(x, y) \mapsto \phi(x + y)$ does not have compact support. Therefore, it is crucial to assume that one of the two distributions in Equation (A.7.1) lies in $\mathcal{E}'(\mathbb{R}^d)$.*

Remark A.7.3: *We remark that u and v can be switched in Definition A.7.1. This implies that the convolution is commutative.*

In the following, we recall some properties of the convolution. We begin with two upper bounds for the support and for the singular support respectively.

Theorem A.7.4: *Let $u \in \mathcal{E}'(\mathbb{R}^d)$ and $v \in \mathcal{D}'(\mathbb{R}^d)$. Then*

$$\text{supp}(u * v) \subset \text{supp}(u) + \text{supp}(v), \quad \text{singsupp}(u * v) \subset \text{singsupp}(u) + \text{singsupp}(v).$$

Let $h \in \mathbb{R}^d$. Given a function ϕ, we define the translation map τ_h by

$$(\tau_h \phi)(x) = \phi(x - h), \quad x \in \mathbb{R}^d.$$

By duality, the action of the map τ_h can be extended to distributions.

Definition A.7.5: *Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$. Then $\tau_h u$ is the distribution defined by*

$$(\tau_h u)(\phi) = u(\tau_{-h}\phi), \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d).$$

Theorem A.7.6: *Let $u \in \mathcal{E}'(\mathbb{R}^d)$ and $v \in \mathcal{D}'(\mathbb{R}^d)$. Then the following statements hold true:*

- $\partial^\ell (u * v) = u * (\partial^\ell v) = (\partial^\ell u) * v$, *for any multi-index* $\ell \in \mathbb{N}_0^d$.
- $\tau_h(u * v) = (\tau_h u) * v = u * (\tau_h v)$, *for any* $h \in \mathbb{R}^d$.
- $\delta * v = v$, *where δ is the Dirac delta centered at the origin.*

Next we recall that the convolution with a compactly supported smooth function yields an element of $C^\infty(\mathbb{R}^d)$.

Theorem A.7.7: *Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\rho \in \mathcal{D}(\mathbb{R}^d)$. Then $\rho * u$ lies in $C^\infty(\mathbb{R}^d)$ and*

$$(\rho * u)(x) = u(\rho(x - \cdot)), \quad x \in \mathbb{R}^d.$$

Definition A.7.8: *Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\rho \in \mathcal{D}(\mathbb{R}^d)$. We say that $\rho * u$ is a* ***regularization*** *of u.*

On account of Theorem A.7.7, one can prove the following density result.

Theorem A.7.9: *Let $\Omega \subseteq \mathbb{R}^d$, then $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}'(\Omega)$.*

At last, we prove a result of distribution theory which shall play a leading rôle in the proof of Theorem 4.2.1 - see [RS21, Lemma 24]. The proof of this result is based on the following lemma.

Lemma A.7.10: *Let $\rho \in \mathcal{D}(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} \rho(x)dx = 1$. For any $\phi \in \mathcal{D}(\mathbb{R}^d)$, then $\phi * \rho^\lambda \xrightarrow{\mathcal{D}} \phi$ as per Definition A.1.2, where we set $\rho^\lambda(\cdot) := \lambda^{-d}\rho(\lambda^{-1}\cdot)$.*

Lemma A.7.11: *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $u \in \mathcal{D}'(\Omega)$ be such that, for any compact set $\mathfrak{K} \subset \Omega$ and any $\rho \in \mathcal{D}(\mathfrak{K})$ such that $\int \rho(x)dx = 1$, $\sup_{x\in\mathfrak{K}}|u(\rho_x^\lambda)| \to 0$ as $\lambda \to 0^+$. Then, $u(\phi) = 0$ for any $\phi \in \mathcal{D}(\Omega)$.*

Proof. Let $\mathfrak{K} \subset \Omega$ be a compact set and let $\rho \in \mathcal{D}(\mathfrak{K})$ such that $\int_{\mathbb{R}^d} \rho(x)dx = 1$. Given $\phi \in \mathcal{D}(\mathfrak{K})$, on account of Lemma A.7.10, it descends that $\phi * \rho^\lambda \xrightarrow{\mathcal{D}} \phi$ as $\lambda \to 0^+$. By sequential continuity of u as per Definition A.2.4, we infer that $u(\phi * \rho^\lambda) \to u(\phi)$ as $\lambda \to 0^+$. Moreover, it descends that

$$|u(\phi * \rho^\lambda)| = \left| \int_{\mathbb{R}^d} \phi(x) u(\rho_x^\lambda) dx \right| \leq \mathcal{L}^d(\text{supp}(\phi)) \|\phi\|_{L^\infty(\Omega)} \sup_{x\in\mathfrak{K}} |u(\rho_x^\lambda)|,$$

where $\mathcal{L}^d(\text{supp}(\phi))$ denotes the d-dimensional volume of $\text{supp}(\phi)$. Since $\sup_{x\in\mathfrak{K}}|u(\rho_x^\lambda)| \to 0$ as $\lambda \to 0^+$ per hypotesis, it descends that $u(\phi) = 0$. Since this argument holds true for any compact set $\mathfrak{K} \subset \Omega$, we infer that $u(\phi) = 0$ for any $\phi \in \mathcal{D}(\Omega)$. □

A.8 The Schwartz kernel theorem

In this section, we recall the Schwartz kernel theorem. For this topic, we refer to [Hör03, Sect. 5.2]. Given two open sets $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$, we begin by observing that any function $K \in C^0(\Omega_1 \times \Omega_2)$ defines an integral operator $\mathcal{K}\colon \mathcal{D}(\Omega_2) \to C^0(\Omega_1)$ given by

$$(\mathcal{K}\phi)(x_1) = \int_{\Omega_2} K(x_1, x_2)\phi(x_2)dx_2, \quad \forall \phi \in \mathcal{D}(\Omega_2),\ \forall x_1 \in \Omega_1.$$

The kernel theorem states that the definition of $\mathcal{K}$ can be extended to $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$. As a matter of fact, we remark that, if $K \in C^0(\Omega_1 \times \Omega_2)$, then

$$(\mathcal{K}\phi)(\psi) = K(\psi \otimes \phi), \quad \forall \psi \in \mathcal{D}(\Omega_1),\ \forall \phi \in \mathcal{D}(\Omega_2). \tag{A.8.1}$$

Theorem A.8.1: *Let $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets. A linear map $\mathcal{K}\colon \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ is sequentially continuous if and only if there exists a unique distribution $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ such that Equation (A.8.1) is valid. The distribution K is called the kernel of $\mathcal{K}$.*

Remark A.8.2: *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $u \in \mathcal{D}'(\Omega)$. In this manuscript, with a slight abuse of notation, we sometimes shall denote u by means of a formal* integral kernel *$u(x)$, namely*

$$u(\phi) = \int_{\Omega} u(x)\phi(x)dx, \quad \forall \phi \in \mathcal{D}(\Omega). \tag{A.8.2}$$

A.9 Pullback of a distribution along a smooth function

This section shall be devoted to defining the composition of distributions with smooth functions. Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets and let $f\colon \Omega_1 \to \Omega_2$ be a smooth function. If $u \in C^0(\Omega_2)$, then u can always be pulled back along f. As a matter of fact, the pullback of u along f is the composition $u \circ f \in C^0(\Omega_1)$. At the same time, this operation can be extended to all distributions if the differential of f, written as df, is surjective. In this respect we state the following theorem of which we sketch the proof - see [Hör03, Th. 6.1.2].

Definition A.9.1: *Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets where $d_1 \geq d_2$. A smooth map $f\colon \Omega_1 \to \Omega_2$ is said to be a* ***submersion*** *if $df(x)$ is surjective for every $x \in \Omega_1$.*

Theorem A.9.2: *Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets where $d_1 \geq d_2$ and let $f\colon \Omega_1 \to \Omega_2$ be a submersion. Then there exists a unique continuous linear map $f^*\colon \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$ such that $f^*u = u \circ f$ for any $u \in C^0(\Omega_2)$. We call f^*u the* ***pullback*** *of u along f.*

Proof. Uniqueness follows immediately from Theorem A.7.9. At the same time, existence is more involved. Given $x_0 \in \Omega_1$, choose a function $g\colon \Omega_1 \to \mathbb{R}^{d_1-d_2}$ such that the pair (f, g), defined by

$$\Omega_1 \ni x \mapsto (f(x), g(x)) \in \mathbb{R}^{d_1} = \mathbb{R}^{d_2} \oplus \mathbb{R}^{d_1-d_2},$$

has a bijective differential at x_0. On account of the inverse function theorem, there exist an open neighborhood $\Omega'_1 \subset \Omega_1$ of x_0 and an open neighborhood Ω'_2 of $(f(x_0), g(x_0))$ such that

$$f \oplus g|_{\Omega'_1}\colon \Omega'_1 \to \Omega'_2.$$

is a diffeomorphism. Let then h be the local inverse of $f \oplus g|_{\Omega'_1}$. If $u \in C^0(\Omega_2)$ and $\phi \in \mathcal{D}(\Omega'_1)$, then

$$\int_{\Omega'_1} (f^*u)(x)\phi(x)dx = \int_{\Omega'_1} u(f(x))\phi(x)dx = \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1-d_2}} u(y')\phi(h(y', y''))|\det dh(y', y'')|dy'dy'',$$

where we applied the change of variable $f(x) \mapsto y'$ with $(y', y'') \in \mathbb{R}^{d_2} \times \mathbb{R}^{d_1 - d_2}$ in the second equality. Therefore we have

$$(f^*u)(\phi) = (u \otimes 1)(\Phi), \quad \Phi(y) = \phi(h(y))|\det dh(y', y'')|, \tag{A.9.1}$$

where 1 is the constant function 1 in $\mathbb{R}^{d_1-d_2}$. To conclude, by Theorem A.7.9, Equation (A.9.1) can be extended to $u \in \mathcal{D}'(\Omega_2)$. Moreover, Equation (A.9.1) is a localization of f^*u and it entails its continuity. □

Remark A.9.3: *Note that if $d_1 = d_2$ and the map $f\colon \Omega_1 \to \Omega_2$ is a diffeomorphism, then the pullback f^*u is given by*

$$(f^*u)(\phi) = \langle u(y), (f^{-1})^*\phi(y)|\det df^{-1}(y)|\rangle, \quad \phi \in \mathcal{D}(\Omega_1).$$

Since f^*u has been defined by a continuous extension of the composition for functions, then the following rules still hold true:

$$\partial_j f^*u = \sum_{i=1}^{d_2} \partial_j f_i f^* \partial_i u, \quad u \in \mathcal{D}'(\Omega_2) \text{ (chain rule)},$$

$$f^*(\psi u) = (f^*\psi)(f^*u), \quad \psi \in C^\infty(\Omega_2),\ u \in \mathcal{D}'(\Omega_2),$$

where we set $f = (f_1, \ldots, f_{d_2})$.

A.10 Distributions on Smooth Manifolds

This section is devoted to recalling the basic notions and results concerning distribution theory on smooth manifolds. The following notions play a key rôle in Chapter 4. For further details concerning this topic, the reader may refer to [Hör03, Sect. 6.3]. We start by giving the definition of distribution supported on an arbitrary smooth manifold M.

Definition A.10.1: *Let M be a d-dimensional smooth manifold. For any local chart (U, h) on M, let $u_{h(U)} \in \mathcal{D}'(h(U))$ be such that the overlapping condition,*

$$u_{h'(U')} = (h \circ h'^{-1})^* u_{h(U)} \quad \text{on} \quad h'(U \cap U'), \tag{A.10.1}$$

holds true for any pair of local charts $(U, h), (U', h')$ on M, where $(h \circ h'^{-1})^*$ is the pullback along $h \circ h'^{-1}$ as per Remark A.9.3. We call the family $\{u_{h(U)}\}_{(U,h)}$ a distribution u on M. We denote the set of all distributions on M by $\mathcal{D}'(M)$.

The following theorem provides a rather useful characterization of the concept of distribution on a smooth manifold. As a matter of fact, it asserts that Equation (A.10.1) can be checked only on one atlas in order to construct an element lying in $\mathcal{D}'(M)$ instead of considering all possible local charts on M - see [Hör03, Th. 6.3.4].

Theorem A.10.2: *Let M be a d-dimensional smooth manifold and let $\mathcal{A} = \{(U_i, h_i)\}_{i \in I}$ be a smooth atlas thereon. Assume that for any local chart $(U, h) \in \mathcal{A}$ there exists a distribution $u_{h(U)} \in \mathcal{D}'(h(U))$ such that Equation (A.10.1) holds true for any pair of local charts $(U, h), (U', h') \in \mathcal{A}$. Then there exists a unique $u \in \mathcal{D}'(M)$ such that $(h^{-1})^* u = u_{h(U)}$ for any $(U, h) \in \mathcal{A}$, where $(h^{-1})^*$ is the pullback along h^{-1} as per Remark A.9.3.*

A.11 Tempered distributions and Fourier transforms

In this section, we shall mainly introduce the space of *tempered distributions* and the theory of the Fourier transform. For further information on the following concepts, refer to [FJ99, Chap. 8] and to [Hör03, Chap. VII]. We start by introducing the space of *rapidly decreasing functions*. Henceforth, we shall use the differential operators

$$D_j := -i\partial_j, \quad j = 1, \dots, d,$$

where $i^2 = -1$. Moreover, if $\ell = (\ell_1, \dots, \ell_d) \in \mathbb{N}_0^d$ is a multi-index and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we set

$$x^\ell = x_1^{\ell_1} \cdots x_d^{\ell_d}.$$

Definition A.11.1: *A smooth function $\phi\colon \mathbb{R}^d \to \mathbb{C}$ is called **rapidly decreasing** if*

$$\|\phi\|_{\ell,k} := \sup_{x\in\mathbb{R}^d} |x^\ell D^k \phi(x)| < \infty, \tag{A.11.1}$$

for every multi-indexes $\ell, k \in \mathbb{N}_0^d$. We denote the space of rapidly decreasing functions by $\mathcal{S}(\mathbb{R}^d)$. Moreover, we say that a sequence $(\phi_j)_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^d)$ converges in $\mathcal{S}(\mathbb{R}^d)$ to ϕ if $\|\phi_j - \phi\|_{\ell,k} \to 0$ as $j \to \infty$ for all $\ell, k \in \mathbb{N}_0^d$. In this case, we write $\phi_j \xrightarrow{\mathcal{S}} \phi$.

The space $\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space with the topology induced by the semi-norms in the left-hand side of Equation (A.11.1). Subsequently, we list a few remarkable consequences of Definition A.11.1 in the following theorem.

Theorem A.11.2: *The following statements hold true:*

a) The space $\mathcal{S}(\mathbb{R}^d)$ is stable under differentiation and multiplication, that is $D_j\mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ and $x_j\mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ for all $j = 1, \dots, d$.

b) $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$.

c) $\mathcal{S}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ for any $1 \le p \le \infty$.

Given $f \in L^1(\mathbb{R}^d)$, we define the *Fourier transform* of f by

$$\mathcal{F}u(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}dx, \quad \forall \xi \in \mathbb{R}^d, \tag{A.11.2}$$

where $x \cdot \xi = \sum_{i=1}^d x_i\xi_i$ is the Euclidean inner product on $\mathbb{R}^d$.

Lemma A.11.3: *The Fourier transform $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is a continuous map, such that*

$$\widehat{D_j\phi} = \xi_j\widehat{\phi}, \quad \widehat{x_j\phi} = -D_j\widehat{\phi}$$

for every $\phi \in \mathcal{S}(\mathbb{R}^d)$.

Theorem A.11.4: *The Fourier transform $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is a continuous isomorphism with inverse given by Fourier's inversion formula:*

$$\mathcal{F}^{-1}\phi(x) = \check{\phi}(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \phi(\xi)e^{ix\cdot\xi}d\xi, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d). \tag{A.11.3}$$

Remark A.11.5: *Observe that we can write Equation (A.11.3) as*

$$\check{\phi}(\cdot) = (2\pi)^{-d}\widehat{\phi}(-\cdot)$$

Next we list some basic properties of the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$.

Theorem A.11.6: *Let $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$. Then*

$$\int_{\mathbb{R}^d} \widehat{\phi}(\xi)\psi(\xi)d\xi = \int_{\mathbb{R}^d} \phi(x)\widehat{\psi}(x)dx, \tag{A.11.4}$$

$$\int_{\mathbb{R}^d} \phi(x)\overline{\psi}(x)dx = (2\pi)^{-d}\int_{\mathbb{R}^d} \widehat{\phi}(\xi)\overline{\widehat{\psi}}(\xi)d\xi \quad \textit{(Parseval's formula)}, \tag{A.11.5}$$

$$\widehat{\phi * \psi} = \widehat{\phi}\widehat{\psi}, \tag{A.11.6}$$

$$\widehat{\phi\psi} = (2\pi)^{-d}\widehat{\phi} * \widehat{\psi}. \tag{A.11.7}$$

The topological dual of $\mathcal{S}(\mathbb{R}^d)$ is the space of tempered distributions.

Definition A.11.7: *A linear map $u\colon \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ is a **tempered distribution** if there exist a constant $C > 0$ and a nonnegative integer N such that*

$$|u(\phi)| \le C \sum_{|\ell|,|k|\le N} \sup_{x\in\mathbb{R}^d} |x^\ell D^k \phi(x)|, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d). \tag{A.11.8}$$

The space of tempered distribution is denoted by $\mathcal{S}'(\mathbb{R}^d)$.

The continuity property in Equation (A.11.8) can be equivalently characterized in terms of sequential continuity, that is, if $\phi_j \overset{\mathcal{S}}{\to} \phi$ then $u(\phi_j) \to u(\phi)$ as $j \to \infty$.
On account of statement b) of Theorem A.11.2, the restriction of a tempered distribution to $\mathcal{D}(\mathbb{R}^d)$ individuates an element in $\mathcal{D}'(\mathbb{R}^d)$. Therefore, the following set of inclusions holds true:

$$\mathcal{E}'(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \hookrightarrow \mathcal{D}'(\mathbb{R}^d).$$

By duality, we define the Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$.

Definition A.11.8: *Let $u \in \mathcal{S}'(\mathbb{R}^d)$. The Fourier transform $\hat{u}$ is the tempered distribution defined by*

$$\widehat{u}(\phi) = u(\widehat{\phi}), \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d). \tag{A.11.9}$$

Theorem A.11.9: *The Fourier transform $\mathcal{F}\colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, $u \mapsto \widehat{u}$, is an isomorphism. Moreover, the Fourier's inversion formula,*

$$u(\phi(-\cdot)) = (2\pi)^{-d}\widehat{u}(\widehat{\phi}) \quad \phi \in \mathcal{S}'(\mathbb{R}^d),$$

is valid for every $u \in \mathcal{S}'(\mathbb{R}^d)$.

For the sake of completeness, we recall the *Plancherel's theorem.* Taking into account that $L^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$, such a result states that the Fourier transform also extends to an isomorphism on $L^2(\mathbb{R}^d)$.

Theorem A.11.10: *If $u \in L^2(\mathbb{R}^d)$, then its Fourier transform $\hat{u}$ also lies in $L^2(\mathbb{R}^d)$. In addition, it holds true the Parseval's identity:*

$$\|\widehat{u}\|^2_{L^2(\mathbb{R}^d)} = (2\pi)^d \|u\|^2_{L^2(\mathbb{R}^d)}.$$

Again by duality, we can obtain the following identities:

$$\widehat{D^\ell u} = \xi^\ell \widehat{u}, \tag{A.11.10}$$

$$\widehat{x^\ell u} = (-1)^{|\ell|} D^\ell \widehat{u}, \tag{A.11.11}$$

$$\widehat{\tau_h u}(\xi) = \widehat{u}(\xi)e^{-i\xi\cdot h}, \quad h \in \mathbb{R}^d, \tag{A.11.12}$$

$$\widehat{ue^{ix\cdot h}} = \tau_h \widehat{u}, \quad h \in \mathbb{R}^d. \tag{A.11.13}$$

The following theorem states that the Fourier transform of a compactly supported distribution is a smooth function.

Theorem A.11.11: *Let $u \in \mathcal{E}'(\mathbb{R}^d)$. Then $\widehat{u}$ lies in $C^\infty(\mathbb{R}^d)$ and*

$$\widehat{u}(\xi) = \langle u(x), e^{-ix\cdot\xi}\rangle \quad \forall \xi \in \mathbb{R}^d.$$

In particular, there exists a constant $C > 0$ such that

$$|\widehat{u}(\xi)| \leq C\langle\xi\rangle^{\mathrm{ord}(u)}, \quad \forall \xi \in \mathbb{R}^d,$$

where $\langle\xi\rangle := (1 + |\xi|^2)^{\frac{1}{2}}$ and $\mathrm{ord}(u)$ is the order of u, as per Definition A.2.5.

In the following, we state an extension of Equation (A.11.6).

Theorem A.11.12: *Let $u \in \mathcal{S}'(\mathbb{R}^d)$ and $v \in \mathcal{E}'(\mathbb{R}^d)$. Then $u * v \in \mathcal{S}'(\mathbb{R}^d)$ and $\widehat{u * v} = \widehat{u}\widehat{v}$.*

At last, we recall a result concerning the behaviour of the Fourier transform of a *homogeneous* distribution.

Definition A.11.13: *Let $\alpha \in \mathbb{R}$. We say that $u \in \mathcal{D}'(\mathbb{R}^d)$ is **homogeneous** of degree α if*

$$u(\phi^\lambda) = \lambda^\alpha u(\phi), \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d),\ \forall \lambda \in (0, 1],$$

where $\phi^\lambda(\cdot) := \lambda^{-d}\phi(\lambda^{-1}\cdot)$.

Proposition A.11.14: *Let $\alpha \in \mathbb{R}$ and let $u \in \mathcal{S}'(\mathbb{R}^d)$. If u is homogeneous of degree α, then its Fourier transform $\widehat{u}$ is a homogeneous distribution of degree $-\alpha - d$.*

Appendix **B**

Coherence on an open set

In this Appendix, we shall give a local formulation of concepts and results discussed in Subsection 2.4.1. More precisely, we introduce the notions of coherence and of enhanced coherence on an open set $\Omega \subset \mathbb{R}^d$. Lastly, we shall prove that the notion of coherence is stable with respect to restrictions - see Proposition B.0.8. The following concepts and results shall be play a leading rôle in Chapter 4, where we generalize the theory of germs of distributions on smooth manifolds. We shall mainly refer to [RS21, Appendix B]. Throughout this Subsection, we denote by $\lesssim$ an inequality holding true up to a multiplicative finite constant. Given a compact set $\mathfrak{K} \subset \mathbb{R}^d$ and $R > 0$, its R-enlargement of $\mathfrak{K}$, denoted by $\overline{\mathfrak{K}}_R$, is defined as per Equation (2.4.1). We denote by $B(0, R)$ the open ball centered at the origin of radius R. Given an open set $\Omega \subset \mathbb{R}^d$, its boundary is denoted by $\partial\Omega$. Moreover, given a function $f \colon \Omega \to \mathbb{R}$ and a point $x \in U$, we recall that $f_x^\lambda \colon \Omega \to \mathbb{R}^d$ denotes the rescaled version of f, defined as

$$f_x^\lambda(y) := \lambda^{-d} f(\lambda^{-1}(y - x)), \quad y \in \Omega,$$

for $\lambda \in (0, 1]$.

We start by introducing the notion of coherence on an open set $\Omega \subset \mathbb{R}^d$.

Definition B.0.1: *Let $\Omega \subset \mathbb{R}^d$ be an open set. A family $F = (F_x)_{x\in\Omega}$ of distributions, $F_x \in \mathcal{D}'(\Omega)$ for any $x \in \Omega$, is said to be a* ***germ*** *on Ω if, for any $\psi \in \mathcal{D}(\Omega)$, the map $x \mapsto F_x(\psi)$ is measurable.*

Definition B.0.2: *Let $\Omega \subset \mathbb{R}^d$ be an open set. Let $\gamma \in \mathbb{R}$ and let $F = (F_x)_{x\in\Omega}$ be a germ as per Definition B.0.1. F is called γ-**coherent** on Ω if there exists $\underline{\kappa} \in \mathcal{D}(\Omega)$ with $\check{\underline{\kappa}}(0) \neq 0$ such that for any compact set $\mathfrak{K} \subset \Omega$ there exists $\zeta_{\mathfrak{K}} \leq \min\{0, \gamma\}$ such that*

$$|(F_y - F_x)(\underline{\kappa}_x^\lambda)| \lesssim \lambda^{\zeta_{\mathfrak{K}}}(|x - y| + \lambda)^{\gamma - \zeta_{\mathfrak{K}}}, \tag{B.0.1}$$

uniformly for $x, y \in \mathfrak{K}$ and for $\lambda \in (0, D_{\mathfrak{K}}/4]$, where we set $D_{\mathfrak{K}} := \operatorname{dist}(\partial\Omega, \mathfrak{K})$. We say that F is $(\boldsymbol{\zeta}, \gamma)$-coherent where $\boldsymbol{\zeta} = (\zeta_{\mathfrak{K}})_{\mathfrak{K}}$ is the family of exponents in Equation (2.4.3). In particular, if $\zeta_{\mathfrak{K}} = \zeta$ for any compact set $\mathfrak{K}$, F is said to be (ζ, γ)-coherent.

Remark B.0.3: *Let $\Omega \subset \mathbb{R}^d$ and let $\mathfrak{K} \subset \Omega$ be a compact set. Since $\partial\Omega$ is a closed set and $\partial\Omega \cap \mathfrak{K} = \emptyset$, then $D_{\mathfrak{K}} > 0$.*

Remark B.0.4: *As mentioned in Remark 2.4.5, we could replace the constraint $\lambda \in (0, D_{\mathfrak{K}}/4]$ by $\lambda \in (0, \epsilon]$, for any fixed $\epsilon > 0$. As a matter of fact, the bound in Equation (B.0.1) is established up to a multiplicative constant. The choice of $\frac{D_{\mathfrak{K}}}{4}$ as a supremum among all possibile values of λ shall be clear by what follows.*

Analogously to Subsection 2.4.1, we introduce the notion of *enhanced coherence*. In the same spirit of Subsection 2.4.1, the main idea is to remove the dependence on the test function $\underline{\kappa}$ from the notion of coherence as per Definition B.0.2. This can be achieved by promoting the coherence condition in Equation (B.0.1) to a uniform condition on arbitrary test functions, provided that the exponents $\zeta_{\mathfrak{K}}$ are suitably adjusted. First of all, we state the following proposition, which is an adaptation to our setting of [CZ20, Prop. 12.6].

Proposition B.0.5: *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $u \in \mathcal{D}'(\Omega)$. Suppose that there exist a compact set $\mathfrak{K} \subset \Omega$ and a test function $\underline{\kappa} \in \mathcal{D}(\Omega)$ with $\check{\underline{\kappa}}(0) \neq 0$ such that, for any $x \in \overline{\mathfrak{K}}_{\frac{D}{2}}$ and for any $\varepsilon \in \{2^{-n}\}_{n\in\mathbb{N}}$*

$$|u(\underline{\kappa}_x^{\varepsilon})| \leq \varepsilon^{\zeta} g(\varepsilon, x), \tag{B.0.2}$$

where $\zeta \leq 0$, $g\colon (0, \frac{D}{4}] \times \overline{\mathfrak{K}}_{\frac{D}{2}} \to (0,\infty]$ is an arbitrary function while we set $D := \mathrm{dist}(\mathfrak{K}, \partial\Omega)$. Then, for any integer $r > -\zeta$, it holds true that

$$\forall x \in \mathfrak{K}, \forall \phi \in \mathcal{D}(B(0,1)) \quad |u(\phi_x^{\lambda})| \lesssim \|\phi\|_{C^r(\mathbb{R}^d)} \lambda^{\zeta} \tilde{g}(\lambda, x), \quad \forall \lambda \in \left(0, \frac{D}{4}\right], \tag{B.0.3}$$

where $\tilde{g}\colon (0, \frac{D}{4}] \times \mathfrak{K} \to [0,\infty)$ is defined as

$$\tilde{g}(\lambda, x) := \sup_{\substack{\lambda' \in (0,\lambda], \\ x' \in \overline{B}(x, 2\lambda)}} g(\lambda', x') \tag{B.0.4}$$

while $\|\cdot\|_{C^r(\mathbb{R}^d)}$ has been defined in Equation (A.1.3).

Proof. We omit the proof since it is similar to that of [CZ20, Prop. 12.6]. For this reason, we stress only the main difference. Since this result is the localization on an open set Ω of [CZ20, Prop. 12.6], we consider the $\frac{D}{2}$-enlargement of $\mathfrak{K}$ to make sure that $\mathrm{supp}(\phi_x^{\lambda})$ is contained in Ω for any $\phi \in \mathcal{D}(B(0,1))$ and for any $\lambda \in (0, \frac{D}{4}]$. □

Enhanced coherence is a consequence of the previous proposition.

Proposition B.0.6: *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $F = (F_x)_{x\in\Omega}$ be a γ-coherent germ on Ω as per Definition B.0.2, i.e. there exist $\underline{\kappa} \in \mathcal{D}(\Omega)$ with $\check{\underline{\kappa}}(0) \neq 0$ and a family $\boldsymbol{\zeta} = (\zeta_{\mathfrak{K}})_{\mathfrak{K}}$ such that Equation (B.0.1) holds true. We set $\tilde{\zeta}_{\mathfrak{K}} := \zeta_{\overline{\mathfrak{K}}_{\frac{D_{\mathfrak{K}}}{2}}}$ where $D_{\mathfrak{K}} = \mathrm{dist}(\mathfrak{K}, \partial\Omega)$. Then, for any compact set $\mathfrak{K} \subset \Omega$ and any integer $r > -\tilde{\zeta}_{\mathfrak{K}}$, it holds true that*

$$|(F_x - F_y)(\phi_y^{\lambda})| \lesssim \|\phi\|_{C^r(\mathbb{R}^d)} \lambda^{\tilde{\zeta}_{\mathfrak{K}}} (|x-y| + \lambda)^{\gamma - \tilde{\zeta}_{\mathfrak{K}}}, \tag{B.0.5}$$

uniformly for $\phi \in \mathcal{D}(B(0,1))$, $\lambda \in (0, \frac{D_{\mathfrak{K}}}{4}]$ and $x, y \in \mathfrak{K}$.

Proof. The proof of this result is similar to that of [CZ20, Prop. 13.1]. For this reason, we omit it. In the following, we stress only the main difference. Being this result the local formulation of [CZ20, Prop. 13.1] and on account of the definition of $D_{\mathfrak{K}}$, we need to make sure that the $\frac{D_{\mathfrak{K}}}{2}$-enlargement of $\mathfrak{K}$ is contained in Ω. In addition, Definition B.0.2 and Proposition B.0.5 entail the coherence bound in Equation B.0.5 with exponents $\zeta_{\overline{\mathfrak{K}}_{\frac{D_{\mathfrak{K}}}{2}}}$. □

On account of Proposition B.0.6, we give the following equivalent definition of coherence on an open set $\Omega \subset \mathbb{R}^d$.

Definition B.0.7: *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\gamma \in \mathbb{R}$. A germ of distributions $F = (F_x)_{x\in\Omega}$ is said to be γ-**coherent** on Ω if for any compact set $\mathfrak{K} \subset \Omega$ there exists $\zeta_{\mathfrak{K}} \leq \min\{0, \gamma\}$ such that, for any integer $r > -\zeta_{\mathfrak{K}}$,*

$$|(F_x - F_y)(\phi_x^\lambda)| \lesssim \|\phi\|_{C^r(\mathbb{R}^d)} \lambda^{\zeta_{\mathfrak{K}}} (|x-y| + \lambda)^{\gamma - \zeta_{\mathfrak{K}}}, \tag{B.0.6}$$

uniformly for $\phi \in \mathcal{D}(B(0,1))$, $x, y \in \mathfrak{K}$ and $\lambda \in (0, \frac{D_{\mathfrak{K}}}{4}]$, where we set $D_{\mathfrak{K}} := \mathrm{dist}(\partial\Omega, \mathfrak{K})$.

At last, we prove that the notion of coherence on an open set is stable with respect to restrictions - see [RS21, Prop 32].

Proposition B.0.8: *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\Omega' \subset \Omega$ be an open subset. In addition, let $F = (F_x)_{x\in\Omega}$ be a $(\boldsymbol{\zeta}, \gamma)$-coherent germ on Ω with $\gamma \in \mathbb{R}$ and $\boldsymbol{\zeta} = (\zeta_{\mathfrak{K}})_{\mathfrak{K}}$ as per Definition B.0.7. Then it is also $(\boldsymbol{\zeta}, \gamma)$-coherent on Ω'.*

Proof. Let $\mathfrak{K} \subset \Omega'$ be a compact set. In addition, we set $D_{\mathfrak{K}}^{\Omega'} := \mathrm{dist}(\partial\Omega', \mathfrak{K})$ and $D_{\mathfrak{K}}^{\Omega} := \mathrm{dist}(\partial\Omega, \mathfrak{K})$. Being $\mathfrak{K} \subset \Omega$ and on account of coherence on Ω as per Definition 2.4.4, there exists $\zeta_{\mathfrak{K}} \leq \min\{0, \gamma\}$ such that, for any integer $r > -\zeta_{\mathfrak{K}}$,

$$|(F_x - F_y)(\phi_y^\lambda)| \lesssim \|\phi\|_{C^r(\mathbb{R}^d)} \lambda^{\zeta_{\mathfrak{K}}} (|x-y| + \lambda)^{\gamma - \zeta_{\mathfrak{K}}}, \tag{B.0.7}$$

uniformly for $x, y \in \mathfrak{K}$, $\lambda \in (0, \frac{D_{\mathfrak{K}}^{\Omega}}{4}]$ and $\phi \in \mathcal{D}(B(0,1))$. Since $D_{\mathfrak{K}}^{\Omega'} \leq D_{\mathfrak{K}}^{\Omega}$, the bound in Equation (B.0.7) holds true uniformly for $\lambda \in (0, \frac{D_{\mathfrak{K}}^{\Omega'}}{4}]$. Therefore, F is γ-coherent on Ω' with exponents $\boldsymbol{\zeta} = (\zeta_{\mathfrak{K}})_{\mathfrak{K}}$. □

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