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Modeling of socio-economic phenomena by Fokker-Planck Equations.

Supervisor
Prof. Giuseppe Toscani

PhD candidate
Marco Torregrossa

Contents

Abstract

This thesis is centred on Fokker-Planck Equation and its application in Econophysics.
Part I is devoted to the modeling of wealth distribution and it is based on two works that were developed under the supervision of Professor G.Toscani [85, 86]. Within this Part, in Chapter 4, we study a Fokker-Planck equation with variable coefficient of diffusion and boundary conditions which appears in the study of the wealth distribution in a multi-agent society [17, 32, 76]. In particular, we analyze the large-time behaviour of the solution, by showing that convergence to the steady state can be obtained in various norms at different rates. In Chapter 5, we consider the same Fokker-Planck equation with variable coefficient of diffusion which appears in Chapter 4. At difference with previous studies, to describe a society in which agents can have debts, we allow the wealth variable to be negative. It is shown that even starting with debts, if the initial mean wealth is assumed positive, the solution of the Fokker-Planck equation is such that debts are absorbed in time, and a unique equilibrium density located in the positive part of the real axis will be reached.
Part II shows an application of Fokker-Planck equation in the formulation of rating model. It is based on a joint work with Professor B.Düring and with Dr.Marie-Therese Wolfram [40]. In this paper, we propose and study a new kinetic rating model for a large number of players, which is motivated by the well-known Elo rating system. Each player is characterised by an intrinsic strength and a rating, which are both updated after each game. We state and analyse the respective Boltzmann type equation and derive the corresponding nonlinear, nonlocal Fokker-Planck equation. We investigate the existence of solutions to the Fokker-Planck equation and discuss their behaviour in the long-time limit. Furthermore, we illustrate the dynamics of the Boltzmann and Fokker-Planck equation with various numerical experiments.

Chapter 1

Introduction

In the last three decades, different theories were developed in the attempt to explain complex phenomena through the use of Theoretical Physics and Statistical Mechanics techniques. In fact, various research communities, from Biology to Sociology and Economics, have developed increasing interest in the description of collective phenomena, made of large numbers of individuals. It is interesting to show and to clarify the ideas on which these new theories were built.
Many physical phenomena can be described by analogy with particles dynamic. The modeling of these phenomena is based on the search of universal properties and the laws that regulate their evolution. The main idea (that is also the most relevant issue) at the base of these theories is to find their analogous in the description of the human phenomena and in the Social Sciences. We can describe complex human multiagent systems as sets of autonomous individuals who show a collective behaviour as the result of their interactions.
The attempts to study social events using physical laws are ancient, in particular in Finance and Economy. During the nineteenth century, the Italian economist Vilfredo Pareto investigated the distribution of wealth in a population: he predicted that the 20% of the population had the 80% of total wealth or, in other words, he postulated the existence of a small fraction of very rich people [78]. This criterion is known as Pareto's principle (or also as $80-20$ rule) and was the first power-law ever discovered for the description of both natural and social phenomena. In the twentieth century, the engineer J.M.Juran applied the Pareto's principle to quality issues and since then it was applied in several fields. The currently used version of the Pareto's Principle affirms that the fraction of population having revenue greater than some amount $w >> 1$ is proportional to $w^{-\alpha}$, where the parameter $\alpha > 1$ is known as Pareto's index [28]. In other words, if $f(w)$ is the probability density function of the wealth and $F(w) = \int_0^w f(x)dx$ is the corresponding cumulative distribution function,

$$1 - F(w) \sim w^{-\alpha}.$$

This kind of behaviour is not common in comparison with many natural phenomena. Indeed, the more frequent distribution is the Gaussian, that presents an exponential decay at infinity. In other words, we can observe the most relevant difference if we compare the fraction of population that is characterized by a value of the quantity in the object of study greater than a certain $w >> 1$: it results very greater in the case of power-law decays with respect to the Gaussian distribution. In consequence of this, it is said that a population with beahviour as Pareto's prediction exhibits *"heavy tails"*. Models involving heavy-tailed distributions are used in many fields of applications as economics, telecommunications, physics, and biology and have been confirmed by data [69, 70, 91]. The Levy theory in probability, the concept of scaling of thermodynamic function and correlation function in physics allowed the development of

new distribution with heavy tails. A stochastic process is defined as Levy stable if it is governed by a generalized central limit theorem. Mandelbrot in [71] postulated that price changes obey to a Levy stable distribution. The sum of i.i.d. Levy stable random variables is a Levy distribution, that is characterized by a density distribution with power decays. This fact indicates that the distribution of a Levy stable process is a power-law distribution when the stochastic variable v assumes large values.
In the first half of 1900s, other links to the bond between the physics and the social sciences were highlighted by Majorana [67, 68]. He stressed, in particular, the importance of the non-deterministic aspects in the modelling of social phenomena. Majorana wrote: "It is known that the laws of the mechanics, in a particular way, have longly appeared as the insuperable type of our knowledge of nature, and...also all the other sciences should have been brought back to such type (of laws)." Moreover, Majorana underlined the success achieved by the deterministic conception in the development of modern science not only in physics but also in unexpected fields of application such as social science. More precisely, Majorana identified the analogy between thermodynamic and social science,in particular between particles and individuals: "For example, when a statistic law is enunciated on a population, the investigation on the biography of the single individuals is dropped; in the same way, if we define the (macroscopic) state of a gas with pressure and volume, we abdicate to investigate position and velocity of a single particle". In the second part of his works, Majorana underlined the role of the non-deterministic aspects in consideration of the new knowledge of the quantum mechanics: "also the laws that concern the elementary (atomic systems) phenomena have statistic character...These statistic laws point out a reality defect of deterministic conception". The same considerations are valid for social sciences: "if we remember what we have said above on the mortality of the radioactive atoms, we are induced to ask us if there is an analogy with social facts, that are rather described with similar language". Nevertheless, the unconventional Majorana's point of view was considered of marginal interest for several decades and the true development of this way of thinking began only in the late 1900s.
In the second part of 1900, several new attempts were made to apply the methods of physics and statistic mechanics to social sciences. Among these, the Black-Scholes rational option-pricing formula had a key role in introducing random effects in human phenomena, despite it doesn't represent precisely the reality [15]. They established that under certain "ideal condition", the value of the option will depend only on the price of the stock and on time. One of this condition provides that the stock price follows a random walk in continuous time. The random walk is a very common concept in natural sciences. This concept had been introduced in the doctoral thesis of Bachelier in 1900 [5], five years before Einstein's paper on the Brownian motion. Bachelier's work is considered pioneering, even if it is not entirely rigorous from the mathematical point of view. The intuitions of Majorana and Bachelier may be considered the roots of modern Econophysics [72].
The word "Econophysics" was introduced in 1996 by H.E. Stanley in [84], in analogy with Biophyisics and Astophysics. The examples of Pareto and Mandelbrot showed that there is a bidirectional relation between both physics and economics; however, the contribution of Physics to Economics is more relevant than the reverse. There are several definitions of Econophysics. According to Burda [20], "Econophysics is an approach to quantitative economy using ideas, models, conceptual and computational methods of statistical physics". It also explains a large spectrum of problems of Economy. Its target is to explain phenomena of finance and economy as the evolution of complex systems by using instruments and principles of Statistical Physics (microscopic models, scaling laws). The development of this new discipline was triggered by two principal reasons: a) certain difficulties met by the classical Economics approach and b) the benefits of the new method. In [61], Keen underlines how the theories of the classical economy were based on hypothesis (as Completeness, Transitivity, Non-satiation, and Convexity) that turned out to be not trustworthy. Furthermore, the fundamental laws that characterize the models of economic systems were not yet completely understood [92]. Otherwise, Physics seeks universal laws and its models are supported and confirmed by experimental data [12]. As economic phenomena show certain empirical

regularities, a relevant proportion of social organizations may be included in Physics schemes.
One of the major intuition of Econophysics has been to understand how asset prices evolve in terms of fractional Brownian motion. This kind of description can explain the behaviour of several similar structure as model asset price dynamics, stock prices, stock indexes, currency exchange rates and the DAX data series. For example, by analyzing the correlation among different stock returns, we can note its resemblance with the correlation among spectroscopic data on energy levels: both phenomena were interpreted with the use of random matrix theory, that had been developed in nuclear physics. In the description of stock market movements, the physical concept of entropy plays a significant role. This idea arose in the nineteenth century in Thermodynamics to explain the evolution of an isolated system toward equilibrium and it was incorporated in the framework of Statistical Mechanics by Boltzmann and Gibbs [16, 53] (For a brief presentation of the Boltzmann theory, see Chapter 2). The concept of Entropy came into knowledge of information theory and probability with the work of Shannon and Kolmogorov [81, 63]. More precisely, it became an instrument to study the equilibrium of systems that are driven by space or temporal interactions. However, the evolution of many phenomena is not governed by the Shannon entropy. This necessarily led to the introduction of new forms of entropy [90]. Independently from its expression, the entropy is an index of disorder and of lack of knowledge and information about a system. The correspondent economic meaning is the uncertainty about the values that characterize the population [55]. Entropy communicates more information about probability distribution than standard deviation; in consequence of this, entropy has replaced the standard deviation in many applications, such as index of the volatility of the stock market.
In the context of Econophysics, Fokker-Plank equations play an important role. This type of equations is useful to describe systems that are affected by a certain noise. For this reason, it was applied to describe several human phenomena, such as opinion formation [88] and wealth distribution [17, 32]. The aim of this thesis is to show some applications of Fokker-Planck equations and related mathematical problems.
In the next chapter, I will introduce the most important mathematical tools that were the basis for the development of Econophysics and that are still used in the description of economic phenomena. More precisely, I will introduce the Boltzmann Equation with the famous H-Theorem and some results on the classical Fokker-Planck equation.
Part I of the thesis is focused on models of wealth distribution. In particular, in chapter 3, I will describe some kinetics models and some results about the convergence towards equilibrium of the solutions of Fokker Planck equation. In the other two chapters, I will present two works that I developed during my doctorate under the supervision of Professor G.Toscani [85, 86].
Kinetics models are used also to describe human phenomena that are far away from Economy. Sometimes these theories are indicated with the term Sociophysics. In this set, in addition to opinion formation models that I recalled before, there are ranking systems. In part II, I will deal with the argument of ranking systems. I will present my work on an Elo model with dynamical strength, that I wrote with Professor B.Düring and with the Dr.Marie-Therese Wolfram [40].

Chapter 2

Preliminaries

2.1 Boltzmann equation

In 1872, Boltzmann formulated the most famous kinetic equation in the context of gas theory [16]. A kinetic equation describes the dynamical state of a single particle, that is characterized at any time by coordinates and velocity. Boltzmann equation is a non-linear partial integro-differential equation that describes statistical behaviour of a thermodynamic system. This equation correlates the microscopic properties of particles (atom and molecules) to the macroscopic qualities of the system, such as temperature. More precisely, the Boltzmann equation predicts the evolution of the density of a rarefied monoatomic gas, determined by mutual interactions among its particles. The description of a system in terms of statistical mechanics relies on the identification of the fraction of particles positioned a time $t > 0$ in a particular position $x \in \mathbb{R}^3$ (or in the infinitesimal range between the position x and $x + dx$) having velocity $v \in \mathbb{R}^3$ (or between v and $v + dv$). Let us indicate with $f(x, v, t)$ the probability density of particles that at time t are in position $x \in \mathbb{R}^3$ with velocity $v \in \mathbb{R}^3$. Thus, the evolution equation for an isolated system (on which external strengths do not act) is

$$\frac{\partial}{\partial t} f(x, v, t) = -v \cdot \nabla_x f(x, v, t) + Q(f, f)(x, v, t). \tag{2.1}$$

The term $-v \cdot \nabla_x f(x, v, t)$ describes the transport of particles, i.e. the movement of a particle in the short time lag which occurs between two successive interactions. In this context, the only admissible interactions are the binary collisions, due to the hypotesis of rarefaction of the gas, and we only consider elastic collisions that preserve both momentum and energy. If we indicate with (x, v) and (y, w) the positions and the velocities of the two particles before the collision, the respective velocities after collision v^* and w^* are determined by the rules

$$v^* = v - \mathbf{n}[\mathbf{n} \cdot (v - w)] \quad w^* = w - \mathbf{n}[\mathbf{n} \cdot (v - w)],$$

where $\mathbf{n}$ is the unitary vector from position x to position y [25]. According to the pervious rules, we have

$$v^* + w^* = v + w \quad |v^*|^2 + |w^*|^2 = |v|^2 + |w|^2,$$

that are precisely the momentum and energy preservation. The operator $Q(f, f)$ is defined as

$$Q(f, f)(v) = \int_{\mathbb{R}^3 \times S^+} [f(v^*)f(w^*) - f(v)f(w)]|(v - w) \cdot \mathbf{n}| dw dn,$$

where S^+ is the unitary emisphere and dn is the normalized surface measure on S^+.

Homogeneous case

Let us consider the homogeneous Boltzmann equation in which the distribution f does not depend on the position x, i.e. $f = f(v,t)$. With this assumption, Boltzmann equation reads as

$$\frac{\partial}{\partial t} f(v,t) = Q(f,f)(v,t), \tag{2.2a}$$

$$f(v,0) = f_0(v). \tag{2.2b}$$

The problem can be written as a differential equation in an appropriate Banach space. Let us introduce the space

$$L^1_s = \{f : \mathbb{R}^3 \to \mathbb{R} \ : \ \|f\|_{L^1_s} < +\infty\}$$

where

$$\|f\|_{L^1_s} = \int_{\mathbb{R}^3} (1+|v|^2)^{\frac{s}{2}} |f(v)| dv.$$

It is possible to apply standard arguments of O.D.E. in Banach space that guarantee existence and uniqueness of the solution.

Theorem 1. *([19]) Assume that $f_0 \in L^1_4$ and $f_0 \geq 0$ for a.e. $v \in \mathbb{R}^3$. Then the Cauchy problem* (2.2a)-(2.2b) *has a unique solution, defined for all $t \geq 0$. The map $t \mapsto f(t)$ is continuously differentiable as a map with values in L^1_2. Moreover, for every $t \geq 0$ there holds*

$$\int_{\mathbb{R}^3} f(v)dv = \int_{\mathbb{R}^3} f_0(v)dv, \quad \|f\|_{L^1_2} = \|f_0\|_{L^1_2}.$$

By direct computation, it is not hard to show that homogeneous Boltzmann equation presents some invariants. Let us consider functional of the form

$$\Phi(f) = \int_{\mathbb{R}^3} \phi(v) f(v,t) dv,$$

where $\phi(v)$ is a (smooth enough) function. This functional is invariant of Boltmzann equation if

$$\int_{\mathbb{R}^3} \phi(v) Q(f) dv = 0,$$

for every f solution of (2.2a). If $f \in C^\infty_c$, the previous condition is equivalent to

$$\phi(v) + \phi(w) = \phi(v^*) + \phi(w^*). \tag{2.3}$$

A function ϕ for which previous condition holds, has the form

$$\phi(v) = a + bv + c|v|^2,$$

where $a, c \in \mathbb{R}$ and $b \in \mathbb{R}^3$. For suitable choice of this constants, we obtain the preservation of mass $\phi(v) = 1$, momentum $\phi(v) = v$ and energy $\phi(v) = \frac{|v|^2}{2}$ that are the three desired invariants.

Maxwellian distribution and H-theorem

Operator $Q(f,f)$ describes the total effect on the system of all binary collisions. So it is important to individuate a distribution f for which $Q(f,f)=0$ and if such a distribution exists. Let us start with the homogeneous case. Direct computation show that for each $x,y>0$, $(y-x)log\left(\frac{x}{y}\right)\geq 0$ and the equivalence holds if and only if $x=y$. This proves the following Boltzmann inequality

$$\int_{\mathbb{R}^3} Q(f,f)log(f)dv \leq 0,$$

and it is equal to zero if and only if $f(v^*)f(w^*)=f(v)f(w)$ [25]. In other words, the function $\phi(f)=log(f)$ satisfies (2.3). In the homogeneous case, it implies that the function f has the form

$$f(v,t)=Ce^{-\alpha|v-y|^2},$$

where $C,\alpha\in\mathbb{R}^+$ and $y\in\mathbb{R}^3$. In the general case of inhomogeneous Boltzmann equation (2.1), we define Maxwellian distribution a function $M(x,v,t)$ such that $Q(M,M)=0$. The Maxwellian equilibrium distribution $M(x,v,t)$ may be written as function of macroscopic quantities as mass ρ, velocity V and temperature T which are expressed by using the first three moments of the solution of the Boltzmann equation:

$$\begin{aligned}
\rho(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)dv,\\
V(x,t) &= \rho(x,t)^{-1}\int_{\mathbb{R}^3} vf(x,v,t)dv,\\
T(x,t) &= (3\rho(x,t))^{-1}\int_{\mathbb{R}^3}\left[v-V\right]^2 f(x,v,t)dv.
\end{aligned}$$

With this notation, it results

$$M(x,v,t)=\rho(x,t)\big(2\pi T(x,t)\big)^{-\frac{3}{2}}e^{-\frac{|v-V(x,t)|}{2T(x,t)}}.$$

This fact helps to clarify the link between microscopic and macroscopic properites.
Furthermore, as a consequence of the analisys of the properties of the solution of (2.1), Boltzmann formulated the famous H-theorem: it affirms that for an isolated rarefied gas the function

$$H(t)=\int f(x,v,t)lnf(x,v,t)dxdv$$

is non-increasing in time and the minimum is achieved in correspondance of a Maxwell distribution. This minimum is the same value (except for the sign) computed by Gibbs [53]. The Boltzmann H-theorem was the first molecular-kinetic interpretation of the second law of thermodynamics and provided a statistical explanation of the concept of entropy.
The importance of the Boltzmann work is not limited to the interpretation of entropy. In every system in which the number of agents is huge, the agents are indistinguishable and they don't have memory of previous collision, it is possible to replicate Boltzmann's approach. In particular, the collisional bilinear operator of Boltzmann equation becomes the model on which it is possible to construct the interaction operator of the system.

2.2 Fokker-Planck equation

Fokker-Planck equation was introduced by Fokker [48] and Planck [79] to study the Brownian motion of particles. Fokker-Planck equation furnishes a strong instrument to study the effects of fluctuations close to transition points. It has turned out that this approach is more efficient than others. Fokker-Planck equation can adequately treat problems that are affected by noise and is used not only in physics (e.g. in studies of electrical circuits or quantum optics) but also in a large number of different fields, such as natural sciences, chemical physics, theoretical biology and social sciences [80].
Let us consider a small but macroscopic particle of mass m, e.g. a small sphere, that is placed in a fluid. If the mass m is big enough (with respect to the mass of fluid particles), the thermal velocity of the sphere, defined as $v_t = \sqrt{\frac{KT}{m}}$ (where K is the Boltzmann's constant and T is the temperature), is small and negligible. Nevertheless, if the mass m is not so big, it is necessary to take into account count v_t in the equation of the velocity of the small sphere. There are several forces that act on the sphere. We can observe a friction force F_v and a fluctuation force F_f. These forces are the result of impacts of the molecules of the fluid on the surface of the sphere. Due to large number of these collisions, it is not possible to predict the exact position of the sphere, because it would be necessary to solve the coupled equations of motion for all the molecules of the fluid (that are about 10^{23}) and for the sphere. Furthermore, we do not know the initial positions and velocities of all the molecules of the fluid, so it is not possible to compute their exact trajectory. If we have changed the initial values of the fluid, we would observe a different motion of the sphere. We can look for the probability that the particle has to be in a certain area of the fluid. In thermodynamics, one usual approach is to consider a set of systems (sphere in a fluid) of the same type (Gibbs'ensemble). In this context, since the force F_f and, consequently, velocities are not the same from system to system, we consider them as stochastic quantities and compute the average force on the entire ensemble. It means to normalize F_f and to introduce the Langevin force $\mathcal{F} = \frac{F_f}{m}$. We may assume that $\mathcal{F}$ is distributed as a Gaussian of mean 0 and variance σ^2, i.e. $\mathcal{F} \sim N(0, \sigma^2)$.
We ask for the fraction of sphere of the ensemble whose velocities are in the interval $(v, v + dv)$, or, in other words, we look for the probability $W(v, t)$ that, at time $t > 0$, the velocity of one sphere is in the interval $(v, v + dv)$. The equation for the distribution function is

$$\frac{\partial}{\partial t} W(v,t) = \gamma \frac{\partial}{\partial v}[vW(v,t)] + \gamma(KT/m)\frac{\partial^2}{\partial v^2} W(v,t),$$

where K is the Boltzmann's constant, T is the temperature and γ is the coefficient of friction per unit mass. The second order term arises from the stochastic force $\mathcal{F}$. Previous equation is one of the simplest Fokker-Planck equation. Fokker-Planck equation in general form reads as

$$\begin{aligned} \frac{\partial}{\partial t} W(v,t) = -\sum_{j=1}^{d} \frac{\partial}{\partial v_j}(a_j(v)W(v,t)) + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial v_j v_i}(D_{ij}(v)W(v,t)), \\ W(v,0) = W_0(v), \end{aligned} \tag{2.4}$$

where $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$. From a mathematical point of view, it is a parabolic linear second-order partial differential equation and is also called a forward Kolmogorov equation. It represents a diffusion process on $\mathbb{R}^d$ with an additional first-order derivative in which drift and diffusion coefficients are time-independent and in which the diffusion matrix is always non-negative.
The easiest Fokker-Planck equation is obtained for $D_{ij}(v) = \sigma^2 I_n$, where I_n is the unitary $n-$ dimensional matrix. In this case, and in general in every case in which the operator of the right side of (2.4) is uniformly elliptic for each t, the existence and uniqueness of a weak solution are the consequence of standard arguments in the theory of parabolic equations [45].

Part I

Fokker-Planck description of weatlh distribution

Chapter 3

Wealth distribution models

Fokker-Planck equations are an important model in the description of wealth distribution. One of the first attempts to describe wealth distribution with a Fokker-Planck equation was done by J.P.Bouchaud and M.Mézard. In 2000 [17], they introduced a simple model characterized by the presence of a random speculative trading. The aim of this model is to describe the formation of Pareto's tails with a general simple model which takes into account economic growth, taxes and redistribution mechanisms. The Bouchaud-Mézard model founds his roots in the physical problem of 'directed polymer'. The model is based on an evolution stochastic differential equation that considers the trading between individuals together with the natural oscillations of the economy due to investments effects. These oscillations are described with a Gaussian random variable of mean m and variance $2\sigma^2$. In the simplest case, the rate J at which agent i trades with agent j is constant or, in other words, all agents' behaviours have the same effect on the system evolution. In the assumption of a large number of agents, the population may be described by a deterministic Fokker-Plank equation

$$\frac{\partial}{\partial t} f(v,t) = \frac{\partial}{\partial v}\left[\big(J(v-1)+\sigma^2 v\big) f(v,t)\right] + \sigma^2 \frac{\partial}{\partial v}\left[v \frac{\partial}{\partial v}\big(v f(v,t)\big)\right], \tag{3.1}$$

where $f(v,t)$ is the fraction of population that has wealth w at time $t > 0$. The asymptotic distribution for equation (3.1) reads as

$$f_\infty(v) = \frac{(\mu-1)^\mu}{\Gamma(\mu)} \frac{e^{-\frac{\mu-1}{v}}}{v^{1+\mu}},$$

where $\mu = 1 + \frac{J}{\sigma^2}$ and J is the rate of exchange between agents. This distribution exhibits Pareto's heavy tails for $v >> 1$. More precisely, Pareto's index μ increases when J increases or when σ^2 decreases: from an economic point of view, it means that the distribution of wealth is more equitable when the exchanges are encouraged and when the oscillations in individual wealth, due to investments, are small. From a mathematical point of view, equation (3.1) may be brought back to the class of Fokker-Plack equations that are explicitly solved in [56]. It is possible to define the model as a stochastic process in a discrete time. For this discrete model too, the steady state has a power-law decay, with an exponent $\mu > 0$. For $\mu > 1$, the behaviour of the solution is the same as the continuous model. Conversely, if $\mu < 1$, we observe a wealth codensation, i.e. a small fraction of the population has a significant part of total wealth.

Even if this model is not completely realistic, this work reaches two important outcomes. Firstly, it underlines the existence of a phase transition with respect to the chosen value of the parameter μ. Secondly, it correlates a simple economic model to a large class of equations of type (3.1).

In the same year, Dragulescu and Yakovenko in [35] and Chakraborti and Chakrabarti in [27] introduced simple models of a closed economy. Dragulescu and Yakovenko assumed that at every interaction a certain amount of money Δm is simply moved from agent i to agent j (or vice-versa), i.e.

$$v^* = v + \Delta m, \quad w^* = w - \Delta m,$$

where v and w are the pre-interaction money of the agents and v^* and w^* are the post interaction ones. These interaction rules are in accordance with the hypothesis of preservation of thr total amount of money, i.e.

$$v^* + w^* = v + w.$$

This claim is postulated in analogy with energy conservation in gas dynamics. The correspondence with statistical mechanics is also reflected in the identification of a stationary state. In statistical mechanics, the Boltzmann-Gibbs law establishes that the probability distribution of energy E is $P(E) = Ce^{-E/T}$, where T is the temperature, and C is a normalizing constant. Due to conservation property, Dragulescu and Yakovenko claimed that, in their model, the steady state for the money w is given by distribution function $P(w) = Ce^{-w/T}$, where T represents the average of money, owned by each agent. This claim is supported by some observations about the model. Indeed, the total amount of money w is preserved and money is an additive quantity, i.e. if we split the population into two subsets and their respective amount of money is w_i, it results $w = w_1 + w_2$. On the other hand, the fraction of population having wealth $w = w_1 + w_2$ is $P(w)$ by definition but it may also be described as the probability of intersection of the of having money w_1 (for an agent in the first subset) and w_2 (for an agent in the second subset), i.e. $P(w_1)P(w_2)$. These observations lead to write the equation $P(w_1)P(w_2) = P(w_1 + w_2)$ and an exponential law solves it. Another way to justify the Boltzmann-Gibbs' distribution as an equilibrium state for this economic model is to use an entropy argument. More precisely, the distribution $P(w) = Ce^{-w/T}$ maximize the entropy $S = -\int_0^{+\infty} P(w) lnP(w) dw$, under the condition of money preservation. These results, that are obtained in comparison with gas dynamics, are confirmed by numerical simulations for a large class of initial data [35]. According to [27], one of the most important outcomes achieved in [35] has been to explain the meaning of T as the average of money of each agent. Nevertheless, in this model, a huge fraction of agents has few money in the steady state. For this reason, this prediction is not very realistic, as the authors themselves underlined. Starting from this model, Chakraborti and Chakrabarti in [27] wrote more general interaction rules that consider the saving propensity λ (called "marginal propensity to save" in the economic context) of each agent. In fact, in each trading, every agent uses a fraction of his total wealth and saves a fraction that is proportional to his pre-interaction wealth. Although each agent is characterized by a personal saving propensity, in [27] it is assumed to be constant and strictly positive for all the agent. The model of Dragulescu and Yakovenko, that corresponds in [27] to the case $\lambda = 0$, reflects the behaviour of non-cooperative market.
Moreover, Chakraborti and Chakrabarti assumed that the portion of wealth that is involved in the trading is totally split among the agents or, in other words, they assumed that the market is strictly conservative. With these assumptions, the interaction rules read as

$$v^* = \lambda v + (1-\eta)(1-\lambda)(v+w), \quad w^* = \lambda w + \eta(1-\lambda)(v+w), \tag{3.2}$$

where $\lambda \in (0,1)$ and η is a random variable in the interval $(0,1)$. The presence of a random variable underlines that the not-saved portion of wealth is randomly shared among the agents. Furthermore, in [27] debts are not allowed, i.e. $w, v \geq 0$. Saving propensity changes the collective behaviour of the agents, that becomes more cooperative. Moreover, it also disrupts the multiplicative property of the distribution function. Even if it is not possible to compute explicitly the steady state, it is robustly obtained with numerical results. Due to the effect of saving propensity, most of the agents assembled very close to 0 cannot be observed in the steady state, as in [35], and the maximum value of the distribution function

(the most probable quantity of money held by an agent) becomes closer to the average amount of wealth T as λ becomes closer to 1. In addition, stationary state for each $\lambda > 0$ does not show heavy tails. The numerical results are in accordance with the theoretical results in the simple case in which the portion of wealth that is involved in the trading is equally split among the agents, i.e. $\eta = \frac{1}{2}$. In this simple case, the interaction rules become

$$v^* = \lambda v + \frac{1}{2}(1-\lambda)(v+w), \quad w^* = \lambda w + \frac{1}{2}(1-\lambda)(v+w).$$

The steady state becomes a Dirac-δ concentrated in T. This kind of singularity is natural in this econophysics context, although there is not an equivalent of saving propensity in gas dynamics. In consequence of this, there is no similar distribution in the physical framework where we observe more regular equilibrium such as Gibbs' distribution.
Some analysis, such as the two previously described, show that the wealth presents a power-law decay at infinity and an exponential decay at 0. Moreover, it is plausible to consider a model in which there is, at the macroscopic level, a net increase of total wealth, as an effect of human activities. This behaviour has many similarities with inelastically scattering particles of granular gas. There was a lot of numerical results that have confirmed this intuition. In this direction, Slanina obtained an analytical result in a simple case [83]. Although the structure of the equations for the granular gas and wealth are the same, it is not possible to transfer directly the solution from physical to economical framework. This is due to the different sign in the time variations of the energy of granular gas (that is decreasing) and the average value of the wealth (that is increasing). Slanina considered an open market, in which the net gain of wealth is the effect of the contribution of an external source (energy of Earth and Sun). The interaction rules read as

$$v^* = (1+\gamma-\beta)v + \beta w, \quad w^* = (1+\gamma-\beta)w + \beta v,$$

where $\beta \in (0,1)$ represents the exchanged fraction of wealth and $\gamma > 0$ is the income from the external environment. These interaction rules lead to a Boltzmann-like equation for the distribution function of wealth

$$\frac{\partial}{\partial t}P(v,t) + P(v,t) = \int_{\mathbf{R}_+^2} P(w_i,t)P(w_j,t)\delta((1-\beta+\gamma)w_i + \beta w_j - v)dw_i dw_j, \tag{3.3}$$

which is an exact description of the phenomenon in the limit of infinite number of particles and has the same structure of the distribution of inelastically scattering particles. The first moment of solution of (3.3) $m(t) = \int_0^{+\infty} v(P(v,t)dv$ is exponentially increasing. In consequence of it, in [83] a scaling (self-similar) function $\Phi(v,t)$ is introduced such that

$$P(v,t) = \frac{1}{m(t)}\Phi\left(\frac{v}{m(t)}, t\right).$$

With this scaling,the first moment of $\Phi(v,t)$ is constant in time. The properties of the solution are investigated by using Laplace transform $\widehat{\Phi}(x,t)$ of $\Phi(v,t)$, defined as $\widehat{\Phi}(x,t) = \int_0^{+\infty} e^{-xv}\Phi(v,t)dv$ where $x \in \mathbb{C}$. Firstly, in the limit $x \to 0$, the behaviour for large v is investigated. The result confirms the presence, in the distribution $\Phi(v,t)$, of heavy tails of index $\alpha \in (1,2)$ which depends on γ and β. Other results are obtained in the contiunuos trading limit, i.e. to perform the limit $\gamma \to 0$ and $\beta \to 0$. The unique identified stationary state (that presents power-law decay at infnity) has the form

$$\Phi(v) = Cv^{-\alpha-1}e^{-\frac{\alpha-1}{v}},$$

where $\alpha \in (1,2)$ and C is the normalization constant and it depends on α. One of the outcomes of Slanina's model was to underline the correlation between the Pareto's index and the rates of both

increasing and exchanging wealth. However, in this model, we can observe an important element that was missing in the aforesaid models, i.e. the derivation, in the continuous trading limit, of an evolution equation starting from the interactions rules.
In 2005, Cordier et al. [32] proposed the CPT model for an open market, that takes into account trading between agents, risk investment and speculation. Due to their strong interconnection, money and wealth, in this model, assume the same meaning even they are not completely interchangeable. The change of wealth that is due to investments is not governed by a deterministic mechanism; so it generates a random gain or loss of wealth that, in [32], is assumed to be proportional to pre-interaction wealth. Furthermore, the agents interact without taking on debt. With this assumptions, the interaction rules are

$$\begin{aligned} v^* &= (1-\gamma+\eta)v+\gamma w, \\ w^* &= (1-\gamma+\eta^*)w+\gamma v, \end{aligned} \tag{3.4}$$

where $\gamma \in (0, \frac{1}{2})$ is a constant that is related to the saving propensity of each agent and η, η^* are i.i.d. random variables which describe risk investment. In order to maintain non-negative post-interaction wealth of each agent, we should assume $\eta, \eta^* \geq -(1-\gamma)$. Firstly, in [32] the authors analyze the case in which η is distributed as Gaussian, centered in the origin with variance σ^2, $\eta \sim \mathcal{N}(0, \sigma^2)$. In this case, the market, even if it is not pointwise conservative, results conservative on the average, i.e.

$$\langle v^* + w^* \rangle = \langle v + w \rangle.$$

Otherwise, the model describes a non-conservative economy. In the general case, by using standard techniques of statistical mechanics, the authors write the Boltzmann-like equation for the distribution function of the wealth $f(v,t)$. In weak form, it reads for all smooth test function $\phi(v)$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^+} \phi(v) f(v,t) dv = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2_+} \beta f(w) f(v)(\phi(w^*) - \phi(w)) dw dv d\eta d\eta^*, \tag{3.5}$$

where $\beta = \beta(\eta, \eta^*, v, w)$ is a kernel interaction that guarantees the assumption of an economy without debts. It is possible to choose a particular form of this kernel, such as

$$\beta(\eta, \eta^*, v, w) = \Theta(\eta)\Theta(\eta^*)\chi(w^* \geq 0)\chi(v^* \geq 0),$$

where $\chi(\cdot)$ is the characteristic function and $\Theta(\cdot)$ is a symmetric probability density which has mean zero and variance σ^2. This choice may be simplified by taking $\eta \in (-(1-\gamma), 1-\gamma)$. With this semplification, the interaction kernel does not depend on w and v and corresponds to the case of preservation of average wealth. In the general case, the increase of wealth has an exponential rate and it depends on the choice of $\Theta(\cdot)$. In this case, as in [83], the more conventional method to obtain information on steady state is to scale the solution to have a distribution with constant average wealth. Let us assume that the initial distribution $f_0(v)$ is smooth enough, i.e. the moments of $f_0(v)$ are bounded up to the order $2+\delta$ with $\delta > 0$, and define the solution to the weak Boltzmann-like equation $g_\gamma(v,\tau) = f(v,t)$ where $\tau = \gamma t$. With this further assumption, in [32] the authors prove that in the limit $\sigma, \gamma \to 0$ in such way that $\sigma^2 = \gamma\lambda$, the sequence $g_\gamma(v,\tau)$ converges to a density $g(v,\tau)$ that is a weak solution of a Fokker-Planck equation

$$\frac{\partial}{\partial t} g = \frac{\lambda}{2}\frac{\partial^2}{\partial v^2}\left(v^2 g\right) + \left((v-m)g\right).$$

Stationary solution (explicitly computed) is the same as the model of Bouchaud and Mezard and shows heavy tail as Pareto's prediction. Numerical results confirm both this behaviour of steady state and the rate of growth of the average wealth in the general case before the scaling.

Although these results are interesting from a modellistic point of view, it was necessary to complete some theoretical gaps. With this aim, in 2006 D. Matthes and G.Toscani investigated the hypothesys under which on microscopic interactions in a conservative economy an unique (non-trivial) steady state exists and it exhibits Pareto's tails [74]. Binary collisions are defined, in a very general notation, as

$$v^* = p_1 v + q_1 w, \quad w^* = p_2 v + q_2 w, \tag{3.6}$$

where p_i and q_i, for $i = 1, 2$, can be constant parameters or random variables, with the only constraint of the maintence of positivity of post-interaction values of wealth. In [74], mean wealth preservation means both the concept of *"pointwise preservation"*, i.e. $p_1 + p_2 = q_1 + q_2 = 1$, and *"preservation in the mean"*, i.e. $\langle p_1 + p_2 \rangle = \langle q_1 + q_2 \rangle = 1$. However, p_i and q_i are choosen as independent on time and variables v and w. With these interaction rules, Boltzmann-likes equation for the density of wealth $f(v, t)$, in weak form, reads as

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^+} \phi(v) f(v) dv = \frac{1}{2} \left\langle \int_{\mathbb{R}_+} \int_{\mathbb{R}_+^2} f(w) f(v) (\phi(v^*) + \phi(w^*) - \phi(v) - \phi(w)) dw dv \right\rangle. \tag{3.7}$$

The aim of this work is to relate the existence of the stationary state and the formation of heavy tails with the quantity $\mathfrak{G}(s)$, that is defined for each $s > 0$ as

$$\mathfrak{G}(s) = \frac{1}{2} \sum_{i=1}^{2} \langle p_i^s + q_i^s \rangle - 1. \tag{3.8}$$

By using the Fourier transform of (3.7), it is possible to write the Fourier distance $d_s(f_1, f_2)$ between two solutions of (3.7) $d_s(f_1, f_2)$. With direct computations, the authors of [74] obtained the following result:

Theorem 2. *([74]). Let $f_1(t)$ and $f_2(t)$ be two solutions of the Boltzmann equation* (3.7)*, corresponding to initial values $f_{1,0}$ and $f_{2,0}$ satisfying conditions of normalization of mass and mean value*

$$\int_0^{+\infty} f_{i,0}(v) dv = 1, \quad \int_0^{+\infty} v f_{i,0}(v) dv = M, \tag{3.9}$$

for $i = 1, 2$. Let $s \geq 1$ be such that $d_s(f_{1,0}, f_{2,0})$ is finite. Then, for all times $t \geq 0$,

$$d_s(f_1(t), f_2(t)) \leq d_s(f_{1,0}, f_{2,0}) exp\{\mathfrak{G}(s)t\}. \tag{3.10}$$

In particular, if $\mathfrak{G}(s)$ is negative, then the d_s-distance between f_1 and f_2 decays exponentially in time.

Putting $f_{1,0} = f_{2,0} = f_0$ in (3.10) and using $s = 1$ yields

Corollary 1. *([74]) If f_0 is a nonnegative density satisfying conditions* (3.9)*, then there exists an unique weak solution $f(t)$ of the Boltzmann equation with $f(0) = f_0$.*

The aforesaid analysis shows the relation between the sign of $\mathfrak{G}(s)$ and the uniqueness of solution of (3.7). The authors in [74] analyze its relation with the existence of the steady state. Due to definition of $\mathfrak{G}(s)$ and average conservation of wealth, it results $\mathfrak{G}(1) = 0$. In consequence of this, $\mathfrak{G}(s)$ is well defined for $s \in (0, 1]$ but it is possible that for each $s > s_\infty \geq 1$ $\mathfrak{G}(s) = +\infty$. Furthermore, convexity of $\mathfrak{G}(s)$ allows only three different cases. In the first case, $\mathfrak{G}(s) \geq 0$ for all $s > 0$. In this case, the previous analysis does not provide information about the existence of a steady state. In the second case, $\mathfrak{G}(s) < 0$ for some $0 < s < 1$ and $\mathfrak{G}(s) > 0$ for all $s > 1$. It is possible to prove that an unique stationary state

exists and it is precisely a Dirac-δ centrated in the origin. In the third case, $\mathfrak{G}(s) < 0$ for $1 < s < \overline{s}$ and $\mathfrak{G}(s) > 0$ for the other value of s. Let us consider an $s \in (1, \min\{2, \overline{s}\})$ and let us assume that the initial distribution has finite moments up to order $S > s$. With this assumptions, the unique solution of Boltzmann-like equation (3.7) $f(v,t)$ converges in the metric d_s to a limit distribution $f_\infty(v)$ with exponential rate $\mathfrak{G}(s)$. The distribution $f_\infty(v)$ results as the (unique) stationary solution for the same equation and it presents the same first moment of initial distribution $f_0(v)$. In this third case, there are two different situation:

(PT) $\overline{s} < +\infty$ and $\mathfrak{G}(\overline{s}) = 0$,

(ST) $\overline{s} = +\infty$, i.e. $\mathfrak{G}(s) < 0$ for all $s > 1$.

In case (PT), steady distribution $f_\infty(v)$ exhibits Pareto's tails of index that is exactly $\overline{s}$. In case (ST), $f_\infty(v)$ has slim tail.
We can apply these criteria to previous models. Pointwise conservative models, as the ones of Dragulescu and Yakovenko or Chakraborti and Chakrabarti, are examples of the cases (ST). A general way to write pointwise conservative interaction rules is to set $p_1 = q_2 = \lambda$ and $p_2 = q_1 = 1 - \lambda$, where λ is a random variable $\lambda \in [0,1]$. In this case, it results $\mathfrak{G}(s) = \lambda^s + (1-\lambda)^s - 1 \leq 0$ for all $s > 1$. If $\mathfrak{G}(s) \neq 0$, the assumptions of case (ST) hold and the stationary solution has slim tail. If $\lambda = 0, 1$ a.s. , i.e. $\mathfrak{G}(s) \equiv 0$, the interaction rules become trivial and the unique solution is $f(v,t) = f_0(v)$, for all $t \geq 0$. If $\mathfrak{G}(s) \equiv 0$ with $\lambda \neq 0, 1$ a.s., all the moments of the solution are increasing and the only stationary state is Dirac-δ, centered in the origin. The model of Chakraborti and Chakrabarti gives one example. Although in [27] the interaction rules depend on a random variable η (see (3.2)), the choice of this variable does not affect the behaviour of the steady state. Indeed, by inserting (3.2) in (3.8), it results

$$\mathfrak{G}(s) = \frac{1}{2}\left(\langle[\lambda + (1-\eta)(1-\lambda)]^s\rangle + \langle[1-(1-\eta)(1-\lambda)]^s\rangle + [\langle(1-\eta)^s\rangle + \langle\eta^s\rangle](1-\lambda)^s\right) - 1.$$

Due to its convexity,for each $s > 1$, $\mathfrak{G}(s)$ results negative. Thus, as previously indicated in case (ST), the steady state has slim tails. In consequence of these observations, it is possible to conclude that only the models that are conservative in mean, but not pointwise, may have heavy tails. One example is furnished by CPT model. It may be useful to underline the role of the random variable that is involved in interaction rules together with the necessity of a well balance between random effects and saving propensity. The easiest choice for centred η in (3.4) is $\eta = \pm\mu$ where the random variable assumes each value with probability $\frac{1}{2}$ and $0 < \mu < 1 - 2\gamma$. Observe that this condition is enough to maintain the positivity of post interaction wealth. With this choice, it results

$$\mathfrak{G}(s) = \frac{1}{2}\left[(1-\gamma+\mu)^s + (1-\gamma-\mu)^s\right] + \gamma^s - 1.$$

If $\gamma > \frac{1}{4}$ or $\mu < 2\gamma$ the steady state shows slim tails. This situation corresponds to a market with both low risk and low saving propensity (the parameter γ has the opposite meaning). By increasing saving propensity and risk of the market, there is a region of the plane in which wealth distribution shows Pareto's tails. Finally, when γ is very close to 0 and μ is very close to 1, a small fraction of the population is very rich while a huge fraction loses his wealth. From an economic point of view, a sufficient saving propensity is necessary for the formation of wealth. Nevertheless, it is not possible to accumulate richness in the hands of few agents until the investment risk is big enough.
Previous results underline the interest of Econophysics community on wealth distribution and the role of Fokker-Planck Equation. In particular, aforesaid theoretical results represent the base on which my work on wealth distribution model was developed. In chapter 4, a Fokker-Planck equation with variable coefficient is used as a model of wealth distribution on the positive half-line, i.e. a model in absence of debts. In chapter 5, this model is generalized in a context in which debts are allowed.

Chapter 4

Wealth distibution without debts

4.1 Introduction

Among the mathematical models introduced in recent years to study the evolution of wealth distribution in a multi-agent society [76], Fokker–Planck type equations play an important role. Let $f(v,t)$ denote the density of agents with personal wealth $v \geq 0$ at time $t \geq 0$. The prototype of these Fokker–Planck equations reads

$$\frac{\partial f}{\partial t} = J(h) = \frac{\sigma}{2}\frac{\partial^2}{\partial v^2}\left(v^2 f\right) + \lambda \frac{\partial}{\partial v}\left((v-1)f\right), \tag{4.1}$$

where λ and σ denote two positive constants related to essential properties of the trade rules of the agents. Equation (4.1) has been first derived by Bouchaud and Mezard [17] through a mean field limit procedure applied to a stochastic dynamical equation for the wealth density. The same equation was subsequently obtained by one of the present authors with Cordier and Pareschi [32] via an asymptotic procedure from a Boltzmann-type kinetic model for trading agents. This procedure also furnished the existence (without uniqueness) of a weak solution to equation (4.1).

One of the main features of equation (4.1) is that it possesses a unique stationary solution of unit mass, given by the (inverse) Γ-like distribution [17, 32]

$$f_\infty(v) = \frac{(\mu-1)^\mu}{\Gamma(\mu)} \frac{\exp\left(-\frac{\mu-1}{v}\right)}{v^{1+\mu}}, \tag{4.2}$$

where

$$\mu = 1 + 2\frac{\lambda}{\sigma} > 1.$$

This stationary distribution, as predicted by the analysis of the italian economist Vilfredo Pareto [78], exhibits a power-law tail for large values of the wealth variable.

Equation (4.1) differs from the classical Fokker–Planck equation in two important points. First, the domain of the wealth variable v takes only values in $\mathbb{R}_+$. Second, the coefficient of diffusion depends on the wealth variable. This makes the analysis of the large-time behavior of the solution to equation (4.1) very different from the analogous one studied in [87] for the classical Fokker–Planck equation.

Indeed, Fokker–Planck equations with variable coefficients and in presence of boundary conditions have been rarely studied. Maybe the first result in this direction can be found in a paper by Feller [46], who treated the case $v \in \mathbb{R}_+$ and coefficient of diffusion v, with a general drift term (cf. also the book [47] for a general view about boundary conditions for diffusion equations). In particular, the importance of the boundary conditions has been shown in [46] to be related to the action of the drift term.

More recently, Fokker-Planck type equations with almost general coefficient of diffusions have been studied by Le Bris and Lions in [65]. Unlikely, their analysis does not apply to equation (4.1).

As far as the large-time behavior is concerned, the main argument in the standard Fokker–Planck equation is to resort to entropy decay, and to logarithmic Sobolev inequalities [4]. However, as discussed in [73], this type of inequalities do not seem available in presence of variable diffusion coefficients.

In the following, we will try to give a satisfactory answer to some of the open questions. As we shall see, various properties of the solution to equation (4.1) can be extracted from the limiting relationship between the Fokker–Planck description and its kinetic level, given by the bilinear Boltzmann-type equation introduced in [32]. We will discuss this aspect in Section 4.2, by means of a detailed Fourier analysis. In particular, we will show that, at least for some range of the parameters λ and σ, the Fokker–Planck equation (4.1) can be rigorously obtained in the asymptotic limit procedure known as quasi-invariant trade limit. Then, convergence to equilibrium will be discussed in Section 4.3. The essential argument here will be to resort to an inequality of Chernoff type [30, 62], that allows to prove convergence with exponential rate in the case of initial data sufficiently close to the steady state, and to a rate at least $1/t$ for a large class of initial data.

4.2 Kinetic model and Fokker-Planck equation

4.2.1 Main properties of the Fokker–Planck equation

To start with the analysis of the initial value problem for the Fokker–Planck equation (4.1), it is essential to consider, together with a suitable decay of the solution at infinity, physical boundary conditions at the point $v = 0$. Let $\phi(v)$ be a smooth function, bounded at $v = 0$. Then, a simple computation shows that

$$\frac{d}{dt}\int_{\mathbb{R}_+}\phi(v)f(v,t)dv = \int_{\mathbb{R}_+}\left[\frac{\sigma}{2}v^2\phi''(v) - \lambda(v-1)\phi'(v)\right] f(v,t)\,dv + \left[\frac{\sigma}{2}\left(\phi(v)\frac{\partial}{\partial v}(v^2 f(v,t)) - v^2\phi'(v)f(v,t)\right) + \lambda\phi(v)(v-1)f(v,t)\right]_0^\infty .$$

While the vanishing of the boundary term at infinity follows by choosing initial data with a smooth and rapid decay, at the boundary $v = 0$ it is required that

$$v^2 f(v,t)\,|_{v=0} = 0, \qquad t > 0 \tag{4.3}$$

and

$$\lambda(v-1)f(v,t) + \frac{\sigma}{2}\frac{\partial}{\partial v}\left(v^2 f(v,t)\right)\Big|_{v=0} = 0, \quad t > 0. \tag{4.4}$$

Condition (4.3) is automatically satisfied for a sufficiently regular density f. On the contrary condition (4.4) requires an exact balance between the so-called advective and diffusive fluxes on the boundary $v = 0$. This condition is usually referred to as the *no-flux* boundary condition. If both conditions (4.3) and (4.4) hold, we can pass from equation (4.1) to its weak form, given by

$$\frac{d}{dt}\int_{\mathbb{R}_+}\phi(v)f(v,t)dv = (\phi, J(f)) = \int_{\mathbb{R}_+}\left[\frac{\sigma}{2}v^2\phi''(v) - \lambda(v-1)\phi'(v)\right] f(v,t)\,dv. \tag{4.5}$$

By choosing $\phi(v) = 1, v$ shows that the solution to (4.1) satisfies

$$\frac{d}{dt}\int_{\mathbb{R}_+} f(v,t)dv = 0, \quad \frac{d}{dt}\int_{\mathbb{R}_+} vf(v,t)dv = \lambda\left(-\int_{\mathbb{R}_+} vf(v,t)\,dv + \int_{\mathbb{R}_+} f(v,t)\,dv\right).$$

Therefore, if the (nonnegative) initial value $\varphi(v)$ of equation (4.1) is a density function satisfying the normalization conditions

$$\int_{\mathbb{R}_+} \varphi(v)\, dv = 1; \quad \int_{\mathbb{R}_+} v\varphi(v)\, dv = 1 \tag{4.6}$$

the solution to (4.1) still satisfies conditions (4.6). In other words, if the initial datum is a probability density with unit mean, then the solution at any subsequent time remains a probability density with unit mean. For $n \in \mathbb{N}_+$ let us define

$$M_n(t) = \int_{\mathbb{R}_+} v^n f(v,t) dv.$$

An elementary computation shows that, if φ satisfies conditions (4.6) and its second moment is bounded, the second moment of the solution follows the law

$$\frac{d}{dt} M_2(t) = (\sigma - 2\lambda) M_2(t) + 2\lambda. \tag{4.7}$$

Hence, the value of the second moment stays bounded when $\sigma < 2\lambda$, while it diverges in the opposite case. In the former case, solving equation (5.36) we obtain

$$M_2(t) = e^{(\sigma-2\lambda)t} \left(M_2(0) + \frac{2\lambda}{\sigma - 2\lambda} \right) + \frac{2\lambda}{2\lambda - \sigma}, \tag{4.8}$$

which implies

$$\lim_{t\to\infty} M_2(t) = \frac{2\lambda}{2\lambda - \sigma}.$$

It is clear that the principal moments of the solution to the Fokker–Planck equation can be obtained recursively, and explicitly evaluated at the price of increasing length of computations. Since it will be useful in the following, we evaluate here the third moment $M_3(t)$. We obtain

$$\frac{d}{dt} M_3(t) = 3(\sigma - \lambda) M_3(t) + 3\lambda M_2(t).$$

Then, if the initial density $\varphi(v)$ has the third moment bounded, the evolution law for $M_3(t)$ is given by

$$M_3(t) = e^{3(\sigma-\lambda)t} \left\{ M_3(0) + 3\lambda \int_0^t e^{-3(\sigma-\lambda)r} M_2(r) dr \right\}. \tag{4.9}$$

Using (5.37), the third moment is evaluated as

$$\begin{aligned} M_3(t) = M_3(0) e^{3(\sigma-\lambda)t} &+ \left(\frac{3\lambda}{(\lambda - 2\sigma)} M_2(0) + \frac{6\lambda^2}{(\sigma - 2\lambda)(\lambda - 2\sigma)} \right) \left(e^{(\sigma-2\lambda)t} - e^{3(\sigma-\lambda)t} \right) \\ &+ \frac{2\lambda^2}{(\sigma - 2\lambda)(\lambda - \sigma)} \left(1 - e^{3(\sigma-\lambda)t} \right). \end{aligned} \tag{4.10}$$

Therefore the third moment is uniformly bounded in time if $\sigma < \lambda$ and it grows to $+\infty$ in the opposite case.

Last, choosing $\phi(v) = e^{-i\xi v}$ we obtain the Fourier transformed version of the Fokker–Planck equation (4.1)

$$\frac{\partial}{\partial t} \widehat{f}(\xi,t) = \widehat{J}(\widehat{f}) = \frac{\sigma}{2} \xi^2 \frac{\partial^2}{\partial \xi^2} \widehat{f}(\xi,t) - \lambda \xi \frac{\partial}{\partial \xi} \widehat{f}(\xi,t) - i\lambda\xi \widehat{f}(\xi,t), \tag{4.11}$$

where, as usual $\widehat{g}(\xi)$ denotes the Fourier transform of $g(v)$, $v \in \mathbb{R}_+$. In this case

$$\widehat{g}(\xi) = \int_{\mathbb{R}_+} e^{-i\xi v} g(v)\, dv.$$

4.2.2 The kinetic model

The basic model discussed in this section has been introduced in [32] within the framework of classical models of wealth distribution in economy.

As shown in [32], the Fokker–Planck equation (4.1) is strongly related to a bilinear kinetic model of Boltzmann type, modelling the evolution of wealth in a multi-agent society in which agents interact through binary trades [76]. This model belongs to a class of models in which the interacting agents are indistinguishable. The agent's *state* at any instant of time $t \geq 0$ is completely characterized by his current wealth $v \geq 0$ [38, 39]. When two agents encounter in a trade, their *pre-trade wealths* v, w change into the *post-trade wealths* v^*, w^* according to the rule [26, 27, 29]

$$v^* = p_1 v + q_1 w, \quad w^* = q_2 v + p_2 w.$$

The *interaction coefficients* p_i and q_i are non-negative random variables. While q_1 denotes the fraction of the second agent's wealth transferred to the first agent, the difference $p_1 - q_2$ is the relative gain (or loss) of wealth of the first agent due to market risks. It is usually assumed that p_i and q_i have fixed laws, which are independent of v and w, and of time. This means that the amount of wealth an agent contributes to a trade is (on the average) proportional to the respective agent's wealth.

In [32] the trade has been modelled to include the idea that wealth changes hands for a specific reason: one agent intends to *invest* his wealth in some asset, property etc. in possession of his trade partner. Typically, such investments bear some risk, and either provide the buyer with some additional wealth, or lead to the loss of wealth in a non-deterministic way. An easy realization of this idea consists in coupling the saving propensity parameter [26, 27] with some *risky investment* that yields an immediate gain or loss proportional to the current wealth of the investing agent. The interactions rules for this model are obtained by fixing

$$\begin{aligned} p_1 &= 1 - \varepsilon\lambda + \eta_\varepsilon, \quad q_1 = \varepsilon\lambda \\ p_2 &= \varepsilon\lambda, \quad q_2 = 1 - \varepsilon\lambda + \tilde{\eta}_\varepsilon, \end{aligned} \tag{4.12}$$

where $0 \leq \lambda \leq 1$ is the parameter which identifies the saving propensity, namely the intuitive behavior which prevents the agent to put in a single trade the whole amount of his money, while ε is a small positive parameter, which measures the quantity of money exchanged in a single trade. The coefficients $\eta_\varepsilon, \tilde{\eta}_\varepsilon$ are independent and identically distributed random parameters, such that always $\eta_\varepsilon, \tilde{\eta}_\varepsilon \geq \varepsilon\lambda - 1$. This clearly implies $v^*, w^* \geq 0$. Therefore, the collision rule in [32] reads

$$\begin{aligned} v^* &= (1 - \varepsilon\lambda)v + \varepsilon\lambda w + \eta_\varepsilon v, \\ w^* &= (1 - \varepsilon\lambda)w + \varepsilon\lambda v + \tilde{\eta}_\varepsilon w. \end{aligned} \tag{4.13}$$

Remark 2. *In the rest of the paper, the mean value of a random quantity θ will be denoted by $\langle\theta\rangle$. A simple way to characterize the ε-dependence of the random parameters is to define η_ε and $\tilde{\eta}_\varepsilon$ as independent copies of a random variable η with finite variance σ, with $\eta_\varepsilon = \tilde{\eta}_\varepsilon = \sqrt{\varepsilon}\eta$. If the random parameters are even, so that $\langle\eta_\varepsilon\rangle = \langle\tilde{\eta}_\varepsilon\rangle = 0$*

$$\langle v^* + w^*\rangle = (1 + \langle\eta_\varepsilon\rangle)v + (1 + \langle\tilde{\eta}_\varepsilon\rangle)w = v + w, \tag{4.14}$$

implying conservation of the average wealth. In the remaining cases, it is immediately seen that the mean wealth is not preserved, but it increases or decreases exponentially (see the computations in [32]). Various specific choices for the random parameters have been discussed in [74]. Note that, when $\langle\eta\rangle = 0$, $\langle\eta_\varepsilon^2\rangle = \langle\tilde{\eta}_\varepsilon^2\rangle = \varepsilon\sigma$.

Owing to classical arguments of kinetic theory [76], it has been shown in [32] that the evolution of the wealth density consequent to the binary interactions (4.13) obeys to a Boltzmann-type equation.

To outline the dependence on ε, let us denote with $h_\varepsilon(v,\tau)$ the distribution of the agents wealth $v \geq 0$ at time $\tau > 0$. Then, the equation for the evolution of h_ε can be fruitfully written in weak form. It corresponds to say that, for any smooth function ϕ, h_ε satisfies the equation

$$\frac{d}{d\tau}\int_{\mathbb{R}_+}\phi(v)h_\varepsilon(v,\tau)dv = \frac{1}{2}\left\langle \int_{\mathbb{R}_+\times\mathbb{R}_+} h_\varepsilon(v,\tau)h_\varepsilon(w,\tau)\big(\phi(v^*)+\phi(w^*)-\phi(v)-\phi(w)\big)dvdw\right\rangle. \tag{4.15}$$

Existence and uniqueness of the solution to equation (4.15) has been proven in [74]. We will detail later on some of these results for their connection with the Fokker–Planck equation (4.1). The weak form (4.15) allows to evaluate moments of the solution in a closed form. The choice $\phi(v) = 1$ immediately gives mass conservation. In addition, if $\phi(v) = v$, in view of (4.14) one obtains that the mean value of the solution is preserved in time. Therefore, if the initial value satisfies the normalization conditions (4.6) it follows that the solution $h_\varepsilon(v,\tau)$ still satisfies the same conditions at any subsequent time $\tau > 0$.

Let us choose now $\phi(v) = v^2$. A simple computation gives

$$\langle v^{*2} + w^{*2} - v^2 - w^2\rangle = 2(\varepsilon^2\lambda^2 - \varepsilon\lambda)(v-w)^2 + \varepsilon\sigma(v^2+w^2).$$

Therefore

$$\frac{d}{d\tau}\int_{\mathbb{R}_+} v^2 h_\varepsilon(v,t)dv = \varepsilon\left[\sigma - 2\left(\lambda - \varepsilon\lambda^2\right)\right]\int_{\mathbb{R}_+} v^2 h_\varepsilon(v,\tau)dv + 2\varepsilon(\lambda-\varepsilon\lambda^2). \tag{4.16}$$

The evolution law of the second moment of $h_\varepsilon(v,\tau)$ depends explicitly on ε, and clearly changes with ε. This is due to the fact that changing the value of ε in the binary collision (4.13) we change the quantity of wealth which is involved into the trade. In the limit case $\varepsilon \to 0$, we have a trade in which the post interaction wealths are left unchanged. This suggests to scale time in such a way to maintain an effective law of evolution of the second moment even in the limit $\varepsilon \to 0$. This can be easily done by setting $t = \varepsilon\tau$, while $h_\varepsilon(v,\tau) = f_\varepsilon(v,t)$ [32]. One then obtains that $f_\varepsilon(v,t)$ satisfies

$$\frac{d}{dt}\int_{\mathbb{R}_+}\phi(v)f_\varepsilon(v,t)dv = \left(\phi, Q_\varepsilon(f_\varepsilon,f_\varepsilon)\right), \tag{4.17}$$

where we defined

$$(\phi, Q_\varepsilon(f_\varepsilon,f_\varepsilon)) = \frac{1}{2\varepsilon}\left\langle \int_{\mathbb{R}_+\times\mathbb{R}_+} f_\varepsilon(v,t)f_\varepsilon(w,t)\big(\phi(v^*)+\phi(w^*)-\phi(v)-\phi(w)\big)dvdw\right\rangle. \tag{4.18}$$

In this case, (4.16) changes into

$$\frac{d}{dt}\int_{\mathbb{R}_+} v^2 f_\varepsilon(v,t)dv = \left[\sigma - \left(\lambda-\varepsilon\lambda^2\right)\right]\int_{\mathbb{R}_+} v^2 f_\varepsilon(v,t)dv + 2(\lambda-\varepsilon\lambda^2). \tag{4.19}$$

Remark 3. *Note that the conservation of the mean value is not modified by this scaling. However, in (4.19) the dependence on ε remains in the factor $\lambda - \varepsilon\lambda^2$. One can easily eliminate this dependence by choosing in (4.13) a different value (ε-dependent) of the saving propensity λ. This can be obtained by the choice*

$$\tilde{\lambda} = \tilde{\lambda}(\varepsilon) = \frac{2\lambda}{1+\sqrt{1-4\varepsilon\lambda}}, \tag{4.20}$$

which is such that $\tilde{\lambda} - \varepsilon\tilde{\lambda}^2 = \lambda$. Moreover

$$\tilde{\lambda} > \lambda, \quad \lim_{\varepsilon\to 0}\tilde{\lambda} = \lambda.$$

Clearly, definition (4.20) requires to choose ε small enough.

4.2.3 The invariant trade limit of the Boltzmann equation

The close relation between the kinetic equation (4.17) and the Fokker–Planck equation (4.5) has been outlined in [32], where it was shown that in the limit $\varepsilon \to 0$ a subsequence of solutions $f_\varepsilon(v,t)$ to (4.17) converges to $f(v,t)$, solution of (4.5). In this section we aim in improving these results.

In the rest, we will fix a time $T > 0$, and we will consider both equations (4.17) and (4.1) in the time interval $0 \le t \le T$. In addition, let the even random variable η which defines the random part of the trade possess the third moment bounded, and let us set $\langle |\eta|^3 \rangle = \sigma_3$. Analogously, let $\langle |\eta| \rangle = \sigma_1$. Using a Taylor's formula at the second order, one can write $\phi(v^*)$ as

$$\langle \phi(v^*) - \phi(v) \rangle = \phi'(v)\langle v^* - v \rangle + \frac{1}{2}\phi''(v)\langle v^* - v \rangle^2 + \frac{1}{3!}\langle \phi'''(\tilde{v})(v^* - v)^3 \rangle$$

where, for some α such that $0 \le \alpha \le 1$, $\tilde{v} = \alpha v + (1-\alpha)v^*$. Then, by (4.13) and the properties of the random variable η it holds

$$\begin{aligned} \langle v^* - v \rangle &= \varepsilon\lambda(w - v), \\ \langle (v^* - v)^2 \rangle &= \varepsilon^2\lambda^2(w-v)^2 + \varepsilon\sigma v^2, \\ \langle |v^* - v|^3 \rangle &\le \varepsilon^{3/2}\left[\sigma_3 v^3 + 3\varepsilon^{1/2}\sigma\lambda v^2|w - v| + 3\varepsilon\sigma_1\lambda^2 v(w-v)^2 + \varepsilon^{3/2}\lambda^3|v-w|^3\right]. \end{aligned} \tag{4.21}$$

In particular, (4.17) can be rewritten as

$$\frac{d}{dt}\int_I \phi(v) f_\varepsilon(v,t)dv = \int_{\mathbb{R}_+} f_\varepsilon(v,t)\left[\frac{\sigma}{2}v^2\phi''(v) - \lambda(v-1)\phi'(v)\right]dv + \int_{\mathbb{R}_+} R_\varepsilon(\phi(v),t)\, f_\varepsilon(v,t)\, dv, \tag{4.22}$$

where the last integral (the remainder) accounts for all the higher order ε-dependent terms in the expansion

$$R_\varepsilon(\phi(v),t) = \int_{\mathbb{R}_+}\left[\frac{1}{2}\varepsilon\lambda^2(w-v)^2\phi''(v) + \frac{1}{\varepsilon\, 3!}\langle \phi'''(\tilde{v})(v^* - v)^3 \rangle\right] f_\varepsilon(w,t)\, dw. \tag{4.23}$$

Therefore, for any given (smooth function) $\phi(v)$ and density $f(v)$, $v \in \mathbb{R}_+$, we have the identity

$$(\phi, Q_\varepsilon(f,f) - J(f)) = \int_{\mathbb{R}_+} R_\varepsilon(\phi(v),t)\, f(v,t)\, dv. \tag{4.24}$$

Remark 4. *In reason of the fact that the solution to the kinetic model (4.17) satisfies conditions (4.6) for all times $t \ge 0$, it follows from (4.22) that*

$$\int_{\mathbb{R}_+} R_\varepsilon(\phi(v))\, f_\varepsilon(v,t)\, dv = 0$$

whenever $\phi(v) = 1, v$.

For $m \in \mathbb{N}_+$, let $C^m(\mathbb{R}_+)$ be the set of m-times continuously differentiable functions, endowed with its natural norm $\|\cdot\|_m$. Then for $f = f(v)$, $v \in \mathbb{R}_+$, let us define

$$\|f\|_m^* = \sup\left\{|(\phi, f)|\,, \phi \in C^m(\mathbb{R}_+), \|\phi\|_m \le 1\right\}. \tag{4.25}$$

Let $m \ge 3$. Thanks to (4.21) , whenever $f(v)$ is a probability density with the third moment bounded

$$\|Q_\varepsilon(f,f) - J(f)\|_m^* = \sup_\phi \left|\int_{\mathbb{R}_+} R_\varepsilon(\phi(v))\, f(v)\, dv\right| \le \varepsilon^{1/2} C_\varepsilon(M_1(f), M_2(f), M_3(f)),$$

and

$$\lim_{\varepsilon\to 0} \|Q_\varepsilon(f,f) - J(f)\|_m^* = 0. \tag{4.26}$$

Let the initial datum of the Fokker–Planck equation possess moments bounded up to the order three. Thanks to (4.10) the moments up to the third order of the solution to the Fokker-Planck equation remain uniformly bounded in the time interval $0 \le t \le T$. Therefore, if $f(v,t)$ is a solution to the Fokker–Planck equation (4.5), for $0 \le t \le T$

$$\|Q_\varepsilon(f,f) - J(f)\|_m^*(t) = \sup_\phi \left| \int_{\mathbb{R}_+} R_\varepsilon(\phi(v))\, f_\varepsilon(v,t)\, dv \right| \le \varepsilon^{1/2} C_\varepsilon(T), \tag{4.27}$$

where the constant $C_\varepsilon(T)$ depends on moments of $f(v,t)$ up to the order three.

Let us consider a family of metrics that has been introduced in the paper [50] to study the trend to equilibrium of solutions to the space homogeneous Boltzmann equation for Maxwell molecules, and subsequently applied to a variety of problems related to kinetic models of Maxwell type. For a more detailed description, we address the interested reader to the lecture notes [24].

Given $s > 0$ and two probability densities f_1 and f_2, their Fourier based distance $d_s(f_1, f_2)$ is given by the quantity

$$d_s(f_1, f_2) := \sup_{\xi\in\mathbb{R}^n} \frac{\left|\widehat{f}_1(\xi) - \widehat{f}_2(\xi)\right|}{|\xi|^s}. \tag{4.28}$$

The distance is finite, provided that f_1 and f_2 have the same moments up to order $[s]$, where, if $s \notin \mathbb{N}_+$, $[s]$ denotes the entire part of s, or up to order $s-1$ if $s \in \mathbb{N}_+$. Moreover d_s is an ideal metric. Its main properties are the following

1. Let X_1, X_2, X_3, with X_3 independent of the pair X_1, X_2 be random variables with probability distributions f_1, f_2, f_3. Then

   $$d_s(f_1 * f_3, f_2 * f_3) \le d_s(f_1, f_2)$$

   where the symbol $*$ denotes convolution;

2. Define for a given nonnegative constant a the dilation

   $$f_a(x) = \frac{1}{a} f\left(\frac{v}{a}\right).$$

   Then, given two probability densities f_1 and f_2, for any nonnegative constant a

   $$d_s(aX_1, aX_2) = d_s(f_{1,a}, f_{2,a}) \le a^s\, d_s(f_1, f_2) = a^s\, d_s(X_1, X_2).$$

3. Let $d_s(f_1, f_2)$ be finite for some $s > 0$. Then the following interpolation property holds [24]

   $$d_p(f_1, f_2) \le 2\left(\frac{s-p}{2p}\right)^{p/s} \frac{s}{s-p} [d_s(f_1, f_2)]^{p/s} = C_{p,s}[d_s(f_1, f_2)]^{p/s}, \tag{4.29}$$

   for any $0 < p < s$.

Since the equation for the Fourier transform of the density $f_\varepsilon(v,t)$, solution of the kinetic equation (4.17) takes the form [74]

$$\frac{\partial}{\partial t}\widehat{f}_\varepsilon(\xi,t) = \widehat{Q}_\varepsilon(f_\varepsilon, f_\varepsilon)(\xi,t) = \frac{1}{\varepsilon}\left[\langle \widehat{f}_\varepsilon\big((1-\varepsilon\lambda+\eta_\varepsilon)\xi\big)\rangle \widehat{f}_\varepsilon(\varepsilon\lambda\xi) - \widehat{f}_\varepsilon(\xi,t)\right], \tag{4.30}$$

in reason of (4.24) we can write

$$\widehat{J}(\widehat{f}) = \widehat{Q}_\varepsilon(\widehat{f},\widehat{f}) + \widehat{R}_\varepsilon(\widehat{f}), \tag{4.31}$$

where

$$\widehat{R}_\varepsilon(\widehat{f}) = \widehat{J}(\widehat{f}) - \widehat{Q}_\varepsilon(\widehat{f},\widehat{f}).$$

As proven in [89], the metric d_2 is equivalent to $\|\cdot\|_m^*$, $m \in \mathbb{N}_+$, that is

$$d_2(f,g) \to 0 \quad \text{if and only if} \quad \|f-g\|_m^* \to 0.$$

Let the initial datum of the Fokker–Planck equation possess moments bounded up to the order three. Thanks to (4.27), if $f(v,t)$ is the corresponding solution to the Fokker–Planck equation, in the time interval $0 \le t \le T$, for a suitable constant r_ε

$$d_2\left(\widehat{J}(\widehat{f}(t)), \widehat{Q}_\varepsilon(\widehat{f}(t),\widehat{f}(t))\right) = \sup_\xi \frac{1}{|\xi|^2}|\widehat{R}_\varepsilon(\widehat{f})(t)| \le r_\varepsilon(T), \tag{4.32}$$

where

$$\lim_{\varepsilon\to 0} r_\varepsilon(T) = 0.$$

Using the expression (4.31) for the Fokker–Planck operator in (4.5), we obtain that the difference between the solution f_ε of (4.30) and the solution f to the Fokker–Planck equation (4.11) satisfies

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\widehat{f}_\varepsilon - \widehat{f}}{|\xi|^2}\right) + \frac{\widehat{f}_\varepsilon - \widehat{f}}{|\xi|^2} = \\ &\frac{1}{\varepsilon}\frac{\widehat{f}_\varepsilon\big((1-\lambda+\eta)\xi\big)\widehat{f}_\varepsilon(\lambda\xi) - \widehat{f}\big((1-\lambda+\eta)\xi\big)\widehat{f}(\lambda\xi)}{|\xi|^2} + \frac{1}{|\xi|^2}\widehat{R}_\varepsilon(\widehat{f}). \end{aligned} \tag{4.33}$$

Let $f_\varepsilon(v,t)$ and $f(v,t)$ solutions departing from initial value $\tilde{f}_0$ and f_0 satisfying conditions (4.6) and such that their distance $d_2(\tilde{f}_0, f_0)$ is finite. Let us set

$$h_\varepsilon = \frac{\widehat{f}_\varepsilon - \widehat{f}}{|\xi|^2},$$

which is such that $\|h_\varepsilon(\cdot,t)\|_\infty = d_2(f_\varepsilon, f)$. Since $|\widehat{f}_\varepsilon| = |\widehat{f}| = 1$ we obtain for any $0 \le t \le T$

$$\begin{aligned} \left|\frac{\partial}{\partial t}h_\varepsilon + \frac{1}{\varepsilon}h_\varepsilon\right| \le& \frac{1}{\varepsilon}\left|\frac{\widehat{f}_\varepsilon\big((1-\lambda\varepsilon+\eta_\varepsilon)\xi\big) - \widehat{f}\big((1-\lambda\varepsilon+\eta_\varepsilon)\xi\big)}{\langle|1-\lambda\varepsilon+\eta_\varepsilon\xi|^2\rangle}\right|\langle|1-\lambda\varepsilon+\eta_\varepsilon|^2\rangle \\ &+ \frac{1}{\varepsilon}\left|\frac{\widehat{f}_\varepsilon(\lambda\xi) - \widehat{f}(\lambda\xi)}{|\lambda\varepsilon\xi|^2}\right|(\lambda\varepsilon)^2 + \frac{1}{|\xi|^2}|\widehat{R}_\varepsilon(\widehat{f})| \\ \le& \frac{1}{\varepsilon}\|h_\varepsilon(\cdot,t)\|_\infty\left[\langle(1-\lambda\varepsilon+\eta_\varepsilon)^2\rangle + (\lambda\varepsilon)^2\right] + r_\varepsilon(T). \end{aligned} \tag{4.34}$$

Consider that, if $\sigma < 2\lambda$, for ε sufficiently small

$$\langle(1-\lambda\varepsilon+\eta_\varepsilon)^2\rangle + (\lambda\varepsilon)^2 = 1 + \varepsilon\left[\sigma - 2\lambda(1-\lambda\varepsilon)\right] \le 1. \tag{4.35}$$

If this is the case, $h_\varepsilon(t)$ satisfies

$$\left|\frac{\partial}{\partial t}h_\varepsilon + \frac{1}{\varepsilon}h_\varepsilon\right| \le \frac{1}{\varepsilon}\|h_\varepsilon(\cdot,t)\|_\infty + r_\varepsilon(T) \tag{4.36}$$

Proceeding as in [89], Theorem 5, we conclude by Gronwall inequality that (4.36) implies

$$\|h_\varepsilon(\cdot,t)\|_\infty \le \|h_\varepsilon(\cdot,0)\|_\infty + r_\varepsilon(T)\,t. \tag{4.37}$$

Letting ε going to 0 we obtain

$$\lim_{\varepsilon\to 0} d_2(f_\varepsilon, f)(t) \le d_2(\tilde{f}_0, f).$$

Hence, if we start with the same initial value $\tilde{f}_0 = f_0$, we conclude with $\lim_{\varepsilon\to 0} d_2(f_\varepsilon, f)(t) = 0$ for $0 \le t \le T$.

Analogous reasoning can be used to prove uniqueness of the solution to the Fokker–Planck equation (5.12). By resorting to the approximation (4.31) of the Fokker–Planck operator in (4.5), we obtain that the difference between two solutions $f(v,t)$ and $g(v,t)$ to the Fokker–Planck equation (5.13), for any given small value of ε satisfies

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\widehat{f}-\widehat{g}}{|\xi|^2}\right) + \frac{\widehat{f}-\widehat{g}}{|\xi|^2} = \\
&\frac{1}{\varepsilon}\frac{\widehat{f}\big((1-\lambda+\eta)\xi\big)\widehat{f}(\lambda\xi) - \widehat{g}\big((1-\lambda+\eta)\xi\big)\widehat{g}(\lambda\xi)}{|\xi|^2} - \frac{1}{|\xi|^2}\widehat{R}_\varepsilon(\widehat{f}) + \frac{1}{|\xi|^2}\widehat{R}_\varepsilon(\widehat{g}).
\end{aligned} \tag{4.38}$$

Let $f(v,t)$ and $g(v,t)$ solutions departing from initial value f_0 and g_0 satisfying conditions (4.6) and such that their distance $d_2(f_0, g_0)$ is finite. If the third moments of f_0 and g_0 are finite, proceeding as before we conclude with the bound

$$d_2(f,g)(t) \le d_2(f_0, g_0),$$

that clearly implies uniqueness of the solution. We can resume the previous result in the following.

Theorem 5. *Let $f_0(v)$ be a probability density in $\mathbb{R}_+$ satisfying conditions (4.6), and such that its third moment is finite. Assume moreover that the random part η in the binary collision (4.13) is such that $\langle|\eta|^3\rangle$ is finite. Then, for any finite time T, as $\varepsilon \to 0$, the unique solution $f_\varepsilon(v,t)$ to the kinetic model (4.17) with initial datum f_0 converges to the solution $f(v,t)$ of the Fokker–Planck equation (5.12) with the same initial datum f_0, and*

$$d_2(f_\varepsilon, f)(t) \to 0, \quad 0 < t \le T.$$

Moreover, the solution to the Fokker–Planck equation is unique.

4.2.4 A regularity result

Theorem 5 shows that, for any given datum with a suitable decay at infinity, the Fokker–Planck equation (5.12) possesses a unique solution. As usual, the homogeneous Sobolev space $\dot{H}_s$, for $s > 0$ is defined by the norm

$$\|f\|_{\dot{H}_s} = \int_{\mathbb{R}} |\xi|^{2s}\,|\widehat{f}|^2(\xi)\,d\xi.$$

However, if the initial datum belongs to $\dot{H}_p(\mathbb{R}_+)$, we can conclude that the solution maintains the same regularity for any subsequent positive time. Let $\widehat{f}(\xi,t) = a(\xi,t) + ib(\xi,t)$. Starting from the Fourier transform version of the Fokker-Planck equation (5.13), let us split it into the real and imaginary part. We obtain

$$\begin{aligned}
\frac{\partial}{\partial t}a(\xi,t) &= \frac{\sigma}{2}\xi^2\frac{\partial^2}{\partial\xi^2}a(\xi,t) - \lambda\xi\frac{\partial}{\partial\xi}a(\xi,t) + \lambda\xi b(\xi,t), \\
\frac{\partial}{\partial t}b(\xi,t) &= \frac{\sigma}{2}\xi^2\frac{\partial^2}{\partial\xi^2}b(\xi,t) - \lambda\xi\frac{\partial}{\partial\xi}b(\xi,t) - \lambda\xi a(\xi,t).
\end{aligned} \tag{4.39}$$

Multiplying equations (4.39) respectively by $2a$ and $2b$ and summing up, we get

$$\frac{\partial}{\partial t}|\widehat{f}|^2 = \sigma\xi^2\left[a\frac{\partial^2}{\partial\xi^2}a + b\frac{\partial^2}{\partial\xi^2}b\right] - \lambda\xi\frac{\partial|\widehat{f}|^2}{\partial\xi}. \tag{4.40}$$

Hence, multiplying by $|\xi|^p$ and integrating over $\mathbb{R}$ with respect to ξ, we obtain the evolution equation of the $\dot{H}_{p/2}-$norm of $f(v,t)$. We have

$$\frac{\partial}{\partial t}\int_{\mathbb{R}}|\xi|^p\,|\widehat{f}|^2\,d\xi = \sigma\int_{\mathbb{R}}|\xi|^{2+p}\left[a\frac{\partial^2}{\partial\xi^2}a + b\frac{\partial^2}{\partial\xi^2}b\right]d\xi - \lambda\int_{\mathbb{R}}\xi|\xi|^p\frac{\partial|\widehat{f}|^2}{\partial\xi}d\xi. \tag{4.41}$$

Integrating by parts the two integrals, it results

$$\frac{\partial}{\partial t}\int_{\mathbb{R}}|\xi|^p\,|\widehat{f}|^2\,d\xi = (p+1)\left[\frac{\sigma}{2}(p+2)+\lambda\right]\int_{\mathbb{R}}|\xi|^p|\widehat{f}|^2d\xi - \sigma\int_{\mathbb{R}}|\xi|^{2+p}\left[\left|\frac{\partial}{\partial\xi}a\right|^2 + \left|\frac{\partial}{\partial\xi}b\right|^2\right]d\xi. \tag{4.42}$$

The last integral in (4.42) can be estimated from below as follows

$$\int_{\mathbb{R}}|\xi|^{2+p}\left[\left|\frac{\partial}{\partial\xi}a\right|^2 + \left|\frac{\partial}{\partial\xi}b\right|^2\right]d\xi \geq \frac{(p+1)^2}{2}\int_{\mathbb{R}}|\xi|^p|\widehat{f}|^2.$$

Indeed, for all $\mu > 0$ it holds

$$0 \leq \int_{\mathbb{R}}|\xi|^p\left(\xi\,\frac{\partial a}{\partial\xi} + \mu a\right)^2 d\xi = \int_{\mathbb{R}}|\xi|^{2+p}\left|\frac{\partial a}{\partial\xi}\right|^2 + \left(\mu^2 - \mu(p+1)\right)\int_{\mathbb{R}}|\xi|^{2+p}a^2\,d\xi.$$

Optimizing over μ we get

$$\int_{\mathbb{R}}|\xi|^{2+p}\left|\frac{\partial a}{\partial\xi}\right|^2 d\xi \geq \frac{(p+1)^2}{4}\int_{\mathbb{R}}|\xi|^{2+p}a^2\,d\xi.$$

An analogous computation works for b. Hence

$$\frac{\partial}{\partial t}\int_{\mathbb{R}}|\widehat{f}|^2|\xi|^p d\xi \leq \frac{p+1}{2}\left[\sigma + 2\lambda\right]\int_{\mathbb{R}}|\xi|^p|\widehat{f}|^2d\xi. \tag{4.43}$$

The inequality (4.43) implies that if the initial data has finite $\dot{H}_p-$norm, then for all finite $t > 0$, the $\dot{H}_p-$norm of the solution remains finite and it grows up to $+\infty$ for $t \to +\infty$.

4.3 Convergence to equilibrium

4.3.1 L_1-convergence

In this section, we will be concerned with the study of the large-time behaviour of the Fokker–Planck equation (4.1). The main argument here will be the study of the time evolution of various Lyapunov functionals, starting from Shannon entropy of the solution $f(v,t)$ relative to the steady state $f_\infty(v)$. We recall that the relative Shannon entropy of the two probability density functions f and g is defined by the formula

$$H(f,g) = \int_I f(v)\log\frac{f(v)}{g(v)}\,dv. \tag{4.44}$$

As a first step in this analysis, we will introduce in the following equivalent formulations of the Fokker–Planck equation, that result to be very useful to justify rigorously the behaviour of these Lyapunov functionals.

Indeed, equation (4.1) admits many equivalent formulations, each of them well adapted to different purposes [49]. To this extent, recall that the equilibrium distribution f_∞ defined in (4.2) satisfies

$$\frac{\partial}{\partial v}(v^2 f_\infty) + (v-1) f_\infty = 0, \tag{4.45}$$

or, equivalently

$$\frac{\partial}{\partial v}\log(v^2 f_\infty) = -\frac{v-1}{v^2}. \tag{4.46}$$

Then, for $v > 0$

$$\frac{\partial}{\partial v}\left(v^2 f\right) + (v-1)\, f = v^2 f\left(\frac{\partial}{\partial v}\log(v^2 f) + \frac{v-1}{v^2}\right) =$$

$$v^2 f\left(\frac{\partial}{\partial v}\log(v^2 f) - \frac{\partial}{\partial v}\log(v^2 f_\infty)\right) = v^2 f \frac{\partial}{\partial v}\log\frac{f}{f_\infty} = v^2 f_\infty \frac{\partial}{\partial v}\frac{f}{f_\infty}.$$

Hence, we can write the Fokker–Planck equation (4.1) in the equivalent form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left[v^2 f \frac{\partial}{\partial v}\log\frac{f}{f_\infty}\right], \tag{4.47}$$

which enlightens the role of the logarithm of the quotient f/f_∞, and

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left[v^2 f_\infty \frac{\partial}{\partial v}\frac{f}{f_\infty}\right]. \tag{4.48}$$

In particular, owing to (4.45), the form (4.48) allows us to obtain the evolution equation for the quotient $F = f/f_\infty$. Indeed

$$\frac{\partial f}{\partial t} = f_\infty \frac{\partial F}{\partial t} = v^2 f_\infty \frac{\partial^2}{\partial v^2}\frac{f}{f_\infty} + \frac{\partial}{\partial v}(v^2 f_\infty)\frac{\partial}{\partial v}\frac{f}{f_\infty} =$$

$$= v^2 f_\infty \frac{\partial^2 F}{\partial v^2} - (v-1) f_\infty \frac{\partial F}{\partial v},$$

which shows that F satisfies the equation

$$\frac{\partial F}{\partial t} = v^2 \frac{\partial^2 F}{\partial v^2} - (v-1)\frac{\partial F}{\partial v}. \tag{4.49}$$

If mass conservation is imposed on equation (4.1), we obtain at $v = 0$ the boundary conditions (4.3) and (4.4). In analogous way, the boundary conditions of the equivalent form (4.49) are now written in the form

$$v^2 f_\infty(v) F(v,t)\big|_{v=0} = 0, \tag{4.50}$$

and

$$v^2 f_\infty(v)\frac{\partial}{\partial v}\frac{f(v,t)}{f_\infty(v)}\bigg|_{v=0} = v^2 f_\infty(v)\frac{\partial F(v,t)}{\partial v}\bigg|_{v=0} = 0. \tag{4.51}$$

In view of the decay property at $v = 0$ of the steady state f_∞, the boundary conditions (4.50) and (4.51) are satisfied any time the solution to equation (4.49) is bounded together with its derivative at zero.

To proceed, and to avoid inessential difficulties, for any given initial density $f_0(v)$ and positive constant $\delta \ll 1$, let us consider a regular approximation $f_0^\delta(v)$ satisfying the conditions

$$f_0^\delta(v) = f_\infty(v) \quad \text{if } v \le \delta \text{ and } v \ge 1/\delta, \quad \delta^2 \le f_0^\delta(v) \le 1/\delta^2 \quad \text{if } \delta \le v \le 1/\delta, \tag{4.52}$$

while

$$\int_{\mathbb{R}_+} f_0^\delta(v)\, dv = 1. \tag{4.53}$$

Then, in a time interval $(0, T)$ the (unique) solution $F^\delta(v,t)$ of the initial-boundary value problem for equation (4.49) corresponding to the initial value $F_0^\delta = f_0^\delta / f_\infty$ is such that

$$F^\delta(v,t) = 1 \quad \text{if } v \le \delta \text{ and } v \ge 1/\delta, \quad \delta^2/\Delta_+ \le F^\delta(v,t) \le \delta^2/\Delta_- \text{if } \delta \le v \le 1/\delta, \tag{4.54}$$

where we denoted

$$\Delta_+ = \max_v f_\infty(v), \quad \Delta_- = \min_{\delta \le v \le 1/\delta} f_\infty(v) > 0.$$

Indeed, a solution constant in the interval $v \le \delta$ satisfies both the boundary conditions (4.50) and (4.51), and equation (4.49), and converges to the right initial value as $t \to 0$. Analogous conclusion can be drawn in the interval $v \ge 1/\delta$. Consider now the solution to equation (4.49) in a bounded interval (v_-, v_+), where $v_- \le \delta$ and $v_+ \ge 1/\delta$, with boundary conditions $F(v_-, t) = 1$ and $F(v_+, t) = 1$ for $t \le T$, and initial value $F_0^\delta = f_0^\delta / f_\infty$. Since in this interval the coefficient of the second-order term is strictly positive, the second condition in (4.54) follows from the maximum principle for the solution to a uniformly parabolic equation.

The previous discussion shows that, since the initial datum F_0^δ satisfies

$$m_\delta \le F_0^\delta(v) \le M_\delta \tag{4.55}$$

for some positive constants $m_\delta < M_\delta$, the same condition holds at any subsequent time $t \le T$, so that

$$m_\delta \le F^\delta(v,t) \le M_\delta. \tag{4.56}$$

As remarked in [49], condition (4.56) allows to recover rigorously the time decay of various Lyapunov functionals. Indeed, the following holds (cf. Theorem 3.1 of [49])

Theorem 6. *Let the smooth function $\Phi(x)$, $x \in \mathbb{R}_+$ be convex. Then, if $F(v,t)$ is the solution to equation (4.49) in $\mathbb{R}_+$, and $c \le F(v,t) \le C$ for some positive constants $c < C$, the functional*

$$\Theta(F(t)) = \int_I f_\infty(v) \Phi(F(v,t))\, dv$$

is monotonically decreasing in time, and the following equality holds

$$\frac{d}{d\tau} \Theta(F(t)) = -I_\Theta(F(t)), \tag{4.57}$$

where I_Θ denotes the nonnegative quantity

$$I_\Theta(F(t)) = \int_{\mathbb{R}_+} v^2 f_\infty(v) \Phi''(F(v,t)) \left| \frac{\partial F(v,t)}{\partial v} \right|^2 dv. \tag{4.58}$$

We can couple Theorem 6 with the so-called Chernoff inequality with weight, recently proven in [49] (cf. Theorem 3.3). In our case, this result reads

Theorem 7. *Let X be a random variable distributed with density $f_\infty(v)$, $v \in \mathbb{R}_+$, where the probability density function f_∞ satisfies the differential equality*

$$\frac{\partial}{\partial v}\left(v^2 f_\infty\right) + (v-1)\, f_\infty = 0, \quad v \in \mathbb{R}_+. \tag{4.59}$$

If the function ϕ is absolutely continuous on $\mathbb{R}_+$ and $\phi(X)$ has finite variance, then

$$Var[\phi(X)] \leq E\left\{X^2[\phi'(X)]^2\right\} \tag{4.60}$$

with equality if and only if $\phi(X)$ is linear in X.

Choose now $\Phi(x) = x \log x$, $x \geq 0$. Then, $\Theta(F^\delta(t))$ coincides with the entropy of f^δ relative to f_∞. If the relative entropy is finite at time $t = 0$, by Theorem 6 it decays, and its rate of decay is given by the expression

$$I(F^\delta(t)) = \int_{\mathbb{R}_+} v^2 f_\infty(v) \frac{1}{F^\delta(v,t)} \left|\frac{\partial F^\delta(v,t)}{\partial v}\right|^2 dv = 4 \int_{\mathbb{R}_+} v^2 f_\infty(v) \left|\frac{\partial \sqrt{F^\delta(v,t)}}{\partial v}\right|^2 dv. \tag{4.61}$$

If we apply inequality (4.60) with $\phi(v) = \sqrt{F^\delta(v,t)}$ with fixed $t > 0$ we get

$$\begin{aligned} I(F^\delta(t)) &= 4 \int_{\mathbb{R}_+} v^2 f_\infty(v) \left(\partial_v \sqrt{\frac{f^\delta(v,t)}{f_\infty(v)}}\right)^2 dv \geq \\ 4 &\left(\int_{\mathbb{R}_+} \frac{f^\delta(v,t)}{f_\infty(v)} f_\infty(v)\, dv - \left(\int_{\mathbb{R}_+} \sqrt{\frac{f(v,t)}{f_\infty(v)}} f_\infty(v)\, dv\right)^2\right) = \\ 4 &\left(1 - \left(\int_{\mathbb{R}_+} \sqrt{f^\delta(v,t)\, f_\infty(v)}\, dv\right)^2\right). \end{aligned} \tag{4.62}$$

On the other hand, as remarked in [59], whenever f and g are probability density functions, the square of their Hellinger distance

$$d_H(f,g) = \left[\int_{\mathbb{R}_+} \left(\sqrt{f} - \sqrt{g}\right)^2 dv\right]^{1/2} \tag{4.63}$$

satisfies

$$\begin{aligned} &= d_H(f,g)^2 = \int_{\mathbb{R}} \left(f(v) + g(v) - 2\sqrt{f(v)\, g(v)}\right) dv = \\ 2 &\left(1 - \int_{\mathbb{R}} \sqrt{f(v)\, g(v)}\, dv\right) \leq 2 \left(1 - \left(\int_{\mathbb{R}} \sqrt{f(v)\, g(v)}\, dv\right)^2\right). \end{aligned} \tag{4.64}$$

The last inequality in (4.64) follows by Cauchy–Schwartz inequality. Finally, for $t > 0$

$$I(F^\delta(t)) \geq 2 d_H(f^\delta(t), f_\infty)^2, \quad t > 0. \tag{4.65}$$

that implies the differential inequality

$$\frac{d}{dt} H(f^\delta(t)|f_\infty) \leq -2 d_H(f^\delta(t), f_\infty)^2, \tag{4.66}$$

and, consequently, the bound

$$\int_0^\infty d_H(f^\delta(t), f_\infty)^2\, dt \le \frac{1}{2} H(f_0^\delta | f_\infty). \tag{4.67}$$

Now, let us apply again Theorem 6 to the convex function $\phi(x) = (\sqrt{x} - 1)^2$. In this case $\Theta(F^\delta(t))$ coincides with the square of the Hellinger distance (4.63) between f^δ and f_∞, which in consequence of (4.57) is shown to decay in time.

Therefore, inequality (4.67) coupled with the time decay of the Hellinger distance shows that for large times

$$d_H(f^\delta(t), f_\infty)^2 = o(1/t). \tag{4.68}$$

Note that to obtain the decay we need the boundedness of the relative entropy $H(f_0^\delta | f_\infty)$. Last, by Cauchy–Schwartz inequality we can bound the L^1 distance between two densities f and g in terms of the Hellinger distance $d_H(f, g)$. Indeed

$$\begin{aligned}
\int_I |f(v) - g(v)|\, dv &= \int_I \left|\sqrt{f(v)} - \sqrt{g(v)}\right| \left(\sqrt{f(v)} + \sqrt{g(v)}\right)\, dv \\
&\le \left(\int_I \left(\sqrt{f(v)} - \sqrt{g(v)}\right)^2\, dv\right)^{\frac{1}{2}} \left(\int_I \left(\sqrt{f(v)} + \sqrt{g(v)}\right)^2\, dv\right)^{\frac{1}{2}} \\
&= d_H(f, g) \left(\int_I \left(f(v) + g(v) + 2\sqrt{f(v)g(v)}\right)\, dv\right)^{\frac{1}{2}} \\
&= \sqrt{2} d_H(f, g) \left(1 + \int_I \sqrt{f(v)g(v)}\, dv\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore

$$\|f - g\|_{L^1} \le 2 d_H(f, g). \tag{4.69}$$

Finally, in view of (4.68), the L_1-distance between $f^\delta(t)$ and f_∞ decays to zero as time goes to infinity,

$$\|f^\delta(t) - f_\infty\|_{L^1}^2 \le o(1/t), \tag{4.70}$$

and

$$\int_0^\infty \|f^\delta(t) - f_\infty)\|_{L_1}^2\, dt \le 2H(f_0^\delta | f_\infty). \tag{4.71}$$

Let us now proceed to remove the lifting of the initial value. The following lemma will be useful

Lemma 8. *Let $f(v, t)$ be a solution of the initial-boundary value problem for the Fokker–Planck equation (4.1), corresponding to an initial value $f_0(v)$ such that, as in Theorem 5 $|f_0(v)| \in L_1(\mathbb{R}_+)$ and $v^3 |f_0(v)| \in L_1(\mathbb{R}_+)$. Then, the L_1-norm of $f(v, t)$ is non-increasing for $t \ge 0$.*

Proof. For a given $\gamma > 0$, let us consider a regularized increasing approximation of the sign function $sign_\gamma(z)$, with $z \in \mathbb{R}$, and let us define the regularized approximation $|f|_\gamma(z)$ of $|f|(z)$ via the primitive of $sign_\gamma(f)(z)$. We now multiply equation (4.1) by $sign_\gamma(f(t))$ to obtain, after integrating by parts

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}_+} sign_\gamma(f(t)) f(t)\, dv &= -\int_{\mathbb{R}_+} sign'_\gamma(f) \left[\frac{\sigma}{2} \frac{\partial f}{\partial v} \frac{\partial (v^2 f)}{\partial v} + \lambda(v - 1) f \frac{\partial f}{\partial v}\right] dv \\
= -\int_{\mathbb{R}_+} sign'_\gamma(f) \frac{\sigma}{2} v^2 \left|\frac{\partial f}{\partial v}\right|^2 dv &- \int_{\mathbb{R}_+} \frac{(\lambda + \sigma)v - \lambda}{2} sign'_\gamma(f) f \frac{\partial f}{\partial v}\, dv.
\end{aligned} \tag{4.72}$$

Indeed, the border term contribution vanishes in view of condition (4.4). Moreover, since we have the equality

$$sign'_\gamma(f) f \frac{\partial f}{\partial v} = \frac{\partial}{\partial v}\left[f\, sign_\gamma(f) - |f|_\gamma\right]$$

after another integration by parts in the last term of the right-hand side of (4.72) we obtain

$$-\int_{\mathbb{R}_+} \frac{(\lambda+\sigma)v-\lambda}{2}\, sign'_\gamma(f) f \frac{\partial f}{\partial v}\, dv =$$
$$\frac{\lambda}{2}\left[\left(f\, sign_\gamma(f) - |f|_\gamma\right)(v=0) + \int_{\mathbb{R}_+} \left(f\, sign_\gamma(f) - |f|_\gamma\right) dv\right],$$

and this contribution, in the limit $\gamma \to 0$ vanishes. Hence

$$\frac{d}{dt}\int_{\mathbb{R}_+} |f(v,t)|\, dv \le 0.$$

□

In reason of lemma 8, for any given initial datum $f_0(v)$ satisfying the hypotheses of theorem 5, and its modification (4.52), we have that, at any subsequent time $t > 0$

$$\|f(v,t) - f^\delta(v,t)\|_{L_1} \le \|f_0(v) - f_0^\delta(v)\|_{L_1}.$$

Hence, since $f_0^\delta(v)$ converges to $f_0(v)$ in L_1-norm and in relative entropy, letting $\delta \to 0$ inequality (4.70) implies

$$\|f(t) - f_\infty\|_{L^1}^2 \le o(1/t). \tag{4.73}$$

Moreover, for each finite time T, inequality (4.71) yields

$$\int_0^T \|f(t) - f_\infty)\|_{L_1}^2\, dt \le 2\int_0^T \|f^\delta(t) - f_\infty)\|_{L_1}^2\, dt + 2\int_0^T \|f^\delta(t) - f(t))\|_{L_1}^2\, dt \le$$
$$2\int_0^T \|f^\delta(t) - f_\infty)\|_{L_1}^2\, dt + 2T\|f_0^\delta - f_0\|_{L_1}^2 \le 4H(f_0^\delta|f_\infty) + 2T\|f_0^\delta - f_0\|_{L_1}^2.$$

Hence, letting $\delta \to 0$ we obtain, for each $T > 0$ the inequality

$$\int_0^T \|f(t) - f_\infty)\|_{L_1}^2\, dt \le 4H(f_0|f_\infty). \tag{4.74}$$

We proved

Theorem 9. *Let $f(v,t)$ be a solution of the initial-boundary value problem for the Fokker–Planck equation (4.1), corresponding to an initial density $f_0(v)$ such that, as in Theorem 5 $f_0(v) \in L_1(\mathbb{R}_+)$ and $v^3 f_0(v) \in L_1(\mathbb{R}_+)$. Then, if the relative entropy between f_0 and f_∞ is bounded, $f(v,t)$ converges in $L_1(\mathbb{R}_+)$ towards the steady state f_∞, and both (4.73) and (4.74) hold.*

4.3.2 Further convergence results

The analysis of the previous section shows that the solution to the Fokker–Planck equation converges towards the stationary state in the L_1-norm for a large class of initial data, but with a polynomial rate of decay. However, stronger results of convergence can be obtained by suitably restricting the allowed initial data, or, in alternative, by relaxing the distance in which the decay holds.

Let us apply Theorem 6 to the convex function $\phi(x) = (x-1)^2$. In this case $\Theta(F^\delta(t))$ coincides with the weighted (with weight f_∞) L_2-norm between f^δ and f_∞

$$\Theta(F^\delta(t)) = \int_{\mathbb{R}_+} |f^\delta(v,t) - f_\infty(v)|^2 f_\infty^{-1}(v)\, dv.$$

Then

$$I_\Theta(F^\delta(t)) = 2\int_{\mathbb{R}_+} v^2 f_\infty(v) \left|\frac{\partial F^\delta(v,t)}{\partial v}\right|^2 dv,$$

and application of Chernoff inequality with weight (4.60) gives

$$I_\Theta(F^\delta(t)) \geq 2\Theta(F^\delta(t)).$$

Hence, exponential decay follows, and

$$\int_{\mathbb{R}_+} |f^\delta(v,t) - f_\infty(v)|^2 f_\infty^{-1}(v)\, dv \leq e^{-2t} \int_{\mathbb{R}_+} |f_0^\delta(v) - f_\infty(v)|^2 f_\infty^{-1}(v)\, dv. \tag{4.75}$$

Proceeding as before, and removing the lifting on the initial data we obtain the following

Theorem 10. *Let $f(v,t)$ be a solution of the initial-boundary value problem for the Fokker–Planck equation (4.1), corresponding to an initial density $f_0(v)$ such that, as in Theorem 5 $f_0(v) \in L_1(\mathbb{R}_+)$ and $v^3 f_0(v) \in L_1(\mathbb{R}_+)$. Then, if the the weighted L_2-norm*

$$\|f_0 - f_\infty\|_2^* = \int_{\mathbb{R}_+} |f_0(v) - f_\infty(v)|^2 f_\infty^{-1}(v)\, dv$$

is bounded, $f(v,t)$ converges in $L_2^*(\mathbb{R}_+)$ towards the steady state f_∞, and

$$\|f(t) - f_\infty\|_2^* \leq e^{-2t}\|f_0 - f_\infty\|_2^*. \tag{4.76}$$

Last, we analyze the rate of decay towards equilibrium in the weak d_2-distance defined in (4.28). Proceeding as in Section 4.2.3, it is immediate to compute the rate of convergence of two different solution of Boltzmann equation (4.15). Let $f_\varepsilon(t)$ and $g_\varepsilon(t)$ denote two solutions of the kinetic model (4.15), departing from initial data $\tilde{f}_0$ and $\tilde{g}_0$ respectively. Let us suppose in addition that the distance in the Fourier metric $d_3(f_\varepsilon, g_\varepsilon)$ is initially bounded, and let us define

$$h_\varepsilon(\xi,t) = \frac{\widehat{f}_\varepsilon(\xi,t) - \widehat{g}_\varepsilon(\xi,t)}{|\xi|^3}.$$

Then, proceeding as in section 4.2.3, we obtain

$$\left|\frac{\partial}{\partial t} h_\varepsilon + \frac{1}{\varepsilon} h_\varepsilon\right| \leq \frac{1}{\varepsilon}\|h_\varepsilon\|_\infty \left[\langle |1 - \lambda\varepsilon + \tilde{\eta}_\varepsilon|^3\rangle + |\lambda\varepsilon|^3\right].$$

Since by construction the quantity $1-\lambda\varepsilon+\tilde{\eta}_\varepsilon$ is nonnegative, and $\langle\tilde{\eta}_\varepsilon^3\rangle=0$

$$\langle|1-\lambda\varepsilon+\tilde{\eta}_\varepsilon|^3\rangle+|\lambda\varepsilon|^3=1-3\varepsilon(\lambda-\sigma)(1-\lambda\varepsilon).$$

Therefore Gronwall inequality yields

$$d_3(f_\varepsilon(t),g_\varepsilon(t))\leq d_3(\tilde{f}_0,\tilde{g}_0)e^{-3(\lambda-\sigma)(1-\lambda\varepsilon)t}. \tag{4.77}$$

Clearly, if $\lambda>\sigma$, and $\varepsilon\ll 1$ so that $\lambda\varepsilon\leq\delta<1$, the distance between the solutions $d_3(f_\varepsilon,g_\varepsilon)$, for each $\varepsilon>0$ decays exponentially with a rate bigger than $3(\lambda-\sigma)(1-\delta)$. The decay of two different solutions to the Kinetic Boltzmann-type equation (4.15) allows to prove a similar result for the Fokker–Planck equation. To this aim, we recall a result on the Fourier distance proven in [23], adapted to the present situation

Lemma 11. *Let $\{f_n(v)\}_{n\geq 0}$ and $\{g_n(v)\}_{n\geq 0}$, $v\in\mathbb{R}_+$, be two sequences of probability density functions with moments bounded up to the second order, such that $f_n\rightharpoonup f$ and $g_n\rightharpoonup g$. Suppose in addition that, for some $r>2$*

$$\int_{\mathbb{R}_+}|v|^r f(v)<+\infty,\quad \int_{\mathbb{R}_+}|v|^r g(v)<+\infty.$$

If

$$d_r(f_n,g_n)<+\infty,$$

then for all $s<r$,

$$d_s(f,g)\leq\liminf d_s(f_n,g_n).$$

Thanks to the interpolation formula (4.29) with $s=3$ and $p=2$, we obtain from (4.77) the bound

$$d_2(f_\varepsilon(t),g_\varepsilon(t))\leq\left[d_3(\tilde{f}_0,\tilde{g}_0)\right]^{2/3}e^{-2(1-\delta)(\lambda-\sigma)t}. \tag{4.78}$$

Thus, by Lemma 11, letting $\varepsilon\to 0$, we conclude that, if $f(t)$ and $g(t)$ are solutions of the Fokker-Planck equation (4.1), corresponding to initial values $\tilde{f}_0$ and $\tilde{g}_0$ such that $d_3(\tilde{f}_0,\tilde{g}_0)$ is finite, the distance $d_2(f(t),g(t))$ decays to zero with the explicit exponential rate $2(1-\delta)(\lambda-\sigma)$. We can resume the previous result in the following.

Theorem 12. *Let $f(v,t)$ be a solution of the initial-boundary value problem for the Fokker–Planck equation (4.1), corresponding to an initial density $f_0(v)$ such that, as in Theorem 5 $f_0(v)\in L_1(\mathbb{R}_+)$ and $v^3f_0(v)\in L_1(\mathbb{R}_+)$. Then, provided $d_3(f_0,f_\infty)$ is finite, the solution $f(v,t)$ converges towards the equilibrium density in the Fourier distance d_2, and for each constant δ such that $0<\delta<1$, the following bound holds*

$$d_2(f(t),f_\infty)\leq\left[d_3(f_0,f_\infty)\right]^{2/3}e^{-2(1-\delta)(\lambda-\sigma)t}.$$

4.4 Conclusions

In this paper, we studied existence, uniqueness and asymptotic behavior of a Fokker–Planck equation for wealth distribution first derived in [17]. In particular, we investigated the connections between a bilinear kinetic model for wealth distribution introduced in [32] and the Fokker–Planck equation (4.1). In various cases, these connections allow to pass results which have been found for the kinetic model to the Fokker–Planck equation.

Some problems, however, remain open. In particular, the invariant trade limit which allows to obtain the Fokker–Planck equation has been proven to hold only when the initial values in the kinetic equation

possess moments bounded up to the order three. This condition reflects also on the limit Fokker–Planck equation, where, for example, the exponential decay towards equilibrium obtained in Theorem 12 requires the boundedness of the d_3-distance between the initial datum and the corresponding equilibrium density. Thus, convergence results towards equilibrium in absence of a sufficiently high number of moments initially bounded is unknown.

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Chapter 5

Wealth distribution with debts

5.1 Introduction

Mathematical modeling of wealth distribution has seen in recent years a remarkable development, mainly linked to the understanding of the mechanisms responsible of the formation of Pareto tails [78] (cf. Chapter 5 of [76] for a recent survey). Among the various kinetic and mean field models considered so far [26, 27, 29, 38, 39], the Fokker–Planck type description of the evolution of the personal wealth revealed to be successful. In [17] Bouchaud and Mezard introduced a simple model of economy, where the time evolution of wealth is described by an equation capturing both exchange between individuals and random speculative trading, in such a way that the fundamental symmetry of the economy under an arbitrary change of monetary units is insured. A Fokker–Planck type model was then derived through a mean field limit procedure, with a solution becoming in time a Pareto (power-law) type distribution. Let $f(v,t)$ denote the probability density at time $t \geq 0$ of agents with personal wealth $v \geq 0$, departing from an initial density $f_0(v)$ with a mean value fixed equal to one

$$m(f_0) = \int_{\mathbb{R}^+} v f_0(v)\, dv = 1. \tag{5.1}$$

The evolution in time of the density $f(v,t)$ was described in [17] by the Fokker–Planck equation

$$\frac{\partial f}{\partial t} = J(h) = \frac{\sigma}{2}\frac{\partial^2}{\partial v^2}\left(v^2 f\right) + \lambda \frac{\partial}{\partial v}\left((v-1)f\right), \tag{5.2}$$

where λ and σ denote two positive constants related to essential properties of the trade rules of the agents.

The key features of equation (5.2) is that, in presence of suitable boundary conditions at the point $v = 0$, the solution is mass and momentum preserving, and approaches in time a unique stationary solution of unit mass [85]. This stationary state is given by the (inverse) Γ-like distribution [17]

$$f_\infty(v) = \frac{(\mu-1)^\mu}{\Gamma(\mu)} \frac{\exp\left(-\frac{\mu-1}{v}\right)}{v^{1+\mu}}, \tag{5.3}$$

where the positive constant $\mu > 1$ is given by

$$\mu = 1 + 2\frac{\lambda}{\sigma}.$$

As predicted by the observations of the Italian economist Vilfredo Pareto [78], (5.3) exhibits a power-law tail for large values of the wealth variable.

The explicit form of the equilibrium density, which represents one of the main aspects linked to the validity of the model in its economic setting, is indeed very difficult to achieve at the Boltzmann kinetic level, where only few relatively simple models can be treated analytically [9, 10, 60].

In addition to [17], the Fokker–Planck equation (5.2) appears as limit of different kinetic models. It was obtained by one of the present authors with Cordier and Pareschi [32] via an asymptotic procedure applied to a Boltzmann-type kinetic model for binary trading in presence of risks. Also, the same equation with a modified drift term appears when considering suitable asymptotics of Boltzmann-type equations for binary trading in presence of taxation [13], in the case in which taxation is described by the redistribution operator introduced in [14]. Systems of Fokker–Planck equations of type (5.2) have been considered in [42] to model wealth distribution in different countries which are coupled by mixed trading. Further, the operator $J(f)$ in equation (5.2) and its equilibrium kernel density have been considered in a nonhomogeneous setting to obtain Euler-type equations describing the joint evolution of wealth and propensity to trading [41], and to study the evolution of wealth in a society with agents using personal knowledge to trade [77].

These results contributed to retain that this limit model represents a quite satisfactory description of the time-evolution of wealth density towards a Pareto-type equilibrium in a trading society.

Existence, uniqueness and asymptotic behavior of the solution to equation (5.2) have been recently addressed in [85]. In this paper, by resorting in part to the strategy outlined in [49], a precise relationship between the solution of the kinetic model considered in [32] and the solution to the Fokker–Planck equation (5.2) was obtained, together with an exhaustive study of the large-time behavior of the latter. Various properties of the solution to equation (5.2) can in fact be extracted from the limiting relationship between the Fokker–Planck description and its kinetic level, given by the bilinear Boltzmann-type equation introduced in [32]. It is essential to remark that, in reason of the fact that the domain of the wealth variable v takes values in $\mathbb{R}_+$, and that the coefficient of diffusion depends on the wealth variable, the analysis of the large-time behavior of the solution to equation (5.2) appears very different from the analogous one studied in [4, 87] for the classical Fokker–Planck equation. In particular, the essential argument in [85] was to resort to an inequality of Chernoff type [30, 62], recently revisited in [49], that allows to prove convergence to equilibrium in various settings.

All the previous results describe a society in which all agents have initially a non negative wealth, and do not consider the unpleasant but realistic possibility that part of the agents would have debts, clearly expressed by a negative wealth. Recent results on one-dimensional kinetic models [8, 7] showed however that there are no mathematical obstacles in considering the Boltzmann-type equation introduced in [32] with initial values supported on the whole real line.

Following the idea of [8, 7], we will study in this paper the initial value problem for the Fokker–Planck equation (5.2) posed on the whole real line $\mathbb{R}$, by assuming that the initial datum satisfies condition (5.1), that is by assuming that part of the agents of the society could initially have debts, while the initial (conserved) mean wealth is positive. As we shall see, also in this situation, the positivity of the mean wealth will be enough to drive the solution towards the (unique) equilibrium density, still given by (5.3). Also, the forthcoming analysis will clearly indicate that the initial-boundary value problem considered in [85], in which the initial density is supported on the positive half-line, is simply a particular case of the general situation studied here. However, while the analysis of [85] allows to conclude that the solution to the initial-boundary value problem for (5.2) converges strongly towards the equilibrium density (5.3) with an explicit rate, in the general situation discussed in this paper, we are able to show that exponential in time convergence to equilibrium takes place only in a weak setting, well described by resorting to Fourier based metrics.

As discussed in Section 5.3.3, the usual approach to convergence to equilibrium via entropy arguments

fails in reason of the fact that in this situation the initial density and consequently the solution at each time $t > 0$ is supported on the whole real line $\mathbb{R}$, while the equilibrium density is supported only on the positive half-line $\mathbb{R}_+$. This problem can be bypassed by resorting to entropy functionals different from the standard relative Shannon entropy. However, a detailed evaluation of the entropy production of the new entropy functional allows to conclude only with a result convergence in the classical L_1 setting, without rate.

5.2 Main results

5.2.1 Existence and uniqueness

Existence of a (unique) solution for the initial value problem for the Fokker–Planck equation can be recovered by means of the analysis done in [85], which is based on the strong connection between equation (5.2) and the kinetic equation of Boltzmann type introduced in [32]. Indeed, the existence proof in [85] is based on the Fourier transformed version of the kinetic equation, and applies without any change even if the wealth variable takes values on the whole real line. However, while Fokker–Planck equations with variable coefficients and in presence of boundary conditions have been rarely studied [46] (cf. also the book [47] for a general view about boundary conditions for diffusion equations), in absence of boundaries, other results are available, which apply directly to the Fokker–Planck equation (5.2).

Particular cases of Fokker–Planck type equations with variable coefficient of diffusion, mainly related to the linearization of fast diffusion equations have been studied in details (cf. [23] and the references therein). Then, the initial value problem for Fokker-Planck type equations with general coefficients has been recently investigated by Le Bris and Lions in [65]. Their results allow to conclude that the initial value problem for equation (5.2) has a unique solution for a large class of initial values. In one-dimension of space Le Bris and Lions consider Fokker-Planck equations in one of the the forms

$$\frac{\partial}{\partial t}p(v,t) = \frac{1}{2}\frac{\partial}{\partial v^2}\left(\sigma^2(v)p(v,t)\right) + \frac{\partial}{\partial v}\left(b(v)p(v,t)\right), \tag{5.4}$$

which corresponds to our case, equations in divergence form

$$\frac{\partial}{\partial t}p(v,t) = \frac{\partial}{\partial v}\left(\frac{1}{2}\sigma^2(v)\frac{\partial}{\partial v}p(v,t) + b(v)p(v,t)\right), \tag{5.5}$$

and the so-called backward Kolmogorov equation

$$\frac{\partial}{\partial t}p(v,t) = \frac{1}{2}\sigma^2(v)\frac{\partial}{\partial v^2}p(v,t) - b(v)\frac{\partial}{\partial v}p(v,t). \tag{5.6}$$

Let b^σ and the Stratonovich drift b^s be defined as

$$b^\sigma = b - \frac{1}{2}\frac{\partial}{\partial v}\sigma^2, \quad b^s = b - \frac{1}{2}\sigma\frac{\partial}{\partial v}\sigma.$$

Furthermore, for $1 \le p, q \le +\infty$, let us define the space

$$L^p + L^q(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \ /\varphi = g + h, \text{ where } g \in L^p \text{and } h \in L^q\}.$$

Then, the following holds

Theorem 13. ([65]) *Let us assume that any one of the three drift functions b, b^σ or b^{Strat} satisfies*

$$b(v) \in W^{1,1}_{loc}(\mathbb{R}), \qquad \frac{\partial}{\partial v}b(v) \in L^\infty(\mathbb{R}), \qquad \frac{b(v)}{1+|v|} \in L^1 + L^\infty(\mathbb{R}), \tag{5.7}$$

and that σ satisfies

$$\sigma(v) \in W^{1,2}_{loc}(\mathbb{R}), \qquad \frac{\sigma(v)}{1+|v|} \in L^2 + L^\infty(\mathbb{R}). \tag{5.8}$$

Then for each initial condition in $L^1 \cap L^\infty(\mathbb{R})$ (resp. $L^2 \cap L^\infty(\mathbb{R})$), the Fokker-Planck equation (5.4)*, the Fokker-Planck equation of divergence form* (5.5)*, and the backward Kolmogorov equation* (5.6) *all have a unique solution in the space*

$$p \in L^\infty\big([0,T], L^1 \cap L^\infty\big) \ \big(resp.\ L^\infty\big([0,T], L^2 \cap L^\infty\big)\big), \ \ \sigma \frac{\partial}{\partial v} p \in L^2\big([0,T], L^2\big). \tag{5.9}$$

The natural condition for the Fokker–Planck equation (5.2) is to apply Theorem 13 considering as initial value a probability density in $L^1 \cap L^\infty(\mathbb{R})$. To this extent, it is sufficient to rewrite equation (5.2) in the divergence form

$$\frac{\partial}{\partial t} f = \frac{1}{2}\frac{\partial}{\partial v}\left(\sigma v^2 \frac{\partial}{\partial v} f\right) + \frac{\partial}{\partial v}\left[\Big((\sigma+\lambda)v - \lambda\Big) f\right], \tag{5.10}$$

that is the analogous of equation (5.5), and to remark that in our case $b(v) = (\sigma+\lambda)v - \lambda$ and $\sigma(v) = \sigma^{1/2} v$.

We obtain

Theorem 14. *Let $f_0(v)$ belong to $L^1 \cap L^\infty(\mathbb{R})$. Then, the the Fokker–Planck equation (5.2), for $t \le T$, has a unique solution $f(v,t)$ in the space*

$$f(v,t) \in L^\infty\big([0,T], L^1 \cap L^\infty\big), \ \ v \frac{\partial}{\partial v} f(v,t) \in L^2\big([0,T], L^2\big). \tag{5.11}$$

5.2.2 Regularity

The regularity of the solution to the initial-boundary value problem for equation (5.2) has been studied in [85]. For the sake of completeness, and for its consequences on the large-time behavior of the solution, we give here a short proof.

For any given smooth function $\varphi(v)$, $v \in \mathbb{R}$ let us consider the weak form of equation (5.2)

$$\frac{d}{dt}\int_{\mathbb{R}} \varphi(v) f(v,t) dv = (\varphi, J(f)) = \int_{\mathbb{R}} \left[\frac{\sigma}{2} v^2 \varphi''(v) - \lambda(v-1)\varphi'(v)\right] f(v,t)\, dv. \tag{5.12}$$

Under the hypotheses of Theorem 14, by choosing $\varphi(v) = e^{-i\xi v}$ we obtain the Fourier transformed version of the Fokker–Planck equation (5.2)

$$\frac{\partial}{\partial t}\widehat{f}(\xi,t) = \widehat{J}(\widehat{f}) = \frac{\sigma}{2}\xi^2 \frac{\partial^2}{\partial \xi^2}\widehat{f}(\xi,t) - \lambda\xi \frac{\partial}{\partial \xi}\widehat{f}(\xi,t) - i\lambda\xi \widehat{f}(\xi,t), \tag{5.13}$$

where, as usual $\widehat{g}(\xi)$ denotes the Fourier transform of $g(v)$, $v \in \mathbb{R}$

$$\widehat{g}(\xi) = \int_{\mathbb{R}} e^{-i\xi v} g(v)\, dv.$$

Let $\widehat{f}(\xi,t) = a(\xi,t) + ib(\xi,t)$. Then the real and imaginary parts of $\widehat{f}$ satisfy

$$\begin{aligned} \frac{\partial}{\partial t} a(\xi,t) &= \frac{\sigma}{2}\xi^2 \frac{\partial^2}{\partial \xi^2} a(\xi,t) - \lambda\xi \frac{\partial}{\partial \xi} a(\xi,t) + \lambda\xi b(\xi,t), \\ \frac{\partial}{\partial t} b(\xi,t) &= \frac{\sigma}{2}\xi^2 \frac{\partial^2}{\partial \xi^2} b(\xi,t) - \lambda\xi \frac{\partial}{\partial \xi} b(\xi,t) - \lambda\xi a(\xi,t). \end{aligned} \tag{5.14}$$

Let us multiply equations (5.14) respectively by $2a$ and $2b$. Summing up we get the evolution equation satisfied by $|\widehat{f}(\xi,t)|^2$.

$$\frac{\partial}{\partial t}|\widehat{f}|^2 = \sigma\xi^2\left[a\frac{\partial^2}{\partial\xi^2}a + b\frac{\partial^2}{\partial\xi^2}b\right] - \lambda\xi\frac{\partial|\widehat{f}|^2}{\partial\xi}. \tag{5.15}$$

Hence, multiplying by $|\xi|^p$ and integrating over $\mathbb{R}$ with respect to ξ, we obtain the evolution equation of the $\dot{H}_{p/2}-$norm of $f(v,t)$, where, as usual, the homogeneous Sobolev space $\dot{H}_s$, is defined by the norm

$$\|f\|_{\dot{H}_s} = \int_{\mathbb{R}} |\xi|^{2s}\,|\widehat{f}|^2(\xi)\,d\xi.$$

We obtain

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} |\xi|^p\,|\widehat{f}|^2\,d\xi = \sigma\int_{\mathbb{R}} |\xi|^{2+p}\left[a\frac{\partial^2}{\partial\xi^2}a + b\frac{\partial^2}{\partial\xi^2}b\right]d\xi - \lambda\int_{\mathbb{R}} \xi|\xi|^p\frac{\partial|\widehat{f}|^2}{\partial\xi}d\xi, \tag{5.16}$$

and integrating by parts the two integrals, it results

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} |\xi|^p\,|\widehat{f}|^2\,d\xi = (p+1)\left[\frac{\sigma}{2}(p+2)+\lambda\right]\int_{\mathbb{R}} |\xi|^p|\widehat{f}|^2 d\xi - \sigma\int_{\mathbb{R}} |\xi|^{2+p}\left[\left|\frac{\partial}{\partial\xi}a\right|^2 + \left|\frac{\partial}{\partial\xi}b\right|^2\right]d\xi. \tag{5.17}$$

Since the last integral in (5.17) can be bounded from below [85]

$$\int_{\mathbb{R}} |\xi|^{2+p}\left[\left|\frac{\partial}{\partial\xi}a\right|^2 + \left|\frac{\partial}{\partial\xi}b\right|^2\right]d\xi \geq \frac{(p+1)^2}{4}\int_{\mathbb{R}} |\xi|^p|\widehat{f}|^2\,d\xi.$$

we finally obtain

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} |\xi|^p|\widehat{f}|^2\,d\xi \leq \frac{p+1}{2}\left[\sigma\frac{p+3}{2}+2\lambda\right]\int_{\mathbb{R}} |\xi|^p|\widehat{f}|^2 d\xi. \tag{5.18}$$

The inequality (5.18) implies that if the initial data has bounded $\dot{H}_p-$norm, then for all $t>0$, the $\dot{H}_p-$norm of the solution remains bounded, even if not uniformly bounded with respect to time. We proved

Theorem 15. ([85]) *Let $f_0(v)$ be a probability density in $\mathbb{R}$ that belongs to $\dot{H}_r(\mathbb{R})$. Then, the $\dot{H}_r-$norm of the solution $f(v,t)$ to the Fokker–Planck equation (5.2), for $t\leq T$, still belongs to $\dot{H}_r(\mathbb{R})$, and*

$$\int_{\mathbb{R}} |\xi|^{2r}|\widehat{f}|^2(t)\,d\xi \leq \exp\left\{\frac{2r+1}{2}\left[\sigma\frac{2r+3}{2}+2\lambda\right]t\right\}\int_{\mathbb{R}} |\xi|^{2r}|\widehat{f_0}|^2\,d\xi. \tag{5.19}$$

Remark 16. *The difficulty of recovering the uniform boundedness of the $\dot{H}_r(\mathbb{R})$-norm of the solution to the Fokker–Planck equation (5.2) is strictly related to the singularity of the coefficient of diffusion σv^2, which vanishes in correspondence to the point $v=0$. Indeed, as proven in [23] for a similar Fokker–Planck equation with coefficient of diffusion $1+\sigma v^2$, the uniform boundedness of the $\dot{H}_r(\mathbb{R})$-norm of the solution holds.*

5.2.3 Further properties

The analysis of [65] do not care about the eventual preservation of positivity of the solutions to (5.4). However, this property can be easily proved for equation (5.2), by resorting to the same argument used in [85] for the same equation posed in $\mathbb{R}_+$. Indeed, as proven in [85], the solution to the Fokker–Planck equation (5.2) is the limit of the solution to a kinetic equation of Boltzmann type, for which it is elementary to obtain the positivity property.

Positivity can however be proven directly by working on the Fokker–Planck equation, by resorting to the following argument [57]. Suppose the initial data $f_0(v)$ (and hence the unique solution) to the Fokker–Planck equation (5.2) are smooth and vanish for $v = \pm\infty$. Suppose moreover that $f_0(v) \geq 0$. Since the (smooth) initial value is non negative, for $t \geq 0$, every point $v_m(t)$ in which $f(v_m(t), t) = 0$ is either a local minimum, and

$$\frac{\partial}{\partial v} f(v,t)\bigg|_{v=v_m(t)} = 0, \quad \frac{\partial^2}{\partial v^2} f(v,t)\bigg|_{v=v_m(t)} > 0. \tag{5.20}$$

or a stationary point, and in this case

$$\frac{\partial}{\partial v} f(v,t)\bigg|_{v=v_m(t)} = 0, \quad \frac{\partial^2}{\partial v^2} f(v,t)\bigg|_{v=v_m(t)} = 0. \tag{5.21}$$

Computing derivatives, the Fokker–Planck equation (5.2) can be written in the form

$$\frac{\partial}{\partial t} f(v,t) = \frac{\sigma}{2} v^2 \frac{\partial^2}{\partial v^2} f(v,t) + [(2\sigma + \lambda)v - \lambda] \frac{\partial}{\partial v} f(v,t) + (\lambda + \sigma) f(v,t). \tag{5.22}$$

Hence, evaluating (5.22) at the point $v = v_m(t)$, and using (5.20) shows that, if $v_m(t) \neq 0$ is a local minimum

$$\frac{\partial}{\partial t} f(v,t) \mid_{v=v_m(t)} = \frac{\sigma}{2} v_m(t)^2 \frac{\partial^2}{\partial v^2} f(v,t) \mid_{v=v_m(t)} > 0.$$

This entails that the function $f(v,t)$ is increasing in time at the point $v = v_m(t)$, unless $v_m(t) = 0$. Indeed, if the local minimum is attained at $v_m(t) = 0$

$$\frac{\partial}{\partial t} f(v,t) \mid_{v=0} = 0,$$

and $f(0,t)$ remains equal to zero at any subsequent time.

If now $v_m(t)$ is a stationary point, so that (5.21) holds,

$$\frac{\partial}{\partial t} f(v,t) \mid_{v=v_m(t)} = 0,$$

and $f(v,t)$ remains equal to zero. Therefore

$$\min_{x \in \mathbb{R}} f(v,t) \geq 0, \tag{5.23}$$

and positivity follows. The proof for initial data satisfying the conditions of Theorem 14 then follows first considering a suitable smoothing of the initial data, and then using the fact that at any subsequent time $t > 0$, the solution corresponding to the smoothed initial data converges to the solution of the original data when eliminating the initial smoothing.

Remark 17. *A further consequence of this analysis is that, if the initial datum vanishes on the half-line $v \leq 0$, in reason of the properties of the solution at the point $v = 0$, the solution at any subsequent time $t \geq 0$ will remain equal to zero on the half-line $v \leq 0$.*

Further results in this direction follows by studying the evolution of the mass located in the negative part of the real axis. To start with, consider that by evaluating (5.12) with test functions $\phi(v) = 1, v$ one obtains that, if the initial value $f_0(v)$ vanishes for $v = \pm\infty$, the solution to (5.2) satisfies

$$\frac{d}{dt}\int_{\mathbb{R}} f(v,t)dv = 0, \quad \frac{d}{dt}\int_{\mathbb{R}} vf(v,t)dv = \lambda\left(-\int_{\mathbb{R}} vf(v,t)\,dv + \int_{\mathbb{R}} f(v,t)\,dv\right).$$

Therefore, if the (nonnegative) initial value of the Fokker–Planck equation (5.2) is a density function satisfying the normalization conditions

$$\int_{\mathbb{R}} f_0(v)\,dv = 1; \quad \int_{\mathbb{R}} vf_0(v)\,dv = 1 \tag{5.24}$$

the solution $f(v,t)$ to (5.2) still satisfies conditions (5.24). In other words, if the initial datum is a probability density with unit mean, then the solution at any subsequent time remains a probability density with unit mean.

A further interesting property of the solution can be extracted by analyzing the behaviour of the mass and the mean value separately on the left and right half-line. Let us denote by $\rho_+(t)$ (respectively $\rho_-(t)$) the fraction of the mass distributed on the positive half-line (respectively on the negative half-line) at time $t \geq 0$, that is

$$\rho_+(t) = \int_0^{+\infty} f(v,t)\,dv; \quad \rho_-(t) = \int_{-\infty}^0 f(v,t)\,dv. \tag{5.25}$$

Let the initial value $f_0(v) \in C(\mathbb{R})$ satisfy conditions (5.24). Let $H_n(v)$ be a smooth approximation to the Heaviside step function, for example the logistic function

$$H_n(v) = \frac{1}{1+e^{-2nv}}.$$

Then, equation (5.12) implies, for any $t > 0$

$$\int_{\mathbb{R}} H_n(v)f(v,t)\,dv = \int_{\mathbb{R}} H_n(v)f_0(v)\,dv +$$

$$+\int_0^t\int_{\mathbb{R}}\left[\frac{\sigma}{2}v^2H_n''(v) - \lambda(v-1)H_n'(v)\right] f(v,s)\,dv\,ds.$$

Letting $n \to +\infty$, and considering that $H_n'(v)$ converges to a Dirac delta in zero, while $v^2H_n''(v)$ is a uniformly bounded function that converges pointwise to zero, we obtain

$$\lim_{n\to+\infty}\int_0^t\int_{\mathbb{R}}\left[\frac{\sigma}{2}v^2H_n''(v) - \lambda(v-1)H_n'(v)\right] f(v,s)\,dv\,ds = \int_0^t f(0,s)\,ds.$$

Therefore, since for ant $t \geq 0$

$$\lim_{n\to+\infty}\int_{\mathbb{R}} H_n(v)f(v,t)\,dv = \int_0^{+\infty} f(v,t)\,dv = \rho_+(t),$$

it follows that

$$\rho_+(t) = \rho_+(0) + \int_0^t f(0,s)\,ds, \tag{5.26}$$

namely that the mass in the positive half-line can not decrease if the mean value is positive.

With similar arguments it is possible to analyze the time behaviour of the parts of the mean value located on the positive and negative parts of the real line. Let us indicate these parts by $m_+(t)$ and $m_-(t)$, where

$$m_+(t) = \int_0^{+\infty} vf(v,t)\,dv, \quad m_-(t) = \int_{-\infty}^0 vf(v,t)\,dv \tag{5.27}$$

A direct computation shows that, for each time $t > 0$

$$\begin{aligned} m_+(t) &= m_+(0) + \int_0^t \left(-\rho_-(s)m_+(s) + \rho_+(s)m_-(s)\right)\,ds, \\ m_-(t) &= m_-(0) + \int_0^t \left(\rho_-(s)m_+(s) - \rho_+(s)m_-(s)\right)\,ds. \end{aligned} \tag{5.28}$$

The choice of mean value $m = 1 > 0$ implies $m_+(t) = |m_-|(t) + 1$. Therefore using this equality into the second equation in (5.28) we obtain

$$\frac{d}{dt}|m_-(t)| = -\left((\rho_-(t)m_+(t) - \rho_+(t)m_-(t)\right) = -\left(|m_-(t)| + \rho_-(t)|m_-(t)|\right) \leq -|m_-(t)|\,.$$

Consequently, by Gronwall inequality we conclude that

$$|m_-(t)| \leq |m_-(0)|e^{-t}, \tag{5.29}$$

and the negative part of the mean value decays exponentially fast towards zero. We can group the previous results into the following

Theorem 18. *Let $f_0(v)$ be a probability density in $\mathbb{R}$, satisfying the normalization conditions (5.24). Then, the solution $f(v,t)$ to the Fokker–Planck equation (5.2) remains a probability density for each subsequent time $t \geq 0$, and satisfies conditions (5.24). Moreover, the mass $\rho_+(t)$ located on the positive part of the real line is non decreasing in time and (5.26) holds. Also, the part of the mean value $m_-(t)$ located on the negative part of the real axis is exponentially decreasing in time, and (5.29) holds.*

In the economic context, the consequences of Theorem 18 appear relevant.

Remark 19. *Equation (5.26), coupled with the property of mass conservation, implies that, in the particular case in which the initial data is a smooth probability density which takes values only in the region $v \geq 0$, since the mass in this region can only increase, the solution at any subsequent time $t > 0$ remains a smooth probability density distributed on the same region $v \geq 0$. This independently of any boundary condition one can introduce to justify mass and momentum conservation [49, 85]. This property can be easily relaxed to general probability measures initially taking values on the set $v \geq 0$. In other words, the lack of diffusion at the point $v = 0$, as outlined in Remark 17, is enough to maintain the whole mass, initially located on the positive part of the real line, on the same set.*

Remark 20. *The previous results about the time evolution of the mass and mean value located on the set $v \geq 0$ show that the part of mass that is initially distributed on the negative half-space (the debts) moves to the region $v \geq 0$, and this process is exponentially rapid in terms of the negative part of the mean value. However, since the regularity results of Theorem 15 are not uniform with respect to time, it could happen that there is accumulation of the negative fraction of the mass at the point $v = 0$, with the eventual formation of a Dirac delta in $v = 0$, namely the point in which there is no diffusion.*

5.2.4 The stationary state

Let us consider the Γ-like distribution (5.3), continuously extended to zero for $v < 0$

$$f_\infty(v) = \frac{(\mu-1)^\mu}{\Gamma(\mu)} \frac{\exp\left(-\frac{\mu-1}{v}\right)}{v^{1+\mu}} \quad \text{if } v \geq 0; \qquad f_\infty(v) = 0 \quad \text{if } v < 0. \tag{5.30}$$

where

$$\mu = 1 + 2\frac{\lambda}{\sigma} > 1. \tag{5.31}$$

It can be easily verified that the equilibrium distribution (5.30) achieves its maximum value

$$\bar{f}_\infty = \frac{(\mu+1)^{\mu+1}}{\Gamma(\mu)(\mu-1)} \exp\left(-(\mu+1)\right) \tag{5.32}$$

at the point

$$\bar{v} = \frac{\mu-1}{\mu+1}. \tag{5.33}$$

Therefore it is increasing in the interval $(0, \bar{v})$ and decreasing on $(\bar{v}, +\infty)$. Note that the value $1 + \mu$ defines the rate of decay at infinity of the power tailed distribution (5.30). Consequently

$$\int_{\mathbb{R}} |v|^r f_\infty(v)\, dv < \infty \tag{5.34}$$

if and only if $r < \mu$.

Then, owing to elementary properties of the Gamma function, it is immediate to conclude that, provided $\mu > 2$, the second moment of the steady state is bounded, and

$$\int_{\mathbb{R}} f_\infty(v)\, dv = 1; \quad \int_{\mathbb{R}} v\, f_\infty(v)\, dv = 1; \quad \int_{\mathbb{R}} v^2 f_\infty(v)\, dv = \frac{\mu-1}{\mu-2}. \tag{5.35}$$

It follows that, if the initial value for the Fokker–Planck equation (5.2) posed in the whole space $\mathbb{R}$ is a probability density function of mean value equal to one, $f_\infty(v)$ is a smooth probability density with the same mean value, which in addition satisfies the Fokker–Planck equation (5.2) on $\mathbb{R}$. If in addition $\mu > 2$, and the initial value has the second moment bounded, then the second moment of the solution converges exponentially towards the second moment of f_∞.

For $n \in \mathbb{N}_+$ let us define

$$M_n(t) = \int_{\mathbb{R}_+} v^n f(v,t) dv.$$

Then [85]

$$\frac{d}{dt} M_2(t) = (\sigma - 2\lambda) M_2(t) + 2\lambda. \tag{5.36}$$

Hence, the value of the second moment stays bounded when $\sigma < 2\lambda$ (or, what is the same $\mu > 2$), while it diverges in the opposite case. In the former case, solving equation (5.36) we obtain

$$M_2(t) = e^{(\sigma-2\lambda)t}\left(M_2(0) + \frac{2\lambda}{\sigma - 2\lambda}\right) + \frac{2\lambda}{2\lambda - \sigma}, \tag{5.37}$$

which implies

$$\lim_{t\to\infty} M_2(t) = \frac{2\lambda}{2\lambda - \sigma}.$$

Thus, $f_\infty(v)$ is the (unique) steady state of the Fokker–Planck equation with moments satisfying (5.35). This clearly indicates that one could expect that, even starting with a probability density defined on the whole $\mathbb{R}$, but with positive mean value (equal to one in our case), the solution to the initial value problem will converge in time towards the equilibrium (5.30). A rigorous proof of this property will be presented in the next section.

Remark 21. *It is clear that the evolution of the principal moments of the solution to the Fokker–Planck equation (5.2) can be obtained recursively, and explicitly evaluated at the price of an increasing length of computations.*

5.3 Convergence to equilibrium

5.3.1 Fourier based metrics

As shown in Section 5.2, in reason of the positivity property, and mass and momentum conservation of the solution of the Fokker–Planck equation (5.2), one can always assume that both the solution and the steady state are probability densities satisfying (5.24). This remark allows to use metrics for probability distributions to study convergence to equilibrium. This is a method that in kinetic theory of rarefied gases goes back to [50], where convergence to equilibrium for the Boltzmann equation for Maxwell pseudo-molecules was studied in terms of a metric for Fourier transforms (cf. also [24, 74, 89] for further applications).

For a given constant $s > 0$ let $\mathcal{M}_s$ be the set of probability measures μ on the Borel subsets of $\mathbb{R}$ such that

$$\int_{\mathbb{R}} |v|^s \, \mu(dv) < \infty,$$

and let $\mathcal{F}_s$ be the set of Fourier transforms of probability distributions μ in $\mathcal{M}_s$. A useful metric in $\mathcal{F}_s$ has been introduced in [50] in connection with the Boltzmann equation for Maxwell molecules, and subsequently applied in various contexts, which include kinetic models for wealth distribution [76]. For a given pair of random variables X and Y distributed according to ϕ and ψ this metric reads

$$d_s(X,Y) = d_s(\phi,\psi) = \sup_{\xi\in\mathbb{R}} \frac{|\widehat{\phi}(\xi) - \widehat{\psi}(\xi)|}{|\xi|^s}, \tag{5.38}$$

As shown in [50], the metric $d_s(\phi,\psi)$ is finite any time the probability distributions ϕ and ψ have equal moments up to $[s]$, namely the entire part of $s \in \mathbb{R}_+$, or equal moments up to $s-1$ if $s \in \mathbb{N}$, and it is equivalent to the weak* convergence of measures for all $s > 0$. Among other properties, it is easy to see [50, 76] that, for any random variable Z independent of X and Y and for any constant c

$$\begin{aligned} &d_s(X+Z, Y+Z) \le d_s(X,Y), \\ &d_s(cX, cY) = |c|^s d_s(X,Y). \end{aligned} \tag{5.39}$$

These properties classify d_s as an ideal probability metric in the sense of Zolotarev [95].

Few years after the publication of [50], Baringhaus and Grübel [6], in connection with the study of convex combinations of random variables with random coefficients, considered a Fourier metric similar to (5.38), defined by

$$D_s(X,Y) = D_s(\phi,\psi) = \int_{\mathbb{R}} \frac{|\widehat{\phi}(\xi) - \widehat{\psi}(\xi)|}{|\xi|^{1+s}}\, d\xi. \tag{5.40}$$

As shown in [6], also D_s as an ideal probability metric in the sense of Zolotarev, and for $1 < s < 2$ the space $\tilde{\mathcal{F}}_s \subset \mathcal{F}$ of probability distributions satisfying (5.24) endowed with the metric D_s is complete.

It can be verified that the metrics d_s and D_s are strictly connected. In particular, if $r < s$, $D_r(\phi,\psi) \leq c(r,s) d_s(\phi,\psi)^{r/s}$, where $c(r,s)$ is a positive constant which depends only on r, s.

Indeed, since $|\widehat{\phi}(\xi)| \leq 1$, $|\widehat{\psi}(\xi)| \leq 1$, for any given positive constant R

$$\int_{|\xi|>R} \frac{|\widehat{\phi}(\xi) - \widehat{\psi}(\xi)|}{|\xi|^{1+r}}\, d\xi \leq \int_{|\xi|>R} \frac{2}{|\xi|^{1+r}}\, d\xi = \frac{4}{rR^r}.$$

On the other hand, on the interval $|\xi| \leq R$, for $s > r$ it holds

$$\int_{|\xi|\leq R} \frac{|\widehat{\phi}(\xi) - \widehat{\psi}(\xi)|}{|\xi|^{1+r}}\, d\xi = \int_{|\xi|\leq R} \frac{|\widehat{\phi}(\xi) - \widehat{\psi}(\xi)|}{|\xi|^{s}} \cdot \frac{1}{|\xi|^{1+r-s}}\, d\xi \leq$$

$$d_s(\phi,\psi) \int_{|\xi|\leq R} \frac{1}{|\xi|^{1+r-s}}\, d\xi = 2\, d_s(\phi,\psi) \frac{R^{s-r}}{s-r}.$$

Therefore, for for any given positive constant R

$$D_r(\phi,\psi) \leq 2\, d_s(\phi,\psi) \frac{R^{s-r}}{s-r} + \frac{4}{rR^r},$$

and, optimizing over R we obtain, for $s > r$

$$D_r(\phi,\psi) \leq c(r,s) d_s(\phi,\psi)^{r/s}, \tag{5.41}$$

where

$$c(r,s) = 2^{2-r/s} \frac{s}{r(s-r)}. \tag{5.42}$$

This allows to conclude that for $1 < r < 2$, and for $s > r$, the space $\tilde{\mathcal{F}}_s$ endowed with the metric d_s is complete.

New metrics on $\mathcal{F}_s$ can be introduced according to the following definition. Let $p \geq 1$, and $s > 0$. For a given pair of random variables X and Y distributed according to ϕ and ψ we define

$$D_{s,p}(X,Y) = D_{s,p}(\phi,\psi) = \left[\int_{\mathbb{R}} |\xi|^{-(ps+1)} |\widehat{\phi}(\xi) - \widehat{\psi}(\xi)|^p d\xi \right]^{\frac{1}{p}}. \tag{5.43}$$

The metric D_s corresponds to $D_{s,1}$, while the metric d_s is obtained by taking the limit $p \to \infty$ of $D_{s,p}$. Moreover, for any given value of the constant p, the $D_{s,p}$ metric is an ideal probability metric in the sense of Zolotarev. Proceeding as before, it is immediate to show that these metrics satisfy an inequality similar to (5.41)

$$D_{s,p}(\phi,\psi) \leq c(p,r,s) d_s(\phi,\psi)^{r/s}, \tag{5.44}$$

where

$$c(p,r,s) = 2^{1-r/s}\left[\frac{2s}{pr(s-r)}\right]^{\frac{1}{p}}. \tag{5.45}$$

In addition, it can be shown that the $D_{s,p}$-metrics are strictly related each other. In fact, if $p < q$ and $r < s$, by similar methods one proves that there exists a finite explicitly computable constant such that the following estimate holds

$$D_{r,p}(\phi,\psi) \le c(p,q,r,s) D_{s,q}(\phi,\psi)^{r/s}. \tag{5.46}$$

A distinguished case is obtained by fixing $p = 2$. Then the $D_{s,2}$ metric

$$D_{s,2}(\phi,\psi) = \left[\int_{\mathbb{R}} |\xi|^{-(2s+1)} |\widehat{\phi}(\xi) - \widehat{\psi}(\xi)|^2 d\xi\right]^{\frac{1}{2}}. \tag{5.47}$$

coincides with the distance between ϕ and ψ in the homogeneous Sobolev space of fractional order with negative index $\dot{H}_{-q}$, with $q = s + 1/2$, where, for $h \in \dot{H}_{-q}$

$$\|h\|_{\dot{H}_{-q}} = \int_{\mathbb{R}} |\xi|^{-2q} |\widehat{h}(\xi)|^2 \, d\xi. \tag{5.48}$$

5.3.2 Convergence in Fourier metric

Convergence to equilibrium of the solution to the Fokker–planck equation (5.2) in the metric $D_{s,2}$ is an easy consequence of the result of Theorem 15. Indeed, looking at its proof it is immediate to notice that all computations leading to formula (5.19) still holds when $r < 0$. Moreover, thanks to the linearity of the Fokker–Planck equation (5.2), formula (5.19) remains valid if we substitute $f(v,t)$ with the difference $f(v,t) - f_\infty(v)$. Hence, by setting $r = -(s + 1/2)$ we obtain

$$\begin{aligned} D_{s,2}(f(t), f_\infty) &= \left[\int_{\mathbb{R}} |\xi|^{-(2s+1)} |\widehat{f}(\xi) - \widehat{f}_\infty(\xi)|^2 d\xi\right]^{\frac{1}{2}} \le \\ &\exp\left\{-\frac{s}{2}\left((1-s)\sigma) + 2\lambda\right)\right\} \left[\int_{\mathbb{R}} |\xi|^{-(2s+1)} |\widehat{f}_0(\xi) - \widehat{f}_\infty(\xi)|^2 d\xi\right]^{\frac{1}{2}} = \\ &\exp\left\{-\frac{s}{2}\left((1-s)\sigma) + 2\lambda\right)\right\} D_{s,2}(f_0, f_\infty). \end{aligned} \tag{5.49}$$

Therefore, if the exponent is negative, there is exponential convergence in $D_{s,2}$-metric of the solution $f(v,t)$ towards the steady distribution $f_\infty(v)$. This happens if

$$s < 1 + 2\frac{\lambda}{\sigma} = \mu \tag{5.50}$$

where the constant μ has been defined in (5.31), and characterizes the decay at infinity of the stationary distribution $f_\infty(v)$. Note that, since $\mu > 1$, by taking $s = 1$ we obtain that convergence to equilibrium holds for all initial values satisfying (5.24) at a rate 2λ, which results to be independent of the coefficient of diffusion σ of the Fokker–Planck equation (5.2). Hence we have

Theorem 22. *Let $f_0(v)$ be a probability density in $\mathbb{R}$ satisfying (5.24), and such that $D_{s,2}(f_0, f_\infty)$ is finite for some $s < \mu$, where μ is defined in (5.31). Then, the solution to the Fokker–Planck equation (5.2) posed in the whole space $\mathbb{R}$ is exponentially converging to the equilibrium density f_∞ in $D_{2,s}$-metric and the following decay holds*

$$D_{2,s}(f(t), f_\infty) \le \exp\left\{-\frac{s}{2}\left((1-s)\sigma) + 2\lambda\right)\right\} D_{2,s}(f_0, f_\infty). \tag{5.51}$$

It is immediate to verify that the rate of decay to equilibrium is maximum when $s = \mu/2$. In this case, provided $D_{2,\mu/2}(f_0, f_\infty) < \infty$

$$D_{2,\mu/2}(f(t), f_\infty) \leq \exp\left\{-\frac{\sigma\mu^2}{8}\right\} D_{2,\mu/2}(f_0, f_\infty). \tag{5.52}$$

5.3.3 The monotonicity of relative entropy

The result of Section 5.3.2 shows that, at least in a weak sense, there is exponential convergence of the solution to the Fokker–Planck equation (5.2) posed in the whole space towards the unique steady state $f_\infty(v)$ defined in (5.30). As usual for this type of equations [49], to obtain stronger convergence results, tipically in $L^1(\mathbb{R})$, a classical method is to resort to the study of the time decay of various Lyapunov functionals involving the solution $f(v,t)$ and the steady state.

Among these functionals, a leading rule is usually assumed by the relative Shannon entropy $H(f,h)$, where, for any given pair of probability densities f, h

$$H(f,h) = \int_{\mathbb{R}} f(v) \log \frac{f(v)}{h(v)}\, dv. \tag{5.53}$$

However, since in our case the steady state (5.30) is supported on the positive real line, while the initial value is in general supported on the whole real line, the Shannon entropy $H(f(t), f_\infty)$ of the solution $f(v,t)$ to the Fokker–Planck equation (5.2) relative to the steady state $f_\infty(v)$ is unbounded, and consequently useless.

A related relative entropy which appears more appropriate to treat the present problem is the so-called Jensen-Shannon entropy, introduced by Lin in [66]. Given the pair of probability densities f, h, and a constant $0 < \alpha < 1$, the Jensen-Shannon entropy H_α of f relative to h is defined by

$$H_\alpha(f,h) = \int_{\mathbb{R}} f(v) \log \frac{f(v)}{\alpha f(v) + (1-\alpha)h(v)}\, dv. \tag{5.54}$$

Note that, since the convex combination $\alpha f + (1-\alpha)h$ of the two probability densities f and h is still a probability density, say h_α, the Jensen-Shannon entropy H_α of f relative to h is simply the Shannon entropy of f relative to h_α. The main properties of these entropies have been studied in [66] (cf. also [49]). In particular, thanks to Lemma 27 in [49] the Jensen–Shannon entropy of two probability densities is always bounded.

Let $g(v,t) = \alpha f(v,t) + (1-\alpha) f_\infty(v)$. Thanks to (5.24) and (5.35), it follows that $g(v,t)$ is a probability density of unit mean. Moreover, since both $f(v,t)$ and $f_\infty(v)$ are solutions to the linear Fokker–Planck equation (5.2), $g(v,t)$ is itself a solution to (5.2). Note that for $v \leq 0$, $f(v)/g(v) = 1/\alpha$. Moreover, if $v > 0$, $f(v)/g(v) \leq 1/\alpha$. Since for $r \geq 0$ the function $r \log r$ is bounded from below, writing

$$H_\alpha(f, f_\infty) = \int_{\mathbb{R}} \left(\frac{f(v)}{g(v)} \log \frac{f(v)}{g(v)} \right) g(v)\, dv,$$

it is immediate to conclude that $H_\alpha(f, f_\infty)$ is well defined and bounded from above and below independently of the regularity of the initial data.

In what follows, to avoid inessential difficulties in the forthcoming computations, we will assume that the initial density $f_0(v)$ (and consequently the solution $f(v,t)$), is smooth and has enough moments bounded.

To compute the evolution of the Jensen–Shannon entropy, let us first remark that

$$\int_{-\infty}^{0} f(v,t) \log \frac{f(v,t)}{\alpha f(v,t) + (1-\alpha) f_\infty(v)}\, dv = \log \frac{1}{\alpha} \int_{-\infty}^{0} f(v,t)\, dv.$$

Since, according to (5.26) the mass in the negative half-line can not increase, and $\log \frac{1}{\alpha} > 0$, the part of Jensen–Shannon entropy relative to the domain $v \leq 0$ is nonincreasing in time.

On the set $(\gamma, +\infty)$ we have

$$\begin{aligned} \frac{d}{dt}\int_\gamma^\infty f \log \frac{f}{g}\, dv = &\int_\gamma^\infty \left\{ \log \frac{f}{g} \frac{\partial^2}{\partial v^2}(v^2 f) - \frac{f}{g}\frac{\partial^2}{\partial v^2}(v^2 g)\right\} dv \\ &+ \int_\gamma^\infty \left\{ \log \frac{f}{g} \frac{\partial}{\partial v}[(v-1)f] - \frac{f}{g}\frac{\partial}{\partial v}[(v-1)g]\right\} dv. \end{aligned} \tag{5.55}$$

Using the identity $f/g = (v^2 f)/(v^2 g)$, that clearly holds when $v \in (\gamma, +\infty)$, integration by parts gives (cf. the proof of Proposition 25 in [49])

$$\begin{aligned} &\int_\gamma^{+\infty} \left\{ \log \frac{v^2 f}{v^2 g} \frac{\partial^2}{\partial v^2}(v^2 f) - \frac{v^2 f}{v^2 g}\frac{\partial^2}{\partial v^2}(v^2 g)\right\} dv = \\ &\left[\log \frac{f}{g}\frac{\partial}{\partial v}(v^2 f) - \frac{f}{g}\frac{\partial}{\partial v}(v^2 g)\right]_\gamma^{+\infty} - \int_\gamma^{+\infty} v^2 f \left[\frac{\partial}{\partial v}\log\frac{f}{g}\right]^2 dv. \end{aligned} \tag{5.56}$$

The contribution of the border term at infinity can be easily shown to vanish provided $f(v,t)$ has moments bounded of order $2+\delta$ for some $\delta > 0$. Indeed, given p, q conjugate exponents, namely such that $1/p + 1/q = 1$

$$\left| \log \frac{f}{g}\frac{\partial}{\partial v}(v^2 f)\right| = \left| \left(\frac{f}{g}\right)^{1/q} \log \frac{f}{g} \left(\frac{v^2 g}{v^2 f}\right)^{1/q} \frac{\partial}{\partial v}(v^2 f)\right| \leq C_q (v^2 g)^{1/q} p \left| \frac{\partial (v^2 f)^{1/p}}{\partial v}\right|.$$

In the previous inequality we defined

$$C_q = \sup \left| \left(\frac{f}{g}\right)^{1/q} \log \frac{f}{g}\right|,$$

which is bounded in reason of the fact that $f/g \leq 1/\alpha$. Moreover, by Hölder inequality, whenever $p/q \leq \delta/2$

$$\begin{aligned} &\int_{\mathbb{R}} (v^2 f)^{1/p}\, dv = \int_{\mathbb{R}} (v^2 f)^{1/p} (1+v^2)^{1/q} (1+v^2)^{-1/q}\, dv \leq \\ &\left(\int_{\mathbb{R}} v^2 (1+v^2)^{p/q} f\, dv\right)^{1/p} \left(\int_{\mathbb{R}} (1+v^2)^{-1}\, dv\right)^{1/q} \leq C. \end{aligned}$$

Consequently, as soon as $p/q \leq \delta/2$, both the smooth functions $v^2 g$ and $(v^2 f)^{1/p}$ are integrable, and

$$\lim_{v\to\infty} (v^2 g)^{1/q} \left| \frac{\partial (v^2 f)^{1/p}}{\partial v}\right| = 0. \tag{5.57}$$

Analogous arguments can be used to prove that

$$\lim_{v\to\infty} \frac{f}{g}\frac{\partial}{\partial v}(v^2 g) = 0. \tag{5.58}$$

On the other extremal point, the choice $p = q = 2$ gives

$$\left| \log \frac{f}{g}\frac{\partial}{\partial v}(v^2 f)\right| \leq 2C_2 (v^2 g)^{1/2} \left| \frac{\partial (v f^{1/2})}{\partial v}\right| = 2C_2 \left| v g^{1/2} \left(v \frac{\partial f^{1/2}}{\partial v} + f^{1/2}\right)\right|.$$

Then, considering that both f and g are smooth, and $f^{1/2} \in L^2(\mathbb{R})$, one obtains

$$\lim_{v\to 0}(v^2 g)^{1/2}\left|\frac{\partial (v^2 f)^{1/2}}{\partial v}\right| = 0, \tag{5.59}$$

and

$$\lim_{v\to 0}\frac{f}{g}\frac{\partial}{\partial v}(v^2 g) = 0. \tag{5.60}$$

Let us consider now the second integral into (5.55). Integrating by parts first on the interval $(\gamma, 1-\gamma)$, and using the identity $f/g = (v-1)f/[(v-1)g]$ we obtain

$$\begin{aligned}&\int_\gamma^{1-\gamma}\left(\log\frac{f}{g}\frac{\partial}{\partial v}[(v-1)f] - \frac{f}{g}\frac{\partial}{\partial v}[(v-1)g]\right)dv = \\ &\left[\log\frac{f}{g}(v-1)f - (v-1)f\right]_\gamma^{1-\gamma} = \left[\frac{f}{g}\log\frac{f}{g}(v-1)g - (v-1)f.\right]_\gamma^{1-\gamma}\end{aligned} \tag{5.61}$$

Hence, since the quantity $(f/g)\log(f/g)$ is uniformly bounded from above and below

$$\lim_{v\to 1}\left(\frac{f}{g}\log\frac{f}{g}(v-1)g - (v-1)f\right) = 0.$$

Moreover, since

$$\lim_{v\to 0}\frac{f(v)}{g(v)} = \frac{1}{\alpha},$$

we obtain

$$\lim_{v\to 0}\left(\log\frac{f(v)}{g(v)}(v-1)f(v) - (v-1)f(v)\right) = f(0)\left(1+\log\alpha\right).$$

This implies

$$\int_0^1\left(\log\frac{f}{g}\frac{\partial}{\partial v}[(v-1)f] - \frac{f}{g}\frac{\partial}{\partial v}[(v-1)g]\right)dv = -f(0)\left(1+\log\alpha\right)$$

Similar computations then give

$$\int_1^{+\infty}\left(\log\frac{f}{g}\frac{\partial}{\partial v}[(v-1)f] - \frac{f}{g}\frac{\partial}{\partial v}[(v-1)g]\right)dv = 0.$$

Grouping the various pieces, we conclude that the Jensen–Shannon entropy $H_\alpha(f(t), f_\infty)$ is nonincreasing in time. We proved

Theorem 23. *Let $f_0(v)$ be a smooth probability density in $\mathbb{R}$ satisfying (5.24), and such that its moments up to $2+\delta$ are finite for some $\delta > 0$. Then, for any $0 < \alpha < 1$, the Jensen–Shannon entropy $H_\alpha(f(t), f_\infty)$ of the solution to the Fokker–Planck equation (5.2) relative to the equilibrium solution is monotonically nonincreasing, and the following decay holds*

$$H_\alpha(f(t), f_\infty) = H_\alpha(f_0, f_\infty) - \int_0^t f(0,s)\,ds - \int_0^t\int_0^{+\infty} v^2 f(v,s)\left[\frac{\partial}{\partial v}\log\frac{f(v,s)}{g(v,s)}\right]^2 dv\,ds. \tag{5.62}$$

5.3.4 The monotonicity of Hellinger distance

A second interesting functional that has been shown to be monotonically decreasing along the solution to Fokker–Planck type equations [49] is the Hellinger distance. For any given pair of probability densities f and h defined on $\mathbb{R}$, the Hellinger distance $d_H(f,h)$ is [95]

$$d_H(f,h) = \left(\int_{\mathbb{R}} \left(\sqrt{f(v)} - \sqrt{h(v)} \right)^2 \, dv \right)^{\frac{1}{2}}. \tag{5.63}$$

In what follows, in analogy with the definition of Jensen-Shannon entropy, defined in (5.54), we will define, for $0 < \alpha < 1$ the α-Hellinger distance of f and h by

$$d_{H,\alpha}(f,h)^2 = \int_{\mathbb{R}} \left(\sqrt{f(v)} - \sqrt{\alpha f(v) + (1-\alpha)h(v)} \right)^2 \, dv, \tag{5.64}$$

and we will study the time-evolution of the square of the α-Hellinger distance between the solution $f(v,t)$ of the Fokker–Planck equation (5.2), and the equilibrium density $f_\infty(v)$, namely the square of the Hellinger distance between $f(v,t)$ and $g(v,t) = \alpha f(v,t) + (1-\alpha) f_\infty(v)$.

As in Section 5.3.3, we will assume that the initial density $f_0(v)$ (and consequently the solution $f(v,t)$), is smooth and has enough moments bounded. Moreover, since most of the computations that follow are analogous to the computations of Section 5.3.3, we will only outline the differences.

To compute the the evolution of the square of the α-Hellinger distance, let us first remark that

$$\int_{-\infty}^{0} \left(\sqrt{f(v)} - \sqrt{\alpha f(v) + (1-\alpha) f_\infty(v)} \right)^2 \, dv = \left(1 - \sqrt{\alpha}\right)^2 \int_{-\infty}^{0} f(v,t) \, dv.$$

Therefore, since according to (5.26) the mass in the negative half-line can not increase, and $\left(1-\sqrt{\alpha}\right)^2 > 0$, the part of the square of the α-Hellinger distance relative to the domain $v \le 0$ is nonincreasing in time.

On the set $(\gamma, +\infty)$ we have

$$\begin{aligned} &\frac{d}{dt} \int_{\gamma}^{\infty} (\sqrt{f} - \sqrt{g})^2 \, dv = \\ &\int_{\gamma}^{\infty} \left\{ \left(1 - \sqrt{\frac{g}{f}}\right) \frac{\partial^2}{\partial v^2}(v^2 f) + \left(1 - \sqrt{\frac{f}{g}}\right) \frac{\partial^2}{\partial v^2}(v^2 g) \right\} dv \\ &+ \int_{\gamma}^{\infty} \left\{ \left(1 - \sqrt{\frac{g}{f}}\right) \frac{\partial}{\partial v}[(v-1)f] + \left(1 - \sqrt{\frac{f}{g}}\right) \frac{\partial}{\partial v}[(v-1)g] \right\} dv. \end{aligned} \tag{5.65}$$

Using the identity $f/g = (v^2 f)/(v^2 g)$, that clearly holds when $v \in (\gamma, +\infty)$, integration by parts gives (cf. the proof of Proposition 25 in [49])

$$\begin{aligned} &\int_{\gamma}^{+\infty} \left\{ \left(1 - \sqrt{\frac{v^2 g}{v^2 f}}\right) \frac{\partial^2}{\partial v^2}(v^2 f) + \left(1 - \sqrt{\frac{v^2 f}{v^2 g}}\right) \frac{\partial^2}{\partial v^2}(v^2 g) \right\} dv = \\ &\left[\left(1 - \sqrt{\frac{v^2 g}{v^2 f}}\right) \frac{\partial}{\partial v}(v^2 f) + \left(1 - \sqrt{\frac{v^2 f}{v^2 g}}\right) \frac{\partial}{\partial v}(v^2 g) \right]_{\gamma}^{+\infty} - \frac{1}{2} \int_{\gamma}^{+\infty} v^2 \sqrt{fg} \left[\frac{\partial}{\partial v} \log \frac{f}{g} \right]^2 dv. \end{aligned} \tag{5.66}$$

Proceeding as in the proof of monotonicity of Jensen-Shannon entropy, the contribution of the border term at infinity in (5.66) can be easily shown to vanish provided $f(v,t)$ possesses moments bounded of order $3+\delta$ for some $\delta > 0$. Indeed, it is enough to follow the proof of Section 5.3.3 by choosing $p = q = 2$.

On the other extremal point, considering that both f and g are smooth, and $f^{1/2} \in L^2(\mathbb{R})$, one obtains

$$\lim_{v\to 0}(v^2 g)^{1/2}\left|\frac{\partial (v^2 f)^{1/2}}{\partial v}\right| = 0, \tag{5.67}$$

and, since f/g is bounded,

$$\lim_{v\to 0}\sqrt{\frac{f}{g}}\frac{\partial}{\partial v}(v^2 g) = 0. \tag{5.68}$$

Let us consider now the second integral into (5.65). Integrating by parts first on the interval $(\gamma, 1-\gamma)$, and using the identity $f/g = (v-1)f/[(v-1)g]$ we obtain

$$\begin{aligned}&\int_\gamma^{1-\gamma}\left(\left(1-\sqrt{\frac{g}{f}}\right)\frac{\partial}{\partial v}[(v-1)f] + \left(1-\sqrt{\frac{f}{g}}\right)\frac{\partial}{\partial v}[(v-1)g]\right)dv = \\ &\left[(v-1)(f+g) - 2(v-1)\sqrt{fg}\right]_\gamma^{1-\gamma}.\end{aligned} \tag{5.69}$$

Hence

$$\lim_{v\to 1}\left((v-1)(f+g) - 2(v-1)\sqrt{fg}\right) = 0,$$

and

$$\lim_{v\to 0}\left((v-1)(f+g) - 2(v-1)\sqrt{fg}\right) = -f(0)\left(1-\sqrt{\alpha}\right)^2.$$

This implies

$$\int_0^1\left((v-1)(f+g) - 2(v-1)\sqrt{fg}\right)dv = f(0)\left(1-\sqrt{\alpha}\right)^2.$$

Similar computations then give

$$\int_1^{+\infty}\left((v-1)(f+g) - 2(v-1)\sqrt{fg}\right)dv = 0.$$

Grouping the various pieces, we conclude that the square of the α-Hellinger distance is nonincreasing in time. We have

Theorem 24. *Let $f_0(v)$ be a smooth probability density in $\mathbb{R}$ satisfying (5.24), and such that its moments up to $3+\delta$ are finite for some $\delta > 0$. Then, for any $0 < \alpha < 1$, the α-Hellinger distance $d_{H,\alpha}(f(t), f_\infty)$ between the solution to the Fokker–Planck equation (5.2) and the equilibrium solution is monotonically nonincreasing, and the following decay holds*

$$d_{H,\alpha}(f(t), f_\infty) = d_{H,\alpha}(f_0, f_\infty) - \frac{1}{2}\int_0^t\int_0^{+\infty} v^2\sqrt{f(v,s)g(v,s)}\left[\frac{\partial}{\partial v}\log\frac{f(v,s)}{g(v,s)}\right]^2 dv\,ds. \tag{5.70}$$

Note that, at difference with the Jensen–Shannon entropy, the behavior of the solution at the point $v = 0$ does not enter into the expression of the entropy production.

As we shall see in the next Section, the monotonicity of Hellinger distance can coupled with the monotonicity of Jensen–Shannon entropy to obtain decay without rate of some α-Hellinger distance towards zero.

5.3.5 The decay of the α-Hellinger distance

In general, precise lower bounds for the entropy production of the Jensen–Shannon entropy are difficult to obtain. The main obstacle comes from the fact that, at difference with the case treated in [85], where the support of the initial value coincides with the support of the steady state a Chernoff-type inequality [30, 62, 49] connecting the relative entropy production (5.62) found in Theorem 23 with the Hellinger distance (5.64) (cf. [59, 49]) is not available here. Nevertheless, we can still resort to Chernoff inequality to obtain a convergence result in α-Hellinger distance. Thanks to the identity

$$f\left(\frac{\partial}{\partial v}\log\frac{f}{g}\right)^2 = f\left(\frac{1}{g}\frac{\partial}{\partial v}g - \frac{1}{f}\frac{\partial}{\partial v}f\right)^2 = \frac{(1-\alpha)^2}{\alpha^2}g\left(\frac{1}{g}\frac{\partial}{\partial v}g - \frac{1}{f_\infty}\frac{\partial}{\partial v}f_\infty\right)^2\frac{f_\infty^2}{fg}, \tag{5.71}$$

and to the upper bound

$$f = \frac{1}{\alpha}\cdot\alpha f \le \frac{1}{\alpha}g,$$

the integral part of the entropy production of the Jensen–Shannon entropy can be bounded as

$$\begin{aligned}
&\int_0^\infty v^2 f(v,s)\left[\frac{\partial}{\partial v}\log\frac{f(v,s)}{g(v,s)}\right]^2 dv = \\
&\frac{(1-\alpha)^2}{\alpha^2}\int_0^\infty v^2 g(v,s)\left[\frac{\partial}{\partial v}\log\frac{g(v,s)}{f_\infty(v)}\right]^2\frac{f_\infty(v)^2}{f(v,s)g(v,s)}\,dv \ge \\
&\frac{(1-\alpha)^2}{\alpha}\int_0^\infty v^2 g(v,s)\left[\frac{\partial}{\partial v}\log\frac{g(v,s)}{f_\infty(v)}\right]^2\left(\frac{f_\infty(v)}{g(v,s)}\right)^2 dv = \\
&4\,\frac{(1-\alpha)^2}{\alpha}\int_0^\infty v^2 f_\infty(v)\left[\frac{\partial}{\partial v}\sqrt{\frac{g(v,s)}{f_\infty(v)}}\right]^2\left(\frac{f_\infty(v)}{g(v,s)}\right)^2 dv = \\
&4\,\frac{(1-\alpha)^2}{\alpha}\int_0^\infty v^2 f_\infty(v)\left[\frac{\partial}{\partial v}\sqrt{\frac{f_\infty(v)}{g(v,s)}}\right]^2 dv.
\end{aligned} \tag{5.72}$$

For the last equality in (5.72) we refer to [49, 59].

We can now apply Chernoff inequality with weight, in the form proven in [49].

Theorem 25 ([49]). *Let X be a random variable distributed with density $f_\infty(v)$, $v \in I \subseteq \mathbb{R}$, where the probability density function f_∞ satisfies the differential equality*

$$\frac{\partial}{\partial v}\left(\kappa(v)f_\infty\right) + (v-m)\,f_\infty = 0, \quad v \in I. \tag{5.73}$$

If the function ϕ is absolutely continuous on I and $\phi(X)$ has finite variance, then

$$Var[\phi(X)] \le E\left\{\kappa(X)[\phi'(X)]^2\right\} \tag{5.74}$$

with equality if and only if $\phi(X)$ is linear in X.

We apply Theorem 25 with $I = \mathbb{R}_+$, $\kappa(v) = \sigma/(2\lambda)v^2$, and density $f_\infty(v)$, which is such that (5.73) holds in $\mathbb{R}_+$. Moreover

$$\phi(v) = \sqrt{\frac{f_\infty(v)}{g(v)}}.$$

By (5.74)

$$\begin{aligned}
&\int_0^\infty \kappa(v) f_\infty(v) \left[\frac{\partial}{\partial v}\sqrt{\frac{f_\infty(v)}{g(v,s)}}\right]^2 dv \geq \\
&\int_0^\infty \left[\sqrt{\frac{f_\infty(v)}{g(v,s)}} - \int_0^\infty \sqrt{\frac{f_\infty(w)}{g(w,s)}}\, f_\infty(w)\, dw\right]^2 f_\infty(v)\, dv = \\
&\|h(s)\|_{L_1}\left[1 - \left(\int_0^\infty \sqrt{\bar{h}(v,s) f_\infty(v)}\right)^2\right].
\end{aligned} \tag{5.75}$$

In (5.75) we defined

$$h(v,s) = \frac{f_\infty(v)^2}{g(v,s)}, \tag{5.76}$$

and with $\bar{h}(v,s)$ the probability density on $\mathbb{R}_+$ given by

$$\bar{h}(v,s) = \frac{h(v,s)}{\|h(s)\|_{L_1}}. \tag{5.77}$$

Note that, since by definition $f_\infty(v)/g(v,s) \leq 1/(1-\alpha)$, and by Cauchy–Schwartz inequality

$$1 = \int_0^\infty f_\infty(v)\, dv = \int_0^\infty \frac{f_\infty(v)}{\sqrt{g(v,s)}}\sqrt{g(v,s)}\, dv \leq \|h(s)\|_{L_1}^{1/2}\left(\int_0^\infty g(v,s)\, dv\right)^{1/2},$$

for all $s \geq 0$ it holds

$$1 \leq \|h(s)\|_{L_1} \leq \frac{1}{1-\alpha}. \tag{5.78}$$

On the other hand, as proven in [59], for any given pair of probability densities f, g

$$1 - \left(\int_0^\infty \sqrt{f(v)g(v)}\, dv\right)^2 \geq \frac{1}{2}\, d_H(f,g)^2. \tag{5.79}$$

In conclusion we obtain that on $\mathbb{R}_+$ the entropy production of the Jensen–Shannon entropy satisfies the lower bound

$$\int_0^\infty v^2 f(v,s)\left[\frac{\partial}{\partial v}\log\frac{f(v,s)}{g(v,s)}\right]^2 dv \geq \frac{\alpha\sigma}{4(1-\alpha)^2\lambda}\, d_H(h(s), \|h(s)\|_{L_1} f_\infty(s))^2. \tag{5.80}$$

Note that in (5.80) the coefficient is independent of time. Substituting into (5.62), (5.80) implies that

$$\int_0^\infty d_H(h(s), \|h(s)\|_{L_1} f_\infty(s))^2\, ds \leq \frac{\alpha\sigma}{2(1-\alpha)^2\lambda} H_\alpha(f_0, f_\infty).$$

Consequently, the sequence $\{d_H(h(t), \|h(t)\|_{L_1} f_\infty(t))\}_{t\geq 0}$ contains a subsequence $\{d_H(h(t_n), \|h(t_n)\|_{L_1} f_\infty(t_n))\}_{n\geq 0}$ such that, as $n \to \infty$, $t_n \to \infty$, and

$$\lim_{n\to\infty} d_H(h(t_n), \|h(t_n)\|_{L_1} f_\infty) = 0. \tag{5.81}$$

Now, consider that, for any given nonnegative L_1-functions $p(v)$ and $q(v)$, $v \in \mathbb{R}$ it holds

$$\begin{aligned}
&\int_{\mathbb{R}} |p(v) - q(v)| \, dv = \int_{\mathbb{R}} \left| \sqrt{p(v)} - \sqrt{q(v)} \right| \cdot \left| \sqrt{p(v)} + \sqrt{q(v)} \right| \, dv \leq \\
&\left[\int_{\mathbb{R}} \left(\sqrt{p(v)} - \sqrt{q(v)} \right)^2 \, dv \right]^{1/2} \cdot \left[\int_{\mathbb{R}} \left(\sqrt{p(v)} + \sqrt{q(v)} \right)^2 \, dv \right]^{1/2} \leq \\
&d_H(p,q) \left[2 \int_{\mathbb{R}} (p(v) + q(v)) \, dv \right]^{1/2} = \sqrt{2} \, d_H(p,q) \left(\|p\|_{L_1} + \|q\|_{L_1} \right)^{1/2}.
\end{aligned} \tag{5.82}$$

Hence, using (5.78) we obtain

$$\int_{\mathbb{R}} |h(t_n) - \|h(t_n)\|_{L_1} f_\infty| \, dv \leq \frac{2}{1-\alpha} d_H(h(t_n), \|h(t_n)\|_{L_1} f_\infty),$$

namely the L_1-convergence to zero of the sequence $\{h(t_n) - \|h(t_n)\|_{L_1} f_\infty\}_{n \geq 0}$. This implies that we can extract from the above sequence of times a subsequence, still denoted by t_n, such that on this subsequence

$$h(v, t_n) - \|h(t_n)\|_{L_1} f_\infty(v) \to 0 \qquad \text{a.s. in } \mathbb{R}_+.$$

Since $f_\infty(v) > 0$, $v \in \mathbb{R}_+$, it holds

$$\frac{f_\infty(v)}{\|h(t_n)\|_{L_1} g(v, t_n)} \to 1 \qquad \text{a.s. in } \mathbb{R}_+,$$

or, what is the same

$$\frac{f_\infty(v)}{\|h(t_n)\|_{L_1}} - g(v, t_n) \to 0 \qquad \text{a.s. in } \mathbb{R}_+.$$

Integrating on $\mathbb{R}_+$, and recalling that by Theorem 22

$$\int_0^\infty g(v, t_n) \, dv \to \int_0^\infty f_\infty(v) \, dv = 1,$$

shows that

$$\lim_{n \to \infty} \|h(t_n)\|_{L_1} \to 1. \tag{5.83}$$

The validity of (5.81) and (5.83) then imply

$$\lim_{n \to \infty} d_H(h(t_n), f_\infty) = 0. \tag{5.84}$$

Indeed

$$\begin{aligned}
&d_H(h(t_n), f_\infty)^2 = d_H(h(t_n), \|h(t_n)\|_{L_1} f_\infty)^2 + \\
&1 - \|h(t_n)\|_{L_1} + 2 \left(\|h(t_n)\|_{L_1} f_\infty)^{1/2} - 1 \right) \int_0^\infty \sqrt{h f_\infty} \, dv,
\end{aligned} \tag{5.85}$$

and

$$\int_0^\infty \sqrt{h f_\infty} \, dv = \int_0^\infty \sqrt{\frac{f_\infty}{g(t_n)}} f_\infty \, dv \leq \frac{1}{\sqrt{1-\alpha}},$$

Last, consider that

$$d_H(h(t_n), f_\infty)^2 = \int_0^\infty \left(\frac{f_\infty}{\sqrt{g(t_n)}} - \sqrt{f_\infty} \right)^2 dv = \\ \int_0^\infty \frac{f_\infty}{g(t_n)} \left(\sqrt{g(t_n)} - \sqrt{f_\infty} \right)^2 dv \geq \int_{\{f_\infty \geq g(t_n)\}} \left(\sqrt{g(t_n)} - \sqrt{f_\infty} \right)^2 dv. \tag{5.86}$$

Thanks to (5.82), from (5.86) we obtain

$$d_H(h(t_n), f_\infty) \geq \frac{1}{2} \int_{\{f_\infty \geq g(t_n)\}} (f_\infty - g(t_n))\, dv. \tag{5.87}$$

Taking into account that both f_∞ and $g(t_n)$ are probability density functions, it holds

$$\int_{\{f_\infty \geq g(t_n)\}} (f_\infty - g(t_n))\, dv = \frac{1}{2} \|g(t_n) - f_\infty\|_{L_1}. \tag{5.88}$$

Also, since for $a > b > 0$

$$(a-b)^2 \leq a^2 - b^2,$$

one obtains easily the inequality

$$d_H(g(t_n), f_\infty)^2 \leq \|g(t_n) - f_\infty\|_{L_1}. \tag{5.89}$$

Grouping all these inequalities we finally get

$$d_H(h(t_n), f_\infty) \geq \frac{1}{4} \|g(t_n) - f_\infty\|_{L_1} \geq \frac{1}{4} d_H(g(t_n), f_\infty)^2. \tag{5.90}$$

It follows that, along the subsequence $\{t_n\}_{n \geq 0}$

$$\lim_{n \to \infty} d_H(g(t_n), f_\infty) = \lim_{n \to \infty} d_{H,\alpha}(f(t_n), f_\infty) = 0 \tag{5.91}$$

However, in view of Theorem 24, the sequence $d_{H,\alpha}(f(t), f_\infty)$, $t \geq 0$ is monotonically nonincreasing. This implies that the whole sequence converges to zero as time goes to infinity.

Theorem 26. *Let $f_0(v)$ be a smooth probability density in $\mathbb{R}$ satisfying (5.24), and such that its moments up to $3 + \delta$ are finite for some $\delta > 0$. Then, for $0 < \alpha < 1$, the solution to the Fokker–Planck equation (5.2) converges towards the equilibrium density f_∞ in α-Hellinger distance.*

Theorem 26 has important consequences. First, in view of the inequality

$$\|f - g\|_{L_1} \leq 2 d_H(f, g),$$

that holds for any pair of probability densities f, g, we get, for $0 < \alpha < 1$

$$(1-\alpha) \int_{\mathbb{R}} |f(v) - g(v)|\, dv = \int_{\mathbb{R}} |f - (\alpha f(v) + (1-\alpha) g(v))|\, dv \leq d_{H,\alpha}(f, g).$$

Hence, under the same conditions of Theorem 26 the convergence to zero in α-Hellinger distance implies the L_1-convergence of the solution to the Fokker–Planck equation (5.2) towards its equilibrium density.

Moreover, as proven in [85], Lemma 3.3, the condition of smoothness of the initial value can be dropped as soon as convergence is restricted to L^1.

Corollary 27. *Let $f_0(v)$ be a probability density in $\mathbb{R}$ satisfying (5.24), and such that its moments up to $3 + \delta$ are finite for some $\delta > 0$. Then, the solution to the Fokker–Planck equation (5.2) converges towards the equilibrium density f_∞ in L^1.*

5.4 Conclusions

The Fokker–Planck equation (5.2) studied in this paper appears as a useful and consistent model to study the evolution in time of the distribution of wealth in a population, even in the realistic case in which part of the agents can have debts. If the total mean wealth of the population is positive, it is shown that the unique equilibrium density, supported in half-line of positive wealths, is still attracting any density, with part of the mass located on the negative half-line. At difference with the case studied in [85], where convergence to the equilibrium density has been shown in L^1-norm, here convergence with rate has been proven only in terms of a Fourier-based metric, equivalent to the weak*-convergence of measures. A rigorous study of the time evolution of relative entropy functionals, Jensen–Shannon entropy [66] and α-Hellinger distance, shows that these functionals are monotonically nonincreasing in time, and can be coupled to furnish convergence without rate in α-Hellinger distance and consequently in L_1. A challenging problem which remains open is to be able to quantify the rate of decay of the solution with respect to the L_1-norm.

Part II

Elo rating model

Introduction

As briefly introduced above, the word Econophysics arises from the merge between Economics and Physics. Unexpectedly, its fields off application are various and not only related to Economics. Historically speaking, economical phenomena have been the first to be investigated by Econophysics. However, in the last three decades new econophysical models have been developed for human phenomena, such as opinion formation and knowledge development. One of the first work in this field was presented by S.Galam et al. in 1982 [52], which used the word *Sociophysics* in this context. In this work, they describe the use of physics techniques in the explanation of collective behaviour of worker in a plant, which is based on the analysis of a discomfort function. The model is characterized by the presence of a critical point in which the system can go from an "individual phase" to a "collective phase". The collective phase is split into two regions, a "work state" and a "strike state". Near to the critical point, either changes in the parameters or a perturbation of the system can cause relevant changes in the state of the plant. The authors in [52] individuated the existence of metastable states of equilibruim. Indipendently from its real possibility of application, this model presents an intresting approach. Starting from this first paper, in the last three decades Galam produced a huge number of works in the field of Sociophysics. I refer to [51] and its reference. In [51], the author underlines the connections between works that are related to very different problems, such as political and social issues. Furthermore, he underlines how the study of sociophysics models leads to develop new results in statistical mechanics. Despite of its field of application, we can consider Sociophysics as a branch of Econophysics.
Fokker-Plack equation plays an important role also in the application of Econophysics to social sciences. In this context, it is possible to recall, as examples, opinion formation and knowledge development models. A Fokker-Planck description of opinion formation is in [88]. In this model, the opinion of an agent is a compactly supported variable, i.e. $v \in [-1, 1]$. In [88], the author assumes that it is not possible to cross the boundaries of the domain. Fokker-Planck equation arises from the balance between the compromise effects and the tendency to conserve an opinion, that is higher if the opinion is more extreme. Due to this assumption, the diffusion terms depends on the absolute value of the opinion. The most important goals of this work are to obtain a closed form for the evolution of the moments and to show a rigorous proof of the validity of limit procedure to obtain a Fokker-Planck equation in a compact support. This work represents an important starting point for the analysis of the model when the domain is bounded (see also Section 6.4 for its use).
Among the problems of interest of Econophysics, we can include the rating system of competitive sports, such as chess or football. In this field, one of the most important models is the Elo Model, introduced by Arpad Elo in 1950 for chess players [44]. The next chapter is the transposition of a paper that I wrote toghether with Professor B.Düring and Dr. Marie-Therese Wolfram [40]. In this work there is a generalization of of the Elo Model. In section 6.2, I will add a brief presentation of the work of Junca and Jabin [58], that represents the first rigorous mathematical analysis of the Elo Model.

Chapter 6

On a Kinetic Elo rating model for players with dynamical strength

6.1 Introduction

In 1950 the Hungarian physicist Arpad Elo developed a rating system to calculate the relative skill level of players in competitor versus competitor games, see [44]. The Elo rating system was initially used in chess competitions, but was quickly adopted by the US Chess Federation as well as the World Chess Federation, and the National Football Foundation. In June 2018, FIFA announced switching their world football ranking to an Elo system, following two years of reviews and studies of different alternatives. The Elo rating system assigns each player a rating, which is updated according to the wins and losses as well as the difference of the ratings. It is hoped that the rating converges to the relative strength level and is a valid measure of the player's skills. However, assigning an initial rating to a new player is a delicate issue, since it is not clear how an inaccurate initial rating influences the latter performance. Elo himself tried to validate the model using computational experiments, while Glickman used statistical techniques to understand the dynamics [54]. The first rigorous proof of convergence of the ratings to the individual strength was presented by Junca and Jabin in [58], who introduced a continuous version of the Elo rating system. In this continuous model every player is characterised by its intrinsic strength ρ and their rating R. The intrinsic strength is fixed in time. If two players with rating R_i and R_j meet in a game, their ratings after the game, R_i^* and R_j^* are given by

$$R_i^* = R_i + K(S_{ij} - b(R_i - R_j)), \tag{6.1a}$$

$$R_j^* = R_j + K(-S_{ij} - b(R_j - R_i)). \tag{6.1b}$$

In (6.1) the random variable S_{ij} is the score result of the game, it takes the value 1 if player i wins and the value -1 if player j wins. The mean score (i.e. expected value of S_{ij}) is assumed to be equal to $b(\rho_i - \rho_j)$, hence the result of each game depends on the difference of the player's intrinsic strengths. The rating of each player in- or decreases proportionally with the outcome of the game, relative to the predicted mean score $b(R_i - R_j)$. The speed of the adjustment is controlled by the constant parameter K. The function b is chosen in such a way that extreme differences are moderated; a typical choice is

$$b(z) = \tanh(cz), \tag{6.2}$$

where c is a suitably chosen positive constant. This choice weighs the impact of the outcome with respect to the relative rating. If a player with a high rating wins a game against a player with a low rating, the

players' ratings change little. However, if the player with the low rating wins against a the highly rated player, the ratings are strongly adjusted.

Junca and Jabin proposed the following equation to describe the evolution of the distribution of players $f = f(r, R, t)$ with respect to their strengths and ratings

$$\partial_t f(\rho, r, t) + \partial_r(a(f)f) = 0 \text{ with } a(f) = \int_{\mathbb{R}^2} w(r - r')(b(\rho - \rho') - b(r - r'))f(t, r', \rho')d\rho' dr'. \tag{6.3}$$

This equation describes a more general setup than in the microscopic equations. Here two players only interact according to the interaction rate function w, which depends on the difference of their ratings. The function w is assumed to be an even and nonnegative. Junca and Jabin analysed the long time behaviour of solutions to (6.3). They proved that in the case $w = 1$, a so-called 'all-play-all' tournament, the ratings converge exponentially fast to the intrinsic strength. In the case of local interactions, that is individuals only play if their ratings are close, the ratings may not converge to the intrinsic strength and the rating fails to give a fair representation of the player's strength distribution.

Rather recently Krupp [64] proposed an extension of the model by Jabin and Junca [58]. In her model not only the rating, but also the intrinsic strength changes as players continuously compete in games. In particular, she assumes that the intrinsic strength ρ changes in every game according to

$$\rho_i^* = \rho_i + Z_{ij}\tilde{K}, \tag{6.4a}$$

$$\rho_j^* = \rho_j + Z_{ij}\tilde{K}, \tag{6.4b}$$

where $\tilde{K}$ is a positive constant and Z_{ij} takes the value $z_1 \in \mathbf{N}$ and $z_2 \in \mathbf{N}$ depending on which player wins. If $z_1 < z_2$ the looser benefits more from the game, while if $z_1 > z_2$ the winner learns more. If $z_1 = z_2$ both learn the same. The corresponding equation for the distribution of the players $f = f(r, \rho, t)$ with respect to their strength and rating reads as

$$\partial_t f(r, \rho, t) + \partial_r(a(f)f) + \partial_\rho(c(f)f) = 0, \tag{6.5}$$

where

$$a(f) = \int_{\mathbb{R}^2} w(r - r')[b(\rho - \rho') - b(r - r')]f(r', \rho', t)d\rho' dr'$$

and

$$c(f) = \int_{\mathbb{R}^2} w(r - r')[\frac{z_1}{2}(b(\rho - \rho') + 1) - \frac{z_2}{2}(b(r - r') - 1)]f(r', \rho', t)d\rho' dr'.$$

Krupp analysed the qualitative behaviour of solutions to (6.5). Due to the continuous increase in strength, the ratings increase in time. Therefore, an appropriately shifted problem was studied, in which the ratings converged exponentially fast to the intrinsic strength in the case $w = 1$.

In this paper we propose a more general approach to describe how a player's strength changes in encounters. We assume that individuals benefit from every game and increase their strength because of these interactions. However, the extent of the benefit depends on several factors – first, players with a lower ratings benefit more. Second, the stronger the opponent, the more a win pushes the intrinsic strength. Furthermore, the individual performance changes due to small fluctuations, accounting for variations in the mental strength or personal fitness on a day. Based on the microscopic interaction laws we derive the corresponding kinetic Boltzmann type and limiting Fokker-Planck equations and analyse their behaviour. In the case of no diffusion we can show that the strength and ratings of the appropriately

shifted PDE converge, while we observe the formation of non-measure valued steady states in the case of diffusion. We illustrate our analytic results with numerical simulations of the kinetic as well as the limiting Fokker-Planck equation. The simulations give important insights into the dynamics, especially in situations where we are not able to proof rigorous results.

The proposed interaction laws are a first step to develop and analyse more complicated rating models with dynamic strength. The next developments of the model should include losses in the player's strength to ensure that the strength stays within certain bounds.

The kinetic description of the Elo rating system gave novel insights into the qualitative behaviour of solutions. In the last decades kinetic models have been used successfully to describe the behaviour of large multiagent systems in socio-economic applications. In all these applications interactions among individuals are modeled as 'collisions', in which agents exchange goods [34, 36, 21], wealth [41, 38, 11, 33], opinion [88, 18, 37, 75, 2, 43] or knowledge [77, 22]. For a general overview on interacting multi-agent systems and kinetic equations we refer to the book of Pareschi and Toscani [76].

This paper is organised as follows. We introduce a generalization of the kinetic Elo model with variable intrinsic strength due to learning in Section 6.3. In Section 6.4 we derive the corresponding Fokker-Planck type equation as the quasi-invariant limit of the Boltzmann type model. Convergence towards steady states of a suitable shifted Fokker-Planck model is analysed in Section 6.5. We conclude by presenting various numerical simulations of the Boltzmann and the Fokker-Planck type equation in Section 6.6.

6.2 Previous result on Elo model

The first rigouros result on Elo model was obtained by P.E. Jabin and S.Junca in 2014 [58]. In this work, the authors assume that each player i has a constant intrinsic strength ρ_i. A rating system is valid if, after a certain number of matches, his ranking R_i converges (or is close enough) in some sense to the strength. The authors of [58] considered two different cases:

(i) "All meet all case", in which every match between two players is possible;

(ii) "Local interactions", in which a match between two players is allowed only if their rankings are close enough.

In the case (i), the function $w(\rho)$, that appears in (6.3), is strictly positive for each $\rho \in \mathbb{R}$ and for simplicity we can assume without loss of generality $w \equiv 1$. The case (ii) is characterized by a choice of a compactly supported $w(\cdot)$, as, for example,

$$w > 0 \text{ on } (-1,1), \quad w(\rho) = 0 \quad \forall |\rho| \geq 1.$$

The derivation of the model in [58] is different from the limit procedure of Chapters 4 and 5 and it is beyond the scope of this section.
A first analysis in [58] shows some properties of the solution of (6.3), such as mass and mean preservation. Among these, the preservation of the average value of ranking is equivalent to the invariance with respect to traslation. This result allows to normalize these invariants,

$$\int_{\mathbb{R}^2} f(\rho, r, t) d\rho dr = 1,$$
$$\int_{\mathbb{R}^2} \rho f(\rho, r, t) d\rho dr = \int_{\mathbb{R}^2} r f(\rho, r, t) d\rho dr = 0.$$

In order to study the convergence towards the equilibrium, the authors in [58] introduced the family of entropies

$$E_\phi(t) = \int_{\mathbb{R}^2} \phi(\rho - r) f(\rho, r, t) d\rho dr,$$

where ϕ is a convex function. By direct computation, due to the symmetry of $b(\cdot)$, the previous functional decays in time if $\phi \in C^1(\mathbb{R})$, i.e.

$$\frac{d}{dt} E_\phi(t) \leq 0, \quad \forall \phi \in C^1(\mathbb{R}). \tag{6.6}$$

It is useful to underline that the rate of decay depends on the choice of the function $w(\cdot)$. Anyway, inequality (6.6) allows to prove a uniform control in time of the compact support of the solution.

Corollary 28. *[58] If $f_0(\rho, r)$ is compactly supported in $[-\alpha, \alpha] \times [-\beta, \beta]$, then for all t, $(\rho, r) \mapsto f(\rho, r, t)$ is compactly supported in $[-\alpha, \alpha] \times [\beta - 2\alpha, \beta + 2\alpha]$.*

Among all possible choices of $\phi(\cdot)$, the most relevant for further analysis is $\phi(\rho - r) = (\rho - r)^2$, which allows to define the Energy $E(t)$ as

$$E(t) = \int_{\mathbb{R}^2} (\rho - r)^2 f(\rho, r, t) d\rho dr.$$

Results on "All meet all" case.

The aim of the authors in [58] was to prove that with the choice $w > 0$ the solution converge exponentially to a steady state of the form $f_\infty(\rho, r) = h(\rho)\delta(\rho - r)$. Let us assume that $w(\cdot)$ is smooth enough, i.e. $w \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and $f_0(\rho, r)$ is compactly supported. Let us indicate with $\mathcal{K}$ the uniformly support obtained in corollary 28. Due to hypotesis on $w(\cdot)$, $\inf_{\mathcal{K}} w \geq \underline{w} > 0$. By direct computation, it results

$$\frac{d}{dt} E(t) = -\int_{\mathbb{R}^4} (r - r' + \rho - \rho')(b(\rho - \rho') - b(r - r')) f' f d\rho' dr' d\rho dr,$$

where $f = f(\rho, r, t)$ and $f' = f(\rho', r', t)$. Since b is non decreasing and due to the assumption on w, we obtain

$$\frac{d}{dt} E(t) \leq -C\underline{w} \int_{\mathbb{R}^2} |\rho - r|^2 f(\rho, r, t) d\rho dr.$$

The following results follows from the previous computation by using Gronwall's inequality.

Proposition 29. *[58] If $f_0(\rho, r)$ is compactly supported and w is positive everywhere then the energy associated to the solution of (6.3) decays exponentially*

$$E(t) \leq E(0) exp\{-2C\underline{w}t\},$$

where C depends on the support of f_0 and $\underline{w}$ depends on the support of f_0 and the function $w(\cdot)$.

Propostion 29 implies that, for $t \to +\infty$, $f(\rho, r, t) \to f_\infty(\rho, r)$ and the support of f_∞ is the diagonal $\rho = r$. Hence, the steady state could be written as

$$f_\infty(\rho, r) = h(\rho)\delta(\rho - r).$$

Furthermore, the total intrinsic strength is preserved, i.e.

$$\int_{\mathbb{R}} f(\rho, r, t) dr = \int_{\mathbb{R}} f_0(\rho, r) dr,$$

and this equality is valid also in the limit $t \to +\infty$. So, the unique possibility is that

$$h(\rho) = \int_{\mathbb{R}} f_0(\rho, r)dr.$$

The steady state is unique and well identified. The authors also computed directly the rate of convergence to equilibrium state in the norm $W^{-1,1}$, defined as

$$\|f\|_{W^{-1,1}} = \sup_{\|\phi\|_{W^{1,\infty}} \leq 1} \int_{\mathbb{R}^2} \phi(\rho, r) f(\rho, r) d\rho dr,$$

and they estabilished an exponential rate that depends on energy decay

$$\|f - f_\infty\|_{W^{-1,1}} \leq \sqrt{E(t)}.$$

All previous results suggest that the rating of each player in the Elo system can precisely describe his strength. However, the exponential rate of convergence indicates that a good estimation is achieved in a short time.

Result on "Local interactions" case.

In the case of "Local interactions", an entropy or energy argument does not perform well as in the "All meet all" case, properly because in this case $\inf_{\mathcal{K}} w \geq \underline{w} > 0$. Hence, the convergence to an equilibrium along the diagonal is obtained with a very different strategy. The details of the proof are beyond the scopes of this section, thus I will report only a sketch. The authors of [58] assumed that the interaction function is regular enough, i.e. $w \in C^2(\mathbb{R}^2)$. They also assumed that f_0 is compactly supported and indicated with $R_\rho \times R_r$ the uniform support of the solution. Their analysis starts with the study of the properties of the set $\Omega \subset M^1(R_\rho \times R_r)$ of all the (weak*-)limit of a certain sequence of measures. Firstly, the authors proved that each element $\overline{f}$ of the set Ω is a sum of probability measures, supported in a line of the plane $\rho - r$, i.e.

$$\overline{f}(\rho, r) = \sum_{i=1}^{n} \delta(r - (\rho + c_i))h_i(\rho), \ \forall \overline{f} \in \Omega.$$

Furthermore, in each strip $R_\rho \times (r-1, r+1)$ there is at most one i such that the previous equivalence holds. Secondly, the authors proved that, for this function, the entropy is constant in time, i.e.

$$\frac{d}{dt} E_\phi(t) = 0, \ \forall \overline{f} \in \Omega, \forall \phi \in C^2(\mathbb{R}) \text{ convex}.$$

It also implies that

$$\int_{\mathbb{R}} \overline{f}(\rho + r, r) d\rho = g_\infty(r).$$

These results identify the structure of the limit probability measure of set Ω. Furthermore, the support of the possible $g_\infty(r)$, that depends on the initial data, is an attractive set for the total mass. In other words, the mass at time $t > T$ is concentrated along a tubular neighborhood of all the diagonals. The uniqueness of the element $\overline{f} \in \Omega$ completes the proof. We can summarize the results in the following

Theorem 3. *[58] Assume that $w \in C^2(\mathbb{R})$ and that $f_0(\rho, r)$ is compactly supported then there exists distinct constants $c_1, \ldots, c_n$ and $M^1(\mathbb{R})$ measures $h_1, \ldots, h_n$ with disjoint supports s.t.*

$$f(\rho, r, t) \to \sum_{i=1}^{n} h_i(\rho)\delta(r - \rho - c_i) \quad \textit{in weak-* } M^1(\mathbb{R}^2), \ \textit{as } t \to +\infty.$$

Thus, the authors in [58] concluded that, in the "Local interactions" case, if two players interact then they must have the same constant c_i. Moreover, the Elo system does not perform well if there is a gap in the initial distribution, that means that a group of players is isolated.

6.3 An Elo model with learning

In this section we introduce an Elo model, in which the rating and the intrinsic strength of the players change in time. The dynamics are driven by similar microscopic binary interactions as in the original model by Jabin and Junca [58] and Krupp [64]. We state the specific microscopic interaction rules in each encounter and derive the corresponding limiting Fokker-Planck equation.

6.3.1 Kinetic model

We follow the notation introduced in Section 6.1 and denote the individual strength by ρ and the rating by R. If two players with ratings R_i and R_j meet, their ratings and strength after the game are given by:

$$R_i^* = R_i + \gamma(S_{ij} - b(R_i - R_j)), \tag{6.7a}$$
$$R_j^* = R_j + \gamma(-S_{ij} - b(R_j - R_i)), \tag{6.7b}$$
$$\rho_i^* = \rho_i + \gamma h(\rho_j - \rho_i) + \eta, \tag{6.7c}$$
$$\rho_j^* = \rho_j + \gamma h(\rho_i - \rho_j) + \tilde{\eta}. \tag{6.7d}$$

The interaction rules are motivated by the following considerations: player ratings change with the outcome of each game (as in the original model (6.1) proposed by Jabin and Junca [58]). The random variable S_{ij} corresponds to the score of the match and depends on the difference in strength of the two players. We assume that S_{ij} takes the values ± 1 with an expectation $\langle S_{ij} \rangle = b(\rho_i - \rho_j)$. Note that one could also assume that S_{ij} is continuous, for example $S_{ij} \in [-1, +1]$. The constant parameter $\gamma > 0$ controls the speed of adjustment.

The variables η and $\tilde{\eta}$ are independent identically distributed random variables with mean zero and variance σ^2 which model small fluctuations due to day-linked performance in the mental strength or personal fitness.

The function h describes the learning mechanism. We assume that h takes the following form,

$$h(\rho_j - \rho_i) = \big[\alpha h_1(\rho_j - \rho_i) + \beta h_2(\rho_j - \rho_i)\big]. \tag{6.8}$$

The function h_1 corresponds to the increase in knowledge or skills because of interactions. We assume that each player learns in a game, however players with a lower strength benefit more. A possible choice for h_1, which we shall use throughout this paper, is

$$h_1(\rho_j - \rho_i) = 1 + b(\rho_j - \rho_i), \tag{6.9}$$

where b is given by (6.2). Note that b is an odd function. Since h_1 is positive, both players are able to learn and improve in each game, to an extent which depends on the difference in strengths, with a player with lower strength benefiting more.

The second function, h_2, models a change of strength due to gain or loss of self-confidence due to winning or being defeated in a game. We assume that the loss of the stronger player is the same as the gain for the weaker one. Hence, we choose $h_2(\rho_j - \rho_i) = S_{ij} l(\rho_j - \rho_i)$ to be an odd, regular,

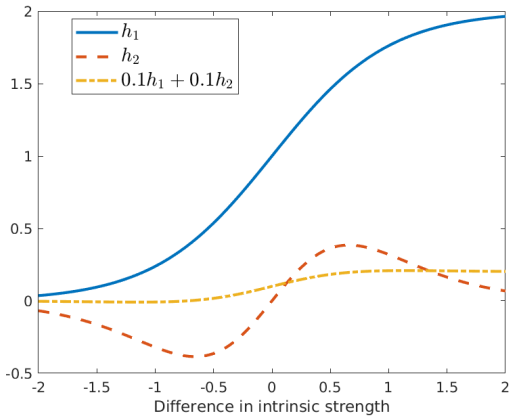


Figure 6.1: Possible choices of h_1 and h_2.

bounded function which is vanishing at infinity, where the function l corresponds to the net change of self-confidence. A possible choice which we adopt in the following corresponds to

$$h_2(\rho_j - \rho_i) = S_{ij}[1 - \tanh^2(\rho_j - \rho_i)]. \tag{6.10}$$

Note that the expectation for the learning function function is given by

$$\langle h(\rho_j - \rho_i)\rangle = \big[\alpha h_1(\rho_j - \rho_i) + \beta\langle h_2(\rho_j - \rho_i)\rangle\big] = \big[\alpha h_1(\rho_j - \rho_i) + \beta b(\rho_i - \rho_j)(1 - \tanh^2(\rho_j - \rho_i))\big]. \tag{6.11}$$

Figure 6.1 shows the function h_1, $\langle h_2\rangle$ and $\langle h\rangle$ for the particular choice of $\alpha = \beta = 0.1$ and $c = 1$. If $\alpha > \beta$ players always improve in strength. In this case the strength and subsequently the rating will always increase in time. We see that, as in the original Elo model, the choices of interaction rules and the function $b(\cdot)$ preserve the total value of the rating pointwise and in mean, that is

$$\langle R_i^* + R_j^*\rangle = R_i + R_j.$$

The evolution of the total strength depends on the choices of the function h_1 and h_2. Note that the function h_2 does not affect the total strength since

$$\langle \rho_j^* + \rho_j^*\rangle - (\rho_j + \rho_j) = 2\gamma\alpha.$$

We see that that the proposed interaction rules result in a net increase of the total knowledge in every interactions. Therefore, we expect to see on overall increase in strength for all times.

Now we are able to state the evolution equation for the distribution of players $f_\varepsilon = f_\varepsilon(\rho, R, t)$ with respect to their rating R and intrinsic strength ρ. The evolution of f_ε can be described by the following Boltzmann type equation which can be obtained by standard methods of kinetic theory:

$$\begin{aligned}\frac{d}{dt}\int_\Omega \phi(\rho_i, R_j) f_\varepsilon(\rho_i, R_i, t) dR_i d\rho_i = \frac{1}{2}\Bigg\langle \int_\Omega\int_\Omega \Big(\phi(\rho_i^*, R_j^*) + \phi(\rho_j^*, R_j^*) - \phi(\rho_i, R_i) - \phi(\rho_j, R_j)\Big) \\ \times w(R_i - R_j) f_\varepsilon(\rho_i, R_i, t) f_\varepsilon(\rho_j, R_j, t)\, dR_j d\rho_j dR_i d\rho_i \Bigg\rangle,\end{aligned} \tag{6.12}$$

where $\phi(\cdot)$ is a (smooth) test function, with support $\mathrm{supp}(\phi) \subseteq \Omega$. The function $w(\cdot)$ corresponds to the interaction rate function which depends on the difference of the ratings. If $w \equiv 1$ we consider a so-called *all-play-all* game. If w has compact support, we consider a so-called *Swiss-system tournament*, where only subsets of players compete with each other. Possible choices for w are

$$w(R_i - R_j) = e^{\frac{\log 2}{1+(\rho_j-\rho_i)^2}} - 1 \text{ or } w(R_i - R_j) = \chi_{\{|R_i - R_j| \leq c\}}. \tag{6.13}$$

where χ denotes the indicator function.

In the following we shall analyse (6.12) as well as different asymptotic limits of it. The presented analysis is based on the following assumptions:

($\mathcal{A}1$) Let $\Omega = \mathbb{R}^2$ or a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$.

($\mathcal{A}2$) Let $f_0 \in H^1(\Omega)$ with $f_0 \geq 0$ and compact support. Furthermore we assume that it has mean value zero, and bounded moments up to order two. Hence

$$\int_\Omega f_0(\rho, R)\, d\rho dR = 1, \quad \int_\Omega R f_0(\rho, R)\, d\rho dR = 0, \text{ and } \int_\Omega \rho f_0(\rho, R)\, d\rho dR = 0.$$

($\mathcal{A}3$) The random variables $\eta, \tilde{\eta}$ in (6.7) have the same distribution, zero mean, $\langle \eta \rangle = 0$, and variance σ_η^2.

($\mathcal{A}4$) Let the interaction rate function w be an even non-negative function with $w \in C^2(\Omega) \cap L^\infty(\Omega)$.

The kinetic Elo model can be formulated on the whole space as well as on a bounded domain. In reality, the Elo ratings of top chess players vary between 2000 to 3000, which provides evidence for the assumption of a bounded domain Ω. However, sometimes it is easier to study the dynamics of models on the whole space, i.e. without boundary effects. We will generally work on the bounded domain, and clearly state where we deviate from this assumption, e.g. when we study the asymptotic behaviour of moments. The second assumption states the necessary regularity assumptions on the initial data, which we shall use in the analysis of the moments and the existence proof. We assume that the interaction rate function among individuals is symmetric and bounded from above which implies the last assumption.

6.3.2 Analysis of the moments

We start by studying basic properties of the Boltzmann type equation (6.12) such as mass conservation and the evolution of the first and second moments with respect to the strength and the ratings. Throughout this section we consider the problem in the whole space.

Conservation of mass:

Setting $\phi(\rho_i, R_i) = 1$ in the equation (6.12) we see that

$$\frac{d}{dt} \int_{\mathbb{R}^2} f_\varepsilon(\rho_i, \mathbb{R}, t)\, dR d\rho = 0.$$

Therefore, the total mass is conserved, that is

$$\int_{\mathbb{R}^2} f_\varepsilon(R, \rho, t)\, dR d\rho = 1, \quad \text{for all times } t \geq 0. \tag{6.14}$$

Moments with respect to the rating.

The s-th moment, for $s \in \mathbb{N}$, with respect to R_i is defined as

$$m_{R_i}(t) = \int_{\mathbb{R}^2} R_i f_\varepsilon(\rho_i, R_i, t)\, dR_i d\rho_i \text{ and } \mathcal{M}_{s,R_i}(t) = \int_{\mathbb{R}^2} {R_i}^s f_\varepsilon(\rho_i, R_i, t)\, dR_i d\rho_i,$$

where $m_{R_i}(t) = \mathcal{M}_{1,R_i}$. We choose $\phi(\rho_i, R_i) = R_i$. Due to $(\mathcal{A}2)$ and the symmetry of $b(\cdot)$ we obtain

$$\begin{aligned}\frac{d}{dt} m_{R_i}(t) =& \frac{1}{2}\gamma \int_{\mathbb{R}^4} f_\varepsilon(\rho_i, R_i, t) f_\varepsilon(\rho_j, R_j, t) \times \\ &\times (b(\rho_i - \rho_j) - b(R_i - R_j) + b(\rho_j - \rho_i) - b(R_j - R_i)) w(R_i - R_j)\, dR_j d\rho_j dR_i d\rho_i = 0.\end{aligned}$$

Hence the mean value w.r.t. the rating is preserved in time and therefore

$$m_{R_i}(t) = 0, \quad \text{for all times } t \geq 0.$$

The evolution of the second moment can be obtained by setting $\phi(\rho_i, R_i) = {R_i}^2$. We see that

$$\begin{aligned}\frac{d}{dt}\mathcal{M}_{2,R_i}(t) =& \frac{1}{2}\int_{\mathbb{R}^4} f_\varepsilon(\rho_i, R_i, t) f_\varepsilon(\rho_j, R_j, t) w(R_i - R_j) \times \\ &\times \Bigg[\gamma^2 \Big(\big(b(\rho_i - \rho_j) - b(R_i - R_j)\big)^2 + \big(b(\rho_j - \rho_i) - b(R_j - R_i)\big)^2 \Big) \\ &+ 2\gamma \Big(R_i(b(\rho_i - \rho_j) - b(R_i - R_j)) + R_j(b(\rho_j - \rho_i) - b(R_j - R_i)) \Big)\Bigg] dR_j d\rho_j dR_i d\rho_i.\end{aligned}$$

The second term in the integral is non-positive and we obtain the bound

$$\frac{d}{dt}\mathcal{M}_{2,R_i}(t) \leq 4\gamma^2 \|b\|_\infty^2.$$

Hence, the second moment grows at most linearly and remains bounded for finite times. Note that the integral is negative for γ small enough, which implies a decreasing second moment.

Moments with respect to the strength

The moments with respect to strength are defined in an analogous way, that is

$$m_{\rho_i}(t) = \int_{\mathbb{R}^2} \rho_i f(\rho_i, R_i, t)\, dR_i d\rho_i \text{ and } \mathcal{M}_{s,\rho_i}(t) = \int_{\mathbb{R}^2} {\rho_i}^s f(\rho_i, R_i, t)\, dR_i d\rho_i,$$

for $s \in \mathbb{N}$ and using again $m_{\rho_i}(t) = \mathcal{M}_{1,\rho_i}$. Since $(\mathcal{A}2)$ holds, we see that for $\phi(\rho_i, R_i) = \rho_i$, we have

$$\frac{d}{dt} m_{\rho_i}(t) = \frac{1}{2}\gamma \int_{\mathbb{R}^4} f_\varepsilon(\rho_i, R_i, t) f_\varepsilon(\rho_j, R_j, t) w(R_i - R_j) [\langle h(\rho_j - \rho_i) + h(\rho_i - \rho_j)\rangle]\, d\rho_j dR_j d\rho_i dR_i.$$

Therefore,

$$-\gamma \|\langle h\rangle\|_\infty \leq \frac{d}{dt} m_{\rho_i}(t) \leq \frac{1}{2}\gamma \int_{\mathbb{R}^4} 2\|\langle h\rangle\|_\infty f_\varepsilon(\rho_i, R_i, t) f_\varepsilon(\rho_j, R_j, t) d\rho_j dR_j d\rho_i dR_i \leq \gamma \|\langle h\rangle\|_\infty, \tag{6.15}$$

which implies that the mean value is bounded for all times $t \in [0,T]$ and that $|m_{\rho_i}(t)|$ grows at most linearly in time if $h(\cdot)$ is bounded. If we consider the specific interaction rules (6.9)-(6.13), we obtain

$$\frac{d}{dt}m_{\rho_i}(t) = \gamma\alpha \int_{\mathbb{R}^4} w(R_i - R_j) f_\varepsilon(\rho_i, R_i, t) f_\varepsilon(\rho_j, R_j, t) d\rho_j dR_j d\rho_i dR_i \leq \gamma\alpha,$$

with equality holding in the "all-play-all" case $w = 1$. The evolution of the second moment $\mathcal{M}_{2,\rho_i}$ can be computed by setting $\phi(\rho_i, R_i) = {\rho_i}^2$. We see that

$$\begin{aligned}\frac{d}{dt}\mathcal{M}_{2,\rho_i}(t) =& \frac{1}{2}\int_{\mathbb{R}^4} \big(\gamma^2[\langle h(\rho_j - \rho_i)^2\rangle + \langle h(\rho_i - \rho_j)^2\rangle] + 2\gamma[\rho_i\langle h(\rho_j - \rho_i)\rangle + \rho_j\langle h(\rho_i - \rho_j)\rangle] \\ &\qquad + 2\sigma^2(\gamma)\big) w(R_i - R_j) f_\varepsilon(\rho_i, R_i, t) f_\varepsilon(\rho_j, R_j, t)\, d\rho_j dR_j d\rho_i dR_i \\ \leq& \gamma^2 \|\langle h^2\rangle\|_\infty + \sigma^2(\gamma) + 4\gamma|m_{\rho_i}(t)|.\end{aligned} \tag{6.16}$$

If $h(\cdot)$ is bounded the second moment grows at most at polynomial rate. Since the second moment of f_0 is bounded (see assumption ($\mathcal{A}2$)), it remains finite for all times $t \in [0,T]$.

6.4 The Fokker-Planck limit

In the last section we analysed the evolution of moments to the Boltzmann type equation (6.12). However, it is often more useful to study the dynamics of simplified models (generally of Fokker-Planck type), which can be derived in particular asymptotic limits. These asymptotics provide a good approximation of the stationary profiles of the kinetic equation. In what follows we consider the so-called *quasi-invariant* limit, in which diffusion and the outcome of the game influence the long-time dynamics. More specifically, we consider the limit

$$\gamma \to 0,\ \sigma_\eta \to 0 \text{ such that } \frac{\sigma_\eta^2}{\gamma} =: \sigma^2 \text{ is kept fixed.}$$

In Appendix 7 we derive the following Fokker-Planck limit: The differential form of (7.4) is given by (writing t instead of τ)

$$\frac{\partial f(\rho,R,t)}{\partial t} = -\frac{\partial}{\partial R}\left(a[f]f(\rho,R,t)\right) - \frac{\partial}{\partial \rho}\left(c[f]f(\rho,R,t)\right) + \frac{\sigma^2}{2}d[f]\frac{\partial^2}{\partial \rho^2}f(\rho,R,t) \text{ in } \Omega \times (0,T), \tag{6.17}$$

where

$$\begin{aligned} a[f] = a[f](\rho,R,t) &= \int_{\mathbb{R}^2} w(R - R_j)(b(\rho - \rho_j) - b(R - R_j)) f(\rho_j, R_j, t)\, d\rho_j dR_j, \\ c[f] = c[f](\rho,R,t) &= \int_{\mathbb{R}^2} w(R - R_j)\big(\alpha h_1(\rho_j - \rho) + \beta\langle h_2(\rho_j - \rho)\rangle\big) f(\rho_j, R_j, t)\, d\rho_j dR_j, \\ d[f] = d[f](R,t) &= \int_{\mathbb{R}^2} w(R - R_j) f(\rho_j, R_j, t)\, d\rho_j dR_j. \end{aligned}$$

We consider equation (6.17) with initial datum f_0 satisfying assumption ($\mathcal{A}2$) in the following. Note that (6.17) includes the nonlocal operator $a[f]$, corresponding to the change of the ratings, similar as in the Fokker-Planck equations (6.3) and (6.5) obtained in [58] and [64], respectively. The nonlocal operator $c[f]$ in the transport terms corresponds to the change of the individual strengths while the operator $d[f]$ describes the fluctuations of the individual strength due to encounters.

6.4.1 Qualitative properties of the Fokker-Planck equation

We continue by discussing qualitative properties of the Fokker-Planck equation (6.17). We shall see that several properties, which we observed for the Boltzmann type equation (6.12), can be transferred.

Conservation of mass and positivity of solution: Due to mass conservation and ($\mathcal{A}2$) we have that

$$\int_{\mathbb{R}^2} f(\rho, R, t)\, d\rho dR = \int_{\mathbb{R}^2} f_0(\rho, R)\, d\rho dR = 1 \text{ for all } t \geq 0.$$

Using similar arguments as in [86], we can directly prove that the Fokker-Planck equation maintains the positivity of the solution. Let $v_m(\tilde{t}) = (\rho_m(t), R_m(t))$ denote the point in which one assumes that f reaches its minimum, which is obtained at time $\tilde{t}$. Clearly, if at certain time $\tilde{t} \geq 0$ the function equals zero, i.e. $f(\rho, R, \tilde{t}) = 0$, this point is a stationary point or a local minimum, hence

$$\frac{\partial}{\partial R} f(v_m, \tilde{t}) = 0, \quad \frac{\partial}{\partial \rho} f(v_m, \tilde{t}) = 0, \quad \frac{\partial^2}{\partial R^2} f(v_m, \tilde{t}) \geq 0, \quad \frac{\partial^2}{\partial \rho^2} f(v_m, \tilde{t}) \geq 0.$$

Evaluating (6.17) in $(v_m, \tilde{t})$ gives

$$\begin{aligned}\frac{\partial}{\partial t} f(v_m, \tilde{t}) =& f(v_m, \tilde{t})\big(- \frac{\partial}{\partial R} a[f](v_m, \tilde{t}) - \frac{\partial}{\partial \rho} c[f](v_m, \tilde{t})\big) \\ &- a[f](v_m, \tilde{t}) \frac{\partial}{\partial R} f(v_m, \tilde{t}) - c[f](v_m, \tilde{t}) \frac{\partial}{\partial r} f(v_m, \tilde{t}) + \frac{\sigma^2}{2}(v_m, \tilde{t}) d[f] \frac{\partial^2}{\partial \rho^2} \big(f(v_m, \tilde{t})\big) \geq 0,\end{aligned}$$

which implies that the function f is non-decreasing in time and cannot assume negative values.

Evolution of the moments: We now consider the evolution of the moments of the solution of (6.17) using the interaction rules (6.9) and (6.10). Similar calculations as in Section 6.3.2 confirm the expected behaviour —due to the continuous increase in strength in each game the system does not converge to a steady state and therefore the respective mean of the solution is non-decreasing in time. Summarising the results, we have

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^2} R f(\rho, R, t)\, dR d\rho = 0 \tag{6.18}$$

$$\begin{aligned}\frac{\partial}{\partial t} \int_{\mathbb{R}^2} \rho f(\rho, R, t)\, dR d\rho &= \alpha \int_{\mathbb{R}^2} c[f] f(\rho, R, t)\, dR d\rho \\ &= \alpha \int_{\mathbb{R}^4} w(R - R_j) f(\rho, R, t) f(\rho_j, R_j, t)\, d\rho_j dR_j d\rho dR.\end{aligned} \tag{6.19}$$

The previous results confirm that due to the continuous increase in strength in each game, rating and skills tend to become increasingly distant from each other. Therefore, we adopt an idea by Krupp [64] and study the evolution of a suitably shifted problem instead. We define

$$g(\rho, R, t) = f(\rho + H(\rho, R, t), R, t), \tag{6.20}$$

where the scaling function H is given by

$$\frac{\partial H(\rho, R, t)}{\partial t} = \int_{\mathbb{R}^2} \alpha w(R - R_j) f(\rho_j, R_j, t)\, d\rho_j dR_j = \alpha d[f]. \tag{6.21}$$

This scaling ensures that the mean value is preserved in time. The corresponding evolution equation for $g(\rho, R, t)$ is given by

$$\frac{\partial g(\rho, R, t)}{\partial t} = -\frac{\partial}{\partial R}(a[g]g(\rho, R, t)) - \frac{\partial}{\partial \rho}(\tilde{c}[g]g(\rho, R, t)) + \frac{\sigma^2}{2}d[g]\frac{\partial^2}{\partial \rho^2}g(\rho, R, t),$$

where

$$\tilde{c}[g] = \tilde{c}[g](\rho, R, t) = \int_{\mathbb{R}^2} \big(\alpha b(\rho_j - \rho) + \beta\langle h_2(\rho_j - \rho)\rangle\big)w(R - R_j)g(\rho_j, R_j, t)\, d\rho_j dR_j.$$

Now, the mean value of $g(\rho, R, t)$ is constant w.r.t. both R and ρ and we can normalize

$$\int_{\mathbb{R}^2} Rg(\rho, R, t)\, d\rho dR = 0, \text{ and } \int_{\mathbb{R}^2} \rho g(\rho, R, t)\, d\rho dR = 0.$$

In a general setting it is not possible to compute scaling function explicitly. However, in 'all-meet-all' tournaments, that is $w(R - R_j) = 1$, and in case of the specific interaction rules (6.9)-(6.10), we obtain that

$$H(\rho, R, t) = \alpha t.$$

Therefore, in the rest of this paper, we consider the following problem on a bounded domain $\Omega \subset \mathbb{R}^2$, with no-flux boundary condition

$$\frac{\partial g(\rho, R, t)}{\partial t} = -\frac{\partial}{\partial R}(a[g]g(\rho, R, t)) - \frac{\partial}{\partial \rho}(\tilde{c}[g]g(\rho, R, t)) + \frac{\sigma^2}{2}d[g]\frac{\partial^2}{\partial \rho^2}g(\rho, R, t) \quad \text{in } \Omega \times (0, T), \tag{6.22a}$$

$$\frac{\partial}{\partial \nu}g = 0 \quad \text{on } \partial\Omega, \tag{6.22b}$$

$$g(\rho, R, 0) = g_0(\rho, R) \quad \text{in } \Omega. \tag{6.22c}$$

Note that the existence of solutions to (6.22a) on the whole domain is more involved, since we would need to proof that the solution decays sufficiently as R and ρ tend to infinity. Therefore, we consider the equation on a bounded domain only.

6.4.2 Analysis of the Fokker-Planck equation

In the section we prove existence of weak solutions to (6.22). The main result reads as follows.

Theorem 4. *Let ($\mathcal{A}$1) be satisfied, $g_0 \in H^1(\Omega)$ and $0 \le g_0 \le M_0$ for some $M_0 > 0$ and assume h_1, $\langle h_2\rangle$, $b \in L^\infty(\Omega) \cap C^2(\Omega)$. Then there exists a weak solution $g \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$ to* (6.22a)–(6.22c)*, satisfying $0 \le g \le M_0 e^{\lambda t}$ for all $(\rho, R) \in \Omega$, $t > 0$, with a constant $\lambda > 0$ depending on the functions $h_1, \langle h_2\rangle, b$ and w.*

The presented existence proof was adapted from a similar argument for a nonlinear Fokker-Planck equation describing the dynamics of agents in an economic market, see [36]. However, equation (6.22a) has an additional nonlinearity in the derivative w.r.t. the rating R. We divide the proof in several steps for the ease of presentation. In *Step 0* we regularize the non-linear Fokker Planck equation (6.22a) by adding a Laplace operator with small diffusivity $\mu \ge 0$. We linearise the equation in *Step 1* and show existence of a unique solution for this problem. In *Step 2* we derive the necessary L^∞ estimates to use Leray-Schauder's fixed point theorem and show existence of solutions to the nonlinear regularised problem. In *Step 3* we present additional H^1 estimates, which allow us to pass to the limit $\mu \to 0$ in *Step 4*.

Proof. Step 0: the regularised problem. For $M > 0$, let us denote by $g_M = \max\{0, \min\{g, M\}\}$ and define

$$K_M[g] = \int_\Omega [\alpha h_1(\rho_j - \rho) + \beta\langle h_2(\rho_j - \rho)\rangle] w(R - R_j) g_M(\rho_j, R_j, t)\, d\rho_j dR_j,$$
$$L_M[g] = \int_\Omega [b(\rho - \rho_j) - b(R - R_j)] w(R - R_j) g_M(\rho_j, R_j)\, d\rho_j dR_j.$$

Next we consider the regularised non linear problem for $0 < \mu < 1$,

$$\begin{aligned}\frac{\partial}{\partial t} g_\mu = -\frac{\partial}{\partial R}(L_M[g_\mu] g_\mu(\rho, R, t)) - \frac{\partial}{\partial \rho}(K_M[g_\mu] g_\mu(\rho, R, t)) \\ + \frac{\sigma^2}{2} d[g_\mu] \frac{\partial^2}{\partial \rho^2}(g_\mu(\rho, R, t)) + \mu\Delta(g_\mu(\rho, R, t)) \text{ in } \Omega \times (0, T),\end{aligned} \tag{6.23a}$$

with boundary and initial conditions given by

$$\frac{\partial}{\partial \nu} g_\mu = 0 \text{ on } \partial\Omega, \text{ and } g_\mu(\rho, R, 0) = g_0 \text{ on } \Omega. \tag{6.23b}$$

The weak formulation of (6.23) is given by

$$\int_0^T \left\langle \frac{\partial}{\partial t} g_\mu, v \right\rangle dt = \int_0^T \int_\Omega \left(L_M[g_\mu] g_\mu \frac{\partial}{\partial R} v + K_M[g_\mu] g_\mu \frac{\partial}{\partial \rho} v - \frac{\sigma^2}{2} d[g_\mu] \frac{\partial}{\partial \rho} g_\mu \frac{\partial}{\partial \rho} v - \mu \frac{\partial}{\partial R} g_\mu \frac{\partial}{\partial R} v \right) dR d\rho dt, \tag{6.24}$$

where $\langle \cdot, \cdot \rangle$ is the dual product between $H^1(\Omega)$ and $H^{-1}(\Omega)$ and $v \in H^1(\Omega)$.

Step 1: solution of the linearised regularised problem. Next we want to apply Leray-Schauder's fixed point theorem. Let $\tilde{g} \in L^2(0, T; L^2(\Omega))$, $\theta \in [0, 1]$ and $g^+ = \max(g, 0)$. We introduce the operators $A : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ and $F : H^1(\Omega) \to \mathbb{R}$:

$$A(g_\mu, v) = \int_\Omega \mu \left(\frac{\partial}{\partial R} g_\mu \frac{\partial}{\partial R} v + \frac{\partial}{\partial \rho} g_\mu \frac{\partial}{\partial \rho} v \right) dR d\rho, \tag{6.25}$$

$$F(v) = \theta \int_\Omega \left(L_M[\tilde{g}] \tilde{g}^+ \frac{\partial}{\partial R} v + K_M[\tilde{g}] \tilde{g}^+ \frac{\partial}{\partial \rho} v - \frac{\sigma^2}{2} d[\tilde{g}] \frac{\partial}{\partial \rho} \tilde{g}^+ \frac{\partial}{\partial \rho} v \right) dR d\rho. \tag{6.26}$$

The operator $A(\cdot, \cdot)$ is bilinear and continuous on $H^1(\Omega) \times H^1(\Omega)$. The quantities $|K_M[\tilde{g}]|$ and $|L_M[\tilde{g}]|$ are bounded (because of the assumption made on $h_1, \langle h_2 \rangle$ and b), therefore F is continuous in $H^1(\Omega)$. Because of Poincaré's inequality, for some constant C_1 and C_2

$$A(g_\mu, g_\mu) = \mu \int_\Omega \left(\left| \frac{\partial}{\partial \rho} g_\mu \right|^2 + \left| \frac{\partial}{\partial R} g_\mu \right|^2 \right) dR d\rho \geq C_1 \mu \|g_\mu\|_{H^1(\Omega)} - C_2 \|g_\mu\|_2.$$

By corollary 23.26 in [94] (see Appendix for details), there exists a unique solution $g_\mu \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$ to

$$\left\langle \frac{\partial}{\partial t} g_\mu, v \right\rangle + A(g_\mu, v) = F(v), \ t > 0, \ g_\mu(0) = \theta g_0. \tag{6.27}$$

This defines the fixed-point operator $T : L^2(0, T; L^2(\Omega)) \times [0, 1] \to L^2(0, T; L^2(\Omega))$, $(\tilde{g}, \theta) \mapsto T(\tilde{g}, \theta) = g_\mu$, where g_μ solves (6.27). This operator satisfies $T(\tilde{g}, 0) = 0$. Standard arguments, including Galerkin's method and estimates on $\|\frac{\partial}{\partial t} g_\mu\|_{L^2(0,T;H^{-1}(\Omega))}$, show that the operator T is continuous (with constants depending on the regularisation parameter μ). The operator is also compact, because $L^2(0, T; H^1(\Omega)) \cap$

$H^1(0,T;H^{-1}(\Omega))$ is compactly embedded in $L^2(0,T;L^2(\Omega))$, see [82]. In order to apply the fixed-point theorem of Leray-Schauder, we need to show uniform estimates.

Step 2: uniform L^∞ bound & existence of a fixed point. We start by proving upper and lower bounds for the function g_μ. Let g_μ be a fixed point of $T(\cdot,\theta)$, i.e. g_μ solves (6.27) with $\tilde{g}=g_\mu$, and $\theta\in[0,1]$. For a lower bound, choosing $v=g_\mu^-=\min\{0,g_\mu\}\in L^2(0,T;H^1(\Omega))$ as test function in (6.27) and integrating in time, we obtain

$$\frac{d}{dt}\|g_\mu^-\|_{L^2(\Omega)}^2=-2A(g_\mu,g_\mu^-)\le -C_1\|g_\mu^-\|_2^2\le 0.$$

This shows that if $g_\mu(0)^-=0$, then $g_\mu(t)^-=0$ for all $t>0$. Hence, in all previous computations and in (6.25)-(6.26), we can replace g_μ^+ with g_μ.

Now we show an upper bound. Let $g_*=(g_\mu-M)^+$, where $M=M_0e^{\lambda t}$, for some $\lambda>0$ to be determined below. We choose $v=g_*\in L^2(0,T;H^1(\Omega))$ as test function in (6.24). By assumption, $g_0\le M_0$, i.e. $g_*(0)=(g_0-M_0)^+=0$. We note that $\frac{\partial}{\partial t}M=\lambda M$ and $\frac{1}{2}\frac{\partial}{\partial\rho}(g_*^2)=(g_\mu-M)\frac{\partial}{\partial\rho}g_*$. Then

$$\begin{aligned}\frac{1}{2}\int_\Omega g_*(t)^2\,dRd\rho&=\int_0^t\left[-\lambda\int_\Omega Mg_*\,dRd\rho-A(g_\mu,g_*)+F(g_*)\right]ds\\&=\int_0^t\frac{\sigma^2}{2}\int_\Omega d[g_\mu]\frac{\partial}{\partial\rho}((g_\mu-M)+M)\frac{\partial}{\partial\rho}g_*\,dRd\rho-\mu\int_\Omega|\nabla g_*|^2\,dRd\rho+\theta(I+J)\,ds\\&\le\int_0^t\theta(I+J)\,ds,\end{aligned}$$

where $I=\int_\Omega L_M[g_\mu]g_\mu\frac{\partial}{\partial R}g_*\,dRd\rho$ and $J=\int_\Omega K_M[g_\mu]g_\mu\frac{\partial}{\partial\rho}g_*\,dRd\rho$. Let us consider I and J separately:

$$\begin{aligned}I&=\int_\Omega L_M[g_\mu](g_\mu-M)\frac{\partial}{\partial R}g_*\,dRd\rho+\int_\Omega L_M[g_\mu]M\frac{\partial}{\partial R}g_*\,dRd\rho\\&=-\frac{1}{2}\int_\Omega\frac{\partial}{\partial R}[L_M[g_\mu]]g_*^2\,dRd\rho-\int_\Omega\frac{\partial}{\partial R}[L_M[g_\mu]]Mg_*\,dRd\rho\\J&=\int_\Omega K_M[g_\mu](g_\mu-M)\frac{\partial}{\partial\rho}g_*\,dRd\rho+\int_\Omega L_M[g_\mu]M\frac{\partial}{\partial\rho}g_*\,dRd\rho\\&=-\frac{1}{2}\int_\Omega\frac{\partial}{\partial\rho}[K_M[g_\mu]]g_*^2\,dRd\rho-\int_\Omega\frac{\partial}{\partial\rho}[K_M[g_\mu]]Mg_*\,dRd\rho.\end{aligned}$$

The assumptions on h_1, $\langle h_2\rangle$ and b ensure that $\frac{\partial}{\partial R}[L_M[g_\mu]]$ and $\frac{\partial}{\partial\rho}[K_M[g_\mu]]$ are bounded. Hence

$$\begin{aligned}\frac{1}{2}\int_\Omega g_*^2\,dRd\rho&=\int_\Omega\Big(\frac{\partial}{\partial t}g_*\Big)g_*\,dRd\rho\\&\le C(L_M[g_\mu],K_M[g_\mu])\int_\Omega g_*^2\,dRd\rho+(C(L_M[g_\mu],K_M[g_\mu])-\lambda)\int_\Omega Mg_*\,dRd\rho.\end{aligned}$$

Choosing λ large enough and using Gronwall's lemma, we obtain

$$\int_\Omega g_*(t)^2\,dRd\rho\le\int_\Omega g_*(0)^2\exp[2C(L_M[g_\mu],K_M[g_\mu])t]\,dRd\rho=0.$$

Therefore $g_*(t)=0$ for all $t>0$, which implies $g_\mu(t)\le M$ for all $t>0$. This allows us to replace $L_M[g_\mu]$ with $a[g_\mu]$ and $K_M[g_\mu]$ with $\tilde{c}[g_\mu]$ in (6.24). The uniform L^∞ bound provides the necessary bound for the fixed-point operator in $L^2(0,T;L^2(\Omega))$. This implies existence of a weak solution to (6.24).

Step 3: uniform H^1 bound. Our aim is to derive an H^1 bound which is independent of μ. Choosing $v = g_\mu$ in (6.24) with t instead of T, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega g_\mu(t)^2\,dRd\rho &= \int_\Omega a[g_\mu]g_\mu\frac{\partial}{\partial R}g_\mu\,dRd\rho + \int_\Omega \tilde{c}[g_\mu]g_\mu\frac{\partial}{\partial\rho}g\,dRd\rho \\ &\quad - \int_\Omega\Big(\frac{\sigma^2}{2}d[g_\mu]+\mu\Big)\Big|\frac{\partial}{\partial\rho}g_\mu\Big|^2\,dRd\rho - \mu\int_\Omega\Big|\frac{\partial}{\partial R}g_\mu\Big|^2\,dRd\rho \\ &= -\frac{1}{2}\int_\Omega\frac{\partial}{\partial R}a[g_\mu]g_\mu^2\,dRd\rho - \frac{1}{2}\int_\Omega\frac{\partial}{\partial\rho}\tilde{c}[g_\mu]g_\mu^2\,dRd\rho - \int_\Omega\Big(\frac{\sigma^2}{2}d[g_\mu]+\mu\Big)\Big|\frac{\partial}{\partial\rho}g_\mu\Big|^2\,dRd\rho \\ &\quad - \mu\int_\Omega\Big|\frac{\partial}{\partial R}g_\mu\Big|^2\,dRd\rho.\end{aligned}$$

Because of the assumptions on h_1, $\langle h_2\rangle$ and b we have that $\Big|-\frac{1}{2}\Big(\frac{\partial}{\partial R}a[g_\mu]+\frac{\partial}{\partial\rho}\tilde{c}[g_\mu]\Big)\Big| < C$. Therefore, we can rewrite the above estimate as

$$\begin{aligned}\frac{1}{2}\int_\Omega g_\mu(t)^2\,dRd\rho + \int_0^t\Big[\int_\Omega\Big(\frac{\sigma^2}{2}d[g_\mu]+\mu\Big)\Big|\frac{\partial}{\partial\rho}g_\mu\Big|^2\,dRd\rho + \mu\int_\Omega\Big|\frac{\partial}{\partial R}g_\mu\Big|^2\,dRd\rho\Big]ds \\ \le C\int_0^t\int_\Omega g_\mu(t)^2\,dRd\rho dt + \frac{1}{2}\int_\Omega g(0)^2\,dRd\rho.\end{aligned} \tag{6.28}$$

Using Gronwall's lemma, the previous estimate guarantees (independent by μ) estimates for $g_\mu(t)$, i.e.

$$\|g_\mu\|_{L^\infty(0,T;L^2(\Omega))} \le C.$$

However, this does not ensure an (independent of μ) estimate for $\frac{\partial}{\partial R}g_\mu$ and $\frac{\partial}{\partial\rho}g_\mu$. In order to obtain it, we differentiate (6.23a) with respect to R and ρ in the sense of distributions. This gives us estimates for $y := \frac{\partial}{\partial R}g_\mu$ and $z := \frac{\partial}{\partial\rho}g_\mu$. We obtain

$$\frac{\partial}{\partial t}y = -\frac{\partial}{\partial R}\big(d[g_\mu]g_\mu + a[g_\mu]y\big) - \frac{\partial}{\partial\rho}(\tilde{c}[g_\mu]y) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial\rho^2}y + \gamma\frac{\partial^2}{\partial R^2}y \quad \text{in } \Omega\times(0,T). \tag{6.29}$$

Due to no-flux boundary condition (6.22b), equation (6.29) is complemented with

$$\frac{\partial}{\partial\nu_R}y(\rho,R,t) = 0 \quad \text{on } \partial\Omega,$$

where ν_R is the component w.r.t. variable R of the normal vector ν to Ω. Furthermore $y(\rho,R,0) = \frac{\partial}{\partial R}g_0(\rho,R)$. Choosing $v\in L^2(0,T;H_0^1(\Omega))$ and setting $d'[g_\mu] = \frac{\partial}{\partial R}d[g_\mu]$, $\tilde{c}_R[g_\mu] = \frac{\partial}{\partial R}\tilde{c}[g_\mu]$ and $a_R[g_\mu] = \frac{\partial}{\partial R}a[g_\mu]$, we obtain the weak formulation of equation (6.29):

$$\begin{aligned}\int_0^T\Big\langle\frac{\partial}{\partial t}y, v\Big\rangle ds = \int_0^T\int_\Omega\Big(a_R[g_\mu]g_\mu\frac{\partial}{\partial R}v + a[g_\mu]y\frac{\partial}{\partial R}v + \tilde{c}_R[g_\mu]g_\mu\frac{\partial}{\partial\rho}v + \tilde{c}[g_\mu]y\frac{\partial}{\partial\rho}v \\ - \frac{\sigma^2}{2}\frac{\partial}{\partial\rho}\big(d'[g_\mu]g_\mu + d[g_\mu]y\big)\frac{\partial}{\partial\rho}v - \mu\Big(\frac{\partial}{\partial\rho}y\frac{\partial}{\partial\rho}v + \frac{\partial}{\partial R}y\frac{\partial}{\partial R}v\Big)\Big)\,dRd\rho ds.\end{aligned} \tag{6.30}$$

We introduce the operators

$$\begin{aligned}B_y(y,v) &= \int_\Omega -a[g_\mu]y\frac{\partial}{\partial R}v - \tilde{c}[g_\mu]y\frac{\partial}{\partial\rho}v + \frac{\sigma^2}{2}d[g_\mu]\frac{\partial}{\partial\rho}y\frac{\partial}{\partial\rho}v + \mu\Big(\frac{\partial}{\partial\rho}y\frac{\partial}{\partial\rho}v + \frac{\partial}{\partial R}y\frac{\partial}{\partial R}v\Big)\,dRd\rho \\ G_y(v) &= \int_\Omega \tilde{c}_R[g_\mu]g_\mu\frac{\partial}{\partial\rho}v + a_R[g_\mu]g_\mu\frac{\partial}{\partial R}v - \frac{\sigma^2}{2}d'[g_\mu]\frac{\partial}{\partial\rho}g_\mu\frac{\partial}{\partial\rho}v\,dRd\rho.\end{aligned}$$

Both operators $B_y : L^2(0,T;H_0^1(\Omega)) \times L^2(0,T;H_0^1(\Omega)) \to \mathbb{R}$ and $G_y : L^2(0,T;H_0^1(\Omega)) \to \mathbb{R}$ are linear and continuous. Garding's inequality implies

$$\begin{aligned} B_y(y,y) &= \int_\Omega \mu|\nabla y|^2\,dRd\rho + \frac{1}{2}\int_\Omega (\tilde{c}_R[g_\mu] + a_R[g_\mu])y^2 d\rho dR + \frac{\sigma^2}{2}\int_\Omega d[g_\mu]\Big|\frac{\partial}{\partial\rho}y\Big|^2\,dRd\rho \\ &\geq \mu\|y\|^2_{H^1(\Omega)} - \Big(\mu + \frac{1}{2}\|a[g_\mu]\|_\infty + \frac{1}{2}\|\tilde{c}[g_\mu]\|_\infty\Big)\|y\|_2^2. \end{aligned}$$

Then corollary 23.26 in [94] gives existence of a unique solution $y \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ to

$$\Big\langle \frac{\partial}{\partial t}y, v\Big\rangle + B_y(y,v) = G_y(v),\ t>0,\ y(0) = y_0. \tag{6.31}$$

Choosing $v = y$ in (6.30), we obtain (using Young's and Gardin's inequality)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega y(t)^2\,dRd\rho = -B_y(y,y) + G_y(y) \\ &\leq -\mu\|y\|^2_{H^1(\Omega)} + C\|y\|_2^2 + \frac{1}{2}\Big(\|\frac{\partial^2}{\partial R^2}a[g_\mu]\|_\infty + \|\frac{\partial}{\partial\rho}\Big(\frac{\partial}{\partial R}\tilde{c}[g_\mu]\Big)\|_\infty\Big)\int_\Omega g_\mu^2 + y^2 + \Big|\frac{\partial}{\partial\rho}g_\mu\Big|^2\,dRd\rho \\ &\quad - \frac{\sigma^2}{2}\int_\Omega d'[g_\mu]\frac{\partial}{\partial\rho}g_\mu\frac{\partial}{\partial\rho}y\,dRd\rho. \end{aligned}$$

Considering the last integral, we calculate

$$\begin{aligned} -\frac{\sigma^2}{2}\int_\Omega d'[g_\mu]\frac{\partial}{\partial\rho}g_\mu\frac{\partial}{\partial\rho}y\,dRd\rho &= -\frac{\sigma^2}{2}\int_\Omega d'[g_\mu]\frac{\partial}{\partial\rho}g_\mu\frac{\partial}{\partial\rho}\Big(\frac{\partial}{\partial R}g_\mu\Big)\,dRd\rho \\ &= \frac{\sigma^2}{2}\int_\Omega \frac{\partial}{\partial R}d'[g_\mu]\Big|\frac{\partial}{\partial\rho}g_\mu\Big|^2\,dRd\rho + \frac{\sigma^2}{2}\int_\Omega d'[g_\mu]\frac{\partial}{\partial R}\Big(\frac{\partial}{\partial\rho}g_\mu\Big)\frac{\partial}{\partial\rho}g_\mu\,dRd\rho, \end{aligned}$$

and therefore,

$$-\frac{\sigma^2}{2}\int_\Omega d'[g_\mu]\frac{\partial}{\partial\rho}g_\mu\frac{\partial}{\partial\rho}y\,dRd\rho = \frac{\sigma^2}{4}\int_\Omega \frac{\partial}{\partial R}d'[g_\mu]\Big|\frac{\partial}{\partial\rho}g_\mu\Big|^2\,dRd\rho.$$

This gives us the following estimate for $\|y\|_{L^2(\Omega)}$ (with a constant depending on $a[g_\mu]$, $\tilde{c}[g_\mu]$ and their derivatives)

$$\int_\Omega y(t)^2\,dRd\rho \leq \int_\Omega h(0)^2\,dRd\rho + C\int_0^t\int_\Omega y^2 + g_\mu^2 + \Big|\frac{\partial}{\partial\rho}g_\mu\Big|^2\,dRd\rho ds. \tag{6.32}$$

We use similar arguments for $z = \frac{\partial}{\partial\rho}g_\mu$. For a suitable C, which depends on $a[g_\mu]$, $\tilde{c}[g_\mu]$, $d[g_\mu]$ and their derivatives (but not on μ), we obtain an estimate for the L^2 norm of z:

$$\int_\Omega z(t)^2\,dRd\rho \leq \int_\Omega h(0)^2\,dRd\rho + C\int_0^t\int_\Omega z^2 + g_\mu^2 + \Big|\frac{\partial}{\partial R}g_\mu\Big|^2\,dRd\rho ds. \tag{6.33}$$

We add (6.28), (6.32) and (6.33) to obtain

$$\begin{aligned} &\int_\Omega g_\mu(\rho,R,t)^2 + y(\rho,R,t)^2 + z(\rho,R,t)^2\,dRd\rho + \frac{\sigma^2}{2}\int_0^t\int_\Omega z(\rho,R,s)^2\,dRd\rho ds \\ &\leq C\int_0^t\int_\Omega y(\rho,R,s)^2 + g_\mu(\rho,R,s)^2 + z(\rho,R,s)^2\,dRd\rho ds + \int_\Omega g(\rho,R,0)^2 + y(\rho,R,0)^2 + z(\rho,R,0)^2\,dRd\rho, \end{aligned} \tag{6.34}$$

where C does not depend on μ. Using Gronwall's lemma gives the following estimates (independent of μ)

$$\|g_\mu\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \quad \|\frac{\partial}{\partial\rho}g_\mu\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \quad \|\frac{\partial}{\partial R}g_\mu\|_{L^\infty(0,T;L^2(\Omega))} \leq C. \tag{6.35}$$

Step 4: The limit $\mu \to 0$. Let g_μ solution of (6.23a)-(6.23b) with $L[g_\mu] = a[g_\mu]$ and $K[g_\mu] = \tilde{c}[g_\mu]$. We can estimate $\|\frac{\partial}{\partial t}g_\mu\|_{L^2(0,T;H^{-1}(\Omega))}$, using the norm of operators $\|\frac{\partial}{\partial t}g_\mu\|_{H^{-1}(\Omega)} = \sup_{\|v\|_{H^1(\Omega)}=1} |\langle \frac{\partial}{\partial t}g_\mu, v\rangle|$. For a suitable $C \geq (\|\frac{\partial}{\partial R}a[g]\|_\infty)^{\frac{1}{2}} + (\|\frac{\partial}{\partial\rho}\tilde{c}[g]\|_\infty)^{\frac{1}{2}} + \frac{\sigma^2}{2} + 1$, we obtain

$$\begin{aligned}
\Big|\Big\langle \frac{\partial}{\partial t}g_\mu, v\Big\rangle\Big| &\leq \|a[g_\mu]\|_\infty \int_\Omega \Big(g_\mu^2 + \Big|\frac{\partial}{\partial R}v\Big|^2\Big)\, dRd\rho + \|\tilde{c}[g_\mu]\|_\infty \int_\Omega \Big(g_\mu^2 + \Big|\frac{\partial}{\partial\rho}v\Big|^2\Big)\, dRd\rho \\
&\quad + \frac{\sigma^2}{2}\|d[g_\mu]\|_\infty \int_\Omega \Big|\frac{\partial}{\partial\rho}g_\mu\Big|^2 + \Big|\frac{\partial}{\partial\rho}v\Big|^2\, dRd\rho + \mu\int_\Omega |\nabla g_\mu|^2 + |\nabla v|^2\, dRd\rho \\
&\leq C(\|g_\mu\|_{H^1(\Omega)})\|v\|_{H^1(\Omega)}.
\end{aligned}$$

This implies

$$\|\frac{\partial}{\partial t}g_\mu\|_{L^2(0,T;H^{-1}(\Omega))} \leq C \text{ and } \int_0^T \|g_\mu\|^2_{H^1(\Omega)}dt = C\|g_\mu\|_{L^2(0,T;H^1(\Omega))} \leq C, \tag{6.36}$$

where C does not depend on μ. Estimates (6.35) and (6.36) allow us to apply Aubin-Lions lemma and conclude the existence of a subsequence of (g_μ) such that for $\mu \to 0$,

$$\begin{aligned}
g_\mu &\to g \text{ strongly in } L^2(0,T;L^2(\Omega)), \\
g_\mu &\rightharpoonup g \text{ weakly in } L^2(0,T;H^1(\Omega)), \\
\frac{\partial}{\partial t}g_\mu &\rightharpoonup \frac{\partial}{\partial t}g \text{ weakly in } L^2(0,T;H^{-1}(\Omega)).
\end{aligned}$$

Furthermore, by direct computation, we obtain

$$\|\tilde{c}[g]g - \tilde{c}[g_\mu]g_\mu\|_{L^2(0,T;L^2(\Omega))} \leq \|\tilde{c}[g](g-g_\mu)\|_{L^2(0,T;L^2(\Omega))} + \|(\tilde{c}[g]-\tilde{c}[g_\mu])g_\mu\|_{L^2(0,T;L^2(\Omega))}.$$

The first term on the right side of the previous inequality goes to 0 when $\mu \to 0$ because $\tilde{c}[g_\mu]$ is bounded and $g_\mu \to g$ strongly in $L^2(0,T;L^2(\Omega))$. Using Cauchy-Schwartz's inequality and that the domain Ω is bounded, yields

$$\begin{aligned}
\|(\tilde{c}[g]-\tilde{c}[g_\mu])g_\mu\|_{L^1(0,T;L^1(\Omega))} &= \int_0^T\int_\Omega \Big|\int_\Omega \big(\alpha h_1(\rho_j-\rho) + \beta\langle h_2(\rho_j-\rho)\rangle\big)\times \\
&\quad \times w(R-R_j)\big(g(\rho_j,R_j,t) - g_\mu(\rho_j,R_j,t)d\rho_j dR_j\big)\Big| g_\mu(\rho,R,t)d\rho dR dt \\
&\leq C\int_0^T \Big(\int_\Omega g(\rho_j,R_j,t) - g_\mu(\rho_j,R_j,t)d\rho_j dR_j\Big)\Big(\int_\Omega g_\mu(\rho,R,t)d\rho dR\Big)dt \\
&\leq C|\Omega|^{\frac{1}{2}}\|g_\mu - g\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

The constant is bounded from above by the L^∞-norm of h and w, hence this term goes to 0 as $\mu \to 0$.

Since $c[g_\mu]g_\mu$ is bounded, convergence holds in L^p for all $p < \infty$. The same argument holds for the difference $\|a[g_\mu]g_\mu - a[g]g\|_{L^2(0,T;L^2(\Omega))}$. So, we have shown that

$$\tilde{c}[g_\mu]g_\mu \to \tilde{c}[g]g \text{ strongly in } L^2(0,T;L^2(\Omega)),$$
$$a[g_\mu]g_\mu \to a[g]g \text{ strongly in } L^2(0,T;L^2(\Omega)).$$

Therefore, we can pass to the limit $\mu \to 0$ in the equation (6.24) and obtain for all $v \in L^2(0,T;H^1(\Omega))$

$$\int_0^T \left\langle \frac{\partial}{\partial t} g, v \right\rangle dt = \int_0^T \int_\Omega a[g]g\frac{\partial}{\partial R}v + \tilde{c}[g]g\frac{\partial}{\partial \rho}v - \frac{\sigma^2}{2}\frac{\partial}{\partial \rho}g\frac{\partial}{\partial \rho}v \, dRd\rho dt. \tag{6.37}$$

This completes the proof. □

6.5 Long time behaviour of ratings and strength

In this section we study possible steady states of the proposed Elo model and discuss the convergence of the ratings to the strength. We recall that Junca and Jabin [58] showed that the ratings of players converge to their intrinsic strength in the case $w = 1$. This corresponds to the concentration of mass along the diagonal. In our model the intrinsic strength is continuously increasing in time. Hence, to be able to identify steady states, we consider the shifted Fokker-Planck equation (6.22a). Throughout this section we consider the problem in the whole space.

Since the diffusion part in (6.22a) is singular, the equation is degenerate parabolic. Degenerate Fokker-Planck equations frequently, despite their lack of coercivity, exhibit exponential convergence to equilibrium, a behaviour which has been referred to by Villani as hypocoercivity in [93]. For subsequent research on hypercoercity in linear Fokker-Planck equations, see [3, 1]. Since (6.22a) is a nonlinear, nonlocal Fokker-Planck equation these results do not apply here, but it is conceivable that generalisations of this approach can be used in studying the decay to equilibrium for (6.22a), which is however beyond the scope of the present paper. In the following, we present some results on the longterm behaviour of solutions to (6.22a).

Due to normalisation of the mean value, the only point in which the formation of a steady state is possible are $R_0 = 0$ and $\rho_0 = 0$. Let us assume that we have measure valued steady state in $(0,0)$, that is $g_\infty(\rho, R) = \delta(\rho)\delta(R)$. Then direct computations using the weak form of (6.22a) give

$$0 = \frac{\partial}{\partial \rho}(\phi(\rho_0, R_0))[\alpha b(0) + \beta\langle h_2\rangle(0)] + \frac{\sigma^2}{2}w(0)\frac{\partial^2}{\partial \rho^2}(\phi(\rho_0, R_0)) = \frac{\sigma^2}{2}w(0)\frac{\partial^2}{\partial \rho^2}(\phi(\rho_0, R_0)).$$

This equation is not satisfied for all test functions ϕ. Therefore, we investigate the possibility of having more complex steady states, which have a similar form as the one identified by Junca and Jabin. Let us assume that g_∞ is of the form

$$g_\infty(\rho, R) = \delta(\rho)\tilde{g}(R), \tag{6.38}$$

or alternatively

$$g_\infty(\rho, R) = \delta(R)\tilde{g}(\rho), \tag{6.39}$$

where $\tilde{g}(\cdot)$ in both cases is not a $\delta-$Dirac.
By direct computation in weak form of (6.22a) with $\phi(\rho, R) = \rho^2$ and $\phi(\rho, R) = R^2$ respectively, we

compute the following expressions for the second moments of the density function $g(\rho, R, t)$:

$$\begin{aligned}\frac{d}{dt}M_{g,2,\rho}(t) = \frac{\sigma^2}{2}\int_{\mathbb{R}^4} w(R-R_j)g(\rho,R,t)g(\rho_j,R_j,t)\,dR_j d\rho_i dR d\rho \\ - \int_{\mathbb{R}^4}(\rho_j-\rho)\big[\alpha b(\rho_j-\rho)+\beta\langle h_2(\rho_j-\rho)\rangle\big]w(R-R_j)g(\rho,R,t)g(\rho_j,R_j,t)\,dR_j d\rho_i dR d\rho,\end{aligned} \tag{6.40}$$

$$\frac{d}{dt}M_{g,2,R}(t) = \int_{\mathbb{R}^4} 2R(b(\rho-\rho_j)-b(R-R_j))w(R-R_j)g(\rho,R,t)g(\rho_j,R_j,t)\,dR_j d\rho_j dR d\rho. \tag{6.41}$$

The analysis of the second moment w.r.t. ρ leads us to conclude that the diffusion prevents the formation of a steady state as in (6.38) if $w = 1$. Indeed, in this case, the first integral in (6.40) equals σ^2. If at certain time $\bar{t} > 0$, $\rho \simeq \rho_j$ or $g(\rho, R, \bar{t}) = \delta(\rho - \rho_0)\tilde{g}(R, \bar{t})$, the integral becomes small or vanishes (anyhow smaller than σ^2) and then $\frac{d}{dt}M_{2,\rho_i}(\bar{t}) \geq 0$ .Thus, we can conclude that the diffusion prevents the accumulation of the mass in in $\rho = 0$. For a general choice of w, the long time behaviour of solutions is less clear.
Conversely, the second moment w.r.t. R is decreasing. Due to the symmetry of the functions b and w, we can rewrite (6.41) as

$$\frac{d}{dt}M_{g,2,R}(t) = -\int_{\mathbb{R}^4}(R-R_j)b(R-R_j)w(R-R_j)g(\rho,R,t)g(\rho_j,R_j,t)\,dR_j d\rho_j dR d\rho \leq 0.$$

This inequality does not contradict the assumption of a steady state of form (6.39).

In order to evaluate if, with the scaling (6.21), the rating converges to the intrinsic strength, let us define the energy

$$E_2(t) = \int_{\mathbb{R}^2}(\rho-R)^2 g(\rho,R,t)\,d\rho dR. \tag{6.42}$$

We are interested in the evolution of E_2 and compute

$$\begin{aligned}\frac{d}{dt}E_2(t) = &-2\int_{\mathbb{R}^4}(\rho-R)w(R-R_j)b(R-R_j)g(\rho,R,t)g(\rho_j,R_j,t)\,d\rho_j dR_j d\rho dR \\ &+2\int_{\mathbb{R}^4}(\rho-R)w(R-R_j)b(\rho-\rho_j)g(\rho,R,t)g(\rho_j,R_j,t)\,d\rho_j dR_j d\rho dR \\ &+2\alpha\int_{\mathbb{R}^4}(\rho-R)w(R-R_j)b(\rho-\rho_j)g(\rho,R,t)g(\rho_j,R_j,t)\,d\rho_j dR_j d\rho dR \\ &+2\beta\int_{\mathbb{R}^4}(\rho-R)w(R-R_j)\langle h_2(\rho-\rho_j)\rangle g(\rho,R,t)g(\rho_j,R_j,t)\,d\rho_j dR_j d\rho dR \\ &+\sigma^2\int_{\mathbb{R}^2} d[g]g(\rho,R,t)\,d\rho dR.\end{aligned} \tag{6.43}$$

For general functions w it is not possible to determine the signs of the respective integrals. Therefore, we consider the case $w = 1$ only. For all odd functions $b(\cdot)$ (the same holds true for $\langle h_2(\rho-\rho_j)\rangle$) we are able to show that

$$\begin{aligned}&\int_{\mathbb{R}^4}\rho b(\rho_j-\rho)g(\rho,R,t)g(\rho_j,R_j,t)\,d\rho_j dR_j d\rho dR \\ &\qquad = \frac{1}{2}\int_{\mathbb{R}^4}\rho(b(\rho_j-\rho)-b(\rho-\rho_j))g(\rho,R,t)g(\rho_j,R_j,t)\,d\rho_j dR_j d\rho dR \\ &\qquad = -\frac{1}{2}\int_{\mathbb{R}^4}(\rho_j-\rho)b(\rho_j-\rho)g(\rho,R,t)g(\rho_j,R_j,t)\,d\rho_j dR_j d\rho dR \\ &\qquad \leq 0,\end{aligned}$$

and $\int_{\Omega^2} \rho b(R-R_j)g(\rho,R,t)g(\rho_j,R_j,t)\,d\rho_j dR_j d\rho dR = 0$. In this case we can rewrite the equation (6.43) as

$$\begin{aligned}\frac{d}{dt}E_2(t) = &-\int_{\mathbb{R}^4}(R-R_j)b(R-R_j)g(\rho,R,t)g(\rho_j,R_j,t)\,d\rho_j dR_j d\rho dR\\ &-\int_{\mathbb{R}^4}(\rho-\rho_j)b(\rho-\rho_j)g(\rho,R,t)g(\rho_j,R_j,t)\,d\rho_j dR_j d\rho dR\\ &-\alpha\int_{\mathbb{R}^4}(\rho-\rho_j)b(\rho-\rho_j)g(\rho,R,t)g(\rho_j,R_j,t)\,d\rho_j dR_j d\rho dR\\ &-2\beta\int_{\mathbb{R}^4}(\rho-\rho_j)\langle h_2(\rho-\rho_j)\rangle g(\rho,R,t)g(\rho_j,R_j,t)\,d\rho_j dR_j d\rho dR\\ &+\sigma^2.\end{aligned} \tag{6.44}$$

Again we would like to know if a concentration of mass along the diagonal is possible. Let us assume that at certain time the solution is $g(\rho,R,t) = \delta(\rho-R)\tilde{g})(\rho,R,t)$. Insert this claim in (6.44), we obtain

$$\frac{d}{dt}E_2(t) = \sigma^2 > 0.$$

It shows that the diffusion counteracts the accumulation of the mass along the diagonal. On the other hand, the four integrals in (6.44) are strictly negative. Hence if σ^2 is small enough, the distance between rating and intrinsic strength becomes small, and the diffusive term can be controlled. This indicates concentration of the mass in a certain neighbourhood of the diagonal in the long run.

6.6 Numerical simulations

In this section we discuss the numerical discretisation of the Boltzmann equation (6.12) and the shifted Fokker-Planck equation (6.22a). We initialise the distribution of players with respect to their strength and rating with values from the unit interval and consider appropriately shifted interaction rules to ensure that the distribution remains inside the unit square for all times $t > 0$.

6.6.1 Monte Carlo simulations of the Boltzmann equation

We use the classical Monte Carlo method to compute a series of realisations of the Boltzmann equation (6.12). In the direct Monte Carlo method, also known as Bird's scheme, pairs of players are randomly and non-exclusively selected for two-player games. The outcome of the game is determined by (6.7). Note that we consider the following shifted interaction rules for the ratings, to ensure that $\rho \in [0,1]$ and $R \in [0,1]$:

$$\rho_i^* = \rho_i + \gamma\tilde{h}(\rho_j-\rho_i)w(R_i-R_j) + \eta \tag{6.45a}$$

$$\rho_j^* = \rho_j + \gamma\tilde{h}(\rho_i-\rho_j)w(R_i-R_j) + \tilde{\eta}, \tag{6.45b}$$

where $\tilde{h} = b(\rho_j-\rho_i)$. The microscopic interactions are simulated as follows: the outcome of the game S_{ij} is the realisation of a discrete distribution function, which takes the value $\{-1,1\}$ with probability $\{b(\rho_i-\rho_j), 1-b(\rho_i-\rho_j)\}$. The random variables η are generated such that they assume values $\eta = \pm 0.025$ with equal probability, the parameter γ is set to 0.05. Further information on Monte Carlo methods for Boltzmann type equations can be found in [76].

In each simulation we consider $N = 5000$ players and compute the steady state distribution by performing 10^8 time steps. The result is then averaged over another 10^5 time steps. We perform $M = 10$ realizations and compute the density from the averaged steady states.

6.6.2 Finite volume discretisation and simulations of the nonlinear Fokker-Planck equation

The solver for the Fokker-Planck equation is based on a Strang splitting and an upwind finite volume scheme. We recall that we discretise the shifted Fokker-Planck equation (6.22a), which allows us to perform simulations on a bounded domain. Because of the splitting we consider the interactions in the rating and the strength variable separately. We define two operators, which correspond to

($\mathcal{S}_1$): Interaction step in the strength variable R:

$$\frac{\partial g^*}{\partial t}(\rho, R, t) = -\frac{\partial}{\partial \rho}(c[\tilde{g}]g^*(\rho, R, t)) + \frac{\sigma^2}{2} d[\tilde{g}] \frac{\partial^2}{\partial \rho^2}(g^*(\rho, R, t))$$

subject to the initial condition $g^*(\rho, R, t) = \tilde{g}(\rho, R, t)$. Note that we compute the interaction integrals using $\tilde{g}$, which corresponds to the solution at the previous time step in the full splitting scheme.

($\mathcal{S}_2$): Interaction step in the rating variable ρ:

$$\frac{\partial g^\diamond}{\partial t}(\rho, R, t) = -\frac{\partial}{\partial R}(a[g^*]g^\diamond(\rho, R, t))$$

We approximate all integrals, which appear in the interaction coefficients using the trapezoidal rule.

Let $\hat{g}^k$ denote the solution at time $t^k = k\Delta t$, where Δt corresponds to the time step size. Then the Strang splitting results in the scheme

$$\hat{g}^{k+1}(\rho, R) = \mathcal{S}_2\Big(\hat{g}^{*,k+1}, \frac{\Delta t}{2}\Big) \circ \mathcal{S}_1\Big(\hat{g}^{\diamond,k+\frac{1}{2}}, \Delta t\Big) \circ \mathcal{S}_2\Big(\hat{g}^k, \frac{\Delta t}{2}\Big),$$

where the superscripts denote the solutions of g^* and $g^\diamond$ at the discrete time steps $t^{k+1} = (k+1)\Delta t$ and $t^{k+\frac{1}{2}} = (k+\frac{1}{2})\Delta t$. We use a conservative upwind finite volume discretisation to discretise the respective operators. The corresponding explicit-in-time upwind finite volume methods is given by

$$\hat{g}_j^{n+1} = \hat{g}_j^n + \lambda_1(\hat{c}_{j+\frac{1}{2}} - \hat{c}_{j-\frac{1}{2}}) + \lambda_2(\hat{d}_{j+\frac{1}{2}} - \hat{d}_{j-\frac{1}{2}}),$$

where $\hat{c}$ is the upwind flux and the diffusive flux is given $\hat{d}_{j+\frac{1}{2}} = D(\hat{g}_{j+1})\hat{g}_{j+1} - D(\hat{g}_j)\hat{g}_j$. Here $\lambda_1 = \Delta t/\Delta x$ and $\lambda_2 = \Delta t/\Delta x^2$.

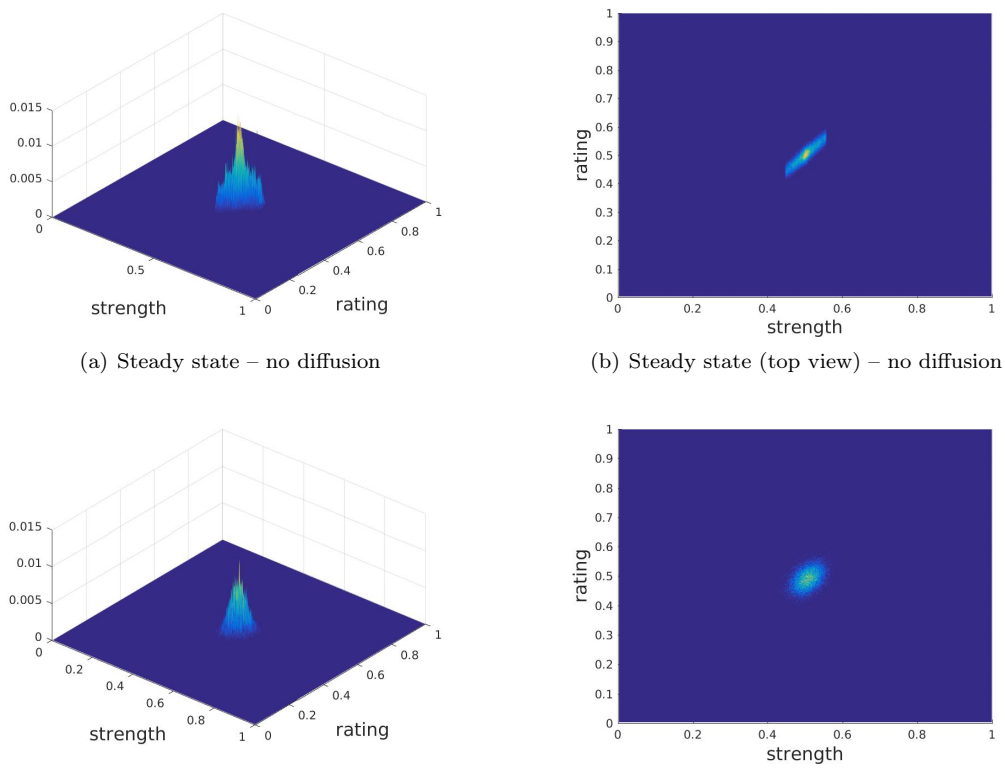
6.6.3 Computational experiments

All micro- and macroscopic simulations are performed on the domain $[0,1] \times [0,1]$ with no-flux boundary conditions. In the case of a general interaction function, the interaction rate function $w(r_i - r_j)$ is a piecewise constant function given by

$$w(z) = \begin{cases} 1 & \text{if } |z| \leq 0.1 \\ 0 & \text{otherwise.} \end{cases}$$

All-play-all tournaments:

We start by investigating the long time behaviour of the Elo model with $w = 1$, $\alpha = 0.1$ and $\beta = 0$ in (6.8). Hence players have the same probability to play against each another independent of their respective ratings. We have seen in Section 6.5 that we expect a measure valued solution in the case of no diffusion. However, we can not show convergence of solutions to a measure valued steady state if stochastic fluctuations influence the intrinsic strength. In the following we compare computed steady states of the Boltzmann as well as the Fokker-Planck equation in the case of diffusion and no diffusion. We start with a uniform distribution of agents in the micro- as well as the macroscopic situation. Figure 6.2 as well as Figure 6.3 confirm the expected formation of a Delta Dirac at the center of mass in the case of no diffusion. If the individual strength is also influenced by stochastic fluctuations, the steady state is smoothed out with respect to the rating as well. The resulting steady states are Gaussian like profiles in the micro- as well as macroscopic simulations, see Figures 6.2 and 6.3. Figure 6.3 also shows the decay of the energy E_2 in time.



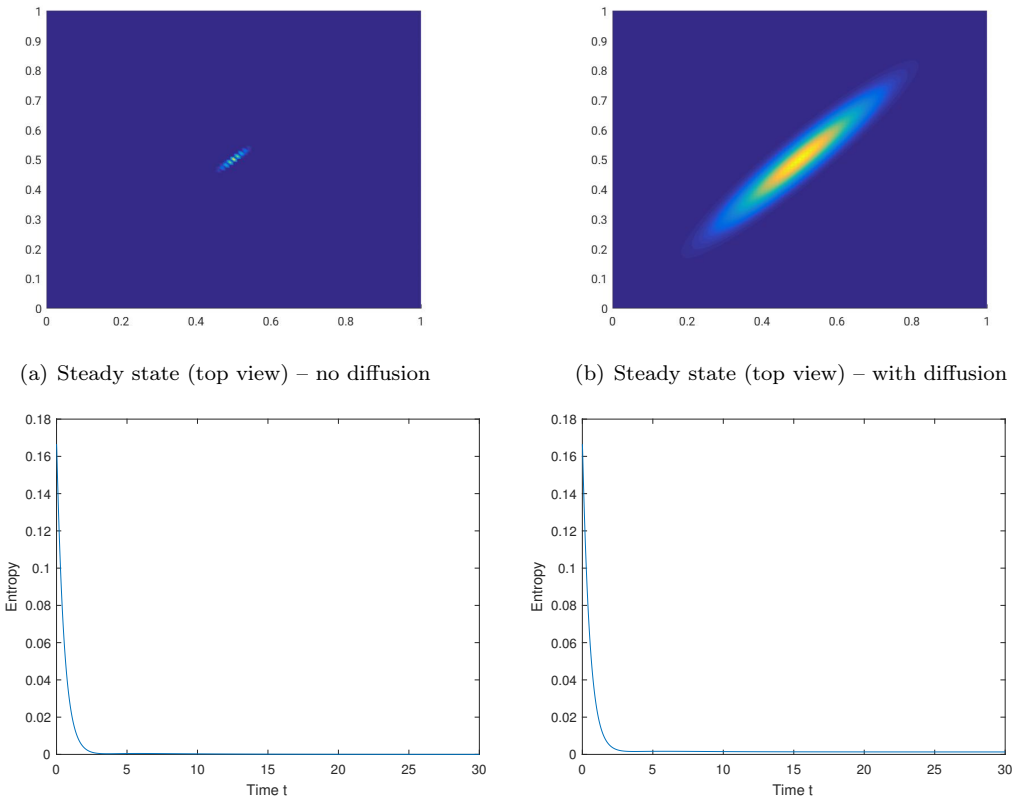
(a) Steady state – no diffusion

(b) Steady state (top view) – no diffusion

(c) Steady state – diffusion $\nu = 0.025$

(d) Steady state (top view) – diffusion $\nu = 0.025$

Figure 6.2: Computational steady state of the Boltzmann model for $w = 1$ in the case of no diffusion, $\eta = \bar{\eta} = 0$, and small diffusion in the strength $\eta = \bar{\eta} = 0.025$.



(a) Steady state (top view) – no diffusion

(b) Steady state (top view) – with diffusion

(c) Energy decay in the case of no diffusion

(d) Energy decay in the case of diffusion

Figure 6.3: Computational steady state of the Fokker-Planck model and energy decay for $w = 1$ in the case of no and little diffusion strength.

Swiss-system tournaments

Assigning initial ratings to players in the Elo rating is a delicate issue, since inaccurate initial ratings may influence the ability of the rating to converge to a 'good' rating of players reflecting their intrinsic strengths. We illustrate this by studying the following situation of a Swiss-system tournament.

In a Swiss-system tournament players are paired using a set of rules, which ensure that only players with a similar rating compete. We set the interaction rate function to $w(z) = \chi_{\{|z|\leq 0.1\}}$ – hence individuals only play against each other, if the difference between their ratings is small.

We consider two groups of players with different strength and rating levels as initial distribution. The first group is underrated, that is all players have rating $R = 0.2$ but their strength is distributed as $\rho \in \mathcal{N}(0.75, 0.1)$. The second group is overrated, with rating $R = 0.9$ and a uniform distribution in strength. We use this initial configuration in two computational experiments.

In the first, we choose the learning parameters $\alpha = 0.1$ and $\beta = 0$. We see that the two groups remain separated due to their different ratings in this case, see Figure 6.4. However, players compete within their own group and since $\beta = 0$ the overall rating improves. In the overrated group the strongest players accumulate at the highest possible rating, while the underrated group forms a diagonal pattern.

Here the underrated players evolve to the maximum possible rating level.

In the second experiment, using the same initial configuration, but $\alpha = 0.1$ and $\beta = 0.05$ the steady state profile looks totally different. In this setting stronger players loose strength, when loosing against a weaker opponent. Therefore, the ratings of the overrated group decrease, while the ratings of the underrated group increases. After a while the two groups merge, accumulating on a diagonal which underestimates the intrinsic strength of players by approximately 0.1, see Figure 6.5.

These examples show the importance of the initial ratings as well as the influence of the adapted learning mechanism.

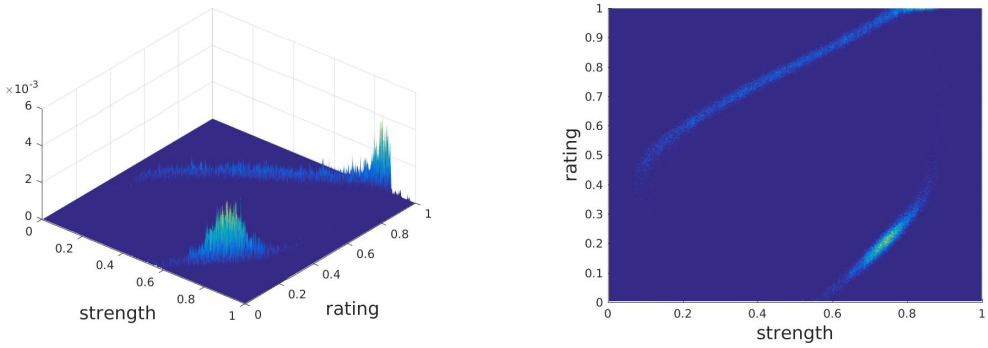


Figure 6.4: Computed stationary profiles in a Swiss-system tournament in case of two initially separated groups (one underrated with high strength but low rating and one overrated with variable strength but rating 0.9). Due to the limited interaction between the groups and the chosen learning mechanism, they remain separated.

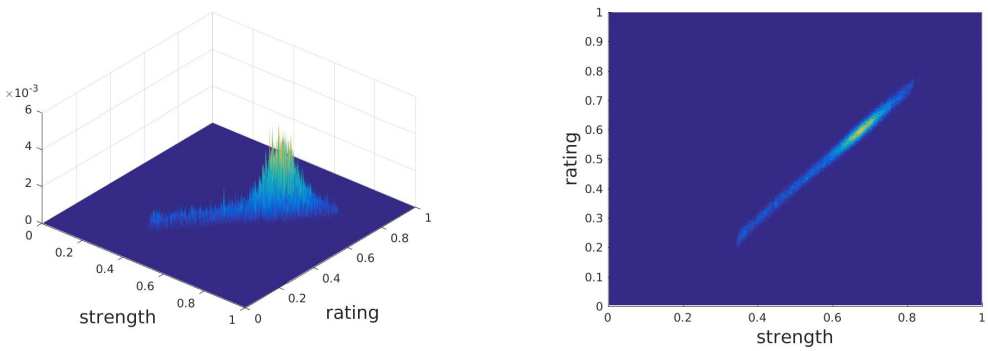


Figure 6.5: Computed stationary profiles in a Swiss-system tournament in case of two initially separated groups (one underrated with high strength but low rating and one overrated with variable strength but rating 0.9). Despite the limited interaction between the groups, the adapted learning mechanism leads to convergence of the ratings to a slightly shifted diagonal.

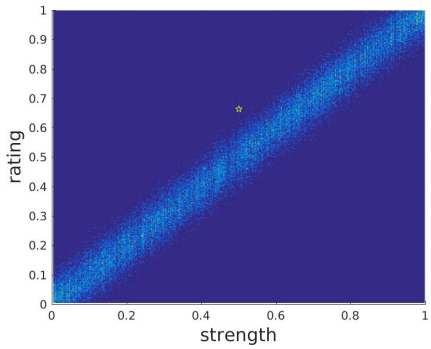


Figure 6.6: Computed stationary profile in a foul play where the first player has an unfair advantage in each game. We observe that the ratings and strength all players except the first one converge. The cheating player (indicated by a star) ends up with a higher rating than it is supposed to have.

Foul play

Finally, we consider a series of games, in which one player, without loss of generality the first one, is playing unfairly, e.g. through cheating, doping or bribing of referees. This means that the outcome of every game which involves this player is biased in his/her favor. In particular we assume that the probability of winning is increased by a factor $\tilde{b}$ for player 1 and decreased by $\tilde{b}$ for the other contestant. Figure 6.6 shows the stationary profile in the case of a uniform initial distribution of agents, $\alpha = 0.1$, $\beta = 0$, $w = 1$ and $\tilde{b} = 0.2$. The star indicates the position of the unfair first player. While the distribution of players with respect to their ratings and their strengths accumulates along the diagonal, we see that the first player is rated higher than implied by their strength.

Chapter 7

Appendix

Derivation of the Fokker-Planck equation

In this section we derive the limiting Fokker-Planck equation in the case $\gamma \to 0$, $\sigma_\eta \to 0$ such that $\sigma_\eta^2/\gamma =: \sigma^2$ is kept fixed. Based on the interaction rules (6.7), which define the outcome of a game, we compute the expected values of the following quantities:

$$\begin{aligned}
\langle (R_i^* - R_i)\rangle &= \gamma(b(\rho_i - \rho_j) - b(R_i - R_j)) \\
\langle ((R_i^* - R_i)^2)\rangle &= \gamma^2(b(\rho_i - \rho_j) - b(R_i - R_j))^2; \\
\langle (\rho_i^* - \rho_i)\rangle &= \gamma(\alpha h_1(\rho_j - \rho_i) + \beta\langle h_2(\rho_j - \rho_i)\rangle) \\
\langle (\rho_i^* - \rho_i)^2\rangle &= \gamma^2\left(\alpha h_1(\rho_j - \rho_i) + \beta\langle h_2(\rho_j - \rho_i)\rangle\right)^2 + \sigma_\eta^2 \\
\langle (\rho_i^* - \rho_i)(R_i^* - R_i)\rangle &= \gamma^2(\alpha h_1(\rho_j - \rho_i) + \beta\langle h_2(\rho_j - \rho_i)\rangle)(b(\rho_i - \rho_j) - b(R_i - R_j)).
\end{aligned}$$

Using Taylor expansion of $\phi(\rho_i^*, R_i^*)$ up to order two around (ρ_i, R_i), we obtain

$$\begin{aligned}
&\langle \phi(\rho_i^*, R_i^*) - \phi(\rho_i, R_i)\rangle \\
&= \langle R_i^* - R_i\rangle \frac{\partial}{\partial R_i}\phi(\rho_i, R_i) + \langle \rho_i^* - \rho_i\rangle \frac{\partial}{\partial \rho_i}\phi(\rho_i, R_j) \\
&+ \frac{1}{2}\left[\langle (R_i^* - R_i)^2\rangle \frac{\partial^2}{\partial R_i^2}\phi(\rho_i, R_i) + \langle (\rho_i^* - \rho_i)^2\rangle \frac{\partial^2}{\partial \rho_i^2}\phi(\rho_i, R_i) + 2\langle (\rho_i^* - \rho_i)(R_i^* - R_i)\rangle \frac{\partial^2}{\partial \rho_i \partial R_i}\phi(\rho_i, R_i)\right] \\
&+ \mathcal{R}_\gamma(\phi, \rho_i^*, R_i^*, \rho_i, R_i, \tau),
\end{aligned}$$

where the remainder term $\mathcal{R}_\gamma$ is given by

$$\mathcal{R}_\gamma = \begin{pmatrix} \rho_i^* - \rho_i \\ R_i^* - R_i \end{pmatrix}^T \begin{pmatrix} \frac{\partial^2}{\partial \rho_i^2}\phi(\overline{\rho_i}, \overline{R_i}) - \frac{\partial^2}{\partial \rho_i^2}\phi(\rho_i, R_i) & \frac{\partial^2}{\partial \rho_i \partial R_i}\phi(\overline{\rho_i}, \overline{R_i}) - \frac{\partial^2}{\partial \rho_i \partial R_i}\phi(\rho_i, R_i) \\ \frac{\partial^2}{\partial \rho_i \partial R_i}\phi(\overline{\rho_i}, \overline{R_i}) - \frac{\partial^2}{\partial \rho_i \partial R_i}\phi(\rho_i, R_i) & \frac{\partial^2}{\partial R_i^2}\phi(\overline{\rho_i}, \overline{R_i}) - \frac{\partial^2}{\partial R_i^2}\phi(\rho_i, R_i) \end{pmatrix} \begin{pmatrix} \rho_i^* - \rho_i \\ R_i^* - R_i \end{pmatrix},$$

for some $0 \le \theta_1, \theta_2 \le 1$ with $\overline{\rho_i}$ and $\overline{R_i}$ defined as

$$\overline{\rho_i} = \theta_1 \rho_i + (1 - \theta_1)\rho_i^* \text{ and } \overline{R_i} = \theta_2 R_i + (1 - \theta_2)R_i^*.$$

Next we rescale time as $\tau = \gamma t$ and insert the expansion in (6.12). This yields

$$\begin{aligned}
&\frac{d}{d\tau}\int_{\mathbb{R}^2}\phi(\rho_i,R_j)f_\varepsilon(\rho_i,R_i,\tau)dR_id\rho_i = \frac{1}{2\gamma}\int_{\mathbb{R}^2}\tilde{\mathcal{R}}_\gamma(\phi,\rho_i^*,R_i^*,\rho_i,R_i,\tau)f_\varepsilon(\rho_i,R_i,\tau)dR_id\rho_i\\
&+\int_{\mathbb{R}^4}\Bigg[\frac{\partial}{\partial R_i}\phi(\rho_i,R_j)(b(\rho_i-\rho_j)-b(R_i-R_j))+\frac{\partial}{\partial\rho_i}\phi(\rho_i,R_j)(\alpha h_1(\rho_j-\rho_i)+\beta\langle h_2(\rho_j-\rho_i)\rangle)\\
&\qquad+\frac{\sigma_\eta^2}{2\gamma}\frac{\partial^2}{\partial\rho_i^2}\phi(\rho_i,R_j)\Bigg]w(R_i-R_j)f_\varepsilon(\rho_i,R_i,\tau)f_\varepsilon(\rho_j,R_j,\tau)dR_jd\rho_jdR_id\rho_i,
\end{aligned}\tag{7.1}$$

where

$$\begin{aligned}
\tilde{\mathcal{R}}_\gamma(\phi,\rho_i^*,R_i^*,\rho_i,R_i,\tau) &= \gamma^2\int_{\mathbb{R}^2}\frac{\partial^2}{\partial R_i^2}\phi(\rho_i,R_i)(b(\rho_i-\rho_j)-b(R_i-R_j))^2w(R_i-R_j)f_\varepsilon(\rho_j,R_j,\tau)dR_jd\rho_j\\
&+\gamma^2\int_{\mathbb{R}^2}\frac{\partial^2}{\partial\rho_i^2}\phi(\rho_i,R_i)\big(\alpha h_1(\rho_j-\rho_i)+\beta\langle h_2(\rho_j-\rho_i)\rangle\big)^2w(R_i-R_j)f_\varepsilon(\rho_j,R_j,\tau)dR_jd\rho_j\\
&+2\gamma^2\int_{\mathbb{R}^2}\frac{\partial}{\partial\rho_i\partial R_i}\phi(\rho_i,R_i)(b(\rho_i-\rho_j)-b(R_i-R_j))\big(\alpha h_1(\rho_j-\rho_i)+\beta\langle h_2(\rho_j-\rho_i)\rangle\big)w(R_i-R_j)f_\varepsilon(\rho_j,R_j,\tau)dR_jd\\
&+\int_{\mathbb{R}^2}R_\gamma w(R_i-R_j)f_\varepsilon(\rho_j,R_j,\tau)dR_jd\rho_j.
\end{aligned}$$

Next we show that the remainder $\frac{1}{2\gamma}\int_{\mathbb{R}^2}\tilde{\mathcal{R}}_\gamma(\phi,\rho_i^*,R_i^*,\rho_i,R_i,\tau)f_\varepsilon(\rho_i,R_i,\tau)dR_id\rho_i$ vanishes for $\gamma\to 0$. Let us assume that $\phi(\rho_i,R_i)$ belongs to the space $\mathcal{C}_{2+\delta}(\mathbb{R}^2)=\{h:\mathbb{R}^2\to\mathbb{R},\ \|D^\zeta h\|_\delta<+\infty\}$, where $0<\delta\le 1$, ζ is a multi-index with $|\zeta|\le 2$ and the seminorm $\|\cdot\|_\delta$ is the usual Hölder seminorm

$$\|f\|_\delta=\sup_{x,y\in\mathbb{R}^2}\frac{|f(x)-f(y)|}{|x-y|^\delta}.$$

With this choice of $\phi(\rho_i,R_i)$, all the terms wich contain $\frac{\partial^2}{\partial\rho_i^2}\phi$ and $\frac{\partial^2}{\partial R_i^2}\phi$ vanish using the same arguments as in [88, 31]. Hence, we focus on the mixed derivative $\frac{\partial^2}{\partial\rho_i\partial R_i}\phi(\rho_i,R_i)$. Since $\phi(\rho_i,R_i)\in\mathcal{C}_{2+\delta}(\mathbb{R}^2)$ and $\|(\overline{\rho_i},\overline{R_i})-(\rho_i,R_i)\|\le\|(\rho_i^*,R_i^*)-(\rho_i,R_i)\|$, we have

$$\left|\frac{\partial^2}{\partial\rho_i\partial R_i}\phi(\overline{\rho_i},\overline{R_i})-\frac{\partial^2}{\partial\rho_i\partial R_i}\phi(\rho_i,R_i)\right|\le\|\phi\|_{2+\delta}\|(\rho_i^*,R_i^*)-(\rho_i,R_i)\|^\delta.$$

Furthermore, due to (6.2), (6.9) and (6.10),

$$\|(\rho_i^*,R_i^*)-(\rho_i,R_i)\|=\Big[\gamma^2\left(\alpha h_1(\rho_j-\rho_i)+\beta\langle h_2(\rho_j-\rho_i)\rangle\right)^2+\gamma^2\left(b(\rho_i-\rho_j)-b(R_i-R_j)\right)^2\Big]^{\frac{1}{2}}\le C\gamma.$$

Using the previous inequalities we estimate the mixed term as

$$\begin{aligned}
&\frac{1}{2\gamma}\left|\int_{\mathbb{R}^4}\left(\frac{\partial^2\phi(\overline{\rho_i},\overline{R_i})}{\partial\rho_i\partial R_i}-\frac{\partial^2\phi(\rho_i,R_i)}{\partial\rho_i\partial R_i}\right)\left\|\begin{pmatrix}\overline{\rho_i}\\ \overline{R_i}\end{pmatrix}-\begin{pmatrix}\rho_i\\ R_i\end{pmatrix}\right\|^2 w(R_i-R_j)f_\varepsilon(\rho_j,R_j,\tau)f_\varepsilon(\rho_i,R_i,\tau)\,dR_id\rho_idR_jd\rho_j\right|\\
&\qquad\le\frac{1}{2\gamma}\int_{\mathbb{R}^4}\|\phi\|_{2+\delta}\|(\rho_i^*,R_i^*)-(\rho_i,R_i)\|^\delta\|(\rho_i^*,R_i^*)-(\rho_i,R_i)\|^2f_\varepsilon(\rho_j,R_j,\tau)f_\varepsilon(\rho_i,R_i,\tau)\,dR_id\rho_idR_jd\rho_j\\
&\qquad\le\frac{1}{2\gamma}\int_{\mathbb{R}^4}C^\delta\|\phi\|_{2+\delta}\gamma^{2+\delta}f_\varepsilon(\rho_j,R_j,\tau)f_\varepsilon(\rho_i,R_i,\tau)\,dR_id\rho_idR_jd\rho_j\le\frac{C^\delta}{2}\|\phi\|_{2+\delta}\gamma^{1+\delta}.
\end{aligned}$$

Hence the remainder term converges to 0 as $\gamma \to 0$. Therefore, the density $f_\varepsilon(\rho_i, R_i, \tau)$ converges to $f(\rho_i, R_i, \tau)$ which solves

$$\begin{aligned}
&\frac{d}{d\tau}\int_{\mathbb{R}^2} \phi(\rho_i, R_j) f(\rho_i, R_i, \tau) dR_i d\rho_i = \\
&\quad \int_{\mathbb{R}^2} f(\rho_i, R_i, \tau)\Bigg\{ \frac{\partial}{\partial R_i}\phi(\rho_i, R_j)\Bigg[\int_{\mathbb{R}^2} w(R_i - R_j)(b(\rho_i - \rho_j) - b(R_i - R_j)) f(\rho_j, R_j, \tau) d\rho_j dR_j\Bigg] \\
&\quad + \frac{\partial}{\partial \rho_i}\phi(\rho_i, R_j)\Bigg[\int_{\mathbb{R}^2} w(R_i - R_j)\big(\alpha h_1(\rho_j - \rho_i) + \beta\langle h_2(\rho_j - \rho_i)\rangle\big) f(\rho_j, R_j, \tau) d\rho_j dR_j\Bigg] \\
&\quad + \frac{\sigma^2}{2}\frac{\partial^2}{\partial \rho_i^2}\phi(\rho_i, R_j)\Bigg[\int_{\mathbb{R}^2} w(R_i - R_j) f(\rho_j, R_j, \tau) d\rho_j dR_j\Bigg]\Bigg\} dR_i d\rho_i. \qquad (7.2)
\end{aligned}$$

It remains to show that under suitable boundary conditions equation (7.2) gives the desired weak formulation of the Fokker Planck equation. We split the boundary terms BT into the different parts BT_i, $i = 1, 2, 3$ that arises respectively from each integral. They are given by

$$\begin{aligned}
B_1 &= \int_{\mathbb{R}} \Bigg[f(\rho_i, R_i, \tau)\phi(\rho_i, R_i)\Bigg(\int_{\mathbb{R}^2} w(R_i - R_j)(b(\rho_i - \rho_j) - b(R_i - R_j)) f(\rho_j, R_j, \tau) dR_j d\rho_j\Bigg)\Bigg]_{R_i=-\infty}^{R_i=+\infty} d\rho_i \\
B_2 &= \int_{\mathbb{R}} \Bigg[f(\rho_i, R_i, \tau)\phi(\rho_i, R_i)\Bigg(\int_{\mathbb{R}^2} w(R_i - R_j)(\alpha h_1(\rho_j - \rho_i) + \beta\langle h_2(\rho_j - \rho_i)\rangle) f(\rho_j, R_j, \tau) dR_j d\rho_j\Bigg)\Bigg]_{\rho_i=-\infty}^{\rho_i=+\infty} dR_i \\
B_3 &= \frac{\sigma^2}{2}\int_{\mathbb{R}} \Bigg[\frac{\partial}{\partial \rho_i}\phi(\rho_i, R_i) f(\rho_i, R_i, \tau)\Bigg(\int_{\mathbb{R}^2} w(R_i - R_j) f(\rho_j, R_j, \tau) d\rho_j dR_j\Bigg) \\
&\quad - \phi(\rho_i, R_i)\frac{\partial}{\partial \rho_i}\Bigg[f(\rho_i, R_i, \tau)\Bigg(\int_{\mathbb{R}^2} w(R_i - R_j) f(\rho_j, R_j, \tau) d\rho_j dR_j\Bigg)\Bigg]\Bigg]_{\rho_i=-\infty}^{\rho_i=+\infty} dR_i.
\end{aligned}$$

These three terms are zero, if the following boundary conditions are satisfied:

$$\lim_{|R_i|\to+\infty} f(\rho_i, R_i, \tau) = 0, \quad \lim_{|\rho_i|\to+\infty} f(\rho_i, R_i, \tau) = 0, \quad \lim_{|\rho_i|\to+\infty} \frac{\partial}{\partial \rho_i} f(\rho_i, R_i, \tau) = 0. \qquad (7.3)$$

These boundary condition are guaranteed for the Boltzmann equation $f_\varepsilon(\rho_i, R_i, \tau)$ by mass conservation and the upper and lower bounds on the mean, see (6.15). Therefore, (7.2) is the weak form of the Fokker-Planck equation

$$\begin{aligned}
&\frac{d}{d\tau}\int_{\mathbb{R}^2} \phi(\rho_i, R_i) f(\rho_i, R_i, \tau) dR_i d\rho_i = \\
&\quad \int_{\mathbb{R}^2} \phi(\rho_i, R_i)\Bigg\{ -\frac{\partial}{\partial R_i}\Bigg[f(\rho_i, R_i, \tau)\int_{\mathbb{R}^2} w(R_i - R_j)(b(\rho_i - \rho_j) - b(R_i - R_j)) f(\rho_j, R_j) d\rho_j dR_j\Bigg] \\
&\quad - \frac{\partial}{\partial \rho_i}\Bigg[f(\rho_i, R_i, \tau)\int_{\mathbb{R}^2} w(R_i - R_j)\big(\alpha h_1(\rho_j - \rho_i) + \beta\langle h_2(\rho_j - \rho_i)\rangle\big) f(\rho_j, R_j) d\rho_j dR_j\Bigg] \\
&\quad + \frac{\sigma^2}{2}\Bigg[\int_{\mathbb{R}^2} w(R_i - R_j) f(\rho_j, R_j, \tau) d\rho_j dR_j\Bigg]\frac{\partial^2}{\partial \rho_i^2}\Big[f(\rho_i, R_i, \tau)\Big]\Bigg\} d\rho_i dR_i. \qquad (7.4)
\end{aligned}$$

Some tools on Parabolic PDE

In this Section there are some results on existence and regularity of solution of Parabolic PDE, in particular some appplications of Galerkin method that are used in chapter 6.4.

Let "$V \subset H \subset V^*$" be an evolution triple. The model of parabolic PDE is

$$\frac{\partial}{\partial t}(u(t), v)_H + a(u(t), v) = \langle b(t), v\rangle_V, \tag{7.5a}$$
$$u(0) = u_0 \in H, \tag{7.5b}$$
$$u \in W_2^1(0, T; V, H), \tag{7.5c}$$

where (7.5a) is valid for all $v \in V$ and almost all $t \in (0, T)$. Observe that the map a does not depend on time.

Theorem 30. *[94] We make the following assumptions:*

(H1) $V \subseteq H \subseteq V'$ *is an evolution triple with* $dimV = \infty$, $0 < T < \infty$. *The spaces* V *and* H *are real H-spaces.*

(H2) The mapping $a : V \times V \to \mathbb{R}$ *is bilinear, bounded, and strongly positive. Moreover. we are given* $u_0 \in H$ *and* $b \in L^2(0, T; V^*)$.

(H3) $\{w_1, w_2, \dots\}$ *is a basis in* V, *and* (u_{n0}) *is a sequence from* H *with*

$$u_{n,0} \to u_0 \text{ in } H \text{ as } n \to +\infty$$

where

$$u_{n,0} \in span\{w_1, \dots, w_n\} \text{ for all } n.$$

If the assumptions $(H1), (H2), (H3)$ *hold, then,*

(a) Existence and uniqueness. The equation (7.5a)-(7.5c) *has exactly one solution* u.

(b) Continuous dependence on the data. The map

$$(u_0, b) \mapsto u$$

is linear and continuous from $H \times L^2(0, T; V^*)$ *to* $W_2^l(0, T; V, H)$, *i.e., there is a constant* $D > 0$ *such that*

$$\|u\|_{W_2^l} \leq D(\|u_0\|_H + \|b\|_{L^2(0,T;V^*)}),$$

for all $u_0 \in H$ *and* $b \in L^2(0, T; V^*)$.

(c) Convergence of the Galerkin method. For all $n = 1, 2, \dots$ *the Galerkin equation has exactly one solution* $u_n \in W_2^l(0, T; V, H)$. *The sequence* (u_n) *converges as* $n \to +\infty$ *to the solution of [?]-[?] in the following sense*

$$u_n \to u \text{ in } L^2(0, T; V) \tag{7.6a}$$
$$max_{0 \leq t \leq T}\|u_n(t) - u(t)\|_H \to 0. \tag{7.6b}$$

The equation (7.5a)-(7.5c) *is equivalent to the following operator equation [94]*

$$u'(t) + Au(t) = b(t) \text{ for almost all } t \in (0, T) \tag{7.7a}$$
$$u(0) = u_0 \in H. \tag{7.7b}$$

Here, the operator $A : V \to V^*$ is linear, continuous, and strongly monotone with

$$\langle Au, v\rangle_V = a(u, v) \text{ for all } u, v \in V.$$

The theorem 30 is true for a problem with A that does not depend on time. The generalization is the following problem 7.8a. We now replace the orignal problem with

$$\frac{\partial}{\partial t}(u(t), v)_H + a(u(t), v, t) = \langle b(t), v\rangle_V, \rangle_V, \tag{7.8a}$$

$$u(0) = u_0, \tag{7.8b}$$

$$u \in W_2^1(0, T; V, H), \tag{7.8c}$$

where (7.8a) *is valid for all $v \in V$ and almost all $t \in (0, T)$. Observe that the operator a depends on time.*

Corollary 31. *([94]) Suppose that $(H1), (H2^*), (H3)$ hold true, where*

(H2) For all $t \in]0, T[$, the mapping*

$$a(\cdot, \cdot, t) : V \times V \to \mathbb{R}$$

is bilinear, bounded and satisfies the abstract Gardin inequality, where the constant are independent of time t. That is, there exist constant $C, c > 0$ and $d \geq 0$, such that

$$|a(u, v, t)| \leq C\|u\|_V \|v\|_V,$$

$$|a(u, u)| \geq c\|u\|_V^2 - d\|u\|_H^2,$$

for all $u, v \in V$ and $t \in]0, T[$.
Moreover, the function

$$t \mapsto a(u, v, t)$$

is measurable on $]0, T[$ for all $u, v \in V$.
We are give $u_0 \in H$ and $b \in L^2(0, T; V')$. Then all the assertions of Theorem 30 are also valide for equation (7.8a)-(7.8c).
Moreover (7.8a)-(7.8c) *is equivalent to the following operator equation*

$$u'(t) + A(t)u(t) = b(t) \text{ for almost all } t \in (0, T) \tag{7.9a}$$

$$u(0) = u_0 \in H, \tag{7.9b}$$

where the operator $A(t) : V \to V^*$ is defined by

$$\langle A(t)u, v\rangle_V = a(u, v; t) \text{ for all } u, v \in V.$$

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